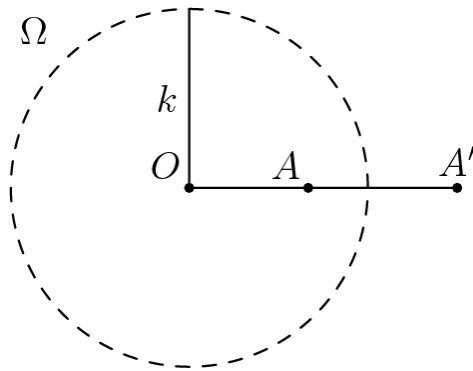# Circular Inversion

Circular Inversion, sometimes called **Geometric Inversion** or simply **Inversion**, is a transformation where point A in the Cartesian plane is transformed based on a circle Ω with radius k and center O such that $\overline{OA} \cdot \overline{OA'} = k^2$, where A' is the transformed point on the ray extending from O through A.

Note that A', when inverted, transforms back to A. All points outside of Ω are transformed inside Ω, and vice versa. Points on Ω transform to themselves, meaning $A = A'$. Finally, the transformation of O is debated on its existence. Some call the transformation the *ideal point*, which is infinitely far away and in every direction. Others claim that this point does not have an inverse.
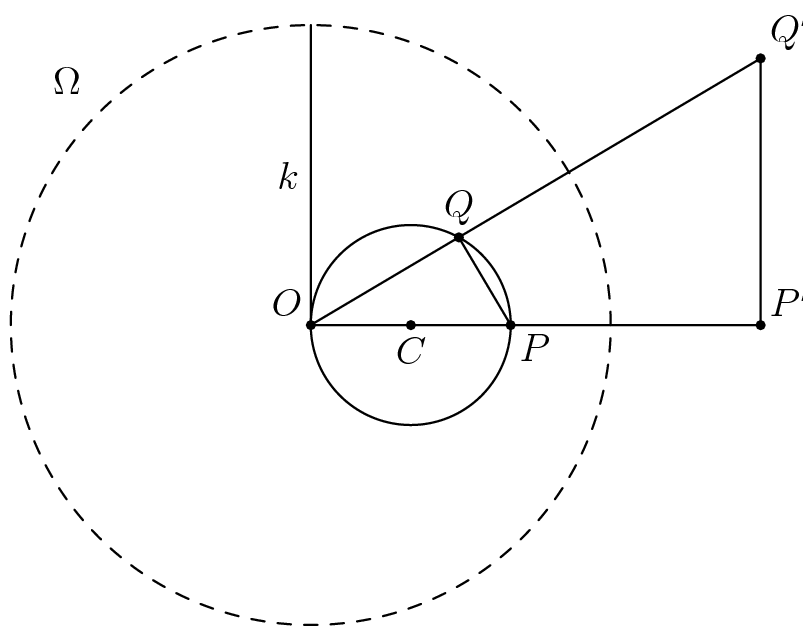
Geometric Inversion technically refers to many different types of inversions, however, if Geometric Inversion is used without clarification, Circular Inversion is usually assumed.

Circular Inversion can be a very useful tool in solving problems involving many tangent circles and/or lines.

Basics of Circular Inversion

1. Inversion of a Circle intersecting O

The first thing that we must learn about inversion is what happens when a circle which intersects the center of the inversion, O, is inverted. Let us have circle C, with diameter $\overline{OP}$. Q is chosen arbitrarily on circle C. Points P' and Q' represent the inversions of P and Q, respectively. k is the radius of Ω. We seek to show that circle C inverts to a line perpendicular to $\overline{OP}$ through P'.

By the definition of inversion, we have $\overline{OP} \cdot \overline{OP'} = k^2$ and $\overline{OQ} \cdot \overline{OQ'} = k^2$.

We can combine the two equations to get $\overline{OP} \cdot \overline{OP'} = \overline{OQ} \cdot \overline{OQ'}$. Rewriting this gives:

$$\frac{\overline{OP}}{\overline{OQ}} = \frac{\overline{OQ'}}{\overline{OP'}}.$$

Also, since $\overline{OP}$ is a diameter of circle C, $\angle OQP$ must be right.

Now, we consider $\triangle OQP$ and $\triangle OP'Q'$. They share an angle - $\angle QOP$, and we know that $\frac{\overline{OP}}{\overline{OQ}} = \frac{\overline{OQ'}}{\overline{OP'}}$. Therefore, we have SAS similarity. Therefore, $\angle OP'Q'$ must be right. From there, it follows that all points on circle C will be inverted onto the line perpendicular to $\overline{OP}$ at P'.

Therefore, the inversion of circle C becomes a line.

Note that, if circle C extends beyond Ω, the argument still holds. All one needs to do is shuffle things around.

2. Inversion of a Circle not intersecting O
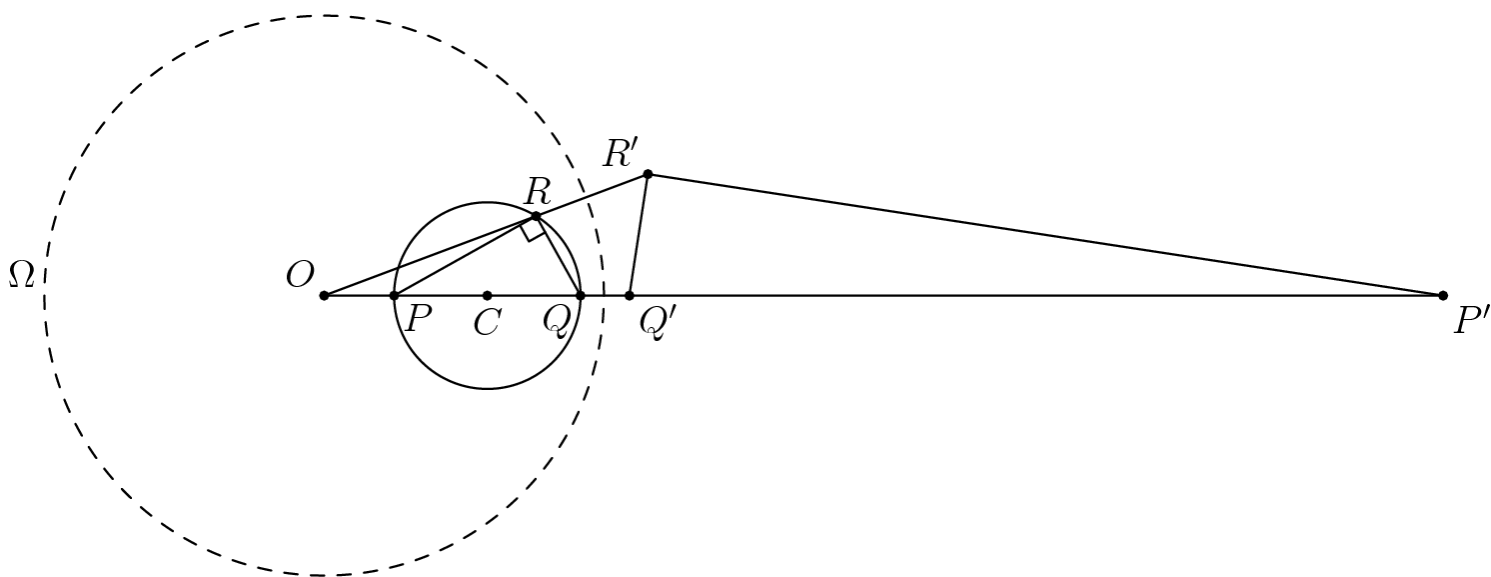


Now, we study the inversion of a circle not intersecting the center of inversion. Let us have circle C not intersecting O, the center of Ω, the circle which we invert around. The points where $\overline{OC}$ intersect circle C are points P and Q, respectively. Point R is arbitrary and on circle C. We invert points P, Q, and R, producing P', Q', and R', respectively. We draw $\triangle PRQ$ and $\triangle OR'P'$. Because $\overline{PQ}$ is a diameter, $\angle PRQ$ must be right. We wish to show that circle C inverts to another circle.

The definition of inversion tells us that $\overline{OP} \cdot \overline{OP'} = \overline{OQ} \cdot \overline{OQ'} = \overline{OR} \cdot \overline{OR'} = k^2$. From here, we obtain that $\dfrac{\overline{OP}}{\overline{OR}} = \dfrac{\overline{OR'}}{\overline{OP'}}$ and $\dfrac{\overline{OQ}}{\overline{OR}} = \dfrac{\overline{OR'}}{\overline{OQ'}}$. By SAS symmetry (exploiting $\angle ROQ$), the ratios tell us that:

$$\triangle ORP \sim \triangle OP'R', \triangle ORQ \sim \triangle OQ'R'.$$

Therefore, we have $\angle OPR = \angle OR'P'$ and $\angle OQR = \angle OR'Q'$. Note that $\angle OPR = \angle PRQ + \angle PQR = 90° + \angle PQR$, which must equal $\angle OR'P'$. Therefore, $\angle Q'R'P' = \angle OR'P' - \angle OR'Q' = 90° + \angle PQR - \angle OQR$. But $\angle OQR = \angle PQR$. Therefore, $\angle Q'R'P' = 90°$. As this holds for any R, all points on circle C will invert to a point on a circle with diameter $\overline{Q'P'}$.

3. General Formula for the Radius of a Circle in Terms of the Radius of its Inverse Circle
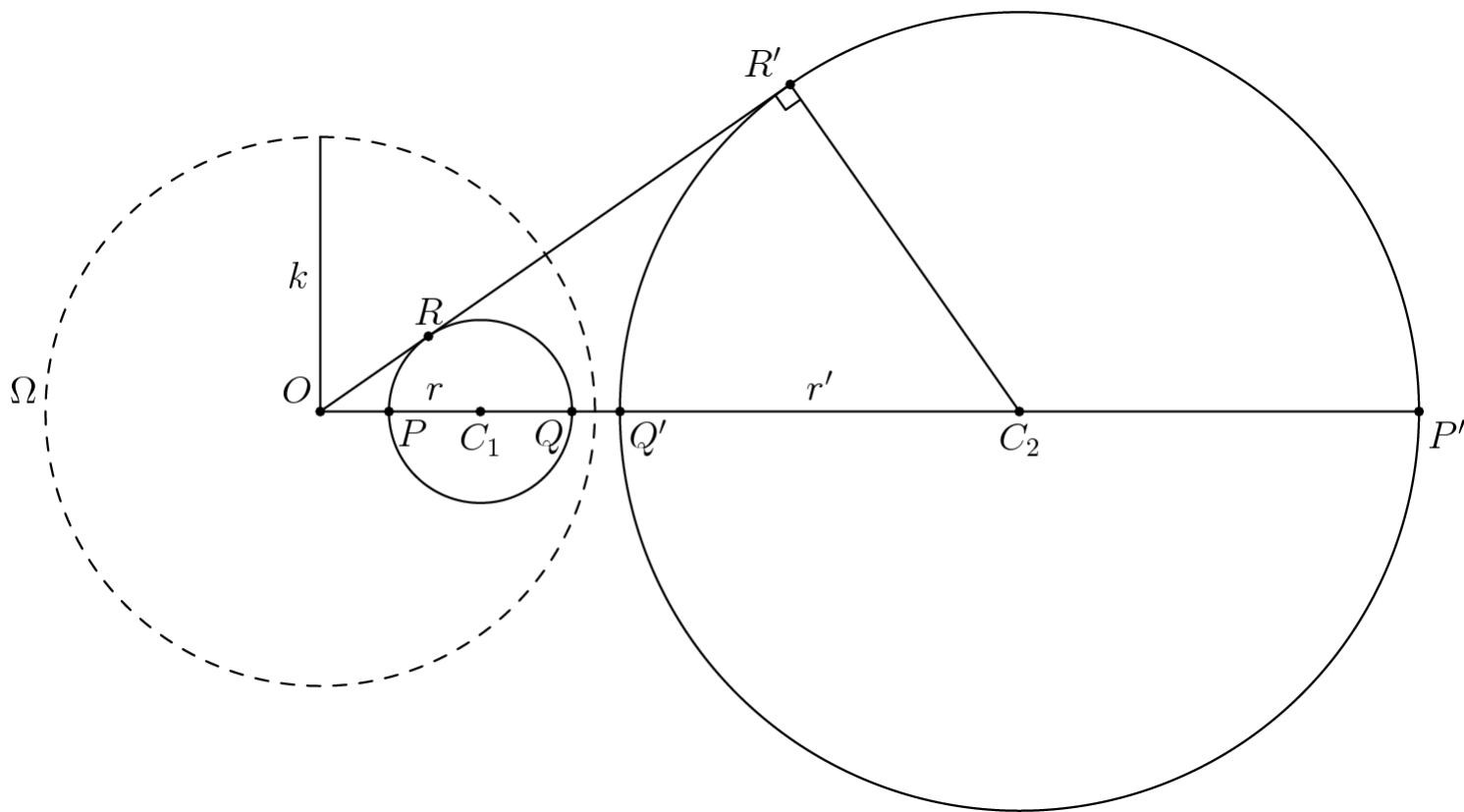
This is how circular inversion is useful in the first place - we find the radius of an inverted circle to find the radius of the original circle. Let the original circle be C_1 and the inverted circle be C_2, with radii of r and r', respectively. The radius of the circle of inversion is k. We draw the tangent line of circle C_1 intersecting O. We know that this is also a tangent line to circle C_2 from the result from part 2 - the tangent line, by definition, intersects circle C_1 at exactly one point, and for every intersection point, part 2 says that there will be another intersection point. Therefore, the tangent line to circle C_1 intersects circle C_2 at exactly one point, necessitating this line to be a tangent line. Call the intersections R and R', respectively. We have $\angle OR'C_2 = 90°$.

We have $\overline{OQ} - \overline{OP} = 2r$ and $\overline{OP'} - \overline{OQ'} = 2r'$. We can write an equation for $\dfrac{r}{r'}$ by dividing:

$$\frac{r}{r'} = \frac{\overline{OQ} - \overline{OP}}{\overline{OP'} - \overline{OQ'}}.$$

From the definition of inversion, we have $\overline{OQ} = \dfrac{k^2}{\overline{OQ'}}, \overline{OP} = \dfrac{k^2}{\overline{OP'}}$. Subsituting yields:

$$\frac{r}{r'} = \frac{\frac{k^2}{\overline{OQ'}} - \frac{k^2}{\overline{OP'}}}{\overline{OP'} - \overline{OQ'}} = \frac{k^2 \frac{\overline{OP'} - \overline{OQ'}}{\overline{OP'} \cdot \overline{OQ'}}}{\overline{OP'} - \overline{OQ'}} = \frac{k^2}{\overline{OP'} \cdot \overline{OQ'}}.$$

From Power of a Point, we know that $\overline{OP'} \cdot \overline{OQ'} = \overline{OR'}^2$, which equals $\overline{OC_2}^2 - r'^2$. Subsistuting gives $\dfrac{r}{r'} = \dfrac{k^2}{\overline{OC_2}^2 - r'^2}$, and solving for r gives:

$$r = \frac{k^2 \cdot r'}{\overline{OC_2}^2 - r'^2}.$$

Alternately,

$$\frac{r}{r'} = \frac{\overline{OR}}{\overline{OR'}} = \frac{k^2}{\overline{OR'}^2} = \frac{k^2}{\overline{OC_2}^2 - r'^2}.$$

Problems that use Circular Inversion

Problem 1 (AMC12)

Source: 2017 AMC12A Problem 16

In the figure below, semicircles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semicircle with diameter JK. The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semicircles and internally tangent to the largest semicircle. What is the radius of the circle centered at P?
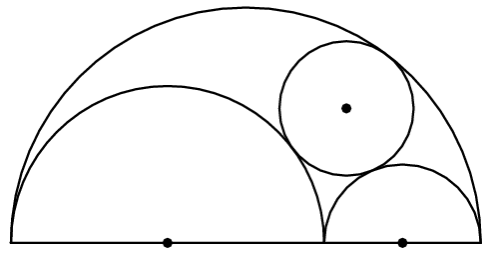
(A) $\dfrac{3}{4}$ (B) $\dfrac{6}{7}$ (C) $\dfrac{1}{2}\sqrt{3}$ (D) $\dfrac{5}{8}\sqrt{2}$ (E) $\dfrac{11}{12}$

Solution using Circular Inversion

Let Ω be a circle with radius of 6 and centered at the left corner of the semi-circle (O) with radius 3. Extend the three semicircles to full circles. Label the resulting four circles as shown in the diagram:
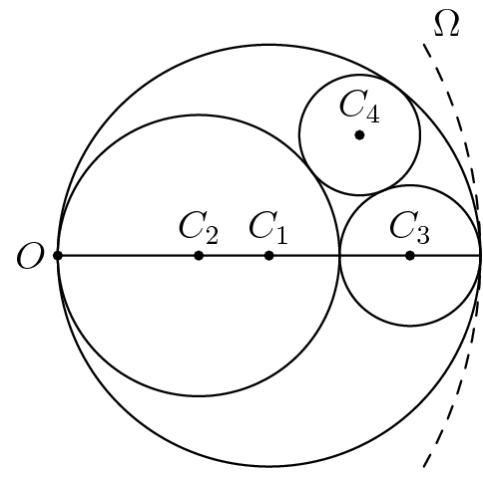
C_1 has radius 3, C_2 has radius 2, and C_3 has radius 1. We want to find the radius of C_4.

We now invert the four circles. C_1 inverts to a line. Given that one point is on Ω, and all points on Ω invert to themselves, we know that the resulting line must intersect that intersection point. C_2 also inverts to a line. C_2 has radius 2, and since Ω has radius of 6, the resulting line must be $\dfrac{36}{4} = 9$ units away from O. C_3 inverts to a circle. By observing the diagram, we note that C_3''s center must be on $\overline{OC_3}$ and be between the two inverted lines, because C_3 is tangent to C_1 and C_2 (Remeber that tangency still holds in inverted diagrams). Therefore, we must have a circle with radius $\dfrac{3}{2}$ that is $\dfrac{15}{2}$ units from O.
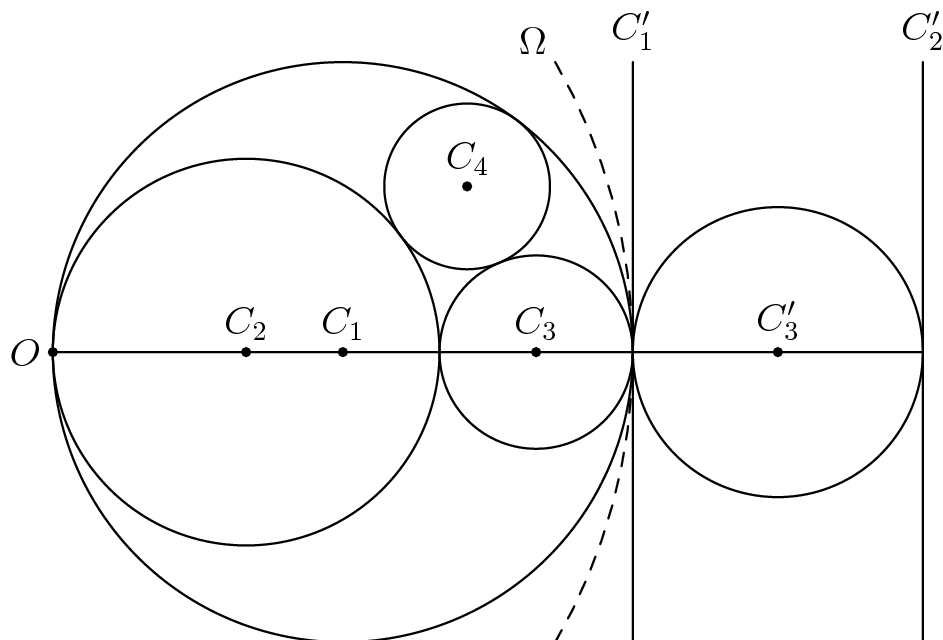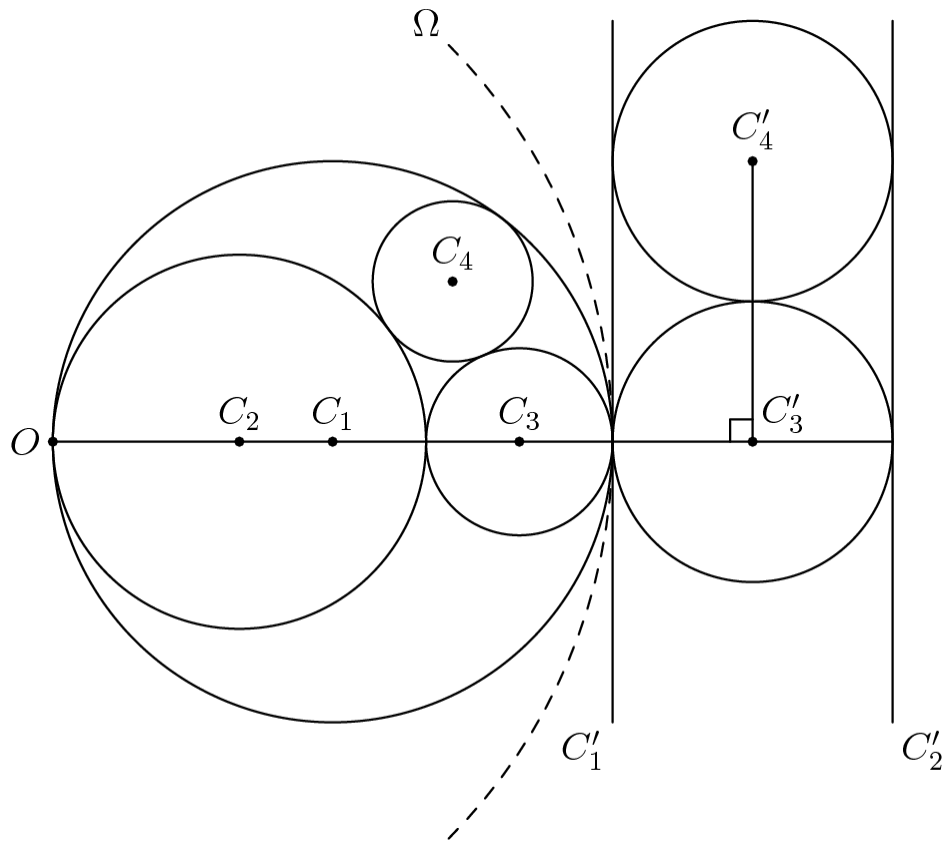
Now, we invert C_4. Note that C_4 is tangent to the three other original circles. So, in the inversion, C_4' must be tangent to the two lines *and* C_3'. It is then quickly seen that C_3' and C_4' have the same radius: $\dfrac{3}{2}$.

Now, we can determine the radius of C_4 using the formula $r = \dfrac{k^2 \cdot r'}{\overline{OC_2}^2 - r'^2}$. $k^2 = 36$, and $r' = \dfrac{3}{2}$. $\overline{OC}$ is just the distance from the center of the inverted circle to the center of inversion. The center of C_4' is 3 units above the center of C_3'. Since $\overline{OC_3'} = \dfrac{15}{2}$, we use Pythagoras to learn that $\overline{OC_4'}^2 = \left(\dfrac{15}{2}\right)^2 + 9$. We do not take the square root because our relationship formula takes $\overline{OC_4'}^2$.

Therefore, we have:

$$r = \frac{36 \cdot \frac{3}{2}}{\left(\frac{15}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 9} = \frac{54}{9 \cdot 6 + 9} = \frac{6}{7} = \boxed{B}.$$

Here is the diagram with C_4'.