16.2 Circuit Element Models

Having mastered how to obtain the Laplace transform and its inverse, we are now prepared to employ the Laplace transform to analyze circuits. This usually involves three steps.

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the s-domain.
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Only the first step is new and will be discussed here. As we did in phasor analysis, we transform a circuit in the time domain to the frequency or s-domain by Laplace transforming each term in the circuit.

As one can infer from step 2, all the circuit analysis techniques applied for dc circuits are applicable to the s-domain.

For a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t) \tag{16.1}$$

Taking the Laplace transform, we get

$$\boxed{V(s) = RI(s)} \tag{16.2}$$

For an inductor,

$$v(t) = L\frac{di(t)}{dt} \tag{16.3}$$

Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-) \tag{16.4}$$

or

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}} \tag{16.5}$$

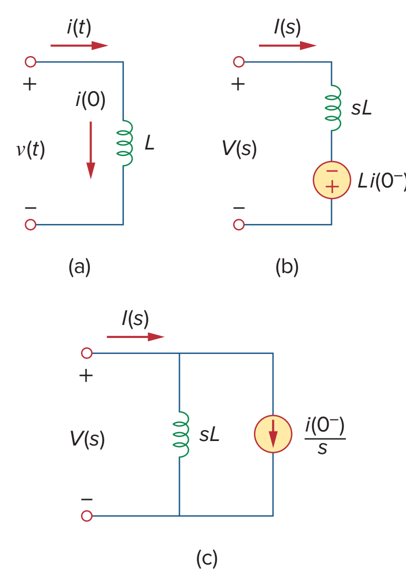

Figure 16.1 Representation of an inductor: (a) time-domain, (b,c) s-domain equivalents.

The s-domain equivalents are shown in Fig. 16.1, where the initial condition is modeled as a voltage or current source.

For a capacitor,

$$i(t) = C\frac{dv(t)}{dt} \tag{16.6}$$

which transforms into the s-domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-) \tag{16.7}$$

or

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}} \tag{16.8}$$

The elegance of using the Laplace transform in circuit analysis lies in the automatic inclusion of the initial conditions in the transformation process, thus providing a complete (transient and steady-state) solution.

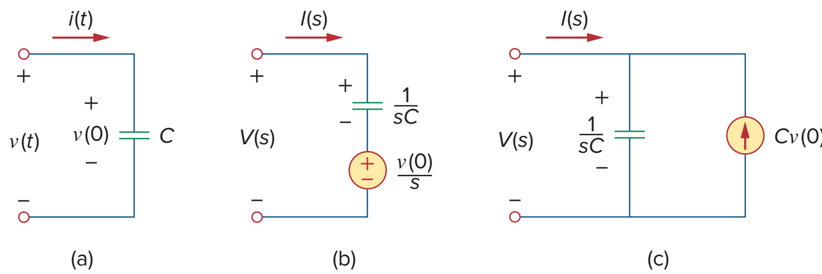

Figure 16.2
Representation of a capacitor: (a) time-domain, (b,c) s-domain equivalents.

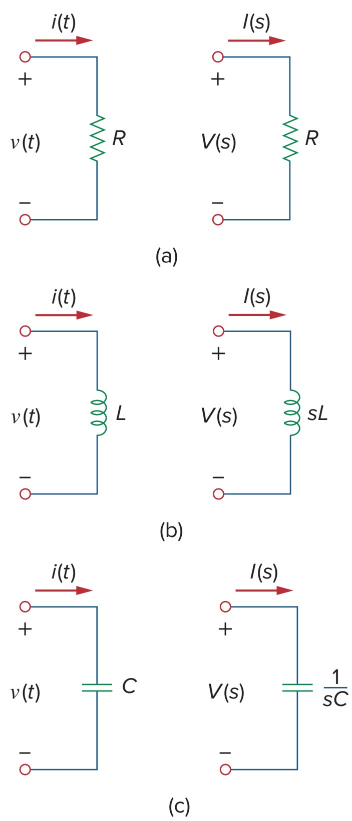

Figure 16.3
Time-domain and s-domain representations of passive elements under zero initial conditions.

The s-domain equivalents are shown in Fig. 16.2. With the s-domain equivalents, the Laplace transform can be used readily to solve first- and second-order circuits such as those we considered in Chapters 7 and 8. We should observe from Eqs. (16.3) to (16.8) that the initial conditions are part of the transformation. This is one advantage of using the Laplace transform in circuit analysis. Another advantage is that a complete response—transient and steady state—of a network is obtained. We will illustrate this with Examples 16.2 and 16.3. Also, observe the duality of Eqs. (16.5) and (16.8), confirming what we already know from Chapter 8 (see Table 8.1), namely, that L and C, $I(s)$ and $V(s)$, and $v(0)$ and $i(0)$ are dual pairs.

If we assume zero initial conditions for the inductor and the capacitor, the above equations reduce to:

$$\begin{aligned} &\text{Resistor:} && V(s) = RI(s) \\ &\text{Inductor:} && V(s) = sLI(s) \\ &\text{Capacitor:} && V(s) = \frac{1}{sC}I(s) \end{aligned} \tag{16.9}$$

The s-domain equivalents are shown in Fig. 16.3.

We define the impedance in the s-domain as the ratio of the voltage transform to the current transform under zero initial conditions; that is,

$$Z(s) = \frac{V(s)}{I(s)} \tag{16.10}$$

Thus, the impedances of the three circuit elements are

$$\begin{aligned} &\text{Resistor:} && Z(s) = R \\ &\text{Inductor:} && Z(s) = sL \\ &\text{Capacitor:} && Z(s) = \frac{1}{sC} \end{aligned} \tag{16.11}$$

Table 16.1 summarizes these. The admittance in the s-domain is the reciprocal of the impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \tag{16.12}$$

The use of the Laplace transform in circuit analysis facilitates the use of various signal sources such as impulse, step, ramp, exponential, and sinusoidal.

The models for dependent sources and op amps are easy to develop drawing from the simple fact that if the Laplace transform of $f(t)$ is $F(s)$,

TABLE 16.1

Impedance of an element in the s-domain.*

Element	$Z(s) = V(s)/I(s)$
Resistor	R
Inductor	sL
Capacitor	$1/sC$

* Assuming zero initial conditions

then the Laplace transform of $af(t)$ is $aF(s)$—the linearity property. The dependent source model is a little easier in that we deal with a single value. The dependent source can have only two controlling values, a constant times either a voltage or a current. Thus,

$$\mathcal{L}[av(t)] = aV(s) \tag{16.13}$$

$$\mathcal{L}[ai(t)] = aI(s) \tag{16.14}$$

The ideal op amp can be treated just like a resistor. Nothing within an op amp, either real or ideal, does anything more than multiply a voltage by a constant. Thus, we only need to write the equations as we always do using the constraint that the input voltage to the op amp has to be zero and the input current has to be zero.

Example 16.1

Find $v_o(t)$ in the circuit of Fig. 16.4, assuming zero initial conditions.

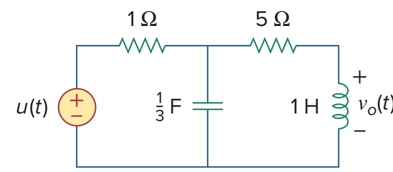

Figure 16.4
For Example 16.1.

Solution:
We first transform the circuit from the time domain to the s-domain.

$$u(t) \quad \Rightarrow \quad \frac{1}{s}$$

$$1\text{ H} \quad \Rightarrow \quad sL = s$$

$$\frac{1}{3}\text{F} \quad \Rightarrow \quad \frac{1}{sC} = \frac{3}{s}$$

The resulting s-domain circuit is in Fig. 16.5. We now apply mesh analysis. For mesh 1,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)I_1 - \frac{3}{s}I_2 \tag{16.1.1}$$

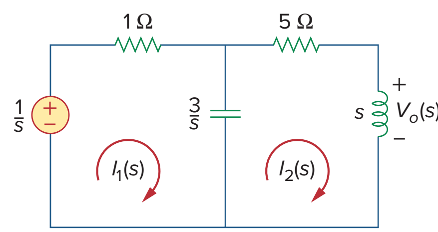

Figure 16.5
Mesh analysis of the frequency-domain equivalent of the same circuit.

For mesh 2,

$$0 = -\frac{3}{s}I_1 + \left(s + 5 + \frac{3}{s}\right)I_2$$

or

$$I_1 = \frac{1}{3}(s^2 + 5s + 3)I_2 \tag{16.1.2}$$

Substituting this into Eq. (16.1.1),

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2$$

Multiplying through by $3s$ gives

$$3 = (s^3 + 8s^2 + 18s)I_2 \qquad \Rightarrow \qquad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}}\,\frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}t\text{ V}, \qquad t \geq 0$$

Practice Problem 16.1

Determine $v_o(t)$ in the circuit of Fig. 16.6, assuming zero initial conditions.

Answer: $10(1 - e^{-2t} - 2te^{-2t})u(t)$ V.

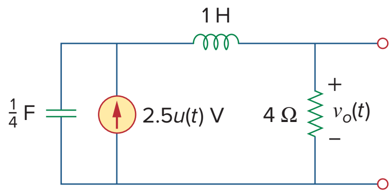

Figure 16.6
For Practice Prob. 16.1.

Example 16.2

Find $v_o(t)$ in the circuit of Fig. 16.7. Assume $v_o(0) = 5$ V.

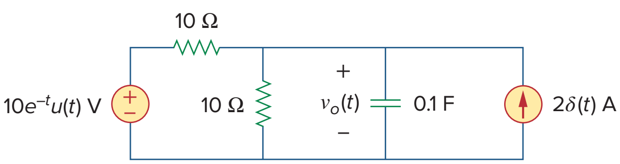

Figure 16.7
For Example 16.2.

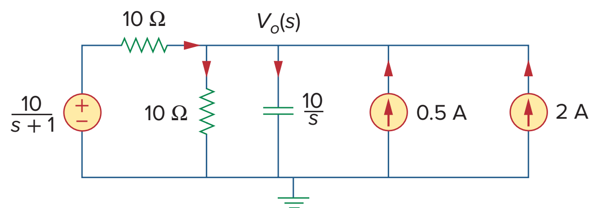

Figure 16.8
Nodal analysis of the equivalent of the circuit in Fig. 16.7.

Solution:
We transform the circuit to the s-domain as shown in Fig. 16.8. The initial condition is included in the form of the current source $Cv_o(0) = 0.1(5) = 0.5$ A. [See Fig. 16.2(c).] We apply nodal analysis. At the top node,

$$\frac{10/(s+1) - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}$$

or

$$\frac{1}{s+1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10}V_o(s+2)$$

Multiplying through by 10,

$$\frac{10}{s+1} + 25 = V_o(s+2)$$

or

$$V_o = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

where

$$A = (s+1)V_o(s)\big|_{s=-1} = \frac{25s+35}{(s+2)}\bigg|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2)V_o(s)\big|_{s=-2} = \frac{25s+35}{(s+1)}\bigg|_{s=-2} = \frac{-15}{-1} = 15$$

Thus,

$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t) \text{ V}$$

Practice Problem 16.2

Find $v_o(t)$ in the circuit shown in Fig. 16.9. Note that, since the voltage input is multiplied by $u(t)$, the voltage source is a short for all $t < 0$ and $i_L(0) = 0$.

Answer: $(12e^{-2t} - 2e^{-t/3})u(t)$ V.

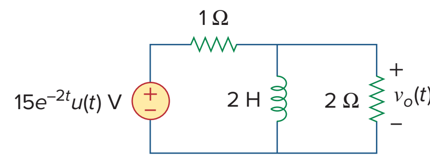

Figure 16.9
For Practice Prob. 16.2.

Example 16.3

In the circuit of Fig. 16.10(a), the switch moves from position a to position b at $t = 0$. Find $i(t)$ for $t > 0$.

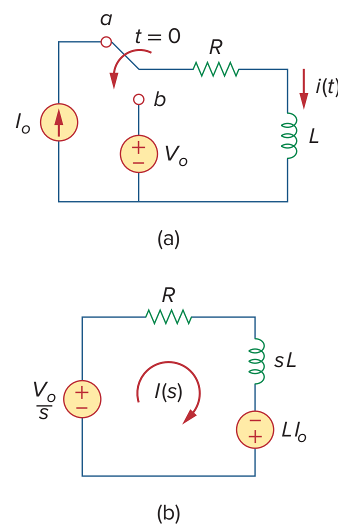

Figure 16.10
For Example 16.3.

Solution:
The initial current through the inductor is $i(0) = I_o$. For $t > 0$, Fig. 16.10(b) shows the circuit transformed to the s-domain. The initial condition is incorporated in the form of a voltage source as $Li(0) = LI_o$. Using mesh analysis,

$$I(s)(R + sL) - LI_o - \frac{V_o}{s} = 0 \tag{16.3.1}$$

or

$$I(s) = \frac{LI_o}{R + sL} + \frac{V_o}{s(R + sL)} = \frac{I_o}{s + R/L} + \frac{V_o/L}{s(s + R/L)} \tag{16.3.2}$$

Applying partial fraction expansion on the second term on the right-hand side of Eq. (16.3.2) yields

$$I(s) = \frac{I_o}{s + R/L} + \frac{V_o/R}{s} - \frac{V_o/R}{(s + R/L)} \tag{16.3.3}$$

The inverse Laplace transform of this gives

$$i(t) = \left(I_o - \frac{V_o}{R}\right)e^{-t/\tau} + \frac{V_o}{R}, \qquad t \geq 0 \tag{16.3.4}$$

where $\tau = R/L$. The term in parentheses is the transient response, while the second term is the steady-state response. In other words, the final value is $i(\infty) = V_o/R$, which we could have predicted by applying the final-value theorem on Eq. (16.3.2) or (16.3.3); that is,

$$\lim_{s\to 0} sI(s) = \lim_{s\to 0}\left(\frac{sI_o}{s + R/L} + \frac{V_o/L}{s + R/L}\right) = \frac{V_o}{R} \tag{16.3.5}$$

Equation (16.3.4) may also be written as

$$i(t) = I_oe^{-t/\tau} + \frac{V_o}{R}(1 - e^{-t/\tau}), \qquad t \geq 0 \tag{16.3.6}$$

The first term is the natural response, while the second term is the forced response. If the initial condition $I_o = 0$, Eq. (16.3.6) becomes

$$i(t) = \frac{V_o}{R}(1 - e^{-t/\tau}), \qquad t \geq 0 \tag{16.3.7}$$

which is the step response, since it is due to the step input V_o with no initial energy.

Practice Problem 16.3

The switch in Fig. 16.11 has been in position b for a long time. It is moved to position a at $t = 0$. Determine $v(t)$ for $t > 0$.

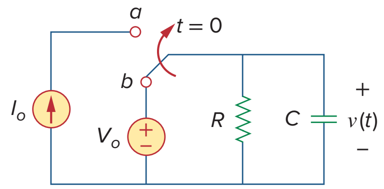

Figure 16.11
For Practice Prob. 16.3.

Answer: $v(t) = (V_o - I_oR)e^{-t/\tau} + I_oR$, $t > 0$, where $\tau = RC$.

16.3 Circuit Analysis

Circuit analysis is again relatively easy to do when we are in the s-domain. We merely need to transform a complicated set of mathematical relationships in the time domain into the s-domain where we convert operators (derivatives and integrals) into simple multipliers of s and $1/s$. This now allows us to use algebra to set up and solve our circuit equations. The exciting thing about this is that *all* of the circuit theorems and relationships we developed for dc circuits are perfectly valid in the s-domain.

> Remember, **equivalent circuits,** with capacitors and inductors, only exist in the s-domain; they cannot be transformed back into the time domain.

Example 16.4

Consider the circuit in Fig. 16.12(a). Find the value of the voltage across the capacitor assuming that the value of $v_s(t) = 10u(t)$ V and assume that at $t = 0$, -1 A flows through the inductor and $+5$ V is across the capacitor.

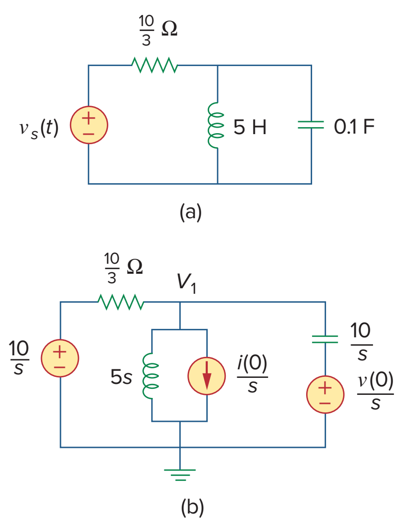

Figure 16.12
For Example 16.4.

Solution:
Figure 16.12(b) represents the entire circuit in the s-domain with the initial conditions incorporated. We now have a straightforward nodal analysis problem. Because the value of V_1 is also the value of the capacitor voltage in the time domain and is the only unknown node voltage, we only need to write one equation.

$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + \frac{i(0)}{s} + \frac{V_1 - [v(0)/s]}{1/(0.1s)} = 0 \qquad \textbf{(16.4.1)}$$

or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s} + \frac{1}{s} + 0.5 \qquad \textbf{(16.4.2)}$$

where $v(0) = 5$ V and $i(0) = -1$ A. Simplifying we get

$$(s^2 + 3s + 2)\ V_1 = 40 + 5s$$

or

$$V_1 = \frac{40 + 5s}{(s+1)(s+2)} = \frac{35}{s+1} - \frac{30}{s+2} \qquad \textbf{(16.4.3)}$$

Taking the inverse Laplace transform yields

$$v_1(t) = (35e^{-t} - 30e^{-2t})u(t)\ \text{V} \qquad \textbf{(16.4.4)}$$

Practice Problem 16.4

For the circuit shown in Fig. 16.12 with the same initial conditions, find the current through the inductor for all time $t > 0$.

Answer: $i(t) = (3 - 7e^{-t} + 3e^{-2t})u(t)$ A.

Example 16.5

For the circuit shown in Fig. 16.12, and the initial conditions used in Example 16.4, use superposition to find the value of the capacitor voltage.

Solution:
Inasmuch as the circuit in the s-domain actually has three independent sources, we can look at the solution one source at a time. Figure 16.13 presents the circuits in the s-domain considering one source at a time. We now have three nodal analysis problems. First, let us solve for the capacitor voltage in the circuit shown in Fig. 16.13(a).

$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + 0 + \frac{V_1 - 0}{1/(0.1s)} = 0$$

or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_1 = \frac{3}{s}$$

Simplifying we get

$$(s^2 + 3s + 2)V_1 = 30$$

$$V_1 = \frac{30}{(s+1)(s+2)} = \frac{30}{s+1} - \frac{30}{s+2}$$

or

$$v_1(t) = (30e^{-t} - 30e^{-2t})u(t) \text{ V} \qquad \textbf{(16.5.1)}$$

For Fig. 16.13(b) we get,

$$\frac{V_2 - 0}{10/3} + \frac{V_2 - 0}{5s} - \frac{1}{s} + \frac{V_2 - 0}{1/(0.1s)} = 0$$

or

$$0.1\left(s + 3 + \frac{2}{s}\right)V_2 = \frac{1}{s}$$

This leads to

$$V_2 = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2}$$

Taking the inverse Laplace transform, we get

$$v_2(t) = (10e^{-t} - 10e^{-2t})u(t) \text{ V} \qquad \textbf{(16.5.2)}$$

For Fig. 16.13(c),

$$\frac{V_3 - 0}{10/3} + \frac{V_3 - 0}{5s} - 0 + \frac{V_3 - 5/s}{1/(0.1s)} = 0$$

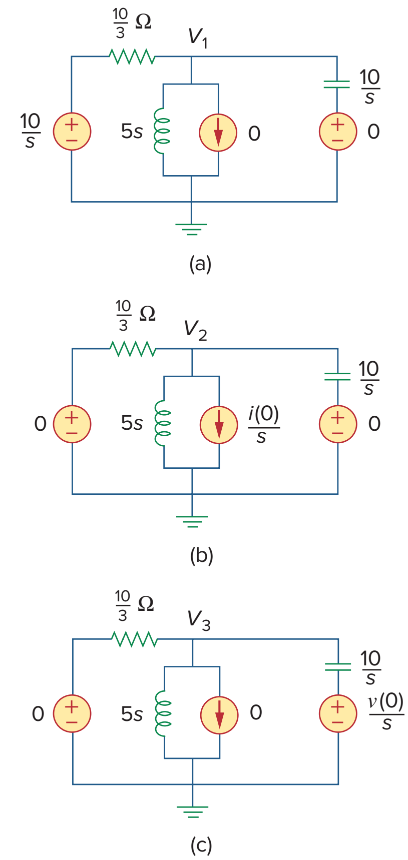

Figure 16.13
For Example 16.5.

or

$$0.1\left(s+3+\frac{2}{s}\right)V_3 = 0.5$$

$$V_3 = \frac{5s}{(s+1)(s+2)} = \frac{-5}{s+1} + \frac{10}{s+2}$$

This leads to

$$v_3(t) = (-5e^{-t} + 10e^{-2t})u(t) \text{ V} \quad \textbf{(16.5.3)}$$

Now all we need to do is to add Eqs. (16.5.1), (16.5.2), and (16.5.3):

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= \{(30 + 10 - 5)e^{-t} + (-30 + 10 - 10)e^{-2t}\}u(t) \text{ V} \end{aligned}$$

or

$$v(t) = (35e^{-t} - 30e^{-2t})u(t) \text{ V}$$

which agrees with our answer in Example 16.4.

Practice Problem 16.5

For the circuit shown in Fig. 16.12, and the same initial conditions in Example 16.4, find the current through the inductor for all time $t > 0$ using superposition.

Answer: $i(t) = (3 - 7e^{-t} + 3e^{-2t})u(t)$ A.

Example 16.6

Assume that there is no initial ener gy stored in the circuit of Fig. 16.14 at $t = 0$ and that $i_s = 10u(t)$ A. (a) Find $V_o(s)$ using Thevenin's theorem. (b) Apply the initial- and final-value theorems to find $v_o(0^+)$ and $v_o(\infty)$. (c) Determine $v_o(t)$.

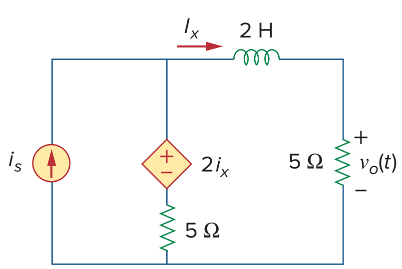

Figure 16.14
For Example 16.6.

Solution:
Because there is no initial energy stored in the circuit, we assume that the initial inductor current and initial capacitor voltage are zero at $t = 0$.

(a) To find the Thevenin equivalent circuit, we remove the 5-Ω resistor and then find V_{oc} (V_{Th}) and I_{sc}. To find V_{Th}, we use the Laplace-transformed circuit in Fig. 16.15(a). Since $I_x = 0$, the dependent voltage source contributes nothing, so

$$V_{oc} = V_{Th} = 5\left(\frac{10}{s}\right) = \frac{50}{s}$$

To find Z_{Th}, we consider the circuit in Fig. 16.15(b), where we first find I_{sc}. We can use nodal analysis to solve for V_1 which then leads to $I_{sc}(I_{sc} = I_x = V_1/2s)$.

$$-\frac{10}{s} + \frac{(V_1 - 2I_x) - 0}{5} + \frac{V_1 - 0}{2s} = 0$$

along with

$$I_x = \frac{V_1}{2s}$$

leads to

$$V_1 = \frac{100}{2s + 3}$$

Hence,

$$I_{sc} = \frac{V_1}{2s} = \frac{100/(2s + 3)}{2s} = \frac{50}{s(2s + 3)}$$

and

$$Z_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{50/s}{50/[s(2s + 3)]} = 2s + 3$$

The given circuit is replaced by its Thevenin equivalent at terminals *a-b* as shown in Fig. 16.16. From Fig. 16.16,

$$V_o = \frac{5}{5 + Z_{Th}} V_{Th} = \frac{5}{5 + 2s + 3}\left(\frac{50}{s}\right) = \frac{250}{s(2s + 8)} = \frac{125}{s(s + 4)}$$

(b) Using the initial-value theorem we find

$$v_o(0) = \lim_{s\to\infty} sV_o(s) = \lim_{s\to\infty} \frac{125}{s + 4} = \lim_{s\to\infty} \frac{125/s}{1 + 4/s} = \frac{0}{1} = 0$$

Using the final-value theorem we find

$$v_o(\infty) = \lim_{s\to 0} sV_o(s) = \lim_{s\to 0} \frac{125}{s + 4} = \frac{125}{4} = 31.25 \text{ V}$$

(c) By partial fraction,

$$V_o = \frac{125}{s(s + 4)} = \frac{A}{s} + \frac{B}{s + 4}$$

$$A = sV_o(s)\Big|_{s=0} = \frac{125}{s + 4}\Big|_{s=0} = 31.25$$

$$B = (s + 4)V_o(s)\Big|_{s=-4} = \frac{125}{s}\Big|_{s=-4} = -31.25$$

$$V_o = \frac{31.25}{s} - \frac{31.25}{s + 4}$$

Taking the inverse Laplace transform gives

$$v_o(t) = 31.25(1 - e^{-4t})u(t) \text{ V}$$

Notice that the values of $v_o(0)$ and $v_o(\infty)$ obtained in part (b) are confirmed.

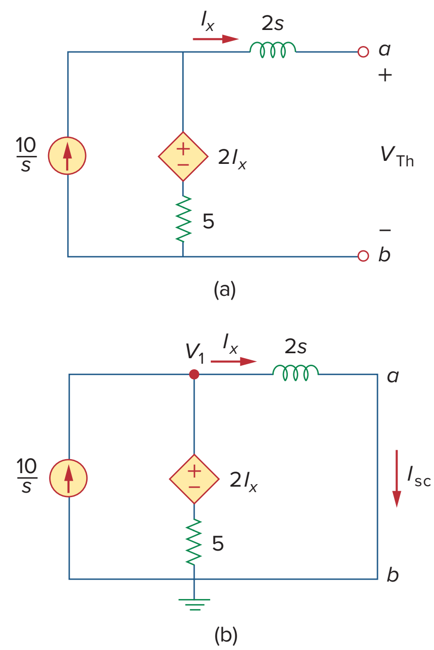

Figure 16.15
For Example 16.6: (a) finding V_{Th}, (b) determining Z_{Th}.

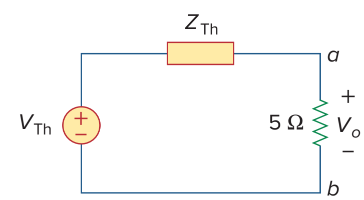

Figure 16.16
The Thevenin equivalent of the circuit in Fig. 16.14 in the *s*-domain.

Practice Problem 16.6

The initial energy in the circuit of Fig. 16.17 is zero at $t = 0$. Assume that $v_s = 360u(t)$ V. (a) Find $V_o(s)$ using the Thevenin theorem. (b) Apply the initial- and final-value theorems to find $v_o(0)$ and $v_o(\infty)$. (c) Obtain $v_o(t)$.

Answer: (a) $V_o(s) = \frac{288(s+0.25)}{s(s+0.3)}$, (b) 288 V, 240 V,

(c) $(240 + 48e^{-0.3t})u(t)$ V.

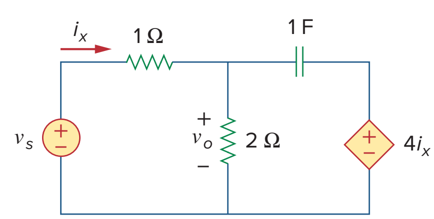

Figure 16.17
For Practice Prob. 16.6.