Linearity

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace transforms of $f_1(t)$ and $f_2(t)$, then

$$\boxed{\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)} \tag{15.7}$$

where a_1 and a_2 are constants. Equation 15.7 expresses the linearity property of the Laplace transform. The proof of Eq. (15.7) follows readily from the definition of the Laplace transform in Eq. (15.1).

For example, by the linearity property in Eq. (15.7), we may write

$$\mathcal{L}[\cos \omega t\, u(t)] = \mathcal{L}\left[\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right] = \frac{1}{2}\mathcal{L}[e^{j\omega t}] + \frac{1}{2}\mathcal{L}[e^{-j\omega t}] \tag{15.8}$$

But from Example 15.1(b), $\mathcal{L}[e^{-at}] = 1/(s + a)$. Hence,

$$\mathcal{L}[\cos \omega t\, u(t)] = \frac{1}{2}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right) = \frac{s}{s^2 + \omega^2} \tag{15.9}$$

Scaling

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f(at)] = \int_{0-}^{\infty} f(at) e^{-st}\, dt \tag{15.10}$$

where a is a constant and $a > 0$. If we let $x = at$, $dx = a\, dt$, then

$$\mathcal{L}[f(at)] = \int_{0-}^{\infty} f(x)\, e^{-x(s/a)} \frac{dx}{a} = \frac{1}{a}\int_{0-}^{\infty} f(x) e^{-x(s/a)} dx \tag{15.11}$$

Comparing this integral with the definition of the Laplace transform in Eq. (15.1) shows that s in Eq. (15.1) must be replaced by s/a while the dummy variable t is replaced by x. Hence, we obtain the scaling property as

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)} \tag{15.12}$$

For example, we know from Example 15.2 that

$$\mathcal{L}[\sin \omega t\, u(t)] = \frac{\omega}{s^2 + \omega^2} \tag{15.13}$$

Using the scaling property in Eq. (15.12),

$$\mathcal{L}[\sin 2\omega t\, u(t)] = \frac{1}{2}\frac{\omega}{(s/2)^2 + \omega^2} = \frac{2\omega}{s^2 + 4\omega^2} \tag{15.14}$$

which may also be obtained from Eq. (15.13) by replacing ω with 2ω.

Time Shift

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}[f(t - a)u(t - a)] = \int_{0-}^{\infty} f(t - a)u(t - a)e^{-st}\, dt \tag{15.15}$$

$$a \geq 0$$

But $u(t-a) = 0$ for $t < a$ and $u(t-a) = 1$ for $t > a$. Hence,

$$\mathcal{L}[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-st}\,dt \tag{15.16}$$

If we let $x = t - a$, then $dx = dt$ and $t = x + a$. As $t \to a$, $x \to 0$ and as $t \to \infty$, $x \to \infty$. Thus,

$$\begin{aligned}\mathcal{L}[f(t-a)u(t-a)] &= \int_{0^-}^\infty f(x)e^{-s(x+a)}\,dx \\ &= e^{-as}\int_{0^-}^\infty f(x)e^{-sx}\,dx = e^{-as}F(s)\end{aligned}$$

or

$$\boxed{\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)} \tag{15.17}$$

In other words, if a function is delayed in time by a, the result in the s-domain is found by multiplying the Laplace transform of the function (without the delay) by e^{-as}. This is called the *time-delay* or *time-shift property* of the Laplace transform.

As an example, we know from Eq. (15.9) that

$$\mathcal{L}[\cos\omega t\,u(t)] = \frac{s}{s^2+\omega^2}$$

Using the time-shift property in Eq. (15.17),

$$\mathcal{L}[\cos\omega(t-a)u(t-a)] = e^{-as}\frac{s}{s^2+\omega^2} \tag{15.18}$$

Frequency Shift

If $F(s)$ is the Laplace transform of $f(t)$, then

$$\begin{aligned}\mathcal{L}[e^{-at}f(t)u(t)] &= \int_0^\infty e^{-at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s+a)t}\,dt = F(s+a)\end{aligned}$$

or

$$\boxed{\mathcal{L}[e^{-at}f(t)u(t)] = F(s+a)} \tag{15.19}$$

That is, the Laplace transform of $e^{-at}f(t)$ can be obtained from the Laplace transform of $f(t)$ by replacing every s with $s + a$. This is known as *frequency shift* or *frequency translation.*

As an example, we know that

$$\cos\omega t\,u(t) \quad\Leftrightarrow\quad \frac{s}{s^2+\omega^2}$$

and

$$\sin\omega t\,u(t) \quad\Leftrightarrow\quad \frac{\omega}{s^2+\omega^2} \tag{15.20}$$

Using the shift property in Eq. (15.19), we obtain the Laplace transform of the damped sine and damped cosine functions as

$$\mathcal{L}[e^{-at}\cos\omega t\, u(t)] = \frac{s+a}{(s+a)^2+\omega^2} \tag{15.21a}$$

$$\mathcal{L}[e^{-at}\sin\omega t\, u(t)] = \frac{\omega}{(s+a)^2+\omega^2} \tag{15.21b}$$

Time Differentiation

Given that $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of its derivative is

$$\mathcal{L}\left[\frac{df}{dt}u(t)\right] = \int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt \tag{15.22}$$

To integrate this by parts, we let $u = e^{-st}$, $du = -se^{-st}\,dt$, and $dv = (df/dt)\,dt = df(t)$, $v = f(t)$. Then

$$\begin{aligned}\mathcal{L}\left[\frac{df}{dt}u(t)\right] &= f(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)[-se^{-st}]\,dt \\ &= 0 - f(0^-) + s\int_{0^-}^{\infty} f(t)e^{-st}\,dt = sF(s) - f(0^-)\end{aligned}$$

or

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0^-)} \tag{15.23}$$

The Laplace transform of the second derivative of $f(t)$ is a repeated application of Eq. (15.23) as

$$\begin{aligned}\mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= s\mathcal{L}[f'(t)] - f'(0^-) = s[sF(s) - f(0^-)] - f'(0^-) \\ &= s^2F(s) - sf(0^-) - f'(0^-)\end{aligned}$$

or

$$\boxed{\mathcal{L}[f''(t)] = s^2F(s) - sf(0^-) - f'(0^-)} \tag{15.24}$$

Continuing in this manner, we can obtain the Laplace transform of the nth derivative of $f(t)$ as

$$\boxed{\begin{aligned}\mathcal{L}\left[\frac{d^nf}{dt^n}\right] &= s^nF(s) - s^{n-1}f(0^-) \\ &\quad - s^{n-2}f'(0^-) - \cdots - s^0 f^{(n-1)}(0^-)\end{aligned}} \tag{15.25}$$

As an example, we can use Eq. (15.23) to obtain the Laplace transform of the sine from that of the cosine. If we let $f(t) = \cos\omega t\, u(t)$, then $f(0) = 1$ and $f'(t) = -\omega\sin\omega t\, u(t)$. Using Eq. (15.23) and the scaling property,

$$\begin{aligned}\mathcal{L}[\sin\omega t\, u(t)] &= -\frac{1}{\omega}\mathcal{L}[f'(t)] = -\frac{1}{\omega}[sF(s) - f(0^-)] \\ &= -\frac{1}{\omega}\left(s\frac{s}{s^2+\omega^2} - 1\right) = \frac{\omega}{s^2+\omega^2}\end{aligned} \tag{15.26}$$

as expected.

Time Integration

If $F(s)$ is the Laplace transform of $f(t)$, the Laplace transform of its integral is

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \int_{0^-}^{\infty}\left[\int_0^t f(x)dx\right]e^{-st}\,dt \tag{15.27}$$

To integrate this by parts, we let

$$u = \int_0^t f(x)dx, \qquad du = f(t)dt$$

and

$$dv = e^{-st}\,dt, \qquad v = -\frac{1}{s}e^{-st}$$

Then

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \left[\int_0^t f(x)dx\right]\left(-\frac{1}{s}e^{-st}\right)\Bigg|_{0^-}^{\infty} - \int_0^{\infty}\left(-\frac{1}{s}\right)e^{-st}f(t)dt$$

For the first term on the right-hand side of the equation, evaluating the term at $t = \infty$ yields zero due to $e^{-s\infty}$ and evaluating it at $t = 0$ gives $\frac{1}{s}\int_0^0 f(x)\,dx = 0$. Thus, the first term is zero, and

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}\,dt = \frac{1}{s}F(s)$$

or simply,

$$\boxed{\mathcal{L}\left[\int_0^t f(x)dx\right] = \frac{1}{s}F(s)} \tag{15.28}$$

As an example, if we let $f(t) = u(t)$, from Example 15.1(a), $F(s) = 1/s$. Using Eq. (15.28),

$$\mathcal{L}\left[\int_0^t f(x)dx\right] = \mathcal{L}[t] = \frac{1}{s}\left(\frac{1}{s}\right)$$

Thus, the Laplace transform of the ramp function is

$$\mathcal{L}[t] = \frac{1}{s^2} \tag{15.29}$$

Applying Eq. (15.28), this gives

$$\mathcal{L}\left[\int_0^t x\,dx\right] = \mathcal{L}\left[\frac{t^2}{2}\right] = \frac{1}{s}\,\frac{1}{s^2}$$

or

$$\mathcal{L}[t^2] = \frac{2}{s^3} \tag{15.30}$$

Repeated applications of Eq. (15.28) lead to

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \tag{15.31}$$

Similarly, using integration by parts, we can show that

$$\mathcal{L}\left[\int_{-\infty}^{t} f(x)dx\right] = \frac{1}{s}F(s) + \frac{1}{s}f^{-1}(0^-) \tag{15.32}$$

where

$$f^{-1}(0^-) = \int_{-\infty}^{0^-} f(t)dt$$

Frequency Differentiation

If $F(s)$ is the Laplace transform of $f(t)$, then

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

Taking the derivative with respect to s,

$$\frac{dF(s)}{ds} = \int_{0^-}^{\infty} f(t)(-te^{-st})\,dt = \int_{0^-}^{\infty} (-tf(t))e^{-st}\,dt = \mathcal{L}[-tf(t)]$$

and the frequency differentiation property becomes

$$\boxed{\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}} \tag{15.33}$$

Repeated applications of this equation lead to

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n} \tag{15.34}$$

For example, we know from Example 15.1(b) that $\mathcal{L}[e^{-at}] = 1/(s+a)$. Using the property in Eq. (15.33),

$$\mathcal{L}[te^{-at}u(t)] = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2} \tag{15.35}$$

Note that if $a = 0$, we obtain $\mathcal{L}[t] = 1/s^2$ as in Eq. (15.29), and repeated applications of Eq. (15.33) will yield Eq. (15.31).

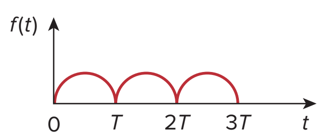

Figure 15.3
A periodic function.

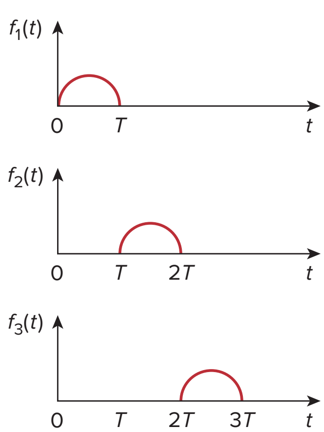

Figure 15.4
Decomposition of the periodic function in Fig. 15.3.

Time Periodicity

If function $f(t)$ is a periodic function such as sho wn in Fig. 15.3, it can be represented as the sum of time-shifted functions sho wn in Fig. 15.4. Thus,

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) + \cdots \\ &= f_1(t) + f_1(t-T)u(t-T) \\ &\quad + f_1(t-2T)u(t-2T) + \cdots \end{aligned} \tag{15.36}$$

where $f_1(t)$ is the same as the function $f(t)$ g ated o ver the interv al $0 < t < T$, that is,

$$f_1(t) = f(t)[u(t) - u(t-T)] \tag{15.37a}$$

or

$$f_1(t) = \begin{cases} f(t), & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \tag{15.37b}$$

We now transform each term in Eq. (15.36) and apply the time-shift property in Eq. (15.17). We obtain

$$\begin{aligned} F(s) &= F_1(s) + F_1(s)e^{-Ts} + F_1(s)e^{-2Ts} + F_1(s)e^{-3Ts} + \cdots \\ &= F_1(s)[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots] \end{aligned} \tag{15.38}$$

But

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x} \tag{15.39}$$

if $|x| < 1$. Hence,

$$\boxed{F(s) = \frac{F_1(s)}{1 - e^{-Ts}}} \tag{15.40}$$

where $F_1(s)$ is the Laplace transform of $f_1(t)$; in other words, $F_1(s)$ is the transform $f(t)$ defined over its first period only. Equation (15.40) shows that the Laplace transform of a periodic function is the transform of the first period of the function divided by $1 - e^{-Ts}$.

Initial and Final Values

The initial-value and final-value properties allow us to find the initial value $f(0)$ and the final value $f(\infty)$ of $f(t)$ directly from its Laplace transform $F(s)$. To obtain these properties, we begin with the differentiation property in Eq. (15.23), namely,

$$sF(s) - f(0) = \mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt \tag{15.41}$$

If we let $s \to \infty$, the integrand in Eq. (15.41) vanishes due to the damping exponential factor, and Eq. (15.41) becomes

$$\lim_{s\to\infty} [sF(s) - f(0)] = 0$$

Because $f(0)$ is independent of s, we can write

$$\boxed{f(0) = \lim_{s\to\infty} sF(s)} \tag{15.42}$$

This is known as the *initial-value theorem*. For example, we know from Eq. (15.21a) that

$$f(t) = e^{-2t}\cos 10t\, u(t) \quad \Leftrightarrow \quad F(s) = \frac{s + 2}{(s + 2)^2 + 10^2} \tag{15.43}$$

Using the initial-value theorem,

$$\begin{aligned} f(0) = \lim_{s\to\infty} sF(s) &= \lim_{s\to\infty} \frac{s^2 + 2s}{s^2 + 4s + 104} \\ &= \lim_{s\to\infty} \frac{1 + 2/s}{1 + 4/s + 104/s^2} = 1 \end{aligned}$$

which confirms what we would expect from the given $f(t)$.

In Eq. (15.41), we let $s \to 0$; then

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \int_{0^-}^{\infty} \frac{df}{dt} e^{0t}\, dt = \int_{0^-}^{\infty} df = f(\infty) - f(0^-)$$

or

$$\boxed{f(\infty) = \lim_{s\to 0} sF(s)} \qquad \textbf{(15.44)}$$

This is referred to as the *final-value theorem*. In order for the final-value theorem to hold, all poles of $F(s)$ must be located in the left half of the s plane (see Fig. 15.1 or 15.9); that is, the poles must have negative real parts. The only exception to this requirement is the case in which $F(s)$ has a simple pole at $s = 0$, because the effect of $1/s$ will be nullified by $sF(s)$ in Eq. (15.44). For example, from Eq. (15.21b),

$$f(t) = e^{-2t} \sin 5t\, u(t) \qquad \Leftrightarrow \qquad F(s) = \frac{5}{(s+2)^2 + 5^2} \qquad \textbf{(15.45)}$$

Applying the final-value theorem,

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{5s}{s^2 + 4s + 29} = 0$$

as expected from the given $f(t)$. As another example,

$$f(t) = \sin t\, u(t) \qquad \Leftrightarrow \qquad f(s) = \frac{1}{s^2 + 1} \qquad \textbf{(15.46)}$$

so that

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s^2 + 1} = 0$$

This is incorrect, because $f(t) = \sin t$ oscillates between $+1$ and -1 and does not have a limit as $t \to \infty$. Thus, the final-value theorem cannot be used to find the final value of $f(t) = \sin t$, because $F(s)$ has poles at $s = \pm j$, which are not in the left half of the s plane. In general, the final-value theorem does not apply in finding the final values of sinusoidal functions—these functions oscillate forever and do not have final values.

The initial-value and final-value theorems depict the relationship between the origin and infinity in the time domain and the s-domain. They serve as useful checks on Laplace transforms.

Table 15.1 provides a list of the properties of the Laplace transform. The last property (on convolution) will be proved in Section 15.5. There are other properties, but these are enough for present purposes. Table 15.2 summarizes the Laplace transforms of some common functions. We have omitted the factor $u(t)$ except where it is necessary.

We should mention that many software packages, such as Mathcad, *MATLAB*, Maple, and Mathematica, offer symbolic math. For example, Mathcad has symbolic math for the Laplace, Fourier, and Z transforms as well as the inverse function.

TABLE 15.1

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)dx$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

TABLE 15.2

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Example 15.3

Obtain the Laplace transform of $f(t) = \delta(t) + 2u(t) - 3e^{-2t}\,u(t)$.

Solution:

By the linearity property,

$$F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}\,u(t)]$$

$$= 1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2+s+4}{s(s+2)}$$

Practice Problem 15.3

Find the Laplace transform of $f(t) = (\cos(2t) + e^{-4t})u(t)$.

Answer: $\frac{2s^2+4s+4}{(s+4)(s^2+4)}$.