Streaming Erasure Codes over Multi-Link Multi-hop Network

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***Abstract*—Utilizing multiple paths between a source and a destination to increase throughput and/or robustness is a well-known approach. This paper analyzes the transmission of a sequence of messages (streaming messages) through a packet erasure network with a strict delay constraint. Further, we assume that some of the paths are going through nodes, which can apply coding operations. The streaming capacity of the simplest network, which is a three-node single link network, was first considered by Fong et al., [1] and later extended by Domanovitz et al., [2] to any number of nodes (both networks have a single path between the source and destination). We start by studying the streaming codes' performance over multi-hop multi-link networks for which closed-form expressions for both the upper bound and achievable are derived. The achievable scheme we propose uses the concatenation of capacity-achieving codes for each path. We further show this scheme can be improved by concatenating codes with different multiplicities over each path. Formalizing the problem of finding the optimal multiplicity per path as a convex optimization problem, we show it is tractable for practical network configurations. We then extend the network model to a directed acyclic graph between the source and the destination, for which we extend both the upper bound and the achievable rate.**

I. Introduction

Real-time interactive video streaming is becoming an integral part of people's lives throughout the world. Where traditionally, Internet traffic is not sensitive to latency, remote learning, remote working and remote monitoring become more and more in use. According to [3], live video will grow 15-fold to reach 17 percent of Internet video traffic by 2022.

A fundamental difference between real-time interactive video streaming and other popular services is the latency constraint imposed on each packet to result in a good user experience. While for other services (such as file downloading and even non-real-time video streaming) buffering can be used to result in a good user experience, in real-time interactive video, buffering is precluded.

As each transmission over each network is susceptible to errors (which we model as packet erasures), there are two main mechanisms to handle it. Automatic repeat request (ARQ) is one popular method in which the receiver acknowledges the transmitter which packets arrived and which did not, and the transmitter sends the erased packets again. See, e.g. [4]. Many variants of ARQ were suggested to reduce the memory utilization at the sender or to reduce overhead traffic (for example, Stop-and-wait ARQ, Go-Back-N ARQ, and Selective Repeat ARQ/Selective Reject ARQ, see, e.g. [5], [6]).

While many works focused on improving the efficiency of ARQ, when considering a stingiest latency constraints, none can overcome its basic requirement, which is that the overall latency will be higher than a three-time one-way trip delay. Another mechanism to handle erasures is forward error correction (FEC). A plurality of works analyzed the guaranteed rate for streaming codes while assuming a maximal number of erasures (either a burst or arbitrary). See e.g., [7]–[11].

In [1], the capacity of a three-node network with a single link between each node was analyzed. It was shown that better modelling of the network results in a better-guaranteed streaming rate, and further that the capacity can be achieved by using symbol-wise decode and forward (SDWF) streaming code. In [2], the network model was extended to multiple nodes with a single link connecting each node. It was shown that when the used field size goes to infinity, state-dependent symbol-wise decode and forward (SD-SWDF) can achieve capacity.

Originally, routing protocols select only a single forwarding path for the traffic between each source-destination pair. It is well recognized that using multiple paths between the source and destination can improve throughput and/or robustness of transmission. See e.g., [12], [13].

Utilization of multiple paths for improving throughput was discussed in a plurality of papers [14], [15]. Utilization of multiple paths for low latency communication was discussed [16] where reducing the average delay was the focus. Combination of multiple paths and forward error correction was discussed in [17], but no guaranteed performance was derived.

In this paper, we first extend the analysis done in [2] from a multi-node single link-network to a multi-node multi-link network (which we denote as relayed network). Using the network structure, we derive an upper bound on the guaranteed rate of streaming code, assuming a (different) maximal number of erasures per link. We then suggest an extension for the SD-SWDF scheme, which builds upon it. Using concatenation of

SD-SWDF codes used on each of the different paths from the source to the destination, we derive a closed-form expression for an achievable rate. We further show that this scheme can be improved by using different codes with different multiplicities. Finding the optimal amount of multiplicity per path can be formalized as a convex optimization problem. We further demonstrate the trade-off between the achievable rate and the (maximal) size of transmitted packets of the network.

We then extend this analysis to a more general network - any acyclic directed graph between the source and destination. For this network, the upper bound and the achievable rate are expressed as an optimization over the network's parameters, which is tractable for practical networks.

The rest of this paper is organized as follows. Section I-A outlines the network model of a relayed network. Section I-B described the known results for some specific relayed networks. Section I-C outlines the main results of this paper. As both the upper bound and the achievable scheme use concatenation of streaming codes, concatenation is defined in Section II. The upper bound of the relayed network is derived in Section III, and the achievable scheme is described in Section V. The network model is extended to directed acyclic graph in Section VI, where the network model is described in Section VI-A, the upper bound is described in Section VI-B and the achievable scheme is described in Section VI-C. Finally, we provide detailed examples for three networks in Section VII.

A. Relayed Network Model

A source node wants to send a sequence of messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ to a destination node with the help of H middle nodes $r_1, \ldots, r_H$. To ease notation we denote the source node as r_0, and destination node as r_{H+1}. We assume there exists links only between node j to node $j+1$. We assume L_1 links between the source the first relay (which we also denote as the first hop). In general, we assume L_j links between node r_{j-1} and r_j (which we denote that the j'th hop).

We assume that, on the discrete timeline, each link i in the first hop introduces at most $N_i^{(1)}$ erasures. In general, each link i in the j'th hop introduces at most $N_i^{(j)}$ erasures.

We denote by $\mathbf{N}^{(1)} = [N_1^{(1)}, \ldots, N_{L_1}^{(1)}]$ the erasures introduces in the first hop and by $\mathbf{N}^{(j)} = [N_1^{(j)}, \ldots, N_{L_j}^{(j)}]$ the erasures introduced in the j'th hop. For simplicity, we denote the field $\mathbb{F}_e^n \triangleq \mathbb{F}^n \cup \{*\}$. This network is depicted in Fig 1 and formalized below.

Definition 1. *Let $\mathbf{n}^{(j)} = [n_1^{(j)}, \ldots, n_{L_j}^{(j)}]$. An $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code consists of the following:*

1) *A sequence of source messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ where $\mathbf{s}_i \in \mathbb{F}^k$.*
2) *A list of L_1 encoding functions $f_{t,i}^{(1)} : \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{t+1 \text{ times}} \to \mathbb{F}^{n_1^{(1)}}$, $i \in \{1, \ldots, L_1\}$ used by the source at time t to generate*
$$\mathbf{x}_{t,i}^{(1)} = f_{t,i}^{(1)}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_t).$$
3) *A list of relaying functions for node $j \in [1, \ldots, H]$,*
$$f_{t,i}^{(j+1)} : \underbrace{\mathbb{F}_e^{n_1^{(j)}} \times \ldots \times \mathbb{F}_e^{n_1^{(j)}}}_{t+1 \text{ times}} \ldots \underbrace{\mathbb{F}_e^{n_{L_j}^{(j)}} \times \ldots \times \mathbb{F}_e^{n_{L_j}^{(j)}}}_{t+1 \text{ times}} \to \mathbb{F}^{n_i^{(j+1)}},$$
$i \in \{1, \ldots, L_{j+1}\}$ used by node j at time t to generate
$$\mathbf{x}_{t,i}^{(j+1)} = f_{t,i}^{(j+1)}\left(\{\mathbf{y}_{0,l}^{(j)}\}_{l=1}^{L_j}, \ldots, \{\mathbf{y}_{t,l}^{(j)}\}_{l=1}^{L_j}\right).$$
4) *A decoding function*
$$\varphi_{t+T} : \underbrace{\mathbb{F}_e^{n_1^{(H+1)}} \times \ldots \times \mathbb{F}_e^{n_1^{(H+1)}}}_{T+t+1 \text{ times}} \ldots \underbrace{\mathbb{F}_e^{n_{L_{H+1}}^{(H+1)}} \times \ldots \times \mathbb{F}_e^{n_{L_{H+1}}^{(H+1)}}}_{T+t+1 \text{ times}} \to \mathbb{F}^k,$$
$i \in \{1, \ldots, L_{H+1}\}$ is used by the destination (node r_{H+1}) at time $t+T$ to estimate $\mathbf{s_i}$ according to $\hat{\mathbf{s}}_t = \varphi_{t+T}(\{\mathbf{y}_{0,l}^{(H+1)}\}_{l=1}^{L_{H+1}}, \ldots, \{\mathbf{y}_{t+T,l}^{(H+1)}\}_{l=1}^{L_{H+1}})$.

Definition 2. *An erasure sequence is a binary sequence denoted by $e^\infty \triangleq \{e_t\}_{t=0}^{\infty}$, where $e_t = \mathbf{1}\{$an erasure occurs at time $t\}$.*

An N-erasure sequence is an erasure sequence e^∞ that satisfies $\sum_{t=0}^{\infty} e_t^\infty = N$. In other words, an N-erasure sequence specifies N arbitrary erasures on the discrete timeline. The set of N-erasure sequences is denoted by Ω_N.

Definition 3. *The mapping $g_n : \mathbb{F}^n \times \{0,1\} \to \mathbb{F}_e^n$ of an erasure channel is defined as*
$$g_n(\mathbf{x}, e) = \begin{cases} \mathbf{x} & \text{if } e = 0 \\ * & \text{if } e = 1. \end{cases} \tag{1}$$
For any erasure sequence e^∞ and any $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code, the following input-output relation holds for the i'th link in the j'th hop ($j \in \{1, \ldots, H+1\}$), for each $t \in \mathbb{Z}_+$:
$$\mathbf{y}_{t,i}^{(j)} = g_{n_i^{(j)}}(\mathbf{x}_{t,i}^{(j)}, e_{t,i}^{(j)}), \tag{2}$$
where $e_{t,i}^{(j)} \in \Omega_{N_i^{(j)}}$, $i \in \{1, \ldots, L_j\}$.

Definition 4. *An $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code is said to be $(\mathbf{N}^{(1)}, \ldots \mathbf{N}^{(H+1)})$-achievable if, for any $e_{t,i}^{(j)} \in \Omega_{N_i^{(j)}}$, for all $j \in \{1, \ldots, H+1\}$, for all $i \in \{1, \ldots, L_j\}$ and for all $t \in \mathbb{Z}_+$ and all $\mathbf{s}_t \in \mathbb{F}^k$, we have $\hat{\mathbf{s}}_t = \mathbf{s}_t$.*

Definition 5. *The rate of an $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code is $\frac{k}{\max\left(\max(\mathbf{n}^{(1)}), \ldots \max(\mathbf{n}^{(H+1)})\right)}$.*

Definition 6. *The $(T, \mathbf{N}^{(1)}, \ldots \mathbf{N}^{(H+1)})$-capacity, denoted by $C_{T,\mathbf{N}^{(1)},\ldots\mathbf{N}^{(H+1)}}$, is the maximum achievable rate by $(\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming codes that are $(\mathbf{N}^{(1)}, \ldots \mathbf{N}^{(H+1)})$-achievable.*

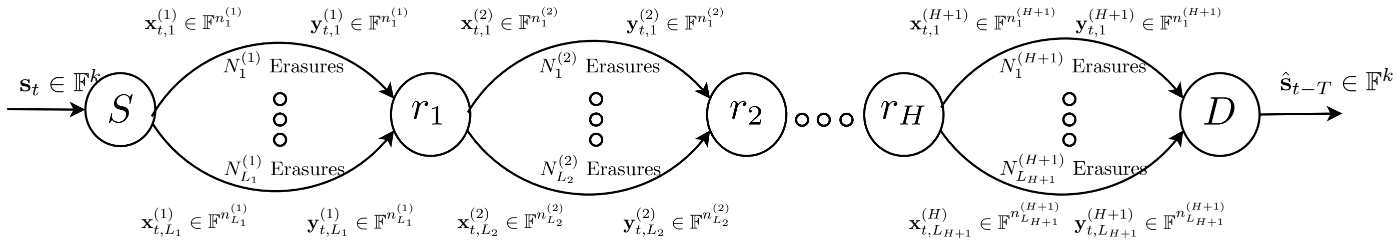

Fig. 1: Symbols generated in the $H+2$-node relay network at time i.

B. Known results

For several (relayed) network configurations the capacity is known.

1) **Point-to-point link** ($H=1,\ L_1=1$): It was previously shown [8] that the maximum achievable rate of the point-to-point packet erasure channel with $N_1^{(1)}$ arbitrary erasures and delay of $T \geq N_1^{(1)}$ denoted by $C_{T,N_1^{(1)}}$ satisfies

$$C_{T,N_1^{(1)}} = \frac{T-N_1^{(1)}+1}{T+1}. \tag{3}$$

2) **Point-to-point network with multiple links** ($H=1,\ L$): The analysis of [8] can be easily extended for the case of multiple links between the source and destination. For completeness, it is provided in Appendix**??**. Assuming N_i arbitrary erasures on the ith link and delay of $T \geq \min(\mathbf{N}^{(1)})$, denoting $\mathbf{N}^{(1)} = [N_1^{(1)}, \ldots, N_L^{(1)}]$, the capacity of this network is

$$C_{T,N_1^{(1)},\ldots,N_L^{(1)}} = \frac{\sum_{i=1}^{L}\left(T-N_i+1\right)^+}{T+1}. \tag{4}$$

3) **Three-node-network with a single link between the nodes** ($H=2,\ L_1=L_2=1$): In [1], a three-node relay network was analyzed, and it was shown than for any $(T, N_1^{(1)}, N_1^{(2)})$, when $T \geq N_1^{(1)} + N_1^{(2)}$ we have

$$C_{T,N_1^{(1)},N_1^{(1)}} = \min\left(C_{T-N_1^{(2)},N_1^{(1)}}, C_{T-N_1^{(1)},N_1^{(2)}}\right). \tag{5}$$

4) $h+1$**-node-network with a single link between the nodes** ($H=h,\ L_j=1\ \forall j\in\{1,\ldots,h\}$): In [2] it was shown that for a target overall delay of $T \geq \sum_{l=1}^{h+1} N_1^{(l)}$, the achievable rate is upper bounded by

$$R \leq \frac{T-\sum_{l=1}^{h+1} N_1^{(l)}+1}{T-\min_j \sum_{l=1,l\neq j}^{h+1} N_1^{(l)}+1}. \tag{6}$$

Denoting

$$n_{\max} \triangleq \max_{j\in\{1,\ldots,h+1\}} \left(T - \sum_{l=1,l\neq j}^{h+1} N_1^{(l)} + 1\right), \tag{7}$$

it was shown that when $|\mathbb{F}| \geq n_{\max}$, the following rate is achievable

$$R \geq \frac{T-\sum_{l=1}^{h+1} N_1^{(l)}+1}{T-\min_j\left\{\sum_{l=1,l\neq j}^{h+1} N_1^{(l)}\right\}+1+\frac{n_{\max}\lceil\log(n_{\max})\rceil}{\log(|\mathbb{F}|)}}. \tag{8}$$

which approaches the upper bound as $|\mathbb{F}| \to \infty$

C. Main Results

In this paper, we first derive an upper bound for the achievable rate in $H+1$-node relayed network where each hop has L_j links. Denote the average number of maximal erasures in the j'th hop as

$$\bar{\mathbf{N}}^{(j)} = \frac{1}{L_j}\sum_{i=1}^{L_j} N_i^{(j)}. \tag{9}$$

We thus have the following Lemma

Lemma 1. *Assume a relayed network with H relays, each having L_j links in the j'th hop. For a target overall delay of $T \geq \sum_{l=1}^{H+1}\min(\mathbf{N}^{(l)})$ we have*

$$R \leq \min_j \left(\frac{\sum_{i=1}^{L_j}\left(T-\sum_{l=1,l\neq j}^{H+1}\min(\mathbf{N}^{(l)})-N_i^{(j)}+1\right)^+}{T-\sum_{l=1,l\neq j}^{H+1}\min(\mathbf{N}^{(l)})+1}\right) \tag{10}$$

Corollary 1. *When*

$$T \geq \max_j\left(\sum_{l=1,l\neq j}^{H+1}\min(\mathbf{N}^{(l)})+\max(\mathbf{N}^{(j)})\right),$$

the overall rate is upper bounded by

$$R \leq \min_j\left(\frac{T+1-\sum_{l\neq j}\min(\mathbf{N}^{(l)})-\bar{\mathbf{N}}^{(j)}}{\frac{1}{L_j}\left(T+1-\sum_{l\neq j}\min(\mathbf{N}^{(l)})\right)}\right). \tag{11}$$

Denoting

$$n_{\max} \triangleq \max_{j\in\{1,\ldots,L+1\}}\left(T-\sum_{l=1,l\neq j}^{L+1}\min(\mathbf{N}^{(l)})+1\right), \tag{12}$$

we then show the following Lemma on the achievable rate.

Lemma 2. *Assume a path with H relays having L_J links between each relay. For a target overall delay of T, where the maximal number of arbitrary erasures in the i'th link in the j'th hop is $N_i^{(j)}$ and $T \geq \sum_{l=1}^{H+1}\max(\mathbf{N}^{(l)})$.*

When $|\mathbb{F}| \geq n_{\max}$, The following rate is achievable.

$$R_i^{(j)} \geq \frac{T+1-\sum_{l=1,l\neq j}^{h+1} \bar{\mathbf{N}}^{(l)} - N_i^{(j)}}{T+1-\sum_{l\neq j} \bar{\mathbf{N}}^{(l)} + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}} \tag{13}$$

where $n_{\max}$ is defined in (12).

Remark 1. *It can be shown that the achievable rate equals the upper bound only when the number of maximal erasures per link is equal for all links in the same hop (and thus the average (of the maximal) number of erasures per link in a hop equals the minimum (of the maximal) number of erasures per link in a hop.*

Corollary 2. *When $T \geq \sum_{l=1}^{H+1} \max(\mathbf{N}^{(l)})$, the following rate is achievable*

$$R \geq \min_j \left(\frac{T+1-\sum_j \bar{\mathbf{N}}^{(l)}}{\frac{1}{L_j}\left(T+1-\sum_{l\neq j} \bar{\mathbf{N}}^{(l)} + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}\right)} \right). \tag{14}$$

We further provide an optimization method that can be used to improve the achievable rate.

Remark 2. *We show next that the suggested achievable scheme holds for any $T \geq \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$ albeit, the overall rate can not be expressed in a compact form.*

We then extend the analysis to any directed acyclic graph with a single source, single destination and a total of L links. While there are no compact expressions derived for both the upper bound and the achievable scheme, we show that, as expected, the derived expressions converge to the expressions derived for the relayed network.

II. Concatenation of streaming codes

As concatenation of streaming codes is a building block in showing both the upper bound and achievable schemes, we start by defining it.

Definition 7. *A concatenation of an $(n', k', T')_{\mathbb{F}}$ point-to-point code with an $(n'', k'', T'')_{\mathbb{F}}$ point-to-point code is an $(n'+n'', k'+k'', [T', T''])_{\mathbb{F}}$ point-to-point code with the following properties*

- *Let $\mathbf{s} = [\mathbf{s'}_i \ \mathbf{s''}_i]^T$ be the the input to the concatenated code where $s'_i \in \mathbb{F}^{k'}$ and $s''_i \in \mathbb{F}^{k''}$.*
- *Let $\{f_t^{(1)'}\}$ be the encoding function of the first code and $\{f_t^{(1)''}\}$ be the encoding function of the second code. The encoding function of the concatenated code outputs $[f_t^{(1)'}(s'_0, \ldots, s'_t) \ f_t^{(1)''}(s''_0, \ldots, s''_t)]^T$*
- *Let $\mathbf{y}_t$ denote the input to the decoder of the concatenated code. Denote $\{\varphi'_{t+T'[j]}\}_{j=1}^{k'}$ as the list of decoding functions of the first code and $\{\varphi''_{t+T''[j]}\}_{j=1}^{k''}$ as the list of decoding functions of the second code. The output of the concatenated code is*

$$\begin{aligned}\hat{\mathbf{s}}_t &= [\hat{\mathbf{s}}'_t \ \hat{\mathbf{s}}''_t]^T \\ &= [\varphi_{i+T'}(\mathbf{y}_0, \ldots, \mathbf{y}_{t+T'}) \ \varphi_{i+T''}(\mathbf{y}_0, \ldots, \mathbf{y}_{t+T''})]^T\end{aligned}$$

We denote the concatenated code as $\frac{k'}{n'} \frown \frac{k''}{n''}$. We note that from this definition the rate of the concatenated code is $\frac{k'+k''}{n'+n''}$. The achievable scheme we describe next extends this definition to multi-hop codes (state-dependent symbol-wise decode and forward code) and analyzes the resulting rate when the relaying functions are restricted to generate a concatenated version of these codes (i.e., each stream is forwarded to a different link).

The following Proposition follows immediately from definition 4 (for the case of $H = 0$ and $L_1 = 1$) for concatenated codes.

Proposition 1. *Concatenation of two N achievable streaming codes $(n', k', T')_{\mathbb{F}}$ and $(n'', k'', T'')_{\mathbb{F}}$, $(n'+n'', k'+k'', [T', T''])_{\mathbb{F}}$, is also N achievable streaming code.*

Following Proposition 1, we have that if $(n, k, T)_{\mathbb{F}}$-streaming code is N achievable than concatenating it c times (denoted as $(c \cdot n, c \cdot k, [T, T, \ldots, T])_{\mathbb{F}}$-streaming code) is also N achievable for any $c \in \mathbb{Z}^+$. In the sequel we sometime refer to concatenation of a code c times as transmitting the code with multiplicity c.

III. Upper bound

Lemma 3. *Assume a point-to-point link with L_1 links connecting the source and destination. We have*

$$\frac{k}{\max(\mathbf{n})} \leq \begin{cases} \frac{\sum_{l=1}^{L_1}\left(T+1-N_l^{(1)}\right)^+}{T+1} & \text{if } T \geq \min(\mathbf{N}^{(1)}) \\ 0 \quad \text{Otherwise} \end{cases} \tag{15}$$

Proof. When assuming systematic codes, the Lemma follows from analyzing the periodic erasure channel induced by the parameters of the link. Any $(N_1^{(1)}, N_2^{(1)} \ldots, N_{L_1}^{(1)})$-achievable $(n_1^{(1)}, n_2^{(1)}, \ldots, n_{L_1}^{(1)}, k, T)_{\mathbb{F}}$-streaming code can recover the following periodic erasure pattern

$$\tilde{e}_{i,l} = \begin{cases} 0 & \text{if } 0 \leq i \bmod (T+1) \leq T - N_l^{(l)} + 1 \\ 1 & \text{otherwise} \end{cases} \tag{16}$$

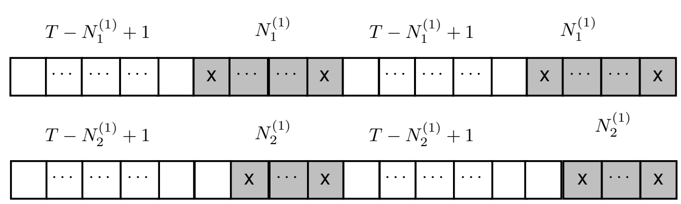

Fig. 2: A periodic erasure sequence with period T.

While such a channel is not included in our model, any$(N_1^{(1)}, N_2^{(1)} \ldots, N_{L_1}^{(1)})$-achievable

$(n_1^{(1)}, n_2^{(1)}, \ldots, n_{L_1}^{(1)}, k, T)_{\mathbb{F}}$-streaming code is also feasible for the proposed periodic erasure channel.[1]

Thus, when $T \geq \min(\mathbf{N}^{(1)})$, the achievable rate of the code is upper bounded as

$$R \leq \sum_{l=1}^{L_j} \left(1 - \frac{N_i^{(1)}}{T+1}\right)^+ = \frac{\sum_{l=1}^{L_j} \left(T + 1 - N_i^{(1)}\right)^+}{T+1}$$

□

Lemma 4. *Assume a relayed network with H relays having a single link in the j'th hop (and $L_l \geq 1$ links in the l'th hop for any $l \neq j$). We have*

$$\frac{k}{n_1^{(j)}} \leq \begin{cases} \frac{T - \sum_{l=1, l\neq j}^{H+1} \min(\mathbf{N}^{(l)}) - N_1^{(j)} + 1}{T - \sum_{l=1, l \neq j}^{H+1} \min(\mathbf{N}^{(l)}) + 1} & \text{if } T \geq \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)}) \\ 0 \quad \text{Otherwise} \end{cases} \tag{17}$$

Proof of Lemma 4. The proof follows the footsteps of Theorem 1 in [2]. We show all the steps for the case of $j = 1$ (i.e. having a single link at the first hop).

Suppose we are given an $(N_1^{(1)}, \mathbf{N}^{(2)} \ldots, \mathbf{N}^{(H+1)})$-achievable $(n_1^{(1)}, \mathbf{n}^{(2)} \ldots, \mathbf{n}^{(H+1)}, k, T)_{\mathbb{F}}$-streaming code for some $n_1^{(1)}, \mathbf{n}^{(2)} \ldots, \mathbf{n}^{(H+1)}, k$ and $\mathbb{F}$.

To this end, we let $\{\mathbf{s}_i\}_{i \in \mathbb{Z}_+}$ be i.i.d. random variables where $\mathbf{s}_0$ is uniform on $\mathbb{F}^k$. Since the $n_1^{(1)}, \mathbf{n}^{(2)} \ldots, \mathbf{n}^{(H+1)}, k$ and $\mathbb{F}$-streaming code is $(N_1^{(1)}, \mathbf{N}^{(2)} \ldots, \mathbf{N}^{(H+1)})$-achievable, it follows from Definition 4 that

$$H\Big(\mathbf{s}_i \,\Big|\, \{\mathbf{y}_{0,l}^{(L+1)}\}_{l=1}^{l=L_{H+1}}, \{\mathbf{y}_1^{(L+1)}\}_{l=1}^{l=L_{H+1}}, \ldots, \{\mathbf{y}_T^{(L+1)}\}_{l=1}^{l=L_{H+1}}\Big) = 0 \tag{18}$$

for any $i \in \mathbb{Z}_+$ and any $e_i^{(j),\infty} \in \Omega_{N_i^{(j)}}$. Consider the two cases.

Case $T < \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$: Let $e_i^{(j),\infty} \in \Omega_{N_i^{(j)}}$ be the error sequence on the l'th link in the j'th hop such that the first $\min(\mathbf{N}^{(j)})$ erasures are

$$e_i^{(j),\infty} = 1 \quad \text{if } \sum_{l=1}^{j-1} \min(\mathbf{N}^{(j)}) \leq i \leq \sum_{l=1}^{j} \min(\mathbf{N}^{(l)}) - 1 \tag{19}$$

and the other erasures are located arbitrarily.

Assuming without loss of generality that the first link in each hop has the minimal number of erasures. We note that (19) means that packet $\mathbf{y}_{\min(\mathbf{N}^{(1)}),1}^{(1)}$ is the first packet which can be used to recover $\mathbf{s}_0$ at r_1. Further, $\mathbf{y}_{\min(\mathbf{N}^{(1)})+\min(\mathbf{N}^{(2)}),1}^{(r_2)}$ is the first packet which can be used to recover $\mathbf{s}_0$ at r_2. Continuing the transmission across all other relays, it follows that $\mathbf{y}_{\sum_{\min(\mathbf{N}^{(j)})},1}^{(r_{H+1})}$ is the first packet which can be used to recover $\mathbf{s}_0$. Since $T < \sum_{l=1}^{H+1} \min(\mathbf{N}^{(j)})$ it follows the delay constraint can not be met.

Hence, due to (19) and Definition 1, we have

$$I\Big(\mathbf{s}_0; \{\mathbf{y}_{0,l}^{(L+1)}\}_{l=1}^{l=L_{H+1}}, \{\mathbf{y}_1^{(L+1)}\}_{l=1}^{l=L_{H+1}}, \ldots, \{\mathbf{y}_T^{(L+1)}\}_{l=1}^{l=L_{H+1}}\Big) = 0. \tag{20}$$

Combining (29), (20) and the assumption that $T < \sum_{l=1}^{L+1} N_l$, we obtain $H(\mathbf{s}_0) = 0$. Since $\mathbf{s}_0$ consists of k uniform random variables in $\mathbb{F}$, the only possible value of k is zero, which implies

$$\frac{k}{n_1} = 0. \tag{21}$$

Case $T \geq \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$:

We note that for every $i \in \mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered by node r_1 by time $i + T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)})$ given that $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{i-1}$ have been correctly decoded by node r_1, or otherwise a length $\min(\mathbf{N}^{(2)})$ burst erasure from time $i + T - \sum_{l=2}^{L+1} \min(\mathbf{N}^{(l)}) + 1$ to $i + T - \sum_{l=3}^{L+1} \min(\mathbf{N}^{(l)})$ introduced on all channels between r_1 to r_2 followed by a length $\min(N^{(3)})$ burst erasure from time $i + T - \sum_{l=3}^{L+1} \min(\mathbf{N}^{(l)}) + 1$ to $i + T - \sum_{l=4}^{L+1} \min(\mathbf{N}^{(l)})$ introduced on all channels between (r_3, r_4) and so on until a length $\min(\mathbf{N}^{(H+1)})$ burst erasure from time $i + T - \min(\mathbf{N}^{H+1}) + 1$ to $i + T$ would result in a decoding failure for node r_1, node r_2 and all the nodes up to the destination r_{H+1}.

It then follows that

$$H\Big(\mathbf{s}_i \,\Big|\, \Big\{\mathbf{x}_{i,1}^{(1)}, \mathbf{x}_{i+1,1}^{1)}, \ldots, \mathbf{x}_{i+T-\sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}),1}^{(1)}\Big\} \setminus \Big\{\mathbf{x}_{\theta_1,1}^{(1)}, \ldots, \mathbf{x}_{\theta_{N_1^{(1)}},1}^{(1)}\Big\}, \mathbf{s}_0, \ldots, \mathbf{s}_{i-1}\Big) = 0 \tag{22}$$

for any $i \in \mathbb{Z}_+$ and $N_1^{(1)}$ non-negative integers denoted by $\theta_1, \ldots, \theta_{N_1^{(1)}}$.

We note that for all $q \in \mathbb{Z}_+$,

$$\begin{aligned} &\Big|\Big\{q, q+1, \ldots, q + T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)})\Big\} \bigcap \\ &\Big\{m \cdot (T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}) + 1), \\ &1 + m \cdot (T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}) + 1) \ldots, \\ &T - N_1^{(1)} - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}) + m \cdot (T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}) + 1)\Big\}_{m=0}^{j}\Big| \\ &= T - N_1^{(1)} - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}) + 1. \end{aligned} \tag{23}$$

Hence, from (22), (23) and the chain rule we have

$$H\Big(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}) + (j-1)(T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)}) + 1)} \,\Big|$$

[1] A similar converse argument involving periodic erasure channel for the burst-erasure channel is also presented in [7], and [18]. For a rigorous information theoretic argument, we refer the reader to [19], [20], and [21] for the case of burst erasure channel. A similar approach can be used in the present setup, but it will not be presented

$$\Big\{\mathbf{x}^{(1)}_{m\cdot(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1),1},\mathbf{x}^{(1)}_{1+m\cdot(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1),1},\ldots$$
$$,\mathbf{x}^{(1)}_{T-N_1-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+m\cdot(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1),1}\Big\}_{m=0}^{j}\Big)=0. \tag{24}$$

Therefore, following the arguments in [1], we note that the (24) involves $j(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1)$ source messages and $(j+1)(T-\sum_{l=1}^{H+1}\min(\mathbf{N}^{(l)})+1)$ source packets. Therefore, the $(N_1^{(1)},\ldots,\mathbf{N}^{H+1})$-achievable $(n_1,\ldots,\mathbf{n}^{HL+1},k,T)_{\mathbb{F}}$-streaming code restricted to channel (r_0,r_1) can be viewed as a point-to-point streaming code with rate k/n_1 and delay $T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})$ which can correct any $N_1^{(1)}$ erasures. In particular the point-to-point code can correct the periodic erasure sequence $\tilde{e}^{\infty}$ depicted in Figure 3, which is formally defined as

$$\tilde{e}_i=\begin{cases}0 \ \text{ if } 0 & \leq i \bmod (T-\sum_{l=2}^{L+1}\min(\mathbf{N}^l)+1)\leq \\ & T-\sum_{l=1}^{L+1}\min(\mathbf{N}^l)+1 \\ 1 \ \text{ otherwise} & \end{cases} \tag{25}$$

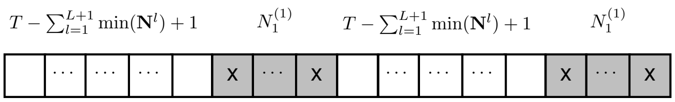

Fig. 3: A periodic erasure sequence with period $T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})$.

By standard arguments, we conclude that

$$\begin{aligned}\frac{k}{n_1^{(1)}}&\leq\frac{T-\sum_{l=2}^{L+1}\min(\mathbf{N}^l)-N_1^{(1)}+1}{T-\sum_{l=2}^{L+1}\min(\mathbf{N}^l)+1}\\&=C_{T-\sum_{l=2}^{L+1}\min(\mathbf{N}^l),N_1^{(1)}}.\end{aligned} \tag{26}$$

Remark 3. *Following proposition 1, using multiplicity $c\in\mathbb{Z}^+$ does not change the behaviour of the code, thus for a code with multiplicity c we have*

$$\frac{k}{n_1^{(1)}}\leq\frac{c\left(T-\sum_{l=2}^{L+1}\min(\mathbf{N}^l)-N_1^{(1)}+1\right)}{c\left(T-\sum_{l=2}^{L+1}\min(\mathbf{N}^l)+1\right)}. \tag{27}$$

□

We next show the following Lemma

Lemma 5. *Assume a relayed network with H relays with L_j links in the j'th hop . If $T\geq\sum_{l=1}^{H+1}\min(\mathbf{N}^{(l)})$ we have*

$$\frac{k}{\max(\mathbf{n}^{(j)})}\leq\frac{\sum_{l=1}^{L_j}\left(T-\sum_{l=1,l\neq j}^{H+1}\min(\mathbf{N}^{(l)})-N_l^{(j)}+1\right)^+}{T-\sum_{l=1,l\neq j}^{H+1}\min(\mathbf{N}^{(l)})+1} \tag{28}$$

Proof of Lemma 5. We show all the steps for the case of $j=1$.

Suppose we are given an $(\mathbf{N}^{(1)},\mathbf{N}^{(2)}\ldots,\mathbf{N}^{(H+1)})$-achievable $(\mathbf{n}^{(1)},\mathbf{n}^{(2)}\ldots,\mathbf{n}^{(H+1)},k,T)_{\mathbb{F}}$-streaming code for some $\mathbf{n}^{(1)},\mathbf{n}^{(2)}\ldots,\mathbf{n}^{(H+1)},k$ and $\mathbb{F}$. it follows from Definition 4 that

$$\begin{aligned}H\Big(\mathbf{s}_i\,\Big|\,&\{\mathbf{y}_{0,l}^{(L+1)}\}_{l=1}^{l=L_{H+1}},\{\mathbf{y}_{1}^{(L+1)}\}_{l=1}^{l=L_{H+1}},\\&\ldots,\{\mathbf{y}_{T}^{(L+1)}\}_{l=1}^{l=L_{H+1}}\Big)=0\end{aligned} \tag{29}$$

for any $i\in\mathbb{Z}_+$ and any $e_i^{(j),\infty}\in\Omega_{N_i^{(j)}}$. Consider the following two cases

Case $T\geq\max(\mathbf{N}^{(1)})+\sum_{l=2}^{H+1}\min\mathbf{N}^{(l)}$:

Repeating the same arguments as presented in Lemma 4 We note that for every $i\in\mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered by node r_1 by time $i+T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})$ given that $\mathbf{s}_0,\mathbf{s}_1,\ldots,\mathbf{s}_{i-1}$ have been correctly decoded by node r_1, or otherwise a length $\min(\mathbf{N}^{(2)})$ burst erasure from time $i+T-\sum_{l=2}^{L+1}\min(\mathbf{N}^{(l)})+1$ to $i+T-\sum_{l=3}^{L+1}\min(\mathbf{N}^{(l)})$ introduced on all channels between r_1 to r_2 followed by a length $\min(N^{(3)})$ burst erasure from time $i+T-\sum_{l=3}^{L+1}\min(\mathbf{N}^{(l)})+1$ to $i+T-\sum_{l=4}^{L+1}\min(\mathbf{N}^{(l)})$ introduced on all channels between (r_3,r_4) and so on until a length $\min(\mathbf{N}^{(H+1)})$ burst erasure from time $i+T-\min(\mathbf{N}^{H+1})+1$ to $i+T$ would result in a decoding failure for node r_1, node r_2 and all the nodes up to the destination r_{H+1}.

It then follows that

$$\begin{aligned}H\Big(\mathbf{s}_i\,\Big|\,&\Big\{\Big\{\mathbf{x}_{i,l}^{(1)},\mathbf{x}_{i+1,l}^{1)},\ldots,\mathbf{x}^{(1)}_{i+T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)}),l}\Big\}\setminus\\&\Big\{\mathbf{x}^{(1)}_{\theta_1^{(l)},l},\ldots,\mathbf{x}^{(1)}_{\theta^{(l)}_{N_1^{(1)}},l}\Big\}\Big\}_{l=1}^{L_1},\mathbf{s}_0,\ldots,\mathbf{s}_{i-1}\Big)=0\end{aligned} \tag{30}$$

for any $i\in\mathbb{Z}_+$ and $N_l^{(1)}$ non-negative integers denoted by $\theta_1^{(l)},\ldots,\theta^{(l)}_{N_1^{(1)}}$.

Hence, from (30), (23) and the chain rule we have

$$\begin{aligned}&H\Big(\mathbf{s}_0,\mathbf{s}_1,\ldots,\mathbf{s}_{T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+(j-1)(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1)}\,\Big|\\&\Big\{\Big\{\mathbf{x}^{(1)}_{m\cdot(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1)},\mathbf{x}^{(1)}_{1+m\cdot(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1),l},\ldots\\&,\mathbf{x}^{(1)}_{T-N_1-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+m\cdot(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1),l}\Big\}_{m=0}^{j}\Big\}_{l=1}^{L_1}\Big)=0.\end{aligned} \tag{31}$$

Therefore, following the arguments in [1], we note that the (31) involves $j(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})+1)$ source messages and $(j+1)(T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})^{L+1}-N_l^{(1)}+1)$ channel packets sent over the l'th link.

Therefore, the $(\mathbf{N}^{(1)},\ldots,\mathbf{N}^{(H+1)})$-achievable $(n_1,\ldots,\mathbf{n}^{HL+1},k,T)_{\mathbb{F}}$-streaming code restricted to channel (r_0,r_1) can be viewed as a multi point-to-point streaming code with rate $k/\max(\mathbf{n}^{(1)})$ and delay $T-\sum_{l=2}^{H+1}\min(\mathbf{N}^{(l)})$ which can correct any $N_l^{(1)}$ erasures on the l'th link. In particular the multi point-to-point code can correct the periodic erasure sequences $\tilde{e}_i^{\infty}$ depicted in Figure 3, which is formally defined as

$$\tilde{e}_{i,l} = \begin{cases} 0 \;\; \text{if } 0 \leq & i \bmod (T - \sum_{l=2}^{L+1} \min(\mathbf{N}^l) + 1) \leq \\ & T - \sum_{l=2}^{L+1} \min(\mathbf{N}^l) - N_l^{(l)} + 1 \\ 1 \;\; \text{otherwise} & \end{cases} \tag{32}$$

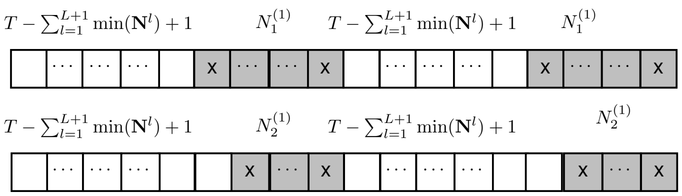

Fig. 4: A periodic erasure sequence with period $T - \sum_{l=2}^{H+1} \min(\mathbf{N}^{(l)})$.

Following Lemma 3 we conclude that

$$\frac{k}{\max(\mathbf{n}^{(1)})} \leq \frac{\sum_{l=1}^{L_1} \left(T - \sum_{l=2}^{L+1} \min(\mathbf{N}^l) - N_l^{(1)} + 1 \right)^+}{T - \sum_{l=2}^{L+1} \min(\mathbf{N}^l) + 1} \tag{33}$$

IV. Motivating Example

Consider a network with two hops, where $\mathbf{N}^{(1)} = [3\ 2]$, $\mathbf{N}^{(2)} = [2\ 1]$ and $T = 5$ which is depicted in Figure 7. Using (**??**) we have $R < \min\left(\frac{2+3}{5}, \frac{2+3}{4}\right) = 1$. Figure 2 also presents the four paths the network is composed of. Transmitting only on the best path (which is path 2 in this example) results in $R = 2/4$.

The suggested coding scheme, which we describe next, decompose the relayed network into multi-node single-link paths, transmit on each path capacity-achieving streaming code (SWDF in this example) and concatenates the codes transmitted over each link. SWDF, presented in [1] transmits diagonally interleaved block codes both at the source and at the relay while utilizing the fact that information symbols sent at the same time (each belong to a different block code), are guaranteed to be available at the relay at different times. For further details see [1].

Assume a source that sends eight (information) bits at each time instance denoted as $\mathbf{s}_i = [a_i,\ b_i,\ c_i,\ A_i,\ B_i,\ C_i,\ D_i,\ E_i,\ F_i]$ where bits denoted with small letters are transmitted over the upper link from the source to the relay, and the bits denoted with capital letters are transmitted over the lower link from the source to the relay. Table I depicts the coding scheme used by the source and Table **??** depicts the scheme used by the relay for the network in Figure 7. Each streaming code used over a specific path is denoted with a unique color (symbols belong to the same MDS block code, which is applied diagonally, are marked with the same frame type). To save space, we denote with “X” a parity symbol, noting that this is a place holder.

To demonstrate that this scheme meets the delay constraint (and to give a brief example for SWDF), let’s focus on the code

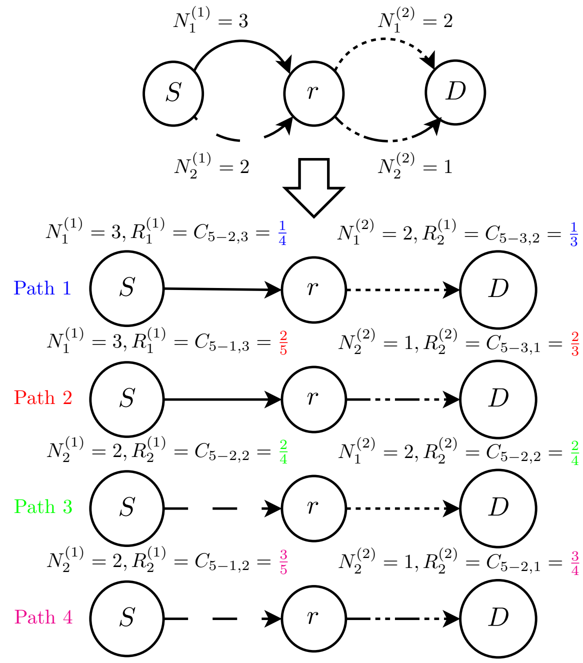

Fig. 5: Three-node network with two links between each node, $\mathbf{N}^{(1)} = [3\ 2]$, $\mathbf{N}^{(2)} = [2\ 1]$, $T = 5$, each link is marked with a unique line type, decomposed into four paths.

Time	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
$x_{i,1}^{(1)}[1] = a_i$	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}	a_{i+5}
$x_{i,1}^{(1)}[2] = X$	X	X	X	X	X	X
$x_{i,1}^{(1)}[3] = X$	X	X	X	X	X	X
$x_{i,1}^{(1)}[4] = X$	X	X	X	X	X	X
$x_{i,1}^{(1)}[5] = b_i$	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}	b_{i+5}
$x_1^{(1)}[6] = c_i$	c_i	c_{i+1}	c_{i+2}	c_{i+3}	c_{i+2}	c_{i+3}
$x_{i,1}^{(1)}[7] = X$	X	X	X	X	X	X
$x_{i,1}^{(1)}[8] = X$	X	X	X	X	X	X
$x_{i,1}^{(1)}[9] = X$	X	X	X	X	X	X

(a) Symbols transmitted by the source node s over link $L_1^{(1)}$ where $N_1^{(1)} = 3$ which are the outcome of concatenating SWDF codes sent in the first hop over paths 1 and 2.

Time	i	$i+1$	$i+2$	$i+3$	$i+4$	$i+5$
$x_{i,2}^{(1)}[1] = A_i$	A_i	A_{i+1}	A_{i+2}	A_{i+3}	A_{i+4}	A_{i+5}
$x_{i,2}^{(1)}[2] = B_i$	B_i	B_{i+1}	B_{i+2}	B_{i+3}	B_{i+4}	B_{i+5}
$x_{i,2}^{(1)}[3] = X$	X	X	X	X	X	X
$x_{i,2}^{(1)}[4] = X$	X	X	X	X	X	X
$x_{i,2}^{(1)}[5] = C_i$	C_i	C_{i+1}	C_{i+2}	C_{i+3}	C_{i+4}	C_{i+5}
$x_{i,2}^{(1)}[6] = D_i$	D_i	D_{i+1}	D_{i+2}	D_{i+3}	D_{i+4}	D_{i+5}
$x_{i,2}^{(1)}[7] = E_i$	E_i	E_{i+1}	E_{i+2}	E_{i+3}	E_{i+4}	E_{i+5}
$x_{i,2}^{(1)}[8] = X$	X	X	X	X	X	X
$x_{i,2}^{(1)}[9] = X$	X	X	X	X	X	X

(b) Symbols transmitted by the source node s over link $L_2^{(1)}$ where $N_2^{(1)} = 2$ which are the outcome of concatenating SWDF codes sent in the first hop over paths 3 and 4.

TABLE I: Symbols transmitted by the source node s.

sent over the second path (marked in red) which is composed of the first link in the first hop and the second link in the second hop. At time i the last five symbols the source sends (over the upper link) are $x_{i,2}^{(1)}[5:9] = [b_i, c_i, X = b_{i-2} + c_{i-1}, X = b_{i-3} + c_{i-2}, X = b_{i-4} + c_{i-3}]$ (note that each information symbol belongs to a different $(5,2)$ MDS block code).

Given a maximum of $N_2^{(1)} = 2$ erasures on this link, it is guaranteed that b_i is available at the relay at time $i+4$ and c_i is available at the relay at time $i+3$. Thus, the first three symbols the relay sends at time $i+3$ over the upper link from the relay to the destination are $x_{i+3,1}^{(2)}[1:3] = [c_i, b_{i-1}, X = c_{i-2} + b_{i-2}]$. Given a maximum of $N_1^{(2)} = 1$ erasures on this link, it is guaranteed that $[b_i, c_i]$ can be recovered at time $i+5$.

Since each of these codes is a capacity-achieving code, it is guaranteed that all the information symbols sent at time i can be recovered at time $i+5$. Recalling Definition 5, the achieved rate is $R = \frac{8}{\max(9,7)} = \frac{8}{9}$, thus improving significantly compared to using the best (yet a single) path.

Remark 4. *This scheme's performance can be improved when a different multiplicity for each of the codes applied on different paths is considered. Noting that applying multiplicity does not increase the overhead size, finding the optimal multiplicity amount can be formalized as a convex optimization problem. For the network in Figure 7, not transmitting the code on path number 1, and transmit the codes on paths* $2,3,4$ *with multiplicities of* $5,8,4$ *results in* $R = 0.95$.

V. Achievable Scheme

The two main building blocks of the achievable scheme are a concatenation of streaming codes described in Section II (that will now be used to concatenate different codes) and state-dependent symbol-wise decode and forward, which we briefly recall.

A. State-dependent symbol-wise decode and forward

Presented in [2], state-dependent symbol-wise decode and forward is an extension of symbol-wise decode and forward that was shown to achieve the upper bound in a three-node network with a single link between each node (see, e.g. [1]). Symbol-wise decode and forward does not wait for the entire message to be decoded at the relay but rather, utilizing the structure of the messages sent from the source (which uses diagonal interleaving), it was shown that some of the information symbols composing the message sent at time t from the source can be forwarded earlier to the destination and thus the upper bound can be achieved.

When trying to extend this idea to a network with more than three nodes (with a single link between each node), it can be shown that the upper bound can no longer be achieved. Nevertheless, as was shown in [2], given that the number of erasures does not exceed the maximal number of erasures per link for which the code was designed for, there are always available symbols (either information symbols or parity symbols) that can be forwarded earlier. Since this transmission method depends on the actual erasure pattern, an additional header needs to be added to the packet to allow the receiving node to process the received symbols.

It was shown that the achievable rate in the j'th hop of the multi-mode network with a single link between the nodes is

$$R^{(j)} = \frac{T + 1 - \sum_{l=1, l\neq j}^{h+1} N^{(l)} - N^{(j)}}{T + 1 - \sum_{l\neq j} N^{(l)} + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}}, \tag{34}$$

where $n_{\max}$ is defined in (12) where the overall rate is derived in (8) as the minimum of the achievable rate per hop over all hops.

It was shown in [2] that state-dependent symbol-wise decode and forward could achieve the upper bound up to a gap which depends on the field size used and further this gap vanishes as the field size increases. Further details on state-dependent symbol-wise decode and forward, along with a detailed definition of diagonal interleaving and some detailed examples, can be found in [2].

B. Suggested basic coding scheme

The suggested coding scheme treats the multi-node multi-link network as a group of paths, where each path is a multi-node single-link network.

Definition 8. *For a network with* $H+1$ *nodes with* L_j *links in the* j*'th hop* $(1 \leq j \leq H+1)$

- *The (unique)* m*'th path from the source to destination is defined as the set of indices* $\{l_{1,m}, l_{2,m}, \ldots, l_{H+1,m}\}$ *where* $1 \leq l_{1,m} \leq L_1$, $1 \leq l_{2,m} \leq L_2$, $\ldots$ $1 \leq l_{H+1,m} \leq L_{H+1}$*, i.e.,* $l_{j,m}$ *is the index of the link used in the* j*'th hop in a single path from the source to the destination. For* $m \neq m'$ *there exists* $1 \leq h \leq H+1$ *such that* $l_{h,m} \neq l_{h,m'}$.
- *Denote with* $\mathcal{M}_l^{(j)}$ *the set of all paths in which* $l_j = l$*, i.e., set of all paths which include link* $1 \leq l \leq L_j$ *in hop* j*. We note that that* $|\mathcal{M}_l^{(j)}| = \prod_{s\neq j} L_s$ *and further note that* $\mathcal{M}_l^{(j)} \cap \mathcal{M}_w^{(j)} = \emptyset$, $\forall 1 \leq l \neq w \leq L_j$.
- *Denote with* $\mathcal{M}_l^{(j)}[w]$ *the* w*'th path in* $\mathcal{M}_l^{(j)}$ *(noting that* $1 \leq w \leq \prod_{s\neq j} L_s$ *and* $1 \leq \mathcal{M}_l^{(j)}[w] \leq \prod L_s$*).*

Since each path is a multi-node network with a single link between each node, the upper bound for transmitting over this path is given in (**??**). Further, using state-dependent symbol-wise decode and forward over this path achieves the upper bound when $\mathbb{F} \to \infty$.

For completeness, we state the full definition of streaming code for a multi-mode network with a single link between each node for the m'th path, which is identical to Definition 1 in [2].

Definition 9. *An* $(n_m^{(1)}, n_m^{(2)}, \ldots, n_m^{(H+1)}, k_m, T)_{\mathbb{F}}$*-streaming code transmitted over the* m*'th path consists of the following:*

1) *A sequence of source messages* $\{\mathbf{s}_i\}_{i=0}^{\infty}$ *where* $\mathbf{s}_i \in \mathbb{F}^k$.

2) *An encoding function* $f_{t,m}^{(1)}: \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{t+1 \text{ times}} \to \mathbb{F}^{n_m^{(1)}}$ *used by the source at time* t *to generate*

$$\mathbf{x}_{t,m}^{(1)} = f_{t,m}^{(1)}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_t).$$

3) *A relaying function for node* $j \in [1, \ldots, H]$,

$$f_{t,m}^{(j+1)}: \underbrace{\mathbb{F}_e^{n_m^{(j)}} \times \ldots \times \mathbb{F}_e^{n_m^{(j)}}}_{t+1 \text{ times}} \to \mathbb{F}^{n_m^{(j+1)}},$$

4) *A decoding function*

$$\varphi_{t+T,m}: \underbrace{\mathbb{F}_e^{n_m^{(H+1)}} \times \ldots \times \mathbb{F}_e^{n_m^{(H+1)}}}_{T+t+1 \text{ times}} \to \mathbb{F}^k,$$

The suggested coding scheme transmits on each path capacity-achieving scheme (as if there were no additional links between each node). It uses concatenation to account for all possible paths for a given link while limiting the relaying functions to perform operations only in the "path level" (i.e. without "mixing" information symbols from different paths). While this limitation to the relay operation is not optimal, it allows us to bound the maximal size of the needed overhead and to derive closed-form expressions for the achievable rate.

The coding scheme can be described as follows

- At the source, transmit on the i'th link ($1 \le i \le L_1$) the concatenation of the output of all $f_{t,m}^{(1)}$ encoding functions $\forall m \in \mathcal{M}_i^{(1)}$ where $f_{t,m}^{(1)}$ is the encoding function of state-dependent symbol-wise decode and forward code designed for the m'th path ($m \in \mathcal{M}_i^{(1)}$ means that path m includes link i in the first hop).
- Each node j gets the packets transmitted from all Lj links from node $j-1$. The relay transmits on the i'th link ($1 \le i \le L_{j+1}$) the concatenation of the output of all $f_{t,m}^{(j+1)}$ relaying functions $\forall m \in \mathcal{M}_i^{(j+1)}$ where $f_{t,m}^{(j+1)}$ is the relaying function (at node $j+1$) of state-dependent symbol-wise decode and forward code designed for the m'th path ($m \in \mathcal{M}_i^{(j+1)}$ means that path m includes link i in the $j+1$'th hop).
- The decoder gets the packets transmitted from all L_{H+1} links from node r_H. The decoder applies the $\prod L_i$ decoding functions $\varphi_{t+T,m}$ for all paths m where $\varphi_{t+T,m}$ is the decoding function of state-dependent symbol-wise decode and forward code designed for the m'th path.

Proposition 2. *The suggested coding scheme is* $(\mathbf{N}^{(1)}, \ldots \mathbf{N}^{(H+1)})$*-achievable.*

Proof. Since the coding scheme suggested above can be viewed as transmitting simultaneously $\prod_l L_l$ codes on $\prod_l L_l$ different paths without interaction between the codes, and since each code used on the m'th path is $(N_{l_{1,m}}^{(1)}, \ldots N_{l_{H+1,m}}^{(H+1)})$-achievable it follows that the overall code is $(\mathbf{N}^{(1)}, \ldots \mathbf{N}^{(H+1)})$-achievable. □

Proof of Lemma 5. Following Proposition 2, the suggested coding scheme is $(\mathbf{N}^{(1)}, \ldots \mathbf{N}^{(H+1)})$-achievable. Therefore, to show Lemma 5 we analyze the achievable rate of the scheme.

The code transmitted on the i'th link in the j'th hop ($1 \le i \le L_j$) which belongs to path $m \in \mathcal{M}_i^{(j)}$ is at rate of

$$R_{i,m}^{(j)} = \frac{T - \sum_{h=1, h\neq j}^{H+1} N_{l_{h,m}}^{(h)} - N_i^{(j)} + 1}{T - \sum_{h=1, l\neq j}^{H+1} N_{l_{h,m}}^{(h)} + 1 + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}} \triangleq \frac{k_{i,m}^{(j)}}{n_{i,m}^{(j)}}. \tag{35}$$

We note that there is a (different) header for each code used over each path. Nevertheless, the size of each of these headers is upper bounded by $\frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}$.

The overall rate to be transmitted over this link is the concatenation of the codes to be used on all possible $m \in \mathcal{M}_i^{(j)}$ paths, i.e.

$$R_i^{(j)} = \frac{k_{i,\mathcal{M}_i^{(j)}[1]}^{(j)}}{n_{i,\mathcal{M}_i^{(j)}[1]}^{(j)}} \frown \frac{k_{i,\mathcal{M}_i^{(j)}[2]}^{(j)}}{n_{i,\mathcal{M}_i^{(j)}[2]}^{(j)}} \frown \cdots \frown \frac{k_{i,\mathcal{M}_i^{(j)}[\prod_{h\neq j} L_l]}^{(j)}}{n_{i,\mathcal{M}_i^{(j)}[\prod_{h\neq j} L_h]}^{(j)}} \tag{36}$$

$$= \frac{(\prod_{h\neq j} L_h)\cdot(T+1-N_i^{(j)}) - \sum_{s\neq j}(\prod_{h\neq s,j} L_h)\cdot \sum \mathbf{N}^{(s)}}{(\prod_{h\neq j} L_h)\cdot\left(T+1+\frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}\right) - \sum_{s\neq j}(\prod_{h\neq s,j} L_h)\cdot \sum \mathbf{N}^{(s)}} \tag{37}$$

$$= \frac{T+1-N_{l_{i,m}}^{(j)} - \sum_{s\neq j}\bar{\mathbf{N}}^{(s)}}{T+1-\sum_{s\neq j}\bar{\mathbf{N}}^{(s)} + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}}. \tag{38}$$

□

To show corollary 2, we note that the total rate transmitted over this hop is

$$\begin{aligned} R^{(j)} &= \sum_{l_{i,m}=1}^{L_j} R_i^{(j)} \\ &= \frac{L_j\left(T+1-\sum_{s\neq j}\bar{\mathbf{N}}^{(s)}\right) - \sum_{l_{i,m}=1}^{L_j} N_{l_{i,m}}^{(j)}}{T+1-\sum_{s\neq j}\bar{\mathbf{N}}^{(s)}} \\ &= \frac{T+1-\sum \bar{\mathbf{N}}^{(s)}}{\frac{1}{L_j}\left(T+1-\sum_{s\neq j}\bar{\mathbf{N}}^{(s)} + \frac{n_{\max}\lceil \log(n_{\max})\rceil}{\log(|\mathbb{F}|)}\right)}. \end{aligned}$$

Following definition 5, the overall rate transmitted over the network is the minimum across all hops which is given in (14).

While Lemma 5 and Corollary 2 assume $T \ge \sum_{l=1}^{H+1} \max(\mathbf{N}^{(l)})$ the same coding scheme can be used for any $T \ge \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$. We note that that in this case, there might be paths with zero upper bound. Hence, the scheme will not transmit over these paths.

Denoting

$$\left[\frac{k_{i,m}^{(j)}}{n_{i,m}^{(j)}}\right]^+ = \begin{cases} \frac{k_{i,m}^{(j)}}{n_{i,m}^{(j)}}, & \text{if } k_{i,m}^{(j)} > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $k_{i,m}^{(j)}$ and $n_{i,m}^{(j)}$ are defined in (35). The transmitted rate at the i'th link in the j'th hop is

$$\left[R_i^{(j)}\right]^+ = \left[\frac{k_{i,\mathcal{M}_i^{(j)}[1]}^{(j)}}{n_{i,\mathcal{M}_i^{(j)}[1]}^{(j)}}\right]^+ \frown \left[\frac{k_{i,\mathcal{M}_i^{(j)}[2]}^{(j)}}{n_{i,\mathcal{M}_i^{(j)}[2]}^{(j)}}\right]^+ \frown \cdots \frown \left[\frac{k_{i,\mathcal{M}_i^{(j)}[\prod_{h\neq j} L_l]}^{(j)}}{n_{i,\mathcal{M}_i^{(j)}[\prod_{h\neq j} L_h]}^{(j)}}\right]^+,$$

and thus we have

$$\left[R^{(j)}\right]^+ = \sum_{l_{i,m}=1}^{L_j} \left[R_i^{(j)}\right]^+,$$

Therefore,

$$R \geq \min_j \left[R^{(j)}\right]^+ \tag{39}$$

which establishes explicit expressions for the achievable rate for any $T \geq \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$.

C. Improved coding scheme

The basic scheme suggested above uses all paths with multiplicity 1. This scheme's natural extension is to transmit on different paths different amounts of concatenated multi-node single-link codes. Note that we suggest two levels of concatenations. The first is to use a different multiplicity of the *s*ame multi-node single-link codes across *all* hops (where different codes can be concatenated at different multiplicities). The second is to concatenate the codes to be used on the same link but belong to different paths.

To simplify the description of the improved scheme we assume $T \geq \sum_{l=1}^{H+1} \max(\mathbf{N}^{(l)})$ (i.e., potentially, all paths are active), but note that it holds for any $T \geq \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$.

We note that the straight-forward benefit to this improved scheme is that no additional headers are required when state-dependent symbol-wise decode and forward code is used per path (applying different multiplicity for the code using over a path) since when an erasure occurs, it'll occur simultaneously to all concatenated codes thus a single header is needed (per path) to allow each node to perform its operations. Specifically, the number of the required headers is the same as in the basic scheme (one header per one path).

Explicitly, denoting with c_m the multiplicity of the code used in the m'th path, Equation (36) becomes

$$R_i^{(j)} = \frac{c_{\mathcal{M}_i^{(j)}[1]} \cdot k_{i,\mathcal{M}_i^{(j)}[1]}^{(j)}}{c_{\mathcal{M}_i^{(j)}[1]} \cdot n_{i,\mathcal{M}_i^{(j)}[1]}^{(j)}} \frown \frac{c_{\mathcal{M}_i^{(j)}[2]} \cdot k_{i,\mathcal{M}_i^{(j)}[2]}^{(j)}}{c_{\mathcal{M}_i^{(j)}[2]} \cdot n_{i,\mathcal{M}_i^{(j)}[2]}^{(j)}} \frown \cdots \frown \frac{c_{\mathcal{M}_i^{(j)}[\prod_{l\neq j} L_l]} \cdot k_{i,\mathcal{M}_i^{(j)}[\prod_{l\neq j} L_l]}^{(j)}}{c_{\mathcal{M}_i^{(j)}[\prod_{l\neq j} L_l]} \cdot n_{i,\mathcal{M}_i^{(j)}[\prod_{l\neq j} L_l]}^{(j)}}.$$

From (35) (specifically, the definition of $k_{i,m}^{(j)}$) we note that per path, the number of information symbols transmitted in each hop is equal, i.e.

$$k_{l_{1,m},m}^{(1)} = k_{l_{2,m},m}^{(2)} = \cdots = k_{l_{H+1,m},m}^{(H+1)} = T - \sum_{h=1}^{H+1} N_{l_{h,m}}^{(h)} + 1 \triangleq k_m.$$

Therefore, the total number of information symbols sent through the j'th hop is defined as

$$k_{\text{tot}}^{(j)} = \sum_{i=1}^{L_j} \sum_{w=1}^{\prod_{s\neq j} L_s} c_{\mathcal{M}_i^{(j)}[w]} \cdot k_{\mathcal{M}_i^{(j)}[w]}^{(j)} \stackrel{(a)}{=} \sum_{w=1}^{\prod_s L_s} c_w \cdot k_w, \tag{40}$$

where (a) follows since $\mathcal{M}_l^{(j)} \cap \mathcal{M}_w^{(j)} = \emptyset, \forall 1 \leq l \neq w \leq L_j$ and $\bigcup_l \mathcal{M}_l^{(j)}$ equals to all $\prod_l L_l$ paths.

We note that (40) means that total number of information symbols sent through each of the hops is the same, i.e.,

$$k_{\text{tot}}^{(1)} = k_{\text{tot}}^{(2)} = \cdots = k_{\text{tot}}^{(H+1)} = \sum_{m=1}^{\prod_s L_s} c_m \cdot k_m \triangleq k_{\text{tot}}.$$

The total size of the packet sent over the i'th link in the j'th hop is

$$n_i^{(j)} = \sum_{w=1}^{\prod_{s\neq j} L_s} c_{\mathcal{M}_i^{(j)}[w]} \cdot n_{\mathcal{M}_i^{(j)}[w]}^{(j)}.$$

Therefore, denoting $\mathbf{n}^{(j)} = \left[n_1^{(j)}, \ldots, n_{L_j}^{(j)}\right]$, using Definition 5 the achievable rate is

$$R = \max_{c_m, 1\leq m\leq \prod L_l} \min\left(\frac{k_{\text{tot}}}{n_1^{(1)}}, \frac{k_{\text{tot}}}{n_2^{(1)}}, \ldots, \frac{k_{\text{tot}}}{n_1^{(2)}}, \ldots, \frac{k_{\text{tot}}}{n_{L_{H+1}}^{(H+1)}}\right) = \max_{c_m, 1\leq m\leq \prod L_l} \frac{k_{\text{tot}}}{\max_j\left(\max(\mathbf{n}^{(j)})\right)}. \tag{41}$$

This is can converted to the following (convex) optimization problem

$$R \geq \frac{k_{\text{tot}}}{D}$$
$$\text{Where } : \ 0 \leq \tilde{c}_m \leq 1, \quad 1 \leq m \leq \prod_l L_l$$
$$\sum_{m=1}^{\prod_l L_l} \tilde{c_m} = 1$$
$$D \geq n_i^{(j)}, \quad 1 \leq j \leq H+1, \ 1 \leq i \leq L_j,$$

with $\prod_l L_l$ parameters (the amount of multiplicity c_m to apply on each path), which is tractable for some practical examples as we demonstrate below. We note that this is a convex problem since both k_{tot} and D are positive (all $n_i^{(j)}$

are positive and there is at least one $n_i^{(j)} > 0$). Thus the objective function is a convex function.

While the outcome of this optimization are $0 \le \tilde{c}_i \le 1$, the multiplicity amounts to be used can be derived by taking $c_i = \lfloor \tilde{c}_i \cdot c \rfloor$ where c is a constant (same constant for all $\tilde{c}_i$) chosen to trade performance with overall packet size.

VI. Directed acyclic graph

The upper bound and achievable scheme described above for a relayed network can be extended to a general directed acyclic graph. Since the network cannot be decoupled to different hops, we briefly repeat the network model's definition where each link gets a unique index l.

A. Network model

A source node wants to send a sequence of messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ to a destination node through L links where some of the links may start or end at an "interim nodes" (which is capable of performing processing on its received packets). We assume there are $J-2$ interim nodes, and we denote the source as node (0) and the destination as node (J). To prevent circles node j is allowed to send information only to a node in $\{j+1,\dots,\}$. We assume on the discrete timeline, each link l introduces at most N_l erasures. We denote with $\mathcal{L}_{\text{out}}^{(j)}$ the (sorted) set of links going out of node j and with $\mathcal{L}_{\text{in}}^{(j)}$ the (sorted) set of links going into of node j. We denote with $\mathcal{L}_{\text{out}}^{(j)}[w]$ the w'th link in $\mathcal{L}_{\text{out}}^{(j)}[w]$.

Definition 10. *Let $\mathbf{n} = [n_1 \dots, n_L]$. An $(n_1, n_2, \dots, n_L, k, T)_{\mathbb{F}}$-streaming code consists of the following:*

1) *A sequence of source messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ where $\mathbf{s}_i \in \mathbb{F}^k$.*
2) *A list of $L_{out}^{(0)}$ encoding functions $f_{t,i} : \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{t+1 \text{ times}} \to \mathbb{F}^{n_i}$, $i \in \mathcal{L}_{out}^{(0)}$ used by the source at time t to generate*
$$\mathbf{x}_{t,i} = f_{t,i}(\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_t).$$
3) *A list of relaying functions for node j*
$$f_{t,i} : \underbrace{\mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(j)}[1]}} \times \ldots \times \mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(j)}[1]}}}_{t+1 \text{ times}} \times \\ \ldots \times \underbrace{\mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(j)}[|\mathcal{L}_{\text{in}}^{(j)}|]}} \times \ldots \times \mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(j)}[|\mathcal{L}_{\text{in}}^{(j)}|]}}}_{t+1 \text{ times}} \to \mathbb{F}^{n_i},$$
$i \in \mathcal{L}_{out}^{(j)}$ used by node j at time t to generate
$$\mathbf{x}_{t,i} = f_{t,i}\left(\{\mathbf{y}_{0,l}\}_{l \in \mathcal{L}_{\text{in}}^{(j)}}, \dots, \{\mathbf{y}_{t,l}\}_{l \in \mathcal{L}_{\text{in}}^{(j)}}\right).$$
4) *A decoding function*
$$\varphi_{t+T} : \underbrace{\mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(J)}[1]}} \times \ldots \times \mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(J)}[1]}}}_{T+t+1 \text{ times}} \times \\ \ldots \times \underbrace{\mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(J)}[|\mathcal{L}_{\text{in}}^{(J)}|]}} \times \ldots \times \mathbb{F}_e^{n_{\mathcal{L}_{\text{in}}^{(J)}[|\mathcal{L}_{\text{in}}^{(J)}|]}}}_{T+t+1 \text{ times}} \to \mathbb{F}^k,$$
is used by the destination at time $t+T$ to estimate $\mathbf{s_i}$ according to $\hat{\mathbf{s}}_t = \varphi_{t+T}(\{\mathbf{y}_{0,l}\}_{l \in \mathcal{L}_{\text{in}}^{(J)}}, \dots, \{\mathbf{y}_{t+T,l}\}_{l \in \mathcal{L}_{\text{in}}^{(J)}})$.

Definition 11. *An $(n_1, n_2, \dots, n_L, k, T)_{\mathbb{F}}$-streaming code is said to be $N_1, \dots N_L$-achievable if, for any $e_{t,i} \in \Omega_{N_i}$, for all $i \in \{1, \dots, L\}$ and $t \in \mathbb{Z}_+$ and all $\mathbf{s}_t \in \mathbb{F}^k$, we have $\hat{\mathbf{s}}_t = \mathbf{s}_t$.*

Definition 12. *The rate of an $(n_1, n_2, \dots, n_L, k, T)_{\mathbb{F}}$-streaming code is $\frac{k}{\max(\mathbf{n})}$.*

An example of a directed graph is given in Figure 6 below, where there are five links in the network.

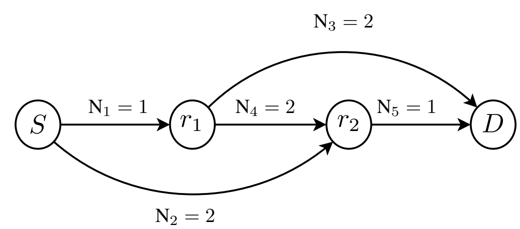

Fig. 6: Directed graph, $T = 4$, the maximal number of erasures per link is denoted on each link.

We further modify Definition 8 to

Definition 13. *In a directed network with a single source, single destination and with L links*

- *The (unique) m'th path from the source to destination is defined as the set of k_m indices $m = \{l_1, \dots, l_{k_m}\}$, where l_1 start at the source, l_{k_m} ends at the destination and $l_i \neq l_j$ for any $i \neq j$. We note that different paths may have different size (i.e. k_m is not fixed). Further we require that the paths are disjoint, i.e. for $m \neq m'$, either $k_m \neq k_{m'}$ or there exists $1 \le h \le \max k_m, k_{m'}$ such that $l_h \neq l_h$.*
- *Denote with $\mathcal{M}_l$ the set of all paths in which include l. We note that $\mathcal{M}_l \cap \mathcal{M}_{l'} = \emptyset$, $\forall l \neq l'$.*
- *Denote with $\mathcal{M}_l[w]$ the w'th path in $\mathcal{M}_l$.*

For example, in Figure 11 there are three paths from the source to the destination $\{\{1, 4, 5\}, \{1, 3\}, \{2, 5\}\}$. As before, each path is a multi-node network with a single link between each node.

We further define a cut in the graph, which we use later to derive an upper bound for the achievable streaming rate in this graph

Definition 14. *An s-d cut $(\mathcal{S}, \mathcal{D})$ is a partition of the vertices of the graph such that $s \in \mathcal{S}$ and $d \in \mathcal{D}$. That is, s-d cut is a division of the network's vertices into two groups, with the source in one group and the destination in the other.*

Denote with $\mathcal{L}_{(\mathcal{S},\mathcal{D})}$ all the links which connect any of the nodes in $\mathcal{S}$ to any of the nodes in $\mathcal{D}$, the capacity of an s-d *cut is the total rate the flows from any of the nodes in $\mathcal{S}$ to any of the nodes in $\mathcal{D}$, denoted as*

$$C(\mathcal{S},\mathcal{D}) = \sum_{l\in\mathcal{L}_{(\mathcal{S},\mathcal{D})}} \tilde{R}_l.$$

where $\tilde{R}$ is the maximal rate possible on link l.

Remark 5. *We note that in a standard network (not streaming code with an overall delay constraint), the sum of rates incoming to a node equals the sum of rates going out of the node (in this case, it can be shown that the minimum cut equals the maximal flow). We will use the definition above to analyze a network's capacity with a maximal achievable rate on each link, which is a function of the overall delay-constraint (where the bound may be derived based on different paths in the network). Hence, the sum of rates incoming to a node does not necessarily equal the sum of rates going out of the node. Therefore, we do not claim that the max flow equals the min-cut, but only that the minimum cut is an upper bound for the maximal flow.*

B. upper bound

We note that each path defined in Definition 12 is a single-link multi-node network. Thus, the upper bound on each link (assuming only this path exists) is given in (**??**). In the current notations we have

$$R_{l_j,m} \leq \frac{T+1-\sum_{i=1,i\neq j}^{k_m} N_{l_i} - N_{l_j}}{T+1-\sum_{i=1}^{k_m} N_{l_i}}.$$

The following proposition derives a (simple) upper bound for the maximal rate achievable per link.

Proposition 3. *The rate transmitted in link l is upper bounded by*

$$R_l \leq \max_{m\in\mathcal{M}_l} R_{l,m}. \tag{42}$$

The proposition immediately holds since it can be interpreted as taking the worst upper bound over all paths, which l is part of. Alternatively, it can be viewed as if the derived bound assumes that the network is composed only of a single path, which results in the worst upper bound on the achievable rate for link l. Since it does not assume any utilization of additional links (and paths) that exists, it is an upper bound for the maximal rate achievable on this link.

Remark 6. *We note that that it can be easily shown that* (**??**) *and* (42) *converge for the case of relayed network.*

We thus have the following lemma.

Lemma 6. *The achievable rate in a directed acyclic graph with a single source and single destination and L links is upper bounded by the minimum cut of the directed acyclic graph with R_l defined in* (42)*.*

Proof. The capacity of a cut can be interpreted as the achievable rate when all nodes in $\mathcal{S}$ are fully cooperating and all nodes in $\mathcal{D}$ are fully cooperating.

Since cooperation is equivalent to assuming no erasures between the cooperating nodes and since the upper bound depends on the number of erasures on all other links $l' \neq l$, it follows that the upper bound on the achievable rate on link l can be improved when cooperation is assumed. Nevertheless, in analyzing the upper bound, we take the rate defined in (42) as the upper bound for the achievable rate over link l even though cooperation is assumed. Thus, we have that for a specific cut. The overall achievable rate is upper bounded by the cut's capacity when the rate of each link is the upper bound possible on this link as defined in (42).

To show that $C(\mathcal{S},\mathcal{D})$ is indeed the upper bound of cut $(\mathcal{S},\mathcal{D})$ we note that recalling Definition 12, the rate over a cut $(\mathcal{S},\mathcal{D})$ is defined as $R_{C(\mathcal{S},\mathcal{D})} = \frac{k_{C(\mathcal{S},\mathcal{D})}}{n_{C(\mathcal{S},\mathcal{D})}}$ where $k_{C(\mathcal{S},\mathcal{D})}$ is the number of information symbols transmitted through the cut and $n_{C(\mathcal{S},\mathcal{D})} = \max_{l\in\mathcal{L}_{(\mathcal{S},\mathcal{D})}} n_l$ is the maximal overall packet size sent each of the links.

Since we may use the upper bound of each link with different multiplicity, we can arrive at a common denominator for all links thus conclude that the overall rate sent through the cut is upper bounded by

$$R_{C(\mathcal{S},\mathcal{D})} \leq \sum_{l\in\mathcal{L}_{(\mathcal{S},\mathcal{D})}} R_l,$$

where R_l id defined in (42).

Further, using Definition 12 we have that for any two cuts with upper bounds $R_{C(\mathcal{S}_1,\mathcal{D}_1)} \leq \frac{k_1}{n_1}$ and $R_{C(\mathcal{S}_2,\mathcal{D}_2)} \leq \frac{k_2}{n_2}$ we have

$$R_{C(\mathcal{S}_1,\mathcal{D}_1)} \leq \frac{k_1\cdot k_2}{n_1\cdot k_2},\ R_{C(\mathcal{S}_2,\mathcal{D}_2)} \leq \frac{k_2\cdot k_1}{n_2\cdot k_1}.$$

Hence, we have that the number of information symbols sent over these two cuts is the same, where the overall packet size is different. Therefore we have

$$\begin{aligned} R &\leq \frac{k_1\cdot k_2}{\max\left(n_1\cdot k_2, n_2\cdot k_1\right)} \\ &= \min\left(R_{C(\mathcal{S}_1,\mathcal{D}_1)}, R_{C(\mathcal{S}_2,\mathcal{D}_2)}\right). \end{aligned}$$

Hence, the overall rate of a streaming code over a directed acyclic graph is upper bounded as

$$R \leq \min_{\text{c}-\text{d cut}} C(\mathcal{S},\mathcal{D}).$$

□

C. Achievable scheme

Assuming $T \geq \sum_{l=1}^{H+1} \max(\mathbf{N}^{(l)})$, denoting

$$n_{\max} \triangleq \max_m \left(T - \sum_{j=1}^{m_k} N_{l_j} + 1\right), \tag{43}$$

using state-dependent symbol-wise decode and forward over each path, the rate on link l_j in path m (assuming only path m exists) can be expressed as

$$R_{l_j,m} = \frac{T+1-\sum_{i=1,i\neq j}^{k_m} N_{l_i} - N_{l_j}}{T+1-\sum_{i=1}^{k_m} N_{l_i} + \frac{n_{\max}\lceil\log(n_{\max})\rceil}{\log(|\mathbb{F}|)}}$$

$$\triangleq \frac{k_{l_j,m}}{n_{l_j,m}}.$$

and the overall code used in link l is a concatenation of the all codes used in link l in all paths $m \in \mathcal{M}_l$, i.e.

$$\begin{aligned} R_l &= R_{l,\mathcal{M}_l[1]} \frown R_{l,\mathcal{M}_l[2]} \frown \cdots \frown R_{l,\mathcal{M}_l[|\mathcal{M}_l|]} \\ &= \frac{\sum_{w=1}^{|\mathcal{M}_l|} k_{l_j,m}}{\sum_{w=1}^{|\mathcal{M}_l|} n_{l_j,m}} \\ &\triangleq \frac{k_l}{n_l}. \end{aligned} \tag{44}$$

We note that since the total number of information symbols used by the code is the sum of information symbols used by all the links going out from the source (equivalently, the sum of information symbols used by all the links going into the destination), using Definition 12 we have that

$$R = \frac{\sum_{l \in \mathcal{L}_{\text{out}}^{(0)}} k_l}{\max_l (n_l)} \tag{45}$$

where k_l and k_l defined in (44) is achievable.

Remark 7. *Using the equivalence between Definition 5 and Definition 12 it can be shown that* (45) *equals to* (14) *when the network is a relayed network.*

Remark 8. *Similar to the case of relayed network, the achievable scheme holds also for* $T \geq \sum_{l=1}^{H+1} \min(\mathbf{N}^{(l)})$ *(where not all paths are "active") and the same optimization for the multiplicity of the codes used over each path can be performed to improve the achievable rate as can be shown in Example 3 in Section VII-C.*

VII. Examples

A. Example 1 - simple relayed network

Consider a network with two hops, where $\mathbf{N}^{(1)} = [3\ 2]$, $\mathbf{N}^{(2)} = [2\ 1]$ and $T = 5$ which is depicted in Figure 7. As can be seen the achievable rate in this network is upper bounded by $R \leq \min(\frac{2}{5} + \frac{3}{5}, \frac{2}{4} + \frac{3}{4}) = 1$.

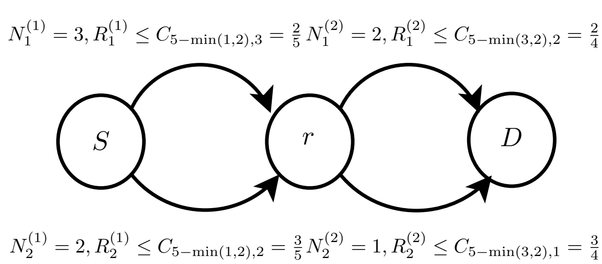

Fig. 7: Three-node network with two links between each node. $\mathbf{N}^{(1)} = [3\ 2],\ \mathbf{N}^{(2)} = [2\ 1],\ T = 5.$

We note that there are four paths from the source to destination depicted in Figure 8 below. For each path, the upper bound is achievable when symbol-wise decode and forward is used.

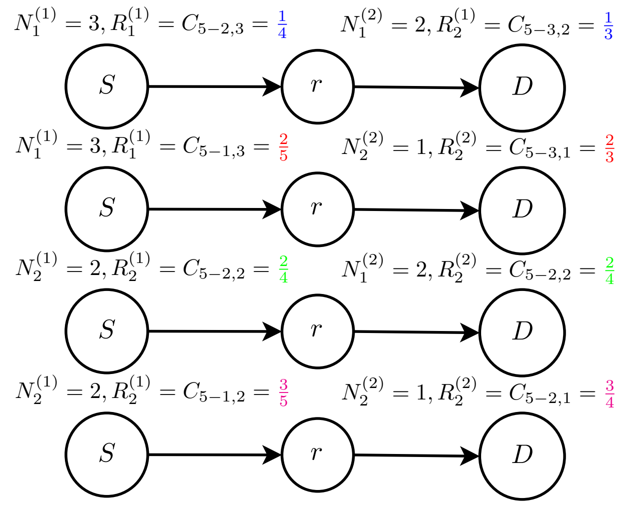

Fig. 8: Four paths composing the network of Figure 7.

The suggested basic coding scheme concatenates (per path) capacity achieving codes and is depicted in Figure 9. The achieved rate is

$$\begin{aligned} R &= \min\left(\frac{3}{9} + \frac{5}{9}, \frac{3}{7} + \frac{5}{7}\right) \\ &= \frac{8}{9}. \end{aligned}$$

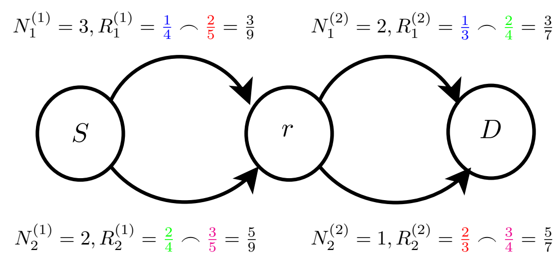

Fig. 9: Basic coding scheme which concatenates single path capacity-achieving codes with the same multiplicity.

As mentioned above, this scheme's performance can be improved when a different multiplicity of the codes applied on different paths is considered. Since in this example there are four paths, there are four variables to optimize over $\{c_1, c_2, c_3, c_4\}$ as depicted in Figure 10.

Following (41) the achievable rate is

$$R = \max_{c_1,c_2,c_3,c_4} \frac{c_1 + 2c_2 + 2c_3 + 3c_4}{\max\left(4c_1 + 5c_3, 4c_2 + 5c_4, 3c_1 + 4c_2, 3c_3 + 4c_4\right)}.$$

Solving it, we note that substituting $c_1 = 0, c_2 = 5, c_3 = 8, c_4 = 4$ we get

$$\begin{aligned} R &= \frac{38}{\max\left(40, 40, 20, 40\right)} \\ &= 0.95. \end{aligned}$$

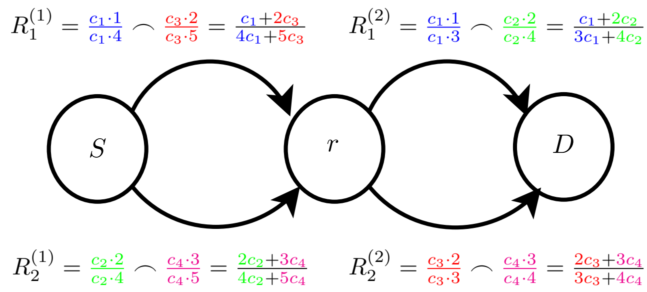

Fig. 10: Improved coding scheme which concatenates single path capacity-achieving codes with different multiplicity.

Hence we managed to improve the achievable rate compared to the basic scheme (from 8/9 to 0.95). Nevertheless, we still have a gap from the upper bound (which is 1).

B. Example 2 - more complicated relayed network

Consider a network with three hops, where $\mathbf{N}^{(1)} = [1\ 2\ 3]$, $\mathbf{N}^{(2)} = [1\ 3\ 3]$, $\mathbf{N}^{(3)} = [7\ 8\ 9]$ and $T = 15$ which is depicted in Figure 11.

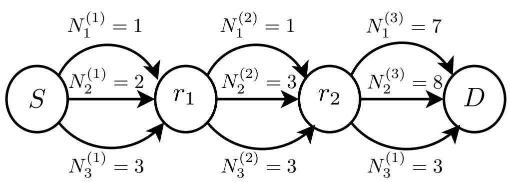

Fig. 11: Four node network with three links between each node. The maximal number of erasures per link and the upper bound per link is depicted on each link.

It can be shown that the achievable rate is upper bounded by

$$R \leq 18/14 \approx 1.2857.$$

When $|\mathbb{F}| \to \infty$, the basic scheme achieves

$$R \geq 99/105 \approx 0.9428.$$

We denote the link index of the first hop as the "LSB" of the path index and the link index of the last hop as the "MSB" of the path index. For example, path number 13 is composed of links $\{1, 2, 2\}$.

Setting $c_2 = 6, c_7 = 29,\ c_{10} = 10, c_{11} = 22, c_{19} = 31$ (and all other $c_w = 0$) results in

$$R \geq 506/434 \approx 1.16590.$$

Setting $c_2 = 0,\ c_7 = 298,\ c_{10} = 101, c_{11} = 227, c_{19} = 312$ (and all other $c_w = 0$) results in

$$R \geq 5151/4368 \approx 1.1792.$$

Thus, we demonstrated the trade-off between the achievable rate and the packet size. In fact it can be shown that in the limit of large packet size the achievable rate is

$$R \approx 1.18077.$$

To give an indication to the overall performance of the achievable scheme we plot the ratio of the achievable rate and the upper bound (for both the basic and the improved scheme) over 1000 four-node networks with 3 links between the nodes. The maximal number of erasures per link is randomly chosen (between 1 and 10) and the overall delay constraint is taken as $T = \sum_j \max(\mathbf{N}^j)$. We further show the performance of using a single path (the best path) from source to destination which result in a much higher gap from the upper bound.

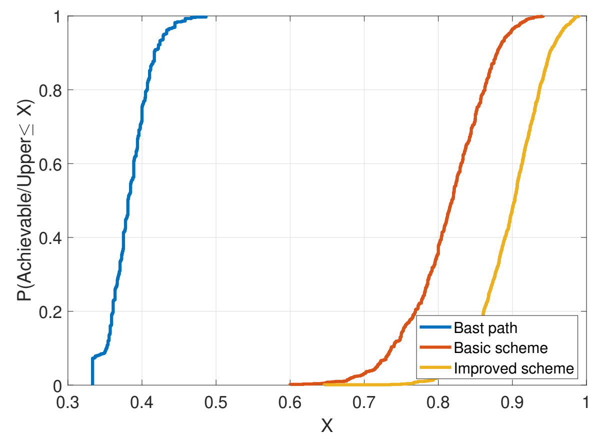

Fig. 12: CDF of the ratio of the achievable schemes and the upper bound for 1000 4-node network with 3 links between the nodes. As a reference we also plot the rate of the using a single path (the best path) with the upper bound.

C. Example 3 - directed graph

We analyze into more depth the network depicted in Figure 6.

We note that in this network there are four possible cuts. We denote below the different cuts and and the upper bound for achievable rate over each cut.

1) $R_{(\{s\},\{r_1,r_2,d\})} \leq \frac{2}{3} + \frac{2}{4} = \frac{7}{6}$
2) $R_{(\{s,r_1\},\{r_2,d\})} \leq \frac{2}{4} + \frac{1}{3} + \frac{2}{4} = \frac{4}{3}$
3) $R_{(\{s,r_2\},\{r_1,d\})} \leq \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$
4) $R_{(\{s,r_1,r_2\},\{d\})} \leq \frac{2}{3} + \frac{2}{4} = \frac{7}{6}$

The minimal cut is achieved (for example) when splitting the graph to $(\{s\}, \{r_1, r_2, d\})$ which is depicted in Figure 13 below, thus we have $R \leq \frac{7}{6}$

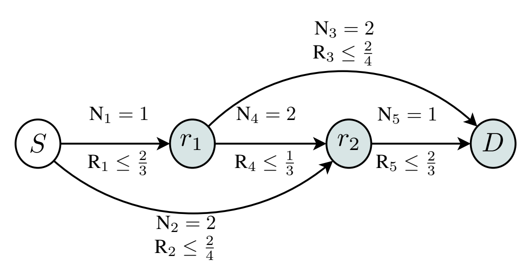

Fig. 13: Directed graph, $T = 4$ divided to $\mathcal{S} = \{s\}$, $\mathcal{D} = \{r_1, r_2, d\}$. Maximal number of erasures per link and the upper bound per link are denoted on the links.

The achievable scheme is depicted in Figure 14. The basic scheme assumes $c_1 = c_2 = c_3 = 1$. Assuming $|\mathbb{F}| \to \infty$, using (45) and noting that

$$\sum_{l \in \mathcal{L}^{(0)}_{\text{out}}} k_l = 1 + 2 + 2 = 5$$

$$\max_l (n_l) = \max(5, 4, 3, 3, 4) = 5.$$

Thus,

$$R = \frac{5}{5} = 1$$

is achievable.

We further note that using $c_1 = 1,\ c_2 = 2\ c_3 = 2$ we get

$$\sum_{l \in \mathcal{L}^{(0)}_{\text{out}}} k_l = 1 + 4 + 4 = 9$$

$$\max_l (n_l) = \max(8, 8, 6, 3, 8) = 8.$$

Thus

$$R = \frac{9}{8}$$

is achievable.

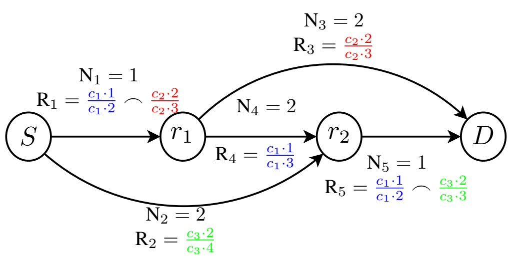

Fig. 14: Directed graph, $T = 4$, the code used on each link is depicted on the link.

VIII. Concluding Remarks

While using multiple paths for transmission is known to improve the transmission rate and/or robustness, in this paper, we have established the benefit of using multiple paths for the guaranteed rate of streaming codes. As was shown, the gain over using a single path can be very substantial.

While describing the model and derive the upper bound and the achievable rate over a deterministic erasure model (i.e., assuming a global limit on the maximal number of erasures that can occur over a link), we note that similarly to other works on the guaranteed rate of streaming codes (for example, [2]), the results also hold in case of assuming a maximal number of erasures over a sliding window (at length $T + 1$).

Finally, one of the main tools used in deriving the achievable scheme is using SD-SWDF, which requires an additional header to allow nodes to perform their coding operation. As was shown, using concatenation across the paths (i.e., the interim node performs operations on the path level codes without "mixing" codes across paths) results in a reasonable (and tractable) amount of overall overhead needed. Using enhanced coding schemes will probably result in an improved performance at the expense of a larger overhead. Studying this trade-off is the next avenue with respect to streaming codes over the multi-path transmission.

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