Explicit Lower Bounds on the Outage Probability of Integer Forcing over $N_r \times 2$ Channels

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***Abstract*—The performance of integer-forcing equalization for communication over the compound multiple-input multiple-output channel is investigated. An upper bound on the resulting outage probability as a function of the gap to capacity has been derived previously, assuming a random precoding matrix drawn from the circular unitary ensemble is applied prior to transmission. In the present work a simple and explicit lower bound on the worst-case outage probability is derived for the case of a system with two transmit antennas and two or more receive antennas, leveraging the properties of the Jacobi ensemble. The derived lower bound is also extended to random space-time precoding, and may serve as a useful benchmark for assessing the relative merits of various algebraic space-time precoding schemes.**

I. Introduction

This paper addresses communication over a compound multiple-input multiple output (MIMO) channel, where the transmitter only knows the number of transmit antennas and the mutual information. More specifically, the goal of this work is to assess the performance of (randomly precoded) integer-forcing (IF) equalization for such a scenario.

Communication over the compound MIMO channel using an architecture employing space-time linear processing at the transmitter side and IF equalization at the receiver side was proposed in [1]. It was shown that such an architecture *universally* achieves capacity up to a constant gap, provided that the precoding matrix corresponds to a linear perfect space-time code [2], [3].

Recently, in [4], the outage probability of IF where random unitary precoding is applied over the spatial dimension only was considered and an explicit universal upper bound on the outage probability for a given target rate and gap to capacity was derived.

In the present work we derive an explicit lower bound on this outage probability for the case of a system with two transmit antennas. We further extend the framework of [4] by considering also space-time random unitary precoding (rather than space-only).

II. Problem Formulation and Preliminaries

A. Channel Model

The (complex) MIMO channel is described by[1]

$$\boldsymbol{y}_c = \mathbf{H}_c \boldsymbol{x}_c + \boldsymbol{z}_c, \tag{1}$$

where $\mathbf{x}_c \in \mathbb{C}^{N_t}$ is the channel input vector, $\mathbf{y}_c \in \mathbb{C}^{N_r}$ is the channel output vector, $\mathbf{H}_c$ is an $N_r \times N_t$ complex channel matrix, and $\boldsymbol{z}_c$ is an additive noise vector of i.i.d. unit-variance circularly symmetric complex Gaussian random variables. We assume that the channel is fixed throughout the transmission period. Further, we may assume without loss of generality that the input vector $\boldsymbol{x}_c$ is subject to the power constraint[2]

$$\mathbb{E}(\boldsymbol{x}_c^H \boldsymbol{x}_c) \leq N_t.$$

Consider the mutual information achievable with a Gaussian isotropic or "white" input (WI)

$$C = \log\det\left(\mathbf{I} + \mathbf{H}_c\mathbf{H}_c^H\right). \tag{2}$$

We define the set of all channels with N_t transmit antennas (and arbitrary N_r) having the same WI mutual information C

$$\mathbb{H}(C) = \left\{\mathbf{H}_c : \log\det\left(\mathbf{I} + \mathbf{H}_c\mathbf{H}_c^H\right) = C\right\}. \tag{3}$$

The corresponding compound channel model is defined by (1) with the channel matrix $\mathbf{H}_c$ arbitrarily chosen from the set $\mathbb{H}(C)$. The matrix $\mathbf{H}_c$ is known to the receiver, but not to the transmitter. Clearly, the capacity of this compound channel is C, and is achieved with an isotropic Gaussian input.

Applying the singular-value decomposition (SVD) to the channel matrix, $\mathbf{H}_c = \mathbf{U}_c\mathbf{\Sigma}_c\mathbf{V}_c^H$, we note that the unitary matrices have no impact on the mutual information. Let $\mathbf{D}_c$ be defined by

$$\left(\mathbf{I} + \mathbf{H}_c^H\mathbf{H}_c\right) = \mathbf{U}_c\mathbf{D}_c\mathbf{U}_c^H. \tag{4}$$

and note that $\mathbf{D}_c = \mathbf{I} + \mathbf{\Sigma}_c^H\mathbf{\Sigma}_c$. Thus, the compound set (3) may equivalently be described by constraining $\mathbf{D}_c$ to belong to the set

$$\mathbb{D}(C) = \left\{N_t \times N_t \text{ diagonal } \mathbf{D}_c : \sum \log(d_{i,i}) = C\right\}. \tag{5}$$

We turn now to the performance of IF. It has been observed that employing the IF receiver allows approaching C for "most" but not all matrices $\mathbf{H}_c \in \mathbb{H}(C)$. In the present work, we quantify the measure of the set of bad channel matrices by considering outage events, i.e., those events (channels) where integer forcing fails even though the channel has sufficient mutual information. The probability space here is induced by considering a randomized scheme where a random unitary

[1]We denote all complex variables with c to distinguish them from their real-valued representation.

[2]We denote by $[\cdot]^T$, the transpose of a vector/matrix and by $[\cdot]^H$, the Hermitian transpose.

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precoding matrix $\mathbf{P}_c$ is applied prior to transmission over the channel.

More specifically, denoting by $R_{\mathrm{IF}}(\mathbf{H}_c)$ the rate achievable with IF over a channel $\mathbf{H}_c$, the achievable rate of the randomized scheme is $R_{\mathrm{IF}}(\mathbf{H}_c \cdot \mathbf{P}_c)$. As $\mathbf{P}_c$ is drawn at random, the latter rate is also random. Following [4], we define the worst-case (WC) outage probability of randomized IF as

$$P_{\mathrm{out}}^{\mathrm{WC,IF}}(C, \Delta C) = \sup_{\mathbf{H}_c \in \mathbb{H}(C)} \Pr\left(R_{\mathrm{IF}}(\mathbf{H}_c \cdot \mathbf{P}_c) < C - \Delta C\right), \tag{6}$$

where the probability is with respect to the ensemble of precoding matrices and $R_{\mathrm{IF}}(\cdot)$ is the achievable rate of IF as given in [5].

Note that in (6), we take the supremum over the entire compound class rather than taking the average with respect to some putative distribution over $\mathbb{H}(C)$. It follows that $P_{\mathrm{out}}^{\mathrm{WC,IF}}(C, R)$ provides an upper bound on the outage probability that holds for any such distribution.

Clearly (6) is not an explicit bound. Nonetheless, by restricting attention to a uniform (Haar) measure over the unitary precoding matrices, we are able to obtain closed-form upper as well as lower bounds.

Specifically, we consider precoding matrices $\mathbf{P}_c$ drawn from the circular unitary ensemble (CUE), see e.g., [6]. Applying the SVD to the effective channel, we have $\mathbf{H}_c\mathbf{P}_c = \mathbf{U}_c\mathbf{\Sigma}_c\mathbf{V}_c^T\mathbf{P}_c$. From the properties of the CUE it follows that $\mathbf{V}_c^T\mathbf{P}_c$ has the same (CUE) distribution as $\mathbf{P}_c$. Thus, $\mathbf{V}_c$ (and of course $\mathbf{U}_c$) plays no role in (6) and we may rewrite the latter as

$$P_{\mathrm{out}}^{\mathrm{WC,IF}}(C, \Delta C) = \sup_{\mathbf{D}_c \in \mathbb{D}(C)} \Pr\left(R_{\mathrm{IF}}(\mathbf{D}_c \cdot \mathbf{P}_c) < C - \Delta C\right), \tag{7}$$

and thus the analysis for CUE precoding is greatly simplified.[3]

Both the upper and lower bounds for (7) developed below heavily rely on the well-studied properties of the CUE. For the lower bound we utilize, following the approach of [7], the Jacobi distribution [8] which gives the eigenvalue distribution of submatrices of such matrices.

We similarly denote by $P_{\mathrm{out}}^{\mathrm{WC,IF-SIC}}(C, \Delta C)$ the WC outage probability of IF with successive interference cancellation (SIC), the rate of which we denote by $R_{\mathrm{IF-SIC}}(\mathbf{H}_c)$ and for which we give an explicit expression next.

B. Integer-Forcing Equalization: Achievable Rates

We begin by recalling the achievable rates of the IF equalization scheme, where the reader is referred to [5] and [9] for the derivation, details and proofs. Furthermore, we follow the notation of these works, and in particular we present IF over the reals. We also focus our attention on IF receivers employing successive interference cancellation (SIC).

[3]We note that in many natural statistical scenarios, including that of an i.i.d. Rayleigh fading environment, the random transformation is actually performed by nature.

For a given choice of (invertible) integer matrix $\mathbf{A}$, let $\mathbf{L}$ be defined by the following Cholesky decomposition

$$\mathbf{A}\left(\mathbf{I} + \mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T = \mathbf{L}\mathbf{L}^T. \tag{8}$$

Denoting by $\ell_{m,m}$ the diagonal entries of $\mathbf{L}$, IF-SIC can achieve [9] any rate satisfying $R < R_{\mathrm{IF-SIC}}(\mathbf{H})$ where

$$R_{\mathrm{IF-SIC}}(\mathbf{H}) = 2N_t \frac{1}{2} \max_{\mathbf{A}} \min_{m=1,\ldots,2N_t} \log\left(\frac{1}{\ell_{m,m}^2}\right)$$

and the maximization is over all $2N_t \times 2N_t$ full-rank integer matrices.

III. Closed-Form Bounds for $N_r \times 2$ Channels

A. Space-Only Precoded Integer-Forcing

1) Upper Bound: We recall known upper bounds for the achievable WC outage probability of CUE-precoded IF-SIC for $N_r \times 2$ channels. The following theorem combines Theorem 2, Lemma 4 and Corollary 2 of [4].

Theorem 1. *[4] For any $N_r \times 2$ complex channel $\mathbf{H}_c$ with white-input mutual information $C > 1$, i.e., $\mathbf{D} \in \mathbb{D}(C)$, and for $\mathbf{P}_c$ drawn from the CUE (which induces a real-valued precoding matrix $\mathbf{P}$), we have*

$$P_{\mathrm{out}}^{\mathrm{WC,IF-SIC}}(C, \Delta C) \leq 81\pi^2 2^{-\Delta C}, \tag{9}$$

for $\Delta C > 1$. A tighter yet less explicit bound is

$$P_{\mathrm{out}}^{\mathrm{WC,IF-SIC}}(C, \Delta C) \leq \max_{d_{\max}} \sum_{\mathbf{a} \in \mathbb{B}(\beta, d_{\max})} \frac{2\pi^2 2^{-3/4(C+\Delta C)}}{\pi^2 \frac{\|\mathbf{a}\|^3}{2^C} \sqrt{d_{\max}}},$$

where $d_{\max} = \max_i d_i$ and

$$\mathbb{B}(\beta, d_{\max}) = \left\{\mathbf{a} : 0 < \|\mathbf{a}\| < \sqrt{\beta d_{\max}} \text{ and } \nexists 0 < c < 1 \,:\, c\mathbf{a} \in \mathbb{Z}^n\right\}$$

with $\beta = 2^{-1/2(C+\Delta C)}$.

2) Lower Bound on the Outage Probability via Maximum-Likelihood Decoding: It is natural to compare the performance attained by an IF receiver with that of an optimal maximum likelihood (ML) decoder for the same precoding scheme but where each stream is coded using an independent Gaussian codebook. Since we are confining the encoders to operate in parallel (independent streams), we are in fact considering coding over a MIMO multiple-access channel (MAC).

Thus, a simple upper bound on the achievable rate of integer-forcing is the capacity of the MIMO MAC with independent Gaussian codebooks of equal rates [5]. Specifically, let $\mathbf{H}_S$ denote the submatrix of $\mathbf{H}_c\mathbf{P}_c$ formed by taking the columns with indices in $S \subseteq \{1, 2, ..., N_t\}$. For a joint ML decoder, the maximal achievable rate over the considered MIMO multiple-access channel is

$$R_{\mathrm{JOINT}} = \min_{S \subseteq \{1,2,\ldots,N_t\}} \frac{N_t}{|S|} \log\det\left(\mathbf{I}_{N_r} + \mathbf{H}_S\mathbf{H}_S^H\right). \tag{10}$$

Note that since $\mathbf{H}_c\mathbf{P}_c$ and thus also $\mathbf{H}_S$ depends on the random precoding matrix $\mathbf{P}_c$, R_{JOINT} is a random variable.

We next derive the exact WC scheme outage for ML decoding when CUE precoding is applied (with independent Gaussian codebooks) over a MIMO channel with two transmit antennas.

When $N_t = 2$, the SVD decomposition of $\mathbf{H}_c\mathbf{P}_c$ can be written as

$$\mathbf{H}_c\mathbf{P}_c = \mathbf{U}_c \begin{bmatrix} \sqrt{\rho_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\rho_2} & 0 & \cdots & 0 \end{bmatrix}^H \mathbf{V}_c^H\mathbf{P}_c, \quad (11)$$

where $\rho_i = \mathbf{\Sigma}_{i,i}^2$. Substituting the latter in (2) yields

$$C = \log(1+\rho_1) + \log(1+\rho_2). \quad (12)$$

Theorem 2. *For a CUE-precoded $N_r \times 2$ compound MIMO channel with white-input mutual information C and $N_r \geq 2$, we have*

$$P_{\text{out,JOINT}}^{\text{WC}}(C, \Delta C) = 1 - \sqrt{1 - 2^{-\Delta C}}. \quad (13)$$

Proof. The capacity (10) of the $N_r \times 2$ MIMO MAC channel with equal user rates is given by

$$\begin{aligned} R_{\text{JOINT}}(\mathbf{H}_c\mathbf{P}_c) &= \min_k \min_{S \in \mathcal{S}_k} \frac{2}{k} \log\det\left(\mathbf{I}_{N_r} + \mathbf{H}_S\mathbf{H}_S^H\right) \\ &\triangleq \min_k \min_{S \in \mathcal{S}_k} R(S) \end{aligned} \quad (14)$$

where $\mathcal{S}_k$ is the set of all the subsets of cardinality k from $\{1, 2\}$. Hence $\mathbf{H}_S$ is a submatrix of $\mathbf{H}_c\mathbf{P}_c$ formed by taking k columns (k equals 1 or 2). Since we assume that $\mathbf{P}_c$ is drawn from the CUE, it follows that $\mathbf{P}_c$ is equal in distribution to $\mathbf{V}_c^H\mathbf{P}_c$. Hence, taking k columns from $\mathbf{H}_c\mathbf{P}_c$ is equivalent to multiplying $\mathbf{H}_c$ with k columns of $\mathbf{P}_c$. Therefore (14) can be written as

$$R_{\text{JOINT}}(\mathbf{H}_c\mathbf{P}_c) = \min\{R(\{1\}), R(\{2\}), R(\{1,2\})\}. \quad (15)$$

When $k = 2$, we have $R(\{1,2\}) = C$. Plugging this into (15), we get

$$R_{\text{JOINT}}(\mathbf{H}_c\mathbf{P}_c) = \min\{R(\{1\}), R(\{2\}), C\}. \quad (16)$$

We now turn to study $R(\{1\})$. Note that

$$\log\det\left(\mathbf{I}_{N_r} + \mathbf{H}_S\mathbf{H}_S^H\right) = \log\left(1 + \mathbf{H}_S^H\mathbf{H}_S\right), \quad (17)$$

so that

$$\begin{aligned} R(\{1\}) &= 2\log\left(1 + \begin{bmatrix} \mathbf{P}_{1,1} \\ \mathbf{P}_{1,2} \end{bmatrix}^H \begin{bmatrix} \rho_1 & \mathbf{0} \\ \mathbf{0} & \rho_2 \end{bmatrix} \begin{bmatrix} \mathbf{P}_{1,1} \\ \mathbf{P}_{1,2} \end{bmatrix}\right) \\ &= 2\log\left(1 + \rho_1\mathbf{P}_{1,1}^H\mathbf{P}_{1,1} + \rho_2\mathbf{P}_{1,2}^H\mathbf{P}_{1,2}\right). \end{aligned} \quad (18)$$

Also, since $\mathbf{P}_{1,1}$ and $\mathbf{P}_{1,2}$ form a vector in a unitary matrix,

$$\mathbf{P}_{1,1}^H\mathbf{P}_{1,1} + \mathbf{P}_{1,2}^H\mathbf{P}_{1,2} = 1, \quad (19)$$

and hence

$$\begin{aligned} R(\{1\}) &= 2\log\left(1 + \rho_1|\mathbf{P}_{1,1}|^2 + \rho_2(1 - |\mathbf{P}_{1,1}|^2)\right) \\ &= 2\log\left(1 + \rho_2 + |\mathbf{P}_{1,1}|^2(\rho_1 - \rho_2)\right). \end{aligned} \quad (20)$$

Without loss of generality we assume that $\rho_2 \leq \rho_1$. Therefore,

$$\begin{aligned} &\Pr(R(\{1\}) < R) = \\ &\Pr\left(2\log\left(1 + \rho_2 + |\mathbf{P}_{1,1}|^2(\rho_1 - \rho_2)\right) < R\right) = \\ &\Pr\left(|\mathbf{P}_{1,1}|^2 < \frac{2^{R/2} - 1 - \rho_2}{\rho_1 - \rho_2}\right), \end{aligned} \quad (21)$$

where $0 \leq \rho_2 \leq 2^{C/2} - 1$.

The probability density function of the squared magnitude of any entry of an $M \times M$ matrix drawn from the circular unitary ensemble is [10]:[4]

$$f_{|\mathbf{P}_{1,1}|^2}(\mu) \begin{cases} (M-1)(1-\mu)^{M-2} & 0 \leq \mu \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad (22)$$

where the expression holds for $M \geq 2$. In our case, $M = 2$, and thus $|\mathbf{P}_{1,1}|^2 \sim U(0, 1)$. Hence,

$$\begin{aligned} \Pr(R(\{1\}) < R) &= \max\left(\frac{2^{R/2} - 1 - \rho_2}{\rho_1 - \rho_2}, 0\right) \\ &\geq \frac{2^{R/2} - 1 - \rho_2}{\rho_1 - \rho_2}. \end{aligned} \quad (23)$$

As from (12) we have $\rho_1 = \frac{2^C}{1+\rho_2} - 1$, it follows that

$$\Pr(R(\{1\}) < R)) = \frac{2^{R/2} - 1 - \rho_2}{\frac{2^C}{1+\rho_2} - 1 - \rho_2}. \quad (24)$$

Now, by symmetry, it is clear that

$$\Pr(R(\{2\}) < R)) = \Pr(R(\{1\}) < R)). \quad (25)$$

Furthermore, it is not difficult to show (a proof appears in Appendix B of [11]) that the events $\{R(\{1\}) < R\}$ and $\{R(\{2\}) < R\}$ are disjoint. Due to this and by (24) and (16), it follows that[5]

$$\begin{aligned} \Pr\left(R_{\text{JOINT}}(\mathbf{H}_c\mathbf{P}_c) < R\right) &= 2 \cdot \Pr(R(\{1\}) < R) \\ &= 2 \cdot \frac{2^{R/2} - 1 - \rho_2}{\frac{2^C}{1+\rho_2} - 1 - \rho_2}, \end{aligned} \quad (26)$$

which implies that

$$P_{\text{out,JOINT}}^{\text{WC}}(C, R) = \max_{0 \leq \rho_2 \leq 2^{C/2}-1} 2 \cdot \frac{2^{R/2} - 1 - \rho_2}{\frac{2^C}{1+\rho_2} - 1 - \rho_2}. \quad (27)$$

It is readily verified that the derivative of the expression that is maximized with respect to ρ_2 is zero for (and only for)

$$\rho_2^* = 2^{-R/2-1}\left(2^{C+1} - 2^{R/2+1} - 2\sqrt{2^{2C} - 2^{C+R}}\right),$$

and moreover, that the second derivative at this point is negative, and hence this is a global maximum. Finally, by plugging $\rho_2 = \rho_2^*$ (and noting that $R = C - \Delta C$), we obtain

$$P_{\text{out,JOINT}}^{\text{WC}}(C, \Delta C) = 1 - \sqrt{1 - 2^{-\Delta C}}. \quad (28)$$

[4]It is readily seen that this distribution is a special case of the Jacobi distribution.

[5]For the case of $N_r = 1$, the exact outage probability is given by (26), setting $\rho_2 = 0$.

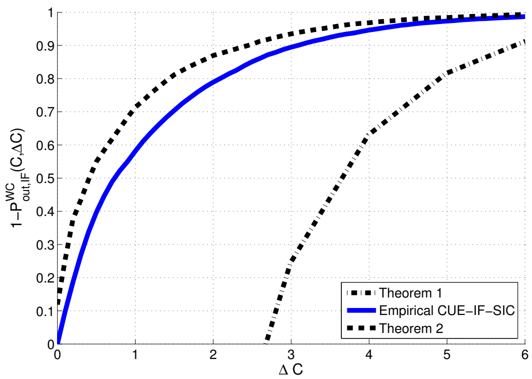

Fig. 1. Theorem 1 (upper bound on outage probability/lower bound on achievable rate) and Theorem 2 (lower bound on WC outage probability/upper bound on achievable rate) for $N_r \times 2$ MIMO channels ($N_r \geq 2$) with mutual information $C = 14$.

□

3) Comparison of Bounds and Empirical Results: Figure 1 depicts the lower and upper bounds as well as results of an empirical simulation of the scheme.[6] We observe that for $N_r \times 2$ channels, the empirical performance of randomly precoded IF-SIC is very close to the upper (ML) bound. This suggests that one can expect that the ML bound may serve as a useful design tool for more general cases ($N_t > 2$).

B. Space-Time Precoding

Hitherto the role of random precoding was limited to facilitating performance evaluation. Namely, applying CUE precoding results in performance being dictated solely by the singular values of the channel, so that one can then consider the worst case performance only with respect to the latter.

In contrast, applying random precoding over time as well as space has operational significance, allowing to improve the guaranteed performance as we quantify next.

1) Background: A block of T channel uses is processed jointly so that the $N_r \times N_t$ physical MIMO channel (1) is transformed into a "time-extended" $N_rT \times N_tT$ MIMO channel. A unitary precoding matrix $\mathbf{P}_{st,c} \in \mathbb{C}^{N_tT\times N_tT}$ that can be either deterministic or random is then applied to the time-extended channel. At the receiver, IF equalization is employed.

Hence, the equivalent channel takes the form

$$\mathcal{H}_c = \mathbf{I}_{T\times T} \otimes \mathbf{H}_c, \tag{29}$$

where $\otimes$ is the Kronecker product. Let $\bar{\boldsymbol{x}}_c \in \mathbb{C}^{N_tT\times 1}$ be the input vector to the time-extended channel. It follows that the output of the time-extended channel is given by

$$\bar{\boldsymbol{y}}_c^P = \mathcal{H}_c\mathbf{P}_{st,c}\bar{\boldsymbol{x}}_c + \bar{\boldsymbol{z}}_c, \tag{30}$$

where $\bar{\boldsymbol{z}}_c$ is i.i.d. unit-variance circularly symmetric complex Gaussian noise. As we assume that the precoding matrix is unitary (for both deterministic or random cases), the WI mutual information of this channel (normalized per channel use) remains unchanged, i.e.,

$$\frac{1}{T}\log\det\left(\mathbf{I} + (\mathcal{H}_c\mathbf{P}_{st,c})(\mathcal{H}_c\mathbf{P}_{st,c})^H\right) = C. \tag{31}$$

When using a given space-time precoding ensemble, the WC scheme outage is defined as

$$P_{\text{out}}^{\text{WC,scheme}}(C,\Delta C) = \sup_{\mathbf{H}_c\in\mathbb{H}(C)} \Pr\left(\frac{1}{T}R_{\text{scheme}}(\mathcal{H}_c\cdot\mathbf{P}_{st,c}) < C - \Delta C\right). \tag{32}$$

2) Upper Bound: For $N_t = 2$, space-time CUE precoding results in a $N_rT \times 2T$ MIMO channel. An upper bound on the WC outage probability can be obtained from Theorem 1 in [4], by substituting $N_t = 2T$.

3) Lower Bound: Define

$$\mathcal{B}_1(T,k,R,\rho_1,\rho_2) = \left\{\underline{\lambda} : \prod_{i=1}^{k}(1+\rho_1\lambda_i+\rho_2(1-\lambda_i)) \leq 2^{R\frac{k}{2}}\right\}$$

$$\mathcal{B}_2(T,k,\tilde{R},\rho_1,\rho_2) = \left\{\underline{\lambda} : \prod_{i=1}^{2T-k}(1+\rho_1\lambda_i+\rho_2(1-\lambda_i)) \leq 2^{\tilde{R}}\right\}$$

$$\kappa_{m_1,m_2,n} = \prod_{j=1}^{n}\frac{\Gamma(m_1-n+j)\Gamma(m_2-n+j)\Gamma(1+j)}{\Gamma(2)\Gamma(m_1+m_2-n+j)}.$$

where $\tilde{R} = \frac{k}{2}\max(R-(k-T),0)$.

Theorem 3. *For an $N_r \times 2$ compound channel with WI-MI equal C, and CUE precoding over T time extensions, we have*

$$P_{\text{out}}^{\text{WC}}(C,R) \geq \max_{0\leq\rho_2\leq 2^C/2}\max_k P_{\text{out}} \tag{33}$$

where $P_{\text{out}} = P_{\text{out}}(k,T,R,\rho_1,\rho_2)$ and

- *For $1 \leq k \leq T$:*

$$P_{\text{out}} = \kappa_1\int_{\mathcal{B}_1}\prod_{i=1}^{k}\lambda_i^{T-k}(1-\lambda_i)^{T-k}\prod_{i<j}(\lambda_j-\lambda_i)^2 d\underline{\lambda}$$

- *For $T+1 \leq k \leq 2T$:*

$$P_{\text{out}} = \kappa_2\int_{\mathcal{B}_2}\prod_{i=1}^{2T-k}\lambda_i^{k-T}(1-\lambda_i)^{k-T}\prod_{i<j}(\lambda_j-\lambda_i)^2 d\underline{\lambda}.$$

and where $\kappa_1 = \kappa_{T,T,k}^{-1}$ and $\kappa_2 = \kappa_{T,T,2T-k}^{-1}$.

Proof. The proof depends on the eigenvalue distribution of submatrices of $\mathbf{P}_c$. As mentioned above, these eigenvalues follow the Jacobi distribution. The full description of the distribution and proof can be found in [11]. □

4) Comparison of Bounds and Empirical Results: We compare the obtained upper and lower bounds with the empirical performance results of CUE-precoded IF-SIC. In addition, for a $T = 2$ time-extended $N_r \times 2$ channel, it is natural to also compare performance with that obtained by replacing CUE

[6] Rather than plotting the WC outage probability, we plot its complement.

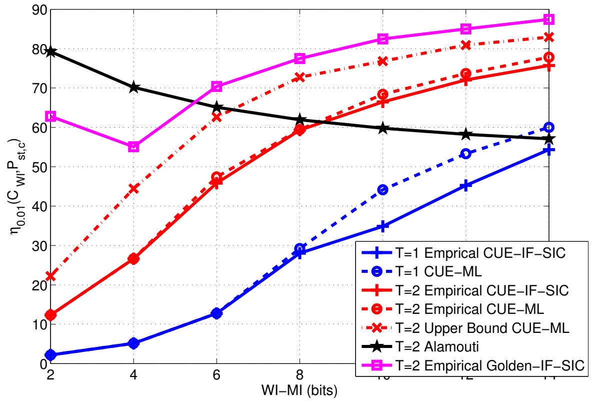

Fig. 2. Guaranteed efficiency at 1% outage probability for the $N_r \times 2$ MIMO channel with various precoding and decoding options, for $T = 1, 2$.

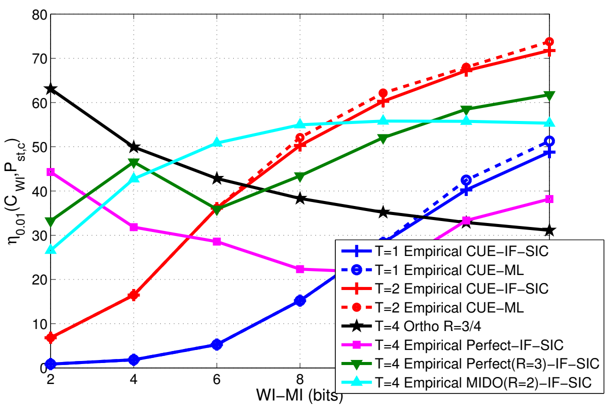

Fig. 3. Guaranteed efficiency at 1% outage probability for the $N_r \times 4$ MIMO channel with various precoding and decoding options, for $T = 1, 2, 4$.

precoding with algebraic precoding. Specifically, we consider Alamouti and golden code precoding.[7]

To that end, let us define the ε-outage capacity of a scheme $R_{\text{scheme}}(\mathbf{P}_{st,c}; \varepsilon)$ as the rate for which

$$P_{\text{out}}^{\text{WC,scheme}}\left(C, R_{\text{scheme}}(\mathbf{P}_{st,c}; \varepsilon)\right) = \varepsilon. \tag{34}$$

Further, the guaranteed transmission efficiency of a scheme, at a given outage probability ε and WI mutual information C, is defined as

$$\eta_\varepsilon(C, \mathbf{P}_{st,c}) = R_{\text{scheme}}(\mathbf{P}_{st,c}; \varepsilon)/C. \tag{35}$$

Figure 2 depicts the guaranteed efficiency at 1% outage for several precoding options for an $N_r \times 2$ channel and $T = 1, 2$. We plot the empirical efficiency for both IF-SIC and ML receivers. It can be seen that for CUE precoding, the performance of IF-SIC is very close to that of ML.

We also present empirical results for an $N_r \times 4$ channel.

[7]When using a fixed space-time precoding matrix, we apply in addition CUE precoding to the physical channel.

Figure 3 depicts the guaranteed efficiency at 1% for several precoding and receiver topologies, where the algebraic codes considered are orthogonal space-time block precoding (rate $3/4$), the perfect code [3], the latter punctured to rate 3, and also the MIDO (rate 2) code [12].

IV. Discussion and Outlook

For the $N_r \times 2$ compound MIMO channel, using CUE precoding over a time-extend channel offers significant benefit over space-only precoding. However, space-time CUE precoding falls short when compared to algebraic space-time precoding. Specifically, the combination of Alamouti precoding at low rates and golden code precoding (with IF-SIC) at high rates is superior to CUE precoding.

Nonetheless, for the $N_r \times 4$ compound MIMO channel, we observe from the empirical results (Figure 3) that there is a region where using random space-time CUE precoding results in the highest guaranteed efficiency. This provides motivation for searching for fixed precoding matrices that yield better results than perfect codes at the price of a small outage probability.

As a concluding remark, we note that the derived lower bound holds only for the case of a maximum of two distinct singular values in the SVD decomposition of $\mathbf{H}_c$. Nevertheless, the treatment holds also for the important case of a open-loop MAC channel with a single receive antenna, where the transmitters (also equipped with a single antenna) know only the sum capacity of the channel, where the results are modified as described in footnote 5.

References

[1] O. Ordentlich and U. Erez, "Precoded integer-forcing universally achieves the MIMO capacity to within a constant gap," *Information Theory, IEEE Transactions on*, vol. 61, no. 1, pp. 323–340, Jan 2015.

[2] P. Elia, B. Sethuraman, and P. V. Kumar, "Perfect space–time codes for any number of antennas," *IEEE Transactions on Information Theory*, vol. 53, no. 11, pp. 3853–3868, 2007.

[3] F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, "Perfect space–time block codes," *IEEE Transactions on Information Theory*, vol. 52, no. 9, p. 3885, 2006.

[4] E. Domanovitz and U. Erez, "Outage behavior of randomly precoded integer forcing over MIMO channels," *CoRR*, vol. abs/1608.01588, 2016. [Online]. Available: http://arxiv.org/abs/1608.01588

[5] J. Zhan, B. Nazer, U. Erez, and M. Gastpar, "Integer-forcing linear receivers," *Information Theory, IEEE Transactions on*, vol. 60, no. 12, pp. 7661–7685, Dec 2014.

[6] M. Mehta, "Random matrices and the statistical theory of energy levels," *Academic, New York*, 1967.

[7] R. Dar, M. Feder, and M. Shtaif, "The Jacobi MIMO channel," *IEEE Transactions on Information Theory*, vol. 59, pp. 2426–2441, 2013.

[8] A. Edelman and N. R. Rao, "Random matrix theory," *Acta Numerica*, vol. 14, pp. 233–297, 2005.

[9] O. Ordentlich, U. Erez, and B. Nazer, "Successive integer-forcing and its sum-rate optimality," in *Communication, Control, and Computing, 2013 51st Annual Allerton Conference on.* IEEE, 2013, pp. 282–292.

[10] A. Narula, M. Trott, and G. W. Wornell, "Performance limits of coded diversity methods for transmitter antenna arrays," *Information Theory, IEEE Transactions on*, vol. 45, no. 7, pp. 2418–2433, Nov 1999.

[11] E. Domanovitz and U. Erez, "Universal performance bounds for randomly precoded integer forcing." [Online]. Available: https://www.eng.tau.ac.il/~uri/papers/ExplicitIFNrx2_Full.pdf

[12] F. Oggier, C. Hollanti, and R. Vehkalahti, "An algebraic MIDO-MISO code construction," in *2010 International Conference on Signal Processing and Communications (SPCOM)*, July 2010, pp. 1–5.