High Rate Streaming Codes Over the Three-Node Relay Network

Abstract—**In this paper, we investigate streaming codes over a three-node relay network. Source node transmits a sequence of message packets to the destination via a relay. Source-to-relay and relay-to-destination links are unreliable and introduce at most N_1 and N_2 packet erasures, respectively. Destination needs to recover each message packet with a strict decoding delay constraint of T time slots. We propose streaming codes under this setting for all feasible parameters $\{N_1, N_2, T\}$. Relay naturally observes erasure patterns occurring in the source-to-relay link. In our code construction, we employ a channel-state-dependent relaying strategy, which rely on these observations. In a recent work, Fong et al. provide streaming codes featuring channel-state-independent relaying strategies, for all feasible parameters $\{N_1, N_2, T\}$. Our schemes offer a strict rate improvement over the schemes proposed by Fong et al., whenever $N_1 < N_2$.**

I. Introduction

Reliable communication with low-latency is critical in many applications, such as audio/video streaming, virtual gaming and tele-medicine. Data packets, which are generated in a sequential fashion, need to be communicated to the receivers over an unreliable packet erasure channel, with strict decoding deadlines. Due to high round-trip delays, methods involving retransmission, such as automatic repeat request (ARQ) are not suitable. For that reason, the literature has considered forward error correction (FEC) schemes a more appropriate solution. In particular, recent literature has studied streaming codes, which are packet-level FECs designed to ensure reliability against packet erasures under a tight decoding delay constraint.

The first paper to consider such codes was [1], where the authors study a point-to-point system consisting of a source and destination. In this work, the authors consider streaming codes which tolerate a packet erasure burst of length at most B and derive an upper bound on the achievable rate of streaming codes. A family of optimal codes, namely, maximally-short codes have been proposed for a wide range of parameters $\{B, T\}$. In a subsequent work [2], the authors prove tightness of the rate upper bound in [1] by providing optimal streaming codes for all $\{B, T\}$. Badr et al. considers a more general, sliding-window-based, packet erasure model where, in any sliding window of W consecutive time slots, there can be either (i) at most N erasures at arbitrary positions or else (ii) an erasure burst which affects at most B consecutive time slots. The paper provides a rate upper bound for streaming codes over the sliding window model and also proposes near-optimal streaming codes. The works [3]–[6] provide rate-optimal streaming codes under the sliding window erasure model. There are several works such as [7]–[10] which study various other models for low-delay communication systems. In contrast to the existing literature which focusses on point-to-point networks, our focus in this paper, is on a topology introduced in a recent paper [11]. Fong et al. [11] propose a generalization of the point-to-point networks to a three-node architecture for streaming codes, which consists of a source, a relay and a destination. This kind of a topology is often present in content delivery networks [11], [12]. An extension of the three-node relay network to a multi-hop network may be found in [13].

A. Our Contributions:

We present a family of streaming codes over the three-node relay network for all parameters $\{N_1, N_2, T\}$. We employ a channel-state-dependent encoding strategy at the relay to generate relay packets which need to be transmitted to the destination. This results in rate improvement, i.e., lesser error control overhead, over the existing coding scheme presented in [11].

II. Setting

In this section, we formally introduce the problem setting. We use the following notation throughout the paper. The set of non-negative integers is denoted by $\mathbb{Z}_+$. The finite field with q elements is denoted by $\mathbb{F}_q$. The set of l-dimensional column vectors over $\mathbb{F}_q$ is denoted by $\mathbb{F}_q^l$. For $a, b \in \mathbb{Z}_+$, we use $[a : b]$ to denote $\{i \in \mathbb{Z}_+ \mid a \leq i \leq b\}$. Naturally, we set $[a : \infty] \triangleq \{i \in \mathbb{Z}_+ \mid i \geq a\}$.

Consider a three node setup consisting of a source, relay and destination. All packet communication happening in source-to-relay and relay-to-destination links are assumed to be instantaneous, i.e., with no propagation delays. In each discrete time slot $t \in [0 : \infty]$, soure has a *message packet* $\underline{m}(t) \in \mathbb{F}_q^k$ available, which needs to be communicated to the destination via relay. Towards this, in time-t, source invokes a source-side encoder:

$$\mathcal{E}_{\mathrm{S}}(t) : \underbrace{\mathbb{F}_q^k \times \cdots \mathbb{F}_q^k}_{t+1 \text{ times}} \to \mathbb{F}_q^{n_1}$$

to produce a *source packet* $\underline{x}(t) \in \mathbb{F}_q^{n_1}$, which is obtained as a function of message packets $\{\underline{m}(t')\}_{t' \in [0:t]}$. Source transmits $\underline{x}(t)$ to the relay over a packet erasure channel. Let $\underline{x}_{\mathrm{R}}(t)$ denote the packet received by relay. We have:

$$\underline{x}_{\mathrm{R}}(t) = \begin{cases} *, & \text{if } \underline{x}(t) \text{ is erased,} \\ \underline{x}(t), & \text{otherwise.} \end{cases}$$

In time-t, once relay receives $\underline{x}_{\mathrm{R}}(t)$, it produces a *relay packet* $\underline{y}(t) \in \mathbb{F}_q^{n_2}$ by invoking a relay-side encoder:

$$\mathcal{E}_{\mathrm{R}}(t) : \underbrace{\mathbb{F}_q^{n_1} \cup \{*\} \times \cdots \times \mathbb{F}_q^{n_1} \cup \{*\}}_{t+1 \text{ times}} \to \mathbb{F}_q^{n_2}.$$

The relay packet $\underline{y}(t)$ is a function of packets $\{\underline{x}_{\mathrm{R}}(t')\}_{t' \in [0:t]}$. Relay transmits $\underline{y}(t)$ to the destination in time-t. Owing to erasures in relay-to-destination link, the packet $\underline{y}_{\mathrm{D}}(t)$ received by destination in time-t is given by:

$$\underline{y}_{\mathrm{D}}(t) \quad = \quad \begin{cases} *, & \text{if } \underline{y}(t) \text{ is erased,} \\ \underline{y}(t), & \text{otherwise.} \end{cases}$$

At time-$(t+T)$, destination uses decoder:

$$\mathcal{D}(t) : \underbrace{\mathbb{F}_q^{n_2} \cup \{*\} \times \cdots \times \mathbb{F}_q^{n_2} \cup \{*\}}_{t+1+T \text{ times}} \to \mathbb{F}_q^{k}$$

to obtain an estimate $\underline{\hat{m}}(t) \in \mathbb{F}_q^{k \times 1}$ of $\underline{m}(t)$ as a function of received packets $\{\underline{y}_{\mathrm{D}}(t')\}_{t' \in [0:t+T]}$. The decoder is delay-constrained as $\underline{m}(t)$ has to be estimated by time-$(t+T)$. The tuple $(\{\mathcal{E}_{\mathrm{S}}(t)\}, \{\mathcal{E}_{\mathrm{R}}(t)\}, \{\mathcal{D}(t)\})$ constitutes an $(n_1, n_2, k, T)_q$ *streaming code*. Rate of an $(n_1, n_2, k, T)_q$ streaming code is naturally defined to be $\frac{k}{\max\{n_1, n_2\}}$.

Definition II.1 (Erasure Sequences)**.** A source-relay erasure sequence denoted by $e_{\mathrm{S}}^{\infty} \triangleq \{e_{\mathrm{S},t}\}_{t \in [0:\infty]}$ is a binary sequence, where $e_{\mathrm{S},t} = 1$ iff $x_{\mathrm{R}}(t) = *$. Similarly, a relay-destination erasure sequence $e_{\mathrm{R}}^{\infty} \triangleq \{e_{\mathrm{R},t}\}_{t \in [0:\infty]}$ will have $e_{\mathrm{R},t} = 1$ iff $y_{\mathrm{D}}(t) = *$

Definition II.2 (N-Erasure Sequences)**.** Let $N \in \mathbb{Z}_+$. A source-relay erasure sequence e_{S}^{∞} is defined to be an N-erasure sequence if $\sum_{t \in [0:\infty]} e_{\mathrm{S},t} \leq N$. Similarly, e_{R}^{∞} is an N-erasure sequence if $\sum_{t \in [0:\infty]} e_{\mathrm{R},t} \leq N$.

Definition II.3 ((N_1, N_2)-Achievability)**.** An $(n_1, n_2, k, T)_q$ streaming code is defined to be (N_1, N_2)-achievable if it is possible to perfectly reconstruct all message packets (i.e., $\underline{\hat{m}}(t) = \underline{m}(t)$ for all t) at the destination in presence of (i) any N_1-erasure sequence e_{S}^{∞} and (ii) any N_2-erasure sequence e_{R}^{∞}.

It may be noted that for an (N_1, N_2)-achievable $(n_1, n_2, k, T)_q$ streaming code, we have $N_1 + N_2 \leq T$. This is because, if $N_1 + N_2 > T$, in presence of erasure of erasure patterns $e_{\mathrm{S}}^{\infty} = \{\underbrace{0, \ldots, 0}_{i}, \underbrace{1 \ldots, 1}_{N_1}, 0, \ldots\}$ and $e_{\mathrm{R}}^{\infty} = \{\underbrace{0, \ldots, 0}_{i+N_1}, \underbrace{1 \ldots, 1}_{N_2}, 0, \ldots\}$, it is impossible for the destination to recover $\underline{m}(i)$ by time-$(i+T)$.

In a recent work [11], the authors provide $(n_1, n_2, k, T)_q$ streaming codes for all parameters $\{T, N_1, N_2\}$ which yield rate:

$$R_{T,N_1,N_2} \triangleq \frac{T+1-N_1-N_2}{T+1-\max\{N_1,N_2\}}. \tag{1}$$

The codes presented in [11] are *state-independent* in the sense that relay-side encoding at time-t performed by $\mathcal{E}_{\mathrm{R}}(t)$ does not depend on the erasure pattern $\{e_{\mathrm{S},t'}\}_{t' \in [0:t]}$ observed thus far by the relay. In contrast, in the present paper, we consider *state-dependent* streaming codes for all parameters $\{T, N_1, N_2\}$. If $N_1 \geq N_2$, rate achievable by our codes match (1). However, when $N_1 < N_2$, our codes offer a strict rate improvement over (1).

Remark II.1. Error protection provided by (N_1, N_2)-achievable (n_1, n_2, k, T) streaming codes may appear to be limiting, as they consider only N_1 erasures across all time slots $[0 : \infty]$ in source-relay link and N_2 erasures across all time slots $[0 : \infty]$ in relay-destination link. However, owing to the delay-constrained decoder, these codes can in fact recover from any e_S^{∞}, e_R^{∞} which satisfy: $\sum_{t'=i}^{i+T} e_{S,t} \leq N_1$ and $\sum_{t'=i}^{i+T} e_{R,t} \leq N_2$ for all $i \in [0 : \infty]$. i.e., in any sliding window of $T+1$ consecutive time slots, source-relay and relay-destination links see at most N_1 and N_2 erasures, respectively.

III. Code Construction

For given parameters $\{N_1, N_2, T\}$, we set message packet, source packet sizes as the following:

$$k \quad \triangleq \quad \prod_{i=0}^{N_1} T+1-N_2-i, \tag{2}$$

$$n_1 \quad \triangleq \quad (T+1-N_2) \prod_{i=0}^{N_1-1} T+1-N_2-i, \tag{3}$$

$$\begin{aligned} n_2 \quad \triangleq \quad & (T+1-N_1) \prod_{i=1}^{N_1} T+1-N_2-i \\ & + \sum_{l=1}^{N_1} \prod_{i=0, i \neq l}^{N_1} T+1-N_2-i. \end{aligned} \tag{4}$$

Let message packet $\underline{m}(t)$ be represented as a column vector of the form:

$$\underline{m}(t) \triangleq \begin{bmatrix} m_0(t) \\ m_1(t) \\ \vdots \\ m_{k-1}(t) \end{bmatrix}.$$

For consistency in notation, we assume that $\underline{m}(t) \triangleq \underline{0}$, if $t < 0$.

A. Source-to-Relay Encoding

In our code construction, each source packet $\underline{x}(t)$ takes the following systematic form (i.e., message packet is embedded within the source packet) up to a permutation of symbols:

$$\underline{x}(t) \triangleq \begin{bmatrix} \underline{m}(t) \\ \underline{p}(t) \end{bmatrix},$$

where $\underline{p}(t) \triangleq [p_0(t)\ p_1(t)\ \cdots\ p_{n-k-1}(t)]^{\top}$ is referred to as a parity packet. Each $\underline{p}(t)$ is computed as a function of message packets $\{\underline{m}(0), \underline{m}(1), \ldots, \underline{m}(t-1)\}$. The generation of source packets can be described in three steps.

1. The source partitions each $\underline{m}(t)$ into $\ell' \triangleq \prod_{i=0}^{N_1-1} (T+1-N_2-i)$ message sub-packets $\{\underline{m}^{(i)}(t)\}_{i \in [0:\ell'-1]}$, each of size $k' \triangleq T+1-N_2-N_1$. Let $\underline{m}^{(0)}(t) \triangleq [m_0(t)\ \ m_1(t)\ \ \cdots\ \ m_{k'-1}(t)]^{\top}$,

$\underline{m}^{(1)}(t) \triangleq [m_{k'}(t)\ m_{k'+1}(t)\ \cdots\ m_{2k'-1}(t)]^\top$ and so on.

2. Source applies diagonal interleaving involving $[n' \triangleq T+1-N_2, k']$-systematic-MDS codes on each sequence of message sub-packets $\{\underline{m}^{(i)}(t)\}_{t\in[0:\infty]} \subseteq \mathbb{F}_q^{k'}$ (see Fig. 1) to produce a corresponding sequence of source sub-packets $\{\underline{x}^{(i)}(t)\}_{t\in[0:\infty]} \subseteq \mathbb{F}_q^{n'}$. As can be noted in Fig. 1, the first k' symbols of each $\underline{x}^{(i)}(t) \in \mathbb{F}_q^{n'}$ constitute the message sub-packet $\underline{m}^{(i)}(t)$.
3. Finally, each source packet $\underline{x}(t)$ is obtained by vertically stacking coded sub-packets $\{\underline{x}^{(i)}(t)\}_{i\in[0:\ell'-1]}$, i.e.,

$$\underline{x}(t) = \begin{bmatrix} \underline{x}^{(0)}(t) \\ \vdots \\ \underline{x}^{(\ell'-1)}(t) \end{bmatrix}.$$

Thus, each source packet $\underline{x}(t)$ has size $n_1 = \ell' n'$ as in (3). It is straightforward to verify that if up to N_1 source packets are erased in source-to-relay link, each erased $\underline{x}(i)$ can be recovered by time-$(i+T-N_2)$.

Comparison With [11]: Source-to-relay encoding steps which we discussed are precisely as in [11] except for the following minor difference. In [11], the authors have only a "single layer" of diagonally interleaved $[n', k']$-MDS codes producing source packets (i.e., $k = k' = T+1-N_2-N_1$, $n = n' = T+1-N_2$) and hence, only steps 1 and 2 are there. Using Lemmas 3 and 4 of [11], the authors of [11] make the following observation:

- O1: If a source packet $\underline{x}(i)$ is erased, one symbol of $\underline{m}(i)$ can be decoded by relay by time-$(i+N_1)$, one more by time-$(i+N_1+1)$ and finally all the $k = T+1-N_2-N_1$ symbols by time-$(i+T-N_2)$.

This observation is true as long as e_S^∞ is an N_1-erasure sequence and exact time slots where erasures happened does not matter, i.e., the observation is inherently state-independent. As we use ℓ' layers of the coding scheme employed in [11], the observation translates to the following.

- O2: If a source packet $\underline{x}(i)$ is erased, ℓ' symbols of $\underline{m}(i)$ can be decoded by relay by time-$(i+N_1)$, ℓ' more symbols by time-$(i+N_1+1)$ and finally all the $k = (T+1-N_2-N_1)\ell$ symbols by time-$(i+T-N_2)$.

In [11], motivated by observation O1, relay transmits relay packets which contain "information" of $\underline{m}(i)$ from time-$(i+N_1+1)$ onwards. In a delay-constrained setup (each $\underline{m}(i)$ has to be recovered at destination by time-$(i+T)$), intuitively, it is desirable if relay can transmit relay packets containing information pertaining to $\underline{m}(i)$ as early as possible. Towards this, we make the following observation (see Appendix A for more details):

- O3: If a source packet $\underline{x}(i)$ is erased, *estimates* of ℓ' symbols of $\underline{m}(i)$ can be obtained by relay, with the reception of each subsequent non-erased source packet. i.e., ℓ' symbols of $\underline{m}(i)$ can be estimated by relay with the reception of the first non-erased source packet after time-i, ℓ' more symbols with the reception of second non-erased source packet after time-i and so on. Finally, estimates all the $k = (T+1-N_2-N_1)\ell'$ symbols are available to relay after receiving the $T+1-N_2-N_1$)-th non-erased source packet post time-i.

Here, by estimate, we mean the following. The symbol $\underline{\tilde{m}}_j(i) \in \mathbb{F}_q$ is an estimate of a message symbol $\underline{m}_j(i) \in \mathbb{F}_q$ if $\underline{m}_j(i)$ can be determined as a function of $\underline{\tilde{m}}_j(i)$ and "past" message packets $\{\underline{m}(t)\}_{t\in[0:i-1]}$. i.e., there exists a *recovery function* $\Psi_{i,j}$ such that: $\Psi_{i,j}(\{\underline{m}(t)\}_{t\in[0:i-1]}, \underline{\tilde{m}}_j(i)) = \underline{m}_j(i)$. Recovery functions are known to the decoder at destination. It may be noted that observation O3 is inherently state-dependent, as time slots in which relay can obtain estimates depends on the erasure pattern seen in source-relay link. In our relay-to-destination encoding which we will discuss next, the encoder make use of these estimates to produce relay packets. Since $\underline{x}(i)$ is erased by assumption and there can be at most N_1-1 more erasures after time-i, first set of ℓ' estimates of message symbols in $\underline{m}(i)$ are available on or before time-$i+N_1$, next set of ℓ' estimates are available on or before time-$(i+N_1+1)$. Hence, by making use of estimates instead of actual message symbols (as in [11]), relay can potentially transmit relay packets containing information on $\underline{m}(i)$ earlier than in [11]. Eventually this translates to a strict rate gain as we will note later in Remark III.2.

B. Relay-to-Destination Encoding

Relay employs two different encoding mechanisms depending on whether the source packet $\underline{x}(t)$ sent from source is successfully received (non-erased) or not (erased). In each time-t, relay transmits a relay packet $\underline{y}(t)$ which is a function of all non-erased source-to-relay source packets within the set $\{\underline{x}(t')\}_{t'\in[0:t]}$. For ease of exposition, we will view each $\underline{y}(t)$ as an unordered set of n_2 symbols, rather than a column vector.

1) $\underline{x}(t)$ is Non-Erased: If a source packet $\underline{x}(t)$ is successfully received by the relay, owing to the use of systematic source-to-relay encoding, the whole message packet $\underline{m}(t)$ of size k (see (2)) is known to the relay in time-t itself. Relay will partition $\underline{m}(t)$ into $\ell'' \triangleq \prod_{i=1}^{N_1}(T+1-N_2-i)$ message sub-packets $\{\underline{m}'^{(i)}(t)\}_{i\in[0:\ell'-1]}$, each of size $k'' \triangleq T+1-N_2$. Relay will then employ diagonal interleaving involving $[n'' \triangleq T+1, k'']$-systematic-MDS codes for each of $\{\underline{m}'^{(i)}(t)\}_{i\in[0:\ell'-1]}$ in the following manner. Let $G \triangleq [I_{k''}\ P]$ denote the generator matrix of the $[n'', k'']$-MDS code, $\underline{m}'^{(0)}(t) \triangleq [m_0'^{(0)}(t)\ m_1'^{(0)}(t)\ \cdots\ m_{k''-1}'^{(0)}(t)]^\top = [m_0(t)\ m_1(t)\ \cdots\ m_{k''-1}(t)]^\top$, $\underline{m}'^{(1)}(t) \triangleq [m_0'^{(1)}(t)\ m_1'^{(1)}(t)\ \cdots\ m_{k''-1}'^{(1)}(t)]^\top = [m_{k''}(t)\ m_{k''+1}(t)\ \cdots\ m_{2k''-1}(t)]^\top$ and so on. Let

$$[p^{(i)}(t+k'')\ p^{(i)}(t+k''+1)\ \cdots\ p^{(i)}(t+n''-1)] = \underline{m}'^{(i)}(t)^\top P.$$

Then, for all $i \in [0 : \ell''-1]$, the relay adds $m_1'^{(i)}(t)\ \cdots\ m_{k''-1}'^{(i)}(t), p^{(i)}(t+k'')\ p^{(i)}(t+k''+1)\ \cdots\ p^{(i)}(t+n''-1)$ to $\underline{y}(t), \underline{y}(t+1), \ldots, y(t+n'-1) \triangleq \underline{y}(t+T)\}$, respectively. Thus, each non-erased source packet $\underline{x}(t)$ contributes ℓ'' symbols to each of the relay packets $\underline{y}(t), \underline{y}(t+$

$m_0(0)$	$m_0(1)$	$\cdots$	$\cdots$	$\cdots$	$m_0(k'-1)$	$m_0(k')$	$\cdots$	$m_0(n'-1)$	$m_0(n')$
$m_1(0)$	$m_1(1)$	$\cdots$	$\cdots$	$\cdots$	$m_1(k'-1)$	$m_1(k')$	$\cdots$	$m_1(n'-1)$	$m_1(n')$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m_{k'-2}(0)$	$m_{k'-2}(1)$	$\cdots$	$\cdots$	$\cdots$	$m_{k'-2}(k'-1)$	$m_{k'-2}(k')$	$\cdots$	$m_{k'-2}(n'-1)$	$m_{k'-2}(n')$
$m_{k'-1}(0)$	$m_{k'-1}(1)$	$\cdots$	$\cdots$	$\cdots$	$m_{k'-1}(k'-1)$	$m_{k'-1}(k')$	$\cdots$	$m_{k'-1}(n'-1)$	$m_{k'-1}(n')$
$p_0(0)$	$p_0(1)$	$\cdots$	$\cdots$	$\cdots$	$p_0(k'-1)$	$p_0(k')$	$\cdots$	$p_0(n'-1)$	$p_0(n')$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p_{r'-1}(0)$	$p_{r'-1}(1)$	$\cdots$	$p_{r'-1}(r')$	$\cdots$	$p_{r'-1}(k'-1)$	$p_{r'-1}(k')$	$\cdots$	$p_{r'-1}(n'-1)$	$p_{r'-1}(n')$
$\underline{x}^{(0)}(0)$	$\underline{x}^{(0)}(1)$	$\cdots$	$\underline{x}^{(0)}(r')$	$\cdots$	$\underline{x}^{(0)}(k'-1)$	$\underline{x}^{(0)}(k')$	$\cdots$	$\underline{x}^{(0)}(n'-1)$	$\underline{x}^{(0)}(n')$

Fig. 1: An illustration of diagonal interleaving technique applied by source-side encoder to produce source sub-packets $\{\underline{x}^{(i)}(t)\}_{t\in[0:\infty]}$. We illustrate here the case $i=0$ and the same procedure will be applied for all $i \in [0:\ell'-1]$. Let $k' \triangleq T+1-N_2-N_1, n' \triangleq T+1-N_2, r' \triangleq n'-k' = N_1$ and $\underline{m}^{(0)}(t) \triangleq [m_0(t)\ m_1(t)\ \cdots\ m_{k'-1}(t)]^\top$. Each diagonal is a codeword of a systematic $[n',k']$-MDS code, whose initial k' symbols are message symbols. Coded packet $\underline{x}(t)$ is obtained by vertically stacking ℓ' coded sub-packets $\{\underline{x}^{(i)}(t)\}_{i\in[0:\ell'-1]}$.

$1),\ldots,y(t+n'-1)$. Note that there is a slight difference (apart from the difference in MDS code parameters) in the way message symbols are arranged in the diagonal interleaving techniques employed at source-side encoder (see Fig. 1) and relay-side encoder. In Fig. 1, symbols of each message sub-packet $\underline{m}^{(i)}(t)$ appear vertically (within the same coded sub-packet). However, in relay-side diagonal interleaving, symbols of each message sub-packet appear diagonally, i.e., they are part of the same MDS codeword.

If $\underline{x}(t)$ is erased, relay has no information of $\underline{m}(t)$ in time-t and relay will follow a different encoding mechanism. Relay will include $\{C(t;1),C(t;2),\ldots,C(t;T)\}$ as a part of $\underline{y}(t+1),\underline{y}(t+2),\ldots,\underline{y}(t+T)$, respectively. Here, each $C(t;j)$ is a set of code symbols (to be viewed as a column vector) computed by relay, as a function of all non-erased source packets in time slots $[0:t+j]$ (see Fig. 2 for an illustration). The size of each $C(t;j)$ can vary anywhere in $[0:\ell']$. In the remainder of this section, we will discuss (i) how to determine $C(t;j)$'s, (ii) how we obtain a relay packet size which matches (4) and (iii) how recoverability of each $\underline{m}(t)$ is guaranteed at destination by time-$(t+T)$ despite the possibility of N_2 erasures in relay-to-destination link.

2) $\underline{x}(t)$ is Erased: If source packet $\underline{x}(t)$ is erased, as source-to-relay encoding is causal, the relay clearly does not receive any information regarding message packet $\underline{m}(t)$ in time-t. Let $\mathcal{I}_t \triangleq \{t_1,t_2,\ldots,t_{T+1-N_2-N_1}\}$ denote the first $T+1-N_2-N_1$ time slots in $[t+1:t+T]$ during which source packets are non-erased in source-to-relay link. Based on our discussion in Sec. III-A, relay has access to estimates of $\ell' j$ message symbols in $\underline{m}(t)$ by time-t_j, $j\in[1:T{+}1{-}N_2{-}N_1]$.

Recall that $C(t;i)$, $i\in[1:T]$ consists of a set of code symbols which are to be included a part of relay packet $\underline{y}(t+i)$. Each $C(t;i)$ has size $\alpha_{t,i}\in[0:\ell']$ and is computed purely as a function of estimates of $\underline{m}(t)$ received in time slots $\{t_j, j\in[1:T+1-N_2-N_1] \mid t_j \le t+i\}$. Each $\alpha_{t,i}$ is determined on-the-fly by relay in time-$(t+i)$ based on erasure pattern in the source-relay link in time slots $[t:t+i]$. $C(t;i)$ is obtained by "slicing" a codeword of a systematic MDS code in the following manner. Consider a "long" systematic $[n_{\text{long}},k_{\text{long}}]$-MDS code, where $n_{\text{long}} \triangleq \sum_{i\in[1:T]}\alpha_{t,i}, k_{\text{long}} \triangleq (T+1-N_2-N_1)\ell' = k$. The length-$n_{\text{long}}$ row-vector $C(t)^\top \triangleq [C(t;1)^\top\ C(t;2)^\top\ \cdots\ C(t;T)^\top]$ is then a codeword of this $[n_{\text{long}},k_{\text{long}}]$-MDS code. Initial k code symbols of $C(t)^\top$ are k estimates of the symbols in $\underline{m}(t)$. Precisely, the first ℓ' code symbols of $C(t)^\top$ are the ℓ' estimates of $\underline{m}(t)$ determined by relay in time-t_1, the next ℓ' code symbols are the ℓ' estimates determined in time-t_2 and so on. The last $n_{\text{long}}-k$ code symbols of $C(t)^\top$ are MDS parity symbols obtained as a function of the initial k code symbols of $C(t)^\top$. In the following, we discuss how $\{\alpha_{t,i}\}$ are determined, which essentially completes the description of relay-to-destination encoder.

Consider time slots $[t:t{+}T]$. By assumption, $\underline{x}(t)$ is erased and there can be at most N_1-1 more erasures in time slots $[t+1:t+T]$ (in source-to-relay link). For $j\in[1:N_1-1]$, let $t+v_j$ denote the j-th time slot within $[t{+}1:t{+}T]$ where there is an erasure (see Fig. 3). If there are only $l<N_1-1$ erasures in time slots $[t+1:t+T]$, we set $v_{j'} \triangleq T+1$, $j'\in[l{+}1:N_1{-}1]$. Also, let $v_0 \triangleq 1$, $v_{N_1} \triangleq T$. Let $\kappa_t(t{+}i)$ denote

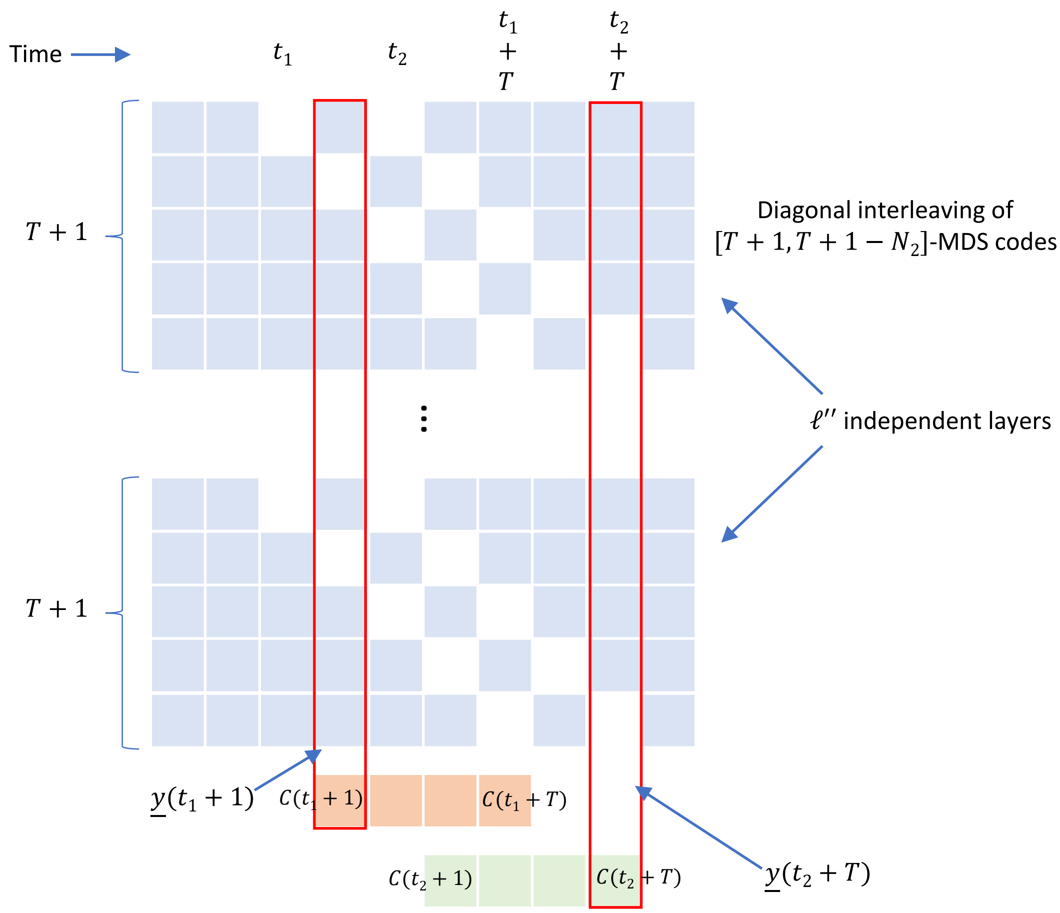

Fig. 2: Let $k'' = T+1-N_2$, $n'' = T+1$ and $r'' = n'' - k'' = N_2$. In the figure, source packets $\underline{x}(t_1)$ and $\underline{x}(t_2)$ are assumed to be erased. If a source packet $\underline{x}(t)$ is non-erased, diagonal interleaving using systematic $[n'', k'']$-MDS codes will be applied to each of the corresponding ℓ'' message sub-packets. Each non-erased source packet thus contributes ℓ'' symbols to each of relay packets $\{\underline{y}(t), \ldots, \underline{y}(t+T)\}$. If $\underline{x}(t)$ is erased, relay adds the sets of code symbols $C(t;1), \ldots, C(t;T)$ as part of relay packets $\underline{y}(t+1), \ldots, \underline{y}(t+T)$, respectively. Each $C(t;i)$ may be viewed as a column vector of size ranging in $[0:\ell']$. As illustrated in the figure, symbols appearing in a column (say, corresponding to time-t) constitute the relay packet $\underline{y}(t)$.

the cumulative number of estimations of message symbols in $\underline{m}(t)$ available to relay by time-$\{t+i\}$. The values of $\alpha_{t,i}$'s are obtained by relay in the following manner:

- *Step-1*: Initialize $i=1$. Go to next step.
- *Step-2*: Let the number of erasures in time slots $[t+1 : t+i-1]$ be $j^* \in [0:N_1-1]$. If $\underline{x}(t+i)$ is not erased or $\kappa_t(t+i) = k$, $\alpha_{t,i} = \frac{k}{T-N_2-j^*} \triangleq \ell_{j^*}$. Go to next step.
- *Step-3* If $\underline{x}(t+i)$ is erased and $\kappa_t(t+i) < k$, $\alpha_{t,i} = \min\{\ell_{j^*}, \kappa_t(t+i) - \sum_{a\in[1:i-1]} \alpha_{t,a}\}$. Go to next step.
- *Step-4* Increment i by 1. If $i \le T$, go to Step-2.

This is just a greedy algorithm such that as much symbols are included in $C(t;i)$ subject to following constraints:

1) $\alpha_{t,i} \le \ell_{j^*}$,
2) $C(t;i)$ is a function of message symbol estimates of $\underline{m}(t)$ obtained by relay in non-erased time slots among $[t+1 : t+i]$,
3) $C(t)^\top \triangleq [C(t;1)^\top \; C(t;2)^\top \; \cdots \; C(t;T)^\top]$ is a codeword of a systematic $[n_{\text{long}}, k_{\text{long}}]$-MDS code. Initial k code symbols k message symbol estimates of $\underline{m}(t)$.

3) Worst-Case Length of Relay Packets: Recall from Sec. III-B1 that if $\underline{x}(t')$ is non-erased, it contributes ℓ'' symbols to the packet-size of relay packets in time slots $[t' : t'+T]$, where $\ell'' = \frac{k}{T+1-N_2}$. At a given time-$t$, each non-erased source packet in time slots $[t-T:t]$ contributes ℓ'' symbols to the length of $\underline{y}(t)$. Assume that there are $i \le N_1$ erasures at time slots $\{\tau_1, \tau_2, \ldots, \tau_i\} \subseteq [t-T:t]$. Using the definition of j^* in Step-2 of the algorithm in Sec. III-B2 and monotonicity of ℓ_{j^*}'s; $\ell_0 < \ell_1 < \cdots < \ell_{N_1-1}$, we have that $C(\tau_1; t-\tau_1)$ has size $\alpha_{\tau_1, t-\tau_1} \le \ell_{i-1}$. Here $C(\tau_1; t-\tau_1)$ is the set of symbols getting added to $\underline{y}(t)$ owing to erasure at time-τ_1. Similarly, we have: $\alpha_{\tau_2, t-\tau_2} \le \ell_{i-2}, \ldots, \alpha_{\tau_i, t-\tau_i} \le \ell_0$. Thus, packet length of $\underline{y}(t)$ is at most $(T+1-i)\ell'' + \sum_{j\in[0:i-1]} \ell_j$. As $\ell'' < \ell_0 < \cdots < \ell_{N_1-1}$, $i = N_1$ maximizes the packet-length and hence, worst-case packet length is given by $(T+1-N_1)\ell'' + \sum_{j\in[0:N_1-1]} \ell_j \triangleq n_2$ (see (4)). If relay packet length is less than n_2, we use zero padding to make the effective length equal to n_2.

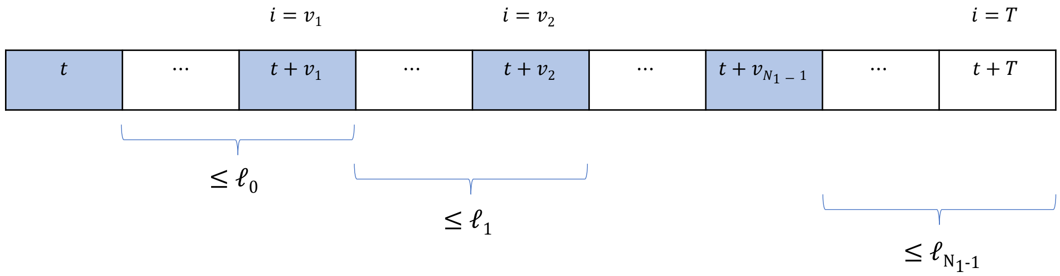

Fig. 3: An erasure pattern observed by relay in time slots $[t : t+T]$. Let $\ell_j = \frac{k}{T-N_2-j}$. $\{\alpha_{t,i}\}_{i\in[1:T]}$ are bounded as shown in the figure.

Remark III.1 (Relay Packet Overhead)**.** As destination needs to know the exact locations where erasures happened in the source-to-relay link, we add some fixed overhead to the relay packets. By adding $\log_q \binom{T+1}{N_1}$ symbols to each relay packet, each relay packet $\underline{y}(t)$ can communicate the erasure pattern observed in time slots $[t-T : t]$ to the destination.

Remark III.2 (Comparison of Achievable Rates)**.** Using (2), (3) and (4), we verified via computer simulations that the rate achieved by our code construction is at least (1) for all $N_1, N_2, T \leq 150$ (T is capped at 150 owing to overflow errors). There is a strict rate improvement when $N_2 > N_1$.

4) Recoverability of $\underline{m}(t)$ at Destination by Time-$(t+T)$: If $\underline{x}(t)$ is non-erased, owing to the use of diagonal interleaving involving $[T+1, T+1-N_2]$-MDS codes, clearly, $\underline{m}(t)$ is recoverable at destination by time-$(t+T)$ even in presence of N_2 erasures in relay-to-destination link. If $\underline{x}(t)$ is erased, as discussed earlier, we pack scalar code symbols of a long MDS code of dimension k into T vector code symbols of sizes $\{\alpha_{t,1}, \alpha_{t,2}, \ldots, \alpha_{t,T}\}$. We will initially show that even if there are N_2 erasures in relay-to-destination link, the destination will have access to at least k code symbols of the long MDS code by time-$(t+T)$. In other words, we need to show that for any $\mathcal{J} \in [1:T]$ such that $|\mathcal{J}| = T - N_2$, we have $\sum_{j\in\mathcal{J}} \alpha_{t,j} \geq k$. We show this in the following.

Case-1: Let $\alpha_{t,i} = \ell_{j^*}$ for all $i \in [1:T]$, where j^* is as defined in Step-2 of the algorithm in Sec. III-B2. As $\ell_0 < \ell_1 < \cdots < \ell_{N_1-1}$, in this case, $\alpha_{t,i} \geq \ell_0 = \frac{k}{T-N_2}$ $\forall i \in [1:T]$. Thus, clearly, $\sum_{j\in\mathcal{J}} \alpha_{t,j} \geq k$.

Case-2: There exists an $i \in [1:T]$ such that $\alpha_{t,i} < \ell_{j^*}$. Let i^* denote the largest such i. Note that condition in Step-3 is true in this case. Clearly, $i^* = v_{j^*}$ for some $j^* \in [1:N_1-1]$. For $j \in [i^*+1 : T]$, we have:

$$\alpha_{t,j} \geq \frac{k}{T-N_2-j^*}. \tag{5}$$

Moreover, we have:

$$\alpha_{t,i} \leq \frac{k}{T-N_2-(j^*-1)}, \tag{6}$$

for all $i \in [1:i^*]$. Since $\kappa_t(t+i^*) < k$ (as condition in Step-3 is true), there has to be strictly less than $T+1-N_2-N_1$ non-erased packets in time slots $[t+1 : t+i^*]$ (since relay recovers $\ell' s$ estimates of message symbols if there are s non-erased packets and $k = \ell'(T+1-N_2-N_1)$). On the other hand, the number of non-erased packets in time slots $[t+1 : t+i^*]$ is precisely given by $i^* - j^*$. Thus $i^* - j^* < T+1-N_2-N_1$ and hence:

$$T - i^* - N_2 > 0. \tag{7}$$

From (5),(6) and (7), we have: $\sum_{j\in\mathcal{J}} \alpha_{t,j} \geq \sum_{i\in[1:i^*]} \alpha_{t,j} + (T-i^*-N_2)\frac{k}{T-N_2-j^*}$. Since $\alpha_{t,i} < \ell_{j^*}$, all message symbol estimates recovered within time slots $[t+1 : t+i^*]$ are transmitted via $C(t;1), \ldots, C(t;i^*)$. Thus, we have $\sum_{i\in[1:i^*]} \alpha_{t,j} = (i^*-j^*)\ell' = (i^*-j^*)\frac{k}{T+1-N_2-N_1}$. Thus:

$$\begin{aligned}
\sum_{j\in\mathcal{J}} \alpha_{t,j} &\geq (i^*-j^*)\frac{k}{T+1-N_2-N_1} \\
&\quad + (T-i^*-N_2)\frac{k}{T-N_2-j^*} \\
&\geq (i^*-j^*)\frac{k}{T-N_2-j^*} \\
&\quad + (T-i^*-N_2)\frac{k}{T-N_2-j^*} \\
&= k.
\end{aligned}$$

Owing to the use of the long $[n_{\text{long}}, k_{\text{long}} = k]$-MDS code, k estimates of message symbols in $\underline{m}(t)$ can thus be recovered by time-$(t+T)$. Let t' denote the first time slot where source-to-link faced an erasure. Clearly, all message packets $\{\underline{m}(0), \underline{m}(1), \ldots, \underline{m}(t'-1)\}$ are available to the destination by time-$(t'-1+T)$ (because of diagonal interleaving). In time-$(t'+T)$, using these "past" message packets, $\underline{m}(t')$ can be recovered from its k message symbol estimates. This process can clearly be repeated and thus, destination recovers all $\underline{m}(t)$ by time-$(t+T)$.

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Appendix A
Recovery of Estimates of Message Symbols at Relay

Assume that $\underline{x}(t)$ is erased. If prior message packets $\{\underline{m}(t')\}_{t'\in[0:t-1]}$ are known, with the reception of each non-erased $\underline{x}(.)$ after time-t, relay will be able to uncover ℓ' symbols of $\underline{m}(t)$. As $k = k'\ell'$, with the reception of the k'-th non-erased source packet after time-t, relay will thus have decoded the full message packet $\underline{m}(t)$. In order to see this, consider the sequence of source sub-packets $\{\underline{x}^{(0)}(t)\}_{t\in[0:\infty]}$ produced by interleaving systematic $[n', k']$-MDS codes (Fig. 1).

The set of code symbols: $\{m_0(t-k'+1), m_1(t-k'+2), \cdots, m_{k'-1}(t), p_0(t+1), \cdots, p_{r'-1}(t+r')\}$ forms a codeword of the $[n', k']$-MDS code. As prior message packets $\{\underline{m}(t')\}_{t'\in[0:t-1]}$ are available, the $k'-1$ message symbols $m_0(t-k'+1), m_1(t-k'+2), \cdots, m_{k'-2}(t-1)$ are already known. As $r' = n' - k' = N_1$, there will be at most $N_1 - 1 = r' - 1$ more erasures in time slots $[t+1 : t+r']$. Hence, there exists a non-erased source sub-packet $\underline{x}^{(0)}(t+l)$ for some $l \in [1 : r']$. From $\underline{x}^{(0)}(t+l)$, relay obtains the symbol $p_{l-1}(t+l)$. Clearly, symbol $m_{k'-1}(t)$ can now be recovered as (ignoring the coefficients of linear combination):

$$\begin{aligned} m_{k'-1}(t) &= \underbrace{\sum_{l=1}^{k'-1} m_{k'-1-l}(t-l)}_{(k'-1) \text{ message symbols from prior time slots}} \\ &+ \underbrace{p_{l-1}(t+l)}_{\text{from first non-erased coded sub-packet after time-}t}. \end{aligned}$$

Similarly, $m_\ell(t), \ell \in [0 : k'-1]$ can be expressed as $m^*_\ell(t) + \tilde{m}_\ell(t)$. Here, $m^*_\ell(t)$ is a sum of ℓ message symbols $\{m_0(t-\ell), \ldots, m_{\ell-1}(t-1)\}$ from the past and $\tilde{m}_\ell(t)$ is a sum of $k'-\ell$ symbols which are obtained from first $k'-\ell$ non-erased source sub-packets received by relay after time-t. We will refer to this second term as an estimate of $m_\ell(t)$. In presence of prior message packets, estimate $\tilde{m}_\ell(t)$ can be used to recover $m_\ell(t)$. As each source packet is obtained by vertically stacking ℓ' source sub-packets, with the reception of each non-erased source packet after time-t, relay can compute estimates of ℓ' message symbols in $\underline{m}(t)$.