Simple Bounds for the Symmetric Capacity of the Rayleigh Fading Multiple Access Channel

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Abstract—**Communication over the i.i.d. Rayleigh slow-fading MAC is considered, where all terminals are equipped with a single antenna. Further, a communication protocol is considered where all users transmit at (just below) the symmetric capacity (per user) of the channel, a rate which is fed back (dictated) to the users by the base station. Tight bounds are established on the distribution of the rate attained by the protocol. In particular, these bounds characterize the probability that the dominant face of the MAC capacity region contains a symmetric rate point, i.e., that the considered protocol strictly attains the sum capacity of the channel. The analysis provides a non-asymptotic counterpart to the diversity-multiplexing tradeoff of the multiple access channel. Finally, a practical scheme based on integer-forcing and space-time precoding is shown to be an effective coding architecture for this communication scenario.**

I. INTRODUCTION

The problem of communication over the slow (block) fading Rayleigh multiple access channel (MAC) is addressed, where all nodes are equipped with a single antenna. Specifically, the performance of a simple protocol is considered, where all users transmit at a rate just below the symmetric capacity (per user) of the channel. The underlying assumption is that the latter rate is dictated to the users by the base station, utilizing a minimal amount of feedback. It will be shown that such a protocol is quite effective in terms of the fraction of capacity achieved. The analysis is largely based on the approach developed in the recent work [1].

Some insight into the performance of the protocol may be obtained by considering the diversity-multiplexing trade-off (DMT) of the channel. Specifically, the DMT of the Rayleigh MAC was studied in [2]. As a special case, the scenario where all users transmit at the same rate was highlighted and a simple expression for the DMT in this case was derived.

Although the DMT analysis is asymptotic in nature, instructive lessons may nonetheless be drawn from it. First, it is clear that in the limit of high SNR, the ratio of the symmetric capacity and sum capacity approaches one. More importantly, the analysis of the typical error events in the Rayleigh-fading MAC reveals that with high probability (again, at high SNR) outage occurs either due to the sum capacity being the bottleneck or a single user being the bottleneck [2]. That is, for the scalar MAC with symmetric rate transmission, the DMT is the intersection of only two lines, see Figure 1.

This in turn suggests that in the protocol considered, where the sum capacity is given, the bottleneck will correspond to
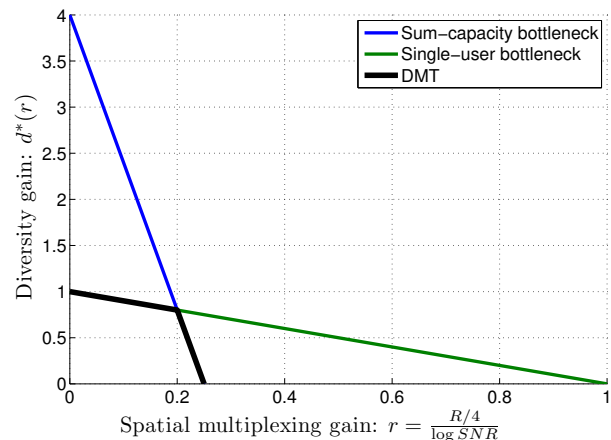

Fig. 1. DMT curve for a four-user Rayleigh fading MAC.

a single-user constraint. The analysis to follow, characterizing the distribution of the rate attained by the transmission protocol, will indeed establish this intuition. We further characterize the probability that the dominant face of the MAC capacity region contains a symmetric-rate point, i.e., the probability that the scheme strictly attains the sum capacity of the channel.

The analysis provides a non-asymptotic counterpart to the diversity-multiplexing tradeoff of the MAC and will serve to obtain tight bounds on the average throughput of the described protocol. The latter turns out to be quite close to the average sum capacity.

As a candidate practical scheme, it is shown that the integer-forcing receiver combined with space-time precoding performs well in the considered scenario.

II. PROBLEM FORMULATION AND PRELIMINARIES

The channel is described by

$$y = \sum_{i=1}^{N} h_i x_i + n \qquad (1)$$

where $h_i \sim \mathcal{CN}(0, \mathsf{SNR})$ and $n \sim \mathcal{CN}(0, 1)$, and where there is no statistical dependence between any of these random variables. Without loss of generality we assume throughout the analysis to follow that the signal x_i of each user satisfies a

power constraint of 1, i.e., the SNR is absorbed in the channel gains.

The capacity region of the channel is given by (see, e.g., [3]) all rate vectors $(R_1, \ldots, R_N)$ satisfying the constraints

$$\sum_{i \in S} R_i < \log \left(1 + \sum_{i \in S} |h_i|^2 \right), \tag{2}$$

for all $S \subseteq \{1, 2, \ldots, N\}$. We denote the sum capacity by

$$C = \log \left(1 + \sum_{i=1}^{N} |h_i|^2 \right). \tag{3}$$

If we impose the constraint that all users transmit at the same rate, then the maximal achievable rate is given by substituting $R_i = C_{\text{sym}}/N$ in (2), from which it follows that the symmetric capacity is dictated by the bottleneck:

$$C_{\text{sym}} = \min_{S \subseteq \{1,2,\ldots,N\}} \frac{N}{|S|} \log \left(1 + \sum_{i \in S} |h_i|^2 \right). \tag{4}$$

We next analyze the conditional "cumulative distribution function"

$$\Pr(C_{\text{sym}} < R | C) \tag{5}$$

for i.i.d. Rayleigh fading.[1] The latter quantity provides a full statistical characterization for the performance of the transmission protocol considered. Another interpretation of (5) is as a conditional outage probability in an open-loop scenario. That is, in a scenario where all users (when they are active) transmit at a common target rate R. For a given number of active users N, the outage probability is then given by $\mathbb{E}[\Pr(C_{\text{sym}} < N \cdot R | C)]$ where the expectation is over C and is computed w.r.t. an i.i.d. Rayleigh distribution.

III. TWO-USER I.I.D. RAYLEIGH FADING MAC

We start by analyzing the simplest case of a two-user MAC, for which we obtain an exact of characterization of (5).

Theorem 1. *For a two-user i.i.d. Rayleigh fading MAC with sum capacity C, for any rate $R \leq C$,*

$$\Pr(C_{\text{sym}} < R | C) = 2 \cdot \frac{2^{R/2} - 1}{2^C - 1}. \tag{6}$$

Proof. Given C, $\mathbf{h} \triangleq (h_1, h_2)$ is uniformly distributed over a two-dimensional complex sphere of radius $\sqrt{2^C - 1}$. Hence, $\mathbf{h}/\|\mathbf{h}\|$ can be viewed as the first row of a random (Haar) unitary matrix $\mathbf{U}$.

By (4) we obtain

$$C_{\text{sym}} = \min \left\{ C(\{1\}), C(\{2\}), C \right\}. \tag{7}$$

We start by analyzing $C(\{1\})$, which is given by

$$\begin{aligned} C(\{1\}) &= 2 \log \left(1 + |h_1|^2 \right) \\ &= 2 \log \left(1 + |\mathbf{U}_{1,1}|^2 (2^C - 1) \right). \end{aligned} \tag{8}$$

[1]We use quotation marks since we impose strict inequality in $C_{\text{sym}} < R$.

It follows that

$$\Pr(C(\{1\}) < R | C) = \Pr \left(|\mathbf{U}_{1,1}|^2 < \frac{2^{R/2} - 1}{2^C - 1} \right) \tag{9}$$

$$= \Pr \left(|\mathbf{U}_{1,1}|^2 \in \left[0, \frac{2^{R/2} - 1}{2^C - 1} \right) \right) \tag{10}$$

Since (see, e.g., [4]) for a 2×2 matrix drawn uniformly with respect to the Haar measure, we have $|\mathbf{U}_{1,1}|^2 \sim \text{Unif}([0,1])$, we obtain

$$\Pr(C(\{1\}) < R | C) = \frac{2^{R/2} - 1}{2^C - 1}. \tag{11}$$

Now, since $\mathbf{U}_{1,1}$ and $\mathbf{U}_{1,2}$ are the elements of a row in a unitary matrix, we have

$$|\mathbf{U}_{1,1}|^2 + |\mathbf{U}_{1,2}|^2 = 1. \tag{12}$$

Hence,

$$\begin{aligned} \Pr(C(\{2\}) < R | C) &= \Pr \left(|\mathbf{U}_{1,2}|^2 < \frac{2^{R/2} - 1}{2^C - 1} \right) \\ &= \Pr \left(1 - |\mathbf{U}_{1,1}|^2 < \frac{2^{R/2} - 1}{2^C - 1} \right) \\ &= \Pr \left(|\mathbf{U}_{1,1}|^2 \in \left(1 - \frac{2^{R/2} - 1}{2^C - 1} \right] \right) \end{aligned} \tag{13}$$

Since for any rate $R \leq C$, the intervals appearing in (10) and (13) are disjoint and of the same length, it follows that

$$\Pr(C_{\text{sym}} < R | C) = 2 \cdot \frac{2^{R/2} - 1}{2^C - 1}. \tag{14}$$

$\square$

We note that the probability in (14) is strictly smaller than 1 at $R = C$. Thus, the probability that the symmetric capacity coincides with the sum capacity is strictly positive.

Figure 2 depicts the capacity region for three different channel realizations for which the sum capacity equals 2. The probability that the symmetric capacity coincides with the sum capacity amounts to the probability that the symmetric rate line passes through the dominant face of the capacity region and is given by

$$\begin{aligned} \Pr \left(C_{\text{sym}} = C | C \right) &= 1 - \Pr \left(C_{\text{sym}} < C | C \right) \\ &= 1 - 2 \cdot \frac{2^{C/2} - 1}{2^C - 1}. \end{aligned} \tag{15}$$

As an example, for $C = 2$, this probability is $1/3$.

Figure 3 depicts the probability density function of the symmetric capacity of a two-user i.i.d. Rayleigh fading MAC given that the sum capacity is $C = 2$. The probability in (15) manifests itself as a delta function at the sum capacity.

IV. EXTENSION TO N-USER I.I.D. RAYLEIGH FADING MAC ($N \geq 2$)

Theorem 1 may be extended to the case of $N > 2$ users. However, rather than obtaining an exact characterization of the distribution of the symmetric capacity, we will now be content
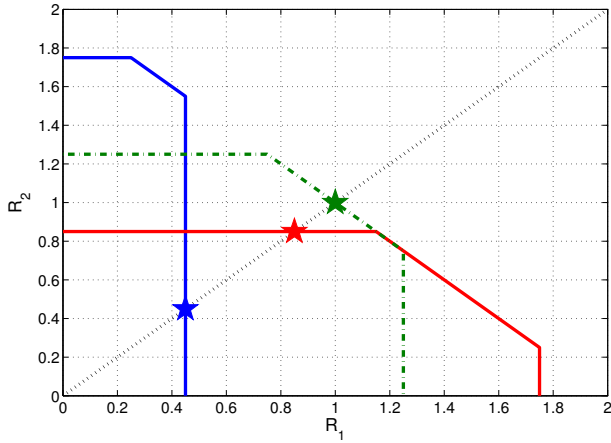
Fig. 2. Different capacity regions corresponding to a two-user MAC with sum capacity $C = 2$. For the channel depicted with a dashed-dotted line, the dominant face constitutes the bottleneck and $C_{\text{sym}} = C$.
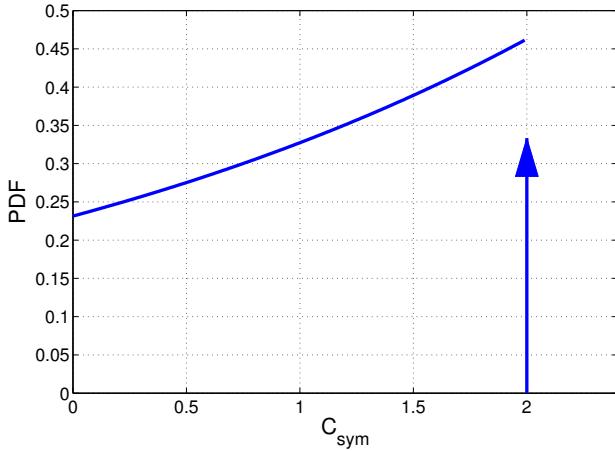

Fig. 3. Probability density function of the symmetric capacity of a two-user i.i.d. Rayleigh fading MAC given that the sum capacity is $C = 2$.

with deriving lower and upper bounds for it. We begin with the following lemma from which Theorem 2 follows.

Lemma 1. *For an N-user i.i.d. Rayleigh fading MAC with sum capacity C, and for any subset of users $S \subseteq \{1, 2, \ldots, N\}$ with cardinality k, we have*

$$\Pr\left(C(\{S\}) < R|C\right) = \frac{\mathcal{B}(\frac{2^{R|S|/N}-1}{2^C-1}; |S|, N-|S|)}{\mathcal{B}(1; |S|, N-|S|)}$$
$$\triangleq P_{\text{out}}(k, R|C) \qquad (16)$$

where $0 \leq R \leq C$ and

$$\mathcal{B}(x; a, b) = \int_0^x u^{a-1}(1-u)^{b-1} du \qquad (17)$$

is the incomplete beta function.

Proof. Similar to the case of two users, $\mathbf{h} \triangleq (h_1, \ldots, h_N)$ is

uniformly distributed over an N-dimensional complex sphere of radius $\sqrt{2^C - 1}$ and hence $\mathbf{h}/\|\mathbf{h}\|$ may be viewed as the first row of a unitary matrix $\mathbf{U}$ drawn at random according to the Haar measure.

By symmetry, for any set S with cardinally k, the distribution of $C(\{S\})$ is equal to that of

$$C(\{1, 2, \ldots, k\}) = \frac{N}{k} \log\left(1 + \sum_{i=1}^{k} |h_i|^2\right)$$
$$= \frac{N}{k} \log\left(1 + \left(2^C - 1\right) \sum_{i=1}^{k} |U_{1,i}|^2\right). \qquad (18)$$

Denoting the partial sum of k entries as $X = \sum_{i=1}^{k} |U_{1,i}|^2$, we therefore have

$$\Pr(C(\{S\}) < R|C) = \Pr\left(1 + \left(2^C - 1\right) X < 2^{R\frac{N}{k}}\right)$$
$$= \Pr\left(X < \frac{2^{R\frac{N}{k}} - 1}{2^C - 1}\right). \qquad (19)$$

We note that the vector $(|\mathbf{U}_{1,1}|^2, \ldots, |\mathbf{U}_{1,N}|^2)$ follows the Dirichlet distribution and a partial sum of its entries has a Jacobi (also referred to as MANOVA) distribution. To see this, we note that (18) can be written as

$$C(\{1, 2, \ldots, k\}) = \frac{N}{k} \log\left(1 + (2^C - 1)\mathbf{U}(k)_1 \mathbf{U}(k)_1^H\right) \qquad (20)$$

where $\mathbf{U}(k)_1$ is a vector which contains the first k elements of the first row of $\mathbf{U}$. Noting that since $\mathbf{U}(k)_1$ is a submatrix of a unitary matrix, its singular values follow (see, e.g., [5]) the Jacobi distribution, and more specifically, X has Jacobi distribution with rank 1. We thus obtain

$$\Pr\left(C(\{S\}) < R|C\right) = \int_0^{\frac{2^{Rk/N}-1}{2^C-1}} x^{k-1} x^{N-k-1} d\lambda$$
$$= \frac{\mathcal{B}(\frac{2^{Rk/N}-1}{2^C-1}; k, N-k)}{\mathcal{B}(1; k, N-k)},$$

where $\mathcal{B}(x; a, b)$ is the incomplete beta function defined in (17).

$\square$

Theorem 2. *For an N-user i.i.d. Rayleigh MAC, we have*

$$\max_k P_{\text{out}}(k, R|C) \leq \Pr\left(C_{\text{sym}} < R|C\right) \qquad (21)$$
$$\leq \sum_{k=1}^{N} \binom{N}{k} P_{\text{out}}(k, R|C),$$

where $P_{\text{out}}(k, R|C)$ is defined Lemma 1.

Proof. To establish the left hand side of the theorem, first note that $C_{\text{sym}} \leq C(\{S\})$ for any subset S and hence

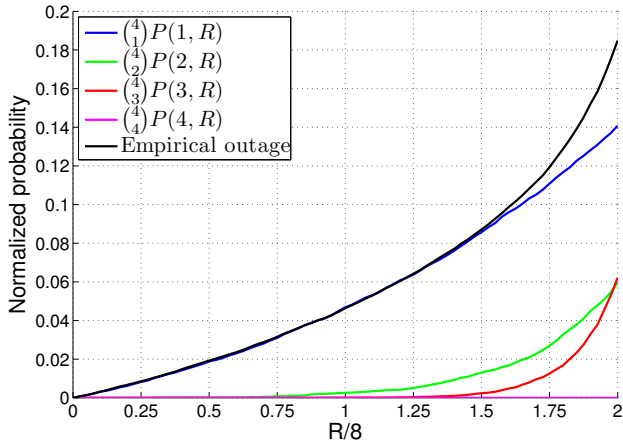$$C_{\text{sym}} \leq \min_k C(\{1, 2, \ldots, k\}). \qquad (22)$$

Fig. 4. Demonstration of the quantities appearing in the bounds appearing in Theorem 2 for the case of a 4-user i.i.d. Rayleigh fading MAC with sum capacity $C = 8$.
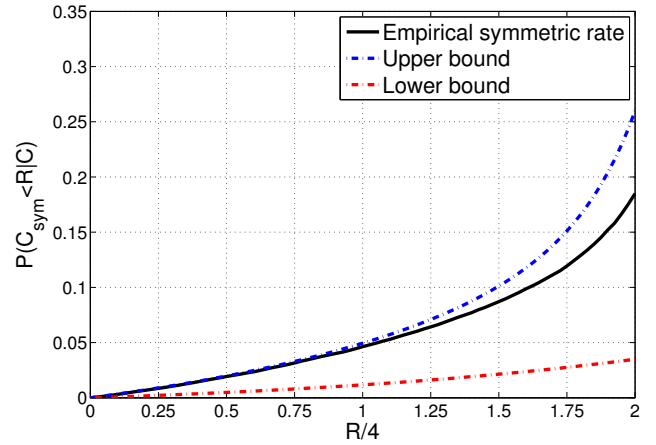

Fig. 5. Comparison of empirical evaluation of (4) and Theorem 5 (upper and lower bounds for the outage probability) for a 4-user i.i.d. Rayleigh fading MAC with sum capacity $C = 8$.

It follows that

$$\Pr\left(C_{\text{sym}} < R \Big| C\right) \geq \Pr\left(\min_k C(\{1, 2, \ldots, k\}) < R \Big| C\right)$$

$$= \Pr\left(\bigcup_k \{C(\{1, 2, \ldots, k\}) < R\} \Big| C\right)$$

$$\geq \max_k \Pr\left(C(\{1, 2, \ldots, k\}) < R \Big| C\right)$$

$$= \max_k P_{\text{out}}(k, R | C). \tag{23}$$

The right hand side follows by the union bound. □

Figures 4 and 5 illustrate the theorem for the case of four users. As can be seen from Figure 4, already at not very high values of capacity, the single-user constraints already constitute the bottleneck. We further observe from Figure 5 that the union bound is quite tight.

V. PRACTICAL REALIZATION OF THE COMMUNICATION PROTOCOL VIA PRECODED INTEGER FORCING

In this section we empirically demonstrate the effectiveness of the integer-forcing (IF) receiver when used in conjunction with space-time precoding as a practical transmission scheme for the considered communication protocol. Due to space limitations, we refer the reader to [6] for a description of the integer forcing framework and its implementation.

When it comes to fading channels, it has been shown in [6] that the IF receiver achieves the DMT over i.i.d. Rayleigh fading channels where the number of receive antennas is greater or equal to the number of transmit antennas.

We observe that this does not hold in the general case; in particular, IF does not achieve the DMT for the case of a MAC where all terminals are equipped with a single antenna. Specifically, Figure 6 depicts (in logarithmic scale) the empirical outage probability of the IF receiver and the exact outage probability for optimal communication (Gaussian

codebooks and ML decoding), as given by Theorem 1, for the two-user i.i.d. Rayleigh fading MAC. The symmetric rate achieved by a given scheme is denoted by R_{scheme}.

It is evident that the slopes are different. This raises the question of whether IF is inherently ill-suited for the MAC channel. A negative answer to this question may be inferred by recalling some further lessons from the DMT analysis of the MAC.

While the optimal DMT for the i.i.d. Rayleigh fading MAC was derived in [2] using Gaussian codebooks of sufficient length, it was subsequently shown that the MAC DMT can be achieved using structured codebooks by combining uncoded QAM constellations with space-time unitary precoding (and ML decoding). Specifically, such a MAC-DMT achieving construction is given in [7]. This raises the possibility that the sub-optimality of the IF receiver observed in Figure 6 may at least in part be remedied by applying unitary space-time precoding at each of the transmitters. We note that each transmitter applies precoding only to its own data streams so the distributed nature of the problem is not violated.

Following this approach, we have implemented the IF receiver with unitary space-time precoding applied at each transmitter. We have employed random (Haar) precoding (with independent matrices drawn for the different users) over two ($T = 2$) time instances as well as deterministic precoding using the matrices proposed in [8].[2]

These matrices can be expressed as

$$\mathbf{P}_{st,c}^1 = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \alpha\phi \\ \bar{\alpha} & \bar{\alpha}\bar{\phi} \end{bmatrix}, \qquad \mathbf{P}_{st,c}^2 = \frac{1}{\sqrt{5}} \begin{bmatrix} j\alpha & j\alpha\phi \\ \bar{\alpha} & \bar{\alpha}\bar{\phi} \end{bmatrix} \tag{24}$$

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[2]When using an ML receiver, this space-time code is known to achieve the DMT for multiplexing rates $r \leq \frac{1}{5}$. As detailed in [7], whether this code achieves the optimal MAC-DMT also when $r > \frac{1}{5}$ remains an open question.
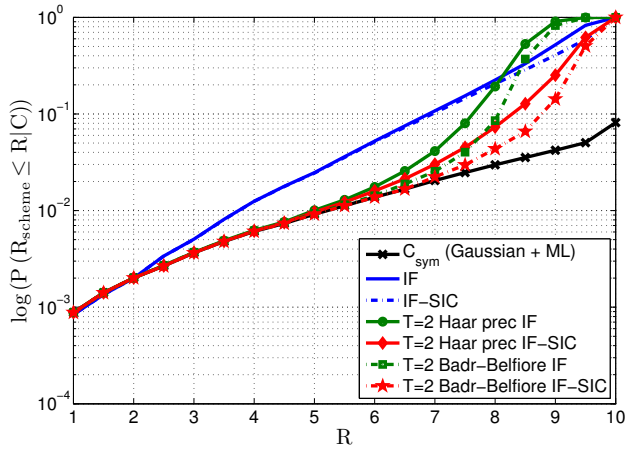
Fig. 6. Outage probability of linear codes (with and without space-time precoding) with IF equalization versus Gaussian codebooks with ML decoding for a two-user i.i.d. Rayleigh fading MAC with sum capacity $C = 10$.

Fig. 8. Average rate conditioned on the sum capacity when using linear codes (with and without space-time precoding) with IF equalization versus Gaussian codebooks with ML decoding, over a two-user normalized (conditioned) i.i.d. Rayleigh fading MAC.

used.

In Figure 8 we study the average symmetric rate achieved by different schemes w.r.t. a two-user i.i.d. Rayleigh fading channel when we condition on the sum capacity of the channel. We plot the fraction of the sum capacity attained by the various schemes. We first observe that IF-SIC combined with space-time precoded linear codes achieves a large fraction of the symmetric capacity. Further, as can be seen, the fraction of the sum capacity achieved by all the different schemes considered approaches one as the sum capacity grows.
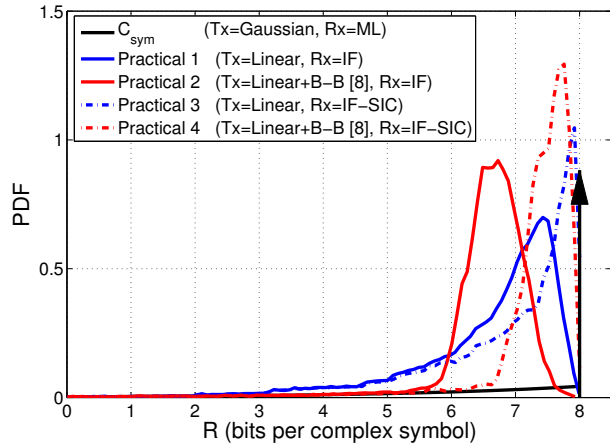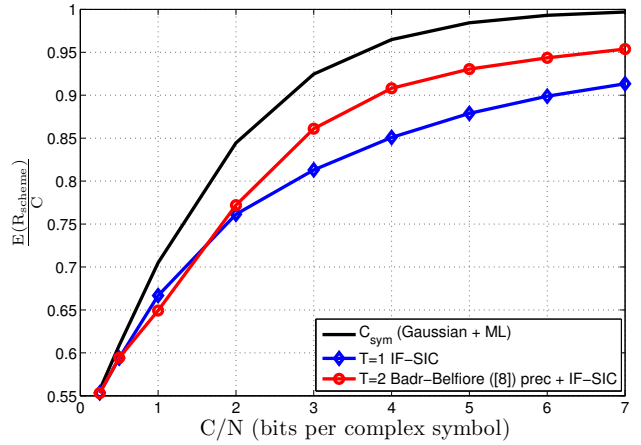

Fig. 7. Probability distribution function of the rate achieved with linear codes (with and without space-time precoding) in conjunction with IF equalization versus that achieved Gaussian codebooks with ML decoding for a two-user i.i.d. Rayleigh fading MAC with sum capacity $C = 10$.

where

$$\phi = \frac{1 + \sqrt{5}}{2}, \qquad \bar{\phi} = \frac{1 - \sqrt{5}}{2}$$
$$\alpha = 1 + j - j\phi, \qquad \bar{\alpha} = 1 + j - j\bar{\phi}. \qquad (25)$$

We also replot Figure 6 in terms of PDF (rather than CDF) as Figure 7, but without random Haar space-time precoding (so as to avoid "clutter"). As can be seen, the precoding matrices in (24) improve the outage probability for most target rates.

We further note that in addition to standard IF, we also implemented a variant that incorporates successive interference cancellation, referred to as IF-SIC [9]. As can be seen, IF-SIC results in a significant improvement for all precoding schemes

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