Streaming Erasure Codes over Multi-hop Relay Network

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***Abstract*—A typical path over the internet is composed of multiple hops. When considering the transmission of a sequence of messages (streaming messages) through packet erasure channel over a three-node network, it has been shown that taking into account the erasure pattern of each segment can result in improved performance compared to treating the channel as a point-to-point link. Since rarely there is only a single relay between the sender and the destination, it calls for trying to extend this scheme to more than a single relay. In this paper, we first extend the upper bound on the rate of transmission of a sequence of messages for any number of relays. We further suggest an achievable adaptive scheme that is shown to achieve the upper bound up to the size of an additional header that is required to allow each receiver to meet the delay constraints.**

I. Introduction

Real-time interactive video streaming is one of the fastest-growing types of internet traffic. Traditionally, most of the traffic on the internet is not extremely sensitive to latency. However, as networks evolved, more and more people are using the network for real-time conversations, video conferencing, and on-line monitoring. According to [1], IP video traffic will account for 82 percent of traffic by 2022. Further, live video will grow 15-fold to reach 17 percent of Internet video traffic by 2022.

The fundamental difference between real-time interactive video streaming and other services is the (much more stringent) latency requirement each packet has to meet in order to provide a good user experience. In multiple streaming codes works [2]–[6], using automatic repeat request (ARQ) for handling errors in transmission was discussed, and it was demonstrated that it might not be adequate for streaming applications. Using ARQ means that the latency (in case of an error) is at least three times the one-way delay, which in many cases may violate the latency requirements.

An alternative method for handling error in transmission is forward error correction (FEC). Using FEC codes has the potential to lower the recovery latency since it does not require communication between the receiver and transmitter. In [7], streaming codes for three-node networks were analyzed. For these networks, an explicit expression for capacity was derived. Analyzing the resulting capacity for the three-node network shows that when constraints are taken into consideration per segment rather than globally (while meeting the same global requirements), the capacity increases. As we demonstrate next, treating the network as a single-hop link with $N = N_1 + N_2$ erasures and total delay constraint of T symbols is worse than analyzing a three-node network where N_1 erasures are expected in the first segment, and N_2 are expected in the second segment with a total delay of T symbols.

However, Internet paths almost never consist of only a single relay (see, e.g., [8], [9]). Hence, analyzing streaming codes while taking into account the error behavior of each link rather than aggregate some of the links is expected to result in improved performance.

The achievable scheme suggested in [7] does not depend on the location of erasures (i.e. "state-independent"). Unfortunately, there is no straight-forward extension of this scheme to a more general case (a network with more than three nodes). The scheme suggested in this paper depends on the specific erasure pattern (i.e., "state-dependent" scheme) that occurred in the current and previous hops. While requiring an additional header to allow the receiver to recover the actual code that was used (over each diagonal), we show that it can be easily extended to any number of relays.

In this paper, we first extend the upper bound derived in [7] to the general case of a multi-hop relay network. We then describe the state-dependent scheme for the general L relay scenario and show it achieves the upper bound while requiring an additional overhead in the packet header. We further show that the size of this header does not depend on the field size used by the code; therefore, the gap from the upper bound decreases as the field size increases.

To simplify notation, we sometimes denote $N_a^b = \sum_{l=a}^{b} N_l$. We will take all logarithms to base 2 throughout this paper.

II. Network Model

A source node wants to send a sequence of messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ to a destination node with the help of L middle nodes $r_1, \ldots, r_L$. To ease notation we denote the source node as r_0, and destination node as r_{L+1}. Let k be a non-negative integer, and $n_1, n_2, \ldots, n_{L+1}$ be $L+1$ natural numbers.

Each $\mathbf{s}_i$ is an element in $\mathbb{F}^k$ where $\mathbb{F}$ is some finite field. In each time slot $i \in \mathbb{Z}_+$, the source message $\mathbf{s}_i$ is encoded into a length-n_1 packet $\mathbf{x}_i^{(r_0)} \in \mathbb{F}^{n_1}$ to be transmitted to the first

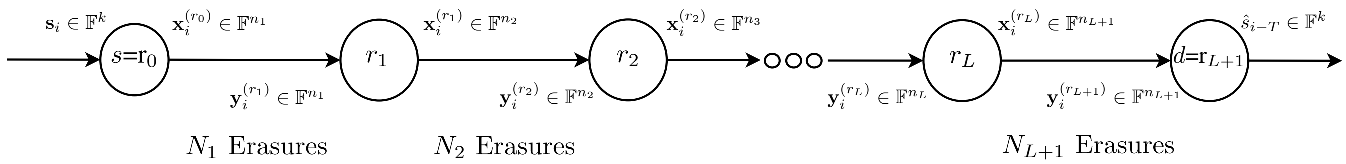

Fig. 1: Symbols generated in the L-node relay network at time i.

relay through the erasure channel (r_0, r_1). The relay receives $\mathbf{y}_i^{(r_1)} \in \mathbb{F}^{n_1} \cup \{*\}$ where $\mathbf{y}_i^{(r_1)}$ equals either $\mathbf{x}_i^{(r_0)}$ or the erasure symbol "$*$". In the same time slot, relay r_1 transmits $\mathbf{x}_i^{(r_1)} \in \mathbb{F}^{n_2}$ to relay r_2 through the erasure channel (r_1, r_2). Relay r_2 receives $\mathbf{y}_i^{(r_2)} \in \mathbb{F}^{n_2} \cup \{*\}$ where $\mathbf{y}_i^{(r_2)}$ equals either $\mathbf{x}_i^{(r_1)}$ or the erasure symbol "$*$". The same process continues (in the same time slot) until relay r_L transmits $\mathbf{x}_i^{(r_L)} \in \mathbb{F}^{n_{L+1}}$ to the destination r_{L+1} through the erasure channel (r_L, r_{L+1}).

We assume that on the discrete timeline, each channel (r_{j-1}, r_j) introduces up to N_j arbitrary erasures respectively. The symbols generated in the L-node relay network at time i are illustrated in Figure 1.

III. Standard Definitions and Known Results

Definition 1. *An $(n_1, n_2, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code consists of the following:*

1) *A sequence of source messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ where $\mathbf{s}_i \in \mathbb{F}^k$.*
2) *An encoding function $f_i^{(r_0)} : \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{i+1 \text{ times}} \to \mathbb{F}^{n_1}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_0)}$ is used by node r_0 at time i to encode $\mathbf{s}_i$ according to $\mathbf{x}_i^{(r_0)} = f_i^{(r_0)}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_i)$.*
3) *A relaying function for node $j \in [1, \ldots, L]$, $f_i^{(r_j)} : \underbrace{\mathbb{F}^{n_j} \cup \{*\} \times \ldots \times \mathbb{F}^{n_j} \cup \{*\}}_{i+1 \text{ times}} \to \mathbb{F}^{n_{j+1}}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_j)}$ is used by node r_j at time i to construct $\mathbf{x}_i^{(r_j)} = f_i^{(r_j)}(\mathbf{y}_0^{(r_j)}, \mathbf{y}_1^{(r_j)}, \ldots, \mathbf{y}_i^{(r_j)})$.*
4) *For each $i \in \mathbb{Z}_+$, a decoding function ϕ_{i+T} : $\underbrace{\mathbb{F}^{n_L} \cup \{*\} \times \ldots \times \mathbb{F}^{n_L} \cup \{*\}}_{i+T+1 \text{ times}} \to \mathbb{F}^{n_{L+1}}$ is used by node n_{L+1} at time $i+T$ to estimate $\mathbf{s_i}$ according to $\hat{\mathbf{s}}_i = \phi_{i+T}(\mathbf{y}_0^{(r_{L+1})}, \mathbf{y}_1^{(r_{L+1})}, \ldots, \mathbf{y}_{i+T}^{(r_{L+1})})$.*

Definition 2. *An erasure sequence is a binary sequence denoted by $e^{\infty} \triangleq \{e_i\}_{i=0}^{\infty}$ where* $e_i = \mathbf{1}\{\text{erasure occurs at time i}\}$.

An N-erasure sequence is an erasure sequence e^{∞} that satisfies $\sum_{l=0}^{\infty} e_l = N$. In other words, an N-erasure sequence introduces N arbitrary erasures on the discrete timeline. The set of N-erasure sequences is denoted by Ω_N.

Definition 3. *The mapping $g_{n_j} : \mathbb{F}^{n_j} \times \{0,1\} \to \mathbb{F}^{n_j} \cup \{*\}$ of an erasure channel is defined as*

$$g_{n_j}(\mathbf{x}^{(r_j)}, e_i) = \begin{cases} \mathbf{x}^{(r_j)} & \text{if } e_i = 0, \\ * & \text{if } e_i = 1. \end{cases} \tag{1}$$

For any erasure sequence e^{∞} and any $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code, the following input-output relations holds for the erasure channel (r_j, r_{j+1}) for each $i \in \mathbb{Z}_+$ $\mathbf{y}_i^{(r_{j+1})} = g_{n_j}(\mathbf{x}_i^{(r_j)}, e_i)$.

Definition 4. *An $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code is said to be $(N_1, N_2, \ldots, N_{L+1})$-achievable if the following holds for any N_j-erasure sequence $e^{\infty} \in \Omega_{N_j}$, for all $i \in \mathbb{Z}_+$ and all $\mathbf{s}_i \in \mathbb{F}^k$, we have $\hat{\mathbf{s}}_i = \mathbf{s}_i$ where*

$$\hat{\mathbf{s}}_i = \phi_{i+T}(g_{n_{L+1}}(\mathbf{x}_0^{(r_L)}, e_0), \ldots, g_{n_{L+1}}(\mathbf{x}_{i+T}^{(r_L)}, e_{i+T})) \tag{2}$$

and for previous nodes

$$\mathbf{x}_i^{(r_{j+1})} = f_i(g_{n_j}(\mathbf{x}_0^{(r_j)}, e_0), \ldots, g_{n_j}(\mathbf{x}_i^{(r_j)}, e_i)). \tag{3}$$

Definition 5. *The rate of an $(n_1, n_2, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code is $\frac{k}{\max\{n_1, n_2, \ldots, n_{L+1}\}}$.*

It was previously shown [3] that the maximum achievable rate of the point-to-point packet erasure channel with N arbitrary erasures and delay of $T \geq N$ denoted by $C_{T,N}$ satisfies

$$C_{T,N} = \frac{T - N + 1}{T + 1}. \tag{4}$$

In [7], a three node relay network was analyzed, and it was shown than for any (T, N_1, N_2), when $T \geq N_1 + N_2$ we have

$$C_{T,N_1,N_2} = \min\left(C_{T-N_2,N_1}, C_{T-N_1,N_2}\right)) \tag{5}$$

We recall that for any natural numbers L and M, a systematic maximum-distance separable (MDS) $(L+M, L)$-code is characterized by an $L \times M$ parity matrix $\mathbf{V}^{L \times M}$ where any L columns of $\left[\mathbf{I}_L \ \mathbf{V}^{L \times M}\right] \in \mathbb{F}^{L \times (L+M)}$ are independent. It is well known that a systematic MDS $(L+M, L)$-code always exists as long as $|\mathbb{F}| \geq L + M$ [10]. We further define a point-to-point $(n, k, T)_{\mathbb{F}}$-block code as

Definition 6. *A point-to-point $(n, k, T)_{\mathbb{F}}$-block code consists of the following*

1) *A sequence of k symbols $\{u[l]\}_{l=0}^{k-1}$ where $u[l] \in \mathbb{F}$.*
2) *A generator matrix $\mathbf{G} \in \mathbb{F}^{k \times n}$. The source codeword is generated according to*

$$[x[0] \ x[1] \ \ldots \ x[n-1]] = [u[0] \ u[1] \ \ldots \ u[k-1]] \, \mathbf{G}$$

3) *A decoding function $\varphi_{l+T} : \mathbb{F} \cup \{*\} \times \ldots \mathbb{F} \cup \{*\} \to \mathbb{F}$ for each $l \in \{0, 1, \ldots, k-1\}$, where φ_{l+T} is used by the destination at time $\min\{l+T, n-1\}$ to estimate $u[l]$ according to*

$$\hat{u}[l] = \begin{cases} \varphi_{l+T}(y[0], y[1], \ldots, y[l+T]) & \text{if } l + T \leq n - 1 \\ \varphi_{l+T}(y[0], y[1], \ldots, y[n-1]) & \text{if } l + T > n - 1 \end{cases}$$

IV. Main Results

In this paper we first derive an upper bound for the achievable rate in $L+1$-node relay network.

Theorem 1. *Assume a path with L relays. For a target overall delay of T, where the maximal number of arbitrary erasures in link (r_{j-1}, r_j), $j \in [1, \dots, L+1]$ is N_j and $T \geq \sum_{l=1}^{L+1} N_l$. The achievable rate is upper bounded by*

$$R \leq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \sum_{l=1, l\neq j}^{L+1} N_l + 1} \triangleq C^{+}_{T, N_1, \dots, N_{L+1}}. \quad (6)$$

Denoting

$$n_{\max} \triangleq \max_{j \in \{1, \dots, L+1\}} \left(T - \sum_{l=1, l\neq j}^{L+1} N_l + 1 \right), \quad (7)$$

we then show the following Theorem on the achievable rate.

Theorem 2. *Assume a link with L relays. For a target overall delay of T, where the maximal number of arbitrary erasures in link (r_j, r_{j+1}), $j \in \{0, \dots, L\}$ is N_{j+1} and $T \geq \sum_{l=1}^{L+1} N_l$. When $|\mathbb{F}| \geq n_{\max}$, The following rate is achievable.*

$$R \geq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \left\{ \sum_{l=1, l\neq j}^{L+1} N_l \right\} + 1 + \frac{n_{\max} \lceil \log(n_{\max}) \rceil}{\log(|\mathbb{F}|)}} \quad (8)$$

where $n_{\max}$ is defined in (7).

V. Motivating Example

Consider a link with up to $N = 2$ arbitrary erasures, where the delay constraint the transmission has to meet is $T = 3$ symbols. The capacity of this link according to (4) is $C_{3,2} = 2/4$. Now, assume that in fact, this link is a three-node network ($L = 1$), where up to $N_1 = 1$ erasures occur in link (r_0, r_1) and up to $N_2 = 1$ erasures occur in link (r_1, r_2), where transmission has to be decoded with the same overall delay of $T = 3$ symbols. The capacity of this link according to (5) is $C_{3,1,1} = 2/3$, which is better than the point-to-point link.

Suppose node s transmits two bits a_i and b_i at each discrete time $i \geq 0$ to node d with delay 3. We start by describing the suggested scheme for the three-node network and show it achieves capacity up to the size of the header. We then show how it can be extended for the four-node network.

In the proposed scheme, the source r_0 uses the same code suggested in [7], i.e., a $(3, 2)$ MDS code that can recover any arbitrary single erasure in a delay of two symbols combined with diagonal interleaving (i.e. the code is applied on the diagonals). Denoting the two information bits transmitted at time i as $[a_i, b_i]$. Applying the code on the diagonals means, for example, that one diagonal is $[a_i, b_{i+1}, a_i + b_{i+1}]$ (where $a_i + b_{i+1}$ is the parity symbol). The transmission from the source is depicted in Table I (different diagonals are marked with different colors).

When there are no erasures, relay r_1 uses the same codes as the source r_0 *while delaying it by one symbol*. For example,

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	123	123	123	123
a_i	a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
b_i	b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
$a_{i-2}+$ b_{i-1}	$a_{i-3}+$ b_{i-2}	$a_{i-2}+$ b_{i-1}	$a_{i-1}+$ b_i	a_i+ b_{i+1}	$a_{i+1}+$ b_{i+2}	$a_{i+2}+$ b_{i+3}

TABLE I: Transmission of the Source in (r_0, r_1).

in Table II below $[a_{i-1}, b_i, a_{i-1} + b_i]$ is transmitted on the diagonal starting time i.

If an erasure occurred, the relay sends first any available symbols (per diagonal) in the order they were received until it can decode the information symbols. Then, the erased symbols (or additional linear combinations) are sent. For example, assuming that a symbol at time i is erased when transmitted from the sender to relay r_1, the suggested transmission scheme of relay r_1 is given in Table II below.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	$\boxed{223}$	$\boxed{113}$	123	123
	a_{i-2}	a_{i-1}	$\mathbf{b_{i+1}}$	a_{i+1}	a_{i+2}	a_{i+3}
	b_{i-2}	b_{i-1}	b_i	$\mathbf{a_i}$	b_{i+2}	b_{i+3}
	$a_{i-4}+$ b_{i-3}	$a_{i-3}+$ b_{i-2}	$a_{i-2}+$ b_{i-1}	$a_{i-1}+$ b_i	a_i+ b_{i+1}	$a_{i+1}+$ b_{i+2}

TABLE II: Transmission of relay r_1 in (r_1, r_2), given that symbol i was erased when transmitted in link (r_0, r_1).

We note that the erasure in time i in link (r_0, r_1) caused a change in the order of the symbols in packets $i+1$ and $i+2$. Denoting the order of symbols in the code $[a_i, b_{i+1}, a_i + b_{i+1}]$ as $[1, 2, 3]$, the header of the packet in time i is composed of the order of the symbols from each block code (the order of the symbols in each diagonal) at time i. At time $i+1$, for example, the relay can send symbol b_{i+1} from the code applied on $[a_i, b_{i+1}]$ (which is symbol with label "2" in this code) rather than symbol a_i that has been erased, symbol b_i from the code applied on $[a_{i-1}, b_i]$ (label "2") and symbol $a_{i-2} + b_{i-1}$ (label "3") from the code applied on $[a_{i-2}, b_{i-1}]$ hence the header used is "223".

At time $i+2$, the relay can recover the information symbols from code $[a_i, b_{i+1}, a_i + b_{i+1}]$ and send a_i (as the second symbol in x_{i+1}). Hence, the header used at time $i+2$ is "113". It can be easily verified that the destination can recover the original data at a delay of $T = 3$ (assuming any arbitrary one erasure in the link between the relay and destination).

This concept can be applied to additional relays if they exist. For example consider four-node network ($L = 2$). The transmission scheme on the next relay r_2 (in this specific example), is the same as the transmission scheme of the first relay, i.e., in case there is no erasure, transmit (on each diagonal) the symbols in the same order as received, delayed by one symbol (i.e., a total delay of two symbols from the sender). If an erasure occurred before r_2 has decoded the information symbols, transmit the available symbols (again, it is guaranteed that there will be enough symbols). When the information symbols can be decoded, transmit the erased symbols.

For example, in case the symbol transmitted from relay r_1 to r_2 at time $i+2$ is erased, the suggested transmission scheme of relay r_2 in given in Table III below.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	123	223	$\boxed{213}$	$\boxed{113}$
	a_{i-3}	a_{i-2}	a_{i-1}	b_{i+1}	b_{i+2}	a_{i+2}
	b_{i-3}	b_{i-2}	b_{i-1}	b_i	a_i	a_{i+1}
	$a_{i-5}+$ b_{i-4}	$a_{i-4}+$ b_{i-3}	$a_{i-3}+$ b_{i-2}	$a_{i-2}+$ b_{i-1}	a_{i-1} b_i	a_i+ b_{i+1}

TABLE III: Transmission of relay r_2 in (r_2, r_3), given that symbol $i+2$ was erased when transmitted in link (r_1, r_2).

Since, basically, the same $(3, 2)$ code is used (with a different order of symbols which is communicated to the receiver), it can be easily verified that each packet can be recovered up to delay of $T = 4$ symbols for any arbitrary erasure in the link between the relay and the destination.

In this example, the header is composed of three numbers, each taken from $\{1, 2, 3\}$, hence, its size is $3\lceil\log(3)\rceil$ bits. Since the block code used in each link transmits two information bits using three bits in every channel use, we conclude that the scheme achieves a rate of $R = \frac{2}{3+3\lceil\log(3)\rceil}$. In case the sender sends symbols taken from a field $\mathbb{F}$ it can shown that this scheme can achieve a rate of

$$R = \frac{2 \cdot \log(|\mathbb{F}|)}{3 \cdot \log(|\mathbb{F}|) + 3\lceil\log(\lceil 3)\rceil}, \tag{9}$$

which approaches $2/3$ as the field size increases. As we show next, the upper bound for this scenario is indeed $2/3$.

VI. Sketch of the Proof of the Upper Bound

Due to space limitations we only provide the main idea of the proof. The proof follows the footsteps of [7] and appears in [11]. Generalizing the arguments given in [7] for the first and second segments to the first and last segments in our case are presented in [11] when erasures are "concatenated", i.e., if N_1 erasures occur in link (r_0, r_1), at time $[i, \dots, i + N_1 - 1]$, we assume N_2 erasures occur in link (r_1, r_2) at time $[i + N_1, \dots, i + N_1 + N_2 - 1]$ and so on.

A similar technique is used to derive a constraint on the code to be used in link (r_{j-1}, r_j). The analyzed erasure pattern appears in Figure 2. In [11] it is shown for every $i \in \mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered from $\left\{\mathbf{x}^{(r_{j-1})}_{i+\sum_{l=1}^{j-1} N_l}, \mathbf{x}^{(r_{j-1})}_{i+\sum_{l=1}^{j-1} N_l+1}, \dots, \mathbf{x}^{(r_{j-1})}_{i+T-\sum_{l=j+1}^{L+1} N_l}\right\}$ by node j given that $\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{i-1}$ have been correctly decoded by node j.

Now denote with $\tilde{N}_{j,1} = \sum_{l=1}^{j-1} N_l$, and with $\tilde{N}_{j,2} = \sum_{l=j+1}^{L+1} N_l$ it follows that

$$H\Big(\mathbf{s}_i \;\Big|\; \left\{\mathbf{x}^{(r_{j-1})}_{i+\tilde{N}_{j,1}}, \mathbf{x}^{(r_{j-1})}_{i+\tilde{N}_{j,1}+1}, \dots, \mathbf{x}^{(r_{j-1})}_{i+T-\tilde{N}_{j,2}}\right\} \setminus \left\{\mathbf{x}_{\theta_1}, \dots, \mathbf{x}_{\theta_{N_j}}\right\}, \mathbf{s}_0, \dots, \mathbf{s}_{i-1}\Big) = 0, \tag{10}$$

for any $i \in \mathbb{Z}_+$ and any N_j non-negative integers denoted by $\theta_1, \dots, \theta_{N_j}$.

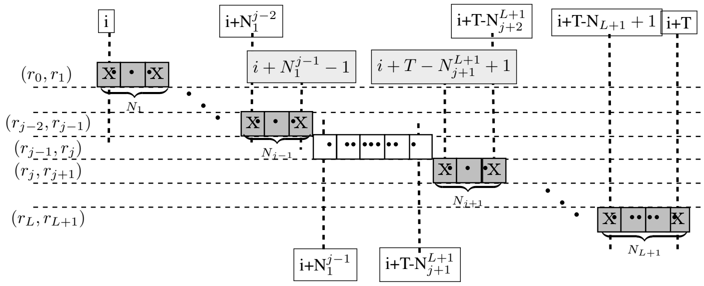

Fig. 2: Constrains imposed on transmission in link (r_{j-1}, r_j).

Denoting $T_{\tilde{N}_1,\tilde{N}_2} = T - \tilde{N}_{j,1} - \tilde{N}_{j,2}$ and using (10) and the chain rule we note that for all $q \in \mathbb{Z}_+$ (the derivation is detailed in [11])

$$H\Big(\mathbf{s}_0, \dots, \mathbf{s}_{T_{\tilde{N}_1,\tilde{N}_2}+(j-1)(T_{\tilde{N}_1,\tilde{N}_2}+1)} \;\Big|\; \Big\{\mathbf{x}^{(r_{j-1})}_{\tilde{N}_{j,1}+N_j+m(T_{\tilde{N}_1,\tilde{N}_2}+1)}, \mathbf{x}^{(r_{j-1})}_{\tilde{N}_{j,1}+N_j+1+m(T_{\tilde{N}_1,\tilde{N}_2}+1)}, \dots, \mathbf{x}^{(r_{j-1})}_{T-\tilde{N}_{j,2}+m(T_{\tilde{N}_1,\tilde{N}_2}+1)}{}_{m=0}^{j}\Big) = 0.$$

Again, following the arguments in [7] when restricted to the channel (r_{j-1}, r_j), any $(N_1, \dots, N_{L+1})$ achievable $(n_1, \dots n_{L+1}, k, T)_{\mathbb{F}}$ code can be viewed as a point-to-point streaming code with rate k/n_j, delay $T - \sum_{l=1, l\neq j}^{L} N_l$ which can correct any N_j erasures. We therefore have

$$\frac{k}{n_j} \le \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=1, l\neq j}^{L+1} N_l + 1} = C_{T-\sum_{l=1, l\neq j}^{L+1} N_l, N_j}. \tag{11}$$

Since (11) hold for any $j \in [1, \dots, L+1]$ we have

$$R \le \frac{k}{\max_j(n_j)} = \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \sum_{l=1, l\neq j}^{L+1} N_l + 1}. \tag{12}$$

VII. Suggested Coding Scheme

As mentioned above, the suggested coding scheme is a adaptive symbol-wise decode and forward scheme. Each relay is using a $(n_{j,j+1}, k) \triangleq (T - \sum_{l=1, l\neq j+1}^{L+1} N_l + 1, T - \sum_{l=1}^{L+1} N_l + 1)$ MDS block code which is diagonally interleaved. More formally, let $s_i[l]$ be the l'th symbol of the source message $\mathbf{s}_i$ and let $x_i^{(r_j)}[l]$ be the l'th symbol of the output of encoding function $f_i^{(r_j)}$. We may say for each $i \in \mathbb{Z}_+$, a single transmission function of relay r_j constructs

$$\left[x^{(r_j)}_{i+N_1^j}[0] \;\; x^{(r_j)}_{i+N_1^j+1}[1] \;\; \cdots \;\; x^{(r_j)}_{i+N_1^j+n_{j+1}-1}[n_{j+1}-1]\right] = \underbrace{\left[s_i[0] \;\; s_{i+1}[1] \;\; \cdots \;\; s_{i+k-1}[k-1]\right]}_{\triangleq \tilde{\mathbf{s}}_i} \times \mathbf{G}_i^{(r_j)}, \tag{13}$$

where $\mathbf{G}_i^{(r_j)}$ is the generator matrix of this code.

While this is similar to the encoder originally suggested in [2] and used in many other streaming settings, it is important to note that while the codes applied on each diagonal in the suggested scheme share the same rate, each $\mathbf{G}_i^{(r_j)}$ can be different (based on current and previous erasure pattern).

We describe next the process of generating $\mathbf{G}_i^{(r_j)}$ (required to transmit $\tilde{\mathbf{s}}_i$ in each one of the relays).

- At the sender (r_0), use an (n_1, k) MDS code. Hence, $\mathbf{G}_i^{(r_0)}$ is the generator matrix of an (n_1, k) MDS code.
- Each encoder at relay r_j ($j \in \{1, \dots, L\}$) performs the following
  1) Store any non-erased symbols from the first N_j received symbols from link (r_{j-1}, r_j), i.e., all non-erased symbols from $\left\{x_{i+N_1{}^{j-1}}^{(r_{j-1})}[0], \dots, x_{i+N_1{}^{j-1}+N_j-1}^{(r_{j-1})}[N_j - 1]\right\}$.
  2) Start transmitting at time $i + N_1^j$ (while continuing to store the received symbols from link (r_{j-1}, r_j)). Until time $i + N_1^j + k - 2$, forward the $k-1$ symbols received from link (r_{j-1}, r_j) by the order they were received, i.e., forward the $k-1$ non-erased symbols from $\left\{x_{i+N_1{}^{j-1}}^{(r_{j-1})}[0], \dots, x_{i+N_1{}^{j-1}+N_j+k-2}^{(r_{j-1})}[N_j + k - 2]\right\}$. Noting the $N_j + k - 2 = n_j - 1$ means that relay forwards the $k-1$ symbols received by the order they were received, i.e., all non-erased symbols from $\left\{x_{i+N_1{}^{j-1}}^{(r_{j-1})}[0], \dots, x_{i+N_1{}^{j-1}+N_j-1}^{(r_{j-1})}[n_j - 1]\right\}$.
  3) At time $N_1^j + k - 1$, recover $\tilde{\mathbf{s}}_\mathbf{i}$. In Lemma 1 below we prove that it is feasible for any $N_1, \dots, N_{L+1}$-erasure sequence.
  4) Transmit until time $N_1^j + n_{j+1} - 1$ encoded symbols. The encoded symbols should be non-received symbols from (n_{max}, k) MDS code defined below.
  5) For each transmitted symbol, attach a header which will be defined next.

Recalling the definition of $n_{\max}$ (7), the following Proposition sheds light on the suggested method of encoding $\tilde{\mathbf{s}}_\mathbf{i}$ once it is recovered at r_j (step (4)) in order to guarantee that each $\mathbf{G}_i^{(r_j)}$ is a generator matrix of (n_{j+1}, k) MDS code and further simplify the header required to allow decoding.

Proposition 1. *All block codes used by nodes r_j where $j \in \{0, \dots, L\}$ to transmit $\tilde{\mathbf{S}}_i$ can be generated by puncturing and applying a permutation to the $(n_{\max}, k)$ MDS code which is associated with rate $C^+_{T,N_1,\dots,N_{L+1}}$.*

See [11] for further details. In fact, denoting with $\mathbf{G}_{\max}$ the generator matrix of $(n_{\max}, k)$ MDS code, $\mathbf{G}_i^{(r_j)}$ can be viewed as taking n_{j+1} columns from $\mathbf{G}_{\max}$ and apply permutation on the order of the columns. The specific columns taken from $\mathbf{G}_{\max}$ and their order is defined by the specific erasure pattern which occurs.

Further, following Proposition 1 we define the header as a number which indicates the location of the column from $\mathbf{G}_{\max}$ that was used to generate this symbol. Thus, the header attached to each symbol transmitted from each relay is a number in the range $[1, \dots, n_{\max}]$. Next, we have the following Lemmas where due to space limitations we omit their proofs (the proofs can be found in [11]).

Lemma 1. *For any $N_1, \dots, N_{L+1}$-erasure sequence, any $j \in \{0, \dots, L\}$ and any $i \in \mathbb{Z}_+$, $\mathbf{G}_i^{(r_j)}$ is a generator matrix of (n_{j+1}, k) MDS code.*

Lemma 2 (Based on Lemma 3 in [7])**.** *Suppose $T \geq N$, and let $k \triangleq T - N + 1$ and $n \triangleq k + N$. For any $\mathbb{F}$ such that $\mathbb{F} \geq n$, there exists an N-achievable point-to-point $(n, k, T)_\mathbb{F}$-block code.*

Lemma 3. *The streaming code resulting from using $\mathbf{G}_i^{(r_j)}$ defined above in each relay $j \in \{0, \dots, L\}$ for every $i \in \mathbb{Z}_+$ is a $(n_1, \dots, n_{L+1}, k, T)_\mathbb{F}$ streaming code which is also $N_1, \dots, N_{L+1}$-achievable.*

Corollary 1. *Recalling that relay r_j (for any $j \in \{0, \dots, L\}$) starts transmitting the coded symbols of $\tilde{\mathbf{s}}_i$ at time $i + N_1^j$, it follows that for any $N_1, \dots, N_{L+1}$-erasure sequence the code used in each relay r_j to transmit $\tilde{\mathbf{s}}_i$ is N_{j+1}-achievable $(n_{j+1}, k, N_1^j + T_{j+1})_\mathbb{F}$ point-to-point block code, i.e., all the symbols of $\tilde{\mathbf{s}}_i$ can be decoded at relay r_{j+1} by delay of $i + T - \sum_{l=j+2}^{L+1} N_l$*

Thus to prove Theorem 2, we need to show that when $|\mathbb{F}| \geq n_{\max}$, $\mathbf{G}_i^{(r_j)}$ can be generated at each relay r_j for $j \in \{0, \dots, L\}$ and analyze the overall rate (by bounding the rate of the additional header).

Proof of Theorem 2. We first note that, as mentioned in Section II, an (n_{max}, k) MDS code exists as long as $|\mathbb{F}| \geq n_{\max}$. Therefore, following Lemma 1, when $|\mathbb{F}| \geq n_{\max}$ it follows that for any $N_1, \dots, N_{L+1}$-erasure sequence, any $j \in \{0, \dots, L\}$ and any $i \in \mathbb{Z}_+$, there exists $\mathbf{G}_i^{(r_j)}$ which is a generator matrix of (n_{j+1}, k) MDS code.

Following Lemma 3 it follows that streaming code resulting from using $\mathbf{G}_i^{(r_j)}$ defined above in each relay $j \in \{0, \dots, L\}$ for every $i \in \mathbb{Z}_+$ is a $(n_1, \dots, n_{L+1}, k, T)_\mathbb{F}$ streaming code which is also $N_1, \dots, N_{L+1}$-achievable. The rate in relay r_j, without taking the size of the header into account, is $\frac{k}{n_{j+1}}$. Thus, from Definition 5, the overall rate of transmission is upper bounded by

$$R \leq \min_{j \in \{0,1,\dots,L\}} \frac{k}{n_{j+1}}. \tag{14}$$

The header attached to each packet sent from relay r_j is composed from stacking the n_{j+1} headers used by each symbol generated from a $(n_{j+1}, k, N_1^j + T_j)_\mathbb{F}$ block code which is part of each transmission packet. As we defined above, this header is a number from $[1, \dots, n_{\max}]$. Hence the size of the header is $n_{j+1} \log(n_{\max})$ bits. We further note that the size of the header is upper bounded by $n_{\max} \log(n_{\max})$.

To conclude, each node transmits n_{j+1} coded symbols (each taken from field $\mathbb{F}$) along with $n_{j+1} \lceil \log(n_{j+1}) \rceil$ bits of header to transfer k information symbols (each taken from field $\mathbb{F}$). The overall rate is

$$\begin{aligned} R &\geq \min_j \frac{k \cdot \log(|\mathbb{F}|)}{n_{j+1} \cdot \log(|\mathbb{F}|) + n_{\max} \lceil \log(n_{\max}) \rceil} \\ &= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \left\{ \sum_{l=1, l \neq j}^{L+1} N_l \right\} + 1 + \frac{n_{\max} \lceil \log(n_{\max}) \rceil}{\log(|\mathbb{F}|)}} \end{aligned} \tag{15}$$

where $n_{\max}$ is defined in (7).

□

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