Streaming Erasure Codes over the Any-Node Relay Network

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Abstract

A typical path over the internet is composed of multiple hops. When considering the transmission of a sequence of messages (streaming messages) through packet erasure channel over a three-node network, it has been shown that taking into account the erasure pattern of each segment can result in improved performance compared to treating the channel as a point-to-point link. Since rarely there is only a single relay between the sender and the destination, it calls for trying to extend this scheme to more than a single relay. In this paper, we first extend the upper bound on the rate of transmission of a sequence of messages for any number of relays. We further suggest an achievable adaptive scheme that is shown to achieve the upper bound up to the size of an additional header that is required to allow each receiver to meet the delay constraints.

I. Introduction

Real-time interactive video streaming is one of the fastest-growing types of internet traffic. Traditionally, most of the traffic on the internet is not extremely sensitive to latency. However, as networks evolved, more and more people are using the network for real-time conversations, video conferencing, and on-line monitoring. According to [1], IP video traffic will account for 82 percent of traffic by 2022. Further, live video will grow 15-fold to reach 17 percent of Internet video traffic by 2022.

The fundamental difference between real-time video streaming and other services is the (much more stringent) latency requirement each packet has to meet in order to provide a good user experience. In multiple streaming codes works [2]–[6], using automatic repeat request (ARQ) for handling errors in transmission was discussed, and it was demonstrated that it might not be adequate for streaming applications. Using ARQ means that the latency (in case of an error) is at least three times the one-way delay, which in many cases may violate the latency requirements.

An alternative method for handling error in transmission is forward error correction (FEC). Using FEC can lower recovery latency since FEC does not depend on propagation delay. In [7], streaming codes for three-node networks were analyzed. For these networks, an explicit expression for capacity was derived. The concept of "symbol-wise" decode and forward introduced in [7] was shown to outperform any "message-wise" decode and forward strategy. Further, analyzing the resulting capacity for the three-node network shows that when constraints are imposed per segment rather than globally (while meeting the same global requirements), the capacity increases. As we demonstrate next, treating the network as a single-hop link with $N = N_1 + N_2$ erasures and total delay constraint of T symbols is worse than analyzing a three-node network where N_1 erasures are expected in the first segment, and N_2 are expected in the second segment with a total delay of T symbols.

However, Internet paths almost never consists of only a single relay (see, e.g., [8], [9]). Hence, designing a streaming code for a path consisting of multiple links when possible (i.e., take into account the error behavior of each link rather than rather than aggregate across all of the links) is expected to result in improved performance.

The scheme suggested in [7] does not depend on the location of erasures. Unfortunately, there is no straight-forward extension of this scheme to a more general case (a network with more than three nodes). The scheme suggested in this paper is an adaptive scheme; hence, an adaptive scheme which reorders the symbols for decoding based on erasure patterns encountered in the previous link. While requiring an additional header to notify the receiver about the order that was used (in each diagonal), we show that it can be easily extended to any number of relays.

In this paper, we first extend the upper bound derived in [7] to the general case of multi-hop relay network. We then describe an adaptive scheme for the general L relay scenario and show it achieves the upper bound an additional overhead in the packet header for dictating the symbol decoding order. We further show that the size of the header is a function of the required delay and the erasure pattern (hence it does not depend on the field size used by the code), therefore, the gap from the upper bound decreases as the field size increases.

II. Channel Model

A source node wants to send a sequence of messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ to a destination node with the help of L middle nodes $r_1, \ldots, r_L$. To ease notation we denote the source node as r_0, and destination node as r_{L+1}. Let k be a non-negative integer, and $n_1, n_2, \ldots, n_{L+1}$ be $L+1$ natural numbers.

Each $\mathbf{s}_i$ is an element in $\mathbb{F}^k$ where $\mathbb{F}$ is some finite field. In each time slot $i \in \mathbb{Z}_+$, the source message $\mathbf{s}_i$ is encoded into a length-n_1 packet $\mathbf{x}_i^{(r_0)} \in \mathbb{F}^{n_1}$ to be transmitted to the first relay through the erasure channel (r_0, r_1). The relay receives

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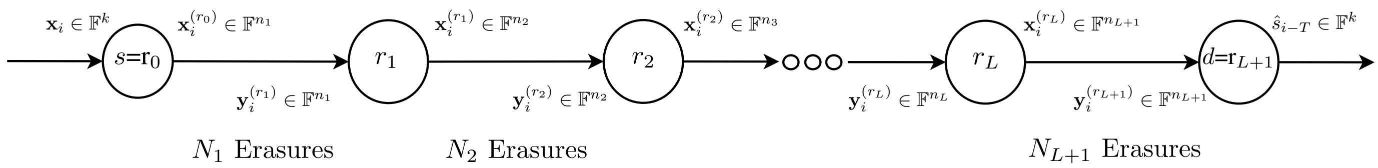

Fig. 1: Symbols generated in the L-node relay network at time i.

$\mathbf{y}_i^{(r_1)} \in \mathbb{F}^{n_1} \cup \{*\}$ where $\mathbf{y}_i^{(r_1)}$ equals either $\mathbf{x}_i^{(r_0)}$ or the erasure symbol "$*$". In the same time slot, relay r_1 transmits $\mathbf{x}_i^{(r_1)} \in \mathbb{F}^{n_2}$ to relay r_2 through the erasure channel (r_1, r_2). Relay r_2 receives $\mathbf{y}_i^{(r_2)} \in \mathbb{F}^{n_2} \cup \{*\}$ where $\mathbf{y}_i^{(r_2)}$ equals either $\mathbf{x}_i^{(r_1)}$ or the erasure symbol "$*$". The same process continues (in the same time slot) until relay r_L transmits $\mathbf{x}_i^{(r_L)} \in \mathbb{F}^{n_{L+1}}$ to the destination r_{L+1} through the erasure channel (r_L, r_{L+1}).

We assume that on the discrete timeline, each channel (r_{j-1}, r_j) introduces up to N_j arbitrary erasures respectively. The symbols generated in the L-node relay network at time i are illustrated in Figure 1.

We recall that for any natural numbers L and M, a systematic maximum-distance separable (MDS) $(L+M, L)$-code is characterized by an $L \times M$ parity matrix $\mathbf{V}^{L\times M}$ where any L columns of $\begin{bmatrix}\mathbf{I}_L & \mathbf{V}^{L\times M}\end{bmatrix} \in \mathbb{F}^{L\times(L+M)}$ are independent. It is well known that a systematic MDS $(L+M, L)$-code always exists as long as $|\mathbb{F}| \geq L+M$ [10].

To simplify notation, we sometimes denote $N_a^b = \sum_{l=a}^{b} N_l$. We will take all logarithms to base 2 throughout this paper.

III. Standard Definitions and Known Results

Definition 1. *An $(n_1, n_2, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code consists of the following:*

1) *A sequence of source messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ where $\mathbf{s}_i \in \mathbb{F}^k$.*
2) *An encoding function $f_i^{(r_0)} : \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{i+1 \text{ times}} \to \mathbb{F}^{n_1}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_0)}$ is used by node r_0 at time i to encode $\mathbf{s}_i$ according to $\mathbf{x}_i^{(r_0)} = f_i^{(r_0)}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_i)$.*
3) *A relaying function for node $j \in [1, \ldots, L]$, $f_i^{(r_j)} : \underbrace{\mathbb{F}^{n_j} \cup \{*\} \times \ldots \times \mathbb{F}^{n_j} \cup \{*\}}_{i+1 \text{ times}} \to \mathbb{F}^{n_{j+1}}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_j)}$ is used by node r_j at time i to construct $\mathbf{x}_i^{(r_j)} = f_i^{(r_j)}(\mathbf{y}_0^{(r_j)}, \mathbf{y}_1^{(r_j)}, \ldots, \mathbf{y}_i^{(r_j)})$.*
4) *A decoding function $\phi_{i+T} : \underbrace{\mathbb{F}^{n_L} \cup \{*\} \times \ldots \times \mathbb{F}^{n_L} \cup \{*\}}_{i+T+1 \text{ times}} \to \mathbb{F}^{n_{L+1}}$ for each $i \in \mathbb{Z}_+$ is used by node n_{L+1} at time $i+T$ to estimate $\mathbf{s_i}$ according to $\hat{\mathbf{s}}_i = \phi_{i+T}(\mathbf{y}_0^{(r_{L+1})}, \mathbf{y}_1^{(r_{L+1})}, \ldots, \mathbf{y}_{i+T}^{(r_{L+1})})$.*

Definition 2. *An erasure sequence is a binary sequence denoted by $e^\infty \triangleq \{e_i\}_{i=0}^{\infty}$ where $e_i = \mathbf{1}\{\text{erasure occurs at time i}\}$.*

An N-erasure sequence is an erasure sequence e^∞ that satisfies $\sum_{l=0}^{\infty} e_l = N$. In other words, an N-erasure sequence introduces N arbitrary erasures on the discrete timeline. The set of N-erasure sequences is denoted by Ω_N.

Definition 3. *The mapping $g_{n_j} : \mathbb{F}^{n_j} \times \{0,1\} \to \mathbb{F}^{n_j} \cup \{*\}$ of an erasure channel is defined as*

$$g_{n_j}(\mathbf{x}^{(r_j)}, e_i) = \begin{cases} \mathbf{x}^{(r_j)} & \text{if } e{=}0, \\ * & \text{if } e{=}1. \end{cases} \tag{1}$$

For any erasure sequence e^∞ and any $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code, the following input-output relations holds for the erasure channel (r_j, r_{j+1}) for each $i \in \mathbb{Z}_+$ $\mathbf{y}_i^{(r_{j+1})} = g_{n_j}(\mathbf{x}_i^{(r_j)}, e_i)$.

Definition 4. *An $(n_1, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code is said to be $(N_1, N_2, \ldots, N_{L+1})$-achievable if the following holds for any N_j-erasure sequence $e^\infty \in \Omega_{N_j}$, for all $i \in \mathbb{Z}_+$ and all $\mathbf{s}_i \in \mathbb{F}^k$, we have $\hat{\mathbf{s}}_i = \mathbf{s}_i$ where*

$$\hat{\mathbf{s}}_i = \phi_{i+T}(g_{n_{L+1}}(\mathbf{x}_0^{(r_L)}, e_0), \ldots, g_{n_{L+1}}(\mathbf{x}_{i+T}^{(r_L)}, e_{i+T})) \tag{2}$$

and for previous nodes

$$\mathbf{x}_i^{(r_{j+1})} = f_i(g_{n_j}(\mathbf{x}_0^{(r_j)}, e_0), \ldots, g_{n_j}(\mathbf{x}_i^{(r_j)}, e_i)). \tag{3}$$

Definition 5. *The rate of an $(n_1, n_2, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code is $\frac{k}{\max\{n_1, n_2, \ldots, n_{L+1}\}}$.*

If $N_{l\neq j}=0$, then the L-node relay network with erasures reduces to a point-to-point packet erasure channel. It was previously shown [3] that the maximum achievable rate of the point-to-point packet erasure channel with $N_j = N$ arbitrary erasures and delay of T denoted by $C_{T,N}$ satisfies

$$C_{T,N} = \begin{cases} \frac{T-N+1}{T+1} & T \geq N \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

It was further shown that the capacity of point-to-point channel with N arbitrary erasures and delay of T can be achieved by diagonally interleaving $(T+1, T-N+1)$ MDS code.

In [7], a three node relay network was analyzed, and the following theorem was shown.

Theorem 1 (Theorem 1 in [7]). *Fix any (T, N_1, N_2). Recalling that the point-to-point capacity satisfies (4), we have*

$$C_{T,N_1,N_2} = \min\left(C_{T-N_2,N_1}, C_{T-N_1,N_2}\right). \tag{5}$$

IV. Main Results

In this paper we first derive an upper bound for the achievable rate in $L+1$-node relay network.

Theorem 2. *Assume a path with L relays. For a target overall delay of T, where the maximal number of arbitrary erasures in link $(r_{j-1}, r_j),\ j \in [1, \ldots, L+1]$ is N_j and $T \geq \sum_{l=1}^{L+1} N_l$.*

The achievable rate is upper bounded by

$$R \leq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \sum_{l=1, l\neq j}^{L+1} N_l + 1} \triangleq C^{+}_{T,N_1,\ldots,N_{L+1}}. \tag{6}$$

We then suggest an achievable scheme which achieves the upper bound up to a size of an overhead which is required by the scheme. To assist the reader, we denote the parameters of the code used by relay r_j with indices $j, j+1$. The rate used in link (r_j, r_{j+1}) can be expressed as

$$R_{j,j+1} \triangleq \frac{(T - \sum_{l=1}^{L+1} N_l + 1)|\mathbb{F}|}{(T - \sum_{l=1, l\neq j+1}^{L+1} N_l + 1)|\mathbb{F}| + O_{j,j+1}} \tag{7}$$

where $O_{j,j+1}$ is the size of the header required by node j.

Denoting with $n_{j,j+1} \triangleq T - \sum_{l=1, l\neq j+1}^{L+1} N_l + 1$ the block size of the MDS code used by node j we have

$$n_{\max} \triangleq \max_j (n_{j,j+1}). \tag{8}$$

We show next the following Proposition.

Proposition 1. *The size of the header added by relay j can be upper bounded by $O_{j,j+1} \leq (n_{\max}+1)\log(n_{\max}+1)$.*

Using this proposition we show the following Theorem.

Theorem 3. *Assume a link with L relays. For a target overall delay of T, where the maximal number of arbitrary erasures in link $(r_j, r_{j+1}),\ j \in [0, \ldots, L]$ is N_{j+1} and $T \geq \sum_{l=1}^{L+1} N_l$, When $|\mathbb{F}| \geq n_{\max}$, The following rate is achievable.*

$$R \geq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{\max_j \left\{ T - \sum_{l=1, l\neq j}^{L+1} N_l + 1 \right\} + \frac{(n_{\max}+1)\log(n_{\max}+1)}{|\mathbb{F}|}} \tag{9}$$

where $n_{\max}$ is defined in (8).

Remark 1. *Although the deterministic erasure model is formulated in such a way that link (r_{j-1}, r_j) introduce only a finite number of erasures over the discrete timeline, the maximum coding rate remains unchanged for the following sliding window model that can introduce infinite erasures: Every message must be perfectly recovered with delay T as long as the numbers of erasures introduced by in link (r_{j-1}, r_j) in every sliding window of size T + 1 do not exceed N_j.*

V. MOTIVATING EXAMPLE

Consider a link with up to $N = 2$ arbitrary erasures, where the delay constraint the transmission has to meet is $T = 3$ symbols. The capacity of this link (4) is $C_{3,2} = 2/4$. Now, assume that in fact, this link is a three-node network ($L = 1$), where up to $N_1 = 1$ erasures occur in link (r_0, r_1) and up to $N_2 = 1$ erasures occur in link (r_1, r_2), where transmission has to be decoded with the same overall delay of $T = 3$ symbols. The capacity of this link (5) is $C_{3,1,1} = 2/3$, which is better than the point-to-point link.

We suggest an adaptive scheme, i.e., a scheme in which the order of symbols in each diagonal is set according to the erasure pattern in the previous link. The order of the symbols is transmitted to the receiver to allow decoding. Hence, additional overhead is required. We first show an example of the suggested scheme to the three-node network and then show how to extend it to the four-node network.

In the proposed scheme, the source r_0 uses the same code suggested in [7], i.e., a $(3, 2)$ MDS code that can recover any arbitrary single erasure in a delay of two symbols combined with diagonal interleaving (i.e. the code is applied on the diagonals). Denoting the two information symbols transmitted at time i as $[a_i, b_i] \in \mathbb{F}$. Applying the code on the diagonals means, for example, that one diagonal is $[a_i, b_{i+1}, a_i + b_{i+1}]$ (where $a_i + b_{i+1} \in \mathbb{F}$ is the parity symbol). The transmission from the source is depicted in I (different diagonals are marked with different colors).

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	$\cancel{123}$	123	123	123	123
a_i	a_{i-1}	$\cancel{a_i}$	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
b_i	b_{i-1}	$\cancel{b_i}$	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
$a_{i-2}+ b_{i-1}$	$a_{i-3}+ b_{i-2}$	$\cancel{a_{i-2}+ b_{i-1}}$	$a_{i-1}+ b_i$	$a_i+ b_{i+1}$	$a_{i+1}+ b_{i+2}$	$a_{i+2}+ b_{i+3}$

TABLE I: Transmission of the Source in (r_0, r_1).

When there are no erasures, relay r_1 uses the same codes as the source r_0 *while delaying it by one symbol*, i.e. code $[a_i, b_{i+1}, a_i + b_{i+1}]$ will be transmitted on the diagonal starting time $i + 1$.

If an erasure occurred, the relay would send any available symbols (per diagonal) in the received order until it can decode the information symbols. Then, the erased symbols will be sent. For example, assuming that a symbol at time i is erased when transmitted from the sender to relay r_1, the suggested transmission scheme of relay r_1 is given in Table II below.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	$\boxed{223}$	$\boxed{113}$	123	123
	a_{i-2}	a_{i-1}	$\mathbf{b_{i+1}}$	a_{i+1}	a_{i+2}	a_{i+3}
	b_{i-2}	b_{i-1}	b_i	$\mathbf{a_i}$	b_{i+2}	b_{i+3}
	$a_{i-4}+ b_{i-3}$	$a_{i-3}+ b_{i-2}$	$a_{i-2}+ b_{i-1}$	$a_{i-1}+ b_i$	$a_i+ b_{i+1}$	$a_{i+1}+ b_{i+2}$

TABLE II: Transmission of relay r_1 in (r_1, r_2), given that symbol i was erased when transmitted in link (r_0, r_1).

We note that the erasure in time i in link (r_0, r_1) caused a change in the order of the symbols in packets $i + 1$ and $i + 2$. Denoting the order of symbols sent from r_0 as $[1, 2, 3]$, the header of the packet in time i is composed of the order of the symbols from each block code (the order of the symbols in each diagonal) at time i. At time $i + 1$, for example, the relay can send symbol b_{i+1} from the code applied on $[a_i, b_{i+1}]$ rather than symbol a_i that has been erased. At time $i + 2$, the relay can recover the information symbols and send a_i. Hence the parts of the header which are relevant to this code will be $2xx$ at time $i + 1$ and $x1x$ at time $i + 2$. It can be easily verified that the destination can recover the original data at a delay of $T = 3$ (assuming any arbitrary one erasure in the link between the relay and destination).

This concept can be applied to additional relays if they exist. For example consider four-node network ($L = 2$). The transmission scheme on the next relay r_2 (in this specific example), is the same as the transmission scheme of the first relay, i.e., in case there is no erasure, transmit (on each diagonal) the symbols in the same order as received, delayed by one symbol (i.e., total delay of two symbols from the sender). If an erasure occurred before r_2 decoded the information symbols, transmit the available symbols (again, it is guaranteed that there will be enough symbols). When the information symbols can be decoded, transmit the erased symbols.

For example, in case the symbol transmitted from relay r_1 to r_2 at time $i + 2$ is erased, the suggested transmission scheme of relay r_2 in given in Table III below.

We note that the erasure in time $i + 2$ in link (r_1, r_2) caused a change in order of the symbols in packets $i + 3$ and $i + 4$ (the order of symbols in time $i + 2$ is not the original order, yet it is the same as was transmitted from r_1 at time $i + 1$).

Since, basically, the same $(3, 2)$ code is used (with a different order of symbols which is communicated to the receiver), it can be easily verified that each packet can be recovered up to delay of $T = 4$ symbols for any arbitrary erasure in the link between the relay and the destination.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	123	223	$\boxed{213}$	$\boxed{113}$
	a_{i-3}	a_{i-2}	a_{i-1}	b_{i+1}	b_{i+2}	a_{i+2}
	b_{i-3}	b_{i-2}	b_{i-1}	b_i	a_i	a_{i+1}
	$a_{i-5}+$ b_{i-4}	$a_{i-4}+$ b_{i-3}	$a_{i-3}+$ b_{i-2}	$a_{i-2}+$ b_{i-1}	a_{i-1} b_i	a_i+ b_{i+1}

TABLE III: Transmission of relay r_2 in (r_2, r_3), given that symbol $i+2$ was erased when transmitted in link (r_1, r_2).

In this example, the maximal size of the header is three numbers, each taken from $\{1, 2, 3\}$. Hence, its maximal size of the header is $3\log(3)$ bits. Since the block code used in each link transmits two information symbols (taken from $\mathbb{F}$) using three symbols (again, taken from $\mathbb{F}$) in every channel use, we conclude that the scheme achieves a rate of

$$R = \frac{2 \cdot |\mathbb{F}|}{3 \cdot |\mathbb{F}| + 3\log(3)}, \tag{10}$$

which approaches $2/3$ as the field size increases. As we show next, the upper bound for this scenario is indeed $2/3$.

VI. Proof of upper bound

Fix any $(N_1, \ldots, N_{L+1}, T)$. Suppose we are given an $(N_1, \ldots, N_{L+1})$-achievable $(n_1, \ldots, n_{j+1}, k, T)_{\mathbb{F}}$-streaming code for some $n_1, \ldots, n_{j+1}, k$ and $\mathbb{F}$. Our goal is to show that

$$\frac{k}{\max\{n_1, \ldots, n_{L+1}\}} \leq \min_j \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=1, l\neq j}^{L+1} N_l + 1}. \tag{11}$$

To this end, we let $\{\mathbf{s}_i\}_{i\in\mathbb{Z}_+}$ be i.i.d. random variables where $\mathbf{s}_0$ is uniform on $\mathbb{F}^k$. Since the $(n_1, \ldots, n_{j+1}, k, T)_{\mathbb{F}}$-streaming code is $(N_1, \ldots, N_{L+1})$-achievable, it follows from Definition **??** that

$$H\left(\mathbf{s}_i \,\middle|\, \mathbf{y}_0^{(L+1)}, \mathbf{y}_1^{(L+1)}, \ldots, \mathbf{y}_{i+T}^{(L+1)}\right) \tag{12}$$

for any $i \in \mathbb{Z}_+$ and any $e_j \in \Omega_{N_j}$. Consider the two cases.

Case $T < \sum_{l=1}^{L+1} N_l$:

Let $e_j \in \Omega_{N_j}$ such that

$$e_j = \begin{cases} 1 & if \sum_{l=1}^{j-1} \leq \sum_{l=1}^{j} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Due to (13) and Definition **??**, we have

$$I\left(\mathbf{s}_0; \mathbf{y}_0^{(L+1)}, \mathbf{y}_1^{(L+1)}, \ldots, \mathbf{y}_T^{(L+1)}\right) = 0. \tag{14}$$

Combining (12), (14) and the assumption that $T < \sum_{l=1}^{L+1} N_l$, we obtain $H(\mathbf{s}_0)$. Since $\mathbf{s}_0$ consists of k uniform random variables in $\mathbb{F}$, the only possible value of k is zeros, which implies

$$\frac{k}{\max\{n_1, \ldots, n_{L+1}\}} = 0. \tag{15}$$

Case $T \geq \sum_{l=1}^{L+1} N_l$:

The proof follows that footsteps of [7]. We start by generalizing the arguments given in [7] for the first and second segments to the first and last segments in our case. Than we show how similar technique can be used to derive a constraint on the code to be used in an intermediate segment.

First Segment (link (r_0, r_1)):

First we note that for every $i \in \mathbb{Z}_+$, message $\mathbf{x}_i$ has to be perfectly recovered by node r_1 by time $i + T - \sum_{l=2}^{L+1} N_l$ given that $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ have been correctly decoded by node r_1, or otherwise a length N_2 burst erasure from time $i + T - \sum_{l=2}^{L+1} N_l + 1$ to $i + T - \sum_{l=3}^{L+1} N_l$ introduced on channel (r_1, r_2) followed by a length N_3 burst erasure from time $i + T - \sum_{l=3}^{L+1} N_l + 1$ to $i + T - \sum_{l=4}^{L+1} N_l$ introduced on channel (r_2, r_3) and so on until a length N_{L+1} burst erasure from time $i + T - N_{L+1} + 1$ to $i + T$ would result in a decoding failure for node r_1, node r_2 and all the nodes up to the destination r_{L+1}.

Denoting $\tilde{N}_2 = \sum_2^{L+1} N_l$ it then follows that

$$H\left(\mathbf{s}_i \,\middle|\, \left\{\mathbf{x}_i^{(r_0)}, \mathbf{x}_{i+1}^{(r_0)}, \ldots, \mathbf{x}_{i+T-\tilde{N}_2}^{(r_0)}\right\} \setminus \left\{\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_{N_1}}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{i-1}\right) = 0 \tag{16}$$

for any $\in \mathbb{Z}_+$ and N_1 non-negative integers denoted by $\theta_1, \ldots, \theta_{N_1}$. We note that the following holds (by assuming, for example, the last N_1 symbols in every window of $T - \tilde{N}_2$ symbols starting time $i = 0$ are erased):

$$
\begin{aligned}
&H\left(\mathbf{s}_0 \,\middle|\, \left\{\mathbf{x}_0^{(r_0)}, \mathbf{x}_1^{(r_0)}, \ldots, \mathbf{x}_{T-\tilde{N}_2-N_1}^{(r_0)}\right\}\right) = 0\\
&H\left(\mathbf{s}_1 \,\middle|\, \left\{\mathbf{x}_1^{(r_0)}, \mathbf{x}_2^{(r_0)}, \ldots, \mathbf{x}_{T-\tilde{N}_2-N_1}^{(r_0)}, \mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)}\right\}, \mathbf{s}_0\right) = 0\\
&\qquad\vdots\\
&H\left(\mathbf{s}_{T-\tilde{N}_2-N_1} \,\middle|\, \left\{\mathbf{x}_{T-\tilde{N}_2-N_1}^{(r_0)}, \mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)}, \mathbf{x}_{1(T-\tilde{N}_2+1)+1}^{(r_0)}, \ldots, \mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1-1}^{(r_0)}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_2-N_1-1}\right) = 0\\
&H\left(\mathbf{s}_{T-\tilde{N}_2-N_1+1} \,\middle|\, \left\{\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)}, \ldots, \mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}^{(r_0)}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_2-N_1}\right) = 0\\
&H\left(\mathbf{s}_{T-\tilde{N}_2-N_1+2} \,\middle|\, \left\{\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)}, \ldots, \mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}^{(r_0)}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_2-N_1+1}\right) = 0\\
&\qquad\vdots\\
&H\left(\mathbf{s}_{T-\tilde{N}_2-1} \,\middle|\, \left\{\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)}, \ldots, \mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}^{(r_0)}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_2-2}\right) = 0\\
&H\left(\mathbf{s}_{T-\tilde{N}_2} \,\middle|\, \left\{\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)}, \ldots, \mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}^{(r_0)}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_2-1}\right) = 0\\
&\qquad\vdots
\end{aligned}
\tag{17}
$$

Using the chain rule we have the following for each $j \in \mathbb{N}$

$$
H\left(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{T-\tilde{N}_2+(j-1)(T-\tilde{N}_2+1)} \mid \left\{\mathbf{x}_{m(T-\tilde{N}_2+1)}^{(r_0)}, \mathbf{x}_{1+m(T-\tilde{N}_2+1)}^{(r_0)}, \ldots, \mathbf{x}_{T-N_1-\tilde{N}_2+m(T-\tilde{N}_2+1)}^{(r_0)}\right\}_{m=0}^{j}\right) = 0. \tag{18}
$$

Alternatively we note that for all $q \in \mathbb{Z}_+$,

$$
\begin{aligned}
&\left|\left\{q, q+1, \ldots, T-\tilde{N}_2\right\} \bigcap\right.\\
&\left.\left\{m(T-\tilde{N}_2+1), 1+m(T-\tilde{N}_2+1) \ldots, T-N_1+m(T-\tilde{N}_2+1)\right\}_{m=0}^{j}\right|\\
&= T - N_1 - \tilde{N}_2 + 1.
\end{aligned}
\tag{19}
$$

Hence, (18) follow from (16), (19) and the chain rule.

Therefore, following the arguments in [7] any $(N_1, \ldots, N_{L+1})$ achievable $(n_1, \ldots n_{L+1}, k, T)_{\mathbb{F}}$ code restricted to channel (r_0, r_1) can be viewed as a point-to-point streaming code with rate k/n_1, delay $T - \sum_{l=2}^{L+1} N_l$ which can correct any N_1 erasures. We therefore have

$$
\begin{aligned}
\frac{k}{n_1} &\leq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=2}^{L+1} N_l + 1}\\
&= C_{T-\sum_{l=2}^{L+1} N_l, N_1}.
\end{aligned}
\tag{20}
$$

Last Segment (link (r_L, r_{L+1})):

In addition, for every $i \in \mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered from

$$
\left\{\mathbf{x}_{i+\sum_{l=1}^{L} N_l}^{(r_{L+1})}, \mathbf{x}_{i+\sum_{l=1}^{L} N_l+1}^{(r_{L+1})}, \ldots, \mathbf{x}_{T}^{(r_{L+1})}\right\} \tag{21}
$$

by node r_{L+1} given that $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ have been correctly decoded by node r_{L+1}, or otherwise a length N_1 burst erasure from time i to $i+N_1-1$ induced on channel (r_0, r_1) followed by a length N_2 burst erasure from time $i+N_1$ to $i+N_1+N_2-1$ induced on channel (r_1, r_2) and so on until a length N_L burst erasure from time $i + \sum_{l=1}^{L-1} N_l$ to $i + \sum_{l=1}^{L} N_l - 1$ induced on channel (r_{L-1}, r_L) would result in a decoding failure for node r_{L+1}.

Denoting $\tilde{N}_1 = \sum_1^L N_l$ it then follows that

$$
H\left(\mathbf{s}_i \,\middle|\, \left\{\mathbf{x}_{i+\tilde{N}_1}^{(r_L)}, \mathbf{x}_{i+\tilde{N}_1+1}^{(r_L)}, \ldots, \mathbf{x}_{i+T}^{(r_L)}\right\} \setminus \left\{\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_{N_{L+1}}}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{i-1}\right) = 0 \tag{22}
$$

for any $\in \mathbb{Z}_+$ and N_{L+1} non-negative integers denoted by $\theta_1, \ldots, \theta_{N_{L+1}}$.

We note that for all $q \in \mathbb{Z}_+$,

$$
\begin{aligned}
&\left|\left\{q+\tilde{N}_1, q+\tilde{N}_1+1, \ldots, q+T\right\} \bigcap\right.\\
&\left.\left\{\tilde{N}_1+N_{L+1}+m(T-\tilde{N}_1+1), \tilde{N}_1+N_{L+1}+1+m(T-\tilde{N}_1+1) \ldots, T+m(T-\tilde{N}_1+1)\right\}_{m=0}^{j}\right|
\end{aligned}
$$

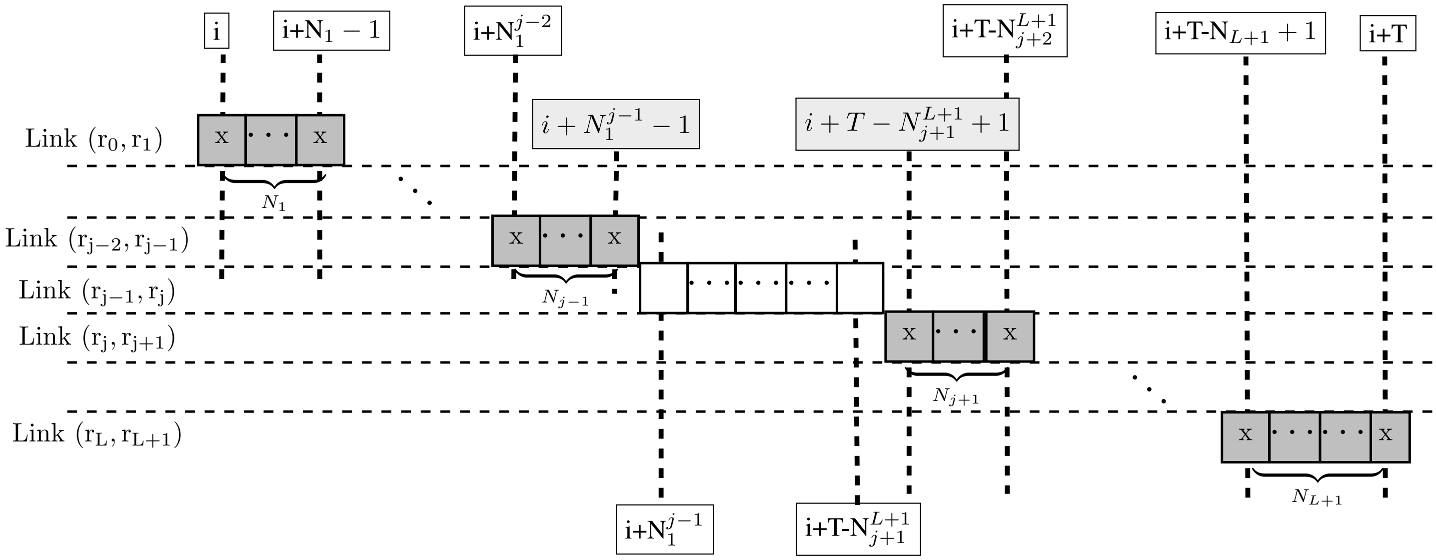

Fig. 2: Constrains imposed on transmission in link (r_{j-1}, r_j).

$$= T - \tilde{N}_1 - N_{L+1} + 1 \tag{23}$$

Using (22), (23) and the chain rule, we have

$$H\left(\mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_1+(j-1)(T-\tilde{N}_1+1)} \mid \left\{\mathbf{x}^{(r_L)}_{\tilde{N}_1+N_{L+1}+m(T-\tilde{N}_1+1)}, \mathbf{x}^{(r_L)}_{\tilde{N}_1+N_{L+1}+1+m(T-\tilde{N}_1+1)}, \ldots, \mathbf{x}^{(r_L)}_{T+m(T-\tilde{N}_1+1)}\right\}_{m=0}^{j}\right)$$
$$= 0. \tag{24}$$

Again, following the arguments in [7] when restricted to the channel (r_L, r_{L+1}), any $(N_1, \ldots, N_{L+1})$ achievable $(n_1, \ldots n_{L+1}, k, T)_{\mathbb{F}}$ code

$$\frac{k}{n_{L+1}} \leq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=1}^{L} N_l + 1} = C_{T - \sum_{l=1}^{L} N_l, N_{L+1}}.$$

The j'th Segment (link (r_{j-1}, r_j)):

Now, when considering a j'th segment (the channel (r_{j-1}, r_j)), we show again that any $(N_1, \ldots, N_{L+1})$ achievable $(n_1, \ldots n_{L+1}, k, T)_{\mathbb{F}}$ code restricted to the channel (r_{j-1}, r_j) can be viewed as a point-to-point code which should handle any N_J erasures with a delay which we define next.

Combining the arguments given above we note that first, for every $i \in \mathbb{Z}_+$, message $\mathbf{x}_i$ has to be perfectly recovered by node r_j by time $i + T - \sum_{l=j+1}^{L+1} N_l$ given that $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ have been correctly decoded by node r_j, or otherwise a length N_{j+1} burst from time $i + T - \sum_{l=j+1}^{L+1} N_l + 1$ to $i + T - \sum_{l=j+2}^{L+1} N_l$ introduced on channel (r_j, r_{j+1}), followed by a length N_{j+2} burst from time $i + T - \sum_{l=j+2}^{L+1} N_l + 1$ to $i + T - \sum_{l=j+3}^{L+1} N_l$ introduced on channel (r_{j+1}, r_{j+2}) and so on, up to a length N_{L+1} burst from time $i - N_{L+1} + 1$ to $i + T$ introduced on channel (r_L, r_{L+1}) would result in a decoding failure for node r_j and all the nodes up to the destination r_{L+1}.

Further, in addition, for every $i \in \mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered from

$$\left(\mathbf{x}^{(r_{j-1})}_{i+\sum_{l=1}^{j-1} N_l}, \mathbf{x}^{(r_{j-1})}_{i+\sum_{l=1}^{j-1} N_l+1}, \ldots, \mathbf{x}^{(r_{j-1})}_{i+T-\sum_{l=j+1}^{L+1} N_l}\right) \tag{25}$$

by node j given that $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{i-1}$ have been correctly decoded by node j, or otherwise a length N_1 burst erasure from time i to $i + N_1 - 1$ induced on channel (r_0, r_1) followed by a length N_2 burst erasure from time $i + N_1$ to $i + N_1 + N_2 - 1$ induced on channel (r_0, r_2) and so on (up to a burst erasure N_{j-1} from time $i + \sum_{l=1}^{j-2} N_l$ to $i + \sum_{l=1}^{j-1} N_l - 1$ induced on channel (r_{j-2}, r_{j-1})) would result in a decoding failure for node j. These constraints are depicted in Figure 2.

Now denote with $\tilde{N}_{j,1} = \sum_{l=1}^{j-1} N_l$, and with $\tilde{N}_{j,2} = \sum_{l=j+1}^{L+1} N_l$ it follows that

$$H\left(\mathbf{s}_i \,\middle|\, \left\{\mathbf{x}^{(r_{j-1})}_{i+\tilde{N}_{j,1}}, \mathbf{x}^{(r_{j-1})}_{i+\tilde{N}_{j,1}+1}, \ldots, \mathbf{x}^{(r_{j-1})}_{i+T-\tilde{N}_{j,2}}\right\} \setminus \left\{\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_{N_j}}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{i-1}\right) = 0 \tag{26}$$

We note that for all $q \in \mathbb{Z}_+$,

$$\left|\left\{q + \tilde{N}_1, q + \tilde{N}_1 + 1, \ldots, q + T - \tilde{N}_2\right\} \bigcap\right.$$

$$\left\{\tilde{N}_1+N_j+m(T-\tilde{N}_1-\tilde{N}_2+1),\tilde{N}_1+N_j+1+m(T-\tilde{N}_1+\tilde{N}_2+1)\ldots,T-\tilde{N}_2+m(T-\tilde{N}_1-\tilde{N}_2+1)\right\}_{m=0}^{j}\Bigg|$$
$$=T-\tilde{N}_1-N_j-\tilde{N}_2+1 \tag{27}$$

Using (26), (27) and the chain rule, we have

$$\begin{aligned}&H\Big(\mathbf{s}_0,\ldots,\mathbf{s}_{T-\tilde{N}_1-\tilde{N}_2+(j-1)(T-\tilde{N}_1-\tilde{N}_2+1)}\mid\\&\left\{\mathbf{x}^{(r_L)}_{\tilde{N}_1+N_j+m(T-\tilde{N}_1-\tilde{N}_2+1)},\mathbf{x}^{(r_L)}_{\tilde{N}_1+N_j+1+m(T-\tilde{N}_1-\tilde{N}_2+1)},\ldots,\mathbf{x}^{(r_L)}_{T-\tilde{N}_2+m(T-\tilde{N}_1-\tilde{N}_2+1)}\right\}_{m=0}^{j}\Big)\\&=0.\end{aligned} \tag{28}$$

Again, following the arguments in [7] when restricted to the channel (r_{j-1},r_j), any $(N_1,\ldots,N_{L+1})$ achievable $(n_1,\ldots n_{L+1},k,T)_{\mathbb{F}}$ code can be viewed as a point-to-point streaming code with rate k/n_j, delay $T-\sum_{l=1,l\neq j}^{L}N_l$ which can correct any N_j erasures. We therefore have

$$\begin{aligned}\frac{k}{n_j}&\leq\frac{T-\sum_{l=1}^{L+1}N_l+1}{T-\sum_{l=1,l\neq j}^{L+1}N_l+1}\\&=C_{T-\sum_{l=1,l\neq j}^{L+1}N_l,N_j}.\end{aligned} \tag{29}$$

Therefore we have

$$\begin{aligned}R&\leq\frac{k}{\max_j(n_j)}\\&=\frac{T-\sum_{l=1}^{L+1}N_l+1}{T-\min_j\sum_{l=1,l\neq j}^{L+1}N_l+1}\\&=\min_j C_{T-\sum_{l=1,l\neq j}^{L+1}N_l,N_j}\\&=C_{T,N_1,\ldots,N_{L+1}}.\end{aligned} \tag{30}$$

VII. Suggested Coding Scheme

As mentioned above, the suggested coding scheme is an adaptive symbol-wise decode and forward scheme. Specifically the suggested scheme is as follows.

- From the sender (link (r_0,r_1)), transmit a $C_{T-\sum_{l=2}^{L+1}N_l,N_1}$ block code combined with diagonal interleaving (i.e., the symbols of the code are transmitted over the diagonals).
- From the first relay (link (r_1,r_2)):
  - Delay transmission by N_1 symbols.
  - Transmit a $C_{T-\sum_{l=1,l\neq 2}^{L+1}N_l,N_2}$ block code combined with diagonal interleaving. The order of symbols over each diagonal is a function of the error pattern in link (r_0,r_1).
  - For each packet, attach a header which indicates the order of the symbols in it.
- In general, in link (r_j,r_{j+1}), where $j\in[0,\ldots,L]$:
  - Delay transmission by $\sum_{l=1}^{j}N_j$ symbols.
  - Transmit a $C_{T-\sum_{l=1,l\neq j+1}^{L+1}N_l,N_{j+1}}$ block code combined with diagonal interleaving. The order of the symbols over each diagonal (compared to the transmitted packets from relay relay r_{j-1}), is a function of the error pattern in the link (r_{j-1},r_j).
  - For each packet, attach a header which indicates the order of the symbols in it.

We note that as the field size $|\mathbb{F}|$ increases, the loss compared to the upper bound is reduced. Before showing the proof of the Theorem we show the following Proposition.

Proposition 2. *The block code used by relay r_j can be viewed as sub-code of* $\min_j C_{j,j+1}$ *(i.e., the outcome of puncturing* $\min_j C_{j,j+1}$*) which can be viewed as the “bottleneck” in the chain of relays.*

This proposition holds since k (the number of information symbols) is the same for all codes. Recalling that (MDS) code $\min_j C_{j,j+1}$ can correct any N_{max} erasures with delay of $\max_j T_{j,j+1}$, we note that puncturing any $N_{\max}-N_j$ columns from the generator matrix of this code results with a code that can correct any N_j erasure with delay of $T_{j,j+1}$.

Proof. We focus on a single block code (single diagonal) and show that the data sent on it can be decoded in overall delay of T assuming the maximal number of arbitrary erasures in a window of $T+1$ symbols in link (r_j,r_j+1) is N_{j+1} ($\forall j\in[0,\ldots,L]$) and $T\geq\sum_{l=1}^{L+1}N_l$.

Since all diagonals are subjected to the same constraints (delay constraint of T symbols and maximal number of erasures in a window of $T+1$ symbols), showing that one block code (one diagonal) can be decoded in the delay constraint means all diagonals can be decoded within this constraint.

Analyzing the suggested scheme we note that each node r_j transmits using a block code at rate $C_{j,j+1}$. Therefore, it is straightforward that the overall rate of transmission (up to the additional header required starting r_1) is

$$\begin{aligned} C_{T,N_1,\ldots,N_{L+1}} &= \min_j C_{j,j+1} \\ &= \min_j C_{T-\sum_{l=1,l\neq j}^{L+1} N_l, N_j}. \end{aligned} \tag{31}$$

We therefore need to show that it is feasible to transmit at rate $C_{j,j+1}$, defined in (7), in link (r_j, r_{j+1}), and that the packet sent at time i from the sender can be recovered up to delay T at the receiver. Analyzing the actual size of the overhead will conclude the proof.

With respect to feasibility, we note that delaying transmission by N_j symbols prior to transmitting from relay r_j (compared to the beginning of transmission in relay r_{j-1}), means that relay r_j will always have symbols to transmit as it can be thought of as if relay r_j "buffer" up to N_j symbols potentially received from r_{j-1} (which is the maximal number of symbols that can be erased in link (r_{j-1}, r_j) in any window of $T+1$ symbols).

Then we note that when relay r_{j-1} ends transmitting its block code diagonally (which is $n_{j-1,j}$ symbols after it started at time $i+\sum_{l=1}^{j-1} N_l$) it is guaranteed that relay r_j has successfully decoded the original k information symbols (as at this time, relay relay r_{j-1} has finished transmitting its code). This time index can be denoted as

$$\begin{aligned} i+\sum_{l=1}^{j-1} N_l + n_{j-1,j} =& i+\sum_{l=1}^{j-1} N_l + T - \sum_{l=1,l\neq j}^{L+1} N_l \\ &= i+T-\sum_{j+1}^{L+1} N_l \end{aligned} \tag{32}$$

Therefore, for each relay r_j, transmission is performed differently in the following two regions

- $t \in \left[i+\sum_{l=1}^{j} N_l, \ldots, i+T-\sum_{l=j+1}^{L+1} N_l\right]$: Until node j can recover the original information symbols, transmit the non-erased symbols received from node relay r_{j-1}. Delaying the beginning of transmission by N_j symbols (compared to the beginning of transmission by r_{j-1}) guarantees that there will be enough available symbols for transmission.
- $t \in \left[i+T-\sum_{l=j+1}^{L+1} N_l + 1, \ldots, i+T-\sum_{l=j+2}^{L+1} N_l\right]$: the k information symbols can be recovered by node j. Thus (in case there were erasures in the link (r_{j-1}, r_j)
  - In case $C_{j,j+1} > C_{j-1,j}$ transmit sub-group of the erased symbols.
  - In case $C_{j,j+1} < C_{j-1,j}$ transmit the erased symbols and additional (independent) linear combinations of the information symbols.

These two regions are depicted in Figure 3. Following proposition 2 we assume all symbols used by all codes are taken from block code $\min_j C_{j,j+1}$.

We show examples for the following two cases:

- $C_{j+1,j+2} > C_{j,j+2}$. This means that $n_{j+1,j+2} < n_{j,j+1}$, i.e., that the block size of the MDS code used by relay r_{j+1} is smaller than the block size used by relay r_j. At time $i+T-\sum_{j+1}^{L+1} N_l + 1$, node $j+1$ can recover the original data and send any of the erased symbols of the code used by r_j.
  An example is given in Table IV for $N_{j+1}=2,\ N_{j+2}=1,\ T'=4$ (where $T' = T - \sum_{l=1,l\neq j+1,j+2}$). We note that in this example $k=2$ and indeed, at $i+N_1^j+3$ the relay can recover the original data.

$i+N_1^j$	$i+N_1^j+1$	$i+N_1^j+2$	$i+N_1^j+3$	$i+N_1^j+4$
Link (r_j, r_{j+1})				
a_i				
	b_{i+1}			
		$f^1(a_i, b_{i+1})$		
			$f^2(a_i, b_{i+1})$	
Link (r_{j+1}, r_{j+2})				
		b_{i+1}		
			a_i	
				$f^1(a_i, b_{i+1})$

TABLE IV: Example of increasing the rate between links. In this example, $N_{j+1}=2,\ N_{j+2}=1,\ T'=4$, hence $C_{j+1,j+2} = 2/4 < 2/3 = C_{j,j+1}$. Assuming symbol $i+N_1^j$ and $i+N_1^j+2$ were erased when transmitted in link (r_j, r_{j+1}).

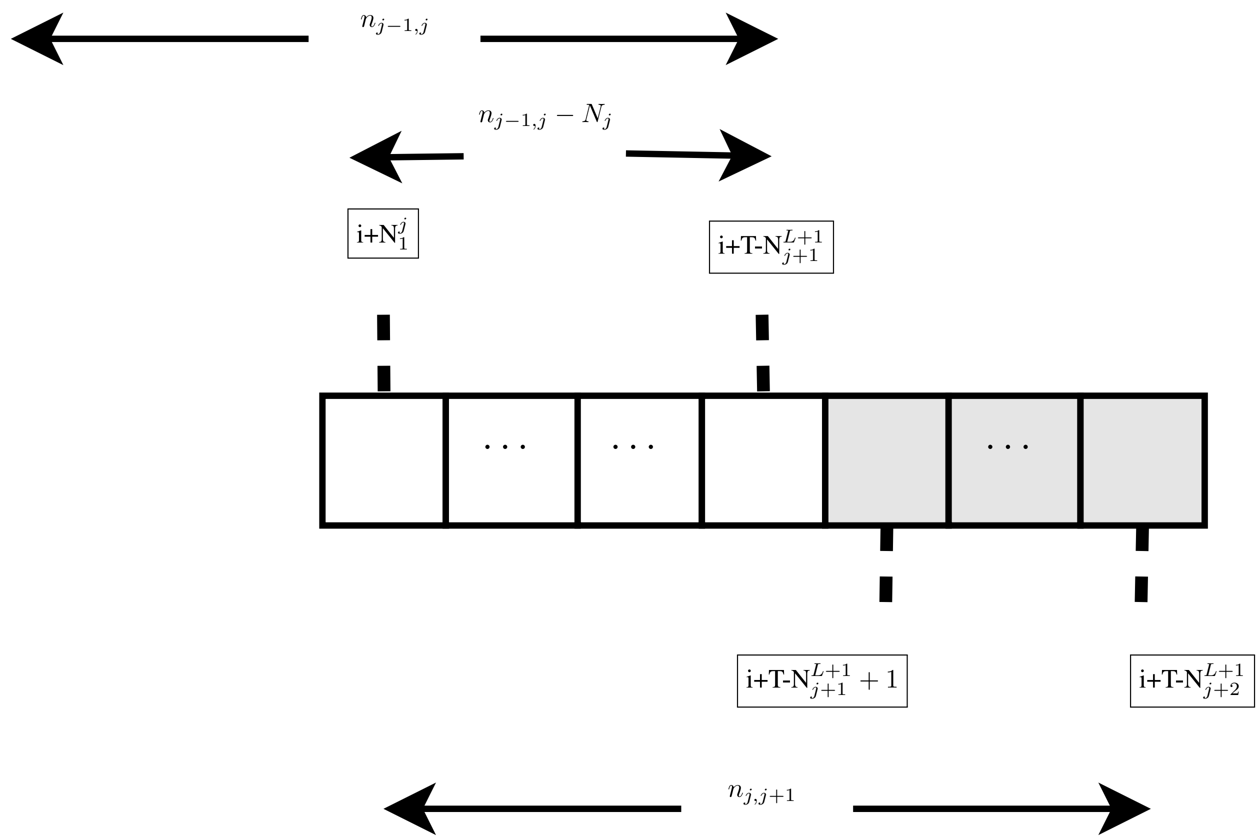

Fig. 3: Two regions of transmission in link (r_j, r_{j+1}). The symbols with white background are the symbols received from in link (r_{j-1}, r_j). The shaded symbols are transmitted after the k information symbols are decoded hence they are either symbols erased in link (r_{j-1}, r_j) or additional (independent) linear combinations of the information symbols.

- $C_{j+1,j+2} < C_{j,j+1}$. This means that $n_{j+1,j+2} > n_{j,j+1}$, i.e., that the block size of the code used by relay r_{j+1} is larger than the block size used by relay r_j. At time $i + T - \sum_{j+1}^{L+1} N_l + 1$, relay r_{j+1} can again recover the original data and hence transmit additional $n_{j+1,j+2} - k$ symbols needed to allow handling any N_{j+2} erasures in the link (r_{j+1}, r_{j+2}). An example is given in Table V for $N_{j+1} = 1,\ N_{j+2} = 2,\ T' = 4$ (where $T' = T - \sum_{l=1, l\neq j+1, j+2}$). We note that in this example at time $i + N_1^j + 2$ relay r_{j+1} can recover the original data, hence it can send any of the erased symbols of the code used by r_j. Further, since $n_{j+1,j+2} > n_{j,j+1}$, it can add (independent) linear combinations of the information symbols as required.

$i+N_1^j$	$i+N_1^j+1$	$i+N_1^j+2$	$i+N_1^j+3$	$i+N_1^j+4$
Link (r_j, r_{j+1})				
a_i				
	b_{i+1}			
		$f^1(a_i, b_{i+1})$		
Link (r_{j+1}, r_{j+2})				
	b_{i+1}			
		a_i		
			$f^1(a_i, b_{i+1})$	
				$f^2(a_i, b_{i+1})$

TABLE V: Example of reducing rate between nodes. In this example, $N_{j+1} = 1,\ N_{j+2} = 2,\ T' = 4$, hence $C_{j,j+1} = 2/3 > 2/4 = C_{j+1,j_2}$. Assuming symbol $i + N_1^j$ was erased when transmitted in link (r_j, r_{j+1}).

With respect to meeting the delay constraint, we note that using the suggested construction, relay r_j starts its transmission at time $i + \sum_{l=1}^{j} N_l$. This means that the final relay r_L starts it transmission at $i + \sum_{l=1}^{L} N_l$. Following (7), we note that the block size of this code is $T - \sum_{l=1}^{L} N_l$, hence the packet can be decoded (assuming any N_{L+1} erasures in the link (r_L, r_{L+1}) at time $i + T$ hence meeting the overall delay constraint.

The assumption that each node can decode the information sent from the previous node holds since the order of the symbols in the code is sent in a header. Therefore we need to analyze the overhead of this header. From Proposition 2 it follows all codes can be viewed as sub codes of $\min_j C_{j,j+1}$. The block size of this code is $n_{\max}$ hence, in the worst case, the header should consist of $n_{\max}$ elements, each one is chosen from $[1, \ldots, n_{\max}]$ (where repetitions are allowed per packet). Therefore the size of the header is upper bounded by $n_{\max} \log(n_{\max})$.

To conclude, each node transmits $n_{j,j+1}$ coded symbols (each taken from field $\mathbb{F}$) along with $O_{j,j+1}$ bits of header to transfer k information symbols (each taken from field $\mathbb{F}$). The overall rate is

$$\begin{aligned} R &= \min_j R_j \\ &= \min_j \frac{k \cdot |\mathbb{F}|}{n_{j,j+1} \cdot |\mathbb{F}| + n_{j,j+1} \log(n_{j,j+1})} \\ &= \min_j \frac{k}{n_{j,j+1} \left(1 + \frac{\log(n_{j,j+1})}{|\mathbb{F}|}\right)} \\ &= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{\max_j \left\{ (T_{j,j+1} + 1) \left(1 + \frac{\log((T_{j,j+1}+1))}{|\mathbb{F}|}\right) \right\}} \end{aligned} \tag{33}$$

where $T_{j,j+1}$ is defined in (8). $\square$

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