Streaming Erasure Codes over the Any-Node Relay Network

Elad Domanovitz and Ashish Khisti

Abstract

A typical connection over the internet is composed from several hops. When considering transmission of a sequence of messages (streaming messages) through packet erasure channel over a three node network it has been shown that taking into account the erasure pattern of each segment can result with an improved performance compared to treating the channel as a point-to-point link. Further, it was shown that a universal capacity scheme exists (in the sense that it does not depend on specific erasure patterns). Since rarely there is only a single relay between the sender and the destination, it calls for trying to extend this scheme to more than a single relay. In this paper we first extend the upper bound on the rate of transmission of a sequence of messages for any number of relays. We further suggest an achievable adaptive scheme which is shown to achieve the upper bound up to the size of an additional header that is required to allow each receiver to meet the delay constraints.

I. Introduction

Real-time video streaming is one of the fastest growing types of internet traffic. Traditionally, most of the traffic in the internet is not extremely sensitive to latency. However, as networks evolved, more and more people are using the network for real-time conversations, video conferencing and on-line monitoring. According to [1], IP video traffic will account for 82 percent of traffic by 2022. Further, live video will grow 15-fold to reach 17 percent of Internet video traffic by 2022.

The fundamental difference between real-time video streaming and other services is the (much more stringiest) latency requirement each packet has to meet, in order to provide good user experience. In multiple streaming codes works [2]–[6] using automatic repeat request (ARQ) for handling errors in transmission was discussed and showed that it may not be adequate for streaming applications. Using ARQ means that the latency (in case of an error) is at least three times the round-trip delay which in many cases may violate the latency requirements.

The alternative method for handling error in transmission is forward error correction (FEC). It was demonstrated that using FEC, lower latency can be achieved.

In [7], streaming codes for three-node networks were analyzed. For this type of networks, explicit expression for capacity was derived. The concept of "symbol-wise" decode and forward introduced in [7] was shown to outperform any "message-wise" decode and forward strategy. Further, analyzing the resulting capacity for the three-node network shows that when constraints are imposed per segment rather than globally (while meeting the same global requirements), it outperforms amplify and forward strategy. As we demonstrate next, treat the network as a single hop link with $N = N_1 + N_2$ erasures and total delay constraint of T symbols is worst than analyzing a three-node network where N_1 erasures are expected in the first segment and N_2 are expected in the second segment with a total delay of T symbols.

However, closer inspection of any link over which real-time video (or any kind of traffic) is sent, shows that it is almost never consists of a single hop (i.e. direct connection). In 2001, in [8], the average number of hops was indicated as greater than 14. This was validated 10 years later in [9]. Hence, designing a streaming code more carefully when possible (i.e., take into account the error behavior of each link rather than aggregate some of the links) is expected to result in an improved performance.

The scheme suggested in [7] is a universal scheme, i.e., it does not depend on the location of erasures. Unfortunately, there is no straight-forward extension to this scheme to a more general case (a network with more than three nodes).

In this paper, we first extend the upper bound derived in [7] to the general case of any-node relay network. We than describe an adaptive scheme for the general L relay scenario and show it achieves the upper bound up to an additional overhead which is the size of a header which is required by this scheme to be added to each packet. We further show that the size of the header is a function of the required delay and the erasure pattern (hence it does not depend on the field size used by the code), therefore, the gap from the upper bound decreases as the field size increases.

II. Channel Model

Let k be a non-negative integer, and $n_1, n_2, \ldots, n_{L+1}$ be $L+1$ natural numbers. Node s wants to send a sequence of messages $\{\mathbf{x}_i\}_{i=0}^{\infty}$ to node d with the help of L middle nodes $r_1, \ldots, r_L$. To ease notation we denote the source node s as r_0, and destination node d as r_{L+1}.

E. Domanovitz and A. Khisti are with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (email: silas.fong, akhisti@utoronto.ca

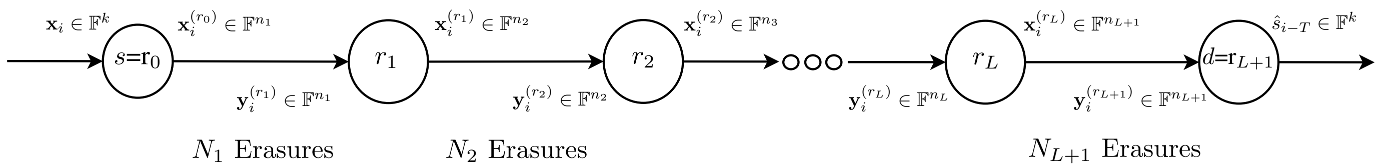

Fig. 1: Symbols generated in the L-node relay network at time i.

Each $\mathbf{x}_i$ is an element in $\mathbb{F}^k$ where $\mathbb{F}$ is some finite field. In each time slot $i \in \mathbb{Z}_+$, the source message $\mathbf{x}_i$ is encoded into a length-n_1 packet $\mathbf{x}_i^{(r_0)} \in \mathbb{F}^{n_1}$ to be transmitted to the first relay through the erasure channel (r_0, r_1). The relay receives $\mathbf{y}_i^{(r_1)} \in \mathbb{F}^{n_1} \cup \{*\}$ where $\mathbf{y}_i^{(r_1)}$ equals either $\mathbf{x}_i^{(r_0)}$ or the erasure symbol '$*'$. slot, relay r_1 transmits $\mathbf{x}_i^{(r_1)} \in \mathbb{F}^{n_2}$ to relay r_2 through the erasure channel (r_1, r_2). Relay r_2 receives $\mathbf{y}_i^{(r_2)} \in \mathbb{F}^{n_2} \cup \{*\}$ where $\mathbf{y}_i^{(r_2)}$ equals either $\mathbf{x}_i^{(r_1)}$ or the erasure symbol '$*'$. The same process continues (in the same time slot) until relay r_L transmits $\mathbf{x}_i^{(r_L)} \in \mathbb{F}^{n_{L+1}}$ to the destination r_{L+1} through the erasure channel (r_L, r_{L+1}). Destination r_{L+1} receives $\mathbf{y}_i^{(r_{L+1})} \in \mathbb{F}^{n_{L+1}} \cup \{*\}$ where $\mathbf{y}_i^{(r_{L+1})}$ equals either $\mathbf{x}_i^{(r_L)}$ or the erasure symbol '$*'$.

The fraction $\frac{k}{\max(n_1,\ldots,n_{L+1})}$ specifies the rate of the code. Every code is subject to a delay constraint of T time slots, meaning that the destination must produce an estimate of $\mathbf{x}_i$, denoted by $\hat{s}_i$, upon receiving $\mathbf{y}_i^{(r_{L+1})} + T$.

We assume that on the discrete timeline, each channel (r_{j-1}, r_j) introduces up to N_j arbitrary erasures in a window of $T+1$ symbols respectively. The symbols generated in the L-node relay network at time i are illustrated in Figure 1.

If $N_{l \neq j} = 0$, then the L-node relay network with erasures reduces to a point-to-point packet erasure channel. It was previously shown [3] that the maximum achievable rate of the point-to-point packet erasure channel with $N_j = N$ arbitrary erasures and delay of T denoted by $C_{T,N}$ satisfies

$$C_{T,N} = \begin{cases} \frac{T-N+1}{T+1} & T \geq N \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Further, it was shown that the capacity of point-to-point channel with N arbitrary erasures and delay of T can be achieved by diagonally interleaving $(T+1, T-N+1)$ systematic MDS code.

We recall that for any natural numbers L and N, a systematic maximum-distance separable (MDS) $(L+N, L)$-code is characterized by an $L \times N$ parity matrix $\mathbf{V}^{L\times N}$ where any L columns of $\begin{bmatrix}\mathbf{I}_L & \mathbf{V}^{L\times N}\end{bmatrix} \in \mathbb{F}^{L\times(L+N)}$ are independent. It is well known that a systematic MDS $(L+N, L)$-code always exists as long as $|\mathbb{F}| \geq L+N$ [10].

We denote the (independent) linear combination applied on L symbols $a_1, a_2, \ldots, a_L$ as $f^{(w)}(a_1, a_2, \ldots, a_k)$ where $w \in [1, \ldots, N]$.

In [7], a three node relay network was analyzed, and the following theorem was shown.

Theorem 1 (Theorem 1 in [7])**.** *Fix any (T, N_1, N_2). Recalling that the point-to-point capacity satisfies (1), we have*

$$C_{T,N_1,N_2} = \min\left(C_{T-N_2,N_1}, C_{T-N_1,N_2}\right). \tag{2}$$

In particular, for any $\mathbb{F}$ with $|\mathbb{F}| \geq T+1$, there exists an (N_1, N_2)-achievable $(n, m, k, T)_{\mathbb{F}}$-streaming code with $k = T - N_1 - N_2 + 1$, $n = T - N_2 + 1$, $m = T - N_1 + 1$ and rate

$$\frac{k}{\max(n,m)} = C_{T,N_1,N_2}. \tag{3}$$

To simplify notation, we sometimes denote $N_a^b = \sum_{l=a}^{b} N_l$. We will take all logarithms to base 2 throughout this paper.

III. Standard Definitions and Main Result

Definition 1. *An $(n_1, n_2, \ldots, n_{L+1}, k, T)_{\mathbb{F}}$-streaming code consists of the following:*

1) *A sequence of source messages $\{\mathbf{s}_i\}_{i=0}^{\infty}$ where $\mathbf{s}_i \in \mathbb{F}^k$.*
2) *An encoding function $f_i^{(r_0)} : \underbrace{\mathbb{F}^k \times \ldots \times \mathbb{F}^k}_{i+1 \text{ times}} \to \mathbb{F}^{n_1}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_0)}$ is used by node r_0 at time i to encode $\mathbf{s}_i$ according to $\mathbf{x}_i^{(r_0)} = f_i^{(r_0)}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_i)$.*
3) *A relaying function for node $j \in [1, \ldots, L]$, $f_i^{(r_j)} : \underbrace{\mathbb{F}^{n_j} \cup \{*\} \times \ldots \times \mathbb{F}^{n_j} \cup \{*\}}_{i+1 \text{ times}} \to \mathbb{F}^{n_{j+1}}$ for each $i \in \mathbb{Z}_+$, where $f_i^{(r_j)}$ is used by node r_j at time i to construct $\mathbf{x}_i^{(r_j)} = f_i^{(r_j)}(\mathbf{y}_0^{(r_j)}, \mathbf{y}_1^{(r_j)}, \ldots, \mathbf{y}_i^{(r_j)})$.*

4) *A decoding function* $\phi_{i+T} : \underbrace{\mathbb{F}^{n_L} \cup \{*\} \times \ldots \times \mathbb{F}^{n_L} \cup \{*\}}_{i+T+1 \text{ times}} \rightarrow \mathbb{F}^{n_{L+1}}$ *for each* $i \in \mathbb{Z}_+$ *is used by node* n_{L+1} *at time* $i+T$ *to estimate* $\mathbf{s_i}$ *according to* $\hat{\mathbf{s}}_i = \phi_{i+T}(\mathbf{y}_0^{(r_{L+1})}, \mathbf{y}_1^{(r_{L+1})}, \ldots, \mathbf{y}_{i+T}^{(r_{L+1})})$.

Definition 2. *An erasure sequence is a binary sequence denoted by* $e^\infty \triangleq \{e_i\}_{i=0}^\infty$ *where* $e_i = \mathbf{1}\{\text{erasure occurs at time i}\}$.

An N-erasure sequence is an erasure sequence e^∞ that satisfies $\sum_{l=0}^\infty e_l = N$. In other words, an N-erasure sequence introduces N arbitrary erasures on the discrete timeline. The set of N-erasure sequences is denoted by Ω_N.

Definition 3. *The mapping* $g_{n_j} : \mathbb{F}^{n_j} \times \{0, 1\} \rightarrow \mathbb{F}^{n_j} \cup \{*\}$ *of an erasure channel is defined as*

$$g_{n_j}(\mathbf{x}^{r_j}, e_i) = \begin{cases} \mathbf{x} & \text{if } e = 0, \\ * & \text{if } e = 1. \end{cases} \tag{4}$$

For any erasure sequence e^∞ *and any* $(n_1, \ldots, n_{L+1}, k, T)_\mathbb{F}$*-streaming code, the following input-output relations holds for the erasure channel* (r_j, r_{j+1}) *for each* $i \in \mathbb{Z}_+$ $\mathbf{y}_i^{(r_{j+1})} = g_{n_j}(\mathbf{x}_i^{(r_j)}, e_i)$.

Definition 4. *An* $(n_1, \ldots, n_{L+1}, k, T)_\mathbb{F}$*-streaming code is said to be* $(N_1, N_2, \ldots, N_{L+1})$*-achievable if the following holds for any* N_j*-erasure sequence* $e^\infty \in \Omega_{N_j}$*, for all* $i \in \mathbb{Z}_+$ *and all* $\mathbf{s}_i \in \mathbb{F}^k$*, we have* $\hat{\mathbf{s}}_i = \mathbf{s}_i$ *where*

$$\hat{\mathbf{s}}_i = \phi_i(g_{n_{L+1}}(\mathbf{x}_0^{(r_L)}, e_0), \ldots, g_{n_{L+1}}(\mathbf{x}_{i+T}^{(r_L)}, e_{i+T})) \tag{5}$$

and for previous nodes

$$\mathbf{y}_i^{(r_{j+1})} = f_i(g_{n_j}(\mathbf{x}_0^{(r_j)}, e_0), \ldots, g_{n_j}(\mathbf{x}_i^{(r_j)}, e_i)). \tag{6}$$

Definition 5. *The rate of an* $(n_1, n_2, \ldots, n_{j+1}, k, T)_\mathbb{F}$*-streaming code is* $\frac{k}{\max\{n_1, n_2, \ldots, n_{j+1}\}}$.

In this paper we first derive an upper bound for the achievable rate in L relay network.

Theorem 2. *Assume a link with* L *relays (where* L *is fixed). For a target overall delay of* T*, where the maximal number of arbitrary erasures in any window of* $T+1$ *symbols in link* $(r_{j-1}, r_j),\ j \in [1, \ldots, L+1]$ *is* N_j *and* $T \geq \sum_{l=1}^{L+1} N_l$*.*

The achievable rate is upper bounded by

$$\begin{aligned} R &\leq C_{T, N_1, \ldots, N_{L+1}} \\ &\triangleq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \min_j \sum_{l=1, l \neq j}^{L+1} N_l + 1}. \end{aligned} \tag{7}$$

We then suggest an achievable scheme which achieves the upper bound up to a size of an overhead which is required by the scheme. The rate used in link (r_j, r_{j+1}) can be expressed as

$$R_{j,j+1} \triangleq \frac{(T - \sum_{l=1}^{L+1} N_l + 1)|\mathbb{F}|}{(T - \sum_{l=1, l \neq j+1}^{L+1} N_l + 1)|\mathbb{F}| + O_{j,j+1}} \tag{8}$$

where $O_{j,j+1}$ is the size of the header required by node j. We denote with $T_{j,j+1}$ the delay of the code used by node j where

$$T_{j,j+1} \triangleq T - \sum_{l=1, l \neq j+1}^{L+1} N_l, \tag{9}$$

and with $C_{j,j+1}$ the maximal attainable rate of link (r_j, r_{j+1}) as

$$\begin{aligned} C_{j,j+1} &\triangleq C_{T - \sum_{l=1, l \neq j+1}^{L+1} N_l, N_j} \\ &= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=1, l \neq j+1}^{L+1} N_l + 1} \\ &= \frac{T_{j,j+1} - N_{j+1} + 1}{T_{j,j+1} + 1} \\ &\triangleq \frac{k}{n_{j,j+1}}. \end{aligned} \tag{10}$$

We show next that

$$O_{j,j+1} \leq n_{j,j+1} \log(n_{j,j+1}). \tag{11}$$

Denoting

$$
\begin{aligned}
N_{\max} &= \max_j N_j \\
n_{\max} &= \max_j n_{j,j+1} \\
&= \max_j T_{j,j+1} + 1,
\end{aligned} \tag{12}
$$

we show the following Theorem.

Theorem 3. *Assume a link with L relays (where L is fixed). For a target overall delay of T, where the maximal number of arbitrary erasures in any window of $T+1$ symbols in link $(r_j, r_{j+1},\ j \in [0, \dots, L])$ is N_{j+1} and $T \geq \sum_{l=1}^{L+1} N_l$.*
When $|\mathbb{F}| \geq T+1$, The following rate is achievable

$$
\begin{aligned}
R &= \min_j R_{j,j+1} \\
&= \frac{T - \sum_{l=1}^{L+1} N_l + 1}{\max_j \left\{ (T_{j,j+1}+1) \cdot \left(1 + \frac{\log(T_{j,j+1}+1)}{|\mathbb{F}|}\right) \right\}}
\end{aligned} \tag{13}
$$

where $T_{j,j+1}$ is defined in (9).

We further show that the gap from the upper bound vanishes as the field size increases.

IV. MOTIVATING EXAMPLE

Consider a link with up to $N = 2$ arbitrary erasure in any window of $T+1$ symbols, where the delay constraint the transmission has to meet is $T = 3$ symbols. The capacity of this link (1) is

$$C_{P2P} = 2/4. \tag{14}$$

Now, assume the in fact, this link is a three-node network ($L = 1$), where up to $N_1 = 1$ erasures in the link (r_0, r_1) and up to $N_2 = 1$ erasures in the link (r_1, r_2) in any window of $T+1$ symbols, where transmission has to be decoded with the same delay of $T = 3$ symbols. The capacity of this link (2) is

$$C_{3,1,1} = 2/3, \tag{15}$$

which is better than the point-to-point link.

The achievable scheme presented in [7] is shown to achieve this rate universally (i.e. it does not adapt the transmission at the relay based on the location of the erasure). This is achieved by cleverly analyze the delay profile of point-to-point streaming codes per link at rate which equals the upper bound per link and apply a proper permutation on the order of symbols in the second link.

We now show that using point-to-point streaming code per link at a rate which equals the upper bound per link can be used in a different way. We suggest an adaptive scheme, i.e., a scheme in which the order of symbols is defined based on the erasure pattern in the previous link. To allow decoding, the order of the symbols has to be transmitted to the receiver, hence, resulting with an additional overhead. We first show an example of the suggested scheme to the three-node network and show it achieves capacity up to the overhead of the header. Than we show how to extend it to four-node network.

In the proposed scheme, the source r_0 uses the same code suggested in [7], i.e, C_{T-N_2,N_1}, a streaming code that can recover from a single erasure in delay of two symbols is applied on the diagonals. This is depicted in Table I.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	123	123	123	123
a_i	a_{i-1}	a_i	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}
b_i	b_{i-1}	b_i	b_{i+1}	b_{i+2}	b_{i+3}	b_{i+4}
$a_{i-2}+$ b_{i-1}	$a_{i-3}+$ b_{i-2}	$a_{i-2}+$ b_{i-1}	$a_{i-1}+$ b_i	a_i+ b_{i+1}	$a_{i+1}+$ b_{i+2}	$a_{i+2}+$ b_{i+3}

TABLE I: Symbols transmitted by the source in link (r_0, r_1) from time $i-1$ to time $i+4$.

The relay r_1 uses the same transmission scheme as the source r_0 *while delaying it by one symbol*. When there are no erasure, it transmits identically to r_0 (shifted by one symbol). As we show, in case an erasure occurs in link (r_0, r_1), this delay guarantees that for each diagonal, at least one symbol from the code is available at the relay. This means that the same code can be used, albeit, the order of the symbols is not fixed (and in fact is determined by the actual pattern of erasures). Since we assume the order of symbols is also transmitted in the header, the receiver can recover the information symbols from any single erasure while meeting its delay constraint.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	223	113	123	123
	a_{i-2}	a_{i-1}	$\mathbf{b_{i+1}}$	a_{i+1}	a_{i+2}	a_{i+3}
	b_{i-2}	b_{i-1}	b_i	$\mathbf{a_i}$	b_{i+2}	b_{i+3}
	$a_{i-4}+ b_{i-3}$	$a_{i-3}+ b_{i-2}$	$a_{i-2}+ b_{i-1}$	$a_{i-1}+ b_i$	$a_i+ b_{i+1}$	$a_{i+1}+ b_{i+2}$

TABLE II: Symbols transmitted by relay r_1 in (r_1, r_2) from time $i-1$ to time $i+4$, given that symbol i was erased when transmitted in link (r_0, r_1).

For example, assuming that a symbol at time i is erased when transmitted from the sender to relay r_1 (the highlighted packet in Table I), the suggested transmission scheme of relay r_1 is given in Table II below.

We note that the erasure in time i in link (r_0, r_1) caused a change in order if the symbols in packets $i+1$ and $i+2$. At time $i+1$, for example, the relay can send b_{i+1} from the code applied on $[a_i, b_{i+1}]$. At time $i+2$, since the code applied by the source s on each diagonal is designed to allow recovery from any single erasure in delay of two symbols, the relay can recover the information symbols and send a_i from the code applied on $[a_i, b_{i+1}]$. Again, each change in order is reflected in the header of these packets (to allow the receiver to decode it properly).

To show that packet $\mathbf{x}_i$ (which is composed from a_i and b_i) can be recovered at time $i+3$ (at the latest), we look at the two relevant potential erasures (which affect this packet) in the link (r_1, r_2). If the symbol at time $i+1$ is erased, both symbols a_i and b_i can be recovered at time $i+2$. Further, If symbol $i+2$ is erased, symbol b_i will be recovered at time $i+1$ and symbol a_i will be recovered at time $i+3$ hence meeting the delay constraint.

The true merit of the suggested scheme is revealed when another relay, r_2, is considered (i.e., $L=2$). As we show next, in this case, while assuming maximum of one erasure in each of the links (i.e., $N_1 = N_2 = N_3 = 1$) in any window of $T+1$ symbols and in case the maximal delay is $T=4$ symbols, the achievable rate is upper bounded by

$$\begin{aligned} R &\leq C_{4,1,1,1} \\ &= 2/3. \end{aligned} \tag{16}$$

We describe next that how the suggested scheme achieves this rate up to an additional overhead required by the header.

The transmission scheme on the next relay (in this specific example) is the same as the transmission scheme of the first relay, i.e., in case there is no erasure, transmit (on the diagonal) the symbols in the same order as received, delayed by one symbol (i.e., total delay of two symbols from the sender). If an erasure occurred, it is again guaranteed (per diagonal) that some of the symbols are already available at r_2, therefore transmit them and update the header such that the receiver will know the modified order.

For example, the symbol transmitted from relay r_1 to r_2 at time $i+2$ is erased (the highlighted packet in Table II), the suggested transmission scheme of relay r_2 in given in Table III below.

Time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
Header	123	123	123	223	213	113
	a_{i-3}	a_{i-2}	a_{i-1}	b_{i+1}	b_{i+2}	a_{i+2}
	b_{i-3}	b_{i-2}	b_{i-1}	b_i	a_i	a_{i+1}
	$a_{i-5}+ b_{i-4}$	$a_{i-4}+ b_{i-3}$	$a_{i-3}+ b_{i-2}$	$a_{i-2}+ b_{i-1}$	a_{i-1} b_i	$a_i+ b_{i+1}$

TABLE III: Symbols transmitted by relay r_2 in (r_2, r_3) from time $i-1$ to time $i+4$, given that symbol $i+2$ was erased when transmitted in link (r_1, r_2).

We note that the erasure in time $i+2$ in link (r_1, r_2) caused a change in order of the symbols in packets $i+3$ and $i+4$ (the order of symbols in time $i+2$ is not the original order, yet it is the same comparing the transmission at time $i+1$ from r_1 and from r_2).

To show that packet $\mathbf{x}_i$ (which is composed from a_i and b_i) can be recovered at time $i+4$ (at the latest), we look at the two relevant potential erasures (which affect this packet) in link (r_2, r_3). If the symbol at time $i+2$ is erased, both symbols a_i and b_i can be recovered at time $i+3$. Further, If symbol $i+3$ is erased, symbol b_i will be recovered at time $i+2$ and symbol a_i will be recovered at time $i+4$ hence meeting the delay constraint.

In this example, the maximal size of the header is three numbers, each taken from $\{1,2,3\}$. Hence, its maximal size is $3\log(3)$ bits. Since the block code used in each link transmits two information symbols (taken from $\mathbb{F}$) using three symbols (again, taken from $\mathbb{F}$) in every channel use, we conclude that the scheme achieves rate of

$$\begin{aligned} R &= \frac{2 \cdot |\mathbb{F}|}{3 \cdot |\mathbb{F}| + 3\log(3)} \\ &= \frac{2}{3\left(1 + \frac{\log(3)}{|\mathbb{F}|}\right)}. \end{aligned} \tag{17}$$

V. Proof of upper bound

Fix any $(N_1,\ldots,N_{L+1},T)$. Suppose we are given an $(N_1,\ldots,N_{L+1})$-achievable $(n_1,\ldots,n_{j+1},k,T)_{\mathbb{F}}$-streaming code for some $n_1,\ldots,n_{j+1},k$ and $\mathbb{F}$. Our goal is to show that

$$\frac{k}{\max\{n_1,\ldots,n_{L+1}\}} \le \min_j \frac{T-\sum_{l=1}^{L+1} N_l+1}{T-\sum_{l=1,l\neq j}^{L+1} N_l+1}. \tag{18}$$

To this end, we let $\{\mathbf{s}_i\}_{i\in\mathbb{Z}_+}$ be i.i.d. random variables where $\mathbf{s}_0$ is uniform on $\mathbb{F}^k$. Since the $(n_1,\ldots,n_{j+1},k,T)_{\mathbb{F}}$-streaming code is $(N_1,\ldots,N_{L+1})$-achievable, it follows from Definition 4 that

$$H\left(\mathbf{s}_i \,\middle|\, \mathbf{y}_0^{(L+1)},\mathbf{y}_1^{(L+1)},\ldots,\mathbf{y}_{i+T}^{(L+1)}\right) \tag{19}$$

for any $i\in\mathbb{Z}_+$ and any $e_j\in\Omega_{N_j}$. Consider the two cases.

Case $T<\sum_{l=1}^{L+1} N_l$:

Let $e_j\in\Omega_{N_j}$ such that

$$e_j=\begin{cases}1 & if\sum_{l=1}^{j-1}\le\sum_{l=1}^{j}\\ 0 & \text{otherwise.}\end{cases} \tag{20}$$

Due to (20) and Definition 1, we have

$$I\left(\mathbf{s}_0;\mathbf{y}_0^{(L+1)},\mathbf{y}_1^{(L+1)},\ldots,\mathbf{y}_T^{(L+1)}\right)=0. \tag{21}$$

Combining (19), (21) and the assumption that $T<\sum_{l=1}^{L+1} N_l$, we obtain $H(\mathbf{s}_0)$. Since $\mathbf{s}_0$ consists of k uniform random variables in $\mathbb{F}$, the only possible value of k is zeros, which implies

$$\frac{k}{\max\{n_1,\ldots,n_{L+1}\}}=0. \tag{22}$$

Case $T\ge\sum_{l=1}^{L+1} N_l$:

The proof follows that footsteps of [7]. We start by generalizing the arguments given in [7] for the first and second segments to the first and last segments in our case. Than we show how similar technique can be used to derive a constraint on the code to be used in an intermediate segment.

First Segment (link (r_0,r_1)):

First we note that for every $i\in\mathbb{Z}_+$, message $\mathbf{x}_i$ has to be perfectly recovered by node r_1 by time $i+T-\sum_{l=2}^{L+1}N_l$ given that $\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{i-1}$ have been correctly decoded by node r_1, or otherwise a length N_2 burst erasure from time $i+T-\sum_{l=2}^{L+1}N_l+1$ to $i+T-\sum_{l=3}^{L+1}N_l$ introduced on channel (r_1,r_2) followed by a length N_3 burst erasure from time $i+T-\sum_{l=3}^{L+1}N_l+1$ to $i+T-\sum_{l=4}^{L+1}N_l$ introduced on channel (r_2,r_3) and so on until a length N_{L+1} burst erasure from time $i+T-N_{L+1}+1$ to $i+T$ would result in a decoding failure for node r_1, node r_2 and all the nodes up to the destination r_{L+1}.

Denoting $\tilde{N}_2=\sum_2^{L+1}N_l$ it then follows that

$$H\left(\mathbf{s}_i \,\middle|\, \left\{\mathbf{x}_i^{(r_0)},\mathbf{x}_{i+1}^{(r_0)},\ldots,\mathbf{x}_{i+T-\tilde{N}_2}^{(r_0)}\right\}\setminus\left\{\mathbf{x}_{\theta_1},\ldots,\mathbf{x}_{\theta_{N_1}}\right\},\mathbf{s}_0,\ldots,\mathbf{s}_{i-1}\right)=0 \tag{23}$$

for any $\in\mathbb{Z}_+$ and N_1 non-negative integers denoted by $\theta_1,\ldots,\theta_{N_1}$. We note that the following holds (by assuming, for example, the last N_1 symbols in every window of $T-\tilde{N}_2$ symbols starting time $i=0$ are erased):

$$H\left(\mathbf{s}_0 \,\middle|\, \left\{\mathbf{x}_0^{(r_0)},\mathbf{x}_1^{(r_0)},\ldots,\mathbf{x}_{T-\tilde{N}_2-N_1}^{(r_0)}\right\}\right)=0$$

$$H\left(\mathbf{s}_1 \,\middle|\, \left\{\mathbf{x}_1^{(r_0)},\mathbf{x}_2^{(r_0)},\ldots,\mathbf{x}_{T-\tilde{N}_2-N_1}^{(r_0)},\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)}\right\},\mathbf{s}_0\right)=0$$

$$\vdots$$

$$H\left(\mathbf{s}_{T-\tilde{N}_2-N_1} \,\middle|\, \left\{\mathbf{x}_{T-\tilde{N}_2-N_1}^{(r_0)},\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)},\mathbf{x}_{1(T-\tilde{N}_2+1)+1}^{(r_0)},\ldots,\mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1-1}^{(r_0)}\right\},\mathbf{s}_0,\ldots,\mathbf{s}_{T-\tilde{N}_2-N_1-1}\right)=0$$

$$H\left(\mathbf{s}_{T-\tilde{N}_2-N_1+1} \,\middle|\, \left\{\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)},\ldots,\mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}^{(r_0)}\right\},\mathbf{s}_0,\ldots,\mathbf{s}_{T-\tilde{N}_2-N_1}\right)=0$$

$$H\left(\mathbf{s}_{T-\tilde{N}_2-N_1+2} \,\middle|\, \left\{\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)},\ldots,\mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}^{(r_0)}\right\},\mathbf{s}_0,\ldots,\mathbf{s}_{T-\tilde{N}_2-N_1+1}\right)=0$$

$$\vdots$$

$$H\left(\mathbf{s}_{T-\tilde{N}_2-1} \,\middle|\, \left\{\mathbf{x}_{1(T-\tilde{N}_2+1)}^{(r_0)},\ldots,\mathbf{x}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}^{(r_0)}\right\},\mathbf{s}_0,\ldots,\mathbf{s}_{T-\tilde{N}_2-2}\right)=0$$

$$H\left(\mathbf{s}_{T-\tilde{N}_2} \;\Big|\; \left\{\mathbf{x}^{(r_0)}_{1(T-\tilde{N}_2+1)}, \ldots, \mathbf{x}^{(r_0)}_{1(T-\tilde{N}_2+1)+T-\tilde{N}_2-N_1}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_2-1}\right) = 0$$
$$\vdots \tag{24}$$

Using the chain rule we have the following for each $j \in \mathbb{N}$

$$H\left(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{T-\tilde{N}_2+(j-1)(T-\tilde{N}_2+1)} \mid \left\{\mathbf{x}^{(r_0)}_{m(T-\tilde{N}_2+1)}, \mathbf{x}^{(r_0)}_{1+m(T-\tilde{N}_2+1)}, \ldots, \mathbf{x}^{(r_0)}_{T-N_1-\tilde{N}_2+m(T-\tilde{N}_2+1)}\right\}_{m=0}^{j}\right) = 0. \tag{25}$$

Alternatively we note that for all $q \in \mathbb{Z}_+$,

$$\begin{aligned} &\Big|\left\{q, q+1, \ldots, T-\tilde{N}_2\right\} \bigcap \\ &\left\{m(T-\tilde{N}_2+1), 1+m(T-\tilde{N}_2+1) \ldots, T-N_1+m(T-\tilde{N}_2+1)\right\}_{m=0}^{j}\Big| \\ &= T - N_1 - \tilde{N}_2 + 1. \end{aligned} \tag{26}$$

Hence, (25) follow from (23), (26) and the chain rule.

Therefore, following the arguments in [7] any $(N_1, \ldots, N_{L+1})$ achievable $(n_1, \ldots n_{L+1}, k, T)_{\mathbb{F}}$ code restricted to channel (r_0, r_1) can be viewed as a point-to-point streaming code with rate k/n_1, delay $T - \sum_{l=2}^{L+1} N_l$ which can correct any N_1 erasures. We therefore have

$$\begin{aligned} \frac{k}{n_1} &\leq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=2}^{L+1} N_l + 1} \\ &= C_{T-\sum_{l=2}^{L+1} N_l, N_1}. \end{aligned} \tag{27}$$

Last Segment (link (r_L, r_{L+1})):

In addition, for every $i \in \mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered from

$$\left\{\mathbf{x}^{(r_{L+1})}_{i+\sum_{l=1}^{L} N_l}, \mathbf{x}^{(r_{L+1})}_{i+\sum_{l=1}^{L} N_l+1}, \ldots, \mathbf{x}^{(r_{L+1})}_{T}\right\} \tag{28}$$

by node r_{L+1} given that $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ have been correctly decoded by node r_{L+1}, or otherwise a length N_1 burst erasure from time i to $i+N_1-1$ induced on channel (r_0, r_1) followed by a length N_2 burst erasure from time $i+N_1$ to $i+N_1+N_2-1$ induced on channel (r_1, r_2) and so on until a length N_L burst erasure from time $i + \sum_{l=1}^{L-1} N_l$ to $i + \sum_{l=1}^{L} N_l - 1$ induced on channel (r_{L-1}, r_L) would result in a decoding failure for node r_{L+1}.

Denoting $\tilde{N}_1 = \sum_1^L N_l$ it then follows that

$$H\left(\mathbf{s}_i \;\Big|\; \left\{\mathbf{x}^{(r_L)}_{i+\tilde{N}_1}, \mathbf{x}^{(r_L)}_{i+\tilde{N}_1+1}, \ldots, \mathbf{x}^{(r_L)}_{i+T}\right\} \setminus \left\{\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_{N_{L+1}}}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{i-1}\right) = 0 \tag{29}$$

for any $\in \mathbb{Z}_+$ and N_{L+1} non-negative integers denoted by $\theta_1, \ldots, \theta_{N_{L+1}}$.

We note that for all $q \in \mathbb{Z}_+$,

$$\begin{aligned} &\Big|\left\{q+\tilde{N}_1, q+\tilde{N}_1+1, \ldots, q+T\right\} \bigcap \\ &\left\{\tilde{N}_1 + N_{L+1} + m(T-\tilde{N}_1+1), \tilde{N}_1 + N_{L+1} + 1 + m(T-\tilde{N}_1+1) \ldots, T + m(T-\tilde{N}_1+1)\right\}_{m=0}^{j}\Big| \\ &= T - \tilde{N}_1 - N_{L+1} + 1 \end{aligned} \tag{30}$$

Using (29), (30) and the chain rule, we have

$$\begin{aligned} &H\left(\mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_1+(j-1)(T-\tilde{N}_1+1)} \mid \left\{\mathbf{x}^{(r_L)}_{\tilde{N}_1+N_{L+1}+m(T-\tilde{N}_1+1)}, \mathbf{x}^{(r_L)}_{\tilde{N}_1+N_{L+1}+1+m(T-\tilde{N}_1+1)}, \ldots, \mathbf{x}^{(r_L)}_{T+m(T-\tilde{N}_1+1)}\right\}_{m=0}^{j}\right) \\ &= 0. \end{aligned} \tag{31}$$

Again, following the arguments in [7] when restricted to the channel (r_L, r_{L+1}), any $(N_1, \ldots, N_{L+1})$ achievable $(n_1, \ldots n_{L+1}, k, T)_{\mathbb{F}}$ cod

$$\frac{k}{n_{L+1}} \leq \frac{T - \sum_{l=1}^{L+1} N_l + 1}{T - \sum_{l=1}^{L} N_l + 1} \quad = C_{T-\sum_{l=1}^{L} N_l, N_{L+1}}.$$

The j'th Segment (link (r_{j-1}, r_j)):

Now, when considering a j'th segment (the channel (r_{j-1}, r_j)), we show again that any $(N_1, \ldots, N_{L+1})$ achievable $(n_1, \ldots n_{L+1}, k, T)_{\mathbb{F}}$ code restricted to the channel (r_{j-1}, r_j) can be viewed as a point-to-point code which should handle any N_J erasures with a delay which we define next.

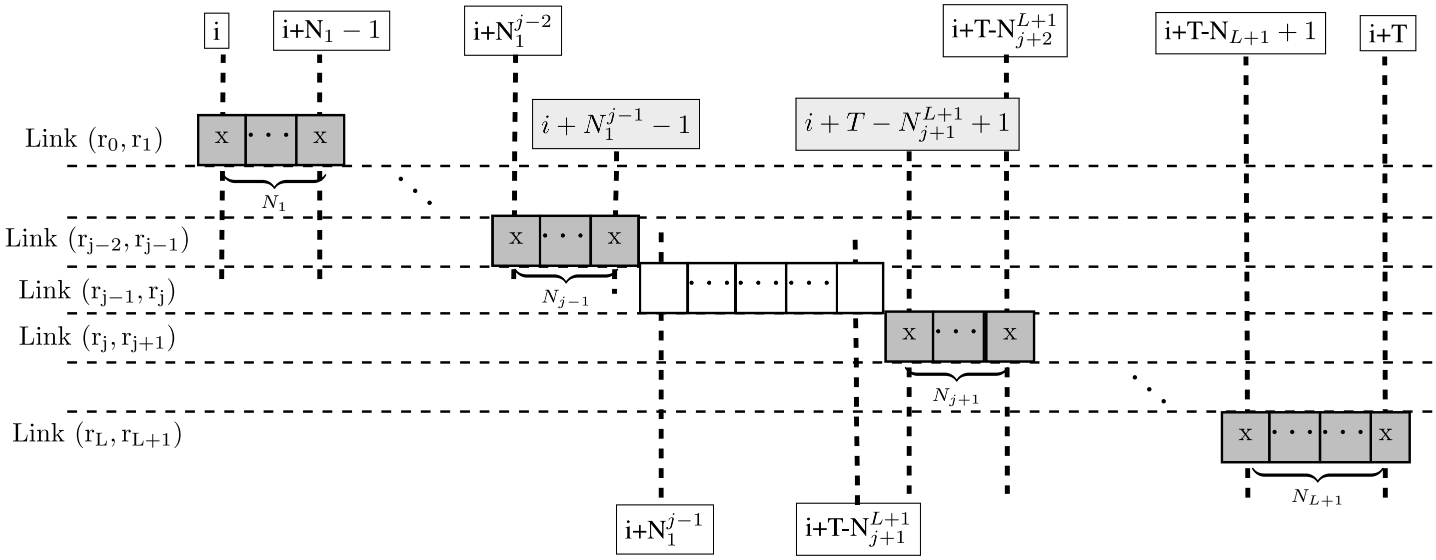

Fig. 2: Constrains imposed on transmission in link (r_{j-1}, r_j).

Combining the arguments given above we note that first, for every $i \in \mathbb{Z}_+$, message $\mathbf{x}_i$ has to be perfectly recovered by node r_j by time $i+T-\sum_{l=j+1}^{L+1} N_l$ given that $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ have been correctly decoded by node r_j, or otherwise a length N_{j+1} burst from time $i+T-\sum_{l=j+1}^{L+1} N_l + 1$ to $i+T-\sum_{l=j+2}^{L+1} N_l$ introduced on channel (r_j, r_{j+1}), followed by a length N_{j+2} burst from time $i+T-\sum_{l=j+2}^{L+1} N_l + 1$ to $i+T-\sum_{l=j+3}^{L+1} N_l$ introduced on channel (r_{j+1}, r_{j+2}) and so on, up to a length N_{L+1} burst from time $i - N_{L+1}+1$ to $i+T$ introduced on channel (r_L, r_{L+1}) would result in a decoding failure for node r_j and all the nodes up to the destination r_{L+1}.

Further, in addition, for every $i \in \mathbb{Z}_+$, message $\mathbf{s}_i$ has to be perfectly recovered from

$$\left(\mathbf{x}^{(r_{j-1})}_{i+\sum_{l=1}^{j-1} N_l}, \mathbf{x}^{(r_{j-1})}_{i+\sum_{l=1}^{j-1} N_l+1}, \ldots, \mathbf{x}^{(r_{j-1})}_{i+T-\sum_{l=j+1}^{L+1} N_l}\right) \tag{32}$$

by node j given that $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{i-1}$ have been correctly decoded by node j, or otherwise a length N_1 burst erasure from time i to $i+N_1-1$ induced on channel (r_0, r_1) followed by a length N_2 burst erasure from time $i+N_1$ to $i+N_1+N_2-1$ induced on channel (r_0, r_2) and so on (up to a burst erasure N_{j-1} from time $i+\sum_{l=1}^{j-2} N_l$ to $i+\sum_{l=1}^{j-1} N_l - 1$ induced on channel (r_{j-2}, r_{j-1})) would result in a decoding failure for node j. These constraints are depicted in Figure 2.

Now denote with $\tilde{N}_{j,1} = \sum_{l=1}^{j-1} N_l$, and with $\tilde{N}_{j,2} = \sum_{l=j+1}^{L+1} N_l$ it follows that

$$H\left(\mathbf{s}_i \,\middle|\, \left\{\mathbf{x}^{(r_{j-1})}_{i+\tilde{N}_{j,1}}, \mathbf{x}^{(r_{j-1})}_{i+\tilde{N}_{j,1}+1}, \ldots, \mathbf{x}^{(r_{j-1})}_{i+T-\tilde{N}_{j,2}}\right\} \setminus \left\{\mathbf{x}_{\theta_1}, \ldots, \mathbf{x}_{\theta_{N_j}}\right\}, \mathbf{s}_0, \ldots, \mathbf{s}_{i-1}\right) = 0 \tag{33}$$

We note that for all $q \in \mathbb{Z}_+$,

$$\begin{aligned}&\Bigg|\left\{q+\tilde{N}_1, q+\tilde{N}_1+1, \ldots, q+T-\tilde{N}_2\right\} \bigcap \\ &\left\{\tilde{N}_1+N_j+m(T-\tilde{N}_1-\tilde{N}_2+1), \tilde{N}_1+N_j+1+m(T-\tilde{N}_1+\tilde{N}_2+1) \ldots, T-\tilde{N}_2+m(T-\tilde{N}_1-\tilde{N}_2+1)\right\}_{m=0}^{j}\Bigg| \\ &= T-\tilde{N}_1-N_j-\tilde{N}_2+1 \end{aligned} \tag{34}$$

Using (33), (34) and the chain rule, we have

$$\begin{aligned}&H\Big(\mathbf{s}_0, \ldots, \mathbf{s}_{T-\tilde{N}_1-\tilde{N}_2+(j-1)(T-\tilde{N}_1-\tilde{N}_2+1)} \,| \\ &\left\{\mathbf{x}^{(r_L)}_{\tilde{N}_1+N_j+m(T-\tilde{N}_1-\tilde{N}_2+1)}, \mathbf{x}^{(r_L)}_{\tilde{N}_1+N_j+1+m(T-\tilde{N}_1-\tilde{N}_2+1)}, \ldots, \mathbf{x}^{(r_L)}_{T-\tilde{N}_2+m(T-\tilde{N}_1-\tilde{N}_2+1)}\right\}_{m=0}^{j}\Big) \\ &= 0. \end{aligned} \tag{35}$$

Again, following the arguments in [7] when restricted to the channel (r_{j-1}, r_j), any $(N_1, \ldots, N_{L+1})$ achievable $(n_1, \ldots n_{L+1}, k, T)_{\mathbb{F}}$ code can be viewed as a point-to-point streaming code with rate k/n_j, delay $T-\sum_{l=1, l\neq j}^{L} N_l$ which can correct any N_j erasures. We therefore have

$$\frac{k}{n_j} \leq \frac{T-\sum_{l=1}^{L+1} N_l + 1}{T-\sum_{l=1, l\neq j}^{L+1} N_l + 1}$$

$$= C_{T-\sum_{l=1,l\neq j}^{L+1} N_l, N_j}. \tag{36}$$

Therefore we have

$$\begin{aligned} R &\leq \frac{k}{\max_j(n_j)} \\ &= \frac{T-\sum_{l=1}^{L+1} N_l+1}{T-\min_j \sum_{l=1,l\neq j}^{L+1} N_l+1} \\ &= \min_j C_{T-\sum_{l=1,l\neq j}^{L+1} N_l, N_j} \\ &= C_{T,N_1,\ldots,N_{L+1}}. \end{aligned} \tag{37}$$

VI. Suggested Coding Scheme

As mentioned above, the suggested coding scheme is an adaptive symbol-wise decode and forward scheme. Specifically the suggested scheme is as follows.

- From the sender (link (r_0, r_1)), transmit a $C_{T-\sum_{l=2}^{L+1} N_l, N_1}$ block code combined with diagonal interleaving (i.e., the symbols of the code are transmitted over the diagonals).
- From the first relay (link (r_1, r_2)):
  - Delay transmission by N_1 symbols.
  - Transmit a $C_{T-\sum_{l=1,l\neq 2}^{L+1} N_l, N_2}$ block code combined with diagonal interleaving. The order of symbols over each diagonal is a function of the error pattern in link (r_0, r_1).
  - For each packet, attach a header which indicates the order of the symbols in it.
- In general, in link (r_j, r_{j+1}), where $j \in [0, \ldots, L]$:
  - Delay transmission by $\sum_{l=1}^{j} N_j$ symbols.
  - Transmit a $C_{T-\sum_{l=1,l\neq j+1}^{L+1} N_l, N_{j+1}}$ block code combined with diagonal interleaving. The order of the symbols over each diagonal (compared to the transmitted packets from relay relay r_{j-1}), is a function of the error pattern in the link (r_{j-1}, r_j).
  - For each packet, attach a header which indicates the order of the symbols in it.

We note that as the field size $|\mathbb{F}|$ increases, the loss compared to the upper bound is reduced. Before showing the proof of the Theorem we show the following Proposition.

Proposition 1. *The block code used by relay r_j can be viewed as sub-code of $\min_j C_{j,j+1}$ (i.e., the outcome of puncturing $\min_j C_{j,j+1}$) which can be viewed as the "bottleneck" in the chain of relays.*

This proposition holds since k (the number of information symbols) is the same for all codes. Recalling that (MDS) code $\min_j C_{j,j+1}$ can correct any N_{max} erasures with delay of $\max_j T_{j,j+1}$, we note that puncturing any $N_{\max} - N_j$ columns from the generator matrix of this code results with a code that can correct any N_j erasure with delay of $T_{j,j+1}$.

Proof. We focus on a single block code (single diagonal) and show that the data sent on it can be decoded in overall delay of T assuming the maximal number of arbitrary erasures in a window of $T+1$ symbols in link (r_j, r_j+1) is N_{j+1} ($\forall j \in [0, \ldots, L]$) and $T \geq \sum_{l=1}^{L+1} N_l$.

Since all diagonals are subjected to the same constraints (delay constraint of T symbols and maximal number of erasures in a window of $T+1$ symbols), showing that one block code (one diagonal) can be decoded in the delay constraint means all diagonals can be decoded within this constraint.

Analyzing the suggested scheme we note that each node r_j transmits using a block code at rate $C_{j,j+1}$. Therefore, it is straightforward that the overall rate of transmission (up to the additional header required starting r_1) is

$$\begin{aligned} C_{T,N_1,\ldots,N_{L+1}} &= \min_j C_{j,j+1} \\ &= \min_j C_{T-\sum_{l=1,l\neq j}^{L+1} N_l, N_j}. \end{aligned} \tag{38}$$

We therefore need to show that it is feasible to transmit at rate $C_{j,j+1}$, defined in (8), in link (r_j, r_{j+1}), and that the packet sent at time i from the sender can be recovered up to delay T at the receiver. Analyzing the actual size of the overhead will conclude the proof.

With respect to feasibility, we note that delaying transmission by N_j symbols prior to transmitting from relay r_j (compared to the beginning of transmission in relay r_{j-1}), means that relay r_j will always have symbols to transmit as it can be thought of as if relay r_j "buffer" up to N_j symbols potentially received from r_{j-1} (which is the maximal number of symbols that can be erased in link (r_{j-1}, r_j) in any window of $T+1$ symbols).

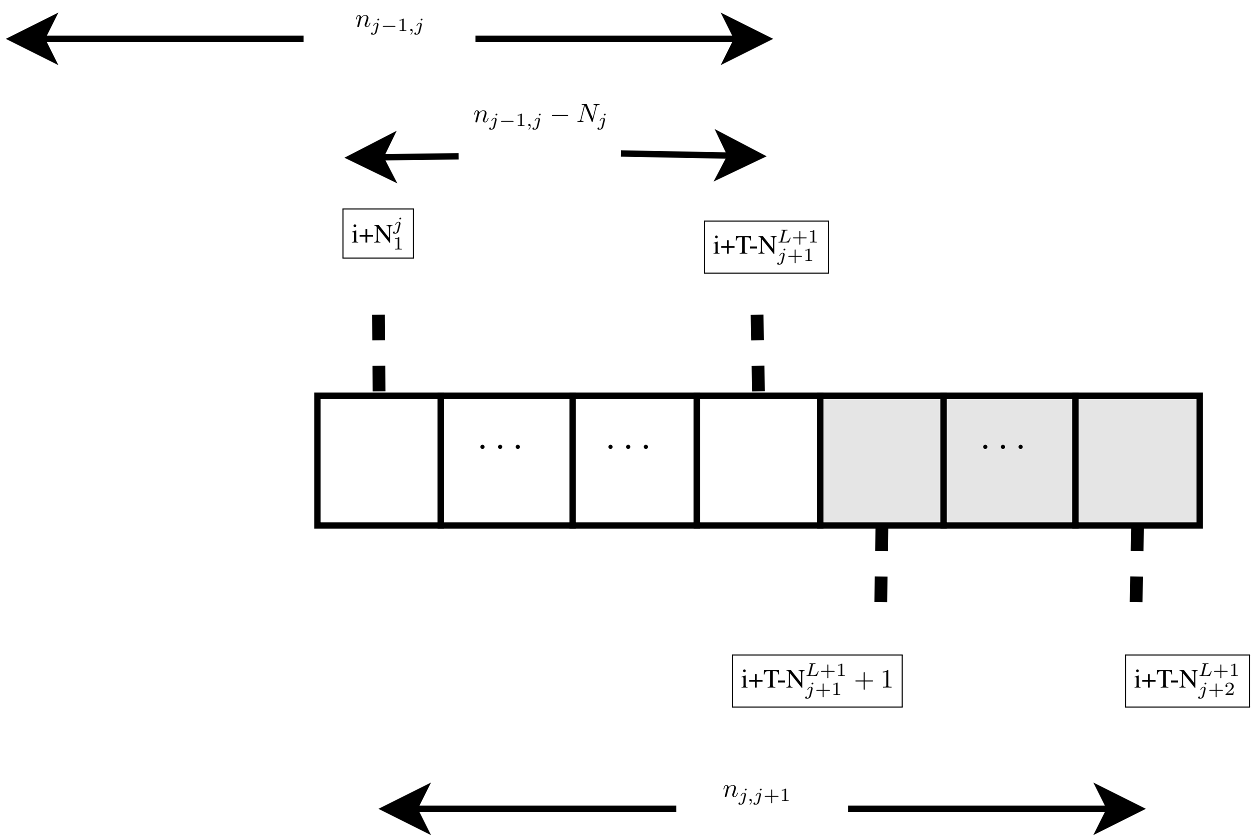

Fig. 3: Two regions of transmission in link (r_j, r_{j+1}). The symbols with white background are the symbols received from in link (r_{j-1}, r_j). The shaded symbols are transmitted after the k information symbols are decoded hence they are either symbols erased in link (r_{j-1}, r_j) or additional (independent) linear combinations of the information symbols.

Then we note that when relay r_{j-1} ends transmitting its block code diagonally (which is $n_{j-1,j}$ symbols after it started at time $i + \sum_{l=1}^{j-1} N_l$) it is guaranteed that relay r_j has successfully decoded the original k information symbols (as at this time, relay relay r_{j-1} has finished transmitting its code). This time index can be denoted as

$$\begin{aligned} i + \sum_{l=1}^{j-1} N_l + n_{j-1,j} =& i + \sum_{l=1}^{j-1} N_l + T - \sum_{l=1, l\neq j}^{L+1} N_l \\ =& i + T - \sum_{j+1}^{L+1} N_l \end{aligned} \tag{39}$$

Therefore, for each relay r_j, transmission is performed differently in the following two regions

- $t \in \left[i + \sum_{l=1}^{j} N_l, \ldots, i + T - \sum_{l=j+1}^{L+1} N_l \right]$: Until node j can recover the original information symbols, transmit the non-erased symbols received from node relay r_{j-1}. Delaying the beginning of transmission by N_j symbols (compared to the beginning of transmission by r_{j-1}) guarantees that there will be enough available symbols for transmission.
- $t \in \left[i + T - \sum_{l=j+1}^{L+1} N_l + 1, \ldots, i + T - \sum_{l=j+2}^{L+1} N_l \right]$: the k information symbols can be recovered by node j. Thus (in case there were erasures in the link (r_{j-1}, r_j)
  - In case $C_{j,j+1} > C_{j-1,j}$ transmit sub-group of the erased symbols.
  - In case $C_{j,j+1} < C_{j-1,j}$ transmit the erased symbols and additional (independent) linear combinations of the information symbols.

These two regions are depicted in Figure 3. Following proposition 1 we assume all symbols used by all codes are taken from block code $\min_j C_{j,j+1}$.

We show examples for the following two cases:

- $C_{j+1,j+2} > C_{j,j+2}$. This means that $n_{j+1,j+2} < n_{j,j+1}$, i.e., that the block size of the MDS code used by relay r_{j+1} is smaller than the block size used by relay r_j. At time $i+T-\sum_{j+1}^{L+1} N_l + 1$, node $j+1$ can recover the original data and send any of the erased symbols of the code used by r_j.
  An example is given in Table IV for $N_{j+1}=2,\ N_{j+2}=1,\ T'=4$ (where $T' = T-\sum_{l=1,l\neq j+1,j+2}$). We note that in this example $k=2$ and indeed, at $i+N_1^j+3$ the relay can recover the original data.

$i+N_1^j$	$i+N_1^j+1$	$i+N_1^j+2$	$i+N_1^j+3$	$i+N_1^j+4$
Link (r_j, r_{j+1})				
a_i				
	b_{i+1}			
		$f^1(a_i,b_{i+1})$		
			$f^2(a_i,b_{i+1})$	
Link (r_{j+1}, r_{j+2})				
		b_{i+1}		
			a_i	
				$f^1(a_i,b_{i+1})$

TABLE IV: Example of increasing the rate between links. In this example, $N_{j+1}=2,\ N_{j+2}=1,\ T'=4$, hence $C_{j+1,j+2} = 2/4 < 2/3 = C_{j,j+1}$. Assuming symbol $i+N_1^j$ and $i+N_1^j+2$ were erased when transmitted in link (r_j, r_{j+1}).

- $C_{j+1,j+2} < C_{j,j+1}$. This means that $n_{j+1,j+2} > n_{j,j+1}$, i.e., that the block size of the code used by relay r_{j+1} is larger than the block size used by relay r_j. At time $i+T-\sum_{j+1}^{L+1} N_l+1$, relay r_{j+1} can again recover the original data and hence transmit additional $n_{j+1,j+2}-k$ symbols needed to allow handling any N_{j+2} erasures in the link (r_{j+1}, r_{j+2}). An example is given in Table V for $N_{j+1}=1,\ N_{j+2}=2,\ T'=4$ (where $T'=T-\sum_{l=1,l\neq j+1,j+2}$). We note that in this example at time $i+N_1^j+2$ relay r_{j+1} can recover the original data, hence it can send any of the erased symbols of the code used by r_j. Further, since $n_{j+1,j+2} > n_{j,j+1}$, it can add (independent) linear combinations of the information symbols as required.

$i+N_1^j$	$i+N_1^j+1$	$i+N_1^j+2$	$i+N_1^j+3$	$i+N_1^j+4$
Link (r_j, r_{j+1})				
a_i				
	b_{i+1}			
		$f^1(a_i,b_{i+1})$		
Link (r_{j+1}, r_{j+2})				
	b_{i+1}			
		a_i		
			$f^1(a_i,b_{i+1})$	
				$f^2(a_i,b_{i+1})$

TABLE V: Example of reducing rate between nodes. In this example, $N_{j+1}=1,\ N_{j+2}=2,\ T'=4$, hence $C_{j,j+1}=2/3 > 2/4 = C_{j+1,j_2}$. Assuming symbol $i+N_1^j$ was erased when transmitted in link (r_j, r_{j+1}).

With respect to meeting the delay constraint, we note that using the suggested construction, relay r_j starts its transmission at time $i+\sum_{l=1}^{j} N_l$. This means that the final relay r_L starts it transmission at $i+\sum_{l=1}^{L} N_l$. Following (8), we note that the block size of this code is $T-\sum_{l=1}^{L} N_l$, hence the packet can be decoded (assuming any N_{L+1} erasures in the link (r_L, r_{L+1}) at time $i+T$ hence meeting the overall delay constraint.

The assumption that each node can decode the information sent from the previous node holds since the order of the symbols in the code is sent in a header. Therefore we need to analyze the overhead of this header. From Proposition 1 it follows all codes can be viewed as sub codes of $\min_j C_{j,j+1}$. The block size of this code is $n_{\max}$ hence, in the worst case, the header should consist of $n_{\max}$ elements, each one is chosen from $[1,\ldots,n_{\max}]$ (where repetitions are allowed per packet). Therefore the size of the header is upper bounded by $n_{\max}\log(n_{\max})$.

To conclude, each node transmits $n_{j,j+1}$ coded symbols (each taken from field $\mathbb{F}$) along with $O_{j,j+1}$ bits of header to transfer k information symbols (each taken from field $\mathbb{F}$). The overall rate is

$$\begin{aligned} R &= \min_j R_j \\ &= \min_j \frac{k\cdot|\mathbb{F}|}{n_{j,j+1}\cdot|\mathbb{F}| + n_{j,j+1}\log(n_{j,j+1})} \\ &= \min_j \frac{k}{n_{j,j+1}\left(1+\frac{\log(n_{j,j+1})}{|\mathbb{F}|}\right)} \\ &= \frac{T-\sum_{l=1}^{L+1} N_l + 1}{\max_j\left\{(T_{j,j+1}+1)\left(1+\frac{\log((T_{j,j+1}+1))}{|\mathbb{F}|}\right)\right\}} \end{aligned} \tag{40}$$

where $T_{j,j+1}$ is defined in (9). □

REFERENCES

[1] V. Cisco, “Cisco visual networking index: Forecast and trends, 2017–2022,” *White Paper*, vol. 1, 2018.

[2] E. Martinian and C.-E. Sundberg, “Burst erasure correction codes with low decoding delay,” *IEEE Transactions on Information theory*, vol. 50, no. 10, pp. 2494–2502, 2004.

[3] A. Badr, A. Khisti, W.-T. Tan, and J. Apostolopoulos, “Streaming codes for channels with burst and isolated erasures,” in *2013 Proceedings IEEE INFOCOM*. IEEE, 2013, pp. 2850–2858.

[4] S. L. Fong, A. Khisti, B. Li, W.-T. Tan, X. Zhu, and J. Apostolopoulos, “Optimal streaming codes for channels with burst and arbitrary erasures,” *IEEE Transactions on Information Theory*, 2019.

[5] M. N. Krishnan and P. V. Kumar, “Rate-optimal streaming codes for channels with burst and isolated erasures,” in *2018 IEEE International Symposium on Information Theory (ISIT)*. IEEE, 2018, pp. 1809–1813.

[6] E. Domanovitz, S. L. Fong, and A. Khisti, “An explicit rate-optimal streaming code for channels with burst and arbitrary erasures,” *arXiv preprint arXiv:1904.06212*, 2019.

[7] S. L. Fong, A. Khisti, B. Li, W.-T. Tan, X. Zhu, and J. Apostolopoulos, “Optimal streaming erasure codes over the three-node relay network,” 2018.

[8] F. Begtasevic and P. Van Mieghem, “Measurements of the hopcount in internet,” in *PAM2001, A workshop on Passive and Active Measurements, Amsterdam, the Netherlands, April 23-24, 2001*, 2001.

[9] A. Mukaddam and I. H. Elhajj, “Hop count variability,” in *2011 International Conference for Internet Technology and Secured Transactions*. IEEE, 2011, pp. 240–244.

[10] F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes*. Elsevier, 1977, vol. 16.