# Uniformly Randomized Markov Inequality

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BST 235 Final Presentation

# Markov Inequality Refresher

**Theorem (Markov Inequality (MI))**

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In general: tighter characterization of bounds $\implies$ tighter uncertainty quantification.

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$$\begin{aligned} X \sim a \cdot \mathrm{Bern}(p) \implies \mathbb{E}X &= ap \quad \text{(for any } p \in (0,1)\text{)} \\ \Pr(X \geq a) &= \Pr(X = a) \\ &= p \\ &= a^{-1}\mathbb{E}X. \end{aligned}$$

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What about in other senses? Is there any way to retain the generality (i.e. for all non-negative $X, a$) but also give almost surely tighter bounds in downstream applications?

# Uniformly Randomized Markov Inequality

**Theorem (Uniformly Randomized Markov Inequality (URMI) [RM23])**

*Let $X : \Omega \to \mathbb{R}^{\geq 0}$, $a \geq 0$, and $U \sim Unif(0,1)$ such that $U \perp X$. Then*

$$\Pr\left(XU^{-1} \geq a\right) \leq a^{-1}\mathbb{E}X.$$

*Proof.*

$$\Pr\left(XU^{-1} \geq a\right) = \mathbb{E}\Pr\left(XU^{-1} \geq a|X\right) = \mathbb{E}\Pr\left(U \leq Xa^{-1}|X\right) = \mathbb{E}\min(1, Xa^{-1}) \leq a^{-1}\mathbb{E}X$$

□

This trivially holds more generally by replacing $U$ with any $B \perp X$ such that $B$ is stochastically larger than $U$, which we denote by $U \preceq B$.

That is, $\forall y \geq 0 : \Pr(B \leq y) \leq \Pr(U \leq y)$.

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$$\text{Naïve } \mathrm{CI}_{\text{Naïve}} = \overline{X}_n \pm \frac{\sigma}{\sqrt{\alpha n}}$$

$$\text{“Typical” Random CI} = \overline{X}_n \pm \frac{\sigma\sqrt{U}}{\sqrt{\alpha n}}$$

$$\text{Randomized Hoeffding CI} = \overline{X}_n \pm \left(\sigma\sqrt{\frac{2\log(2/\alpha)}{n}} + \frac{\log(U)}{\sqrt{2n\log(2/\alpha)}}\right)$$

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We wondered if it was possible to use this idea to actually tighten URMI further.

Instead of using

$$\Pr\left(XU^{-1} \geq a\right) \leq a^{-1}\mathbb{E}X,$$

could we beat it by choosing a $\phi$ such that

$$\Pr\left(X\phi\left(\min_{i\in[k]} U_i^{-1}\right) \geq a\right) \leq a^{-1}\mathbb{E}X$$

and $\phi\left(\min_{i\in[k]} U_i^{-1}\right) \leq U$ (in some reasonable sense)?

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$$\text{“Adjusted” Random CI} = \overline{X}_n \pm \frac{\sigma\sqrt{kU_{(1)}}}{\sqrt{\alpha n}}$$

Over B=2000 iterations, sampled n=1000 $X_i \overset{iid}{\sim} N(0,1)$

# Width

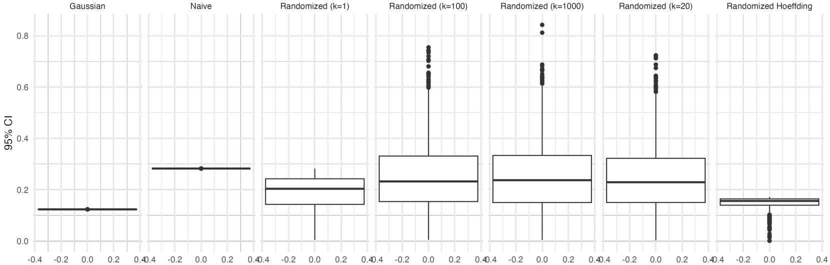


Figure: Comparison of $\overline{X}_n$ CI Width by Randomization Strategy (n=1000)

# Theoretical Contribution: URMI is Tight!

**Theorem (Stochastic upper bounds on $U$ are valid MI randomizers [RM23])**

*Let $U \sim \mathit{Unif}(0,1)$. If $R \succeq U$, then $R$ is a valid MI randomizer.*

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*Let $U \sim Unif(0,1)$. If $R \not\succeq U$, $R$ cannot be a valid MI randomizer.*

**Corollary ("Optimality" of $U$ among valid MI randomizers)**

*For any valid MI randomizer $V$ such that $V \overset{D}{\neq} U$:*

$$\forall y \in (0,1) : \Pr(V \leq y) \leq \Pr(U \leq y)$$
$$\exists y \in (0,1) : \Pr(V \leq y) < \Pr(U \leq y)$$

# References I

[CD23] Christian Covington and Dominic DiSanto. Bst235 project, 2023.

[RM23] Aaditya Ramdas and Tudor Manole. Randomized and exchangeable improvements of markov's, chebyshev's and chernoff's inequalities. *arXiv preprint arXiv:2304.02611*, 2023.

# Supp: Coverage Table

| Type | Coverage Probability |
|---|---|
| Gaussian | 0.95 |
| Naïve | 1.00 |
| Randomized Markov's (k=1) | 0.95 |
| Randomized Markov's (k=100) | 0.95 |
| Randomized Markov's (k=1000) | 0.96 |
| Randomized Markov's (k=20) | 0.95 |
| Randomized Hoeffding | 0.97 |

Table: Table of Empirical Confidence Interval Coverage Probability over B=2000 Iterations