Sum_1

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This file contains the details on the $n = 1$ case, to be merged into the main file as soon as they become polished enough.

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Let us denote by $\mathrm{Fam}_0(\mathbb{C})$ the set of equivalence classes of finite groupoids equipped with functors to $\mathbb{C}$ (seen as a trivial category). In other words, a representative of an element of $\mathrm{Fam}_0(\mathbb{C})$ is a pair (X, ρ) consisting of a finite groupoid X together with a map $\rho\colon \pi_0(X) \to \mathbb{C}$. The map

$$\mathrm{Sum}_0\colon \mathrm{Fam}_0 \to \mathbb{C}$$

is defined as

$$\mathrm{Sum}_0(X, \rho) = \sum_{[x]\in\pi_0(X)} \frac{\rho(x)}{|\mathrm{Aut}(x)|}.$$

We will also use the more evocative notation

$$\int_X \rho(x) d\mu(x)$$

to denote $\mathrm{Sum}_0(X, \rho)$. Notice that Sum_0 is linear in the ρ variable.

A nice property of the morphism Sum_0 (which we are going to use later) is the following: let X be a finite set with an action of a finite group G, and let $X//G$ denotes the action groupoid for this action; finally, let $\rho\colon X//G \to \mathbb{C}$ any functor, i.e., let $\rho\colon X \to \mathbb{C}$ be a function that is constant on the G-orbits in X. Then we have

$$\mathrm{Sum}_0(X//G, \rho) = \frac{1}{|G|} \sum_{x\in X} \rho(x). \tag{1.1}$$

Namely, the set $\pi_0(X/\!/G)$ is the quotient set X/G and for any $x \in X$ seen as an object in $X/\!/G$ we have $\mathrm{Aut}(x) = \mathrm{Stab}_G(x)$. Therefore

$$\begin{aligned}
\mathrm{Sum}_0(X/\!/G, \rho) &= \sum_{[x]\in X/G} \frac{\rho(x)}{|\mathrm{Stab}_G(x)|} \\
&= \frac{1}{|G|} \sum_{[x]\in X/G} \frac{|G|}{|\mathrm{Stab}_G(x)|}\rho(x) \\
&= \frac{1}{|G|} \sum_{[x]\in X/G} |[x]|\, \rho(x) \\
&= \frac{1}{|G|} \sum_{[x]\in X/G} \sum_{x\in[x]} \rho(x) \\
&= \frac{1}{|G|} \sum_{x\in X/G} \rho(x)
\end{aligned}$$

Remark 1.1. A function $f\colon \pi_0(X) \to \mathbb{C}$ can equivalently be seen as an element in $\mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X)$, where $\mathbf{1}_X\colon X \to \mathrm{Vect}$ is the constant functor taking the value $\mathbb{C}$ on X. The map Sum_0 can therefore be seen as the datum of a map

$$\mathrm{Sum}_0\colon \mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X) \to \mathbb{C},$$

for any finite groupoid X.

Definition 1.2. The category $\mathrm{Fam}_1(\mathrm{Vect})$ has as its objects finite groupoids equipped with functors to Vect, i.e., pairs (X, ρ) consisting of a finite groupoid X together with a functor $\rho\colon X \to \mathrm{Vect}$. Morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ are spans of groupoids over Vect, i.e., commutative diagrams

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\rho_X"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\alpha}"] & & Y \arrow[dl, "\rho_Y"] \\
 & \mathrm{Vect} &
\end{tikzcd}$$

where the natural transformation $\alpha\colon \rho_X \circ \pi_X \to \rho_Y \circ \pi_Y$ is part of the data of the commutative diagram.

Remark 1.3. The identity mophisms and the composition of morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ will be defined later.

For every object (X, ρ) in $\mathrm{Fam}_1(\mathrm{Vect})$ we can consider the vector space

$$[\mathbf{1}_X, \rho] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\mathbf{1}_X, \rho).$$

If $f\colon Y \to X$ is a functor between finite groupoids, and $\rho\colon X \to \mathrm{Vect}$ is a functor, we write $f^*\rho\colon Y \to \mathrm{Vect}$ for the functor $\rho \circ f$. If $\alpha\colon \mathbf{1}_X \to \rho$ is a natural transformation, then $f^*\alpha$ is a natural transformation between $f^*\mathbf{1}_X = \mathbf{1}_Y$ and $f^*\rho$. This defines a linear map

$$f^*\colon [\mathbf{1}_X, \rho] \to [\mathbf{1}_Y, f^*\rho].$$

Similarly, we can consider the vector space

$$[\rho, \mathbf{1}_X] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\rho, \mathbf{1}_X)$$

together with the linear map

$$f^*\colon [\rho, \mathbf{1}_X] \to [f^*\rho, \mathbf{1}_Y].$$

Passing to the linear duals we get the linear map

$$(f^*)^\vee\colon [f^*\rho, \mathbf{1}_Y]^\vee \to [\rho, \mathbf{1}_X]^\vee.$$

Next, notice that composition of morphisms gives a bilinear map

$$[\rho, \mathbf{1}_X] \otimes [\mathbf{1}_X, \rho] \to [\mathbf{1}_X, \mathbf{1}_X]$$

and so composing this with the linear map $\mathrm{Sum}_0(X, -)\colon [\mathbf{1}_X, \mathbf{1}_X] \to \mathbb{C}$ we get a bilinear pairing $\langle\,|\,\rangle_X$ between $[\rho, \mathbf{1}_X]$ and $[\mathbf{1}_X, \rho]$, for any ρ. Notice that, by its very definition, the pairing $\langle\,|\,\rangle_X$ satisfies

$$\langle v|\alpha \circ w\rangle_X = \langle v \circ \alpha|w\rangle_X,$$

for any $v \in [\rho_2, \mathbf{1}_X]$, any $w \in [\mathbf{1}_X, \rho_1]$ and any natural transformation $\alpha\colon \rho_1 \to \rho_2$. Equivalently, the bilinear pairing $\langle\,|\,\rangle_X$ is a linear map

$$\delta_{(X,\rho)}\colon [\mathbf{1}_X, \rho] \to [\rho, \mathbf{1}_X]^\vee.$$

Our main *duality assumption* will be that $\delta_{(X,\rho)}$ is a linear isomorphism for any (X, ρ).

Remark 1.4. The duality assumption holds for every finite groupoid X if and only if it holds for finite connected groupoids. Up to equivalence we are therefore reduced to considering the case $X = \mathbf{B}G$, with G a finite group. In this case $\rho\colon \mathbf{B}G \to \mathrm{Vect}$ is the datum of a linear representation V of G, and the vector spaces $[\mathbf{1}_{\mathbf{B}G}, \rho]$ and $[\rho, \mathbf{1}_{\mathbf{B}G}]$ are isomorphic to the space V^G of G-invariants and the linear dual $(V_G)^\vee$ of the space of G-coinvariants, respectively. The map $\delta_{(\mathbf{B}G,\rho)}$ is therefore a map

$$\delta_{(\mathbf{B}G,\rho)}\colon V^G \to (V_G)^{\vee\vee} \cong V_G,$$

from G-invariants to G-coinvariants. Let $v \mapsto [v]$ be the canonical projection $V \to V_G = V/\langle v - g \cdot v\rangle_{v\in V; g\in G}$. By composing with the inclusion $V^G \hookrightarrow V$ we get the morphism $\iota\colon v \mapsto [v]$ from $V^G \to V_G$. This is an isomorphism. Indeed, the canonical projection $\pi\colon V \to V^G$ given by

$$\pi\colon v \mapsto \frac{1}{|G|}\sum_{g\in G} g\cdot v$$

is zero on the elements of the form $v - g\cdot v$ and so induces a linear map $\tilde{\pi}\colon V_G \to V^G$, which is immediate to see is the inverse of ι. For any $w \in (V_G)^\vee$ seen as an element in $[\rho, \mathbf{1}_{\mathbf{B}G}]$ and any $v \in V^G$ seen as an element in $[\mathbf{1}_{\mathbf{B}G}, \rho]$ we have $w \circ v = w([v])$ Therefore

$$\langle w|v\rangle_{\mathbf{B}G} = \frac{1}{|G|} w \circ v = w\left(\frac{1}{|G|}\iota(v)\right).$$

This means that $\delta_{(\mathbf{B}G,\rho)}v$ is the evaluation on $\frac{1}{|G|}\iota(v)$. Therefore, under the canonical isomorphism $(V_G)^{\vee\vee} \cong V_G$, we have $\delta_{(\mathbf{B}G,\rho)} = \frac{1}{|G|}\iota$. As ι is an isomorphism, so is $\delta_{(\mathbf{B}G,\rho)}$.

Under the duality assumption we may define a linear morphism

$$f_*\colon [\mathbf{1}_Y, f^*\rho] \to [\mathbf{1}_X, \rho]$$

as the composition

$$[\mathbf{1}_Y, f^*\rho] \xrightarrow{\delta_{(Y,f^*\rho)}} [f^*\rho, \mathbf{1}_Y]^\vee \xrightarrow{(f^*)^\vee} [\rho, \mathbf{1}_X]^\vee \xrightarrow{\delta^{-1}_{(X,\rho)}} [\mathbf{1}_X, \rho].$$

In other words, if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_*v \in [\mathbf{1}_X, \rho]$ is defined by the equation

$$\langle w|f_*v\rangle_X = \langle f^*w|v\rangle_Y$$

for any $w \in [\rho, \mathbf{1}_X]$. Similarly, we have a morphism $f_*\colon [f^*\rho, \mathbf{1}_Y] \to [\rho, \mathbf{1}_X]$ defined by

$$\langle f_*w|v\rangle_X = \langle w|f^*v\rangle_Y.$$

We can now define our wannabe functor

$$\mathrm{Sum}_1\colon \mathrm{Fam}_1(\mathrm{Vect}) \to \mathrm{Vect}$$

on objects as

$$\mathrm{Sum}_1(X,\rho) = [\mathbf{1}_X, \rho]$$

and on a morphism

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "\rho_X"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\alpha}"] & & Y \arrow[dl, "\rho_Y"] \\ & \mathrm{Vect} & \end{tikzcd}$$

as the composition

$$[\mathbf{1}_X, \rho_X] \xrightarrow{\pi_X^*} [\mathbf{1}_Z, \pi_X^*\rho_X] \xrightarrow{\alpha\circ -} [\mathbf{1}_Z, \pi_Y^*\rho_Y] \xrightarrow{\pi_{Y*}} [\mathbf{1}_Y, \rho_Y].$$

Remark 1.5. Before giving a more explicit expression of the above morphism, let us notice that $[\mathbf{1}_X, \rho]$ is a model for the limit of the functor $\rho\colon X \to \mathrm{Vect}$, in accordance with the prescrition in [**?**]. Also, notice that the morphism associated to the span $X \leftarrow Z \to Y$ together with the natural transformation α has the form "pull-compose-push", and so it is an instance of a Fourier-Mukai-type transform.

Lemma 1.6. (The projection formula) Let $f\colon Y \to X$ be a functor between finite groupoids, let $\rho_1, \rho_2\colon X \to \mathrm{Vect}$ be functors and let $\alpha\colon \rho_1 \to \rho_2$ be a natural transformation. We have an identity of morphisms $[\mathbf{1}_Y, f^*\rho_1] \to [\mathbf{1}_X, \rho_2]$

$$f_*(f^*\alpha \circ -) = \alpha \circ f_*$$

Proof. The identity

$$f_*(f^*\alpha \circ v) = \alpha \circ f_* v$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ is equivalent to the identity

$$\langle w | f_*(f^*\alpha \circ v)\rangle_X = \langle w | \alpha \circ f_* v\rangle_X$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ and every $w \in [\rho_2, \mathbf{1}_X]$. By the properties of the pairings $\langle\,|\,\rangle_X$ and $\langle\,|\,\rangle_X$, and by definition of f_*, we have

$$\begin{aligned}
\langle w | f_*(f^*\alpha \circ v)\rangle_X &= \langle f^*w | f^*\alpha \circ v\rangle_Y \\
&= \langle f^*w \circ f^*\alpha | v\rangle_Y \\
&= \langle f^*(w \circ \alpha) | v\rangle_Y \\
&= \langle w \circ \alpha | f_* v\rangle_X \\
&= \langle w | \alpha \circ f_* v\rangle_X
\end{aligned}$$

□

For later use, let us derive an explicit formla for f_* in case $f\colon \mathbf{B}G \to \mathbf{B}H$ is a morphism of finite groups (seen as groupoids with a single object).

Recall that if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_* v \in [\mathbf{1}_X, \rho]$ is defined by the equation $\langle w | f_* v\rangle_{\mathbf{B}H} = \langle f^*w | v\rangle_{\mathbf{B}G}$ for any $w \in [\rho, \mathbf{1}_X]$. As both $\mathbf{B}G$ and $\mathbf{B}H$ have a single object, this equation reduces to

$$\frac{w \circ f_* v}{|H|} = \frac{f^* w \circ v}{|G|}$$

The morphism $\rho\colon \mathbf{B}H \to \mathrm{Vect}$ is the datum of a linear representation of H on a vector space V, and $f^*\rho$ is the datum of representation of G obtained by pull-back, i.e., the group G acts on V by $g \cdot \xi := f(g) \cdot \xi$. The element $v \in [\mathbf{1}_Y, f^*\rho]$ is therefore a G-invariant vector of V, while $w \in [\rho, \mathbf{1}_X]$ is a linear functional on V factoring through the space of H-coinvariants. The element f^*w is the same linear functional on V, but this time seen as factoring through the space of H-coinvariants. Therefore the composition $f^*w \circ v$ is just the evaluation of $w\colon V \to \mathbb{C}$ on the vector $v \in V$, under the canonical isomorphism $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$. Finally, $f_*v \in [\mathbf{1}_X, \rho]$ is an H-invariant vector of V, and $w \circ f_*v$ is the evaluation of $w\colon V \to \mathbb{C}$ on the vector $f_*v \in V$. The defining equation for f_*v then becomes

$$w\left(\frac{f_*v}{|H|}\right) = w\left(\frac{v}{|G|}\right)$$

for any $w\colon V \to \mathbb{C}$ factoring through the space of H-coinvariants. From $v/|G|$ we get the H-invariant vector

$$\frac{1}{|H|}\sum_{h\in H} h \cdot \frac{v}{|G|}$$

and, since w factors through the H-coinvariants,

$$w\left(\frac{v}{|G|}\right) = w\left(\frac{1}{|H|}\sum_{h\in H} h \cdot \frac{v}{|G|}\right)$$

The defining equation for f_*v can therefore be rewritten as

$$\langle w|f_*v\rangle_{\mathbf{B}H} = \langle w|\sum_{h\in H}\frac{h\cdot v}{|G|}\rangle_{\mathbf{B}H},$$

for any $w \in [\rho, \mathbf{1}_X]$. As the pairing $\langle\,|\,\rangle_{\mathbf{B}H}$ is nondegenerate, this finally gives

$$f_*v = \frac{1}{|G|}\sum_{h\in H} h \cdot v.$$

Remark 1.7. This formula simplifies a bit in case the group homomorphism $f\colon G \to H$ is surjective. Indeed, in that case, the G-invariant vector v will be automatically H-invariant and we find

$$f_*v = \frac{|H|}{|G|}v$$

in this case.

Lemma 1.8. Consider a homotopy pullback diagram of finite groupoids

$$\begin{tikzcd} & X \times_T Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[rr, phantom, "\underset{\eta}{\Rightarrow}"] \arrow[dr, "f_1"'] & & Y \arrow[dl, "f_2"] \\ & T & \end{tikzcd}$$

and a morphism $\rho\colon T \to \mathrm{Vect}$. Then the Beck-Chevalley condition is satisfied, i.e., we have an identity of morphisms $f_2^* f_{1*} = p_{2*}(\eta^*\rho \circ -)p_1^*$ from $[\mathbf{1}_X, f_1^*\rho]$ to $[\mathbf{1}_Y, f_2^*\rho]$.

Proof. By restricting to connected components, and working up to equivalence of finite groupoids, we may assume that the given pullback diagram has the form

$$\begin{tikzcd} & G\backslash\backslash H//K \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbf{B}G \arrow[rr, phantom, "\underset{\eta}{\Rightarrow}"] \arrow[dr, "f_1"'] & & \mathbf{B}K \arrow[dl, "f_2"] \\ & \mathbf{B}H & \end{tikzcd}$$

where the natural transformation η is defined by $\eta_h = h$; this precisely expresses the fact that a morphism from h to h' in the action groupoid $G\backslash\backslash H//K$ is given by a commutative diagram

$$\begin{tikzcd} * \arrow[r, "h"] \arrow[d, "f_1(g)"'] & * \arrow[d, "f_2(k)"] \\ * \arrow[r, "h'"'] & * \end{tikzcd} .$$

The equation $f_2^* f_{1*} = p_{2*}(\eta^*\rho \circ -)p_1^*$ is equivalent to $\langle w | f_2^* f_{1*} v\rangle_{\mathbf{B}K} = \langle w | p_{2*}(\eta^*\rho \circ p_1^* v)\rangle_{\mathbf{B}K}$ for any $v \in [\mathbf{1}_{\mathbf{B}G}, f_1^*\rho]$ and any $w \in [f_2^*\rho, \mathbf{1}_{\mathbf{B}K}]$. This is equivalent to the equation

$$\langle f_{2*} w | f_{1*} v\rangle_{\mathbf{B}H} = \langle p_2^* w | \eta^*\rho \circ p_1^* v\rangle_{G\backslash\backslash H//K}.$$

The morphism $\rho\colon \mathbf{B}H \to \mathrm{Vect}$ is the datum of a linear representation of the finite group H on a vector space V. The element v is a G-invariant vector of V (where G acts on V via f) and w is a linear functional on V which factors through the K-coinvariants (with K

acting via g). By the above explicit formula for f_* in the case of a group morphism, we have

$$f_{1*}v = \frac{1}{|G|}\sum_{h\in H} h\cdot v; \qquad f_{2*}w = \frac{1}{|K|}\sum_{h\in H} h\cdot w,$$

where on the right we have the H-action on linear functionals on V, i.e., $(h\cdot w) = w(h^{-1}\cdot -)$. We therefore find

$$\begin{aligned}
\langle f_{2*}w|f_{1*}v\rangle_{\mathbf{B}H} &= \frac{1}{|H||G||K|}\sum_{h_1,h_2\in H} w(h_1^{-1}h_2\cdot v)\\
&= \frac{1}{|H||G||K|}\sum_{h\in H}\sum_{h_1^{-1}h_2=h} w(h_1^{-1}h_2\cdot v)\\
&= \frac{1}{|H||G||K|}\sum_{h\in H}\sum_{h_1\in H} w(h\cdot v)\\
&= \frac{1}{|G||K|}\sum_{h\in H} w(h\cdot v)
\end{aligned}$$

To every object h of $G\backslash\backslash H//K$, the functors $p_1^*f_1^*\rho$ and $p_2^*f_2^*\rho$ both assign the vector space V, so p_1^*v and p_2^*w consist in a copy of the vector v and of the linear functional $w\colon V\to\mathbb{C}$ for each of the copies of V associated with the elements of H. Moreover, since $\eta_h = h$, the natural transformation $\eta^*\rho = \rho\circ\eta$ at the object h is precisely the multiplication by h as a linear map from V to V. This means that $\eta^*\rho\circ p_1^*v = (h\cdot v)_{h\in H}$, i.e, $\eta^*\rho\circ p_1^*v$ consists of the choice of the vector $h\cdot v$ in the copy of V associated with the object h of $G\backslash\backslash H//K$. We therefore find

$$\langle p_2^*w|\eta^*\rho\circ p_1^*v\rangle_{G\backslash\backslash H//K} = \frac{1}{|G||K|}\sum_{h\in H} w(h\cdot v),$$

by formula 1.1. $\square$