# **Boundaries in finite gauge theory**

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### **Contents**

## 1 Introduction

Let

- $\bullet$  G be a finite group,
- $[\omega] \in H^n(G, \mathbb{C}^{\times}),$
- and  $\mathcal{C}$  a symmetric monoidal  $(\infty, n)$ -category for some  $n \in \mathbb{Z}_+$ .

On the one hand, [?] sketches the construction of a fullly extended TQFT

$$\operatorname{Bord}_n \xrightarrow{I} \operatorname{Fam}_n(\mathcal{C}) \xrightarrow{\operatorname{Sum}_n} \mathcal{C}$$
 (1.1)

based on the data  $(G, \omega)$ , which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [? ? ] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

**Goal.** We want to reformulate the general aspects of the work [??] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [??] as the classical part I in (??) with  $C = B^n \mathbb{C}^{\times}$  (the n-fold delooping of the group  $\mathbb{C}^{\times}$ ) and then

- (i) identify classical boundary conditions as 1-morphisms with source \* in  $\operatorname{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times),$
- (ii) subsume the boundary conditions of [??] as special cases,
- (iii) map classical boundary conditions to quantum ones via  $Sum_n$ , and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs-Schweigert-Valentino in the case n=3,

(iv) do the above not only for framed, but also for oriented, spin,...TQFTs.

Since [?] is very light on details, we first will work out explicitly how (??) recovers the known bulk theory for  $n \in \{1,2\}$  and hopefully also n=3. We also want to explain in detail how  $(G,\omega)$  gives rise to an object in  $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$ , and how group extensions  $1 \to K \to H \xrightarrow{r} G \to 1$  give rise to 1-morphisms  $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)(*,\mathbf{B}G)$ .

(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [?, Sect. 4.3] instead of  $Bord_n$ , but we leave that for another project.)

## **2** Special objects and 1-morphisms in $\operatorname{Fam}_n(\boldsymbol{B}^n\mathbb{C}^{\times})$

TODO:

- spell out definition of  $\operatorname{Fam}_n(\mathcal{C})$
- check that n-functor  $BG \to B^n \mathbb{C}^{\times}$  is precisely an n-cocycle <u>Issue</u>: For n > 3, what is a weak n-functor, i. e. precisely what data and constraints are needed?

<u>Idea</u>: Since both n-categories in  $BG \to B^n\mathbb{C}^{\times}$  are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For  $n \in \{1, 2, 3\}$  this is true, see Section ??.

- clarify precisely how  $BG \to B^n \mathbb{C}^{\times}$  induces an *n*-functor  $BG \to n$ -Vect, at least for  $n \in \{1, 2, 3\}$
- check whether natural transformation from trivial n-functor to  $\omega \circ r$  is trivialisation of  $r^*\omega$

Issue: induced from issue in second item

## 3 Bulk theory

#### **3.1** n=1

Since we take the action of G on  $\mathbb{C}^{\times}$  to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \operatorname{Hom}_{\operatorname{Grp}}(G, \mathbb{C}^\times)$$
(3.1)

is just a group homomorphism. We consider  $B^1\mathbb{C}^{\times}$  as a subcategory of  $\mathrm{Vect}_{\mathbb{C}}$ , and we will use the pair  $(G, \omega)$  to construct a functor

$$\operatorname{Bord}_1 \xrightarrow{I} \operatorname{Fam}_1(\operatorname{Vect}_{\mathbb{C}}) \xrightarrow{\operatorname{Sum}_1} \operatorname{Vect}_{\mathbb{C}}.$$
 (3.2)

According to the cobordism hypothesis, the composite  $\operatorname{Sum}_1 \circ I$  is determined by its value on the point  $\operatorname{pt}_+$ , so we only need to specify  $I(\operatorname{pt}_+)$  and then compute  $\operatorname{Sum}_1(I(\operatorname{pt}_+))$ . Note that  $\mathbf{B}^1G = */\!\!/ G$  is the classical action groupoid. We set  $I(\operatorname{pt}_+)$  to be the functor

$$\chi^{\omega} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto \mathbb{C}, \quad g \longmapsto \omega(g) \in \operatorname{Aut}(\mathbb{C})$$
 (3.3)

for all  $g \in G$ . Then (??) recovers the known result reviewed in [?, Sect. 1]:

**Lemma 3.1.** Sum<sub>1</sub>( $\chi^{\omega}$ ) =  $\mathbb{C}$  if  $\omega(g) = 1$  for all  $g \in G$ , and Sum<sub>1</sub>( $\chi^{\omega}$ ) = 0 otherwise.

*Proof.* The statement is a particular case of this more general one: let  $(V, \rho)$  be a linear representation of the group G, seen as the functor

$$\chi^{\rho} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto V, \quad q \longmapsto \rho(q) \in \operatorname{Aut}(V).$$
 (3.4)

Then the universal cocone of  $\chi^{\rho}$ , viewed as a diagram of shape  $*/\!\!/ G$  in Vect<sub>C</sub> is the pair  $(V_G, \pi)$ , where  $V_G$  is the vector space of coinvariants for the representation  $\rho$ , i.e.,

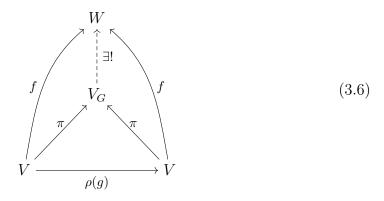
$$V_G = V/\langle v - \rho(g)v \rangle_{g \in G, v \in V}$$

and  $\pi: V \to V_G$  is the projection to the quotient. Namely, let (W, f) be a cocone for  $\chi^{\rho}$ , i.e., a pair consisting of a vector space W together with a linear map  $f: \chi^{\omega}(*) = V \to W$  such that

$$\chi^{\omega}(*) = V \xrightarrow{\chi^{\rho}(g) = \rho(g)} V = \chi^{\omega}(*)$$

$$(3.5)$$

Then, by definition of  $V_G$ , the morphism f uniquely factors through  $V_G$  and so we have a commutative diagram



showing that  $(V_G, \pi)$  enjoys the universal property of the universal cocone.

Corollary 3.2. The linear dual  $\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_{1}(\chi^{\omega}), \mathbb{C})$  of  $\operatorname{Sum}_{1}(\chi^{\omega})$  is naturally isomorphic to the vector space of natural transformations  $\operatorname{Hom}_{[*/\!\!/ G,\operatorname{Vect}_{\mathbb{C}}]}(\chi^{\omega},\chi^{1})$ , where  $\chi^{1}\colon */\!\!/ G \to \operatorname{Vect}_{\mathbb{C}}$  is the trivial representation of G on the vector space  $\mathbb{C}$ . In other words, we have

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_{1}(\chi^{\omega}),\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \chi^{\omega} & \Longrightarrow \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}.$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines  $\operatorname{Sum}_1(\chi^{\omega})$ .

*Proof.* Again, the statement is true for an aritrary linear representation  $(V, \rho)$  of the group G. The natural isomorphism

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(V_G,\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \downarrow^{\rho} \Longrightarrow * \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}$$

is then nothing but the universal property of  $V_G$ . Namely, an element in the right hand side is a morphism  $f: V = \chi^{\rho}(*) \to \chi^{1}(*) = \mathbb{C}$  such that all the diagrams

$$V \xrightarrow{f} \mathbb{C}$$

$$\rho(g) \downarrow \qquad \qquad \downarrow \mathrm{id}_{\mathbb{C}}$$

$$V \xrightarrow{f} \mathbb{C}$$

commute, for any  $g \in G$ . This is the same as requiring that all the diagrams

$$V \xrightarrow{f} V$$

$$V \xrightarrow{\rho(g)} V$$

commute, and so it is precisely the datum of a morphism  $V_G \to \mathbb{C}$  by the argumenti in the proof of Lemma ??.

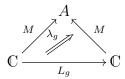
#### **3.2** n=2

TODO:

- The only non-trivial part of the 2-functor  $\chi^{\omega} : \mathbf{B}^2 G \to \mathrm{Alg}_{\mathbb{C}}$  is the coherence 2-morphism  $\chi^{\omega}_{g,h} : \chi^{\omega}(g) \otimes \chi^{\omega}(h) \to \chi^{\omega}(gh)$ , and the constraints (see e.g. [?, Sect. 1.1] for the definitions) on the  $\chi^{\omega}_{g,h}$  precisely say that they are the components of a 2-cocycle. So we can take  $\chi^{\omega}_{g,h} = \omega(g,h)$ .
- compute 2-colimit  $\operatorname{Sum}_2(\chi)$  for  $\chi$  as above, obtain twisted group algebra  $\mathbb{C}^{\omega}[G]$

**Lemma 3.3.**  $\operatorname{Sum}_2(\chi^{\omega}) \cong \mathbb{C}^{\omega}[G]$ , the twisted group algebra of G.

*Proof.* Let (A, M) be a cocone for  $\chi^{\omega}$ , i.e., a pair consisting of a C-algebra A together with a left A-module M (representing a linear functor  ${}_{\mathbb{C}}Mod \to {}_{A}Mod$ ) and homotopy commutative diagrams



where the  $\lambda_g$ 's are isomorphisms of left A-modules

$$\lambda_g \colon M \to M \otimes_{\mathbb{C}} L_g$$

such that ... (here  $L_g$  is the line associated to  $g \in G$  by the 2-character  $\omega$ :  $L_g \otimes L_h \cong L_{gh}$  etc.; this has to be written in detail before this proof in doing the first part of the TODO). The isomorphisms

$$\rho_{g,h} \colon L_g \otimes L_h \cong L_{gh}$$

endow the direct sum

$$\bigoplus_{g \in G} L_g$$

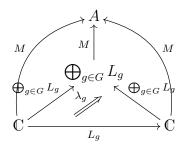
of a C-algebra structure, and the isomorphisms  $\lambda_g$  make M a right  $\bigoplus_{g \in G} L_g$ )module. Therefore, we can see M as a linear functor

$$(\bigoplus_{g \in G} L_g) \operatorname{Mod} \to {}_{A} \operatorname{Mod}$$

i.e., as a morphism from  $(\bigoplus_{g\in G} L_g)$  to A in 2-Vect<sub>C</sub>. The algebra  $\bigoplus_{g\in G} L_g$  seen as a left  $(\bigoplus_{g\in G} L_g)$ -module is a morphism from  $\mathbb C$  to  $\bigoplus_{g\in G} L_g$  in 2-Vect<sub>C</sub> and the natural isomorphism

$${}_{A}M_{\mathbb{C}} \cong {}_{A}M_{\bigoplus_{g \in G} L_g)} \otimes (\bigoplus_{g \in G} L_g) (\bigoplus_{g \in G} L_g)_{\mathbb{C}}$$

gives a canonical factorization



exhibiting the algebra  $\bigoplus_{g \in G} L_g$  together with itself seen as a left module over itself as the universal cocone.

Finally notice that choosing a linear basis  $x_g$  for each line  $L_g$ , the isomorphisms  $\rho_{g,h} \colon L_g \otimes L_h \cong L_{gh}$  (and so the multiplication in  $\bigoplus_{g \in G} L_g$ ) read

$$x_g \cdot x_h = \omega(g, h) x_{gh},$$

thus identifying the algebra  $\bigoplus_{g\in G} L_g$  with the twisted group algebra  $\mathbb{C}^{\omega}[G]$  of G.

#### **3.3** n = 3

TODO:

- The only non-trivial part of the 3-functor  $\chi^{\omega} \colon \boldsymbol{B}^{3}G \to \mathrm{TC}_{\mathbb{C}}$  are the coherence 3-morphisms which assemble into the modification (also) called  $\omega$  in [?, Def. A.4.3], and the constraints on  $\omega$  (see [?, Page 219]) seem to be precisely the 3-cocycle condition.
- <u>Issue</u>: The notion of a 3-colimit (to really compute Sum<sub>3</sub>) is scary. At least we should make some hand-wavy arguments...

## 4 Boundary conditions

**4.1** 
$$n=2$$

TODO

**4.2** 
$$n=3$$

TODO

## References

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