Boundaries in finite gauge theory

Ilka Brunner Nils Carqueville Domenico Fiorenza

Contents

1	Introduction
2	Special objects and 1-morphisms in $\operatorname{Fam}_n({m B}^n{\mathbb C}^ imes)$
3	Bulk theory
	$3.1 n=1 \dots \dots$
	$3.2 n=2 \ldots \ldots \ldots \ldots \ldots \ldots$
	3.3 $n=3$
4	Boundary conditions
	4.1 $n=2$
	4.2 $n = 3$

1 Introduction

Let

- G be a finite group,
- $[\omega] \in H^n(G, \mathbb{C}^\times),$
- and \mathcal{C} a symmetric monoidal (∞, n) -category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fullly extended TQFT

$$\operatorname{Bord}_n \xrightarrow{I} \operatorname{Fam}_n(\mathcal{C}) \xrightarrow{\operatorname{Sum}_n} \mathcal{C}$$
 (1.1)

based on the data (G, ω) , which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $C = \mathbf{B}^n \mathbb{C}^{\times}$ (the n-fold delooping of the group \mathbb{C}^{\times}) and then

- (i) identify classical boundary conditions as 1-morphisms with source * in $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$,
- (ii) subsume the boundary conditions of [Wi, WWW] as special cases,
- (iii) map classical boundary conditions to quantum ones via Sum_n , and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs-Schweigert-Valentino in the case n=3,
- (iv) do the above not only for framed, but also for oriented, spin,... TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1,2\}$ and hopefully also n=3. We also want to explain in detail how (G,ω) gives rise to an object in $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)(*,\mathbf{B}G)$.

(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [Lu, Sect. 4.3] instead of $Bord_n$, but we leave that for another project.)

2 Special objects and 1-morphisms in $\operatorname{Fam}_n(\boldsymbol{B}^n\mathbb{C}^{\times})$

TODO:

- spell out definition of $\operatorname{Fam}_n(\mathcal{C})$
- check that n-functor $BG \to B^n \mathbb{C}^{\times}$ is precisely an n-cocycle <u>Issue</u>: For n > 3, what is a weak n-functor, i. e. precisely what data and constraints are needed?

<u>Idea</u>: Since both n-categories in $BG \to B^n\mathbb{C}^{\times}$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section 3.

- clarify precisely how $BG \to B^n \mathbb{C}^{\times}$ induces an *n*-functor $BG \to n$ -Vect, at least for $n \in \{1, 2, 3\}$
- check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$

<u>Issue</u>: induced from issue in second item

3 Bulk theory

3.1 n=1

Since we take the action of G on \mathbb{C}^{\times} to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \operatorname{Hom}_{\operatorname{Grp}}(G, \mathbb{C}^\times)$$
(3.1)

is just a group homomorphism. We consider $B^1\mathbb{C}^{\times}$ as a subcategory of $\mathrm{Vect}_{\mathbb{C}}$, and we will use the pair (G, ω) to construct a functor

$$\operatorname{Bord}_1 \xrightarrow{I} \operatorname{Fam}_1(\operatorname{Vect}_{\mathbb{C}}) \xrightarrow{\operatorname{Sum}_1} \operatorname{Vect}_{\mathbb{C}}.$$
 (3.2)

According to the cobordism hypothesis, the composite $\operatorname{Sum}_1 \circ I$ is determined by its value on the point pt_+ , so we only need to specify $I(\operatorname{pt}_+)$ and then compute $\operatorname{Sum}_1(I(\operatorname{pt}_+))$. Note that $\mathbf{B}^1G = */\!\!/ G$ is the classical action groupoid. We set $I(\operatorname{pt}_+)$ to be the functor

$$\chi^{\omega} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto \mathbb{C}, \quad g \longmapsto \omega(g) \in \operatorname{Aut}(\mathbb{C})$$
 (3.3)

for all $g \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. Sum₁(χ^{ω}) = \mathbb{C} if $\omega(g) = 1$ for all $g \in G$, and Sum₁(χ^{ω}) = 0 otherwise.

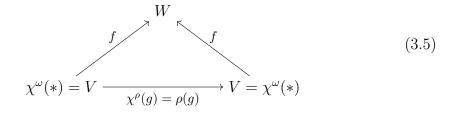
Proof. The statement is a particular case of this more general one: let (V, ρ) be a linear representation of the group G, seen as the functor

$$\chi^{\rho} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto V, \quad q \longmapsto \rho(q) \in \operatorname{Aut}(V).$$
 (3.4)

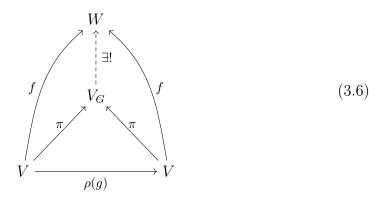
Then the universal cocone of χ^{ρ} , viewed as a diagram of shape $*/\!\!/ G$ in $\text{Vect}_{\mathbb{C}}$ is the pair (V_G, π) , where V_G is the vector space of coinvariants for the representation ρ , i.e.,

$$V_G = V/\langle v - \rho(g)v \rangle_{g \in G, v \in V}$$

and $\pi: V \to V_G$ is the projection to the quotient. Namely, let (W, f) be a cocone for χ^{ρ} , i.e., a pair consisting of a vector space W together with a linear map $f: \chi^{\omega}(*) = V \to W$ such that



Then, by definition of V_G , the morphism f uniquely factors through V_G and so we have a commutative diagram



showing that (V_G, π) enjoys the universal property of the universal cocone. \square Corollary 3.2. The linear dual $\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_1(\chi^{\omega}), \mathbb{C})$ of $\operatorname{Sum}_1(\chi^{\omega})$ is naturally

isomorphic to the vector space of natural transformations $\operatorname{Hom}_{[*/\!/G,\operatorname{Vect}_{\mathbb{C}}]}(\chi^{\omega},\chi^{1})$, where $\chi^{1}\colon */\!/G \to \operatorname{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space \mathbb{C} . In other words, we have

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_{1}(\chi^{\omega}),\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \chi^{\omega} \Longrightarrow * \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}.$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\operatorname{Sum}_1(\chi^{\omega})$.

Proof. Again, the statement is true for an aritrary linear representation (V, ρ) of the group G. The natural isomorphism

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(V_G,\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \downarrow \\ \downarrow \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}$$

is then nothing but the universal property of V_G . Namely, an element in the right hand side is a morphism $f: V = \chi^{\rho}(*) \to \chi^{1}(*) = \mathbb{C}$ such that all the diagrams

$$V \xrightarrow{f} \mathbb{C}$$

$$\rho(g) \downarrow \qquad \qquad \downarrow \mathrm{id}_{\mathbb{C}}$$

$$V \xrightarrow{f} \mathbb{C}$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams

$$V \xrightarrow{\rho(q)} V$$

commute, and so it is precisely the datum of a morphism $V_G \to \mathbb{C}$ by the argumenti in the proof of Lemma 3.5.

3.2 n=2

TODO:

- The only non-trivial part of the 2-functor $\chi^{\omega} : \mathbf{B}^2 G \to \mathrm{Alg}_{\mathbb{C}}$ is the coherence 2-morphism $\chi_{g,h}^{\omega} : \chi^{\omega}(g) \otimes \chi^{\omega}(h) \to \chi^{\omega}(gh)$, and the constraints (see e. g. [Le, Sect. 1.1] for the definitions) on the $\chi_{g,h}^{\omega}$ precisely say that they are the components of a 2-cocycle. So we can take $\chi_{g,h}^{\omega} = \omega(g,h)$.
- compute 2-colimit $\operatorname{Sum}_2(\chi)$ for χ as above, obtain twisted group algebra $\mathbb{C}^{\omega}[G]$

The datum of the 2-functor $\chi^{\omega} \colon \mathbf{B}^2 G \to \mathrm{Alg}_{\mathbb{C}}$ is a collection of lines (1-dimensional complex vector spaces) L_g , indexed by elements in the group G, together with isomorphisms

$$\rho_{g,h} \colon L_g \otimes L_h \xrightarrow{\sim} L_{gh}$$

subject to the associativity constraint given by the commutativity of the diagrams

$$\begin{array}{ccc} L_g \otimes L_h \otimes L_k & \xrightarrow{\rho_{g,h} \otimes \mathrm{id}} & L_{gh} \otimes L_k \\ & & \downarrow^{\rho_{gh,k}} & & \downarrow^{\rho_{gh,k}} \\ L_g \otimes L_{hk} & \xrightarrow{\rho_{g,hk}} & L_{ghk} \end{array}$$

where the tensor products are over \mathbb{C} . We also require that $L_1 = \mathbb{C}$ and that the isomorphisms $L_1 \otimes L_g \xrightarrow{\sim} L_g$ and $L_g \otimes L_1 \xrightarrow{\sim} L_g$ are the structure isomorphisms for the unit object \mathbb{C} of the monoidal category $\operatorname{Vect}_{\mathbb{C}}$. The associativity constraints imply, and are in fact equivalent to this, that the vector space

$$A_{\chi^{\omega}} = \bigoplus_{g \in G} L_g$$

has an associative C-algebra structure.

Lemma 3.3. A choice of a basis element x_g for every line L_g defines a \mathbb{C}^{\times} -valued 2-cocycle ω on the group G. A different choice of basis elements leads to a cohomologous cocycle, so that the class $[\omega] \in H^2(G, \mathbb{C}^{\times})$ is well defines.

Proof. As $\rho_{g,h}(x_g \otimes x_h)$ is a nonzero element in L_{gh} we have $\rho_{g,h}(x_g \otimes x_h) = \omega(g,h)x_{gh}$ for a unique element $\omega(g,h)$ in \mathbb{C}^{\times} . The associativity constraints are then immediately seen to be equivalent to the cocycle equation

$$\omega(gh, k)\omega(g, h) = \omega(g, hk)\omega(h, k).$$

Finally, if $\{y_g\}_{g\in G}$ is a different basis choice, and $\tilde{\omega}$ is the corresponding 2-cocycle, then we have $y_g = \eta_g x_g$ for some η_g in \mathbb{C}^{\times} and

$$\omega(\tilde{g},h)y_{gh} = \rho_{g,h}(y_g \otimes y_h) = \eta_g \eta_h \rho_{g,h}(x_g \otimes x_h) = \eta_g \eta_h \omega(g,h) x_{gh} = \eta_g \eta_h \omega(g,h) \eta_{gh}^{-1} y_{gh}.$$

Therefore

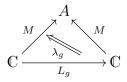
$$\tilde{\omega}(g,h) = \eta_g \eta_{gh}^{-1} \eta_h \omega(g,h),$$

i.e. ω and $\tilde{\omega}$ are cohomologous.

Corollary 3.4. The algebra $A_{\chi^{\omega}}$ is isomorphic to the twisted group algebra $\mathbb{C}^{\omega}[G]$, defined as the C-vector space on the basis $\{x_g\}_{g\in G}$ with the product $x_g \cdot x_h = \omega(g,h)x_{gh}$ and with $x_1 = 1$.

Lemma 3.5. $\operatorname{Sum}_2(\chi^{\omega}) \cong \mathbb{C}^{\omega}[G]$, the twisted group algebra of G.

Proof. Let (A, M) be a cocone for χ^{ω} , i.e., a pair consisting of a C-algebra A together with a left A-module M (representing a linear functor ${}_{\mathbb{C}}\mathrm{Mod} \to {}_{A}\mathrm{Mod}$) and homotopy commutative diagrams



where the λ_g 's are isomorphisms of left A-modules

$$\lambda_q \colon M \otimes_{\mathbb{C}} L_q \xrightarrow{\sim} M$$

such that the diagrams

$$\begin{array}{ccc} M \otimes L_g \otimes L_h & \xrightarrow{\lambda_g \otimes \mathrm{id}} & M \otimes L_h \\ & & \downarrow^{\lambda_h} & & \downarrow^{\lambda_h} \\ M \otimes L_{gh} & \xrightarrow{\lambda_{gh}} & M \end{array}$$

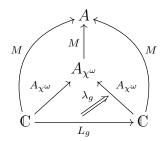
commute. This implies that the isomorphisms λ_g make M a right $A_{\chi^{\omega}}$ -module. Therefore, we can see M as a linear functor

$$_{A_{\gamma^{\omega}}}\mathrm{Mod} \to {}_{A}\mathrm{Mod}$$

i.e., as a morphism from $A_{\chi^{\omega}}$ to A in 2-Vect_C. The algebra $A_{\chi^{\omega}}$ seen as a left $A_{\chi^{\omega}}$ -module is a morphism from $\mathbb C$ to $A_{\chi^{\omega}}$ in 2-Vect_C and the natural isomorphism

$$_{A}M_{\mathbb{C}} \cong {_{A}M_{A_{\chi^{\omega}}}} \otimes {_{A_{\chi^{\omega}}}A_{\chi^{\omega}}}_{\mathbb{C}}$$

gives a canonical factorization



exhibiting the algebra $A_{\chi^{\omega}}$ together with itself seen as a left module over itself as the universal cocone.

3.3 n=3

TODO:

- The only non-trivial part of the 3-functor $\chi^{\omega} \colon \mathbf{B}^3 G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.
- <u>Issue</u>: The notion of a 3-colimit (to really compute Sum₃) is scary. At least we should make some hand-wavy arguments...

4 Boundary conditions

4.1
$$n=2$$

TODO

4.2
$$n = 3$$

TODO

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