Boundaries in finite gauge theory

Ilka Brunner Nils Carqueville Domenico Fiorenza

Contents

1 Introduction

Let

- G be a finite group,
- $[\omega] \in H^n(G, \mathbb{C}^\times)$,
- and $\mathcal{C}$ a symmetric monoidal (∞, n)-category for some $n \in \mathbb{Z}_+$.

On the one hand, [?] sketches the construction of a fullly extended TQFT

$$\mathrm{Bord}_n \xrightarrow[\text{“classical”}]{I} \mathrm{Fam}_n(\mathcal{C}) \xrightarrow[\text{“quantum”}]{\mathrm{Sum}_n} \mathcal{C} \tag{1.1}$$

based on the data (G, ω), which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [? ?] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [? ?] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [? ?] as the classical part I in (??) with $\mathcal{C} = \boldsymbol{B}^n\mathbb{C}^\times$ (the n-fold delooping of the group $\mathbb{C}^\times$) and then

(i) identify *classical* boundary conditions as 1-morphisms with source $*$ in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$,

(ii) subsume the boundary conditions of [? ?] as special cases,

(iii) map classical boundary conditions to quantum ones via Sum_n, and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs–Schweigert–Valentino in the case $n = 3$,

(iv) do the above not only for framed, but also for oriented, spin,... TQFTs.

Since [?] is very light on details, we first will work out explicitly how (??) recovers the known bulk theory for $n \in \{1, 2\}$ and hopefully also $n = 3$. We also want to explain in detail how (G, ω) gives rise to an object in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)(*, \boldsymbol{B}G)$.
(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [? , Sect. 4.3] instead of Bord_n, but we leave that for another project.)

2 Special objects and 1-morphisms in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$

TODO:

- spell out definition of $\mathrm{Fam}_n(\mathcal{C})$
- check that n-functor $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ is precisely an n-cocycle
  Issue: For $n > 3$, what is a weak n-functor, i. e. precisely what data and constraints are needed?
  Idea: Since both n-categories in $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section ??.
- clarify precisely how $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ induces an n-functor $\boldsymbol{B}G \to n$-Vect, at least for $n \in \{1, 2, 3\}$
- check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$
  Issue: induced from issue in second item

3 Bulk theory

3.1 $n = 1$

Since we take the action of G on $\mathbb{C}^\times$ to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{C}^\times) \tag{3.1}$$

is just a group homomorphism. We consider $\boldsymbol{B}^1\mathbb{C}^\times$ as a subcategory of $\mathrm{Vect}_\mathbb{C}$, and we will use the pair (G, ω) to construct a functor

$$\mathrm{Bord}_1 \xrightarrow{I} \mathrm{Fam}_1(\mathrm{Vect}_\mathbb{C}) \xrightarrow{\mathrm{Sum}_1} \mathrm{Vect}_\mathbb{C} \ . \tag{3.2}$$

According to the cobordism hypothesis, the composite $\mathrm{Sum}_1 \circ I$ is determined by its value on the point pt_+, so we only need to specify $I(\mathrm{pt}_+)$ and then compute $\mathrm{Sum}_1(I(\mathrm{pt}_+))$. Note that $\boldsymbol{B}^1 G = * /\!\!/ G$ is the classical action groupoid. We set $I(\mathrm{pt}_+)$ to be the functor

$$\chi^\omega \colon * /\!\!/ G \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,, \qquad * \longmapsto \mathbb{C}\,, \quad g \longmapsto \omega(g) \in \mathrm{Aut}(\mathbb{C}) \tag{3.3}$$

for all $g \in G$. Then (**??**) recovers the known result reviewed in [**?** , Sect. 1]:

Lemma 3.1. $\mathrm{Sum}_1(\chi^\omega) = \mathbb{C}$ if $\omega(g) = 1$ for all $g \in G$, and $\mathrm{Sum}_1(\chi^\omega) = 0$ otherwise.

Proof. The statement is a particular case of this more general one: let (V, ρ) be a linear representation of the group G, seen as the functor

$$\chi^\rho \colon * /\!\!/ G \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,, \qquad * \longmapsto V\,, \quad g \longmapsto \rho(g) \in \mathrm{Aut}(V). \tag{3.4}$$

Then the universal cocone of χ^ρ, viewed as a diagram of shape $*/\!\!/G$ in $\mathrm{Vect}_{\mathbb{C}}$ is the pair (V_G, π), where V_G is the vector space of coinvariants for the representation ρ, i.e.,

$$V_G = V / \langle v - \rho(g)v \rangle_{g \in G, v \in V}$$

and $\pi \colon V \to V_G$ is the projection to the quotient. Namely, let (W, f) be a cocone for χ^ρ, i.e., a pair consisting of a vector space W together with a linear map $f \colon \chi^\omega(*) = V \to W$ such that

$$\begin{tikzcd} & W & \\ \chi^\omega(*) = V \arrow[ur, "f"] \arrow[rr, "\chi^\rho(g) = \rho(g)"'] & & V = \chi^\omega(*) \arrow[ul, "f"'] \end{tikzcd} \tag{3.5}$$

Then, by definition of V_G, the morphism f uniquely factors through V_G and so we have a commutative diagram

$$\begin{tikzcd} & W & \\ & V_G \arrow[u, dashed, "\exists!"'] & \\ V \arrow[uur, bend left, "f"] \arrow[ur, "\pi"] \arrow[rr, "\rho(g)"'] & & V \arrow[uul, bend right, "f"'] \arrow[ul, "\pi"'] \end{tikzcd} \tag{3.6}$$

showing that (V_G, π) enjoys the universal property of the universal cocone. $\square$

Corollary 3.2. The linear dual $\mathrm{Hom}_{\mathrm{Vect}_\mathbb{C}}(\mathrm{Sum}_1(\chi^\omega), \mathbb{C})$ of $\mathrm{Sum}_1(\chi^\omega)$ is naturally isomorphic to the vector space of natural transformations $\mathrm{Hom}_{[*/\!\!/G, \mathrm{Vect}_\mathbb{C}]}(\chi^\omega, \chi^1)$, where $\chi^1\colon * /\!\!/G \to \mathrm{Vect}_\mathbb{C}$ is the trivial representation of G on the vector space $\mathbb{C}$. In other words, we have

$$\mathrm{Hom}_{\mathrm{Vect}_\mathbb{C}}(\mathrm{Sum}_1(\chi^\omega), \mathbb{C}) \cong \left\{ \begin{tikzcd} */\!\!/G \arrow[dd, "\chi^\omega"'] \arrow[dr] & \\ \arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl, "\mathbb{C}"] \\ \mathrm{Vect}_\mathbb{C} & \end{tikzcd} \right\}.$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\mathrm{Sum}_1(\chi^\omega)$.

Proof. Again, the statement is true for an aritrary linear representation (V, ρ) of the group G. The natural isomorphism

$$\mathrm{Hom}_{\mathrm{Vect}_\mathbb{C}}(V_G, \mathbb{C}) \cong \left\{ \begin{tikzcd} */\!\!/G \arrow[dd, "\chi^\rho"'] \arrow[dr] & \\ \arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl, "\mathbb{C}"] \\ \mathrm{Vect}_\mathbb{C} & \end{tikzcd} \right\}$$

is then nothing but the universal property of V_G. Namely, an element in the right hand side is a morphism $f\colon V = \chi^\rho(*) \to \chi^1(*) = \mathbb{C}$ such that all the diagrams

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & \mathbb{C} \arrow[d, "\mathrm{id}_\mathbb{C}"] \\ V \arrow[r, "f"'] & \mathbb{C} \end{tikzcd}$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams

$$\begin{tikzcd} & \mathbb{C} & \\ V \arrow[ur, "f"] \arrow[rr, "\rho(g)"'] & & V \arrow[ul, "f"'] \end{tikzcd}$$

commute, and so it is precisely the datum of a morphism $V_G \to \mathbb{C}$ by the argumenti in the proof of Lemma **??**. $\square$

3.2 $n = 2$

TODO:

- The only non-trivial part of the 2-functor $\chi^\omega \colon \boldsymbol{B}^2 G \to \mathrm{Alg}_{\mathbb{C}}$ is the coherence 2-morphism $\chi^\omega_{g,h} \colon \chi^\omega(g) \otimes \chi^\omega(h) \to \chi^\omega(gh)$, and the constraints (see e. g. [**?** , Sect. 1.1] for the definitions) on the $\chi^\omega_{g,h}$ precisely say that they are the components of a 2-cocycle. So we can take $\chi^\omega_{g,h} = \omega(g,h)$.
- compute 2-colimit $\mathrm{Sum}_2(\chi)$ for χ as above, obtain twisted group algebra $\mathbb{C}^\omega[G]$

Lemma 3.3. $\mathrm{Sum}_2(\chi^\omega) \cong \mathbb{C}^\omega[G]$, the twisted group algebra of G.

Proof. Let (A, M) be a cocone for χ^ω, i.e., a pair consisting of a $\mathbb{C}$-algebra A together with a left A-module M (representing a linear functor ${}_{\mathbb{C}}\mathrm{Mod} \to {}_A\mathrm{Mod}$) and homotopy commutative diagrams

$$\begin{tikzcd} & A & \\ \mathbb{C} \arrow[ur, "M"] \arrow[rr, "L_g"'] & \arrow[u, phantom, "\overset{\lambda_g}{\Rightarrow}"] & \mathbb{C} \arrow[ul, "M"'] \end{tikzcd}$$

where the λ_g's are isomorphisms of left A-modules

$$\lambda_g \colon M \to M \otimes_{\mathbb{C}} L_g$$

such that ... (here L_g is the line associated to $g \in G$ by the 2-character ω: $L_g \otimes L_h \cong L_{gh}$ etc. ; this has to be written in detail before this proof in doing the first part of the TODO). The isomorphisms

$$\rho_{g,h} \colon L_g \otimes L_h \cong L_{gh}$$

endow the direct sum

$$\bigoplus_{g \in G} L_g$$

of a $\mathbb{C}$-algebra structure, and the isomorphisms λ_g make M a right $(\bigoplus_{g\in G} L_g)$-module. Therefore, we can see M as a linear functor

$${}_{(\bigoplus_{g\in G} L_g)}\mathrm{Mod} \to {}_A\mathrm{Mod}$$

i.e., as a morphism from $(\bigoplus_{g\in G} L_g)$ to A in 2-$\mathrm{Vect}_{\mathbb{C}}$. The algebra $\bigoplus_{g\in G} L_g$ seen as a left $(\bigoplus_{g\in G} L_g)$-module is a morphism from $\mathbb{C}$ to $\bigoplus_{g\in G} L_g$ in 2-$\mathrm{Vect}_{\mathbb{C}}$ and the natural isomorphism

$${}_A M_{\mathbb{C}} \cong {}_A M_{(\bigoplus_{g\in G} L_g)} \otimes_{(\bigoplus_{g\in G} L_g)} (\bigoplus_{g\in G} L_g)_{\mathbb{C}}$$

gives a canonical factorization

$$\begin{tikzcd}
 & A & \\
 & \bigoplus_{g\in G} L_g \arrow[u, "M"] & \\
\mathbb{C} \arrow[ur, "\bigoplus_{g\in G} L_g"] \arrow[uur, bend left, "M"] \arrow[rr, "L_g"'] \arrow[r, phantom, "\Nearrow \lambda_g"] & & \mathbb{C} \arrow[ul, "\bigoplus_{g\in G} L_g"'] \arrow[uul, bend right, "M"']
\end{tikzcd}$$

exhibiting the algebra $\bigoplus_{g\in G} L_g$ together with itself seen as a left module over itself as the universal cocone.

Finally notice that choosing a linear basis x_g for each line L_g, the isomorphisms $\rho_{g,h}\colon L_g \otimes L_h \cong L_{gh}$ (and so the multiplication in $\bigoplus_{g\in G} L_g$) read

$$x_g \cdot x_h = \omega(g,h) x_{gh},$$

thus identifying the algebra $\bigoplus_{g\in G} L_g$ with the twisted group algebra $\mathbb{C}^\omega[G]$ of G. $\square$

3.3 $n = 3$

TODO:

- The only non-trivial part of the 3-functor $\chi^\omega\colon \boldsymbol{B}^3 G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [**?** , Def. A.4.3], and the constraints on ω (see [**?** , Page 219]) seem to be precisely the 3-cocycle condition.
- **Issue**: The notion of a 3-colimit (to really compute Sum_3) is scary. At least we should make some hand-wavy arguments. . .

4 Boundary conditions

4.1 $n = 2$

TODO

4.2 $n = 3$

TODO

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