Boundaries in finite gauge theory

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1 Introduction

Let

- \bullet G be a finite group,
- $\bullet \ \omega \in Z^n(G,\mathbb{C}^\times),$
- and \mathcal{C} a symmetric monoidal (∞, n) -category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fullly extended TQFT

$$\operatorname{Bord}_n \xrightarrow{I} \operatorname{Fam}_n(\mathcal{C}) \xrightarrow{\operatorname{Sum}_n} \mathcal{C}$$
 (1.1)

based on the data (G, ω) , which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $C = \mathbf{B}^n \mathbb{C}^{\times}$ (the n-fold delooping of the group \mathbb{C}^{\times}) and then

- (i) identify classical boundary conditions as 1-morphisms with source * in $\operatorname{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$,
- (ii) subsume the boundary conditions of [Wi, WWW] as special cases,
- (iii) map classical boundary conditions to quantum ones via Sum_n , and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs-Schweigert-Valentino in the case n=3,
- (iv) do the above not only for framed, but also for oriented, spin,...TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1,2\}$ and hopefully also n=3. We also want to explain in detail how (G,ω) gives rise to an object in $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)(*,\mathbf{B}G)$.

(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [Lu, Sect. 4.3] instead of $Bord_n$, but we leave that for another project.)

2 Special objects and 1-morphisms in $\operatorname{Fam}_n(\boldsymbol{B}^n\mathbb{C}^{\times})$

TODO:

- spell out definition of $\operatorname{Fam}_n(\mathcal{C})$
- check that n-functor $BG \to B^n \mathbb{C}^{\times}$ is precisely an n-cocycle <u>Issue</u>: For n > 3, what is a weak n-functor, i. e. precisely what data and constraints are needed?

<u>Idea</u>: Since both n-categories in $BG \to B^n\mathbb{C}^{\times}$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section 3.

• clarify precisely how $BG \to B^n \mathbb{C}^{\times}$ induces an *n*-functor $BG \to n$ -Vect, at least for $n \in \{1, 2, 3\}$

• check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$

Issue: induced from issue in second item

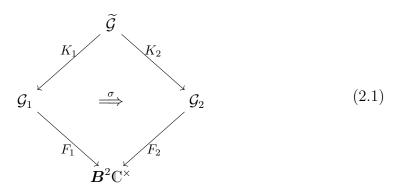
2.1 Classical bulk theories

TODO

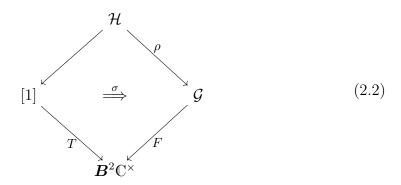
2.2 Classical boundary conditions

2.2.1 n=2

Let $\mathcal{G}_1, \mathcal{G}_2$ be 2-groupoids and let $F_1: \mathcal{G}_1 \to \mathbf{B}^2\mathbb{C}^{\times}$ and $F_2: \mathcal{G}_2 \to \mathbf{B}^2\mathbb{C}^{\times}$ be 2-functors, i. e. objects in $\operatorname{Fam}_2(\mathbf{B}^2\mathbb{C}^{\times})$. A 1-morphism $F_1 \to F_2$ in $\operatorname{Fam}_2(\mathbf{B}^2\mathbb{C}^{\times})$ is a span of 2-groupoids $\mathcal{G}_1 \stackrel{K_1}{\longleftarrow} \widetilde{\mathcal{G}} \stackrel{K_2}{\longrightarrow} \mathcal{G}_2$ together with a pseudonatural transformation $\sigma: F_1 \circ K_1 \to F_2 \circ K_2$:



Let [1] denote the final 2-category with only one object, 1- and 2-morphism, and let $T: [1] \to \mathbf{B}^2 \mathbb{C}^{\times}$ be the unique, trivial 2-functor. A classical boundary condition for $F: \mathcal{G} \to \mathbf{B}^2 \mathbb{C}^{\times}$ is a 1-morphism $T \to F$. Thus it comprises a 2-groupoid \mathcal{H} , a 2-functor $\rho: \mathcal{H} \to \mathcal{G}$, and a pseudonatural transformation



Concretely, since $\mathbf{B}^2\mathbb{C}^{\times}$ has only one object and one 1-morphism, σ is given by a family of 2-morphisms $\sigma_h \colon (F \circ \rho)(h) = 1_* \to 1_*$, indexed by 1-morphisms h

in \mathcal{H} , such that

$$F(\rho(h)) \otimes F(\rho(h')) \xrightarrow{1 \otimes \sigma_{h'}} F(\rho(h))$$

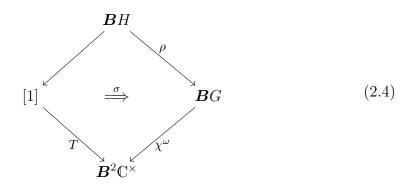
$$(F \circ \rho)_{h,h'} \downarrow \qquad \qquad \downarrow \sigma_{h}$$

$$F(\rho(hh')) \xrightarrow{\sigma_{hh'}} 1_{*}$$

$$(2.3)$$

commutes for al composable 1-morphisms h, h' in \mathcal{H} .

We now choose $\mathcal{H} = \mathbf{B}H$ for some finite group H, $\mathcal{G} = \mathbf{B}G$, and $F = \chi^{\omega}$ as in (3.8) — and we are not careful to distinguish the target $\mathbf{B}^2\mathbb{C}^{\times}$ from $\mathrm{Alg}_{\mathbb{C}}$. Then a 2-functor $\rho \colon \mathbf{B}H \to \mathbf{B}G$ is just a group homomorphism $\rho \colon H \to G$, and a boundary condition



for χ^{ω} is a family $\{\sigma_h\}_{h\in H}\subset \mathbb{C}^{\times}$ such that

$$L_{\rho(h)} \otimes_{\mathbf{C}} L_{\rho(h')} \xrightarrow{1 \otimes \sigma_{h'}} L_{\rho(h)}$$

$$\omega(\rho(h), \rho(h')) \downarrow \qquad \qquad \downarrow \sigma_{h}$$

$$L_{\rho(hh')} \xrightarrow{\sigma_{hh'}} 1_{*}$$

$$(2.5)$$

commutes, i. e. $\omega(\rho(h), \rho(h')) = \sigma_h \cdot \sigma_{hh'}^{-1} \cdot \sigma_{h'}$ for all $h, h' \in H$. Viewing σ as a map $H \to \mathbb{C}^{\times}$, $h \mapsto \sigma_h$, this condition precisely means that σ trivialises the pullback of ω :

$$d\sigma = \rho^* \omega \,. \tag{2.6}$$

3 Bulk theory

3.1 n=1

Since we take the action of G on \mathbb{C}^{\times} to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \operatorname{Hom}_{\operatorname{Grp}}(G, \mathbb{C}^\times)$$
 (3.1)

is just a group homomorphism. We consider $B^1\mathbb{C}^{\times}$ as a subcategory of $\mathrm{Vect}_{\mathbb{C}}$, and we will use the pair (G, ω) to construct a functor

$$\operatorname{Bord}_1 \xrightarrow{I} \operatorname{Fam}_1(\operatorname{Vect}_{\mathbb{C}}) \xrightarrow{\operatorname{Sum}_1} \operatorname{Vect}_{\mathbb{C}}.$$
 (3.2)

According to the cobordism hypothesis, the composite $\operatorname{Sum}_1 \circ I$ is determined by its value on the point pt_+ , so we only need to specify $I(\operatorname{pt}_+)$ and then compute $\operatorname{Sum}_1(I(\operatorname{pt}_+))$. Note that $\mathbf{B}^1G = */\!\!/ G$ is the classical action groupoid. We set $I(\operatorname{pt}_+)$ to be the functor

$$\chi^{\omega} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto \mathbb{C}, \quad g \longmapsto \omega(g) \in \operatorname{Aut}(\mathbb{C})$$
 (3.3)

for all $g \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. Sum₁(χ^{ω}) = \mathbb{C} if $\omega(g) = 1$ for all $g \in G$, and Sum₁(χ^{ω}) = 0 otherwise.

Proof. The statement is a particular case of this more general one: let (V, ρ) be a linear representation of the group G, seen as the functor

$$\chi^{\rho} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto V, \quad g \longmapsto \rho(g) \in \operatorname{Aut}(V).$$
 (3.4)

Then the universal cocone of χ^{ρ} , viewed as a diagram of shape $*/\!\!/ G$ in Vect_C is the pair (V_G, π) , where V_G is the vector space of coinvariants for the representation ρ , i.e.,

$$V_G = V/\langle v - \rho(g)v \rangle_{g \in G, v \in V}$$

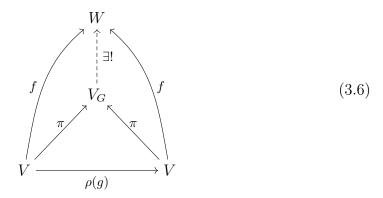
and $\pi: V \to V_G$ is the projection to the quotient. Namely, let (W, f) be a cocone for χ^{ρ} , i.e., a pair consisting of a vector space W together with a linear map $f: \chi^{\rho}(*) = V \to W$ such that

$$\chi^{\rho}(*) = V \xrightarrow{\chi^{\rho}(g) = \rho(g)} V = \chi^{\rho}(*)$$

$$(3.5)$$

Then, by definition of V_G , the morphism f uniquely factors through V_G and so

we have a commutative diagram



showing that (V_G, π) enjoys the universal property of the universal cocone.

Corollary 3.2. The linear dual $\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_{1}(\chi^{\omega}),\mathbb{C})$ of $\operatorname{Sum}_{1}(\chi^{\omega})$ is naturally isomorphic to the vector space of natural transformations $\operatorname{Hom}_{[*/\!/G,\operatorname{Vect}_{\mathbb{C}}]}(\chi^{\omega},\chi^{1})$, where $\chi^{1}\colon */\!\!/G \to \operatorname{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space \mathbb{C} . In other words, we have

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_{1}(\chi^{\omega}),\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \chi^{\omega} \Longrightarrow * \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}.$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\operatorname{Sum}_1(\chi^{\omega})$.

Proof. Again, the statement is true for an aritrary linear representation (V, ρ) of the group G. The natural isomorphism

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(V_G,\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \downarrow^{\rho} \Longrightarrow * \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}$$

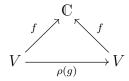
is then nothing but the universal property of V_G . Namely, an element in the right hand side is a morphism $f: V = \chi^{\rho}(*) \to \chi^{1}(*) = \mathbb{C}$ such that all the diagrams

$$V \xrightarrow{f} \mathbb{C}$$

$$\rho(g) \downarrow \qquad \qquad \downarrow \mathrm{id}_{\mathbb{C}}$$

$$V \xrightarrow{f} \mathbb{C}$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams



commute, and so it is precisely the datum of a morphism $V_G \to \mathbb{C}$ by the argumenti in the proof of Lemma 3.8.

3.2 n=2

3.2.1 "Without thinking" (Nils)

Now let $\omega \in Z^2(G, \mathbb{C}^{\times})$. We will construct a 2-functor

$$\operatorname{Bord}_2 \xrightarrow{I} \operatorname{Fam}_2(\operatorname{Alg}_{\mathbb{C}}) \xrightarrow{\operatorname{Sum}_2} \operatorname{Alg}_{\mathbb{C}}$$
 (3.7)

by computing what it does to $pt_+ \in Bord_2$. The first step is to set $I(pt_+)$ to be the 2-functor

$$\chi^{\omega} \colon \mathbf{B}G \longrightarrow \operatorname{Alg}_{\mathbb{C}} \qquad \text{with} \quad \chi_{g,h}^{\omega} := \omega(g,h) \colon L_{g} \otimes_{\mathbb{C}} L_{h} \to L_{gh} \qquad (3.8)$$

$$* \longmapsto \mathbb{C}$$

$$G \ni g \longmapsto L_{g} := {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$$

$$1_{g} \longmapsto 1_{\mathbb{C}}$$

(Recall from [Le, Sect. 1.1] that the constraint on the coherence 2-morphisms $\chi_{g,h}^{\omega} = \omega(g,h) \in \operatorname{End}_{\mathbb{C}}(\mathbb{C})$ is precisely the 2-cocycle condition on ω . Also note that all our 2-functors are strictly unital, so in particular (3.8) completely specifies the 2-functor χ^{ω} .)

The second step is to compute the 2-colimit $\operatorname{Sum}_2(\chi^{\omega})$. There are several versions of 2-limit, bilimit, weighted bilimit etc. in the literature. We will use the notion of [Bo, Def. 7.4.4] and call it 2-limit. To give the details, recall the definitions from [Le, Sect. 1.2–3], and that the *opposite* \mathcal{B}^{op} of a bicategory \mathcal{B} has reversed horizontal composition, but the same vertical composition as \mathcal{B} .

Definition 3.3. Let $F: \mathcal{A} \to \mathcal{B}$ be a 2-functor, and let $B \in \mathcal{B}$.

- (i) The constant 2-functor $\Delta_B \colon \mathcal{A} \to \mathcal{B}$ sends every object in \mathcal{A} to B, 1- and 2-morphisms in \mathcal{A} to 1_B and 1_{1_B} , respectively, and all constraints $(\Delta_B)_{g,h}$ are trivial.
- (ii) The category 2-Cone(B, F) of 2-cones on F with vertex B is the category whose objects are pseudonatural transformations $\Delta_B \to F$ and whose morphisms are modifications between them.

(iii) A 2-limit of F is a pair (L, π) with $L \in \mathcal{B}$ and $\pi: \Delta_L \to F$ a pseudonatural transformation such that the functor

$$\mathcal{B}(B, L) \longrightarrow 2\text{-Cone}(B, F)$$

 $(B \xrightarrow{f} L) \longmapsto (\Delta_B \xrightarrow{f_*} \Delta_L \xrightarrow{\pi} F)$

is an equivalence of categories for all $B \in \mathcal{B}$.

(iv) Now let $G: \mathcal{A} \to \mathcal{B}^{\text{op}}$. The category 2-Cocone(B, G) of 2-cocones on G with vertex $B \in \mathcal{B}$ is defined to be 2-Cone(B, G), but with B viewed as an object in \mathcal{B}^{op} . A 2-limit of $G: \mathcal{A} \to \mathcal{B}^{\text{op}}$ is also called a 2-colimit of $G: \mathcal{B}$.

Spelling out the definition, we see that an object $\sigma: \Delta_B \to F$ of 2-Cone(B, F) is a family of 1- and 2-morphisms

$$\{B \xrightarrow{\sigma_A} F(A)\}_{A \in \mathcal{A}} \quad \text{and} \quad \{F(g) \otimes \sigma_A \xrightarrow{\sigma_g} \sigma_{A'}\}_{g \in \mathcal{A}(A, A')}$$
 (3.9)

such that

$$F(g) \otimes F(h) \otimes \sigma_{A} \xrightarrow{1 \otimes \sigma_{h}} F(g) \otimes \sigma_{A'}$$

$$F_{g,h} \otimes 1 \qquad \qquad \qquad \downarrow \sigma_{g} \qquad (3.10)$$

$$F(gh) \otimes \sigma_{A} \xrightarrow{\sigma_{gh}} \sigma_{A''}$$

commutes for all $A \xrightarrow{h} A' \xrightarrow{g} A''$. Furthermore, a morphism $\Gamma \colon \sigma \to \widetilde{\sigma}$ in 2-Cone(B, F) is a family

$$\{\sigma_{A} \xrightarrow{\Gamma_{A}} \widetilde{\sigma}_{A}\}_{A \in \mathcal{A}} \quad \text{such that} \qquad \begin{array}{c} F(g) \otimes \sigma_{A} \xrightarrow{1 \otimes \Gamma_{A}} F(g) \otimes \widetilde{\sigma}_{A} \\ \\ \sigma_{g} \downarrow & \downarrow \widetilde{\sigma}_{g} \\ \\ \sigma_{A'} \xrightarrow{\Gamma_{A'}} \widetilde{\sigma}_{A'} \end{array}$$
(3.11)

commutes for all $A \stackrel{g}{\longrightarrow} A'$.

Note that above " \otimes " is exclusively used for the horizontal composition in the target \mathcal{B} (while juxtaposition is used for the horizontal composition in \mathcal{A}). Hence going to \mathcal{B}^{op} amounts to reversing all the tensor products above.

Lemma 3.4. Let $B \in Alg_{\mathbb{C}}$ and $\chi^{\omega} \colon BG \to Alg_{\mathbb{C}}$ as in (3.8). Then

$$2-\operatorname{Cone}(B,\chi^{\omega}) \cong {}_{\mathbb{C}^{\omega}[G]}\operatorname{Mod}_{B},$$

$$2-\operatorname{Cocone}(B,\chi^{\omega}) \cong {}_{B}\operatorname{Mod}_{\mathbb{C}^{\omega}[G]}$$
(3.12)

where $\mathbb{C}^{\omega}[G]$ is the ω -twisted group algebra of G.

Proof. We only prove the the statement about 2-cones in detail, the one about 2-cocones follows analogously by reversing all horizontal compositions in \mathcal{B} .

By (3.9) and (3.10), an object in 2-Cone (B, χ^{ω}) is a 1-morphism $M \in Alg_{\mathbb{C}}(B, \mathbb{C})$ together with linear maps $\sigma_g \colon L_g \otimes_{\mathbb{C}} M \to M$ such that

$$L_{g} \otimes_{\mathbb{C}} L_{h} \otimes_{\mathbb{C}} M \xrightarrow{1 \otimes \sigma_{h}} L_{g} \otimes_{\mathbb{C}} M$$

$$\omega(g,h) \otimes 1 \downarrow \qquad \qquad \downarrow \sigma_{g}$$

$$L_{gh} \otimes_{\mathbb{C}} M \xrightarrow{\sigma_{gh}} M$$

$$(3.13)$$

commutes for all $g, h \in G$. But this means that $(M, \sum_{g \in G} \sigma_g)$ is a left $\mathbb{C}^{\omega}[G]$ -module, while by definition of $\mathrm{Alg}_{\mathbb{C}}$, the 1-morphism $M \in \mathrm{Alg}_{\mathbb{C}}(B, \mathbb{C})$ is a right B-module.

According to (3.11), a morphism $(M, \{\sigma_g\}) \to (\widetilde{M}, \{\widetilde{\sigma}_g\})$ in 2-Cone (B, χ^{ω}) is a 2-morphism $\Gamma \colon {}_{\mathbb{C}}M_B \to {}_{\mathbb{C}}\widetilde{M}_B$ in $\mathrm{Alg}_{\mathbb{C}}$ such that

$$L_{g} \otimes_{\mathbb{C}} M \xrightarrow{1 \otimes \Gamma} L_{g} \otimes_{\mathbb{C}} \widetilde{M}$$

$$\sigma_{g} \downarrow \qquad \qquad \downarrow \widetilde{\sigma}_{g}$$

$$M \xrightarrow{\Gamma} \widetilde{M}$$

$$(3.14)$$

commutes. But that means that Γ is both a map of left $\mathbb{C}^{\omega}[G]$ -modules and a map of right B-modules.

Proposition 3.5. (i) A 2-limit of $\chi^{\omega} : \mathbf{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^{\omega}[G], \mathbb{C}^{\omega}[G])$.

(ii) A 2-colimit of $\chi^{\omega} \colon \mathbf{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^{\omega}[G], \mathbb{C}^{\omega}[G])$.

Proof. According to Definition 3.3 and Lemma 3.4, a 2-limit of χ^{ω} is an algebra $L \in \text{Alg}_{\mathbb{C}}$ together with a module $\mathbb{C}^{\omega[G]}M_L$ such that the functor

$$Alg_{\mathbb{C}}(B, L) \longrightarrow_{\mathbb{C}^{\omega}[G]} Mod_B$$

 $N \longmapsto M \otimes_L N$

is an equivalence for every $B \in Alg_{\mathbb{C}}$. This is the case for $L = \mathbb{C}^{\omega}[G]$ as an algebra and $M = \mathbb{C}^{\omega}[G]$ as a bimodule over itself.

The statement about the 2-colimit follows analogously.

3.2.2 "With thinking" (Domenico)

The datum of the 2-functor $\chi^{\omega} \colon \mathbf{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is a collection of lines (1-dimensional complex vector spaces) L_g , indexed by elements in the group G, together with isomorphisms

$$\rho_{q,h} \colon L_q \otimes L_h \xrightarrow{\sim} L_{qh}$$

subject to the associativity constraint given by the commutativity of the diagrams

$$L_{g} \otimes L_{h} \otimes L_{k} \xrightarrow{\rho_{g,h} \otimes \mathrm{id}} L_{gh} \otimes L_{k}$$

$$\downarrow^{\rho_{gh,k}} \downarrow \qquad \qquad \downarrow^{\rho_{gh,k}}$$

$$L_{g} \otimes L_{hk} \xrightarrow{\rho_{g,hk}} L_{ghk}$$

where the tensor products are over \mathbb{C} . We also require that $L_1 = \mathbb{C}$ and that the isomorphisms $L_1 \otimes L_g \xrightarrow{\sim} L_g$ and $L_g \otimes L_1 \xrightarrow{\sim} L_g$ are the structure isomorphisms for the unit object \mathbb{C} of the monoidal category $\mathrm{Vect}_{\mathbb{C}}$. The associativity constraints imply, and are in fact equivalent to this, that the vector space

$$A_{\chi^{\omega}} = \bigoplus_{g \in G} L_g$$

has an associative C-algebra structure.

Lemma 3.6. A choice of a basis element x_g for every line L_g defines a \mathbb{C}^{\times} -valued 2-cocycle ω on the group G. A different choice of basis elements leads to a cohomologous cocycle, so that the class $[\omega] \in H^2(G, \mathbb{C}^{\times})$ is well defines.

Proof. As $\rho_{g,h}(x_g \otimes x_h)$ is a nonzero element in L_{gh} we have $\rho_{g,h}(x_g \otimes x_h) = \omega(g,h)x_{gh}$ for a unique element $\omega(g,h)$ in \mathbb{C}^{\times} . The associativity constraints are then immediately seen to be equivalent to the cocycle equation

$$\omega(gh, k)\omega(g, h) = \omega(g, hk)\omega(h, k).$$

Finally, if $\{y_g\}_{g\in G}$ is a different basis choice, and $\tilde{\omega}$ is the corresponding 2-cocycle, then we have $y_g = \eta_g x_g$ for some η_g in \mathbb{C}^{\times} and

$$\omega(\tilde{g},h)y_{gh} = \rho_{g,h}(y_g \otimes y_h) = \eta_g \eta_h \rho_{g,h}(x_g \otimes x_h) = \eta_g \eta_h \omega(g,h) x_{gh} = \eta_g \eta_h \omega(g,h) \eta_{gh}^{-1} y_{gh}.$$

Therefore

$$\tilde{\omega}(g,h) = \eta_g \eta_{gh}^{-1} \eta_h \omega(g,h),$$

i.e. ω and $\tilde{\omega}$ are cohomologous.

Corollary 3.7. The algebra $A_{\chi^{\omega}}$ is isomorphic to the twisted group algebra $\mathbb{C}^{\omega}[G]$, defined as the C-vector space on the basis $\{x_g\}_{g\in G}$ with the product $x_g \cdot x_h = \omega(g,h)x_{gh}$ and with $x_1 = 1$.

Lemma 3.8. $\operatorname{Sum}_2(\chi^{\omega}) \cong \mathbb{C}^{\omega}[G]$, the twisted group algebra of G.

Proof. Let (A, M) be a cocone for χ^{ω} , i.e., a pair consisting of a C-algebra A together with a left A-module M (representing a linear functor ${}_{\mathbb{C}}Mod \to {}_{A}Mod$) and homotopy commutative diagrams

$$\mathbb{C} \xrightarrow{A \atop \lambda_g} \mathbb{C}$$

where the λ_g 's are isomorphisms of left A-modules

$$\lambda_g \colon M \otimes_{\mathbb{C}} L_g \xrightarrow{\sim} M$$

such that the diagrams

$$M \otimes L_{g} \otimes L_{h} \xrightarrow{\lambda_{g} \otimes \mathrm{id}} M \otimes L_{h}$$

$$\downarrow^{\lambda_{h}} \qquad \qquad \downarrow^{\lambda_{h}}$$

$$M \otimes L_{gh} \xrightarrow{\lambda_{gh}} M$$

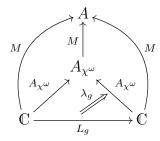
commute. This implies that the isomorphisms λ_g make M a right $A_{\chi^{\omega}}$ -module. Therefore, we can see M as a linear functor

$$A_{\chi^{\omega}} \operatorname{Mod} \to A \operatorname{Mod}$$

i.e., as a morphism from $A_{\chi^{\omega}}$ to A in 2-Vect_C. The algebra $A_{\chi^{\omega}}$ seen as a left $A_{\chi^{\omega}}$ -module is a morphism from $\mathbb C$ to $A_{\chi^{\omega}}$ in 2-Vect_C and the natural isomorphism

$$_{A}M_{\mathbb{C}}\cong {_{A}M_{A_{\chi^{\omega}}}}\otimes {_{A_{\chi^{\omega}}}A_{\chi^{\omega}}}_{\mathbb{C}}$$

gives a canonical factorization



exhibiting the algebra $A_{\chi^{\omega}}$ together with itself seen as a left module over itself as the universal cocone.

3.3 n=3

TODO:

- The only non-trivial part of the 3-functor $\chi^{\omega} : \mathbf{B}G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.
- <u>Issue</u>: The notion of a 3-colimit (to really compute Sum₃) is scary. At least we should make some hand-wavy arguments...

4 Boundary conditions

4.1 n=2

TODO

4.2 n = 3

TODO

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