Sum_1

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This file contains the details on the $n = 1$ case, to be merged into the main file as soon as they become polished enough.

Let us denote by $\mathrm{Fam}_0(\mathbb{C})$ the set of equivalence classes of finite groupoids equipped with functors to $\mathbb{C}$ (seen as a trivial category). In other words, a representative of an element of $\mathrm{Fam}_0(\mathbb{C})$ is a pair (X, ρ) consisting of a finite groupoid X together with a map $\rho\colon \pi_0(X) \to \mathbb{C}$. The map

$$\mathrm{Sum}_0\colon \mathrm{Fam}_0 \to \mathbb{C}$$

is defined as

$$\mathrm{Sum}_0(X, \rho) = \sum_{[x]\in\pi_0(X)} \frac{\rho(x)}{|\mathrm{Aut}(x)|}.$$

We will also use the more evocative notation

$$\int_X \rho(x) d\mu(x)$$

to denote $\mathrm{Sum}_0(X, \rho)$. Notice that Sum_0 is linear in the ρ variable.

A nice property of the morphism Sum_0 (which we are going to use later) is the following: let X be a finite set with an action of a finite group G, and let $X/\!/G$ denotes the action groupoid for this action; finally, let $\rho\colon X/\!/G \to \mathbb{C}$ any functor, i.e., let $\rho\colon X \to \mathbb{C}$ be a function that is constant on the G-orbits in X. Then we have

$$\mathrm{Sum}_0(X/\!/G, \rho) = \frac{1}{|G|} \sum_{x\in X} \rho(x). \tag{0.1}$$

Namely, the set $\pi_0(X/\!/G)$ is the quotient set X/G and for any $x \in X$ seen as an object in $X/\!/G$ we have $\mathrm{Aut}(x) = \mathrm{Stab}_G(x)$. Therefore

$$\mathrm{Sum}_0(X/\!/G, \rho) = \sum_{[x]\in X/G} \frac{\rho(x)}{|\mathrm{Stab}_G(x)|}$$

$$
\begin{aligned}
&= \frac{1}{|G|} \sum_{[x]\in X/G} \frac{|G|}{|\mathrm{Stab}_G(x)|}\rho(x) \\
&= \frac{1}{|G|} \sum_{[x]\in X/G} |[x]|\, \rho(x) \\
&= \frac{1}{|G|} \sum_{[x]\in X/G} \sum_{x\in[x]} \rho(x) \\
&= \frac{1}{|G|} \sum_{x\in X} \rho(x)
\end{aligned}
$$

Remark 0.1. A function $f\colon \pi_0(X) \to \mathbb{C}$ can equivalently be seen as an element in $\mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X)$, where $\mathbf{1}_X\colon X \to \mathrm{Vect}$ is the constant functor taking the value $\mathbb{C}$ on X. The map Sum_0 can therefore be seen as the datum of a map

$$\mathrm{Sum}_0\colon \mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X) \to \mathbb{C},$$

for any finite groupoid X.

Definition 0.2. The category $\mathrm{Fam}_1(\mathrm{Vect})$ has as its objects finite groupoids equipped with functors to Vect, i.e., pairs (X, ρ) consisting of a finite groupoid X together with a functor $\rho\colon X \to \mathrm{Vect}$. Morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ are spans of groupoids over Vect, i.e., commutative diagrams

$$
\begin{tikzcd}
 & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\rho_X"'] \arrow[rr, phantom, "\overset{}{\underset{\alpha}{\Longrightarrow}}"] & & Y \arrow[dl, "\rho_Y"] \\
 & \mathrm{Vect} &
\end{tikzcd}
$$

where the natural transformation $\alpha\colon \rho_X \circ \pi_X \to \rho_Y \circ \pi_Y$ is part of the data of the commutative diagram.

Remark 0.3. The identity mophisms and the composition of morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ will be defined later.

For every object (X, ρ) in $\mathrm{Fam}_1(\mathrm{Vect})$ we can consider the vector space

$$[\mathbf{1}_X, \rho] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\mathbf{1}_X, \rho).$$

If $f\colon Y \to X$ is a functor between finite groupoids, and $\rho\colon X \to \mathrm{Vect}$ is a functor, we write $f^*\rho\colon Y \to \mathrm{Vect}$ for the functor $\rho\circ f$. If $\alpha\colon \mathbf{1}_X \to$

ρ is a natural transformation, then $f^*\alpha$ is a natural transformation between $f^*\mathbf{1}_X = \mathbf{1}_Y$ and $f^*\rho$. This defines a linear map

$$f^*\colon [\mathbf{1}_X, \rho] \to [\mathbf{1}_Y, f^*\rho].$$

Similarly, we can consider the vector space

$$[\rho, \mathbf{1}_X] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\rho, \mathbf{1}_X)$$

together with the linear map

$$f^*\colon [\rho, \mathbf{1}_X] \to [f^*\rho, \mathbf{1}_Y].$$

Passing to the linear duals we get the linear map

$$(f^*)^\vee\colon [f^*\rho, \mathbf{1}_Y]^\vee \to [\rho, \mathbf{1}_X]^\vee.$$

Next, notice that composition of morphisms gives a bilinear map

$$[\rho, \mathbf{1}_X] \otimes [\mathbf{1}_X, \rho] \to [\mathbf{1}_X, \mathbf{1}_X]$$

and so composing this with the linear map $\mathrm{Sum}_0(X, -)\colon [\mathbf{1}_X, \mathbf{1}_X] \to \mathbb{C}$ we get a bilinear pairing $\langle\,|\,\rangle_X$ between $[\rho, \mathbf{1}_X]$ and $[\mathbf{1}_X, \rho]$, for any ρ. Notice that, by its very definition, the pairing $\langle\,|\,\rangle_X$ satisfies

$$\langle v|\alpha \circ w\rangle_X = \langle v \circ \alpha|w\rangle_X,$$

for any $v \in [\rho_2, \mathbf{1}_X]$, any $w \in [\mathbf{1}_X, \rho_1]$ and any natural transformation $\alpha\colon \rho_1 \to \rho_2$. Equivalently, the bilinear pairing $\langle\,|\,\rangle_X$ is a linear map

$$\delta_{(X,\rho)}\colon [\mathbf{1}_X, \rho] \to [\rho, \mathbf{1}_X]^\vee.$$

Our main *duality assumption* will be that $\delta_{(X,\rho)}$ is a linear isomorphism for any (X, ρ).

Remark 0.4. The duality assumption holds for every finite groupoid X if and only if it holds for finite connected groupoids. Up to equivalence we are therefore reduced to considering the case $X = \mathbf{B}G$, with G a finite group. In this case $\rho\colon \mathbf{B}G \to \mathrm{Vect}$ is the datum of a linear representation V of G, and the vector spaces $[\mathbf{1}_{\mathbf{B}G}, \rho]$ and $[\rho, \mathbf{1}_{\mathbf{B}G}]$ are isomorphic to the space V^G of G-invariants and the linear dual $(V_G)^\vee$ of the space of G-coinvariants, respectively. The map $\delta_{(\mathbf{B}G,\rho)}$ is therefore a map

$$\delta_{(\mathbf{B}G,\rho)}\colon V^G \to (V_G)^{\vee\vee} \cong V_G,$$

from G-invariants to G-coinvariants. Let $v \mapsto [v]$ be the canonical projection $V \to V_G = V/\langle v - g \cdot v\rangle_{v \in V; g \in G}$. By composing with the inclusion $V^G \hookrightarrow V$ we get the morphism $\iota\colon v \mapsto [v]$ from $V^G \to V_G$. This is an isomorphism. Indeed, the canonical projection $\pi\colon V \to V^G$ given by

$$\pi\colon v \mapsto \frac{1}{|G|}\sum_{g\in G} g\cdot v$$

is zero on the elements of the form $v - g\cdot v$ and so induces a linear map $\tilde{\pi}\colon V_G \to V^G$, which is immediate to see is the inverse of ι. For any $w \in (V_G)^\vee$ seen as an element in $[\rho, \mathbf{1}_{\mathbf{B}G}]$ and any $v \in V^G$ seen as an element in $[\mathbf{1}_{\mathbf{B}G}, \rho]$ we have $w \circ v = w([v])$ Therefore

$$\langle w|v\rangle_{\mathbf{B}G} = \frac{1}{|G|} w\circ v = w\left(\frac{1}{|G|}\iota(v)\right).$$

This means that $\delta_{(\mathbf{B}G,\rho)}v$ is the evaluation on $\frac{1}{|G|}\iota(v)$. Therefore, under the canonical isomorphism $(V_G)^{\vee\vee} \cong V_G$, we have $\delta_{(\mathbf{B}G,\rho)} = \frac{1}{|G|}\iota$. As ι is an isomorphism, so is $\delta_{(\mathbf{B}G,\rho)}$.

Under the duality assumption we may define a linear morphism

$$f_*\colon [\mathbf{1}_Y, f^*\rho] \to [\mathbf{1}_X, \rho]$$

as the composition

$$[\mathbf{1}_Y, f^*\rho] \xrightarrow{\delta_{(Y,f^*\rho)}} [f^*\rho, \mathbf{1}_Y]^\vee \xrightarrow{(f^*)^\vee} [\rho, \mathbf{1}_X]^\vee \xrightarrow{\delta^{-1}_{(X,\rho)}} [\mathbf{1}_X, \rho].$$

In other words, if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_*v \in [\mathbf{1}_X, \rho]$ is defined by the equation

$$\langle w|f_*v\rangle_X = \langle f^*w|v\rangle_Y$$

for any $w \in [\rho, \mathbf{1}_X]$. Similarly, we have a morphism $f_*\colon [f^*\rho, \mathbf{1}_Y] \to [\rho, \mathbf{1}_X]$ defined by

$$\langle f_*w|v\rangle_X = \langle w|f^*v\rangle_Y.$$

We can now define our wannabe functor

$$\mathrm{Sum}_1\colon \mathrm{Fam}_1(\mathrm{Vect}) \to \mathrm{Vect}$$

on objects as

$$\mathrm{Sum}_1(X,\rho) = [\mathbf{1}_X, \rho]$$

and on a morphism

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[rr, phantom, "\underset{\alpha}{\Rrightarrow}"] \arrow[dr, "\rho_X"'] & & Y \arrow[dl, "\rho_Y"] \\ & \mathrm{Vect} & \end{tikzcd}$$

as the composition

$$[\mathbf{1}_X, \rho_X] \xrightarrow{\pi_X^*} [\mathbf{1}_Z, \pi_X^* \rho_X] \xrightarrow{\alpha \circ -} [\mathbf{1}_Z, \pi_Y^* \rho_Y] \xrightarrow{\pi_{Y*}} [\mathbf{1}_Y, \rho_Y].$$

Remark 0.5. Before giving a more explicit expression of the above morphism, let us notice that $[\mathbf{1}_X, \rho]$ is a model for the limit of the functor $\rho\colon X \to \mathrm{Vect}$, in accordance with the prescrition in [**?**]. Also, notice that the morphism associated to the span $X \leftarrow Z \to Y$ together with the natural transformation α has the form "pull-compose-push", and so it is an instance of a Fourier-Mukai-type transform.

Lemma 0.6. (The projection formula) Let $f\colon Y \to X$ be a functor between finite groupoids, let $\rho_1, \rho_2\colon X \to \mathrm{Vect}$ be functors and let $\alpha\colon \rho_1 \to \rho_2$ be a natural transformation. We have an identity of morphisms $[\mathbf{1}_Y, f^*\rho_1] \to [\mathbf{1}_X, \rho_2]$

$$f_*(f^*\alpha \circ -) = \alpha \circ f_*$$

Proof. The identity

$$f_*(f^*\alpha \circ v) = \alpha \circ f_* v$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ is equivalent to the identity

$$\langle w | f_*(f^*\alpha \circ v)\rangle_X = \langle w | \alpha \circ f_* v\rangle_X$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ and every $w \in [\rho_2, \mathbf{1}_X]$. By the properties of the pairings $\langle\,|\,\rangle_X$ and $\langle\,|\,\rangle_X$, and by definition of f_*, we have

$$\begin{aligned}
\langle w | f_*(f^*\alpha \circ v)\rangle_X &= \langle f^* w | f^*\alpha \circ v\rangle_Y \\
&= \langle f^* w \circ f^*\alpha | v\rangle_Y \\
&= \langle f^*(w \circ \alpha) | v\rangle_Y \\
&= \langle w \circ \alpha | f_* v\rangle_X \\
&= \langle w | \alpha \circ f_* v\rangle_X
\end{aligned}$$

□

For later use, let us derive an explicit formla for f_* in case $f\colon \mathbf{B}G \to \mathbf{B}H$ is a morphism of finite groups (seen as groupoids with a single object).

Recall that if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_* v \in [\mathbf{1}_X, \rho]$ is defined by the equation $\langle w | f_* v\rangle_{\mathbf{B}H} = \langle f^* w | v\rangle_{\mathbf{B}G}$ for any $w \in [\rho, \mathbf{1}_X]$. As both $\mathbf{B}G$ and $\mathbf{B}H$ have a single object, this equation reduces to

$$\frac{w \circ f_* v}{|H|} = \frac{f^* w \circ v}{|G|}$$

The morphism $\rho\colon \mathbf{B}H \to \mathrm{Vect}$ is the datum of a linear representation of H on a vector space V, and $f^*\rho$ is the datum of representation of G obtained by pull-back, i.e., the group G acts on V by $g \cdot \xi := f(g) \cdot \xi$. The element $v \in [\mathbf{1}_Y, f^*\rho]$ is therefore a G-invariant vector of V, while $w \in [\rho, \mathbf{1}_X]$ is a linear functional on V factoring through the space of H-coinvariants. The element f^*w is the same linear functional on V, but this time seen as factoring through the space of H-coinvariants. Therefore the composition $f^*w \circ v$ is just the evaluation of $w\colon V \to \mathbb{C}$ on the vector $v \in V$, under the canonical isomorphism $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$. Finally, $f_*v \in [\mathbf{1}_X, \rho]$ is an H-invariant vector of V, and $w \circ f_*v$ is the evaluation of $w\colon V \to \mathbb{C}$ on the vector $f_*v \in V$. The defining equation for f_*v then becomes

$$w\left(\frac{f_*v}{|H|}\right) = w\left(\frac{v}{|G|}\right)$$

for any $w\colon V \to \mathbb{C}$ factoring through the space of H-coinvariants. From $v/|G|$ we get the H-invariant vector

$$\frac{1}{|H|}\sum_{h\in H} h \cdot \frac{v}{|G|}$$

and, since w factors through the H-coinvariants,

$$w\left(\frac{v}{|G|}\right) = w\left(\frac{1}{|H|}\sum_{h\in H} h \cdot \frac{v}{|G|}\right)$$

The defining equation for f_*v can therefore be rewritten as

$$\langle w|f_*v\rangle_{\mathbf{B}H} = \langle w|\sum_{h\in H}\frac{h\cdot v}{|G|}\rangle_{\mathbf{B}H},$$

for any $w \in [\rho, \mathbf{1}_X]$. As the pairing $\langle\,|\,\rangle_{\mathbf{B}H}$ is nondegenerate, this finally gives

$$f_*v = \frac{1}{|G|}\sum_{h\in H} h \cdot v.$$

Remark 0.7. This formula simplifies a bit in case the group homomorphism $f\colon G \to H$ is surjective. Indeed, in that case, the G-invariant vector v will be automatically H-invariant and we find

$$f_*v = \frac{|H|}{|G|}v$$

in this case.

Lemma 0.8. Consider a homotopy pullback diagram of finite groupoids

$$\begin{tikzcd} & X \times_T Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "f_1"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\eta}"] & & Y \arrow[dl, "f_2"] \\ & T & \end{tikzcd}$$

and a morphism $\rho\colon T \to \mathrm{Vect}$. Then the Beck-Chevalley condition is satisfied, i.e., we have an identity of morphisms $f_2^* f_{1*} = p_{2*}(\eta^*\rho \circ -)p_1^*$ from $[\mathbf{1}_X, f_1^*\rho]$ to $[\mathbf{1}_Y, f_2^*\rho]$.

Proof. By restricting to connected components, and working up to equivalence of finite groupoids, we may assume that the given pullback diagram has the form

$$\begin{tikzcd} & G\backslash\backslash H//K \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbf{B}G \arrow[dr, "f_1"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\eta}"] & & \mathbf{B}K \arrow[dl, "f_2"] \\ & \mathbf{B}H & \end{tikzcd}$$

where the natural transformation η is defined by $\eta_h = h$; this precisely expresses the fact that a morphism from h to h' in the action groupoid $G\backslash\backslash H//K$ is given by a commutative diagram

$$\begin{tikzcd} * \arrow[r, "h"] \arrow[d, "f_1(g)"'] & * \arrow[d, "f_2(k)"] \\ * \arrow[r, "h'"'] & * \end{tikzcd}.$$

The equation $f_2^* f_{1*} = p_{2*}(\eta^*\rho \circ -)p_1^*$ is equivalent to $\langle w | f_2^* f_{1*} v\rangle_{\mathbf{B}K} = \langle w | p_{2*}(\eta^*\rho \circ p_1^* v)\rangle_{\mathbf{B}K}$ for any $v \in [\mathbf{1}_{\mathbf{B}G}, f_1^*\rho]$ and any $w \in [f_2^*\rho, \mathbf{1}_{\mathbf{B}K}]$. This is equivalent to the equation

$$\langle f_{2*} w | f_{1*} v\rangle_{\mathbf{B}H} = \langle p_2^* w | \eta^*\rho \circ p_1^* v\rangle_{G\backslash\backslash H//K}.$$

The morphism $\rho\colon \mathbf{B}H \to \mathrm{Vect}$ is the datum of a linear representation of the finite group H on a vector space V. The element v is a G-invariant vector of V (where G acts on V via f_1) and w is a linear functional on V which factors through the K-coinvariants (with K

acting via f_2). By the above explicit formula for f_* in the case of a group morphism, we have

$$f_{1*}v = \frac{1}{|G|}\sum_{h\in H} h\cdot v; \qquad f_{2*}w = \frac{1}{|K|}\sum_{h\in H} h\cdot w,$$

where on the right we have the H-action on linear functionals on V, i.e., $(h\cdot w) = w(h^{-1}\cdot -)$. We therefore find

$$\begin{aligned}
\langle f_{2*}w|f_{1*}v\rangle_{\mathbf{B}H} &= \frac{1}{|H||G||K|}\sum_{h_1,h_2\in H} w(h_1^{-1}h_2\cdot v)\\
&= \frac{1}{|H||G||K|}\sum_{h\in H}\sum_{h_1^{-1}h_2=h} w(h_1^{-1}h_2\cdot v)\\
&= \frac{1}{|H||G||K|}\sum_{h\in H}\sum_{h_1\in H} w(h\cdot v)\\
&= \frac{1}{|G||K|}\sum_{h\in H} w(h\cdot v)
\end{aligned}$$

To every object h of $G\backslash\backslash H//K$, the functors $p_1^*f_1^*\rho$ and $p_2^*f_2^*\rho$ both assign the vector space V, so p_1^*v and p_2^*w consist in a copy of the vector v and of the linear functional $w\colon V\to\mathbb{C}$ for each of the copies of V associated with the elements of H. Moreover, since $\eta_h = h$, the natural transformation $\eta^*\rho = \rho\circ\eta$ at the object h is precisely the multiplication by h as a linear map from V to V. This means that $\eta^*\rho\circ p_1^*v = (h\cdot v)_{h\in H}$, i.e, $\eta^*\rho\circ p_1^*v$ consists of the choice of the vector $h\cdot v$ in the copy of V associated with the object h of $G\backslash\backslash H//K$. We therefore find

$$\langle p_2^*w|\eta^*\rho\circ p_1^*v\rangle_{G\backslash\backslash H//K} = \frac{1}{|G||K|}\sum_{h\in H} w(h\cdot v),$$

by the formula (0.1). □

Proposition 0.9. Sum_1 is a functor $\mathrm{Fam}_1(\mathrm{Vect})\to\mathrm{Vect}$.

Proof. We have the verify that Sum_1 is compatible with units and composition. Let $\alpha\colon\rho_1\to\rho_2$ and $\beta\colon\rho_2\to\rho_3$ be composable morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$:

$$\begin{tikzcd}
& Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 \arrow[dr, "\rho_1"'] \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] & & X_2 \arrow[dl, "\rho_2"] \\
& \mathrm{Vect} &
\end{tikzcd} , \quad
\begin{tikzcd}
& Y' \arrow[dl, "p_2'"'] \arrow[dr, "p_3'"] & \\
X_2 \arrow[dr, "\rho_2"'] \arrow[rr, phantom, "\overset{\beta}{\Longrightarrow}"] & & X_3 \arrow[dl, "\rho_3"] \\
& \mathrm{Vect} &
\end{tikzcd} .$$

Then by definition of $\mathrm{Fam}_1(\mathrm{Vect})$, their composition $\beta \circ_\eta \alpha$ is the natural transformation $(\beta * 1_{\pi_2}) \cdot (1_{\rho_2} * \eta) \cdot (\alpha * 1_{\pi_1})$:

$$\beta \circ_\eta \alpha \overset{\mathrm{def}}{=}
\begin{tikzcd}
& & Y \times_{X_2} Y' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
& Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[rr, phantom, "\overset{\eta}{\Longrightarrow}"] & & Y' \arrow[dl, "p_2'"'] \arrow[dr, "p_3'"] & \\
X_1 \arrow[dr, "\rho_1"'] \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] & & X_2 \arrow[dl, "\rho_2"] \arrow[dr, "\rho_2"'] \arrow[rr, phantom, "\overset{\beta}{\Longrightarrow}"] & & X_3 \arrow[dl, "\rho_3"] \\
& \mathrm{Vect} \arrow[dr, equal] \arrow[rr, phantom, "\overset{1_{\rho_2}}{\Longrightarrow}"] & & \mathrm{Vect} \arrow[dl, equal] & \\
& & \mathrm{Vect} & &
\end{tikzcd} .$$

Furthermore, by definition of Sum_1, its action on the three morphisms is as follows:

$$\begin{aligned}
\mathrm{Sum}_1(\alpha) &= p_{2*} \cdot (\alpha \circ (-)) \cdot p_1^* , \\
\mathrm{Sum}_1(\beta) &= p'_{3*} \cdot (\beta \circ (-)) \cdot p_2'^* , \\
\mathrm{Sum}_1(\beta \circ_\eta \alpha) &= (p_3' \circ \pi_2)_* \cdot (\beta \circ_\eta \alpha \circ (-)) \cdot (p_1 \circ \pi_1)^* .
\end{aligned}$$

To show that $\mathrm{Sum}_1(\beta \circ_\eta \alpha) = \mathrm{Sum}_1(\beta) \circ \mathrm{Sum}_1(\alpha)$, we first observe that the left-hand side is the clock-wise composition of the outmost arrows from the top left to the bottom left vertex in the following diagram:

$$
\begin{tikzcd}
{[\mathbf{1}_{X_1}, \rho_1]} \arrow[r, "p_1^*"] \arrow[rrr, bend left, "(p_1 \circ \pi_1)^*"] \arrow[ddddd, "{[p'_{3*} \cdot (\beta \circ (-)) \cdot {p'_2}^*] \cdot [p_{2*} \cdot (\alpha \circ (-)) \cdot p_1^*] = \mathrm{Sum}_1(\beta) \cdot \mathrm{Sum}_1(\alpha)}"'] & {[\mathbf{1}_Y, p_1^*\rho_1]} \arrow[rr, "\pi_1^*"] \arrow[d, "\alpha \circ (-)"'] & & {[\mathbf{1}_{Y \times_{X_2} Y'}, \pi_1^* p_1^* \rho_1]} \arrow[ddl, "\pi_1^*\alpha \circ (-)"'] \arrow[ddddd, "\beta \circ_\eta \alpha"] \\
& {[\mathbf{1}_Y, p_2^*\rho_2]} \arrow[dr, "\pi_1^*"] \arrow[ddd, "{p'_2}^* \, p_{2*}"'] & & \\
& & {[\mathbf{1}_{Y \times_{X_2} Y'}, \pi_1^* p_2^* \rho_2]} \arrow[d, "\eta^*\rho_2 \circ (-)"'] & \\
& & {[\mathbf{1}_{Y \times_{X_2} Y'}, \pi_2^* {p'_2}^* \rho_2]} \arrow[dl, "\pi_{2*}"'] \arrow[ddr, "\pi_2^*\beta \circ (-)"] & \\
& {[\mathbf{1}_{Y'}, {p'_2}^* \rho_2]} \arrow[d, "\beta \circ (-)"'] & & \\
{[\mathbf{1}_{X_3}, \rho_3]} & {[\mathbf{1}_{X_3}, {p'_3}^* \rho_3]} \arrow[l, "p'_{3*}"] & & {[\mathbf{1}_{Y \times_{X_2} Y'}, \pi_2^* {p'_3}^* \rho_3]} \arrow[ll, "\pi_{2*}"] \arrow[lll, bend left, "(p'_3 \circ \pi_2)_*"]
\end{tikzcd}
$$

The leftmost vertical arrow is $\mathrm{Sum}_1(\beta) \circ \mathrm{Sum}_1(\alpha)$, hence we are done if we can show that the diagram commutes. This is indeed the case: the leftmost and rightmost subdiagrams commute by definition, as does the other subdiagram involving $\pi_1^*\alpha \circ (-)$; that the top and bottom diagrams commute follows from the definition of pullback and push-forward; finally, the subdiagram involving $\pi_2^*\beta \circ (-)$ and π_{2*} commutes by the projection formula of Lemma 0.6, and the middle subdiagram commutes thanks to the Beck-Chevalley condition of Lemma 0.8.

Finally, if $\rho_2 = \rho_3$, $Y = Y'$ and β is the identity on ρ_2 in $\mathrm{Fam}_1(\mathrm{Vect})$, then $Y \times_{X_2} Y'$ is canonically equivalent to Y, and we see that Sum_1 sends units to units. $\square$