

Boundaries in finite gauge theory

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1 Introduction

Let

- G be a finite group,
- $\omega \in Z^n(G, \mathbb{C}^\times)$,
- and \mathcal{C} a symmetric monoidal (∞, n) -category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fully extended TQFT

$$\text{Bord}_n \xrightarrow[\text{“classical”}]{I} \text{Fam}_n(\mathcal{C}) \xrightarrow[\text{“quantum”}]{\text{Sum}_n} \mathcal{C} \quad (1.1)$$

based on the data (G, ω) , which is viewed as a fully extended version of Dijkgraaf–Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $\mathcal{C} = \mathbf{B}^n \mathbb{C}^\times$ (the n -fold delooping of the group \mathbb{C}^\times) and then

- (i) identify *classical* boundary conditions as 1-morphisms with source $*$ in $\text{Fam}_n(\mathbf{B}^n \mathbb{C}^\times)$,
- (ii) subsume the boundary conditions of [Wi, WWW] as special cases,
- (iii) map classical boundary conditions to quantum ones via Sum_n , and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs–Schweigert–Valentino in the case $n = 3$,
- (iv) do the above not only for framed, but also for oriented, spin, ... TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1, 2\}$ and hopefully also $n = 3$. We also want to explain in detail how (G, ω) gives rise to an object in $\text{Fam}_n(\mathbf{B}^n \mathbb{C}^\times)$, and how group extensions $1 \rightarrow K \rightarrow H \xrightarrow{r} G \rightarrow 1$ give rise to 1-morphisms $\text{Fam}_n(\mathbf{B}^n \mathbb{C}^\times)(*, BG)$.

(A more conceptual approach to boundary conditions and defects would be to start with a bordism category “with singularities” [Lu, Sect. 4.3] instead of Bord_n , but we leave that for another project.)

2 Special objects and 1-morphisms in $\text{Fam}_n(\mathbf{B}^n \mathbb{C}^\times)$

TODO:

- spell out definition of $\text{Fam}_n(\mathcal{C})$
- check that n -functor $BG \rightarrow \mathbf{B}^n \mathbb{C}^\times$ is precisely an n -cocycle
Issue: For $n > 3$, what is a weak n -functor, i. e. precisely what data and constraints are needed?
Idea: Since both n -categories in $BG \rightarrow \mathbf{B}^n \mathbb{C}^\times$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section 3.
- clarify precisely how $BG \rightarrow \mathbf{B}^n \mathbb{C}^\times$ induces an n -functor $BG \rightarrow n\text{-Vect}$, at least for $n \in \{1, 2, 3\}$

- check whether natural transformation from trivial n -functor to $\omega \circ r$ is trivialisation of $r^*\omega$
Issue: induced from issue in second item

3 Bulk theory

3.1 $n = 1$

Since we take the action of G on \mathbb{C}^\times to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \text{Hom}_{\text{Grp}}(G, \mathbb{C}^\times) \quad (3.1)$$

is just a group homomorphism. We consider $\mathbf{B}^1\mathbb{C}^\times$ as a subcategory of $\text{Vect}_{\mathbb{C}}$, and we will use the pair (G, ω) to construct a functor

$$\text{Bord}_1 \xrightarrow{I} \text{Fam}_1(\text{Vect}_{\mathbb{C}}) \xrightarrow{\text{Sum}_1} \text{Vect}_{\mathbb{C}}. \quad (3.2)$$

According to the cobordism hypothesis, the composite $\text{Sum}_1 \circ I$ is determined by its value on the point pt_+ , so we only need to specify $I(\text{pt}_+)$ and then compute $\text{Sum}_1(I(\text{pt}_+))$. Note that $\mathbf{B}^1G = *//G$ is the classical action groupoid. We set $I(\text{pt}_+)$ to be the functor

$$\chi^\omega: *//G \longrightarrow \text{Vect}_{\mathbb{C}}, \quad * \longmapsto \mathbb{C}, \quad g \longmapsto \omega(g) \in \text{Aut}(\mathbb{C}) \quad (3.3)$$

for all $g \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. $\text{Sum}_1(\chi^\omega) = \mathbb{C}$ if $\omega(g) = 1$ for all $g \in G$, and $\text{Sum}_1(\chi^\omega) = 0$ otherwise.

Proof. The statement is a particular case of this more general one: let (V, ρ) be a linear representation of the group G , seen as the functor

$$\chi^\rho: *//G \longrightarrow \text{Vect}_{\mathbb{C}}, \quad * \longmapsto V, \quad g \longmapsto \rho(g) \in \text{Aut}(V). \quad (3.4)$$

Then the universal cocone of χ^ρ , viewed as a diagram of shape $*//G$ in $\text{Vect}_{\mathbb{C}}$ is the pair (V_G, π) , where V_G is the vector space of coinvariants for the representation ρ , i.e.,

$$V_G = V / \langle v - \rho(g)v \rangle_{g \in G, v \in V}$$

and $\pi: V \rightarrow V_G$ is the projection to the quotient. Namely, let (W, f) be a cocone for χ^ρ , i.e., a pair consisting of a vector space W together with a linear map $f: \chi^\omega(*) = V \rightarrow W$ such that

$$\begin{array}{ccc} & W & \\ f \nearrow & & \nwarrow f \\ \chi^\omega(*) = V & \xrightarrow{\chi^\rho(g) = \rho(g)} & V = \chi^\omega(*) \end{array} \quad (3.5)$$

Then, by definition of V_G , the morphism f uniquely factors through V_G and so we have a commutative diagram

$$\begin{array}{ccc}
 & W & \\
 f \nearrow & \uparrow \exists! & \nwarrow f \\
 & V_G & \\
 \pi \nearrow & & \nwarrow \pi \\
 V & \xrightarrow{\rho(g)} & V
 \end{array} \tag{3.6}$$

showing that (V_G, π) enjoys the universal property of the universal cocone. \square

Corollary 3.2. The linear dual $\text{Hom}_{\text{Vect}_{\mathbb{C}}}(\text{Sum}_1(\chi^\omega), \mathbb{C})$ of $\text{Sum}_1(\chi^\omega)$ is naturally isomorphic to the vector space of natural transformations $\text{Hom}_{[*//G, \text{Vect}_{\mathbb{C}}]}(\chi^\omega, \chi^1)$, where $\chi^1: *//G \rightarrow \text{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space \mathbb{C} . In other words, we have

$$\text{Hom}_{\text{Vect}_{\mathbb{C}}}(\text{Sum}_1(\chi^\omega), \mathbb{C}) \cong \left\{ \begin{array}{ccc} *//G & & \\ \chi^\omega \downarrow & \Rightarrow & * \\ \text{Vect}_{\mathbb{C}} & \swarrow \mathbb{C} & \end{array} \right\}.$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\text{Sum}_1(\chi^\omega)$.

Proof. Again, the statement is true for an arbitrary linear representation (V, ρ) of the group G . The natural isomorphism

$$\text{Hom}_{\text{Vect}_{\mathbb{C}}}(V_G, \mathbb{C}) \cong \left\{ \begin{array}{ccc} *//G & & \\ \chi^\rho \downarrow & \Rightarrow & * \\ \text{Vect}_{\mathbb{C}} & \swarrow \mathbb{C} & \end{array} \right\}$$

is then nothing but the universal property of V_G . Namely, an element in the right hand side is a morphism $f: V = \chi^\rho(*) \rightarrow \chi^1(*) = \mathbb{C}$ such that all the diagrams

$$\begin{array}{ccc}
 V & \xrightarrow{f} & \mathbb{C} \\
 \rho(g) \downarrow & & \downarrow \text{id}_{\mathbb{C}} \\
 V & \xrightarrow{f} & \mathbb{C}
 \end{array}$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams

$$\begin{array}{ccc} & \mathbb{C} & \\ f \swarrow & & \nwarrow f \\ V & \xrightarrow{\rho(g)} & V \end{array}$$

commute, and so it is precisely the datum of a morphism $V_G \rightarrow \mathbb{C}$ by the argument in the proof of Lemma 3.8. \square

3.2 $n = 2$

3.2.1 “Without thinking” (Nils)

Now let $\omega \in Z^2(G, \mathbb{C}^\times)$. We will construct a 2-functor

$$\text{Bord}_2 \xrightarrow{I} \text{Fam}_2(\text{Alg}_{\mathbb{C}}) \xrightarrow{\text{Sum}_2} \text{Alg}_{\mathbb{C}} \quad (3.7)$$

by computing what it does to $\text{pt}_+ \in \text{Bord}_2$. The first step is to set $I(\text{pt}_+)$ to be the 2-functor

$$\begin{aligned} \chi^\omega: \mathbf{BG} &\longrightarrow \text{Alg}_{\mathbb{C}} & \text{with } \chi_{g,h}^\omega &:= \omega(g, h): L_g \otimes_{\mathbb{C}} L_h \rightarrow L_{gh} \\ * &\longmapsto \mathbb{C} \\ G \ni g &\longmapsto L_g := {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \\ 1_g &\longmapsto 1_{\mathbb{C}} \end{aligned} \quad (3.8)$$

(Recall from [Le, Sect. 1.1] that the constraint on the coherence 2-morphisms $\chi_{g,h}^\omega = \omega(g, h) \in \text{End}_{\mathbb{C}}(\mathbb{C})$ is precisely the 2-cocycle condition on ω . Also note that all our 2-functors are strictly unital, so in particular (3.8) completely specifies the 2-functor χ^ω .)

The second step is to compute the 2-colimit $\text{Sum}_2(\chi^\omega)$. There are several versions of 2-limit, bilimit, weighted bilimit etc. in the literature. We will use the notion of [Bo, Def. 7.4.4] and call it 2-limit. To give the details, recall the definitions from [Le, Sect. 1.2–3], and that the *opposite* \mathcal{B}^{op} of a bicategory \mathcal{B} has reversed horizontal composition, but the same vertical composition as \mathcal{B} .

Definition 3.3. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a 2-functor, and let $B \in \mathcal{B}$.

- (i) The *constant 2-functor* $\Delta_B: \mathcal{A} \rightarrow \mathcal{B}$ sends every object in \mathcal{A} to B , 1- and 2-morphisms in \mathcal{A} to 1_B and 1_{1_B} , respectively, and all constraints $(\Delta_B)_{g,h}$ are trivial.
- (ii) The category $2\text{-Cone}(B, F)$ of *2-cones on F with vertex B* is the category whose objects are pseudonatural transformations $\Delta_B \rightarrow F$ and whose morphisms are modifications between them.

- (iii) A 2-limit of F is a pair (L, π) with $B \in \mathcal{B}$ and $\pi: \Delta_L \rightarrow F$ a pseudonatural transformation such that the functor

$$\begin{aligned} \mathcal{B}(B, L) &\longrightarrow 2\text{-Cone}(B, F) \\ (B \xrightarrow{f} L) &\longmapsto (\Delta_B \xrightarrow{f_*} \Delta_L \xrightarrow{\pi} F) \end{aligned}$$

is an equivalence of categories for all $B \in \mathcal{B}$.

- (iv) Now let $G: \mathcal{A} \rightarrow \mathcal{B}^{\text{op}}$. The category $2\text{-Cocone}(B, G)$ of 2-cocones on G with vertex $B \in \mathcal{B}$ is defined to be $2\text{-Cone}(B, G)$, but with B viewed as an object in \mathcal{B}^{op} . A 2-limit of $G: \mathcal{A} \rightarrow \mathcal{B}^{\text{op}}$ is also called a 2-colimit of G in \mathcal{B} .

Spelling out the definition, we see that an object $\sigma: \Delta_B \rightarrow F$ of $2\text{-Cone}(B, F)$ is a family of 1- and 2-morphisms

$$\{B \xrightarrow{\sigma_A} F(A)\}_{A \in \mathcal{A}} \quad \text{and} \quad \{F(g) \otimes \sigma_A \xrightarrow{\sigma_g} \sigma_{A'}\}_{g \in \mathcal{A}(A, A')} \quad (3.9)$$

such that

$$\begin{array}{ccc} F(g) \otimes F(h) \otimes \sigma_A & \xrightarrow{1 \otimes \sigma_h} & F(g) \otimes \sigma_{A'} \\ \downarrow F_{g,h} \otimes 1 & & \downarrow \sigma_g \\ F(gh) \otimes \sigma_A & \xrightarrow{\sigma_{gh}} & \sigma_{A''} \end{array} \quad (3.10)$$

commutes for all $A \xrightarrow{h} A' \xrightarrow{g} A''$. Furthermore, a morphism $\Gamma: \sigma \rightarrow \tilde{\sigma}$ in $2\text{-Cone}(B, F)$ is a family

$$\{\sigma_A \xrightarrow{\Gamma_A} \tilde{\sigma}_A\}_{A \in \mathcal{A}} \quad \text{such that} \quad \begin{array}{ccc} F(g) \otimes \sigma_A & \xrightarrow{1 \otimes \Gamma_A} & F(g) \otimes \tilde{\sigma}_A \\ \downarrow \sigma_g & & \downarrow \tilde{\sigma}_g \\ \sigma_{A'} & \xrightarrow{\Gamma_{A'}} & \tilde{\sigma}_{A'} \end{array} \quad (3.11)$$

commutes for all $A \xrightarrow{g} A'$.

Note that above “ \otimes ” is exclusively used for the horizontal composition in the target \mathcal{B} (while juxtaposition is used for the horizontal composition in \mathcal{A}). Hence going to \mathcal{B}^{op} amounts to reversing all the tensor products above.

Lemma 3.4. Let $B \in \text{Alg}_{\mathbb{C}}$ and $\chi^\omega: \mathbf{B}G \rightarrow \text{Alg}_{\mathbb{C}}$ as in (3.8). Then

$$\begin{aligned} 2\text{-Cone}(B, \chi^\omega) &\cong_{\mathbb{C}^\omega[G] \text{Mod}_B} \mathbb{C}^\omega[G] \text{Mod}_B, \\ 2\text{-Cocone}(B, \chi^\omega) &\cong_{B \text{Mod}_{\mathbb{C}^\omega[G]}} B \text{Mod}_{\mathbb{C}^\omega[G]} \end{aligned} \quad (3.12)$$

where $\mathbb{C}^\omega[G]$ is the ω -twisted group algebra of G .

Proof. We only prove the the statement about 2-cones in detail, the one about 2-cocones follows analogously by reversing all horizontal compositions in \mathcal{B} .

By (3.9) and (3.10), an object in $2\text{-Cone}(B, \chi^\omega)$ is a 1-morphism $M \in \text{Alg}_{\mathbb{C}}(B, \mathbb{C})$ together with linear maps $\sigma_g: L_g \otimes_{\mathbb{C}} M \rightarrow M$ such that

$$\begin{array}{ccc} L_g \otimes_{\mathbb{C}} L_h \otimes_{\mathbb{C}} M & \xrightarrow{1 \otimes \sigma_h} & L_g \otimes_{\mathbb{C}} M \\ \omega(g, h) \otimes 1 \downarrow & & \downarrow \sigma_g \\ L_{gh} \otimes_{\mathbb{C}} M & \xrightarrow{\sigma_{gh}} & M \end{array} \quad (3.13)$$

commutes for all $g, h \in G$. But this means that $(M, \sum_{g \in G} \sigma_g)$ is a left $\mathbb{C}^\omega[G]$ -module, while by definition of $\text{Alg}_{\mathbb{C}}$, the 1-morphism $M \in \text{Alg}_{\mathbb{C}}(B, \mathbb{C})$ is a right B -module.

According to (3.11), a morphism $(M, \{\sigma_g\}) \rightarrow (\widetilde{M}, \{\tilde{\sigma}_g\})$ in $2\text{-Cone}(B, \chi^\omega)$ is a 2-morphism $\Gamma: {}_{\mathbb{C}}M_B \rightarrow {}_{\mathbb{C}}\widetilde{M}_B$ in $\text{Alg}_{\mathbb{C}}$ such that

$$\begin{array}{ccc} L_g \otimes_{\mathbb{C}} M & \xrightarrow{1 \otimes \Gamma} & L_g \otimes_{\mathbb{C}} \widetilde{M} \\ \sigma_g \downarrow & & \downarrow \tilde{\sigma}_g \\ M & \xrightarrow{\Gamma} & \widetilde{M} \end{array} \quad (3.14)$$

commutes. But that means that Γ is both a map of left $\mathbb{C}^\omega[G]$ -modules and a map of right B -modules. \square

Proposition 3.5. (i) A 2-limit of $\chi^\omega: \mathbf{BG} \rightarrow \text{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^\omega[G], \mathbb{C}^\omega[G])$.
(ii) A 2-colimit of $\chi^\omega: \mathbf{BG} \rightarrow \text{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^\omega[G], \mathbb{C}^\omega[G])$.

Proof. According to Definition 3.3 and Lemma 3.4, a 2-limit of χ^ω is an algebra $L \in \text{Alg}_{\mathbb{C}}$ together with a module ${}_{\mathbb{C}^\omega[G]}M_L$ such that the functor

$$\begin{aligned} \text{Alg}_{\mathbb{C}}(B, L) &\longrightarrow {}_{\mathbb{C}^\omega[G]}\text{Mod}_B \\ N &\longmapsto M \otimes_L N \end{aligned}$$

is an equivalence for every $B \in \text{Alg}_{\mathbb{C}}$. This is the case for $L = \mathbb{C}^\omega[G]$ as an algebra and $M = \mathbb{C}^\omega[G]$ as a bimodule over itself.

The statement about the 2-colimit follows analogously. \square

3.2.2 “With thinking” (Domenico)

The datum of the 2-functor $\chi^\omega: \mathbf{BG} \rightarrow \mathbf{Alg}_{\mathbb{C}}$ is a collection of lines (1-dimensional complex vector spaces) L_g , indexed by elements in the group G , together with isomorphisms

$$\rho_{g,h}: L_g \otimes L_h \xrightarrow{\sim} L_{gh}$$

subject to the associativity constraint given by the commutativity of the diagrams

$$\begin{array}{ccc} L_g \otimes L_h \otimes L_k & \xrightarrow{\rho_{g,h} \otimes \text{id}} & L_{gh} \otimes L_k \\ \text{id} \otimes \rho_{h,k} \downarrow & & \downarrow \rho_{gh,k} \\ L_g \otimes L_{hk} & \xrightarrow{\rho_{g,hk}} & L_{ghk} \end{array}$$

where the tensor products are over \mathbb{C} . We also require that $L_1 = \mathbb{C}$ and that the isomorphisms $L_1 \otimes L_g \xrightarrow{\sim} L_g$ and $L_g \otimes L_1 \xrightarrow{\sim} L_g$ are the structure isomorphisms for the unit object \mathbb{C} of the monoidal category $\mathbf{Vect}_{\mathbb{C}}$. The associativity constraints imply, and are in fact equivalent to this, that the vector space

$$A_{\chi^\omega} = \bigoplus_{g \in G} L_g$$

has an associative \mathbb{C} -algebra structure.

Lemma 3.6. A choice of a basis element x_g for every line L_g defines a \mathbb{C}^\times -valued 2-cocycle ω on the group G . A different choice of basis elements leads to a cohomologous cocycle, so that the class $[\omega] \in H^2(G, \mathbb{C}^\times)$ is well defines.

Proof. As $\rho_{g,h}(x_g \otimes x_h)$ is a nonzero element in L_{gh} we have $\rho_{g,h}(x_g \otimes x_h) = \omega(g,h)x_{gh}$ for a unique element $\omega(g,h)$ in \mathbb{C}^\times . The associativity constraints are then immediately seen to be equivalent to the cocycle equation

$$\omega(gh, k)\omega(g, h) = \omega(g, hk)\omega(h, k).$$

Finally, if $\{y_g\}_{g \in G}$ is a different basis choice, and $\tilde{\omega}$ is the corresponding 2-cocycle, then we have $y_g = \eta_g x_g$ for some η_g in \mathbb{C}^\times and

$$\omega(\tilde{g}, h)y_{gh} = \rho_{g,h}(y_g \otimes y_h) = \eta_g \eta_h \rho_{g,h}(x_g \otimes x_h) = \eta_g \eta_h \omega(g, h)x_{gh} = \eta_g \eta_h \omega(g, h)\eta_{gh}^{-1}y_{gh}.$$

Therefore

$$\tilde{\omega}(g, h) = \eta_g \eta_{gh}^{-1} \eta_h \omega(g, h),$$

i.e. ω and $\tilde{\omega}$ are cohomologous. \square

Corollary 3.7. The algebra A_{χ^ω} is isomorphic to the twisted group algebra $\mathbb{C}^\omega[G]$, defined as the \mathbb{C} -vector space on the basis $\{x_g\}_{g \in G}$ with the product $x_g \cdot x_h = \omega(g, h)x_{gh}$ and with $x_1 = 1$.

Lemma 3.8. $\text{Sum}_2(\chi^\omega) \cong \mathbb{C}^\omega[G]$, the twisted group algebra of G .

Proof. Let (A, M) be a cocone for χ^ω , i.e., a pair consisting of a \mathbb{C} -algebra A together with a left A -module M (representing a linear functor ${}_{\mathbb{C}}\text{Mod} \rightarrow {}_A\text{Mod}$) and homotopy commutative diagrams

$$\begin{array}{ccc} & A & \\ M \nearrow & & \nwarrow M \\ \mathbb{C} & \xrightarrow{L_g} & \mathbb{C} \\ & \lambda_g \nearrow & \end{array}$$

where the λ_g 's are isomorphisms of left A -modules

$$\lambda_g: M \otimes_{\mathbb{C}} L_g \xrightarrow{\sim} M$$

such that the diagrams

$$\begin{array}{ccc} M \otimes L_g \otimes L_h & \xrightarrow{\lambda_g \otimes \text{id}} & M \otimes L_h \\ \text{id} \otimes \rho_{g,h} \downarrow & & \downarrow \lambda_h \\ M \otimes L_{gh} & \xrightarrow{\lambda_{gh}} & M \end{array}$$

commute. This implies that the isomorphisms λ_g make M a right A_{χ^ω} -module. Therefore, we can see M as a linear functor

$${}_{A_{\chi^\omega}}\text{Mod} \rightarrow {}_A\text{Mod}$$

i.e., as a morphism from A_{χ^ω} to A in $2\text{-Vect}_{\mathbb{C}}$. The algebra A_{χ^ω} seen as a left A_{χ^ω} -module is a morphism from \mathbb{C} to A_{χ^ω} in $2\text{-Vect}_{\mathbb{C}}$ and the natural isomorphism

$${}_A M_{\mathbb{C}} \cong {}_A M_{A_{\chi^\omega}} \otimes_{A_{\chi^\omega}} A_{\chi^\omega} {}_{\mathbb{C}}$$

gives a canonical factorization

$$\begin{array}{ccc} & A & \\ M \nearrow & & \nwarrow M \\ & A_{\chi^\omega} & \\ A_{\chi^\omega} \nearrow & & \nwarrow A_{\chi^\omega} \\ \mathbb{C} & \xrightarrow{L_g} & \mathbb{C} \\ & \lambda_g \nearrow & \end{array}$$

exhibiting the algebra A_{χ^ω} together with itself seen as a left module over itself as the universal cocone. \square

3.3 $n = 3$

TODO:

- The only non-trivial part of the 3-functor $\chi^\omega: \mathbf{BG} \rightarrow \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.
- **Issue:** The notion of a 3-colimit (to really compute Sum_3) is scary. At least we should make some hand-wavy arguments...

4 Boundary conditions

4.1 $n = 2$

TODO

4.2 $n = 3$

TODO

References

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