Boundaries in finite gauge theory

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Contents

1 Introduction

Let

- G be a finite group,
- $\omega \in Z^n(G, \mathbb{C}^\times)$,
- and $\mathcal{C}$ a symmetric monoidal (∞, n)-category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fullly extended TQFT

$$\mathrm{Bord}_n \xrightarrow[\text{“classical”}]{I} \mathrm{Fam}_n(\mathcal{C}) \xrightarrow[\text{“quantum”}]{\mathrm{Sum}_n} \mathcal{C} \tag{1.1}$$

based on the data (G, ω), which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $\mathcal{C} = \boldsymbol{B}^n\mathbb{C}^\times$ (the n-fold delooping of the group $\mathbb{C}^\times$) and then

(i) identify *classical* boundary conditions as 1-morphisms with source $*$ in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$,

(ii) subsume the boundary conditions of [Wi, WWW] as special cases,

(iii) map classical boundary conditions to quantum ones via Sum_n, and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs–Schweigert–Valentino in the case $n = 3$,

(iv) do the above not only for framed, but also for oriented, spin,... TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1, 2\}$ and hopefully also $n = 3$. We also want to explain in detail how (G, ω) gives rise to an object in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)(*, \boldsymbol{B}G)$.
(A more conceptual approach to boundary conditions and defects would be to start with a bordism category “with singularities” [Lu, Sect. 4.3] instead of Bord_n, but we leave that for another project.)

2 Special objects and 1-morphisms in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$

TODO:

- spell out definition of $\mathrm{Fam}_n(\mathcal{C})$
- check that n-functor $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ is precisely an n-cocycle
  Issue: For $n > 3$, what is a weak n-functor, i. e. precisely what data and constraints are needed?
  Idea: Since both n-categories in $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section 3.

- clarify precisely how $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ induces an n-functor $\boldsymbol{B}G \to n$-Vect, at least for $n \in \{1,2,3\}$
- check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$
  Issue: induced from issue in second item

2.1 Classical bulk theories

TODO

2.2 Classical boundary conditions

2.2.1 $n = 2$

Let $\mathcal{G}_1, \mathcal{G}_2$ be 2-groupoids and let $F_1 \colon \mathcal{G}_1 \to \boldsymbol{B}^2\mathbb{C}^\times$ and $F_2 \colon \mathcal{G}_2 \to \boldsymbol{B}^2\mathbb{C}^\times$ be 2-functors, i. e. objects in $\mathrm{Fam}_2(\boldsymbol{B}^2\mathbb{C}^\times)$. A 1-morphism $F_1 \to F_2$ in $\mathrm{Fam}_2(\boldsymbol{B}^2\mathbb{C}^\times)$ is a span of 2-groupoids $\mathcal{G}_1 \xleftarrow{K_1} \widetilde{\mathcal{G}} \xrightarrow{K_2} \mathcal{G}_2$ together with a pseudonatural transformation $\sigma \colon F_1 \circ K_1 \to F_2 \circ K_2$:

$$
\begin{tikzcd}
 & \widetilde{\mathcal{G}} \arrow[dl, "K_1"'] \arrow[dr, "K_2"] & \\
\mathcal{G}_1 \arrow[dr, "F_1"'] \arrow[rr, phantom, "\overset{\sigma}{\Longrightarrow}"] & & \mathcal{G}_2 \arrow[dl, "F_2"] \\
 & \boldsymbol{B}^2\mathbb{C}^\times &
\end{tikzcd}
\tag{2.1}
$$

Let $[1]$ denote the final 2-category with only one object, 1- and 2-morphism, and let $T \colon [1] \to \boldsymbol{B}^2\mathbb{C}^\times$ be the unique, trivial 2-functor. A *classical boundary condition* for $F \colon \mathcal{G} \to \boldsymbol{B}^2\mathbb{C}^\times$ is a 1-morphism $T \to F$. Thus it comprises a 2-groupoid $\mathcal{H}$, a 2-functor $\rho \colon \mathcal{H} \to \mathcal{G}$, and a pseudonatural transformation

$$
\begin{tikzcd}
 & \mathcal{H} \arrow[dl] \arrow[dr, "\rho"] & \\
{[1]} \arrow[dr, "T"'] \arrow[rr, phantom, "\overset{\sigma}{\Longrightarrow}"] & & \mathcal{G} \arrow[dl, "F"] \\
 & \boldsymbol{B}^2\mathbb{C}^\times &
\end{tikzcd}
\tag{2.2}
$$

Concretely, since $\boldsymbol{B}^2\mathbb{C}^\times$ has only one object and one 1-morphism, σ is given by a family of 2-morphisms $\sigma_h\colon (F\circ\rho)(h) = 1_* \to 1_*$, indexed by 1-morphisms h in $\mathcal{H}$, such that

$$\begin{tikzcd} F\big(\rho(h)\big) \otimes F\big(\rho(h')\big) \arrow[r, "1\otimes\sigma_{h'}"] \arrow[d, "(F\circ\rho)_{h,h'}"'] & F\big(\rho(h)\big) \arrow[d, "\sigma_h"] \\ F\big(\rho(hh')\big) \arrow[r, "\sigma_{hh'}"'] & 1_* \end{tikzcd} \tag{2.3}$$

commutes for al composable 1-morphisms h, h' in $\mathcal{H}$.

We now choose $\mathcal{H} = \boldsymbol{B}H$ for some finite group H, $\mathcal{G} = \boldsymbol{B}G$, and $F = \chi^\omega$ as in (3.9) — and we are not careful to distinguish the target $\boldsymbol{B}^2\mathbb{C}^\times$ from $\mathrm{Alg}_\mathbb{C}$. Then a 2-functor $\rho\colon \boldsymbol{B}H \to \boldsymbol{B}G$ is just a group homomorphism $\rho\colon H\to G$, and a boundary condition

$$\begin{tikzcd} & \boldsymbol{B}H \arrow[dl] \arrow[dr, "\rho"] & \\ {[1]} \arrow[dr, "T"'] \arrow[rr, phantom, "\overset{\sigma}{\Longrightarrow}"] & & \boldsymbol{B}G \arrow[dl, "\chi^\omega"] \\ & \boldsymbol{B}^2\mathbb{C}^\times & \end{tikzcd} \tag{2.4}$$

for χ^ω is a family $\{\sigma_h\}_{h\in H} \subset \mathbb{C}^\times$ such that

$$\begin{tikzcd} L_{\rho(h)} \otimes_\mathbb{C} L_{\rho(h')} \arrow[r, "1\otimes\sigma_{h'}"] \arrow[d, "\omega(\rho(h),\rho(h'))"'] & L_{\rho(h)} \arrow[d, "\sigma_h"] \\ L_{\rho(hh')} \arrow[r, "\sigma_{hh'}"'] & 1_* \end{tikzcd} \tag{2.5}$$

commutes, i. e. $\omega(\rho(h),\rho(h')) = \sigma_h\cdot\sigma_{hh'}^{-1}\cdot\sigma_{h'}$ for all $h,h'\in H$. Viewing σ as a map $H\to\mathbb{C}^\times$, $h\mapsto\sigma_h$, this condition precisely means that σ trivialises the pullback of ω:

$$\mathrm{d}\sigma = \rho^*\omega\,. \tag{2.6}$$

3 Bulk theory

3.1 $n = 1$

Since we take the action of G on $\mathbb{C}^\times$ to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{C}^\times) \tag{3.1}$$

is just a group homomorphism. We consider $\boldsymbol{B}^1\mathbb{C}^\times$ as a subcategory of $\mathrm{Vect}_{\mathbb{C}}$, and we will use the pair (G, ω) to construct a functor

$$\mathrm{Bord}_1 \xrightarrow{\quad I \quad} \mathrm{Fam}_1(\mathrm{Vect}_{\mathbb{C}}) \xrightarrow{\quad \mathrm{Sum}_1 \quad} \mathrm{Vect}_{\mathbb{C}} \,. \tag{3.2}$$

According to the cobordism hypothesis, the composite $\mathrm{Sum}_1 \circ I$ is determined by its value on the point pt_+, so we only need to specify $I(\mathrm{pt}_+)$ and then compute $\mathrm{Sum}_1(I(\mathrm{pt}_+))$. Note that $\boldsymbol{B}^1 G = */\!\!/G$ is the classical action groupoid. We set $I(\mathrm{pt}_+)$ to be the functor

$$\chi^\omega \colon\ */\!\!/G \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,, \qquad * \longmapsto \mathbb{C}\,, \quad g \longmapsto \omega(g) \in \mathrm{Aut}(\mathbb{C}) \tag{3.3}$$

for all $g \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. $\mathrm{Sum}_1(\chi^\omega) = \mathbb{C}$ if $\omega(g) = 1$ for all $g \in G$, and $\mathrm{Sum}_1(\chi^\omega) = 0$ otherwise.

Proof. The statement is a particular case of this more general one: let (V, ρ) be a linear representation of the group G, seen as the functor

$$\chi^\rho \colon\ */\!\!/G \longrightarrow \mathrm{Vect}_{\mathbb{C}}\,, \qquad * \longmapsto V\,, \quad g \longmapsto \rho(g) \in \mathrm{Aut}(V). \tag{3.4}$$

Then the universal cocone of χ^ρ, viewed as a diagram of shape $*/\!\!/G$ in $\mathrm{Vect}_{\mathbb{C}}$ is the pair (V_G, π), where V_G is the vector space of coinvariants for the representation ρ, i.e.,

$$V_G = V/\langle v - \rho(g)v \rangle_{g\in G, v\in V}$$

and $\pi \colon V \to V_G$ is the projection to the quotient. Namely, let (W, f) be a cocone for χ^ρ, i.e., a pair consisting of a vector space W together with a linear map $f \colon \chi^\rho(*) = V \to W$ such that

$$\begin{tikzcd} & W & \\ \chi^\rho(*) = V \arrow[ur, "f"] \arrow[rr, "\chi^\rho(g) = \rho(g)"'] & & V = \chi^\rho(*) \arrow[ul, "f"'] \end{tikzcd} \tag{3.5}$$

Then, by definition of V_G, the morphism f uniquely factors through V_G and so we have a commutative diagram

$$
\begin{tikzcd}
& W & \\
& V_G \arrow[u, dashed, "\exists!"] & \\
V \arrow[uur, bend left, "f"] \arrow[ur, "\pi"] \arrow[rr, "\rho(g)"'] & & V \arrow[uul, bend right, "f"'] \arrow[ul, "\pi"']
\end{tikzcd}
\tag{3.6}
$$

showing that (V_G, π) enjoys the universal property of the universal cocone. $\square$

Corollary 3.2. The linear dual $\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(\mathrm{Sum}_1(\chi^\omega), \mathbb{C})$ of $\mathrm{Sum}_1(\chi^\omega)$ is naturally isomorphic to the vector space of natural transformations $\mathrm{Hom}_{[*/\!\!/G, \mathrm{Vect}_{\mathbb{C}}]}(\chi^\omega, \chi^1)$, where $\chi^1\colon * /\!\!/ G \to \mathrm{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space $\mathbb{C}$. In other words, we have

$$
\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(\mathrm{Sum}_1(\chi^\omega), \mathbb{C}) \cong \left\{
\begin{tikzcd}
* /\!\!/ G \arrow[dd, "\chi^\omega"'] \arrow[dr] & \\
\arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl, "\mathbb{C}"] \\
\mathrm{Vect}_{\mathbb{C}} &
\end{tikzcd}
\right\}.
$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\mathrm{Sum}_1(\chi^\omega)$.

Proof. Again, the statement is true for an aritrary linear representation (V, ρ) of the group G. The natural isomorphism

$$
\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(V_G, \mathbb{C}) \cong \left\{
\begin{tikzcd}
* /\!\!/ G \arrow[dd, "\chi^\rho"'] \arrow[dr] & \\
\arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl, "\mathbb{C}"] \\
\mathrm{Vect}_{\mathbb{C}} &
\end{tikzcd}
\right\}
$$

is then nothing but the universal property of V_G. Namely, an element in the right hand side is a morphism $f\colon V = \chi^\rho(*) \to \chi^1(*) = \mathbb{C}$ such that all the diagrams

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & \mathbb{C} \arrow[d, "\mathrm{id}_{\mathbb{C}}"] \\
V \arrow[r, "f"'] & \mathbb{C}
\end{tikzcd}
$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams

$$\begin{tikzcd} & \mathbb{C} & \\ V \arrow[ur, "f"] \arrow[rr, "\rho(g)"'] & & V \arrow[ul, "f"'] \end{tikzcd}$$

commute, and so it is precisely the datum of a morphism $V_G \to \mathbb{C}$ by the argumenti in the proof of Lemma 3.18. □

3.1.1 Quantisation

We now show that Sum_1 is a (TODO: symmetric monoidal!) functor.

Let us denote by $\mathrm{Fam}_0(\mathbb{C})$ the set of equivalence classes of finite groupoids equipped with functors to $\mathbb{C}$ (seen as a trivial category). In other words, a representative of an element of $\mathrm{Fam}_0(\mathbb{C})$ is a pair (X, ρ) consisting of a finite groupoid X together with a map $\rho\colon \pi_0(X) \to \mathbb{C}$. The map

$$\mathrm{Sum}_0\colon \mathrm{Fam}_0 \to \mathbb{C}$$

is defined as

$$\mathrm{Sum}_0(X, \rho) = \sum_{[x]\in\pi_0(X)} \frac{\rho(x)}{|\mathrm{Aut}(x)|}.$$

We will also use the more evocative notation

$$\int_X \rho(x) d\mu(x)$$

to denote $\mathrm{Sum}_0(X, \rho)$. Notice that Sum_0 is linear in the ρ variable.

A nice property of the morphism Sum_0 (which we are going to use later) is the following: let X be a finite set with an action of a finite group G, and let $X//G$ denotes the action groupoid for this action; finally, let $\rho\colon X//G \to \mathbb{C}$ any functor, i.e., let $\rho\colon X \to \mathbb{C}$ be a function that is constant on the G-orbits in X. Then we have

$$\mathrm{Sum}_0(X//G, \rho) = \frac{1}{|G|} \sum_{x\in X} \rho(x). \tag{3.7}$$

Namely, the set $\pi_0(X//G)$ is the quotient set X/G and for any $x \in X$ seen as an object in $X//G$ we have $\mathrm{Aut}(x) = \mathrm{Stab}_G(x)$. Therefore

$$\begin{aligned} \mathrm{Sum}_0(X//G, \rho) &= \sum_{[x]\in X/G} \frac{\rho(x)}{|\mathrm{Stab}_G(x)|} \\ &= \frac{1}{|G|} \sum_{[x]\in X/G} \frac{|G|}{|\mathrm{Stab}_G(x)|} \rho(x) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G|} \sum_{[x] \in X/G} |[x]| \, \rho(x) \\
&= \frac{1}{|G|} \sum_{[x] \in X/G} \sum_{x \in [x]} \rho(x) \\
&= \frac{1}{|G|} \sum_{x \in X} \rho(x)
\end{aligned}
$$

Remark 3.3. A function $f: \pi_0(X) \to \mathbb{C}$ can equivalently be seen as an element in $\mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X)$, where $\mathbf{1}_X : X \to \mathrm{Vect}$ is the constant functor taking the value $\mathbb{C}$ on X. The map Sum_0 can therefore be seen as the datum of a map

$$\mathrm{Sum}_0 \colon \mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X) \to \mathbb{C},$$

for any finite groupoid X.

Definition 3.4. The category $\mathrm{Fam}_1(\mathrm{Vect})$ has as its objects finite groupoids equipped with functors to Vect, i.e., pairs (X, ρ) consisting of a finite groupoid X together with a functor $\rho \colon X \to \mathrm{Vect}$. Morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ are spans of groupoids over Vect, i.e., commutative diagrams

$$
\begin{tikzcd}
 & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\rho_X"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\alpha}"] & & Y \arrow[dl, "\rho_Y"] \\
 & \mathrm{Vect} &
\end{tikzcd}
$$

where the natural transformation $\alpha \colon \rho_X \circ \pi_X \to \rho_Y \circ \pi_Y$ is part of the data of the commutative diagram.

Remark 3.5. The identity mophisms and the composition of morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ will be defined later.

For every object (X, ρ) in $\mathrm{Fam}_1(\mathrm{Vect})$ we can consider the vector space

$$[\mathbf{1}_X, \rho] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\mathbf{1}_X, \rho).$$

If $f \colon Y \to X$ is a functor between finite groupoids, and $\rho \colon X \to \mathrm{Vect}$ is a functor, we write $f^*\rho \colon Y \to \mathrm{Vect}$ for the functor $\rho \circ f$. If $\alpha \colon \mathbf{1}_X \to \rho$ is a natural transformation, then $f^*\alpha$ is a natural transformation between $f^*\mathbf{1}_X = \mathbf{1}_Y$ and $f^*\rho$. This defines a linear map

$$f^* \colon [\mathbf{1}_X, \rho] \to [\mathbf{1}_Y, f^*\rho].$$

Similarly, we can consider the vector space

$$[\rho, \mathbf{1}_X] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\rho, \mathbf{1}_X)$$

together with the linear map

$$f^*\colon [\rho, \mathbf{1}_X] \to [f^*\rho, \mathbf{1}_Y].$$

Passing to the linear duals we get the linear map

$$(f^*)^\vee\colon [f^*\rho, \mathbf{1}_Y]^\vee \to [\rho, \mathbf{1}_X]^\vee.$$

Next, notice that composition of morphisms gives a bilinear map

$$[\rho, \mathbf{1}_X] \otimes [\mathbf{1}_X, \rho] \to [\mathbf{1}_X, \mathbf{1}_X]$$

and so composing this with the linear map $\mathrm{Sum}_0(X, -)\colon [\mathbf{1}_X, \mathbf{1}_X] \to \mathbb{C}$ we get a bilinear pairing $\langle\,|\,\rangle_X$ between $[\rho, \mathbf{1}_X]$ and $[\mathbf{1}_X, \rho]$, for any ρ. Notice that, by its very definition, the pairing $\langle\,|\,\rangle_X$ satisfies

$$\langle v|\alpha \circ w\rangle_X = \langle v \circ \alpha|w\rangle_X,$$

for any $v \in [\rho_2, \mathbf{1}_X]$, any $w \in [\mathbf{1}_X, \rho_1]$ and any natural transformation $\alpha\colon \rho_1 \to \rho_2$. Equivalently, the bilinear pairing $\langle\,|\,\rangle_X$ is a linear map

$$\delta_{(X,\rho)}\colon [\mathbf{1}_X, \rho] \to [\rho, \mathbf{1}_X]^\vee.$$

Our main *duality assumption* will be that $\delta_{(X,\rho)}$ is a linear isomorphism for any (X, ρ).

Remark 3.6. The duality assumption holds for every finite groupoid X if and only if it holds for finite connected groupoids. Up to equivalence we are therefore reduced to considering the case $X = \mathbf{B}G$, with G a finite group. In this case $\rho\colon \mathbf{B}G \to \mathrm{Vect}$ is the datum of a linear representation V of G, and the vector spaces $[\mathbf{1}_{\mathbf{B}G}, \rho]$ and $[\rho, \mathbf{1}_{\mathbf{B}G}]$ are isomorphic to the space V^G of G-invariants and the linear dual $(V_G)^\vee$ of the space of G-coinvariants, respectively. The map $\delta_{(\mathbf{B}G,\rho)}$ is therefore a map

$$\delta_{(\mathbf{B}G,\rho)}\colon V^G \to (V_G)^{\vee\vee} \cong V_G,$$

from G-invariants to G-coinvariants. Let $v \mapsto [v]$ be the canonical projection $V \to V_G = V/\langle v - g \cdot v\rangle_{v\in V; g\in G}$. By composing with the inclusion $V^G \hookrightarrow V$ we get the morphism $\iota\colon v \mapsto [v]$ from $V^G \to V_G$. This is an isomorphism. Indeed, the canonical projection $\pi\colon V \to V^G$ given by

$$\pi\colon v \mapsto \frac{1}{|G|} \sum_{g\in G} g \cdot v$$

is zero on the elements of the form $v - g \cdot v$ and so induces a linear map $\tilde{\pi}\colon V_G \to V^G$, which is immediate to see is the inverse of ι. For any $w \in (V_G)^\vee$ seen as an element in $[\rho, \mathbf{1}_{\mathbf{B}G}]$ and any $v \in V^G$ seen as an element in $[\mathbf{1}_{\mathbf{B}G}, \rho]$ we have $w \circ v = w([v])$ Therefore

$$\langle w|v\rangle_{\mathbf{B}G} = \frac{1}{|G|} w \circ v = w\left(\frac{1}{|G|}\iota(v)\right).$$

This means that $\delta_{(\mathbf{B}G,\rho)} v$ is the evaluation on $\frac{1}{|G|}\iota(v)$. Therefore, under the canonical isomorphism $(V_G)^{\vee\vee} \cong V_G$, we have $\delta_{(\mathbf{B}G,\rho)} = \frac{1}{|G|}\iota$. As ι is an isomorphism, so is $\delta_{(\mathbf{B}G,\rho)}$.

Under the duality assumption we may define a linear morphism

$$f_*\colon [\mathbf{1}_Y, f^*\rho] \to [\mathbf{1}_X, \rho]$$

as the composition

$$[\mathbf{1}_Y, f^*\rho] \xrightarrow{\delta_{(Y,f^*\rho)}} [f^*\rho, \mathbf{1}_Y]^\vee \xrightarrow{(f^*)^\vee} [\rho, \mathbf{1}_X]^\vee \xrightarrow{\delta^{-1}_{(X,\rho)}} [\mathbf{1}_X, \rho].$$

In other words, if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_*v \in [\mathbf{1}_X, \rho]$ is defined by the equation

$$\langle w|f_*v\rangle_X = \langle f^*w|v\rangle_Y$$

for any $w \in [\rho, \mathbf{1}_X]$. Similarly, we have a morphism $f_*\colon [f^*\rho, \mathbf{1}_Y] \to [\rho, \mathbf{1}_X]$ defined by

$$\langle f_*w|v\rangle_X = \langle w|f^*v\rangle_Y.$$

For later use, let us derive an explicit formla for f_* in case $f\colon \mathbf{B}G \to \mathbf{B}H$ is a morphism of finite groups (seen as groupoids with a single object).

Lemma 3.7. If G, H are finite groups and $f\colon \mathbf{B}G \to \mathbf{B}H$ is a functor, then for every $v \in [\mathbf{1}_{\mathbf{B}G}, f^*\rho]$ we have

$$f_*v = \frac{1}{|G|}\sum_{h \in H} h \cdot v\,.$$

Proof. Recall that if $v \in [\mathbf{1}_{\mathbf{B}G}, f^*\rho]$, the element $f_*v \in [\mathbf{1}_{\mathbf{B}H}, \rho]$ is defined by the equation $\langle w|f_*v\rangle_{\mathbf{B}H} = \langle f^*w|v\rangle_{\mathbf{B}G}$ for any $w \in [\rho, \mathbf{1}_{\mathbf{B}H}]$. As both $\mathbf{B}G$ and $\mathbf{B}H$ have a single object, this equation reduces to

$$\frac{w \circ f_*v}{|H|} = \frac{f^*w \circ v}{|G|}$$

The morphism $\rho\colon \mathbf{B}H \to \mathrm{Vect}$ is the datum of a linear representation of H on a vector space V, and $f^*\rho$ is the datum of representation of G obtained by pullback, i.e., the group G acts on V by $g \cdot \xi := f(g) \cdot \xi$. The element $v \in [\mathbf{1}_{\mathbf{B}G}, f^*\rho]$

is therefore a G-invariant vector of V, while $w \in [\rho, \mathbf{1}_{\mathbf{B}H}]$ is a linear functional on V factoring through the space of H-coinvariants. The element f^*w is the same linear functional on V, but this time seen as factoring through the space of G-coinvariants. Therefore the composition $f^*w \circ v$ is just the evaluation of $w\colon V \to \mathbb{C}$ on the vector $v \in V$, under the canonical isomorphism $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$. Finally, $f_*v \in [\mathbf{1}_{\mathbf{B}H}, \rho]$ is an H-invariant vector of V, and $w \circ f_*v$ is the evaluation of $w\colon V \to \mathbb{C}$ on the vector $f_*v \in V$. The defining equation for f_*v then becomes

$$w\left(\frac{f_*v}{|H|}\right) = w\left(\frac{v}{|G|}\right)$$

for any $w\colon V \to \mathbb{C}$ factoring through the space of H-coinvariants. From $v/|G|$ we get the H-invariant vector

$$\frac{1}{|H|}\sum_{h\in H} h \cdot \frac{v}{|G|}$$

and, since w factors through the H-coinvariants,

$$w\left(\frac{v}{|G|}\right) = w\left(\frac{1}{|H|}\sum_{h\in H} h \cdot \frac{v}{|G|}\right)$$

The defining equation for f_*v can therefore be rewritten as

$$\langle w|f_*v\rangle_{\mathbf{B}H} = \langle w|\sum_{h\in H}\frac{h\cdot v}{|G|}\rangle_{\mathbf{B}H},$$

for any $w \in [\rho, \mathbf{1}_X]$. As the pairing $\langle\,|\,\rangle_{\mathbf{B}H}$ is nondegenerate, this finally gives

$$f_*v = \frac{1}{|G|}\sum_{h\in H} h \cdot v.$$

$\square$

Remark 3.8. This formula simplifies a bit in case the group homomorphism $f\colon G \to H$ is surjective. Indeed, in that case, the G-invariant vector v will be automatically H-invariant and we find

$$f_*v = \frac{|H|}{|G|}v$$

in this case.

We can now define our wannabe functor

$$\mathrm{Sum}_1\colon \mathrm{Fam}_1(\mathrm{Vect}) \to \mathrm{Vect}$$

on objects as

$$\mathrm{Sum}_1(X,\rho) = [\mathbf{1}_X, \rho]$$

and on a morphism

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\rho_X"'] & \overset{\Longrightarrow}{\alpha} & Y \arrow[dl, "\rho_Y"] \\
 & \mathrm{Vect} &
\end{tikzcd}$$

as the composition

$$[\mathbf{1}_X, \rho_X] \xrightarrow{\pi_X^*} [\mathbf{1}_Z, \pi_X^*\rho_X] \xrightarrow{\alpha\circ -} [\mathbf{1}_Z, \pi_Y^*\rho_Y] \xrightarrow{\pi_{Y*}} [\mathbf{1}_Y, \rho_Y].$$

Remark 3.9. Before giving a more explicit expression of the above morphism, let us notice that $[\mathbf{1}_X, \rho]$ is a model for the limit of the functor $\rho\colon X \to \mathrm{Vect}$, in accordance with the prescription in [FHLT]. Also, notice that the morphism associated to the span $X \leftarrow Z \rightarrow Y$ together with the natural transformation α has the form "pull-compose-push", and so it is an instance of a Fourier-Mukai-type transform.

Lemma 3.10. (The projection formula) Let $f\colon Y \to X$ be a functor between finite groupoids, let $\rho_1, \rho_2\colon X \to \mathrm{Vect}$ be functors and let $\alpha\colon \rho_1 \to \rho_2$ be a natural transformation. We have an identity of morphisms $[\mathbf{1}_Y, f^*\rho_1] \to [\mathbf{1}_X, \rho_2]$

$$f_*(f^*\alpha \circ -) = \alpha \circ f_*$$

Proof. The identity

$$f_*(f^*\alpha \circ v) = \alpha \circ f_* v$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ is equivalent to the identity

$$\langle w | f_*(f^*\alpha \circ v)\rangle_X = \langle w | \alpha \circ f_* v\rangle_X$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ and every $w \in [\rho_2, \mathbf{1}_X]$. By the properties of the pairings $\langle\,|\,\rangle_X$ and $\langle\,|\,\rangle_Y$, and by definition of f_*, we have

$$\begin{aligned}
\langle w | f_*(f^*\alpha \circ v)\rangle_X &= \langle f^*w | f^*\alpha \circ v\rangle_Y \\
&= \langle f^*w \circ f^*\alpha | v\rangle_Y \\
&= \langle f^*(w \circ \alpha) | v\rangle_Y \\
&= \langle w \circ \alpha | f_* v\rangle_X \\
&= \langle w | \alpha \circ f_* v\rangle_X
\end{aligned}$$

□

Lemma 3.11. Consider a homotopy pullback diagram of finite groupoids

$$
\begin{tikzcd}
 & X \times_T Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f_1"'] \arrow[rr, phantom, "\overset{}{\underset{\eta}{\Rrightarrow}}"] & & Y \arrow[dl, "f_2"] \\
 & T &
\end{tikzcd}
$$

and a morphism $\rho\colon T \to \mathrm{Vect}$. Then the Beck-Chevalley condition is satisfied, i.e., we have an identity of morphisms $f_2^* f_{1*} = p_{2*}(\eta^*\rho \circ -)p_1^*$ from $[\mathbf{1}_X, f_1^*\rho]$ to $[\mathbf{1}_Y, f_2^*\rho]$.

Proof. By restricting to connected components, and working up to equivalence of finite groupoids, we may assume that the given pullback diagram has the form

$$
\begin{tikzcd}
 & G\backslash\backslash H//K \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathbf{B}G \arrow[dr, "f_1"'] \arrow[rr, phantom, "\underset{\eta}{\Rrightarrow}"] & & \mathbf{B}K \arrow[dl, "f_2"] \\
 & \mathbf{B}H &
\end{tikzcd}
$$

where the natural transformation η is defined by $\eta_h = h$; this precisely expresses the fact that a morphism from h to h' in the action groupoid $G\backslash\backslash H//K$ is given by a commutative diagram

$$
\begin{tikzcd}
* \arrow[r, "h"] \arrow[d, "f_1(g)"'] & * \arrow[d, "f_2(k)"] \\
* \arrow[r, "h'"'] & *
\end{tikzcd}
.
$$

The equation $f_2^* f_{1*} = p_{2*}(\eta^*\rho \circ -)p_1^*$ is equivalent to $\langle w | f_2^* f_{1*} v\rangle_{\mathbf{B}K} = \langle w | p_{2*}(\eta^*\rho \circ p_1^* v)\rangle_{\mathbf{B}K}$ for any $v \in [\mathbf{1}_{\mathbf{B}G}, f_1^*\rho]$ and any $w \in [f_2^*\rho, \mathbf{1}_{\mathbf{B}K}]$. This is equivalent to the equation

$$\langle f_{2*} w | f_{1*} v\rangle_{\mathbf{B}H} = \langle p_2^* w | \eta^*\rho \circ p_1^* v\rangle_{G\backslash\backslash H//K}.$$

The morphism $\rho\colon \mathbf{B}H \to \mathrm{Vect}$ is the datum of a linear representation of the finite group H on a vector space V. The element v is a G-invariant vector of V (where G acts on V via f_1) and w is a linear functional on V which factors through the K-coinvariants (with K acting via f_2). By Lemma 3.7 we have

$$f_{1*} v = \frac{1}{|G|} \sum_{h \in H} h \cdot v; \qquad f_{2*} w = \frac{1}{|K|} \sum_{h \in H} h \cdot w,$$

where on the right we have the H-action on linear functionals on V, i.e., $(h \cdot w) = w(h^{-1} \cdot -)$. We therefore find

$$\begin{aligned}
\langle f_{2*}w | f_{1*}v \rangle_{\mathbf{B}H} &= \frac{1}{|H||G||K|} \sum_{h_1, h_2 \in H} w(h_1^{-1} h_2 \cdot v) \\
&= \frac{1}{|H||G||K|} \sum_{h \in H} \sum_{h_1^{-1} h_2 = h} w(h_1^{-1} h_2 \cdot v) \\
&= \frac{1}{|H||G||K|} \sum_{h \in H} \sum_{h_1 \in H} w(h \cdot v) \\
&= \frac{1}{|G||K|} \sum_{h \in H} w(h \cdot v)
\end{aligned}$$

To every object h of $G\backslash\backslash H//K$, the functors $p_1^* f_1^* \rho$ and $p_2^* f_2^* \rho$ both assign the vector space V, so $p_1^* v$ and $p_2^* w$ consist of a copy of the vector v and of the linear functional $w \colon V \to \mathbb{C}$ for each of the copies of V associated with the elements of H. Moreover, since $\eta_h = h$, the natural transformation $\eta^* \rho = \rho \circ \eta$ at the object h is precisely the multiplication by h as a linear map from V to V. This means that $\eta^* \rho \circ p_1^* v = (h \cdot v)_{h \in H}$, i.e, $\eta^* \rho \circ p_1^* v$ consists of the choice of the vector $h \cdot v$ in the copy of V associated with the object h of $G\backslash\backslash H//K$. We therefore find

$$\langle p_2^* w | \eta^* \rho \circ p_1^* v \rangle_{G\backslash\backslash H//K} = \frac{1}{|G||K|} \sum_{h \in H} w(h \cdot v),$$

by the formula (3.7). $\square$

Proposition 3.12. Sum_1 is a functor $\mathrm{Fam}_1(\mathrm{Vect}) \to \mathrm{Vect}$.

Proof. We have the verify that Sum_1 is compatible with units and composition. Let $\alpha \colon \rho_1 \to \rho_2$ and $\beta \colon \rho_2 \to \rho_3$ be composable morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$:

$$\begin{tikzcd}
 & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X_1 \arrow[dr, "\rho_1"'] \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] & & X_2 \arrow[dl, "\rho_2"] \\
 & \mathrm{Vect} &
\end{tikzcd}, \quad
\begin{tikzcd}
 & Y' \arrow[dl, "p_2'"'] \arrow[dr, "p_3'"] & \\
X_2 \arrow[dr, "\rho_2"'] \arrow[rr, phantom, "\overset{\beta}{\Longrightarrow}"] & & X_3 \arrow[dl, "\rho_3"] \\
 & \mathrm{Vect} &
\end{tikzcd}.$$

Then by definition of $\mathrm{Fam}_1(\mathrm{Vect})$, their composition $\beta \circ_\eta \alpha$ is the natural transformation $(\beta * 1_{\pi_2}) \cdot (1_{\rho_2} * \eta) \cdot (\alpha * 1_{\pi_1})$:

$$
\begin{tikzcd}[column sep=small]
 & & & Y \times_{X_2} Y' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & \\
 & & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[rr, phantom, "\xRightarrow{\eta}"] & & Y' \arrow[dl, "p_2'"'] \arrow[dr, "p_3'"] & & \\
\beta \circ_\eta \alpha \overset{\text{def}}{=} & X_1 \arrow[dr, "\rho_1"'] \arrow[rr, phantom, "\xRightarrow{\alpha}"] & & X_2 \arrow[dl, "\rho_2"] \arrow[dr, "\rho_2"'] \arrow[rr, phantom, "\xRightarrow{\beta}"] & & X_3 \arrow[dl, "\rho_3"] & . \\
 & & \mathrm{Vect} \arrow[dr, equal] \arrow[rr, phantom, "\xRightarrow{1_{\rho_2}}"] & & \mathrm{Vect} \arrow[dl, equal] & & \\
 & & & \mathrm{Vect} & & &
\end{tikzcd}
$$

Furthermore, by definition of Sum_1, its action on the three morphisms is as follows:

$$
\begin{aligned}
\mathrm{Sum}_1(\alpha) &= p_{2*} \cdot (\alpha \circ (-)) \cdot p_1^* \,, \\
\mathrm{Sum}_1(\beta) &= p'_{3*} \cdot (\beta \circ (-)) \cdot {p'_2}^* \,, \\
\mathrm{Sum}_1(\beta \circ_\eta \alpha) &= (p'_3 \circ \pi_2)_* \cdot (\beta \circ_\eta \alpha \circ (-)) \cdot (p_1 \circ \pi_1)^* \,.
\end{aligned}
$$

To show that $\mathrm{Sum}_1(\beta \circ_\eta \alpha) = \mathrm{Sum}_1(\beta) \circ \mathrm{Sum}_1(\alpha)$, we first observe that the left-hand side is the clock-wise composition of the outmost arrows from the top left to the bottom left vertex in the following diagram:

$$
\begin{tikzcd}
{[\mathbf{1}_{X_1},\rho_1]} \arrow[r, "p_1^*"] \arrow[rrr, bend left, "(p_1\circ\pi_1)^*"] \arrow[ddddd, "{[p'_{3*}\cdot(\beta\circ(-))\cdot {p'_2}^*]\cdot[p_{2*}\cdot(\alpha\circ(-))\cdot p_1^*] = \mathrm{Sum}_1(\beta)\cdot\mathrm{Sum}_1(\alpha)}"] & {[\mathbf{1}_Y, p_1^*\rho_1]} \arrow[rr, "\pi_1^*"] \arrow[d, "\alpha\circ(-)"'] & & {[\mathbf{1}_{Y\times_{X_2}Y'}, \pi_1^*p_1^*\rho_1]} \arrow[ddl, "\pi_1^*\alpha\circ(-)"'] \arrow[ddddd, "\beta\circ_\eta\alpha"] \\
& {[\mathbf{1}_Y, p_2^*\rho_2]} \arrow[dr, "\pi_1^*"] \arrow[ddd, "{p'_2}^* p_{2*}"'] & & \\
& & {[\mathbf{1}_{Y\times_{X_2}Y'}, \pi_1^*p_2^*\rho_2]} \arrow[d, "\eta^*\rho_2\circ(-)"'] & \\
& & {[\mathbf{1}_{Y\times_{X_2}Y'}, \pi_2^*{p'_2}^*\rho_2]} \arrow[dl, "\pi_{2*}"'] \arrow[ddr, "\pi_2^*\beta\circ(-)"] & \\
& {[\mathbf{1}_{Y'}, {p'_2}^*\rho_2]} \arrow[d, "\beta\circ(-)"'] & & \\
{[\mathbf{1}_{X_3},\rho_3]} & {[\mathbf{1}_{X_3}, {p'_3}^*\rho_3]} \arrow[l, "p'_{3*}"] & & {[\mathbf{1}_{Y\times_{X_2}Y'}, \pi_2^*{p'_3}^*\rho_3]} \arrow[ll, "\pi_{2*}"] \arrow[lll, bend left, "(p'_3\circ\pi_2)_*"]
\end{tikzcd}
$$

The leftmost vertical arrow is $\mathrm{Sum}_1(\beta)\circ\mathrm{Sum}_1(\alpha)$, hence we are done if we can show that the diagram commutes. This is indeed the case: the leftmost and rightmost subdiagrams commute by definition, as does the other subdiagram involving $\pi_1^*\alpha\circ(-)$; that the top and bottom diagrams commute follows from the definition of pullback and pushforward; finally, the subdiagram involving $\pi_2^*\beta\circ(-)$ and π_{2*} commutes by the projection formula of Lemma 3.10, and the middle subdiagram commutes thanks to the Beck-Chevalley condition of Lemma 3.11.

Finally, if $\rho_2 = \rho_3$, $Y = Y'$ and β is the identity on ρ_2 in $\mathrm{Fam}_1(\mathrm{Vect})$, then $Y \times_{X_2} Y'$ is canonically equivalent to Y, and we see that Sum_1 sends units to units. $\square$

3.2 $n = 2$

3.2.1 "Without thinking" (Nils)

Now let $\omega \in Z^2(G, \mathbb{C}^\times)$. We will construct a 2-functor

$$\mathrm{Bord}_2 \xrightarrow{\ I\ } \mathrm{Fam}_2(\mathrm{Alg}_\mathbb{C}) \xrightarrow{\mathrm{Sum}_2} \mathrm{Alg}_\mathbb{C} \tag{3.8}$$

by computing what it does to $\mathrm{pt}_+ \in \mathrm{Bord}_2$. The first step is to set $I(\mathrm{pt}_+)$ to be the 2-functor

$$\begin{aligned} \chi^\omega \colon \boldsymbol{B}G &\longrightarrow \mathrm{Alg}_\mathbb{C} \\ * &\longmapsto \mathbb{C} \\ G \ni g &\longmapsto L_g := {}_\mathbb{C}\mathbb{C}_\mathbb{C} \\ 1_g &\longmapsto 1_\mathbb{C} \end{aligned} \qquad \text{with} \quad \chi^\omega_{g,h} := \omega(g,h) \colon L_g \otimes_\mathbb{C} L_h \to L_{gh} \tag{3.9}$$

(Recall from [Le, Sect. 1.1] that the constraint on the coherence 2-morphisms $\chi^\omega_{g,h} = \omega(g,h) \in \mathrm{End}_\mathbb{C}(\mathbb{C})$ is precisely the 2-cocycle condition on ω. Also note that all our 2-functors are strictly unital, so in particular (3.9) completely specifies the 2-functor χ^ω.)

The second step is to compute the 2-colimit $\mathrm{Sum}_2(\chi^\omega)$. There are several versions of 2-limit, bilimit, weighted bilimit etc. in the literature. We will use the notion of [Bo, Def. 7.4.4] and call it 2-limit. To give the details, recall the definitions from [Le, Sect. 1.2–3], and that the *opposite* $\mathcal{B}^{\mathrm{op}}$ of a bicategory $\mathcal{B}$ has reversed horizontal composition, but the same vertical composition as $\mathcal{B}$.

Definition 3.13. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a 2-functor, and let $B \in \mathcal{B}$.

(i) The *constant 2-functor* $\Delta_B \colon \mathcal{A} \to \mathcal{B}$ sends every object in $\mathcal{A}$ to B, 1- and 2-morphisms in $\mathcal{A}$ to 1_B and 1_{1_B}, respectively, and all constraints $(\Delta_B)_{g,h}$ are trivial.

(ii) The category $2\text{-Cone}(B, F)$ of *2-cones on F with vertex B* is the category whose objects are pseudonatural transformations $\Delta_B \to F$ and whose morphisms are modifications between them.

(iii) A *2-limit* of F is a pair (L, π) with $L \in \mathcal{B}$ and $\pi \colon \Delta_L \to F$ a pseudonatural transformation such that the functor

$$\mathcal{B}(B, L) \longrightarrow 2\text{-Cone}(B, F)$$

$$(B \xrightarrow{f} L) \longmapsto (\Delta_B \xrightarrow{f_*} \Delta_L \xrightarrow{\pi} F)$$

is an equivalence of categories for all $B \in \mathcal{B}$.

(iv) Now let $G\colon \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$. The category $2\text{-Cocone}(B, G)$ of *2-cocones on G with vertex $B \in \mathcal{B}$* is defined to be $2\text{-Cone}(B, G)$, but with B viewed as an object in $\mathcal{B}^{\mathrm{op}}$. A 2-limit of $G\colon \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$ is also called a *2-colimit of G in $\mathcal{B}$*.

Spelling out the definition, we see that an object $\sigma\colon \Delta_B \to F$ of $2\text{-Cone}(B, F)$ is a family of 1- and 2-morphisms

$$\left\{B \xrightarrow{\sigma_A} F(A)\right\}_{A\in\mathcal{A}} \quad \text{and} \quad \left\{F(g) \otimes \sigma_A \xrightarrow{\sigma_g} \sigma_{A'}\right\}_{g\in\mathcal{A}(A,A')} \tag{3.10}$$

such that

$$\begin{tikzcd}
F(g) \otimes F(h) \otimes \sigma_A \arrow[r, "1 \otimes \sigma_h"] \arrow[d, "F_{g,h} \otimes 1"'] & F(g) \otimes \sigma_{A'} \arrow[d, "\sigma_g"] \\
F(gh) \otimes \sigma_A \arrow[r, "\sigma_{gh}"'] & \sigma_{A''}
\end{tikzcd} \tag{3.11}$$

commutes for all $A \xrightarrow{h} A' \xrightarrow{g} A''$. Furthermore, a morphism $\Gamma\colon \sigma \to \widetilde{\sigma}$ in $2\text{-Cone}(B, F)$ is a family

$$\left\{\sigma_A \xrightarrow{\Gamma_A} \widetilde{\sigma}_A\right\}_{A\in\mathcal{A}} \quad \text{such that} \quad \begin{tikzcd}
F(g) \otimes \sigma_A \arrow[r, "1 \otimes \Gamma_A"] \arrow[d, "\sigma_g"'] & F(g) \otimes \widetilde{\sigma}_A \arrow[d, "\widetilde{\sigma}_g"] \\
\sigma_{A'} \arrow[r, "\Gamma_{A'}"'] & \widetilde{\sigma}_{A'}
\end{tikzcd} \tag{3.12}$$

commutes for all $A \xrightarrow{g} A'$.

Note that above "$\otimes$" is exclusively used for the horizontal composition in the target $\mathcal{B}$ (while juxtaposition is used for the horizontal composition in $\mathcal{A}$). Hence going to $\mathcal{B}^{\mathrm{op}}$ amounts to reversing all the tensor products above.

Lemma 3.14. Let $B \in \mathrm{Alg}_{\mathbb{C}}$ and $\chi^\omega\colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ as in (3.9). Then

$$\begin{aligned}
2\text{-Cone}(B, \chi^\omega) &\cong {}_{\mathbb{C}^\omega[G]}\mathrm{Mod}_B\,, \\
2\text{-Cocone}(B, \chi^\omega) &\cong {}_B\mathrm{Mod}_{\mathbb{C}^\omega[G]}
\end{aligned} \tag{3.13}$$

where $\mathbb{C}^\omega[G]$ is the ω-twisted group algebra of G.

Proof. We only prove the the statement about 2-cones in detail, the one about 2-cocones follows analogously by reversing all horizontal compositions in $\mathcal{B}$.

By (3.10) and (3.11), an object in 2-Cone(B,χ^ω) is a 1-morphism $M \in \mathrm{Alg}_{\mathbb{C}}(B,\mathbb{C})$ together with linear maps $\sigma_g\colon L_g \otimes_{\mathbb{C}} M \to M$ such that

$$\begin{tikzcd}
L_g \otimes_{\mathbb{C}} L_h \otimes_{\mathbb{C}} M \arrow[r, "1\otimes\sigma_h"] \arrow[d, "\omega(g,h)\otimes 1"'] & L_g \otimes_{\mathbb{C}} M \arrow[d, "\sigma_g"] \\
L_{gh} \otimes_{\mathbb{C}} M \arrow[r, "\sigma_{gh}"'] & M
\end{tikzcd} \tag{3.14}$$

commutes for all $g,h \in G$. But this means that $(M, \sum_{g\in G}\sigma_g)$ is a left $\mathbb{C}^\omega[G]$-module, while by definition of $\mathrm{Alg}_{\mathbb{C}}$, the 1-morphism $M \in \mathrm{Alg}_{\mathbb{C}}(B,\mathbb{C})$ is a right B-module.

According to (3.12), a morphism $(M,\{\sigma_g\}) \to (\widetilde{M},\{\widetilde{\sigma}_g\})$ in 2-Cone(B,χ^ω) is a 2-morphism $\Gamma\colon {}_{\mathbb{C}}M_B \to {}_{\mathbb{C}}\widetilde{M}_B$ in $\mathrm{Alg}_{\mathbb{C}}$ such that

$$\begin{tikzcd}
L_g \otimes_{\mathbb{C}} M \arrow[r, "1\otimes\Gamma"] \arrow[d, "\sigma_g"'] & L_g \otimes_{\mathbb{C}} \widetilde{M} \arrow[d, "\widetilde{\sigma}_g"] \\
M \arrow[r, "\Gamma"'] & \widetilde{M}
\end{tikzcd} \tag{3.15}$$

commutes. But that means that Γ is both a map of left $\mathbb{C}^\omega[G]$-modules and a map of right B-modules. $\square$

Proposition 3.15. (i) A 2-limit of $\chi^\omega\colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^\omega[G], \mathbb{C}^\omega[G])$.

(ii) A 2-colimit of $\chi^\omega\colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^\omega[G], \mathbb{C}^\omega[G])$.

Proof. According to Definition 3.13 and Lemma 3.14, a 2-limit of χ^ω is an algebra $L \in \mathrm{Alg}_{\mathbb{C}}$ together with a module ${}_{\mathbb{C}^\omega[G]}M_L$ such that the functor

$$\begin{aligned}
\mathrm{Alg}_{\mathbb{C}}(B,L) &\longrightarrow {}_{\mathbb{C}^\omega[G]}\mathrm{Mod}_B \\
N &\longmapsto M \otimes_L N
\end{aligned}$$

is an equivalence for every $B \in \mathrm{Alg}_{\mathbb{C}}$. This is the case for $L = \mathbb{C}^\omega[G]$ as an algebra and $M = \mathbb{C}^\omega[G]$ as a bimodule over itself.

The statement about the 2-colimit follows analogously. $\square$

3.2.2 "With thinking" (Domenico)

The datum of the 2-functor $\chi^\omega\colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is a collection of lines (1-dimensional complex vector spaces) L_g, indexed by elements in the group G, together with isomorphisms

$$\rho_{g,h}\colon L_g \otimes L_h \xrightarrow{\sim} L_{gh}$$

subject to the associativity constraint given by the commutativity of the diagrams

$$\begin{tikzcd} L_g \otimes L_h \otimes L_k \arrow[r, "\rho_{g,h}\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\rho_{h,k}"'] & L_{gh} \otimes L_k \arrow[d, "\rho_{gh,k}"] \\ L_g \otimes L_{hk} \arrow[r, "\rho_{g,hk}"'] & L_{ghk} \end{tikzcd}$$

where the tensor products are over $\mathbb{C}$. We also require that $L_1 = \mathbb{C}$ and that the isomorphisms $L_1 \otimes L_g \xrightarrow{\sim} L_g$ and $L_g \otimes L_1 \xrightarrow{\sim} L_g$ are the structure isomorphisms for the unit object $\mathbb{C}$ of the monoidal category $\mathrm{Vect}_{\mathbb{C}}$. The associativity constraints imply, and are in fact equivalent to this, that the vector space

$$A_{\chi^\omega} = \bigoplus_{g\in G} L_g$$

has an associative $\mathbb{C}$-algebra structure.

Lemma 3.16. A choice of a basis element x_g for every line L_g defines a $\mathbb{C}^\times$-valued 2-cocycle ω on the group G. A different choice of basis elements leads to a cohomologous cocycle, so that the class $[\omega] \in H^2(G, \mathbb{C}^\times)$ is well defines.

Proof. As $\rho_{g,h}(x_g \otimes x_h)$ is a nonzero element in L_{gh} we have $\rho_{g,h}(x_g \otimes x_h) = \omega(g, h)x_{gh}$ for a unique element $\omega(g, h)$ in $\mathbb{C}^\times$. The associativity constraints are then immediately seen to be equivalent to the cocycle equation

$$\omega(gh, k)\omega(g, h) = \omega(g, hk)\omega(h, k).$$

Finally, if $\{y_g\}_{g\in G}$ is a different basis choice, and $\tilde{\omega}$ is the corresponding 2-cocycle, then we have $y_g = \eta_g x_g$ for some η_g in $\mathbb{C}^\times$ and

$$\widetilde{\omega(g, h)}y_{gh} = \rho_{g,h}(y_g\otimes y_h) = \eta_g\eta_h\rho_{g,h}(x_g\otimes x_h) = \eta_g\eta_h\omega(g, h)x_{gh} = \eta_g\eta_h\omega(g, h)\eta_{gh}^{-1}y_{gh}.$$

Therefore

$$\tilde{\omega}(g, h) = \eta_g\eta_{gh}^{-1}\eta_h\omega(g, h),$$

i.e. ω and $\tilde{\omega}$ are cohomologous. □

Corollary 3.17. The algebra A_{χ^ω} is isomorphic to the twisted group algebra $\mathbb{C}^\omega[G]$, defined as the $\mathbb{C}$-vector space on the basis $\{x_g\}_{g\in G}$ with the product $x_g \cdot x_h = \omega(g, h)x_{gh}$ and with $x_1 = 1$.

Lemma 3.18. $\mathrm{Sum}_2(\chi^\omega) \cong \mathbb{C}^\omega[G]$, the twisted group algebra of G.

Proof. Let (A, M) be a cocone for χ^ω, i.e., a pair consisting of a $\mathbb{C}$-algebra A together with a left A-module M (representing a linear functor ${}_{\mathbb{C}}\mathrm{Mod} \to {}_A\mathrm{Mod}$) and homotopy commutative diagrams

$$\begin{tikzcd} & A & \\ \mathbb{C} \arrow[ur, "M"] \arrow[rr, "L_g"'] \arrow[rr, phantom, shift left=3ex, "\Leftarrow \lambda_g"] & & \mathbb{C} \arrow[ul, "M"'] \end{tikzcd}$$

where the λ_g's are isomorphisms of left A-modules

$$\lambda_g \colon M \otimes_{\mathbb{C}} L_g \xrightarrow{\sim} M$$

such that the diagrams

$$\begin{tikzcd}
M \otimes L_g \otimes L_h \arrow[r, "\lambda_g \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \rho_{g,h}"'] & M \otimes L_h \arrow[d, "\lambda_h"] \\
M \otimes L_{gh} \arrow[r, "\lambda_{gh}"'] & M
\end{tikzcd}$$

commute. This implies that the isomorphisms λ_g make M a right A_{χ^ω}-module. Therefore, we can see M as a linear functor

$${}_{A_{\chi^\omega}}\mathrm{Mod} \to {}_A\mathrm{Mod}$$

i.e., as a morphism from A_{χ^ω} to A in 2-$\mathrm{Vect}_{\mathbb{C}}$. The algebra A_{χ^ω} seen as a left A_{χ^ω}-module is a morphism from $\mathbb{C}$ to A_{χ^ω} in 2-$\mathrm{Vect}_{\mathbb{C}}$ and the natural isomorphism

$${}_AM_{\mathbb{C}} \cong {}_AM_{A_{\chi^\omega}} \otimes {}_{A_{\chi^\omega}}A_{\chi^\omega}{}_{\mathbb{C}}$$

gives a canonical factorization

$$\begin{tikzcd}
& A & \\
& A_{\chi^\omega} \arrow[u, "M"] & \\
\mathbb{C} \arrow[ur, "A_{\chi^\omega}"] \arrow[uur, bend left, "M"] \arrow[rr, "L_g"'] \arrow[rr, phantom, bend left=20, "\Rightarrow \lambda_g"] & & \mathbb{C} \arrow[ul, "A_{\chi^\omega}"'] \arrow[uul, bend right, "M"']
\end{tikzcd}$$

exhibiting the algebra A_{χ^ω} together with itself seen as a left module over itself as the universal cocone. $\square$

3.3 $n = 3$

TODO:

- The only non-trivial part of the 3-functor $\chi^\omega \colon \boldsymbol{B}G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.

- **Issue**: The notion of a 3-colimit (to really compute Sum_3) is scary. At least we should make some hand-wavy arguments...

4 Boundary conditions

4.1 $n = 2$

TODO

4.2 $n = 3$

TODO

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