Boundaries in finite gauge theory

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1 Introduction

Let

- G be a finite group,
- $\bullet \ [\omega] \in H^n(G,\mathbb{C}^\times),$
- and \mathcal{C} a symmetric monoidal (∞, n) -category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fullly extended TQFT

$$\operatorname{Bord}_n \xrightarrow{I} \operatorname{Fam}_n(\mathcal{C}) \xrightarrow{\operatorname{Sum}_n} \mathcal{C}$$
 (1.1)

based on the data (G, ω) , which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $C = \mathbf{B}^n \mathbb{C}^{\times}$ (the n-fold delooping of the group \mathbb{C}^{\times}) and then

- (i) identify classical boundary conditions as 1-morphisms with source * in $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$,
- (ii) subsume the boundary conditions of [Wi, WWW] as special cases,
- (iii) map classical boundary conditions to quantum ones via Sum_n , and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs-Schweigert-Valentino in the case n=3,
- (iv) do the above not only for framed, but also for oriented, spin,... TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1,2\}$ and hopefully also n=3. We also want to explain in detail how (G,ω) gives rise to an object in $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)(*,\mathbf{B}G)$.

(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [Lu, Sect. 4.3] instead of $Bord_n$, but we leave that for another project.)

2 Special objects and 1-morphisms in $\operatorname{Fam}_n(\boldsymbol{B}^n\mathbb{C}^{\times})$

TODO:

- spell out definition of $\operatorname{Fam}_n(\mathcal{C})$
- check that n-functor $BG \to B^n \mathbb{C}^{\times}$ is precisely an n-cocycle <u>Issue</u>: For n > 3, what is a weak n-functor, i. e. precisely what data and constraints are needed?

<u>Idea</u>: Since both n-categories in $BG \to B^n\mathbb{C}^{\times}$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section 3.

- clarify precisely how $BG \to B^n \mathbb{C}^{\times}$ induces an *n*-functor $BG \to n$ -Vect, at least for $n \in \{1, 2, 3\}$
- check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$

Issue: induced from issue in second item

3 Bulk theory

3.1 n=1

Since we take the action of G on \mathbb{C}^{\times} to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \operatorname{Hom}_{\operatorname{Grp}}(G, \mathbb{C}^\times)$$
(3.1)

is just a group homomorphism. We consider $B^1\mathbb{C}^{\times}$ as a subcategory of $\mathrm{Vect}_{\mathbb{C}}$, and we will use the pair (G, ω) to construct a functor

$$\operatorname{Bord}_1 \xrightarrow{I} \operatorname{Fam}_1(\operatorname{Vect}_{\mathbb{C}}) \xrightarrow{\operatorname{Sum}_1} \operatorname{Vect}_{\mathbb{C}}.$$
 (3.2)

According to the cobordism hypothesis, the composite $\operatorname{Sum}_1 \circ I$ is determined by its value on the point pt_+ , so we only need to specify $I(\operatorname{pt}_+)$ and then compute $\operatorname{Sum}_1(I(\operatorname{pt}_+))$. Note that $\mathbf{B}^1G = */\!\!/ G$ is the classical action groupoid. We set $I(\operatorname{pt}_+)$ to be the functor

$$\chi^{\omega} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto \mathbb{C}, \quad g \longmapsto \omega(g) \in \operatorname{Aut}(\mathbb{C})$$
 (3.3)

for all $g \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. Sum₁(χ^{ω}) = \mathbb{C} if $\omega(g) = 1$ for all $g \in G$, and Sum₁(χ^{ω}) = 0 otherwise.

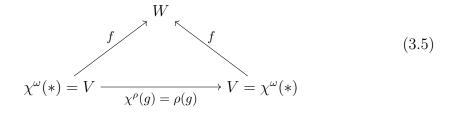
Proof. The statement is a particular case of this more general one: let (V, ρ) be a linear representation of the group G, seen as the functor

$$\chi^{\rho} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto V, \quad q \longmapsto \rho(q) \in \operatorname{Aut}(V).$$
 (3.4)

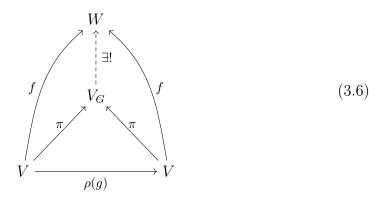
Then the universal cocone of χ^{ρ} , viewed as a diagram of shape $*/\!\!/ G$ in $\text{Vect}_{\mathbb{C}}$ is the pair (V_G, π) , where V_G is the vector space of coinvariants for the representation ρ , i.e.,

$$V_G = V/\langle v - \rho(g)v \rangle_{g \in G, v \in V}$$

and $\pi: V \to V_G$ is the projection to the quotient. Namely, let (W, f) be a cocone for χ^{ρ} , i.e., a pair consisting of a vector space W together with a linear map $f: \chi^{\omega}(*) = V \to W$ such that



Then, by definition of V_G , the morphism f uniquely factors through V_G and so we have a commutative diagram



showing that (V_G, π) enjoys the universal property of the universal cocone. \square Corollary 3.2. The linear dual $\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_1(\chi^{\omega}), \mathbb{C})$ of $\operatorname{Sum}_1(\chi^{\omega})$ is naturally isomorphic to the vector space of natural transformations $\operatorname{Hom}_{\chi(G,V)}(\chi^{\omega})$

isomorphic to the vector space of natural transformations $\operatorname{Hom}_{[*/\!/G,\operatorname{Vect}_{\mathbb{C}}]}(\chi^{\omega},\chi^{1})$, where $\chi^{1}\colon */\!/G \to \operatorname{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space \mathbb{C} . In other words, we have

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_{1}(\chi^{\omega}),\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \chi^{\omega} \Longrightarrow * \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}.$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\operatorname{Sum}_1(\chi^{\omega})$.

Proof. Again, the statement is true for an aritrary linear representation (V, ρ) of the group G. The natural isomorphism

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(V_G,\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \downarrow \\ \downarrow \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}$$

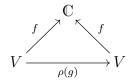
is then nothing but the universal property of V_G . Namely, an element in the right hand side is a morphism $f: V = \chi^{\rho}(*) \to \chi^{1}(*) = \mathbb{C}$ such that all the diagrams

$$V \xrightarrow{f} \mathbb{C}$$

$$\rho(g) \downarrow \qquad \qquad \downarrow \mathrm{id}_{\mathbb{C}}$$

$$V \xrightarrow{f} \mathbb{C}$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams



commute, and so it is precisely the datum of a morphism $V_G \to \mathbb{C}$ by the argumenti in the proof of Lemma 3.1.

3.2 n=2

TODO:

- The only non-trivial part of the 2-functor $\chi^{\omega} : \mathbf{B}^2 G \to \mathrm{Alg}_{\mathbb{C}}$ is the coherence 2-morphism $\chi_{g,h}^{\omega} : \chi^{\omega}(g) \otimes \chi^{\omega}(h) \to \chi^{\omega}(gh)$, and the constraints (see e. g. [Le, Sect. 1.1] for the definitions) on the $\chi_{g,h}^{\omega}$ precisely say that they are the components of a 2-cocycle. So we can take $\chi_{g,h}^{\omega} = \omega(g,h)$.
- compute 2-colimit $\operatorname{Sum}_2(\chi)$ for χ as above, obtain twisted group algebra $\mathbb{C}^{\omega}[G]$

3.3 n = 3

TODO:

- The only non-trivial part of the 3-functor $\chi^{\omega} \colon \mathbf{B}^{3}G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.
- <u>Issue</u>: The notion of a 3-colimit (to really compute Sum₃) is scary. At least we should make some hand-wavy arguments...

4 Boundary conditions

4.1
$$n=2$$

TODO

4.2
$$n=3$$

TODO

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