Boundaries in finite gauge theory

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Contents

1 Introduction

Let

- G be a finite group,
- $\omega \in Z^n(G, \mathbb{C}^\times)$,
- and $\mathcal{C}$ a symmetric monoidal (∞, n)-category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fullly extended TQFT

$$\mathrm{Bord}_n \xrightarrow[\text{“classical”}]{I} \mathrm{Fam}_n(\mathcal{C}) \xrightarrow[\text{“quantum”}]{\mathrm{Sum}_n} \mathcal{C} \tag{1.1}$$

based on the data (G,ω), which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $\mathcal{C} = \boldsymbol{B}^n\mathbb{C}^\times$ (the n-fold delooping of the group $\mathbb{C}^\times$) and then

(i) identify *classical* boundary conditions as 1-morphisms with source $*$ in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$,

(ii) subsume the boundary conditions of [Wi, WWW] as special cases,

(iii) map classical boundary conditions to quantum ones via Sum_n, and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs–Schweigert–Valentino in the case $n = 3$,

(iv) do the above not only for framed, but also for oriented, spin,. . . TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1,2\}$ and hopefully also $n = 3$. We also want to explain in detail how (G,ω) gives rise to an object in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)(*, \boldsymbol{B}G)$.
(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [Lu, Sect. 4.3] instead of Bord_n, but we leave that for another project.)

2 Special objects and 1-morphisms in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$

TODO:

- spell out definition of $\mathrm{Fam}_n(\mathcal{C})$
- check that n-functor $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ is precisely an n-cocycle
  Issue: For $n > 3$, what is a weak n-functor, i. e. precisely what data and constraints are needed?
  Idea: Since both n-categories in $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1,2,3\}$ this is true, see Section 3.
- clarify precisely how $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ induces an n-functor $\boldsymbol{B}G \to n\text{-Vect}$, at least for $n \in \{1,2,3\}$

- check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$
  Issue: induced from issue in second item

2.1 Classical bulk theories

TODO

2.2 Classical boundary conditions

2.2.1 $n = 2$

Let $\mathcal{G}_1, \mathcal{G}_2$ be 2-groupoids and let $F_1 \colon \mathcal{G}_1 \to \boldsymbol{B}^2\mathbb{C}^\times$ and $F_2 \colon \mathcal{G}_2 \to \boldsymbol{B}^2\mathbb{C}^\times$ be 2-functors, i. e. objects in $\mathrm{Fam}_2(\boldsymbol{B}^2\mathbb{C}^\times)$. A 1-morphism $F_1 \to F_2$ in $\mathrm{Fam}_2(\boldsymbol{B}^2\mathbb{C}^\times)$ is a span of 2-groupoids $\mathcal{G}_1 \xleftarrow{K_1} \widetilde{\mathcal{G}} \xrightarrow{K_2} \mathcal{G}_2$ together with a pseudonatural transformation $\sigma \colon F_1 \circ K_1 \to F_2 \circ K_2$:

$$
\begin{tikzcd}
 & \widetilde{\mathcal{G}} \arrow[dl, "K_1"'] \arrow[dr, "K_2"] & \\
\mathcal{G}_1 \arrow[dr, "F_1"'] \arrow[rr, phantom, "\overset{\sigma}{\Longrightarrow}"] & & \mathcal{G}_2 \arrow[dl, "F_2"] \\
 & \boldsymbol{B}^2\mathbb{C}^\times &
\end{tikzcd}
\tag{2.1}
$$

Let $[1]$ denote the final 2-category with only one object, 1- and 2-morphism, and let $T \colon [1] \to \boldsymbol{B}^2\mathbb{C}^\times$ be the unique, trivial 2-functor. A *classical boundary condition* for $F \colon \mathcal{G} \to \boldsymbol{B}^2\mathbb{C}^\times$ is a 1-morphism $T \to F$. Thus it comprises a 2-groupoid $\mathcal{H}$, a 2-functor $\rho \colon \mathcal{H} \to \mathcal{G}$, and a pseudonatural transformation

$$
\begin{tikzcd}
 & \mathcal{H} \arrow[dl] \arrow[dr, "\rho"] & \\
{[1]} \arrow[dr, "T"'] \arrow[rr, phantom, "\overset{\sigma}{\Longrightarrow}"] & & \mathcal{G} \arrow[dl, "F"] \\
 & \boldsymbol{B}^2\mathbb{C}^\times &
\end{tikzcd}
\tag{2.2}
$$

Concretely, since $\boldsymbol{B}^2\mathbb{C}^\times$ has only one object and one 1-morphism, σ is given by a family of 2-morphisms $\sigma_h \colon (F \circ \rho)(h) = 1_* \to 1_*$, indexed by 1-morphisms h

in $\mathcal{H}$, such that

$$\begin{tikzcd}
F\big(\rho(h)\big) \otimes F\big(\rho(h')\big) \arrow[r, "1 \otimes \sigma_{h'}"] \arrow[d, "(F\circ\rho)_{h,h'}"'] & F\big(\rho(h)\big) \arrow[d, "\sigma_h"] \\
F\big(\rho(hh')\big) \arrow[r, "\sigma_{hh'}"'] & 1_*
\end{tikzcd} \tag{2.3}$$

commutes for al composable 1-morphisms h, h' in $\mathcal{H}$.

We now choose $\mathcal{H} = \boldsymbol{B}H$ for some finite group H, $\mathcal{G} = \boldsymbol{B}G$, and $F = \chi^\omega$ as in (3.8) — and we are not careful to distinguish the target $\boldsymbol{B}^2\mathbb{C}^\times$ from $\mathrm{Alg}_\mathbb{C}$. Then a 2-functor $\rho\colon \boldsymbol{B}H \to \boldsymbol{B}G$ is just a group homomorphism $\rho\colon H \to G$, and a boundary condition

$$\begin{tikzcd}
& \boldsymbol{B}H \arrow[dl] \arrow[dr, "\rho"] & \\
{[1]} \arrow[dr, "T"'] \arrow[rr, phantom, "\overset{\sigma}{\Longrightarrow}"] & & \boldsymbol{B}G \arrow[dl, "\chi^\omega"] \\
& \boldsymbol{B}^2\mathbb{C}^\times &
\end{tikzcd} \tag{2.4}$$

for χ^ω is a family $\{\sigma_h\}_{h\in H} \subset \mathbb{C}^\times$ such that

$$\begin{tikzcd}
L_{\rho(h)} \otimes_\mathbb{C} L_{\rho(h')} \arrow[r, "1 \otimes \sigma_{h'}"] \arrow[d, "\omega(\rho(h),\rho(h'))"'] & L_{\rho(h)} \arrow[d, "\sigma_h"] \\
L_{\rho(hh')} \arrow[r, "\sigma_{hh'}"'] & 1_*
\end{tikzcd} \tag{2.5}$$

commutes, i. e. $\omega(\rho(h), \rho(h')) = \sigma_h \cdot \sigma_{hh'}^{-1} \cdot \sigma_{h'}$ for all $h, h' \in H$. Viewing σ as a map $H \to \mathbb{C}^\times$, $h \mapsto \sigma_h$, this condition precisely means that σ trivialises the pullback of ω:

$$\mathrm{d}\sigma = \rho^*\omega\,. \tag{2.6}$$

3 Bulk theory

3.1 $n = 1$

Since we take the action of G on $\mathbb{C}^\times$ to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{C}^\times) \tag{3.1}$$

is just a group homomorphism. We consider $\boldsymbol{B}^1\mathbb{C}^\times$ as a subcategory of $\mathrm{Vect}_\mathbb{C}$, and we will use the pair (G,ω) to construct a functor

$$\mathrm{Bord}_1 \xrightarrow{\quad I \quad} \mathrm{Fam}_1(\mathrm{Vect}_\mathbb{C}) \xrightarrow{\mathrm{Sum}_1} \mathrm{Vect}_\mathbb{C}\ . \tag{3.2}$$

According to the cobordism hypothesis, the composite $\mathrm{Sum}_1 \circ I$ is determined by its value on the point pt_+, so we only need to specify $I(\mathrm{pt}_+)$ and then compute $\mathrm{Sum}_1(I(\mathrm{pt}_+))$. Note that $\boldsymbol{B}^1G = */\!\!/G$ is the classical action groupoid. We set $I(\mathrm{pt}_+)$ to be the functor

$$\chi^\omega\colon\ */\!\!/G \longrightarrow \mathrm{Vect}_\mathbb{C}\,, \qquad * \longmapsto \mathbb{C}\,, \quad g \longmapsto \omega(g) \in \mathrm{Aut}(\mathbb{C}) \tag{3.3}$$

for all $g \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. $\mathrm{Sum}_1(\chi^\omega) = \mathbb{C}$ if $\omega(g) = 1$ for all $g \in G$, and $\mathrm{Sum}_1(\chi^\omega) = 0$ otherwise.

Proof. The statement is a particular case of this more general one: let (V,ρ) be a linear representation of the group G, seen as the functor

$$\chi^\rho\colon\ */\!\!/G \longrightarrow \mathrm{Vect}_\mathbb{C}\,, \qquad * \longmapsto V\,, \quad g \longmapsto \rho(g) \in \mathrm{Aut}(V). \tag{3.4}$$

Then the universal cocone of χ^ρ, viewed as a diagram of shape $*/\!\!/G$ in $\mathrm{Vect}_\mathbb{C}$ is the pair (V_G,π), where V_G is the vector space of coinvariants for the representation ρ, i.e.,

$$V_G = V/\langle v - \rho(g)v\rangle_{g\in G, v\in V}$$

and $\pi\colon V \to V_G$ is the projection to the quotient. Namely, let (W,f) be a cocone for χ^ρ, i.e., a pair consisting of a vector space W together with a linear map $f\colon \chi^\rho(*) = V \to W$ such that

$$\begin{tikzcd} & W & \\ \chi^\rho(*) = V \arrow[ur, "f"] \arrow[rr, "\chi^\rho(g) = \rho(g)"'] & & V = \chi^\rho(*) \arrow[ul, "f"'] \end{tikzcd} \tag{3.5}$$

Then, by definition of V_G, the morphism f uniquely factors through V_G and so we have a commutative diagram

$$
\begin{tikzcd}
& W & \\
& V_G \arrow[u, dashed, "\exists!"'] & \\
V \arrow[ur, "\pi"] \arrow[uur, bend left, "f"] \arrow[rr, "\rho(g)"'] & & V \arrow[ul, "\pi"'] \arrow[uul, bend right, "f"']
\end{tikzcd}
\tag{3.6}
$$

showing that (V_G, π) enjoys the universal property of the universal cocone. $\square$

Corollary 3.2. The linear dual $\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(\mathrm{Sum}_1(\chi^\omega), \mathbb{C})$ of $\mathrm{Sum}_1(\chi^\omega)$ is naturally isomorphic to the vector space of natural transformations $\mathrm{Hom}_{[*/\!\!/G, \mathrm{Vect}_{\mathbb{C}}]}(\chi^\omega, \chi^1)$, where $\chi^1\colon * /\!\!/G \to \mathrm{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space $\mathbb{C}$. In other words, we have

$$
\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(\mathrm{Sum}_1(\chi^\omega), \mathbb{C}) \cong \left\{
\begin{tikzcd}
*/\!\!/G \arrow[dd, "\chi^\omega"'] \arrow[dr] & \\
\arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl, "\mathbb{C}"] \\
\mathrm{Vect}_{\mathbb{C}} &
\end{tikzcd}
\right\}.
$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\mathrm{Sum}_1(\chi^\omega)$.

Proof. Again, the statement is true for an aritrary linear representation (V, ρ) of the group G. The natural isomorphism

$$
\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(V_G, \mathbb{C}) \cong \left\{
\begin{tikzcd}
*/\!\!/G \arrow[dd, "\chi^\rho"'] \arrow[dr] & \\
\arrow[r, phantom, "\Longrightarrow"] & * \arrow[dl, "\mathbb{C}"] \\
\mathrm{Vect}_{\mathbb{C}} &
\end{tikzcd}
\right\}
$$

is then nothing but the universal property of V_G. Namely, an element in the right hand side is a morphism $f\colon V = \chi^\rho(*) \to \chi^1(*) = \mathbb{C}$ such that all the diagrams

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\rho(g)"'] & \mathbb{C} \arrow[d, "\mathrm{id}_{\mathbb{C}}"] \\
V \arrow[r, "f"'] & \mathbb{C}
\end{tikzcd}
$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams

$$\begin{tikzcd} & \mathbb{C} & \\ V \arrow[ur, "f"] \arrow[rr, "\rho(g)"'] & & V \arrow[ul, "f"'] \end{tikzcd}$$

commute, and so it is precisely the datum of a morphism $V_G \to \mathbb{C}$ by the argumenti in the proof of Lemma 3.8. □

3.2 $n = 2$

3.2.1 "Without thinking" (Nils)

Now let $\omega \in Z^2(G, \mathbb{C}^\times)$. We will construct a 2-functor

$$\mathrm{Bord}_2 \xrightarrow{I} \mathrm{Fam}_2(\mathrm{Alg}_{\mathbb{C}}) \xrightarrow{\mathrm{Sum}_2} \mathrm{Alg}_{\mathbb{C}} \tag{3.7}$$

by computing what it does to $\mathrm{pt}_+ \in \mathrm{Bord}_2$. The first step is to set $I(\mathrm{pt}_+)$ to be the 2-functor

$$\begin{aligned} \chi^\omega \colon \boldsymbol{B}G &\longrightarrow \mathrm{Alg}_{\mathbb{C}} \\ * &\longmapsto \mathbb{C} \\ G \ni g &\longmapsto L_g := {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \\ 1_g &\longmapsto 1_{\mathbb{C}} \end{aligned} \qquad \text{with} \quad \chi^\omega_{g,h} := \omega(g,h) \colon L_g \otimes_{\mathbb{C}} L_h \to L_{gh} \tag{3.8}$$

(Recall from [Le, Sect. 1.1] that the constraint on the coherence 2-morphisms $\chi^\omega_{g,h} = \omega(g,h) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C})$ is precisely the 2-cocycle condition on ω. Also note that all our 2-functors are strictly unital, so in particular (3.8) completely specifies the 2-functor χ^ω.)

The second step is to compute the 2-colimit $\mathrm{Sum}_2(\chi^\omega)$. There are several versions of 2-limit, bilimit, weighted bilimit etc. in the literature. We will use the notion of [Bo, Def. 7.4.4] and call it 2-limit. To give the details, recall the definitions from [Le, Sect. 1.2–3], and that the *opposite* $\mathcal{B}^{\mathrm{op}}$ of a bicategory $\mathcal{B}$ has reversed horizontal composition, but the same vertical composition as $\mathcal{B}$.

Definition 3.3. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a 2-functor, and let $B \in \mathcal{B}$.

(i) The *constant 2-functor* $\Delta_B \colon \mathcal{A} \to \mathcal{B}$ sends every object in $\mathcal{A}$ to B, 1- and 2-morphisms in $\mathcal{A}$ to 1_B and 1_{1_B}, respectively, and all constraints $(\Delta_B)_{g,h}$ are trivial.

(ii) The category 2-Cone(B, F) of *2-cones on F with vertex B* is the category whose objects are pseudonatural transformations $\Delta_B \to F$ and whose morphisms are modifications between them.

(iii) A *2-limit* of F is a pair (L, π) with $L \in \mathcal{B}$ and $\pi \colon \Delta_L \to F$ a pseudonatural transformation such that the functor

$$\begin{aligned} \mathcal{B}(B, L) &\longrightarrow \text{2-Cone}(B, F) \\ (B \xrightarrow{f} L) &\longmapsto (\Delta_B \xrightarrow{f_*} \Delta_L \xrightarrow{\pi} F) \end{aligned}$$

is an equivalence of categories for all $B \in \mathcal{B}$.

(iv) Now let $G \colon \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$. The category 2-Cocone$(B, G)$ of *2-cocones on G with vertex $B \in \mathcal{B}$* is defined to be 2-Cone(B, G), but with B viewed as an object in $\mathcal{B}^{\mathrm{op}}$. A 2-limit of $G \colon \mathcal{A} \to \mathcal{B}^{\mathrm{op}}$ is also called a *2-colimit of G in $\mathcal{B}$*.

Spelling out the definition, we see that an object $\sigma \colon \Delta_B \to F$ of 2-Cone(B, F) is a family of 1- and 2-morphisms

$$\big\{B \xrightarrow{\sigma_A} F(A)\big\}_{A \in \mathcal{A}} \quad \text{and} \quad \big\{F(g) \otimes \sigma_A \xrightarrow{\sigma_g} \sigma_{A'}\big\}_{g \in \mathcal{A}(A, A')} \tag{3.9}$$

such that

$$\begin{tikzcd} F(g) \otimes F(h) \otimes \sigma_A \arrow[r, "1 \otimes \sigma_h"] \arrow[d, "F_{g,h} \otimes 1"'] & F(g) \otimes \sigma_{A'} \arrow[d, "\sigma_g"] \\ F(gh) \otimes \sigma_A \arrow[r, "\sigma_{gh}"'] & \sigma_{A''} \end{tikzcd} \tag{3.10}$$

commutes for all $A \xrightarrow{h} A' \xrightarrow{g} A''$. Furthermore, a morphism $\Gamma \colon \sigma \to \widetilde{\sigma}$ in 2-Cone(B, F) is a family

$$\big\{\sigma_A \xrightarrow{\Gamma_A} \widetilde{\sigma}_A\big\}_{A \in \mathcal{A}} \quad \text{such that} \quad \begin{tikzcd} F(g) \otimes \sigma_A \arrow[r, "1 \otimes \Gamma_A"] \arrow[d, "\sigma_g"'] & F(g) \otimes \widetilde{\sigma}_A \arrow[d, "\widetilde{\sigma}_g"] \\ \sigma_{A'} \arrow[r, "\Gamma_{A'}"'] & \widetilde{\sigma}_{A'} \end{tikzcd} \tag{3.11}$$

commutes for all $A \xrightarrow{g} A'$.

Note that above "$\otimes$" is exclusively used for the horizontal composition in the target $\mathcal{B}$ (while juxtaposition is used for the horizontal composition in $\mathcal{A}$). Hence going to $\mathcal{B}^{\mathrm{op}}$ amounts to reversing all the tensor products above.

Lemma 3.4. Let $B \in \mathrm{Alg}_{\mathbb{C}}$ and $\chi^\omega \colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ as in (3.8). Then

$$\begin{aligned} \text{2-Cone}(B, \chi^\omega) &\cong {}_{\mathbb{C}^\omega[G]}\mathrm{Mod}_B \,, \\ \text{2-Cocone}(B, \chi^\omega) &\cong {}_B\mathrm{Mod}_{\mathbb{C}^\omega[G]} \end{aligned} \tag{3.12}$$

where $\mathbb{C}^\omega[G]$ is the ω-twisted group algebra of G.

Proof. We only prove the the statement about 2-cones in detail, the one about 2-cocones follows analogously by reversing all horizontal compositions in $\mathcal{B}$.

By (3.9) and (3.10), an object in $2\text{-Cone}(B, \chi^\omega)$ is a 1-morphism $M \in \mathrm{Alg}_{\mathbb{C}}(B, \mathbb{C})$ together with linear maps $\sigma_g \colon L_g \otimes_{\mathbb{C}} M \to M$ such that

$$\begin{tikzcd}
L_g \otimes_{\mathbb{C}} L_h \otimes_{\mathbb{C}} M \arrow[r, "1 \otimes \sigma_h"] \arrow[d, "\omega(g,h) \otimes 1"'] & L_g \otimes_{\mathbb{C}} M \arrow[d, "\sigma_g"] \\
L_{gh} \otimes_{\mathbb{C}} M \arrow[r, "\sigma_{gh}"'] & M
\end{tikzcd} \tag{3.13}$$

commutes for all $g, h \in G$. But this means that $(M, \sum_{g \in G} \sigma_g)$ is a left $\mathbb{C}^\omega[G]$-module, while by definition of $\mathrm{Alg}_{\mathbb{C}}$, the 1-morphism $M \in \mathrm{Alg}_{\mathbb{C}}(B, \mathbb{C})$ is a right B-module.

According to (3.11), a morphism $(M, \{\sigma_g\}) \to (\widetilde{M}, \{\widetilde{\sigma}_g\})$ in $2\text{-Cone}(B, \chi^\omega)$ is a 2-morphism $\Gamma \colon {}_{\mathbb{C}}M_B \to {}_{\mathbb{C}}\widetilde{M}_B$ in $\mathrm{Alg}_{\mathbb{C}}$ such that

$$\begin{tikzcd}
L_g \otimes_{\mathbb{C}} M \arrow[r, "1 \otimes \Gamma"] \arrow[d, "\sigma_g"'] & L_g \otimes_{\mathbb{C}} \widetilde{M} \arrow[d, "\widetilde{\sigma}_g"] \\
M \arrow[r, "\Gamma"'] & \widetilde{M}
\end{tikzcd} \tag{3.14}$$

commutes. But that means that Γ is both a map of left $\mathbb{C}^\omega[G]$-modules and a map of right B-modules. $\square$

Proposition 3.5. (i) A 2-limit of $\chi^\omega \colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^\omega[G], \mathbb{C}^\omega[G])$.

(ii) A 2-colimit of $\chi^\omega \colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^\omega[G], \mathbb{C}^\omega[G])$.

Proof. According to Definition 3.3 and Lemma 3.4, a 2-limit of χ^ω is an algebra $L \in \mathrm{Alg}_{\mathbb{C}}$ together with a module ${}_{\mathbb{C}^\omega[G]}M_L$ such that the functor

$$\begin{aligned}
\mathrm{Alg}_{\mathbb{C}}(B, L) &\longrightarrow {}_{\mathbb{C}^\omega[G]}\mathrm{Mod}_B \\
N &\longmapsto M \otimes_L N
\end{aligned}$$

is an equivalence for every $B \in \mathrm{Alg}_{\mathbb{C}}$. This is the case for $L = \mathbb{C}^\omega[G]$ as an algebra and $M = \mathbb{C}^\omega[G]$ as a bimodule over itself.

The statement about the 2-colimit follows analogously. $\square$

3.2.2 "With thinking" (Domenico)

The datum of the 2-functor $\chi^\omega \colon \boldsymbol{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is a collection of lines (1-dimensional complex vector spaces) L_g, indexed by elements in the group G, together with isomorphisms

$$\rho_{g,h} \colon L_g \otimes L_h \xrightarrow{\sim} L_{gh}$$

subject to the associativity constraint given by the commutativity of the diagrams

$$\begin{tikzcd} L_g \otimes L_h \otimes L_k \arrow[r, "\rho_{g,h}\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\rho_{h,k}"'] & L_{gh} \otimes L_k \arrow[d, "\rho_{gh,k}"] \\ L_g \otimes L_{hk} \arrow[r, "\rho_{g,hk}"'] & L_{ghk} \end{tikzcd}$$

where the tensor products are over $\mathbb{C}$. We also require that $L_1 = \mathbb{C}$ and that the isomorphisms $L_1 \otimes L_g \xrightarrow{\sim} L_g$ and $L_g \otimes L_1 \xrightarrow{\sim} L_g$ are the structure isomorphisms for the unit object $\mathbb{C}$ of the monoidal category $\mathrm{Vect}_{\mathbb{C}}$. The associativity constraints imply, and are in fact equivalent to this, that the vector space

$$A_{\chi^\omega} = \bigoplus_{g\in G} L_g$$

has an associative $\mathbb{C}$-algebra structure.

Lemma 3.6. A choice of a basis element x_g for every line L_g defines a $\mathbb{C}^\times$-valued 2-cocycle ω on the group G. A different choice of basis elements leads to a cohomologous cocycle, so that the class $[\omega] \in H^2(G, \mathbb{C}^\times)$ is well defines.

Proof. As $\rho_{g,h}(x_g \otimes x_h)$ is a nonzero element in L_{gh} we have $\rho_{g,h}(x_g \otimes x_h) = \omega(g,h)x_{gh}$ for a unique element $\omega(g,h)$ in $\mathbb{C}^\times$. The associativity constraints are then immediately seen to be equivalent to the cocycle equation

$$\omega(gh,k)\omega(g,h) = \omega(g,hk)\omega(h,k).$$

Finally, if $\{y_g\}_{g\in G}$ is a different basis choice, and $\tilde{\omega}$ is the corresponding 2-cocycle, then we have $y_g = \eta_g x_g$ for some η_g in $\mathbb{C}^\times$ and

$$\omega(\tilde{g,h})y_{gh} = \rho_{g,h}(y_g \otimes y_h) = \eta_g\eta_h\rho_{g,h}(x_g \otimes x_h) = \eta_g\eta_h\omega(g,h)x_{gh} = \eta_g\eta_h\omega(g,h)\eta_{gh}^{-1}y_{gh}.$$

Therefore

$$\tilde{\omega}(g,h) = \eta_g\eta_{gh}^{-1}\eta_h\omega(g,h),$$

i.e. ω and $\tilde{\omega}$ are cohomologous. □

Corollary 3.7. The algebra A_{χ^ω} is isomorphic to the twisted group algebra $\mathbb{C}^\omega[G]$, defined as the $\mathbb{C}$-vector space on the basis $\{x_g\}_{g\in G}$ with the product $x_g \cdot x_h = \omega(g,h)x_{gh}$ and with $x_1 = 1$.

Lemma 3.8. $\mathrm{Sum}_2(\chi^\omega) \cong \mathbb{C}^\omega[G]$, the twisted group algebra of G.

Proof. Let (A, M) be a cocone for χ^ω, i.e., a pair consisting of a $\mathbb{C}$-algebra A together with a left A-module M (representing a linear functor ${}_{\mathbb{C}}\mathrm{Mod} \to {}_A\mathrm{Mod}$) and homotopy commutative diagrams

$$\begin{tikzcd}
& A \arrow[d, phantom, "\Leftarrow \lambda_g"] & \\
\mathbb{C} \arrow[ur, "M"] \arrow[rr, "L_g"'] & {} & \mathbb{C} \arrow[ul, "M"']
\end{tikzcd}$$

where the λ_g's are isomorphisms of left A-modules

$$\lambda_g \colon M \otimes_{\mathbb{C}} L_g \xrightarrow{\sim} M$$

such that the diagrams

$$\begin{tikzcd}
M \otimes L_g \otimes L_h \arrow[r, "\lambda_g \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \rho_{g,h}"'] & M \otimes L_h \arrow[d, "\lambda_h"] \\
M \otimes L_{gh} \arrow[r, "\lambda_{gh}"'] & M
\end{tikzcd}$$

commute. This implies that the isomorphisms λ_g make M a right A_{χ^ω}-module. Therefore, we can see M as a linear functor

$${}_{A_{\chi^\omega}}\mathrm{Mod} \to {}_A\mathrm{Mod}$$

i.e., as a morphism from A_{χ^ω} to A in $2\text{-}\mathrm{Vect}_{\mathbb{C}}$. The algebra A_{χ^ω} seen as a left A_{χ^ω}-module is a morphism from $\mathbb{C}$ to A_{χ^ω} in $2\text{-}\mathrm{Vect}_{\mathbb{C}}$ and the natural isomorphism

$${}_A M_{\mathbb{C}} \cong {}_A M_{A_{\chi^\omega}} \otimes_{A_{\chi^\omega}} {A_{\chi^\omega}}_{\mathbb{C}}$$

gives a canonical factorization

$$\begin{tikzcd}
& A & \\
& A_{\chi^\omega} \arrow[u, "M"] \arrow[d, phantom, "\Rightarrow \lambda_g"] & \\
\mathbb{C} \arrow[ur, "A_{\chi^\omega}"] \arrow[uur, bend left, "M"] \arrow[rr, "L_g"'] & {} & \mathbb{C} \arrow[ul, "A_{\chi^\omega}"'] \arrow[uul, bend right, "M"']
\end{tikzcd}$$

exhibiting the algebra A_{χ^ω} together with itself seen as a left module over itself as the universal cocone. $\square$

3.3 $n = 3$

TODO:

- The only non-trivial part of the 3-functor $\chi^\omega : \boldsymbol{B}G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.
- **Issue**: The notion of a 3-colimit (to really compute Sum_3) is scary. At least we should make some hand-wavy arguments. . .

4 Boundary conditions

4.1 $n = 2$

TODO

4.2 $n = 3$

TODO

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