Boundaries in finite gauge theory

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Contents

1 Introduction

Let

- G be a finite group,
- $[\omega] \in H^n(G, \mathbb{C}^\times)$,
- and $\mathcal{C}$ a symmetric monoidal (∞, n)-category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fullly extended TQFT

$$\mathrm{Bord}_n \xrightarrow[\text{“classical”}]{I} \mathrm{Fam}_n(\mathcal{C}) \xrightarrow[\text{“quantum”}]{\mathrm{Sum}_n} \mathcal{C} \tag{1.1}$$

based on the data (G, ω), which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $\mathcal{C} = \boldsymbol{B}^n\mathbb{C}^\times$ (the n-fold delooping of the group $\mathbb{C}^\times$) and then

(i) identify *classical* boundary conditions as 1-morphisms with source $*$ in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$,

(ii) subsume the boundary conditions of [Wi, WWW] as special cases,

(iii) map classical boundary conditions to quantum ones via Sum_n, and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs–Schweigert–Valentino in the case $n = 3$,

(iv) do the above not only for framed, but also for oriented, spin,... TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1, 2\}$ and hopefully also $n = 3$. We also want to explain in detail how (G, ω) gives rise to an object in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)(*, \boldsymbol{B}G)$.
(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [Lu, Sect. 4.3] instead of Bord_n, but we leave that for another project.)

2 Special objects and 1-morphisms in $\mathrm{Fam}_n(\boldsymbol{B}^n\mathbb{C}^\times)$

TODO:

- spell out definition of $\mathrm{Fam}_n(\mathcal{C})$
- check that n-functor $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ is precisely an n-cocycle
  Issue: For $n > 3$, what is a weak n-functor, i. e. precisely what data and constraints are needed?
  Idea: Since both n-categories in $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section 3.
- clarify precisely how $\boldsymbol{B}G \to \boldsymbol{B}^n\mathbb{C}^\times$ induces an n-functor $\boldsymbol{B}G \to n\text{-Vect}$, at least for $n \in \{1, 2, 3\}$
- check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$
  Issue: induced from issue in second item

3 Bulk theory

3.1 $n = 1$

Since we take the action of G on $\mathbb{C}^\times$ to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{C}^\times) \tag{3.1}$$

is just a group homomorphism. We consider $\boldsymbol{B}^1\mathbb{C}^\times$ as a subcategory of $\mathrm{Vect}_\mathbb{C}$, and we will use the pair (G, ω) to construct a functor

$$\mathrm{Bord}_1 \xrightarrow{\quad I \quad} \mathrm{Fam}_1(\mathrm{Vect}_\mathbb{C}) \xrightarrow{\quad \mathrm{Sum}_1 \quad} \mathrm{Vect}_\mathbb{C}\ . \tag{3.2}$$

According to the cobordism hypothesis, the composite $\mathrm{Sum}_1 \circ I$ is determined by its value on the point pt_+, so we only need to specify $I(\mathrm{pt}_+)$ and then compute $\mathrm{Sum}_1(I(\mathrm{pt}_+))$. Note that $\boldsymbol{B}^1 G = {*}/\!\!/G$ is the classical action groupoid. We set $I(\mathrm{pt}_+)$ to be the functor

$$\chi^\omega \colon\ {*}/\!\!/G \longrightarrow \mathrm{Vect}_\mathbb{C}\,, \qquad * \longmapsto \mathbb{C}\,, \quad g \longmapsto \omega(g) \in \mathrm{Aut}(\mathbb{C}) \tag{3.3}$$

for all $g \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. $\mathrm{Sum}_1(\chi^\omega) = \mathbb{C}$ if $\omega(g) = 1$ for all $g \in G$, and $\mathrm{Sum}_1(\chi^\omega) = 0$ otherwise.

Proof. By definition $\mathrm{Sum}_1(\chi^\omega)$ is the universal cocone of χ^ω, viewed as a diagram of shape ${*}/\!\!/G$ in $\mathrm{Vect}_\mathbb{C}$. Hence $\mathrm{Sum}_1(\chi^\omega)$ is in particular a cocone of χ^ω, i.e. a vector space V together with a linear map $f \colon \chi^\omega(*) = \mathbb{C} \to V$ such that

$$\begin{tikzcd}
 & V & \\
\chi^\omega(*) = \mathbb{C} \arrow[ur, "f"] \arrow[rr, "\chi^\omega(g) = \omega(g)"'] & & \mathbb{C} = \chi^\omega(*) \arrow[ul, "f"']
\end{tikzcd} \tag{3.4}$$

commutes for all $g \in G$. If $\omega(g) \neq 1$ for some g, then this forces $f = 0$, and the colimit is trivial:

$$\begin{tikzcd}
 & 0 & \\
 & V \arrow[u, dashed, "\exists!"'] & \\
\mathbb{C} \arrow[ur, "f"] \arrow[uur, bend left, "0"] \arrow[rr, "\omega(g)"'] & & \mathbb{C} \arrow[ul, "f"'] \arrow[uul, bend right, "0"']
\end{tikzcd} \tag{3.5}$$

On the other hand, if ω is trivial, every pair (V, f) is a cocone, but the universal cocone is $(\mathbb{C}, 1)$. $\square$

3.2 $n = 2$

TODO:

- The only non-trivial part of the 2-functor $\chi^\omega \colon \boldsymbol{B}^2 G \to \mathrm{Alg}_{\mathbb{C}}$ is the coherence 2-morphism $\chi^\omega_{g,h} \colon \chi^\omega(g) \otimes \chi^\omega(h) \to \chi^\omega(gh)$, and the constraints (see e. g. [Le, Sect. 1.1] for the definitions) on the $\chi^\omega_{g,h}$ precisely say that they are the components of a 2-cocycle. So we can take $\chi^\omega_{g,h} = \omega(g,h)$.
- compute 2-colimit $\mathrm{Sum}_2(\chi)$ for χ as above, obtain twisted group algebra $\mathbb{C}^\omega[G]$

3.3 $n = 3$

TODO:

- The only non-trivial part of the 3-functor $\chi^\omega \colon \boldsymbol{B}^3 G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.
- <u>**Issue**</u>: The notion of a 3-colimit (to really compute Sum_3) is scary. At least we should make some hand-wavy arguments. . .

4 Boundary conditions

4.1 $n = 2$

TODO

4.2 $n = 3$

TODO

References

[FHLT] D. Freed, M. Hopkins, J. Lurie, and C. Teleman, *Topological Quantum Field Theories from Compact Lie Groups*, [arXiv:0905.0731].

[Le] T. Leinster, *Basic Bicategories*, [math/9810017].

[Lu] J. Lurie, *On the Classification of Topological Field Theories*, Current Developments in Mathematics **2008** (2009), 129–280, [arXiv:0905.0465].

[Sc] G. Schaumann, *Duals in tricategories and in the tricategory of bimodule categories*, PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (2013), urn:nbn:de:bvb:29-opus4-37321.

[Wi] E. Witten, *The "Parity" Anomaly On An Unorientable Manifold*, [arXiv:1605.02391].

[WWW] J. Wang, X.-G. Wen and E. Witten, *Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions*, [arXiv:1705.06728].

[BaeW] J. Baez and D. Wise, Quantum Gravity Seminar at University of California, Riverside, 2005, `http://math.ucr.edu/home/baez/qg-winter2005`.

[BalK] B. Balsam and A. Kirillov, Jr., *Turaev-Viro invariants as an extended TQFT*, [arXiv:1004.1533].

[BBCW] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, *Symmetry, Defects, and Gauging of Topological Phases*, [arXiv:1410.4540].

[BCP1] I. Brunner, N. Carqueville, and D. Plencner, *Orbifolds and topological defects*, Comm. Math. Phys. **315** (2012) 739–769, [arXiv:1307.3141].

[BCP2] I. Brunner, N. Carqueville, and D. Plencner, *Discrete torsion defects*, Comm. Math. Phys. **337** (2015), 429–453, [arXiv:1404.7497].

[BJQ] M. Barkeshli, C.-M. Jian, and X.-L. Qi, *Genons, twist defects, and projective non-Abelian braiding statistics*, Phys. Rev. B **87** (2013), 045130, [arXiv:1208.4834].

[BMS] J. Barrett, C. Meusburger, and G. Schaumann, *Gray categories with duals and their diagrams*, [arXiv:1211.0529].

[BP] C. Bachas and M. Petropoulos, *Topological Models on the Lattice and a Remark on String Theory Cloning*, Commun. Math. Phys. **152** (1993),191–202, [hep-th/9205031].

[BW1] J. Barrett and B. Westbury, *Spherical Categories*, Adv. Math. **143** (1999), 357–375, [hep-th/9310164].

[BW2] J. Barrett and B. Westbury, *Invariants of piecewise-linear 3-manifolds*, Trans. Amer. Math. Soc. **348** (1996), 3997–4022, [hep-th/9311155].

[Ca] N. Carqueville, *Lecture notes on 2-dimensional defect TQFT*, [arXiv:1607.05747].

[CGPW] S. X. Cui, C. Galindo, J. Yael Plavnik, and Z. Wang, *On Gauging Symmetry of Modular Categories*, Communications in Mathematical Physics **348**:3 (2016), 1043–1064, [arXiv:1510.03475].

[CMS] N. Carqueville, C. Meusburger, and G. Schaumann, *3-dimensional defect TQFTs and their tricategories*, [arXiv:1603.01171].

[CQV] N. Carqueville and A. Quintero Vélez, *Calabi-Yau completion and orbifold equivalence*, [arXiv:1509.00880].

[CRCR] N. Carqueville, A. Ros Camacho, and I. Runkel, *Orbifold equivalent potentials*, Journal of Pure and Applied Algebra **220** (2016), 759–781, [arXiv:1311.3354].

[CR1] N. Carqueville and I. Runkel, *Orbifold completion of defect bicategories*, Quantum Topology **7**:2 (2016) 203–279, [arXiv:1210.6363].

[CR2] N. Carqueville and I. Runkel, *Introductory lectures on topological quantum field theory*, [arXiv:1705.05734].

[CRS] N. Carqueville, I. Runkel, and G. Schaumann, in preparation.

[DKR] A. Davydov, L. Kong, and I. Runkel, *Field theories with defects and the centre functor*, Mathematical Foundations of Quantum Field Theory and Perturbative String Theory, Proceedings of Symposia in Pure Mathematics, AMS, 2011, [arXiv:1107.0495].

[ENO] P. Etingof, D. Nikshych, and V. Ostrik, *Fusion categories and homotopy theory*, Quantum Topology **1** (2010) 209–273, [arXiv:0909.3140].

[FFRS] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Defect lines, dualities, and generalised orbifolds*, Proceedings of the XVI International Congress on Mathematical Physics, Prague, August 3–8, 2009, [arXiv:0909.5013].

[FHK] M. Fukuma, S. Hosono, and H. Kawai, *Lattice Topological Field Theory in Two Dimensions*, Comm. Math. Phys. **161** (1994), 157–176, [hep-th/9212154].

[FPSV] J. Fuchs, J. Priel, C. Schweigert, and A. Valentino, *On the Brauer groups of symmetries of abelian Dijkgraaf-Witten theories*, Communications in Mathematical Physics **339**:2 (2015), 385–405, [arXiv:1404.6646].

[FRS] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators. 3. Simple currents*, Nucl. Phys. B **694** (2004) 277–353, [hep-th/0403157].

[FS1] J. Fuchs and C. Schweigert, *Category theory for conformal boundary conditions*, Fields Institute Communications **39** (2003), 25–71, [math/0106050].

[FS2] J. Fuchs and C. Schweigert, *A note on permutation twist defects in topological bilayer phases*, Letters in Mathematical Physics **104**:11 (2014), 1385–1405, [arXiv:1310.1329].

[FSV] J. Fuchs, C. Schweigert, and A. Valentino, *Bicategories for boundary conditions and for surface defects in 3-d TFT*, Communications in Mathematical Physics **321**:2 (2013), 543–575, [arXiv:1203.4568].

[Gu] N. Gurski, *Coherence in Three-Dimensional Category Theory*, *Cambridge Tracts in Mathematics* **201**, Cambridge University Press, 2013.

[Hi] M. W. Hirsch, *Differential topology*, Springer Graduate Texts in Mathematics **33**, Springer, 1976.

[KK] A. Kitaev and L. Kong, *Models for gapped boundaries and domain walls*, Commun. Math. Phys. **313** (2012) 351–373, [arXiv:1104.5047].

[KR] A. Kapustin and L. Rozansky, *Three-dimensional topological field theory and symplectic algebraic geometry II*, Communications of Number Theory and Physics **4** (2010), 463–549, [arXiv:0909.3643].

[KRS] A. Kapustin, L. Rozansky, and N. Saulina, *Three-dimensional topological field theory and symplectic algebraic geometry I*, Nuclear Physics B **816** (2009), 295–355, [arXiv:0810.5415].

[KS] A. Kapustin and N. Saulina, *Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory*, Mathematical Foundations of Quantum Field Theory and Perturbative String Theory, Proceedings of Symposia in Pure Mathematics **83**, 175–198, American Mathematical Society, 2011, [arXiv:1012.0911].

[La] R. J. Lawrence, *An Introduction to Topological Field Theory*, Proc. Symp. Appl. Math., **51** (1996), 89–128.

[MW] S. Morrison and K. Walker, *Blob homology*, Geometry & Topology **16** (2012), 1481–1607, [arXiv:1009.5025].

[Mu] J. R. Munkres, *Elementary Differential Topology*, Annals of Mathematics Studies **54**, Princeton University Press, 1967.

[NS] S.-H. Ng and P. Schauenburg, *Higher Frobenius-Schur indicators for pivotal categories*, Contemporary Mathematics **441** (2007), 63–90, [math.QA/0503167].

[Pa] U. Pachner, *P.L. Homeomorphic Manifolds are Equivalent by Elementary Shellings*, European Journal of Combinatorics **12**:2 (1991), 129–145.

[Qu] F. Quinn, *Lectures on axiomatic topological quantum field theory*, IAS/Park City Mathematics Series **1** (1995), 325–433.

[Sc] G. Schaumann, *Duals in tricategories and in the tricategory of bimodule categories*, PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (2013), urn:nbn:de:bvb:29-opus4-37321.

[SW] C. Schweigert and L. Woike, *Orbifold Construction for Topological Field Theories*, [arXiv:1705.05171].

[TV] V. Turaev and O. Viro, *State sum invariants of 3-manifolds and quantum $6j$-symbols*, Topology **31**:4 (1992), 865–902.