

Sum₁

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This file contains the details on the $n = 1$ case, to be merged into the main file as soon as they become polished enough.

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Let us denote by $\text{Fam}_0(\mathbb{C})$ the set of equivalence classes of finite groupoids equipped with functors to \mathbb{C} (seen as a trivial category). In other words, a representative of an element of $\text{Fam}_0(\mathbb{C})$ is a pair (X, f) consisting of a finite groupoid X together with a map $f: \pi_0(X) \rightarrow \mathbb{C}$. The map

$$\text{Sum}_0: \text{Fam}_0 \rightarrow \mathbb{C}$$

is defined as

$$\text{Sum}_0(X, f) = \sum_{[x] \in \pi_0(X)} \frac{f(x)}{|\text{Aut}(x)|}.$$

We will also use the more evocative notation

$$\int_X f(x) d\mu(x)$$

to denote $\text{Sum}_0(X, f)$. Notice that Sum_0 is linear in the f variable.

Remark 1.1. A function $f: \pi_0(X) \rightarrow \mathbb{C}$ can equivalently be seen as an element in $\text{End}_{[X, \text{Vect}]}(\mathbf{1}_X)$, where $\mathbf{1}_X: X \rightarrow \text{Vect}$ is the constant functor taking the value \mathbb{C} on X . The map Sum_0 can therefore be seen as the datum of a map

$$\text{Sum}_0: \text{End}_{[X, \text{Vect}]}(\mathbf{1}_X) \rightarrow \mathbb{C},$$

for any finite groupoid X .

Definition 1.2. The category $\text{Fam}_1(\text{Vect})$ has as its objects finite groupoids equipped with functors to Vect , i.e., pairs (X, ρ) consisting of a finite groupoid X together with a functor $\rho: X \rightarrow \text{Vect}$. Morphisms in $\text{Fam}_1(\text{Vect})$ are spans of groupoids over Vect , i.e., commutative diagrams

$$\begin{array}{ccc} & Z & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & \xRightarrow[\alpha]{} & Y \\ \rho_X \searrow & & \swarrow \rho_Y \\ & \text{Vect} & \end{array}$$

where the natural transformation $\alpha: \rho_X \circ \pi_X \rightarrow \rho_Y \circ \pi_Y$ is part of the data of the commutative diagram.

Remark 1.3. The identity morphisms and the composition of morphisms in $\text{Fam}_1(\text{Vect})$ will be defined later.

For every object (X, ρ) in $\text{Fam}_1(\text{Vect})$ we can consider the vector space

$$[\mathbf{1}_X, \rho] = \text{Hom}_{[X, \text{Vect}]}(\mathbf{1}_X, \rho).$$

If $f: Y \rightarrow X$ is a functor between finite groupoids, and $\rho: X \rightarrow \text{Vect}$ is a functor, we write $f^*\rho: Y \rightarrow \text{Vect}$ for the functor $\rho \circ f$. If $\alpha: \mathbf{1}_X \rightarrow \rho$ is a natural transformation, then $f^*\alpha$ is a natural transformation between $f^*\mathbf{1}_X = \mathbf{1}_Y$ and $f^*\rho$. This defines a linear map

$$f^*: [\mathbf{1}_X, \rho] \rightarrow [\mathbf{1}_Y, f^*\rho].$$

Similarly, we can consider the vector space

$$[\rho, \mathbf{1}_X] = \text{Hom}_{[X, \text{Vect}]}(\rho, \mathbf{1}_X)$$

together with the linear map (also denoted f^* by slight abuse of notation)

$$f^*: [\rho, \mathbf{1}_X] \rightarrow [f^*\rho, \mathbf{1}_Y].$$

Passing to the linear duals we get the linear map

$$(f^*)^\vee: [f^*\rho, \mathbf{1}_Y]^\vee \rightarrow [\rho, \mathbf{1}_X]^\vee.$$

Next, notice that composition of morphisms gives a bilinear map

$$[\mathbf{1}_X, \rho] \otimes [\rho, \mathbf{1}_X] \rightarrow [\mathbf{1}_X, \mathbf{1}_X]$$

and so composing this with the linear map $\text{Sum}_0(X, -): [\mathbf{1}_X, \mathbf{1}_X] \rightarrow \mathbb{C}$ we get a bilinear pairing between $[\mathbf{1}_X, \rho]$ and $[\rho, \mathbf{1}_X]$. Equivalently, this is a linear map

$$\delta_{(X, \rho)}: [\mathbf{1}_X, \rho] \rightarrow [\rho, \mathbf{1}_X]^\vee.$$

Our main *duality assumption* will be that $\delta_{(X,\rho)}$ is a linear isomorphism for any (X,ρ) . Under this assumption we may define a linear morphism

$$f_* : [\mathbf{1}_Y, f^* \rho] \rightarrow [\mathbf{1}_X, \rho]$$

as the composition

$$[\mathbf{1}_Y, f^* \rho] \xrightarrow{\delta_{(Y, f^* \rho)}} [f^* \rho, \mathbf{1}_Y]^\vee \xrightarrow{(f^*)^\vee} [\rho, \mathbf{1}_X]^\vee \xrightarrow{\delta_{(X, \rho)}^{-1}} [\mathbf{1}_X, \rho].$$

In other words, if $v \in [\mathbf{1}_Y, f^* \rho]$, the element $f_*(v) \in [\mathbf{1}_X, \rho]$ is defined by the equation

$$\int_X (w \circ f_*(v)) d\mu(x) = \int_Y ((f^* w) \circ v) d\mu(y)$$

for any $w \in [\rho, \mathbf{1}_X]$. Each such w can be uniquely written as a sum of elements w_x vanishing outside the isomorphism class of $x \in X$, for all such x . For such a w_x the above equation becomes

$$\frac{w_x \circ f_*(v)}{|\text{Aut}(x)|} = \sum_{f([y])=[x]} \frac{(f^* w_x) \circ v}{|\text{Aut}(y)|}.$$

Similarly, v can be uniquely written as a sum of elements v_y with nontrivial image only in the isomorphism class of $y \in Y$, i. e. $v_{y'}(1) = 0 \in \rho(f(y'))$ unless $y' \in [y]$. Choosing such a v_y with $f(y) = x$ we get

$$\frac{w_x \circ f_*(v_y)}{|\text{Aut}(x)|} = \frac{w_x \circ v_y}{|\text{Aut}(y)|},$$

where we used the fact that, as $v \in [\mathbf{1}_Y, f^* \rho]$ and $f(y) = x$, the source of w_x coincides with the target of v_y and $(f^* w_x) \circ v_y = w_x \circ v_y$. As this has to be true for any w_x , we finally find

$$f_*(v_y) = \frac{|\text{Aut}(x)|}{|\text{Aut}(y)|} v_y,$$

for any v_y concentrated on y , with $f(y) = x$. From this we get the general formula

$$(f_*(v))_x = |\text{Aut}(x)| \sum_{f([y])=[x]} \frac{v_y}{|\text{Aut}(y)|}.$$

We can now define our wannabe functor

$$\text{Sum}_1 : \text{Fam}_1(\text{Vect}) \rightarrow \text{Vect}$$

on objects as

$$\text{Sum}_1(X, \rho) = [\mathbf{1}_X, \rho]$$

and on a morphism

$$\begin{array}{ccc} & Z & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & \xRightarrow[\alpha]{} & Y \\ \rho_X \searrow & & \swarrow \rho_Y \\ & \text{Vect} & \end{array}$$

as the composition

$$[\mathbf{1}_X, \rho_X] \xrightarrow{\pi_X^*} [\mathbf{1}_Z, \pi_X^* \rho_X] \xrightarrow{\alpha \circ -} [\mathbf{1}_Z, \pi_Y^* \rho_Y] \xrightarrow{\pi_Y^*} [\mathbf{1}_Y, \rho_Y].$$

Remark 1.4. Before giving a more explicit expression of the above morphism, let us notice that $[\mathbf{1}_X, \rho]$ is a model for the limit of the functor $\rho: X \rightarrow \text{Vect}$, in accordance with the prescription in [?]. Also, notice that the morphism associated to the span $X \leftarrow Z \rightarrow Y$ together with the natural transformation α has the form “pull-compose-push”, and so it is an instance of a Fourier-Mukai-type transform.

Let us now give an explicit description of the functor Sum_1 . To begin with, notice that the datum of a functor $\rho: X \rightarrow \text{Vect}$ is equivalent to the datum of a linear representation V_x of $\text{Aut}_X(x)$ for every x in a set of representatives for the isomorphism classes of objects in X . So

$$[\mathbf{1}_X, \rho] = \bigoplus_{[x] \in \pi_0(X)} (V_x)^{\text{Aut}_X(x)},$$

where V^G denotes the subspace of G -invariants for a linear G -representation on a vector space V . A linear map $[\mathbf{1}_X, \rho_X] \rightarrow [\mathbf{1}_Y, \rho_Y]$ can therefore be described by its “matrix coefficients” $(V_x)^{\text{Aut}_X(x)} \rightarrow (W_y)^{\text{Aut}_Y(y)}$ with $[x]$ ranging in $\pi_0(X)$ and $[y]$ ranging in $\pi_0(Y)$. The map $\pi_X^*: [\mathbf{1}_X, \rho_X] \rightarrow [\mathbf{1}_Z, \pi_X^* \rho_X]$ maps the invariant vector v_x in V_x to the tuple (v_x, v_x, \dots, v_x) in $\bigoplus_{\pi_X([z]=[x]} V_x$. In this vector space, look at the direct sum over those $[z]$ with $\pi_Y([z]) = [y]$. These are precisely those summands for which α_z is a linear morphism $V_x \rightarrow W_y$. The corresponding tuple $(\alpha_{z_1}(v_x), \alpha_{z_2}(v_x), \dots, \alpha_{z_k}(v_x))$ is then the component of $(\alpha \circ \pi_X^*)(v_x)$ in $[\mathbf{1}_Z, \pi_Y^* \rho_Y]$ going into W_y via π_{Y*} . As shown above, the morphism π_{Y*} acts on this component by mapping it to

$$|\text{Aut}(y)| \sum_{\substack{\pi_X([z])=[x] \\ \pi_Y([z])=[y]}} \frac{\alpha_z(v_x)}{|\text{Aut}(z)|}.$$

In other words, the matrix coefficient

$$(\Phi_\alpha)_x^y: (V_x)^{\text{Aut}_X(x)} \rightarrow (W_y)^{\text{Aut}_Y(y)}$$

is

$$v_x \mapsto |\text{Aut}(y)| \sum_{\substack{\pi_X([z])=[x] \\ \pi_Y([z])=[y]}} \frac{\alpha_z(v_x)}{|\text{Aut}(z)|}.$$