Sum_1

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This file contains the details on the $n = 1$ case, to be merged into the main file as soon as they become polished enough.

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Let us denote by $\mathrm{Fam}_0(\mathbb{C})$ the set of equivalence classes of finite groupoids equipped with functors to $\mathbb{C}$ (seen as a trivial category). In other words, a representative of an element of $\mathrm{Fam}_0(\mathbb{C})$ is a pair (X, f) consisting of a finite groupoid X together with a map $f\colon \pi_0(X) \to \mathbb{C}$. The map

$$\mathrm{Sum}_0\colon \mathrm{Fam}_0 \to \mathbb{C}$$

is defined as

$$\mathrm{Sum}_0(X, f) = \sum_{[x]\in\pi_0(X)} \frac{f(x)}{|\mathrm{Aut}(x)|}.$$

We will also use the more evocative notation

$$\int_X f(x) d\mu(x)$$

to denote $\mathrm{Sum}_0(X, f)$. Notice that Sum_0 is linear in the f variable.

Remark 1.1. A function $f\colon \pi_0(X) \to \mathbb{C}$ can equivalently be seen as an element in $\mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X)$, where $\mathbf{1}_X\colon X \to \mathrm{Vect}$ is the constant functor taking the value $\mathbb{C}$ on X. The map Sum_0 can therefore be seen as the datum of a map

$$\mathrm{Sum}_0\colon \mathrm{End}_{[X,\mathrm{Vect}]}(\mathbf{1}_X) \to \mathbb{C},$$

for any finite groupoid X.

Definition 1.2. The category $\mathrm{Fam}_1(\mathrm{Vect})$ has as its objects finite groupoids equipped with functors to Vect, i.e., pairs (X, ρ) consisting of a finite groupoid X together with a functor $\rho\colon X \to \mathrm{Vect}$. Morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ are spans of groupoids over Vect, i.e., commutative diagrams

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "\rho_X"'] \arrow[rr, phantom, "\underset{\alpha}{\Longrightarrow}"] & & Y \arrow[dl, "\rho_Y"] \\ & \mathrm{Vect} & \end{tikzcd}$$

where the natural transformation $\alpha\colon \rho_X \circ \pi_X \to \rho_Y \circ \pi_Y$ is part of the data of the commutative diagram.

Remark 1.3. The identity mophisms and the composition of morphisms in $\mathrm{Fam}_1(\mathrm{Vect})$ will be defined later.

For every object (X, ρ) in $\mathrm{Fam}_1(\mathrm{Vect})$ we can consider the vector space

$$[\mathbf{1}_X, \rho] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\mathbf{1}_X, \rho).$$

If $f\colon Y \to X$ is a functor between finite groupoids, and $\rho\colon X \to \mathrm{Vect}$ is a functor, we write $f^*\rho\colon Y \to \mathrm{Vect}$ for the functor $\rho \circ f$. If $\alpha\colon \mathbf{1}_X \to \rho$ is a natural transformation, then $f^*\alpha$ is a natural transformation between $f^*\mathbf{1}_X = \mathbf{1}_Y$ and $f^*\rho$. This defines a linear map

$$f^*\colon [\mathbf{1}_X, \rho] \to [\mathbf{1}_Y, f^*\rho].$$

Similarly, we can consider the vector space

$$[\rho, \mathbf{1}_X] = \mathrm{Hom}_{[X,\mathrm{Vect}]}(\rho, \mathbf{1}_X)$$

together with the linear map (also denoted f^* by slight abuse of notation)

$$f^*\colon [\rho, \mathbf{1}_X] \to [f^*\rho, \mathbf{1}_Y].$$

Passing to the linear duals we get the linear map

$$(f^*)^\vee\colon [f^*\rho, \mathbf{1}_Y]^\vee \to [\rho, \mathbf{1}_X]^\vee.$$

Next, notice that composition of morphisms gives a bilinear map

$$[\rho, \mathbf{1}_X] \otimes [\mathbf{1}_X, \rho] \to [\mathbf{1}_X, \mathbf{1}_X]$$

and so composing this with the linear map $\mathrm{Sum}_0(X, -)\colon [\mathbf{1}_X, \mathbf{1}_X] \to \mathbb{C}$ we get a bilinear pairing $\langle\,|\,\rangle_X$ between $[\rho, \mathbf{1}_X]$ and $[\mathbf{1}_X, \rho]$, for any ρ. Notice that, by its very definition, the pairing $\langle\,|\,\rangle_X$ satisfies

$$\langle v | \alpha \circ w \rangle_X = \langle v \circ \alpha | w \rangle_X,$$

for any $v \in [\rho_2, \mathbf{1}_X]$, any $w \in [\mathbf{1}_X, \rho_1]$ and any natural transformation $\alpha\colon \rho_1 \to \rho_2$. Equivalently, the bilinear pairing $\langle\,|\,\rangle_X$ is a linear map

$$\delta_{(X,\rho)}\colon [\mathbf{1}_X, \rho] \to [\rho, \mathbf{1}_X]^\vee.$$

Our main *duality assumption* will be that $\delta_{(X,\rho)}$ is a linear isomorphism for any (X, ρ). Under this assumption we may define a linear morphism

$$f_*\colon [\mathbf{1}_Y, f^*\rho] \to [\mathbf{1}_X, \rho]$$

as the composition

$$[\mathbf{1}_Y, f^*\rho] \xrightarrow{\delta_{(Y,f^*\rho)}} [f^*\rho, \mathbf{1}_Y]^\vee \xrightarrow{(f^*)^\vee} [\rho, \mathbf{1}_X]^\vee \xrightarrow{\delta^{-1}_{(X,\rho)}} [\mathbf{1}_X, \rho].$$

In other words, if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_*v \in [\mathbf{1}_X, \rho]$ is defined by the equation

$$\langle w|f_*v\rangle_X = \langle f^*w|v\rangle_Y$$

for any $w \in [\rho, \mathbf{1}_X]$.

We can now define our wannabe functor

$$\mathrm{Sum}_1\colon \mathrm{Fam}_1(\mathrm{Vect}) \to \mathrm{Vect}$$

on objects as

$$\mathrm{Sum}_1(X, \rho) = [\mathbf{1}_X, \rho]$$

and on a morphism

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[dr, "\rho_X"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\alpha}"] & & Y \arrow[dl, "\rho_Y"] \\
 & \mathrm{Vect} &
\end{tikzcd}$$

as the composition

$$[\mathbf{1}_X, \rho_X] \xrightarrow{\pi_X^*} [\mathbf{1}_Z, \pi_X^*\rho_X] \xrightarrow{\alpha\circ -} [\mathbf{1}_Z, \pi_Y^*\rho_Y] \xrightarrow{\pi_{Y*}} [\mathbf{1}_Y, \rho_Y].$$

Remark 1.4. Before giving a more explicit expression of the above morphism, let us notice that $[\mathbf{1}_X, \rho]$ is a model for the limit of the functor $\rho\colon X \to \mathrm{Vect}$, in accordance with the prescrition in [**?**]. Also, notice that the morphism associated to the span $X \leftarrow Z \to Y$ together with the natural transformation α has the form "pull-compose-push", and so it is an instance of a Fourier-Mukai-type transform.

Lemma 1.5. (The projection formula) Let $f\colon Y \to X$ be a functor between finite groupoids, let $\rho_1, \rho_2\colon X \to \mathrm{Vect}$ be functors and let $\alpha\colon \rho_1 \to \rho_2$ be a natural transformation. We have an identity of morphisms $[\mathbf{1}_Y, f^*\rho_1] \to [\mathbf{1}_X, \rho_2]$

$$f_*(f^*\alpha \circ -) = \alpha \circ f_*$$

Proof. The identity

$$f_*(f^*\alpha \circ v) = \alpha \circ f_* v$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ is equivalent to the identity

$$\langle w | f_*(f^*\alpha \circ v)\rangle_X = \langle w | \alpha \circ f_* v\rangle_X$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ and every $w \in [\rho_2, \mathbf{1}_X]$. By the properties of the pairings $\langle\,|\,\rangle_X$ and $\langle\,|\,\rangle_X$, and by definition of f_*, we have

$$\begin{aligned}
\langle w | f_*(f^*\alpha \circ v)\rangle_X &= \langle f^*w | f^*\alpha \circ v\rangle_Y \\
&= \langle f^*w \circ f^*\alpha | v\rangle_Y \\
&= \langle f^*(w \circ \alpha) | v\rangle_Y \\
&= \langle w \circ \alpha | f_* v\rangle_X \\
&= \langle w | \alpha \circ f_* v\rangle_X
\end{aligned}$$

□

For possible later use, let us derive an explicit formla for f_*v. Recall that if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_*v \in [\mathbf{1}_X, \rho]$ is defined by the equation $\langle w | f_* v\rangle_X = \langle f^* w | v\rangle_Y$ for any $w \in [\rho, \mathbf{1}_X]$. Each such w can be uniquely written as a sum of elements w_x vanishing outside the isomorphism class of $x \in X$, for all such x. For such a w_x the above equation becomes

$$\frac{w_x \circ f_*(v)}{|\mathrm{Aut}(x)|} = \sum_{f([y])=[x]} \frac{(f^*w_x) \circ v}{|\mathrm{Aut}(y)|}.$$

Similarly, v can be uniquely written as a sum of elements v_y with nontrivial image only in the isomorphism class of $y \in Y$, i. e. $v_{y'}(1) = 0 \in \rho(f(y'))$ unless $y' \in [y]$. Choosing such a v_y with $f(y) = x$ we get

$$\frac{w_x \circ f_*(v_y)}{|\mathrm{Aut}(x)|} = \frac{w_x \circ v_y}{|\mathrm{Aut}(y)|},$$

where we used the fact that, as $v \in [\mathbf{1}_Y, f^*\rho]$ and $f(y) = x$, the source of w_x coincides with the target of v_y and $(f^*w_x) \circ v_y = w_x \circ v_y$. As this has to be true for any w_x, we finally find

$$f_*(v_y) = \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(y)|} v_y,$$

for any v_y concentrated on y, with $f(y) = x$. From this we get the general formula

$$(f_*(v))_x = |\mathrm{Aut}(x)| \sum_{f([y])=[x]} \frac{v_y}{|\mathrm{Aut}(y)|}.$$

Let us now give an explicit description of the functor Sum_1. To begin with, notice that the datum of a functor $\rho\colon X \to \mathrm{Vect}$ is equivalent to the datum of a linear representation V_x of $\mathrm{Aut}_X(x)$ for every x in a set of representatives for the isomophism classes of objects in X. So

$$[\mathbf{1}_X, \rho] = \bigoplus_{[x]\in\pi_0(X)} (V_x)^{\mathrm{Aut}_X(x)},$$

where V^G denotes the subspace of G-invariants for a linear G-representation on a vector space V. A linear map $[\mathbf{1}_X, \rho_X] \to [\mathbf{1}_Y, \rho_Y]$ can therefore be described by its "matrix coefficients" $(V_x)^{\mathrm{Aut}_X(x)} \to (W_y)^{\mathrm{Aut}_Y(y)}$ with $[x]$ ranging in $\pi_0(X)$ and $[y]$ ranging in $\pi_0(Y)$. The map $\pi_X^*\colon [\mathbf{1}_X, \rho_X] \to [\mathbf{1}_Z, \pi_X^*\rho_X]$ maps the invariant vector v_x in V_x to the tuple $(v_x, v_x, \ldots, v_x)$ in $\bigoplus_{\pi_X([z])=[x]} V_x$. In this vector space, look at the direct sum over those $[z]$ with $\pi_Y([z]) = [y]$. These are precisely those summands for which α_z is a linear morphism $V_x \to W_y$. The corresponding tuple $(\alpha_{z_1}(v_x), \alpha_{z_2}(v_x), \ldots, \alpha_{z_k}(v_x))$ is then the component of $(\alpha \circ \pi_X^*)(v_x)$ in $[\mathbf{1}_Z, \pi_Y^*\rho_Y]$ going into W_y via π_{Y*}. As shown above, the mophism π_{Y*} acts on this component by mapping it to

$$|\mathrm{Aut}(y)| \sum_{\substack{\pi_X([z])=[x]\\ \pi_Y([z])=[y]}} \frac{\alpha_z(v_x)}{|\mathrm{Aut}(z)|}.$$

In other words, the matrix coefficient

$$(\Phi_\alpha)_x^y\colon (V_x)^{\mathrm{Aut}_X(x)} \to (W_y)^{\mathrm{Aut}_Y(y)}$$

is

$$v_x \mapsto |\mathrm{Aut}(y)| \sum_{\substack{\pi_X([z])=[x]\\ \pi_Y([z])=[y]}} \frac{\alpha_z(v_x)}{|\mathrm{Aut}(z)|}.$$

The prefactor $|\mathrm{Aut}(y)|$ may appear odd, but it becomes transparent if we lower the y-index by using the pairing $\langle\,|\,\rangle_Y$. Namely, what we get this way is

$$(\Phi_\alpha)_{xy} = \int_{Z_{x,y}} \alpha_z d\mu(z),$$

where $Z_{x,y}$ is the groupoid given by the (homotopy) fiber of Z over the point (x, y) of $X \times Y$. This exhibits the "propagator" $(\Phi_\alpha)_{xy}$ as the "sum of α over the histories z going from x to y", and so as a discrete version of the Feynman path integral expression for propagators in quantum field theory.