

Sum₁

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This file contains the details on the $n = 1$ case, to be merged into the main file as soon as they become polished enough.

Let us denote by $\text{Fam}_0(\mathbb{C})$ the set of equivalence classes of finite groupoids equipped with functors to \mathbb{C} (seen as a trivial category). In other words, a representative of an element of $\text{Fam}_0(\mathbb{C})$ is a pair (X, ρ) consisting of a finite groupoid X together with a map $\rho: \pi_0(X) \rightarrow \mathbb{C}$. The map

$$\text{Sum}_0: \text{Fam}_0 \rightarrow \mathbb{C}$$

is defined as

$$\text{Sum}_0(X, \rho) = \sum_{[x] \in \pi_0(X)} \frac{\rho(x)}{|\text{Aut}(x)|}.$$

We will also use the more evocative notation

$$\int_X \rho(x) d\mu(x)$$

to denote $\text{Sum}_0(X, \rho)$. Notice that Sum_0 is linear in the ρ variable.

A nice property of the morphism Sum_0 (which we are going to use later) is the following: let X be a finite set with an action of a finite group G , and let $X//G$ denotes the action groupoid for this action; finally, let $\rho: X//G \rightarrow \mathbb{C}$ any functor, i.e., let $\rho: X \rightarrow \mathbb{C}$ be a function that is constant on the G -orbits in X . Then we have

$$\text{Sum}_0(X//G, \rho) = \frac{1}{|G|} \sum_{x \in X} \rho(x). \quad (0.1)$$

Namely, the set $\pi_0(X//G)$ is the quotient set X/G and for any $x \in X$ seen as an object in $X//G$ we have $\text{Aut}(x) = \text{Stab}_G(x)$. Therefore

$$\text{Sum}_0(X//G, \rho) = \sum_{[x] \in X/G} \frac{\rho(x)}{|\text{Stab}_G(x)|}$$

$$\begin{aligned}
&= \frac{1}{|G|} \sum_{[x] \in X/G} \frac{|G|}{|\text{Stab}_G(x)|} \rho(x) \\
&= \frac{1}{|G|} \sum_{[x] \in X/G} |[x]| \rho(x) \\
&= \frac{1}{|G|} \sum_{[x] \in X/G} \sum_{x \in [x]} \rho(x) \\
&= \frac{1}{|G|} \sum_{x \in X} \rho(x)
\end{aligned}$$

Remark 0.1. A function $f: \pi_0(X) \rightarrow \mathbb{C}$ can equivalently be seen as an element in $\text{End}_{[X, \text{Vect}]}(\mathbf{1}_X)$, where $\mathbf{1}_X: X \rightarrow \text{Vect}$ is the constant functor taking the value \mathbb{C} on X . The map Sum_0 can therefore be seen as the datum of a map

$$\text{Sum}_0: \text{End}_{[X, \text{Vect}]}(\mathbf{1}_X) \rightarrow \mathbb{C},$$

for any finite groupoid X .

Definition 0.2. The category $\text{Fam}_1(\text{Vect})$ has as its objects finite groupoids equipped with functors to Vect , i.e., pairs (X, ρ) consisting of a finite groupoid X together with a functor $\rho: X \rightarrow \text{Vect}$. Morphisms in $\text{Fam}_1(\text{Vect})$ are spans of groupoids over Vect , i.e., commutative diagrams

$$\begin{array}{ccc}
& Z & \\
\pi_X \swarrow & & \searrow \pi_Y \\
X & \xRightarrow{\alpha} & Y \\
\rho_X \searrow & & \swarrow \rho_Y \\
& \text{Vect} &
\end{array}$$

where the natural transformation $\alpha: \rho_X \circ \pi_X \rightarrow \rho_Y \circ \pi_Y$ is part of the data of the commutative diagram.

Remark 0.3. The identity morphisms and the composition of morphisms in $\text{Fam}_1(\text{Vect})$ will be defined later.

For every object (X, ρ) in $\text{Fam}_1(\text{Vect})$ we can consider the vector space

$$[\mathbf{1}_X, \rho] = \text{Hom}_{[X, \text{Vect}]}(\mathbf{1}_X, \rho).$$

If $f: Y \rightarrow X$ is a functor between finite groupoids, and $\rho: X \rightarrow \text{Vect}$ is a functor, we write $f^*\rho: Y \rightarrow \text{Vect}$ for the functor $\rho \circ f$. If $\alpha: \mathbf{1}_X \rightarrow$

ρ is a natural transformation, then $f^*\alpha$ is a natural transformation between $f^*\mathbf{1}_X = \mathbf{1}_Y$ and $f^*\rho$. This defines a linear map

$$f^*: [\mathbf{1}_X, \rho] \rightarrow [\mathbf{1}_Y, f^*\rho].$$

Similarly, we can consider the vector space

$$[\rho, \mathbf{1}_X] = \text{Hom}_{[X, \text{Vect}]}(\rho, \mathbf{1}_X)$$

together with the linear map

$$f^*: [\rho, \mathbf{1}_X] \rightarrow [f^*\rho, \mathbf{1}_Y].$$

Passing to the linear duals we get the linear map

$$(f^*)^\vee: [f^*\rho, \mathbf{1}_Y]^\vee \rightarrow [\rho, \mathbf{1}_X]^\vee.$$

Next, notice that composition of morphisms gives a bilinear map

$$[\rho, \mathbf{1}_X] \otimes [\mathbf{1}_X, \rho] \rightarrow [\mathbf{1}_X, \mathbf{1}_X]$$

and so composing this with the linear map $\text{Sum}_0(X, -): [\mathbf{1}_X, \mathbf{1}_X] \rightarrow \mathbb{C}$ we get a bilinear pairing $\langle | \rangle_X$ between $[\rho, \mathbf{1}_X]$ and $[\mathbf{1}_X, \rho]$, for any ρ . Notice that, by its very definition, the pairing $\langle | \rangle_X$ satisfies

$$\langle v | \alpha \circ w \rangle_X = \langle v \circ \alpha | w \rangle_X,$$

for any $v \in [\rho_2, \mathbf{1}_X]$, any $w \in [\mathbf{1}_X, \rho_1]$ and any natural transformation $\alpha: \rho_1 \rightarrow \rho_2$. Equivalently, the bilinear pairing $\langle | \rangle_X$ is a linear map

$$\delta_{(X, \rho)}: [\mathbf{1}_X, \rho] \rightarrow [\rho, \mathbf{1}_X]^\vee.$$

Our main *duality assumption* will be that $\delta_{(X, \rho)}$ is a linear isomorphism for any (X, ρ) .

Remark 0.4. The duality assumption holds for every finite groupoid X if and only if it holds for finite connected groupoids. Up to equivalence we are therefore reduced to considering the case $X = \mathbf{B}G$, with G a finite group. In this case $\rho: \mathbf{B}G \rightarrow \text{Vect}$ is the datum of a linear representation V of G , and the vector spaces $[\mathbf{1}_{\mathbf{B}G}, \rho]$ and $[\rho, \mathbf{1}_{\mathbf{B}G}]$ are isomorphic to the space V^G of G -invariants and the linear dual $(V_G)^\vee$ of the space of G -coinvariants, respectively. The map $\delta_{(\mathbf{B}G, \rho)}$ is therefore a map

$$\delta_{(\mathbf{B}G, \rho)}: V^G \rightarrow (V_G)^\vee \cong V_G,$$

from G -invariants to G -coinvariants. Let $v \mapsto [v]$ be the canonical projection $V \rightarrow V_G = V / \langle v - g \cdot v \rangle_{v \in V, g \in G}$. By composing with the

inclusion $V^G \hookrightarrow V$ we get the morphism $\iota: v \mapsto [v]$ from $V^G \rightarrow V_G$. This is an isomorphism. Indeed, the canonical projection $\pi: V \rightarrow V^G$ given by

$$\pi: v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v$$

is zero on the elements of the form $v - g \cdot v$ and so induces a linear map $\tilde{\pi}: V_G \rightarrow V^G$, which is immediate to see is the inverse of ι . For any $w \in (V_G)^\vee$ seen as an element in $[\rho, \mathbf{1}_{\mathbf{B}G}]$ and any $v \in V^G$ seen as an element in $[\mathbf{1}_{\mathbf{B}G}, \rho]$ we have $w \circ v = w([v])$. Therefore

$$\langle w|v \rangle_{\mathbf{B}G} = \frac{1}{|G|} w \circ v = w \left(\frac{1}{|G|} \iota(v) \right).$$

This means that $\delta_{(\mathbf{B}G, \rho)} v$ is the evaluation on $\frac{1}{|G|} \iota(v)$. Therefore, under the canonical isomorphism $(V_G)^{\vee\vee} \cong V_G$, we have $\delta_{(\mathbf{B}G, \rho)} = \frac{1}{|G|} \iota$. As ι is an isomorphism, so is $\delta_{(\mathbf{B}G, \rho)}$.

Under the duality assumption we may define a linear morphism

$$f_*: [\mathbf{1}_Y, f^* \rho] \rightarrow [\mathbf{1}_X, \rho]$$

as the composition

$$[\mathbf{1}_Y, f^* \rho] \xrightarrow{\delta_{(Y, f^* \rho)}} [f^* \rho, \mathbf{1}_Y]^\vee \xrightarrow{(f^*)^\vee} [\rho, \mathbf{1}_X]^\vee \xrightarrow{\delta_{(X, \rho)}^{-1}} [\mathbf{1}_X, \rho].$$

In other words, if $v \in [\mathbf{1}_Y, f^* \rho]$, the element $f_* v \in [\mathbf{1}_X, \rho]$ is defined by the equation

$$\langle w|f_* v \rangle_X = \langle f^* w|v \rangle_Y$$

for any $w \in [\rho, \mathbf{1}_X]$. Similarly, we have a morphism $f_*: [f^* \rho, \mathbf{1}_Y] \rightarrow [\rho, \mathbf{1}_X]$ defined by

$$\langle f_* w|v \rangle_X = \langle w|f^* v \rangle_Y.$$

We can now define our wannabe functor

$$\text{Sum}_1: \text{Fam}_1(\text{Vect}) \rightarrow \text{Vect}$$

on objects as

$$\text{Sum}_1(X, \rho) = [\mathbf{1}_X, \rho]$$

and on a morphism

$$\begin{array}{ccc} & Z & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & \xRightarrow[\alpha]{} & Y \\ \rho_X \searrow & & \swarrow \rho_Y \\ & \text{Vect} & \end{array}$$

as the composition

$$[\mathbf{1}_X, \rho_X] \xrightarrow{\pi_X^*} [\mathbf{1}_Z, \pi_X^* \rho_X] \xrightarrow{\alpha \circ -} [\mathbf{1}_Z, \pi_Y^* \rho_Y] \xrightarrow{\pi_Y^*} [\mathbf{1}_Y, \rho_Y].$$

Remark 0.5. Before giving a more explicit expression of the above morphism, let us notice that $[\mathbf{1}_X, \rho]$ is a model for the limit of the functor $\rho: X \rightarrow \mathbf{Vect}$, in accordance with the prescription in [?]. Also, notice that the morphism associated to the span $X \leftarrow Z \rightarrow Y$ together with the natural transformation α has the form “pull-compose-push”, and so it is an instance of a Fourier-Mukai-type transform.

Lemma 0.6. (The projection formula) Let $f: Y \rightarrow X$ be a functor between finite groupoids, let $\rho_1, \rho_2: X \rightarrow \mathbf{Vect}$ be functors and let $\alpha: \rho_1 \rightarrow \rho_2$ be a natural transformation. We have an identity of morphisms $[\mathbf{1}_Y, f^* \rho_1] \rightarrow [\mathbf{1}_X, \rho_2]$

$$f_*(f^* \alpha \circ -) = \alpha \circ f_*$$

Proof. The identity

$$f_*(f^* \alpha \circ v) = \alpha \circ f_* v$$

for every $v \in [\mathbf{1}_Y, f^* \rho_1]$ is equivalent to the identity

$$\langle w | f_*(f^* \alpha \circ v) \rangle_X = \langle w | \alpha \circ f_* v \rangle_X$$

for every $v \in [\mathbf{1}_Y, f^* \rho_1]$ and every $w \in [\rho_2, \mathbf{1}_X]$. By the properties of the pairings $\langle | \rangle_X$ and $\langle | \rangle_X$, and by definition of f_* , we have

$$\begin{aligned} \langle w | f_*(f^* \alpha \circ v) \rangle_X &= \langle f^* w | f^* \alpha \circ v \rangle_Y \\ &= \langle f^* w \circ f^* \alpha | v \rangle_Y \\ &= \langle f^*(w \circ \alpha) | v \rangle_Y \\ &= \langle w \circ \alpha | f_* v \rangle_X \\ &= \langle w | \alpha \circ f_* v \rangle_X \end{aligned}$$

□

For later use, let us derive an explicit formula for f_* in case $f: \mathbf{BG} \rightarrow \mathbf{BH}$ is a morphism of finite groups (seen as groupoids with a single object).

Recall that if $v \in [\mathbf{1}_Y, f^* \rho]$, the element $f_* v \in [\mathbf{1}_X, \rho]$ is defined by the equation $\langle w | f_* v \rangle_{\mathbf{BH}} = \langle f^* w | v \rangle_{\mathbf{BG}}$ for any $w \in [\rho, \mathbf{1}_X]$. As both \mathbf{BG} and \mathbf{BH} have a single object, this equation reduces to

$$\frac{w \circ f_* v}{|H|} = \frac{f^* w \circ v}{|G|}$$

The morphism $\rho: \mathbf{B}H \rightarrow \mathbf{Vect}$ is the datum of a linear representation of H on a vector space V , and $f^*\rho$ is the datum of representation of G obtained by pull-back, i.e., the group G acts on V by $g \cdot \xi := f(g) \cdot \xi$. The element $v \in [\mathbf{1}_Y, f^*\rho]$ is therefore a G -invariant vector of V , while $w \in [\rho, \mathbf{1}_X]$ is a linear functional on V factoring through the space of H -coinvariants. The element f_*w is the same linear functional on V , but this time seen as factoring through the space of H -coinvariants. Therefore the composition $f_*w \circ v$ is just the evaluation of $w: V \rightarrow \mathbb{C}$ on the vector $v \in V$, under the canonical isomorphism $\text{End}(\mathbb{C}) \cong \mathbb{C}$. Finally, $f_*v \in [\mathbf{1}_X, \rho]$ is an H -invariant vector of V , and $w \circ f_*v$ is the evaluation of $w: V \rightarrow \mathbb{C}$ on the vector $f_*v \in V$. The defining equation for f_*v then becomes

$$w\left(\frac{f_*v}{|H|}\right) = w\left(\frac{v}{|G|}\right)$$

for any $w: V \rightarrow \mathbb{C}$ factoring through the space of H -coinvariants. From $v/|G|$ we get the H -invariant vector

$$\frac{1}{|H|} \sum_{h \in H} h \cdot \frac{v}{|G|}$$

and, since w factors through the H -coinvariants,

$$w\left(\frac{v}{|G|}\right) = w\left(\frac{1}{|H|} \sum_{h \in H} h \cdot \frac{v}{|G|}\right)$$

The defining equation for f_*v can therefore be rewritten as

$$\langle w | f_*v \rangle_{\mathbf{B}H} = \langle w | \sum_{h \in H} \frac{h \cdot v}{|G|} \rangle_{\mathbf{B}H},$$

for any $w \in [\rho, \mathbf{1}_X]$. As the pairing $\langle | \rangle_{\mathbf{B}H}$ is nondegenerate, this finally gives

$$f_*v = \frac{1}{|G|} \sum_{h \in H} h \cdot v.$$

Remark 0.7. This formula simplifies a bit in case the group homomorphism $f: G \rightarrow H$ is surjective. Indeed, in that case, the G -invariant vector v will be automatically H -invariant and we find

$$f_*v = \frac{|H|}{|G|} v$$

in this case.

Lemma 0.8. Consider a homotopy pullback diagram of finite groupoids

$$\begin{array}{ccc}
 & X \times_T Y & \\
 p_1 \swarrow & & \searrow p_2 \\
 X & \xRightarrow{\eta} & Y \\
 f_1 \searrow & & \swarrow f_2 \\
 & T &
 \end{array}$$

and a morphism $\rho: T \rightarrow \mathbf{Vect}$. Then the Beck-Chevalley condition is satisfied, i.e., we have an identity of morphisms $f_2^* f_{1*} = p_{2*}(\eta^* \rho \circ -) p_1^*$ from $[\mathbf{1}_X, f_1^* \rho]$ to $[\mathbf{1}_Y, f_2^* \rho]$.

Proof. By restricting to connected components, and working up to equivalence of finite groupoids, we may assume that the given pullback diagram has the form

$$\begin{array}{ccc}
 & G \backslash \backslash H // K & \\
 p_1 \swarrow & & \searrow p_2 \\
 \mathbf{B}G & \xRightarrow{\eta} & \mathbf{B}K \\
 f_1 \searrow & & \swarrow f_2 \\
 & \mathbf{B}H &
 \end{array}$$

where the natural transformation η is defined by $\eta_h = h$; this precisely expresses the fact that a morphism from h to h' in the action groupoid $G \backslash \backslash H // K$ is given by a commutative diagram

$$\begin{array}{ccc}
 * & \xrightarrow{h} & * \\
 f_1(g) \downarrow & & \downarrow f_2(k) \\
 * & \xrightarrow{h'} & *
 \end{array}
 .$$

The equation $f_2^* f_{1*} = p_{2*}(\eta^* \rho \circ -) p_1^*$ is equivalent to $\langle w | f_2^* f_{1*} v \rangle_{\mathbf{B}K} = \langle w | p_{2*}(\eta^* \rho \circ p_1^* v) \rangle_{\mathbf{B}K}$ for any $v \in [\mathbf{1}_{\mathbf{B}G}, f_1^* \rho]$ and any $w \in [f_2^* \rho, \mathbf{1}_{\mathbf{B}K}]$. This is equivalent to the equation

$$\langle f_{2*} w | f_{1*} v \rangle_{\mathbf{B}H} = \langle p_{2*} w | \eta^* \rho \circ p_1^* v \rangle_{G \backslash \backslash H // K}.$$

The morphism $\rho: \mathbf{B}H \rightarrow \mathbf{Vect}$ is the datum of a linear representation of the finite group H on a vector space V . The element v is a G -invariant vector of V (where G acts on V via f_1) and w is a linear functional on V which factors through the K -coinvariants (with K

acting via f_2). By the above explicit formula for f_* in the case of a group morphism, we have

$$f_{1*}v = \frac{1}{|G|} \sum_{h \in H} h \cdot v; \quad f_{2*}w = \frac{1}{|K|} \sum_{h \in H} h \cdot w,$$

where on the right we have the H -action on linear functionals on V , i.e., $(h \cdot w) = w(h^{-1} \cdot -)$. We therefore find

$$\begin{aligned} \langle f_{2*}w | f_{1*}v \rangle_{\mathbf{B}H} &= \frac{1}{|H||G||K|} \sum_{h_1, h_2 \in H} w(h_1^{-1}h_2 \cdot v) \\ &= \frac{1}{|H||G||K|} \sum_{h \in H} \sum_{h_1^{-1}h_2=h} w(h_1^{-1}h_2 \cdot v) \\ &= \frac{1}{|H||G||K|} \sum_{h \in H} \sum_{h_1 \in H} w(h \cdot v) \\ &= \frac{1}{|G||K|} \sum_{h \in H} w(h \cdot v) \end{aligned}$$

To every object h of $G \backslash\backslash H // K$, the functors $p_1^* f_1^* \rho$ and $p_2^* f_2^* \rho$ both assign the vector space V , so $p_1^* v$ and $p_2^* w$ consist in a copy of the vector v and of the linear functional $w: V \rightarrow \mathbb{C}$ for each of the copies of V associated with the elements of H . Moreover, since $\eta_h = h$, the natural transformation $\eta^* \rho = \rho \circ \eta$ at the object h is precisely the multiplication by h as a linear map from V to V . This means that $\eta^* \rho \circ p_1^* v = (h \cdot v)_{h \in H}$, i.e., $\eta^* \rho \circ p_1^* v$ consists of the choice of the vector $h \cdot v$ in the copy of V associated with the object h of $G \backslash\backslash H // K$. We therefore find

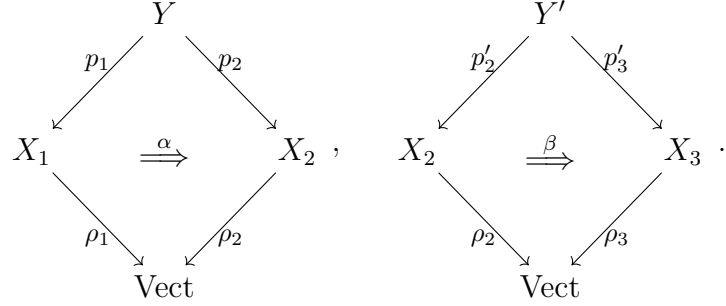
$$\langle p_2^* w | \eta^* \rho \circ p_1^* v \rangle_{G \backslash\backslash H // K} = \frac{1}{|G||K|} \sum_{h \in H} w(h \cdot v),$$

by the formula (0.1). □

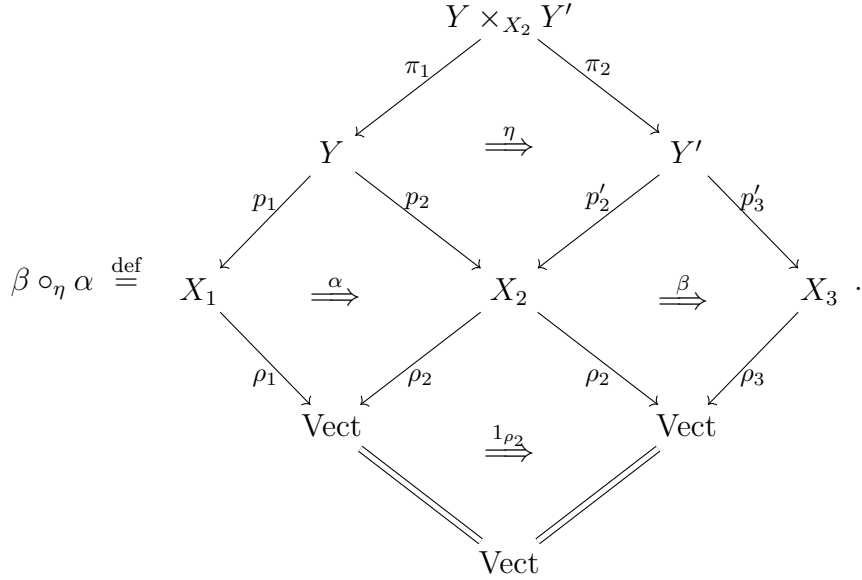
Proposition 0.9. Sum_1 is a functor $\text{Fam}_1(\text{Vect}) \rightarrow \text{Vect}$.

Proof. We have to verify that Sum_1 is compatible with units and composition. Let $\alpha: \rho_1 \rightarrow \rho_2$ and $\beta: \rho_2 \rightarrow \rho_3$ be composable mor-

phisms in $\text{Fam}_1(\text{Vect})$:



Then by definition of $\text{Fam}_1(\text{Vect})$, their composition $\beta \circ_\eta \alpha$ is the natural transformation $(\beta * 1_{\pi_2}) \cdot (1_{\rho_2} * \eta) \cdot (\alpha * 1_{\pi_1})$:

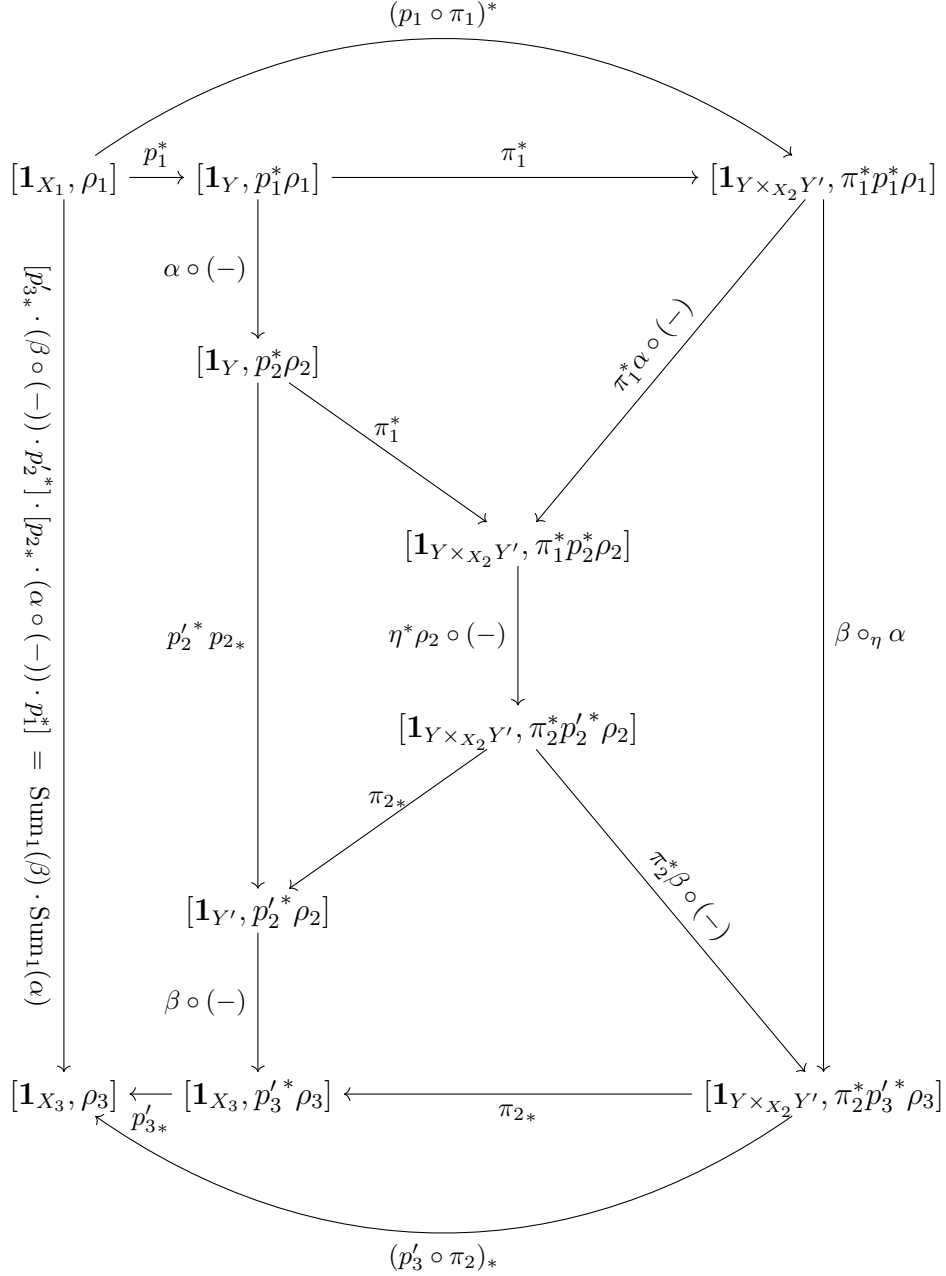


Furthermore, by definition of Sum_1 , its action on the three morphisms is as follows:

$$\begin{aligned} \text{Sum}_1(\alpha) &= p_{2*} \cdot (\alpha \circ (-)) \cdot p_1^*, \\ \text{Sum}_1(\beta) &= p'_{3*} \cdot (\beta \circ (-)) \cdot p'_2{}^*, \\ \text{Sum}_1(\beta \circ_\eta \alpha) &= (p'_3 \circ \pi_2)_* \cdot (\beta \circ_\eta \alpha \circ (-)) \cdot (p_1 \circ \pi_1)^*. \end{aligned}$$

To show that $\text{Sum}_1(\beta \circ_\eta \alpha) = \text{Sum}_1(\beta) \circ \text{Sum}_1(\alpha)$, we first observe that the left-hand side is the clock-wise composition of the outmost arrows from the top left to the bottom left vertex in the following

diagram:



The leftmost vertical arrow is $\text{Sum}_1(\beta) \circ \text{Sum}_1(\alpha)$, hence we are done if we can show that the diagram commutes. This is indeed the case: the leftmost and rightmost subdiagrams commute by definition, as do the other two subdiagrams involving $\pi_1^* \alpha \circ (-)$ and $\pi_2^* \beta \circ (-)$; that the top and bottom diagrams commute follows from the definition of pullback and pushforward; and the middle subdiagram commutes thanks to the Beck-Chevalley condition of Lemma 0.8.

Finally, if $\rho_2 = \rho_3$, $Y = Y'$ and β is the identity on ρ_2 in $\text{Fam}_1(\text{Vect})$, then $Y \times_{X_2} Y'$ is canonically equivalent to Y , and we see that Sum_1 sends units to units. \square