Boundaries in finite gauge theory

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1 Introduction

Let

- \bullet G be a finite group,
- $\omega \in Z^n(G, \mathbb{C}^{\times}),$
- and \mathcal{C} a symmetric monoidal (∞, n) -category for some $n \in \mathbb{Z}_+$.

On the one hand, [FHLT] sketches the construction of a fully extended TQFT

$$\operatorname{Bord}_n \xrightarrow{I} \operatorname{Fam}_n(\mathcal{C}) \xrightarrow{\operatorname{Sum}_n} \mathcal{C}$$
 (1.1)

based on the data (G, ω) , which is viewed as a fully extended version of Dijkgraaf-Witten theory. On the other hand, in [Wi, WWW] a class of boundary conditions for (classical!?) DW theory is proposed in terms of group extensions.

Goal. We want to reformulate the general aspects of the work [Wi, WWW] in the setting of fully extended TQFT. More precisely, we view the bulk theories of [Wi, WWW] as the classical part I in (1.1) with $C = \mathbf{B}^n \mathbb{C}^{\times}$ (the n-fold delooping of the group \mathbb{C}^{\times}) and then

- (i) identify classical boundary conditions as 1-morphisms with source * in $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$,
- (ii) subsume the boundary conditions of [Wi, WWW] as special cases,
- (iii) map classical boundary conditions to quantum ones via Sum_n , and connect them to the known quantum boundary conditions, following the work of Ostrik and Fuchs-Schweigert-Valentino in the case n=3,
- (iv) do the above not only for framed, but also for oriented, spin,...TQFTs.

Since [FHLT] is very light on details, we first will work out explicitly how (1.1) recovers the known bulk theory for $n \in \{1,2\}$ and hopefully also n=3. We also want to explain in detail how (G,ω) gives rise to an object in $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)$, and how group extensions $1 \to K \to H \xrightarrow{r} G \to 1$ give rise to 1-morphisms $\operatorname{Fam}_n(\mathbf{B}^n\mathbb{C}^\times)(*,\mathbf{B}G)$.

(A more conceptual approach to boundary conditions and defects would be to start with a bordism category "with singularities" [Lu, Sect. 4.3] instead of $Bord_n$, but we leave that for another project.)

2 Special objects and 1-morphisms in $\operatorname{Fam}_n(\boldsymbol{B}^n\mathbb{C}^{\times})$

TODO:

- spell out definition of $\operatorname{Fam}_n(\mathcal{C})$
- check that n-functor $BG \to B^n \mathbb{C}^{\times}$ is precisely an n-cocycle <u>Issue</u>: For n > 3, what is a weak n-functor, i. e. precisely what data and constraints are needed?

<u>Idea</u>: Since both n-categories in $BG \to B^n\mathbb{C}^{\times}$ are close to trivial, most data of the functor will be trivial, and the constraints (whatever they are) will be trivially satisfied. For $n \in \{1, 2, 3\}$ this is true, see Section 3.

- clarify precisely how $BG \to B^n \mathbb{C}^{\times}$ induces an *n*-functor $BG \to n$ -Vect, at least for $n \in \{1, 2, 3\}$
- check whether natural transformation from trivial n-functor to $\omega \circ r$ is trivialisation of $r^*\omega$

Issue: induced from issue in second item

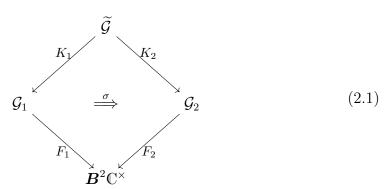
2.1 Classical bulk theories

TODO

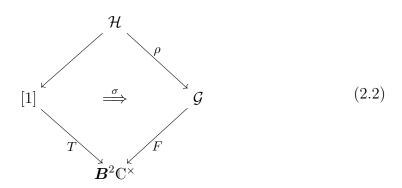
2.2 Classical boundary conditions

$2.2.1 \ n=2$

Let $\mathcal{G}_1, \mathcal{G}_2$ be 2-groupoids and let $F_1: \mathcal{G}_1 \to \boldsymbol{B}^2\mathbb{C}^{\times}$ and $F_2: \mathcal{G}_2 \to \boldsymbol{B}^2\mathbb{C}^{\times}$ be 2-functors, i. e. objects in $\operatorname{Fam}_2(\boldsymbol{B}^2\mathbb{C}^{\times})$. A 1-morphism $F_1 \to F_2$ in $\operatorname{Fam}_2(\boldsymbol{B}^2\mathbb{C}^{\times})$ is a span of 2-groupoids $\mathcal{G}_1 \xleftarrow{K_1} \widetilde{\mathcal{G}} \xrightarrow{K_2} \mathcal{G}_2$ together with a pseudonatural transformation $\sigma: F_1 \circ K_1 \to F_2 \circ K_2$:



Let [1] denote the final 2-category with only one object, 1- and 2-morphism, and let $T: [1] \to \mathbf{B}^2 \mathbb{C}^{\times}$ be the unique, trivial 2-functor. A classical boundary condition for $F: \mathcal{G} \to \mathbf{B}^2 \mathbb{C}^{\times}$ is a 1-morphism $T \to F$. Thus it comprises a 2-groupoid \mathcal{H} , a 2-functor $\rho: \mathcal{H} \to \mathcal{G}$, and a pseudonatural transformation



Concretely, since $\mathbf{B}^2\mathbb{C}^{\times}$ has only one object and one 1-morphism, σ is given by a family of 2-morphisms $\sigma_h \colon (F \circ \rho)(h) = 1_* \to 1_*$, indexed by 1-morphisms h in \mathcal{H} , such that

$$F(\rho(h)) \otimes F(\rho(h')) \xrightarrow{1 \otimes \sigma_{h'}} F(\rho(h))$$

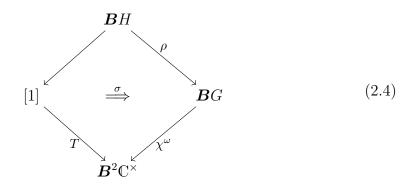
$$(F \circ \rho)_{h,h'} \downarrow \qquad \qquad \downarrow \sigma_{h}$$

$$F(\rho(hh')) \xrightarrow{\sigma_{hh'}} 1_{*}$$

$$(2.3)$$

commutes for al composable 1-morphisms h, h' in \mathcal{H} .

We now choose $\mathcal{H} = \mathbf{B}H$ for some finite group H, $\mathcal{G} = \mathbf{B}G$, and $F = \chi^{\omega}$ as in (3.9) — and we are not careful to distinguish the target $\mathbf{B}^2\mathbb{C}^{\times}$ from $\mathrm{Alg}_{\mathbb{C}}$. Then a 2-functor $\rho \colon \mathbf{B}H \to \mathbf{B}G$ is just a group homomorphism $\rho \colon H \to G$, and a boundary condition



for χ^{ω} is a family $\{\sigma_h\}_{h\in H}\subset \mathbb{C}^{\times}$ such that

$$L_{\rho(h)} \otimes_{\mathbb{C}} L_{\rho(h')} \xrightarrow{1 \otimes \sigma_{h'}} L_{\rho(h)}$$

$$\omega(\rho(h), \rho(h')) \downarrow \qquad \qquad \downarrow \sigma_{h}$$

$$L_{\rho(hh')} \xrightarrow{\sigma_{hh'}} 1_{*}$$

$$(2.5)$$

commutes, i. e. $\omega(\rho(h), \rho(h')) = \sigma_h \cdot \sigma_{hh'}^{-1} \cdot \sigma_{h'}$ for all $h, h' \in H$. Viewing σ as a map $H \to \mathbb{C}^{\times}$, $h \mapsto \sigma_h$, this condition precisely means that σ trivialises the pullback of ω :

$$d\sigma = \rho^* \omega \,. \tag{2.6}$$

3 Bulk theory

3.1 n=1

Since we take the action of G on \mathbb{C}^{\times} to be trivial, the cocycle

$$\omega = [\omega] \in H^1(G, \mathbb{C}^\times) = \operatorname{Hom}_{\operatorname{Grp}}(G, \mathbb{C}^\times)$$
(3.1)

is just a group homomorphism. We consider $B^1\mathbb{C}^{\times}$ as a subcategory of $\mathrm{Vect}_{\mathbb{C}}$, and we will use the pair (G, ω) to construct a functor

$$\operatorname{Bord}_1 \xrightarrow{I} \operatorname{Fam}_1(\operatorname{Vect}_{\mathbb{C}}) \xrightarrow{\operatorname{Sum}_1} \operatorname{Vect}_{\mathbb{C}}.$$
 (3.2)

According to the cobordism hypothesis, the composite $\operatorname{Sum}_1 \circ I$ is determined by its value on the point pt_+ , so we only need to specify $I(\operatorname{pt}_+)$ and then compute $\operatorname{Sum}_1(I(\operatorname{pt}_+))$. Note that $\mathbf{B}^1G = */\!\!/ G$ is the classical action groupoid. We set $I(\operatorname{pt}_+)$ to be the functor

$$\chi^{\omega} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto \mathbb{C}, \quad g \longmapsto \omega(g) \in \operatorname{Aut}(\mathbb{C})$$
 (3.3)

for all $q \in G$. Then (3.2) recovers the known result reviewed in [FHLT, Sect. 1]:

Lemma 3.1. Sum₁(χ^{ω}) = \mathbb{C} if $\omega(g) = 1$ for all $g \in G$, and Sum₁(χ^{ω}) = 0 otherwise.

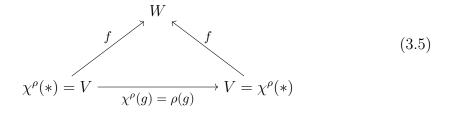
Proof. The statement is a particular case of this more general one: let (V, ρ) be a linear representation of the group G, seen as the functor

$$\chi^{\rho} \colon * /\!\!/ G \longrightarrow \operatorname{Vect}_{\mathbb{C}}, \quad * \longmapsto V, \quad q \longmapsto \rho(q) \in \operatorname{Aut}(V).$$
 (3.4)

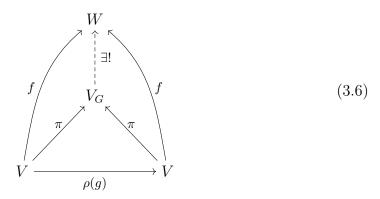
Then the universal cocone of χ^{ρ} , viewed as a diagram of shape $*/\!\!/ G$ in $\text{Vect}_{\mathbb{C}}$ is the pair (V_G, π) , where V_G is the vector space of coinvariants for the representation ρ , i.e.,

$$V_G = V/\langle v - \rho(g)v \rangle_{g \in G, v \in V}$$

and $\pi: V \to V_G$ is the projection to the quotient. Namely, let (W, f) be a cocone for χ^{ρ} , i.e., a pair consisting of a vector space W together with a linear map $f: \chi^{\rho}(*) = V \to W$ such that



Then, by definition of V_G , the morphism f uniquely factors through V_G and so we have a commutative diagram



showing that (V_G, π) enjoys the universal property of the universal cocone. \square Corollary 3.2. The linear dual $\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_1(\chi^{\omega}), \mathbb{C})$ of $\operatorname{Sum}_1(\chi^{\omega})$ is naturally isomorphic to the vector space of natural transformations $\operatorname{Hom}_{[*//G,\operatorname{Vect}_{\mathbb{C}}]}(\chi^{\omega},\chi^1)$, where $\chi^1: *///G \to \operatorname{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space

where χ^1 : $*/\!\!/ G \to \operatorname{Vect}_{\mathbb{C}}$ is the trivial representation of G on the vector space \mathbb{C} . In other words, we have

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(\operatorname{Sum}_{1}(\chi^{\omega}),\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \chi^{\omega} \Longrightarrow * \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}.$$

As a finite dimensional vector space is completely determined by its linear dual, this actually defines $\operatorname{Sum}_1(\chi^{\omega})$.

Proof. Again, the statement is true for an aritrary linear representation (V, ρ) of the group G. The natural isomorphism

$$\operatorname{Hom}_{\operatorname{Vect}_{\mathbb{C}}}(V_G,\mathbb{C}) \cong \left\{ \begin{array}{c} */\!\!/ G \\ \downarrow \\ \downarrow \\ \operatorname{Vect}_{\mathbb{C}} \end{array} \right\}$$

is then nothing but the universal property of V_G . Namely, an element in the right hand side is a morphism $f: V = \chi^{\rho}(*) \to \chi^{1}(*) = \mathbb{C}$ such that all the diagrams

$$V \xrightarrow{f} \mathbb{C}$$

$$\rho(g) \downarrow \qquad \qquad \downarrow \mathrm{id}_{\mathbb{C}}$$

$$V \xrightarrow{f} \mathbb{C}$$

commute, for any $g \in G$. This is the same as requiring that all the diagrams

$$V \xrightarrow{f} V$$

$$V \xrightarrow{\rho(q)} V$$

commute, and so it is precisely the datum of a morphism $V_G \to \mathbb{C}$ by the argumenti in the proof of Lemma 3.18.

3.1.1 Quantisation

We now show that Sum_1 is a (TODO: symmetric monoidal!) functor.

Let us denote by $\operatorname{Fam}_0(\mathbb{C})$ the set of equivalence classes of finite groupoids equipped with functors to \mathbb{C} (seen as a trivial category). In other words, a representative of an element of $\operatorname{Fam}_0(\mathbb{C})$ is a pair (X, ρ) consisting of a finite groupoid X together with a map $\rho \colon \pi_0(X) \to \mathbb{C}$. The map

$$Sum_0: Fam_0 \to \mathbb{C}$$

is defined as

$$\operatorname{Sum}_0(X, \rho) = \sum_{[x] \in \pi_0(X)} \frac{\rho(x)}{|\operatorname{Aut}(x)|}.$$

We will also use the more evocative notation

$$\int_{X} \rho(x) d\mu(x)$$

to denote $Sum_0(X, \rho)$. Notice that Sum_0 is linear in the ρ variable.

A nice property of the morphism Sum_0 (which we are going to use later) is the following: let X be a finite set with an action of a finite group G, and let $X/\!/G$ denotes the action groupoid for this action; finally, let $\rho\colon X/\!/G \to \mathbb{C}$ any functor, i.e., let $\rho\colon X \to \mathbb{C}$ be a function that is constant on the G-orbits in X. Then we have

$$\operatorname{Sum}_{0}(X/\!/G, \rho) = \frac{1}{|G|} \sum_{x \in X} \rho(x).$$
 (3.7)

Namely, the set $\pi_0(X//G)$ is the quotient set X/G and for any $x \in X$ seen as an object in X//G we have $\operatorname{Aut}(x) = \operatorname{Stab}_G(x)$. Therefore

$$\operatorname{Sum}_{0}(X//G, \rho) = \sum_{[x] \in X/G} \frac{\rho(x)}{|\operatorname{Stab}_{G}(x)|}$$
$$= \frac{1}{|G|} \sum_{[x] \in X/G} \frac{|G|}{|\operatorname{Stab}_{G}(x)|} \rho(x)$$

$$= \frac{1}{|G|} \sum_{[x] \in X/G} |[x]| \rho(x)$$

$$= \frac{1}{|G|} \sum_{[x] \in X/G} \sum_{x \in [x]} \rho(x)$$

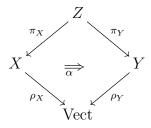
$$= \frac{1}{|G|} \sum_{x \in X} \rho(x)$$

Remark 3.3. A function $f: \pi_0(X) \to \mathbb{C}$ can equivalently be seen as an element in $\operatorname{End}_{[X,\operatorname{Vect}]}(\mathbf{1}_X)$, where $\mathbf{1}_X\colon X \to \operatorname{Vect}$ is the constant functor taking the value \mathbb{C} on X. The map Sum_0 can therefore be seen as the datum of a map

$$\operatorname{Sum}_0 \colon \operatorname{End}_{[X,\operatorname{Vect}]}(\mathbf{1}_X) \to \mathbb{C},$$

for any finite groupoid X.

Definition 3.4. The category $\operatorname{Fam}_1(\operatorname{Vect})$ has as its objects finite groupoids equipped with functors to Vect, i.e., pairs (X, ρ) consisting of a finite groupoid X together with a functor $\rho \colon X \to \operatorname{Vect}$. Morphisms in $\operatorname{Fam}_1(\operatorname{Vect})$ are spans of groupoids over Vect, i.e., commutative diagrams



where the natural transformation $\alpha \colon \rho_X \circ \pi_X \to \rho_Y \circ \pi_Y$ is part of the data of the commutative diagram.

Remark 3.5. The identity mophisms and the composition of morphisms in $Fam_1(Vect)$ will be defined later.

For every object (X, ρ) in $Fam_1(Vect)$ we can consider the vector space

$$[\mathbf{1}_X, \rho] = \operatorname{Hom}_{[X, \operatorname{Vect}]}(\mathbf{1}_X, \rho).$$

If $f: Y \to X$ is a functor between finite groupoids, and $\rho: X \to \text{Vect}$ is a functor, we write $f^*\rho: Y \to \text{Vect}$ for the functor $\rho \circ f$. If $\alpha: \mathbf{1}_X \to \rho$ is a natural transformation, then $f^*\alpha$ is a natural transformation between $f^*\mathbf{1}_X = \mathbf{1}_Y$ and $f^*\rho$. This defines a linear map

$$f^* \colon [\mathbf{1}_X, \rho] \to [\mathbf{1}_Y, f^* \rho].$$

Similarly, we can consider the vector space

$$[\rho, \mathbf{1}_X] = \operatorname{Hom}_{[X, \operatorname{Vect}]}(\rho, \mathbf{1}_X)$$

together with the linear map

$$f^* \colon [\rho, \mathbf{1}_X] \to [f^*\rho, \mathbf{1}_Y].$$

Passing to the linear duals we get the linear map

$$(f^*)^{\vee} \colon [f^*\rho, \mathbf{1}_Y]^{\vee} \to [\rho, \mathbf{1}_X]^{\vee}.$$

Next, notice that composition of morphisms gives a bilinear map

$$[\rho, \mathbf{1}_X] \otimes [\mathbf{1}_X, \rho] \rightarrow [\mathbf{1}_X, \mathbf{1}_X]$$

and so composing this with the linear map $\operatorname{Sum}_0(X, -) : [\mathbf{1}_X, \mathbf{1}_X] \to \mathbb{C}$ we get a bilinear pairing $\langle | \rangle_X$ between $[\rho, \mathbf{1}_X]$ and $[\mathbf{1}_X, \rho]$, for any ρ . Notice that, by its very definition, the pairing $\langle | \rangle_X$ satisfies

$$\langle v | \alpha \circ w \rangle_X = \langle v \circ \alpha | w \rangle_X,$$

for any $v \in [\rho_2, \mathbf{1}_X]$, any $w \in [\mathbf{1}_X, \rho_1]$ and any natural transformation $\alpha \colon \rho_1 \to \rho_2$. Equivalently, the bilinear pairing $\langle \, | \, \rangle_X$ is a linear map

$$\delta_{(X,\rho)} \colon [\mathbf{1}_X, \rho] \to [\rho, \mathbf{1}_X]^{\vee}.$$

Our main duality assumption will be that $\delta_{(X,\rho)}$ is a linear isomorphism for any (X,ρ) .

Remark 3.6. The duality assumption holds for every finite groupoid X if and only if it holds for finite connected groupoids. Up to equivalence we are therefore reduced to considering the case $X = \mathbf{B}G$, with G a finite group. In this case $\rho \colon \mathbf{B}G \to \text{Vect}$ is the datum of a linear representation V of G, and the vector spaces $[\mathbf{1}_{\mathbf{B}G}, \rho]$ and $[\rho, \mathbf{1}_{\mathbf{B}G}]$ are isomorphic to the space V^G of G-invariants and the linear dual $(V_G)^{\vee}$ of the space of G-coinvariants, respectively. The map $\delta_{(\mathbf{B}G,\rho)}$ is therefore a map

$$\delta_{(\mathbf{B}G,\rho)} \colon V^G \to (V_G)^{\vee\vee} \cong V_G,$$

from G-invariants to G-coinvariants. Let $v \mapsto [v]$ be the canonical projection $V \to V_G = V/\langle v - g \cdot v \rangle_{v \in V; g \in G}$. By composing with the inclusion $V^G \hookrightarrow V$ we get the morphism $\iota \colon v \mapsto [v]$ from $V^G \to V_G$. This is an isomorphism. Indeed, the canonical projection $\pi \colon V \to V^G$ given by

$$\pi \colon v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v$$

is zero on the elements of the form $v - g \cdot v$ and so induces a linear map $\tilde{\pi} \colon V_G \to V^G$, which is immediate to see is the inverse of ι . For any $w \in (V_G)^{\vee}$ seen as an element in $[\rho, \mathbf{1}_{\mathbf{B}G}]$ and any $v \in V^G$ seen as an element in $[\mathbf{1}_{\mathbf{B}G}, \rho]$ we have $w \circ v = w([v])$ Therefore

$$\langle w|v\rangle_{\mathbf{B}G} = \frac{1}{|G|}w \circ v = w\left(\frac{1}{|G|}\iota(v)\right).$$

This means that $\delta_{(\mathbf{B}G,\rho)}v$ is the evaluation on $\frac{1}{|G|}\iota(v)$. Therefore, under the canonical isomorphism $(V_G)^{\vee\vee} \cong V_G$, we have $\delta_{(\mathbf{B}G,\rho)} = \frac{1}{|G|}\iota$. As ι is an isomorphism, so is $\delta_{(\mathbf{B}G,\rho)}$.

Under the duality assumption we may define a linear morphism

$$f_* \colon [\mathbf{1}_Y, f^* \rho] \to [\mathbf{1}_X, \rho]$$

as the composition

$$[\mathbf{1}_Y, f^*\rho] \xrightarrow{\delta_{(Y,f^*\rho)}} [f^*\rho, \mathbf{1}_Y]^{\vee} \xrightarrow{(f^*)^{\vee}} [\rho, \mathbf{1}_X]^{\vee} \xrightarrow{\delta_{(X,\rho)}^{-1}} [\mathbf{1}_X, \rho].$$

In other words, if $v \in [\mathbf{1}_Y, f^*\rho]$, the element $f_*v \in [\mathbf{1}_X, \rho]$ is defined by the equation

$$\langle w|f_*v\rangle_X = \langle f^*w|v\rangle_Y$$

for any $w \in [\rho, \mathbf{1}_X]$. Similarly, we have a morphism $f_* : [f^*\rho, \mathbf{1}_Y] \to [\rho, \mathbf{1}_X]$ defined by

$$\langle f_* w | v \rangle_X = \langle w | f^* v \rangle_Y.$$

For later use, let us derive an explicit formla for f_* in case $f: \mathbf{B}G \to \mathbf{B}H$ is a morphism of finite groups (seen as groupoids with a single object).

Lemma 3.7. If G, H are finite groups and $f: \mathbf{B}G \to \mathbf{B}H$ is a functor, then for every $v \in [\mathbf{1}_{\mathbf{B}G}, f^*\rho]$ we have

$$f_*v = \frac{1}{|G|} \sum_{h \in H} h \cdot v.$$

Proof. Recall that if $v \in [\mathbf{1}_{\mathbf{B}G}, f^*\rho]$, the element $f_*v \in [\mathbf{1}_{\mathbf{B}H}, \rho]$ is defined by the equation $\langle w|f_*v\rangle_{\mathbf{B}H} = \langle f^*w|v\rangle_{\mathbf{B}G}$ for any $w \in [\rho, \mathbf{1}_{\mathbf{B}H}]$. As both $\mathbf{B}G$ and $\mathbf{B}H$ have a single object, this equation reduces to

$$\frac{w \circ f_* v}{|H|} = \frac{f^* w \circ v}{|G|}$$

The morphism $\rho \colon \mathbf{B}H \to \text{Vect}$ is the datum of a linear representation of H on a vector space V, and $f^*\rho$ is the datum of representation of G obtained by pullback, i.e., the group G acts on V by $g \cdot \xi := f(g) \cdot \xi$. The element $v \in [\mathbf{1}_{\mathbf{B}G}, f^*\rho]$

is therefore a G-invariant vector of V, while $w \in [\rho, \mathbf{1}_{\mathbf{B}H}]$ is a linear functional on V factoring through the space of H-coinvariants. The element f^*w is the same linear functional on V, but this time seen as factoring through the space of G-coinvariants. Therefore the composition $f^*w \circ v$ is just the evaluation of $w: V \to \mathbb{C}$ on the vector $v \in V$, under the canonical isomorphism $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$. Finally, $f_*v \in [\mathbf{1}_{\mathbf{B}H}, \rho]$ is an H-invariant vector of V, and $w \circ f_*v$ is the evaluation of $w: V \to \mathbb{C}$ on the vector $f_*v \in V$. The defining equation for f_*v then becomes

$$w\left(\frac{f_*v}{|H|}\right) = w\left(\frac{v}{|G|}\right)$$

for any $w:V\to\mathbb{C}$ factoring through the space of H-coinvariants. From v/|G| we get the H-invariant vector

$$\frac{1}{|H|} \sum_{h \in H} h \cdot \frac{v}{|G|}$$

and, since w factors through the H-coinvariants,

$$w\left(\frac{v}{|G|}\right) = w\left(\frac{1}{|H|} \sum_{h \in H} h \cdot \frac{v}{|G|}\right)$$

The defining equation for f_*v can therefore be rewritten as

$$\langle w|f_*v\rangle_{\mathbf{B}H} = \langle w|\sum_{h\in H} \frac{h\cdot v}{|G|}\rangle_{\mathbf{B}H},$$

for any $w \in [\rho, \mathbf{1}_X]$. As the pairing $\langle | \rangle_{\mathbf{B}H}$ is nondegenerate, this finally gives

$$f_*v = \frac{1}{|G|} \sum_{h \in H} h \cdot v.$$

Remark 3.8. This formula simplifies a bit in case the group homomorphism $f: G \to H$ is surjective. Indeed, in that case, the G-invariant vector v will be automatically H-invariant and we find

$$f_*v = \frac{|H|}{|G|}v$$

in this case.

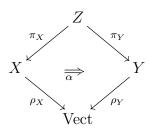
We can now define our wannabe functor

$$Sum_1: Fam_1(Vect) \rightarrow Vect$$

on objects as

$$\operatorname{Sum}_1(X,\rho) = [\mathbf{1}_X,\rho]$$

and on a morphism



as the composition

$$[\mathbf{1}_X, \rho_X] \xrightarrow{\pi_X^*} [\mathbf{1}_Z, \pi_X^* \rho_X] \xrightarrow{\alpha \circ -} [\mathbf{1}_Z, \pi_Y^* \rho_Y] \xrightarrow{\pi_{Y^*}} [\mathbf{1}_Y, \rho_Y].$$

Remark 3.9. Before giving a more explicit expression of the above morphism, let us notice that $[\mathbf{1}_X, \rho]$ is a model for the limit of the functor $\rho \colon X \to \mathrm{Vect}$, in accordance with the prescription in [FHLT]. Also, notice that the morphism associated to the span $X \leftarrow Z \to Y$ together with the natural transformation α has the form "pull-compose-push", and so it is an instance of a Fourier-Mukai-type transform.

Lemma 3.10. (The projection formula) Let $f: Y \to X$ be a functor between finite groupoids, let $\rho_1, \rho_2: X \to \text{Vect}$ be functors and let $\alpha: \rho_1 \to \rho_2$ be a natural transformation. We have an identity of morphisms $[\mathbf{1}_Y, f^*\rho_1] \to [\mathbf{1}_X, \rho_2]$

$$f_*(f^*\alpha \circ -) = \alpha \circ f_*$$

Proof. The identity

$$f_*(f^*\alpha \circ v) = \alpha \circ f_*v$$

for every $v \in [\mathbf{1}_Y, f^* \rho_1]$ is equivalent to the identity

$$\langle w|f_*(f^*\alpha \circ v)\rangle_X = \langle w|\alpha \circ f_*v\rangle_X$$

for every $v \in [\mathbf{1}_Y, f^*\rho_1]$ and every $w \in [\rho_2, \mathbf{1}_X]$. By the properties of the pairings $\langle | \rangle_X$ and $\langle | \rangle_Y$, and by definition of f_* , we have

$$\langle w|f_*(f^*\alpha \circ v)\rangle_X = \langle f^*w|f^*\alpha \circ v\rangle_Y$$

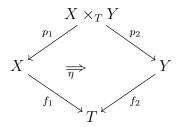
$$= \langle f^*w \circ f^*\alpha|v\rangle_Y$$

$$= \langle f^*(w \circ \alpha)|v\rangle_Y$$

$$= \langle w \circ \alpha|f_*v\rangle_X$$

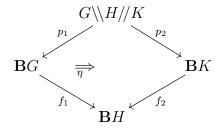
$$= \langle w|\alpha \circ f_*v\rangle_X$$

Lemma 3.11. Consider a homotopy pullback diagram of finite groupoids



and a morphism $\rho: T \to \text{Vect.}$ Then the Beck-Chevalley condition is satisfied, i.e., we have an identity of morphisms $f_2^* f_{1*} = p_{2*}(\eta^* \rho \circ -) p_1^*$ from $[\mathbf{1}_X, f_1^* \rho]$ to $[\mathbf{1}_Y, f_2^* \rho]$.

Proof. By restricting to connected components, and working up to equivalence of finite groupoids, we may assume that the given pullback diagram has the form



where the natural transformation η is defined by $\eta_h = h$; this precisely expresses the fact that a morphism from h to h' in the action groupoid $G \backslash H//K$ is given by a commutative diagram

$$\begin{array}{ccc}
* & \xrightarrow{h} & * \\
f_1(g) \downarrow & & \downarrow f_2(k) \\
* & \xrightarrow{h'} & *
\end{array}$$

The equation $f_2^* f_{1*} = p_{2*}(\eta^* \rho \circ -) p_1^*$ is equivalent to $\langle w | f_2^* f_{1*} v \rangle_{\mathbf{B}K} = \langle w | p_{2*}(\eta^* \rho \circ p_1^* v) \rangle_{\mathbf{B}K}$ for any $v \in [\mathbf{1}_{\mathbf{B}G}, f_1^* \rho]$ and any $w \in [f_2^* \rho, \mathbf{1}_{\mathbf{B}K}]$. This is equivalent to the equation

$$\langle f_{2*}w|f_{1*}v\rangle_{\mathbf{B}H} = \langle p_2^*w|\eta^*\rho\circ p_1^*v\rangle_{G\backslash\backslash H//K}.$$

The morphism $\rho \colon \mathbf{B}H \to \text{Vect}$ is the datum of a linear representation of the finite group H on a vector space V. The element v is a G-invariant vector of V (where G acts on V via f_1) and w is a linear functional on V which factors through the K-coinvariants (with K acting via f_2). By Lemma 3.7 we have

$$f_{1*}v = \frac{1}{|G|} \sum_{h \in H} h \cdot v; \qquad f_{2*}w = \frac{1}{|K|} \sum_{h \in H} h \cdot w,$$

where on the right we have the *H*-action on linear functionals on *V*, i.e., $(h \cdot w) = w(h^{-1} \cdot -)$. We therefore find

$$\langle f_{2*}w|f_{1*}v\rangle_{\mathbf{B}H} = \frac{1}{|H||G||K|} \sum_{h_1,h_2\in H} w(h_1^{-1}h_2 \cdot v)$$

$$= \frac{1}{|H||G||K|} \sum_{h\in H} \sum_{h_1^{-1}h_2=h} w(h_1^{-1}h_2 \cdot v)$$

$$= \frac{1}{|H||G||K|} \sum_{h\in H} \sum_{h_1\in H} w(h \cdot v)$$

$$= \frac{1}{|G||K|} \sum_{h\in H} w(h \cdot v)$$

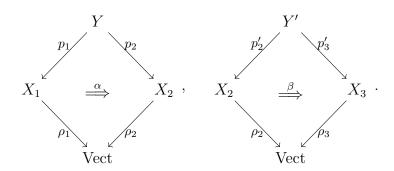
To every object h of $G\backslash H/\!/K$, the functors $p_1^*f_1^*\rho$ and $p_2^*f_2^*\rho$ both assign the vector space V, so p_1^*v and p_2^*w consist of a copy of the vector v and of the linear functional $w: V \to \mathbb{C}$ for each of the copies of V associated with the elements of H. Moreover, since $\eta_h = h$, the natural transformation $\eta^*\rho = \rho \circ \eta$ at the object h is precisely the multiplication by h as a linear map from V to V. This means that $\eta^*\rho \circ p_1^*v = (h \cdot v)_{h \in H}$, i.e, $\eta^*\rho \circ p_1^*v$ consists of the choice of the vector $h \cdot v$ in the copy of V associated with the object h of $G\backslash H/\!/K$. We therefore find

$$\langle p_2^* w | \eta^* \rho \circ p_1^* v \rangle_{G \setminus \backslash H / / K} = \frac{1}{|G||K|} \sum_{h \in H} w(h \cdot v),$$

by the formula (3.7).

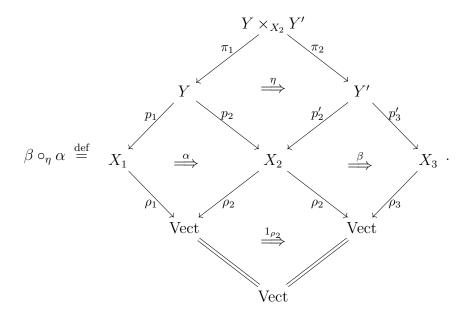
Proposition 3.12. Sum₁ is a functor $Fam_1(Vect) \rightarrow Vect$.

Proof. We have the verify that Sum₁ is compatible with units and composition. Let $\alpha: \rho_1 \to \rho_2$ and $\beta: \rho_2 \to \rho_3$ be composable morphisms in Fam₁(Vect):



Then by definition of Fam₁(Vect), their composition $\beta \circ_{\eta} \alpha$ is the natural trans-

formation $(\beta * 1_{\pi_2}) \cdot (1_{\rho_2} * \eta) \cdot (\alpha * 1_{\pi_1})$:

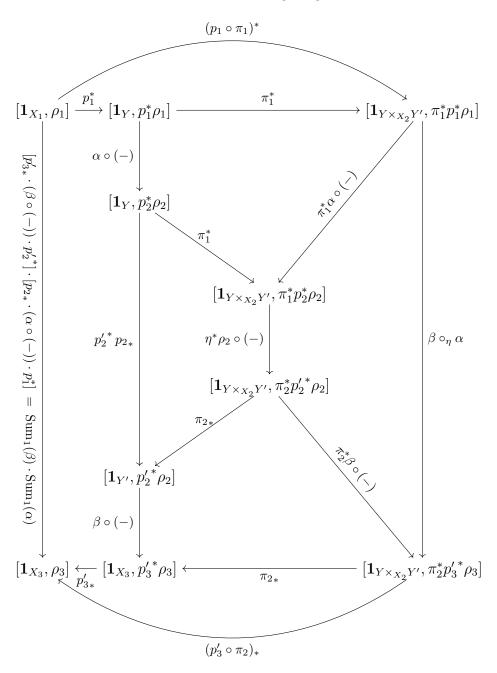


Furthermore, by definition of Sum_1 , its action on the three morphisms is as follows:

$$\begin{aligned} \operatorname{Sum}_{1}(\alpha) &= p_{2*} \cdot (\alpha \circ (-)) \cdot p_{1}^{*}, \\ \operatorname{Sum}_{1}(\beta) &= p_{3*}' \cdot (\beta \circ (-)) \cdot {p_{2}'}^{*}, \\ \operatorname{Sum}_{1}(\beta \circ_{\eta} \alpha) &= (p_{3}' \circ \pi_{2})_{*} \cdot (\beta \circ_{\eta} \alpha \circ (-)) \cdot (p_{1} \circ \pi_{1})^{*}. \end{aligned}$$

To show that $\operatorname{Sum}_1(\beta \circ_{\eta} \alpha) = \operatorname{Sum}_1(\beta) \circ \operatorname{Sum}_1(\alpha)$, we first observe that the left-hand side is the clock-wise composition of the outmost arrows from the top

left to the bottom left vertex in the following diagram:



The leftmost vertical arrow is $\operatorname{Sum}_1(\beta) \circ \operatorname{Sum}_1(\alpha)$, hence we are done if we can show that the diagram commutes. This is indeed the case: the leftmost and rightmost subdiagrams commute by definition, as does the other subdiagram involving $\pi_1^*\alpha \circ (-)$; that the top and bottom diagrams commute follows from the definition of pullback and pushforward; finally, the subdiagram involving $\pi_2^*\beta \circ (-)$ and π_{2*} commutes by the projection formula of Lemma 3.10, and the middle subdiagram commutes thanks to the Beck-Chevalley condition of Lemma 3.11.

Finally, if $\rho_2 = \rho_3$, Y = Y' and β is the identity on ρ_2 in $Fam_1(Vect)$, then $Y \times_{X_2} Y'$ is canonically equivalent to Y, and we see that Sum_1 sends units to units.

3.2 n=2

3.2.1 "Without thinking" (Nils)

Now let $\omega \in Z^2(G, \mathbb{C}^{\times})$. We will construct a 2-functor

$$\operatorname{Bord}_2 \xrightarrow{I} \operatorname{Fam}_2(\operatorname{Alg}_{\mathbb{C}}) \xrightarrow{\operatorname{Sum}_2} \operatorname{Alg}_{\mathbb{C}}$$
 (3.8)

by computing what it does to $\operatorname{pt}_+ \in \operatorname{Bord}_2$. The first step is to set $I(\operatorname{pt}_+)$ to be the 2-functor

$$\chi^{\omega} \colon \boldsymbol{B}G \longrightarrow \operatorname{Alg}_{\mathbb{C}} \quad \text{with} \quad \chi_{g,h}^{\omega} := \omega(g,h) \colon L_{g} \otimes_{\mathbb{C}} L_{h} \to L_{gh} \quad (3.9)$$

$$* \longmapsto \mathbb{C}$$

$$G \ni g \longmapsto L_{g} := {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$$

$$1_{g} \longmapsto 1_{\mathbb{C}}$$

(Recall from [Le, Sect. 1.1] that the constraint on the coherence 2-morphisms $\chi_{g,h}^{\omega} = \omega(g,h) \in \operatorname{End}_{\mathbb{C}}(\mathbb{C})$ is precisely the 2-cocycle condition on ω . Also note that all our 2-functors are strictly unital, so in particular (3.9) completely specifies the 2-functor χ^{ω} .)

The second step is to compute the 2-colimit $\operatorname{Sum}_2(\chi^{\omega})$. There are several versions of 2-limit, bilimit, weighted bilimit etc. in the literature. We will use the notion of [Bo, Def. 7.4.4] and call it 2-limit. To give the details, recall the definitions from [Le, Sect. 1.2–3], and that the *opposite* \mathcal{B}^{op} of a bicategory \mathcal{B} has reversed horizontal composition, but the same vertical composition as \mathcal{B} .

Definition 3.13. Let $F: \mathcal{A} \to \mathcal{B}$ be a 2-functor, and let $B \in \mathcal{B}$.

- (i) The constant 2-functor $\Delta_B \colon \mathcal{A} \to \mathcal{B}$ sends every object in \mathcal{A} to B, 1- and 2-morphisms in \mathcal{A} to 1_B and 1_{1_B} , respectively, and all constraints $(\Delta_B)_{g,h}$ are trivial.
- (ii) The category 2-Cone(B, F) of 2-cones on F with vertex B is the category whose objects are pseudonatural transformations $\Delta_B \to F$ and whose morphisms are modifications between them.
- (iii) A 2-limit of F is a pair (L, π) with $L \in \mathcal{B}$ and $\pi: \Delta_L \to F$ a pseudonatural transformation such that the functor

$$\mathcal{B}(B,L) \longrightarrow 2\text{-Cone}(B,F)$$

$$(B \xrightarrow{f} L) \longmapsto (\Delta_B \xrightarrow{f_*} \Delta_L \xrightarrow{\pi} F)$$

is an equivalence of categories for all $B \in \mathcal{B}$.

(iv) Now let $G: \mathcal{A} \to \mathcal{B}^{op}$. The category 2-Cocone(B, G) of 2-cocones on G with vertex $B \in \mathcal{B}$ is defined to be 2-Cone(B, G), but with B viewed as an object in \mathcal{B}^{op} . A 2-limit of $G: \mathcal{A} \to \mathcal{B}^{op}$ is also called a 2-colimit of $G: \mathcal{B}$.

Spelling out the definition, we see that an object $\sigma: \Delta_B \to F$ of 2-Cone(B, F) is a family of 1- and 2-morphisms

$$\left\{B \xrightarrow{\sigma_A} F(A)\right\}_{A \in \mathcal{A}} \quad \text{and} \quad \left\{F(g) \otimes \sigma_A \xrightarrow{\sigma_g} \sigma_{A'}\right\}_{g \in \mathcal{A}(A, A')}$$
 (3.10)

such that

$$F(g) \otimes F(h) \otimes \sigma_{A} \xrightarrow{1 \otimes \sigma_{h}} F(g) \otimes \sigma_{A'}$$

$$F_{g,h} \otimes 1 \downarrow \qquad \qquad \downarrow \sigma_{g}$$

$$F(gh) \otimes \sigma_{A} \xrightarrow{\sigma_{gh}} \sigma_{A''}$$

$$(3.11)$$

commutes for all $A \xrightarrow{h} A' \xrightarrow{g} A''$. Furthermore, a morphism $\Gamma \colon \sigma \to \widetilde{\sigma}$ in 2-Cone(B,F) is a family

$$\{\sigma_{A} \xrightarrow{\Gamma_{A}} \widetilde{\sigma}_{A}\}_{A \in \mathcal{A}} \quad \text{such that} \quad \begin{array}{c} F(g) \otimes \sigma_{A} \xrightarrow{1 \otimes \Gamma_{A}} F(g) \otimes \widetilde{\sigma}_{A} \\ \\ \sigma_{g} \downarrow & \downarrow \widetilde{\sigma}_{g} \\ \\ \sigma_{A'} \xrightarrow{\Gamma_{A'}} \widetilde{\sigma}_{A'} \end{array}$$
(3.12)

commutes for all $A \xrightarrow{g} A'$.

Note that above " \otimes " is exclusively used for the horizontal composition in the target \mathcal{B} (while juxtaposition is used for the horizontal composition in \mathcal{A}). Hence going to \mathcal{B}^{op} amounts to reversing all the tensor products above.

Lemma 3.14. Let $B \in Alg_{\mathbb{C}}$ and $\chi^{\omega} : BG \to Alg_{\mathbb{C}}$ as in (3.9). Then

$$2-\operatorname{Cone}(B,\chi^{\omega}) \cong_{\mathbb{C}^{\omega}[G]} \operatorname{Mod}_{B},$$

$$2-\operatorname{Cocone}(B,\chi^{\omega}) \cong {}_{B}\operatorname{Mod}_{\mathbb{C}^{\omega}[G]}$$
(3.13)

where $\mathbb{C}^{\omega}[G]$ is the ω -twisted group algebra of G.

Proof. We only prove the statement about 2-cones in detail, the one about 2-cocones follows analogously by reversing all horizontal compositions in \mathcal{B} .

By (3.10) and (3.11), an object in 2-Cone (B, χ^{ω}) is a 1-morphism $M \in Alg_{\mathbb{C}}(B, \mathbb{C})$ together with linear maps $\sigma_g \colon L_g \otimes_{\mathbb{C}} M \to M$ such that

$$L_{g} \otimes_{\mathbb{C}} L_{h} \otimes_{\mathbb{C}} M \xrightarrow{1 \otimes \sigma_{h}} L_{g} \otimes_{\mathbb{C}} M$$

$$\omega(g,h) \otimes 1 \downarrow \qquad \qquad \downarrow \sigma_{g}$$

$$L_{gh} \otimes_{\mathbb{C}} M \xrightarrow{\sigma_{gh}} M$$

$$(3.14)$$

commutes for all $g, h \in G$. But this means that $(M, \sum_{g \in G} \sigma_g)$ is a left $\mathbb{C}^{\omega}[G]$ module, while by definition of $\mathrm{Alg}_{\mathbb{C}}$, the 1-morphism $M \in \mathrm{Alg}_{\mathbb{C}}(B, \mathbb{C})$ is a right B-module.

According to (3.12), a morphism $(M, \{\sigma_g\}) \to (\widetilde{M}, \{\widetilde{\sigma}_g\})$ in 2-Cone (B, χ^{ω}) is a 2-morphism $\Gamma \colon {}_{\mathbb{C}}M_B \to {}_{\mathbb{C}}\widetilde{M}_B$ in $\mathrm{Alg}_{\mathbb{C}}$ such that

$$L_{g} \otimes_{\mathbb{C}} M \xrightarrow{1 \otimes \Gamma} L_{g} \otimes_{\mathbb{C}} \widetilde{M}$$

$$\sigma_{g} \downarrow \qquad \qquad \downarrow \widetilde{\sigma}_{g}$$

$$M \xrightarrow{\Gamma} \widetilde{M}$$

$$(3.15)$$

commutes. But that means that Γ is both a map of left $\mathbb{C}^{\omega}[G]$ -modules and a map of right B-modules. \square

Proposition 3.15. (i) A 2-limit of $\chi^{\omega} : \mathbf{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^{\omega}[G], \mathbb{C}^{\omega}[G])$.

(ii) A 2-colimit of $\chi^{\omega} \colon \mathbf{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is given by $(\mathbb{C}^{\omega}[G], \mathbb{C}^{\omega}[G])$.

Proof. According to Definition 3.13 and Lemma 3.14, a 2-limit of χ^{ω} is an algebra $L \in \text{Alg}_{\mathbb{C}}$ together with a module $\mathbb{C}^{\omega[G]}M_L$ such that the functor

$$\operatorname{Alg}_{\mathbb{C}}(B, L) \longrightarrow_{\mathbb{C}^{\omega}[G]} \operatorname{Mod}_{B}$$
$$N \longmapsto M \otimes_{L} N$$

is an equivalence for every $B \in \mathrm{Alg}_{\mathbb{C}}$. This is the case for $L = \mathbb{C}^{\omega}[G]$ as an algebra and $M = \mathbb{C}^{\omega}[G]$ as a bimodule over itself.

The statement about the 2-colimit follows analogously.

3.2.2 "With thinking" (Domenico)

The datum of the 2-functor $\chi^{\omega} \colon \mathbf{B}G \to \mathrm{Alg}_{\mathbb{C}}$ is a collection of lines (1-dimensional complex vector spaces) L_g , indexed by elements in the group G, together with isomorphisms

$$\rho_{g,h} \colon L_g \otimes L_h \xrightarrow{\sim} L_{gh}$$

subject to the associativity constraint given by the commutativity of the diagrams

$$L_{g} \otimes L_{h} \otimes L_{k} \xrightarrow{\rho_{g,h} \otimes \mathrm{id}} L_{gh} \otimes L_{k}$$

$$\downarrow^{\rho_{gh,k}} \qquad \qquad \downarrow^{\rho_{gh,k}}$$

$$L_{g} \otimes L_{hk} \xrightarrow{\rho_{g,hk}} L_{ghk}$$

where the tensor products are over \mathbb{C} . We also require that $L_1 = \mathbb{C}$ and that the isomorphisms $L_1 \otimes L_g \xrightarrow{\sim} L_g$ and $L_g \otimes L_1 \xrightarrow{\sim} L_g$ are the structure isomorphisms for the unit object \mathbb{C} of the monoidal category $\operatorname{Vect}_{\mathbb{C}}$. The associativity constraints imply, and are in fact equivalent to this, that the vector space

$$A_{\chi^{\omega}} = \bigoplus_{g \in G} L_g$$

has an associative C-algebra structure.

Lemma 3.16. A choice of a basis element x_g for every line L_g defines a \mathbb{C}^{\times} -valued 2-cocycle ω on the group G. A different choice of basis elements leads to a cohomologous cocycle, so that the class $[\omega] \in H^2(G, \mathbb{C}^{\times})$ is well defines.

Proof. As $\rho_{g,h}(x_g \otimes x_h)$ is a nonzero element in L_{gh} we have $\rho_{g,h}(x_g \otimes x_h) = \omega(g,h)x_{gh}$ for a unique element $\omega(g,h)$ in \mathbb{C}^{\times} . The associativity constraints are then immediately seen to be equivalent to the cocycle equation

$$\omega(gh, k)\omega(g, h) = \omega(g, hk)\omega(h, k).$$

Finally, if $\{y_g\}_{g\in G}$ is a different basis choice, and $\tilde{\omega}$ is the corresponding 2-cocycle, then we have $y_g = \eta_g x_g$ for some η_g in \mathbb{C}^{\times} and

$$\omega(g,h)y_{gh} = \rho_{g,h}(y_g \otimes y_h) = \eta_g \eta_h \rho_{g,h}(x_g \otimes x_h) = \eta_g \eta_h \omega(g,h) x_{gh} = \eta_g \eta_h \omega(g,h) \eta_{gh}^{-1} y_{gh}.$$
Therefore

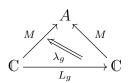
$$\tilde{\omega}(g,h) = \eta_g \eta_{gh}^{-1} \eta_h \omega(g,h),$$

i.e. ω and $\tilde{\omega}$ are cohomologous.

Corollary 3.17. The algebra $A_{\chi^{\omega}}$ is isomorphic to the twisted group algebra $\mathbb{C}^{\omega}[G]$, defined as the C-vector space on the basis $\{x_g\}_{g\in G}$ with the product $x_g \cdot x_h = \omega(g,h)x_{gh}$ and with $x_1 = 1$.

Lemma 3.18. Sum₂ $(\chi^{\omega}) \cong \mathbb{C}^{\omega}[G]$, the twisted group algebra of G.

Proof. Let (A, M) be a cocone for χ^{ω} , i.e., a pair consisting of a C-algebra A together with a left A-module M (representing a linear functor ${}_{\mathbb{C}}\mathrm{Mod} \to {}_{A}\mathrm{Mod}$) and homotopy commutative diagrams



where the λ_g 's are isomorphisms of left A-modules

$$\lambda_q \colon M \otimes_{\mathbb{C}} L_q \xrightarrow{\sim} M$$

such that the diagrams

$$M \otimes L_g \otimes L_h \xrightarrow{\lambda_g \otimes \mathrm{id}} M \otimes L_h$$

$$\downarrow^{\lambda_h} M \otimes L_{gh} \xrightarrow{\lambda_{gh}} M$$

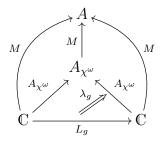
commute. This implies that the isomorphisms λ_g make M a right $A_{\chi^{\omega}}$ -module. Therefore, we can see M as a linear functor

$$_{A,\omega}\operatorname{Mod} \to {}_{A}\operatorname{Mod}$$

i.e., as a morphism from $A_{\chi^{\omega}}$ to A in 2-Vect_C. The algebra $A_{\chi^{\omega}}$ seen as a left $A_{\chi^{\omega}}$ -module is a morphism from $\mathbb C$ to $A_{\chi^{\omega}}$ in 2-Vect_C and the natural isomorphism

$$_{A}M_{\mathbb{C}} \cong {_{A}M_{A_{\chi^{\omega}}}} \otimes {_{A_{\chi^{\omega}}}A_{\chi^{\omega}}}_{\mathbb{C}}$$

gives a canonical factorization



exhibiting the algebra $A_{\chi^{\omega}}$ together with itself seen as a left module over itself as the universal cocone.

3.3 n=3

TODO:

- The only non-trivial part of the 3-functor $\chi^{\omega} \colon \boldsymbol{B}G \to \mathrm{TC}_{\mathbb{C}}$ are the coherence 3-morphisms which assemble into the modification (also) called ω in [Sc, Def. A.4.3], and the constraints on ω (see [Sc, Page 219]) seem to be precisely the 3-cocycle condition.
- <u>Issue</u>: The notion of a 3-colimit (to really compute Sum₃) is scary. At least we should make some hand-wavy arguments...

4 Boundary conditions

4.1 n=2

TODO

4.2 n = 3

TODO

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