

0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of \mathbb{Z} -posets (but just a \mathbb{Z} -equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a \mathbb{Z} -toset, i.e., a totally ordered set together with a monotone action of \mathbb{Z} , that we will denote by $(n, x) \mapsto x + n$. Recall that a J -slicing on a stable ∞ -category \mathcal{D} is a morphism of \mathbb{Z} -posests $\mathfrak{t}: \mathcal{O}(J) \rightarrow \text{ts}(\mathcal{D})$, where $\mathcal{O}(J)$ is the \mathbb{Z} -poset of the slicings of J (i.e., the decompositions of J into a disjoint union of a lower set L and of an upper set U), and $\text{ts}(\mathcal{D})$ is the \mathbb{Z} -poset of t -structures on \mathcal{D} .

Clearly, if $f: J \rightarrow J'$ is a morphism of \mathbb{Z} -tosets, then $f^{-1}: \{\text{subsets of } J'\} \rightarrow \{\text{subsets of } J\}$ induces a \mathbb{Z} -equivariant morphism of \mathbb{Z} -posets $f^{-1}: \mathcal{O}(J') \rightarrow \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$f_*: J\text{-slicings on } \mathcal{D} \rightarrow J'\text{-slicings on } \mathcal{D}$$

$$\mathfrak{t} \mapsto \mathfrak{t} \circ f^{-1}.$$

The slices of $f_*\mathfrak{t}$ are clearly given by $\mathcal{D}_{f_*\mathfrak{t};\phi} = \mathcal{D}_{\mathfrak{t};f^{-1}(\{\phi\})}$. Notice that, as f is monotone, the subset $f^{-1}(\{\phi\})$ is an interval in J . It is immediate to see that f_* restricts to a map

$$f_*: \text{Bridgeland } J\text{-slicings on } \mathcal{D} \rightarrow \text{Bridgeland } J'\text{-slicings on } \mathcal{D}$$

Namely, if $\mathcal{H}_{f_*\mathfrak{t}}^\phi(X) = 0$ for every $\phi \in J'$ then $\mathcal{H}_{\mathfrak{t}}^{f^{-1}(\{\phi\})}(X) = 0$ for every ϕ in J' . As we are assuming the J -slicing \mathfrak{t} is a Bridgeland slicing, this implies that $\mathcal{H}_{\mathfrak{t}}^\psi(X) = 0$ for every ψ in $f^{-1}(\{\phi\})$, for every ϕ . Therefore $\mathcal{H}_{\mathfrak{t}}^\psi(X) = 0$ for every ψ in J and so, again by definition of Bridgeland slicing, $X = 0$. Also, if $\mathcal{H}_{f_*\mathfrak{t}}^\phi(X) \neq 0$ then $\mathcal{H}_{\mathfrak{t}}^{f^{-1}(\{\phi\})}(X) \neq 0$ and so (again by the Bridgeland slicing condition) there exists at least an element ψ in $f^{-1}(\{\phi\})$ such that $\mathcal{H}_{\mathfrak{t}}^\psi(X) \neq 0$. As the J -slicing \mathfrak{t} is Bridgeland, the total of these ψ 's must be finite, so only for finitely many ϕ we can have such a ψ . In other words, the number of indices ϕ in J' such that $\mathcal{H}_{f_*\mathfrak{t}}^\phi(X) \neq 0$ is finite.

Notice that, if \mathfrak{t} is a Bridgeland slicing of \mathcal{D} , and $f: J \rightarrow J'$ is a morphism of \mathbb{Z} -tosets, then for any slicing (L, U) of J' , the lower and the upper categories $\mathcal{D}_{f_*\mathfrak{t}(L)}$ and $\mathcal{D}_{f_*\mathfrak{t}(U)}$ can be equivalently defined as

$$\mathcal{D}_{f_*\mathfrak{t}(L)} = \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$$

$$\mathcal{D}_{f_*\mathfrak{t}(U)} = \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}$$

where $\langle \mathcal{S} \rangle$ denotes the extension-closed subcategory of \mathcal{D} generated by the subcategory \mathcal{S} .¹

¹ At some point it will be useful to recall Lemma 4.21 from ???: Let $\mathcal{S}_1, \mathcal{S}_2$ be two subcategories of \mathcal{D} with $\mathcal{S}_1 \boxtimes \mathcal{S}_2$, i.e., such that $\mathcal{D}(X_1, X_2)$ is contractible for any $X_1 \in \mathcal{S}_1$ and any $X_2 \in \mathcal{S}_2$. Then $\mathcal{S}_1 \boxtimes \langle \mathcal{S}_2 \rangle$ and $\langle \mathcal{S}_1 \rangle \boxtimes \mathcal{S}_2$, and so $\langle \mathcal{S}_1 \rangle \boxtimes \langle \mathcal{S}_2 \rangle$.

The right hand sides of the above two expressions can clearly be defined for every morphism f from J to J' (i.e., not necessarily monotone nor \mathbb{Z} -equivariant), and as soon as f is \mathbb{Z} -equivariant, the assignment

$$(L, U) \mapsto (\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$$

is an equivariant morphism from $\mathcal{O}(J)$ to pairs of subcategories of \mathcal{D} . Clearly, when f is not monotone there is no reason to expect that the pair $(\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$ forms a t -structure on \mathcal{D} . Yet, it is interesting to notice that the condition that f be monotone is only sufficient in order to have this, and can indeed be relaxed.

Definition 0.1. *Let J and J' be \mathbb{Z} -to-sets, and let $f: J \rightarrow J'$ a map of \mathbb{Z} -sets (i.e., a \mathbb{Z} -equivariant map, not necessarily nondecreasing). A Bridgeland J -slicing \mathfrak{t} of \mathcal{D} is f -compatible if $f(\phi) > f(\psi)$ with $\phi \leq \psi$ implies $\mathcal{D}_{\mathfrak{t};\phi} \boxtimes \mathcal{D}_{\mathfrak{t};\psi}$ and $\mathcal{D}_{\mathfrak{t};\phi} \boxtimes \mathcal{D}_{\mathfrak{t};\psi}[1]$*

Remark 0.2. Clearly, if f is monotone, then every J -slicing \mathfrak{t} is order restoring as the condition ' $f(\phi) > f(\psi)$ with $\phi \leq \psi$ ' is empty.

Lemma 0.3. *Let J and J' be \mathbb{Z} -to-sets, and let $f: J \rightarrow J'$ be a \mathbb{Z} -equivariant morphism of \mathbb{Z} -sets (i.e., not necessarily a monotone map) and let \mathfrak{t} be a Bridgeland slicing of \mathcal{D} which is f -compatible. Then, for any slicing (L, U) of J' , the pair of subcategories $(\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$ is a t -structure on \mathcal{D} .*

Proof. As f is \mathbb{Z} -equivariant and $U + 1 \subseteq U$, we have

$$\begin{aligned} \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}[1] &= \langle \mathcal{D}_{\mathfrak{t};\phi}[1] \rangle_{f(\phi) \in U} \\ &= \langle \mathcal{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi) \in U} \\ &= \langle \mathcal{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi+1) \in U+1} \\ &= \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U+1} \\ &\subseteq \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U} \end{aligned}$$

and similarly for the lower subcategory $\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$. To show that

$$\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U} \boxtimes \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$$

it suffices to show that, if $f(\psi) \in L$ and $f(\phi) \in U$ then $\mathcal{D}_{\mathfrak{t};\phi} \boxtimes \mathcal{D}_{\mathfrak{t};\psi}$. As $L \cap U = \emptyset$ we cannot have $\psi = \phi$, so either $\psi < \phi$ or vice versa. In the first case, $\mathcal{D}_{\mathfrak{t};\phi} \boxtimes \mathcal{D}_{\mathfrak{t};\psi}$ by definition of Bridgeland J -slicing. In the second case, we have $\phi < \psi$ and $f(\phi) > f(\psi)$ as $f(\phi) \in U$ and $f(\psi) \in L$. Therefore, since \mathfrak{t} is f -compatible, $\mathcal{D}_{\mathfrak{t};\phi} \boxtimes \mathcal{D}_{\mathfrak{t};\psi}$. Finally, we have to show that every object X in \mathcal{D} fits into a fiber sequence

$$\begin{array}{ccc} X_U & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X_L \end{array}$$

with $X_L \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}$ and $X_U \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}$. As \mathbf{t} is a Bridgeland slicing, we have a factorization of the initial morphism $\mathbf{0} \rightarrow X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{i}}} X_{\bar{i}} \xrightarrow{\alpha_{\bar{i}+1}} X_{\bar{i}+1} \rightarrow \cdots \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \text{cofib}(\alpha_i) = \mathcal{H}_{\mathbf{t}}^{\phi_i}(X) \in \mathcal{D}_{\phi_i}$ for all $i = 1, \dots, n$, with $\phi_i > \phi_{i+1}$. Let us now consider the sequence of symbols L and U obtained putting in the i -th place L if $f(\phi_i) \in L$ and U if $f(\phi_i) \in U$. If this sequence is of the form $(U, U, \dots, U, L, L, \dots, L)$, then there exists an index \bar{i} such that $f(\phi_i) \in U$ for $i \leq \bar{i}$ and $f(\phi_i) \in L$ for $i > \bar{i}$ (with $\bar{i} = -1$ or n when all of the $f(\phi_i)$ are in L or in U , respectively). Then we can consider the pullout diagram

$$\begin{array}{ccc} X_{\bar{i}} & \xrightarrow{\quad} & \mathbf{0} \\ f_L \downarrow & & \downarrow \\ X & \xrightarrow{\quad} & \text{cofib}(f_L) \end{array}$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{i}}} X_{\bar{i}}$$

and

$$X_{\bar{i}} \xrightarrow{\alpha_{\bar{i}+1}} X_{\bar{i}+1} \rightarrow \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that $X_{\bar{i}} \in \langle \cup_{i=0}^{\bar{i}} \mathcal{D}_{\phi_i} \rangle \subseteq \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}$ while the second factorization shows that $\text{cofib}(f_L) \in \langle \cup_{i=\bar{i}+1}^n \mathcal{D}_{\phi_i} \rangle \subseteq \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}$. So we are done in this case. Therefore, we are reduced to showing that we can always avoid a (\dots, L, U, \dots) situation in our sequence of L 's and U 's. Assume we have such a situation. Then we have an index i_0 with $f(\phi_{i_0}) \in L$ and $f(\phi_{i_0+1}) \in U$. This in particular implies $f(\phi_{i_0+1}) > f(\phi_{i_0})$ with $\phi_{i_0+1} < \phi_{i_0}$. As \mathbf{t} is f -compatible, this gives $\mathcal{D}_{\mathbf{t};\phi_{i_0+1}} \boxtimes (\mathcal{D}_{\mathbf{t};\phi_{i_0}}[1])$. In particular, $\mathcal{D}(\mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X)[1])$ is contractible. Now consider the pasting of pullout diagrams

$$\begin{array}{ccccccc} X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\ \downarrow & & \downarrow & & \downarrow \gamma & & \\ \mathbf{0} & \longrightarrow & \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X) & \longrightarrow & Y & \longrightarrow & \mathbf{0} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{0} & \longrightarrow & \mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X)[1]. \end{array}$$

As the arrow $\mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X) \rightarrow \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X)[1]$ factors through $\mathbf{0}$, we have $Y = \mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X) \oplus$

$\mathcal{H}_t^{\phi_{i_0}}(X)$ and the above diagram becomes

$$\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X) & \xrightarrow{\iota_2} & \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \pi_1 & & \downarrow \\
& & 0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X)[1],
\end{array}$$

where ι_2 and π_1 are the canonical inclusion and projection. Let $\beta: X_{i_0+1} \rightarrow \mathcal{H}_t^{\phi_{i_0}}(X)$ be the composition

$$\beta: X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_t^{\phi_{i_0}}(X).$$

Then we have a homotopy commutative diagram

$$\begin{array}{ccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} \\
\downarrow & & \downarrow & & \downarrow \beta \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X) & \xrightarrow{\text{id}} & \mathcal{H}_t^{\phi_{i_0}}(X)
\end{array}$$

(where only the left square is a pullout), and so the composition $\alpha_{i_0+1} \circ \alpha_{i_0}$ factors through the homotopy fiber of β . In other words, we have a homotopy commutative diagram

$$\begin{array}{ccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} \\
\tilde{\alpha}_{i_0} \downarrow & & \downarrow \alpha_{i_0+1} \\
\text{fib}(\beta) & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1}
\end{array}$$

Writing $\tilde{X}_{i_0} = \text{fib}(\beta)$, we get the pasting of pullout diagrams

$$\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\tilde{\alpha}_{i_0}} & \tilde{X}_{i_0} & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0+1}}(X) & \xrightarrow{\iota_1} & \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) & \longrightarrow & \\
& & \downarrow & & \downarrow \pi_2 & & \\
& & 0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X). & &
\end{array}$$

That is, by considering the factorization $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$ we have switched the cofibers with respect to the original factorization $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$. Therefore, writing $\tilde{\phi}_{i_0} = \phi_{i_0+1}$ and $\tilde{\phi}_{i_0+1} = \phi_{i_0}$, we now have $f(\tilde{\phi}_{i_0}) \in U$ and $f(\tilde{\phi}_{i_0+1}) \in L$. That is, we have removed the (\dots, L, U, \dots) situation from the position i_0 , replacing it with a (\dots, U, L, \dots) situation, while keeping all the labels L, U before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the (\dots, L, U, \dots) situations.² \square

Remark 0.4. It follows from the proof of Lemma 0.3 that the objects X_L and X_U in the fiber sequence $X_U \rightarrow X \rightarrow X_L$ associated with the t -struture $(\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U})$ on \mathcal{D} satisfy

$$\begin{aligned} X_L &\in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L \text{ and } \mathcal{H}_{\mathbf{t}}^\phi(X) \neq 0} \\ X_U &\in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U \text{ and } \mathcal{H}_{\mathbf{t}}^\phi(X) \neq 0} \end{aligned}$$

Lemma 0.5. *For any $j \in J'$ we have*

$$\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi)=j} = \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \geq j}.$$

Proof. Clearly, $\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi)=j} \subseteq \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \geq j}$, therefore we only need to prove the converse inclusion. Let $X \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \geq j}$. Then in particular $X \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \geq j}$ and so there exists a factorization of the initial morphism $\mathbf{0} \rightarrow X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \text{cofib}(\alpha_i) \in \mathcal{D}_{\phi_i}$ with $f(\phi_i) \geq i$ for all $i = 1, \dots, n$. As $((-\infty, j], (j, +\infty))$ is a slicing of J' , reasoning as in the proof of Lemma 0.3 we can arrange this factorization is such a way that $f(\phi_i) > j$ for $i \leq \bar{i}$ and $f(\phi_i) = j$ for $i > \bar{i}$. Therefore, again by reasoning as in the proof of Lemma 0.3 we get a fiber sequence of the form

$$\begin{array}{ccc} X_{>j} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbf{0} & \longrightarrow & X_j \end{array}$$

with $X_{>j}$ in $\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) > j}$ and X_j in $\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi)=j}$. This is in particular a fiber sequence of the form $X_{>j} \rightarrow X \rightarrow X_{\leq j}$, with $X_{\leq j} \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \leq j}$. As $(\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \leq j}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) > j})$ is a t -structure on \mathcal{D} by Lemma 0.3, there is (up to equivalence) only one such a fiber sequence. And since $X \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \leq j}$, this is the sequence $\mathbf{0} \rightarrow X \xrightarrow{\text{id}} X$. Therefore $X = X_j$. \square

Proposition 0.6. *Let $f: J \rightarrow J'$ be a \mathbb{Z} -equivariant morphism of \mathbb{Z} -sets (i.e., not necessarily a monotone map) and let \mathbf{t} be a Bridgeland slicing of \mathcal{D} which is f -compatible. The map*

$$\begin{aligned} f_! \mathbf{t}: \mathcal{O}(J') &\rightarrow \text{ts}(\mathcal{D}) \\ (L, U) &\mapsto (\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}) \end{aligned}$$

²This is somehow reminiscent of the ‘bubble sort’ algorithm.

defined by Lemma 0.3 is a Bridgeland J' -slicing of \mathcal{D} , with slices given by

$$\mathcal{D}_{f_!t;j} = \langle \mathcal{D}_{t;\phi} \rangle_{f(\phi)=j}.$$

Proof. The map $f_!t$ is manifestly monotone and \mathbb{Z} -equivariant (see the first part of the proof of Lemma 0.3), so it is a J' slicing of \mathcal{D} , and its slices are given by $\mathcal{D}_{f_!t;j} = \langle \mathcal{D}_{t;\phi} \rangle_{f(\phi)=j}$ by Lemma 0.5. We are therefore left with showing that it is finite and discrete. Given an object X in \mathcal{D} , let now $\{\phi_1, \dots, \phi_n\}$ be the indices in J such that $\mathcal{H}_t^{\phi_i}(X) \neq 0$ and let $\{j_1, \dots, j_k\}$ the image of the set $\{\phi_1, \dots, \phi_n\}$ via f . Up to renaming, we can assume $j_1 > j_2 > \dots > j_k$. Consider now the factorization

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{k-1}} X_{k-1} \xrightarrow{\alpha_k} X_k = X$$

of the initial morphism of X associated to the decreasing sequence $j_1 > j_2 > \dots > j_k$ by the J' -slicing $f_!t$. The cofibers of the morphisms α_i are the cohomologies $\mathcal{H}_{f_!t}^{(j_{i+1}, j_i]}(X)$ and, by Remark 0.4, we have

$$\begin{aligned} \mathcal{H}_{f_!t}^{(j_{i+1}, j_i]}(X) &\in \langle \mathcal{D}_{t;\phi}; \quad f(\phi) \in (j_{i+1}, j_i] \text{ and } \mathcal{H}_t^\phi(X) \neq 0 \rangle \\ &= \langle \mathcal{D}_{t;\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, j_i] \rangle \\ &= \langle \mathcal{D}_{t;\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) = j_i \rangle \\ &\subseteq \langle \mathcal{D}_{t;\phi}; \quad f(\phi) = j_i \rangle = \mathcal{D}_{f_!t;j_i}. \end{aligned}$$

Therefore

$$\mathcal{H}_{f_!t}^{(j_i, j_{i-1}]}(X) = \mathcal{H}_{f_!t}^{j_i} \mathcal{H}_{f_!t}^{(j_i, j_{i-1}]}(X) = \mathcal{H}_{f_!t}^{j_i}(X).$$

This tells us that, if all the cohomologies $\mathcal{H}_{f_!t}^j(X)$ vanish, then also all the $\mathcal{H}_{f_!t}^{(j_i, j_{i-1}]}(X)$ vanish, and so $X = 0$. Finally, if $\tilde{j} \notin \{j_1, \dots, j_k\}$, then there exists an index i such that $j_{i+1} < \tilde{j} < j_i$ and so $(j_{i+1}, \tilde{j}] \cap \{j_1, \dots, j_k\} = \emptyset$. The above argument then shows that

$$\mathcal{H}_{f_!t}^{(j_{i+1}, \tilde{j}]}(X) \in \langle \mathcal{D}_{t;\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, \tilde{j}] \rangle = \{\mathbf{0}\}.$$

It follows that

$$\tilde{\mathcal{H}}_{f_!t}^{\tilde{j}}(X) = \tilde{\mathcal{H}}_{f_!t}^{\tilde{j}} \mathcal{H}_{f_!t}^{(j_{i+1}, \tilde{j}]}(X) = \tilde{\mathcal{H}}_{f_!t}^{\tilde{j}}(\mathbf{0}) = 0,$$

for every $\tilde{j} \notin \{j_1, \dots, j_k\}$. So in particular, for any $X \in \mathcal{D}$, the cohomologies $\mathcal{H}_{f_!t}^j(X)$ are possibly nonzero only for finitely many indices j . \square

Let now J_1 and J_2 be two \mathbb{Z} -tosets, and let $J_1 \times_{\text{lex}} J_2$ and $J_2 \times_{\text{lex}} J_1$ the two \mathbb{Z} -tosets obtained by considering the lexicographic order on the products $J_1 \times J_2$ and $J_2 \times J_1$, respectively, and the diagonal \mathbb{Z} action. The following lemma is immediate.

Lemma 0.7. *The exchange map*

$$\begin{aligned} e: J_1 \times J_2 &\rightarrow J_2 \times J_1 \\ (j_1, j_2) &\mapsto (j_2, j_1) \end{aligned}$$

is \mathbb{Z} -equivariant.

Definition 0.8. *Let J_1 and J_2 be \mathbb{Z} -tosets. A Bridgeland $J_1 \times_{\text{lex}} J_2$ -slicing \mathfrak{t} of \mathcal{D} is said to be gluable if it is e -compatible, where $e: J_1 \times J_2 \rightarrow J_2 \times J_1$ is the exchange map.*

Example 0.9. Let $J_1 = \{0, 1\}$ with the trivial \mathbb{Z} -action and let $J_2 = \mathbb{Z}$ with the standard translation \mathbb{Z} -action. Then a $J_1 \times_{\text{lex}} J_2$ -slicing \mathfrak{t} on \mathcal{D} is the datum of a semiorthogonal decomposition $(\mathcal{D}_0, \mathcal{D}_1)$ of \mathcal{D} together with t -structures \mathfrak{t}_i on \mathcal{D}_i . Spelling out the definition, one sees that the $\{0, 1\} \times_{\text{lex}} \mathbb{Z}$ -slicing \mathfrak{t} is gluable if and only if $\mathcal{D}_{0;\geq 0} \boxtimes \mathcal{D}_{1;-1}$ and $\mathcal{D}_{0;\geq 0} \boxtimes \mathcal{D}_{1;0}$. When this happens, the glued slicing $e!\mathfrak{t}$ is a $\mathbb{Z} \times_{\text{lex}} \{0, 1\}$ slicing on \mathcal{D} , i.e., is the datum of a bounded t -structure on \mathcal{D} together with a torsion theory on its heart. More precisely, the heart of $e!\mathfrak{t}$ is the extension-closed ∞ -subcategory $\mathcal{D}^{\heartsuit_{e!\mathfrak{t}}}$ of \mathcal{D} generated by the hearts of \mathcal{D}_0 and \mathcal{D}_1 , while the torsion theory on $\mathcal{D}^{\heartsuit_{e!\mathfrak{t}}}$ is $(\mathcal{D}_0^{\heartsuit}, \mathcal{D}_1^{\heartsuit})$, i.e., precisely the pair of hearts of the two subcategories in the semiorthogonal decomposition.

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Remark 0.10. In [ref: BBD](#) the authors consider a process consisting in gluing t -structures on a pair of triangulated categories to obtain a t -structure on a third one. In their setting, the three categories are linked by an abstraction of the Grothendieck's six-functors formalism. However, the latter is more or less equivalent to the datum of a semiorthogonal decomposition (i.e., a $\{0, 1\}$ -slicing, where $\{0, 1\}$ is the toset with two elements and trivial \mathbb{Z} -action) on the third category so that the input of the two (bounded) t -structures gives rise to a gluable $\{0, 1\} \times_{\text{lex}} \mathbb{Z}$ -slicing \mathfrak{t} . Then the $\mathbb{Z} \times_{\text{lex}} \{0, 1\}$ -slicing $e!\mathfrak{t}$ is just the new glued t -structure together with the natural torsion pair on its heart. This explains our terminology and links the language of the present paper to the classical theory of t -structures.

To motivate the next section, consider the following isomorphism fo \mathbb{Z} -tosets:

$$\begin{aligned} \mathbb{Z} \times_{\text{lex}} \mathbb{Z} &\rightarrow \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{free}} \\ (n, m) &\mapsto (n, m - n) \end{aligned}$$

...continua ...

Lemma 0.11. *Let $p: J \rightarrow \mathcal{O}(J')^{\text{op}}$ be a monotone map. Then*

$$\Gamma(p) = \{(n, j) \in J \times J'; \quad j \in U_{p(n)}\}.$$

is an upper set of $J \times J'$ with the product order.

Proof. Pick $(n, j) \leq (m, k)$ in $J \times J'$ and suppose $(n, j) \in \Gamma(p)$. Then $j \in p(n) \in \mathcal{O}(J')$ and since $j \leq k$ we have that $k \in p(n)$. But $n \leq m$ and p is monotone, and thus $p(n) \subseteq p(m)$. This gives $k \in p(m)$, as desired. \square

Proposition 0.12. *The map*

$$\begin{aligned} \Gamma: \text{Pos}(J, \mathcal{O}(J')^{\text{op}}) &\rightarrow \mathcal{O}(J \times J') \\ p &\mapsto (J \times J' \setminus \Gamma(p), \Gamma(p)) \end{aligned}$$

is an isomorphism of \mathbb{Z} -posets, where $J \times J'$ carries the product order.

Proof. Pick and upper set U of $J \times J'$ and set, for $n \in J$

$$p_U(n) = \{j \in J'; \quad (n, j) \in U\}.$$

Then the map

$$\begin{aligned} \mathcal{O}(J \times J') &\rightarrow \text{Pos}(J, \mathcal{O}(J')^{\text{op}}) \\ (L, U) &\mapsto p_U \end{aligned}$$

is inverse to $\Gamma(p)$ by a straightforward check. \square

Definition 0.13. *A perversity on J is a monotone map $p: \mathbb{Z} \rightarrow \mathcal{O}(J)^{\text{op}}$ such that*

$$p(n) - n \leq p(m) - m$$

whenever $m \leq n$ in \mathbb{Z} . The set of all perversities on J will be denoted $\Theta(J)$.

Remark 0.14. We can give $\Theta(J)$ the structure of a \mathbb{Z} -poset via the inclusion $\Theta(J) \subseteq \text{Pos}(\mathbb{Z}, \mathcal{O}(J)^{\text{op}})$. In general, it will not be totally ordered. However, by taking joins and meets pointwise $\Theta(J)$ becomes a distributive lattice.

Remark 0.15. Considering the obvious isomorphism $\mathbb{Z} \simeq \mathcal{O}(\mathbb{Z})^{\text{op}}$ a perversity over \mathbb{Z} is just a *monotone and comonotope perversity* in the sense of [referenza](#), [tipo bezrukavnikov o BBD](#).

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Definition 0.16. *Let \mathfrak{t} be a bounded t -structure on \mathcal{D} . An abelian \mathbb{Z} -slicing \mathcal{P} on $\heartsuit_{\mathfrak{t}}$ is called:*

- **perverse filtration** if $\mathcal{P}_{\psi}[n] \subseteq \mathcal{P}_{\phi}^{\perp}$ for $n < \phi - \psi$
- **grading filtration** if it is a perverse filtration and $\mathcal{P}_{\psi}[n] \subseteq \mathcal{P}_{\phi}^{\perp}$ for $2 \leq n = \phi - \psi$
- **mixed filtration** if $\mathcal{P}_{\psi}[n] \subseteq \mathcal{P}_{\phi}^{\perp}$ for $n > \psi - \phi$

The following implications hold:

$$\text{mixed} \implies \text{grading} \implies \text{perverse}$$

proof that mixed implies grading: suppose $n < \phi - \psi$. By definition of slicing, the vanishing is automatic for $n \leq 0$ and we can thus assume $n > 0$. But then $n > 0 > -n > \psi - \phi$ and the required vanishing now follows from the definition of mixed filtration. The same argument also works for $2 \leq n = \phi - \psi$.

Clearly, a torsion pair on \mathcal{V}_t (seen as an abelian \mathbb{Z} -slicing via the inclusion $[1] \subseteq \mathbb{Z}$) is always a grading filtration (proof: it follows from the fact that $\mathcal{P}_\phi[n] = 0$ for $\phi \neq 0, 1$ and thus the only nontrivial case is $n = 0, \phi = 1, \psi = 0$. The Hom-vanishing then follows from the definition of torsion pair). Historically, grading filtrations first appeared in [?] under the name of 'radical filtrations', while mixed filtrations are covered in the last section of [?].

Definition 0.17. A map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is called **perversity** if it is a monotone contraction, i.e. if

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi$$

whenever $\phi \geq \psi$ in \mathbb{Z} .

We denote Ξ the set of perversities.

Clearly, Ξ defines a \mathbb{Z} -poset which is not totally ordered. However, it is a distributive lattice: we have meets and joins given by:

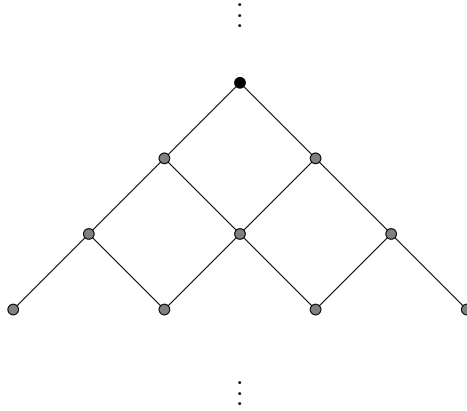
$$(p \wedge q)(\phi) = \min\{p(\phi), q(\phi)\}$$

$$(p \vee q)(\phi) = \max\{p(\phi), q(\phi)\}$$

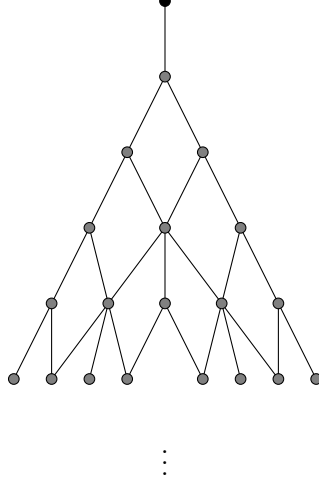
Indeed, if we place on $\mathbb{Z} \times \mathbb{Z}$ the product order, we have an isomorphism of \mathbb{Z} -posets

$$\Xi^{\text{op}} = O(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

given by sending the zero perversity to $A = \{(\phi, \psi)\}_{\phi \geq 0}$ and the identity to $B = \{(\phi, \psi)\}_{\psi \geq 0}$ ($B + i$ and $A + i$ with $i \in \mathbb{Z}$ generate the latter as a complete lattice). Considering the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Z} \subseteq O(\mathbb{Z} \times \mathbb{Z})$ we get a sublattice of Ξ depicted as



This induces the 't-tree' described in [?], as discussed below in a far-reaching generality. By the other hand, each square above contains moreover a copy of the Young lattice



with the root corresponding to the upper vertex of the square.

Proposition 0.18. *Let \mathfrak{t} be a bounded t-structure on \mathcal{D} , \mathcal{P} an abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$, \mathcal{Q} the associated $\mathbb{Z} \times \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , p a perversity. We have:*

1. *if \mathcal{P} is a perverse filtration, then \mathcal{Q} satisfies the assumptions of **Proposition ??** with respect to the map $\mathbb{Z} \times \hat{\mathbb{Z}} \xrightarrow{f_p} \mathbb{Z} \times \hat{\mathbb{Z}}$ given by*

$$f_p(n, \phi) = (n + p(\lfloor \phi/2 \rfloor), -p(\lfloor \phi/2 \rfloor))$$

2. *if \mathcal{P} is a grading filtration, then \mathcal{Q} satisfies the assumptions of **Proposition ??** with respect to the map $\mathbb{Z} \times \hat{\mathbb{Z}} \xrightarrow{g_p} \mathbb{Z} \times \hat{\mathbb{Z}}$ given by*

$$g_p(n, \phi) = (n + p(\phi), -p(\phi))$$

3. *if \mathcal{P} is a mixed filtration, then the abelian \mathbb{Z} -slicing induced on the heart of the bounded t-structure associated to $(g_p)_{\Omega}(\mathcal{Q})$ is split*

Proof. Let's prove (2). Suppose $g_p(n, \phi) > g_p(m, \psi)$. Then either $m - n < p(\phi) - p(\psi)$ or both $m - n = p(\phi) - p(\psi)$ and $p(\psi) < p(\phi)$. Since by definition of t-structure we can assume $m \geq n$, the second case is absurd while in the first case, since p is monotone, we have $\phi \geq \psi$ and thus by definition of perversity

$$m - n < p(\phi) - p(\psi) \leq \phi - \psi$$

and we get the desired Hom-vanishing by definition of grading filtration.

Suppose now $f_p(n, \phi) + 1 > f_p(m, \psi)$ and $(n + 1, \phi) < (m, \psi)$. The only non absurd case is $1 < m - n \leq p(\phi) - p(\psi)$. But then again we have

$$2 \leq m - n \leq p(\phi) - p(\psi) \leq \phi - \psi$$

and we can conclude as above.

To prove (1) consider the monotone map

$$\mathbb{Z} \xrightarrow{\lfloor \cdot / 2 \rfloor} \mathbb{Z}$$

Applying the slice functor to the latter and starting with a perverse filtration, we get a grading filtration by the properties of the floor function and the thesis follows from part (2).

Let's prove (3). We have to show that

$$\mathcal{P}_\psi[1 - p(\psi)] \subseteq \mathcal{P}_\phi[p(\phi)]^\perp$$

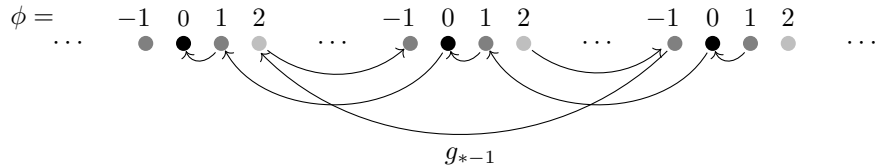
for $-p(\psi) > -p(\phi)$. We denote $n = p(\phi) - p(\psi) - 1$. Assuming again $n \geq 0$, we have $n \geq 0 > p(\psi) - p(\phi) \geq \psi - \phi$ and we can then conclude by definition of mixed filtration. \square

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity p the bounded t-structure coming from $(g_p)_\Omega(\mathcal{Q})$ defines a morphism of \mathbb{Z} -posets

$$\Xi^{\text{op}} \longrightarrow \mathbf{bts}(\mathcal{Q})$$

we can restate this as:

♫ *A grading or perverse filtration on the heart of a bounded t-structure on \mathcal{Q} induces a presheaf of t-structures on $\mathbb{Z} \times \mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z} \times \mathbb{Z}$ -slicing' on \mathcal{Q} .*



Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\text{op}} \hookrightarrow \Xi$$

and the t-structure coming from $(g_{\chi_I})_\Omega(\mathcal{Q})$ is just the tilting of \mathbf{t} with respect to the torsion pair coming from I .

The following proposition gives a characterization of the new heart obtained

by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 0.19. *Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , \mathcal{P} a grading filtration on $\mathfrak{H}_{\mathfrak{t}}$, \mathcal{Q} the associated $\mathbb{Z} \times \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , p a perversity. Denote \mathfrak{q} the bounded t -structure associated to $(g_p)_{\Omega}(\mathcal{Q})$. Then $\mathfrak{H}_{\mathfrak{q}}$ consists of objects $X \in \mathcal{D}$ so that*

$$H_{\mathfrak{t}}^k(X) \in \mathcal{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if \mathcal{P} is a mixed filtration then for each $X \in \mathfrak{H}_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \mathfrak{H}_{\mathfrak{q}}$ if and only if

$$H_{\mathcal{D}}^{\phi}(H_{\mathfrak{t}}^k(X)[-k])[k] = H_{\mathcal{Q}}^{(k, \phi)}(X) = 0$$

for $p(\phi) \neq -k$.

For the second part of the claim, the abelian \mathbb{Z} -slicing induced on $\mathfrak{H}_{\mathfrak{q}}$ is split by **Proposition 0.18** and thus by **Proposition ??**

$$X = \bigoplus_{(k, \phi)} H_{\mathcal{Q}}^{(k, \phi)}(X) = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

where the last equality comes from the first part. □

Example 0.20. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an \mathbb{N} -graded ring with B_0 semisimple. Denote \mathcal{A} the category of \mathbb{Z} -graded B -modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote \mathcal{P}_{ϕ} the full subcategory of \mathcal{A} of modules concentrated in degree ϕ . Clearly, \mathcal{P} defines an abelian \mathbb{Z} -slicing on \mathcal{A} . Following [?] we have

$$\mathrm{Ext}_{\mathcal{A}}^n(\mathcal{P}_{\phi}, \mathcal{P}_{\psi}) = 0$$

for $n > \psi - \phi$. This means that \mathcal{P} is a mixed filtration and the bounded t -structure on $\mathcal{D}^b(\mathcal{A})$ associated to $(g_1)_{\Omega}(\mathcal{Q})$ (where 1 is the identity of \mathbb{Z}) is the 'diagonal' (or 'geometric') t -structure which appears in Koszul duality and other areas.

Example 0.21. Let M be an n -dimensional smooth complex projective variety and consider the n -torsion pair \mathcal{P} on $\mathrm{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that \mathcal{P} , seen as an abelian \mathbb{Z} -slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t -structure associated to $(f_p)_{\Omega}(\mathcal{Q})$ is the one of perverse coherent sheaves as

constructed in [?]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n -slicing on the heart of perverse coherent sheaves as done in [?].

Now we somehow review the gluing construction for t-structures in [?], but generalize it to any slicing. Our language is quite different though, and we formulate the problem very similarly to [?]. We start with two \mathbb{Z} -posets J, J' .

Definition 0.22. *Let \mathcal{P} be a $J \ltimes J'$ -slicing on \mathcal{D} . We call \mathcal{P} **gluable** if it satisfies the assumptions of **Proposition ??** with respect to the map $J \ltimes J' \xrightarrow{e} J' \ltimes J$ that exchanges coordinates. In this case we denote*

$$\overline{\mathcal{P}} = e_{\Omega}(\mathcal{P})$$

By a simple computation, the gluability condition reads: $\mathcal{P}_{(\phi, \psi)} \subseteq \mathcal{P}_{(\phi', \psi')}^{\perp}$ if $\psi' > \psi$ or both $\psi' + 1 > \psi$ and $\phi' + 1 < \phi$. For example, the first condition is automatic when \mathbb{Z} acts trivially on J (in this case, it follows from the second one). In particular, if \mathbb{Z} acts trivially on J , \mathcal{P} is a J -slicing on \mathcal{D} and t_{ϕ} is a bounded t-structure on \mathcal{P}_{ϕ} for each $\phi \in J$, then using **Proposition ??** we get a $J \ltimes \mathbb{Z}$ -slicing \mathcal{Q} which is gluable if and only if

$$\heartsuit_{t_{\psi}}[n] \subseteq \heartsuit_{t_{\phi}}^{\perp}$$

whenever both $n \leq 0$ and $\phi < \psi$.

Remark 0.23. We have a commutative diagram of \mathbb{Z} -posets

$$\begin{array}{ccc} \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \\ \downarrow e & & \downarrow g \\ \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \end{array}$$

where the horizontal isomorphism is the one from **Remark ??**. Indeed, a $\mathbb{Z} \ltimes \mathbb{Z}$ -slicing on \mathcal{D} is gluable if and only if it induces a grading filtration on the heart of the associated t-structure when seen as a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing.

An easy combinatorial calculation finally yields the following two remarks:

- if \mathcal{P} is a gluable $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$ -slicing on \mathcal{D} , then $\overline{\mathcal{P}}$ induces a grading filtration on the associated heart, allowing us again to construct new bounded t-structures depending on a perversity.
- if \mathcal{P} is a mixed filtration on the heart of a bounded t-structure and \mathcal{Q} is the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , then \mathcal{Q} is gluable. In this case, looking at $\overline{\mathcal{Q}}$, we get a baric structure on \mathcal{D} which is usually called **weight decomposition** in literature.

In other words, the chain of implications for a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing refines to

$$\text{mixed} \implies \text{gluable} \implies \text{grading} \implies \text{perverse}$$

