

0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of \mathbb{Z} -posets (but just a \mathbb{Z} -equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a \mathbb{Z} -toset, i.e., a totally ordered set together with a monotone action of \mathbb{Z} , that we will denote by $(n, x) \mapsto x + n$. Recall that a J -slicing on a stable ∞ -category \mathcal{D} is a morphism of \mathbb{Z} -posests $\mathfrak{t}: \mathcal{O}(J) \rightarrow \text{ts}(\mathcal{D})$, where $\mathcal{O}(J)$ is the \mathbb{Z} -poset of the slicings of J (i.e., the decompositions of J into a disjoint union of a lower set L and of an upper set U), and $\text{ts}(\mathcal{D})$ is the \mathbb{Z} -poset of t -structures on \mathcal{D} .

Clearly, if $f: J \rightarrow J'$ is a morphism of \mathbb{Z} -tosets, then $f^{-1}: \{\text{subsets of } J'\} \rightarrow \{\text{subsets of } J\}$ induces a \mathbb{Z} -equivariant morphism of \mathbb{Z} -posets $f^{-1}: \mathcal{O}(J') \rightarrow \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$\begin{aligned} f_*: J\text{-slicings on } \mathcal{D} &\rightarrow J'\text{-slicings on } \mathcal{D} \\ \mathfrak{t} &\mapsto \mathfrak{t} \circ f^{-1}. \end{aligned}$$

The slices of $f_*\mathfrak{t}$ are clearly given by $\mathcal{D}_{f_*\mathfrak{t};j} = \mathcal{D}_{\mathfrak{t};f^{-1}(\{j\})}$, for any $j \in J'$. Notice that, as f is monotone, the subset $f^{-1}(\{j\})$ is an interval in J . It is immediate to see that f_* restricts to a map

$$f_*: \text{Bridgeland } J\text{-slicings on } \mathcal{D} \rightarrow \text{Bridgeland } J'\text{-slicings on } \mathcal{D}$$

Namely, if $\mathcal{H}_{f_*\mathfrak{t}}^j(X) = 0$ for every $j \in J'$ then $\mathcal{H}_{\mathfrak{t}}^{f^{-1}(\{j\})}(X) = 0$ for every j in J . As we are assuming the J -slicing \mathfrak{t} is a Bridgeland slicing, this implies that $\mathcal{H}_{\mathfrak{t}}^\phi(X) = 0$ for every ϕ in $f^{-1}(\{j\})$, for every j . Therefore $\mathcal{H}_{\mathfrak{t}}^\phi(X) = 0$ for every ϕ in J and so, again by definition of Bridgeland slicing, $X = 0$. Also, if $\mathcal{H}_{f_*\mathfrak{t}}^j(X) \neq 0$ then $\mathcal{H}_{\mathfrak{t}}^{f^{-1}(\{j\})}(X) \neq 0$ and so (again by the Bridgeland slicing condition) there exists at least an element ϕ in $f^{-1}(\{j\})$ such that $\mathcal{H}_{\mathfrak{t}}^\phi(X) \neq 0$. As the J -slicing \mathfrak{t} is Bridgeland, the total of these ϕ 's must be finite, so only for finitely many j we can have such a ϕ . In other words, the number of indices j in J' such that $\mathcal{H}_{f_*\mathfrak{t}}^j(X) \neq 0$ is finite.

Notice that, if \mathfrak{t} is a Bridgeland slicing of \mathcal{D} , and $f: J \rightarrow J'$ is a morphism of \mathbb{Z} -tosets, then for any slicing (L, U) of J' , the lower and the upper categories $\mathcal{D}_{f_*\mathfrak{t};L}$ and $\mathcal{D}_{f_*\mathfrak{t};U}$ can be equivalently defined as

$$\begin{aligned} \mathcal{D}_{f_*\mathfrak{t};L} &= \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L} \\ \mathcal{D}_{f_*\mathfrak{t};U} &= \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U} \end{aligned}$$

where $\langle \mathcal{S} \rangle$ denotes the extension-closed subcategory of \mathcal{D} generated by the subcategory \mathcal{S} . In particular the slices of are given by $\mathcal{D}_{f_*\mathfrak{t};j} = \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$.¹

¹ At some point it will be useful to recall Lemma 4.21 from ??: Let $\mathcal{S}_1, \mathcal{S}_2$ be two subcategories of \mathcal{D} with $\mathcal{S}_1 \boxtimes \mathcal{S}_2$, i.e., such that $\mathcal{D}(X_1, X_2)$ is contractible for any $X_1 \in \mathcal{S}_1$ and any $X_2 \in \mathcal{S}_2$. Then $\mathcal{S}_1 \boxtimes \langle \mathcal{S}_2 \rangle$ and $\langle \mathcal{S}_1 \rangle \boxtimes \mathcal{S}_2$, and so $\langle \mathcal{S}_1 \rangle \boxtimes \langle \mathcal{S}_2 \rangle$.

The right hand sides of the above two expressions can clearly be defined for every morphism f from J to J' (i.e., not necessarily monotone nor \mathbb{Z} -equivariant), and as soon as f is \mathbb{Z} -equivariant, the assignment

$$(L, U) \mapsto (\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U})$$

is an equivariant morphism from $\mathcal{O}(J)$ to pairs of subcategories of \mathcal{D} . Clearly, when f is not monotone there is no reason to expect that the pair $(\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U})$ forms a t -structure on \mathcal{D} . Yet, it is interesting to notice that the condition that f be monotone is only sufficient in order to have this, and can indeed be relaxed.

Definition 0.1. Let J and J' be \mathbb{Z} -tosets, and let $f: J \rightarrow J'$ a map of \mathbb{Z} -sets (i.e., a \mathbb{Z} -equivariant map, not necessarily nondecreasing). A Bridgeland J -slicing \mathbf{t} of \mathcal{D} is f -compatible if $f(\phi) > f(\psi)$ with $\phi \leq \psi$ implies $\mathcal{D}_{\mathbf{t};\phi} \boxtimes \mathcal{D}_{\mathbf{t};\psi}$ and $\mathcal{D}_{\mathbf{t};\phi} \boxtimes \mathcal{D}_{\mathbf{t};\psi}[1]$

Remark 0.2. Clearly, if f is monotone, then every J -slicing \mathbf{t} is f -compatible as the condition ' $f(\phi) > f(\psi)$ with $\phi \leq \psi$ ' is empty.

Remark 0.3. Let J and J' be \mathbb{Z} -tosets, let $f: J \rightarrow J'$ a map of \mathbb{Z} -sets, and let $g: J' \rightarrow J''$ be an isomorphism of \mathbb{Z} -posets. Then a Bridgeland J -slicing \mathbf{t} of \mathcal{D} is f -compatible if and only if it is $(g \circ f)$ -compatible. Similarly, if $h: J'' \rightarrow J$ is an isomorphism of \mathbb{Z} -posets, then \mathbf{t} is f -compatible if and only if $(h^{-1})_* \mathbf{t}$ is $(f \circ h)$ -compatible.

Lemma 0.4. Let J and J' be \mathbb{Z} -tosets, and let $f: J \rightarrow J'$ be a \mathbb{Z} -equivariant morphism of \mathbb{Z} -sets (i.e., not necessarily a monotone map) and let \mathbf{t} be a Bridgeland slicing of \mathcal{D} which is f -compatible. Then, for any slicing (L, U) of J' , the pair of subcategories $(\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U})$ is a t -structure on \mathcal{D} .

Proof. As f is \mathbb{Z} -equivariant and $U + 1 \subseteq U$, we have

$$\begin{aligned} \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}[1] &= \langle \mathcal{D}_{\mathbf{t};\phi}[1] \rangle_{f(\phi) \in U} \\ &= \langle \mathcal{D}_{\mathbf{t};\phi+1} \rangle_{f(\phi) \in U} \\ &= \langle \mathcal{D}_{\mathbf{t};\phi+1} \rangle_{f(\phi+1) \in U+1} \\ &= \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U+1} \\ &\subseteq \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U} \end{aligned}$$

and similarly for the lower subcategory $\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}$. To show that

$$\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U} \boxtimes \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}$$

it suffices to show that, if $f(\psi) \in L$ and $f(\phi) \in U$ then $\mathcal{D}_{\mathbf{t};\phi} \boxtimes \mathcal{D}_{\mathbf{t};\psi}$. As $L \cap U = \emptyset$ we cannot have $\psi = \phi$, so either $\psi < \phi$ or vice versa. In the first case, $\mathcal{D}_{\mathbf{t};\phi} \boxtimes \mathcal{D}_{\mathbf{t};\psi}$ by definition of Bridgeland J -slicing. In the second case, we have $\phi < \psi$ and $f(\phi) > f(\psi)$ as $f(\phi) \in U$ and $f(\psi) \in L$. Therefore, since \mathbf{t} is f -compatible,

$\mathcal{D}_{\mathfrak{t};\phi} \boxtimes \mathcal{D}_{\mathfrak{t};\psi}$. Finally, we have to show that every object X in \mathcal{D} fits into a fiber sequence

$$\begin{array}{ccc} X_U & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X_L \end{array}$$

with $X_L \in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$ and $X_U \in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}$. As \mathfrak{t} is a Bridgeland slicing, we have a factorization of the initial morphism $\mathbf{0} \rightarrow X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{i}}} X_{\bar{i}} \xrightarrow{\alpha_{\bar{i}+1}} X_{\bar{i}+1} \rightarrow \cdots \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \text{cofib}(\alpha_i) = \mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \in \mathcal{D}_{\phi_i}$ for all $i = 1, \dots, n$, with $\phi_i > \phi_{i+1}$. Let us now consider the sequence of symbols L and U obtained putting in the i -th place L if $f(\phi_i) \in L$ and U if $f(\phi_i) \in U$. If this sequence is of the form $(U, U, \dots, U, L, L, \dots, L)$, then there exists an index \bar{i} such that $f(\phi_i) \in U$ for $i \leq \bar{i}$ and $f(\phi_i) \in L$ for $i > \bar{i}$ (with $\bar{i} = -1$ or n when all of the $f(\phi_i)$ are in L or in U , respectively). Then we can consider the pullout diagram

$$\begin{array}{ccc} X_{\bar{i}} & \longrightarrow & 0 \\ f_L \downarrow & & \downarrow \\ X & \longrightarrow & \text{cofib}(f_L) \end{array}$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{i}}} X_{\bar{i}}$$

and

$$X_{\bar{i}} \xrightarrow{\alpha_{\bar{i}+1}} X_{\bar{i}+1} \rightarrow \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that $X_{\bar{i}} \in \langle \cup_{i=0}^{\bar{i}} \mathcal{D}_{\phi_i} \rangle \subseteq \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}$ while the second factorization shows that $\text{cofib}(f_L) \in \langle \cup_{i=\bar{i}+1}^n \mathcal{D}_{\phi_i} \rangle \subseteq \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$. So we are done in this case. Therefore, we are reduced to showing that we can always avoid a (\dots, L, U, \dots) situation in our sequence of L 's and U 's. Assume we have such a situation. Then we have an index i_0 with $f(\phi_{i_0}) \in L$ and $f(\phi_{i_0+1}) \in U$. This in particular implies $f(\phi_{i_0+1}) > f(\phi_{i_0})$ with $\phi_{i_0+1} < \phi_{i_0}$. As \mathfrak{t} is f -compatible, this gives $\mathcal{D}_{\mathfrak{t};\phi_{i_0+1}} \boxtimes (\mathcal{D}_{\mathfrak{t};\phi_{i_0}}[1])$. In particular, $\mathcal{D}(\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1])$ is contractible. Now consider the pasting of pullout diagrams

$$\begin{array}{ccccccc} X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\ \downarrow & & \downarrow & & \downarrow \gamma & & \\ 0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]. \end{array}$$

As the arrow $\mathcal{H}_t^{\phi_{i_0+1}}(X) \rightarrow \mathcal{H}_t^{\phi_{i_0}}(X)[1]$ factors through 0, we have $Y = \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X)$ and the above diagram becomes

$$\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X) & \xrightarrow{\iota_2} & \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \pi_1 & & \downarrow \\
& & 0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X)[1],
\end{array}$$

where ι_2 and π_1 are the canonical inclusion and projection. Let $\beta: X_{i_0+1} \rightarrow \mathcal{H}_t^{\phi_{i_0}}(X)$ be the composition

$$\beta: X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_t^{\phi_{i_0}}(X).$$

Then we have a homotopy commutative diagram

$$\begin{array}{ccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} \\
\downarrow & & \downarrow & & \downarrow \beta \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X) & \xrightarrow{\text{id}} & \mathcal{H}_t^{\phi_{i_0}}(X)
\end{array}$$

(where only the left square is a pullout), and so the composition $\alpha_{i_0+1} \circ \alpha_{i_0}$ factors through the homotopy fiber of β . In other words, we have a homotopy commutative diagram

$$\begin{array}{ccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} \\
\tilde{\alpha}_{i_0} \downarrow & & \downarrow \alpha_{i_0+1} \\
\text{fib}(\beta) & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1}
\end{array}$$

Writing $\tilde{X}_{i_0} = \text{fib}(\beta)$, we get the pasting of pullout diagrams

$$\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\tilde{\alpha}_{i_0}} & \tilde{X}_{i_0} & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0+1}}(X) & \xrightarrow{\iota_1} & \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) & \longrightarrow & \\
& & \downarrow & & \downarrow \pi_2 & & \\
& & 0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X) & &
\end{array}$$

That is, by considering the factorization $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$ we have switched the cofibers with respect to the original factorization $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$. Therefore, writing $\tilde{\phi}_{i_0} = \phi_{i_0+1}$ and $\tilde{\phi}_{i_0+1} = \phi_{i_0}$, we now have $f(\tilde{\phi}_{i_0}) \in U$ and $f(\tilde{\phi}_{i_0+1}) \in L$. That is, we have removed the (\dots, L, U, \dots) situation from the position i_0 , replacing it with a (\dots, U, L, \dots) situation, while keeping all the labels L, U before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the (\dots, L, U, \dots) situations.² \square

Remark 0.5. It follows from the proof of Lemma 0.4 that the objects X_L and X_U in the fiber sequence $X_U \rightarrow X \rightarrow X_L$ associated with the t -struture $(\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$ on \mathcal{D} satisfy

$$\begin{aligned} X_L &\in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L \text{ and } \mathcal{H}_{\mathfrak{t}}^\phi(X) \neq 0} \\ X_U &\in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U \text{ and } \mathcal{H}_{\mathfrak{t}}^\phi(X) \neq 0} \end{aligned}$$

Lemma 0.6. *For any $j \in J'$ we have*

$$\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} = \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}.$$

Proof. Clearly, $\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} \subseteq \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}$, therefore we only need to prove the converse inclusion. Let $X \in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}$. Then in particular $X \in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}$ and so there exists a factorization of the initial morphism $\mathbf{0} \rightarrow X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \text{cofib}(\alpha_i) \in \mathcal{D}_{\phi_i}$ with $f(\phi_i) \geq i$ for all $i = 1, \dots, n$. As $((-\infty, j], (j, +\infty))$ is a slicing of J' , reasoning as in the proof of Lemma 0.4 we can arrange this factorization is such a way that $f(\phi_i) > j$ for $i \leq \bar{i}$ and $f(\phi_i) = j$ for $i > \bar{i}$. Therefore, again by reasoning as in the proof of Lemma 0.4 we get a fiber sequence of the form

$$\begin{array}{ccc} X_{>j} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbf{0} & \longrightarrow & X_j \end{array}$$

with $X_{>j}$ in $\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) > j}$ and X_j in $\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$. This is in particular a fiber sequence of the form $X_{>j} \rightarrow X \rightarrow X_{\leq j}$, with $X_{\leq j} \in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j}$. As $(\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j}, \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) > j})$ is a t -structure on \mathcal{D} by Lemma 0.4, there is (up to equivalence) only one such a fiber sequence. And since $X \in \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j}$, this is the sequence $\mathbf{0} \rightarrow X \xrightarrow{\text{id}} X$. Therefore $X = X_j$. \square

Proposition 0.7. *Let J, J' be \mathbb{Z} -toSETS, and let $f: J \rightarrow J'$ be a \mathbb{Z} -equivariant morphism of \mathbb{Z} -sets (i.e., not necessarily a monotone map) and let \mathfrak{t} be a Bridgeland slicing of \mathcal{D} which is f -compatible. The map*

$$\begin{aligned} f_{\mathfrak{t}}: \mathcal{O}(J') &\rightarrow \text{ts}(\mathcal{D}) \\ (L, U) &\mapsto (\langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}) \end{aligned}$$

²This is somehow reminiscent of the ‘bubble sort’ algorithm.

defined by Lemma 0.4 is a Bridgeland J' -slicing of \mathcal{D} , with slices given by

$$\mathcal{D}_{f!t;j} = \langle \mathcal{D}_{t;\phi} \rangle_{f(\phi)=j}.$$

Proof. The map $f!t$ is manifestly monotone and \mathbb{Z} -equivariant (see the first part of the proof of Lemma 0.4), so it is a J' slicing of \mathcal{D} , and its slices are given by $\mathcal{D}_{f!t;j} = \langle \mathcal{D}_{t;\phi} \rangle_{f(\phi)=j}$ by Lemma 0.6. We are therefore left with showing that it is finite and discrete. Given an object X in \mathcal{D} , let now $\{\phi_1, \dots, \phi_n\}$ be the indices in J such that $\mathcal{H}_t^{\phi_i}(X) \neq 0$ and let $\{j_1, \dots, j_k\}$ the image of the set $\{\phi_1, \dots, \phi_n\}$ via f . Up to renaming, we can assume $j_1 > j_2 > \dots > j_k$. Consider now the factorization

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{k-1}} X_{k-1} \xrightarrow{\alpha_k} X_k = X$$

of the initial morphism of X associated to the decreasing sequence $j_1 > j_2 > \dots > j_k$ by the J' -slicing $f!t$. The cofibers of the morphisms α_i are the cohomologies $\mathcal{H}_{f!t}^{(j_{i+1}, j_i]}(X)$ and, by Remark 0.5, we have

$$\begin{aligned} \mathcal{H}_{f!t}^{(j_{i+1}, j_i]}(X) &\in \langle \mathcal{D}_{t;\phi}; \quad f(\phi) \in (j_{i+1}, j_i] \text{ and } \mathcal{H}_t^\phi(X) \neq 0 \rangle \\ &= \langle \mathcal{D}_{t;\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, j_i] \rangle \\ &= \langle \mathcal{D}_{t;\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) = j_i \rangle \\ &\subseteq \langle \mathcal{D}_{t;\phi}; \quad f(\phi) = j_i \rangle = \mathcal{D}_{f!t;j_i}. \end{aligned}$$

Therefore

$$\mathcal{H}_{f!t}^{(j_i, j_{i-1}]}(X) = \mathcal{H}_{f!t}^{j_i} \mathcal{H}_{f!t}^{(j_i, j_{i-1}]}(X) = \mathcal{H}_{f!t}^{j_i}(X).$$

This tells us that, if all the cohomologies $\mathcal{H}_{f!t}^j(X)$ vanish, then also all the $\mathcal{H}_{f!t}^{(j_i, j_{i-1}]}(X)$ vanish, and so $X = 0$. Finally, if $\tilde{j} \notin \{j_1, \dots, j_k\}$, then there exists an index i such that $j_{i+1} < \tilde{j} < j_i$ and so $(j_{i+1}, \tilde{j}] \cap \{j_1, \dots, j_k\} = \emptyset$. The above argument then shows that

$$\mathcal{H}_{f!t}^{(j_{i+1}, \tilde{j}]}(X) \in \langle \mathcal{D}_{t;\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, \tilde{j}] \rangle = \{\mathbf{0}\}.$$

It follows that

$$\mathcal{H}_{f!t}^{\tilde{j}}(X) = \mathcal{H}_{f!t}^{\tilde{j}} \mathcal{H}_{f!t}^{(j_{i+1}, \tilde{j}]}(X) = \mathcal{H}_{f!t}^{\tilde{j}}(\mathbf{0}) = 0,$$

for every $\tilde{j} \notin \{j_1, \dots, j_k\}$. So in particular, for any $X \in \mathcal{D}$, the cohomologies $\mathcal{H}_{f!t}^j(X)$ are possibly nonzero only for finitely many indices j . \square

Remark 0.8. If $f: J \rightarrow J'$ is a morphism of \mathbb{Z} -tosets, then every Bridgeland slicing of t of \mathcal{D} is f -compatible and one has $f!t = f_*t$.

Proposition 0.9. Let J, J', J'' be \mathbb{Z} -tosets, let $f: J \rightarrow J'$ and $g: J' \rightarrow J''$ be \mathbb{Z} -equivariant morphisms of \mathbb{Z} -sets (i.e., not necessarily monotone maps), and let t be a Bridgeland slicing of \mathcal{D} . If t is f -compatible and $f!t$ is g -compatible, then t is $(g \circ f)$ -compatible, and

$$(g \circ f)!t = g!f!t.$$

Proof. Let ϕ, ψ in J be such that $\phi \leq \psi$ and $g(f(\phi)) > g(f(\psi))$, and let $\xi = f(\phi)$ and $\eta = f(\psi)$. As J' is a totally ordered set, either $\xi \leq \eta$ or $\xi > \eta$. In the first case we have $g(\xi) > g(\eta)$ with $\xi \leq \eta$. As $f_!t$ is g -compatible, this implies $\mathcal{D}_{f_!t;\xi} \boxtimes \mathcal{D}_{f_!t;\eta}$ and $\mathcal{D}_{f_!t;\xi} \boxtimes \mathcal{D}_{f_!t;\eta}[1]$. By definition of $f_!t$ we have $\mathcal{D}_{t;\phi} \subseteq \mathcal{D}_{f_!t;\xi}$ and $\mathcal{D}_{t;\psi} \subseteq \mathcal{D}_{f_!t;\eta}$, therefore we have $\mathcal{D}_{t;\phi} \boxtimes \mathcal{D}_{t;\psi}$ and $\mathcal{D}_{t;\phi} \boxtimes \mathcal{D}_{t;\psi}[1]$. In the second case we have $f(\phi) > f(\psi)$ with $\phi \leq \psi$. As t is f -compatible, this again implies $\mathcal{D}_{t;\phi} \boxtimes \mathcal{D}_{t;\psi}$ and $\mathcal{D}_{t;\phi} \boxtimes \mathcal{D}_{t;\psi}[1]$. Finally, for any $j \in J''$ one has

$$\mathcal{D}_{g_!f_!t;j} = \langle \mathcal{D}_{f_!t;\xi} \rangle_{g(\xi)=j} = \langle \langle \mathcal{D}_{t;\phi} \rangle_{f(\phi)=\xi} \rangle_{g(\xi)=j} = \langle \mathcal{D}_{t;\phi} \rangle_{g(f(\phi))=j} = \mathcal{D}_{(g \circ f)_!t;j}.$$

□

Let now J_1 and J_2 be two \mathbb{Z} -tosets, and let $J_1 \times_{\text{lex}} J_2$ and $J_2 \times_{\text{lex}} J_1$ the two \mathbb{Z} -tosets obtained by considering the lexicographic order on the products $J_1 \times J_2$ and $J_2 \times J_1$, respectively, and the diagonal \mathbb{Z} action. The following lemma is immediate.

Lemma 0.10. *The exchange map*

$$\begin{aligned} e: J_1 \times J_2 &\rightarrow J_2 \times J_1 \\ (j_1, j_2) &\mapsto (j_2, j_1) \end{aligned}$$

is \mathbb{Z} -equivariant.

Definition 0.11. *Let J_1 and J_2 be \mathbb{Z} -tosets. A Bridgeland $J_1 \times_{\text{lex}} J_2$ -slicing t of \mathcal{D} is said to be gluable if it is e -compatible, where $e: J_1 \times J_2 \rightarrow J_2 \times J_1$ is the exchange map.*

Example 0.12. Let $J_1 = \{0, 1\}$ with the trivial \mathbb{Z} -action and let $J_2 = \mathbb{Z}$ with the standard translation \mathbb{Z} -action. Then a $J_1 \times_{\text{lex}} J_2$ -slicing t on \mathcal{D} is the datum of a semiorthogonal decomposition $(\mathcal{D}_0, \mathcal{D}_1)$ of \mathcal{D} together with t -structures t_i on \mathcal{D}_i . Spelling out the definition, one sees that the $\{0, 1\} \times_{\text{lex}} \mathbb{Z}$ -slicing t is gluable if and only if $\mathcal{D}_{0;\geq 0} \boxtimes \mathcal{D}_{1;-1}$ and $\mathcal{D}_{0;\geq 0} \boxtimes \mathcal{D}_{1;0}$. When this happens, the glued slicing $e_!t$ is a $\mathbb{Z} \times_{\text{lex}} \{0, 1\}$ slicing on \mathcal{D} , i.e., is the datum of a bounded t -structure on \mathcal{D} together with a torsion theory on its heart. More precisely, the heart of $e_!t$ is the full ∞ -subcategory $\mathcal{D}^{\heartsuit_{e_!t}}$ of \mathcal{D} on those objects X that fall into fiber sequences of the form $X_1 \rightarrow X \rightarrow X_0$ with $X_i \in \mathcal{D}_i^{\heartsuit}$, while the torsion theory on $\mathcal{D}^{\heartsuit_{e_!t}}$ is $(\mathcal{D}_0^{\heartsuit}, \mathcal{D}_1^{\heartsuit})$, i.e., precisely the pair of hearts of the two subcategories in the semiorthogonal decomposition.

Definition 0.13. *Let $U: J \rightarrow \mathcal{O}(J')$ be a map of sets. We denote by Γ_U the subset of $J \times J'$ defined by*

$$\Gamma_U = \{(j, j') \in J \times J'; \quad j' \in U_j\}.$$

For $f: J \rightarrow J'$ a map of sets, we denote by $(< f, \geq f): J \rightarrow \mathcal{O}(J')$ the composition of f with the map

$$\begin{aligned} J' &\rightarrow \mathcal{O}(J') \\ j' &\mapsto ((-\infty, j'), [j', +\infty)). \end{aligned}$$

The upper graphic of f is the subset $\Gamma_{\geq f}$ of $J \times J'$.

Lemma 0.14. *Let $U: J \rightarrow \mathcal{O}(J')$ be a map of sets. Then Γ_U is an upper set of $J \times J'$ with the product order if and only if U is a map of posets $U: J \rightarrow \mathcal{O}(J')^{\text{op}}$. In particular, $\Gamma_{\geq f}$ is an upper set if and only if $f: J \rightarrow J'$ is a nondecreasing map, i.e., a map of posets $J \rightarrow J'^{\text{op}}$.*

Proof. Assume (L, U) is a map of posets from J to $\mathcal{O}(J')^{\text{op}}$. Pick $(j, j') \in \Gamma_U$ and suppose $(j, j') \leq (k, k')$ in $J \times J'$. As (L, U) is monotone and $k \geq j$, we have $U_j \subseteq U_k$; as $(j, j') \in \Gamma_U$, we have $j' \in U_j$. Therefore $j' \in U_k$. Since U_k is an upper set and $k' \geq j'$, we get $k' \in U_k$, i.e. $(k, k') \in \Gamma_U$. Vice versa, assume Γ_U is an upper set, and let $j \leq k$ in J . For any $j' \in U_j$, the element (k, j') in $J \times J'$ satisfies $(k, j') \geq (j, j')$ in the product order and so $(k, j') \in \Gamma_U$. This means $j' \in U_k$ and so $U_j \subseteq U_k$. To prove the second part of the statement, notice that the map $j \mapsto ((-\infty, j'), [j', +\infty))$ is a map of posets from $J' \rightarrow \mathcal{O}(J')$. Therefore, if $f: J \rightarrow J'^{\text{op}}$ is a map of posets, then also $(\leq f, \geq f): J \rightarrow \mathcal{O}(J')^{\text{op}}$ is a map of posets. Vice versa, if $(\leq f, \geq f): J \rightarrow \mathcal{O}(J')^{\text{op}}$ is a map of posets then for every $j \leq k$ in J we have $[f(j), +\infty) \subseteq [f(k), +\infty)$ and so $f(k) \leq f(j)$. \square

Proposition 0.15. *The map*

$$\begin{aligned} \Gamma: \text{Pos}(J, \mathcal{O}(J')^{\text{op}})^{\text{op}} &\rightarrow \mathcal{O}(J \times J') \\ U &\mapsto \Gamma_U \end{aligned}$$

is an isomorphism of posets.

Proof. Let $U_1 \leq U_2$ in the partial order on $\text{Pos}(J, \mathcal{O}(J')^{\text{op}})^{\text{op}}$, and let $(j, j') \in \Gamma_{U_2}$. Then $j' \in U_{2,k} \subseteq U_{1,j}$ and so $(j, j') \in \Gamma_{U_1}$. This means $\Gamma_{U_1} \leq \Gamma_{U_2}$ in $\mathcal{O}(J \times J')$. Next, for any morphism of posets $U: J \rightarrow \mathcal{O}(J')^{\text{op}}$ we have that $(j, j') \in \Gamma_{U+1}$ if and only if $j' \in (U+1)_j$. Pick an upper set \tilde{U} of $J \times J'$ and set, for $j \in J$

$$U_{\tilde{U}}(j) = \{j' \in J'; (j, j') \in \tilde{U}\}.$$

The subset $U_{\tilde{U}}(j) \subseteq J'$ is an upper set. Indeed, if $j' \in U_{\tilde{U}}(j)$ and $k' \geq j'$ in J' , then $(j, k') \geq (j, j')$ in the product order on $J \times J'$ and so $(j, k') \in \tilde{U}$ as \tilde{U} is an upper set. Next, if $j \leq k$ in J and $j' \in U_{\tilde{U}}(j)$, then $(k, j') \in \tilde{U}$ as \tilde{U} is an upper set, and so $j' \in U_{\tilde{U}}(k)$. This shows that $U_{\tilde{U}}(j) \subseteq U_{\tilde{U}}(k)$, and so $U_{\tilde{U}}$ is a map of posets from J to $\mathcal{O}(J')^{\text{op}}$. Moreover, if $\tilde{U}_1 \leq \tilde{U}_2$ in $\mathcal{O}(J \times J')$ then $U_{\tilde{U}_1} \leq U_{\tilde{U}_2}$ in $\text{Pos}(J, \mathcal{O}(J')^{\text{op}})^{\text{op}}$. Therefore we have a map

$$\begin{aligned} \gamma: \mathcal{O}(J \times J') &\rightarrow \text{Pos}(J, \mathcal{O}(J')^{\text{op}})^{\text{op}} \\ \tilde{U} &\mapsto U_{\tilde{U}}. \end{aligned}$$

which is straightforward to see to be the inverse of Γ . \square

Assume J and J' are \mathbb{Z} -posets. Then $\mathcal{O}(J)^{\text{op}}$ is a \mathbb{Z} -poset with 1 acting as $U \mapsto U - 1$. The action of \mathbb{Z} by conjugation on $\text{Pos}(J, \mathcal{O}(J')^{\text{op}})$ is therefore given by $(U \dot{+} 1)_j = U_{j-1} - 1$. To see that $U \dot{+} 1$ is still a morphism of posets from J to $\mathcal{O}(J')^{\text{op}}$, let $j \leq k$ in J . Then $(U \dot{+} 1)_j = U_{j-1} - 1 \subseteq U_{k-1} - 1 = (U \dot{+} 1)_k$. The conjugation action, however, does not define a structure of \mathbb{Z} -poset on

$\text{Pos}(J, \mathcal{O}(J')^{\text{op}})$, as it is generally not true that $U \leq U \dot{+} 1$. This condition is indeed equivalent to $U_j \subseteq (U \dot{+} 1)_j$, i.e., to $U_j + 1 \subseteq U_{j-1}$ for any j in J , and this is not necessarily satisfied by a morphism of posets $U: J \rightarrow \mathcal{O}(J')^{\text{op}}$.

Definition 0.16. A (J, J') -perversity is a map of posets $U: J \rightarrow \mathcal{O}(J')^{\text{op}}$ such that

$$U_j + 1 \subseteq U_{j-1}$$

for any j in J . The set $\text{Perv}(J, J')$ of all (J, J') -perversities inherits a poset structure from the inclusion in $\text{Pos}(J, \mathcal{O}(J')^{\text{op}})^{\text{op}}$. It is a \mathbb{Z} -poset with 1 acting as $U \mapsto U \dot{+} (-1)$.

Remark 0.17. Notice that the shift action on (J, J') -perversities is given by $U \mapsto U \dot{+} (-1)$. Namely, by construction the \mathbb{Z} -action $U \mapsto U \dot{+} 1$ is monotone on the set of perversities with the order induced by the inclusion in $\text{Pos}(J, \mathcal{O}(J')^{\text{op}})$, and the order on $\text{Perv}(J, J')$ is the opposite one. The reason for considering this order is, clearly, Proposition 0.15.

We have seen in Lemma 0.14 that a function $f: J \rightarrow J'$ defines a morphism of posets by $(\geq f): J \rightarrow \mathcal{O}(J')^{\text{op}}$ if and only if f is a morphism of posets from J to J'^{op} . When this happens, $(\geq f)$ is a (J, J') -perversity if and only if $f(j-1) \leq f(j) + 1$, for every $j \in J$. In the particular case $J = \mathbb{Z}$, assuming as usual that J' is a \mathbb{Z} -poset, we can define a new function $p_f: \mathbb{Z} \rightarrow J'$ as $p_f(n) = f(n) + n$. Then the condition $f(n) \leq f(n+1) + 1$ translates into $p_f(n) \leq p_f(n+1)$, while the condition that $f: \mathbb{Z} \rightarrow J'^{\text{op}}$ is a morphism of posets, i.e., $f(n+1) \leq f(n)$ translates to $p_f(n+1) \leq p_f(n) + 1$. As $f \mapsto p_f$ is a bijection of the set of maps from J to J' into itself, we see that the functions $f: \mathbb{Z} \rightarrow J'$ defining perversities correspond bijectively to the set of functions $p: \mathbb{Z} \rightarrow J'$ such that $p(n) \leq p(n+1) \leq p(n) + 1$, for every $n \in \mathbb{Z}$. This motivates the following (see [?]).

Definition 0.18. A (\mathbb{Z}, \mathbb{Z}) -perversity U is said to be defined by a function if there exists $f: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $U = (\geq f)$. The subset of (\mathbb{Z}, \mathbb{Z}) -perversities defined by functions is denoted by $\text{Perv}^\circ(\mathbb{Z}, \mathbb{Z})$.

Lemma 0.19. The subset $\text{Perv}^\circ(\mathbb{Z}, \mathbb{Z})$ is a \mathbb{Z} -sub-poset of $\text{Perv}(\mathbb{Z}, \mathbb{Z})$. It is isomorphic via $f \mapsto (\geq f)$ with the \mathbb{Z} -poset of nonincreasing functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $f(n-1) \leq f(n) + 1$ with the \mathbb{Z} -action given by $(f \dot{+} 1)(n) = f(n+1) + 1$.

Proof. As a function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ is uniquely determined by the collection of upper sets $[f(n), +\infty)$ the set $\text{Perv}^\circ(J, J')$ bijectively corresponds to the set of those functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $(\geq f)$ is a (\mathbb{Z}, \mathbb{Z}) -perversity. As noticed above, these are precisely nonincreasing functions from \mathbb{Z} to itself such that $f(n-1) \leq f(n) + 1$. This bijection is an isomorphism of posets, as $(\geq f_1) \leq (\geq f_2)$ if and only if $f_1 \leq f_2$ (in the standard poset structure on the set of maps from \mathbb{Z} to the poset \mathbb{Z}). Finally, $((\geq f) \dot{+} (-1))_n = (\geq f)_{n+1} + 1 = [f(n+1) + 1, +\infty) = (\geq (f \dot{+} 1))_n$, for any $n \in \mathbb{Z}$. \square

Definition 0.20. A perversity function (on \mathbb{Z}) is a function $p: \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$p(n) \leq p(n+1) \leq p(n) + 1,$$

for every $n \in \mathbb{Z}$. The set $\text{perv}_{\mathbb{Z}}$ of perversity functions is a poset with the partial order induced by the inclusion $\text{perv}_{\mathbb{Z}} \subseteq \text{Pos}(\mathbb{Z}, \mathbb{Z})$. It is a \mathbb{Z} -poset with the action $(p \dot{+} 1)(n) = p(n+1)$.

Remark 0.21. In addition to the \mathbb{Z} -action $p \mapsto p \dot{+} 1$, the poset $\text{perv}_{\mathbb{Z}}$ carries also another natural \mathbb{Z} -action making it a \mathbb{Z} -poset; namely, the action $(p+1)(n) = p(n) + 1$. We will come back to this towards the end of this section.

Remark 0.22. Notice that the \mathbb{Z} -action on perversity functions is a monotone action since, by definition of perversity function, we have $p(n) \leq p(n+1)$ and this precisely means $p(n) \leq (p \dot{+} 1)(n)$.

Lemma 0.23. For every $p: \mathbb{Z} \rightarrow \mathbb{Z}$, let $f_p: \mathbb{Z} \rightarrow \mathbb{Z}$ be the map defined by $f_p(n) = p(n) - n$. Then $p \mapsto (\geq f_p)$ is an isomorphism of \mathbb{Z} -posets between $\text{perv}_{\mathbb{Z}}$ and $\text{Perv}^{\circ}(\mathbb{Z}, \mathbb{Z})$.

Proof. By Lemma 0.19 we only need to show that $p \mapsto f_p$ is a monotone \mathbb{Z} -equivariant bijection between $\text{perv}_{\mathbb{Z}}$ and the set of functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $f(n) \leq f(n-1) \leq f(n) + 1$. That it is a bijection is immediate: the inverse map is $f \mapsto p_f$, where $p_f(n) = f(n) + n$. To see that it is an isomorphism of posets, notice that $f_{p_1} \leq f_{p_2}$ if and only if $p_1(n) - n \leq p_2(n) - n$ for every $n \in \mathbb{Z}$, and so if and only if $p_1(n) \leq p_2(n)$ for every $n \in \mathbb{Z}$. Finally, $f_{p \dot{+} 1}(n) = (p \dot{+} 1)(n) - n = p(n+1) - n = f_p(n+1) + 1 = (f_p \dot{+} 1)(n)$. \square

Definition 0.24. An upper set U in $\mathcal{O}(J \times J')$ is called a kinky upperset if $U \leq U +_{\text{ne}} 1$, where ne is the “northwestern” action of \mathbb{Z} on $J \times J'$ given by $(j, j') +_{\text{ne}} 1 = (j-1, j'+1)$. We denote by $\text{Kink}(J \times J')$ the poset of kinky uppersets of $J \times J'$, with the poset structure induced by the inclusion in $\mathcal{O}(J \times J')$. It is a \mathbb{Z} -poset with the “northwestern” action. We denote by $\text{Kink}^{\circ}(J \times J')$ the \mathbb{Z} -sub-poset of nontrivial kinky uppersets of $J \times J'$, where the trivial upperstes are \emptyset and $J \times J'$.

Lemma 0.25. The map Γ from Proposition 0.15 induces an isomorphism of \mathbb{Z} -posets

$$\Gamma: \text{Perv}(J, J') \rightarrow \text{Kink}(J \times J').$$

Proof. Let $U: J \rightarrow \mathcal{O}(J')^{\text{op}}$ be a perversity, and let $(j, j') \in \Gamma_U$. Then $j' \in U_j$ and so, by definition of perversity, $j' + 1 \in U_{j-1}$. Therefore $(j-1, j'+1) \in \Gamma_U$, i.e., $\Gamma_U \leq \Gamma_U +_{\text{ne}} 1$. Vice versa, if \tilde{U} is a kinky upperset in $J \times J'$, let $U_{\tilde{U}}$ the preimage in $\text{Pos}(J, \mathcal{O}(J')^{\text{op}})^{\text{op}}$ of \tilde{U} via Γ (see the proof of Proposition 0.15). Then for any $j \in J$ and any $j' \in U_{\tilde{U};j}$ we have $j' + 1 \in U_{\tilde{U};j-1}$ and so $U_{\tilde{U};j} + 1 \subseteq U_{\tilde{U};j-1}$. So the isomorphism of posets $\text{Pos}(J, \mathcal{O}(J')^{\text{op}})^{\text{op}} \rightarrow \mathcal{O}(J \times J')$ restricts to an isomorphism of posets $\text{Perv}(J, J') \rightarrow \text{Kink}(J \times J')$. To see that $\Gamma: \text{Perv}(J, J') \rightarrow \text{Kink}(J \times J')$ is also \mathbb{Z} -equivariant, notice that, for every perversity U we have $(j, j') \in \Gamma_{U \dot{+} (-1)}$ if and only if $j' \in U_{j+1} + 1$. i.e., if and

only if $(j+1, j'-1) \in \Gamma_U$. This latter condition is equivalent to $(j, j') \in \Gamma_U +_{\text{nw}} 1$, so we find $\Gamma_{U+(-1)} = \Gamma_U +_{\text{nw}} 1$. \square

Lemma 0.26. *The map Γ from Proposition 0.15 induces an isomorphism of \mathbb{Z} -posets*

$$\Gamma : \text{Perv}^\circ(\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Kink}^\circ(\mathbb{Z} \times \mathbb{Z}).$$

Proof. A kinky upperset U is in the image of Γ if and only if U is of the form $(\geq f)$ for a suitable function $f : \mathbb{Z} \rightarrow \mathbb{Z}$. This is possible if and only if $U_n \neq \emptyset, \mathbb{Z}$ for every $n \in \mathbb{Z}$. As U is kinky, if $U_{n_0} = \emptyset$ for some n_0 , then $U_{n_0+1} + 1 \subseteq U_{n_0} = \emptyset$, and so $U_{n_0+1} = \emptyset$. Inductively, this gives $U_n = \emptyset$ for every $n \geq n_0$. On the other hand, since a kinky upperset is an upperset, if there exists a nonempty U_n with $n < n_0$, then there exist an element (n, m) in U and so, since $(n, m) \leq (n_0, m)$, also $(n_0, m) \in U$. But then $m \in U_{n_0}$, which is impossible. So also the U_n with $n < n_0$ are empty and therefore $U = \emptyset$. Similarly, if $U_{n_0} = \mathbb{Z}$ for some n_0 , then $U_n = \mathbb{Z}$ for every $n > n_0$ as U is an upperset, while the kinkiness condition $U_n + 1 \subseteq U_{n-1}$ implies that also $U_{n_0-1} = \mathbb{Z}$ and so inductively that all U_n with $n < n_0$ are the whole of \mathbb{Z} . That is, $U = \mathbb{Z} \times \mathbb{Z}$ in this case. \square

Lemma 0.27. *The isomorphism of \mathbb{Z} -modules $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ given by $\varphi : (n, n') \mapsto (n + n', n')$ induces an isomorphism of \mathbb{Z} -posets*

$$\varphi : \text{Kink}(\mathbb{Z} \times \mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}),$$

where the \mathbb{Z} -action on $\mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ is the one induced by the “northern” \mathbb{Z} -action on \mathbb{Z}^2 , namely, $(n, n') + 1 = (n, n' + 1)$. In particular φ induces an isomorphism of \mathbb{Z} -posets $\text{Kink}^\circ(\mathbb{Z} \times \mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$.

Proof. A subset U of $\mathbb{Z} \times \mathbb{Z}$ is an upper set (in the product order) if and only if $U + K \subseteq U$, where K is the \mathbb{Z} -cone spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, while U is a kinky upperset if and only if $U + K^{\text{kink}} \subseteq U$, where K^{kink} is the \mathbb{Z} -cone spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Moreover the \mathbb{Z} action on the uppersets is generated by $U \mapsto U + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the \mathbb{Z} action on the kinky uppersets is generated by $U \mapsto U + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, where in both cases the sum on the right hand side is the sum in \mathbb{Z}^2 . As $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is an isomorphism of \mathbb{Z} -modules with $\varphi(K^{\text{kink}}) = K$ and $\varphi\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the statement follows (φ is manifestly inclusion preserving). \square

Corollary 0.28. *We have an isomorphism of \mathbb{Z} -posets*

$$\text{perv}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

mapping a perversity function p to the image via the isomorphism $\varphi : (n, n') \mapsto (n + n', n')$ of the set $\{(n, n') \in \mathbb{Z} \times \mathbb{Z} \text{ such that } n' \geq p(n) - n\}$.

Example 0.29. The zero perversity function $p(n) \equiv 0$ corresponds to the upper set $\{(n, n') \in \mathbb{Z} \times \mathbb{Z} \text{ such that } n \geq 0\}$; the identity perversity function $p(n) \equiv n$ corresponds to the upper set $\{(n, n') \in \mathbb{Z} \times \mathbb{Z} \text{ such that } n' \geq 0\}$.

Remark 0.30. The two “missing” upper sets from $\mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$ can be recovered by adding to the set $\text{perv}_{\mathbb{Z}}$ of perversity functions the two “constant infinite perversities”, i.e., the function $p_{+\infty}: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ defined by $p_{+\infty}(n) = +\infty$ for every $n \in \mathbb{Z}$ and the function $p_{-\infty}: \mathbb{Z} \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ defined by $p_{-\infty}(n) = -\infty$ for every $n \in \mathbb{Z}$. The extended set

$$\widehat{\text{perv}}_{\mathbb{Z}} = \text{perv}_{\mathbb{Z}} \cup \{p_{-\infty}, p_{+\infty}\}$$

is naturally a \mathbb{Z} -poset with $p_{+\infty}$ and $p_{-\infty}$ as maximum and minimum element, respectively (so they are in particular \mathbb{Z} -fixed points), and the inclusion of $\text{perv}_{\mathbb{Z}}$ into $\widehat{\text{perv}}_{\mathbb{Z}}$ is a morphism of \mathbb{Z} -posets. Moreover, the isomorphism of \mathbb{Z} -posets $\text{perv}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$ from Corollary 0.28 extends to an isomorphism of \mathbb{Z} -posets

$$\widehat{\text{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}).$$

As well as the \mathbb{Z} -action $p \mapsto p + 1$, also the action $p \mapsto p + 1$ from Remark 0.21 extends to a \mathbb{Z} -action of $\widehat{\text{perv}}_{\mathbb{Z}}$ (again, $p_{+\infty}$ and $p_{-\infty}$ will be fixed points). The isomorphism of posets $\widehat{\text{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ will then transfer this \mathbb{Z} -action to $\mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ inducing on $\mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ a \mathbb{Z} -action such that $\widehat{\text{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ is an isomorphism of posets with the \mathbb{Z} -action $p \mapsto p + 1$ on the left hand side. More precisely, we have the following result.

Proposition 0.31. *We have an isomorphism of posets*

$$\widehat{\text{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z})$$

mapping a perversity function p to the image via the isomorphism $\varphi: (n, n') \mapsto (n+n', n')$ of the set $S_p = \{(n, n') \in \mathbb{Z} \times \mathbb{Z} \text{ such that } n' \geq p(n) - n\}$, and mapping the “infinite perversity functions” $p_{-\infty}$ and $p_{+\infty}$ to $\mathbb{Z} \times \mathbb{Z}$ and to \emptyset , respectively. Moreover, this is an isomorphism of \mathbb{Z} -posets, where the \mathbb{Z} -action on the left is given by $(p+1)(n) = p(n) + 1$ and the \mathbb{Z} -action on the right is the one induced by the “northeastern” \mathbb{Z} -action on \mathbb{Z}^2 , namely, $(n, n') + 1 = (n+1, n'+1)$.

Proof. After Corollary 0.28 and remark 0.30, the only thing left to prove is the \mathbb{Z} -equivariance of the isomorphism, i.e., that we have

$$\varphi(S_{p+1}) = \varphi(S_p) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

As φ is an isomorphism of \mathbb{Z} -modules, this is equivalent to

$$S_{p+1} = S_p + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

i.e., to the condition $(n, n') \in S_{p+1}$ if and only if $(n, n' - 1) \in S_p$, which is obvious. □

0.2 Slicing the heart

Let \mathcal{D} be a stable ∞ -category. Then, as we recalled above, a bounded t -structure on \mathcal{D} is the datum of a Bridgeland \mathbb{Z} -slicing \mathfrak{t} of \mathcal{D} . Let $\heartsuit_{\mathfrak{t}}$ denote the heart of \mathfrak{t} . Then an abelian \mathbb{Z} -slicing of $\heartsuit_{\mathfrak{t}}$ is the datum of an extension $\tilde{\mathfrak{t}}$ of \mathfrak{t} to a Bridgeland $\mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , where $\hat{\mathbb{Z}}$ denotes the \mathbb{Z} -poset consisting of \mathbb{Z} endowed with the trivial \mathbb{Z} -action, and the morphism $\mathcal{O}(\mathbb{Z}) \rightarrow \mathcal{O}(\mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}})$ is induced by the projection on the first factor $\mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \rightarrow \mathbb{Z}$. See [?, Section?]. We will denote by $\heartsuit_{\mathfrak{t};\phi}$ the ϕ -th slice of the heart of \mathfrak{t} . In other words,

$$\heartsuit_{\mathfrak{t};\phi} = \mathcal{D}_{\tilde{\mathfrak{t}};(0,\phi)}.$$

Notice that we have

$$\heartsuit_{\mathfrak{t};\phi}[n] = \mathcal{D}_{\tilde{\mathfrak{t}};(n,\phi)},$$

where $[n]$ denotes the “shift by n ” functor on \mathcal{D} .

Example 0.32. Via the obvious inclusion of \mathbb{Z} -posets $\{0,1\} \hookrightarrow \hat{\mathbb{Z}}$, any torsion pair $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$ on $\heartsuit_{\mathfrak{t}}$ defines an abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$.

Definition 0.33. Let \mathfrak{t} be a bounded t -structure on \mathcal{D} . An abelian \mathbb{Z} -slicing $\hat{\mathfrak{t}}$ on $\heartsuit_{\mathfrak{t}}$ is called:

- grading (or radical) if $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}[n]$ for $\phi > \psi + n$ and for $\phi = \psi + n$ with $n \geq 2$;
- gluable if $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}[n]$ for $\phi > \psi$ and $n > 0$;

Remark 0.34. The definition of gluable abelian \mathbb{Z} -slicing of $\heartsuit_{\mathfrak{t}}$ is the specialization of Definition 0.11 to $J_1 = \mathbb{Z}$ and $J_2 = \hat{\mathbb{Z}}$.

Remark 0.35. Historically, grading filtrations first appeared in [?] under the name of ‘radical filtrations’.

Example 0.36. Let $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$ be a torsion pair on $\heartsuit_{\mathfrak{t}}$. Then $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$, seen as an abelian \mathbb{Z} -slicing, is grading. Namely, as $\heartsuit_{\mathfrak{t};\phi}[n] = 0$ for $\phi \notin \{0,1\}$, the only nontrivial orthogonality conditions to be checked are:

- $\heartsuit_{\mathfrak{t};0} \boxtimes \heartsuit_{\mathfrak{t};0}[n]$ for $n < 0$;
- $\heartsuit_{\mathfrak{t};0} \boxtimes \heartsuit_{\mathfrak{t};1}[n]$ for $n < -1$;
- $\heartsuit_{\mathfrak{t};1} \boxtimes \heartsuit_{\mathfrak{t};0}[n]$ for $n < 1$;
- $\heartsuit_{\mathfrak{t};1} \boxtimes \heartsuit_{\mathfrak{t};1}[n]$ for $n < 0$.

These all follows from the orthogonality relation $\heartsuit_{\mathfrak{t}} \boxtimes \heartsuit_{\mathfrak{t}}[n]$ for $n < 0$, except for $\heartsuit_{\mathfrak{t};1} \boxtimes \heartsuit_{\mathfrak{t};0}$ which is true by definition of torsion pair.

Proposition 0.37. Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , and let $\hat{\mathfrak{t}}$ be an abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$. If $\tilde{\mathfrak{t}}$ is gluable, then $\tilde{\mathfrak{t}}$ is grading.

Proof. Let ϕ and ψ be in \mathbb{Z} with $\phi > \psi + n$. The orthogonality condition $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}[n]$ is trivially satisfied if $n < 0$, so let us assume $n \geq 0$. If $n = 0$, then $\phi > \psi$ and the orthogonality condition $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}$ is satisfied by definition of slicing. Finally, if $n > 0$, then we have $\phi > \psi$ and $n > 0$, so $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}[n]$ by definition of glueable slicing. If $\phi = \psi + n$ with $n \geq 2$, then in particular $\phi > \psi$ and $n > 0$, so again $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}[n]$. \square

Proposition 0.38. *Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , and let $\tilde{\mathfrak{t}}$ be a grading abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$. Then, $\tilde{\mathfrak{t}}$ is g_p -compatible, where*

$$\begin{aligned} g_p: \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} &\rightarrow \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \\ (n, \phi) &\mapsto (n + p(\phi), -p(\phi)), \end{aligned}$$

for every perversity function $p: \mathbb{Z} \rightarrow \mathbb{Z}$.

Proof. The map g_p is \mathbb{Z} -equivariant, as the action on the second factor is the trivial one. Let (n, ϕ) and (m, ψ) in $\mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}}$ with $(n, \phi) \leq (m, \psi)$ such that $g_p(n, \phi) > g_p(m, \psi)$ in $\mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}}$. By definition of g_p this means that we have

$$(n + p(\phi), -p(\phi)) > (m + p(\psi), -p(\psi))$$

in $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}$, i.e., that $n + p(\phi) > m + p(\psi)$ or that $n + p(\phi) = m + p(\psi)$ and $p(\psi) > p(\phi)$. Similarly, the condition $(n, \phi) \leq (m, \psi)$ means that either $n < m$ or $n = m$ and $\phi \leq \psi$. By considering all possibilities, and taking into account that a perversity function is nondecreasing, one sees that there is actually a single case to deal with:

- $p(\phi) - p(\psi) > m - n$, with $m > n$ and $\phi > \psi$;

As p is a perversity function, if $\phi \geq \psi$ then

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi,$$

so we have

$$\phi \geq \psi + p(\phi) - p(\psi) > \psi + m - n.$$

Since $\tilde{\mathfrak{t}}$ is grading, this implies $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}[m - n]$, and so $\heartsuit_{\mathfrak{t};\phi}[n] \boxtimes \heartsuit_{\mathfrak{t};\psi}[m]$, i.e.,

$$\mathcal{D}_{\tilde{\mathfrak{t}};(n,\phi)} \boxtimes \mathcal{D}_{\tilde{\mathfrak{t}};(m,\psi)}.$$

Since $\phi > \psi + m - n$, either $\phi > \psi + m - n + 1$ or $\phi = \psi + m - n + 1$. In the first case, reasoning as above we find $\mathcal{D}_{\tilde{\mathfrak{t}};(n,\phi)} \boxtimes \mathcal{D}_{\tilde{\mathfrak{t}};(m,\psi)}[1]$. In the second case, as $m > n$ we have $m - n + 1 \geq 2$. As $\tilde{\mathfrak{t}}$ is grading, this gives $\heartsuit_{\mathfrak{t};\phi} \boxtimes \heartsuit_{\mathfrak{t};\psi}[m - n + 1]$, i.e., again

$$\mathcal{D}_{\tilde{\mathfrak{t}};(n,\phi)} \boxtimes \mathcal{D}_{\tilde{\mathfrak{t}};(m,\psi)}[1].$$

\square

Remark 0.39. By taking as perversity function the identity, $\text{id}: \mathbb{Z} \rightarrow \mathbb{Z}$, we see that if $\tilde{\mathfrak{t}}$ is grading then $\tilde{\mathfrak{t}}$ is α -compatible, where $\alpha: \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \rightarrow \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}}$ is the \mathbb{Z} -equivariant map given by

$$\alpha(n, m) = (n + m, -m).$$

The proof of the following lemma is straightforward.

Lemma 0.40. Let $\beta: \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \rightarrow \mathbb{Z} \times_{\text{lex}} \mathbb{Z}$ be the map defined by

$$\beta(n, m) = (n, n + m).$$

Then β is an isomorphism of \mathbb{Z} -tosets. Moreover the diagram

$$\begin{array}{ccc} \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} & \xrightarrow{\alpha} & \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \\ \beta \downarrow & & \downarrow \beta \\ \mathbb{Z} \times_{\text{lex}} \mathbb{Z} & \xrightarrow{e} & \mathbb{Z} \times_{\text{lex}} \mathbb{Z} \end{array}$$

commutes, where $e: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ is the exchange map $e(n, m) = (m, n)$ from Lemma 0.10 and $\alpha: \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \rightarrow \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}}$ is the map $\alpha(n, m) = (n + m, -m)$ from Remark 0.39.

Corollary 0.41. Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , and let $\tilde{\mathfrak{t}}$ be an abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$. Then, in the notation of Lemma 0.40, $\tilde{\mathfrak{t}}$ is α -compatible if and only if $\beta_*\mathfrak{t}$ is gluable.

Proof. By Lemma 0.40 and Remark 0.3, $\tilde{\mathfrak{t}}$ is α -compatible if and only if $\tilde{\mathfrak{t}}$ is $(\beta \circ \alpha)$ -compatible, which happens if and only if $\tilde{\mathfrak{t}}$ is $(e \circ \beta)$ -compatible. As β is an isomorphism of \mathbb{Z} -tosets, we can write $\mathfrak{t} = (\beta^{-1})_*\beta_*\mathfrak{t}$, and so, by Remark 0.3 again, $\tilde{\mathfrak{t}}$ is $(e \circ \beta)$ -compatible if and only if $\beta_*\mathfrak{t}$ is e -compatible. By Remark 0.34, this is equivalent to saying that $\beta_*\mathfrak{t}$ is gluable. \square

From Proposition 0.37 and Remark 0.39 we immediately get the following

Corollary 0.42. Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , and let $\tilde{\mathfrak{t}}$ be an abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$. If $\tilde{\mathfrak{t}}$ is grading, then $\beta_*\mathfrak{t}$ is gluable. In particular, if $\tilde{\mathfrak{t}}$ is grading, then also $\beta_*\mathfrak{t}$ is grading.

Proposition 0.43. Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , and let $\tilde{\mathfrak{t}}$ be a grading abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$. For every perversity p , let $\gamma_p: \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \rightarrow \mathbb{Z}$ the \mathbb{Z} -equivariant morphism given by

$$\gamma_p: (n, \phi) \mapsto n + p(\phi).$$

Then

- $\tilde{\mathfrak{t}}$ is γ_p -compatible;
- $(\gamma_p)_!\tilde{\mathfrak{t}}$ is a bounded t -structure on \mathcal{D} ;

- the map

$$\begin{aligned}\Psi: \text{perv}_{\mathbb{Z}}^{\text{op}} &\rightarrow \text{ts}(\mathcal{D}) \\ p &\mapsto (\gamma_p)_! \tilde{\mathfrak{t}},\end{aligned}$$

is a morphism of \mathbb{Z} -posets, where on the left we have the \mathbb{Z} -action $(p+1)(n) = p(n) + 1$ and on the right the \mathbb{Z} -action given by the shift in \mathcal{D} ;

- Ψ uniquely extends to a morphism of \mathbb{Z} -posets

$$\Psi: \widehat{\text{perv}}_{\mathbb{Z}}^{\text{op}} \rightarrow \text{ts}(\mathcal{D})$$

preserving maxima and minima.

Proof. Let $g_p: \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \rightarrow \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}}$ the map defined in Proposition 0.43. As $\tilde{\mathfrak{t}}$ is grading, by Proposition 0.43, $\tilde{\mathfrak{t}}$ is g_p -compatible. The projection on the first factor, $\pi_1: \mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}} \rightarrow \mathbb{Z}$ is a morphism of \mathbb{Z} -to-sets, so by Remark 0.2 every $\mathbb{Z} \times_{\text{lex}} \hat{\mathbb{Z}}$ -slicing is π_1 -compatible. In particular, $(g_p)_! \tilde{\mathfrak{t}}$ is π_1 -compatible. Therefore, by Proposition 0.9, $\tilde{\mathfrak{t}}$ is $(\pi_1 \circ g_p)$ -compatible. As $\pi_1 \circ g_p = \gamma_p$, this precisely says that $\tilde{\mathfrak{t}}$ is γ_p -compatible. We therefore have a Bridgeland \mathbb{Z} -slicing, i.e., a bounded t -structure, $(\gamma_p)_! \tilde{\mathfrak{t}}$ on \mathcal{D} . The map Ψ is monotone and \mathbb{Z} -equivariant. Indeed, the t -structure Ψ_p is defined by the upper category

$$\mathcal{D}_{(\gamma_p)_! \tilde{\mathfrak{t}}; \geq 0} = \langle \mathcal{D}_{\tilde{\mathfrak{t}}; (n, \phi)} \rangle_{(n, \phi) \in \gamma_p^{-1}([0, +\infty))}.$$

If $p_1 \leq p_2$, then $n + p_1(\phi) \geq 0$ implies $n + p_2(\phi) \geq 0$, and so $\gamma_{p_1}^{-1}([0, +\infty)) \subseteq \gamma_{p_2}^{-1}([0, +\infty))$. This gives $\mathcal{D}_{(\gamma_{p_1})_! \tilde{\mathfrak{t}}; \geq 0} \subseteq \mathcal{D}_{(\gamma_{p_2})_! \tilde{\mathfrak{t}}; \geq 0}$, i.e.,

$$\mathcal{D}_{(\gamma_{p_1})_! \tilde{\mathfrak{t}}} \geq \mathcal{D}_{(\gamma_{p_2})_! \tilde{\mathfrak{t}}}$$

in the partial order on $\text{ts}(\mathcal{D})$. Similarly, $n + (p+1)(\phi) \geq 0$ if and only if $n + p(\phi) \geq -1$ and so

$$\mathcal{D}_{(\gamma_{p+1})_! \tilde{\mathfrak{t}}; \geq 0} = \mathcal{D}_{(\gamma_p)_! \tilde{\mathfrak{t}}; \geq -1} = \mathcal{D}_{(\gamma_p)_! \tilde{\mathfrak{t}}; \geq 0}[-1].$$

Finally, Ψ trivially (and uniquely) extends to $\widehat{\text{perv}}_{\mathbb{Z}}^{\text{op}}$ preserving maxima and minima. \square

-fino a qui- Quanto segue va rivisto alla luce della chiacchierata su wire: dobbiamo introdurre la nozione di strong semiorthogonal decomposition e notare che nel caso in cui abbiamo strong e t -strutture scambiabili allora abbiamo due modi di incollare e questi coincidono. We now recall the celebrated process from BBD of gluing two t -structures into a third one. This will turn out to induce a gluable slicing in our sense, thus showing that the language of the present paper is actually a generalization of a classical piece of literature. This, by the other hand, explains both our terminology and motivation.

Definition 0.44. A recollement datum is a diagram of stable ∞ -categories and exact ∞ -functors

$$\begin{array}{ccccc}
 & & i^* & & j^! \\
 & \swarrow & & \searrow & \\
 \mathcal{D}_{\text{fer}} & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}_{\text{ouv}} \\
 & \nwarrow & & \nearrow & \\
 & & i^! & & j_*
 \end{array}$$

such that the following holds:

1. $(i^*, i_*, i^!)$ and $(j^!, j^*, j_*)$ are adjoint triples,
2. $i_*, j_*, j^!$ are fully faithful and $j^*i_* = 0$,
3. for each object X of \mathcal{D} the (co)units of the above adjunctions give rise to cofiber sequences

$$\begin{array}{ccc}
 i_*i^!X & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & j_*j^*X
 \end{array}
 \qquad
 \begin{array}{ccc}
 j^!j^*X & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & i_*i^*X.
 \end{array}$$

Proposition 0.45. Suppose we are in the situation of 0.44. Then:

1. $(\mathcal{D}_0 = j_*\mathcal{D}_{\text{ouv}}, \mathcal{D}_1 = i_*\mathcal{D}_{\text{fer}})$ is a semiorthogonal decomposition on \mathcal{D} .
2. Given t -structures $\mathbf{t}_0, \mathbf{t}_1$ on \mathcal{D}_{ouv} and \mathcal{D}_{fer} respectively, the associated $\{0, 1\} \times_{\text{lex}} \mathbb{Z}$ -slicing on \mathcal{D} given by

$$\mathcal{D}_{0;\geq 0} = j_*(\mathcal{D}_{\text{ouv};\geq 0}), \quad \mathcal{D}_{1;\geq 0} = i_*(\mathcal{D}_{\text{fer};\geq 0})$$

is gluable.

Proof. Let's prove the first claim. The fact that j^* and j_* are adjoints implies that, for objects $i_*X \in \mathcal{D}_{\text{fer}}$ and $j_*Y \in \mathcal{D}_{\text{ouv}}$,

$$\mathcal{D}(i_*X, j_*Y) = \mathcal{D}(j^*i_*X, Y)$$

and the latter vanishes since $j^*i_* = 0$ by definition. This shows that $\mathcal{D}_1 \boxtimes \mathcal{D}_0$. Considering the left cofiber sequence from Definition 0.44 as a Postnikov tower of length one we see that $(\mathcal{D}_0, \mathcal{D}_1)$ is a $\{0, 1\}$ -slicing on \mathcal{D} , as desired.

□

—FINO A QUI 1.5—

Da qui esempi di Giovanni

1 A zoo of examples

Reminder: da mettere prima di tutto il tilting a la [referenza a collins](#). We now link the notions of the present paper with different work from literature. To begin with, we recall the following definition from [referenza a ekheda](#).

Definition 1.1. A Bridgeland $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}}$ -slicing of \mathcal{D} is called radical filtration if it is α -compatible, where $\alpha: \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}} \rightarrow \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}}$ is defined as $\alpha(n, m) = (n - m, -m)$.

forse conviene mettere la definizione esplicita con gli Ext, che e' ovviamente quella di ekheda.

Remark 1.2. Let \mathfrak{t} be a Bridgeland $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}}$ -slicing of \mathcal{D} and suppose that $\mathcal{D}_{=(n,m)} = \mathbf{0}$ for $m \in \mathbb{Z} \setminus \{0, 1\}$. Then \mathfrak{t} is a radical filtration. To give such a slicing is equivalent to the data of a bounded t -structure of \mathcal{D} together with an abelian $\{0, 1\}$ -slicing (i.e., a torsion pair in the sense of [referenza](#)) on its heart.

Remark 1.3. It is easy to see that the map φ from Lemma ?? is indeed an isomorphism of \mathbb{Z} -posets $\varphi: \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}} \rightarrow \mathbb{Z} \times_{\text{lex}} \mathbb{Z}$ where \mathbb{Z}_{tri} denotes \mathbb{Z} (as a poset) endowed with the trivial action by integers.

The following Lemma shows how natural is the apparently sophisticated definition of a radical filtration.

Lemma 1.4. Let \mathfrak{t} be a Bridgeland $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}}$ -slicing of \mathcal{D} . Then \mathfrak{t} is a radical filtration if and only if $\varphi_! \mathfrak{t}$ is a gluable $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}$ -slicing. In this case,

$$\alpha_! \mathfrak{t} = (\varphi^{-1} e \varphi)_! \mathfrak{t}$$

Proof. This simply follows from the commutativity of the following diagram of \mathbb{Z} -posets:

$$\begin{array}{ccc} \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}} & \xrightarrow{\alpha} & \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}} \\ \downarrow \varphi & & \downarrow \varphi \\ \mathbb{Z} \times_{\text{lex}} \mathbb{Z} & \xrightarrow{e} & \mathbb{Z} \times_{\text{lex}} \mathbb{Z} \end{array}$$

□

Definition 1.5. Let p be a perversity function (on \mathbb{Z}). We define $\alpha_p: \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}} \rightarrow \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}}$ as $\alpha_p(n, m) = (m - p(n), -p(n))$.

Remark 1.6. The map α from Definition 1.1 corresponds simply to α_1 , where 1 denotes the identity of \mathbb{Z} . The identity itself is a special perversity function: it is the only idempotent one [questo dovrebbe essere il motivo combinatorio per la dualita' di Koszul](#).

Lemma 1.7. Let \mathfrak{t} be a Bridgeland $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{tri}}$ -slicing of \mathcal{D} . The following hold:

1. If \mathfrak{t} is gluable, then it is a radical filtration.

2. If \mathfrak{t} is a radical filtration, then it is α_p -compatible for every perversity function p . By considering the t -structure associated to $(\alpha_p)_! \mathfrak{t}$, this defines a morphism of \mathbb{Z} -posets:

$$\mathrm{perv}_{\mathbb{Z}} \rightarrow \mathrm{ts}(\mathcal{D})$$

Proof. Si fa coi conti. Tuttavia mi piacerebbe trovare una prova piu' concettuale che si basi sul Lemma 1.3. \square

Starting with a radical filtration \mathfrak{t} we can then use the isomorphism from Corollary 0.28 to get a morphism of \mathbb{Z} -posets

$$\mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \rightarrow \mathrm{ts}(\mathcal{D})$$

This can finally be regarded as a cohomology theory where dimension is an element of the poset $\mathbb{Z} \times \mathbb{Z}$, which is not totally ordered. In a certain sense, *dimensions are not always comparable*.

Now, the notion of gluability is somehow asymmetric. However, one implication always holds, as shown below.

Lemma 1.8. *Let J_1 and J_2 be \mathbb{Z} -tosets, \mathfrak{t} be a Bridgeland $J_1 \times_{\mathrm{lex}} J_2$ -slicing of \mathcal{D} . If \mathfrak{t} is gluable and \mathbb{Z} acts trivially on J_1 , then $e_! \mathfrak{t}$ is a gluable $J_2 \times_{\mathrm{lex}} J_1$ slicing of \mathcal{D} .*

Remark 1.9. The notion of Bridgeland $\mathbb{Z}_{\mathrm{tri}}$ -slicing already appears in literature: it is no more than an infinite version of a semiorthogonal decomposition in the sense of [referenza](#), or a 'baric structure' as defined in [referenza](#). Thus, we can start with a baric structure of \mathcal{D} and a boudned t -structure of $\mathcal{D}_{=n}$ for every $n \in \mathbb{Z}_{\mathrm{tri}}$. If the associated $\mathbb{Z}_{\mathrm{tri}} \times_{\mathrm{lex}} \mathbb{Z}$ -slicing \mathfrak{t} is gluable, $e_! \mathfrak{t}$ is gluable by Lemma 1.8 and thus, using Lemma 1.8 we get a morphism of \mathbb{Z} -posets

$$\mathrm{perv}_{\mathbb{Z}} \rightarrow \mathrm{ts}(\mathcal{D})$$

We can build like that whole new classes of 'perverse' t -structures on \mathcal{D} . We'll see an instance of this construction in the example below.

Example 1.10. In this example we relate the gluability condition with the Beilinson-Soulé conjecture from motivic topology. We fix a field k and, just for this example, we stick to a more traditional 1-categorical setup. Recall that the existence of motives, which is still an open question in general, was conjectured by Grothendieck in order to build a universal Weyl cohomology (also called 'motivic cohomology') theory for schemes. However, Deligne observed that it could be easier to construct a triangulated category (the 'mixed' motives) which should play the role of the derived category of motives, and later recover the latter as the heart of a bounded t -structure. Voevodskij finally succeeded in constructing a triangulated category of mixed rational motives over k which contains, for $n \in \mathbb{Z}$, a 'Tate object' $\mathbb{Q}(n)$ that represents the n -th motivic cohomology functor. Now, let us consider the triangualted subcategory $\mathcal{D}\mathrm{TM}_k$ generated by the Tate

objects. In other words, \mathcal{DTM}_k is the category of mixed rational Tate motives. We then get isomorphisms of groups

$$\mathcal{DTM}_k(\mathbb{Q}(i), \mathbb{Q}(j)[n]) = K_{2(j-i)-n}(k)^{(j-i)}$$

where $K_a(k)$ is the a -th higher K -theory group of the point $\mathrm{Spec}(k)$ and $K_a(k)^{(b)}$ is the weight b summand of $K_a(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to the Adams action (in other words, $K_a(k)^{(b)}$ is the a -th b -codimensional Bloch's higher Chow group of $\mathrm{Spec}(k)$ with rational coefficients). For dimensional reasons we immediately see that the right hand side vanishes for $i < j$ and, when $i = j$, for $n \neq 0$. In other words, the Tate objects form an infinite exceptional collection ([referenza a decomposizioni semiortogonali](#)) on \mathcal{DTM}_k which is clearly full by definition. By the general theory of semiorthogonal decomposition, this simply means that setting $(\mathcal{DTM}_k)_{=n}$ as the triangulated subcategory generated by $\mathbb{Q}(n)$ defines slices of a baric structure with exact equivalences

$$(\mathcal{DTM}_k)_{=n} \simeq \mathcal{D}^b(\mathbb{Q})$$

where the member on the right is the bounded derived category of finite-dimensional rational vector spaces (which is equivalent to the abelian category graded vector spaces which finite-dimensional homogeneous pieces). The latter, being a derived category, possesses a canonical bounded t -structure and this finally induces a Bridgeland $\mathbb{Z}_{\mathrm{tri}} \times_{\mathrm{lex}} \mathbb{Z}$ slicing on \mathcal{DTM}_k .

Now, the latter slicing is gluable if and only if $K_{2(j-i)-n}(k)^{(j-i)} = \mathbf{0}$ whenever both $i < j$ and $n \leq 0$ hold. This is exactly the Beilinson-Soulé standard vanishing conjecture, which is now known to hold, for example, when k is a number field due to a celebrated computation by Borel. In this case, by applying $e_!$ we get a Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$ -slicing on \mathcal{DTM}_k and thus a bounded t -structure whose heart contains the desired unmixed Tate motives over k . In other words, we recover a well known fact (see [referenza Levine](#)) using a rather abstract and general language: assuming the Beilinson-Soulé conjecture, (Tate) motives exist. Moreover, following the route of Remark 1.9, we get a t -structure for each perversity function. Those are the '*perverse motives*' appearing in [referenza](#).

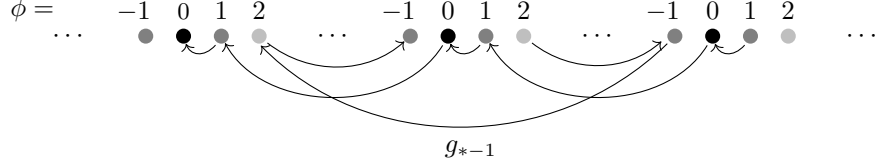
Fino a qui. Tutto cio' che segue e' parte della tesi, e quindi da cancellare alla fine.

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t -structure, associating to a perversity p the bounded t -structure coming from $(g_p)_{\mathcal{Q}}(\mathcal{D})$ defines a morphism of \mathbb{Z} -posets

$$\Xi^{\mathrm{op}} \longrightarrow \mathbf{bts}(\mathcal{D})$$

we can restate this as:

♫ A grading or perverse filtration on the heart of a bounded t -structure on \mathcal{D} induces a presheaf of t -structures on $\mathbb{Z} \times \mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z} \times \mathbb{Z}$ -slicing' on \mathcal{D} .



Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\text{op}} \hookrightarrow \Xi$$

and the t -structure coming from $(g_{\chi_I})_{\Omega}(\mathcal{Q})$ is just the tilting of \mathfrak{t} with respect to the torsion pair coming from I .

The following proposition gives a characterization of the new heart obtained by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 1.11. *Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , \mathcal{P} a grading filtration on $\mathfrak{H}_{\mathfrak{t}}$, \mathcal{Q} the associated $\mathbb{Z} \times \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , p a perversity. Denote \mathfrak{q} the bounded t -structure associated to $(g_p)_{\Omega}(\mathcal{Q})$. Then $\mathfrak{H}_{\mathfrak{q}}$ consists of objects $X \in \mathcal{D}$ so that*

$$H_{\mathfrak{t}}^k(X) \in \mathcal{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if \mathcal{P} is a mixed filtration then for each $X \in \mathfrak{H}_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \mathfrak{H}_{\mathfrak{q}}$ if and only if

$$H_{\mathcal{P}}^{\phi}(H_{\mathfrak{t}}^k(X)[-k])[k] = H_{\mathcal{Q}}^{(k, \phi)}(X) = 0$$

for $p(\phi) \neq -k$.

For the second part of the claim, the abelian \mathbb{Z} -slicing induced on $\mathfrak{H}_{\mathfrak{q}}$ is split by **Proposition 0.43** and thus by **Proposition ??**

$$X = \bigoplus_{(k, \phi)} H_{\mathcal{Q}}^{(k, \phi)}(X) = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

where the last equality comes from the first part. □

Example 1.12. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an \mathbb{N} -graded ring with B_0 semisimple. Denote \mathcal{A} the category of \mathbb{Z} -graded B -modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote \mathcal{P}_ϕ the full subcategory of \mathcal{A} of modules concentrated in degree ϕ . Clearly, \mathcal{P} defines an abelian \mathbb{Z} -slicing on \mathcal{A} . Following [?] we have

$$\mathrm{Ext}_{\mathcal{A}}^n(\mathcal{P}_\phi, \mathcal{P}_\psi) = 0$$

for $n > \psi - \phi$. This means that \mathcal{P} is a mixed filtration and the bounded t-structure on $\mathcal{D}^b(\mathcal{A})$ associated to $(g_1)_\Omega(\mathcal{Q})$ (where 1 is the identity of \mathbb{Z}) is the 'diagonal' (or 'geometric') t-structure which appears in Koszul duality and other areas.

Example 1.13. Let M be an n -dimensional smooth complex projective variety and consider the n -torsion pair \mathcal{P} on $\mathrm{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that \mathcal{P} , seen as an abelian \mathbb{Z} -slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t-structure associated to $(f_p)_\Omega(\mathcal{Q})$ is the one of perverse coherent sheaves as constructed in [?]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n -slicing on the heart of perverse coherent sheaves as done in [?].

Da qui in poi ci sono cose già riportate nella parte sistemata. Le lascio commentate nel file.

