

## 0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of  $\mathbb{Z}$ -posets (but just a  $\mathbb{Z}$ -equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let  $J$  be a  $\mathbb{Z}$ -toset, i.e., a totally ordered set together with a monotone action of  $\mathbb{Z}$ , that we will denote by  $(n, x) \mapsto x + n$ . Recall that a  $J$ -slicing on a stable  $\infty$ -category  $\mathcal{D}$  is a morphism of  $\mathbb{Z}$ -posests  $\mathfrak{t}: \mathcal{O}(J) \rightarrow \text{ts}(\mathcal{D})$ , where  $\mathcal{O}(J)$  is the  $\mathbb{Z}$ -poset of the slicings of  $J$  (i.e., the decompositions of  $J$  into a disjoint union of a lower set  $L$  and of an upper set  $U$ ), and  $\text{ts}(\mathcal{D})$  is the  $\mathbb{Z}$ -poset of  $t$ -structures on  $\mathcal{D}$ .

Clearly, if  $f: J \rightarrow J'$  is a morphism of  $\mathbb{Z}$ -tosets, then  $f^{-1}: \{\text{subsets of } J'\} \rightarrow \{\text{subsets of } J\}$  induces a  $\mathbb{Z}$ -equivariant morphism of  $\mathbb{Z}$ -posets  $f^{-1}: \mathcal{O}(J') \rightarrow \mathcal{O}(J)$ , and so composition with  $f^{-1}$  gives a morphism

$$f_*: J\text{-slicings on } \mathcal{D} \rightarrow J'\text{-slicings on } \mathcal{D}$$

$$\mathfrak{t} \mapsto \mathfrak{t} \circ f^{-1}.$$

The slices of  $f_*\mathfrak{t}$  are clearly given by  $\mathcal{D}_{f_*\mathfrak{t};\phi} = \mathcal{D}_{\mathfrak{t};f^{-1}(\{\phi\})}$ . Notice that, as  $f$  is monotone, the subset  $f^{-1}(\{\phi\})$  is an interval in  $J$ . It is immediate to see that  $f_*$  restricts to a map

$$f_*: \text{Bridgeland } J\text{-slicings on } \mathcal{D} \rightarrow \text{Bridgeland } J'\text{-slicings on } \mathcal{D}$$

Namely, if  $\mathcal{H}_{f_*\mathfrak{t}}^\phi(X) = 0$  for every  $\phi \in J'$  then  $\mathcal{H}_{\mathfrak{t}}^{f^{-1}(\{\phi\})}(X) = 0$  for every  $\phi$  in  $J'$ . As we are assuming the  $J$ -slicing  $\mathfrak{t}$  is a Bridgeland slicing, this implies that  $\mathcal{H}_{\mathfrak{t}}^\psi(X) = 0$  for every  $\psi$  in  $f^{-1}(\{\phi\})$ , for every  $\phi$ . Therefore  $\mathcal{H}_{\mathfrak{t}}^\psi(X) = 0$  for every  $\psi$  in  $J$  and so, again by definition of Bridgeland slicing,  $X = 0$ . Also, if  $\mathcal{H}_{f_*\mathfrak{t}}^\phi(X) \neq 0$  then  $\mathcal{H}_{\mathfrak{t}}^{f^{-1}(\{\phi\})}(X) \neq 0$  and so (again by the Bridgeland slicing condition) there exists at least an element  $\psi$  in  $f^{-1}(\{\phi\})$  such that  $\mathcal{H}_{\mathfrak{t}}^\psi(X) \neq 0$ . As the  $J$ -slicing  $\mathfrak{t}$  is Bridgeland, the total of these  $\psi$ 's must be finite, so only for finitely many  $\phi$  we can have such a  $\psi$ . In other words, the number of indices  $\phi$  in  $J'$  such that  $\mathcal{H}_{f_*\mathfrak{t}}^\phi(X) \neq 0$  is finite.

Notice that, if  $\mathfrak{t}$  is a Bridgeland slicing of  $\mathcal{D}$ , and  $f: J \rightarrow J'$  is a morphism of  $\mathbb{Z}$ -tosets, then for any slicing  $(L, U)$  of  $J'$ , the lower and the upper categories  $\mathcal{D}_{f_*\mathfrak{t}}(L)$  and  $\mathcal{D}_{f_*\mathfrak{t}}(U)$  can be equivalently defined as

$$\mathcal{D}_{f_*\mathfrak{t}}(L) = \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$$

$$\mathcal{D}_{f_*\mathfrak{t}}(U) = \langle \mathcal{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}$$

where  $\langle \mathcal{S} \rangle$  denotes the extension-closed subcategory of  $\mathcal{D}$  generated by the subcategory  $\mathcal{S}$ .<sup>1</sup>

<sup>1</sup> At some point it will be useful to recall Lemma 4.21 from ???: Let  $\mathcal{S}_1, \mathcal{S}_2$  be two subcategories of  $\mathcal{D}$  with  $\mathcal{S}_1 \boxtimes \mathcal{S}_2$ , i.e., such that  $\mathcal{D}(X_2, X_1)$  is contractible for any  $X_1 \in \mathcal{S}_1$  and any  $X_2 \in \mathcal{S}_2$ . Then  $\mathcal{S}_1 \boxtimes \langle \mathcal{S}_2 \rangle$  and  $\langle \mathcal{S}_1 \rangle \boxtimes \mathcal{S}_2$ , and so  $\langle \mathcal{S}_1 \rangle \boxtimes \langle \mathcal{S}_2 \rangle$ .

The right hand sides of the above two expressions can clearly be defined for every morphism  $f$  from  $J$  to  $J'$  (i.e., not necessarily monotone nor  $\mathbb{Z}$ -equivariant), and as soon as  $f$  is  $\mathbb{Z}$ -equivariant, the assignment

$$(L, U \mapsto (\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}))$$

is an equivariant morphism from  $\mathcal{O}(J)$  to pairs of subcategories of  $\mathcal{D}$ . Clearly, when  $f$  is not monotone there is no reason to expect that the pair  $(\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U})$  forms a  $t$ -structure on  $\mathcal{D}$ . Yet, it is interesting to notice that the condition that  $f$  be monotone is only sufficient in order to have this, and can indeed be relaxed.

**Definition 0.1.** *Let  $J$  and  $J'$  be  $\mathbb{Z}$ -posets, and let  $f: J \rightarrow J'$  a map of  $\mathbb{Z}$ -sets (i.e., a  $\mathbb{Z}$ -equivariant map, not necessarily nondecreasing). A Bridgeland  $J$ -slicing  $\mathbf{t}$  of  $\mathcal{D}$  is  $f$ -compatible if  $f(\phi) > f(\psi)$  with  $\phi \leq \psi$  implies  $\mathcal{D}_{\mathbf{t};\psi} \boxtimes \mathcal{D}_{\mathbf{t};\geq \phi-1}$ .*

**Remark 0.2.** Clearly, if  $f$  is monotone, then every  $J$ -slicing  $\mathbf{t}$  is order restoring as the condition ' $f(\phi) > f(\psi)$  with  $\phi \leq \psi$ ' is empty.

**Lemma 0.3.** *Let  $f: J \rightarrow J'$  be a  $\mathbb{Z}$ -equivariant morphism of  $\mathbb{Z}$ -sets (i.e., not necessarily a monotone map) and let  $\mathbf{t}$  be a Bridgeland slicing of  $\mathcal{D}$  which is  $f$ -compatible. Then  $(\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U})$  is a  $t$ -structure on  $\mathcal{D}$ .*

*Proof.* As  $f$  is  $\mathbb{Z}$ -equivariant and  $U + 1 \subseteq U$ , we have

$$\begin{aligned} \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}[1] &= \langle \mathcal{D}_{\mathbf{t};\phi}[1] \rangle_{f(\phi) \in U} \\ &= \langle \mathcal{D}_{\mathbf{t};\phi+1} \rangle_{f(\phi) \in U} \\ &= \langle \mathcal{D}_{\mathbf{t};\phi+1} \rangle_{f(\phi+1) \in U+1} \\ &= \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U+1} \\ &\subseteq \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U} \\ &= \mathcal{D}_{\mathbf{t};\phi} \end{aligned}$$

and similarly for the lower subcategory  $\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}$ . To show that

$$\langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L} \boxtimes \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}$$

it suffices to show that, if  $f(\psi) \in L$  and  $f(\phi) \in U$  then  $\mathcal{D}_{\mathbf{t};\psi} \boxtimes \mathcal{D}_{\mathbf{t};\phi}$ . As  $L \cap U = \emptyset$  we cannot have  $\psi = \phi$ , so either  $\psi < \phi$  or vice versa. In the first case,  $\mathcal{D}_{\mathbf{t};\psi} \boxtimes \mathcal{D}_{\mathbf{t};\phi}$  by definition of Bridgeland  $J$ -slicing. In the second case, we have  $\phi < \psi$  and  $f(\phi) > f(\psi)$  as  $f(\phi) \in U$  and  $f(\psi) \in L$ . Therefore, since  $\mathbf{t}$  is  $f$ -compatible,  $\mathcal{D}_{\mathbf{t};\psi} \boxtimes \mathcal{D}_{\mathbf{t};\geq \phi-1}$  and so in particular  $\mathcal{D}_{\mathbf{t};\psi} \boxtimes \mathcal{D}_{\mathbf{t};\phi}$ . Finally, we have to show that every object  $X$  in  $\mathcal{D}$  fits into a fiber sequence

$$\begin{array}{ccc} X_U & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X_L \end{array}$$

with  $X_L \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}$  and  $X_U \in \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}$ . As  $\mathbf{t}$  is a Bridgeland slicing, we have a factorization of the initial morphism  $\mathbf{0} \rightarrow X$  of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{i}}} X_{\bar{i}} \xrightarrow{\alpha_{\bar{i}+1}} X_{\bar{i}+1} \rightarrow \cdots \xrightarrow{\alpha_n} X_n = X$$

with  $\mathbf{0} \neq \text{cofib}(\alpha_i) = \mathcal{H}_{\mathbf{t}}^{\phi_i}(X) \in \mathcal{D}_{\phi_i}$  for all  $i = 1, \dots, n$ , with  $\phi_i > \phi_{i+1}$ . Let us now consider the sequence of symbols  $L$  and  $U$  obtained putting in the  $i$ -th place  $L$  if  $f(\phi_i) \in L$  and  $U$  if  $f(\phi_i) \in U$ . If this sequence is of the form  $(U, U, \dots, U, L, L, \dots, L)$ , then there exists an index  $\bar{i}$  such that  $f(\phi_i) \in U$  for  $i \leq \bar{i}$  and  $f(\phi_i) \in L$  for  $i > \bar{i}$  (with  $\bar{i} = -1$  or  $n$  when all of the  $f(\phi_i)$  are in  $L$  or in  $U$ , respectively). Then we can consider the pullout diagram

$$\begin{array}{ccc} X_{\bar{i}} & \xrightarrow{\quad} & \mathbf{0} \\ f_L \downarrow & & \downarrow \\ X & \xrightarrow{\quad} & \text{cofib}(f_L) \end{array}$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{i}}} X_{\bar{i}}$$

and

$$X_{\bar{i}} \xrightarrow{\alpha_{\bar{i}+1}} X_{\bar{i}+1} \rightarrow \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that  $X_{\bar{i}} \in \langle \cup_{i=0}^{\bar{i}} \mathcal{D}_{\phi_i} \rangle \subseteq \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in U}$  while the second factorization shows that  $\text{cofib}(f_L) \in \langle \cup_{i=\bar{i}+1}^n \mathcal{D}_{\phi_i} \rangle \subseteq \langle \mathcal{D}_{\mathbf{t};\phi} \rangle_{f(\phi) \in L}$ . So we are done in this case. Therefore, we are reduced to showing that we can always avoid a  $(\dots, L, U, \dots)$  situation in our sequence of  $L$ 's and  $U$ 's. Assume we have such a situation. Then we have an index  $i_0$  with  $f(\phi_{i_0}) \in L$  and  $f(\phi_{i_0+1}) \in U$ . This in particular implies  $f(\phi_{i_0+1}) > f(\phi_{i_0})$  with  $\phi_{i_0+1} < \phi_{i_0}$ . As  $\mathbf{t}$  is  $f$ -compatible, this gives  $\mathcal{D}_{\mathbf{t};\phi_{i_0}} \boxtimes \mathcal{D}_{\mathbf{t};\geq(\phi_{i_0+1})-1}$ , i.e.  $(\mathcal{D}_{\mathbf{t};\phi_{i_0}}[1]) \boxtimes \mathcal{D}_{\mathbf{t};\geq\phi_{i_0+1}}$ . In particular,  $\mathcal{D}(\mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X)[1])$  is contractible. Now consider the pasting of pullout diagrams

$$\begin{array}{ccccccc} X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\ \downarrow & & \downarrow & & \downarrow \gamma & & \\ \mathbf{0} & \longrightarrow & \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X) & \longrightarrow & Y & \longrightarrow & \mathbf{0} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{0} & \longrightarrow & \mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X)[1]. \end{array}$$

As the arrow  $\mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X) \rightarrow \mathcal{H}_{\mathbf{t}}^{\phi_{i_0}}(X)[1]$  factors through  $\mathbf{0}$ , we have  $Y = \mathcal{H}_{\mathbf{t}}^{\phi_{i_0+1}}(X) \oplus$

$\mathcal{H}_t^{\phi_{i_0}}(X)$  and the above diagram becomes

$$\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X) & \xrightarrow{\iota_2} & \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \pi_1 & & \downarrow \\
& & 0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X)[1],
\end{array}$$

where  $\iota_2$  and  $\pi_1$  are the canonical inclusion and projection. Let  $\beta: X_{i_0+1} \rightarrow \mathcal{H}_t^{\phi_{i_0}}(X)$  be the composition

$$\beta: X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_t^{\phi_{i_0}}(X).$$

Then we have a homotopy commutative diagram

$$\begin{array}{ccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} \\
\downarrow & & \downarrow & & \downarrow \beta \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X) & \xrightarrow{\text{id}} & \mathcal{H}_t^{\phi_{i_0}}(X)
\end{array}$$

(where only the left square is a pullout), and so the composition  $\alpha_{i_0+1} \circ \alpha_{i_0}$  factors through the homotopy fiber of  $\beta$ . In other words, we have a homotopy commutative diagram

$$\begin{array}{ccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} \\
\tilde{\alpha}_{i_0} \downarrow & & \downarrow \alpha_{i_0+1} \\
\text{fib}(\beta) & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1}
\end{array}$$

Writing  $\tilde{X}_{i_0} = \text{fib}(\beta)$ , we get the pasting of pullout diagrams

$$\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\tilde{\alpha}_{i_0}} & \tilde{X}_{i_0} & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0+1}}(X) & \xrightarrow{\iota_1} & \mathcal{H}_t^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_t^{\phi_{i_0}}(X) & \longrightarrow & \\
& & \downarrow & & \downarrow \pi_2 & & \\
& & 0 & \longrightarrow & \mathcal{H}_t^{\phi_{i_0}}(X). & & 
\end{array}$$

That is, by considering the factorization  $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$  we have switched the cofibers with respect to the original factorization  $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$ . Therefore, writing  $\tilde{\phi}_{i_0} = \phi_{i_0+1}$  and  $\tilde{\phi}_{i_0+1} = \phi_{i_0}$ , we now have  $f(\tilde{\phi}_{i_0}) \in U$  and  $f(\tilde{\phi}_{i_0+1}) \in L$ . That is, we have removed the  $(\dots, L, U, \dots)$  situation from the position  $i_0$ , replacing it with a  $(\dots, U, L, \dots)$  situation, while keeping all the labels  $L, U$  before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the  $(\dots, L, U, \dots)$  situations.<sup>2</sup>

□

**Proposition 0.4.** *Let  $J \xrightarrow{f} J'$  be a  $\mathbb{Z}$ -equivariant map (not necessarily increasing) between  $\mathbb{Z}$ -posets,  $\mathcal{P}$  a  $J$ -slicing on  $\mathcal{D}$ . Suppose that  $\mathcal{P}_\psi \subseteq \mathcal{P}_\phi^\perp$  whenever one of the following conditions holds:*

- (a)  $f(\phi) > f(\psi)$
- (b)  $f(\phi) + 1 > f(\psi)$  and  $\phi + 1 < \psi$

Then

$$(f_\Omega(\mathcal{P}))_\phi = \mathcal{P}_{f^{-1}(\phi)}$$

defines a  $J'$ -slicing on  $\mathcal{D}$ .

*Proof.* Part (1) of definition of slicing follows from  $\mathbb{Z}$ -equivariance of  $f$ , while part (2) follows from condition (a) above. Pick  $0 \neq X \in \mathcal{D}$  and consider its Postnikov tower with respect to  $\mathcal{P}$ :

$$0 = Y_0 \xrightarrow{\beta_1} \dots \xrightarrow{\beta_n} Y_n = X$$

with  $\text{cone}(\beta_i) = H_{\mathcal{P}}^{\phi_i}(X)$ . Going from left to right, if we encounter an  $i$  so that  $f(\phi_{i+1}) > f(\phi_i)$  then by condition (b)

$$\text{Hom}_{\mathcal{D}}(H_{\mathcal{P}}^{\phi_{i+1}}(X), H_{\mathcal{P}}^{\phi_i}(X)[1]) = 0$$

Using **Proposition ??** and the  $3 \times 3$  **Lemma**, we can complete the identity of  $Y_i$  to get a diagram:

$$\begin{array}{ccccccc}
H_{\mathcal{P}}^{\phi_{i+1}}(X)[-1] & \longrightarrow & Y_i & \longrightarrow & Y_{i+1} & \longrightarrow & H_{\mathcal{P}}^{\phi_{i+1}}(X)[1] \\
\downarrow & & \downarrow 1 & & \downarrow & & \downarrow \\
Y_{i-1} & \longrightarrow & Y_i & \longrightarrow & H_{\mathcal{P}}^{\phi_i}(X) & \longrightarrow & Y_{i-1}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & 0 & \longrightarrow & A[1] & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{\mathcal{P}}^{\phi_{i+1}}(X) & \longrightarrow & Y_i[1] & \longrightarrow & Y_{i+1}[1] & \longrightarrow & H_{\mathcal{P}}^{\phi_{i+1}}(X)[1]
\end{array}$$

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<sup>2</sup>This is somehow reminiscent of the ‘bubble sort’ algorithm.

We then replace  $\beta_i$  with  $Y_{i-1} \longrightarrow A$  and  $\beta_{i+1}$  with  $A \longrightarrow Y_{i+1}$ . Iterating this process<sup>3</sup>, we get a factorization

$$0 = Z_0 \xrightarrow{\gamma_1} \dots \xrightarrow{\gamma_n} Z_n = X$$

with  $\text{cone}(\gamma_i) = H_{\mathcal{D}}^{\phi_{k_i}}(X)$  for some  $k_i$  and  $f(\phi_i) \geq f(\phi_{i+1})$  for all  $i$ . To get a factorization with the strict latter inequality we use the same argument as in the proof of **Proposition ??**.  $\square$

The above proof also shows that, in the same hypotheses and notation,  $(f_{\Omega}(\mathcal{P}))_{\phi}$  consists of objects  $X \in \mathcal{D}$  so that  $H_{\mathcal{D}}^{\psi}(X) = 0$  for  $f(\psi) \neq \phi$ . Also, **Proposition ??** holds when  $f$  is bijective.

**Definition 0.5.** Let  $\mathfrak{t}$  be a bounded  $t$ -structure on  $\mathcal{D}$ . An abelian  $\mathbb{Z}$ -slicing  $\mathcal{P}$  on  $\heartsuit_{\mathfrak{t}}$  is called:

- **perverse filtration** if  $\mathcal{P}_{\psi}[n] \subseteq \mathcal{P}_{\phi}^{\perp}$  for  $n < \phi - \psi$
- **grading filtration** if it is a perverse filtration and  $\mathcal{P}_{\psi}[n] \subseteq \mathcal{P}_{\phi}^{\perp}$  for  $2 \leq n = \phi - \psi$
- **mixed filtration** if  $\mathcal{P}_{\psi}[n] \subseteq \mathcal{P}_{\phi}^{\perp}$  for  $n > \psi - \phi$

The following implications hold:

$$\text{mixed} \implies \text{grading} \implies \text{perverse}$$

Clearly, a torsion pair on  $\heartsuit_{\mathfrak{t}}$  (seen as an abelian  $\mathbb{Z}$ -slicing via the inclusion  $[1] \subseteq \mathbb{Z}$ ) is always a grading filtration. Historically, grading filtrations first appeared in [?] under the name of 'radical filtrations', while mixed filtrations are covered in the last section of [?].

**Definition 0.6.** A map  $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$  is called **perversity** if it is a monotone contraction, i.e. if

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi$$

whenever  $\phi \geq \psi$  in  $\mathbb{Z}$ .

We denote  $\Xi$  the set of perversities.

Clearly,  $\Xi$  defines a  $\mathbb{Z}$ -poset which is not totally ordered. However, it is a distributive lattice: we have meets and joins given by:

$$(p \wedge q)(\phi) = \min\{p(\phi), q(\phi)\}$$

$$(p \vee q)(\phi) = \max\{p(\phi), q(\phi)\}$$

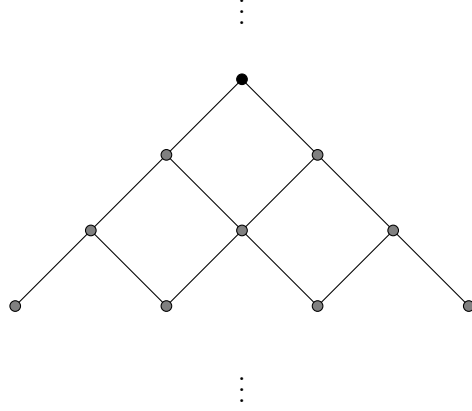
Indeed, if we place on  $\mathbb{Z} \times \mathbb{Z}$  the product order, we have an isomorphism of  $\mathbb{Z}$ -posets

$$\Xi^{\text{op}} = O(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

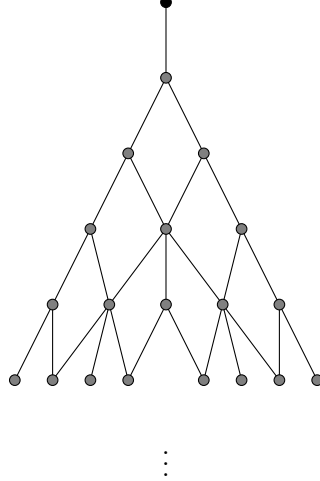
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<sup>3</sup>This is somehow reminiscent of the 'bubble sort' algorithm.

given by sending the zero perversity to  $A = \{(\phi, \psi)\}_{\phi \geq 0}$  and the identity to  $B = \{(\phi, \psi)\}_{\psi \geq 0}$  ( $B + i$  and  $A + i$  with  $i \in \mathbb{Z}$  generate the latter as a complete lattice). Considering the canonical embedding  $\mathbb{Z} \hookrightarrow \mathbb{Z} \subseteq O(\mathbb{Z} \times \mathbb{Z})$  we get a sublattice of  $\Xi$  depicted as



This induces the 't-tree' described in [?], as discussed below in a far-reaching generality. By the other hand, each square above contains moreover a copy of the Young lattice



with the root corresponding to the upper vertex of the square.

**Proposition 0.7.** *Let  $\mathfrak{t}$  be a bounded t-structure on  $\mathcal{D}$ ,  $\mathcal{P}$  an abelian  $\mathbb{Z}$ -slicing on  $\heartsuit_{\mathfrak{t}}$ ,  $\mathcal{Q}$  the associated  $\mathbb{Z} \times \hat{\mathbb{Z}}$ -slicing on  $\mathcal{D}$ ,  $p$  a perversity. We have:*

1. *if  $\mathcal{P}$  is a perverse filtration, then  $\mathcal{Q}$  satisfies the assumptions of **Proposition 0.4** with respect to the map  $\mathbb{Z} \times \hat{\mathbb{Z}} \xrightarrow{f_p} \mathbb{Z} \times \hat{\mathbb{Z}}$  given by*

$$f_p(n, \phi) = (n + p(\lfloor \phi/2 \rfloor), -p(\lfloor \phi/2 \rfloor))$$

2. if  $\mathcal{P}$  is a grading filtration, then  $\mathcal{Q}$  satisfies the assumptions of **Proposition 0.4** with respect to the map  $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{g_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$  given by

$$g_p(n, \phi) = (n + p(\phi), -p(\phi))$$

3. if  $\mathcal{P}$  is a mixed filtration, then the abelian  $\mathbb{Z}$ -slicing induced on the heart of the bounded t-structure associated to  $(g_p)_\Omega(\mathcal{Q})$  is split

*Proof.* Let's prove (2). Suppose  $g_p(n, \phi) > g_p(m, \psi)$ . Then either  $m - n < p(\phi) - p(\psi)$  or both  $m - n = p(\phi) - p(\psi)$  and  $p(\psi) < p(\phi)$ . Since by definition of t-structure we can assume  $m \geq n$ , the second case is absurd while in the first case, since  $p$  is monotone, we have  $\phi \geq \psi$  and thus by definition of perversity

$$m - n < p(\phi) - p(\psi) \leq \phi - \psi$$

and we get the desired Hom-vanishing by definition of grading filtration.

Suppose now  $f_p(n, \phi) + 1 > f_p(m, \psi)$  and  $(n + 1, \phi) < (m, \psi)$ . The only non absurd case is  $1 < m - n \leq p(\phi) - p(\psi)$ . But then again we have

$$2 \leq m - n \leq p(\phi) - p(\psi) \leq \phi - \psi$$

and we can conclude as above.

To prove (1) consider the monotone map

$$\mathbb{Z} \xrightarrow{[* / 2]} \mathbb{Z}$$

Applying the slice functor to the latter and starting with a perverse filtration, we get a grading filtration by the properties of the floor function and the thesis follows from part (2).

Let's prove (3). We have to show that

$$\mathcal{P}_\psi[1 - p(\psi)] \subseteq \mathcal{P}_\phi[p(\phi)]^\perp$$

for  $-p(\psi) > -p(\phi)$ . We denote  $n = p(\phi) - p(\psi) - 1$ . Assuming again  $n \geq 0$ , we have  $n \geq 0 > p(\psi) - p(\phi) \geq \psi - \phi$  and we can then conclude by definition of mixed filtration.  $\square$

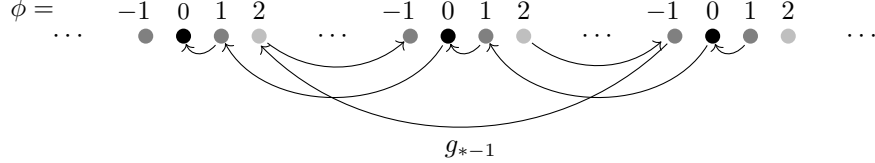
Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity  $p$  the bounded t-structure coming from  $(g_p)_\Omega(\mathcal{Q})$  defines a morphism of  $\mathbb{Z}$ -posets

$$\Xi^{\text{op}} \longrightarrow \mathbf{bts}(\mathcal{Q})$$

we can restate this as:



♫ A grading or perverse filtration on the heart of a bounded  $t$ -structure on  $\mathcal{D}$  induces a presheaf of  $t$ -structures on  $\mathbb{Z} \times \mathbb{Z}$  with the product order, and thus some kind of a 'non totally ordered  $\mathbb{Z} \times \mathbb{Z}$ -slicing' on  $\mathcal{D}$ .



Now, by sending an upper set of  $I \in O(\mathbb{Z})$  to its characteristic function  $\chi_I$  we get an embedding

$$O(\mathbb{Z})^{\text{op}} \hookrightarrow \Xi$$

and the  $t$ -structure coming from  $(g_{\chi_I})_{\Omega}(\mathcal{Q})$  is just the tilting of  $\mathfrak{t}$  with respect to the torsion pair coming from  $I$ .

The following proposition gives a characterization of the new heart obtained by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

**Proposition 0.8.** *Let  $\mathfrak{t}$  be a bounded  $t$ -structure on  $\mathcal{D}$ ,  $\mathcal{P}$  a grading filtration on  $\mathfrak{H}_{\mathfrak{t}}$ ,  $\mathcal{Q}$  the associated  $\mathbb{Z} \times \mathbb{Z}$ -slicing on  $\mathcal{D}$ ,  $p$  a perversity. Denote  $\mathfrak{q}$  the bounded  $t$ -structure associated to  $(g_p)_{\Omega}(\mathcal{Q})$ . Then  $\mathfrak{H}_{\mathfrak{q}}$  consists of objects  $X \in \mathcal{D}$  so that*

$$H_{\mathfrak{t}}^k(X) \in \mathcal{P}_{p^{-1}(-k)}[k]$$

for each  $k \in \mathbb{Z}$ . Moreover, if  $\mathcal{P}$  is a mixed filtration then for each  $X \in \mathfrak{H}_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

*Proof.* This is very similar to **Proposition ??**: we have that  $X \in \mathfrak{H}_{\mathfrak{q}}$  if and only if

$$H_{\mathcal{Q}}^{\phi}(H_{\mathfrak{t}}^k(X)[-k])[k] = H_{\mathcal{Q}}^{(k, \phi)}(X) = 0$$

for  $p(\phi) \neq -k$ .

For the second part of the claim, the abelian  $\mathbb{Z}$ -slicing induced on  $\mathfrak{H}_{\mathfrak{q}}$  is split by **Proposition 0.7** and thus by **Proposition ??**

$$X = \bigoplus_{(k, \phi)} H_{\mathcal{Q}}^{(k, \phi)}(X) = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

where the last equality comes from the first part. □

**Example 0.9.** Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an  $\mathbb{N}$ -graded ring with  $B_0$  semisimple. Denote  $\mathcal{A}$  the category of  $\mathbb{Z}$ -graded  $B$ -modules with only finitely many nonzero graded pieces. For  $\phi \in \mathbb{Z}$ , denote  $\mathcal{P}_\phi$  the full subcategory of  $\mathcal{A}$  of modules concentrated in degree  $\phi$ . Clearly,  $\mathcal{P}$  defines an abelian  $\mathbb{Z}$ -slicing on  $\mathcal{A}$ . Following [?] we have

$$\mathrm{Ext}_{\mathcal{A}}^n(\mathcal{P}_\phi, \mathcal{P}_\psi) = 0$$

for  $n > \psi - \phi$ . This means that  $\mathcal{P}$  is a mixed filtration and the bounded t-structure on  $\mathcal{D}^b(\mathcal{A})$  associated to  $(g_1)_\Omega(\mathcal{Q})$  (where 1 is the identity of  $\mathbb{Z}$ ) is the 'diagonal' (or 'geometric') t-structure which appears in Koszul duality and other areas.

**Example 0.10.** Let  $M$  be an  $n$ -dimensional smooth complex projective variety and consider the  $n$ -torsion pair  $\mathcal{P}$  on  $\mathrm{Coh}(M)$  from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that  $\mathcal{P}$ , seen as an abelian  $\mathbb{Z}$ -slicing via the inclusion  $[n] \subseteq \mathbb{Z}$ , is a perverse filtration. The bounded t-structure associated to  $(f_p)_\Omega(\mathcal{Q})$  is the one of perverse coherent sheaves as constructed in [?]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian  $J_n$ -slicing on the heart of perverse coherent sheaves as done in [?].

Now we somehow review the gluing construction for t-structures in [?], but generalize it to any slicing. Our language is quite different though, and we formulate the problem very similarly to [?]. We start with two  $\mathbb{Z}$ -posets  $J, J'$ .

**Definition 0.11.** Let  $\mathcal{P}$  be a  $J \ltimes J'$ -slicing on  $\mathcal{D}$ . We call  $\mathcal{P}$  **gluable** if it satisfies the assumptions of **Proposition 0.4** with respect to the map  $J \ltimes J' \xrightarrow{e} J' \ltimes J$  that exchanges coordinates. In this case we denote

$$\overline{\mathcal{P}} = e_\Omega(\mathcal{P})$$

By a simple computation, the glubility condition reads:  $\mathcal{P}_{(\phi, \psi)} \subseteq \mathcal{P}_{(\phi', \psi')}^\perp$  if  $\psi' > \psi$  or both  $\psi' + 1 > \psi$  and  $\phi' + 1 < \phi$ . For example, the first condition is automatic when  $\mathbb{Z}$  acts trivially on  $J$  (in this case, it follows from the second one). In particular, if  $\mathbb{Z}$  acts trivially on  $J$ ,  $\mathcal{P}$  is a  $J$ -slicing on  $\mathcal{D}$  and  $\mathfrak{t}_\phi$  is a bounded t-structure on  $\mathcal{P}_\phi$  for each  $\phi \in J$ , then using **Proposition ??** we get a  $J \ltimes \mathbb{Z}$ -slicing  $\mathcal{Q}$  which is gluable if and only if

$$\heartsuit_{\mathfrak{t}_\psi}[n] \subseteq \heartsuit_{\mathfrak{t}_\phi}^\perp$$

whenever both  $n \leq 0$  and  $\phi < \psi$ .

**Remark 0.12.** We have a commutative diagram of  $\mathbb{Z}$ -posets

$$\begin{array}{ccc} \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \\ \downarrow e & & \downarrow g \\ \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \end{array}$$

where the horizontal isomorphism is the one from **Remark ??**. Indeed, a  $\mathbb{Z} \ltimes \mathbb{Z}$ -slicing on  $\mathcal{D}$  is gluable if and only if it induces a grading filtration on the heart of the associated t-structure when seen as a  $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing.

An easy combinatorial calculation finally yields the following two remarks:

- if  $\mathcal{P}$  is a gluable  $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$ -slicing on  $\mathcal{D}$ , then  $\overline{\mathcal{P}}$  induces a grading filtration on the associated heart, allowing us again to construct new bounded t-structures depending on a perversity.
- if  $\mathcal{P}$  is a mixed filtration on the heart of a bounded t-structure and  $\mathcal{Q}$  is the associated  $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing on  $\mathcal{D}$ , then  $\mathcal{Q}$  is gluable. In this case, looking at  $\overline{\mathcal{Q}}$ , we get a baric structure on  $\mathcal{D}$  which is usually called **weight decomposition** in literature.

In other words, the chain of implications for a  $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing refines to

$$\text{mixed} \implies \text{gluable} \implies \text{grading} \implies \text{perverse}$$

