0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of $\mathbb{Z}$-posets (but just a $\mathbb{Z}$-equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a $\mathbb{Z}$-toset, i.e., a totally ordered set together with a monotone action of $\mathbb{Z}$, that we will denote by $(n, x) \mapsto x + n$. Recall that a J-slicing on a stable ∞-category $\mathscr{D}$ is a morphism of $\mathbb{Z}$-posests $\mathfrak{t}\colon \mathcal{O}(J) \to \mathrm{ts}(\mathscr{D})$, where $\mathcal{O}(J)$ is the $\mathbb{Z}$-poset of the slicings of J (i.e., the decompositions of J into a disjoint union of a lower set L and of an upper set U), and $\mathrm{ts}(\mathscr{D})$ is the $\mathbb{Z}$-poset of t-structures on $\mathscr{D}$.
Clearly, if $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then $f^{-1}\colon \{\text{subsets of } J'\} \to \{\text{subsets of } J\}$ induces a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-posets $f^{-1}\colon \mathcal{O}(J') \to \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$\begin{aligned} f_*\colon J\text{-slicings on } \mathscr{D} &\to J'\text{-slicings on } \mathscr{D} \\ \mathfrak{t} &\mapsto \mathfrak{t} \circ f^{-1}. \end{aligned}$$

The slices of $f_*\mathfrak{t}$ are clearly given by $\mathscr{D}_{f_*\mathfrak{t};j} = \mathscr{D}_{\mathfrak{t};f^{-1}(\{j\})}$, for any $j \in J'$. Notice that, as f is monotone, the subset $f^{-1}(\{j\})$ is an interval in J. It is immediate to see that f_* restricts to a map

$$f_*\colon \text{Bridgeland } J\text{-slicings on } \mathscr{D} \to \text{Bridgeland } J'\text{-slicings on } \mathscr{D}$$

Namely, if $\mathcal{H}^j_{f_*\mathfrak{t}}(X) = 0$ for every $j \in J'$ then $\mathcal{H}^{f^{-1}(\{j\})}_{\mathfrak{t}}(X) = 0$ for every j in J'. As we are assuming the J-slicing $\mathfrak{t}$ is a Bridgeland slicing, this implies that $\mathcal{H}^{\phi}_{\mathfrak{t}}(X) = 0$ for every ϕ in $f^{-1}(\{j\})$, for every j. Therefore $\mathcal{H}^{\phi}_{\mathfrak{t}}(X) = 0$ for every ϕ in J and so, again by definition of Bridgeland slicing, $X = 0$. Also, if $\mathcal{H}^j_{f_*\mathfrak{t}}(X) \neq 0$ then $\mathcal{H}^{f^{-1}(\{j\})}_{\mathfrak{t}}(X) \neq 0$ and so (again by the Bridgeland slicing condition) there exists at least an element ϕ in $f^{-1}(\{j\})$ such that $\mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0$. As the J-slicing $\mathfrak{t}$ is Bridgeland, the total of these ϕ's must be finite, so only for finitely many j we can have such a ϕ. In other words, the number of indices j in J' such that $\mathcal{H}^j_{f_*\mathfrak{t}}(X) \neq 0$ is finite.
Notice that, if $\mathfrak{t}$ is a Bridgeland slicing of $\mathscr{D}$, and $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then for any slicing (L, U) of J', the lower and the upper categories $\mathscr{D}_{f_*\mathfrak{t};L}$ and $\mathscr{D}_{f_*\mathfrak{t};U}$ can be equivalently defined as

$$\begin{aligned} \mathscr{D}_{f_*\mathfrak{t};L} &= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L} \\ \mathscr{D}_{f_*\mathfrak{t};U} &= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U} \end{aligned}$$

where $\langle \mathscr{S} \rangle$ denotes the extension-closed subcategory of $\mathscr{D}$ generated by the subcategory $\mathscr{S}$. In particular the slices of are given by $\mathscr{D}_{f_*\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$. [1]

[1] At some point it will be useful to recall Lemma 4.21 from ??: Let $\mathscr{S}_1, \mathscr{S}_2$ be two subcategories of $\mathscr{D}$ with $\mathscr{S}_1 \boxslash \mathscr{S}_2$, i.e., such that $\mathscr{D}(X_1, X_2)$ is contractible for any $X_1 \in \mathscr{S}_1$ and any $X_2 \in \mathscr{S}_2$. Then $\mathscr{S}_1 \boxslash \langle \mathscr{S}_2 \rangle$ and $\langle \mathscr{S}_1 \rangle \boxslash \mathscr{S}_2$, and so $\langle \mathscr{S}_1 \rangle \boxslash \langle \mathscr{S}_2 \rangle$.

The right hand sides of the above two expressions can clearly be defined for every morphism f from J to J' (i.e., not necessarily monotone nor $\mathbb{Z}$-equivariant), and as soon as f is $\mathbb{Z}$-equivariant, the assignment

$$(L, U) \mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$$

is an equivariant morphism from $\mathcal{O}(J)$ to pairs of subcategories of $\mathscr{D}$. Clearly, when f is not monotone there is no reason to expect that the pair $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ forms a t-structure on $\mathscr{D}$. Yet, it interesting to notice that the condition that f be monotone is only only sufficient in order to have this, and can indeed be relaxed.

Definition 0.1. *Let J and J' be $\mathbb{Z}$-tosets, and let $f\colon J \to J'$ a map of $\mathbb{Z}$-sets (i.e., a $\mathbb{Z}$-equivariant map, not necessarily nondecreasing). A Bridgeland J-slicing $\mathfrak{t}$ of $\mathscr{D}$ is* f-compatible *if $f(\phi) > f(\psi)$ with $\phi \leq \psi$ implies $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$*

Remark 0.2. Clearly, if f is monotone, then every J-slicing $\mathfrak{t}$ is f-compatible as the condition '$f(\phi) > f(\psi)$ with $\phi \leq \psi$' is empty.

Remark 0.3. Let J and J' be $\mathbb{Z}$-tosets, let $f\colon J \to J'$ a map of $\mathbb{Z}$-sets, and let $g\colon J' \to J''$ be an isomorphism of $\mathbb{Z}$-posets. Then a Bridgeland J-slicing $\mathfrak{t}$ of $\mathscr{D}$ is f-compatible if and only if it is $(g \circ f)$-compatible. Similarly, if $h\colon J'' \to J$ is an isomorphism of $\mathbb{Z}$-posets, then $\mathfrak{t}$ is f-compatible if and only if $(h^{-1})_*\mathfrak{t}$ is $(f \circ h)$-compatible.

Lemma 0.4. *Let J and J' be $\mathbb{Z}$-tosets, and let $f\colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. Then, for any slicing (L, U) of J', the pair of subcategories $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ is a t-structure on $\mathscr{D}$.*

Proof. As f is $\mathbb{Z}$-equivariant and $U + 1 \subseteq U$, we have

$$\begin{aligned}
\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}[1] &= \langle \mathscr{D}_{\mathfrak{t};\phi}[1] \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi+1)\in U+1} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U+1} \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}
\end{aligned}$$

and similarly for the lower subcategory $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$. To show that

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U} \boxslash \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$$

it suffices to show that, if $f(\psi) \in L$ and $f(\phi) \in U$ then $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. As $L \cap U = \emptyset$ we cannot have $\psi = \phi$, so either $\psi < \phi$ or vice versa. In the first case, $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ by definition of Bridgeland J-slicing. In the second case, we have $\phi < \psi$ and $f(\phi) > f(\psi)$ as $f(\phi) \in U$ and $f(\psi) \in L$. Therefore, since $\mathfrak{t}$ is f-compatible,

$\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. Finally, we have to show that every object X in $\mathscr{D}$ fits into a fiber sequence

$$\begin{tikzcd} X_U \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X_L \end{tikzcd}$$

with $X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$ and $X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}$. As $\mathfrak{t}$ is a Bridgeland slicing, we have a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{\imath}}} X_{\bar{\imath}} \xrightarrow{\alpha_{\bar{\imath}+1}} X_{\bar{\imath}+1} \to \cdots \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) = \mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \in \mathscr{D}_{\phi_i}$ for all $i = 1, \cdots, n$, with $\phi_i > \phi_{i+1}$. Let us now consider the sequence of symbols L and U obtained putting in the i-th place L if $f(\phi_i) \in L$ and U if $f(\phi_i) \in U$. If this sequence is of the form $(U, U, \ldots, U, L, L, \ldots, L)$, then there exists an index $\bar{\imath}$ such that $f(\phi_i) \in U$ for $i \leq \bar{\imath}$ and $f(\phi_i) \in L$ for $i > \bar{\imath}$ (with $\bar{\imath} = -1$ or n when all of the $f(\phi_i)$ are in L or in U, respectively). Then we can consider the pullout diagram

$$\begin{tikzcd} X_{\bar{\imath}} \arrow[r] \arrow[d, "f_L"'] & 0 \arrow[d] \\ X \arrow[r] & \mathrm{cofib}(f_L) \end{tikzcd}$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{\imath}}} X_{\bar{\imath}}$$

and

$$X_{\bar{\imath}} \xrightarrow{\alpha_{\bar{\imath}+1}} X_{\bar{\imath}+1} \to \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that $X_{\bar{\imath}} \in \langle \cup_{i=0}^{\bar{\imath}} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}$ while the second factorization shows that $\mathrm{cofib}(f_L) \in \langle \cup_{i=\bar{\imath}+1}^{n} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$. So we are done in this case. Therefore, we are reduced to showing that we can always avoid a $(\ldots, L, U, \ldots)$ situation in our sequence of L's and U's. Assume we have such a situation. Then we have an index i_0 with $f(\phi_{i_0}) \in L$ and $f(\phi_{i_0+1}) \in U$. This in particular implies $f(\phi_{i_0+1}) > f(\phi_{i_0})$ with $\phi_{i_0+1} < \phi_{i_0}$. As $\mathfrak{t}$ is f-compatible, this gives $\mathscr{D}_{\mathfrak{t};\phi_{i_0+1}} \boxslash (\mathscr{D}_{\mathfrak{t};\phi_{i_0}}[1])$. In particular, $\mathscr{D}(\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1])$ is contractible. Now consider the pasting of pullout diagrams

$$\begin{tikzcd} X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\ 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]. \end{tikzcd}$$

As the arrow $\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]$ factors through 0, we have $Y = \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ and the above diagram becomes

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\iota_2"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d, "\pi_1"] & 0 \arrow[d] \\
 & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1],
\end{tikzcd}$$

where ι_2 and π_1 are the canonical inclusion and projection. Let $\beta\colon X_{i_0+1} \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ be the composition

$$\beta\colon X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).$$

Then we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\beta"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\mathrm{id}"] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)
\end{tikzcd}$$

(where only the left square is a pullout), and so the composition $\alpha_{i_0+1} \circ \alpha_{i_0}$ factors through the homotopy fiber of β. In other words, we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d, "\tilde{\alpha}_{i_0}"'] & X_{i_0} \arrow[d, "\alpha_{i_0+1}"] \\
\mathrm{fib}(\beta) \arrow[r, "\tilde{\alpha}_{i_0+1}"] & X_{i_0+1}
\end{tikzcd}$$

Writing $\tilde{X}_{i_0} = \mathrm{fib}(\beta)$, we get the pasting of pullout diagrams

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\tilde{\alpha}_{i_0}"] \arrow[d] & \tilde{X}_{i_0} \arrow[r, "\tilde{\alpha}_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r, "\iota_1"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[d, "\pi_2"'] \\
 & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).
\end{tikzcd}$$

That is, by considering the factorization $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$ we have switched the cofibers with respect to the original factorization $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$. Therefore, writing $\tilde{\phi}_{i_0} = \phi_{i_0+1}$ and $\tilde{\phi}_{i_0+1} = \phi_{i_0}$, we now have $f(\tilde{\phi}_{i_0}) \in U$ and $f(\tilde{\phi}_{i_0+1}) \in L$. That is, we have removed the $(\dots, L, U, \dots)$ situation from the position i_0, replacing it with a $(\dots, U, L, \dots)$ situation, while keeping all the labels L, U before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the $(\dots, L, U, \dots)$ situations.[2] $\square$

Remark 0.5. It follows from the proof of Lemma 0.4 that the objects X_L and X_U in the fiber sequence $X_U \to X \to X_L$ associated with the t-struture $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ on $\mathscr{D}$ satisfy

$$X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in L \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0 \rangle$$
$$X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in U \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0 \rangle$$

Lemma 0.6. *For any $j \in J'$ we have*

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}.$$

Proof. Clearly, $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}$, therefore we only need to prove the converse inclusion. Let $\in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}$. Then in particular $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}$ and so there exists a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) \in \mathscr{D}_{\phi_i}$ with $f(\phi_i) \geq i$ for all $i = 1, \cdots, n$. As $((-\infty, j], (j, +\infty)$ is a slicing of J', reasoning as in the proof of Lemma 0.4 we can arrange this factorization is such a way that $f(\phi_i) > j$ for $i \leq \bar{\imath}$ and $f(\phi_i) = j$ for $i > \bar{\imath}$. Therefore, again by reasoning as in the proof of Lemma 0.4 we get a fiber sequence of the form

$$\begin{tikzcd} X_{>j} \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X_j \end{tikzcd}$$

with $X_{>j}$ in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j}$ and X_j in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$. This is in particular a fiber sequence of the form $X_{>j} \to X \to X_{\leq j}$, with $X_{\leq j} \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j}$. As $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j})$ is a t-structure on $\mathscr{D}$ by Lemma 0.4, there is (up to equivalence) only one such a fiber sequence. And since $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j}$, this is the sequence $0 \to X \xrightarrow{\mathrm{id}} X$. Therefore $X = X_j$. $\square$

Proposition 0.7. *Let J, J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. The map*

$$\begin{aligned} f_!\mathfrak{t} \colon \mathcal{O}(J') &\to \mathrm{ts}(\mathscr{D}) \\ (L, U) &\mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}) \end{aligned}$$

[2]This is somehow reminiscent of the 'bubble sort' algorithm.

defined by Lemma 0.4 is a Bridgeland J'-slicing of $\mathscr{D}$, with slices given by

$$\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)=j}.$$

Proof. The map $f_!\mathfrak{t}$ is manifestly monotone and $\mathbb{Z}$-equivariant (see the first part of the proof of Lemma 0.4), so it is a J' slicing of $\mathscr{D}$, and its slices are given by $\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)=j}$ by Lemma 0.6. We are therefore left with showing that it is finite and discrete. Given an object X in $\mathscr{D}$, let now $\{\phi_1,\dots,\phi_n\}$ be the indices in J such that $\mathcal{H}_{\mathfrak{t}}^{\phi_i}(X)\neq 0$ and let $\{j_1,\dots,j_k\}$ the image of the set $\{\phi_1,\dots,\phi_n\}$ via f. Up to renaming, we can assume $j_1 > j_2 > \cdots > j_k$. Consider now the factorization

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} X_{k-1} \xrightarrow{\alpha_k} X_k = X$$

of the initial morphism of X associated to the decreasing sequence $j_1 > j_2 > \cdots > j_k$ by the J'-slicing $f_!\mathfrak{t}$. The cofibers of the morphisms α_i are the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X)$ and, by Remark 0.5, we have

$$\begin{aligned}
\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X) \in & \langle \mathscr{D}_{\mathfrak{t};\phi};\quad f(\phi)\in (j_{i+1},j_i] \text{ and } \mathcal{H}_{\mathfrak{t}}^{\phi}(X)\neq 0\rangle\\
&= \langle \mathscr{D}_{\mathfrak{t};\phi};\quad \phi\in\{\phi_1,\dots,\phi_n\} \text{ and } f(\phi)\in (j_{i+1},j_i]\rangle\\
&= \langle \mathscr{D}_{\mathfrak{t};\phi};\quad \phi\in\{\phi_1,\dots,\phi_n\} \text{ and } f(\phi)= j_i\rangle\\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi};\quad f(\phi)= j_i\rangle = \mathscr{D}_{f_!\mathfrak{t};j_i}.
\end{aligned}$$

Therefore

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i}\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i}(X).$$

This tells us that, if all the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ vanish, then also all the $\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X)$ vanish, and so $X=0$. Finally, if $\tilde{j}\notin\{j_1,\dots,j_k\}$, then there exists an index i such that $j_{i+1}<\tilde{j}<j_i$ and so $(j_{i+1},\tilde{j}]\cap\{j_1,\dots,j_k\}=\emptyset$. The above argument then shows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) \in \langle \mathscr{D}_{\mathfrak{t};\phi};\quad \phi\in\{\phi_1,\dots,\phi_n\} \text{ and } f(\phi)\in (j_{i+1},\tilde{j}]\rangle = \{\mathbf{0}\}.$$

It follows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(0) = 0,$$

for every $\tilde{j}\notin\{j_1,\dots,j_k\}$. So in particular, for any $X\in\mathscr{D}$, the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ are possibly nonzero only for finitely many indices j. ☐

Remark 0.8. If $f\colon J\to J'$ is a morphism of $\mathbb{Z}$-tosets, then every Bridgeland slicing of $\mathfrak{t}$ of $\mathscr{D}$ is f-compatible and one has $f_!\mathfrak{t} = f_*\mathfrak{t}$.

Proposition 0.9. *Let J, J', J'' be $\mathbb{Z}$-tosets, let $f\colon J\to J'$ and $g\colon J'\to J''$ be $\mathbb{Z}$-equivariant morphisms of $\mathbb{Z}$-sets (i.e., not necessarily monotone maps), and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$. If $\mathfrak{t}$ is f-compatible and $f_!\mathfrak{t}$ is g-compatible, then $\mathfrak{t}$ is $(g\circ f)$-compatible, and*

$$(g\circ f)_!\mathfrak{t} = g_!f_!\mathfrak{t}.$$

Proof. Let ϕ, ψ in J be such that $\phi \leq \psi$ and $g(f(\phi)) > g(f(\psi))$, and let $\xi = f(\phi)$ and $\eta = f(\psi)$. As J' is a totally ordered set, either $\xi \leq \eta$ or $\xi > \eta$. In the first case we have $g(\xi) > g(\eta)$ with $\xi \leq \eta$. As $f_!\mathfrak{t}$ is g-compatible, this implies $\mathscr{D}_{f_!\mathfrak{t};\xi} \boxslash \mathscr{D}_{f_!\mathfrak{t};\eta}$ and $\mathscr{D}_{f_!\mathfrak{t};\xi} \boxslash \mathscr{D}_{f_!\mathfrak{t};\eta}[1]$. By definition of $f_!\mathfrak{t}$ we have $\mathscr{D}_{\mathfrak{t};\phi} \subseteq \mathscr{D}_{f_!\mathfrak{t};\xi}$ and $\mathscr{D}_{\mathfrak{t};\psi} \subseteq \mathscr{D}_{f_!\mathfrak{t};\eta}$, therefore we have $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$. In the second case we have $f(\phi) > f(\psi)$ with $\phi \leq \psi$. As $\mathfrak{t}$ is f-compatible, this again implies $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$. Finally, for any $j \in J''$ one has

$$\mathscr{D}_{g_!f_!\mathfrak{t};j} = \langle \mathscr{D}_{f_!\mathfrak{t};\xi} \rangle_{g(\xi)=j} = \langle \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=\xi} \rangle_{g(\xi)=j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{g(f(\phi))=j} = \mathscr{D}_{(g\circ f)_!\mathfrak{t};j}.$$

□

Let now J_1 and J_2 be two $\mathbb{Z}$-tosets, and let $J_1 \times_{\mathrm{lex}} J_2$ and $J_2 \times_{\mathrm{lex}} J_1$ the two $\mathbb{Z}$-tosets obtained by considering the lexicogaphic order on the products $J_1 \times J_2$ and $J_2 \times J_1$, respectively, and the diagonal $\mathbb{Z}$ action. The following lemma is immediate.

Lemma 0.10. *The exchange map*

$$\begin{aligned} e\colon J_1 \times J_2 &\to J_2 \times J_1 \\ (j_1, j_2) &\mapsto (j_2, j_1) \end{aligned}$$

is $\mathbb{Z}$-equivariant.

Definition 0.11. *Let J_1 and J_2 be $\mathbb{Z}$-tosets. A Bridgeland $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ of $\mathscr{D}$ is said to be* gluable *if it is e-compatible, where $e\colon J_1 \times J_2 \to J_2 \times J_1$ is the exchange map.*

Example 0.12. Let $J_1 = \{0, 1\}$ with the trivial $\mathbb{Z}$-action and let $J_2 = \mathbb{Z}$ with the standard translation $\mathbb{Z}$-action. Then a $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ on $\mathscr{D}$ is the datum of a semiorthogonal decomposition $(\mathscr{D}_0, \mathscr{D}_1)$ of $\mathscr{D}$ toghether with t-structures $\mathfrak{t}_i$ on $\mathscr{D}_i$. Spelling out the definiton, one sees that the $\{0, 1\} \times_{\mathrm{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$ is gluable if and only if $\mathscr{D}_{0;\geq 0} \boxslash \mathscr{D}_{1;-1}$ and $\mathscr{D}_{0;\geq 0} \boxslash \mathscr{D}_{1;0}$. As $\mathscr{D}_{0;\geq 0}$ is generated by the subcategories $\mathscr{D}_0^\heartsuit[k]$ for $k \geq 0$, this is equivalent to $\mathscr{D}_0^\heartsuit \boxslash \mathscr{D}_1^\heartsuit[n]$ for all $n \leq 0$. When this happens, the glued slicing $e_!\mathfrak{t}$ is a $\mathbb{Z} \times_{\mathrm{lex}} \{0, 1\}$ slicing on $\mathscr{D}$, i.e., is the datum of a bounded t-structure on $\mathscr{D}$ together with a torsion theory on its heart. More precisely, the heart of $e_!\mathfrak{t}$ is the full ∞-subcategory $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ of $\mathscr{D}$ on thise objects X that fall into fiber sequences of the form $X_1 \to X \to X_0$ with $X_i \in \mathscr{D}_i^\heartsuit$, while the torsion theory on $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ is $(\mathscr{D}_0^\heartsuit, \mathscr{D}_1^\heartsuit)$, i.e., precisely the pair of hearts of the two subcategories in the semiorthogonal decomposition. This is exactly the setup of Collins et al, which by the other hand is a particular case of the classical gluing construction of t-structures by BBD, explaining both our nomenclature and motivation.

Lemma 0.13. *Let J_1 and J_2 be $\mathbb{Z}$-tosets, $\mathfrak{t}$ be a Bridgeland $J_1 \times_{\mathrm{lex}} J_2$-slicing of $\mathscr{D}$. If $\mathfrak{t}$ is gluable and $\mathbb{Z}$ acts trivially on J_1, then $e_!\mathfrak{t}$ is a gluable $J_2 \times_{\mathrm{lex}} J_1$ slicing of $\mathscr{D}$.*

Proof. Let $(j_2, j_1) \leq (j_2', j_1')$ in $J_2 \times_{\mathrm{lex}} J_1$ be such that $(j_1, j_2) > (j_1', j_2')$ in $J_1 \times_{\mathrm{lex}} J_2$. As $\mathfrak{t}$ is gluable, we have

$$\mathscr{D}_{e_!\mathfrak{t};(j_2,j_1)} = \langle \mathscr{D}_{\mathfrak{t};(i_1,i_2)} \rangle_{e(i_1,i_2)=(j_2,j_1)} = \langle \mathscr{D}_{\mathfrak{t};(j_1,j_2)} \rangle = \mathscr{D}_{\mathfrak{t};(j_1,j_2)},$$

since slices are extension closed. As $(j_1, j_2) > (j_1', j_2')$, we have $\mathscr{D}_{\mathfrak{t};(j_1,j_2)} \boxslash \mathscr{D}_{\mathfrak{t};(j_1',j_2')}$ by definition of slicing, and so $\mathscr{D}_{e_!\mathfrak{t};(j_2,j_1)} \boxslash \mathscr{D}_{e_!\mathfrak{t};(j_2',j_1')}$. Finally,

$$\mathscr{D}_{e_!\mathfrak{t};(j_2,j_1)}[1] = \mathscr{D}_{\mathfrak{t};(j_1,j_2)}[1] = \mathscr{D}_{\mathfrak{t};(j_1,j_2)+1} = \mathscr{D}_{\mathfrak{t};(j_1,j_2+1)},$$

since the $\mathbb{Z}$-action on J_1 is trivial. The condition $(j_2, j_1) \leq (j_2', j_1')$ with $(j_1, j_2) > (j_1', j_2')$ is actually equivalent to $j_2 < j_2'$ and $j_1 > j_1'$, so in particular we have $(j_1, j_2) > (j_1', j_2' + 1)$.This implies $\mathscr{D}_{\mathfrak{t};(j_1,j_2)} \boxslash \mathscr{D}_{\mathfrak{t};(j_1',j_2'+1)}$ and so $\mathscr{D}_{e_!\mathfrak{t};(j_2,j_1)} \boxslash \mathscr{D}_{e_!\mathfrak{t};(j_2',j_1')}[1]$. $\square$

Definition 0.14. *Let $U\colon J \to \mathcal{O}(J')$ be a map of sets. We denote by Γ_U the subset of $J \times J'$ defined by*

$$\Gamma_U = \{(j, j') \in J \times J'; \quad j' \in U_j\}.$$

For $f\colon J \to J'$ a map of sets, we denote by $(< f, \geq f)\colon J \to \mathcal{O}(J')$ the composition of f with the map

$$\begin{aligned} J' &\to \mathcal{O}(J') \\ j' &\mapsto ((-\infty, j'), [j', +\infty)). \end{aligned}$$

The upper graphic of f is the subset $\Gamma_{\geq f}$ of $J \times J'$.

Lemma 0.15. *Let $U\colon J \to \mathcal{O}(J')$ be a map of sets. Then Γ_U is an upper set of $J \times J'$ with the product order if and only if U is a map of posets $U\colon J \to \mathcal{O}(J')^{\mathrm{op}}$. In particular, $\Gamma_{\geq f}$ is an upper set if and only if $f\colon J \to J'$ is a nondecreasing map, i.e., a map of posets $J \to J'^{\mathrm{op}}$.*

Proof. Assume (L, U) is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Pick $(j, j') \in \Gamma_U$ and suppose $(j, j') \leq (k, k')$ in $J \times J'$. As (L, U) i monotone and $k \geq j$, we have $U_j \subseteq U_k$; as $(j, j') \in \Gamma_U$, we have $j' \in U_j$. Therefore $j \in U_k$. Since U_k is an upper set and $k' \geq j'$, we get $k' \in U_k$, i.e. $(k, k') \in \Gamma_U$. Vice versa, assume Γ_U is an upper set, and let $j \leq k$ in J. For any $j' \in U_j$, the element (k, j') in $J \times J'$ satisfies $(k, j') \geq (j, j')$ in the product order and so $(k, j') \in \Gamma_U$. This means $j' \in U_k$ and so $U_j \subseteq U_k$. To prove the second part of the statement, notice that the map $j \mapsto ((-\infty, j'), [j', +\infty))$ is a map of posets from $J' \to \mathcal{O}(J')$. Therefore, if $f\colon J \to J'^{\mathrm{op}}$ is a map of posets, then also $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets. Vice versa, if $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets then for every $j \leq k$ in J we have $[f(j), +\infty) \subseteq [f(k), +\infty)$ and so $f(k) \leq f(j)$. $\square$

Proposition 0.16. *The map*

$$\begin{aligned} \Gamma\colon \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} &\to \mathcal{O}(J \times J') \\ U &\mapsto \Gamma_U \end{aligned}$$

is an isomorphism of posets.

Proof. Let $U_1 \leq U_2$ in the partial order on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$, and let $(j, j') \in \Gamma_{U_2}$. Then $j' \in U_{2;k} \subseteq U_{1,j}$ and so $(j, j') \in \Gamma_{U_1}$. This means $\Gamma_{U_1} \leq \Gamma_{U_2}$ in $\mathcal{O}(J \times J')$. Next, for any morphism of posets $U \colon J \to \mathcal{O}(J)^{\mathrm{op}}$ we have that $(j, j') \in \Gamma_{U+1}$ if and only if $j' \in (U+1)_j$ Pick and upper set $\tilde{U}$ of $J \times J'$ and set, for $j \in J$

$$U_{\tilde{U}}(j) = \{j' \in J'; \quad (j, j') \in \tilde{U}\}.$$

The subset $U_{\tilde{U}}(j) \subseteq J'$ is an upper set. Indeed, if $j' \in U_{\tilde{U}}(j)$ and $k' \geq j'$ in J', then $(j, k') \geq (j, j')$ in the product order on $J \times J'$ and so $(j, k') \in \tilde{U}$ as $\tilde{U}$ is an upper set. Next, if $j \leq k$ in J and $j' \in U_{\tilde{U}}(j)$, then $(k, j') \in \tilde{U}$ as $\tilde{U}$ is an upper set, and so $j' \in U_{\tilde{U}}(k)$. This shows that $U_{\tilde{U}}(j) \subseteq U_{\tilde{U}}(k)$, and so $U_{\tilde{U}}$ is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Moreover, if $\tilde{U}_1 \leq \tilde{U}_2$ in $\mathcal{O}(J \times J')$ then $U_{\tilde{U}_1} \leq U_{\tilde{U}_2}$ in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. Therefore we have a map

$$\begin{aligned} \gamma \colon \mathcal{O}(J \times J') &\to \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \\ \tilde{U} &\mapsto U_{\tilde{U}}. \end{aligned}$$

which is straightforward to see to be the inverse of Γ. $\square$

Assume J and J' are $\mathbb{Z}$-posets. Then $\mathcal{O}(J)^{\mathrm{op}}$ is a $\mathbb{Z}$-poset with 1 acting as $U \mapsto U - 1$. The action of $\mathbb{Z}$ by conjugation on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$ is therefore given by $(U \dotplus 1)_j = U_{j-1} - 1$. To see that $U \dotplus 1$ is still a morphism of posets from J to $\mathcal{O}(J')$, let $j \leq k$ in J. Then $(U \dotplus 1)_j = U_{j-1} - 1 \subseteq U_{k-1} - 1 = (U \dotplus 1)_k$. The conjugation action, however, does not define a structure of $\mathbb{Z}$-poset on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, as it is generally not true that $U \leq U \dotplus 1$. This condition is indeed equivalent to $U_j \subseteq (U \dotplus 1)_j$, i.e., to $U_j + 1 \subseteq U_{j-1}$ for any j in J, and this is not necessarily satisfied by a morphism of posets $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$.

Definition 0.17. *A* (J, J')-perversity *is a map of posets* $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ *such that*

$$U_j + 1 \subseteq U_{j-1}$$

for any j in J. The set $\mathrm{Perv}(J, J')$ *of all (J, J')-perversities inherits a poset structure from the inclusion in* $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. *It is a $\mathbb{Z}$-poset with* 1 *acting as* $U \mapsto U \dotplus (-1)$.

Remark 0.18. Notice that the shift action on (J, J')-perversities is given by $U \mapsto U \dotplus (-1)$. Namely, by construction the $\mathbb{Z}$-action $U \mapsto U \dotplus 1$ is monotone on the set of perversities with the order induced by the inclusion in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, and the order on $\mathrm{Perv}(J, J')$ is the opposite one. The reason for considering this order is, clearly, Proposition 0.16.

We have seen in Lemma 0.15 that a function $f \colon J \to J'$ defines a morphism of posets by $(\geq f) \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ if and only if f is a morphism of posets from J to J'^{op}. When this happens, $(\geq f)$ is a (J, J')-perversity if and only if $f(j-1) \leq f(j) + 1$, for every $j \in J$. In the particular case $J = \mathbb{Z}$, assuming as usual that J' is a $\mathbb{Z}$-poset, we can define a new function $p_f \colon \mathbb{Z} \to J'$ as $p_f(n) = f(n) + n$. Then the condition $f(n) \leq f(n+1) + 1$ translates into

$p_f(n) \leq p_f(n+1)$, while the condition that $f\colon \mathbb{Z} \to J'^{\mathrm{op}}$ is a morphism of posets, i.e., $f(n+1) \leq f(n)$ translates to $p_f(n+1) \leq p_f(n)+1$. As $f \mapsto p_f$ is a bijection of the set of maps from J to J' into itself, we see that the functions $f\colon \mathbb{Z} \to J'$ defining perversities correspond bijectively to the set of functions $p\colon \mathbb{Z} \to J'$ such that $p(n) \leq p(n+1) \leq p(n)+1$, for every $n \in \mathbb{Z}$. This motivates the following (see [**?**]).

Definition 0.19. *A $(\mathbb{Z},\mathbb{Z})$-perversity U is said to be defined by a function if there exists $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $U = (\geq f)$. The subset of $(\mathbb{Z},\mathbb{Z})$-perversities defined by functions is denoted by* $\mathrm{Perv}^\circ(\mathbb{Z},\mathbb{Z})$.

Lemma 0.20. *The subset* $\mathrm{Perv}^\circ(\mathbb{Z},\mathbb{Z})$ *is a $\mathbb{Z}$-sub-poset of* $\mathrm{Perv}(\mathbb{Z},\mathbb{Z})$*. It is isomorphic via $f \mapsto (\geq f)$ with the $\mathbb{Z}$-poset of nonincreasing functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(n-1) \leq f(n)+1$ with the $\mathbb{Z}$-action given by $(f \dotplus 1)(n) = f(n+1)+1$.*

Proof. As a function $f\colon \mathbb{Z} \to \mathbb{Z}$ is uniquely determined by the collection of upper sets $[f(n),+\infty)$ the set $\mathrm{Perv}^\circ(J,J')$ bijectivley corresponds to the set of those functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $(\geq f)$ is a $(\mathbb{Z},\mathbb{Z})$-perversity. As noticed above, these are precisely nonincreasing functions from $\mathbb{Z}$ to itself such that $f(n-1) \leq f(n)+1$. This bijection is an isomorphism of posets, as $(\geq f_1) \leq (\geq f_2)$ if and only if $f_1 \leq f_2$ (in the standard poset structure on the set of maps from $\mathbb{Z}$ to the poset $\mathbb{Z}$). Finally, $((\geq f) \dotplus (-1))_n = (\geq f)_{n+1} + 1 = [f(n+1)+1,+\infty) = (\geq (f \dotplus 1))_n$, for any $n \in \mathbb{Z}$. $\square$

Definition 0.21. *A* perversity function *(on $\mathbb{Z}$) is a function $p\colon \mathbb{Z} \to \mathbb{Z}$ such that*

$$p(n) \leq p(n+1) \leq p(n)+1,$$

for every $n \in Z$. The set $\mathrm{perv}_{\mathbb{Z}}$ *of perversity functions is a poset with the partial order induced by the inclusion* $\mathrm{perv}_{\mathbb{Z}} \subseteq \mathrm{Pos}(\mathbb{Z},\mathbb{Z})$*. It is a $\mathbb{Z}$-poset with the action $(p \dotplus 1)(n) = p(n+1)$.*

Remark 0.22. In addition to the $\mathbb{Z}$-action $p \mapsto p \dotplus 1$, the poset $\mathrm{perv}_{\mathbb{Z}}$ carries also another natural $\mathbb{Z}$-action making it a $\mathbb{Z}$-poset; namely, the action $(p+1)(n) = p(n)+1$. We will come back to this towards the end of this section.

Remark 0.23. Notice that the $\mathbb{Z}$-action on perversity functions is a monotone action since, by definition of perversity function, we have $p(n) \leq p(n+1)$ and this precisely means $p(n) \leq (p \dotplus 1)(n)$.

Lemma 0.24. *For every $p\colon \mathbb{Z} \to \mathbb{Z}$, let $f_p\colon \mathbb{Z} \to \mathbb{Z}$ be the map defined by $f_p(n) = p(n) - n$. Then $p \mapsto (\geq f_p)$ is an isomorphism of $\mathbb{Z}$-posets between* $\mathrm{perv}_{\mathbb{Z}}$ *and* $\mathrm{Perv}^\circ(\mathbb{Z},\mathbb{Z})$

Proof. By Lemma 0.20 we only need to show that $p \mapsto f_p$ is a monotone $\mathbb{Z}$-equivariant bijection between $\mathrm{perv}_{\mathbb{Z}}$ and the set of functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(n) \leq f(n-1) \leq f(n)+1$. That it is a bijection is immediate: the inverse map is $f \mapsto p_f$, where $p_f(n) = f(n)+n$. To see that it is an isomorphism of posets, notice that $f_{p_1} \leq f_{p_2}$ if and only if $p_1(n) - n \leq p_2(n) - n$ for every $n \in \mathbb{Z}$, and so if and only if $p_1(n) \leq p_2(n)$ for every $n \in \mathbb{Z}$. Finally, $f_{p \dotplus 1}(n) = (p \dotplus 1)(n) - n = p(n+1) - n = f_p(n+1) + 1 = (f_p \dotplus 1)(n)$. $\square$

Definition 0.25. *An upper set U in $\mathcal{O}(J \times J')$ is called a* kinky upperset *if $U \leq U +_{\mathrm{ne}} 1$, where* ne *is the "northwestern" action of $\mathbb{Z}$ on $J \times J'$ given by $(j, j') +_{\mathrm{ne}} 1 = (j-1, j'+1)$. We denote by* $\mathrm{Kink}(J \times J')$ *the poset of kinky uppersets of $J \times J'$, with the poset structure induced by the inclusion in $\mathcal{O}(J \times J')$. it is a $\mathbb{Z}$-poset with the "northwestern" action. We denote by* $\mathrm{Kink}^\circ(J \times J')$ *the $\mathbb{Z}$-sub-poset of nontrivial kinky uppersets of $J \times J'$, where the trivial upperstes are $\emptyset$ and $J \times J'$.*

Lemma 0.26. *The map Γ from Proposition 0.16 induces an isomorphism of $\mathbb{Z}$-posets*

$$\Gamma\colon \mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J').$$

Proof. Let $U\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ be a perversity, and let $(j, j') \in \Gamma_U$. Then $j' \in U_j$ and so, by definition of perversity, $j'+1 \in U_{j-1}$. Therefore $(j-1, j'+1) \in \Gamma_U$, i.e., $\Gamma_U \leq \Gamma_U +_{\mathrm{ne}} 1$. Vice versa, if $\tilde{U}$ is a kinky upperset in $J \times J'$, let $U_{\tilde{U}}$ the preimage in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$ of $\tilde{U}$ via Γ (see the proof of Proposition 0.16). Then for any $j \in J$ and any $j' \in U_{\tilde{U};j}$ we have $j'+1 \in U_{\tilde{U};j-1}$ and so $U_{\tilde{U};j} + 1 \subseteq U_{\tilde{U};j-1}$. So the isomorphism of posets $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{O}(J \times J')$ restricts to an isomorphism of posets $\mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J')$. To see that $\Gamma\colon \mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J')$ is also $\mathbb{Z}$-equivariant, notice that, for every perversity U we have $(j, j') \in \Gamma_{U \dot{+} (-1)}$ if and only if $j' \in U_{j+1} + 1$. i.e., if and only if $(j+1, j'-1) \in \Gamma_U$. This latter condition is equivalent to $(j, j') \in \Gamma_U +_{\mathrm{nw}} 1$, so we find $\Gamma_{U \dot{+} (-1)} = \Gamma_U +_{\mathrm{nw}} 1$. ◻

Lemma 0.27. *The map Γ from Proposition 0.16 induces an isomorphism of $\mathbb{Z}$-posets*

$$\Gamma\colon \mathrm{Perv}^\circ(\mathbb{Z}, \mathbb{Z}) \to \mathrm{Kink}^\circ(\mathbb{Z} \times \mathbb{Z}).$$

Proof. A kinky upperset U is in the image of Γ if and only if U is of the form $(\geq f)$ for a suitable function $f\colon \mathbb{Z} \to \mathbb{Z}$. This is possible if and only if $U_n \neq \emptyset, \mathbb{Z}$ for every $n \in \mathbb{Z}$. As U is kinky, if $U_{n_0} = \emptyset$ for some n_0, then $U_{n_0+1} + 1 \subseteq U_{n_0} = \emptyset$, and so $U_{n_0+1} = \emptyset$. Inductively, this gives $U_n = \emptyset$ for every $n \geq n_0$. On the other hand, since a kinky upperset is an upperset, if there exists a nonempty U_n with $n < n_0$, then there exist an element (n, m) in U and so, since $(n, m) \leq (n_0, m)$, also $(n_0, m) \in U$. But then $m \in U_{n_0}$, which is impossible. So also the U_n with $n < n_0$ are empty and therefore $U = \emptyset$. Similarly, if $U_{n_0} = \mathbb{Z}$ for some n_0, then $U_n = \mathbb{Z}$ for every $n > n_0$ as U is an upperset, while the kinkiness condition $U_n + 1 \subseteq U_{n-1}$ implies that also $U_{n_0-1} = \mathbb{Z}$ and so inductively that all U_n with $n < n_0$ are the whole of $\mathbb{Z}$. That is, $U = \mathbb{Z} \times \mathbb{Z}$ in this case. ◻

Lemma 0.28. *The isomorphism of $\mathbb{Z}$-modules $\varphi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $\varphi\colon (n, n') \mapsto (n+n', n')$ induces an isomorphism of $\mathbb{Z}$-posets*

$$\varphi\colon \mathrm{Kink}(\mathbb{Z} \times \mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}),$$

where the $\mathbb{Z}$-action on $\mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ is the one induced by the "northern" $\mathbb{Z}$-action on $\mathbb{Z}^2$, namely, $(n, n') + 1 = (n, n'+1)$. In particular φ induces an isomorphism of $\mathbb{Z}$-posets $\mathrm{Kink}^\circ(\mathbb{Z} \times \mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$.

Proof. A subset U of $\mathbb{Z} \times \mathbb{Z}$ is an upper set (in the product order) if and only if $U + K \subseteq U$, where K is the $\mathbb{Z}$-cone spanned by $\binom{1}{0}$ and $\binom{0}{1}$, while U is a kinky upperset if and only if $U + K^{\text{kink}} \subseteq U$, where K^{kink} is the $\mathbb{Z}$-cone spanned by $\binom{1}{0}$ and $\binom{-1}{1}$. Morever the $\mathbb{Z}$ action on the uppersets is generated by $U \mapsto U + \binom{0}{1}$ and the the $\mathbb{Z}$ action on the kinky uppersets is generated by $U \mapsto U + \binom{-1}{1}$, where in both cases the sum on the right hand side is the sum in $\mathbb{Z}^2$. As $\varphi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is an isomorphism of $\mathbb{Z}$-modules with $\varphi(K^{\text{kink}}) = K$ and $\varphi\binom{-1}{1} = \binom{0}{1}$, the statement follows (φ is manifestly inclusion preserving). $\square$

Corollary 0.29. *We have an isomorphism of* $\mathbb{Z}$*-posets*

$$\mathrm{perv}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

mapping a perversity function p to the image via the isomorphism $\varphi\colon (n, n') \mapsto (n + n', n')$ of the set $\{(n, n') \in \mathbb{Z} \times \mathbb{Z}$ such that $n' \geq p(n) - n\}$.

Example 0.30. The zero perversity function $p(n) \equiv 0$ corresponds to the upper set $\{(n, n') \in \mathbb{Z} \times \mathbb{Z}$ such that $n \geq 0\}$; the identity perversity function $p(n) \equiv n$ corresponds to the upper set $\{(n, n') \in \mathbb{Z} \times \mathbb{Z}$ such that $n' \geq 0\}$.

Remark 0.31. The two "missing" upper sets from $\mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$ can be recovered by adding to the set $\mathrm{perv}_{\mathbb{Z}}$ of perversity functions the two "constant infinite perversities", i.e., the function $p_{+\infty}\colon \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, +\infty\}$ defined by $p_{+\infty}(n) = +\infty$ for every $n \in \mathbb{Z}$ and the function $p_{-\infty}\colon \mathbb{Z} \to \mathbb{Z} \cup \{-\infty, +\infty\}$ defined by $p_{-\infty}(n) = -\infty$ for every $n \in \mathbb{Z}$. The extended set

$$\widehat{\mathrm{perv}}_{\mathbb{Z}} = \mathrm{perv}_{\mathbb{Z}} \cup \{p_{-\infty}, p_{+\infty}\}$$

is naturally a $\mathbb{Z}$-poset with $p_{+\infty}$ and $p_{-\infty}$ as maximum and minimum element, respectively (so they are in particular $\mathbb{Z}$-fixed points), and the inclusion of $\mathrm{perv}_{\mathbb{Z}}$ into $\widehat{\mathrm{perv}}_{\mathbb{Z}}$ isa morphism of $\mathbb{Z}$-posets. Moreover, the isomorphism of $\mathbb{Z}$-posets $\mathrm{perv}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$ from Corollary 0.29 extends to an isomorphism of $\mathbb{Z}$-posets

$$\widehat{\mathrm{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z}).$$

As well as the $\mathbb{Z}$-action $p \mapsto p \dotplus 1$, also the action $p \mapsto p + 1$ from Remark 0.22 extends to a $\mathbb{Z}$-action of $\widehat{\mathrm{perv}}_{\mathbb{Z}}$ (again, $p_{+\infty}$ and $p_{-\infty}$ will be fixed points). The isomorphism of posets $\widehat{\mathrm{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ will then transfer this $\mathbb{Z}$-action to $\mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ inducing on $\mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ a $\mathbb{Z}$-action such that $\widehat{\mathrm{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z})$ is an isomorphism of posets with the $\mathbb{Z}$-action $p \mapsto p + 1$ on the left hand side. More precisely, we have the following result.

Proposition 0.32. *We have an isomorphism of posets*

$$\widehat{\mathrm{perv}}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z} \times \mathbb{Z})$$

mapping a perversity function p to the image via the isomorphism $\varphi\colon (n, n') \mapsto (n+n', n')$ of the set $S_p = \{(n, n') \in \mathbb{Z}\times\mathbb{Z}$ such that $n' \geq p(n)-n\}$, and mapping the "infinite perversity functions" $p_{-\infty}$ and $p_{+\infty}$ to $\mathbb{Z} \times \mathbb{Z}$ and to $\emptyset$, respectively.

Moreover, this is an isomorphism of $\mathbb{Z}$-posets, where the $\mathbb{Z}$-action on the left is given by $(p+1)(n) = p(n)+1$ and the $\mathbb{Z}$-action on the right is the one induced by the "northeastern" $\mathbb{Z}$-action on $\mathbb{Z}^2$, namely, $(n,n')+1 = (n+1,n'+1)$.

Proof. After Corollary 0.29 and remark 0.31, the only thing left to prove is the $\mathbb{Z}$-equivariancy of the isomorphism, i.e., that we have

$$\varphi(S_{p+1}) = \varphi(S_p) + \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right).$$

As φ is an isomorphism of $\mathbb{Z}$-modules, this is equivalent to

$$S_{p+1} = S_p + \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),$$

i.e., to the condition $(n,n') \in S_{p+1}$ if and only if $(n,n'-1) \in S_p$, which is obvious. $\square$

Takin the opposite of the complement (or, equivalently, the complement of the opposite) gives an isomorphism of posets

$$\begin{aligned}\mathcal{O}(\mathbb{Z}\times\mathbb{Z}) &\xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z})^{\mathrm{op}}\\ U &\mapsto (\mathbb{Z}\times\mathbb{Z})\setminus(-U),\end{aligned}$$

where $-U = \{(-n,-n') \text{ with } (n,n') \in U\}$. This isomorphisms changes the northeastern action in its opposite, i.e., the "southwestern" action $(n,n')+1 = (n-1,n'-1)$, so Proposition 0.32 immediatley gives

Corollary 0.33. *We have an isomorphism of posets*

$$\widehat{\mathrm{perv}}_{\mathbb{Z}}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z})$$

mapping a perversity function p to the complement of the image via the isomorphism $\psi\colon (n,n') \mapsto (-n-n',-n')$ of the set $S_p = \{(n,n') \in \mathbb{Z}\times\mathbb{Z}$ such that $n' \geq p(n)-n\}$, and mapping the "infinite perversity functions" $p_{-\infty}$ and $p_{+\infty}$ to $\emptyset$ and to $\mathbb{Z}\times\mathbb{Z}$, respectively. Moreover, this is an isomorphism of $\mathbb{Z}$-posets, where the $\mathbb{Z}$-action on the left is given by $(p +^{\mathrm{op}} 1)(n) = p(n)-1$ and the $\mathbb{Z}$-action on the right is the one induced by the "northeastern" $\mathbb{Z}$-action on $\mathbb{Z}^2$, namely, $(n,n')+1 = (n+1,n'+1)$.

0.2 Slicing the heart

Let $\mathscr{D}$ be a stable ∞-category. Then, as we recalled above, a bounded t-structure on $\mathscr{D}$ is the datum of a Bridgeland $\mathbb{Z}$-slicing $\mathfrak{t}$ of $\mathscr{D}$. Let $\heartsuit_{\mathfrak{t}}$ denote the heart of $\mathfrak{t}$. Then an abelian $\mathbb{Z}$-slicing of $\heartsuit_{\mathfrak{t}}$ is the datum of an extension $\tilde{\mathfrak{t}}$ of $\mathfrak{t}$ to a Bridgeland $\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, where $\hat{\mathbb{Z}}$ denotes the $\mathbb{Z}$-poset consisting of $\mathbb{Z}$ endowed wth the trivial Z-action, and the morphism $\mathcal{O}(\mathbb{Z}) \to \mathcal{O}(\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}})$ is induced by the projection on the first factor $\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}} \to \mathbb{Z}$. See [**?**, Section??]. We will denote by $\heartsuit_{\mathfrak{t};\phi}$ the ϕ-th slice of the heart of $\mathfrak{t}$. In other words,

$$\heartsuit_{\mathfrak{t};\phi} = \mathscr{D}_{\tilde{\mathfrak{t}};(0,\phi)}.$$

Notice that we have

$$\heartsuit_{\mathfrak{t};\phi}[n] = \mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)},$$

where $[n]$ denotes the "shift by n" functor on $\mathscr{D}$.

Example 0.34. Via the obvious inclusion of $\mathbb{Z}$-posets $\{0,1\} \hookrightarrow \hat{\mathbb{Z}}$, any torsion pair $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$ on $\heartsuit_{\mathfrak{t}}$ defines an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$.

Definition 0.35. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. An abelian $\mathbb{Z}$-slicing $\hat{\mathfrak{t}}$ on $\heartsuit_{\mathfrak{t}}$ is called:*

- grading *(or* radical*) if $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ for $\phi > \psi + n$ and for $\phi = \psi + n$ with $n \geq 2$;*
- gluable *if $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ for $\phi > \psi$ and $n > 0$;*

Remark 0.36. The definition of gluable abelian $\mathbb{Z}$-slicing of $\heartsuit_{\mathfrak{t}}$ is the specialization of Definition 0.11 to $J_1 = \mathbb{Z}$ and $J_2 = \hat{\mathbb{Z}}$.

Remark 0.37. Historically, grading filtrations first appeared in [**?**] under the name of 'radical filtrations'.

Example 0.38. Let $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$ be a torsion pair on $\heartsuit_{\mathfrak{t}}$. Then $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$, seen as an abelian $\mathbb{Z}$-slicing, is grading. Namely, as $\heartsuit_{\mathfrak{t};\phi}[n] = 0$ for $\phi \notin \{0,1\}$, the only nontrivial orthogonality conditions to be checked are:

- $\heartsuit_{\mathfrak{t};0} \boxslash \heartsuit_{\mathfrak{t};0}[n]$ for $n < 0$;
- $\heartsuit_{\mathfrak{t};0} \boxslash \heartsuit_{\mathfrak{t};1}[n]$ for $n < -1$;
- $\heartsuit_{\mathfrak{t};1} \boxslash \heartsuit_{\mathfrak{t};0}[n]$ for $n < 1$;
- $\heartsuit_{\mathfrak{t};1} \boxslash \heartsuit_{\mathfrak{t};1}[n]$ for $n < 0$.

These all follows from the orthogonality relation $\heartsuit_{\mathfrak{t}} \boxslash \heartsuit_{\mathfrak{t}}[n]$ for $n < 0$, except for $\heartsuit_{\mathfrak{t};1} \boxslash \heartsuit_{\mathfrak{t};0}$ which is true by definition of torsion pair.

Proposition 0.39. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let $\tilde{\mathfrak{t}}$ be an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. If $\tilde{\mathfrak{t}}$ is gluable, then $\tilde{\mathfrak{t}}$ is grading.*

Proof. Let ϕ and ψ be in $\mathbb{Z}$ with $\phi > \psi + n$. The orthogonality condition $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ is trivially satisfied if $n < 0$, so let us assume $n \geq 0$. If $n = 0$, then $\phi > \psi$ and the orthogonaliy condition $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}$ is satisfied by definition of slicing. Finally, if $n > 0$, then we have $\phi > \psi$ and $n > 0$, so $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ by definition of gluable slicing. If $\phi = \psi + n$ with $n \geq 2$, then in particular $\phi > \psi$ and $n > 0$, so again $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$. $\square$

Proposition 0.40. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be a grading abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. Then, $\tilde{\mathfrak{t}}$ is g_p-compatible, where*

$$\begin{aligned} g_p \colon \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} &\to \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \\ (n,\phi) &\mapsto (n + p(\phi), -p(\phi)), \end{aligned}$$

for every perversity function $p \colon \mathbb{Z} \to \mathbb{Z}$.

Proof. The map g_p is $\mathbb{Z}$-equivariant, as the action on the second factor is the trivial one. Let (n,ϕ) and (m,ψ) in $\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$ with $(n,\phi)\leq(m,\psi)$ such that $g_p(n,\phi)>g_p(m,\psi)$ in $\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$. By definition of g_p this means that we have

$$(n+p(\phi),-p(\phi))>(m+p(\psi),-p(\psi))$$

in $\mathbb{Z}\times_{\mathrm{lex}}\mathbb{Z}$, i.e., that $n+p(\phi)>m+p(\psi)$ or that $n+p(\phi)=m+p(\psi)$ and $p(\psi)>p(\phi)$. Similarly, the condition $(n,\phi)\leq(m,\psi)$ means that either $n<m$ or $n=m$ and $\phi\leq\psi$. By considering all possibilities, and taking into account that a perversity function is nondecreasing, one sees that there is actually a single case to deal with:

- $p(\phi)-p(\psi)>m-n$, with $m>n$ and $\phi>\psi$;

As p is a perversity function, if $\phi\geq\psi$ then

$$0\leq p(\phi)-p(\psi)\leq\phi-\psi,$$

so we have

$$\phi\geq\psi+p(\phi)-p(\psi)>\psi+m-n.$$

Since $\tilde{\mathfrak{t}}$ is grading, this implies $\heartsuit_{\mathfrak{t};\phi}\boxslash\heartsuit_{\mathfrak{t};\psi}[m-n]$, and so $\heartsuit_{\mathfrak{t};\phi}[n]\boxslash\heartsuit_{\mathfrak{t};\psi}[m]$, i.e.,

$$\mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)}\boxslash\mathscr{D}_{\tilde{\mathfrak{t}};(m,\psi)}.$$

Since $\phi>\psi+m-n$, either $\phi>\psi+m-n+1$ or $\phi=\psi+m-n+1$. In the first case, reasoning as above we find $\mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)}\boxslash\mathscr{D}_{\tilde{\mathfrak{t}};(m,\psi)}[1]$. In the second case, as $m>n$ we have $m-n+1\geq 2$. As $\tilde{\mathfrak{t}}$ is grading, this gives $\heartsuit_{\mathfrak{t};\phi}\boxslash\heartsuit_{\mathfrak{t};\psi}[m-n+1]$, i.e., again

$$\mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)}\boxslash\mathscr{D}_{\tilde{\mathfrak{t}};(m,\psi)}[1].$$

□

Remark 0.41. By taking as perversity function the identity, $\mathrm{id}\colon\mathbb{Z}\to\mathbb{Z}$, we see that if $\tilde{\mathfrak{t}}$ is grading then $\tilde{\mathfrak{t}}$ is α-compatible, where $\alpha\colon\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}\to\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$ is the $\mathbb{Z}$-equivariant map given by

$$\alpha(n,m)=(n+m,-m).$$

The proof of the following lemma is straightforward.

Lemma 0.42. *Let $\beta\colon\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}\to\mathbb{Z}\times_{\mathrm{lex}}\mathbb{Z}$ be the map defined by*

$$\beta(n,m)=(n,n+m).$$

Then β is an isomorphism of $\mathbb{Z}$-tosets. Moreover the diagram

$$\begin{tikzcd}
\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}} \arrow[d, "\beta"] \\
\mathbb{Z}\times_{\mathrm{lex}}\mathbb{Z} \arrow[r, "e"] & \mathbb{Z}\times_{\mathrm{lex}}\mathbb{Z}
\end{tikzcd}$$

commutes, where $e\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ is the exchange map $e(n,m)=(m,n)$ from Lemma 0.10 and $\alpha\colon \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}\to\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$ is the map $\alpha(n,m)=(n+m,-m)$ from Remark 0.41.

Corollary 0.43. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. Then, in the notation of Lemma 0.42, $\tilde{\mathfrak{t}}$ is α-compatible if and only if $\beta_*\mathfrak{t}$ is gluable.*

Proof. By Lemma 0.42 and Remark 0.3, $\tilde{\mathfrak{t}}$ is α-compatible if and only if $\tilde{\mathfrak{t}}$ is $(\beta\circ\alpha)$-compatible, which happens if and only if $\tilde{\mathfrak{t}}$ is $(e\circ\beta)$-compatible. As β is an isomorphism of $\mathbb{Z}$-tosets, we can write $\mathfrak{t}=(\beta^{-1})_*\beta_*\mathfrak{t}$, and so, by Remark 0.3 again, $\tilde{\mathfrak{t}}$ is $(e\circ\beta)$-compatible if and only if $\beta_*\mathfrak{t}$ is e-compatible. By Remark 0.36, this is equivalent to saying that $\beta_*\mathfrak{t}$ is gluable. □

From Proposition 0.39 and Remark 0.41 we immediately get the following

Corollary 0.44. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. If $\tilde{\mathfrak{t}}$ is grading, then $\beta_*\mathfrak{t}$ is gluable. In particular, if $\tilde{\mathfrak{t}}$ is grading, then also $\beta_*\mathfrak{t}$ is grading.*

Proposition 0.45. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be a grading abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. For every perversity p, let $\gamma_p\colon \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}\to\mathbb{Z}$ the $\mathbb{Z}$-equivariant morphism given by*

$$\gamma_p\colon (n,\phi)\mapsto n+p(\phi).$$

Then

- *$\tilde{\mathfrak{t}}$ is γ_p-compatible;*
- *$(\gamma_p)_!\tilde{\mathfrak{t}}$ is a bounded t-structure on $\mathscr{D}$;*
- *the map*

$$\begin{aligned}\Psi\colon \mathrm{perv}_{\mathbb{Z}}^{\mathrm{op}} &\to \mathrm{ts}(\mathscr{D})\\ p &\mapsto (\gamma_p)_!\tilde{\mathfrak{t}},\end{aligned}$$

  is a morphism of $\mathbb{Z}$-posets, where on the left we have the $\mathbb{Z}$-action $(p+1)(n)=p(n)+1$ and on the right the $\mathbb{Z}$-action given by the shift in $\mathscr{D}$;
- *Ψ uniquely extends to a morphism of $\mathbb{Z}$-posets*

$$\Psi\colon \widehat{\mathrm{perv}}_{\mathbb{Z}}^{\mathrm{op}}\to\mathrm{ts}(\mathscr{D})$$

  preserving maxima and minima.

Proof. Let $g_p\colon \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}\to\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$ the map defined in Proposition 0.45 As $\tilde{\mathfrak{t}}$ is grading, by Proposition 0.45, $\tilde{\mathfrak{t}}$ is g_p-compatible. The projection on the first factor, $\pi_1\colon \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}\to\mathbb{Z}$ is a morphism of $\mathbb{Z}$-tosets, so by Remark 0.2 every $\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$-slicing is π_1-compatible. In particular, $(g_p)_!\mathfrak{t}$ is π_1-compatible. Therefore, by Proposition 0.9, $\mathfrak{t}$ is $(\pi_1 \circ g_p)$-compatible. As $\pi_1 \circ g_p = \gamma_p$, this precisely says that $\mathfrak{t}$ is γ_p-compatible. We therefore have a Bridgeland $\mathbb{Z}$-slicing, i.e., a bounded t-structure, $(\gamma_p)_!\mathfrak{t}$ on $\mathscr{D}$. The map Ψ is monotone and $\mathbb{Z}$-equivariant. Indeed, the t-structure Ψ_p is defined by the upper category

$$\mathscr{D}_{(\gamma_p)_!\tilde{\mathfrak{t}};\geq 0} = \langle \mathscr{D}_{\mathfrak{t};(n,\phi)} \rangle_{(n,\phi)\in\gamma_p^{-1}([0,+\infty))}.$$

If $p_1 \geq^{\mathrm{op}} p_2$, then $p_1 \leq p_2$ and so $n + p_1(\phi) \geq 0$ implies $n + p_2(\phi) \geq 0$, and so $\gamma_{p_1}^{-1}([0,+\infty)) \subseteq \gamma_{p_2}^{-1}([0,+\infty))$. This gives $\mathscr{D}_{(\gamma_{p_1})_!\tilde{\mathfrak{t}};\geq 0} \subseteq \mathscr{D}_{(\gamma_{p_2})_!\tilde{\mathfrak{t}};\geq 0}$, i.e.,

$$\mathscr{D}_{(\gamma_{p_1})_!\tilde{\mathfrak{t}}} \geq \mathscr{D}_{(\gamma_{p_2})_!\tilde{\mathfrak{t}}}$$

in the parial order on $\mathrm{ts}(\mathscr{D})$. Similarly, $n + (p +^{\mathrm{op}} 1)(\phi) \geq 0$ if and only if $n + p(\phi) \geq 1$ and so

$$\mathscr{D}_{(\gamma_{p+1})_!\tilde{\mathfrak{t}};\geq 0} = \mathscr{D}_{(\gamma_p)_!\tilde{\mathfrak{t}};\geq 1} = \mathscr{D}_{(\gamma_p)_!\tilde{\mathfrak{t}};\geq 0}[1].$$

Finally, Ψ trivially (and uniquely) extends to $\widehat{\mathrm{perv}}_{\mathbb{Z}}^{\mathrm{op}}$ preserving maxima and minima. $\square$

Recalling Corollary 0.33 and Proposition 0.39 we finally get the result we were aiming to.

Theorem 0.46. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be a glueing abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. Then $\tilde{\mathfrak{t}}$ explicitly induces a natural morphism of $\mathbb{Z}$-posets*

$$\Psi \colon \mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \to \mathrm{ts}(\mathscr{D})$$

such that for every proper upper set in $\mathbb{Z} \times \mathbb{Z}$, the corresponding t-structure on $\mathscr{D}$ is bounded.

Remark 0.47. Theorem 0.46 is actually true under the weaker assumption that $\tilde{\mathfrak{t}}$ is grading. We preferred to state it under the stroinger assumption of a gluable $\tilde{\mathfrak{t}}$ to make its use in the examples below more immediate.

Remark 0.48. The morphism Ψ can be thought of as a $(\mathbb{Z} \times \mathbb{Z})$-slicing of $\mathscr{D}$, but keeping in mind that the poset $\mathbb{Z} \times \mathbb{Z}$ indexing the slices (and so the cohomologies) is now not totally ordered. This is a possibly subtle point, so let us spend a few more words on it. An abelian $\mathbb{Z}$-slicing $\tilde{\mathfrak{t}}$ of $\heartsuit_{\mathfrak{t}}$ is by definition a $(\mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}})$-slicing, and by Lemma 0.42, this is equivalently a $(\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z})$-slicing. So going from an abelian slicing of the heart to a $(\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z})$-slicing of $\mathscr{D}$ is a trivial step. What is highly nontrivial is going from an abelian slicing of the heart to a $(\mathbb{Z} \times \mathbb{Z})$-slicing of $\mathscr{D}$, where now the poset structure on $\mathbb{Z} \times \mathbb{Z}$ is given by the product order and ot by the lexicographic order. And indeed this can generally not be done for an arbitrary abelian slicing of the heart, and here is where the property of the abelian slicing of being grading comes in. Finally, to emphasize once more how going from a $(\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z})$-slicing to a $(\mathbb{Z} \times \mathbb{Z})$-slicing is a nontrivial step, consider how there are in $\mathbb{Z} \times \mathbb{Z}$ many more upper sets than in $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}$

Remark 0.49. Describing the bounded t-structure on $\mathscr{D}$ associated by Theorem 0.46 to a proper upper set U of $\mathbb{Z} \times \mathbb{Z}$ is a bit involved, but it is a completely explicit procedure. To begin with, recall that a bounded t-structure is completely determined by its heart, so we only need to give a description of the heart $\heartsuit_U$ associated with U. To do this, notice that the perveristy function associated to U is

$$p_U(n) = n + \min\{n' \in \mathbb{Z} \text{ such that } (n,n') \notin \psi^{-1}(U)\}$$

where $\psi^{-1}(n,n') = (-n+n', -n')$. The heart $\heartsuit_U$ is then the extension closed subcategory of $\mathscr{D}$ generated by the slices $\mathscr{D}_{(-p_U(n),n)}$ of $\tilde{\mathfrak{t}}$.

Example 0.50. If $U = \{(n,n') \text{ such that } n' \geq 1\}$, then $\psi(U) = \{(n,n') \text{ such that } n' \leq -1\}$ and so $p_U(n) = n$. Therefore, $\heartsuit_U = \langle \mathscr{D}_{(-n,n)} \rangle_{n\in\mathbb{Z}}$ in this case. If $U = \{(n,n') \text{ such that } n \geq 1\}$, then $\psi(U) = \{(n,n') \text{ such that } n' \leq -n-1\}$ and so $p_U(n) = 0$. Therefore, $\heartsuit_U = \langle \mathscr{D}_{(0,n)} \rangle_{n\in\mathbb{Z}}$ in this case.

A more explict description of the *perverse hearts* of $\mathscr{D}$ is as follows.

Theorem 0.51. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be a glueing abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. Let U be an upper set of $\mathbb{Z} \times Z$ and let p be the corresponding perversity. Then the preverse heart $\heartsuit_p = \heartsuit_U$ of $\mathscr{D}$ is the full subcategory of $\mathscr{D}$ on those objects X such that*

$$H^n_{\mathfrak{t}}(X)[-n] = \langle \heartsuit_{\mathfrak{t};n'} \rangle_{n' \in p^{-1}(-n)}$$

for every $n \in \mathbb{Z}$, where $H^n_{\mathfrak{t}}(X)$ is the n-th cohomology object of X in the t-structure $\mathfrak{t}$ and $\{\heartsuit_{\mathfrak{t};n'}\}_{n'\in\mathbb{Z}}$ are the slices of the heart $\heartsuit_{\mathfrak{t}}$ of $\mathfrak{t}$ for the abelian $\mathbb{Z}$-slicing $\tilde{\mathfrak{t}}$.

Proof. Denote by $\mathfrak{t}_p$ the $\mathfrak{t}$-structure on $\mathscr{D}$ associated with the perversity function p. Then the lower subcategory $\mathscr{D}_{\mathfrak{t}_p;<0}$ and the upper subcategory $\mathscr{D}_{\mathfrak{t}_p;\geq 0}$ of $\mathscr{D}$ are defined, by Proposition 0.45, as

$$\mathscr{D}_{\mathfrak{t}_p;<0} = \langle \mathscr{D}_{\tilde{\mathfrak{t}};(n,n')} \rangle_{n+p(n')<0}; \qquad \mathscr{D}_{\mathfrak{t}_p;\geq 0} = \langle \mathscr{D}_{\tilde{\mathfrak{t}};(n,n')} \rangle_{n+p(n')\geq 0}.$$

These can be equivalently described as

$$\mathscr{D}_{\mathfrak{t}_p;<0} = \{X \in \mathscr{D} \text{ such that } H^{(n,n')}_{\tilde{\mathfrak{t}}}(X) = \mathbf{0} \text{ for } n + p(n') \geq 0\};$$
$$\mathscr{D}_{\mathfrak{t}_p;\geq 0} = \{X \in \mathscr{D} \text{ such that } H^{(n,n')}_{\tilde{\mathfrak{t}}}(X) = \mathbf{0} \text{ for } n + p(n') < 0\},$$

see, e.g., [**?**, Remark 4.27]. Therefore

$$\heartsuit_p = \{X \in \mathscr{D} \text{ such that } H^{(n,n')}_{\tilde{\mathfrak{t}}}(X) = \mathbf{0} \text{ for } n + p(n') \neq 0\}$$

Equivalently, this means that

$$\heartsuit_p = \{X \in \mathscr{D} \text{ such that } H^n_{\mathfrak{t}}(X) \in \langle \heartsuit_{\mathfrak{t};n'}[n] \rangle_{p(n')=-n}\}.$$

□

Example 0.52. Let $k \in \mathbb{Z}$ and let $\chi_{[k,+\infty)}\colon \mathbb{Z} \to \{0,1\}$ the characteristic function of the interval $[k,+\infty)$. Seen as a function from $\mathbb{Z}$ to $\mathbb{Z}$, the function $\chi_{[k,+\infty)}$is a perversity function of a very special kind: it is a perversity function taking exactly two values. Moreover, it is easy to see that –up to an additive constant– perverity functions taking exactly two values are precisely characteristic functions of upper intervals in $\mathbb{Z}$. We have

$$\chi^{-1}_{[k,+\infty)}(-n) = \begin{cases} \emptyset & \text{if } n \neq -1, 0 \\ \\ [k,+\infty) & \text{if } n = -1 \\ \\ (-\infty, k) & \text{if } n = 0. \end{cases}$$

Therefore the perverse heart $\heartsuit_{\chi^{-1}_{[k,+\infty)}}$ of $\mathscr{D}$ is the full subcategory of $\mathscr{D}$ on those objects X such that

$$\begin{cases} H^n_{\mathfrak{t}}(X[1]) = \mathbf{0} \qquad \text{if } n \neq 0, 1 \\ \\ H^0_{\mathfrak{t}}(X[1]) = \langle \heartsuit_{\mathfrak{t};n'} \rangle_{n' \in [k,+\infty)} \\ \\ H^1_{\mathfrak{t}}(X[1]) = \langle \heartsuit_{\mathfrak{t};n'}[1] \rangle_{n' \in (-\infty,k)} \end{cases}$$

for every $n \in \mathbb{Z}$. In other words, $\heartsuit_{\chi^{-1}_{[k,+\infty)}}$ is (up to a shift by 1) the heart of the tilted t-structure obrained by tilting $\mathfrak{t}$ with the torsion theory on $\heartsuit_{\mathfrak{t}}$ given by $\mathcal{F} = \langle \heartsuit_{\mathfrak{t};n'} \rangle_{n' \in (-\infty,k)}$ and $\mathcal{T} = \langle \heartsuit_{\mathfrak{t};n'} \rangle_{n' \in [k,+\infty)}$.

1 A zoo of examples

Remark 1.1. The notion of Bridgeland $\hat{\mathbb{Z}}$-slicing already appears in literature: it is no more than an infinite version of a semiorthogonal decomposition in the sense of referenza, or a '*baric structure*' as defined in referenza. Thus, we can start with a baric structure of $\mathscr{D}$ and a boudned t-structure on $\mathscr{D}_n$ for every $n \in \hat{\mathbb{Z}}$. If the associated $\hat{\mathbb{Z}} \times_{\text{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$ is gluable, $e_!\mathfrak{t}$ is gluable by Lemma 0.13 and thus, using Lemma 0.13 we get a morphism of $\mathbb{Z}$-posets

$$\text{perv}_{\mathbb{Z}} \to \text{ts}(\mathscr{D})$$

We can build like that whole new classes of 'perverse' t-structures on $\mathscr{D}$. We'll see an instance of this construction in the example below.

Example 1.2. An instance of a nonstandard construction of a t-structure lies within the theory of Koszul duality. The latter does not preserve indeed the standard t-structures, but but exchanges them with a 'diagonal' one, whose

existence we now relate to the results of the present work.
Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an $\mathbb{N}$-graded ring with B_0 semisimple. Denote $\mathscr{A}$ the category of $\mathbb{Z}$-graded B-modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote $\mathscr{A}_\phi$ the full subcategory of $\mathscr{A}$ of modules concentrated in degree ϕ. Clearly, this defines an abelian $\mathbb{Z}$-slicing $\tilde{\mathfrak{t}}$ on $\mathscr{A}$. Following [**?**] we have

$$\mathrm{Ext}^n_{\mathscr{A}}(\mathscr{A}_\phi, \mathscr{A}_\psi) = 0$$

for $n > \psi - \phi$. This means that $\tilde{\mathfrak{t}}$ is a glauble slicing and the bounded t-structure $(\gamma_1)_! \tilde{\mathfrak{t}}$ (where 1 is the identity of $\mathbb{Z}$) on $\mathscr{D}^b(\mathscr{A})$ is exactly the t-structure involved in Koszul duality.

Example 1.3. In this example we relate the gluability condition with the Beilinson-Soulé conjecture from motivic topology. We fix a field k and, just for this example, we stick to a more traditional 1-categorical setup. Recall that the existence of motives, which is still an open question in general, was conjectured by Grothendieck in order to build a universal Weyl cohomology (also called 'motivic cohomology') theory for schemes. However, Deligne observed that it could be easier to construct a triangulated category (the 'mixed' motives) which should play the role of the derived category of motives, and later recover the latter as the heart of a bounded t-structure. Voedvodskij finally succeded in constructing a triangulated category of mixed rational motives over k which contains, for $n \in \mathbb{Z}$, a 'Tate object' $\mathbb{Q}(n)$ that represents the n-th motivic cohomology functor. Now, let us consider the triangualted subcategory $\mathscr{D}\mathrm{TM}_k$ generated by the Tate objects. In other words, $\mathscr{D}\mathrm{TM}_k$ is the category of mixed rational Tate motives. We then get isomorphisms of groups

$$\mathscr{D}\mathrm{TM}_k(\mathbb{Q}(i), \mathbb{Q}(j)[n]) = K_{2(j-i)-n}(k)^{(j-i)}$$

where $K_a(k)$ is the a-th higher K-theory group of the point $\mathrm{Spec}(k)$ and $K_a(k)^{(b)}$ is the weight b summand of $K_a(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to the Adams action (in other words, $K_a(k)^{(b)}$ is the a-th b-codimensional Bloch's higher Chow group of $\mathrm{Spec}(k)$ with rational coefficients). For dimensional reasons we immediately see that the right hand side vanishes for $i < j$ and, when $i = j$, for $n \neq 0$. In other words, the Tate objects form an infinite exceptional collection (referenza a decomposizioni semiortogonali) on $\mathscr{D}\mathrm{TM}_k$ which is clearly full by definition. By the general theory of semiorthogonal decomposition, this simply means that setting $(\mathscr{D}\mathrm{TM}_k)_{=n}$ as the triangulated subcategory generated by $\mathbb{Q}(n)$ defines slices of a baric structure with exact equivalences

$$(\mathscr{D}\mathrm{TM}_k)_{=n} \simeq \mathscr{D}^b(\mathbb{Q})$$

where the member on the right is the bounded derived category of finite-dimensional rational vector spaces (which is equivalent to the abelian category

graded vector spaces which finite-dimensional homogeneous pieces). The latter, being a derived category, posseses a canonical bounded t-structure and this finally induces a Bridgeland $\mathbb{Z}_{\mathrm{tri}} \times_{\mathrm{lex}} \mathbb{Z}$ slicing on $\mathscr{D}\mathrm{TM}_k$.
Now, the latter slicing is gluable if and only if $K_{2(j-i)-n}(k)^{(j-i)} = \mathbf{0}$ whenever both $i < j$ and $n \leq 0$ hold. This is exactly the Beilinson-Soulé standard vanishing conjecture, which is now known to hold, for example, when k is a number field due to a celebrated computation by Borel. In this case, by applying $e_!$ we get a Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$-slicing on $\mathscr{D}\mathrm{TM}_k$ and thus a bounded t-structure whose heart contains the desired unmixed Tate motives over k. In other words, we recover a well known fact (see referenza Levine) using a rather abstract and general language: assuming the Beilinson-Soulé conjecture, (Tate) motives exist. Moreover, following the route of Remark 1.1, we get a t-structure for each perversity function. Those are the '*perverse motives*' appearing in referenza.

Riflessione finale: a che serve la definizione di 'grading'? Non la usiamo mai negli esempi. I suoi vantaggi sono: si ricollega alla letteratura (le filtrazioni radicali), e' equivalente a essere gluable tramite l'isomorfismo β con $\mathbb{Z} \times_{lex} \mathbb{Z}$ e la Proposition 0.40 e' un se e solo se (e' implicito nel Remark 0.41). Si potrebbe anche togliere tale definizione e usare solo l'incollabilita', renderebbe anche il tutto piu' uniforme.