0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of $\mathbb{Z}$-posets (but just a $\mathbb{Z}$-equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a $\mathbb{Z}$-toset, i.e., a totally ordered set together with a monotone action of $\mathbb{Z}$, that we will denote by $(n, x) \mapsto x + n$. Recall that a J-slicing on a stable ∞-category $\mathscr{D}$ is a morphism of $\mathbb{Z}$-posests $\mathfrak{t}\colon \mathcal{O}(J) \to \mathrm{ts}(\mathscr{D})$, where $\mathcal{O}(J)$ is the $\mathbb{Z}$-poset of the slicings of J (i.e., the decompositions of J into a disjoint union of a lower set L and of an upper set U), and $\mathrm{ts}(\mathscr{D})$ is the $\mathbb{Z}$-poset of t-structures on $\mathscr{D}$.
Clearly, if $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then $f^{-1}\colon \{\text{subsets of } J'\} \to \{\text{subsets of } J\}$ induces a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-posets $f^{-1}\colon \mathcal{O}(J') \to \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$\begin{aligned} f_*\colon J\text{-slicings on } \mathscr{D} &\to J'\text{-slicings on } \mathscr{D} \\ \mathfrak{t} &\mapsto \mathfrak{t} \circ f^{-1}. \end{aligned}$$

The slices of $f_*\mathfrak{t}$ are clearly given by $\mathscr{D}_{f_*\mathfrak{t};\phi} = \mathscr{D}_{\mathfrak{t};f^{-1}(\{\phi\})}$. Notice that, as f is monotone, the subset $f^{-1}(\{\phi\})$ is an interval in J. It is immediate to see that f_* restricts to a map

$$f_*\colon \text{Bridgeland } J\text{-slicings on } \mathscr{D} \to \text{Bridgeland } J'\text{-slicings on } \mathscr{D}$$

Namely, if $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) = 0$ for every $\phi \in J'$ then $\mathcal{H}^{f^{-1}(\{\phi\})}_{\mathfrak{t}}(X) = 0$ for every ϕ in J'. As we are assuming the J-slicing $\mathfrak{t}$ is a Bridgeland slicing, this implies that $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) = 0$ for every ψ in $f^{-1}(\{\phi\})$, for every ϕ. Therefore $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) = 0$ for every ψ in J and so, again by definition of Bridgeland slicing, $X = 0$. Also, if $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) \neq 0$ then $\mathcal{H}^{f^{-1}(\{\phi\})}_{\mathfrak{t}}(X) \neq 0$ and so (again by the Bridgeland slicing condition) there exists at least an element ψ in $f^{-1}(\{\phi\})$ such that $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) \neq 0$. As the J-slicing $\mathfrak{t}$ is Bridgeland, the total of these ψ's must be finite, so only for finitely many ϕ we can have such a ψ. In other words, the number of indices ϕ in J' such that $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) \neq 0$ is finite.
Notice that, if $\mathfrak{t}$ is a Bridgeland slicing of $\mathscr{D}$, and $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then for any slicing (L, U) of J', the lower and the upper categories $\mathscr{D}_{f_*\mathfrak{t}(L)}$ and $\mathscr{D}_{f_*\mathfrak{t}(U)}$ can be equivalently defined as

$$\begin{aligned} \mathscr{D}_{f_*\mathfrak{t}(L)} &= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L} \\ \mathscr{D}_{f_*\mathfrak{t}(U)} &= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U} \end{aligned}$$

where $\langle \mathscr{S} \rangle$ denotes the extension-closed subcategory of $\mathscr{D}$ generated by the subcategory $\mathscr{S}$.[1]

[1] At some point it will be useful to recall Lemma 4.21 from ??: Let $\mathscr{S}_1, \mathscr{S}_2$ be two subcategories of $\mathscr{D}$ with $\mathscr{S}_1 \boxslash \mathscr{S}_2$, i.e., such that $\mathscr{D}(X_1, X_2)$ is contractible for any $X_1 \in \mathscr{S}_1$ and any $X_2 \in \mathscr{S}_2$. Then $\mathscr{S}_1 \boxslash \langle \mathscr{S}_2 \rangle$ and $\langle \mathscr{S}_1 \rangle \boxslash \mathscr{S}_2$, and so $\langle \mathscr{S}_1 \rangle \boxslash \langle \mathscr{S}_2 \rangle$.

The right hand sides of the above two expressions can clearly be defined for every morphism f from J to J' (i.e., not necessarily monotone nor $\mathbb{Z}$-equivariant), and as soon as f is $\mathbb{Z}$-equivariant, the assignment

$$(L,U) \mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$$

is an equivariant morphism from $\mathcal{O}(J)$ to pairs of subcategories of $\mathscr{D}$. Clearly, when f is not monotone there is no reason to expect that the pair $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ forms a t-structure on $\mathscr{D}$. Yet, it interesting to notice that the condition that f be monotone is only only sufficient in order to have this, and can indeed be relaxed.

Definition 0.1. *Let J and J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ a map of $\mathbb{Z}$-sets (i.e., a $\mathbb{Z}$-equivariant map, not necessarily nondecreasing). A Bridgeland J-slicing $\mathfrak{t}$ of $\mathscr{D}$ is* f-compatible *if $f(\phi) > f(\psi)$ with $\phi \leq \psi$ implies $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$*

Remark 0.2. Clearly, if f is monotone, then every J-slicing $\mathfrak{t}$ is order restoring as the condition '$f(\phi) > f(\psi)$ with $\phi \leq \psi$' is empty.

Lemma 0.3. *Let J and J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. Then, for any slicing (L,U) of J', the pair of subcategories $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ is a t-structure on $\mathscr{D}$.*

Proof. As f is $\mathbb{Z}$-equivariant and $U + 1 \subseteq U$, we have

$$\begin{aligned}
\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}[1] &= \langle \mathscr{D}_{\mathfrak{t};\phi}[1] \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi+1)\in U+1} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U+1} \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}
\end{aligned}$$

and similarly for the lower subcategory $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$. To show that

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U} \boxslash \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$$

it suffices to show that, if $f(\psi) \in L$ and $f(\phi) \in U$ then $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. As $L \cap U = \emptyset$ we cannot have $\psi = \phi$, so either $\psi < \phi$ or vice versa. In the first case, $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ by definition of Bridgeland J-slicing. In the second case, we have $\phi < \psi$ and $f(\phi) > f(\psi)$ as $f(\phi) \in U$ and $f(\psi) \in L$. Therefore, since $\mathfrak{t}$ is f-compatible, $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. Finally, we have to show that every object X in $\mathscr{D}$ fits into a fiber sequence

$$\begin{tikzcd}
X_U \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & X_L
\end{tikzcd}$$

with $X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$ and $X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}$. As $\mathfrak{t}$ is a Bridgeland slicing, we have a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar\imath}} X_{\bar\imath} \xrightarrow{\alpha_{\bar\imath+1}} X_{\bar\imath+1} \to \cdots \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) = \mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \in \mathscr{D}_{\phi_i}$ for all $i = 1, \cdots, n$, with $\phi_i > \phi_{i+1}$. Let us now consider the sequence of symbols L and U obtained putting in the i-th place L if $f(\phi_i) \in L$ and U if $f(\phi_i) \in U$. If this sequence is of the form $(U, U, \dots, U, L, L, \dots, L)$, then there exists an index $\bar\imath$ such that $f(\phi_i) \in U$ for $i \leq \bar\imath$ and $f(\phi_i) \in L$ for $i > \bar\imath$ (with $\bar\imath = -1$ or n when all of the $f(\phi_i)$ are in L or in U, respectively). Then we can consider the pullout diagram

$$\begin{tikzcd}
X_{\bar\imath} \arrow[r] \arrow[d, "f_L"'] & 0 \arrow[d] \\
X \arrow[r] & \mathrm{cofib}(f_L)
\end{tikzcd}$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar\imath}} X_{\bar\imath}$$

and

$$X_{\bar\imath} \xrightarrow{\alpha_{\bar\imath+1}} X_{\bar\imath+1} \to \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that $X_{\bar\imath} \in \langle \cup_{i=0}^{\bar\imath} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}$ while the second factorization shows that $\mathrm{cofib}(f_L) \in \langle \cup_{i=\bar\imath+1}^{n} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$. So we are done in this case. Therefore, we are reduced to showing that we can always avoid a $(\dots, L, U, \dots)$ situation in our sequence of L's and U's. Assume we have such a situation. Then we have an index i_0 with $f(\phi_{i_0}) \in L$ and $f(\phi_{i_0+1}) \in U$. This in particular implies $f(\phi_{i_0+1}) > f(\phi_{i_0})$ with $\phi_{i_0+1} < \phi_{i_0}$. As $\mathfrak{t}$ is f-compatible, this gives $\mathscr{D}_{\mathfrak{t};\phi_{i_0+1}} \boxslash (\mathscr{D}_{\mathfrak{t};\phi_{i_0}}[1])$. In particular, $\mathscr{D}(\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1])$ is contractible. Now consider the pasting of pullout diagrams

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1].
\end{tikzcd}$$

As the arrow $\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]$ factors through 0, we have $Y = \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus$

$\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ and the above diagram becomes

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\iota_2"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d, "\pi_1"] & 0 \arrow[d] \\
 & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1],
\end{tikzcd}$$

where ι_2 and π_1 are the canonical inclusion and projection. Let $\beta\colon X_{i_0+1} \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ be the composition

$$\beta\colon X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).$$

Then we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\beta"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\mathrm{id}"] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)
\end{tikzcd}$$

(where only the left square is a pullout), and so the composition $\alpha_{i_0+1} \circ \alpha_{i_0}$ factors through the homotopy fiber of β. In other words, we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d, "\tilde{\alpha}_{i_0}"'] & X_{i_0} \arrow[d, "\alpha_{i_0+1}"] \\
\mathrm{fib}(\beta) \arrow[r, "\tilde{\alpha}_{i_0+1}"] & X_{i_0+1}
\end{tikzcd}$$

Writing $\tilde{X}_{i_0} = \mathrm{fib}(\beta)$, we get the pasting of pullout diagrams

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\tilde{\alpha}_{i_0}"] \arrow[d] & \tilde{X}_{i_0} \arrow[r, "\tilde{\alpha}_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r, "\iota_1"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[d, "\pi_2"'] \\
 & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).
\end{tikzcd}$$

That is, by considering the factorization $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$ we have switched the cofibers with respect to the original factorization $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$. Therefore, writing $\tilde{\phi}_{i_0} = \phi_{i_0+1}$ and $\tilde{\phi}_{i_0+1} = \phi_{i_0}$, we now have $f(\tilde{\phi}_{i_0}) \in U$ and $f(\tilde{\phi}_{i_0+1}) \in L$. That is, we have removed the $(\dots, L, U, \dots)$ situation from the position i_0, replacing it with a $(\dots, U, L, \dots)$ situation, while keeping all the labels L, U before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the $(\dots, L, U, \dots)$ situations.[2] $\square$

Remark 0.4. It follows from the proof of Lemma 0.3 that the objects X_L and X_U in the fiber sequence $X_U \to X \to X_L$ associated with the t-struture $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ on $\mathscr{D}$ satisfy

$$X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in L \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0 \rangle$$
$$X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in U \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0 \rangle$$

Lemma 0.5. *For any $j \in J'$ we have*

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}.$$

Proof. Clearly, $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}$, therefore we only need to prove the converse inclusion. Let $\in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}$. Then in particular $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\geq j}$ and so there exists a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) \in \mathscr{D}_{\phi_i}$ with $f(\phi_i) \geq i$ for all $i = 1, \cdots, n$. As $((-\infty, j], (j, +\infty)$ is a slicing of J', reasoning as in the proof of Lemma 0.3 we can arrange this factorization is such a way that $f(\phi_i) > j$ for $i \leq \bar{\imath}$ and $f(\phi_i) = j$ for $i > \bar{\imath}$. Therefore, again by reasoning as in the proof of Lemma 0.3 we get a fiber sequence of the form

$$\begin{tikzcd}
X_{>j} \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & X_j
\end{tikzcd}$$

with $X_{>j}$ in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j}$ and X_j in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$. This is in particular a fiber sequence of the form $X_{>j} \to X \to X_{\leq j}$, with $X_{\leq j} \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j}$. As $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j})$ is a t-structure on $\mathscr{D}$ by Lemma 0.3, there is (up to equivalence) only one such a fiber sequence. And since $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\leq j}$, this is the sequence $0 \to X \xrightarrow{\mathrm{id}} X$. Therefore $X = X_j$. $\square$

Proposition 0.6. *Let $f\colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. The map*

$$\begin{aligned} f_!\mathfrak{t}\colon \mathcal{O}(J') &\to \mathrm{ts}(\mathscr{D}) \\ (L, U) &\mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}) \end{aligned}$$

[2] This is somehow reminiscent of the 'bubble sort' algorithm.

defined by Lemma 0.3 is a Bridgeland J'-slicing of $\mathscr{D}$, with slices given by

$$\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}.$$

Proof. The map $f_!\mathfrak{t}$ is manifestly monotone and $\mathbb{Z}$-equivariant (see the first part of the proof of Lemma 0.3), so it is a J' slicing of $\mathscr{D}$, and its slices are given by $\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$ by Lemma 0.5. We are therefore left with showing that it is finite and discrete. Given an object X in $\mathscr{D}$, let now $\{\phi_1, \ldots, \phi_n\}$ be the indices in J such that $\mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \neq 0$ and let $\{j_1, \ldots, j_k\}$ the image of the set $\{\phi_1, \ldots, \phi_n\}$ via f. Up to renaming, we can assume $j_1 > j_2 > \cdots > j_k$. Consider now the factorization

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} X_{k-1} \xrightarrow{\alpha_k} X_k = X$$

of the initial morphism of X associated to the decreasing sequence $j_1 > j_2 > \cdots > j_k$ by the J'-slicing $f_!\mathfrak{t}$. The cofibers of the morphisms α_i are the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X)$ and, by Remark 0.4, we have

$$\begin{aligned}
\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X) \in &\langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in (j_{i+1}, j_i] \text{ and } \mathcal{H}_{\mathfrak{t}}^{\phi}(X) \neq 0 \rangle \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \ldots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, j_i] \rangle \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \ldots, \phi_n\} \text{ and } f(\phi) = j_i \rangle \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) = j_i \rangle = \mathscr{D}_{f_!\mathfrak{t};j_i}.
\end{aligned}$$

Therefore

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i} \mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i}(X).$$

This tells us that, if all the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ vanish, then also all the $\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X)$ vanish, and so $X = 0$. Finally, if $\tilde{j} \notin \{j_1, \ldots, j_k\}$, then there exists an index i such that $j_{i+1} < \tilde{j} < j_i$ and so $(j_{i+1}, \tilde{j}] \cap \{j_1, \ldots, j_k\} = \emptyset$. The above argument then shows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \ldots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, \tilde{j}] \rangle = \{\mathbf{0}\}.$$

It follows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}} \mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(0) = 0,$$

for every $\tilde{j} \notin \{j_1, \ldots, j_k\}$. So in particular, for any $X \in \mathscr{D}$, the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ are possibly nonzero only for finitely many indices j. $\square$

Let now J_1 and J_2 be two $\mathbb{Z}$-tosets, and let $J_1 \times_{\text{lex}} J_2$ and $J_2 \times_{\text{lex}} J_1$ the two $\mathbb{Z}$-tosets obtained by considering the lexicogaphic order on the products $J_1 \times J_2$ and $J_2 \times J_1$, respectively, and the diagonal $\mathbb{Z}$ action. The following lemma is immediate.

Lemma 0.7. *The exchange map*

$$\begin{aligned} e\colon J_1 \times J_2 &\to J_2 \times J_1 \\ (j_1, j_2) &\mapsto (j_2, j_1) \end{aligned}$$

is $\mathbb{Z}$-equivariant.

Definition 0.8. *Let J_1 and J_2 be $\mathbb{Z}$-tosets. A Bridgeland $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ of $\mathscr{D}$ is said to be* gluable *if it is e-compatible, where $e\colon J_1 \times J_2 \to J_2 \times J_1$ is the exchange map.*

Example 0.9. Let $J_1 = \{0, 1\}$ with the trivial $\mathbb{Z}$-action and let $J_2 = \mathbb{Z}$ with the standard translation $\mathbb{Z}$-action. Then a $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ on $\mathscr{D}$ is the datum of a semiorthogonal decomposition $(\mathscr{D}_0, \mathscr{D}_1)$ of $\mathscr{D}$ toghether with t-structures $\mathfrak{t}_i$ on $\mathscr{D}_i$. Spelling out the definiton, one sees that the $\{0,1\} \times_{\mathrm{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$ is gluable if and only if $\mathscr{D}_{0;\geq 0} \boxslash \mathscr{D}_{1;-1}$ and $\mathscr{D}_{0;\geq 0} \boxslash \mathscr{D}_{1;0}$. When this happens, the glued slicing $e_!\mathfrak{t}$ is a $\mathbb{Z} \times_{\mathrm{lex}} \{0,1\}$ slicing on $\mathscr{D}$, i.e., is the datum of a bounded t-structure on $\mathscr{D}$ together with a torsion theory on its heart. More precisely, the heart of $e_!\mathfrak{t}$ is the full ∞-subcategory $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ of $\mathscr{D}$ on thise objects X that fall into fiber sequences of the form $X_1 \to X \to X_0$ with $X_i \in \mathscr{D}_i^{\heartsuit}$, while the torsion theory on $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ is $(\mathscr{D}_0^{\heartsuit}, \mathscr{D}_1^{\heartsuit})$, i.e., precisely the pair of hearts of the two subcategories in the semiorthogonal decomposition.

-fino a qui- Quanto segue va rivisto alla luce della chiacchierata su wire: dobbiamo introdurre la nozione di stong semiorthogonal decomposition e notare che nel caso in cui abbiamo strong e t-strutture scambiabili allora abbiamo due modi di incollare e questi coincidono. We now recall the celebrated process from BBD of gluing two t-structures into a third one. This will turn out to induce a gluable slicing in our sense, thus showing that the language of the present paper is actually a generalization of a classical piece of literature. This, by the other hand, explains both our terminology and motivation.

Definition 0.10. *A* recollement datum *is a diagram of stable ∞-categories and exact ∞-functors*

$$\begin{tikzcd} \mathscr{D}_{\mathrm{fer}} \arrow[r, "i_*"'] & \mathscr{D} \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*"'] & \mathscr{D}_{\mathrm{ouv}} \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"] \end{tikzcd}$$

such that the following holds:

1. *$(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples,*

2. *$i_*, j_*, j_!$ are fully faithful and $j^* i_* = 0$,*

3. *for each object X of $\mathscr{D}$ the (co)units of the above adjunctions give rise to cofiber sequences*

$$\begin{tikzcd} i_*i^!X \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & j_*j^*X \end{tikzcd} \qquad \begin{tikzcd} j_!j^*X \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & i_*i^*X. \end{tikzcd}$$

Proposition 0.11. *Suppose we are in the situation of 0.10. Then:*

1. $(\mathscr{D}_0 = j_*\mathscr{D}_{\text{ouv}}, \mathscr{D}_1 = i_*\mathscr{D}_{\text{fer}})$ *is a semiorthogonal decomposition on* $\mathscr{D}$.

2. *Given t-structures* $\mathfrak{t}_0, \mathfrak{t}_1$ *on* $\mathscr{D}_{\text{ouv}}$ *and* $\mathscr{D}_{\text{fer}}$ *respectively, the associated* $\{0,1\} \times_{\text{lex}} \mathbb{Z}$*-slicing on* $\mathscr{D}$ *given by*

$$\mathscr{D}_{0;\geq 0} = j_*(\mathscr{D}_{\text{ouv};\geq 0}), \quad \mathscr{D}_{1;\geq 0} = i_*(\mathscr{D}_{\text{fer};\geq 0})$$

   is gluable.

Proof. Let's prove the first claim. The fact that j^* and j_* are adjoints implies that, for objects $i_*X \in \mathscr{D}_{\text{fer}}$ and $j_*Y \in \mathscr{D}_{\text{ouv}}$,

$$\mathscr{D}(i_*X, j_*Y) = \mathscr{D}(j^*i_*X, Y)$$

and the latter vanishes since $j^*i_* = 0$ by definition. This shows that $\mathscr{D}_1 \boxslash \mathscr{D}_0$. Considering the left cofiber sequence from Definition 0.10 as a Postnikov tower of length one we see that $(\mathscr{D}_0, \mathscr{D}_1)$ is a $\{0,1\}$-slicing on $\mathscr{D}$, as desired.

$\square$

—-FINO A QUI 1.5 —-

l'osservazione che segue dovrà trovare la sua collocazione

Remark 0.12. To motivate the next section, consider the following isomorphism fo $\mathbb{Z}$-tosets:

$$\begin{aligned} \mathbb{Z} \times_{\text{lex}} \mathbb{Z} &\to \mathbb{Z} \times_{\text{lex}} \mathbb{Z}_{\text{free}} \\ (n,m) &\mapsto (n, m-n) \end{aligned}$$

... continua ...

Definition 0.13. *Let* $U\colon J \to \mathcal{O}(J')$ *be a map of sets. We denote by* Γ_U *the subset of* $J \times J'$ *defined by*

$$\Gamma_U = \{(j,j') \in J \times J'; \quad j' \in U_j\}.$$

For $f\colon J \to J'$ *a map of sets, we denote by* $(<f, \geq f)\colon J \to \mathcal{O}(J')$ *the composition of* f *with the map*

$$\begin{aligned} &J' \to \mathcal{O}(J') \\ &j' \mapsto ((-\infty, j'), [j', +\infty)). \end{aligned}$$

The upper graphic of f *is the subset* $\Gamma_{\geq f}$ *of* $J \times J'$.

Lemma 0.14. *Let $U\colon J \to \mathcal{O}(J')$ be a map of sets. Then Γ_U is an upper set of $J \times J'$ with the product order if and only if U is a map of posets $U\colon J \to \mathcal{O}(J')^{\mathrm{op}}$. In particular, $\Gamma_{\geq f}$ is an upper set if and only if $f\colon J \to J'$ is a nondecreasing map, i.e., a map of posets $J \to J'^{\mathrm{op}}$.*

Proof. Assume (L, U) is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Pick $(j, j') \in \Gamma_U$ and suppose $(j, j') \leq (k, k')$ in $J \times J'$. As (L, U) i monotone and $k \geq j$, we have $U_j \subseteq U_k$; as $(j, j') \in \Gamma_U$, we have $j' \in U_j$. Therefore $j \in U_k$. Since U_k is an upper set and $k' \geq j'$, we get $k' \in U_k$, i.e. $(k, k') \in \Gamma_U$. Vice versa, assume Γ_U is an upper set, and let $j \leq k$ in J. For any $j' \in U_j$, the element (k, j') in $J \times J'$ satisfies $(k, j') \geq (j, j')$ in the product order and so $(k, j') \in \Gamma_U$. This means $j' \in U_k$ and so $U_j \subseteq U_k$. To prove the second part of the statement, notice that the map $j \mapsto ((-\infty, j'), [j', +\infty))$ is a map of posets from $J' \to \mathcal{O}(J')$. Therefore, if $f\colon J \to J'^{\mathrm{op}}$ is a map of posets, then also $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets. Vice versa, if $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets then for every $j \leq k$ in J we have $[f(j), +\infty) \subseteq [f(k), +\infty)$ and so $f(k) \leq f(j)$. $\square$

Proposition 0.15. *The map*

$$\begin{aligned} \Gamma\colon \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} &\to \mathcal{O}(J \times J') \\ U &\mapsto \Gamma_U \end{aligned}$$

is an isomorphism of posets.

Proof. Let $U_1 \leq U_2$ in the partial order on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$, and let $(j, j') \in \Gamma_{U_2}$. Then $j' \in U_{2;k} \subseteq U_{1,j}$ and so $(j, j') \in \Gamma_{U_1}$. This means $\Gamma_{U_1} \leq \Gamma_{U_2}$ in $\mathcal{O}(J \times J')$. Next, for any morphism of posets $U\colon J \to \mathcal{O}(J)^{\mathrm{op}}$ we have that $(j, j') \in \Gamma_{U+1}$ if and only if $j' \in (U+1)_j$ Pick and upper set $\tilde{U}$ of $J \times J'$ and set, for $j \in J$

$$U_{\tilde{U}}(j) = \{j' \in J'; \quad (j, j') \in \tilde{U}\}.$$

The subset $U_{\tilde{U}}(j) \subseteq J'$ is an upper set. Indeed, if $j' \in U_{\tilde{U}}(j)$ and $k' \geq j'$ in J', then $(j, k') \geq (j, j')$ in the product order on $J \times J'$ and so $(j, k') \in \tilde{U}$ as $\tilde{U}$ is an upper set. Next, if $j \leq k$ in J and $j' \in U_{\tilde{U}}(j)$, then $(k, j') \in \tilde{U}$ as $\tilde{U}$ is an upper set, and so $j' \in U_{\tilde{U}}(k)$. This shows that $U_{\tilde{U}}(j) \subseteq U_{\tilde{U}}(k)$, and so $U_{\tilde{U}}$ is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Moreover, if $\tilde{U}_1 \leq \tilde{U}_2$ in $\mathcal{O}(J \times J')$ then $U_{\tilde{U}_1} \leq U_{\tilde{U}_2}$ in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. Therefore we have a map

$$\begin{aligned} \gamma\colon \mathcal{O}(J \times J') &\to \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \\ \tilde{U} &\mapsto U_{\tilde{U}}. \end{aligned}$$

which is straightforward to see to be the inverse of Γ. $\square$

Assume J and J' are $\mathbb{Z}$-posets. Then $\mathcal{O}(J)^{\mathrm{op}}$ is a $\mathbb{Z}$-poset with 1 acting as $U \mapsto U - 1$. The action of $\mathbb{Z}$ by conjugation on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$ is therefore given by $(U \dotplus 1)_j = U_{j-1} - 1$. To see that $U \dotplus 1$ is still a morphism of posets from J to $\mathcal{O}(J')$, let $j \leq k$ in J. Then $(U \dotplus 1)_j = U_{j-1} - 1 \subseteq U_{k-1} - 1 = (U \dotplus 1)_k$. The conjugation action, however, does not define a structure of $\mathbb{Z}$-poset on

$\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, as it is generally not true that $U \le U \dotplus 1$. This condition is indeed equivalent to $U_j \subseteq (U \dotplus 1)_j$, i.e., to $U_j + 1 \subseteq U_{j-1}$ for any j in J, and this is not necessarily satisfied by a morphism of posets $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$.

Definition 0.16. *A (J, J')*-perversity *is a map of posets $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ such that*

$$U_j + 1 \subseteq U_{j-1}$$

for any j in J. The set $\mathrm{Perv}(J, J')$ of all (J, J')-perversities inherits a poset structure from the inclusion in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. It is a $\mathbb{Z}$-poset with 1 acting as $U \mapsto U \dotplus (-1)$.

Remark 0.17. Notice that the shift action on (J, J')-perversities is given by $U \mapsto U \dotplus (-1)$. Namely, by construction the $\mathbb{Z}$-action $U \mapsto U \dotplus 1$ is monotone on the set of perversities with the order induced by the inclusion in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, and the order on $\mathrm{Perv}(J, J')$ is the opposite one. The reason for considering this order is, clearly, Proposition 0.15.

We have seen in Lemma 0.14 that a function $f \colon J \to J'$ defines a morphism of posets by $(\ge f) \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ if and only if f is a morphism of posets from J to J'^{op}. When this happens, $(\ge f)$ is a (J, J')-perversity if and only if $f(j-1) \le f(j) + 1$, for every $j \in J$. In the particular case $J = \mathbb{Z}$, assuming as usual that J' is a $\mathbb{Z}$-poset, we can define a new function $p_f \colon \mathbb{Z} \to J'$ as $p_f(n) = f(n) + n$. Then the condition $f(n) \le f(n+1) + 1$ translates into $p_f(n) \le p_f(n+1)$, while the condition that $f \colon \mathbb{Z} \to J'^{\mathrm{op}}$ is a morphism of posets, i.e., $f(n+1) \le f(n)$ translates to $p_f(n+1) \le p_f(n) + 1$. As $f \mapsto p_f$ is a bijection of the set of maps from J to J' into itself, we see that the functions $f \colon \mathbb{Z} \to J'$ defining perversities correspond bijectively to the set of functions $p \colon \mathbb{Z} \to J'$ such that $p(n) \le p(n+1) \le p(n) + 1$, for every $n \in \mathbb{Z}$. This motivates the following (see [**?**]).

Definition 0.18. *A $(\mathbb{Z}, \mathbb{Z})$-perversity U is said to be defined by a function if there exists $f \colon \mathbb{Z} \to \mathbb{Z}$ such that $U = (\ge f)$. The subset of $(\mathbb{Z}, \mathbb{Z})$-perversities defined by functions is denoted by $\mathrm{Perv}^{\circ}(\mathbb{Z}, \mathbb{Z})$.*

Lemma 0.19. *The subset $\mathrm{Perv}^{\circ}(\mathbb{Z}, \mathbb{Z})$ is a $\mathbb{Z}$-sub-poset of $\mathrm{Perv}(\mathbb{Z}, \mathbb{Z})$. It is isomorphic via $f \mapsto (\ge f)$ with the $\mathbb{Z}$-poset of nonincreasing functions $f \colon \mathbb{Z} \to \mathbb{Z}$ such that $f(n-1) \le f(n)+1$ with the $\mathbb{Z}$-action given by $(f \dotplus 1)(n) = f(n+1)+1$.*

Proof. As a function $f \colon \mathbb{Z} \to \mathbb{Z}$ is uniquely determined by the collection of upper sets $[f(n), +\infty)$ the set $\mathrm{Perv}^{\circ}(J, J')$ bijectivley corresponds to the set of those functions $f \colon \mathbb{Z} \to \mathbb{Z}$ such that $(\ge f)$ is a $(\mathbb{Z}, \mathbb{Z})$-perversity. As noticed above, these are precisely nonincreasing functions from $\mathbb{Z}$ to itself such that $f(n-1) \le f(n)+1$. This bijection is an isomorphism of posets, as $(\ge f_1) \le (\ge f_2)$ if and only if $f_1 \le f_2$ (in the standard poset structure on the set of maps from $\mathbb{Z}$ to the poset $\mathbb{Z}$). Finally, $((\ge f) \dotplus (-1))_n = (\ge f)_{n+1} + 1 = [f(n+1)+1, +\infty) = (\ge (f \dotplus 1))_n$, for any $n \in \mathbb{Z}$. $\square$

Definition 0.20. *A* perversity function *(on $\mathbb{Z}$) is a function $p\colon \mathbb{Z} \to \mathbb{Z}$ such that*

$$p(n) \le p(n+1) \le p(n) + 1,$$

for every $n \in Z$. The set $\mathrm{perv}_{\mathbb{Z}}$ *of perversity functions is a poset with the partial order induced by the inclusion* $\mathrm{perv}_{\mathbb{Z}} \subseteq \mathrm{Pos}(\mathbb{Z}, \mathbb{Z})$*. It is a $\mathbb{Z}$-poset with the action $(p \dotplus 1)(n) = p(n+1)$.*

Remark 0.21. Notice that the $\mathbb{Z}$-action on perversity functions is a monotone action since, by definition of perversity function, we have $p(n) \le p(n+1)$ and this precisely means $p(n) \le (p \dotplus 1)(n)$.

Lemma 0.22. *For every $p\colon \mathbb{Z} \to \mathbb{Z}$, let $f_p\colon \mathbb{Z} \to \mathbb{Z}$ be the map defined by $f_p(n) = p(n) - n$. Then $p \mapsto (\ge f_p)$ is an isomorphism of $\mathbb{Z}$-posets between* $\mathrm{perv}_{\mathbb{Z}}$ *and* $\mathrm{Perv}^\circ(\mathbb{Z}, \mathbb{Z})$

Proof. By Lemma 0.19 we only need to show that $p \mapsto f_p$ is a monotone $\mathbb{Z}$-equivariant bijection between $\mathrm{perv}_{\mathbb{Z}}$ and the set of functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(n) \le f(n-1) \le f(n) + 1$. That it is a bijection is immediate: the inverse map is $f \mapsto p_f$, where $p_f(n) = f(n) + n$. To see that it is an isomorphism of posets, notice that $f_{p_1} \le f_{p_2}$ if and only if $p_1(n) - n \le p_2(n) - n$ for every $n \in \mathbb{Z}$, and so if and only if $p_1(n) \le p_2(n)$ for every $n \in \mathbb{Z}$. Finally, $f_{p \dotplus 1}(n) = (p \dotplus 1)(n) - n = p(n+1) - n = f_p(n+1) + 1 = (f_p \dotplus 1)(n)$. □

Definition 0.23. *An upper set U in $\mathcal{O}(J \times J')$ is called a* kinky upperset *if $U \le U +_{\mathrm{ne}} 1$, where* ne *is the "northestern" action of $\mathbb{Z}$ on $J \times J'$ given by $(j, j') +_{\mathrm{ne}} 1 = (j-1, j'+1)$. We denote by* $\mathrm{Kink}(J \times J')$ *the poset of kinky uppersets of $J \times J'$, with the poset structure induced by the inclusion in $\mathcal{O}(J \times J')$. it is a $\mathbb{Z}$-poset with the "northestern" action. We denote by* $\mathrm{Kink}^\circ(J \times J')$ *the $\mathbb{Z}$-sub-poset of nontrivial kinky uppersets of $J \times J'$, where the trivial upperstes are $\emptyset$ and $J \times J'$.*

Lemma 0.24. *The map Γ from Proposition 0.15 induces an isomorphism of $\mathbb{Z}$-posets*

$$\Gamma\colon \mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J').$$

Proof. Let $U\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ be a perversity, and let $(j, j') \in \Gamma_U$. Then $j' \in U_j$ and so, by definition of perversity, $j' + 1 \in U_{j-1}$. Therefore $(j-1, j'+1) \in \Gamma_U$, i.e., $\Gamma_U \le \Gamma_U +_{\mathrm{ne}} 1$. Vice versa, if $\tilde{U}$ is a kinky upperset in $J \times J'$, let $U_{\tilde{U}}$ the preimage in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$ of $\tilde{U}$ via Γ (see the proof of Proposition 0.15). Then for any $j \in J$ and any $j' \in U_{\tilde{U};j}$ we have $j' + 1 \in U_{\tilde{U};j-1}$ and so $U_{\tilde{U};j} + 1 \subseteq U_{\tilde{U};j-1}$. So the isomorphism of posets $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{O}(J \times J')$ restricts to an isomorphism of posets $\mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J')$. To see that $\Gamma\colon \mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J')$ is also $\mathbb{Z}$-equivariant, notice that, for every perversity U we have $(j, j') \in \Gamma_{U \dotplus (-1)}$ if and only if $j' \in U_{j+1} + 1$. i.e., if and only if $(j+1, j'-1) \in \Gamma_U$. This latter condition is equivalent to $(j, j') \in \Gamma_U +_{\mathrm{nw}} 1$, so we find $\Gamma_{U \dotplus (-1)} = \Gamma_U +_{\mathrm{nw}} 1$. □

Lemma 0.25. *The map Γ from Proposition 0.15 induces an isomorphism of $\mathbb{Z}$-posets*

$$\Gamma\colon \mathrm{Perv}^{\circ}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Kink}^{\circ}(\mathbb{Z}\times\mathbb{Z}).$$

Proof. A kinky upperset U is in the image of Γ if and only if U is of the form $(\geq f)$ for a suitable function $f\colon \mathbb{Z}\to\mathbb{Z}$. This is possible if and only if $U_n \neq \emptyset, \mathbb{Z}$ for every $n\in\mathbb{Z}$. As U is kinky, if $U_{n_0}=\emptyset$ for some n_0, then $U_{n_0+1}+1\subseteq U_{n_0}=\emptyset$, and so $U_{n_0+1}=\emptyset$. Inductively, this gives $U_n=\emptyset$ for every $n\geq n_0$. On the other hand, since a kinky upperset is an upperset, if there exists a nonempty U_n with $n<n_0$, then there exist an element (n,m) in U and so, since $(n,m)\leq(n_0,m)$, also $(n_0,m)\in U$. But then $m\in U_{n_0}$, which is impossible. So also the U_n with $n<n_0$ are empty and therefore $U=\emptyset$. Similarly, if $U_{n_0}=\mathbb{Z}$ for some n_0, then $U_n=\mathbb{Z}$ for every $n>n_0$ as U is an upperset, while the kinkiness condition $U_n+1\subseteq U_{n-1}$ implies that also $U_{n_0-1}=\mathbb{Z}$ and so inductively that all U_n with $n<n_0$ are the whole of $\mathbb{Z}$. That is, $U=\mathbb{Z}\times\mathbb{Z}$ in this case. $\square$

Lemma 0.26. *The isomorphism of $\mathbb{Z}$-modules $\varphi\colon \mathbb{Z}^2\to\mathbb{Z}^2$ given by $\varphi\colon (n,n')\mapsto (n+n',n')$ induces an isomorphism of $\mathbb{Z}$-posets*

$$\varphi\colon \mathrm{Kink}(\mathbb{Z}\times\mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z}),$$

where the $\mathbb{Z}$-action on $\mathcal{O}(\mathbb{Z}\times\mathbb{Z})$ is the one induced by the "northern" $\mathbb{Z}$-action on $\mathbb{Z}^2$, namely, $(n,n')+1=(n,n'+1)$. In particular φ induces an isomorphism of $\mathbb{Z}$-posets $\mathrm{Kink}^{\circ}(\mathbb{Z}\times\mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z})\setminus\{\emptyset,\mathbb{Z}\times\mathbb{Z}\}$.

Proof. A subset U of $\mathbb{Z}\times\mathbb{Z}$ is an upper set (in the product order) if and only if $U+K\subseteq U$, where K is the $\mathbb{Z}$-cone spanned by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, while U is a kinky upperset if and only if $U+K^{\mathrm{kink}}\subseteq U$, where K^{kink} is the $\mathbb{Z}$-cone spanned by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$. Morever the $\mathbb{Z}$ action on the uppersets is generated by $U\mapsto U+\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and the the $\mathbb{Z}$ action on the kinky uppersets is generated by $U\mapsto U+\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$, where in both cases the sum on the right hand side is the sum in $\mathbb{Z}^2$. As $\varphi\colon\mathbb{Z}^2\to\mathbb{Z}^2$ is an isomorphism of $\mathbb{Z}$-modules with $\varphi(K^{\mathrm{kink}})=K$ and $\varphi\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, the statement follows (φ is manifestly inclusion preserving). $\square$

Corollary 0.27. *We have an isomorphism of $\mathbb{Z}$-posets*

$$\mathrm{perv}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z})\setminus\{\emptyset,\mathbb{Z}\times\mathbb{Z}\}$$

mapping a perversity function p to the image via the isomorphism $\varphi\colon (n,n')\mapsto (n+n',n')$ of the set $\{(n,n')\in\mathbb{Z}\times\mathbb{Z}$ such that $n'\geq p(n)-n\}$.

Example 0.28. The zero perversity function $p(n)\equiv 0$ corresponds to the upper set $\{(n,n')\in\mathbb{Z}\times\mathbb{Z}$ such that $n\geq 0\}$; the identity perversity function $p(n)\equiv n$ corresponds to the upper set $\{(n,n')\in\mathbb{Z}\times\mathbb{Z}$ such that $n'\geq 0\}$.

l'esempio che segue compare in due versioni molto simili (evidentemente

ho fatto un po' di casino col merge). lasciare solo l'ultima versione (che però non so quale sia)

Example 0.29. In this example we relate the gluability condition with the Beilinson-Soulé conjecture from motivic topology. We fix a field k and, just for this example, we stick to a more traditional 1-categorical setup. Recall that the existence of motives, which is still an open question in general, was conjectured by Grothendieck in order to build a universal Weyl cohomology (also called 'motivic cohomology') theory for schemes. However, Deligne observed that it could be easier to construct a triangulated category (the 'mixed' motives) which should play the role of the derived category of motives, and later recover the latter as the heart of a bounded t-structure. Voedvodskij finally succeded in constructing a triangulated category of mixed rational motives over k which contains, for $n \in \mathbb{Z}$, a 'Tate object' $\mathbb{Q}(n)$ that represents the n-th motivic cohomology functor. Now, let us consider the triangualted subcategory $\mathscr{D}\mathrm{TM}_k$ generated by the Tate objects. In other words, $\mathscr{D}\mathrm{TM}_k$ is the category of mixed rational Tate motives. We then get isomorphisms of groups

$$\mathscr{D}\mathrm{TM}_k(\mathbb{Q}(i), \mathbb{Q}(j)[n]) = K_{2(j-i)-n}(k)^{(j-i)}$$

where $K_a(k)$ is the a-th higher K-theory group of the point $\mathrm{Spec}(k)$ and $K_a(k)^{(b)}$ is the weight b summand of $K_a(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to the Adams action (in other words, $K_a(k)^{(b)}$ is the a-th b-codimensional Bloch's higher Chow group of $\mathrm{Spec}(k)$ with rational coefficients). For dimensional reasons we immediately see that the right hand side vanishes for $i < j$ and, when $i = j$, for $n \neq 0$. In other words, the Tate objects form an infinite exceptional collection (referenza a decomposizioni semiortogonali) on $\mathscr{D}\mathrm{TM}_k$ which is clearly full by definition. By the general theory of semiorthogonal decomposition, this simply means that setting $(\mathscr{D}\mathrm{TM}_k)_{=n}$ as the triangulated subcategory generated by $\mathbb{Q}(n)$ defines slices of a Bridgeland $\mathbb{Z}_{\mathrm{free}}$-slicing (or, as it is called in referenza strutture bariche, a 'baric structure') with exact equivalences

$$(\mathscr{D}\mathrm{TM}_k)_{=n} \simeq \mathscr{D}^b(\mathbb{Q})$$

where the member on the right is the bounded derived category of finite-dimensional rational vector spaces (which is equivalent to the abelian category graded vector spaces which finite-dimensional homogeneous pieces). The latter, being a derived category, posseses a canonical bounded t-structure and this finally induces a Bridgeland $\mathbb{Z}_{\mathrm{free}} \times_{\mathrm{lex}} \mathbb{Z}$ slicing on $\mathscr{D}\mathrm{TM}_k$.
Now, the latter slicing is gluable if and only if $K_{2(j-i)-n}(k)^{(j-i)} = \mathbf{0}$ whenever both $i < j$ and $n \leq 0$ hold. This is exactly the Beilinson-Soulé standard vanishing conjecture, which is now known to hold, for example, when k is a number field due to a celebrated computation by Borel. In this case, by applying $e_!$ we get a

Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{free}}$-slicing on $\mathscr{D}\mathrm{TM}_k$ and thus a bounded t-structure whose heart contains the desired unmixed Tate motives over k. In other words, we recover a well known fact (see referenza Levine) using a rather abstract and general language: assuming the Beilinson-Soulé conjecture, (Tate) motives exist. in realta' facciamo di piu': recuperiamo anche la costruzione dei 'motivi perversi', ma questa cosa la scrivo dopo, quando abbiamo messo anche le perversita'..

da qui in poi cè una seconda versione di quanto appena scritto (evidentemente ho fatto un po' di casino col merge)

Example 0.30. In this example we relate the gluability condition with the Beilinson-Soulé conjecture from motivic topology. We fix a field k and, just for this example, we stick to a more traditional 1-categorical setup. Recall that Voedvodskij's triangulated category of mixed rational motives over k contains, for $n \in \mathbb{Z}$, a 'Tate object' $\mathbb{Q}(n)$ that should represent the n-th motivic cohomology functor, whose definition was indeed the initial goal of the development of motives. Indeed, consider the triangualted subcategory $\mathscr{D}\mathrm{TM}_k$ generated by the Tate object. In other words, $\mathscr{D}\mathrm{TM}_k$ is the category of mixed rational Tate motives. We then get isomorphisms of groups

$$\mathscr{D}\mathrm{TM}_k(\mathbb{Q}(i), \mathbb{Q}(j)[n]) = K_{2(j-i)-n}(k)^{(j-i)}$$

where $K_a(k)$ is the a-th higher K-theory group of the point $\mathrm{Spec}(k)$ and $K_a(k)^{(b)}$ is the weight b summand of $K_a(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to the Adams action (in other words, $K_a(k)^{(b)}$ is the a-th b-codimensional Bloch's higher Chow group of $\mathrm{Spec}(k)$ with rational coefficients). For dimensional reasons we immediately see that the right hand side vanishes for $i < j$ and, when $i = j$, for $n \neq 0$. In other words, the Tate objects form an infinite exceptional collection (referenza a decomposizioni semiortogonali) on $\mathscr{D}\mathrm{TM}_k$ which is clearly full by definition. By the general theory of semiorthogonal decomposition, this simply means that setting $(\mathscr{D}\mathrm{TM}_k)_{=n}$ as the triangulated subcategory generated by $\mathbb{Q}(n)$ defines slices of a Bridgeland $\mathbb{Z}_{\mathrm{free}}$-slicing (or, as it is called in referenza strutture bariche, a 'baric structure') with exact equivalences

$$(\mathscr{D}\mathrm{TM}_k)_{=n} \simeq \mathbb{D}^b(\mathbb{Q})$$

where the member on the right is the bounded derived category of rational vector spaces (which is equivalent to the abelian category of graded vector spaces). The latter posseses a canonical bounded t-structure, coming from the fact of being a derived category and this finally induces a Bridgeland $\mathbb{Z}_{\mathrm{free}} \times_{\mathrm{lex}} \mathbb{Z}$ slicing on $\mathscr{D}\mathrm{TM}_k$.
Now, the gluability condition for this slicing means that

———— This is vague and in part conjectural. Fix a field k and denote $\mathscr{D}\mathrm{TM}_k$ the triangulated category of mixed Tate $\mathbb{Q}$-motives over k. Following [?], the Tate objects $\{\mathbb{Q}(n)\}_{n\in\mathbb{Z}}$ form an 'infinite full exceptional collection': this means that

$$\mathscr{P}_\phi = \langle\langle \mathbb{Q}(\phi)\rangle\rangle$$

with $\phi \in \mathbb{Z}$ defines a baric structure (i.e. a $\hat{\mathbb{Z}}$-slicing) with equivalences $\mathscr{P}_\phi = \mathscr{D}(\mathbb{Q})$, and thus a $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$-slicing $\mathscr{Q}$ on $\mathscr{D}\mathrm{TM}_k$. The collection being ext (i.e. $\mathscr{Q}$ being gluable) means that

$$0 = \mathrm{Hom}_{\mathscr{D}\mathrm{TM}_k}(\mathbb{Q}(i), \mathbb{Q}(j)[n]) = K_{2(j-i)-n}(k)^{(j-i)}$$

for $i < j$ and $n \leq 0$, where $K_a(k)$ is the a-th higher K-theory group of the point $\mathrm{Spec}(k)$ and $K_a(k)^{(b)}$ is the weight b summand of $K_a(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to the Adams action (in other words, $K_a(k)^{(b)}$ is the a-th b-codimensional Bloch's higher Chow group of $\mathrm{Spec}(k)$ with rational coefficients).
This means that $\mathscr{Q}$ is gluable if and only if the Beilinson-Soulé conjecture holds over k and the t-structure coming from $\overline{\mathscr{Q}}$ is the conjectural motivic t-structure of Tate motives. Since in this case we get a grading filtration on the heart of the latter, using **Proposition 0.35** one can also define 'perverse motivic' t-structures depending on a perversity, as done in [?].
Moreover, if k is a number field, a celebrated computation due to Borel shows that not only the latter conjecture holds over k but the t-structure of Tate motives (whose heart we denote TM_k) is hereditary and hence, by **Remark ??**, there is a triangulated equivalence

$$\mathscr{D}^b(\mathrm{TM}_k) = \mathscr{D}\mathrm{TM}_k$$

This result, conjectured in [?], appears with a different proof in [?].

fino a qui-Giovanni-

Remark 0.31. We can give $\Theta(J)$ the structure of a $\mathbb{Z}$-poset via the incluion $\Theta(J) \subseteq \mathrm{Pos}(\mathbb{Z}, \mathcal{O}(J)^{\mathrm{op}})$. In general, it will not be totally ordered. However, by taking joins and meets pointwise $\Theta(J)$ becomes a distributive lattice.

Remark 0.32. Considering the obvious isomoprhism $\mathbb{Z} \simeq \mathcal{O}(\mathbb{Z})^{\mathrm{op}}$ a perversity over $\mathbb{Z}$ is just a *monotone and comonotope perversity* in the sense of referenza, tipo bezrukavnikov o BBD.

Definition 0.33. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. An abelian $\mathbb{Z}$-slicing $\mathscr{P}$ on $\heartsuit_{\mathfrak{t}}$ is called:*

- ***perverse filtration*** *if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n < \phi - \psi$*
- ***grading filtration*** *if it is a perverse filtration and $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $2 \leq n = \phi - \psi$*
- ***mixed filtration*** *if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n > \psi - \phi$*

The following implications hold:

$$\text{mixed} \implies \text{grading} \implies \text{perverse}$$

proof that mixed implies grading: suppose $n < \phi - \psi$. By definition of slicing, the vanishing is automatic for $n \leq 0$ and we can thus assume $n > 0$. But then $n > 0 > -n > \psi - \phi$ and the required vanishing now follows from the definition of mixed filtration. The same argument also works for $2 \leq n = \phi - \psi$.
Clearly, a torsion pair on $\heartsuit_{\mathsf{t}}$ (seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[1] \subseteq \mathbb{Z}$) is always a grading filtration (proof: it follows from the fact that $\mathscr{P}_\phi[n] = 0$ for $\phi \neq 0, 1$ and thus the only nontrivial case is $n = 0, \phi = 1, \psi = 0$. The Hom-vanishing then follows from the definition of torsion pair). Historically, grading filtrations first appeared in [**?**] under the name of 'radical filtrations', while mixed filtrations are covered in the last section of [**?**].

Definition 0.34. *A map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is called* ***perversity*** *if it is a monotone contraction, i.e. if*

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi$$

whenever $\phi \geq \psi$ in $\mathbb{Z}$.
We denote Ξ the set of perversities.

Clearly, Ξ definies a $\mathbb{Z}$-poset which is not totally ordered. However, it is a distributive lattice: we have meets and joins given by:

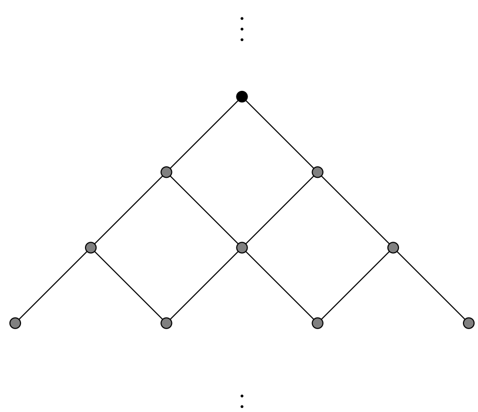
$$(p \wedge q)(\phi) = \min\{p(\phi), q(\phi)\}$$

$$(p \vee q)(\phi) = \max\{p(\phi), q(\phi)\}$$

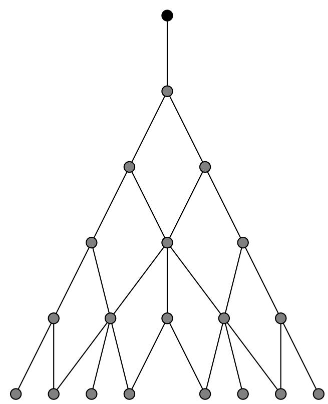
Indeed, if we place on $\mathbb{Z} \times \mathbb{Z}$ the product order, we have an isomorphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} = O(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

given by sending the zero perversity to $A = \{(\phi, \psi)\}_{\phi \geq 0}$ and the identity to $B = \{(\phi, \psi)\}_{\psi \geq 0}$ ($B + i$ and $A + i$ with $i \in \mathbb{Z}$ generate the latter as a complete lattice). Considering the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Z} \subseteq O(\mathbb{Z} \times \mathbb{Z})$ we get a sublattice of Ξ depicted as



This induces the 't-tree' described in [**?**], as discussed below in a far-reaching generality. By the other hand, each square above contains moreover a copy of the Young lattice



$$\vdots$$

with the root corresponding to the upper vertex of the square.

Proposition 0.35. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, $\mathscr{P}$ an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, p a perversity. We have:*

1. *if $\mathscr{P}$ is a perverse filtration, then $\mathscr{Q}$ satisfies the assumptions of **Proposition ??** with respect to the map $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{f_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ given by*

$$f_p(n, \phi) = (n + p(\lfloor \phi/2 \rfloor), -p(\lfloor \phi/2 \rfloor))$$

2. *if $\mathscr{P}$ is a grading filtration, then $\mathscr{Q}$ satisfies the assumptions of **Proposition ??** with respect to the map $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{g_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ given by*

$$g_p(n, \phi) = (n + p(\phi), -p(\phi))$$

3. *if $\mathscr{P}$ is a mixed filtration, then the abelian $\mathbb{Z}$-slicing induced on the heart of the bounded t-structure associated to $(g_p)_{\underline{\Omega}}(\mathscr{Q})$ is split*

Proof. Let's prove *(2)*. Suppose $g_p(n, \phi) > g_p(m, \psi)$. Then either $m - n < p(\phi) - p(\psi)$ or both $m - n = p(\phi) - p(\psi)$ and $p(\psi) < p(\psi)$. Since by definition of t-structure we can assume $m \geq n$, the second case is absurd while in the first case, since p is monotone, we have $\phi \geq \psi$ and thus by definition of perversity

$$m - n < p(\phi) - p(\psi) \leq \phi - \psi$$

and we get the desired Hom-vanishing by definition of grading filtration.

Suppose now $f_p(n, \phi) + 1 > f_p(m, \psi)$ and $(n+1, \phi) < (m, \psi)$. The only non absurd case is $1 < m - n \leq p(\phi) - p(\psi)$. But then again we have

$$2 \leq m - n \leq p(\phi) - p(\psi) \leq \phi - \psi$$

and we can conclude as above.

To prove *(1)* consider the monotone map

$$\mathbb{Z} \xrightarrow{\lfloor */2 \rfloor} \mathbb{Z}$$

Applying the slice functor to the latter and starting with a perverse filtration, we get a grading filtration by the properties of the floor function and the thesis follows from part *(2)*.

Let's prove *(3)*. We have to show that

$$\mathscr{P}_\psi[1 - p(\psi)] \subseteq \mathscr{P}_\phi[p(\phi)]^\perp$$

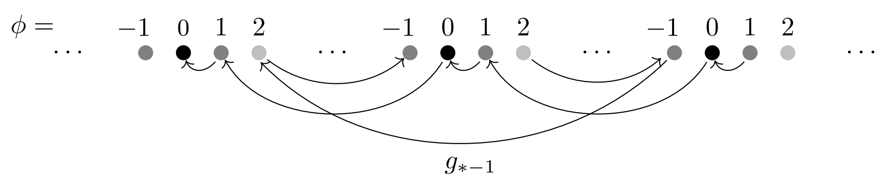
for $-p(\psi) > -p(\phi)$. We denote $n = p(\phi) - p(\psi) - 1$. Assuming again $n \geq 0$, we have $n \geq 0 > p(\psi) - p(\phi) \geq \psi - \phi$ and we can then conclude by definition of mixed filtration. $\square$

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity p the bounded t-structure coming from $(g_p)_{\mathfrak{Q}}(\mathscr{Q})$ defines a morphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} \longrightarrow \mathfrak{bts}(\mathscr{D})$$

we can restate this as:

♫ *A grading or perverse filtration on the heart of a bounded t-structure on $\mathscr{D}$ induces a presheaf of t-structures on $\mathbb{Z} \times \mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z} \times \mathbb{Z}$-slicing' on $\mathscr{D}$.*



Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\mathrm{op}} \hookrightarrow \Xi$$

and the t-structure coming from $(g_{\chi_I})_{\mathfrak{Q}}(\mathscr{Q})$ is just the tilting of $\mathfrak{t}$ with respect to the torsion pair coming from I.
The following proposition gives a characterization of the new heart obtained

by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 0.36. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, $\mathscr{P}$ a grading filtration on $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, p a perversity. Denote $\mathfrak{q}$ the bounded t-structure associated to $(g_p)_{\Omega}(\mathscr{Q})$. Then $\heartsuit_{\mathfrak{q}}$ consists of objects $X \in \mathscr{D}$ so that*

$$H^k_{\mathfrak{t}}(X) \in \mathscr{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if $\mathscr{P}$ is a mixed filtration then for each $X \in \heartsuit_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \heartsuit_{\mathfrak{q}}$ if and only if

$$H^{\phi}_{\mathscr{P}}(H^k_{\mathfrak{t}}(X)[-k])[k] = H^{(k,\phi)}_{\mathscr{Q}}(X) = 0$$

for $p(\phi) \neq -k$.
For the second part of the claim, the abelian $\mathbb{Z}$-slicing induced on $\heartsuit_{\mathfrak{q}}$ is split by **Proposition 0.35** and thus by **Proposition ??**

$$X = \bigoplus_{(k,\phi)} H^{(k,\phi)}_{\mathscr{Q}}(X) = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

where the last equality comes from the first part. □

Example 0.37. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an $\mathbb{N}$-graded ring with B_0 semisimple. Denote $\mathscr{A}$ the category of $\mathbb{Z}$-graded B-modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote $\mathscr{P}_\phi$ the full subcategory of $\mathscr{A}$ of modules concentrated in degree ϕ. Clearly, $\mathscr{P}$ defines an abelian $\mathbb{Z}$-slicing on $\mathscr{A}$. Following [**?**] we have

$$\mathrm{Ext}^n_{\mathscr{A}}(\mathscr{P}_\phi, \mathscr{P}_\psi) = 0$$

for $n > \psi - \phi$. This means that $\mathscr{P}$ is a mixed filtration and the bounded t-structure on $\mathscr{D}^b(\mathscr{A})$ associated to $(g_1)_{\Omega}(\mathscr{Q})$ (where 1 is the identity of $\mathbb{Z}$) is the 'diagonal' (or 'geometric') t-structure which appears in Koszul duality and other areas.

Example 0.38. Let M be an n-dimensional smooth complex projective variety and consider the n-torsion pair $\mathscr{P}$ on $\mathrm{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that $\mathscr{P}$, seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t-structure associated to $(f_p)_{\Omega}(\mathscr{Q})$ is the one of perverse coherent sheaves as constructed in [**?**]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n-slicing on the heart of perverse coherent sheaves as done in [**?**].

Now we somehow review the gluing construction for t-structures in [**?**], but generalize it to any slicing. Our language is quite different though, and we formulate the problem very similarly to [**?**]. We start with two $\mathbb{Z}$-posets J, J'.

Definition 0.39. *Let $\mathscr{P}$ be a $J \ltimes J'$-slicing on $\mathscr{D}$. We call $\mathscr{P}$ **gluable** if it satisties the assumptions of **Proposition ??** with respect to the map $J \ltimes J' \xrightarrow{e} J' \ltimes J$ that exchanges coordinates. In this case we denote*

$$\overline{\mathscr{P}} = e_{\underline{\Omega}}(\mathscr{P})$$

By a simple computation, the gluability condition reads: $\mathscr{P}_{(\phi,\psi)} \subseteq \mathscr{P}^{\perp}_{(\phi',\psi')}$ if $\psi' > \psi$ or both $\psi' + 1 > \psi$ and $\phi' + 1 < \phi$. For example, the first condition is automatic when $\mathbb{Z}$ acts trivially on J (in this case, it follows from the second one). In particular, if $\mathbb{Z}$ acts trivially on J, $\mathscr{P}$ is a J-slicing on $\mathscr{D}$ and $\mathfrak{t}_\phi$ is a bounded t-structure on $\mathscr{P}_\phi$ for each $\phi \in J$, then using **Proposition ??** we get a $J \ltimes \mathbb{Z}$-slicing $\mathscr{Q}$ which is gluable if and only if

$$\heartsuit_{\mathfrak{t}_\psi}[n] \subseteq \heartsuit_{\mathfrak{t}_\phi}^{\perp}$$

whenever both $n \leq 0$ and $\phi < \psi$.

Remark 0.40. We have a commutative diagram of $\mathbb{Z}$-posets

$$\begin{tikzcd} \mathbb{Z} \ltimes \mathbb{Z} \arrow[r, "\sim"] \arrow[d, "e"] & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \arrow[d, "g"] \\ \mathbb{Z} \ltimes \mathbb{Z} \arrow[r, "\sim"] & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \end{tikzcd}$$

where the horizontal isomorphism is the one from **Remark ??**. Indeed, a $\mathbb{Z} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$ is gluable if and only if it induces a grading filtration on the heart of the associated t-structure when seen as a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing.

An easy combinatorial calculation finally yelds the following two remarks:

- if $\mathscr{P}$ is a gluable $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$, then $\overline{\mathscr{P}}$ induces a grading filtration on the associated heart, allowing us again to construct new bounded t-structures depending on a perversity.
- if $\mathscr{P}$ is a mixed filtration on the heart of a bounded t-structure and $\mathscr{Q}$ is the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, then $\mathscr{Q}$ is gluable. In this case, looking at $\overline{\mathscr{Q}}$, we get a baric structure on $\mathscr{D}$ which is usually called **weight decomposition** in literature.

In other words, the chain of implications for a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing refines to

$$\text{mixed} \implies \text{gluable} \implies \text{grading} \implies \text{perverse}$$