0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of $\mathbb{Z}$-posets (but just a $\mathbb{Z}$-equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a $\mathbb{Z}$-poset, i.e., a partially ordered set together with a monotone action of $\mathbb{Z}$, that we will denote by $(n,x) \mapsto x+n$. Recall that a J-slicing on a stable ∞-category $\mathscr{D}$ is a morphism of $\mathbb{Z}$-posests $\mathfrak{t}\colon \mathcal{O}(J) \to \mathrm{ts}(\mathscr{D})$, where $\mathcal{O}(J)$ is the $\mathbb{Z}$-poset of the upper sets of J and $\mathrm{ts}(\mathscr{D})$ is the $\mathbb{Z}$-poset of t-structures on $\mathscr{D}$. Clearly, if $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-posets, then $f^{-1}\colon \{\text{subsets of } J'\} \to \{\text{subsets of } J\}$ induces a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-posets $f^{-1}\colon \mathcal{O}(J') \to \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$\begin{aligned} f_*\colon J\text{-slicings on } \mathscr{D} &\to J'\text{-slicings on } \mathscr{D} \\ \mathfrak{t} &\mapsto \mathfrak{t} \circ f^{-1}. \end{aligned}$$

But actually there is no need for f to be a morphism of posets in order to define a morphism $\mathcal{O}(J') \to \mathcal{O}(J)$: in the general case, just consider the map $\overline{f^{-1}}$ defined by

$$\overline{f^{-1}}(U) = \overline{f^{-1}(U)},$$

where $\overline{S}$ denotes the upperset closure of the subset S of J, i.e., the smallest upperset of J containing S (in other words $\overline{S}$ is the intersection of all the uppersets of J containing S).

Lemma 0.1. *Let J and J' be $\mathbb{Z}$-posets, and let $f\colon J \to J'$ a map of $\mathbb{Z}$-sets (i.e., a $\mathbb{Z}$-equivariant map, not necessarily nondecreasing). The map $\overline{f^{-1}}\colon \mathcal{O}(J') \to \mathcal{O}(J)$ is a morphism of $\mathbb{Z}$-posets.*

Proof. Let U_1 and U_2 two posets in J' with $U_1 \le U_2$, i.e., with $U_2 \subseteq U_1$. Then, $f^{-1}(U_2) \subseteq f^{-1}(U_1)$ and so, by definition of closure, $\overline{f^{-1}(U_2)} \subseteq \overline{f^{-1}(U_1)}$, i.e., $\overline{f^{-1}}(U_1) \le \overline{f^{-1}}(U_2)$. To see that $\overline{f^{-1}}$ is $\mathbb{Z}$-equivariant, notice that for any subset S of J one has $S+1 \subseteq \overline{S}+1$ and so $\overline{S+1} \subseteq \overline{S}+1$, as $\overline{S}+1$ is an upperset and therefore closed. On the other hand, $S+1 \subseteq \overline{S+1}$ and so $S \subseteq \overline{S+1}-1$ which implies $\overline{S} \subseteq \overline{S+1}-1$, i.e., $\overline{S}+1 \subseteq \overline{S+1}$. Therefore, for any upperset U in J' one has

$$\overline{f^{-1}}(U+1) = \overline{f^{-1}(U+1)} = \overline{f^{-1}(U)+1} = \overline{f^{-1}(U)}+1 = \overline{f^{-1}}(U)+1,$$

by the $\mathbb{Z}$-equivariancy of f. $\square$

Corollary 0.2. *Let J and J' be $\mathbb{Z}$-posets. Any $\mathbb{Z}$-equivariant morphism of sets $f\colon J \to J'$ induces a morphism*

$$\begin{aligned} f_*\colon J\text{-slicings on } \mathscr{D} &\to J'\text{-slicings on } \mathscr{D} \\ \mathfrak{t} &\mapsto \mathfrak{t} \circ \overline{f^{-1}}. \end{aligned}$$

If f is a morphism of $\mathbb{Z}$-posets, this reduces the the morphism $\mathfrak{t} \mapsto \mathfrak{t} \circ f^{-1}$ mentioned above.

Proof. The only thing we need to prove is that if f is order preserving then $\overline{f^{-1}} = f^{-1}$. This is immediate, as for an order preserving map the preimage of an upper set is an upper set (and so it is automatically closed). $\square$

Although f_* maps J-slicing to J'-slicings, it does not necessarily map Bridgeland J-slicings to Bridgeland J'-slicings. In order to do so, we need make a few more assumptions on f.

Proposition 0.3. *Let $J \xrightarrow{f} J'$ be a a $\mathbb{Z}$-equivariant map (not necessarily increasing) between $\mathbb{Z}$-posets, $\mathscr{P}$ a J-slicing on $\mathscr{D}$. Suppose that $\mathscr{P}_\psi \subseteq \mathscr{P}_\phi^\perp$ whenever one of the following conditions holds:*

(a) $f(\phi) > f(\psi)$

(b) $f(\phi) + 1 > f(\psi)$ *and* $\phi + 1 < \psi$

Then

$$(f_{\underline{\Omega}}(\mathscr{P}))_\phi = \mathscr{P}_{f^{-1}(\phi)}$$

defines a J'-slicing on $\mathscr{D}$.

Proof. Part *(1)* of definition of slicing follows from $\mathbb{Z}$-equivariance of f, while part *(2)* follows from condition *(a)* above. Pick $0 \neq X \in \mathscr{D}$ and consider its Postnikov tower with respect to $\mathscr{P}$:

$$0 = Y_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} Y_n = X$$

with $\mathrm{cone}(\beta_i) = H^{\phi_i}_{\mathscr{P}}(X)$. Going from left to right, if we encounter an i so that $f(\phi_{i+1}) > f(\phi_i)$ then by condition *(b)*

$$\mathrm{Hom}_{\mathscr{D}}(H^{\phi_{i+1}}_{\mathscr{P}}(X), H^{\phi_i}_{\mathscr{P}}(X)[1]) = 0$$

Using **Proposition ??** and the 3×3 **Lemma**, we can complete the identity of Y_i to get a diagram:

$$\begin{array}{ccccccc}
H^{\phi_{i+1}}_{\mathscr{P}}(X)[-1] & \longrightarrow & Y_i & \longrightarrow & Y_{i+1} & \longrightarrow & H^{\phi_{i+1}}_{\mathscr{P}}(X)[1] \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\
Y_{i-1} & \longrightarrow & Y_i & \longrightarrow & H^{\phi_i}_{\mathscr{P}}(X) & \longrightarrow & Y_{i-1}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & 0 & \longrightarrow & A[1] & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{\phi_{i+1}}_{\mathscr{P}}(X) & \longrightarrow & Y_i[1] & \longrightarrow & Y_{i+1}[1] & \longrightarrow & H^{\phi_{i+1}}_{\mathscr{P}}(X)[1]
\end{array}$$

We then replace β_i with $Y_{i-1} \longrightarrow A$ and β_{i+1} with $A \longrightarrow Y_{i+1}$. Iterating this process[1], we get a factorization

$$0 = Z_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_n} Z_n = X$$

with $\mathrm{cone}(\gamma_i) = H^{\phi_{k_i}}_{\mathscr{P}}(X)$ for some k_i and $f(\phi_i) \geq f(\phi_{i+1})$ for all i. To get a factorization with the strict latter inequality we use the same argument as in the proof of **Proposition ??**. $\square$

The above proof also shows that, in the same hypotheses and notation, $(f_\Omega(\mathscr{P}))_\phi$ consists of objects $X \in \mathscr{D}$ so that $H^{\psi}_{\mathscr{P}}(X) = 0$ for $f(\psi) \neq \phi$. Also, **Proposition ??** holds when f is bijective.

Definition 0.4. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. An abelian $\mathbb{Z}$-slicing $\mathscr{P}$ on $\heartsuit_{\mathfrak{t}}$ is called:*

- ***perverse filtration*** *if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n < \phi - \psi$*
- ***grading filtration*** *if it is a perverse filtration and $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $2 \leq n = \phi - \psi$*
- ***mixed filtration*** *if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n > \psi - \phi$*

The following implications hold:

$$\text{mixed} \implies \text{grading} \implies \text{perverse}$$

Clearly, a torsion pair on $\heartsuit_{\mathfrak{t}}$ (seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[1] \subseteq \mathbb{Z}$) is always a grading filtration. Historically, grading filtrations first appeared in [**?**] under the name of 'radical filtrations', while mixed filtrations are covered in the last section of [**?**].

Definition 0.5. *A map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is called* ***perversity*** *if it is a monotone contraction, i.e. if*

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi$$

whenever $\phi \geq \psi$ in $\mathbb{Z}$.
We denote Ξ the set of perversities.

Clearly, Ξ definies a $\mathbb{Z}$-poset which is not totally ordered. However, it is a distributive lattice: we have meets and joins given by:

$$(p \wedge q)(\phi) = \min\{p(\phi), q(\phi)\}$$

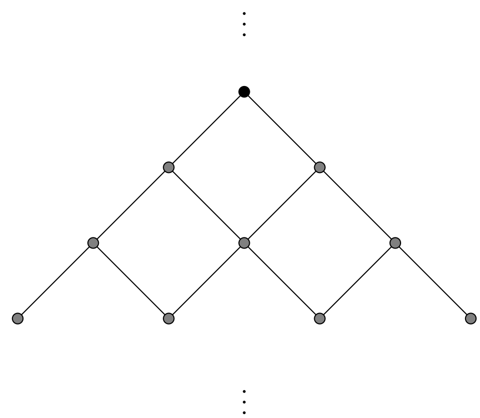
$$(p \vee q)(\phi) = \max\{p(\phi), q(\phi)\}$$

Indeed, if we place on $\mathbb{Z} \times \mathbb{Z}$ the product order, we have an isomorphism of $\mathbb{Z}$-posets

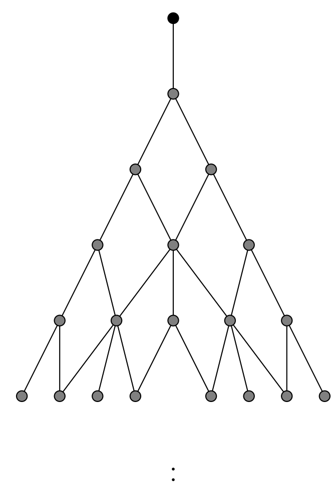
$$\Xi^{\mathrm{op}} = O(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

[1] This is somehow reminiscent of the 'bubble sort' algorithm.

given by sending the zero perversity to $A = \{(\phi, \psi)\}_{\phi \geq 0}$ and the identity to $B = \{(\phi, \psi)\}_{\psi \geq 0}$ ($B + i$ and $A + i$ with $i \in \mathbb{Z}$ generate the latter as a complete lattice). Considering the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Z} \subseteq O(\mathbb{Z} \times \mathbb{Z})$ we get a sublattice of Ξ depicted as



This induces the 't-tree' described in [**?**], as discussed below in a far-reaching generality. By the other hand, each square above contains moreover a copy of the Young lattice



with the root corresponding to the upper vertex of the square.

Proposition 0.6. *Let* $\mathfrak{t}$ *be a bounded t-structure on* $\mathscr{D}$, $\mathscr{P}$ *an abelian* $\mathbb{Z}$*-slicing on* $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ *the associated* $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$*-slicing on* $\mathscr{D}$, p *a perversity. We have:*

1. *if* $\mathscr{P}$ *is a perverse filtration, then* $\mathscr{Q}$ *satisfies the assumptions of* ***Proposition 0.3*** *with respect to the map* $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{f_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ *given by*

$$f_p(n, \phi) = (n + p(\lfloor \phi/2 \rfloor), -p(\lfloor \phi/2 \rfloor))$$

2. *if $\mathscr{P}$ is a grading filtration, then $\mathscr{Q}$ satisfies the assumptions of* ***Proposition 0.3*** *with respect to the map $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{g_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ given by*

$$g_p(n, \phi) = (n + p(\phi), -p(\phi))$$

3. *if $\mathscr{P}$ is a mixed filtration, then the abelian $\mathbb{Z}$-slicing induced on the heart of the bounded t-structure associated to $(g_p)_{\underline{\Omega}}(\mathscr{Q})$ is split*

Proof. Let's prove *(2)*. Suppose $g_p(n, \phi) > g_p(m, \psi)$. Then either $m - n < p(\phi) - p(\psi)$ or both $m - n = p(\phi) - p(\psi)$ and $p(\psi) < p(\psi)$. Since by definition of t-structure we can assume $m \geq n$, the second case is absurd while in the first case, since p is monotone, we have $\phi \geq \psi$ and thus by definition of perversity

$$m - n < p(\phi) - p(\psi) \leq \phi - \psi$$

and we get the desired Hom-vanishing by definition of grading filtration.

Suppose now $f_p(n, \phi) + 1 > f_p(m, \psi)$ and $(n + 1, \phi) < (m, \psi)$. The only non absurd case is $1 < m - n \leq p(\phi) - p(\psi)$. But then again we have

$$2 \leq m - n \leq p(\phi) - p(\psi) \leq \phi - \psi$$

and we can conclude as above.

To prove *(1)* consider the monotone map

$$\mathbb{Z} \xrightarrow{\lfloor */2 \rfloor} \mathbb{Z}$$

Applying the slice functor to the latter and starting with a perverse filtration, we get a grading filtration by the properties of the floor function and the thesis follows from part *(2)*.

Let's prove *(3)*. We have to show that

$$\mathscr{P}_\psi[1 - p(\psi)] \subseteq \mathscr{P}_\phi[p(\phi)]^\perp$$

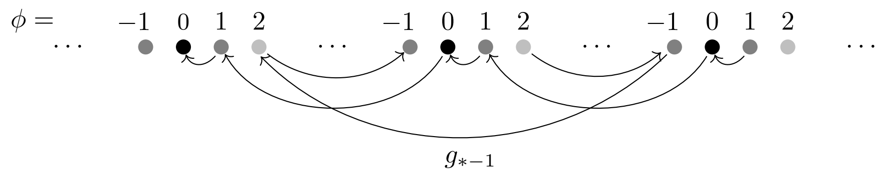
for $-p(\psi) > -p(\phi)$. We denote $n = p(\phi) - p(\psi) - 1$. Assuming again $n \geq 0$, we have $n \geq 0 > p(\psi) - p(\phi) \geq \psi - \phi$ and we can then conclude by definition of mixed filtration. $\square$

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity p the bounded t-structure coming from $(g_p)_{\underline{\Omega}}(\mathscr{Q})$ defines a morphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} \longrightarrow \mathfrak{bts}(\mathscr{D})$$

we can restate this as:

♫ *A grading or perverse filtration on the heart of a bounded t-structure on $\mathscr{D}$ induces a presheaf of t-structures on $\mathbb{Z} \times \mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z} \times \mathbb{Z}$-slicing' on $\mathscr{D}$.*



Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\mathrm{op}} \hookrightarrow \Xi$$

and the t-structure coming from $(g_{\chi_I})_{\Omega}(\mathscr{Q})$ is just the tilting of $\mathfrak{t}$ with respect to the torsion pair coming from I.
The following proposition gives a characterization of the new heart obtained by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 0.7. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, $\mathscr{P}$ a grading filtration on $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, p a perversity. Denote $\mathfrak{q}$ the bounded t-structure associated to $(g_p)_{\Omega}(\mathscr{Q})$. Then $\heartsuit_{\mathfrak{q}}$ consists of objects $X \in \mathscr{D}$ so that*

$$H^k_{\mathfrak{t}}(X) \in \mathscr{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if $\mathscr{P}$ is a mixed filtration then for each $X \in \heartsuit_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \heartsuit_{\mathfrak{q}}$ if and only if

$$H^{\phi}_{\mathscr{P}}(H^k_{\mathfrak{t}}(X)[-k])[k] = H^{(k,\phi)}_{\mathscr{Q}}(X) = 0$$

for $p(\phi) \neq -k$.
For the second part of the claim, the abelian $\mathbb{Z}$-slicing induced on $\heartsuit_{\mathfrak{q}}$ is split by **Proposition 0.6** and thus by **Proposition ??**

$$X = \bigoplus_{(k,\phi)} H^{(k,\phi)}_{\mathscr{Q}}(X) = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

where the last equality comes from the first part. $\square$

Example 0.8. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an $\mathbb{N}$-graded ring with B_0 semisimple. Denote $\mathscr{A}$ the category of $\mathbb{Z}$-graded B-modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote $\mathscr{P}_\phi$ the full subcategory of $\mathscr{A}$ of modules concentrated in degree ϕ. Clearly, $\mathscr{P}$ defines an abelian $\mathbb{Z}$-slicing on $\mathscr{A}$. Following [**?**] we have

$$\operatorname{Ext}^n_{\mathscr{A}}(\mathscr{P}_\phi, \mathscr{P}_\psi) = 0$$

for $n > \psi - \phi$. This means that $\mathscr{P}$ is a mixed filtration and the bounded t-structure on $\mathscr{D}^b(\mathscr{A})$ associated to $(g_1)_\Omega(\mathscr{Q})$ (where 1 is the identity of $\mathbb{Z}$) is the 'diagonal' (or 'geometric') t-structure which appears in Koszul duality and other areas.

Example 0.9. Let M be an n-dimensional smooth complex projective variety and consider the n-torsion pair $\mathscr{P}$ on $\operatorname{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that $\mathscr{P}$, seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t-structure associated to $(f_p)_\Omega(\mathscr{Q})$ is the one of perverse coherent sheaves as constructed in [**?**]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n-slicing on the heart of perverse coherent sheaves as done in [**?**].

Now we somehow review the gluing construction for t-structures in [**?**], but generalize it to any slicing. Our language is quite different though, and we formulate the problem very similarly to [**?**]. We start with two $\mathbb{Z}$-posets J, J'.

Definition 0.10. *Let $\mathscr{P}$ be a $J \ltimes J'$-slicing on $\mathscr{D}$. We call $\mathscr{P}$* ***gluable*** *if it satisties the assumptions of* ***Proposition 0.3*** *with respect to the map $J \ltimes J' \xrightarrow{e} J' \ltimes J$ that exchanges coordinates. In this case we denote*

$$\overline{\mathscr{P}} = e_\Omega(\mathscr{P})$$

By a simple computation, the gluability condition reads: $\mathscr{P}_{(\phi,\psi)} \subseteq \mathscr{P}^\perp_{(\phi',\psi')}$ if $\psi' > \psi$ or both $\psi' + 1 > \psi$ and $\phi' + 1 < \phi$. For example, the first condition is automatic when $\mathbb{Z}$ acts trivially on J (in this case, it follows from the second one). In particular, if $\mathbb{Z}$ acts trivially on J, $\mathscr{P}$ is a J-slicing on $\mathscr{D}$ and $\mathfrak{t}_\phi$ is a bounded t-structure on $\mathscr{P}_\phi$ for each $\phi \in J$, then using **Proposition ??** we get a $J \ltimes \mathbb{Z}$-slicing $\mathscr{Q}$ which is gluable if and only if

$$\heartsuit_{\mathfrak{t}_\psi}[n] \subseteq \heartsuit^\perp_{\mathfrak{t}_\phi}$$

whenever both $n \leq 0$ and $\phi < \psi$.

Remark 0.11. We have a commutative diagram of $\mathbb{Z}$-posets

$$\begin{tikzcd} \mathbb{Z} \ltimes \mathbb{Z} \arrow[r, "\sim"] \arrow[d, "e"] & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \arrow[d, "g"] \\ \mathbb{Z} \ltimes \mathbb{Z} \arrow[r, "\sim"] & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \end{tikzcd}$$

where the horizontal isomorphism is the one from **Remark ??**. Indeed, a $\mathbb{Z} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$ is gluable if and only if it induces a grading filtration on the heart of the associated t-structure when seen as a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing.

An easy combinatorial calculation finally yelds the following two remarks:

- if $\mathscr{P}$ is a gluable $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$, then $\overline{\mathscr{P}}$ induces a grading filtration on the associated heart, allowing us again to construct new bounded t-structures depending on a perversity.

- if $\mathscr{P}$ is a mixed filtration on the heart of a bounded t-structure and $\mathscr{Q}$ is the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, then $\mathscr{Q}$ is gluable. In this case, looking at $\overline{\mathscr{Q}}$, we get a baric structure on $\mathscr{D}$ which is usually called **weight decomposition** in literature.

In other words, the chain of implications for a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing refines to

$$\text{mixed} \implies \text{gluable} \implies \text{grading} \implies \text{perverse}$$