0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of $\mathbb{Z}$-posets (but just a $\mathbb{Z}$-equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a $\mathbb{Z}$-toset, i.e., a totally ordered set together with a monotone action of $\mathbb{Z}$, that we will denote by $(n,x) \mapsto x+n$. Recall that a J-slicing on a stable ∞-category $\mathscr{D}$ is a morphism of $\mathbb{Z}$-posests $\mathfrak{t}\colon \mathcal{O}(J) \to \mathrm{ts}(\mathscr{D})$, where $\mathcal{O}(J)$ is the $\mathbb{Z}$-poset of the slicings of J (i.e., the decompositions of J into a disjoint union of a lower set L and of an upper set U), and $\mathrm{ts}(\mathscr{D})$ is the $\mathbb{Z}$-poset of t-structures on $\mathscr{D}$.
Clearly, if $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then $f^{-1}\colon \{\text{subsets of } J'\} \to \{\text{subsets of } J\}$ induces a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-posets $f^{-1}\colon \mathcal{O}(J') \to \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$f_*\colon J\text{-slicings on } \mathscr{D} \to J'\text{-slicings on } \mathscr{D}$$
$$\mathfrak{t} \mapsto \mathfrak{t} \circ f^{-1}.$$

The slices of $f_*\mathfrak{t}$ are clearly given by $\mathscr{D}_{f_*\mathfrak{t};\phi} = \mathscr{D}_{\mathfrak{t};f^{-1}(\{\phi\})}$. Notice that, as f is monotone, the subset $f^{-1}(\{\phi\})$ is an interval in J. It is immediate to see that f_* restricts to a map

$$f_*\colon \text{Bridgeland } J\text{-slicings on } \mathscr{D} \to \text{Bridgeland } J'\text{-slicings on } \mathscr{D}$$

Namely, if $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) = 0$ for every $\phi \in J'$ then $\mathcal{H}^{f^{-1}(\{\phi\})}_{\mathfrak{t}}(X) = 0$ for every ϕ in J'. As we are assuming the J-slicing $\mathfrak{t}$ is a Bridgeland slicing, this implies that $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) = 0$ for every ψ in $f^{-1}(\{\phi\})$, for every ϕ. Therefore $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) = 0$ for every ψ in J and so, again by definition of Bridgeland slicing, $X = 0$. Also, if $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) \neq 0$ then $\mathcal{H}^{f^{-1}(\{\phi\})}_{\mathfrak{t}}(X) \neq 0$ and so (again by the Bridgeland slicing condition) there exists at least an element ψ in $f^{-1}(\{\phi\})$ such that $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) \neq 0$. As the J-slicing $\mathfrak{t}$ is Bridgeland, the total of these ψ's must be finite, so only for finitely many ϕ we can have such a ψ. In other words, the number of indices ϕ in J' such that $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) \neq 0$ is finite.
Notice that, if $\mathfrak{t}$ is a Bridgeland slicing of $\mathscr{D}$, and $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then for any slicing (L,U) of J', the lower and the upper categories $\mathscr{D}_{f_*\mathfrak{t}(L)}$ and $\mathscr{D}_{f_*\mathfrak{t}(U)}$ can be equivalently defined as

$$\mathscr{D}_{f_*\mathfrak{t}(L)} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$$
$$\mathscr{D}_{f_*\mathfrak{t}(U)} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}$$

where $\langle \mathscr{S} \rangle$ denotes the extension-closed subcategory of $\mathscr{D}$ generated by the subcategory $\mathscr{S}$.[1]

[1] At some point it will be useful to recall Lemma 4.21 from ??: Let $\mathscr{S}_1, \mathscr{S}_2$ be two subcategories of $\mathscr{D}$ with $\mathscr{S}_1 \boxslash \mathscr{S}_2$, i.e., such that $\mathscr{D}(X_1, X_2)$ is contractible for any $X_1 \in \mathscr{S}_1$ and any $X_2 \in \mathscr{S}_2$. Then $\mathscr{S}_1 \boxslash \langle \mathscr{S}_2 \rangle$ and $\langle \mathscr{S}_1 \rangle \boxslash \mathscr{S}_2$, and so $\langle \mathscr{S}_1 \rangle \boxslash \langle \mathscr{S}_2 \rangle$.

The right hand sides of the above two expressions can clearly be defined for every morphism f from J to J' (i.e., not necessarily monotone nor $\mathbb{Z}$-equivariant), and as soon as f is $\mathbb{Z}$-equivariant, the assignment

$$(L, U \mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$$

is an equivariant morphism from $\mathcal{O}(J)$ to pairs of subcategories of $\mathscr{D}$. Clearly, when f is not monotone there is no reason to expect that the pair $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ forms a t-structure on $\mathscr{D}$. Yet, it interesting to notice that the condition that f be monotone is only only sufficient in order to have this, and can indeed be relaxed.

Definition 0.1. *Let J and J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ a map of $\mathbb{Z}$-sets (i.e., a $\mathbb{Z}$-equivariant map, not necessarily nondecreasing). A Bridgeland J-slicing $\mathfrak{t}$ of $\mathscr{D}$ is* f-compatible *if $f(\phi) > f(\psi)$ with $\phi \leq \psi$ implies $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$*

Remark 0.2. Clearly, if f is monotone, then every J-slicing $\mathfrak{t}$ is order restoring as the condition '$f(\phi) > f(\psi)$ with $\phi \leq \psi$' is empty.

Lemma 0.3. *Let J and J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. Then, for any slicing (L, U) of J', the pair of subcategories $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ is a t-structure on $\mathscr{D}$.*

Proof. As f is $\mathbb{Z}$-equivariant and $U + 1 \subseteq U$, we have

$$\begin{aligned}
\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}[1] &= \langle \mathscr{D}_{\mathfrak{t};\phi}[1] \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi+1)\in U+1} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U+1} \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}
\end{aligned}$$

and similarly for the lower subcategory $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$. To show that

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U} \boxslash \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$$

it suffices to show that, if $f(\psi) \in L$ and $f(\phi) \in U$ then $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. As $L \cap U = \emptyset$ we cannot have $\psi = \phi$, so either $\psi < \phi$ or vice versa. In the first case, $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ by definition of Bridgeland J-slicing. In the second case, we have $\phi < \psi$ and $f(\phi) > f(\psi)$ as $f(\phi) \in U$ and $f(\psi) \in L$. Therefore, since $\mathfrak{t}$ is f-compatible, $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. Finally, we have to show that every object X in $\mathscr{D}$ fits into a fiber sequence

$$\begin{tikzcd}
X_U \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & X_L
\end{tikzcd}$$

with $X_L \in \langle \mathscr{D}_{\mathrm{t};\phi} \rangle_{f(\phi)\in L}$ and $X_U \in \langle \mathscr{D}_{\mathrm{t};\phi} \rangle_{f(\phi)\in U}$. As $\mathfrak{t}$ is a Bridgeland slicing, we have a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar\imath}} X_{\bar\imath} \xrightarrow{\alpha_{\bar\imath+1}} X_{\bar\imath+1} \to \cdots \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) = \mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \in \mathscr{D}_{\phi_i}$ for all $i = 1, \cdots, n$, with $\phi_i > \phi_{i+1}$. Let us now consider the sequence of symbols L and U obtained putting in the i-th place L if $f(\phi_i) \in L$ and U if $f(\phi_i) \in U$. If this sequence is of the form $(U, U, \ldots, U, L, L, \ldots, L)$, then there exists an index $\bar\imath$ such that $f(\phi_i) \in U$ for $i \leq \bar\imath$ and $f(\phi_i) \in L$ for $i > \bar\imath$ (with $\bar\imath = -1$ or n when all of the $f(\phi_i)$ are in L or in U, respectively). Then we can consider the pullout diagram

$$\begin{tikzcd}
X_{\bar\imath} \arrow[r] \arrow[d, "f_L"'] & 0 \arrow[d] \\
X \arrow[r] & \mathrm{cofib}(f_L)
\end{tikzcd}$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar\imath}} X_{\bar\imath}$$

and

$$X_{\bar\imath} \xrightarrow{\alpha_{\bar\imath+1}} X_{\bar\imath+1} \to \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that $X_{\bar\imath} \in \langle \cup_{i=0}^{\bar\imath} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathrm{t};\phi} \rangle_{f(\phi)\in U}$ while the second factorization shows that $\mathrm{cofib}(f_L) \in \langle \cup_{i=\bar\imath+1}^{n} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathrm{t};\phi} \rangle_{f(\phi)\in L}$. So we are done in this case. Therefore, we are reduced to showing that we can always avoid a $(\ldots, L, U, \ldots)$ situation in our sequence of L's and U's. Assume we have such a situation. Then we have an index i_0 with $f(\phi_{i_0}) \in L$ and $f(\phi_{i_0+1}) \in U$. This in particular implies $f(\phi_{i_0+1}) > f(\phi_{i_0})$ with $\phi_{i_0+1} < \phi_{i_0}$. As $\mathfrak{t}$ is f-compatible, this gives $\mathscr{D}_{\mathrm{t};\phi_{i_0+1}} \boxslash (\mathscr{D}_{\mathrm{t};\phi_{i_0}}[1])$. In particular, $\mathscr{D}(\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1])$ is contractible. Now consider the pasting of pullout diagrams

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1].
\end{tikzcd}$$

As the arrow $\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]$ factors through 0, we have $Y = \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus$

$\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ and the above diagram becomes

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\iota_2"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d, "\pi_1"] & 0 \arrow[d] \\
& 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1],
\end{tikzcd}$$

where ι_2 and π_1 are the canonical inclusion and projection. Let $\beta\colon X_{i_0+1} \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ be the composition

$$\beta\colon X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).$$

Then we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\beta"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\mathrm{id}"] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)
\end{tikzcd}$$

(where only the left square is a pullout), and so the composition $\alpha_{i_0+1} \circ \alpha_{i_0}$ factors through the homotopy fiber of β. In other words, we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d, "\tilde{\alpha}_{i_0}"'] & X_{i_0} \arrow[d, "\alpha_{i_0+1}"] \\
\mathrm{fib}(\beta) \arrow[r, "\tilde{\alpha}_{i_0+1}"] & X_{i_0+1}
\end{tikzcd}$$

Writing $\tilde{X}_{i_0} = \mathrm{fib}(\beta)$, we get the pasting of pullout diagrams

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\tilde{\alpha}_{i_0}"] \arrow[d] & \tilde{X}_{i_0} \arrow[r, "\tilde{\alpha}_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r, "\iota_1"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[d, "\pi_2"'] \\
& 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).
\end{tikzcd}$$

That is, by considering the factorization $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$ we have switched the cofibers with respect to the original factorization $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$. Therefore, writing $\tilde{\phi}_{i_0} = \phi_{i_0+1}$ and $\tilde{\phi}_{i_0+1} = \phi_{i_0}$, we now have $f(\tilde{\phi}_{i_0}) \in U$ and $f(\tilde{\phi}_{i_0+1}) \in L$. That is, we have removed the $(\dots, L, U, \dots)$ situation from the position i_0, replacing it with a $(\dots, U, L, \dots)$ situation, while keeping all the labels L, U before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the $(\dots, L, U, \dots)$ situations.[2] $\square$

Remark 0.4. It follows from the proof of Lemma 0.3 that the objects X_L and X_U in the fiber sequence $X_U \to X \to X_L$ associated with the t-struture $(\langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\in U})$ on $\mathscr{D}$ satisfy

$$X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in L \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0\rangle$$
$$X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in U \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0\rangle$$

Lemma 0.5. *For any $j \in J'$ we have*

$$\langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)=j} = \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\geq j}.$$

Proof. Clearly, $\langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)=j} \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\geq j}$, therefore we only need to prove the converse inclusion. Let $\in \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\geq j}$. Then in particular $X \in \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\geq j}$ and so there exists a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) \in \mathscr{D}_{\phi_i}$ with $f(\phi_i) \geq i$ for all $i = 1, \cdots, n$. As $((-\infty, j], (j, +\infty)$ is a slicing of J', reasoning as in the proof of Lemma 0.3 we can arrange this factorization is such a way that $f(\phi_i) > j$ for $i \leq \bar{\imath}$ and $f(\phi_i) = j$ for $i > \bar{\imath}$. Therefore, again by reasoning as in the proof of Lemma 0.3 we get a fiber sequence of the form

$$\begin{tikzcd} X_{>j} \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X_j \end{tikzcd}$$

with $X_{>j}$ in $\langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)>j}$ and X_j in $\langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)=j}$. This is in particular a fiber sequence of the form $X_{>j} \to X \to X_{\leq j}$, with $X_{\leq j} \in \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\leq j}$. As $(\langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\leq j}, \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)>j})$ is a t-structure on $\mathscr{D}$ by Lemma 0.3, there is (up to equivalence) only one such a fiber sequence. And since $X \in \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\leq j}$, this is the sequence $0 \to X \xrightarrow{\mathrm{id}} X$. Therefore $X = X_j$. $\square$

Proposition 0.6. *Let $f\colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. The map*

$$\begin{aligned} f_!\mathfrak{t}\colon \mathcal{O}(J') &\to \mathrm{ts}(\mathscr{D}) \\ (L, U) &\mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi}\rangle_{f(\phi)\in U}) \end{aligned}$$

[2]This is somehow reminiscent of the 'bubble sort' algorithm.

defined by Lemma 0.3 is a Bridgeland J'-slicing of $\mathscr{D}$, with slices given by

$$\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}.$$

Proof. The map $f_!\mathfrak{t}$ is manifestly monotone and $\mathbb{Z}$-equivariant (see the first part of the proof of Lemma 0.3), so it is a J' slicing of $\mathscr{D}$, and its slices are given by $\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$ by Lemma 0.5. We are therefore left with showing that it is finite and discrete. Given an object X in $\mathscr{D}$, let now $\{\phi_1, \ldots, \phi_n\}$ be the indices in J such that $\mathcal{H}^{\phi_i}_{\mathfrak{t}}(X) \neq 0$ and let $\{j_1, \ldots, j_k\}$ the image of the set $\{\phi_1, \ldots, \phi_n\}$ via f. Up to renaming, we can assume $j_1 > j_2 > \cdots > j_k$. Consider now the factorization

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} X_{k-1} \xrightarrow{\alpha_k} X_k = X$$

of the initial morphism of X associated to the decreasing sequence $j_1 > j_2 > \cdots > j_k$ by the J'-slicing $f_!\mathfrak{t}$. The cofibers of the morphisms α_i are the cohomologies $\mathcal{H}^{(j_{i+1},j_i]}_{f_!\mathfrak{t}}(X)$ and, by Remark 0.4, we have

$$\begin{aligned}
\mathcal{H}^{(j_{i+1},j_i]}_{f_!\mathfrak{t}}(X) \in & \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in (j_{i+1}, j_i] \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0 \rangle \\
& = \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \ldots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, j_i] \rangle \\
& = \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \ldots, \phi_n\} \text{ and } f(\phi) = j_i \rangle \\
& \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) = j_i \rangle = \mathscr{D}_{f_!\mathfrak{t};j_i}.
\end{aligned}$$

Therefore

$$\mathcal{H}^{(j_i,j_{i-1}]}_{f_!\mathfrak{t}}(X) = \mathcal{H}^{j_i}_{f_!\mathfrak{t}} \mathcal{H}^{(j_i,j_{i-1}]}_{f_!\mathfrak{t}}(X) = \mathcal{H}^{j_i}_{f_!\mathfrak{t}}(X).$$

This tells us that, if all the cohomologies $\mathcal{H}^{j}_{f_!\mathfrak{t}}(X)$ vanish, then also all the $\mathcal{H}^{(j_i,j_{i-1}]}_{f_!\mathfrak{t}}(X)$ vanish, and so $X = 0$. Finally, if $\tilde{j} \notin \{j_1, \ldots, j_k\}$, then there exists an index i such that $j_{i+1} < \tilde{j} < j_i$ and so $(j_{i+1}, \tilde{j}] \cap \{j_1, \ldots, j_k\} = \emptyset$. The above argument then shows that

$$\mathcal{H}^{(j_{i+1},\tilde{j}]}_{f_!\mathfrak{t}}(X) \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \ldots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, \tilde{j}] \rangle = \{\mathbf{0}\}.$$

It follows that

$$\mathcal{H}^{\tilde{j}}_{f_!\mathfrak{t}}(X) = \mathcal{H}^{\tilde{j}}_{f_!\mathfrak{t}} \mathcal{H}^{(j_{i+1},\tilde{j}]}_{f_!\mathfrak{t}}(X) = \mathcal{H}^{\tilde{j}}_{f_!\mathfrak{t}}(0) = 0,$$

for every $\tilde{j} \notin \{j_1, \ldots, j_k\}$. So in particular, for any $X \in \mathscr{D}$, the cohomologies $\mathcal{H}^{j}_{f_!\mathfrak{t}}(X)$ are possibly nonzero only for finitely many indices j. $\square$

Let now J_1 and J_2 be two $\mathbb{Z}$-tosets, and let $J_1 \times_{\text{lex}} J_2$ and $J_2 \times_{\text{lex}} J_1$ the two $\mathbb{Z}$-tosets obtained by considering the lexicogaphic order on the products $J_1 \times J_2$ and $J_2 \times J_1$, respectively, and the diagonal $\mathbb{Z}$ action. The following lemma is immediate.

Lemma 0.7. *The exchange map*

$$\begin{aligned} e\colon J_1 \times J_2 &\to J_2 \times J_1 \\ (j_1, j_2) &\mapsto (j_2, j_1) \end{aligned}$$

is $\mathbb{Z}$-equivariant.

Definition 0.8. *Let J_1 and J_2 be $\mathbb{Z}$-tosets. A Bridgeland $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ of $\mathscr{D}$ is said to be* gluable *if it is e-compatible, where $e\colon J_1 \times J_2 \to J_2 \times J_1$ is the exchange map.*

Example 0.9. Let $J_1 = \{0, 1\}$ with the trivial $\mathbb{Z}$-action and let $J_2 = \mathbb{Z}$ with the standard translation $\mathbb{Z}$-action. Then a $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ on $\mathscr{D}$ is the datum of a semiorthogonal decomposition $(\mathscr{D}_0, \mathscr{D}_1)$ of $\mathscr{D}$ toghether with t-structures $\mathfrak{t}_i$ on $\mathscr{D}_i$. Spelling out the definiton, one sees that the $\{0, 1\} \times_{\mathrm{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$ is gluable if and only if $\mathscr{D}_{0;0} \boxslash \mathscr{D}_{1;0}$ and $\mathscr{D}_{0;0} \boxslash \mathscr{D}_{1;0}[-1]$. When this happens, the glued slicing $e_!\mathfrak{t}$ is a $\mathbb{Z} \times_{\mathrm{lex}} \{0, 1\}$ slicing on $\mathscr{D}$, i.e., is the datum of a bounded t-structure on $\mathscr{D}$ together with a torsion theory on its heart. More precisely, the heart of $e_!\mathfrak{t}$ is the extension-closed ∞-subcategory $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ of $\mathscr{D}$ generated by the hearts of $\mathscr{D}_0$ and $\mathscr{D}_1$, while the torsion theory on $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ is $(\mathscr{D}_0^{\heartsuit}, \mathscr{D}_1^{\heartsuit})$, i.e., precisely the pair of hearts of the two subcategories in the semiorthogonal decomposition.

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Remark 0.10. In ref: BBD the authors consider a process consisting in gluing t-structures on a pair of triangulated categories to obtain a t-structure on a third one. In their setting, the three categories are linked by an abstraction of the Grothendieck's six-functors formalism. However, the latter is more or less equivalent to the datum of a semiorthogonal decomposition (i.e., a $\{0, 1\}$-slicing, where $\{0, 1\}$ is the toset with two elements and trivial $\mathbb{Z}$-action) on the third category so that the input of the two (bounded) t-structures gives rise to a gluable $\{0, 1\} \times_{\mathrm{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$. Then the $\mathbb{Z} \times_{\mathrm{lex}} \{0, 1\}$-slicing $e_!\mathfrak{t}$ is just the new glued t-structure together with the natural torsion pair on its heart. This explains our terminology and links the language of the present paper to the classical theory of t-structures.

To motivate the next section, consider the following isomorphism fo $\mathbb{Z}$-tosets:

$$\begin{aligned} \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z} &\to \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{free}} \\ (n, m) &\mapsto (n, m - n) \end{aligned}$$

...continua ...

Lemma 0.11. *Let $p\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ be a monotone map. Then*

$$\Gamma(p) = \{(n, j) \in J \times J'; \quad j \in U_{p(n)}\}.$$

is an upper set of $J \times J'$ with the product order.

Proof. Pick $(n,j) \leq (m,k)$ in $J \times J'$ and suppose $(n,j) \in \Gamma(p)$. Then $j \in p(n) \in \mathcal{O}(J')$ and since $j \leq k$ we have that $k \in p(n)$. But $n \leq m$ and p is monotone, and thus $p(n) \subseteq p(m)$. This gives $k \in p(m)$, as desired. $\square$

Proposition 0.12. *The map*

$$\begin{aligned} \Gamma \colon \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}}) &\to \mathcal{O}(J \times J') \\ p &\mapsto (J \times J' \setminus \Gamma(p), \Gamma(p)) \end{aligned}$$

is an isomorphism of $\mathbb{Z}$-posets, where $J \times J'$ carries the product order.

Proof. Pick and upper set U of $J \times J'$ and set, for $n \in J$

$$p_U(n) = \{j \in J'; \quad (n,j) \in U\}.$$

Then the map

$$\begin{aligned} \mathcal{O}(J \times J') &\to \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}}) \\ (L, U) &\mapsto p_U \end{aligned}$$

is inverse to $\Gamma(p)$ by a straightforward check. $\square$

Definition 0.13. *A* perversity *on J is a monotone map $p \colon \mathbb{Z} \to \mathcal{O}(J)^{\mathrm{op}}$ such that*

$$p(n) - n \leq p(m) - m$$

whenever $m \leq n$ in $\mathbb{Z}$. The set of all perversities on J will be denoted $\Theta(J)$.

Remark 0.14. We can give $\Theta(J)$ the structure of a $\mathbb{Z}$-poset via the incluion $\Theta(J) \subseteq \mathrm{Pos}(\mathbb{Z}, \mathcal{O}(J)^{\mathrm{op}})$. In general, it will not be totally ordered. However, by taking joins and meets pointwise $\Theta(J)$ becomes a distributive lattice.

Remark 0.15. Considering the obvious isomoprhism $\mathbb{Z} \simeq \mathcal{O}(\mathbb{Z})^{\mathrm{op}}$ a perversity over $\mathbb{Z}$ is just a *monotone and comonotope perversity* in the sense of referenza, tipo bezrukavnikov o BBD.

fino a qui 2

Definition 0.16. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. An abelian $\mathbb{Z}$-slicing $\mathscr{P}$ on $\heartsuit_{\mathfrak{t}}$ is called:*

- ***perverse filtration*** *if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n < \phi - \psi$*
- ***grading filtration*** *if it is a perverse filtration and $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $2 \leq n = \phi - \psi$*
- ***mixed filtration*** *if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n > \psi - \phi$*

The following implications hold:

$$\text{mixed} \implies \text{grading} \implies \text{perverse}$$

proof that mixed implies grading: suppose $n < \phi - \psi$. By definition of slicing, the vanishing is automatic for $n \leq 0$ and we can thus assume $n > 0$. But then $n > 0 > -n > \psi - \phi$ and the required vanishing now follows from the definition of mixed filtration. The same argument also works for $2 \leq n = \phi - \psi$.
Clearly, a torsion pair on $\heartsuit_{\mathfrak{t}}$ (seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[1] \subseteq \mathbb{Z}$) is always a grading filtration (proof: it follows from the fact that $\mathscr{P}_\phi[n] = 0$ for $\phi \neq 0, 1$ and thus the only nontrivial case is $n = 0, \phi = 1, \psi = 0$. The Hom-vanishing then follows from the definition of torsion pair). Historically, grading filtrations first appeared in [**?**] under the name of 'radical filtrations', while mixed filtrations are covered in the last section of [**?**].

Definition 0.17. *A map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is called **perversity** if it is a monotone contraction, i.e. if*

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi$$

whenever $\phi \geq \psi$ in $\mathbb{Z}$.
We denote Ξ the set of perversities.

Clearly, Ξ defines a $\mathbb{Z}$-poset which is not totally ordered. However, it is a distributive lattice: we have meets and joins given by:

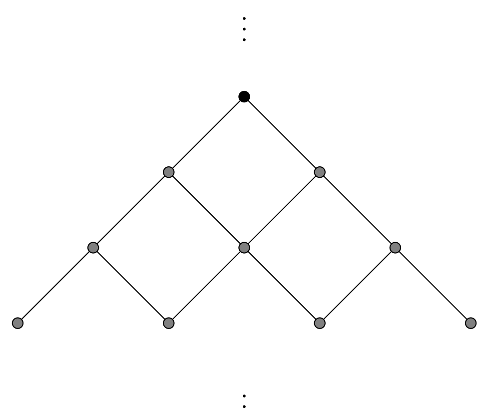
$$(p \wedge q)(\phi) = \min\{p(\phi), q(\phi)\}$$

$$(p \vee q)(\phi) = \max\{p(\phi), q(\phi)\}$$

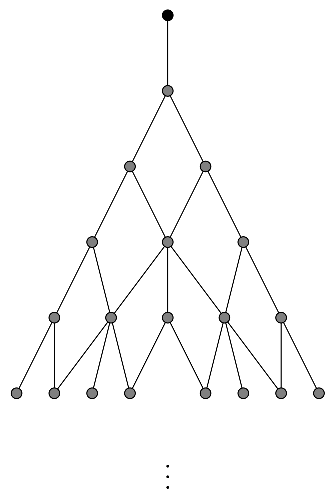
Indeed, if we place on $\mathbb{Z} \times \mathbb{Z}$ the product order, we have an isomorphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} = O(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

given by sending the zero perversity to $A = \{(\phi, \psi)\}_{\phi \geq 0}$ and the identity to $B = \{(\phi, \psi)\}_{\psi \geq 0}$ ($B + i$ and $A + i$ with $i \in \mathbb{Z}$ generate the latter as a complete lattice). Considering the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Z} \subseteq O(\mathbb{Z} \times \mathbb{Z})$ we get a sublattice of Ξ depicted as



This induces the 't-tree' described in [**?**], as discussed below in a far-reaching generality. By the other hand, each square above contains moreover a copy of the Young lattice



with the root corresponding to the upper vertex of the square.

Proposition 0.18. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, $\mathscr{P}$ an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, p a perversity. We have:*

1. *if $\mathscr{P}$ is a perverse filtration, then $\mathscr{Q}$ satisfies the assumptions of **Proposition ??** with respect to the map $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{f_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ given by*

$$f_p(n, \phi) = (n + p(\lfloor \phi/2 \rfloor), -p(\lfloor \phi/2 \rfloor))$$

2. *if $\mathscr{P}$ is a grading filtration, then $\mathscr{Q}$ satisfies the assumptions of **Proposition ??** with respect to the map $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{g_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ given by*

$$g_p(n, \phi) = (n + p(\phi), -p(\phi))$$

3. *if $\mathscr{P}$ is a mixed filtration, then the abelian $\mathbb{Z}$-slicing induced on the heart of the bounded t-structure associated to $(g_p)_{\underline{\Omega}}(\mathscr{Q})$ is split*

Proof. Let's prove *(2)*. Suppose $g_p(n, \phi) > g_p(m, \psi)$. Then either $m - n < p(\phi) - p(\psi)$ or both $m - n = p(\phi) - p(\psi)$ and $p(\psi) < p(\psi)$. Since by definition of t-structure we can assume $m \geq n$, the second case is absurd while in the first case, since p is monotone, we have $\phi \geq \psi$ and thus by definition of perversity

$$m - n < p(\phi) - p(\psi) \leq \phi - \psi$$

and we get the desired Hom-vanishing by definition of grading filtration.

Suppose now $f_p(n,\phi)+1 > f_p(m,\psi)$ and $(n+1,\phi) < (m,\psi)$. The only non absurd case is $1 < m-n \leq p(\phi)-p(\psi)$. But then again we have

$$2 \leq m-n \leq p(\phi)-p(\psi) \leq \phi-\psi$$

and we can conclude as above.

To prove *(1)* consider the monotone map

$$\mathbb{Z} \xrightarrow{\lfloor */2 \rfloor} \mathbb{Z}$$

Applying the slice functor to the latter and starting with a perverse filtration, we get a grading filtration by the properties of the floor function and the thesis follows from part *(2)*.

Let's prove *(3)*. We have to show that

$$\mathscr{P}_\psi[1-p(\psi)] \subseteq \mathscr{P}_\phi[p(\phi)]^\perp$$

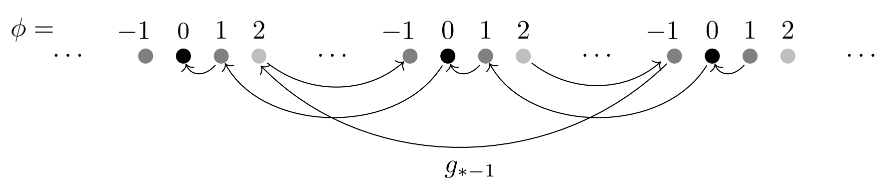
for $-p(\psi) > -p(\phi)$. We denote $n = p(\phi)-p(\psi)-1$. Assuming again $n \geq 0$, we have $n \geq 0 > p(\psi)-p(\phi) \geq \psi-\phi$ and we can then conclude by definition of mixed filtration. $\square$

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity p the bounded t-structure coming from $(g_p)_{\mathfrak{Q}}(\mathscr{Q})$ defines a morphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} \longrightarrow \mathfrak{bts}(\mathscr{D})$$

we can restate this as:

♫ *A grading or perverse filtration on the heart of a bounded t-structure on $\mathscr{D}$ induces a presheaf of t-structures on $\mathbb{Z}\times\mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z}\times\mathbb{Z}$-slicing' on $\mathscr{D}$.*



Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\mathrm{op}} \hookrightarrow \Xi$$

and the t-structure coming from $(g_{\chi_I})_{\mathfrak{Q}}(\mathscr{Q})$ is just the tilting of $\mathfrak{t}$ with respect to the torsion pair coming from I.
The following proposition gives a characterization of the new heart obtained

by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 0.19. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, $\mathscr{P}$ a grading filtration on $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, p a perversity. Denote $\mathfrak{q}$ the bounded t-structure associated to $(g_p)_{\Omega}(\mathscr{Q})$. Then $\heartsuit_{\mathfrak{q}}$ consists of objects $X \in \mathscr{D}$ so that*

$$H^k_{\mathfrak{t}}(X) \in \mathscr{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if $\mathscr{P}$ is a mixed filtration then for each $X \in \heartsuit_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \heartsuit_{\mathfrak{q}}$ if and only if

$$H^{\phi}_{\mathscr{P}}(H^k_{\mathfrak{t}}(X)[-k])[k] = H^{(k,\phi)}_{\mathscr{Q}}(X) = 0$$

for $p(\phi) \neq -k$.
For the second part of the claim, the abelian $\mathbb{Z}$-slicing induced on $\heartsuit_{\mathfrak{q}}$ is split by **Proposition 0.18** and thus by **Proposition ??**

$$X = \bigoplus_{(k,\phi)} H^{(k,\phi)}_{\mathscr{Q}}(X) = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

where the last equality comes from the first part. □

Example 0.20. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an $\mathbb{N}$-graded ring with B_0 semisimple. Denote $\mathscr{A}$ the category of $\mathbb{Z}$-graded B-modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote $\mathscr{P}_\phi$ the full subcategory of $\mathscr{A}$ of modules concentrated in degree ϕ. Clearly, $\mathscr{P}$ defines an abelian $\mathbb{Z}$-slicing on $\mathscr{A}$. Following [**?**] we have

$$\mathrm{Ext}^n_{\mathscr{A}}(\mathscr{P}_\phi, \mathscr{P}_\psi) = 0$$

for $n > \psi - \phi$. This means that $\mathscr{P}$ is a mixed filtration and the bounded t-structure on $\mathscr{D}^b(\mathscr{A})$ associated to $(g_1)_{\Omega}(\mathscr{Q})$ (where 1 is the identity of $\mathbb{Z}$) is the 'diagonal' (or 'geometric') t-structure which appears in Koszul duality and other areas.

Example 0.21. Let M be an n-dimensional smooth complex projective variety and consider the n-torsion pair $\mathscr{P}$ on $\mathrm{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that $\mathscr{P}$, seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t-structure associated to $(f_p)_{\Omega}(\mathscr{Q})$ is the one of perverse coherent sheaves as constructed in [?]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n-slicing on the heart of perverse coherent sheaves as done in [?].

Now we somehow review the gluing construction for t-structures in [?], but generalize it to any slicing. Our language is quite different though, and we formulate the problem very similarly to [?]. We start with two $\mathbb{Z}$-posets J, J'.

Definition 0.22. *Let $\mathscr{P}$ be a $J \ltimes J'$-slicing on $\mathscr{D}$. We call $\mathscr{P}$ **gluable** if it satisties the assumptions of **Proposition ??** with respect to the map $J \ltimes J' \xrightarrow{e} J' \ltimes J$ that exchanges coordinates. In this case we denote*

$$\overline{\mathscr{P}} = e_{\underline{\Omega}}(\mathscr{P})$$

By a simple computation, the gluability condition reads: $\mathscr{P}_{(\phi,\psi)} \subseteq \mathscr{P}^{\perp}_{(\phi',\psi')}$ if $\psi' > \psi$ or both $\psi' + 1 > \psi$ and $\phi' + 1 < \phi$. For example, the first condition is automatic when $\mathbb{Z}$ acts trivially on J (in this case, it follows from the second one). In particular, if $\mathbb{Z}$ acts trivially on J, $\mathscr{P}$ is a J-slicing on $\mathscr{D}$ and $\mathfrak{t}_\phi$ is a bounded t-structure on $\mathscr{P}_\phi$ for each $\phi \in J$, then using **Proposition ??** we get a $J \ltimes \mathbb{Z}$-slicing $\mathscr{Q}$ which is gluable if and only if

$$\heartsuit_{\mathfrak{t}_\psi}[n] \subseteq \heartsuit_{\mathfrak{t}_\phi}^{\perp}$$

whenever both $n \leq 0$ and $\phi < \psi$.

Remark 0.23. We have a commutative diagram of $\mathbb{Z}$-posets

$$\begin{tikzcd}
\mathbb{Z} \ltimes \mathbb{Z} \arrow[r, "\sim"] \arrow[d, "e"] & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \arrow[d, "g"] \\
\mathbb{Z} \ltimes \mathbb{Z} \arrow[r, "\sim"] & \mathbb{Z} \ltimes \hat{\mathbb{Z}}
\end{tikzcd}$$

where the horizontal isomorphism is the one from **Remark ??**. Indeed, a $\mathbb{Z} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$ is gluable if and only if it induces a grading filtration on the heart of the associated t-structure when seen as a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing.

An easy combinatorial calculation finally yelds the following two remarks:

- if $\mathscr{P}$ is a gluable $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$, then $\overline{\mathscr{P}}$ induces a grading filtration on the associated heart, allowing us again to construct new bounded t-structures depending on a perversity.

- if $\mathscr{P}$ is a mixed filtration on the heart of a bounded t-structure and $\mathscr{Q}$ is the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, then $\mathscr{Q}$ is gluable. In this case, looking at $\overline{\mathscr{Q}}$, we get a baric structure on $\mathscr{D}$ which is usually called **weight decomposition** in literature.

In other words, the chain of implications for a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing refines to

$$\text{mixed} \implies \text{gluable} \implies \text{grading} \implies \text{perverse}$$