0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of $\mathbb{Z}$-posets (but just a $\mathbb{Z}$-equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a $\mathbb{Z}$-toset, i.e., a totally ordered set together with a monotone action of $\mathbb{Z}$, that we will denote by $(n, x) \mapsto x + n$. Recall that a J-slicing on a stable ∞-category $\mathscr{D}$ is a morphism of $\mathbb{Z}$-posets $\mathfrak{t} \colon \mathcal{O}(J) \to \mathrm{ts}(\mathscr{D})$, where $\mathcal{O}(J)$ is the $\mathbb{Z}$-poset of the slicings of J (i.e., the decompositions of J into a disjoint union of a lower set L and of an upper set U), and $\mathrm{ts}(\mathscr{D})$ is the $\mathbb{Z}$-poset of t-structures on $\mathscr{D}$.

Clearly, if $f \colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then $f^{-1} \colon \{\text{subsets of } J'\} \to \{\text{subsets of } J\}$ induces a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-posets $f^{-1} \colon \mathcal{O}(J') \to \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$f_* \colon J\text{-slicings on } \mathscr{D} \to J'\text{-slicings on } \mathscr{D}$$

$$\mathfrak{t} \mapsto \mathfrak{t} \circ f^{-1}.$$

The slices of $f_*\mathfrak{t}$ are clearly given by $\mathscr{D}_{f_*\mathfrak{t};\phi} = \mathscr{D}_{\mathfrak{t};f^{-1}(\{\phi\})}$. Notice that, as f is monotone, the subset $f^{-1}(\{\phi\})$ is an interval in J. It is immediate to see that f_* restricts to a map

$$f_* \colon \text{Bridgeland } J\text{-slicings on } \mathscr{D} \to \text{Bridgeland } J'\text{-slicings on } \mathscr{D}$$

Namely, if $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) = 0$ for every $\phi \in J'$ then $\mathcal{H}^{f^{-1}(\{\phi\})}_{\mathfrak{t}}(X) = 0$ for every ϕ in J'. As we are assuming the J-slicing $\mathfrak{t}$ is a Bridgeland slicing, this implies that $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) = 0$ for every ψ in $f^{-1}(\{\phi\})$, for every ϕ. Therefore $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) = 0$ for every ψ in J and so, again by definition of Bridgeland slicing, $X = 0$. Also, if $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) \neq 0$ then $\mathcal{H}^{f^{-1}(\{\phi\})}_{\mathfrak{t}}(X) \neq 0$ and so (again by the Bridgeland slicing condition) there exists at least an element ψ in $f^{-1}(\{\phi\})$ such that $\mathcal{H}^{\psi}_{\mathfrak{t}}(X) \neq 0$. As the J-slicing $\mathfrak{t}$ is Bridgeland, the total of these ψ's must be finite, so only for finitely many ϕ we can have such a ψ. In other words, the number of indices ϕ in J' such that $\mathcal{H}^{\phi}_{f_*\mathfrak{t}}(X) \neq 0$ is finite.

Notice that, if $\mathfrak{t}$ is a Bridgeland slicing of $\mathscr{D}$, and $f \colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then for any slicing (L, U) of J', the lower and the upper categories $\mathscr{D}_{f_*\mathfrak{t}(L)}$ and $\mathscr{D}_{f_*\mathfrak{t}(U)}$ can be equivalently defined as

$$\mathscr{D}_{f_*\mathfrak{t}(L)} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$$
$$\mathscr{D}_{f_*\mathfrak{t}(U)} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}$$

where $\langle \mathscr{S} \rangle$ denotes the extension-closed subcategory of $\mathscr{D}$ generated by the subcategory $\mathscr{S}$.[1]

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[1] At some point it will be useful to recall Lemma 4.21 from ??: Let $\mathscr{S}_1, \mathscr{S}_2$ be two subcategories of $\mathscr{D}$ with $\mathscr{S}_1 \boxdot \mathscr{S}_2$, i.e., such that $\mathscr{D}(X_1, X_2)$ is contractible for any $X_1 \in \mathscr{S}_1$ and any $X_2 \in \mathscr{S}_2$. Then $\mathscr{S}_1 \boxdot \langle \mathscr{S}_2 \rangle$ and $\langle \mathscr{S}_1 \rangle \boxdot \mathscr{S}_2$, and so $\langle \mathscr{S}_1 \rangle \boxdot \langle \mathscr{S}_2 \rangle$.

The right hand sides of the above two expressions can clearly be defined for every morphism f from J to J' (i.e., not necessarily monotone nor $\mathbb{Z}$-equivariant), and as soon as f is $\mathbb{Z}$-equivariant, the assignment

$$(L, U) \mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$$

is an equivariant morphism from $\mathcal{O}(J)$ to pairs of subcategories of $\mathscr{D}$. Clearly, when f is not monotone there is no reason to expect that the pair $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$ forms a t-structure on $\mathscr{D}$. Yet, it interesting to notice that the condition that f be monotone is only only sufficient in order to have this, and can indeed be relaxed.

Definition 0.1. *Let J and J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ a map of $\mathbb{Z}$-sets (i.e., a $\mathbb{Z}$-equivariant map, not necessarily nondecreasing). A Bridgeland J-slicing $\mathfrak{t}$ of $\mathscr{D}$ is f-compatible if $f(\phi) > f(\psi)$ with $\phi \leq \psi$ implies $\mathscr{D}_{\mathfrak{t};\phi} \boxtimes \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxtimes \mathscr{D}_{\mathfrak{t};\psi}[1]$*

Remark 0.2. Clearly, if f is monotone, then every J-slicing $\mathfrak{t}$ is order restoring as the condition '$f(\phi) > f(\psi)$ with $\phi \leq \psi$' is empty.

Lemma 0.3. *Let J and J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. Then, for any slicing (L, U) of J', the pair of subcategories $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$ is a t-structure on $\mathscr{D}$.*

Proof. As f is $\mathbb{Z}$-equivariant and $U + 1 \subseteq U$, we have

$$\begin{aligned}
\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}[1] &= \langle \mathscr{D}_{\mathfrak{t};\phi}[1] \rangle_{f(\phi) \in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi) \in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi+1) \in U+1} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U+1} \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}
\end{aligned}$$

and similarly for the lower subcategory $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$. To show that

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U} \boxtimes \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$$

it suffices to show that, if $f(\psi) \in L$ and $f(\phi) \in U$ then $\mathscr{D}_{\mathfrak{t};\phi} \boxtimes \mathscr{D}_{\mathfrak{t};\psi}$. As $L \cap U = \emptyset$ we cannot have $\psi = \phi$, so either $\psi < \phi$ or vice versa. In the first case, $\mathscr{D}_{\mathfrak{t};\phi} \boxtimes \mathscr{D}_{\mathfrak{t};\psi}$ by definition of Bridgeland J-slicing. In the second case, we have $\phi < \psi$ and $f(\phi) > f(\psi)$ as $f(\phi) \in U$ and $f(\psi) \in L$. Therefore, since $\mathfrak{t}$ is f-compatible, $\mathscr{D}_{\mathfrak{t};\phi} \boxtimes \mathscr{D}_{\mathfrak{t};\psi}$. Finally, we have to show that every object X in $\mathscr{D}$ fits into a fiber sequence

$$\begin{array}{ccc}
X_U & \longrightarrow & X \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X_L
\end{array}$$

2

with $X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$ and $X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}$. As $\mathfrak{t}$ is a Bridgeland slicing, we have a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{\imath}}} X_{\bar{\imath}} \xrightarrow{\alpha_{\bar{\imath}+1}} X_{\bar{\imath}+1} \to \cdots \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) = \mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \in \mathscr{D}_{\phi_i}$ for all $i = 1, \cdots, n$, with $\phi_i > \phi_{i+1}$. Let us now consider the sequence of symbols L and U obtained putting in the i-th place L if $f(\phi_i) \in L$ and U if $f(\phi_i) \in U$. If this sequence is of the form $(U, U, \ldots, U, L, L, \ldots, L)$, then there exists an index $\bar{\imath}$ such that $f(\phi_i) \in U$ for $i \leq \bar{\imath}$ and $f(\phi_i) \in L$ for $i > \bar{\imath}$ (with $\bar{\imath} = -1$ or n when all of the $f(\phi_i)$ are in L or in U, respectively). Then we can consider the pullout diagram

$$
\begin{array}{ccc}
X_{\bar{\imath}} & \longrightarrow & 0 \\
{\scriptstyle f_L} \downarrow & & \downarrow \\
X & \longrightarrow & \mathrm{cofib}(f_L)
\end{array}
$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{\imath}}} X_{\bar{\imath}}$$

and

$$X_{\bar{\imath}} \xrightarrow{\alpha_{\bar{\imath}+1}} X_{\bar{\imath}+1} \to \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that $X_{\bar{\imath}} \in \langle \cup_{i=0}^{\bar{\imath}} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U}$ while the second factorization shows that $\mathrm{cofib}(f_L) \in \langle \cup_{i=\bar{\imath}+1}^{n} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}$. So we are done in this case. Therefore, we are reduced to showing that we can always avoid a $(\ldots, L, U, \ldots)$ situation in our sequence of L's and U's. Assume we have such a situation. Then we have an index i_0 with $f(\phi_{i_0}) \in L$ and $f(\phi_{i_0+1}) \in U$. This in particular implies $f(\phi_{i_0+1}) > f(\phi_{i_0})$ with $\phi_{i_0+1} < \phi_{i_0}$. As $\mathfrak{t}$ is f-compatible, this gives $\mathscr{D}_{\mathfrak{t};\phi_{i_0+1}} \boxtimes (\mathscr{D}_{\mathfrak{t};\phi_{i_0}}[1])$. In particular, $\mathscr{D}(\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1])$ is contractible. Now consider the pasting of pullout diagrams

$$
\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle \gamma} & & \\
0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) & \longrightarrow & Y & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]. &
\end{array}
$$

As the arrow $\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]$ factors through 0, we have $Y = \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus$

$\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ and the above diagram becomes

$$
\begin{array}{ccccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) & \xrightarrow{\iota_2} & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\pi_1} & & \downarrow \\
& & 0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1],
\end{array}
$$

where ι_2 and π_1 are the canonical inclusion and projection. Let $\beta\colon X_{i_0+1} \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ be the composition

$$
\beta\colon X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).
$$

Then we have a homotopy commutative diagram

$$
\begin{array}{ccccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} & \xrightarrow{\alpha_{i_0+1}} & X_{i_0+1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\beta} \\
0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) & \xrightarrow{\mathrm{id}} & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)
\end{array}
$$

(where only the left square is a pullout), and so the composition $\alpha_{i_0+1} \circ \alpha_{i_0}$ factors through the homotopy fiber of β. In other words, we have a homotopy commutative diagram

$$
\begin{array}{ccc}
X_{i_0-1} & \xrightarrow{\alpha_{i_0}} & X_{i_0} \\
{\scriptstyle\tilde{\alpha}_{i_0}}\downarrow & & \downarrow{\scriptstyle\alpha_{i_0+1}} \\
\mathrm{fib}(\beta) & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1}
\end{array}
$$

Writing $\tilde{X}_{i_0} = \mathrm{fib}(\beta)$, we get the pasting of pullout diagrams

$$
\begin{array}{ccccc}
X_{i_0-1} & \xrightarrow{\tilde{\alpha}_{i_0}} & \tilde{X}_{i_0} & \xrightarrow{\tilde{\alpha}_{i_0+1}} & X_{i_0+1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} \\
0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) & \xrightarrow{\iota_1} & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \\
& & \downarrow & & \downarrow{\scriptstyle\pi_2} \\
& & 0 & \longrightarrow & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).
\end{array}
$$

That is, by considering the factorization $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$ we have switched the cofibers with respect to the original factorization $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$. Therefore, writing $\tilde{\phi}_{i_0} = \phi_{i_0+1}$ and $\tilde{\phi}_{i_0+1} = \phi_{i_0}$, we now have $f(\tilde{\phi}_{i_0}) \in U$ and $f(\tilde{\phi}_{i_0+1}) \in L$. That is, we have removed the $(\ldots, L, U, \ldots)$ situation from the position i_0, replacing it with a $(\ldots, U, L, \ldots)$ situation, while keeping all the labels L, U before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the $(\ldots, L, U, \ldots)$ situations.[2] $\qquad\square$

Remark 0.4. It follows from the proof of Lemma 0.3 that the objects X_L and X_U in the fiber sequence $X_U \to X \to X_L$ associated with the t-struture $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$ on $\mathscr{D}$ satisfy

$$X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in L \text{ and } \mathcal{H}_{\mathfrak{t}}^{\phi}(X) \neq 0 \rangle$$
$$X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in U \text{ and } \mathcal{H}_{\mathfrak{t}}^{\phi}(X) \neq 0 \rangle$$

Lemma 0.5. *For any $j \in J'$ we have*

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}.$$

Proof. Clearly, $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}$, therefore we only need to prove the converse inclusion. Let $\in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}$. Then in particular $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \geq j}$ and so there exists a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) \in \mathscr{D}_{\phi_i}$ with $f(\phi_i) \geq i$ for all $i = 1, \cdots, n$. As $((-\infty, j], (j, +\infty))$ is a slicing of J', reasoning as in the proof of Lemma 0.3 we can arrange this factorization is such a way that $f(\phi_i) > j$ for $i \leq \bar{\imath}$ and $f(\phi_i) = j$ for $i > \bar{\imath}$. Therefore, again by reasoning as in the proof of Lemma 0.3 we get a fiber sequence of the form

$$\begin{array}{ccc} X_{>j} & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X_j \end{array}$$

with $X_{>j}$ in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j}$ and X_j in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$. This is in particular a fiber sequence of the form $X_{>j} \to X \to X_{\leq j}$, with $X_{\leq j} \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j}$. As $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j})$ is a t-structure on $\mathscr{D}$ by Lemma 0.3, there is (up to equivalence) only one such fiber sequence. And since $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \leq j}$, this is the sequence $0 \to X \xrightarrow{\mathrm{id}} X$. Therefore $X = X_j$. $\qquad\square$

Proposition 0.6. *Let $f \colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. The map*

$$f_! \mathfrak{t} \colon \mathcal{O}(J') \to \mathrm{ts}(\mathscr{D})$$
$$(L, U) \mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi) \in U})$$

---

[2]This is somehow reminiscent of the 'bubble sort' algorithm.

defined by Lemma 0.3 is a Bridgeland J'-slicing of $\mathscr{D}$, with slices given by

$$\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}.$$

Proof. The map $f_!\mathfrak{t}$ is manifestly monotone and $\mathbb{Z}$-equivariant (see the first part of the proof of Lemma 0.3), so it is a J' slicing of $\mathscr{D}$, and its slices are given by $\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$ by Lemma 0.5. We are therefore left with showing that it is finite and discrete. Given an object X in $\mathscr{D}$, let now $\{\phi_1,\ldots,\phi_n\}$ be the indices in J such that $\mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \neq 0$ and let $\{j_1,\ldots,j_k\}$ the image of the set $\{\phi_1,\ldots,\phi_n\}$ via f. Up to renaming, we can assume $j_1 > j_2 > \cdots > j_k$. Consider now the factorization

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} X_{k-1} \xrightarrow{\alpha_k} X_k = X$$

of the initial morphism of X associated to the decreasing sequence $j_1 > j_2 > \cdots > j_k$ by the J'-slicing $f_!\mathfrak{t}$. The cofibers of the morphisms α_i are the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X)$ and, by Remark 0.4, we have

$$
\begin{aligned}
\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X) \in &\langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in (j_{i+1},j_i] \text{ and } \mathcal{H}_{\mathfrak{t}}^{\phi}(X) \neq 0 \rangle \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1,\ldots,\phi_n\} \text{ and } f(\phi) \in (j_{i+1},j_i] \rangle \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1,\ldots,\phi_n\} \text{ and } f(\phi) = j_i \rangle \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) = j_i \rangle = \mathscr{D}_{f_!\mathfrak{t};j_i}.
\end{aligned}
$$

Therefore

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i}\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i}(X).$$

This tells us that, if all the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ vanish, then also all the $\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X)$ vanish, and so $X = 0$. Finally, if $\tilde{j} \notin \{j_1,\ldots,j_k\}$, then there exists an index i such that $j_{i+1} < \tilde{j} < j_i$ and so $(j_{i+1},\tilde{j}] \cap \{j_1,\ldots,j_k\} = \emptyset$. The above argument then shows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1,\ldots,\phi_n\} \text{ and } f(\phi) \in (j_{i+1},\tilde{j}] \rangle = \{\mathbf{0}\}.$$

It follows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(0) = 0,$$

for every $\tilde{j} \notin \{j_1,\ldots,j_k\}$. So in particular, for any $X \in \mathscr{D}$, the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ are possibly nonzero only for finitely many indices j. $\qquad\square$

Let now J_1 and J_2 be two $\mathbb{Z}$-tosets, and let $J_1 \times_{\mathrm{lex}} J_2$ and $J_2 \times_{\mathrm{lex}} J_1$ the two $\mathbb{Z}$-tosets obtained by considering the lexicogaphic order on the products $J_1 \times J_2$ and $J_2 \times J_1$, respectively, and the diagonal $\mathbb{Z}$ action. The following lemma is immediate.

Lemma 0.7. *The exchange map*

$$e\colon J_1 \times J_2 \to J_2 \times J_1$$
$$(j_1, j_2) \mapsto (j_2, j_1)$$

is $\mathbb{Z}$-equivariant.

Definition 0.8. *Let J_1 and J_2 be $\mathbb{Z}$-tosets. A Bridgeland $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ of $\mathscr{D}$ is said to be* gluable *if it is e-compatible, where $e\colon J_1 \times J_2 \to J_2 \times J_1$ is the exchange map.*

Example 0.9. Let $J_1 = \{0,1\}$ with the trivial $\mathbb{Z}$-action and let $J_2 = \mathbb{Z}$ with the standard translation $\mathbb{Z}$-action. Then a $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ on $\mathscr{D}$ is the datum of a semiorthogonal decomposition $(\mathscr{D}_0, \mathscr{D}_1)$ of $\mathscr{D}$ toghether with t-structures $\mathfrak{t}_i$ on $\mathscr{D}_i$. Spelling out the definiton, one sees that the $\{0,1\} \times_{\mathrm{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$ is gluable if and only if $\mathscr{D}_{0;\geq 0} \boxdot \mathscr{D}_{1;-1}$ and $\mathscr{D}_{0;\geq 0} \boxdot \mathscr{D}_{1;0}$. When this happens, the glued slicing $e_!\mathfrak{t}$ is a $\mathbb{Z} \times_{\mathrm{lex}} \{0,1\}$ slicing on $\mathscr{D}$, i.e., is the datum of a bounded t-structure on $\mathscr{D}$ together with a torsion theory on its heart. More precisely, the heart of $e_!\mathfrak{t}$ is the full ∞-subcategory $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ of $\mathscr{D}$ on thise objects X that fall into fiber sequences of the form $X_1 \to X \to X_0$ with $X_i \in \mathscr{D}_i^{\heartsuit}$, while the torsion theory on $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ is $(\mathscr{D}_0^{\heartsuit}, \mathscr{D}_1^{\heartsuit})$, i.e., precisely the pair of hearts of the two subcategories in the semiorthogonal decomposition.

-fino a qui- Quanto segue va rivisto alla luce della chiacchierata su wire: dobbiamo introdurre la nozione di stong semiorthogonal decomposition e notare che nel caso in cui abbiamo strong e t-strutture scambiabili allora abbiamo due modi di incollare e questi coincidono. We now recall the celebrated process from BBD of gluing two t-structures into a third one. This will turn out to induce a gluable slicing in our sense, thus showing that the language of the present paper is actually a generalization of a classical piece of literature. This, by the other hand, explains both our terminology and motivation.

Definition 0.10. *A* recollement datum *is a diagram of stable ∞-categories and exact ∞-functors*

such that the following holds:

1. *$(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples,*

2. *$i_*, j_*, j_!$ are fully faithful and $j^* i_* = 0$,*

3. *for each object X of $\mathscr{D}$ the (co)units of the above adjunctions give rise to cofiber sequences*

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$$
\begin{CD}
i_* i^! X @>>> X \\
@VVV @VVV \\
0 @>>> j_* j^* X
\end{CD}
\qquad\qquad
\begin{CD}
j_! j^* X @>>> X \\
@VVV @VVV \\
0 @>>> i_* i^* X.
\end{CD}
$$

Proposition 0.11. *Suppose we are in the situation of 0.10. Then:*

1. *$(\mathscr{D}_0 = j_* \mathscr{D}_{\mathrm{ouv}}, \mathscr{D}_1 = i_* \mathscr{D}_{\mathrm{fer}})$ is a semiorthogonal decomposition on $\mathscr{D}$.*

2. *Given t-structures $\mathfrak{t}_0, \mathfrak{t}_1$ on $\mathscr{D}_{\mathrm{ouv}}$ and $\mathscr{D}_{\mathrm{fer}}$ respectively, the associated $\{0,1\} \times_{\mathrm{lex}} \mathbb{Z}$-slicing on $\mathscr{D}$ given by*

$$
\mathscr{D}_{0;\geq 0} = j_*(\mathscr{D}_{\mathrm{ouv};\geq 0}), \quad \mathscr{D}_{1;\geq 0} = i_*(\mathscr{D}_{\mathrm{fer};\geq 0})
$$

   is gluable.

Proof. Let's prove the first claim. The fact that j^* and j_* are adjoints implies that, for objects $i_* X \in \mathscr{D}_{\mathrm{fer}}$ and $j_* Y \in \mathscr{D}_{\mathrm{ouv}}$,

$$
\mathscr{D}(i_* X, j_* Y) = \mathscr{D}(j^* i_* X, Y)
$$

and the latter vanishes since $j^* i_* = 0$ by definition. This shows that $\mathscr{D}_1 \boxslash \mathscr{D}_0$. Considering the left cofiber sequence from Definition 0.10 as a Postnikov tower of length one we see that $(\mathscr{D}_0, \mathscr{D}_1)$ is a $\{0,1\}$-slicing on $\mathscr{D}$, as desired.

$\square$

—-FINO A QUI 1.5 —-

l'osservazione che segue dovrà trovare la sua collocazione

Remark 0.12. To motivate the next section, consider the following isomorphism fo $\mathbb{Z}$-tosets:

$$
\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z} \to \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{free}}
$$
$$
(n, m) \mapsto (n, m - n)
$$

... continua ...

Definition 0.13. *Let $U \colon J \to \mathcal{O}(J')$ be a map of sets. We denote by Γ_U the subset of $J \times J'$ defined by*

$$
\Gamma_U = \{(j, j') \in J \times J'; \quad j' \in U_j\}.
$$

For $f \colon J \to J'$ a map of sets, we denote by $(< f, \geq f) \colon J \to \mathcal{O}(J')$ the composition of f with the map

$$
J' \to \mathcal{O}(J')
$$
$$
j' \mapsto ((-\infty, j'), [j', +\infty)).
$$

The upper graphic of f is the subset $\Gamma_{\geq f}$ of $J \times J'$.

Lemma 0.14. *Let $U\colon J \to \mathcal{O}(J')$ be a map of sets. Then Γ_U is an upper set of $J \times J'$ with the product order if and only if U is a map of posets $U\colon J \to \mathcal{O}(J')^{\mathrm{op}}$. In particular, $\Gamma_{\geq f}$ is an upper set if and only if $f\colon J \to J'$ is a nondecreasing map, i.e., a map of posets $J \to J'^{\mathrm{op}}$.*

Proof. Assume (L, U) is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Pick $(j, j') \in \Gamma_U$ and suppose $(j, j') \leq (k, k')$ in $J \times J'$. As (L, U) i monotone and $k \geq j$, we have $U_j \subseteq U_k$; as $(j, j') \in \Gamma_U$, we have $j' \in U_j$. Therefore $j \in U_k$. Since U_k is an upper set and $k' \geq j'$, we get $k' \in U_k$, i.e. $(k, k') \in \Gamma_U$. Vice versa, assume Γ_U is an upper set, and let $j \leq k$ in J. For any $j' \in U_j$, the element (k, j') in $J \times J'$ satisfies $(k, j') \geq (j, j')$ in the product order and so $(k, j') \in \Gamma_U$. This means $j' \in U_k$ and so $U_j \subseteq U_k$. To prove the second part of the statement, notice that the map $j \mapsto ((-\infty, j'), [j', +\infty))$ is a map of posets from $J' \to \mathcal{O}(J')$. Therefore, if $f\colon J \to J'^{\mathrm{op}}$ is a map of posets, then also $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets. Vice versa, if $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets then for every $j \leq k$ in J we have $[f(j), +\infty) \subseteq [f(k), +\infty)$ and so $f(k) \leq f(j)$. $\square$

Proposition 0.15. *The map*

$$\Gamma\colon \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{O}(J \times J')$$
$$U \mapsto \Gamma_U$$

is an isomorphism of posets.

Proof. Let $U_1 \leq U_2$ in the partial order on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$, and let $(j, j') \in \Gamma_{U_2}$. Then $j' \in U_{2;k} \subseteq U_{1,j}$ and so $(j, j') \in \Gamma_{U_1}$. This means $\Gamma_{U_1} \leq \Gamma_{U_2}$ in $\mathcal{O}(J \times J')$. Next, for any morphism of posets $U\colon J \to \mathcal{O}(J)^{\mathrm{op}}$ we have that $(j, j') \in \Gamma_{U+1}$ if and only if $j' \in (U + 1)_j$ Pick and upper set $\tilde{U}$ of $J \times J'$ and set, for $j \in J$

$$U_{\tilde{U}}(j) = \{j' \in J'; \quad (j, j') \in \tilde{U}\}.$$

The subset $U_{\tilde{U}}(j) \subseteq J'$ is an upper set. Indeed, if $j' \in U_{\tilde{U}}(j)$ and $k' \geq j'$ in J', then $(j, k') \geq (j, j')$ in the product order on $J \times J'$ and so $(j, k') \in \tilde{U}$ as $\tilde{U}$ is an upper set. Next, if $j \leq k$ in J and $j' \in U_{\tilde{U}}(j)$, then $(k, j') \in \tilde{U}$ as $\tilde{U}$ is an upper set, and so $j' \in U_{\tilde{U}}(k)$. This shows that $U_{\tilde{U}}(j) \subseteq U_{\tilde{U}}(k)$, and so $U_{\tilde{U}}$ is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Moreover, if $\tilde{U}_1 \leq \tilde{U}_2$ in $\mathcal{O}(J \times J')$ then $U_{\tilde{U}_1} \leq U_{\tilde{U}_2}$ in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. Therefore we have a map

$$\gamma\colon \mathcal{O}(J \times J') \to \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$$
$$\tilde{U} \mapsto U_{\tilde{U}}.$$

which is straightforward to see to be the inverse of Γ. $\square$

Assume J and J' are $\mathbb{Z}$-posets. Then $\mathcal{O}(J)^{\mathrm{op}}$ is a $\mathbb{Z}$-poset with 1 acting as $U \mapsto U - 1$. The action of $\mathbb{Z}$ by conjugation on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$ is therefore given by $(U \dotplus 1)_j = U_{j-1} - 1$. To see that $U \dotplus 1$ is still a morphism of posets from J to $\mathcal{O}(J')$, let $j \leq k$ in J. Then $(U \dotplus 1)_j = U_{j-1} - 1 \subseteq U_{k-1} - 1 = (U \dotplus 1)_k$. The conjugation action, however, does not define a structure of $\mathbb{Z}$-poset on

$\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, as it is generally not true that $U \leq U \dotplus 1$. This condition is indeed equivalent to $U_j \subseteq (U \dotplus 1)_j$, i.e., to $U_j + 1 \subseteq U_{j-1}$ for any j in J, and this is not necessarily satisfied by a morphism of posets $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$.

Definition 0.16. *A* perversity *is a map of posets $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ such that*

$$U_j + 1 \subseteq U_{j-1}$$

for any j in J. The set $\mathrm{Perv}(J, J')$ *of all perversities from J to $\mathcal{O}(J')$ inherits a poset structure from the inclusion in* $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. *It is a $\mathbb{Z}$-poset with 1 acting as $U \mapsto U \dotplus (-1)$.*

Remark 0.17. Notice that the shift action on perversities is given by $U \mapsto U \dotplus (-1)$. Namely, by construction the $\mathbb{Z}$-action $U \mapsto U \dotplus 1$ is monotone on the set of perversities with the order induced by the inclusion in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, and the order on $\mathrm{Perv}(J, J')$ is the opposite one. The reason for considering this order is, clearly, Proposition 0.15.

Remark 0.18. We have seen in Lemma 0.14 that a function $f \colon J \to J'$ defines a morphism of posets by $(\geq f) \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ if and only if f is a morphism of posets from J to J'^{op}. When this happens, $(\geq f)$ is a perversity if and only if $f(j-1) \leq f(j) + 1$, for every $j \in J$. In the particular case $J = \mathbb{Z}$, assuming as usual that J' is a $\mathbb{Z}$-poset, we can define a new function $p_f \colon \mathbb{Z} \to J'$ as $p_f(n) = f(n) + n$. Then the condition $f(n) \leq f(n+1) + 1$ translates into $p_f(n) \leq p_f(n+1)$, while the condition that $f \colon \mathbb{Z} \to J'^{\mathrm{op}}$ is a morphism of posets, i.e., $f(n+1) \leq f(n)$ translates to $p_f(n+1) \leq p_f(n) + 1$. As $f \mapsto p_f$ is a bijection of the set of maps from J to J' into itself, we see that the functions $f \colon \mathbb{Z} \to J'$ defining perversities correspond bijectively to the set of functions $p \colon \mathbb{Z} \to J'$ such that

$$p(n) \leq p(n+1) \leq p(n) + 1,$$

for every $n \in \mathbb{Z}$. In case $J' = \mathbb{Z}$, this precisely reproduces the definition of perversity function as given in ???.

Definition 0.19. *An upper set U in $\mathcal{O}(J \times J')$ is called a* kinky upperset *if $U \leq U +_{\mathrm{ne}} 1$, where* ne *is the "northestern" action of $\mathbb{Z}$ on $J \times J'$ given by $(j, j') +_{\mathrm{ne}} 1 = (j-1, j'+1)$. We denote by* $\mathrm{Kink}(J \times J')$ *the poset of kinky uppersets of $J \times J'$, with the poset structure induced by the inclusion in $\mathcal{O}(J \times J')$. it is a $\mathbb{Z}$-poset with the "northestern" action.*

Lemma 0.20. *The map Γ from Proposition 0.15 induces an isomorphism of $\mathbb{Z}$-posets*

$$\Gamma \colon \mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J').$$

Proof. Let $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ be a perversity, and let $(j, j') \in \Gamma_U$. Then $j' \in U_j$ and so, by definition of perversity, $j' + 1 \in U_{j-1}$. Therefore $(j-1, j'+1) \in \Gamma_U$, i.e., $\Gamma_U \leq \Gamma_U +_{\mathrm{ne}} 1$. Vice versa, if $\tilde{U}$ is a kinky upperset in $J \times J'$, let $U_{\tilde{U}}$ the preimage in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$ of $\tilde{U}$ via Γ (see the proof of Proposition 0.15). Then for any $j \in J$ and any $j' \in U_{\tilde{U};j}$ we have $j' + 1 \in U_{\tilde{U};j-1}$ and so

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$U_{\tilde U;j}+1 \subseteq U_{\tilde U;j-1}$. So the isomorphism of posets $\mathrm{Pos}(J,\mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{O}(J \times J')$ restricts to an isomorphism of posets $\mathrm{Perv}(J,J') \to \mathrm{Kink}(J \times J')$. To see that $\Gamma \colon \mathrm{Perv}(J,J') \to \mathrm{Kink}(J \times J')$ is also $\mathbb{Z}$-equivariant, notice that, for every perversity U we have $(j,j') \in \Gamma_{U \dot{+}(-1)}$ if and only if $j' \in U_{j+1}+1$. i.e., if and only if $(j+1,j'-1) \in \Gamma_U$. This latter condition is equivalent to $(j,j') \in \Gamma_U +_{\mathrm{nw}} 1$, so we find $\Gamma_{U \dot{+}(-1)} = \Gamma_U +_{\mathrm{nw}} 1$. $\qquad\square$

fino a qui

Remark 0.21. We can give $\Theta(J)$ the structure of a $\mathbb{Z}$-poset via the incluion $\Theta(J) \subseteq \mathrm{Pos}(\mathbb{Z},\mathcal{O}(J)^{\mathrm{op}})$. In general, it will not be totally ordered. However, by taking joins and meets pointwise $\Theta(J)$ becomes a distributive lattice.

Remark 0.22. Considering the obvious isomoprhism $\mathbb{Z} \simeq \mathcal{O}(\mathbb{Z})^{\mathrm{op}}$ a perversity over $\mathbb{Z}$ is just a *monotone and comonotope perversity* in the sense of referenza, tipo bezrukavnikov o BBD.

Definition 0.23. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. An abelian $\mathbb{Z}$-slicing $\mathscr{P}$ on $\heartsuit_{\mathfrak{t}}$ is called:*

- *perverse filtration if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n < \phi - \psi$*

- *grading filtration if it is a perverse filtration and $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $2 \leq n = \phi - \psi$*

- *mixed filtration if $\mathscr{P}_\psi[n] \subseteq \mathscr{P}_\phi^\perp$ for $n > \psi - \phi$*

The following implications hold:

$$\mathrm{mixed} \implies \mathrm{grading} \implies \mathrm{perverse}$$

proof that mixed implies grading: suppose $n < \phi - \psi$. By definition of slicing, the vanishing is automatic for $n \leq 0$ and we can thus assume $n > 0$. But then $n > 0 > -n > \psi - \phi$ and the required vanishing now follows from the definition of mixed filtration. The same argument also works for $2 \leq n = \phi - \psi$.
Clearly, a torsion pair on $\heartsuit_{\mathfrak{t}}$ (seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[1] \subseteq \mathbb{Z}$) is always a grading filtration (proof: it follows from the fact that $\mathscr{P}_\phi[n] = 0$ for $\phi \neq 0,1$ and thus the only nontrivial case is $n = 0, \phi = 1, \psi = 0$. The Hom-vanishing then follows from the definition of torsion pair). Historically, grading filtrations first appeared in [**?**] under the name of 'radical filtrations', while mixed filtrations are covered in the last section of [**?**].

Definition 0.24. *A map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is called **perversity** if it is a monotone contraction, i.e. if*
$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi$$
whenever $\phi \geq \psi$ in $\mathbb{Z}$.
We denote Ξ the set of perversities.

Clearly, Ξ definies a $\mathbb{Z}$-poset which is not totally ordered. However, it is a distributive lattice: we have meets and joins given by:
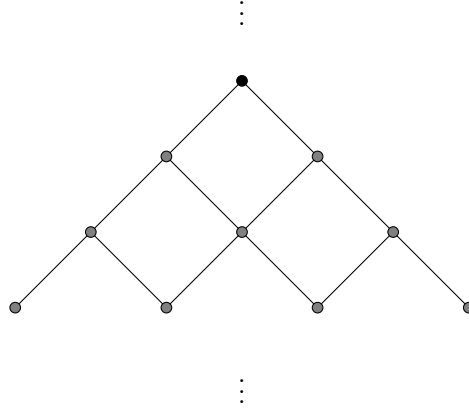
$$(p \wedge q)(\phi) = \min\{p(\phi), q(\phi)\}$$

$$(p \vee q)(\phi) = \max\{p(\phi), q(\phi)\}$$

Indeed, if we place on $\mathbb{Z} \times \mathbb{Z}$ the product order, we have an isomorphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} = O(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

given by sending the zero perversity to $A = \{(\phi, \psi)\}_{\phi \geq 0}$ and the identity to $B = \{(\phi, \psi)\}_{\psi \geq 0}$ ($B + i$ and $A + i$ with $i \in \mathbb{Z}$ generate the latter as a complete lattice). Considering the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Z} \subseteq O(\mathbb{Z} \times \mathbb{Z})$ we get a sublattice of Ξ depicted as

This induces the 't-tree' described in [**?**], as discussed below in a far-reaching generality. By the other hand, each square above contains moreover a copy of the Young lattice

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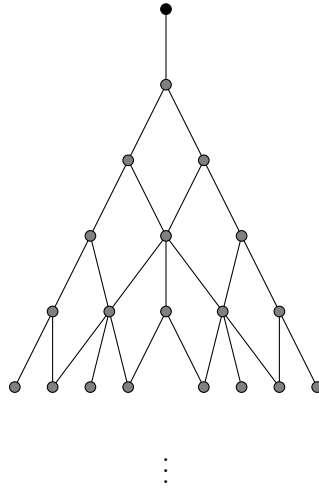
with the root corresponding to the upper vertex of the square.

Proposition 0.25. *Let* $\mathfrak{t}$ *be a bounded t-structure on* $\mathscr{D}$, $\mathscr{P}$ *an abelian* $\mathbb{Z}$*-slicing on* $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ *the associated* $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$*-slicing on* $\mathscr{D}$, p *a perversity. We have:*

1. *if* $\mathscr{P}$ *is a perverse filtration, then* $\mathscr{Q}$ *satisfies the assumptions of* **Proposition ??** *with respect to the map* $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{f_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ *given by*

$$f_p(n, \phi) = (n + p(\lfloor \phi/2 \rfloor), -p(\lfloor \phi/2 \rfloor))$$

2. *if* $\mathscr{P}$ *is a grading filtration, then* $\mathscr{Q}$ *satisfies the assumptions of* **Proposition ??** *with respect to the map* $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{g_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ *given by*

$$g_p(n, \phi) = (n + p(\phi), -p(\phi))$$

3. *if* $\mathscr{P}$ *is a mixed filtration, then the abelian* $\mathbb{Z}$*-slicing induced on the heart of the bounded t-structure associated to* $(g_p)_{\underline{\Omega}}(\mathscr{Q})$ *is split*

Proof. Let's prove *(2)*. Suppose $g_p(n, \phi) > g_p(m, \psi)$. Then either $m - n < p(\phi) - p(\psi)$ or both $m - n = p(\phi) - p(\psi)$ and $p(\psi) < p(\psi)$. Since by definition of t-structure we can assume $m \geq n$, the second case is absurd while in the first case, since p is monotone, we have $\phi \geq \psi$ and thus by definition of perversity

$$m - n < p(\phi) - p(\psi) \leq \phi - \psi$$

and we get the desired Hom-vanishing by definition of grading filtration.

Suppose now $f_p(n, \phi) + 1 > f_p(m, \psi)$ and $(n + 1, \phi) < (m, \psi)$. The only non absurd case is $1 < m - n \leq p(\phi) - p(\psi)$. But then again we have

$$2 \leq m - n \leq p(\phi) - p(\psi) \leq \phi - \psi$$

and we can conclude as above.

To prove *(1)* consider the monotone map

$$\mathbb{Z} \xrightarrow{\lfloor */2 \rfloor} \mathbb{Z}$$

Applying the slice functor to the latter and starting with a perverse filtration, we get a grading filtration by the properties of the floor function and the thesis follows from part *(2)*.

Let's prove *(3)*. We have to show that

$$\mathscr{P}_{\psi}[1 - p(\psi)] \subseteq \mathscr{P}_{\phi}[p(\phi)]^{\perp}$$

for $-p(\psi) > -p(\phi)$. We denote $n = p(\phi) - p(\psi) - 1$. Assuming again $n \geq 0$, we have $n \geq 0 > p(\psi) - p(\phi) \geq \psi - \phi$ and we can then conclude by definition of mixed filtration. $\square$

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity p the bounded t-structure coming from $(g_p)_\Omega(\mathscr{Q})$ defines a morphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} \longrightarrow \mathfrak{bts}(\mathscr{D})$$

we can restate this as:

♩ *A grading or perverse filtration on the heart of a bounded t-structure on $\mathscr{D}$ induces a presheaf of t-structures on $\mathbb{Z} \times \mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z} \times \mathbb{Z}$-slicing' on $\mathscr{D}$.*
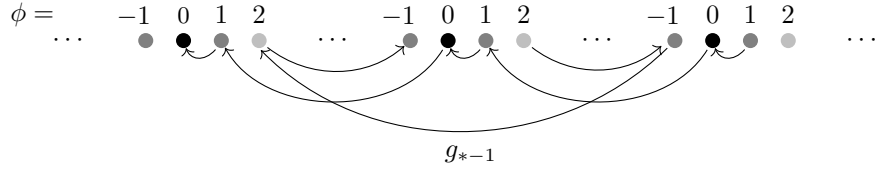
Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\mathrm{op}} \hookrightarrow \Xi$$

and the t-structure coming from $(g_{\chi_I})_\Omega(\mathscr{Q})$ is just the tilting of $\mathfrak{t}$ with respect to the torsion pair coming from I.

The following proposition gives a characterization of the new heart obtained by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 0.26. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, $\mathscr{P}$ a grading filtration on $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, p a perversity. Denote $\mathfrak{q}$ the bounded t-structure associated to $(g_p)_\Omega(\mathscr{Q})$. Then $\heartsuit_{\mathfrak{q}}$ consists of objects $X \in \mathscr{D}$ so that*

$$H_{\mathfrak{t}}^k(X) \in \mathscr{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if $\mathscr{P}$ is a mixed filtration then for each $X \in \heartsuit_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \heartsuit_{\mathfrak{q}}$ if and only if

$$H_{\mathscr{P}}^\phi(H_{\mathfrak{t}}^k(X)[-k])[k] = H_{\mathscr{Q}}^{(k,\phi)}(X) = 0$$

for $p(\phi) \neq -k$.

For the second part of the claim, the abelian $\mathbb{Z}$-slicing induced on $\heartsuit_{\mathfrak{q}}$ is split by **Proposition 0.25** and thus by **Proposition ??**

$$X = \bigoplus_{(k,\phi)} H_{\mathscr{Q}}^{(k,\phi)}(X) = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

where the last equality comes from the first part. $\qquad\qquad\qquad\square$

Example 0.27. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an $\mathbb{N}$-graded ring with B_0 semisimple. Denote $\mathscr{A}$ the category of $\mathbb{Z}$-graded B-modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote $\mathscr{P}_\phi$ the full subcategory of $\mathscr{A}$ of modules concentrated in degree ϕ. Clearly, $\mathscr{P}$ defines an abelian $\mathbb{Z}$-slicing on $\mathscr{A}$. Following [**?**] we have

$$\mathrm{Ext}^n_{\mathscr{A}}(\mathscr{P}_\phi, \mathscr{P}_\psi) = 0$$

for $n > \psi - \phi$. This means that $\mathscr{P}$ is a mixed filtration and the bounded t-structure on $\mathscr{D}^b(\mathscr{A})$ associated to $(g_1)_{\mathfrak{Q}}(\mathscr{Q})$ (where 1 is the identity of $\mathbb{Z}$) is the 'diagonal' (or 'geometric') t-structure which appears in Koszul duality and other areas.

Example 0.28. Let M be an n-dimensional smooth complex projective variety and consider the n-torsion pair $\mathscr{P}$ on $\mathrm{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that $\mathscr{P}$, seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t-structure associated to $(f_p)_{\mathfrak{Q}}(\mathscr{Q})$ is the one of perverse coherent sheaves as constructed in [**?**]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n-slicing on the heart of perverse coherent sheaves as done in [**?**].

Now we somehow review the gluing construction for t-structures in [**?**], but generalize it to any slicing. Our language is quite different though, and we formulate the problem very similarly to [**?**]. We start with two $\mathbb{Z}$-posets J, J'.

Definition 0.29. *Let $\mathscr{P}$ be a $J \ltimes J'$-slicing on $\mathscr{D}$. We call $\mathscr{P}$ **gluable** if it satisties the assumptions of **Proposition ??** with respect to the map $J \ltimes J' \xrightarrow{e} J' \ltimes J$ that exchanges coordinates. In this case we denote*

$$\overline{\mathscr{P}} = e_{\mathfrak{Q}}(\mathscr{P})$$

By a simple computation, the gluability condition reads: $\mathscr{P}_{(\phi,\psi)} \subseteq \mathscr{P}^\perp_{(\phi',\psi')}$ if $\psi' > \psi$ or both $\psi' + 1 > \psi$ and $\phi' + 1 < \phi$. For example, the first condition is automatic when $\mathbb{Z}$ acts trivially on J (in this case, it follows from the second one). In particular, if $\mathbb{Z}$ acts trivially on J, $\mathscr{P}$ is a J-slicing on $\mathscr{D}$ and $\mathfrak{t}_\phi$ is a bounded t-structure on $\mathscr{P}_\phi$ for each $\phi \in J$, then using **Proposition ??** we get a $J \ltimes \mathbb{Z}$-slicing $\mathscr{Q}$ which is gluable if and only if

$$\heartsuit_{\mathfrak{t}_\psi}[n] \subseteq \heartsuit^\perp_{\mathfrak{t}_\phi}$$

whenever both $n \leq 0$ and $\phi < \psi$.

Remark 0.30. We have a commutative diagram of $\mathbb{Z}$-posets

$$\begin{array}{ccc} \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle g} \\ \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \end{array}$$

where the horizontal isomorphism is the one from **Remark ??**. Indeed, a $\mathbb{Z} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$ is gluable if and only if it induces a grading filtration on the heart of the associated t-structure when seen as a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing.

An easy combinatorial calculation finally yelds the following two remarks:

- if $\mathscr{P}$ is a gluable $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$-slicing on $\mathscr{D}$, then $\overline{\mathscr{P}}$ induces a grading filtration on the associated heart, allowing us again to construct new bounded t-structures depending on a perversity.

- if $\mathscr{P}$ is a mixed filtration on the heart of a bounded t-structure and $\mathscr{Q}$ is the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, then $\mathscr{Q}$ is gluable. In this case, looking at $\overline{\mathscr{Q}}$, we get a baric structure on $\mathscr{D}$ which is usually called **weight decomposition** in literature.

In other words, the chain of implications for a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing refines to

$$\text{mixed} \implies \text{gluable} \implies \text{grading} \implies \text{perverse}$$

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