

0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of \mathbb{Z} -posets (but just a \mathbb{Z} -equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a \mathbb{Z} -poset, i.e., a partially ordered set together with a monotone action of \mathbb{Z} , that we will denote by $(n, x) \mapsto x + n$. Recall that a J -slicing on a stable ∞ -category \mathcal{D} is a morphism of \mathbb{Z} -posets $\mathfrak{t}: \mathcal{O}(J) \rightarrow \text{ts}(\mathcal{D})$, where $\mathcal{O}(J)$ is the \mathbb{Z} -poset of the upper sets of J and $\text{ts}(\mathcal{D})$ is the \mathbb{Z} -poset of t -structures on \mathcal{D} . Clearly, if $f: J \rightarrow J'$ is a morphism of \mathbb{Z} -posets, then $f^{-1}: \{\text{subsets of } J'\} \rightarrow \{\text{subsets of } J\}$ induces a \mathbb{Z} -equivariant morphism of \mathbb{Z} -posets $f^{-1}: \mathcal{O}(J') \rightarrow \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$f_*: J\text{-slicings on } \mathcal{D} \rightarrow J'\text{-slicings on } \mathcal{D}$$

$$\mathfrak{t} \mapsto \mathfrak{t} \circ f^{-1}.$$

But actually there is no need for f to be a morphism of posets in order to define a morphism $\mathcal{O}(J') \rightarrow \mathcal{O}(J)$: in the general case, just consider the map f^{-1} defined by

$$\overline{f^{-1}(U)} = \overline{f^{-1}(U)},$$

where \overline{S} denotes the upperset closure of the subset S of J , i.e., the smallest upperset of J containing S (in other words \overline{S} is the intersection of all the uppersets of J containing S).

Lemma 0.1. *Let J and J' be \mathbb{Z} -posets, and let $f: J \rightarrow J'$ a map of \mathbb{Z} -sets (i.e., a \mathbb{Z} -equivariant map, not necessarily nondecreasing). The map $f^{-1}: \mathcal{O}(J') \rightarrow \mathcal{O}(J)$ is a morphism of \mathbb{Z} -posets.*

Proof. Let U_1 and U_2 two posets in J' with $U_1 \leq U_2$, i.e., with $U_2 \subseteq U_1$. Then, $f^{-1}(U_2) \subseteq f^{-1}(U_1)$ and so, by definition of closure, $\overline{f^{-1}(U_2)} \subseteq \overline{f^{-1}(U_1)}$, i.e., $\overline{f^{-1}(U_1)} \subseteq \overline{f^{-1}(U_2)}$. To see that f^{-1} is \mathbb{Z} -equivariant, notice that for any subset S of J one has $S + 1 \subseteq \overline{S} + 1$ and so $\overline{S + 1} \subseteq \overline{S} + 1$, as $\overline{S} + 1$ is an upperset and therefore closed. On the other hand, $S + 1 \subseteq \overline{S + 1}$ and so $S \subseteq \overline{S + 1} - 1$ which implies $\overline{S} \subseteq \overline{S + 1} - 1$, i.e., $\overline{S} + 1 \subseteq \overline{S + 1}$. Therefore, for any upperset U in J' one has

$$\overline{f^{-1}(U + 1)} = \overline{f^{-1}(U + 1)} = \overline{f^{-1}(U)} + 1 = \overline{f^{-1}(U)} + 1 = \overline{f^{-1}(U)} + 1,$$

by the \mathbb{Z} -equivariance of f . \square

Proposition 0.2. *Let $J \xrightarrow{f} J'$ be a \mathbb{Z} -equivariant map (not necessarily increasing) between \mathbb{Z} -posets, \mathcal{P} a J -slicing on \mathcal{D} . Suppose that $\mathcal{P}_\psi \subseteq \mathcal{P}_\phi^\perp$ whenever one of the following conditions holds:*

- (a) $f(\phi) > f(\psi)$
- (b) $f(\phi) + 1 > f(\psi)$ and $\phi + 1 < \psi$

Then

$$(f_{\Omega}(\mathcal{P}))_{\phi} = \mathcal{P}_{f^{-1}(\phi)}$$

defines a J' -slicing on \mathcal{D} .

Proof. Part (1) of definition of slicing follows from \mathbb{Z} -equivariance of f , while part (2) follows from condition (a) above. Pick $0 \neq X \in \mathcal{D}$ and consider its Postnikov tower with respect to \mathcal{P} :

$$0 = Y_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} Y_n = X$$

with $\text{cone}(\beta_i) = H_{\mathcal{P}}^{\phi_i}(X)$. Going from left to right, if we encounter an i so that $f(\phi_{i+1}) > f(\phi_i)$ then by condition (b)

$$\text{Hom}_{\mathcal{D}}(H_{\mathcal{P}}^{\phi_{i+1}}(X), H_{\mathcal{P}}^{\phi_i}(X)[1]) = 0$$

Using **Proposition ??** and the 3×3 **Lemma**, we can complete the identity of Y_i to get a diagram:

$$\begin{array}{ccccccc} H_{\mathcal{P}}^{\phi_{i+1}}(X)[-1] & \longrightarrow & Y_i & \longrightarrow & Y_{i+1} & \longrightarrow & H_{\mathcal{P}}^{\phi_{i+1}}(X)[1] \\ \downarrow & & \downarrow 1 & & \downarrow & & \downarrow \\ Y_{i-1} & \longrightarrow & Y_i & \longrightarrow & H_{\mathcal{P}}^{\phi_i}(X) & \longrightarrow & Y_{i-1}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & 0 & \longrightarrow & A[1] & \longrightarrow & A[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{\mathcal{P}}^{\phi_{i+1}}(X) & \longrightarrow & Y_i[1] & \longrightarrow & Y_{i+1}[1] & \longrightarrow & H_{\mathcal{P}}^{\phi_{i+1}}(X)[1] \end{array}$$

We then replace β_i with $Y_{i-1} \rightarrow A$ and β_{i+1} with $A \rightarrow Y_{i+1}$. Iterating this process¹, we get a factorization

$$0 = Z_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_n} Z_n = X$$

with $\text{cone}(\gamma_i) = H_{\mathcal{P}}^{\phi_{k_i}}(X)$ for some k_i and $f(\phi_i) \geq f(\phi_{i+1})$ for all i . To get a factorization with the strict latter inequality we use the same argument as in the proof of **Proposition ??**. \square

The above proof also shows that, in the same hypotheses and notation, $(f_{\Omega}(\mathcal{P}))_{\phi}$ consists of objects $X \in \mathcal{D}$ so that $H_{\mathcal{P}}^{\psi}(X) = 0$ for $f(\psi) \neq \phi$. Also, **Proposition ??** holds when f is bijective.

Definition 0.3. Let \mathfrak{t} be a bounded t -structure on \mathcal{D} . An abelian \mathbb{Z} -slicing \mathcal{P} on $\heartsuit_{\mathfrak{t}}$ is called:

¹This is somehow reminiscent of the 'bubble sort' algorithm.

- **perverse filtration** if $\mathcal{P}_\psi[n] \subseteq \mathcal{P}_\phi^\perp$ for $n < \phi - \psi$
- **grading filtration** if it is a perverse filtration and $\mathcal{P}_\psi[n] \subseteq \mathcal{P}_\phi^\perp$ for $2 \leq n = \phi - \psi$
- **mixed filtration** if $\mathcal{P}_\psi[n] \subseteq \mathcal{P}_\phi^\perp$ for $n > \psi - \phi$

The following implications hold:

$$\text{mixed} \implies \text{grading} \implies \text{perverse}$$

Clearly, a torsion pair on \heartsuit_t (seen as an abelian \mathbb{Z} -slicing via the inclusion $[1] \subseteq \mathbb{Z}$) is always a grading filtration. Historically, grading filtrations first appeared in [?] under the name of 'radical filtrations', while mixed filtrations are covered in the last section of [?].

Definition 0.4. A map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is called **perversity** if it is a monotone contraction, i.e. if

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi$$

whenever $\phi \geq \psi$ in \mathbb{Z} .

We denote Ξ the set of perversities.

Clearly, Ξ defines a \mathbb{Z} -poset which is not totally ordered. However, it is a distributive lattice: we have meets and joins given by:

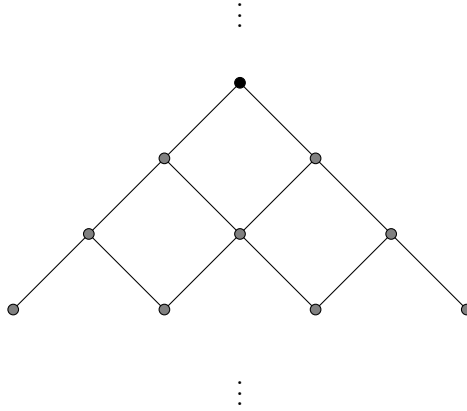
$$(p \wedge q)(\phi) = \min\{p(\phi), q(\phi)\}$$

$$(p \vee q)(\phi) = \max\{p(\phi), q(\phi)\}$$

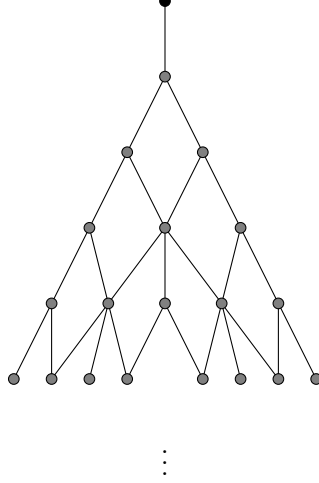
Indeed, if we place on $\mathbb{Z} \times \mathbb{Z}$ the product order, we have an isomorphism of \mathbb{Z} -posets

$$\Xi^{\text{op}} = O(\mathbb{Z} \times \mathbb{Z}) \setminus \{\emptyset, \mathbb{Z} \times \mathbb{Z}\}$$

given by sending the zero perversity to $A = \{(\phi, \psi)\}_{\phi \geq 0}$ and the identity to $B = \{(\phi, \psi)\}_{\psi \geq 0}$ ($B + i$ and $A + i$ with $i \in \mathbb{Z}$ generate the latter as a complete lattice). Considering the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Z} \subseteq O(\mathbb{Z} \times \mathbb{Z})$ we get a sublattice of Ξ depicted as



This induces the 't-tree' described in [?], as discussed below in a far-reaching generality. By the other hand, each square above contains moreover a copy of the Young lattice



with the root corresponding to the upper vertex of the square.

Proposition 0.5. *Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , \mathcal{P} an abelian \mathbb{Z} -slicing on $\heartsuit_{\mathfrak{t}}$, \mathcal{Q} the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , p a perversity. We have:*

1. *if \mathcal{P} is a perverse filtration, then \mathcal{Q} satisfies the assumptions of **Proposition 0.2** with respect to the map $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{f_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ given by*

$$f_p(n, \phi) = (n + p(\lfloor \phi/2 \rfloor), -p(\lfloor \phi/2 \rfloor))$$

2. *if \mathcal{P} is a grading filtration, then \mathcal{Q} satisfies the assumptions of **Proposition 0.2** with respect to the map $\mathbb{Z} \ltimes \hat{\mathbb{Z}} \xrightarrow{g_p} \mathbb{Z} \ltimes \hat{\mathbb{Z}}$ given by*

$$g_p(n, \phi) = (n + p(\phi), -p(\phi))$$

3. *if \mathcal{P} is a mixed filtration, then the abelian \mathbb{Z} -slicing induced on the heart of the bounded t -structure associated to $(g_p)_{\Omega}(\mathcal{Q})$ is split*

Proof. Let's prove (2). Suppose $g_p(n, \phi) > g_p(m, \psi)$. Then either $m - n < p(\phi) - p(\psi)$ or both $m - n = p(\phi) - p(\psi)$ and $p(\psi) < p(\phi)$. Since by definition of t -structure we can assume $m \geq n$, the second case is absurd while in the first case, since p is monotone, we have $\phi \geq \psi$ and thus by definition of perversity

$$m - n < p(\phi) - p(\psi) \leq \phi - \psi$$

and we get the desired Hom-vanishing by definition of grading filtration.

Suppose now $f_p(n, \phi) + 1 > f_p(m, \psi)$ and $(n + 1, \phi) < (m, \psi)$. The only non absurd case is $1 < m - n \leq p(\phi) - p(\psi)$. But then again we have

$$2 \leq m - n \leq p(\phi) - p(\psi) \leq \phi - \psi$$

and we can conclude as above.

To prove (1) consider the monotone map

$$\mathbb{Z} \xrightarrow{\lfloor */2 \rfloor} \mathbb{Z}$$

Applying the slice functor to the latter and starting with a perverse filtration, we get a grading filtration by the properties of the floor function and the thesis follows from part (2).

Let's prove (3). We have to show that

$$\mathcal{P}_\psi[1 - p(\psi)] \subseteq \mathcal{P}_\phi[p(\phi)]^\perp$$

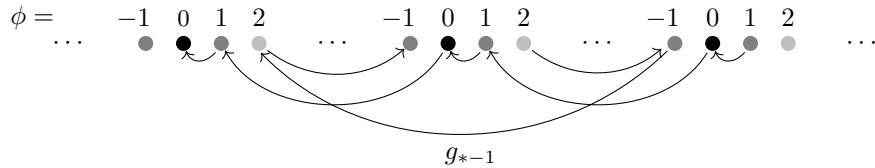
for $-p(\psi) > -p(\phi)$. We denote $n = p(\phi) - p(\psi) - 1$. Assuming again $n \geq 0$, we have $n \geq 0 > p(\psi) - p(\phi) \geq \psi - \phi$ and we can then conclude by definition of mixed filtration. \square

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity p the bounded t-structure coming from $(g_p)_\Omega(\mathcal{Q})$ defines a morphism of \mathbb{Z} -posets

$$\Xi^{\text{op}} \longrightarrow \mathbf{bts}(\mathcal{Q})$$

we can restate this as:

♫ *A grading or perverse filtration on the heart of a bounded t-structure on \mathcal{Q} induces a presheaf of t-structures on $\mathbb{Z} \times \mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z} \times \mathbb{Z}$ -slicing' on \mathcal{Q} .*



Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\text{op}} \hookrightarrow \Xi$$

and the t-structure coming from $(g_{\chi_I})_\Omega(\mathcal{Q})$ is just the tilting of \mathbf{t} with respect to the torsion pair coming from I .

The following proposition gives a characterization of the new heart obtained

by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 0.6. *Let \mathfrak{t} be a bounded t -structure on \mathcal{D} , \mathcal{P} a grading filtration on $\heartsuit_{\mathfrak{t}}$, \mathcal{Q} the associated $\mathbb{Z} \times \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , p a perversity. Denote \mathfrak{q} the bounded t -structure associated to $(g_p)_{\Omega}(\mathcal{Q})$. Then $\heartsuit_{\mathfrak{q}}$ consists of objects $X \in \mathcal{D}$ so that*

$$H_{\mathfrak{t}}^k(X) \in \mathcal{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if \mathcal{P} is a mixed filtration then for each $X \in \heartsuit_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \heartsuit_{\mathfrak{q}}$ if and only if

$$H_{\mathcal{P}}^{\phi}(H_{\mathfrak{t}}^k(X)[-k])[k] = H_{\mathcal{Q}}^{(k, \phi)}(X) = 0$$

for $p(\phi) \neq -k$.

For the second part of the claim, the abelian \mathbb{Z} -slicing induced on $\heartsuit_{\mathfrak{q}}$ is split by **Proposition 0.5** and thus by **Proposition ??**

$$X = \bigoplus_{(k, \phi)} H_{\mathcal{Q}}^{(k, \phi)}(X) = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^n(X)$$

where the last equality comes from the first part. □

Example 0.7. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an \mathbb{N} -graded ring with B_0 semisimple. Denote \mathcal{A} the category of \mathbb{Z} -graded B -modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote \mathcal{P}_{ϕ} the full subcategory of \mathcal{A} of modules concentrated in degree ϕ . Clearly, \mathcal{P} defines an abelian \mathbb{Z} -slicing on \mathcal{A} . Following [?] we have

$$\mathrm{Ext}_{\mathcal{A}}^n(\mathcal{P}_{\phi}, \mathcal{P}_{\psi}) = 0$$

for $n > \psi - \phi$. This means that \mathcal{P} is a mixed filtration and the bounded t -structure on $\mathcal{D}^b(\mathcal{A})$ associated to $(g_1)_{\Omega}(\mathcal{Q})$ (where 1 is the identity of \mathbb{Z}) is the 'diagonal' (or 'geometric') t -structure which appears in Koszul duality and other areas.

Example 0.8. Let M be an n -dimensional smooth complex projective variety and consider the n -torsion pair \mathcal{P} on $\mathrm{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that \mathcal{P} , seen as an abelian \mathbb{Z} -slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t -structure associated to $(f_p)_{\Omega}(\mathcal{Q})$ is the one of perverse coherent sheaves as

constructed in [?]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n -slicing on the heart of perverse coherent sheaves as done in [?].

Now we somehow review the gluing construction for t-structures in [?], but generalize it to any slicing. Our language is quite different though, and we formulate the problem very similarly to [?]. We start with two \mathbb{Z} -posets J, J' .

Definition 0.9. *Let \mathcal{P} be a $J \ltimes J'$ -slicing on \mathcal{D} . We call \mathcal{P} **gluable** if it satisfies the assumptions of **Proposition 0.2** with respect to the map $J \ltimes J' \xrightarrow{e} J' \ltimes J$ that exchanges coordinates. In this case we denote*

$$\overline{\mathcal{P}} = e_{\Omega}(\mathcal{P})$$

By a simple computation, the gluability condition reads: $\mathcal{P}_{(\phi, \psi)} \subseteq \mathcal{P}_{(\phi', \psi')}^{\perp}$ if $\psi' > \psi$ or both $\psi' + 1 > \psi$ and $\phi' + 1 < \phi$. For example, the first condition is automatic when \mathbb{Z} acts trivially on J (in this case, it follows from the second one). In particular, if \mathbb{Z} acts trivially on J , \mathcal{P} is a J -slicing on \mathcal{D} and \mathbf{t}_{ϕ} is a bounded t-structure on \mathcal{P}_{ϕ} for each $\phi \in J$, then using **Proposition ??** we get a $J \ltimes \mathbb{Z}$ -slicing \mathcal{Q} which is gluable if and only if

$$\heartsuit_{\mathbf{t}_{\psi}}[n] \subseteq \heartsuit_{\mathbf{t}_{\phi}}^{\perp}$$

whenever both $n \leq 0$ and $\phi < \psi$.

Remark 0.10. We have a commutative diagram of \mathbb{Z} -posets

$$\begin{array}{ccc} \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \\ \downarrow e & & \downarrow g \\ \mathbb{Z} \ltimes \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} \ltimes \hat{\mathbb{Z}} \end{array}$$

where the horizontal isomorphism is the one from **Remark ??**. Indeed, a $\mathbb{Z} \ltimes \mathbb{Z}$ -slicing on \mathcal{D} is gluable if and only if it induces a grading filtration on the heart of the associated t-structure when seen as a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing.

An easy combinatorial calculation finally yields the following two remarks:

- if \mathcal{P} is a gluable $\hat{\mathbb{Z}} \ltimes \mathbb{Z}$ -slicing on \mathcal{D} , then $\overline{\mathcal{P}}$ induces a grading filtration on the associated heart, allowing us again to construct new bounded t-structures depending on a perversity.
- if \mathcal{P} is a mixed filtration on the heart of a bounded t-structure and \mathcal{Q} is the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing on \mathcal{D} , then \mathcal{Q} is gluable. In this case, looking at $\overline{\mathcal{Q}}$, we get a baric structure on \mathcal{D} which is usually called **weight decomposition** in literature.

In other words, the chain of implications for a $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$ -slicing refines to

$$\text{mixed} \implies \text{gluable} \implies \text{grading} \implies \text{perverse}$$

