0.1 The (he)art of gluing

The slice functor hides greater potential. Indeed, even if we don't have a morphism of $\mathbb{Z}$-posets (but just a $\mathbb{Z}$-equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Let J be a $\mathbb{Z}$-toset, i.e., a totally ordered set together with a monotone action of $\mathbb{Z}$, that we will denote by $(n, x) \mapsto x + n$. Recall that a J-slicing on a stable ∞-category $\mathscr{D}$ is a morphism of $\mathbb{Z}$-posests $\mathfrak{t}\colon \mathcal{O}(J) \to \mathrm{ts}(\mathscr{D})$, where $\mathcal{O}(J)$ is the $\mathbb{Z}$-poset of the slicings of J (i.e., the decompositions of J into a disjoint union of a lower set L and of an upper set U), and $\mathrm{ts}(\mathscr{D})$ is the $\mathbb{Z}$-poset of t-structures on $\mathscr{D}$.
Clearly, if $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then $f^{-1}\colon \{\text{subsets of } J'\} \to \{\text{subsets of } J\}$ induces a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-posets $f^{-1}\colon \mathcal{O}(J') \to \mathcal{O}(J)$, and so composition with f^{-1} gives a morphism

$$\begin{aligned} f_*\colon J\text{-slicings on } \mathscr{D} &\to J'\text{-slicings on } \mathscr{D} \\ \mathfrak{t} &\mapsto \mathfrak{t} \circ f^{-1}. \end{aligned}$$

The slices of $f_*\mathfrak{t}$ are clearly given by $\mathscr{D}_{f_*\mathfrak{t};j} = \mathscr{D}_{\mathfrak{t};f^{-1}(\{j\})}$, for any $j \in J'$. Notice that, as f is monotone, the subset $f^{-1}(\{j\})$ is an interval in J. It is immediate to see that f_* restricts to a map

$$f_*\colon \text{Bridgeland } J\text{-slicings on } \mathscr{D} \to \text{Bridgeland } J'\text{-slicings on } \mathscr{D}$$

Namely, if $\mathcal{H}^j_{f_*\mathfrak{t}}(X) = 0$ for every $j \in J'$ then $\mathcal{H}^{f^{-1}(\{j\})}_{\mathfrak{t}}(X) = 0$ for every j in J'. As we are assuming the J-slicing $\mathfrak{t}$ is a Bridgeland slicing, this implies that $\mathcal{H}^{\phi}_{\mathfrak{t}}(X) = 0$ for every ϕ in $f^{-1}(\{j\})$, for every j. Therefore $\mathcal{H}^{\phi}_{\mathfrak{t}}(X) = 0$ for every ϕ in J and so, again by definition of Bridgeland slicing, $X = 0$. Also, if $\mathcal{H}^j_{f_*\mathfrak{t}}(X) \neq 0$ then $\mathcal{H}^{f^{-1}(\{j\})}_{\mathfrak{t}}(X) \neq 0$ and so (again by the Bridgeland slicing condition) there exists at least an element ϕ in $f^{-1}(\{j\})$ such that $\mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0$. As the J-slicing $\mathfrak{t}$ is Bridgeland, the total of these ϕ's must be finite, so only for finitely many j we can have such a ϕ. In other words, the number of indices j in J' such that $\mathcal{H}^j_{f_*\mathfrak{t}}(X) \neq 0$ is finite.
Notice that, if $\mathfrak{t}$ is a Bridgeland slicing of $\mathscr{D}$, and $f\colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then for any slicing (L, U) of J', the lower and the upper categories $\mathscr{D}_{f_*\mathfrak{t};L}$ and $\mathscr{D}_{f_*\mathfrak{t};U}$ can be equivalently defined as

$$\begin{aligned} \mathscr{D}_{f_*\mathfrak{t};L} &= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L} \\ \mathscr{D}_{f_*\mathfrak{t};U} &= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U} \end{aligned}$$

where $\langle \mathscr{S} \rangle$ denotes the extension-closed subcategory of $\mathscr{D}$ generated by the subcategory $\mathscr{S}$. In particular the slices of are given by $\mathscr{D}_{f_*\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$. [1]

[1] At some point it will be useful to recall Lemma 4.21 from ??: Let $\mathscr{S}_1, \mathscr{S}_2$ be two subcategories of $\mathscr{D}$ with $\mathscr{S}_1 \boxslash \mathscr{S}_2$, i.e., such that $\mathscr{D}(X_1, X_2)$ is contractible for any $X_1 \in \mathscr{S}_1$ and any $X_2 \in \mathscr{S}_2$. Then $\mathscr{S}_1 \boxslash \langle \mathscr{S}_2 \rangle$ and $\langle \mathscr{S}_1 \rangle \boxslash \mathscr{S}_2$, and so $\langle \mathscr{S}_1 \rangle \boxslash \langle \mathscr{S}_2 \rangle$.

The right hand sides of the above two expressions can clearly be defined for every morphism f from J to J' (i.e., not necessarily monotone nor $\mathbb{Z}$-equivariant), and as soon as f is $\mathbb{Z}$-equivariant, the assignment

$$(L, U) \mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$$

is an equivariant morphism from $\mathcal{O}(J)$ to pairs of subcategories of $\mathscr{D}$. Clearly, when f is not monotone there is no reason to expect that the pair $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ forms a t-structure on $\mathscr{D}$. Yet, it interesting to notice that the condition that f be monotone is only only sufficient in order to have this, and can indeed be relaxed.

Definition 0.1. *Let J and J' be $\mathbb{Z}$-tosets, and let $f\colon J \to J'$ a map of $\mathbb{Z}$-sets (i.e., a $\mathbb{Z}$-equivariant map, not necessarily nondecreasing). A Bridgeland J-slicing $\mathfrak{t}$ of $\mathscr{D}$ is* f-compatible *if $f(\phi) > f(\psi)$ with $\phi \leq \psi$ implies $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$*

Remark 0.2. Clearly, if f is monotone, then every J-slicing $\mathfrak{t}$ is f-compatible as the condition '$f(\phi) > f(\psi)$ with $\phi \leq \psi$' is empty.

Remark 0.3. Let J and J' be $\mathbb{Z}$-tosets, let $f\colon J \to J'$ a map of $\mathbb{Z}$-sets, and let $g\colon J' \to J''$ be an isomorphism of $\mathbb{Z}$-posets. Then a Bridgeland J-slicing $\mathfrak{t}$ of $\mathscr{D}$ is f-compatible if and only if it is $(g \circ f)$-compatible. Similarly, if $h\colon J'' \to J$ is an isomorphism of $\mathbb{Z}$-posets, then $\mathfrak{t}$ is f-compatible if and only if $(h^{-1})_*\mathfrak{t}$ is $(f \circ h)$-compatible.

Lemma 0.4. *Let J and J' be $\mathbb{Z}$-tosets, and let $f\colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. Then, for any slicing (L, U) of J', the pair of subcategories $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ is a t-structure on $\mathscr{D}$.*

Proof. As f is $\mathbb{Z}$-equivariant and $U + 1 \subseteq U$, we have

$$\begin{aligned}
\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}[1] &= \langle \mathscr{D}_{\mathfrak{t};\phi}[1] \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi)\in U} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi+1} \rangle_{f(\phi+1)\in U+1} \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U+1} \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}
\end{aligned}$$

and similarly for the lower subcategory $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$. To show that

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U} \boxslash \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$$

it suffices to show that, if $f(\psi) \in L$ and $f(\phi) \in U$ then $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. As $L \cap U = \emptyset$ we cannot have $\psi = \phi$, so either $\psi < \phi$ or vice versa. In the first case, $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ by definition of Bridgeland J-slicing. In the second case, we have $\phi < \psi$ and $f(\phi) > f(\psi)$ as $f(\phi) \in U$ and $f(\psi) \in L$. Therefore, since $\mathfrak{t}$ is f-compatible,

$\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$. Finally, we have to show that every object X in $\mathscr{D}$ fits into a fiber sequence

$$\begin{tikzcd} X_U \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X_L \end{tikzcd}$$

with $X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$ and $X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}$. As $\mathfrak{t}$ is a Bridgeland slicing, we have a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{\imath}}} X_{\bar{\imath}} \xrightarrow{\alpha_{\bar{\imath}+1}} X_{\bar{\imath}+1} \to \cdots \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \operatorname{cofib}(\alpha_i) = \mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \in \mathscr{D}_{\phi_i}$ for all $i = 1, \cdots, n$, with $\phi_i > \phi_{i+1}$. Let us now consider the sequence of symbols L and U obtained putting in the i-th place L if $f(\phi_i) \in L$ and U if $f(\phi_i) \in U$. If this sequence is of the form $(U, U, \ldots, U, L, L, \ldots, L)$, then there exists an index $\bar{\imath}$ such that $f(\phi_i) \in U$ for $i \leq \bar{\imath}$ and $f(\phi_i) \in L$ for $i > \bar{\imath}$ (with $\bar{\imath} = -1$ or n when all of the $f(\phi_i)$ are in L or in U, respectively). Then we can consider the pullout diagram

$$\begin{tikzcd} X_{\bar{\imath}} \arrow[r] \arrow[d, "f_L"'] & 0 \arrow[d] \\ X \arrow[r] & \operatorname{cofib}(f_L) \end{tikzcd}$$

together with the factorizations

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \cdots \xrightarrow{\alpha_{\bar{\imath}}} X_{\bar{\imath}}$$

and

$$X_{\bar{\imath}} \xrightarrow{\alpha_{\bar{\imath}+1}} X_{\bar{\imath}+1} \to \cdots \xrightarrow{\alpha_n} X_n = X.$$

The first factorization shows that $X_{\bar{\imath}} \in \langle \cup_{i=0}^{\bar{\imath}} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}$ while the second factorization shows that $\operatorname{cofib}(f_L) \in \langle \cup_{i=\bar{\imath}+1}^{n} \mathscr{D}_{\phi_i} \rangle \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}$. So we are done in this case. Therefore, we are reduced to showing that we can always avoid a $(\ldots, L, U, \ldots)$ situation in our sequence of L's and U's. Assume we have such a situation. Then we have an index i_0 with $f(\phi_{i_0}) \in L$ and $f(\phi_{i_0+1}) \in U$. This in particular implies $f(\phi_{i_0+1}) > f(\phi_{i_0})$ with $\phi_{i_0+1} < \phi_{i_0}$. As $\mathfrak{t}$ is f-compatible, this gives $\mathscr{D}_{\mathfrak{t};\phi_{i_0+1}} \boxslash (\mathscr{D}_{\mathfrak{t};\phi_{i_0}}[1])$. In particular, $\mathscr{D}(\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X), \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1])$ is contractible. Now consider the pasting of pullout diagrams

$$\begin{tikzcd} X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\ 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]. \end{tikzcd}$$

As the arrow $\mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1]$ factors through 0, we have $Y = \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ and the above diagram becomes

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] & \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\iota_2"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r] \arrow[d, "\pi_1"] & 0 \arrow[d] \\
 & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)[1],
\end{tikzcd}$$

where ι_2 and π_1 are the canonical inclusion and projection. Let $\beta\colon X_{i_0+1} \to \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)$ be the composition

$$\beta\colon X_{i_0+1} \xrightarrow{\gamma} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \xrightarrow{\pi_2} \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).$$

Then we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d] & X_{i_0} \arrow[r, "\alpha_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\beta"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[r, "\mathrm{id}"] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X)
\end{tikzcd}$$

(where only the left square is a pullout), and so the composition $\alpha_{i_0+1} \circ \alpha_{i_0}$ factors through the homotopy fiber of β. In other words, we have a homotopy commutative diagram

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\alpha_{i_0}"] \arrow[d, "\tilde{\alpha}_{i_0}"'] & X_{i_0} \arrow[d, "\alpha_{i_0+1}"] \\
\mathrm{fib}(\beta) \arrow[r, "\tilde{\alpha}_{i_0+1}"] & X_{i_0+1}
\end{tikzcd}$$

Writing $\tilde{X}_{i_0} = \mathrm{fib}(\beta)$, we get the pasting of pullout diagrams

$$\begin{tikzcd}
X_{i_0-1} \arrow[r, "\tilde{\alpha}_{i_0}"] \arrow[d] & \tilde{X}_{i_0} \arrow[r, "\tilde{\alpha}_{i_0+1}"] \arrow[d] & X_{i_0+1} \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \arrow[r, "\iota_1"] \arrow[d] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0+1}}(X) \oplus \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X) \arrow[d, "\pi_2"'] \\
 & 0 \arrow[r] & \mathcal{H}_{\mathfrak{t}}^{\phi_{i_0}}(X).
\end{tikzcd}$$

That is, by considering the factorization $X_{i_0-1} \xrightarrow{\tilde{\alpha}_{i_0}} \tilde{X}_{i_0} \xrightarrow{\tilde{\alpha}_{i_0+1}} X_{i_0+1}$ we have switched the cofibers with respect to the original factorization $X_{i_0-1} \xrightarrow{\alpha_{i_0}}$

$X_{i_0} \xrightarrow{\alpha_{i_0+1}} X_{i_0+1}$. Therefore, writing $\tilde{\phi}_{i_0} = \phi_{i_0+1}$ and $\tilde{\phi}_{i_0+1} = \phi_{i_0}$, we now have $f(\tilde{\phi}_{i_0}) \in U$ and $f(\tilde{\phi}_{i_0+1}) \in L$. That is, we have removed the $(\dots, L, U, \dots)$ situation from the position i_0, replacing it with a $(\dots, U, L, \dots)$ situation, while keeping all the labels L, U before this positions unchanged. Repeating the procedure the needed number of times, we eventually get rid of all the $(\dots, L, U, \dots)$ situations.[2] $\square$

Remark 0.5. It follows from the proof of Lemma 0.4 that the objects X_L and X_U in the fiber sequence $X_U \to X \to X_L$ associated with the t-struture $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U})$ on $\mathscr{D}$ satisfy

$$X_L \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in L \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0 \rangle$$
$$X_U \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in U \text{ and } \mathcal{H}^{\phi}_{\mathfrak{t}}(X) \neq 0 \rangle$$

Lemma 0.6. *For any $j \in J'$ we have*

$$\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\le j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\ge j}.$$

Proof. Clearly, $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j} \subseteq \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\le j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\ge j}$, therefore we only need to prove the converse inclusion. Let $\in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\le j} \cap \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\ge j}$. Then in particular $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\ge j}$ and so there exists a factorization of the initial morphism $\mathbf{0} \to X$ of the form

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_{n-1} \xrightarrow{\alpha_n} X_n = X$$

with $\mathbf{0} \neq \mathrm{cofib}(\alpha_i) \in \mathscr{D}_{\phi_i}$ with $f(\phi_i) \geq i$ for all $i = 1, \cdots, n$. As $((-\infty, j], (j, +\infty)$ is a slicing of J', reasoning as in the proof of Lemma 0.4 we can arrange this factorization is such a way that $f(\phi_i) > j$ for $i \le \bar{\imath}$ and $f(\phi_i) = j$ for $i > \bar{\imath}$. Therefore, again by reasoning as in the proof of Lemma 0.4 we get a fiber sequence of the form

$$\begin{tikzcd} X_{>j} \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X_j \end{tikzcd}$$

with $X_{>j}$ in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j}$ and X_j in $\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$. This is in particular a fiber sequence of the form $X_{>j} \to X \to X_{\le j}$, with $X_{\le j} \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\le j}$. As $(\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\le j}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)>j})$ is a t-structure on $\mathscr{D}$ by Lemma 0.4, there is (up to equivalence) only one such a fiber sequence. And since $X \in \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\le j}$, this is the sequence $0 \to X \xrightarrow{\mathrm{id}} X$. Therefore $X = X_j$. $\square$

Proposition 0.7. *Let J, J' be $\mathbb{Z}$-tosets, and let $f \colon J \to J'$ be a $\mathbb{Z}$-equivariant morphism of $\mathbb{Z}$-sets (i.e., not necessarily a monotone map) and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$ which is f-compatible. The map*

$$\begin{aligned} f_!\mathfrak{t} \colon \mathcal{O}(J') &\to \mathrm{ts}(\mathscr{D}) \\ (L, U) &\mapsto (\langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in L}, \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)\in U}) \end{aligned}$$

[2] This is somehow reminiscent of the 'bubble sort' algorithm.

defined by Lemma 0.4 is a Bridgeland J'-slicing of $\mathscr{D}$, with slices given by

$$\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}.$$

Proof. The map $f_!\mathfrak{t}$ is manifestly monotone and $\mathbb{Z}$-equivariant (see the first part of the proof of Lemma 0.4), so it is a J' slicing of $\mathscr{D}$, and its slices are given by $\mathscr{D}_{f_!\mathfrak{t};j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=j}$ by Lemma 0.6. We are therefore left with showing that it is finite and discrete. Given an object X in $\mathscr{D}$, let now $\{\phi_1, \dots, \phi_n\}$ be the indices in J such that $\mathcal{H}_{\mathfrak{t}}^{\phi_i}(X) \neq 0$ and let $\{j_1, \dots, j_k\}$ the image of the set $\{\phi_1, \dots, \phi_n\}$ via f. Up to renaming, we can assume $j_1 > j_2 > \cdots > j_k$. Consider now the factorization

$$\mathbf{0} = X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{k-1}} X_{k-1} \xrightarrow{\alpha_k} X_k = X$$

of the initial morphism of X associated to the decreasing sequence $j_1 > j_2 > \cdots > j_k$ by the J'-slicing $f_!\mathfrak{t}$. The cofibers of the morphisms α_i are the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X)$ and, by Remark 0.5, we have

$$\begin{aligned}
\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},j_i]}(X) \in &\langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) \in (j_{i+1}, j_i] \text{ and } \mathcal{H}_{\mathfrak{t}}^{\phi}(X) \neq 0 \rangle \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, j_i] \rangle \\
&= \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) = j_i \rangle \\
&\subseteq \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad f(\phi) = j_i \rangle = \mathscr{D}_{f_!\mathfrak{t};j_i}.
\end{aligned}$$

Therefore

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i} \mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{j_i}(X).$$

This tells us that, if all the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ vanish, then also all the $\mathcal{H}_{f_!\mathfrak{t}}^{(j_i,j_{i-1}]}(X)$ vanish, and so $X = 0$. Finally, if $\tilde{j} \notin \{j_1, \dots, j_k\}$, then there exists an index i such that $j_{i+1} < \tilde{j} < j_i$ and so $(j_{i+1}, \tilde{j}] \cap \{j_1, \dots, j_k\} = \emptyset$. The above argument then shows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) \in \langle \mathscr{D}_{\mathfrak{t};\phi}; \quad \phi \in \{\phi_1, \dots, \phi_n\} \text{ and } f(\phi) \in (j_{i+1}, \tilde{j}] \rangle = \{\mathbf{0}\}.$$

It follows that

$$\mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}} \mathcal{H}_{f_!\mathfrak{t}}^{(j_{i+1},\tilde{j}]}(X) = \mathcal{H}_{f_!\mathfrak{t}}^{\tilde{j}}(0) = 0,$$

for every $\tilde{j} \notin \{j_1, \dots, j_k\}$. So in particular, for any $X \in \mathscr{D}$, the cohomologies $\mathcal{H}_{f_!\mathfrak{t}}^{j}(X)$ are possibly nonzero only for finitely many indices j. $\square$

Remark 0.8. If $f \colon J \to J'$ is a morphism of $\mathbb{Z}$-tosets, then every Bridgeland slicing of $\mathfrak{t}$ of $\mathscr{D}$ is f-compatible and one has $f_!\mathfrak{t} = f_*\mathfrak{t}$.

Proposition 0.9. *Let J, J', J'' be $\mathbb{Z}$-tosets, let $f \colon J \to J'$ and $g \colon J' \to J''$ be $\mathbb{Z}$-equivariant morphisms of $\mathbb{Z}$-sets (i.e., not necessarily monotone maps), and let $\mathfrak{t}$ be a Bridgeland slicing of $\mathscr{D}$. If $\mathfrak{t}$ is f-compatible and $f_!\mathfrak{t}$ is g-compatible, then $\mathfrak{t}$ is $(g \circ f)$-compatible, and*

$$(g \circ f)_!\mathfrak{t} = g_! f_! \mathfrak{t}.$$

Proof. Let ϕ, ψ in J be such that $\phi \leq \psi$ and $g(f(\phi)) > g(f(\psi))$, and let $\xi = f(\phi)$ and $\eta = f(\psi)$. As J' is a totally ordered set, either $\xi \leq \eta$ or $\xi > \eta$. In the first case we have $g(\xi) > g(\eta)$ with $\xi \leq \eta$. As $f_!\mathfrak{t}$ is g-compatible, this implies $\mathscr{D}_{f_!\mathfrak{t};\xi} \boxslash \mathscr{D}_{f_!\mathfrak{t};\eta}$ and $\mathscr{D}_{f_!\mathfrak{t};\xi} \boxslash \mathscr{D}_{f_!\mathfrak{t};\eta}[1]$. By definition of $f_!\mathfrak{t}$ we have $\mathscr{D}_{\mathfrak{t};\phi} \subseteq \mathscr{D}_{f_!\mathfrak{t};\xi}$ and $\mathscr{D}_{\mathfrak{t};\psi} \subseteq \mathscr{D}_{f_!\mathfrak{t};\eta}$, therefore we have $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$. In the second case we have $f(\phi) > f(\psi)$ with $\phi \leq \psi$. As $\mathfrak{t}$ is f-compatible, this again implies $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}$ and $\mathscr{D}_{\mathfrak{t};\phi} \boxslash \mathscr{D}_{\mathfrak{t};\psi}[1]$. Finally, for any $j \in J''$ one has

$$\mathscr{D}_{g_!f_!\mathfrak{t};j} = \langle \mathscr{D}_{f_!\mathfrak{t};\xi} \rangle_{g(\xi)=j} = \langle \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{f(\phi)=\xi} \rangle_{g(\xi)=j} = \langle \mathscr{D}_{\mathfrak{t};\phi} \rangle_{g(f(\phi))=j} = \mathscr{D}_{(g\circ f)_!\mathfrak{t};j}.$$

$\square$

Let now J_1 and J_2 be two $\mathbb{Z}$-tosets, and let $J_1 \times_{\mathrm{lex}} J_2$ and $J_2 \times_{\mathrm{lex}} J_1$ the two $\mathbb{Z}$-tosets obtained by considering the lexicogaphic order on the products $J_1 \times J_2$ and $J_2 \times J_1$, respectively, and the diagonal $\mathbb{Z}$ action. The following lemma is immediate.

Lemma 0.10. *The exchange map*

$$\begin{aligned} e\colon J_1 \times J_2 &\to J_2 \times J_1 \\ (j_1, j_2) &\mapsto (j_2, j_1) \end{aligned}$$

is $\mathbb{Z}$-equivariant.

Definition 0.11. *Let J_1 and J_2 be $\mathbb{Z}$-tosets. A Bridgeland $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ of $\mathscr{D}$ is said to be* gluable *if it is e-compatible, where $e\colon J_1 \times J_2 \to J_2 \times J_1$ is the exchange map.*

Example 0.12. Let $J_1 = \{0, 1\}$ with the trivial $\mathbb{Z}$-action and let $J_2 = \mathbb{Z}$ with the standard translation $\mathbb{Z}$-action. Then a $J_1 \times_{\mathrm{lex}} J_2$-slicing $\mathfrak{t}$ on $\mathscr{D}$ is the datum of a semiorthogonal decomposition $(\mathscr{D}_0, \mathscr{D}_1)$ of $\mathscr{D}$ toghether with t-structures $\mathfrak{t}_i$ on $\mathscr{D}_i$. Spelling out the definiton, one sees that the $\{0, 1\} \times_{\mathrm{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$ is gluable if and only if $\mathscr{D}_{0;\geq 0} \boxslash \mathscr{D}_{1;-1}$ and $\mathscr{D}_{0;\geq 0} \boxslash \mathscr{D}_{1;0}$. When this happens, the glued slicing $e_!\mathfrak{t}$ is a $\mathbb{Z} \times_{\mathrm{lex}} \{0, 1\}$ slicing on $\mathscr{D}$, i.e., is the datum of a bounded t-structure on $\mathscr{D}$ together with a torsion theory on its heart. More precisely, the heart of $e_!\mathfrak{t}$ is the full ∞-subcategory $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ of $\mathscr{D}$ on thise objects X that fall into fiber sequences of the form $X_1 \to X \to X_0$ with $X_i \in \mathscr{D}_i^{\heartsuit}$, while the torsion theory on $\mathscr{D}^{\heartsuit_{e_!\mathfrak{t}}}$ is $(\mathscr{D}_0^{\heartsuit}, \mathscr{D}_1^{\heartsuit})$, i.e., precisely the pair of hearts of the two subcategories in the semiorthogonal decomposition.

Definition 0.13. *Let $U\colon J \to \mathcal{O}(J')$ be a map of sets. We denote by Γ_U the subset of $J \times J'$ defined by*

$$\Gamma_U = \{(j, j') \in J \times J'; \quad j' \in U_j\}.$$

For $f\colon J \to J'$ a map of sets, we denote by $(< f, \geq f)\colon J \to \mathcal{O}(J')$ the composition of f with the map

$$\begin{aligned} &J' \to \mathcal{O}(J') \\ &j' \mapsto ((-\infty, j'), [j', +\infty)). \end{aligned}$$

The upper graphic of f is the subset $\Gamma_{\geq f}$ of $J \times J'$.

Lemma 0.14. *Let $U\colon J \to \mathcal{O}(J')$ be a map of sets. Then Γ_U is an upper set of $J \times J'$ with the product order if and only if U is a map of posets $U\colon J \to \mathcal{O}(J')^{\mathrm{op}}$. In particular, $\Gamma_{\geq f}$ is an upper set if and only if $f\colon J \to J'$ is a nondecreasing map, i.e., a map of posets $J \to J'^{\mathrm{op}}$.*

Proof. Assume (L, U) is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Pick $(j, j') \in \Gamma_U$ and suppose $(j, j') \leq (k, k')$ in $J \times J'$. As (L, U) i monotone and $k \geq j$, we have $U_j \subseteq U_k$; as $(j, j') \in \Gamma_U$, we have $j' \in U_j$. Therefore $j \in U_k$. Since U_k is an upper set and $k' \geq j'$, we get $k' \in U_k$, i.e. $(k, k') \in \Gamma_U$. Vice versa, assume Γ_U is an upper set, and let $j \leq k$ in J. For any $j' \in U_j$, the element (k, j') in $J \times J'$ satisfies $(k, j') \geq (j, j')$ in the product order and so $(k, j') \in \Gamma_U$. This means $j' \in U_k$ and so $U_j \subseteq U_k$. To prove the second part of the statement, notice that the map $j \mapsto ((-\infty, j'), [j', +\infty))$ is a map of posets from $J' \to \mathcal{O}(J')$. Therefore, if $f\colon J \to J'^{\mathrm{op}}$ is a map of posets, then also $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets. Vice versa, if $(< f, \geq f)\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ is a map of posets then for every $j \leq k$ in J we have $[f(j), +\infty) \subseteq [f(k), +\infty)$ and so $f(k) \leq f(j)$. $\square$

Proposition 0.15. *The map*

$$\begin{aligned} \Gamma\colon \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} &\to \mathcal{O}(J \times J') \\ U &\mapsto \Gamma_U \end{aligned}$$

is an isomorphism of posets.

Proof. Let $U_1 \leq U_2$ in the partial order on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$, and let $(j, j') \in \Gamma_{U_2}$. Then $j' \in U_{2;k} \subseteq U_{1,j}$ and so $(j, j') \in \Gamma_{U_1}$. This means $\Gamma_{U_1} \leq \Gamma_{U_2}$ in $\mathcal{O}(J \times J')$. Next, for any morphism of posets $U\colon J \to \mathcal{O}(J)^{\mathrm{op}}$ we have that $(j, j') \in \Gamma_{U+1}$ if and only if $j' \in (U+1)_j$ Pick and upper set $\tilde{U}$ of $J \times J'$ and set, for $j \in J$

$$U_{\tilde{U}}(j) = \{j' \in J'; \quad (j, j') \in \tilde{U}\}.$$

The subset $U_{\tilde{U}}(j) \subseteq J'$ is an upper set. Indeed, if $j' \in U_{\tilde{U}}(j)$ and $k' \geq j'$ in J', then $(j, k') \geq (j, j')$ in the product order on $J \times J'$ and so $(j, k') \in \tilde{U}$ as $\tilde{U}$ is an upper set. Next, if $j \leq k$ in J and $j' \in U_{\tilde{U}}(j)$, then $(k, j') \in \tilde{U}$ as $\tilde{U}$ is an upper set, and so $j' \in U_{\tilde{U}}(k)$. This shows that $U_{\tilde{U}}(j) \subseteq U_{\tilde{U}}(k)$, and so $U_{\tilde{U}}$ is a map of posets from J to $\mathcal{O}(J')^{\mathrm{op}}$. Moreover, if $\tilde{U}_1 \leq \tilde{U}_2$ in $\mathcal{O}(J \times J')$ then $U_{\tilde{U}_1} \leq U_{\tilde{U}_2}$ in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. Therefore we have a map

$$\begin{aligned} \gamma\colon \mathcal{O}(J \times J') &\to \mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \\ \tilde{U} &\mapsto U_{\tilde{U}}. \end{aligned}$$

which is straightforward to see to be the inverse of Γ. $\square$

Assume J and J' are $\mathbb{Z}$-posets. Then $\mathcal{O}(J)^{\mathrm{op}}$ is a $\mathbb{Z}$-poset with 1 acting as $U \mapsto U - 1$. The action of $\mathbb{Z}$ by conjugation on $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$ is therefore given by $(U \dotplus 1)_j = U_{j-1} - 1$. To see that $U \dotplus 1$ is still a morphism of posets from J to $\mathcal{O}(J')$, let $j \leq k$ in J. Then $(U \dotplus 1)_j = U_{j-1} - 1 \subseteq U_{k-1} - 1 = (U \dotplus 1)_k$. The conjugation action, however, does not define a structure of $\mathbb{Z}$-poset on

$\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, as it is generally not true that $U \leq U \dotplus 1$. This condition is indeed equivalent to $U_j \subseteq (U \dotplus 1)_j$, i.e., to $U_j + 1 \subseteq U_{j-1}$ for any j in J, and this is not necessarily satisfied by a morphism of posets $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$.

Definition 0.16. *A* (J, J')-perversity *is a map of posets* $U \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ *such that*

$$U_j + 1 \subseteq U_{j-1}$$

for any j in J. The set $\mathrm{Perv}(J, J')$ *of all (J, J')-perversities inherits a poset structure from the inclusion in* $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$. *It is a $\mathbb{Z}$-poset with 1 acting as $U \mapsto U \dotplus (-1)$.*

Remark 0.17. Notice that the shift action on (J, J')-perversities is given by $U \mapsto U \dotplus (-1)$. Namely, by construction the $\mathbb{Z}$-action $U \mapsto U \dotplus 1$ is monotone on the set of perversities with the order induced by the inclusion in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})$, and the order on $\mathrm{Perv}(J, J')$ is the opposite one. The reason for considering this order is, clearly, Proposition 0.15.

We have seen in Lemma 0.14 that a function $f \colon J \to J'$ defines a morphism of posets by $(\geq f) \colon J \to \mathcal{O}(J')^{\mathrm{op}}$ if and only if f is a morphism of posets from J to J'^{op}. When this happens, $(\geq f)$ is a (J, J')-perversity if and only if $f(j-1) \leq f(j) + 1$, for every $j \in J$. In the particular case $J = \mathbb{Z}$, assuming as usual that J' is a $\mathbb{Z}$-poset, we can define a new function $p_f \colon \mathbb{Z} \to J'$ as $p_f(n) = f(n) + n$. Then the condition $f(n) \leq f(n+1) + 1$ translates into $p_f(n) \leq p_f(n+1)$, while the condition that $f \colon \mathbb{Z} \to J'^{\mathrm{op}}$ is a morphism of posets, i.e., $f(n+1) \leq f(n)$ translates to $p_f(n+1) \leq p_f(n)+1$. As $f \mapsto p_f$ is a bijection of the set of maps from J to J' into itself, we see that the functions $f \colon \mathbb{Z} \to J'$ defining perversities correspond bijectively to the set of functions $p \colon \mathbb{Z} \to J'$ such that $p(n) \leq p(n+1) \leq p(n) + 1$, for every $n \in \mathbb{Z}$. This motivates the following (see [**?**]).

Definition 0.18. *A $(\mathbb{Z}, \mathbb{Z})$-perversity U is said to be defined by a function if there exists $f \colon \mathbb{Z} \to \mathbb{Z}$ such that $U = (\geq f)$. The subset of $(\mathbb{Z}, \mathbb{Z})$-perversities defined by functions is denoted by* $\mathrm{Perv}^\circ(\mathbb{Z}, \mathbb{Z})$.

Lemma 0.19. *The subset* $\mathrm{Perv}^\circ(\mathbb{Z}, \mathbb{Z})$ *is a $\mathbb{Z}$-sub-poset of* $\mathrm{Perv}(\mathbb{Z}, \mathbb{Z})$. *It is isomorphic via $f \mapsto (\geq f)$ with the $\mathbb{Z}$-poset of nonincreasing functions $f \colon \mathbb{Z} \to \mathbb{Z}$ such that $f(n-1) \leq f(n)+1$ with the $\mathbb{Z}$-action given by $(f \dotplus 1)(n) = f(n+1)+1$.*

Proof. As a function $f \colon \mathbb{Z} \to \mathbb{Z}$ is uniquely determined by the collection of upper sets $[f(n), +\infty)$ the set $\mathrm{Perv}^\circ(J, J')$ bijectivley corresponds to the set of those functions $f \colon \mathbb{Z} \to \mathbb{Z}$ such that $(\geq f)$ is a $(\mathbb{Z}, \mathbb{Z})$-perversity. As noticed above, these are precisely nonincreasing functions from $\mathbb{Z}$ to itself such that $f(n-1) \leq f(n)+1$. This bijection is an isomorphism of posets, as $(\geq f_1) \leq (\geq f_2)$ if and only if $f_1 \leq f_2$ (in the standard poset structure on the set of maps from $\mathbb{Z}$ to the poset $\mathbb{Z}$). Finally, $((\geq f) \dotplus (-1))_n = (\geq f)_{n+1} + 1 = [f(n+1)+1, +\infty) = (\geq (f \dotplus 1))_n$, for any $n \in \mathbb{Z}$. □

Definition 0.20. *A* perversity function *(on $\mathbb{Z}$) is a function $p\colon \mathbb{Z} \to \mathbb{Z}$ such that*

$$p(n) \le p(n+1) \le p(n) + 1,$$

for every $n \in Z$. The set $\mathrm{perv}_{\mathbb{Z}}$ *of perversity functions is a poset with the partial order induced by the inclusion* $\mathrm{perv}_{\mathbb{Z}} \subseteq \mathrm{Pos}(\mathbb{Z},\mathbb{Z})$*. It is a $\mathbb{Z}$-poset with the action $(p \dotplus 1)(n) = p(n+1)$.*

Remark 0.21. Notice that the $\mathbb{Z}$-action on perversity functions is a monotone action since, by definition of perversity function, we have $p(n) \le p(n+1)$ and this precisely means $p(n) \le (p \dotplus 1)(n)$.

Lemma 0.22. *For every $p\colon \mathbb{Z} \to \mathbb{Z}$, let $f_p\colon \mathbb{Z} \to \mathbb{Z}$ be the map defined by $f_p(n) = p(n) - n$. Then $p \mapsto (\ge f_p)$ is an isomorphism of $\mathbb{Z}$-posets between* $\mathrm{perv}_{\mathbb{Z}}$ *and* $\mathrm{Perv}^\circ(\mathbb{Z},\mathbb{Z})$

Proof. By Lemma 0.19 we only need to show that $p \mapsto f_p$ is a monotone $\mathbb{Z}$-equivariant bijection between $\mathrm{perv}_{\mathbb{Z}}$ and the set of functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(n) \le f(n-1) \le f(n)+1$. That it is a bijection is immediate: the inverse map is $f \mapsto p_f$, where $p_f(n) = f(n) + n$. To see that it is an isomorphism of posets, notice that $f_{p_1} \le f_{p_2}$ if and only if $p_1(n) - n \le p_2(n) - n$ for every $n \in \mathbb{Z}$, and so if and only if $p_1(n) \le p_2(n)$ for every $n \in \mathbb{Z}$. Finally, $f_{p \dotplus 1}(n) = (p \dotplus 1)(n) - n = p(n+1) - n = f_p(n+1) + 1 = (f_p \dotplus 1)(n)$. □

Definition 0.23. *An upper set U in $\mathcal{O}(J \times J')$ is called a* kinky upperset *if $U \le U +_{\mathrm{ne}} 1$, where* ne *is the "northestern" action of $\mathbb{Z}$ on $J \times J'$ given by $(j, j') +_{\mathrm{ne}} 1 = (j-1, j'+1)$. We denote by* $\mathrm{Kink}(J \times J')$ *the poset of kinky uppersets of $J \times J'$, with the poset structure induced by the inclusion in $\mathcal{O}(J \times J')$. it is a $\mathbb{Z}$-poset with the "northestern" action. We denote by* $\mathrm{Kink}^\circ(J \times J')$ *the $\mathbb{Z}$-sub-poset of nontrivial kinky uppersets of $J \times J'$, where the trivial upperstes are $\emptyset$ and $J \times J'$.*

Lemma 0.24. *The map Γ from Proposition 0.15 induces an isomorphism of $\mathbb{Z}$-posets*

$$\Gamma\colon \mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J').$$

Proof. Let $U\colon J \to \mathcal{O}(J')^{\mathrm{op}}$ be a perversity, and let $(j, j') \in \Gamma_U$. Then $j' \in U_j$ and so, by definition of perversity, $j' + 1 \in U_{j-1}$. Therefore $(j-1, j'+1) \in \Gamma_U$, i.e., $\Gamma_U \le \Gamma_U +_{\mathrm{ne}} 1$. Vice versa, if $\tilde{U}$ is a kinky upperset in $J \times J'$, let $U_{\tilde{U}}$ the preimage in $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}}$ of $\tilde{U}$ via Γ (see the proof of Proposition 0.15). Then for any $j \in J$ and any $j' \in U_{\tilde{U};j}$ we have $j' + 1 \in U_{\tilde{U};j-1}$ and so $U_{\tilde{U};j} + 1 \subseteq U_{\tilde{U};j-1}$. So the isomorphism of posets $\mathrm{Pos}(J, \mathcal{O}(J')^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{O}(J \times J')$ restricts to an isomorphism of posets $\mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J')$. To see that $\Gamma\colon \mathrm{Perv}(J, J') \to \mathrm{Kink}(J \times J')$ is also $\mathbb{Z}$-equivariant, notice that, for every perversity U we have $(j, j') \in \Gamma_{U \dotplus (-1)}$ if and only if $j' \in U_{j+1} + 1$. i.e., if and only if $(j+1, j'-1) \in \Gamma_U$. This latter condition is equivalent to $(j, j') \in \Gamma_U +_{\mathrm{nw}} 1$, so we find $\Gamma_{U \dotplus (-1)} = \Gamma_U +_{\mathrm{nw}} 1$. □

Lemma 0.25. *The map* Γ *from Proposition 0.15 induces an isomorphism of* $\mathbb{Z}$*-posets*

$$\Gamma\colon \mathrm{Perv}^{\circ}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Kink}^{\circ}(\mathbb{Z}\times\mathbb{Z}).$$

Proof. A kinky upperset U is in the image of Γ if and only if U is of the form $(\geq f)$ for a suitable function $f\colon \mathbb{Z}\to\mathbb{Z}$. This is possible if and only if $U_n \neq \emptyset, \mathbb{Z}$ for every $n\in\mathbb{Z}$. As U is kinky, if $U_{n_0}=\emptyset$ for some n_0, then $U_{n_0+1}+1\subseteq U_{n_0}=\emptyset$, and so $U_{n_0+1}=\emptyset$. Inductively, this gives $U_n=\emptyset$ for every $n\geq n_0$. On the other hand, since a kinky upperset is an upperset, if there exists a nonempty U_n with $n<n_0$, then there exist an element (n,m) in U and so, since $(n,m)\leq(n_0,m)$, also $(n_0,m)\in U$. But then $m\in U_{n_0}$, which is impossible. So also the U_n with $n<n_0$ are empty and therefore $U=\emptyset$. Similarly, if $U_{n_0}=\mathbb{Z}$ for some n_0, then $U_n=\mathbb{Z}$ for every $n>n_0$ as U is an upperset, while the kinkiness condition $U_n+1\subseteq U_{n-1}$ implies that also $U_{n_0-1}=\mathbb{Z}$ and so inductively that all U_n with $n<n_0$ are the whole of $\mathbb{Z}$. That is, $U=\mathbb{Z}\times\mathbb{Z}$ in this case. □

Lemma 0.26. *The isomorphism of* $\mathbb{Z}$*-modules* $\varphi\colon \mathbb{Z}^2\to\mathbb{Z}^2$ *given by* $\varphi\colon (n,n')\mapsto (n+n',n')$ *induces an isomorphism of* $\mathbb{Z}$*-posets*

$$\varphi\colon \mathrm{Kink}(\mathbb{Z}\times\mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z}),$$

where the $\mathbb{Z}$*-action on* $\mathcal{O}(\mathbb{Z}\times\mathbb{Z})$ *is the one induced by the "northern"* $\mathbb{Z}$*-action on* $\mathbb{Z}^2$*, namely,* $(n,n')+1=(n,n'+1)$*. In particular* φ *induces an isomorphism of* $\mathbb{Z}$*-posets* $\mathrm{Kink}^{\circ}(\mathbb{Z}\times\mathbb{Z}) \xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z})\setminus\{\emptyset,\mathbb{Z}\times\mathbb{Z}\}$.

Proof. A subset U of $\mathbb{Z}\times\mathbb{Z}$ is an upper set (in the product order) if and only if $U+K\subseteq U$, where K is the $\mathbb{Z}$-cone spanned by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, while U is a kinky upperset if and only if $U+K^{\mathrm{kink}}\subseteq U$, where K^{kink} is the $\mathbb{Z}$-cone spanned by $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$. Morever the $\mathbb{Z}$ action on the uppersets is generated by $U\mapsto U+\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and the the $\mathbb{Z}$ action on the kinky uppersets is generated by $U\mapsto U+\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$, where in both cases the sum on the right hand side is the sum in $\mathbb{Z}^2$. As $\varphi\colon\mathbb{Z}^2\to\mathbb{Z}^2$ is an isomorphism of $\mathbb{Z}$-modules with $\varphi(K^{\mathrm{kink}})=K$ and $\varphi\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, the statement follows (φ is manifestly inclusion preserving). □

Corollary 0.27. *We have an isomorphism of* $\mathbb{Z}$*-posets*

$$\mathrm{perv}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z}\times\mathbb{Z})\setminus\{\emptyset,\mathbb{Z}\times\mathbb{Z}\}$$

mapping a perversity function p *to the image via the isomorphism* $\varphi\colon (n,n')\mapsto (n+n',n')$ *of the set* $\{(n,n')\in\mathbb{Z}\times\mathbb{Z}$ *such that* $n'\geq p(n)-n\}$.

Example 0.28. The zero perversity function $p(n)\equiv 0$ corresponds to the upper set $\{(n,n')\in\mathbb{Z}\times\mathbb{Z}$ such that $n\geq 0\}$; the identity perversity function $p(n)\equiv n$ corresponds to the upper set $\{(n,n')\in\mathbb{Z}\times\mathbb{Z}$ such that $n'\geq 0\}$.

0.2 Slicing the heart

Let $\mathscr{D}$ be a stable ∞-category. Then, as we recalled above, a bounded t-structure on $\mathscr{D}$ is the datum of a Bridgeland $\mathbb{Z}$-slicing $\mathfrak{t}$ of $\mathscr{D}$. Let $\heartsuit_{\mathfrak{t}}$ denote the heart of $\mathfrak{t}$. Then an abelian $\mathbb{Z}$-slicing of $\heartsuit_{\mathfrak{t}}$ is the datum of an extension $\tilde{\mathfrak{t}}$ of $\mathfrak{t}$ to a Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, where $\hat{\mathbb{Z}}$ denotes the $\mathbb{Z}$-poset consisting of $\mathbb{Z}$ endowed wth the trivial Z-action, and the morphism $\mathcal{O}(\mathbb{Z}) \to \mathcal{O}(\mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}})$ is induced by the projection on the first factor $\mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \to \mathbb{Z}$. See [**?**, Section??]. We will denote by $\heartsuit_{\mathfrak{t};\phi}$ the ϕ-th slice of the heart of $\mathfrak{t}$. In other words,

$$\heartsuit_{\mathfrak{t};\phi} = \mathscr{D}_{\tilde{\mathfrak{t}};(0,\phi)}.$$

Notice that we have

$$\heartsuit_{\mathfrak{t};\phi}[n] = \mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)},$$

where $[n]$ denotes the "shift by n" functor on $\mathscr{D}$.

Example 0.29. Via the obvious inclusion of $\mathbb{Z}$-posets $\{0,1\} \hookrightarrow \hat{\mathbb{Z}}$, any torsion pair $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$ on $\heartsuit_{\mathfrak{t}}$ defines an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$.

Definition 0.30. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. An abelian $\mathbb{Z}$-slicing $\hat{\mathfrak{t}}$ on $\heartsuit_{\mathfrak{t}}$ is called:*

- grading *(or* radical*) if $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ for $\phi > \psi + n$ and for $\phi = \psi + n$ with $n \geq 2$;*
- gluable *if $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ for $\phi > \psi$ and $n > 0$;*

Remark 0.31. The definition of gluable abelian $\mathbb{Z}$-slicing of $\heartsuit_{\mathfrak{t}}$ is the specialization of Definition 0.11 to $J_1 = \mathbb{Z}$ and $J_2 = \hat{\mathbb{Z}}$.

Remark 0.32. Historically, grading filtrations first appeared in [**?**] under the name of 'radical filtrations'.

Example 0.33. Let $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$ be a torsion pair on $\heartsuit_{\mathfrak{t}}$. Then $(\heartsuit_{\mathfrak{t};0}, \heartsuit_{\mathfrak{t};1})$, seen as an abelian $\mathbb{Z}$-slicing, is grading. Namely, as $\heartsuit_{\mathfrak{t};\phi}[n] = 0$ for $\phi \notin \{0,1\}$, the only nontrivial orthogonality conditions to be checked are:

- $\heartsuit_{\mathfrak{t};0} \boxslash \heartsuit_{\mathfrak{t};0}[n]$ for $n < 0$;
- $\heartsuit_{\mathfrak{t};0} \boxslash \heartsuit_{\mathfrak{t};1}[n]$ for $n < -1$;
- $\heartsuit_{\mathfrak{t};1} \boxslash \heartsuit_{\mathfrak{t};0}[n]$ for $n < 1$;
- $\heartsuit_{\mathfrak{t};1} \boxslash \heartsuit_{\mathfrak{t};1}[n]$ for $n < 0$.

These all follows from the orthogonality relation $\heartsuit_{\mathfrak{t}} \boxslash \heartsuit_{\mathfrak{t}}[n]$ for $n < 0$, except for $\heartsuit_{\mathfrak{t};1} \boxslash \heartsuit_{\mathfrak{t};0}$ which is true by definition of torsion pair.

Proposition 0.34. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let $\hat{\mathfrak{t}}$ be an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. If $\tilde{\mathfrak{t}}$ is gluable, then $\tilde{\mathfrak{t}}$ is grading.*

Proof. Let ϕ and ψ be in $\mathbb{Z}$ with $\phi > \psi + n$. The orthogonality condition $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ is trivially satisfied if $n < 0$, so let us assume $n \geq 0$. If $n = 0$, then $\phi > \psi$ and the orthogonaliy condition $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}$ is satisfied by definition of slicing. Finally, if $n > 0$, then we have $\phi > \psi$ and $n > 0$, so $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$ by definition of gluable slicing. If $\phi = \psi + n$ with $n \geq 2$, then in particular $\phi > \psi$ and $n > 0$, so again $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[n]$. $\square$

Proposition 0.35. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be a grading abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. Then, $\tilde{\mathfrak{t}}$ is g_p-compatible, where*

$$\begin{aligned} g_p \colon \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} &\to \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \\ (n, \phi) &\mapsto (n + p(\phi), -p(\phi)), \end{aligned}$$

for every perversity function $p \colon \mathbb{Z} \to \mathbb{Z}$.

Proof. The map g_p is $\mathbb{Z}$-equivariant, as the action on the second factor is the trivial one. Let (n, ϕ) and (m, ψ) in $\mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}}$ with $(n, \phi) \leq (m, \psi)$ such that $g_p(n, \phi) > g_p(m, \psi)$ in $\mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}}$. By definition of g_p this means that we have

$$(n + p(\phi), -p(\phi)) > (m + p(\psi), -p(\psi))$$

in $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}$, i.e., that $n + p(\phi) > m + p(\psi)$ or that $n + p(\phi) = m + p(\psi)$ and $p(\psi) > p(\phi)$. Similarly, the condition $(n, \phi) \leq (m, \psi)$ means that either $n < m$ or $n = m$ and $\phi \leq \psi$. By considering all possibilities, and taking into account that a perversity function is nondecreasing, one sees that there is actually a single case to deal with:

- $p(\phi) - p(\psi) > m - n$, with $m > n$ and $\phi > \psi$;

As p is a perversity function, if $\phi \geq \psi$ then

$$0 \leq p(\phi) - p(\psi) \leq \phi - \psi,$$

so we have

$$\phi \geq \psi + p(\phi) - p(\psi) > \psi + m - n.$$

Since $\tilde{\mathfrak{t}}$ is grading, this implies $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[m-n]$, and so $\heartsuit_{\mathfrak{t};\phi}[n] \boxslash \heartsuit_{\mathfrak{t};\psi}[m]$, i.e.,

$$\mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)} \boxslash \mathscr{D}_{\tilde{\mathfrak{t}};(m,\psi)}.$$

Since $\phi > \psi + m - n$, either $\phi > \psi + m - n + 1$ or $\phi = \psi + m - n + 1$. In the first case, reasoning as above we find $\mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)} \boxslash \mathscr{D}_{\tilde{\mathfrak{t}};(m,\psi)}[1]$. In the second case, as $m > n$ we have $m - n + 1 \geq 2$. As $\tilde{\mathfrak{t}}$ is grading, this gives $\heartsuit_{\mathfrak{t};\phi} \boxslash \heartsuit_{\mathfrak{t};\psi}[m-n+1]$, i.e., again

$$\mathscr{D}_{\tilde{\mathfrak{t}};(n,\phi)} \boxslash \mathscr{D}_{\tilde{\mathfrak{t}};(m,\psi)}[1].$$

□

Remark 0.36. By taking as perversity function the identity, $\mathrm{id} \colon \mathbb{Z} \to \mathbb{Z}$, we see that if $\tilde{\mathfrak{t}}$ is grading then $\tilde{\mathfrak{t}}$ is α-compatible, where $\alpha \colon \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \to \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}}$ is the $\mathbb{Z}$-equivariant map given by

$$\alpha(n, m) = (n + m, -m).$$

The proof of the following lemma is straightforward.

Lemma 0.37. *Let $\beta\colon \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \to \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}$ be the map defined by*

$$\beta(n, m) = (n, n + m).$$

Then β is an isomorphism of $\mathbb{Z}$-tosets. Moreover the diagram

$$\begin{tikzcd}
\mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \arrow[d, "\beta"] \\
\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z} \arrow[r, "e"] & \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}
\end{tikzcd}$$

commutes, where $e\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is the exchange map $e(n, m) = (m, n)$ from Lemma 0.10 and $\alpha\colon \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \to \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}}$ is the map $\alpha(n, m) = (n + m, -m)$ from Remark 0.36.

Corollary 0.38. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. Then, in the notation of Lemma 0.37, $\tilde{\mathfrak{t}}$ is α-compatible if and only if $\beta_*\mathfrak{t}$ is gluable.*

Proof. By Lemma 0.37 and Remark 0.3, $\tilde{\mathfrak{t}}$ is α-compatible if and only if $\tilde{\mathfrak{t}}$ is $(\beta \circ \alpha)$-compatible, which happens if and only if $\tilde{\mathfrak{t}}$ is $(e \circ \beta)$-compatible. As β is an isomorphism of $\mathbb{Z}$-tosets, we can write $\mathfrak{t} = (\beta^{-1})_*\beta_*\mathfrak{t}$, and so, by Remark 0.3 again, $\tilde{\mathfrak{t}}$ is $(e \circ \beta)$-compatible if and only if $\beta_*\mathfrak{t}$ is e-compatible. By Remark 0.31, this is equivalent to saying that $\beta_*\mathfrak{t}$ is gluable. $\square$

From Proposition 0.34 and Remark 0.36 we immediately get the following

Corollary 0.39. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be an abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. If $\tilde{\mathfrak{t}}$ is grading, then $\beta_*\mathfrak{t}$ is gluable. In particular, if $\tilde{\mathfrak{t}}$ is grading, then also $\beta_*\mathfrak{t}$ is grading.*

Proposition 0.40. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, and let let $\tilde{\mathfrak{t}}$ be a grading abelian $\mathbb{Z}$-slicing on $\heartsuit_{\mathfrak{t}}$. For every perversity p, let $\gamma_p\colon \mathbb{Z} \times_{\mathrm{lex}} \hat{\mathbb{Z}} \to \mathbb{Z}$ the $\mathbb{Z}$-equivariant morphism given by*

$$\gamma_p\colon (n, \phi) \mapsto n + p(\phi).$$

Then

- *$\tilde{\mathfrak{t}}$ is γ_p-compatible;*
- *the map*

$$\begin{aligned} \mathrm{perv}_{\mathbb{Z}} &\to \mathrm{ts}(\mathscr{D}) \\ p &\mapsto (\gamma_p)_!\tilde{\mathfrak{t}}, \end{aligned}$$

  is a morphism of $\mathbb{Z}$-posets.

Proof. Let $g_p\colon \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}} \to \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$ the map defined in Proposition 0.40 As $\tilde{\mathfrak{t}}$ is grading, by Proposition 0.40, $\tilde{\mathfrak{t}}$ is g_p-compatible. The projection on the first factor, $\pi_1\colon \mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}} \to \mathbb{Z}$ is a morphism of $\mathbb{Z}$-tosets, so by Remark 0.2 every $\mathbb{Z}\times_{\mathrm{lex}}\hat{\mathbb{Z}}$-slicing is π_1-compatible. In particular, $(g_p)_!\mathfrak{t}$ is π_1-compatible. Therefore, by Proposition 0.9, $\mathfrak{t}$ is $(\pi_1\circ g_p)$-compatible. As $\pi_1\circ g_p=\gamma_p$, this precisely says that $\mathfrak{t}$ is γ_p-compatible. We therefore have a Bridgeland $\mathbb{Z}$-slicing, i.e., a bounded t-structure, $(\gamma_p)_!\mathfrak{t}$ on $\mathscr{D}$.
Resta da scrivere la parte "$p\mapsto(\gamma_p)_!\tilde{\mathfrak{t}}$ è un morfismo di $\mathbb{Z}$-poset". □

-fino a qui- Quanto segue va rivisto alla luce della chiacchierata su wire: dobbiamo introdurre la nozione di strong semiorthogonal decomposition e notare che nel caso in cui abbiamo strong e t-strutture scambiabili allora abbiamo due modi di incollare e questi coincidono. We now recall the celebrated process from BBD of gluing two t-structures into a third one. This will turn out to induce a gluable slicing in our sense, thus showing that the language of the present paper is actually a generalization of a classical piece of literature. This, by the other hand, explains both our terminology and motivation.

Definition 0.41. *A* recollement datum *is a diagram of stable ∞-categories and exact ∞-functors*

$$\begin{tikzcd} \mathscr{D}_{\mathrm{fer}} \arrow[r, "i_*"'] & \mathscr{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"'] & \mathscr{D}_{\mathrm{ouv}} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that the following holds:

1. *$(i^*,i_*,i^!)$ and $(j_!,j^*,j_*)$ are adjoint triples,*
2. *$i_*,j_*,j_!$ are fully faithful and $j^*i_*=0$,*
3. *for each object X of $\mathscr{D}$ the (co)units of the above adjunctions give rise to cofiber sequences*

$$\begin{tikzcd} i_*i^!X \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & j_*j^*X \end{tikzcd} \qquad \begin{tikzcd} j_!j^*X \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & i_*i^*X. \end{tikzcd}$$

Proposition 0.42. *Suppose we are in the situation of 0.41. Then:*

1. *$(\mathscr{D}_0=j_*\mathscr{D}_{\mathrm{ouv}},\mathscr{D}_1=i_*\mathscr{D}_{\mathrm{fer}})$ is a semiorthogonal decomposition on $\mathscr{D}$.*
2. *Given t-structures $\mathfrak{t}_0,\mathfrak{t}_1$ on $\mathscr{D}_{\mathrm{ouv}}$ and $\mathscr{D}_{\mathrm{fer}}$ respectively, the associated $\{0,1\}\times_{\mathrm{lex}}\mathbb{Z}$-slicing on $\mathscr{D}$ given by*

$$\mathscr{D}_{0;\geq 0}=j_*(\mathscr{D}_{\mathrm{ouv};\geq 0}),\quad \mathscr{D}_{1;\geq 0}=i_*(\mathscr{D}_{\mathrm{fer};\geq 0})$$

is gluable.

Proof. Let's prove the first claim. The fact that j^* and j_* are adjoints implies that, for objects $i_*X \in \mathscr{D}_{\mathrm{fer}}$ and $j_*Y \in \mathscr{D}_{\mathrm{ouv}}$,

$$\mathscr{D}(i_*X, j_*Y) = \mathscr{D}(j^*i_*X, Y)$$

and the latter vanishes since $j^*i_* = 0$ by definition. This shows that $\mathscr{D}_1 \boxslash \mathscr{D}_0$. Considering the left cofiber sequence from Definition 0.41 as a Postnikov tower of length one we see that $(\mathscr{D}_0, \mathscr{D}_1)$ is a $\{0, 1\}$-slicing on $\mathscr{D}$, as desired.

□

—-FINO A QUI 1.5 —-

Da qui esempi di Giovanni

1 A zoo of examples

Reminder: da mettere prima di tutto il tilting a la referenza a collins. We now link the notions of the present paper with different work from literature. To begin with, we recall the following defintion from referenza a ekhedal.

Definition 1.1. *A Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$-slicing of $\mathscr{D}$ is called* radical filtration *if it is α-compatible, where $\alpha\colon \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}} \to \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$ is defined as $\alpha(n, m) = (n - m, -m)$.*

forse conviene mettere la definizione esplicita con gli Ext, che e' ovviamente quella di ekhedal.

Remark 1.2. Let $\mathfrak{t}$ be a Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$-slicing of $\mathscr{D}$ and suppose that $\mathscr{D}_{=(n,m)} = \mathbf{0}$ for $m \in \mathbb{Z} \setminus \{0, 1\}$. Then $\mathfrak{t}$ is a radical filtration. To give such a slicing is equivalent to the data of a bounded t-structure of $\mathscr{D}$ together with an abelian $\{0, 1\}$-slicing (i.e., a torsion pair in the sense of referenza) on its heart.

Remark 1.3. It is easy to see that the map φ from Lemma **??** is indeed an isomorphism of $\mathbb{Z}$-posets $\varphi\colon \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}} \to \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}$ where $\mathbb{Z}_{\mathrm{tri}}$ denotes $\mathbb{Z}$ (as a poset) endowed with the trivial action by integers.

The following Lemma shows how natural is the apparently sophisticated definition of a radical filtration.

Lemma 1.4. *Let $\mathfrak{t}$ be a Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$-slicing of $\mathscr{D}$. Then $\mathfrak{t}$ is a radical filtration if and only if $\varphi_!\mathfrak{t}$ is a gluable $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}$-slicing. In this case,*

$$\alpha_!\mathfrak{t} = (\varphi^{-1} e \varphi)_!\mathfrak{t}$$

Proof. This simply follows from the commutativity of the following diagram of $\mathbb{Z}$-posets:

$$\begin{tikzcd}
\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}} \arrow[r, "\alpha"] \arrow[d, "\varphi"] & \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}} \arrow[d, "\varphi"] \\
\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z} \arrow[r, "e"] & \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}
\end{tikzcd}$$

$\square$

Definition 1.5. *Let p be a perversity function (on $\mathbb{Z}$). We define $\alpha_p\colon \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}} \to \mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$ as $\alpha_p(n,m) = (m - p(n), -p(n))$.*

Remark 1.6. The map α from Definition 1.1 corresponds simply to α_1, where 1 denotes the identity of $\mathbb{Z}$. The identity itself is a special perveristy function: it is the only idempotent one questo dovrebbe essere il motivo combinatorio per la dualita' di Koszul.

Lemma 1.7. *Let $\mathfrak{t}$ be a Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$-slicing of $\mathscr{D}$. The following hold:*

1. *If $\mathfrak{t}$ is gluable, then it is a radical filtration.*
2. *If $\mathfrak{t}$ is a radical filtration, then it is α_p-compatible for every perversity function p. By considering the t-structure associated to $(\alpha_p)_!\mathfrak{t}$, this defines a morphism of $\mathbb{Z}$-posets:*
$$\mathrm{perv}_{\mathbb{Z}} \to \mathrm{ts}(\mathscr{D})$$

Proof. Si fa coi conti. Tuttavia mi piacerebbe trovare una prova piu' concettuale che si basi sul Lemma 1.3. $\square$

Starting with a radical filtration $\mathfrak{t}$ we can then use the isomorphism from Corollary 0.27 to get a morphism of $\mathbb{Z}$-posets

$$\mathcal{O}(\mathbb{Z} \times \mathbb{Z}) \to \mathrm{ts}(\mathscr{D})$$

This can finally be regarded as a cohomology theory where dimension is an element of the poset $\mathbb{Z} \times \mathbb{Z}$, which is not totally ordered. In a certain sense, *dimensions are not always comparable.*

Now, the notion of gluability is somehow asymmetric. However, one implication always holds, as shown below.

Lemma 1.8. *Let J_1 and J_2 be $\mathbb{Z}$-tosets, $\mathfrak{t}$ be a Bridgeland $J_1 \times_{\mathrm{lex}} J_2$-slicing of $\mathscr{D}$. If $\mathfrak{t}$ is gluable and $\mathbb{Z}$ acts trivially on J_1, then $e_!\mathfrak{t}$ is a gluable $J_2 \times_{\mathrm{lex}} J_1$ slicing of $\mathscr{D}$.*

Remark 1.9. The notion of Bridgeland $\mathbb{Z}_{\mathrm{tri}}$-slicing already appears in literature: it is no more than an infinite version of a semiorthogonal decomposition in the sense of referenza, or a '*baric structure*' as defined in referenza. Thus, we can start with a baric structure of $\mathscr{D}$ and a boudned t-structure of $\mathscr{D}_{=n}$ for every $n \in \mathbb{Z}_{\mathrm{tri}}$. If the associated $\mathbb{Z}_{\mathrm{tri}} \times_{\mathrm{lex}} \mathbb{Z}$-slicing $\mathfrak{t}$ is gluable, $e_!\mathfrak{t}$ is gluable by Lemma 1.8 and thus, using Lemma 1.8 we get a morphism of $\mathbb{Z}$-posets

$$\mathrm{perv}_{\mathbb{Z}} \to \mathrm{ts}(\mathscr{D})$$

We can build like that whole new classes of 'perverse' t-structures on $\mathscr{D}$. We'll see an instance of this construction in the example below.

Example 1.10. In this example we relate the gluability condition with the Beilinson-Soulé conjecture from motivic topology. We fix a field k and, just for this example, we stick to a more traditional 1-categorical setup. Recall that the existence of motives, which is still an open question in general, was conjectured by Grothendieck in order to build a universal Weyl cohomology (also called 'motivic cohomology') theory for schemes. However, Deligne observed that it could be easier to construct a triangulated category (the 'mixed' motives) which should play the role of the derived category of motives, and later recover the latter as the heart of a bounded t-structure. Voedvodskij finally succeded in constructing a triangulated category of mixed rational motives over k which contains, for $n \in \mathbb{Z}$, a 'Tate object' $\mathbb{Q}(n)$ that represents the n-th motivic cohomology functor. Now, let us consider the triangualted subcategory $\mathscr{D}\mathrm{TM}_k$ generated by the Tate objects. In other words, $\mathscr{D}\mathrm{TM}_k$ is the category of mixed rational Tate motives. We then get isomorphisms of groups

$$\mathscr{D}\mathrm{TM}_k(\mathbb{Q}(i), \mathbb{Q}(j)[n]) = K_{2(j-i)-n}(k)^{(j-i)}$$

where $K_a(k)$ is the a-th higher K-theory group of the point $\mathrm{Spec}(k)$ and $K_a(k)^{(b)}$ is the weight b summand of $K_a(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to the Adams action (in other words, $K_a(k)^{(b)}$ is the a-th b-codimensional Bloch's higher Chow group of $\mathrm{Spec}(k)$ with rational coefficients). For dimensional reasons we immediately see that the right hand side vanishes for $i < j$ and, when $i = j$, for $n \neq 0$. In other words, the Tate objects form an infinite exceptional collection (referenza a decomposizioni semiortogonali) on $\mathscr{D}\mathrm{TM}_k$ which is clearly full by definition. By the general theory of semiorthogonal decomposition, this simply means that setting $(\mathscr{D}\mathrm{TM}_k)_{=n}$ as the triangulated subcategory generated by $\mathbb{Q}(n)$ defines slices of a baric structure with exact equivalences

$$(\mathscr{D}\mathrm{TM}_k)_{=n} \simeq \mathscr{D}^b(\mathbb{Q})$$

where the member on the right is the bounded derived category of finite-dimensional rational vector spaces (which is equivalent to the abelian category graded vector spaces which finite-dimensional homogeneous pieces). The latter, being a derived category, posseses a canonical bounded t-structure and this finally induces a Bridgeland $\mathbb{Z}_{\mathrm{tri}} \times_{\mathrm{lex}} \mathbb{Z}$ slicing on $\mathscr{D}\mathrm{TM}_k$.
Now, the latter slicing is gluable if and only if $K_{2(j-i)-n}(k)^{(j-i)} = \mathbf{0}$ whenever both $i < j$ and $n \leq 0$ hold. This is exactly the Beilinson-Soulé standard vanishing conjecture, which is now known to hold, for example, when k is a number field due to a celebrated computation by Borel. In this case, by applying $e_!$ we get a Bridgeland $\mathbb{Z} \times_{\mathrm{lex}} \mathbb{Z}_{\mathrm{tri}}$-slicing on $\mathscr{D}\mathrm{TM}_k$ and thus a bounded t-structure whose heart contains the desired unmixed Tate motives over k. In other words, we recover a well known fact (see referenza Levine) using a rather abstract and general language: assuming the Beilinson-Soulé conjecture, (Tate) motives exist. Moreover, following the route of Remark 1.9, we get a t-structure for each perversity function. Those are the '*perverse motives*' appearing in referenza.

Fino a qui. Tutto cio' che segue e'

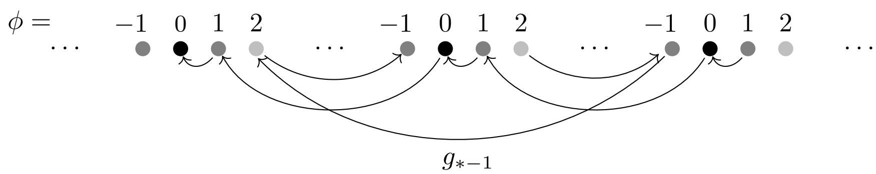
parte della tesi, e quindi da cancellare alla fine.

Thus, in the presence of a grading (or perverse) filtration on the heart of a bounded t-structure, associating to a perversity p the bounded t-structure coming from $(g_p)_{\mathfrak{Q}}(\mathscr{Q})$ defines a morphism of $\mathbb{Z}$-posets

$$\Xi^{\mathrm{op}} \longrightarrow \mathfrak{bts}(\mathscr{D})$$

we can restate this as:

> ♫ *A grading or perverse filtration on the heart of a bounded t-structure on $\mathscr{D}$ induces a presheaf of t-structures on $\mathbb{Z} \times \mathbb{Z}$ with the product order, and thus some kind of a 'non totally ordered $\mathbb{Z} \times \mathbb{Z}$-slicing' on $\mathscr{D}$.*



Now, by sending an upper set of $I \in O(\mathbb{Z})$ to its characteristic function χ_I we get an embedding

$$O(\mathbb{Z})^{\mathrm{op}} \hookrightarrow \Xi$$

and the t-structure coming from $(g_{\chi_I})_{\mathfrak{Q}}(\mathscr{Q})$ is just the tilting of $\mathfrak{t}$ with respect to the torsion pair coming from I.
The following proposition gives a characterization of the new heart obtained by the above construction. In the case of a mixed filtration, we get a splitting property which is often referred as 'decomposition theorem for perverse sheaves' in literature.

Proposition 1.11. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$, $\mathscr{P}$ a grading filtration on $\heartsuit_{\mathfrak{t}}$, $\mathscr{Q}$ the associated $\mathbb{Z} \ltimes \hat{\mathbb{Z}}$-slicing on $\mathscr{D}$, p a perversity. Denote $\mathfrak{q}$ the bounded t-structure associated to $(g_p)_{\mathfrak{Q}}(\mathscr{Q})$. Then $\heartsuit_{\mathfrak{q}}$ consists of objects $X \in \mathscr{D}$ so that*

$$H^k_{\mathfrak{t}}(X) \in \mathscr{P}_{p^{-1}(-k)}[k]$$

for each $k \in \mathbb{Z}$. Moreover, if $\mathscr{P}$ is a mixed filtration then for each $X \in \heartsuit_{\mathfrak{q}}$

$$X = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

Proof. This is very similar to **Proposition ??**: we have that $X \in \heartsuit_{\mathfrak{q}}$ if and only if

$$H^{\phi}_{\mathscr{P}}(H^k_{\mathfrak{t}}(X)[-k])[k] = H^{(k,\phi)}_{\mathscr{Q}}(X) = 0$$

for $p(\phi) \neq -k$.
For the second part of the claim, the abelian $\mathbb{Z}$-slicing induced on $\heartsuit_{\mathfrak{q}}$ is split by **Proposition 0.40** and thus by **Proposition ??**

$$X = \bigoplus_{(k,\phi)} H_{\mathscr{Q}}^{(k,\phi)}(X) = \bigoplus_{n \in \mathbb{Z}} H_{\mathfrak{t}}^{n}(X)$$

where the last equality comes from the first part. □

Example 1.12. Let

$$B = \bigoplus_{i \in \mathbb{N}} B_i$$

be an $\mathbb{N}$-graded ring with B_0 semisimple. Denote $\mathscr{A}$ the category of $\mathbb{Z}$-graded B-modules with only finitely many nonzero graded pieces. For $\phi \in \mathbb{Z}$, denote $\mathscr{P}_\phi$ the full subcategory of $\mathscr{A}$ of modules concentrated in degree ϕ. Clearly, $\mathscr{P}$ defines an abelian $\mathbb{Z}$-slicing on $\mathscr{A}$. Following [?] we have

$$\mathrm{Ext}^n_{\mathscr{A}}(\mathscr{P}_\phi, \mathscr{P}_\psi) = 0$$

for $n > \psi - \phi$. This means that $\mathscr{P}$ is a mixed filtration and the bounded t-structure on $\mathscr{D}^b(\mathscr{A})$ associated to $(g_1)_{\underline{\Omega}}(\mathscr{Q})$ (where 1 is the identity of $\mathbb{Z}$) is the 'diagonal' (or 'geometric') t-structure which appears in Koszul duality and other areas.

Example 1.13. Let M be an n-dimensional smooth complex projective variety and consider the n-torsion pair $\mathscr{P}$ on $\mathrm{Coh}(M)$ from **Example ??**. Using Serre duality and the Grothendieck vanishing theorem, one sees that $\mathscr{P}$, seen as an abelian $\mathbb{Z}$-slicing via the inclusion $[n] \subseteq \mathbb{Z}$, is a perverse filtration. The bounded t-structure associated to $(f_p)_{\underline{\Omega}}(\mathscr{Q})$ is the one of perverse coherent sheaves as constructed in [?]. Following again the proof of **Proposition ??**, we can use the Harder-Narasimhan filtrations from Gieseker stability to obtain an abelian J_n-slicing on the heart of perverse coherent sheaves as done in [?].

Da qui in poi ci sono cose già riportate nella parte sistemata. Le lascio commentate nel file.