Continuous logic for the classical logician

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ABSTRACT. Let $\mathcal{L}$ be a first-order two-sorted language. In general, very little model theory is possible if we restrict to $\mathcal{L}$-structures that have the form $\langle M, I\rangle$ for some fixed structure I. But not everything is lost when I is a compact topological space. Continuous logic deals with this case. Rivisiting old ideas we present a back-to-basics approach to continuous logic.

1. Introduction

As a minimal motivating example, consider a real vector space M. This is are among the most simple strucures considered in model theory. Now, expand it by adding a norm. A norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two-sorted structure $\langle M, \mathbb{R}\rangle$. Now, we have two conflicting aspirations

i. to apply the basic tools of model theory (such as elementarity, compactness, and saturation);

ii. to stay within the realm of normed spaces (hence insist that $\mathbb{R}$ should remain $\mathbb{R}$ throughout).

These two requests are blatantly incompatible, something has to give. Different people have attemped different paths.

1. Nonstandard analysts happily embrace norms that take values in some ${}^*\mathbb{R} \succeq \mathbb{R}$. These normed spaces are nonstandard, but there are tricks to transfer results from nonstandard to standard normed spaces and vice versa. See [G] for a survey. Model theorists, when confronted with similar problems, have used similar approaches (in greater generality and with different notation).

2. Henson and Iovino propose to stick to the standard notion of norm but restrict the notion of elementarity and saturation to a smaller class of formulas. They demostrate that most standard tools of model theory apply in their setting. The work of Henson and Iovino is focussed on Banach spaces and has hardly been applied to broader contexts. See [HI] for a survey.

3. Real valued logicians opt for an esoteric approach. They abandon classical true/false valued logic in favour of a logic valued in the interval $[0,1]$. See [K] or [H] for a survey. Arguably, this esoteric flavor helped the approach with some popularity. Unfortunately, this approach adds notational burden and unnatural restrictions. Our down-to-classical-logic approach is an alternative.

We elaborate on the ideas of Henson and Iovino and generalize them to a larger class of structures. Following Henson and Iovino we restrict the notions of elementarity and saturation though we use a larger class of formulas. We also borrow intuitions and results from nonstandard analisys to an extent that we could claim the title: *nonstandard analysis for the standard logician.*

2. A class of structures

Let I be some fixed first-order structure which is endowed with a Hausdorff compact[1] topology (in particular, a normal topology). The language L_{I} contains relation symbols for the compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \to I$. In particular, there is a constant for

[1] The requirement of compactness seems to exclude the motivational example mentioned above. To include this and others similar examples we have to replace $\mathbb{R}$ with $\mathbb{R} \cup \{\pm\infty\}$.

each element of I. According to the context, C and f denote either the symbols of L_{I} or their interpretation in the stucture I. The most common examples of structures I are the unit interval $[0,1]$, with the usual topology, and its homeomophic copies $\mathbb{R}^+ \cup \{0,\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language L_{H} which we call the language of the home sort.

Definition 1. Let $\mathcal{L}$ be a two sorted language that expands both L_{H} and L_{I}. By model we mean a two-sorted $\mathcal{L}$-structure of the form $\langle M, I\rangle$, where M ranges over all L_{H}-structures, while I is the structure we fixed above. We write H and I to denote the two sorts of $\mathcal{L}$. The new symbols allowed in $\mathcal{L}$ are function of sort $\mathsf{H}^n \to \mathsf{I}$.

Models are denoted by the domain of their home sort.

The notion of model will be (inessentially) restricted once the monster model is introduced in Section 7.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (when I is finite), they are not models in the sense of Definion 1. As a remedy, below we carve out a set of formulas $\mathcal{J}$, such that $L_{\mathsf{H}} \subseteq \mathcal{J} \subseteq \mathcal{L}$ and every model has an $\mathcal{J}$-elementary, $\mathcal{J}$-saturated extension that is also a model.

Warning: as usual, L_{I}, L_{H}, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The symbols $\mathcal{J}$ and $\mathcal{H}$ defined below denote only sets of $\mathcal{L}$-formulas.

Definition 2. Formulas in $\mathcal{J}$ are constructed inductively from two sorts of $\mathcal{J}$-atomic formulas

i. formulas of the form $t(x;\eta) \in C$, where $C \subseteq I^n$ is compact and $t(x;\eta)$ is a tuple of terms of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$;

ii. all formulas in L_{H}.

We require that $\mathcal{J}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{I}}$, $\exists^{\mathsf{I}}$ of sort I. We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort I. Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

We write $\mathcal{H}$ for the set of formulas in $\mathcal{J}$ with only quantifiers of sort H (i.e. without quantifiers of sort I).

Formulas in $\mathcal{H}$ generalize the positive bounded formulas of Henson and Iovino to a larger class of structures. But $\mathcal{H}$ also enlarges the class of Henson and Iovino as it comprises all L_{H}-formulas. This enlargment comes at a price: the failure of the Perturbation Lemma [HI, Proposition 5.15]. However, we will recover part of Perturbation Lemma in Proposition 31.

In Section 6 we prove the Compactness Theorem for $\mathcal{J}$. This class expands $\mathcal{H}$ by allowing quantifiers of sort I. The extra expressivity is convenient and comes almost for free. For instance, it is easy to see (cf. Example 3 for a hint) that $\mathcal{J}$ has at least the same expressive power as real valued logic[2]. Somewhat surprisingly, we will see (cf. Propositions 25 and 26) that the formulas in $\mathcal{J}$ can be approximated by formulas in $\mathcal{H}$.

Example 3. Let $I = [0,1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort $\mathsf{H}^{|x|} \to \mathsf{I}$. Then there is a formula in $\mathcal{J}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall x\, \big[t(x) \dot{-} \tau \in \{0\}\big] \quad \wedge \quad \forall \varepsilon \Big[\varepsilon \in \{0\} \ \vee\ \exists x\, \big[\tau \dot{-} (t(x)+\varepsilon) \in \{0\}\big]\Big]$$

which, in a more legible form, becomes

[2] This unsubstantited claim should be interpreted as concjecture. A painstaking technical comparison with real valued logic is not in the scope of this indroductory exploratory paper.

$$\forall x\left[t(x) \leq \tau\right] \quad \wedge \quad \forall \varepsilon>0\, \exists x\left[\tau \leq t(x)+\varepsilon\right].$$

The following example is inspired by [HPP].

Example 4. Let $\mathcal{U}$ be a monster model of signature L_{H}. Let $\mathcal{L}$ contain, besides L_{H} and L_{I}, symbols for the functions $f:\mathcal{U}^n \to \mathsf{I}$ that are continuous in the following sense: the inverse image of every compact $C \subseteq I$ is L_{H}-type-definable over every $M \preceq \mathcal{U}$. It is easy to verify that the model $\langle \mathcal{U}, I\rangle$ is $\mathcal{J}$-saturated, see definition in Section 6.

We mentioned normed spaces as a motivating example. Essentially, functional analysis and model theory are only interested in the unit ball of normed spaces. To formalize the unit ball of a normed space as a model (where I is the unit interval) is straightforward but pretty ugly. Alternatively one can view a normed space as a many-sorted structure: a sort for each ball of radius $n \in \mathbb{Z}^+$, but this is also a bloated formalism.

A neater formalization is possible if one accepts that some $\mathcal{J}$-elementary extension of a normed space could contain vectors of infinite norm hence not be themselves normed spaces. These infinities are harmless if one is only interested in the unit ball. In fact, inside any ball of finite radius these nonstandard normed spaces are completely standard.

Example 5. Let $I = \mathbb{R} \cup \{\pm\infty\}$. Let L_{H} be the language of real (or complex) vector spaces. The language $\mathcal{L}$ contains a function symbol $\|\text{-}\|$ of sort $\mathsf{H} \to \mathsf{I}$. Normed space are models with the natural interpretation of the language. It is easy to verify that the unit ball of every model is a the unit ball of a normed space. Note that the same remains true if we add to the language any continuous (a.k.a. bounded) operator or functional.

We conclude this introduction with a question. Is it possible to extend the Compactness Theorem 16 to a larger class of formulas? E.g. can we allow in $\mathcal{L}$ (and in $\mathcal{J}$) function symbols of sort $\mathsf{H}^n \times \mathsf{I}^m \to \mathsf{I}$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I.

3. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathcal{J}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained by replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C. Note that $>$ is a dense (pre)order of $\mathcal{J}(M)$. Formulas in (i) of Definition 2 do not occur under the scope of a negation therefore $\varphi \to \varphi'$ in all $\mathcal{L}$-structures.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. Moreover the $\mathcal{J}$-atomic formulas in L_{H} are replaced with their negation and each connective is replaced with its dual i.e., $\vee, \wedge, \exists, \forall$ are replaced with $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde{\varphi}$ is a strong negation of φ. It is clear that $\tilde{\varphi} \to \neg\varphi$.

Lemma 6. For all $\varphi \in \mathcal{J}(M)$

1. for every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$;
2. for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L_{\mathsf{H}}$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in I \smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C.

By the normality of I, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. □

For $p(x) \subseteq \mathcal{I}(A)$, we write

$$p'(x) \quad = \quad \{\varphi'(x) \ : \ \varphi' > \varphi \text{ for some } \varphi(x) \in p\}$$

in particular $\{\varphi(x)\}' = \{\varphi'(x) \ : \ \varphi' > \varphi\}$. The following is an interesting fact which we do not know if it holds for arbitrary I.

At face value the following fact sounds stronger then it is in reality. In fact, though we will prove a form of compactness for $\mathcal{I}$-formulas, this is only applicable in Fact 21. Hence we may not conclude that $\{\varphi(x)\}'$ is equivalent to some finite subset.

Proposition 7. Suppose $I = [0,1]$. Then for every $\varphi(x) \in \mathcal{I}$ there is a formula $\psi(x) \in \mathcal{I}$ such that

$$\psi(x) \quad \leftrightarrow \quad \{\varphi(x)\}'$$

holds in every model.

Proof. ??? □

4. Morphisms

Let M and N be two models. We say that a partial map $f : M \to N$ is $\mathcal{I}$-elementary if for every $\varphi(x) \in \mathcal{I}$ and every $a \in (\operatorname{dom} f)^{|x|}$

1. $$M \models \varphi(a) \quad \Rightarrow \quad N \models \varphi(f a).$$

An $\mathcal{I}$-elementary map that is total is called an $\mathcal{I}$-elementary embedding. When the map $\mathrm{id}_M : M \to N$ is an $\mathcal{I}$-elementary embedding, we write $M \preceq^{\mathcal{I}} N$ and say that M is an $\mathcal{I}$-elementary submodel of N.

The definitions of $\mathcal{H}$-elementary map/embedding/submodel is obtained replacing $\mathcal{I}$ by $\mathcal{H}$.

As $\mathcal{I}$ and $\mathcal{H}$-elementary maps are in particular L_{H}-elementary, they are injective. However, their inverse need not be $\mathcal{I}$, respectively $\mathcal{H}$-elementary. In other words, the converse of the implication in (1) may not hold. We have chosen to work with the classical notion of satisfaction at the cost of this asymmetric notion of elementarity. This differs from the approach of Henson and Iovino. They introduce the notion of *approximate satisfaction*. The approximate $\mathcal{H}$-morphisms we define below are the maps that preserve Henson-Iovino approximate satisfaction. These morphisms are invertible, cf. Fact 8.

We say that the map $f : M \to N$ is an approximate $\mathcal{I}$-elementary if for every formula $\varphi(x) \in \mathcal{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$

$$M \models \varphi(a) \quad \Rightarrow \quad N \models \big\{\varphi(f a)\big\}'.$$

We define approximate $\mathcal{H}$-elementary maps in the same manner (from Proposition 25 it follows that the two notions coincide). It is clear that $\mathcal{I}$-elementarity implies its approximate variant, similarly for $\mathcal{H}$-elementarity. We will see that with a slight amount of saturation also the converse holds (Proposition 18). Finally, under full saturation, all differences disappear as all these morphims becomes $\mathcal{L}$-elementarity maps (Corollary 24).

The following holds in general.

Fact 8. If $f : M \to N$ is approximate $\mathcal{I}$-elementary then

$$M \models \big\{\varphi(a)\big\}' \quad \Leftrightarrow \quad N \models \big\{\varphi(f a)\big\}'.$$

for every $\varphi(x) \in \mathcal{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$. The same holds for approximate $\mathcal{H}$-elementary maps.

Proof. $\Rightarrow$ Fix $\varphi' > \varphi$ and assume $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By the definition of approximate elementarity, $N \models \varphi''(fa)$ and $N \models \varphi'(fa)$ follows.

$\Leftarrow$ Assume the r.h.s. of the equivalence. Fix $\varphi' > \varphi$ and prove $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By Lemma 6 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \rightarrow \neg\tilde{\varphi} \rightarrow \varphi'$. Then $N \models \neg\tilde{\varphi}(fa)$ and therefore $M \models \neg\tilde{\varphi}(a)$. Then $M \models \varphi'(a)$. □

Finally, we introduce the morphisms that corrisponds to the classical partial embeddings. We say that the map $f : M \rightarrow N$ is a partial $\mathcal{I}$-embedding if the implication in (1) holds for all $\mathcal{I}$-atomic formulas $\varphi(x)$. Note that this is the same as requiring that (1) holds for all formulas without quantifiers of sort H (unless they occour in L_{H} formulas).

Fact 9. If $f : M \rightarrow N$ is a partial $\mathcal{I}$-embedding then

$$M \models \varphi(a) \quad \Leftrightarrow \quad N \models \varphi(fa)$$

for every $a \in (\operatorname{dom} f)^{|x|}$ and every $\mathcal{I}$-formulas $\varphi(x)$ without quantifiers of sort H.

Proof. The equivalence is trivial for formulas in L_{H} so we only consider $\mathcal{I}$-atomic formulas as in (i) of Definion 1. Implication $\Rightarrow$ holds by definition. Vice versa, if $M \models t(a) \notin C$ then, by normality, $M \models t(a) \in \tilde{C}$ for some compact $\tilde{C}$ disjoint of C. By the definition of $\mathcal{I}$-embedding, $N \models t(a) \in \tilde{C}$. Hence $N \models t(a) \notin C$. Induction is immediate. □

5. The standard part

Our goal is to prove the Compactness Theorem for $\mathcal{I}$. To bypass lengthy applications of ultrafilters, ultrapowers, and ultralimits, we assume the Classical Compactness Theorem and built on that.

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 14 which in turn is required for the proof of the Compactness Theorem for $\mathcal{I}$ (Theorem 16). The reader willing to accept it without proof may skip this section.

Let $\langle N, {}^*I\rangle$ be an $\mathcal{L}$-structure that extends $\mathcal{L}$-elementarily the model M. Let η be a free variable of sort I. For each $\beta \in I$, we define the type

$$\mathrm{m}_\beta(\eta) \quad = \quad \{\eta \in D \,:\, D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(\eta)$ in ${}^*I$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 10. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Proof. Negate the existence of β. For every $\gamma \in I$ pick some compact neighborhood D_γ of γ, such that ${}^*I \models \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ, with $\gamma \in \Gamma$, cover I. By elementarity these D_γ also cover ${}^*I$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Fact 11. For every $\alpha \in {}^*I$ and every compact $C \subseteq I$

$${}^*I \quad \models \quad \alpha \in C \ \rightarrow\ \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*I \models \alpha \in D \subseteq \neg C$. □

Fact 12. For every $\alpha \in ({}^*I)^{|\alpha|}$ and every function symbol f of sort $\mathsf{I}^{|\alpha|} \to \mathsf{I}$

$$ {}^*I \quad \models \quad \operatorname{st}\big(f(\alpha)\big) = f\big(\operatorname{st}(\alpha)\big). $$

Proof. By Fact 10 and the definition of st(-) it suffices to prove that ${}^*I \models f(\alpha) \in D$ for every compact neighborhood D of $f\big(\operatorname{st}(\alpha)\big)$.

Fix one such D. Then $\operatorname{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\operatorname{st}(\alpha)$. Therefore ${}^*I \models \alpha \in f^{-1}[D]$ and, as $I \preceq {}^*I$ we obtain ${}^*I \models f(\alpha) \in D$. □

The standard part of $\langle N, {}^*I\rangle$ is the model $\langle N, I\rangle$ that interprets the symbols f of sort $\mathsf{H}^n \to \mathsf{I}$ as the functions

$$ f^N(a) \quad = \quad \operatorname{st}\big({}^*f(a)\big) \qquad \text{for all } a \in N^n, $$

where ${}^*f$ is the interpretation of f in $\langle N, {}^*I\rangle$. Symbols in L_{H} maintain the same interpretation.

Fact 13. With the notation as above. Let $t(x;\eta)$ be a term of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$. Then for every $a \in M^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$ t^N\big(a;\operatorname{st}(\alpha)\big) \quad = \quad \operatorname{st}\big({}^*t(a;\alpha)\big) $$

Proof. When a t is function symbol of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{I}^{|\eta|} \to \mathsf{I}$, the claim follows from Fact 12. Now, assume inductively that

$$ t_i^N\big(a;\operatorname{st}(\alpha)\big) \quad = \quad \operatorname{st}\big({}^*t_i(a;\alpha)\big) $$

holds for the terms $t_1(x;\eta),\ldots,t_n(x;\eta)$ and let $t = f(t_1,\ldots,t_n)$ for some function f of sort $\mathsf{I}^n \to \mathsf{I}$. Then the claim follows immediately from the induction hypothesis and Fact 12. □

Lemma 14. With the notation as above. For every $\varphi(x;\eta) \in \mathcal{I}$, $a \in N^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$ \langle N, {}^*I\rangle \models \varphi(a;\alpha) \quad \Rightarrow \quad N \models \varphi\big(a;\operatorname{st}(\alpha)\big) $$

Proof. Suppose $\varphi(x;\eta)$ is $\mathcal{I}$-atomic. If $\varphi(x;\eta)$ is a formula of L_{H} the claim is trivial. Otherwise $\varphi(x;\eta)$ has the form $t(x;\eta) \in C$. Assume that the tuple $t(x;\eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, {}^*I\rangle \models t(a;\alpha) \in C$. Then $\operatorname{st}\big({}^*t(a;\alpha)\big) \in C$ by Fact 11. Therefore $t^N\big(a;\operatorname{st}(\alpha)\big) \in C$ follows from Fact 13. This proves the lemma for $\mathcal{I}$-atomic formulas. Induction is immediate. □

Corollary 15. Let M be an arbitrary model. Let $p(\eta) \subseteq \mathcal{I}(M)$ be a type that does not contain existential quantifiers of sort H. Then, if $p(\eta)$ is finitely consistent in M, it is realized in M.

The corollary has also a direct proof. This goes through the observation that the formulas in $p(\eta)$ define compact subsets of I.

Proof. Let $\langle {}^*M, {}^*I\rangle$ be a $\mathcal{L}$-elementary saturated superstructure of $\langle M, I\rangle$. Then $\langle {}^*M, {}^*I\rangle \models p(\alpha)$ for some $\alpha \in^* I$. By the lemma above, ${}^*M \models p(\operatorname{st}(\alpha))$. Now observe that the truth of formulas without existential quantifiers of sort H is preserved by substructures. □

The exclusion of existential quantifiers of sort H is necessary. For a counterexample take $M = I = [0,1]$. Assume that $\mathcal{L}$ contains a function symbol for the identity map $\iota : M \to I$. Let $p(\eta)$ contain the formulas $\exists x\,(x > 0 \wedge \iota(x) + \eta \in [0,1/n]$ for all positive integers n.

6. Compactness

It is convenient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\langle M, {}^*I\rangle \models T$ for some $\mathcal{L}$-structure. We say that T is consistent when $M \models T$ is for some model M.

Theorem 16 (Compactness Theorem for $\mathfrak{I}$). Let $T \subseteq \mathfrak{I}$ be finitely consistent. Then T is consistent.

Proof. Suppose T is finitely consistent (or finitely $\mathcal{L}$-consistent, for that matter). By the classical Compactness Theorem, there is an $\mathcal{L}$-structure $\langle M, {}^*I\rangle \models T$. Let $\langle M, I\rangle$ be its standard part as defined in Section 5. Then $M \models T$ by Lemma 14. □

A model N is λ-$\mathfrak{I}$-saturated if it realizes all types $p(x;\eta) \subseteq \mathfrak{I}(N)$ with fewer than λ parameters that are finitely consistent in N. When $\lambda = |N|$ we simply say $\mathfrak{I}$-saturated. The existence of $\mathfrak{I}$-saturated models is obtained from the classical case just as for Theorem 16.

Theorem 17. Every model has an $\mathfrak{I}$-elementary extension to a saturated model (possibly of inaccessible cardinality).

The following proposition shows that a slight amount of saturation tames the $\mathfrak{I}$-formulas.

Proposition 18. Let N be an ω-$\mathfrak{I}$-saturated model. Then

$$\{\varphi(x;\eta)\}' \;\leftrightarrow\; \varphi(x;\eta)$$

holds in N for every formula $\varphi(x,;\alpha) \in \mathfrak{I}(N)$.

Proof. We prove $\rightarrow$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

ih. $$\{\varphi(x,z;\eta)\}' \;\rightarrow\; \varphi(x,z;\eta)$$

We need to prove

$$\{\exists z\,\varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\,\varphi(x,z;\eta)$$

From (ih) we have

$$\exists z\,\{\varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\,\varphi(x,z;\eta)$$

Therefore it suffices to prove

$$\{\exists z\,\varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\,\{\varphi(x,z;\eta)\}'$$

Replace the variables $x;\eta$ with parameters, say $a;\alpha$ and assume that the theory $\{\exists z\,\varphi'(a,z,;\alpha) : \varphi' > \varphi\}$ is true in N. We need to prove the consistency of the type $\{\varphi'(a,z,;\alpha) : \varphi' > \varphi\}$. By saturation, finite concistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly. □

Remark 19. By Corollary 15, when $\varphi(x;\eta)$ does not contains existential quantifiers of sort H, the proposition above does not require the assumption of saturation. In general, some saturation is necessary: consider the model presented after Corollary 15 and the formula $\exists x\,(x > 0 \wedge \iota(x) \in \{0\})$.

A consequence of Proposition 18 is that the approximate morphisms defined in Section 4 coincide with their unapproximate version when the codomain is a ω-$\mathfrak{I}$-saturated model. In particular, the following follows from Fact 8.

Corollary 20. Let M be an ω-$\mathfrak{I}$-saturated model. Let $f : M \to N$ be an $\mathfrak{I}$-elementary map. Then

$$M \models \varphi(a) \quad \Leftrightarrow \quad N \models \varphi(fa)$$

for every $\mathfrak{I}$-formula $\varphi(x)$ and every $a \in (\operatorname{dom} f)^{|x|}$.

7. The monster model

We denote by $\mathfrak{U}$ some large $\mathfrak{I}$-saturated model which we call the monster model. For convenience we assume that the cardinality of $\mathfrak{U}$ is an inaccessible cardinal. Truth is evaluated in $\mathfrak{U}$ unless otherwise is specified. Below we say model for $\mathfrak{I}$-elementary submodel of $\mathfrak{U}$. We stress once again that the truth of some $\varphi \in \mathfrak{I}(M)$ in a model M implies the truth of φ (in $\mathfrak{U}$) but not vice versa. If also the converse implication holds we say M a strong model. However, all models agree on the approximated truth, see Fact 8. Hence strong models are those that realize the equivalence in Proposition 18, e.g. ω-$\mathfrak{I}$-saturated models are strong models.

Let $A \subseteq \mathfrak{U}$ be a small set throughout this section. We define a topology on $\mathfrak{U}^{|x|}$ which we call the $\mathfrak{I}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x) \subseteq \mathfrak{I}(A)$. This is a compact topology by the Compactness Theorem for $\mathfrak{I}$. The following fact demostrate how $\mathfrak{I}$-compactness applies in this context. There are some subtle differences from the classical setting.

Fact 21. Let $p(x) \subseteq \mathfrak{I}(A)$ be a type. Then for every $\varphi(x) \in \mathfrak{I}(\mathfrak{U})$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \neg\varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (1) is immediate by saturation; (2) follows from (1) by Lemma 6. □

When $A \subseteq \mathfrak{U}$, we write $S_{\mathfrak{I}}(A)$ for the set of types

$$\mathfrak{I}\text{-tp}(a/A) \quad = \quad \big\{\varphi(x) \ : \ \varphi(x) \in \mathfrak{I}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of $\mathfrak{U}$. We write $S_{\mathfrak{I},x}(A)$ when the tuple of variables x is fixed. The same notation applies also with $\mathcal{H}$ for $\mathfrak{I}$.

The following proposition will be strengthen by Corollary 24 below.

Proposition 22. The types $p(x) \in S_{\mathfrak{I}}(A)$ are maximally consistent subsets of $\mathfrak{I}_x(A)$. That is, for every $\varphi(x) \in \mathfrak{I}(A)$, either $\varphi(x) \in p$ or $p(x) \to \neg\varphi(x)$. The same holds with $\mathcal{H}$ for $\mathfrak{I}$.

Proof. Let $p(x) = \mathfrak{I}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 6 and Proposition 18 we obtain

$$\neg\varphi(x) \quad \to \quad \bigvee_{\tilde\varphi \perp \varphi} \tilde\varphi(x).$$

Hence $\tilde\varphi(a)$ holds for some $\tilde\varphi \perp \varphi$ and $p(x) \to \neg\varphi(x)$ follows. □

8. Approximate elimination of quantifiers of sort I

We show that the quantifiers $\forall^{\mathsf{I}}$ and $\exists^{\mathsf{I}}$ can be eliminated up to some approximation.

Proposition 23. The monster model (or any saturated model, for that matter) is $\mathcal{H}$-homogeneous, that is, every $\mathcal{H}$-elementary map $f:\mathcal{U}\to\mathcal{U}$ of cardinality $<|\mathcal{U}|$ extends to an automorphism.

Proof. By Proposition 22, the inverse of an $\mathcal{H}$-elementary map $f:\mathcal{U}\to\mathcal{U}$ is $\mathcal{H}$-elementary. Then the usual proof by back-and-forth applies. □

As promised, we strengthen Proposition 22. Let $A\subseteq\mathcal{U}$ be a small set throughout this section.

Corollary 24. Let $p(x)\in S_{\mathcal{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathcal{L}_x(A)$. That is, for every $\varphi(x)\in\mathcal{L}(A)$, either $p(x)\to\varphi(x)$ or $p(x)\to\neg\varphi(x)$. Clearly, the same holds for $p'(x)$.

Proof. If $b\models p(x)=\mathcal{H}\text{-tp}(a/A)$ then there is an $\mathcal{H}$-elementary map $f\supseteq \mathrm{id}_A$ such that $fa=b$. Then f exends to an automorphism. As every automorphism is $\mathcal{L}$-elementary, the corollary follows. □

For $a,b\in\mathcal{U}^{|x|}$ we write $a\equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A, bearing in mind that formulas in $\mathcal{H}(A)$ or $\mathcal{I}(A)$, or their approximations, suffice to test the equivalence.

In the classical setting, from an equivalence between types one derives an equivalence between formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence. Next proposition show that formulas in $\mathcal{I}(A)$ are approximated by formulas in $\mathcal{H}(A)$.

Proposition 25. Let $\varphi(x)\in\mathcal{I}(A)$. For every given $\varphi'>\varphi$ there is some formula $\psi(x)\in\mathcal{H}(A)$ such that $\varphi(x)\to\psi(x)\to\varphi'(x)$.

Proof. By Corollary 24 and Proposition 18

$$\neg\varphi(x)\ \to\ \bigvee_{p'(x)\to\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathcal{H}}(A)$. By Fact 21 and Lemma 6

$$\neg\varphi(x)\ \to\ \bigvee_{\neg\tilde\psi(x)\to\neg\varphi(x)} \neg\tilde\psi(x),$$

where $\tilde\psi(x)\in\mathcal{H}(A)$. Equivalently,

$$\varphi(x)\ \leftarrow\ \bigwedge_{\tilde\psi(x)\leftarrow\varphi(x)} \tilde\psi(x).$$

By compactness, see Fact 21, for every $\varphi'>\varphi$ there are some finitely many $\tilde\psi_i(x)\in\mathcal{H}(A)$ such that

$$\varphi'(x)\ \leftarrow\ \bigwedge_{i=1,\dots,n}\tilde\psi_i(x)\ \leftarrow\ \varphi(x)$$

which yields the interpolant required by the proposition. □

We also need an approximation restult for the negation of formulas in $\mathcal{I}(A)$.

Proposition 26. Let $\varphi(x)\in\mathcal{I}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x)\to\neg\varphi(x)$ for some consistent $\psi(x)\in\mathcal{H}(A)$ and some $\psi'>\psi$.

Proof. Let $a\in\mathcal{U}^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x)=\mathcal{H}\text{-tp}(a/A)$. By Corollary 24, $p'(x)\to\neg\varphi(x)$. By compactness $\psi'(x)\to\neg\varphi(x)$ for some $\psi'>\psi\in p(x)$. □

9. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is not literally the Tarski-Vaught test because it only applies when the larger structure is the moster model.

Proposition 27. Let M be a subset of $\mathcal{U}$. Then the following are equivalent

1. M is a model;
2. for every formula $\varphi(x) \in \mathcal{H}(M)$
$$\exists x\, \varphi(x) \quad \Rightarrow \quad \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathcal{I}(M)$
$$\exists x \neg\varphi(x) \quad \Rightarrow \quad \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 6 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\forall x\, \tilde{\varphi}(x)$ hence, by (1), $M \models \neg\forall x\, \varphi(x)$. Then $M \models \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $M \models \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathcal{I}(M)$ be such that $\exists x \neg\varphi(x)$. By Corollary 26, there are a consistent $\psi(x) \in \mathcal{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). By the classical Tarski-Vaught test $M \preceq_{\mathsf{H}} \mathcal{U}$. Then M is the domain of an $\mathcal{L}$-substructure of $\mathcal{U}$. Then $\varphi(a) \Leftrightarrow M \models \varphi(a)$ holds for every $\mathcal{I}$-atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \models \varphi(a, b) \quad \Rightarrow \quad \varphi(a, b).$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$M \models \forall y\, \varphi(a, y) \quad \Rightarrow \quad \forall y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned}
\neg\forall y\, \varphi(a, y) \quad &\Rightarrow \quad \exists y \neg\varphi(a, y) \\
&\Rightarrow \quad \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow \quad M \models \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow \quad M \not\models \forall y\, \varphi(a, y).
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists^{\mathsf{H}}$, $\exists^{\mathsf{I}}$, and $\forall^{\mathsf{I}}$ is straightforward. □

Classically, the main application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all $\mathcal{L}$-structures. In particular every $A \subseteq \mathcal{U}$ is contained in a model of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of $\mathcal{L}$ is eccessively large because of the aboundance of symbols in L_{I}.

We say that $\mathcal{L}$ is separable if there is a countable $\mathcal{H}_0 \subseteq \mathcal{H}$ such that for every $\varphi' > \varphi \in \mathcal{H}$ there is a $\varphi_0 \in \mathcal{H}_0$ such that $\varphi \to \varphi_0 \to \varphi'$ holds in every $\mathcal{L}$-structure.

Proposition 28. Assume that $\mathcal{L}$ is separable. Let $A \subseteq \mathcal{U}$ be countable. Then there is a countable model M containing A.

Proof. As in the classical proof of the Löwenheim-Skolem Theorem, we construct $M \subseteq \mathcal{U}$ that contain a witness for every consistent formula in $\mathcal{H}_0(M)$. By separability and (2) in Proposition 27, M is a model. □

We have defined separability as a property of $\mathcal{L}$. Arguably, it could have been introduced as a property of (the theory of) $\mathcal{U}$. The following proposition provides the examples that motivate the definition of separability (in these examples the property depends soley on $\mathcal{L}$).

Proposition 29. Assume that I is second countable, and that there are at most countably many symbols in $\mathcal{L} \smallsetminus L_{\mathsf{I}}$. Then $\mathcal{L}$ is separable.

Proof. ??? □

10. Cauchy complete models

For $t(x,z)$ a term of sort $\mathsf{H}^{|x|+|z|} \to \mathsf{I}$ we define the formula

$$x \sim_t y \quad = \quad \forall z\ \ t(x,z) = t(y,z).$$

We also define the type

$$x \sim_{\mathsf{I}} y \quad = \quad \Big\{ x \sim_t y \ :\ t(x,z) \text{ as above} \Big\}.$$

Fact 30. For any $a = \langle a_i : i < \lambda \rangle$ and $b = \langle b_i : i < \lambda \rangle$

$$a \sim_{\mathsf{I}} b \quad \Leftrightarrow \quad a_i \sim_{\mathsf{I}} b_i \quad \text{for every } i < \lambda.$$

Proof. Only implication $\Leftarrow$ requires a proof. Assume $a_i \sim_{\mathsf{I}} b_i$ for every i. Let $|a| = |b| = \lambda$ and assume inductively that

$$\forall y, z \ \ t(a,y,z) \ = \ t(b,y,z)$$

holds for every term $t(x,y,z)$, with $|x| = \lambda$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

1. $$\forall z \ \ t(a,a_\lambda,z) \ = \ t(b,a_\lambda,z).$$

As $a_\lambda \sim_{\mathsf{I}} b_\lambda$ then

$$\forall x,z \ \ t(x,a_\lambda,z) \ = \ t(x,b_\lambda,z)$$

and in particular

2. $$\forall z \ \ t(b,a_\lambda,z) \ = \ t(b,b_\lambda,z).$$

From (1) and (2) we obtain

$$\forall z \ \ t(a,a_\lambda,z) \ = \ t(b,b_\lambda,z).$$

For limit ordinals induction is trivial. □

Note that the approximations of the formula $x \sim_t y$ have the form

$$x \sim_{t,D} y \quad = \quad \forall z \ \ \langle t(x,z), t(y,z) \rangle \in D$$

for some compact neighborhood D of Δ, the diagonal of I^2. Hence we define

$$x \sim'_{\mathsf{I}} y \quad = \quad \Big\{ x \sim_{t,D} y \ :\ t(x,z) \text{ parameter-free term, } D \text{ compact neigborhood of } \Delta \Big\}.$$

If $t(x,z) = t_1(x,z),\ldots,t_n(x,z)$ is a tuple of terms, by $x \sim_{t,D} y$ we denote the conjunction of the formulas $x \sim_{t_i,D} y$. As D ranges over the compact neighborhoods of Δ and $t(x,z)$ ranges over the finite tuples parameter-free terms. These formulas $x \sim_{t,D} y$ define a system of entougages on $\mathcal{U}^{|x|}$. We refer to this uniformity and the topology associated as the I-topology. Though not nedded in the sequel, it is worth mentioning that the I-topology on $\mathcal{U}^{|x|}$ coincides with the product of the I-topology on $\mathcal{U}$. This can be verified by an argument similar to what used in the proof of Fact 30.

We say that $p(x) \subseteq \mathcal{I}(\mathcal{U})$ is I-invariant if $x \sim_{\mathsf{I}} y \to [p(x) \leftrightarrow p(y)]$. It is easy to see that all formulas constructed without the use of (ii) of Definion 2 are I-invariant. The following proposition is an immediate consequence of compactness. It corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Proposition 31. The following are equivalent for every $\varphi(x) \in \mathcal{I}(\mathcal{U})$

1. $\varphi(x)$ is I-invariant;
2. for every $\varphi' > \varphi$ there are a tuple of therm t and a compact neighborhood of Δ such that

$$x \sim_{t,D} y \wedge \varphi(x) \quad \rightarrow \quad \varphi'(y).$$

We say that $p(x)$ is a Cauchy type if it is consistent and $p(x) \wedge p(y) \rightarrow x \sim_{\mathsf{I}} y$.

We say that a type $q(x)$ is finitely satisfiable in M if every conjunction of formulas in $q(x)$ has a solution in M. This definition is classical but note that, though classically every type over M is finitely saisfiable in M, here we can only infer that $q'(x)$ is finitely satisfiable.

Lemma 32. The following are equivalent for every model M

1. $p(x) \subseteq \mathcal{I}(M)$ is an I-invariant Cauchy type;
2. $p(x) \leftrightarrow a \sim_{\mathsf{I}} x$ for some/any $a \models p(x)$;
3. $a \sim_{\mathsf{I}}' x$ is finitely satisfiable in M for some/any $a \models p(x)$.

Proof. (1⇒2) Let $a \models p(x)$. Substituting a for y in the definition of Cauchy type we obtain $p(x) \rightarrow a \sim_{\mathsf{I}} x$ Similarly, from the definition of I-invariance we obtain $a \sim_{\mathsf{I}} x \rightarrow p(x)$.

(2⇒3) By compactness, as $p'(x)$ is finitely satisfiable in M.

(3⇒1) Let $a_{t,D} \in M$ be a realization of $a \sim_{t,D} x$ as D ranges over the compact neighborhoods of the diagonal and $t(x,z)$ over the finite tuples parameter-free terms. Then, by the uniqueness of the limit (up to $\sim_I$-equivalence), the type $p(x)$ cointaining the formulas $x \sim_{t,D} a_{t,D}$ is as required by the lemma. □

We say that a set $A \subseteq \mathcal{U}$ is Cauchy complete if every I-invariant Cauchy type $p(x) \subseteq \mathcal{I}(A)$ is realized by some $c \in A^{|x|}$.

Fact 33. Let M be a model that is λ-$\mathcal{I}$-saturated for some $\lambda > |\mathcal{L}|$. Then M is Cauchy complete.

Proof. Let $p(x) \subseteq \mathcal{I}(M)$ be an I-invariant Cauchy type. Pick any $a \models p(x)$. By Lemma 32, $p(x) \leftrightarrow a \sim_{\mathsf{I}} x$. For every formula $\varphi(x)$ in the type $a \sim_{\mathsf{I}} x$ and every $\varphi' > \varphi$ pick some $\psi(x) \in p$ such that $\psi(x) \rightarrow \varphi'(x)$. There are $|\mathcal{L}|$ many formulas in $a \sim_{\mathsf{I}} x$. Then by saturation the formulas $\psi(x)$ we picked above have a common solution b in M. Then $b \sim_{\mathsf{I}} a$ and, by invariance, $b \models p(x)$. □

It is clear from the proof that when $\mathcal{L}$ is separable it suffices to require $\lambda > \omega$.

11. Example: metric spaces

We discuss a simple example. Let I be $\mathbb{R}^+ \cup \{0, \infty\}$, with the topology that makes it homeomorphic to the unit interval. Let M, d is a metric space. Let L_{H} contain symbols for functions $M^n \rightarrow M$ that are uniformly continuous. Let $\mathcal{L}$ contain a symbol for d and possibly for some funtions of $M^n \rightarrow I$ that are uniformly continuous w.r.t. the metric d.

Let $\langle M, I \rangle$ be the structure that interprets the symbols of the language as natural.

Let $\mathcal{U}$ be a monster model that is an $\mathcal{I}$ elementary extension of M. Clearly, d does not define a metric on $\mathcal{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathcal{U}$. Therefore the notion of convergent sequence makes perfectly sense in $\mathcal{U}$. As all functions have been required to be uniformly continuous, it is immediate that $x \sim_{\mathsf{I}} y$ is equivalent to $d(x, y) = 0$.

Fact 34. Let $\mathcal{U}$ be as above. Then for every model N the following are equivalent for every $a \in \mathcal{U}$

1. $p(x) = \mathcal{J}\text{-tp}(a/N)$ is a Cauchy type;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of N that converges to a.

Proof. (2⇒1) Let $\langle \varepsilon_i : i \in \omega\rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then the formulas $d(a_i, x) \leq \varepsilon_i$ are in $p(x)$. Then every element realizing $p(x)$ is at distance 0 from a. Therefore $p(x) \to a \sim_{\mathsf{I}} x$.

(1⇒2) As $p(x)$ is Cauchy type, $p(x) \to a \sim_{\mathsf{I}} x$. Then $p'(x) \to d(a, x) < 2^{-i}$ for all i. By compactness (see Fact 21) there are formulas $\varphi_i(x) \in p$ such that $\varphi'_i(x) \to d(a, x) < 2^{-i}$ for some $\varphi'_i > \varphi_i$. By $\mathcal{J}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega\rangle$ converges to a. □

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12. Approximate elimination of quantifiers

Let T be a $\mathcal{J}$-theory.

We say that T has (approximate) elimination of quantifiers (of sort H) if for every $\varphi(x) \in \mathcal{J}$ and every $\varphi' > \varphi$ there is a formula $\psi(x) \in \mathcal{J}$ without quantifiers of sort H such that $\varphi(x) \to \psi(x) \to \varphi'(x)$ holds modulo T.

The following fact is routinely proved by back-and-forth.

Fact 35. The following are equivalent

1. T has elimination of quantifiers;
2. every finite partial $\mathcal{J}$-embedding $k : M \to N$ beween models of T is an approximate $\mathcal{J}$-elemenrary map;
3. for every finite partial $\mathcal{J}$-embedding $k : M \to N$ between ω-$\mathcal{J}$-saturated models of T, and for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is also a partial $\mathcal{J}$-embedding.

13. Pseudofinite randomizations

Let T be a complete first-order theory of signature L with an infinite model. Let $\mathcal{U}$ be a model of T. Let Ω be a finite set. We denote by $\mathcal{U}^\Omega$ the set of functions $\Omega \to \mathcal{U}$. The elements of $\mathcal{U}^\Omega$ are denoted by letters decorated with a circonflexe accent $\hat{a}, \hat{b}$, etc. The value of $\hat{a}$ at $\omega \in \Omega$ is denoted by a_ω. We now define models of the form $\langle \mathcal{U}^\Omega, I\rangle$, where I is the real interval $[0,1]$ with the usual topology.

The language L_H is empty. The language $\mathcal{L}$ contains functions of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, one for each formula $\varphi(x) \in L$. These are denoted by $\Pr\varphi(\hat{x})$. Variables of sort H are decorated with a circonflexe accent. The interpretation of $\Pr\varphi(\hat{x})$ is

$$\Pr\big(\varphi(\hat{a})\big) \quad = \quad \frac{\#[\![\varphi(\hat{a})]\!]}{\#\Omega},$$

where by $\#$ denotes the finite cardinality, and

$$[\![\varphi(\hat{a})]\!] \quad = \quad \big\{\omega \in \Omega \,:\, \mathcal{U} \models \varphi(\omega)\big\}.$$

The pseudofinite randomization of T is the $\mathcal{J}$-theory

$$T^{\mathrm{pfr}} \quad = \quad \Big\{\varphi \in \mathcal{J} \,:\, \langle \mathcal{U}^\Omega, I\rangle \models \varphi \text{ for all sufficiently large finite set } \Omega\Big\}$$

We prove that T^{pfr} is a compete theory with elimination of quantifiers.

Lemma 36. Let $k : M \to N$ be a finite partial $\mathcal{J}$-embedding between ω-$\mathcal{J}$-saturated models of T^{pfr}. Then for every $\hat{b} \in M$ and there is a $\hat{c} \in N$ such that $k \cup \{\langle \hat{b}, \hat{c}\rangle\} : M \to N$ is an $\mathcal{J}$-partial embedding.

Proof. Let $\hat{a}$ be an enumeration of $\mathrm{dom} k$. Let $p(\hat{x}, \hat{z})$ be the $\mathcal{J}$-atomic type of $\hat{b}, \hat{a}$ in M. It suffices to show that $p(\hat{x}, k\hat{a})$ is finitely consistent in N.

We may assume that the formulas in $p(\hat{x}, \hat{z})$ have the form $\gamma = \Pr\varphi(\hat{x}, \hat{z})$ for some $\gamma \in I$. Pick a finite set of these formulas, say $\gamma_i = \Pr\varphi_i(\hat{x}, \hat{z})$, for $i < n$. By the saturation of N, it suffices to prove that for every ε there is a $\hat{c} \in N$ such that

$$\gamma_i \quad =_\varepsilon \quad \Pr\varphi_i(\hat{c}, k\hat{a})$$

is true in N for all $i < n$.

Some preliminary work is required. For each $J \subseteq n$ define the following L-formulas

$$\psi_J(z) \quad = \quad \bigwedge_{i \in J} \exists x\, \varphi_i(x, z)$$

and

$$\xi_J(z) \quad = \quad \psi_J(z) \wedge \bigwedge_{J' \supset J} \neg\psi_{J'}(z).$$

The consistent formulas among the $\xi_J(z)$ define a partition of $\mathcal{U}^{|z|}$. For $J \subseteq n$ let $\alpha_J \in I$ be such that $p(\hat{x}, \hat{z})$ contains the formula

$$\alpha_J \quad = \quad \Pr\big(\xi_J(\hat{z})\big) \tag{2}$$

When α_J is positive, for some $\beta_{i,J} \in I$, the type $p(\hat{x}, \hat{z})$ contains the formulas

$$\beta_{i,J} \quad = \quad \Pr\Big(\varphi_i(\hat{x}, \hat{z}) \,\Big|\, \xi_J(\hat{z})\Big). \tag{3}$$

As the $\varphi_i(x, z)$ define a partition, the $\beta_{i,J}$ add up to 1 for any fixed J for which they are defined (i.e. such that $\alpha_J \neq 0$). Note that $\beta_{i,J} = 0$ for $i \notin J$.

It is plain that for $i \in J$ the following is a consequence of $p(\hat{x}, \hat{z})$

$$1 \quad = \quad \Pr\Big(\exists x\, \varphi_i(x, \hat{z}) \,\Big|\, \xi_J(\hat{z})\Big) \tag{4}$$

Now we prove that the implication (2) & (4) $\Rightarrow$ (1) holds in $\mathcal{U}^{\Omega}$ when Ω large enough (larger than n/ε suffices). Strictly speaking, this implication is not a $\mathcal{J}$-formula, but the reader can easily verify that a suitable approximation of (2) and (4) suffices.

Let $\hat{a}' \in \mathcal{U}^{\Omega}$ saisfy (2) and (4) for every $i \in J \subseteq n$. We define $\hat{c}' \in \mathcal{U}^{\Omega}$ such that

$$\mathcal{U}^{\Omega} \quad \models \quad \beta_{i,J} =_{\varepsilon} \Pr\Big(\varphi_i(\hat{c}', \hat{a}') \,\Big|\, \xi_J(\hat{a}')\Big).$$

We may define $\hat{c}'$ separately in each event $[\![\xi_J(\hat{a}')]\!]$. Partition $[\![\xi_J(\hat{a}')]\!]$ into events $E_i \subseteq \Omega$, for $i \in J$, such that $\#E_i/\#\Omega =_{\varepsilon} \beta_{i,J}$. Then define c'_{ω} for $\omega \in E_i$ to be any witness of $\exists x\, \varphi_i(x, a_{\omega})$. By (4), we can always find such a witness.

Finally, from elementary probability theory (the theorem of total probability)

$$\mathcal{U}^{\Omega} \quad \models \quad \gamma_i =_{\varepsilon} \Pr\big(\varphi_i(\hat{c}', \hat{a}')\big).$$

As $k\hat{a}$ satisfy (2) and (4), we conclude that some $\hat{c} \in N$ satisfy (1). □

Now, from Fact 35 we obtain that $T^{\rm pfr}$ has elimination of quantifiers. Completeness follows.

Corollary 37. The theory $T^{\rm pfr}$ is compete and has elimination of quantifiers.