CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. Introduzione

As a minimal motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now, expand them by adding a norm. A norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two-sorted structure. Now, we have two conflicting aspirations

i. to apply the basic tools of model theory (such as elementarity and saturation);

ii. to stay within the realm of normed spaces (hence insist that $\mathbb{R}$ should remain $\mathbb{R}$ throughout).

These two requests are blatantly incompatible, something has to give. Different people have attemped different paths.

1. Nonstandard analysts happily embrace norms that take values in some ${}^*\mathbb{R} \succeq \mathbb{R}$. These normed spaces are nonstandard, but there are tricks to transfer results from nonstandard to standard normed spaces and vice versa. Model theorists, when confronted with similar problems, have used similar approaches (in more generality and with different notation).
2. Henson and Iovino propose to stick to the standard notion of norm but restrict the notion of elementarity and saturation to a smaller class of formulas (which here is denoted by $\mathbb{H}$). They demostrate that many standard tools of model theory apply in their setting. The work of Henson and Iovino is focussed on Banach spaces and has hardly been applied to broader contexts.
3. Real valued logicians take the most radical approach. They abandon classical true/false valued logic in favour of a logic valued in the interval $[0, 1]$. Unfortunately, this adds notational burden and restricts the class of structures that can be considered.

We elaborate on the ideas of Henson and Iovino and generalize them to a larger class of structures. Following Henson and Iovino, we restrict the notions of elementarity and saturation. But we use a class of formulas $\mathbb{I}$ which is larger than $\mathbb{H}$. Though we prove that $\mathbb{I}$ has, in a sense, approximatively the same expressive power as $\mathbb{H}$ (cf. Propositions 20 and 21), it is evident that $\mathbb{I}$ is more convinent to use.

We also borrow intuitions and results from nonstandard analisys to an extent that we could claim the title: nonstandard analysis for the finite logician.

2. A class of structures

Let R be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language L_R contains relation symbols for the compact subsets $C \subseteq R^n$ and a function symbol for each continuous functions $f : R^n \to R$. According to the context, C and f denote either the symbols of L_R or their interpretation in R.

Definition 1. Let L be a one-sorted (first-order) language. By model we mean an $\mathcal{L}$-structure of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is the structure above. The language $\mathcal{L}$ is an expansion of both L and L_R. The new symbols allowed in $\mathcal{L}$ are function symbols of sort $\mathsf{M}^n \to \mathsf{R}$.

In some examples we need to replace the sort M with many sorts $\mathsf{M}_1 \ldots, \mathsf{M}_k$. In this case the language L governs the structure $\langle \mathsf{M}_1 \ldots, \mathsf{M}_k\rangle$. Models have the form $\langle \mathsf{M}_1 \ldots, \mathsf{M}_k, R\rangle$. The new symbols in the language $\mathcal{L}$ are of sort $\mathsf{M}_1^{n_1} \ldots, \mathsf{M}_k^{n_k} \to \mathsf{R}$.

The scope of the notion of model will be (inessentially) restricted once the monster model is introduced, in Section 7.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (i.e. when R is finite) they are not models. As a remedy, below we carve out a set of formulas $L \subseteq \mathbb{I} \subseteq \mathcal{L}$, such that every model has an $\mathbb{I}$-elementary, $\mathbb{I}$-saturated extension that is a model.

Warning: as usual L and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The symbols $\mathbb{I}$ and $\mathbb{H}$ defined below denote only sets of $\mathcal{L}$-formulas.

Definition 2. Formulas in $\mathbb{I}$ are defined inductively from two sorts of $\mathbb{I}$-atomic formulas

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact (in the product topology) and $t(x, y)$ is a n-tuple of terms of sort $\mathsf{M}^{|x|} \times \mathsf{R}^{|y|} \to \mathsf{R}$;

ii. all formulas in L.

We require that $\mathbb{I}$ is closed under the Boolean connectives $\wedge, \vee$; the quantifiers $\forall^{\mathsf{M}}, \exists^{\mathsf{M}}$ of sort M; and the quantifiers of sort R which are denoted by $\forall^{\mathsf{R}}, \exists^{\mathsf{R}}$.

We write $\mathbb{H}$ for the set of formulas in $\mathbb{I}$ without quantifiers of sort R.

Formulas in $\mathbb{H}$ are a generalization of the positive bounded formulas of Henson and Iovino. Here we also introduce the larger class $\mathbb{I}$ because it offers some advantages. For instance, it is easy to see (cf. Example 3 for a hint) that $\mathbb{I}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive[1]). Somewhat surprisingly, we will see (cf. Propositions 20 and 21) that the formulas in $\mathbb{I}$ can be approximated by formulas in $\mathbb{H}$.

Example 3. Let $R = [0, 1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort $\mathsf{M}^{|x|} \to \mathsf{R}$. Then there is a formula in $\mathbb{I}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall^{\mathsf{M}} x \left[t(x) \dot{-} \tau \in \{0\}\right] \quad \wedge \quad \forall^{\mathsf{R}} \varepsilon \Big[\varepsilon \in \{0\} \ \vee \ \exists^{\mathsf{M}} x \left[\tau \dot{-} (t(x) + \varepsilon) \in \{0\}\right]\Big]$$

which, in a more legible form, becomes

$$\forall^{\mathsf{M}} x \left[t(x) \leq \tau\right] \quad \wedge \quad \forall^{\mathsf{R}} \varepsilon > 0 \ \exists^{\mathsf{M}} x \left[\tau \leq t(x) + \varepsilon\right].$$

[1]This unsubstatited claim should be interpreted as a concjecture. A painstaking technical comparison with real valued logic is not in the scope of this indroductory exploratory paper.

3. The standard part

This a standard introduction to the sandard part of elements of the nonstandard extension of a compact topological space. Our goal is to prove Lemma 8. Let $R \preceq {}^*R$. For each $\beta \in R$ we define the type

$$\mathrm{m}_\beta(x) \quad = \quad \{x \in D \ : \ D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(x)$ in ${}^*R$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 4. For every $\alpha \in {}^*R$ there is a unique $\beta \in R$ such that $\alpha \vDash \mathrm{m}_\beta(x)$.

Proof. Negate the existence of β. For every $\gamma \in R$ pick some compact neighborhood D_γ of γ, such that ${}^*R \ \vDash \ \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq R$ such that D_γ, with $\gamma \in \Gamma$, cover R. By elementarity these D_γ also cover ${}^*R$. A contradiction. The uniqueness of β follows from normality. $\square$

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in R$ such that $\alpha \vDash \mathrm{m}_\beta(x)$. We write $\alpha \approx \alpha'$ if $\mathrm{st}(\alpha) = \mathrm{st}(\alpha')$.

Fact 5. For every $\alpha \in {}^*R$ and every compact $C \subseteq R$

$${}^*R \quad \vDash \quad \alpha \in C \ \rightarrow \ \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*R \vDash \alpha \in D \subseteq \neg C$. $\square$

Fact 6. For every $\alpha \in ({}^*R)^{|x|}$ and every function symbol f of sort $\mathsf{R}^{|x|} \rightarrow \mathsf{R}$

$${}^*R \quad \vDash \quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big).$$

Proof. By Fact 4 and the definition of st(-) it suffices to prove that ${}^*R \ \vDash \ f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*R \vDash \alpha \in f^{-1}[D]$ and, as $R \preceq {}^*R$ we obtain ${}^*R \vDash f(\alpha) \in D$. $\square$

Let ${}^*\mathcal{M} = \langle M, {}^*R\rangle$ be an $\mathcal{L}$-structure such that $R \preceq {}^*R$. The standard part of ${}^*\mathcal{M}$ is the model $\mathcal{M} = \langle M, R\rangle$ that interprets the symbols f of sort $\mathsf{M}^n \rightarrow \mathsf{R}$ as the functions

$$f^{\mathcal{M}}(a) \quad = \quad \mathrm{st}\big(f^{{}^*\mathcal{M}}(a)\big) \qquad\qquad \text{for all } a \in M^n.$$

Symbols in L maintain the same interpretation.

Fact 7. Let ${}^*\mathcal{M}$ and $\mathcal{M}$ be as above. Let $t(x\,;y)$ be a term of sort $\mathsf{M}^{|x|} \times \mathsf{R}^{|y|} \rightarrow \mathsf{R}$. Then for every $a \in M^{|x|}$ and $\alpha \in \mathsf{R}^{|y|}$

$$t^{\mathcal{M}}\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big(t^{{}^*\mathcal{M}}(a\,;\alpha)\big)$$

Proof. When a t is function symbol of sort $\mathsf{M}^{|x|} \to \mathsf{R}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{R}^{|y|} \to \mathsf{R}$, the claim follows from Fact 6. Now, assume inductively that

$$t_i^{\mathcal{M}}\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big(t_i^{*\mathcal{M}}(a\,;\alpha)\big)$$

holds for the terms $t_1(x\,;y), \ldots, t_n(x\,;y)$ and let $t = f(t_1, \ldots, t_n)$ for some function f of sort $\mathsf{R}^n \to \mathsf{R}$. Then the claim follows immediately from the induction hypothesis and Fact 6. □

Lemma 8. Let ${}^*\mathcal{M}$ and $\mathcal{M}$ be as above. Then for every $\varphi(x\,;y) \in \mathbb{I}$, $a \in M^{|x|}$ and $\alpha \in ({}^*R)^{|y|}$

$${}^*\mathcal{M} \vDash \varphi(a\,;\alpha) \quad \Rightarrow \quad \mathcal{M} \vDash \varphi\big(a\,;\mathrm{st}(\alpha)\big)$$

Proof. Suppose $\varphi(x\,;y)$ is $\mathbb{I}$-atomic. If $\varphi(x\,;y)$ is a formula of L the claim is trivial. Otherwise $\varphi(x\,;y)$ has the form $t(x\,;y) \in C$. Assume that the tuple $t(x\,;y)$ consists of a single term. The general case is left to the reader as it follows easily from this special case.

Assume ${}^*\mathcal{M} \vDash t(a\,;\alpha) \in C$. Then $\mathrm{st}\big(t^{*\mathcal{M}}(a\,;\alpha\big) \in C$ by Fact 5. Therefore $t^{\mathcal{M}}(a\,;\mathrm{st}(\alpha) \in C$ follows from Fact 7.

This proves the lemma for $\mathbb{I}$-atomic formulas. Induction is immediate. □

4. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{I}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained by replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C. Note that $>$ is a dense (pre)order of $\mathbb{I}(M)$. Formulas in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. Moreover the $\mathbb{I}$-atomic formulas in L are replaced with their negation and each connective is replaced with its dual i.e., $\vee, \wedge, \exists, \forall, \exists^{\mathsf{R}}, \forall^{\mathsf{R}}$ are replaced with $\wedge, \vee, \forall, \exists, \forall^{\mathsf{R}}, \exists^{\mathsf{R}}$ respectively. We say that $\tilde{\varphi}$ is a strong negation of φ. It is clear that $\tilde{\varphi} \to \neg\varphi$.

Lemma 9.

1. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$.
2. For every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in R \smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of R, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = t \in C'$ is as required. The lemma follows easily by induction. □

5. Elementarity

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two models. We say that a partial map $f : M \to N$ is $\mathbb{I}$-elementary if for every $\varphi(x) \in \mathbb{I}$ and every $a \in (\mathrm{dom} f)^{|x|}$

1. $$\mathcal{M} \vDash \varphi(a) \;\Rightarrow\; \mathcal{N} \vDash \varphi(fa).$$

An $\mathbb{I}$-elementary map that is total is called an $\mathbb{I}$-elementary embedding. we write $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{I}$-elementary submodel of $\mathcal{N}$.

The definitions of $\mathbb{H}$-elementary map/embedding/submodel is obtained replacing $\mathbb{I}$ by $\mathbb{H}$.

As $\mathbb{I}$ and $\mathbb{H}$-elementary maps are in particular L-elementary, they are injective. However their inverse need not be $\mathbb{I}$, respectively $\mathbb{H}$-elementary. In other words the converse of (1) may not hold. We will see that $\mathbb{I}$ and $\mathbb{H}$-elementary are between saturated models they coincide $\mathcal{L}$-elementary, hence the converse of (1) holds trivially.

Now we introduce a two of weak notions of elementarity that are invertible in all models. We say that $f : M \to N$ is an $\mathbb{I}$-partial embedding if (1) holds when $\varphi(x)$ is $\mathbb{I}$-atomic.

Fact 10. If $f : M \to N$ is a partial embedding then

$$\mathcal{M} \vDash \varphi(a) \;\Leftrightarrow\; \mathcal{N} \vDash \varphi(fa)$$

for every $\mathbb{I}$-atomic formula $\varphi(x)$ and every $a \in (\operatorname{dom} f)^{|x|}$

We say that $f : M \to N$ is approximately $\mathbb{I}$-elementary if for every formula $\varphi(x) \in \mathbb{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$

$$\mathcal{M} \vDash \varphi'(a) \;\text{ for all } \varphi' > \varphi \;\Rightarrow\; \mathcal{N} \vDash \varphi'(fa) \;\text{ for all } \varphi' > \varphi.$$

We define approximately $\mathbb{H}$-elementary maps in the same manner. We will prove that between saturated model approximately $\mathbb{I}$ and $\mathbb{H}$-elementary maps are both $\mathcal{L}$-elementary. In general we have the follwing.

Fact 11. If $f : M \to N$ is approximately $\mathbb{I}$-elementary then

$$\mathcal{M} \vDash \varphi'(a) \text{ for all } \varphi' > \varphi \;\Leftrightarrow\; \mathcal{N} \vDash \varphi'(fa) \text{ for all } \varphi' > \varphi.$$

for every $\varphi(x) \in \mathbb{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$. The same holds for approximately $\mathbb{H}$-elementary maps.

6. $\mathbb{I}$-COMPACTNESS

It is convenient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure $\mathcal{M}$. We say that T is consistent when $\mathcal{M}$ is required to be a model.

Theorem 12. Let $T \subseteq \mathbb{I}$ be finitely consistent. Then T is consistent.

Proof. Suppose T is finitely consistent (or finitely $\mathcal{L}$-consistent, for that matter). By the classical compactness theorem ${}^*\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure ${}^*\mathcal{M} = \langle M, {}^*R\rangle$. Let $\mathcal{M}$ be the model $\langle M, R\rangle$. Then $\mathcal{M} \vDash T$ by Lemma 8. $\square$

A model $\mathcal{N}$ is λ-$\mathbb{I}$-saturated if it realizes all types with fewer than λ parameters that are finitely consistent in $\mathcal{N}$. When $\lambda = |\mathcal{N}|$ we simply say $\mathbb{I}$-saturated. The existence of $\mathbb{I}$-saturated models is obtained from the classical case just as for Theorem 12.

Theorem 13. Every model has an $\mathbb{I}$-elementary extension to a saturated model (possibly of inaccessible cardinality).

7. The monster model

We denote by $\mathcal{U} = \langle U, R\rangle$ some large $\mathbb{I}$-saturated structure which we call the monster model. For convenience we assume that the cardinality of $\mathcal{U}$ is an inaccessible cardinal which we denote by κ. Below we say model for $\mathbb{I}$-elementary submodel of $\mathcal{U}$.

Let $A \subseteq U$ be a small set throughout this section. We define a topology on $U^{|x|} \times R^{|y|}$ which we call the $\mathbb{I}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x\,;y) \subseteq \mathbb{I}(A)$. This is a compact topology by the $\mathbb{I}$-compactness Theorem 12.

The following fact demostrate how $\mathbb{I}$-compactness applies. There are some subtle differences from the classical setting.

Fact 14. Let $p(x) \subseteq \mathbb{I}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{I}(U)$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. Claim (1) is immediate by saturation. Claim (2) follows from the first by Lemma 9. □

Proposition 15. For every $\varphi(x) \in \mathbb{I}(U)$

$$\bigwedge_{\varphi'>\varphi} \varphi'(x\,;y) \;\leftrightarrow\; \varphi(x\,;y)$$

Proof. We prove $\to$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifier is immediate. We consider case of the existential quantifiers of sort M. The case of existential quantifiers of sort R is identical. Assume inductively

$$\text{ih.} \qquad \bigwedge_{\varphi'>\varphi} \varphi'(x, z\,;y) \;\to\; \varphi(x, z\,;y)$$

We need to prove

$$\bigwedge_{\varphi'>\varphi} \exists^{\mathsf{M}} z\, \varphi'(x, z\,;y) \;\to\; \exists^{\mathsf{M}} z\, \varphi(x, z\,;y)$$

From (ih) we have

$$\exists^{\mathsf{M}} z \bigwedge_{\varphi'>\varphi} \varphi'(x, z\,;y) \;\to\; \exists^{\mathsf{M}} z\, \varphi(x, z\,;y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi'>\varphi} \exists^{\mathsf{M}} z\, \varphi'(x, z\,;y) \;\to\; \exists^{\mathsf{M}} z \bigwedge_{\varphi'>\varphi} \varphi'(x, z\,;y)$$

Replace x, y with some fix but arbitrary parameters, say a, α and assume the antecedent, that is, the truth of the theory $\{\exists^{\mathsf{M}} z\, \varphi'(a, z\,;\alpha) \,:\, \varphi' > \varphi\}$. We need to prove the consistency of the type $\{\varphi'(a, z\,;\alpha) \,:\, \varphi' > \varphi\}$. By the saturation of $\mathcal{U}$, finite concistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. $\square$

When $A \subseteq U$, we write $S_{\mathbb{I}}(A)$ for the set of types

$$\mathbb{I}\text{-tp}(a/A) \quad = \quad \big\{\varphi(x) \ : \ \varphi(x) \in \mathbb{I}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of U. We write $S_{\mathbb{I},x}(A)$ when the tuple of variables x is fixed. The same notation applies also with $\mathbb{H}$ for $\mathbb{I}$.

The following corollary will be strengthen by Corollary 19 below.

Corollary 16. The types $p(x) \in S_{\mathbb{I}}(A)$ are maximally consistent subsets of $\mathbb{I}_x(A)$. That is, for every $\varphi(x) \in \mathbb{I}(A)$, either $\varphi(x) \in p$ or $p(x) \to \neg\varphi(x)$. The same holds with $\mathbb{H}$ for $\mathbb{I}$.

Proof. Let $p(x) = \mathbb{I}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 9 and Proposition 15 we obtain

$$\neg\varphi(x) \quad \to \quad \bigvee_{\tilde{\varphi}\perp\varphi} \tilde{\varphi}(x).$$

Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \to \neg\varphi(x)$ follows. $\square$

The following will be useful below

Remark 17. For $p(x) \subseteq \mathbb{I}(U)$, we write $p'(x)$ for the type

$$p'(x) \quad = \quad \big\{\varphi'(x) \ : \ \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}.$$

Note that $p'(x)$ is equivalent to $p(x)$ by Proposition 15.

8. Elimination of quantifiers

We show that the quantifiers $\forall^{\mathsf{R}}$ and $\exists^{\mathsf{R}}$ can be eliminated up to some approximation.

Proposition 18. The monster model (or any saturated model, for that matter) is $\mathbb{H}$-homogeneous, that is, every $\mathbb{H}$-elementary map $f : U \to U$ of cardinality $< \kappa$ extends to an automorphism.

Proof. By Corollary 16, the inverse of an $\mathbb{H}$-elementary map $f : U \to U$ is $\mathbb{H}$-elementary. Then the usual proof by back-and-forth applies. $\square$

Now, as promised, we strengthen Corollary 16. Let $A \subseteq U$ be a small set throughout this section.

Corollary 19. Let $p(x) \in S_{\mathbb{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathcal{L}_x(A)$. That is, for every $\varphi(x) \in \mathcal{L}(A)$, either $p(x) \to \varphi(x)$ or $p(x) \to \neg\varphi(x)$. Clearly, the same holds for $p'(x)$.

Proof. If $b \vDash p(x) = \mathbb{H}\text{-tp}(a/A)$ then there is an $\mathbb{H}$-elementary map $f \supseteq \mathrm{id}_A$ such that $fa = b$. As f exends to an automorphism and every automorphism is $\mathcal{L}$-elementary, the corollary follows. $\square$

For $a, b \in U^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A, bearing in mind that formulas in $\mathbb{H}(A)$ or $\mathbb{I}(A)$ suffices to test the equivalence.

In the classical setting, from equivalence of types one can derive equivalence of formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence. Indeed, the next two propositions show that formulas in $\mathbb{I}(A)$ are approximated by formulas in $\mathbb{H}(A)$.

Proposition 20. Let $\varphi(x) \in \mathbb{I}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathbb{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 19 and Remark 17

$$\neg\varphi(x) \;\to\; \bigvee_{p'(x)\to\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathbb{H},x}(A)$. By Fact 14 and Lemma 9

$$\neg\varphi(x) \;\to\; \bigvee_{\neg\tilde{\psi}(x)\to\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathbb{H}(A)$. Equivalently,

$$\varphi(x) \;\leftarrow\; \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 14, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathbb{H}(A)$ such that

$$\varphi'(x) \;\leftarrow\; \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \;\leftarrow\; \varphi(x)$$

which yields the interpolant required by the proposition. □

Proposition 21. Let $\varphi(x) \in \mathbb{I}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathbb{H}(A)$ and some $\psi' > \psi$.

Proof. Let $a \in U^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$. By Corollary 19, $p'(x) \to \neg\varphi(x)$. By compactness $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. □

9. The Tarski-Vaught test

The following proposition is not literally the Tarski-Vaught test. Infact, it only applies when the larger structure is the moster model.

Proposition 22. Let M be a subset of U. Then the following are equivalent

1. M is the domain of a model $\mathcal{M} = \langle M, R\rangle$;
2. for every formula $\varphi(x) \in \mathbb{H}(M)$
$$\exists^{\mathsf{M}} x\, \varphi(x) \;\Rightarrow\; \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathbb{I}(M)$
$$\exists^{\mathsf{M}} x\, \neg\varphi(x) \;\Rightarrow\; \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists^{\mathsf{M}} x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 9 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\forall^{\mathsf{M}} x\, \tilde{\varphi}(x)$ hence, by (1), $\mathcal{M} \vDash \neg\forall^{\mathsf{M}} x\, \varphi(x)$. Then $\mathcal{M} \vDash \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $\mathcal{M} \vDash \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathbb{I}(M)$ be such that $\exists^{\mathsf{M}} x\, \neg\varphi(x)$. By Corollary 21, there are a consistent $\psi(x) \in \mathbb{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). By the classical Tarski-Vaught test $M \preceq U$. Let $\mathcal{M} = \langle M, R \rangle$ be the unique $\mathcal{L}$-structure that is a substructure of $\mathcal{U}$. Then $\varphi(a) \Leftrightarrow \mathcal{M} \vDash \varphi(a)$ holds for every $a \in M^{|x|}$ and for every $\mathbb{I}$-atomic formula $\varphi(x)$. Now, assume inductively

$$\mathcal{M} \vDash \varphi(a, b) \;\Rightarrow\; \varphi(a, b).$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$\mathcal{M} \vDash \forall^{\mathsf{M}} y\, \varphi(a, y) \;\Rightarrow\; \forall^{\mathsf{M}} y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned} \neg\forall^{\mathsf{M}} y\, \varphi(a, y) &\Rightarrow \exists^{\mathsf{M}} y\, \neg\varphi(a, y) \\ &\Rightarrow \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\ &\Rightarrow \mathcal{M} \vDash \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\ &\Rightarrow \mathcal{M} \nvDash \forall^{\mathsf{M}} y\, \varphi(a, y). \end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists^{\mathsf{M}}$, $\exists^{\mathsf{R}}$, and $\forall^{\mathsf{R}}$ is straightforward. □

10. Completeness

Needs some deeper thinking

For $a, b \in U$ we write $a \sim_R b$ if $t(a, y) = t(b, y)$ holds (universal quantification over the free variables is understood) for every parameter free term $t(x, y)$ of sort $\mathsf{M}^{1+|y|} \to \mathsf{R}$. We write $[a]_R$ for the class of equivalence of a.

Fact 23. If $\bar{a} = \langle a_i : i < \lambda \rangle$ and $\bar{b} = \langle b_i : i < \lambda \rangle$ are such that $a_i \sim_R b_i$ for every $i < \lambda$ then $t(\bar{a}, y) = t(\bar{b}, y)$ holds for every term $t(x, y)$ of sort $\mathsf{M}^{\lambda+|y|} \to \mathsf{R}$ with parameters in U.

Proof. By induction on λ we prove that $t(\bar{a}, y) = t(\bar{b}, y)$ holds for every parameter free term $t(x, y)$. This is sufficient to prove the fact. Assume that

$$t(\bar{a}, z, y) \;=\; t(\bar{b}, z, y)$$

holds for every term $t(\bar{x}, z, y)$. In particular for any a_λ

1. $$t(\bar{a}, a_\lambda, y) \;=\; t(\bar{b}, a_\lambda, y).$$

If $a_\lambda \sim_R b_\lambda$ then

$$t(\bar{x}, a_\lambda, y) \;=\; t(\bar{x}, b_\lambda, y)$$

and in particular

2. $$t(\bar{b}, a_\lambda, y) \;=\; t(\bar{b}, b_\lambda, y).$$

From (1) and (2) we obtain

$$t(\bar{a}, a_\lambda, y) \;=\; t(\bar{b}, b_\lambda, y).$$

For limit ordinals induction is trivial. □

For $a \in U$ and $A \subseteq U$, let $p(x) = \mathbb{I}\text{-tp}(a/A)$. We say that a is a limit point of A if $p(x) \to a \sim_R x$.

Fact 24. Let $\mathcal{M}$ be a model that is λ-$\mathbb{I}$-saturated for some $\lambda > |\mathcal{L}|$. Then $M \cap [a]_R \neq \varnothing$ for every limit point a of M.

Proof. Let $p(x) = \mathbb{I}\text{-tp}(a/A)$. For every formula $\varphi(x)$ in the type $a \sim x$ and every $\varphi' > \varphi$ pick some $\psi(x) \in p$ such that $\psi(x) \rightarrow \varphi'(x)$. There are are at most $|\mathcal{L}|$ such formulas $\psi(x)$. By saturation they have a common solution b in M. Clearly $b \sim_R a$. □

We say that a formula $\varphi(x)$ is $\sim_R$-invariant if $x \sim_R y \rightarrow [\varphi(x) \leftrightarrow \varphi(y)]$. The relation $x \sim_R y$ is defined by a parameter-free $\mathbb{I}$-type.

11. Metric spaces

We discuss a simple example. Let M, d is a metric space. Let L contain symbols for all continuous functions $M^n \rightarrow M$. The language $\mathcal{L}$ has also a symbol d for a function of sort $\mathsf{M}^2 \rightarrow \mathsf{R}$. Let $R = \mathbb{R} \cup \{\pm\infty\}$ be the compactification of $\mathbb{R}$. Let $\mathcal{M} = \langle M, R \rangle$ be the structure that interprets the symbols in $\mathcal{L}$ as natural.

Let $\mathcal{U}$ be a monster model such that $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{U}$. Clearly, d does not define a metric on $\mathcal{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathcal{U}$. Therefore the definition of Cauchy sequence and the corresponding notion of completeness make perfectly sense for $^*d$.

Fact 25. Let $\mathcal{U}$ be as above. Then for every model $\mathcal{N} = \langle N, R \rangle$ the following are equivalent

1. $a \in U$ is definable in the limit over N
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of N that converges to a.

Proof. First note that, by the continuity of the terms with respect to the metric, $a \sim_R b$ is equivalent to $d(a, b) = 0$.

($2 \Rightarrow 1$) Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence in $[0, 1]$ that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_N a$. By the uniqueness of the limit $d(a, b) = 0$.

($1 \Rightarrow 2$) Let $p(x) = \mathbb{I}\text{-tp}(a/N)$. Assume that $p(x) \rightarrow a \sim_R x$. Then $p'(x) \rightarrow d(a, x) < 2^{-i}$ for all i. By compactness (see Fact 14) there are formulas $\varphi_i(x) \in p$ such that $\varphi_i'(x) \rightarrow d(a, x) < 2^{-i}$ for some $\varphi_i' > \varphi_i$. By $\mathbb{I}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega \rangle$ converges to a. □

12. Measure spaces

⚠ Garbage below this point

Let $\mathcal{B}$ be a σ-algebra of subsets of Ω. Let F be the set of $\mathcal{B}$-simple functions, i.e. linear combinations of indicator functions of sets in $\mathcal{B}$. Let M be the set of $\mathbb{R}$-valued $\mathcal{B}$-measures concentrated on finitely many points of Ω.

Let $\mathcal{M} = \langle M, F, R \rangle$, where $R = \mathbb{R} \cup \{\pm\infty\}$ is the compactification of $\mathbb{R}$. The sort M in the previous section is replaced here by two sorts which we call M and F. Here L is the language of the structure $\langle M, F \rangle$. As both F and M are lattice $\mathbb{R}$-algebras, we assume that L has symbols for these lattice algebras. There is also a function symbol of sort $\mathsf{F} \times \mathsf{M} \rightarrow \mathsf{M}$ that is interpreted as the natural action of functions on measures (via multiplication). Finally, in $\mathcal{L}$ there is a function symbol I of sort $\mathsf{M} \to \mathsf{R}$ that is interpreted as the integral over Ω.

Let $\mathcal{U} = \langle {}^*\!M, {}^*\!F, R\rangle$ be a monster model such that $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{U}$.

Let μ be a $\mathbb{R}$-valued $\mathcal{B}$-measure. By saturation, there is some ${}^*\mu \in {}^*\!M$ such that $g \cdot {}^*\mu = \int_\Omega g \, \mathrm{d}\mu$ for every $g \in F$.

Let $\gamma : \Omega \to \mathbb{R}$. By saturation, there is some ${}^*\gamma \in {}^*\!F$ such that ${}^*\gamma \cdot m = \gamma \cdot m$ for every $m \in M$. Note that as m has finite support $\gamma \cdot m$ is well-defined for all γ.

For $X \in \mathcal{B}$ we write $\mathrm{I}_X(\text{-})$ for $\mathrm{I}(1_X \cdot \text{-})$.

Lemma 26. Assume γ is measurable. Then for every $X \in \mathcal{B}$

$$\mathrm{I}_X\big({}^*\gamma \cdot {}^*\mu\big) \quad = \quad \int_X f \, \mathrm{d}\mu$$

Proof. Assume for simplicity that both f and μ are non-negative. The general case follows by decomposition. Let $G_f \subseteq R$ be the following set

$$G_f \quad = \quad \bigg\{ \int_\Omega g \, \mathrm{d}\mu \ : \ 0 \le g \le f, \ g \in F \bigg\}.$$

If G_f is bounded, pick for every $\varepsilon \in \mathbb{R}^+$ some $\alpha_{\varepsilon,f} \in G_f$ such that

$$\int_\Omega f \, \mathrm{d}\mu \quad \le \quad \alpha_{\varepsilon,f} + \varepsilon.$$

If G_f is unbounded, let $\alpha_{\varepsilon,f} = +\infty$ for all ε. Let y be a variable of sort M. Let $p(y)$ be the $\mathbb{I}$-type that contains, for all f as in the lemma, the formulas

$$\mathrm{I}\big(f \cdot y\big) \quad \ge \quad \alpha \qquad\qquad \text{for all } \alpha \in G_f;$$

$$\mathrm{I}\big(f \cdot y\big) \quad \le \quad \alpha_{\varepsilon,f} + \varepsilon \qquad\qquad \text{for all } \varepsilon \in \mathbb{R}^+.$$

By basic measure theory, $p(y)$ is finitely consistent in $\mathcal{M}$. Any realization of $p(y)$ in $\mathcal{U}$ proves the lemma. □

In M and F we use the functions -^+, -^-, $|\text{-}|$, and the relation $\le$ as usually defined in lattice vector spaces. We use the symbol 0 to denote the zeros of M, F, and R. We denote by $1 \in F$ the function that is identically $1 \in R$.

The $\mathbb{I}$-formula $\forall^{\mathsf{F}} f \ge 0 \ \mathrm{I}\big(f \cdot \mu^+\big) \ge 0$ is obviously true in $\mathcal{M}$ and so is the same formula for μ^-. As the action of F on M distributes with the algebraic operations of M and F, properties such as the Hahn decomposition are immediate in $\mathcal{M}$ and are inherited by $\mathcal{U}$.

Note that the L-formula $\forall^{\mathsf{F}} f \in [0,1] \ \ f^2 = f$ defines the indicator functions.

For $\nu, \mu \in {}^*\!M$ we write $\nu \ll \mu$ for the $\mathcal{L}$-formula

$$\forall^{\mathsf{R}}\varepsilon \ \exists^{\mathsf{R}}\delta \ \forall^{\mathsf{F}} f \ \big[\mathrm{I}\big(|f \cdot \mu|\big) < \delta \to \mathrm{I}\big(|f \cdot \nu|\big) < \varepsilon\big].$$

Proposition 27. Let $\nu, \mu \in {}^*\!M$ be bounded and non-negative such that $\nu \ll \mu$. Then there is a $g \in {}^*\!F$ such that $|\nu - g\mu| \sim_R 0$.

Proof. The following $\mathbb{I}$-formula is true in $\mathcal{M}$.

$$\forall \mu, \nu\ \forall \varepsilon > 0 \left[\forall^{\mathsf{F}} f\ \mathrm{I}\big(|f\cdot\nu| \dot{-} |f\cdot\mu|\big) < \varepsilon \to \exists^{\mathsf{F}} g\ \forall^{\mathsf{F}} f\ \mathrm{I}\big(f\cdot|\nu - g\cdot\mu|\big) \leq \varepsilon \right]$$

$\square$

Theorem 28 (Radon-Nikodym). Let ν, μ be bounded non-negative $\mathcal{B}$-measures on Ω such that $\nu \ll \mu$. Then there is a $\mathcal{B}$-measurable $f : \Omega \to \mathbb{R}$ such that

$$\int_X f \, \mathrm{d}\mu = \int_X \mathrm{d}\nu$$

for every $X \in \mathcal{B}$.