Continuous logic for the classical logician

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ABSTRACT. In general, very little model theory is possible if we restrict to a class of two-sorted structures $\langle M, R\rangle$ where M is arbitrary and R is fixed. But not everything is lost when R is a compact topological space. Continuous logic deals with this case. Rivisiting ideas of Henson and Iovino we present an approach to continuous logic that is simple and general.

1. Introduction

As a minimal motivating example, consider a real vector space M. This is are among the most simple strucures considered in model theory. Now, expand it by adding a norm. A norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two-sorted structure $\langle M, \mathbb{R}\rangle$. Now, we have two conflicting aspirations

i. to apply the basic tools of model theory (such as elementarity, compactness, and saturation);

ii. to stay within the realm of normed spaces (hence insist that $\mathbb{R}$ should remain $\mathbb{R}$ throughout).

These two requests are blatantly incompatible, something has to give. Different people have attemped different paths.

1. Nonstandard analysts happily embrace norms that take values in some ${}^*\mathbb{R} \succeq \mathbb{R}$. These normed spaces are nonstandard, but there are tricks to transfer results from nonstandard to standard normed spaces and vice versa. Model theorists, when confronted with similar problems, have used similar approaches (in more generality and with different notation).

2. Henson and Iovino propose to stick to the standard notion of norm but restrict the notion of elementarity and saturation to a smaller class of formulas (which here is denoted by $\mathbb{H}$). They demostrate that many standard tools of model theory apply in their setting. The work of Henson and Iovino is focussed on Banach spaces and has hardly been applied to broader contexts. See [1] for a survey.

3. Real valued logicians take the most radical approach. They abandon classical true/false valued logic in favour of a logic valued in the interval $[0, 1]$. Unfortunately, this adds notational burden and restricts the class of structures that can be considered. See [2] for a survey.

We elaborate on the ideas of Henson and Iovino and generalize them to a larger class of structures. We restrict the notions of elementarity and saturation. But we use a class of formulas $\mathbb{I}$ which is larger than $\mathbb{H}$. Though $\mathbb{I}$ has approximatively the same expressive power as $\mathbb{H}$ (cf. Propositions 19 and 20), it is evident (cf. Example 3) that $\mathbb{I}$ is more convinent to use.

We also borrow intuitions and results from nonstandard analisys to an extent that we could claim the title: nonstandard analysis for the finite logician.

2. A class of structures

Let R be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language L_{R} contains relation symbols for the compact subsets $C \subseteq R^n$ and a function symbol for each continuous functions $f : R^n \to R$. According to the context, C and f denote either the symbols of L_{R} or their interpretation in R.

Definition 1. Let L_{M} be a one-sorted (first-order) language. By model we mean an $\mathcal{L}$-structure of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L_{M}-structures, while R is the structure above. We use M and R to denote the two sorts of $\mathcal{L}$. The language $\mathcal{L}$ is an expansion of both L_{M} and L_{R}. The new symbols allowed in $\mathcal{L}$ are function symbols of sort $\mathsf{M}^n \to \mathsf{R}$.

In some examples we need to replace the sort M with many sorts $\mathsf{M}_1 \dots, \mathsf{M}_k$. In this case the language L_{M} is replaced with $L_{\mathsf{M}_1\dots,\mathsf{M}_k}$, the language that governs the structure $\langle M_1 \dots, M_k\rangle$. Models have the form $\langle M_1 \dots, M_k, R\rangle$. The new symbols in the language $\mathcal{L}$ are of sort $\mathsf{M}_1^{n_1} \dots, \mathsf{M}_k^{n_k} \to \mathsf{R}$.

The notion of model will be (inessentially) restricted once the monster model is introduced, in Section 7.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (i.e. when R is finite), they are not models in the sense of Definion 1. As a remedy, below we carve out a set of formulas $L \subseteq \mathbb{I} \subseteq \mathcal{L}$, such that every model has an $\mathbb{I}$-elementary, $\mathbb{I}$-saturated extension that is also a model.

Warning: as usual L_{R}, L_{M}, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The symbols $\mathbb{I}$ and $\mathbb{H}$ defined below denote only sets of $\mathcal{L}$-formulas.

Definition 2. Formulas in $\mathbb{I}$ are defined inductively from two sorts of $\mathbb{I}$-atomic formulas

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact (in the product topology) and $t(x, y)$ is a n-tuple of terms of sort $\mathsf{M}^{|x|} \times \mathsf{R}^{|y|} \to \mathsf{R}$;

ii. all formulas in L_{M}.

We require that $\mathbb{I}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{M}}$, $\exists^{\mathsf{M}}$ of sort M; and the quantifiers of sort R which are denoted by $\forall^{\mathsf{R}}$, $\exists^{\mathsf{R}}$.

We write $\mathbb{H}$ for the set of formulas in $\mathbb{I}$ without quantifiers of sort R.

Formulas in $\mathbb{H}$ are a generalization of the positive bounded formulas of Henson and Iovino. Here we also introduce the larger class $\mathbb{I}$ because it offers some advantages. For instance, it is easy to see (cf. Example 3 for a hint) that $\mathbb{I}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive[1]). Somewhat surprisingly, we will see (cf. Propositions 19 and 20) that the formulas in $\mathbb{I}$ can be approximated by formulas in $\mathbb{H}$.

Example 3. Let $R = [0, 1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort $\mathsf{M}^{|x|} \to \mathsf{R}$. Then there is a formula in $\mathbb{I}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall^{\mathsf{M}} x \left[t(x) \dot{-} \tau \in \{0\}\right] \quad \wedge \quad \forall^{\mathsf{R}} \varepsilon \Big[\varepsilon \in \{0\} \ \vee\ \exists^{\mathsf{M}} x \left[\tau \dot{-} (t(x) + \varepsilon) \in \{0\}\right]\Big]$$

which, in a more legible form, becomes

$$\forall^{\mathsf{M}} x \left[t(x) \leq \tau\right] \quad \wedge \quad \forall^{\mathsf{R}} \varepsilon > 0\, \exists^{\mathsf{M}} x \left[\tau \leq t(x) + \varepsilon\right].$$

We conclude this introduction with a question. Is it possible extend the theory exposed below to a larger $\mathcal{L}$? E.g. can we allow in $\mathcal{L}$ function symbols of sort $\mathsf{M}^n \times \mathsf{R}^m \to \mathsf{R}$ with $m, n > 0$. This would be a natural option when M is a group acting on a compact set R. Also symbols of sort $\mathsf{M}^n \times \mathsf{R}^m \to \mathsf{M}$ have natural applications (e.g. consider the action of the field of scalars in a vector space).

3. The standard part

In this section we recall the notion of the sandard part of an element in an elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 8 which in turn is required for the proof of the

[1] This unsubstatited claim should be interpreted as a concjecture. A painstaking technical comparison with real valued logic is not in the scope of this indroductory exploratory paper.

Compactness Theorem (the reader that is willing to accept the Compactness Theorem without proof may skip this section).

Let $R \preceq {}^*R$. For each $\beta \in R$ we define the type

$$\mathrm{m}_\beta(x) \quad = \quad \{x \in D \ : \ D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(x)$ in ${}^*R$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 4. For every $\alpha \in {}^*R$ there is a unique $\beta \in R$ such that $\alpha \vDash \mathrm{m}_\beta(x)$.

Proof. Negate the existence of β. For every $\gamma \in R$ pick some compact neighborhood D_γ of γ, such that ${}^*R \ \vDash \ \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq R$ such that D_γ, with $\gamma \in \Gamma$, cover R. By elementarity these D_γ also cover ${}^*R$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in R$ such that $\alpha \vDash \mathrm{m}_\beta(x)$. We write $\alpha \approx \alpha'$ if $\mathrm{st}(\alpha) = \mathrm{st}(\alpha')$.

Fact 5. For every $\alpha \in {}^*R$ and every compact $C \subseteq R$

$${}^*R \quad \vDash \quad \alpha \in C \ \rightarrow \ \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*R \vDash \alpha \in D \subseteq \neg C$. □

Fact 6. For every $\alpha \in ({}^*R)^{|x|}$ and every function symbol f of sort $\mathsf{R}^{|x|} \rightarrow \mathsf{R}$

$${}^*R \quad \vDash \quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big).$$

Proof. By Fact 4 and the definition of st(-) it suffices to prove that ${}^*R \vDash f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*R \vDash \alpha \in f^{-1}[D]$ and, as $R \preceq {}^*R$ we obtain ${}^*R \vDash f(\alpha) \in D$. □

Let ${}^*\mathcal{M} = \langle M, {}^*R\rangle$ be an $\mathcal{L}$-structure such that $R \preceq {}^*R$. The standard part of ${}^*\mathcal{M}$ is the model $\mathcal{M} = \langle M, R\rangle$ that interprets the symbols f of sort $\mathsf{M}^n \rightarrow \mathsf{R}$ as the functions

$$f^{\mathcal{M}}(a) \quad = \quad \mathrm{st}\big(f^{{}^*\mathcal{M}}(a)\big) \qquad \text{for all } a \in M^n.$$

Symbols in L_{M} maintain the same interpretation.

Fact 7. Let ${}^*\mathcal{M}$ and $\mathcal{M}$ be as above. Let $t(x\,;y)$ be a term of sort $\mathsf{M}^{|x|} \times \mathsf{R}^{|y|} \rightarrow \mathsf{R}$. Then for every $a \in M^{|x|}$ and $\alpha \in ({}^*R)^{|y|}$

$$t^{\mathcal{M}}\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big(t^{{}^*\mathcal{M}}(a\,;\alpha)\big)$$

Proof. When a t is function symbol of sort $\mathsf{M}^{|x|} \rightarrow \mathsf{R}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{R}^{|y|} \rightarrow \mathsf{R}$, the claim follows from Fact 6. Now, assume inductively that

$$t_i^{\mathcal{M}}\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big(t_i^{{}^*\mathcal{M}}(a\,;\alpha)\big)$$

holds for the terms $t_1(x\,;y), \dots, t_n(x\,;y)$ and let $t = f(t_1, \dots, t_n)$ for some function f of sort $\mathsf{R}^n \rightarrow \mathsf{R}$. Then the claim follows immediately from the induction hypothesis and Fact 6. □

Lemma 8. Let ${}^*\mathcal{M}$ and $\mathcal{M}$ be as above. Then for every $\varphi(x\,;y) \in \mathbb{I}$, $a \in M^{|x|}$ and $\alpha \in ({}^*R)^{|y|}$

$$ {}^*\mathcal{M} \vDash \varphi(a\,;\alpha) \quad\Rightarrow\quad \mathcal{M} \vDash \varphi\big(a\,;\mathrm{st}(\alpha)\big) $$

Proof. Suppose $\varphi(x\,;y)$ is $\mathbb{I}$-atomic. If $\varphi(x\,;y)$ is a formula of L_M the claim is trivial. Otherwise $\varphi(x\,;y)$ has the form $t(x\,;y) \in C$. Assume that the tuple $t(x\,;y)$ consists of a single term. The general case follows easily from this special case. Assume that ${}^*\mathcal{M} \vDash t(a\,;\alpha) \in C$. Then $\mathrm{st}\big(t^{{}^*\mathcal{M}}(a\,;\alpha)\big) \in C$ by Fact 5. Therefore $t^{\mathcal{M}}(a\,;\mathrm{st}(\alpha)) \in C$ follows from Fact 7. This proves the lemma for $\mathbb{I}$-atomic formulas. Induction is immediate. $\square$

4. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{I}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained by replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C. Note that $>$ is a dense (pre)order of $\mathbb{I}(M)$. Formulas in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde\varphi \perp \varphi$ when $\tilde\varphi$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde C$ where $\tilde C$ is some compact set disjoint from C. Moreover the $\mathbb{I}$-atomic formulas in L_M are replaced with their negation and each connective is replaced with its dual i.e., $\vee, \wedge, \exists, \forall, \exists^\mathsf{R}, \forall^\mathsf{R}$ are replaced with $\wedge, \vee, \forall, \exists, \forall^\mathsf{R}, \exists^\mathsf{R}$ respectively. We say that $\tilde\varphi$ is a strong negation of φ. It is clear that $\tilde\varphi \to \neg\varphi$.

Lemma 9. For all $\varphi \in \mathbb{I}(M)$

1. for every $\varphi' > \varphi$ there is a formula $\tilde\varphi \perp \varphi$ such that $\varphi \to \neg\tilde\varphi \to \varphi'$;
2. for every $\tilde\varphi \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde\varphi$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde\varphi = (t \in R \smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde\varphi$ is of the form $t \in \tilde C$ for some compact $\tilde C$ disjoint from C. By the normality of R, there is C', a compact neighborhood of C disjoint from $\tilde C$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. $\square$

5. Morphisms

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two models. We say that a partial map $f : M \to N$ is $\mathbb{I}$-elementary if for every $\varphi(x) \in \mathbb{I}$ and every $a \in (\mathrm{dom} f)^{|x|}$

1. $$\mathcal{M} \vDash \varphi(a) \quad\Rightarrow\quad \mathcal{N} \vDash \varphi(fa).$$

An $\mathbb{I}$-elementary map that is total is called an $\mathbb{I}$-elementary embedding. We write $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{I}$-elementary submodel of $\mathcal{N}$.

The definitions of $\mathbb{H}$-elementary map/embedding/submodel is obtained replacing $\mathbb{I}$ by $\mathbb{H}$.

As $\mathbb{I}$ and $\mathbb{H}$-elementary maps are in particular L_M-elementary, they are injective. However, their inverse need not be $\mathbb{I}$, respectively $\mathbb{H}$-elementary. In other words the converse implication in (1) may not hold. We have choosen to work with the classical notion of satisfaction at the cost of this asymmetric notion of elementarity. This contrast with [1] where Henson and Iovino introduce a notion of approximated satisfaction. The approximated $\mathbb{H}$-morphisms defined below are the maps that preserve approximated satisfaction. These morphisms are invertible, cf. Fact 10.

We say that the map $f : M \to N$ is approximately $\mathbb{I}$-elementary if for every formula $\varphi(x) \in \mathbb{I}$, and every $a \in (\mathrm{dom} f)^{|x|}$

$$\mathcal{M} \vDash \varphi'(a) \ \text{ for all } \varphi' > \varphi \ \Rightarrow \ \mathcal{N} \vDash \varphi'(fa) \ \text{ for all } \varphi' > \varphi.$$

We define approximately $\mathbb{H}$-elementary maps in the same manner. We will see (Proposition 13) that a slight amount of saturation suffices to make these approximated versions of elementarity equivalent to their unapproximated version. With full saturation, $\mathbb{I}$ and $\mathbb{H}$-elementarity coincide with $\mathcal{L}$-elementarity (Corollary 18).

However, the following holds in general.

Fact 10. If $f : M \to N$ is approximately $\mathbb{I}$-elementary then

$$\mathcal{M} \vDash \varphi'(a) \text{ for all } \varphi' > \varphi \ \Leftrightarrow \ \mathcal{N} \vDash \varphi'(fa) \text{ for all } \varphi' > \varphi.$$

for every $\varphi(x) \in \mathbb{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$. The same holds for approximately $\mathbb{H}$-elementary maps.

Proof. Only implication $\Leftarrow$ requires a proof. Assume the r.h.s. of the equivalence. Fix $\varphi' > \varphi$ and prove $\mathcal{M} \vDash \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By Lemma 9 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \to \neg\tilde{\varphi} \to \varphi'$. Then $\mathcal{N} \vDash \neg\tilde{\varphi}(fa)$ and therefore $\mathcal{M} \vDash \neg\tilde{\varphi}(a)$. Then $\mathcal{M} \vDash \varphi'(a)$. □

6. Compactness

It is convenient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure $\mathcal{M}$. We say that T is consistent when $\mathcal{M}$ is required to be a model.

Theorem 11. Let $T \subseteq \mathbb{I}$ be finitely consistent. Then T is consistent.

Proof. Suppose T is finitely consistent (or finitely $\mathcal{L}$-consistent, for that matter). By the classical compactness theorem ${}^*\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure ${}^*\mathcal{M} = \langle M, {}^*R\rangle$. Let $\mathcal{M} = \langle M, R\rangle$ be the standard part of ${}^*\mathcal{M}$ as defined in Section 3. Then $\mathcal{N} \vDash T$ by Lemma 8. □

A model $\mathcal{N}$ is λ-$\mathbb{I}$-saturated if it realizes all types with fewer than λ parameters that are finitely consistent in $\mathcal{N}$. When $\lambda = |\mathcal{N}|$ we simply say $\mathbb{I}$-saturated. The existence of $\mathbb{I}$-saturated models is obtained from the classical case just as for Theorem 11.

Theorem 12. Every model has an $\mathbb{I}$-elementary extension to a saturated model (possibly of inaccessible cardinality).

The following proposition shows that a slight amount of saturation tames the $\mathbb{I}$-formulas.

Proposition 13. Let $\mathcal{N}$ be an ω-$\mathbb{I}$-saturated model. Then the following holds in $\mathcal{N}$ for every formula $\varphi(x) \in \mathbb{I}(N)$

$$\bigwedge_{\varphi' > \varphi} \varphi'(x\,;y) \ \leftrightarrow \ \varphi(x\,;y)$$

Proof. We prove $\to$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifier is immediate. We consider case of the existential quantifiers of sort M. The case of existential quantifiers of sort R is identical. Assume inductively

ih. $$\bigwedge_{\varphi' > \varphi} \varphi'(x, z\,;y) \ \to \ \varphi(x, z\,;y)$$

We need to prove

$$\bigwedge_{\varphi'>\varphi} \exists^{\mathsf{M}} z\, \varphi'(x,z\,;y) \;\rightarrow\; \exists^{\mathsf{M}} z\, \varphi(x,z\,;y)$$

From (ih) we have

$$\exists^{\mathsf{M}} z \bigwedge_{\varphi'>\varphi} \varphi'(x,z\,;y) \;\rightarrow\; \exists^{\mathsf{M}} z\, \varphi(x,z\,;y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi'>\varphi} \exists^{\mathsf{M}} z\, \varphi'(x,z\,;y) \;\rightarrow\; \exists^{\mathsf{M}} z \bigwedge_{\varphi'>\varphi} \varphi'(x,z\,;y)$$

Replace x, y with some fix but arbitrary parameters, say a, α and assume the antecedent, that is, the truth of the theory $\{\exists^{\mathsf{M}} z\, \varphi'(a,z\,;\alpha) : \varphi' > \varphi\}$. We need to prove the consistency of the type $\{\varphi'(a,z\,;\alpha) : \varphi' > \varphi\}$. By the saturation of $\mathfrak{U}$, finite concistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. □

For $p(x) \subseteq \mathbb{I}(A)$, we write $p'(x)$ for the type

$$p'(x) \;=\; \big\{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}.$$

The following consequence of the proposition above will be frequently used

Remark 14. If evaluated in an ω-$\mathbb{I}$-saturated model, $p'(x)$ is equivalent to $p(x)$.

7. The monster model

We denote by $\mathfrak{U} = \langle U, R\rangle$ some large $\mathbb{I}$-saturated structure which we call the monster model. For convenience we assume that the cardinality of $\mathfrak{U}$ is an inaccessible cardinal which we denote by κ. Below we say model for $\mathbb{I}$-elementary submodel of $\mathfrak{U}$.

Let $A \subseteq U$ be a small set throughout this section. We define a topology on $U^{|x|} \times R^{|y|}$ which we call the $\mathbb{I}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x\,;y) \subseteq \mathbb{I}(A)$. This is a compact topology by the $\mathbb{I}$-compactness Theorem 11. The following fact demostrate how $\mathbb{I}$-compactness applies in this context. There are some subtle differences from the classical setting.

Fact 15. Let $p(x) \subseteq \mathbb{I}(A)$ be a type. Then for every $\varphi(x) \in \mathbb{I}(U)$

1. if $p(x) \rightarrow \neg\varphi(x)$ then $\psi(x) \rightarrow \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \rightarrow \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \rightarrow \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. Claim (1) is immediate by saturation. Claim (2) follows from the first by Lemma 9. □

When $A \subseteq U$, we write $S_{\mathbb{I}}(A)$ for the set of types

$$\mathbb{I}\text{-tp}(a/A) \;=\; \big\{\varphi(x) \;:\; \varphi(x) \in \mathbb{I}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of U. We write $S_{\mathbb{I},x}(A)$ when the tuple of variables x is fixed. The same notation applies also with $\mathbb{H}$ for $\mathbb{I}$.

The following proposition will be strengthen by Corollary 18 below.

Proposition 16. The types $p(x) \in S_{\mathbb{I}}(A)$ are maximally consistent subsets of $\mathbb{I}_x(A)$. That is, for every $\varphi(x) \in \mathbb{I}(A)$, either $\varphi(x) \in p$ or $p(x) \rightarrow \neg\varphi(x)$. The same holds with $\mathbb{H}$ for $\mathbb{I}$.

Proof. Let $p(x) = \mathbb{I}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 9 and Proposition 13 we obtain

$$\neg\varphi(x) \;\rightarrow\; \bigvee_{\tilde{\varphi}\perp\varphi} \tilde{\varphi}(x).$$

Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \rightarrow \neg\varphi(x)$ follows. □

8. Elimination of quantifiers of sort R

We show that the quantifiers $\forall^{\mathsf{R}}$ and $\exists^{\mathsf{R}}$ can be eliminated up to some approximation.

Proposition 17. The monster model (or any saturated model, for that matter) is $\mathbb{H}$-homogeneous, that is, every $\mathbb{H}$-elementary map $f : U \rightarrow U$ of cardinality $< \kappa$ extends to an automorphism.

Proof. By Proposition 16, the inverse of an $\mathbb{H}$-elementary map $f : U \rightarrow U$ is $\mathbb{H}$-elementary. Then the usual proof by back-and-forth applies. □

As promised, we strengthen Proposition 16. Let $A \subseteq U$ be a small set throughout this section.

Corollary 18. Let $p(x) \in S_{\mathbb{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathcal{L}_x(A)$. That is, for every $\varphi(x) \in \mathcal{L}(A)$, either $p(x) \rightarrow \varphi(x)$ or $p(x) \rightarrow \neg\varphi(x)$. Clearly, the same holds for $p'(x)$.

Proof. If $b \vDash p(x) = \mathbb{H}\text{-tp}(a/A)$ then there is an $\mathbb{H}$-elementary map $f \supseteq \mathrm{id}_A$ such that $fa = b$. As f exends to an automorphism and every automorphism is $\mathcal{L}$-elementary, the corollary follows. □

For $a, b \in U^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A, bearing in mind that formulas in $\mathbb{H}(A)$ or $\mathbb{I}(A)$ suffices to test the equivalence.

In the classical setting, from an equivalence between types one derives an equivalence between formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence. Next proposition show that formulas in $\mathbb{I}(A)$ are approximated by formulas in $\mathbb{H}(A)$.

Proposition 19. Let $\varphi(x) \in \mathbb{I}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathbb{H}(A)$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi'(x)$.

Proof. By Corollary 18 and Remark 14

$$\neg\varphi(x) \;\rightarrow\; \bigvee_{p'(x)\rightarrow\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathbb{H},x}(A)$. By Fact 15 and Lemma 9

$$\neg\varphi(x) \;\rightarrow\; \bigvee_{\neg\tilde{\psi}(x)\rightarrow\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathbb{H}(A)$. Equivalently,

$$\varphi(x) \;\leftarrow\; \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 15, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathbb{H}(A)$ such that

$$\varphi'(x) \;\leftarrow\; \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \;\leftarrow\; \varphi(x)$$

which yields the interpolant required by the proposition. □

We also need an approximation restult for the negation of formulas in $\mathbb{I}(A)$.

Proposition 20. Let $\varphi(x) \in \mathbb{I}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathbb{H}(A)$ and some $\psi' > \psi$.

Proof. Let $a \in U^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$. By Corollary 18, $p'(x) \to \neg\varphi(x)$. By compactness $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. □

9. The Tarski-Vaught test

The following proposition is not literally the Tarski-Vaught test. Infact, it only applies when the larger structure is the moster model.

Proposition 21. Let M be a subset of U. Then the following are equivalent

1. M is the domain of a model $\mathcal{M} = \langle M, R\rangle$;
2. for every formula $\varphi(x) \in \mathbb{H}(M)$
$$\exists^{\mathsf{M}} x\, \varphi(x) \quad\Rightarrow\quad \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathbb{I}(M)$
$$\exists^{\mathsf{M}} x\, \neg\varphi(x) \quad\Rightarrow\quad \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists^{\mathsf{M}} x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 9 there is some $\tilde\varphi \perp \varphi$ such that $\varphi(x) \to \neg\tilde\varphi(x) \to \varphi'(x)$. Then $\neg\forall^{\mathsf{M}} x\, \tilde\varphi(x)$ hence, by (1), $\mathcal{M} \vDash \neg\forall^{\mathsf{M}} x\, \varphi(x)$. Then $\mathcal{M} \vDash \neg\tilde\varphi(a)$ for some $a \in M$. Hence $\mathcal{M} \vDash \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathbb{I}(M)$ be such that $\exists^{\mathsf{M}} x\, \neg\varphi(x)$. By Corollary 20, there are a consistent $\psi(x) \in \mathbb{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). By the classical Tarski-Vaught test $M \preceq U$. Let $\mathcal{M} = \langle M, R\rangle$ be the unique $\mathcal{L}$-structure that is a substructure of $\mathcal{U}$. Then $\varphi(a) \Leftrightarrow \mathcal{M} \vDash \varphi(a)$ holds for every $a \in M^{|x|}$ and for every $\mathbb{I}$-atomic formula $\varphi(x)$. Now, assume inductively

$$\mathcal{M} \vDash \varphi(a, b) \quad\Rightarrow\quad \varphi(a, b).$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$\mathcal{M} \vDash \forall^{\mathsf{M}} y\, \varphi(a, y) \quad\Rightarrow\quad \forall^{\mathsf{M}} y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned}
\neg\forall^{\mathsf{M}} y\, \varphi(a, y) \quad&\Rightarrow\quad \exists^{\mathsf{M}} y\, \neg\varphi(a, y) \\
&\Rightarrow\quad \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow\quad \mathcal{M} \vDash \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow\quad \mathcal{M} \nvDash \forall^{\mathsf{M}} y\, \varphi(a, y).
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists^{\mathsf{M}}$, $\exists^{\mathsf{R}}$, and $\forall^{\mathsf{R}}$ is straightforward. □

10. Entourages

For $t(x, z)$ a term of sort $\mathsf{M}^{|x|+|z|} \to \mathsf{R}$ and $C \subseteq R^2$ we define

$$\tau_{t,C}(x, y) \quad=\quad \forall^{\mathsf{M}} z\, \langle t(x, z), t(y, z)\rangle \in C.$$

As C ranges over the compact neighborhoods of the diagonal $\Delta \subseteq R^2$ and $t(x, z)$ rangers over the parameter-free terms, the formulas $\tau_{t,C}(x, y)$ define a system of entougages on $U^{|x|}$. We refer to the topology associated as the R-topology. It is not difficult to verify that the R-topology on $U^{|x|}$ coincides with the product of the R-topology on U. We define the type

$$x \sim_{\mathsf{R}} y \quad = \quad \Big\{ \tau_{t,C}(x,y) \ : \ t(x,z) \text{ a term, } C \text{ a compact neigborhood of } \Delta \Big\}.$$

In other words, the type above says that $\forall^{\mathsf{M}} z \, \big[t(x,z) = t(y,z) \big]$ holds for every term $t(x,z)$. Again, it is easy to verify that if $|x| = |y| = \lambda$ then $x \sim_{\mathsf{R}} y$ if and only if $x_i \sim_{\mathsf{R}} y_i$ for every $i < \lambda$.

Let $p(x) \subseteq \mathbb{I}(A)$ for some small $A \subseteq U$. We say that $p(x)$ is R-invariant if $x \sim_{\mathsf{R}} y \to [p(x) \leftrightarrow p(y)]$. It is immediate that $\mathbb{I}$-formulas constructed without using (ii) of Definition 2 are R-invariant.

We say that $p(x)$ is a Cauchy type if it is consistent and $p(x) \cup p(y) \to x \sim_{\mathsf{R}} y$.

Fact 22. Let $\mathcal{M}$ be a model that is λ-$\mathbb{I}$-saturated for some $\lambda > |\mathcal{L}|$. Then every R-invariant Cauchy type $p(x) \subseteq \mathbb{I}(M)$ is realized in $\mathcal{M}$.

Proof. Pick any $a \vDash p(x)$. For every formula $\varphi(x)$ in the type $a \sim_{\mathsf{R}} x$ and every $\varphi' > \varphi$ pick some $\psi(x) \in p$ such that $\psi(x) \to \varphi'(x)$. There are at most $|\mathcal{L}|$ such formulas $\psi(x)$. By saturation they have a common solution b in M. Then $b \sim_{\mathsf{R}} a$ and, by invariance, $b \vDash p(x)$. □

We say that a model $\mathcal{M}$ is Cauchy saturated (or Cauchy complete) if every R-invariant Cauchy type over M is realized in $\mathcal{M}$. Note that for any type $p(x) \subseteq \mathbb{I}(M)$ the type $\bar{p}(x) = \exists y {\sim_{\mathsf{R}}} x \; p(y)$ is an R-invariant. If $p(x)$ is Cauchy also $\bar{p}(x)$ is Cauchy. Therefore $\mathcal{M}$ is Cauchy saturated if every Cauchy type $p(x) \subseteq \mathbb{I}(M)$ is realized by some $a \sim_{\mathsf{R}} M$.

11. Example: metric spaces

We discuss a simple example. Let M, d is a metric space. Let L_{M} contain symbols only for continuous functions $M^n \to M$. The language $\mathcal{L}$ has also a symbol d for a function of sort $\mathsf{M}^2 \to \mathsf{R}$. Let $R = \mathbb{R} \cup \{\pm\infty\}$ be the compactification of $\mathbb{R}$. Let $\mathcal{M} = \langle M, R \rangle$ be the structure that interprets the symbols of the language as natural.

Let $\mathcal{U}$ be a monster model, $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{U}$. Clearly, d does not define a metric on U as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on U. Therefore the notion of convergent sequence makes perfectly sense in U.

Fact 23. Let $\mathcal{U}$ be as above. Then for every model $\mathcal{N} = \langle N, R \rangle$ the following are equivalent for every $a \in U$

1. $p(x) = \mathbb{I}\text{-tp}(a/N)$ is a Cauchy type;
2. there is a sequence $\langle a_i \ : \ i \in \omega \rangle$ of elements of N that converges to a.

Proof. First note that, by the continuity of the terms with respect to the metric, $a \sim_{\mathsf{R}} b$ is equivalent to $d(a,b) = 0$.

(2⇒1) Let $\langle \varepsilon_i \ : \ i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then the formulas $d(a_i, x) \leq \varepsilon_i$ are in $p(x)$. Then every element realizing $p(x)$ is at distance 0 from a. Therefore $p(x) \to a \sim_{\mathsf{R}} x$.

(1⇒2) As $p(x)$ is Cauchy type, $p(x) \to a \sim_{\mathsf{R}} x$. Then $p'(x) \to d(a,x) < 2^{-i}$ for all i. By compactness (see Fact 15) there are formulas $\varphi_i(x) \in p$ such that $\varphi'_i(x) \to d(a,x) < 2^{-i}$ for some $\varphi'_i > \varphi_i$. By $\mathbb{I}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i \ : \ i \in \omega \rangle$ converges to a. □

References

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12. Measure spaces

⚠ Garbage after this point

Let $\mathcal{B}$ be a σ-algebra of subsets of Ω. Let F be the set of $\mathbb{R}$-valued $\mathcal{B}$-simple functions, i.e. linear combinations of indicator functions of sets in $\mathcal{B}$. Let M be the set of $\mathbb{R}$-valued $\mathcal{B}$-measures concentrated on finitely many points of Ω.

Let $\mathcal{M} = \langle M, F, R\rangle$, where $R = \mathbb{R}\cup\{\pm\infty\}$ is the compactification of $\mathbb{R}$. The sort M in the previous section is replaced here by two sorts which we call M and F. Here L_{MF} is the language of the structure $\langle M, F\rangle$. As both F and M are lattice $\mathbb{R}$-algebras, we assume that L_{MF} has symbols for these lattice algebras. There is also a function symbol of sort $\mathsf{F}\times\mathsf{M}\to\mathsf{M}$ that is interpreted as the natural action of functions on measures (via multiplication). Finally, in $\mathcal{L}$ there is a function symbol I of sort $\mathsf{M}\to\mathsf{R}$ that is interpreted as the integral over Ω.

Let $\mathcal{U} = \langle {}^*M, {}^*F, R\rangle$ be a monster model such that $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{U}$.

Let μ be a $\mathbb{R}$-valued $\mathcal{B}$-measure. By saturation, there is some ${}^*\mu \in {}^*M$ such that $g \cdot {}^*\mu = \int_\Omega g\,\mathrm{d}\mu$ for every $g\in F$.

Let $\gamma : \Omega\to\mathbb{R}$. By saturation, there is some ${}^*\gamma \in {}^*F$ such that ${}^*\gamma\cdot m = \gamma\cdot m$ for every $m\in M$. Note that, as m has finite support, $\gamma\cdot m$ is a well-defined element of M for all γ.

For $X\in\mathcal{B}$ we write $\mathrm{I}_X(\text{-})$ for $\mathrm{I}(1_X\cdot\text{-})$.

Lemma 24. Assume γ is measurable. Then for every $X\in\mathcal{B}$

$$\mathrm{I}_X({}^*\gamma\cdot{}^*\mu) \;=\; \int_X \gamma\,\mathrm{d}\mu$$

Proof. Assume for simplicity that both γ and μ are non-negative. Let $G\subseteq R$ be the following set

$$G \;=\; \Big\{\int_X g\,\mathrm{d}\mu \;:\; 0\le g\le\gamma,\ g\in F\Big\}.$$

If G is bounded, pick for every $\varepsilon\in\mathbb{R}^+$ some $\alpha_\varepsilon\in G$ such that

$$\int_X \gamma\,\mathrm{d}\mu \;\le\; \alpha_{\varepsilon,f}+\varepsilon.$$

If G_f is unbounded, let $\alpha_{\varepsilon,f} = +\infty$ for all ε. Let y be a variable of sort M. Let $p(y)$ be the $\mathbb{I}$-type that contains, for all f as in the lemma, the formulas

$$\begin{aligned}
\mathrm{I}(f\cdot y) &\;\ge\; \alpha && \text{for all } \alpha\in G_f;\\
\mathrm{I}(f\cdot y) &\;\le\; \alpha_{\varepsilon,f}+\varepsilon && \text{for all } \varepsilon\in\mathbb{R}^+.
\end{aligned}$$

By basic measure theory, $p(y)$ is finitely consistent in $\mathcal{M}$. Any realization of $p(y)$ in $\mathcal{U}$ proves the lemma. □

In M and F we use the functions -^+, -^-, $|\text{-}|$, and the relation $\le$ as usually defined in lattice vector spaces. We use the symbol 0 to denote the zeros of M, F, and R. We denote by $1\in F$ the function that is identically $1\in R$.

The $\mathbb{I}$-formula $\forall^{\mathsf{F}} f\ge 0\ \mathrm{I}(f\cdot\mu^+)\ge 0$ is obviously true in $\mathcal{M}$ and so is the same formula for μ^-. As the action of F on M distributes with the algebraic operations of M and F, properties such as the Hahn decomposition are immediate in $\mathcal{M}$ and are inherited by $\mathcal{U}$.

Note that the L_{M}-formula $\forall^{\mathsf{F}} f\in[0,1]\ \ f^2{=}f$ defines the indicator functions.

For $\nu,\mu\in{}^*M$ we write $\nu\ll\mu$ for the $\mathcal{L}$-formula

$$\forall^{\mathsf{R}}\varepsilon\ \exists^{\mathsf{R}}\delta\ \forall^{\mathsf{F}} f\ \big[\mathrm{I}(|f\cdot\mu|)<\delta\to\mathrm{I}(|f\cdot\nu|)<\varepsilon\big].$$

Proposition 25. Let $\nu, \mu \in {}^*M$ be bounded and non-negative such that $\nu \ll \mu$. Then there is a $g \in {}^*F$ such that $|\nu - g\mu| \sim_R 0$.

Proof. The following $\mathbb{I}$-formula is true in $\mathcal{M}$.

$$\forall \mu, \nu\ \forall \varepsilon > 0\ \Big[\forall^{\mathsf{F}} f\ \mathrm{I}\big(|f \cdot \nu| \dot{-} |f \cdot \mu|\big) < \varepsilon \to \exists^{\mathsf{F}} g\ \forall^{\mathsf{F}} f\ \mathrm{I}\big(f \cdot |\nu - g \cdot \mu|\big) \leq \varepsilon\Big]$$

□

Theorem 26 (Radon-Nikodym). Let ν, μ be bounded non-negative $\mathcal{B}$-measures on Ω such that $\nu \ll \mu$. Then there is a $\mathcal{B}$-measurable $f : \Omega \to \mathbb{R}$ such that

$$\int_X f \,\mathrm{d}\mu = \int_X \mathrm{d}\nu$$

for every $X \in \mathcal{B}$.