Continuous logic for the classical logician

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ABSTRACT. Let $\mathcal{L}$ be a first-order two-sorted language. In general, very little model theory is possible if we restrict to $\mathcal{L}$-structures that have the form $\langle M, I\rangle$ for some fixed structure I. But not everything is lost when I is a compact topological space. Continuous logic deals with this case. Rivisiting old ideas we present a back-to-basics approach to continuous logic.

1. Introduction

As a minimal motivating example, consider a real vector space M. This is are among the most simple strucures considered in model theory. Now, expand it by adding a norm. A norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two-sorted structure $\langle M, \mathbb{R}\rangle$. Now, we have two conflicting aspirations

i. to apply the basic tools of model theory (such as elementarity, compactness, and saturation);

ii. to stay within the realm of normed spaces (hence insist that $\mathbb{R}$ should remain $\mathbb{R}$ throughout).

These two requests are blatantly incompatible, something has to give. Different people have attemped different paths.

1. Nonstandard analysts happily embrace norms that take values in some ${}^*\mathbb{R} \succeq \mathbb{R}$. These normed spaces are nonstandard, but there are tricks to transfer results from nonstandard to standard normed spaces and vice versa. Model theorists, when confronted with similar problems, have used similar approaches (in more generality and with different notation). See [G] for a survey.
2. Henson and Iovino propose to stick to the standard notion of norm but restrict the notion of elementarity and saturation to a smaller class of formulas (which here is denoted by $\mathcal{H}$). They demostrate that most standard tools of model theory apply in their setting. The work of Henson and Iovino is focussed on Banach spaces and has hardly been applied to broader contexts. See [HI] for a survey.
3. Real valued logicians opt for an esoteric approach. They abandon classical true/false valued logic in favour of a logic valued in the interval $[0, 1]$. See [K] or [H] for a survey. Arguably, this esoteric flavor is what has helped the approach to its popularity. Unfortunately, this approach adds notational burden and unnatural restrictions. Our down-to-classical-logic approach ia an alternative.

We elaborate on the ideas of Henson and Iovino and generalize them to a larger class of structures. Following Henson and Iovino we also restrict the notions of elementarity and saturation. But we use a class of formulas $\mathcal{I}$ which is larger than $\mathcal{H}$. Though $\mathcal{I}$ has approximatively the same expressive power as $\mathcal{H}$ (cf. Propositions 23 and 24), it is evident (cf. Example 3) that $\mathcal{I}$ is more convinent to use.

We also borrow intuitions and results from nonstandard analisys to an extent that we could claim the title: nonstandard analysis for the standard logician.

2. A class of structures

Let I be some fixed first-order structure which is endowed with a Hausdorff compact[1] topology (in particular, a normal topology). The language $L_{\rm I}$ contains relation symbols for the compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \to I$. In particular, there is a constant for each element of I. According to the context, C and f denote either the symbols of $L_{\rm I}$ or their interpretation in the stucture I. The most common examples of structures I are the unit interval $[0, 1]$, with the usual topology, and its homeomophic copies $\mathbb{R}^+ \cup \{0, \infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language $L_{\sf H}$ which we call the language of the home sort.

Definition 1. Let $\mathcal{L}$ be a two sorted language that expands both $L_{\sf H}$ and $L_{\rm I}$. By model we mean a two-sorted $\mathcal{L}$-structure of the form $\langle M, I\rangle$, where M ranges over all $L_{\sf H}$-structures, while I is the structure we fixed above. We write $\sf H$ and $\rm I$ to denote the two sorts of $\mathcal{L}$. The new symbols allowed in $\mathcal{L}$ are function symbols of sort ${\sf H}^n \to {\rm I}$.

Models are denoted by the domain of their home sort.

The notion of model will be (inessentially) restricted once the monster model is introduced in Section 7.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (i.e. when I is finite), they are not models in the sense of Definion 1. As a remedy, below we carve out a set of formulas $L_{\sf H} \subseteq \mathcal{I} \subseteq \mathcal{L}$, such that every model has an $\mathcal{I}$-elementary, $\mathcal{I}$-saturated extension that is also a model.

Warning: as usual, $L_{\rm I}$, $L_{\sf H}$, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The symbols $\mathcal{I}$ and $\mathcal{H}$ defined below denote only sets of $\mathcal{L}$-formulas.

Definition 2. Formulas in $\mathcal{I}$ are defined inductively from two sorts of $\mathcal{I}$-atomic formulas

i. formulas of the form $t(x\,;\eta) \in C$, where $C \subseteq I^n$ is compact and $t(x\,;\eta)$ is a tuple of terms of sort ${\sf H}^{|x|} \times {\rm I}^{|\eta|} \to {\rm I}$;

ii. all formulas in $L_{\sf H}$.

We require that $\mathcal{I}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\sf H}$, $\exists^{\sf H}$ of sort $\sf H$; and the quantifiers $\forall^{\rm I}$, $\exists^{\rm I}$ of sort $\rm I$. We will use Latin letters x, y, z for variables of sort $\sf H$ and Greek letters η, ε for variables of sort $\rm I$. Therefore we can safely drop the superscript from quantifiers that are followed by variables.

We write $\mathcal{H}$ for the set of formulas in $\mathcal{I}$ with only quantifiers of sort $\sf H$ (i.e. without quantifiers of sort $\rm I$).

Formulas in $\mathcal{H}$ are a generalization of the positive bounded formulas of Henson and Iovino. Here we also introduce the larger class $\mathcal{I}$ because it offers some advantages. For instance, it is easy to see (cf. Example 3 for a hint) that $\mathcal{I}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive[2]). Somewhat surprisingly, we will see (cf. Propositions 23 and 24) that the formulas in $\mathcal{I}$ can be approximated by formulas in $\mathcal{H}$.

Example 3. Let $I = [0, 1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort ${\sf H}^{|x|} \to {\rm I}$. Then there is a formula in $\mathcal{I}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

[1] The requirement of compactness seems to exclude the motivational example mentioned above. To include this and others similar examples we have to replace $\mathbb{R}$ with $\mathbb{R} \cup \{\pm\infty\}$.

[2] This unsubstantited claim should be interpreted as a concjecture. A painstaking technical comparison with real valued logic is not in the scope of this indroductory exploratory paper.

$$\forall x\,\big[t(x) \dot{-} \tau \in \{0\}\big] \quad \wedge \quad \forall \varepsilon \Big[\varepsilon \in \{0\} \;\vee\; \exists x\,\big[\tau \dot{-} (t(x)+\varepsilon) \in \{0\}\big]\Big]$$

which, in a more legible form, becomes

$$\forall x\,\big[t(x) \le \tau\big] \quad \wedge \quad \forall \varepsilon > 0\, \exists x\,\big[\tau \le t(x)+\varepsilon\big].$$

We conclude this introduction with a question. Is it possible extend the theory exposed below to a larger $\mathcal{L}$? E.g. can we allow in $\mathcal{L}$ function symbols of sort $\mathsf{H}^n \times \mathrm{I}^m \to \mathrm{I}$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I.

3. The standard part

In this section we recall the notion of sandard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 8 which in turn is required for the proof of the $\mathcal{I}$-Compactness Theorem. The reader that is willing to accept the $\mathcal{I}$-Compactness Theorem without proof may skip this section, see Theorem 13.

Here, instead of reinventing the wiel (i.e. going back to ultrafilters and ultapowers), we assume the classical compacteness theorem and built on that. Let $I \preceq {}^*I$ be a proper extension. Let η be a free variable of sort I. For each $\beta \in I$, we define the type

$$\mathrm{m}_\beta(\eta) \quad = \quad \{\eta \in D \;:\; D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(\eta)$ in ${}^*I$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 4. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \vDash \mathrm{m}_\beta(\alpha)$.

Proof. Negate the existence of β. For every $\gamma \in I$ pick some compact neighborhood D_γ of γ, such that ${}^*I \vDash \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ, with $\gamma \in \Gamma$, cover I. By elementarity these D_γ also cover ${}^*I$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in I$ such that ${}^*I \vDash \mathrm{m}_\beta(\alpha)$. We write $\alpha \approx \alpha'$ if $\mathrm{st}(\alpha) = \mathrm{st}(\alpha')$.

Fact 5. For every $\alpha \in {}^*I$ and every compact $C \subseteq I$

$$ {}^*I \quad \vDash \quad \alpha \in C \;\rightarrow\; \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*I \vDash \alpha \in D \subseteq \neg C$. □

Fact 6. For every $\alpha \in ({}^*I)^{|\alpha|}$ and every function symbol f of sort $\mathrm{I}^{|\alpha|} \to \mathrm{I}$

$$ {}^*I \quad \vDash \quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big).$$

Proof. By Fact 4 and the definition of st(-) it suffices to prove that ${}^*I \vDash f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*I \vDash \alpha \in f^{-1}[D]$ and, as $I \preceq {}^*I$ we obtain ${}^*I \vDash f(\alpha) \in D$. □

Let $\langle M, {}^*I\rangle$ be an $\mathcal{L}$-structure such that $I \preceq {}^*I$. The standard part of $\langle M, {}^*I\rangle$ is the model $\langle M, I\rangle$ that interprets the symbols f of sort $\mathsf{H}^n \to \mathrm{I}$ as the functions

$$f^M(a) \quad = \quad \mathrm{st}\big({}^*f(a)\big) \qquad\qquad \text{for all } a \in M^n,$$

where ${}^*f$ is the interpretation of f in $\langle M, {}^*I\rangle$. Symbols in L_{H} maintain the same interpretation.

Fact 7. With the notation as above. Let $t(x\,;\eta)$ be a term of sort $\mathsf{H}^{|x|}\times\mathtt{I}^{|\eta|}\to\mathtt{I}$. Then for every $a\in M^{|x|}$ and $\alpha\in({}^*I)^{|\eta|}$

$$t^M\big(a\,;\mathrm{st}(\alpha)\big) \quad=\quad \mathrm{st}\big({}^*t(a\,;\alpha)\big)$$

Proof. When a t is function symbol of sort $\mathsf{H}^{|x|}\to\mathtt{I}$, the claim holds by definition. When t is a function symbol of sort $\mathtt{I}^{|\eta|}\to\mathtt{I}$, the claim follows from Fact 6. Now, assume inductively that

$$t_i^M\big(a\,;\mathrm{st}(\alpha)\big) \quad=\quad \mathrm{st}\big({}^*t_i(a\,;\alpha)\big)$$

holds for the terms $t_1(x\,;\eta),\dots,t_n(x\,;\eta)$ and let $t=f(t_1,\dots,t_n)$ for some function f of sort $\mathtt{I}^n\to\mathtt{I}$. Then the claim follows immediately from the induction hypothesis and Fact 6. □

Lemma 8. With the notation as above. For every $\varphi(x\,;\eta)\in\mathfrak{I}$, $a\in M^{|x|}$ and $\alpha\in({}^*I)^{|\eta|}$

$$\langle M, {}^*I\rangle\vDash\varphi(a\,;\alpha) \quad\Rightarrow\quad M\vDash\varphi\big(a\,;\mathrm{st}(\alpha)\big)$$

Proof. Suppose $\varphi(x\,;\eta)$ is $\mathfrak{I}$-atomic. If $\varphi(x\,;\eta)$ is a formula of L_{H} the claim is trivial. Otherwise $\varphi(x\,;\eta)$ has the form $t(x\,;\eta)\in C$. Assume that the tuple $t(x\,;\eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle M, {}^*I\rangle\vDash t(a\,;\alpha)\in C$. Then $\mathrm{st}\big({}^*t(a\,;\alpha)\big)\in C$ by Fact 5. Therefore $t^M\big(a\,;\mathrm{st}(\alpha)\big)\in C$ follows from Fact 7. This proves the lemma for $\mathfrak{I}$-atomic formulas. Induction is immediate. □

Corollary 9. Let M be a model and let $p(\eta)\subseteq\mathfrak{I}(M)$ be a type that is finitely consistent in M, where η is a tuple of variables of sort $\mathtt{I}$. Then $p(\eta)$ is realized in M.

Proof. By the classical Compactess Theorem, the type $p(\eta)$ is realized in some $\mathcal{L}$-structure $\langle M, {}^*I\rangle$ by some $\alpha\in({}^*I)^{|\eta|}$. Therefore, by the lemma above $M\vDash p\big(\mathrm{st}(\alpha)\big)$. □

4. Henson-Iovino approximations

For $\varphi,\varphi'\in\mathfrak{I}(M)$ (free variables are hidden) we write $\varphi'>\varphi$ if φ' is obtained by replacing each atomic formula of the form $t\in C$ occurring in φ with $t\in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occur in φ, then $\varphi>\varphi$. We also have $\varphi>\varphi$ when $\varphi=(t\in C)$ for some clopen set C. Note that $>$ is a dense (pre)order of $\mathfrak{I}(M)$. Formulas in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi\to\varphi'$.

We write $\tilde\varphi\perp\varphi$ when $\tilde\varphi$ is obtained by replacing each atomic formula $t\in C$ occurring in φ with $t\in\tilde C$ where $\tilde C$ is some compact set disjoint from C. Moreover the $\mathfrak{I}$-atomic formulas in L_{H} are replaced with their negation and each connective is replaced with its dual i.e., $\vee,\wedge,\exists,\forall$ are replaced with $\wedge,\vee,\forall,\exists$, respectively. We say that $\tilde\varphi$ is a strong negation of φ. It is clear that $\tilde\varphi\to\neg\varphi$.

Lemma 10. For all $\varphi\in\mathfrak{I}(M)$

1. for every $\varphi'>\varphi$ there is a formula $\tilde\varphi\perp\varphi$ such that $\varphi\to\neg\tilde\varphi\to\varphi'$;
2. for every $\tilde\varphi\perp\varphi$ there is a formula $\varphi'>\varphi$ such that $\varphi\to\varphi'\to\neg\tilde\varphi$.

Proof. If $\varphi\in L$ the claims are obvious. Suppose φ is of the form $t\in C$. Let φ' be $t\in C'$, for some compact neighborhood of C. Let O be an open set such that $C\subseteq O\subseteq C'$. Then $\tilde\varphi=(t\in I\smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde\varphi$ is of the form $t\in\tilde C$ for some compact $\tilde C$ disjoint from C.

By the normality of I, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. □

For $p(x) \subseteq \mathfrak{I}(A)$, we write

$$p'(x) \quad = \quad \big\{\varphi'(x) \ : \ \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}$$

in particular $\{\varphi(x)\}' = \big\{\varphi'(x) \ : \ \varphi' > \varphi\big\}$.

5. Morphisms

Let M and N be two models. We say that a partial map $f : M \to N$ is $\mathfrak{I}$-elementary if for every $\varphi(x) \in \mathfrak{I}$ and every $a \in (\operatorname{dom} f)^{|x|}$

1. $$M \vDash \varphi(a) \ \Rightarrow \ N \vDash \varphi(fa).$$

An $\mathfrak{I}$-elementary map that is total is called an $\mathfrak{I}$-elementary embedding. When the map $\mathrm{id}_M : M \to N$ is an $\mathfrak{I}$-elementary embedding, we write $M \preceq^{\mathfrak{I}} N$ and say that M is an $\mathfrak{I}$-elementary submodel of N.

The definitions of $\mathcal{H}$-elementary map/embedding/submodel is obtained replacing $\mathfrak{I}$ by $\mathcal{H}$.

As $\mathfrak{I}$ and $\mathcal{H}$-elementary maps are in particular L_{H}-elementary, they are injective. However, their inverse need not be $\mathfrak{I}$, respectively $\mathcal{H}$-elementary. In other words, the converse of the implication in (1) may not hold. We have choosen to work with the classical notion of satisfaction at the cost of this asymmetric notion of elementarity. This contrast with the approach of Henson and Iovino. They introduce the notion of *approximated satisfaction*. The approximated $\mathcal{H}$-morphisms we define below are the maps that preserve Henson-Iovino approximated satisfaction. These morphisms are invertible, cf. Fact 11.

We say that the map $f : M \to N$ is approximately $\mathfrak{I}$-elementary if for every formula $\varphi(x) \in \mathfrak{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$

$$M \vDash \big\{\varphi(a)\big\}' \ \Rightarrow \ N \vDash \big\{\varphi(fa)\big\}'.$$

We define approximately $\mathcal{H}$-elementary maps in the same manner. We leave to the reader to verify (using Lemma 10) that $\mathfrak{I}$-elementarity implies its approximated version, similarly for $\mathcal{H}$-elementarity. We will see (Proposition 15) that with a slight amount of saturation also the converse holds. Finally, under full saturation, all differences disappear as all these morphims becomes $\mathcal{L}$-elementarity maps (Corollary 22).

The following holds in general.

Fact 11. If $f : M \to N$ is approximately $\mathfrak{I}$-elementary then

$$M \vDash \big\{\varphi(a)\big\}' \ \Leftrightarrow \ N \vDash \big\{\varphi(fa)\big\}'.$$

for every $\varphi(x) \in \mathfrak{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$. The same holds for approximately $\mathcal{H}$-elementary maps.

Proof. Only implication $\Leftarrow$ requires a proof. Assume the r.h.s. of the equivalence. Fix $\varphi' > \varphi$ and prove $M \vDash \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By Lemma 10 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \to \neg\tilde{\varphi} \to \varphi'$. Then $N \vDash \neg\tilde{\varphi}(fa)$ and therefore $M \vDash \neg\tilde{\varphi}(a)$. Then $M \vDash \varphi'(a)$. □

Finally, we introduce the morphisms that corrisponds to the classical partial embeddings. We say that $f : M \to N$ is a partial $\mathfrak{I}$-embedding if the implication in (1) holds for all $\mathfrak{I}$-atomic formulas $\varphi(x)$. Note that this is the same as requiring that (1) holds for all formulas without quantifiers of sort H.

Fact 12. If $f : M \to N$ is a partial $\mathfrak{I}$-embedding then

$$M \vDash \varphi(a) \ \Leftrightarrow \ N \vDash \varphi(fa)$$

for every $a \in (\operatorname{dom} f)^{|x|}$ and every $\mathfrak{I}$-formulas $\varphi(x)$ without quantifiers of sort H.

Proof. The equivalence is trivial for formulas in L_{H} so we only consider $\mathfrak{I}$-atomic formulas as in (i) of Definion 1. Implication $\Rightarrow$ holds by definition. Vice versa, if $M \vDash t(a) \notin C$ then, by normality, $M \vDash t(a) \in \tilde{C}$ for some compact $\tilde{C}$ disjoint of C. By the definition of $\mathfrak{I}$-embedding, $N \vDash t(a) \in \tilde{C}$. Hence $N \vDash t(a) \notin C$. Induction is immediate. □

6. Compactness

It is convenient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\langle M, {}^*I\rangle \vDash T$ for some $\mathcal{L}$-structure. We say that T is consistent when $M \vDash T$ is for some model M.

Theorem 13 ($\mathfrak{I}$-Compactness Theorem). Let $T \subseteq \mathfrak{I}$ be finitely consistent. Then T is consistent.

Proof. Suppose T is finitely consistent (or finitely $\mathcal{L}$-consistent, for that matter). By the classical Compactness Theorem, there is an $\mathcal{L}$-structure $\langle M, {}^*I\rangle \vDash T$. Let $\langle M, I\rangle$ be its standard part as defined in Section 3. Then $M \vDash T$ by Lemma 8. □

A model N is λ-$\mathfrak{I}$-saturated if it realizes all types $p(x) \subseteq \mathfrak{I}(N)$ with fewer than λ parameters that are finitely consistent in N. When $\lambda = |N|$ we simply say $\mathfrak{I}$-saturated. The existence of $\mathfrak{I}$-saturated models is obtained from the classical case just as for Theorem 13.

Theorem 14. Every model has an $\mathfrak{I}$-elementary extension to a saturated model (possibly of inaccessible cardinality).

The following proposition shows that a slight amount of saturation tames the $\mathfrak{I}$-formulas.

Proposition 15. Let N be an ω-$\mathfrak{I}$-saturated model. Then

$$\{\varphi(x)\}' \;\leftrightarrow\; \varphi(x)$$

holds in N for every formula $\varphi(x) \in \mathfrak{I}(N)$.

Proof. We prove $\rightarrow$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

ih. $$\{\varphi(x,z)\}' \;\rightarrow\; \varphi(x,z)$$

We need to prove

$$\{\exists z\, \varphi(x,z)\} \;\rightarrow\; \exists z\, \varphi(x,z)$$

From (ih) we have

$$\exists z\, \{\varphi(x,z)\} \;\rightarrow\; \exists z\, \varphi(x,z)$$

Therefore it suffices to prove

$$\{\exists z\, \varphi(x,z)\}' \;\rightarrow\; \exists z\, \{\varphi(x,z)\}'$$

Replace x with some parameters, say a and assume the antecedent, that is, the theory $\{\exists z\, \varphi'(a,z) : \varphi' > \varphi\}$ is true in N. We need to prove the consistency of the type $\{\varphi'(a,z) : \varphi' > \varphi\}$. By saturation, finite concistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly by applying Corollary 9. □

Remark 16. If we restrict to formulas without quantifiers of sort H, the proposition above holds in every model (this is clear from the proof).

Remark 17. The proposition above will be frequently used in this form: if evaluated in an ω-$\mathfrak{I}$-saturated model, $p'(x)$ is equivalent to $p(x)$.

A consequence of Proposition 15 is that between ω-$\mathfrak{I}$-saturated models the approximated morphisms defined in Section 5 coincide with their unapproximated version. In particular, the following follows from Fact 11.

Corollary 18. Let M be an ω-$\mathfrak{I}$-saturated model. Let $f : M \to N$ be an $\mathfrak{I}$-elementary map. Then

$$M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(fa)$$

for every $\mathfrak{I}$-formula $\varphi(x)$ and every $a \in (\mathrm{dom} f)^{|x|}$.

7. The monster model

We denote by $\mathfrak{U}$ some large $\mathfrak{I}$-saturated model which we call the monster model. For convenience we assume that the cardinality of $\mathfrak{U}$ is an inaccessible cardinal. Truth is evaluated in $\mathfrak{U}$ unless otherwise is specified. Below we say model for $\mathfrak{I}$-elementary submodel of $\mathfrak{U}$. We stress once again that the truth of some $\varphi \in \mathfrak{I}(M)$ in a model M implies the truth of φ (in $\mathfrak{U}$) but not vice versa. If also the converse implication holds we say M a strong model. By Corollary 18, strong models are those that realize the equivalence in Proposition 15. Hence ω-$\mathfrak{I}$-saturated models are strong models. Also Cauchy complete models, which we introduce below, are strong models. However, models agree on the approximated truth, see Fact 11.

Let $A \subseteq \mathfrak{U}$ be a small set throughout this section. We define a topology on $\mathfrak{U}^{|x|}$ which we call the $\mathfrak{I}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x) \subseteq \mathfrak{I}(A)$. This is a compact topology by the $\mathfrak{I}$-Compactness Theorem 13. The following fact demostrate how $\mathfrak{I}$-compactness applies in this context. There are some subtle differences from the classical setting.

Fact 19. Let $p(x) \subseteq \mathfrak{I}(A)$ be a type. Then for every $\varphi(x) \in \mathfrak{I}(U)$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \neg\varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (1) is immediate by saturation; (2) follows from (1) by Lemma 10. □

When $A \subseteq \mathfrak{U}$, we write $S_{\mathfrak{I}}(A)$ for the set of types

$$\mathfrak{I}\text{-tp}(a/A) \quad = \quad \big\{\varphi(x) \ : \ \varphi(x) \in \mathfrak{I}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of $\mathfrak{U}$. We write $S_{\mathfrak{I},x}(A)$ when the tuple of variables x is fixed. The same notation applies also with $\mathcal{H}$ for $\mathfrak{I}$.

The following proposition will be strengthen by Corollary 22 below.

Proposition 20. The types $p(x) \in S_{\mathfrak{I}}(A)$ are maximally consistent subsets of $\mathfrak{I}_x(A)$. That is, for every $\varphi(x) \in \mathfrak{I}(A)$, either $\varphi(x) \in p$ or $p(x) \to \neg\varphi(x)$. The same holds with $\mathcal{H}$ for $\mathfrak{I}$.

Proof. Let $p(x) = \mathfrak{I}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 10 and Proposition 15 we obtain

$$\neg\varphi(x) \quad \to \quad \bigvee_{\tilde{\varphi} \perp \varphi} \tilde{\varphi}(x).$$

Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \to \neg\varphi(x)$ follows. □

8. Approximate elimination of quantifiers of sort I

We show that the quantifiers $\forall$ and $\exists$ can be eliminated up to some approximation.

Proposition 21. The monster model (or any saturated model, for that matter) is $\mathcal{H}$-homogeneous, that is, every $\mathcal{H}$-elementary map $f : \mathcal{U} \to \mathcal{U}$ of cardinality $< |\mathcal{U}|$ extends to an automorphism.

Proof. By Proposition 20, the inverse of an $\mathcal{H}$-elementary map $f : \mathcal{U} \to \mathcal{U}$ is $\mathcal{H}$-elementary. Then the usual proof by back-and-forth applies. □

As promised, we strengthen Proposition 20. Let $A \subseteq \mathcal{U}$ be a small set throughout this section.

Corollary 22. Let $p(x) \in S_{\mathcal{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathcal{L}_x(A)$. That is, for every $\varphi(x) \in \mathcal{L}(A)$, either $p(x) \to \varphi(x)$ or $p(x) \to \neg\varphi(x)$. Clearly, the same holds for $p'(x)$.

Proof. If $b \vDash p(x) = \mathcal{H}\text{-tp}(a/A)$ then there is an $\mathcal{H}$-elementary map $f \supseteq \mathrm{id}_A$ such that $fa = b$. Then f exends to an automorphism. As every automorphism is $\mathcal{L}$-elementary, the corollary follows. □

For $a, b \in \mathcal{U}^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A, bearing in mind that formulas in $\mathcal{H}(A)$ or $\mathcal{I}(A)$, or their approximations, suffice to test the equivalence.

In the classical setting, from an equivalence between types one derives an equivalence between formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence. Next proposition show that formulas in $\mathcal{I}(A)$ are approximated by formulas in $\mathcal{H}(A)$.

Proposition 23. Let $\varphi(x) \in \mathcal{I}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathcal{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 22 and Remark 17

$$\neg\varphi(x) \quad \to \quad \bigvee_{p'(x) \to \neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathcal{H},x}(A)$. By Fact 19 and Lemma 10

$$\neg\varphi(x) \quad \to \quad \bigvee_{\neg\tilde{\psi}(x) \to \neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{H}(A)$. Equivalently,

$$\varphi(x) \quad \leftarrow \quad \bigwedge_{\tilde{\psi}(x) \leftarrow \varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 19, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{H}(A)$ such that

$$\varphi'(x) \quad \leftarrow \quad \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \quad \leftarrow \quad \varphi(x)$$

which yields the interpolant required by the proposition. □

We also need an approximation restult for the negation of formulas in $\mathcal{I}(A)$.

Proposition 24. Let $\varphi(x) \in \mathcal{I}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathcal{H}(A)$ and some $\psi' > \psi$.

Proof. Let $a \in \mathcal{U}^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathcal{H}\text{-tp}(a/A)$. By Corollary 22, $p'(x) \rightarrow \neg\varphi(x)$. By compactness $\psi'(x) \rightarrow \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. □

9. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is not literally the Tarski-Vaught test because it only applies when the larger structure is the moster model.

Proposition 25. Let M be a subset of $\mathcal{U}$. Then the following are equivalent

1. M is a model;
2. for every formula $\varphi(x) \in \mathcal{H}(M)$

$$\exists x\, \varphi(x) \quad \Rightarrow \quad \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$

3. for every formula $\varphi(x) \in \mathcal{I}(M)$

$$\exists x\, \neg\varphi(x) \quad \Rightarrow \quad \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 10 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \rightarrow \neg\tilde{\varphi}(x) \rightarrow \varphi'(x)$. Then $\neg\forall x\, \tilde{\varphi}(x)$ hence, by (1), $M \vDash \neg\forall x\, \varphi(x)$. Then $M \vDash \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $M \vDash \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathcal{I}(M)$ be such that $\exists x\, \neg\varphi(x)$. By Corollary 24, there are a consistent $\psi(x) \in \mathcal{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \rightarrow \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). By the classical Tarski-Vaught test $M \preceq_{\mathrm{H}} \mathcal{U}$. Then M is the domain of an $\mathcal{L}$-substructure of $\mathcal{U}$. Then $\varphi(a) \Leftrightarrow M \vDash \varphi(a)$ holds for every $\mathcal{I}$-atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \vDash \varphi(a, b) \quad \Rightarrow \quad \varphi(a, b).$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$M \vDash \forall y\, \varphi(a, y) \quad \Rightarrow \quad \forall y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned} \neg\forall y\, \varphi(a, y) \quad &\Rightarrow \quad \exists y\, \neg\varphi(a, y) \\ &\Rightarrow \quad \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\ &\Rightarrow \quad M \vDash \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\ &\Rightarrow \quad M \nvDash \forall y\, \varphi(a, y). \end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists$, $\exists$, and $\forall$ is straightforward. □

Classically, the main application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all $\mathcal{L}$-structures. In particular every $A \subseteq U$ is contained in a model of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of $\mathcal{L}$ is eccessively large because of the aboundance of symbols in L_{I}.

We say that $\mathcal{L}$ is separable if there is a countable $\mathcal{H}_0 \subseteq \mathcal{H}$ such that for every $\varphi' > \varphi \in \mathcal{H}$ there is a $\varphi_0 \in \mathcal{H}_0$ such that $\varphi \rightarrow \varphi_0 \rightarrow \varphi'$ holds in every $\mathcal{L}$-structure.

Proposition 26. Assume that $\mathcal{L}$ is separable. Let $A \subseteq \mathcal{U}$ be countable. Then there is a countable model M containing A.

Proof. As in the classical proof of the Löwenheim-Skolem Theorem, we construct $M \subseteq \mathfrak{U}$ that contain a witness for every consistent formula in $\mathcal{H}_0(M)$. By separability and (2) in Proposition 25, M is a model. □

We have defined separability as a property of $\mathcal{L}$. Arguably, it could have been introduced as a property of the theory of $\mathfrak{U}$. The following proposition provides the examples that motivate the definition of separability (in these examples the property depends soley on $\mathcal{L}$).

Proposition 27. Assume that I is second countable, and that there are at most countably many symbols in $\mathcal{L} \smallsetminus L_{\mathrm{I}}$. Then $\mathcal{L}$ is separable.

Proof. ??? □

10. Cauchy complete models

For $t(x, z)$ a term of sort $\mathsf{H}^{|x|+|z|} \to \mathrm{I}$ we define the formula

$$x \sim_t y \quad = \quad \forall z \;\; t(x, z) = t(y, z).$$

We also define the type

$$x \sim_{\mathrm{I}} y \quad = \quad \Big\{ x \sim_t y \;\;:\;\; t(x, z) \text{ as above} \Big\}.$$

Fact 28. For any $a = \langle a_i \,:\, i < \lambda \rangle$ and $b = \langle b_i \,:\, i < \lambda \rangle$

$$a \sim_{\mathrm{I}} b \;\;\Leftrightarrow\;\; a_i \sim_{\mathrm{I}} b_i \quad \text{for every } i < \lambda.$$

Proof. Only implication $\Leftarrow$ requires a proof. Assume $a_i \sim_{\mathrm{I}} b_i$ for every i. Let $|a| = |b| = \lambda$ and assume inductively that

$$\forall y, z \quad t(a, y, z) \;=\; t(b, y, z)$$

holds for every term $t(x, y, z)$, with $|x| = \lambda$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

1. $$\forall z \quad t(a, a_\lambda, z) \;=\; t(b, a_\lambda, z).$$

As $a_\lambda \sim_{\mathrm{I}} b_\lambda$ then

$$\forall x, z \quad t(x, a_\lambda, z) \;=\; t(x, b_\lambda, z)$$

and in particular

2. $$\forall z \quad t(b, a_\lambda, z) \;=\; t(b, b_\lambda, z).$$

From (1) and (2) we obtain

$$\forall z \quad t(a, a_\lambda, z) \;=\; t(b, b_\lambda, z).$$

For limit ordinals induction is trivial. □

Note that the approximations of the formula $x \sim_t y$ have the form

$$x \sim_{t,D} y \quad = \quad \forall z \;\; \langle t(x, z), t(y, z) \rangle \in D$$

for some compact neighborhood D of Δ, the diagonal of I^2. Hence we define

$$x \sim'_{\mathrm{I}} y \quad = \quad \Big\{ x \sim_{t,D} y \;\;:\;\; t(x, z) \text{ parameter-free term, } D \text{ compact neigborhood of } \Delta \Big\}.$$

If $t(x, z) = t_1(x, z), \ldots, t_n(x, z)$ is a tuple of terms, by $x \sim_{t,D} y$ we denote the conjunction of the formulas $x \sim_{t_i,D} y$. As D ranges over the compact neighborhoods of Δ and $t(x, z)$ ranges over the finite tuples parameter-free terms. These formulas $x \sim_{t,D} y$ define a system of entougages on $\mathfrak{U}^{|x|}$. We refer to this uniformity and the topology associated as the I-topology. Though not nedded in the sequel, it is worth mentioning that the I-topology on $\mathcal{U}^{|x|}$ coincides with the product of the I-topology on $\mathcal{U}$. This can be verified by an argument similar to what used in the proof of Fact 28.

We say that $p(x) \subseteq \mathfrak{I}(\mathcal{U})$ is I-invariant if $x \sim_{\mathrm{I}} y \rightarrow [p(x) \leftrightarrow p(y)]$. We say that $p(x)$ is a Cauchy type if it is consistent and $p(x) \wedge p(y) \rightarrow x \sim_{\mathrm{I}} y$.

We say that a type $q(x)$ is finitely satisfiable in M if every conjunction of formulas in $q(x)$ has a solution in M. This definition is classical, but note that classically every type over M is finitely saisfiable in M. Not here, as we can only infer that $q'(x)$ is finitely satisfiable.

Lemma 29. The following are equivalent for every model M

1. $p(x) \subseteq \mathfrak{I}(M)$ is an I-invariant Cauchy type;
2. $p(x) \leftrightarrow a \sim_{\mathrm{I}} x$ for some/any $a \vDash p(x)$;
3. $a \sim'_{\mathrm{I}} x$ is finitely satisfiable in M for some/any $a \vDash p(x)$.

Proof. (1⇒2) Let $a \vDash p(x)$. Substituting a for y in the definition of Cauchy type we obtain $p(x) \rightarrow a \sim_{\mathrm{I}} x$ Similarly, from the definition of I-invariance we obtain $a \sim_{\mathrm{I}} x \rightarrow p(x)$.

(2⇒3) By compactness, as $p'(x)$ is finitely satisfiable in M.

(3⇒1) Let $a_{t,D} \in M$ be a realization of $a \sim_{t,D} x$ as D ranges over the compact neighborhoods of the diagonal and $t(x,z)$ over the finite tuples parameter-free terms. Then, by the uniqueness of the limit, the type $p(x)$ cointaining the formulas $x \sim_{t,D} a_{t,D}$ is as required by the lemma. □

We say that M is Cauchy complete if every I-invariant Cauchy type $p(x) \subseteq \mathfrak{I}(M)$ is realized in M. Note that if $p(x)$ is Cauchy type then any consistent type $q(x)$ containing $p(x)$ is also Cauchy. In particular the type $p(x) \cup \mathfrak{I}\text{-Th}(\mathcal{U}/M)$ is Cauchy. The following trivial consequence of this fact that is worth noticing

Remark 30. Cauchy complete models are strong models. I.e., $M \vDash \varphi$ for every true $\varphi \in \mathfrak{I}(M)$.

Fact 31. Let M be a model that is λ-$\mathfrak{I}$-saturated for some $\lambda > |\mathcal{L}|$. Then M is Cauchy complete.

Proof. Let $p(x) \subseteq \mathfrak{I}(M)$ be an I-invariant Cauchy type. Pick any $a \vDash p(x)$. By Lemma 29, $p(x) \leftrightarrow a \sim_{\mathrm{I}} x$. For every formula $\varphi(x)$ in the type $a \sim_{\mathrm{I}} x$ and every $\varphi' > \varphi$ pick some $\psi(x) \in p$ such that $\psi(x) \rightarrow \varphi'(x)$. There are $|\mathcal{L}|$ many formulas in $a \sim_{\mathrm{I}} x$. Then by saturation the formulas $\psi(x)$ we picked above have a common solution b in M. Then $b \sim_{\mathrm{I}} a$ and, by invariance, $b \vDash p(x)$. □

Clearly, when $\mathcal{U}$ is separable we can improve the proposition above by taking $\lambda = \omega_1$.

11. Example: metric spaces

We discuss a simple example. Let I be $\mathbb{R}^+ \cup \{0, \infty\}$, with the topology that makes it homeomorphic to the unit interval. Let M, d is a metric space. Let L_{H} contain symbols for functions $M^n \rightarrow M$ that are uniformly continuous. Let $\mathcal{L}$ contain a symbol for d and possibly for some funtions of $M^n \rightarrow I$ that are uniformly continuous w.r.t. the metric d.

Let $\langle M, I \rangle$ be the structure that interprets the symbols of the language as natural.

Let $\mathcal{U}$ be a monster model that is an $\mathfrak{I}$ elementary extension of M. Clearly, d does not define a metric on $\mathcal{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathcal{U}$. Therefore the notion of convergent sequence makes perfectly sense in $\mathcal{U}$. As all functions have been required to be uniformly continuous, it is immediate that $x \sim_{\mathrm{I}} y$ is equivalent to $d(x, y) = 0$.

Fact 32. Let $\mathcal{U}$ be as above. Then for every model N the following are equivalent for every $a \in \mathcal{U}$

1. $p(x) = \mathcal{I}\text{-tp}(a/N)$ is a Cauchy type;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of N that converges to a.

Proof. (2⇒1) Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then the formulas $d(a_i, x) \leq \varepsilon_i$ are in $p(x)$. Then every element realizing $p(x)$ is at distance 0 from a. Therefore $p(x) \rightarrow a \sim_{\mathrm{I}} x$.

(1⇒2) As $p(x)$ is Cauchy type, $p(x) \rightarrow a \sim_{\mathrm{I}} x$. Then $p'(x) \rightarrow d(a, x) < 2^{-i}$ for all i. By compactness (see Fact 19) there are formulas $\varphi_i(x) \in p$ such that $\varphi_i'(x) \rightarrow d(a, x) < 2^{-i}$ for some $\varphi_i' > \varphi_i$. By $\mathcal{I}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega \rangle$ converges to a. □

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⚠ Garbage after this point

12.

13. Separability

Let $t(x, y)$ and D be a term of sort $\mathsf{H}^{|x|+|z|} \rightarrow \mathrm{I}$, respectively a compact neighborhood of the diagonal. For $\gamma, \xi \in \mathrm{Aut}(\mathcal{U})$ and a a finite tuple of elements of M we write $\gamma \sim_{a,t,C} \xi$ for $\gamma a \sim_{t,D} \xi a$. This defines a system of entourages on $\mathrm{Aut}(\mathcal{U})$.

By $\mathrm{Aut}(\mathcal{U}/\{M\})$ we denote the set of automorphisms of $\mathcal{U}$ that fix M setwise. When $\mathrm{Aut}(\mathcal{U}/\{M\})$ is dense in $\mathrm{Aut}(\mathcal{U})$ with respect to the topology above, we say that $\mathcal{M}$ is approximatively ω-homogeneous. Note that this is the classical notion of homogeneity only with the discrete topology replaced by the I-topology.

We write $\mathcal{I}_\forall$ for the set of $\mathcal{I}$-formulas where the quantifier $\exists^{\mathsf{H}}$ do not occur. Let T_0 be an $\mathcal{I}_\forall$-theory with countable models (L_{H} is countable). Let $\mathcal{C}$ be a category whose objects, which we call $\mathcal{C}$-models, are countable models of T_0. For ease of speaking we identify these models with the domain of their sort H. The morphisms of $\mathcal{C}$, which we call $\mathcal{C}$-morphisms, are finite maps between $\mathcal{C}$-models.

c1. the identity map $\mathrm{id}_A : M \rightarrow M$ is a $\mathcal{C}$-morphism, for any $A \subseteq M$;

c2. $\mathcal{C}$-morphisms are partial $\mathcal{I}$-embeddings;

c3. the inverse of a $\mathcal{C}$-morphism is a $\mathcal{C}$-morphism;

c3. if M is a $\mathcal{C}$-model and $f : M \to N$ and $\mathcal{I}$-elementary map, then N is a $\mathcal{C}$-model and f is a $\mathcal{C}$-morphism.

We write $M \leq N$ if the identity map $\mathrm{id}_M : M \to N$ is a $\mathcal{C}$-morphism. We also assume that $\mathcal{C}$ is connected (c4) and that it has the amalgamation property (c5).

c4. the empty map between two $\mathcal{C}$-models is a $\mathcal{C}$-morphism;

c5. for every $\mathcal{C}$-morphism $k : M \to N$ and every $b \in M$ there is a $\mathcal{C}$-model $N \geq M$ and a $\mathcal{C}$-morphism $h : M \to N'$ that exiends k and is defined on b.