CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. Introduzione

As a minimal motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now, expand them by adding a norm. A norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two-sorted structure. Now, we have two conflicting aspirations: we would like to

i. apply the basic tools of model theory (such as elementarity and saturation);

ii. stay within the realm of normed spaces (hence insist that $\mathbb{R}$ should remain $\mathbb{R}$ throughout).

These two requests are blatantly incompatible, something has to give. Different people have attemped different paths.

1. Nonstandard analysts happily embrace norms that take values in some ${}^*\mathbb{R} \succeq \mathbb{R}$. These normed spaces are nonstandard, but there are tricks to transfer results from nonstandard to standard normed spaces and vice versa. Model theorists, when confronted with similar problems, have used similar approaches (in more generality and with different notation).
2. Henson and Iovino propose to stick to the standard notion of norm but restrict the notion of elementarity and saturation to a smaller class of formulas (which here is denoted by $\mathbb{H}$). They demostrate that many standard tools of model theory apply in their setting. The work of Henson and Iovino is focussed on Banach spaces and has hardly been applied to broader contexts.
3. Real valued logicians have the most radical approach. They abandon classical true/false valued logic altogether in favour of a logic valued in the interval $[0,1]$. Unfortunately, this adds unnecessary notational burden and restricts the class of structures that can be considered.

We elaborate on the ideas of Henson and Iovino and generalize them to a larger class of structures. Following Henson and Iovino, we restrict the notions of elementarity and saturation. But we use a class of formulas $\mathbb{I}$ which is larger than their $\mathbb{H}$. Though we prove that $\mathbb{I}$ has, in a sense, approximatively the same expressive power as $\mathbb{H}$ (cf. Propositions 18 and 19), it is evident that $\mathbb{I}$ is more convinent to use.

We also borrow intuitions and results from nonstandard analisys to an extent that we could also claim as title: nonstandard analysis for the finite logician.

2. A class of structures

Let R be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language L_R contains relation symbols for the compact subsets $C \subseteq R^n$ and a function symbol for each continuous functions $f : R^n \to R$. According to the context, C and f denote either the symbols of L_R or their interpretation in R.

Definition 1. Let L be a one-sorted (first-order) language. By model we mean an $\mathcal{L}$-structure of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is the structure above. The language $\mathcal{L}$ is an expansion of both L and L_R. The new symbols allowed in $\mathcal{L}$ are function symbols of sort $\mathsf{M}^n \to \mathsf{R}$.

In some examples we may need to replace the sort M with many sorts $\mathsf{M}_1 \dots, \mathsf{M}_k$. In this case the language L governs the structure $\langle \mathsf{M}_1 \dots, \mathsf{M}_k\rangle$. The models have the form $\langle \mathsf{M}_1 \dots, \mathsf{M}_k, R\rangle$. The language $\mathcal{L}$ has function symbols of sort $\mathsf{M}_1^{n_1} \dots, \mathsf{M}_k^{n_k} \to \mathsf{R}$.

The scope of the notion of model will be (inessentially) restricted once the monster model is introduced, in Section 7.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (i.e. when R is finite) they are not models. As a remedy, below we carve out a set of formulas $L \subseteq \mathbb{I} \subseteq \mathcal{L}$, such that every model has an $\mathbb{I}$-elementary, $\mathbb{I}$-saturated extension that is a model.

Warning: as usual L and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The symbols $\mathbb{I}$ and $\mathbb{H}$ defined below denote only sets of $\mathcal{L}$-formulas.

Definition 2. Formulas in $\mathbb{I}$ are defined inductively from two sorts of $\mathbb{I}$-atomic formulas

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact (in the product topology) and $t(x, y)$ is a n-tuple of terms of sort $\mathsf{M}^{|x|} \times \mathsf{R}^{|y|} \to \mathsf{R}$;

ii. all formulas in L.

We require that $\mathbb{I}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall$, $\exists$ of sort M; and the quantifiers of sort R which are denoted by $\forall^{\mathsf{R}}$, $\exists^{\mathsf{R}}$.

We write $\mathbb{H}$ for the set of formulas in $\mathbb{I}$ without quantifiers of sort R.

For later reference we remark the following which is an immediate consequence of the restrictions imposed by Definition 1 on the sorts of the function symbols in $\mathcal{L}$.

Formulas in $\mathbb{H}$ are a generalization of the positive bounded formulas of Henson and Iovino. Here we also introduce the larger class $\mathbb{I}$ because it offers some advantages. For instance, it is easy to see (cf. Example 3 for a hint) that $\mathbb{I}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive[1]). Somewhat surprisingly, we will see (cf. Propositions 18 and 19) that the formulas in $\mathbb{I}$ can be very well approximated by formulas in $\mathbb{H}$.

Example 3. Let $R = [0, 1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort $\mathsf{M}^{|x|} \to \mathsf{R}$. Then there is a formula in $\mathbb{I}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall x \left[t(x) \dot{-} \tau \in \{0\}\right] \quad \wedge \quad \forall^{\mathsf{R}}\varepsilon\Big[\varepsilon \in \{0\} \ \vee\ \exists x \left[\tau \dot{-} (t(x) + \varepsilon) \in \{0\}\right]\Big]$$

which, in a more legible form, becomes

$$\forall x \left[t(x) \leq \tau\right] \quad \wedge \quad \forall \varepsilon > 0\, \exists x \left[\tau \leq t(x) + \varepsilon\right].$$

[1]This unsubstatited claim should be interpreted as a concjecture. A painstaking technical comparison with real valued logic is not in the scope of this indroductory exploratory paper.

3. The standard part

Let $R \preceq {}^*R$. For each $\beta \in R$ we define the type

$$\mathrm{m}_\beta(x) \quad = \quad \{x \in D \ : \ D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(x)$ in ${}^*R$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 4. For every $\alpha \in {}^*R$ there is a unique $\beta \in R$ such that $\alpha \vDash \mathrm{m}_\beta(x)$.

Proof. Negate the existence of β. For every $\gamma \in R$ pick some compact neighborhood D_γ of γ, such that ${}^*R \ \vDash \ \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq R$ such that D_γ, with $\gamma \in \Gamma$, cover R. By elementarity these D_γ also cover ${}^*R$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in R$ such that $\alpha \vDash \mathrm{m}_\beta(x)$. We write $\alpha \approx \alpha'$ if $\mathrm{st}(\alpha) = \mathrm{st}(\alpha')$.

Fact 5. For every $\alpha \in {}^*R$ and every compact $C \subseteq R$

$${}^*R \quad \vDash \quad \alpha \in C \ \rightarrow \ \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*R \vDash \alpha \in D \subseteq \neg C$. □

Fact 6. For every $\alpha \in ({}^*R)^{|x|}$ and every function symbol of sort $\mathsf{R}^{|x|} \rightarrow \mathsf{R}$

$${}^*R \quad \vDash \quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big).$$

Proof. By Fact 4 and the definition of st(-) it suffices to prove that ${}^*R \vDash f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*R \vDash \alpha \in f^{-1}[D]$ and, as $R \preceq {}^*R$ we obtain ${}^*R \vDash f(\alpha) \in D$. □

Let ${}^*\mathcal{M} = \langle M, {}^*R\rangle$ be an $\mathcal{L}$-structure such that $R \preceq {}^*R$. The standard part of ${}^*\mathcal{M}$ is the model $\mathcal{M} = \langle M, R\rangle$ that interprets the symbols f of sort $\mathsf{M}^n \rightarrow \mathsf{R}$ as the functions

$$f^{\mathcal{M}}(a) \quad = \quad \mathrm{st}\big(f^{{}^*\mathcal{M}}(a)\big) \qquad \text{for all } a \in M^n.$$

Symbols in L maintain the same interpretation.

Fact 7. Let ${}^*\mathcal{M}$ and $\mathcal{M}$ be as above. Let $t(x\,;y)$ be a term of sort $\mathsf{M}^{|x|} \times \mathsf{R}^{|y|} \rightarrow \mathsf{R}$. Then for every $a \in M^{|x|}$ and $\alpha \in \mathsf{R}^{|y|}$ every term

$$t^{\mathcal{M}}\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big(t^{{}^*\mathcal{M}}(a\,;\alpha)\big)$$

Proof. When a t is function symbol of sort $\mathsf{M}^{|x|} \rightarrow \mathsf{R}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{R}^{|y|} \rightarrow \mathsf{R}$, the claim follows from Fact 6. Now, assume inductively that

$$t_i^{\mathcal{M}}\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big(t_i^{{}^*\mathcal{M}}(a\,;\alpha)\big)$$

holds for the terms $t_1, \dots, t_n$ and let $t = f(t_1, \dots, t_n)$ for some function f of sort $\mathsf{R}^n \to \mathsf{R}$. Then the claim follows immediately from the induction hypothesis and Fact 6. □

Lemma 8. Let ${}^*\mathcal{M}$ and $\mathcal{M}$ be as above. Then for every $\varphi(x\,;y) \in \mathbb{I}$, $a \in M^{|x|}$ and $\alpha \in ({}^*R)^{|y|}$

$${}^*\mathcal{M} \vDash \varphi(a\,;\alpha) \quad\Rightarrow\quad \mathcal{M} \vDash \varphi\big(a\,;\mathrm{st}(\alpha)\big)$$

Proof. Suppose $\varphi(x\,;y)$ is $\mathbb{I}$-atomic. If $\varphi(x\,;y)$ is a formula of L the claim is trivial. Otherwise $\varphi(x\,;y)$ has the form $t(x\,;y) \in C$. Assume that the tuple $t(x\,;y)$ consists of a single term. The general case is left to the reader as it follows easily from this special case.

Assume ${}^*\mathcal{M} \vDash t(a\,;\alpha) \in C$. Then $\mathrm{st}\big(t^{{}^*\mathcal{M}}(a\,;\alpha)\big) \in C$ by Fact 5. Therefore $t^{\mathcal{M}}(a\,;\mathrm{st}(\alpha)) \in C$ follows from Fact 7.

This proves the lemma for $\mathbb{I}$-atomic formulas. Induction is immediate. □

4. $\mathbb{I}$-ELEMENTARITY

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two models. We say that $f : M \to N$, a partial map, is $\mathbb{I}$-elementary if for every $\varphi(x) \in \mathbb{I}$ and every $a \in (\mathrm{dom} f)^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \quad\Rightarrow\quad \mathcal{N} \vDash \varphi(f a).$$

An $\mathbb{I}$-elementary map is in particular L-elementary and therefore it is injective. An $\mathbb{I}$-elementary map that is total is called an $\mathbb{I}$-elementary embedding. When the map $\mathrm{id}_M : M \hookrightarrow N$ is an $\mathbb{I}$-embedding, that is, if for every $\varphi(x) \in \mathbb{I}$ and every $a \in M^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \quad\Rightarrow\quad \mathcal{N} \vDash \varphi(a),$$

we write $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{I}$-elementary submodel of $\mathcal{N}$.

The definitions of $\mathbb{H}$-elementary map/embedding/submodel is obtained replacing $\mathbb{I}$ by $\mathbb{H}$.

5. $\mathbb{I}$-COMPACTNESS

It is convenient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure $\mathcal{M}$. We say that T is consistent when $\mathcal{M}$ is required to be a model.

Theorem 9. Let $T \subseteq \mathbb{I}$ be finitely consistent. Then T is consistent.

Proof. Suppose T is finitely consistent (or finitely $\mathcal{L}$-consistent, for that matter). By the classical compactness theorem ${}^*\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure ${}^*\mathcal{M} = \langle M, {}^*R\rangle$. Let $\mathcal{M}$ be the model $\langle M, R\rangle$. Then $\mathcal{M} \vDash T$ by Lemma 8. □

A model $\mathcal{N}$ is λ-$\mathbb{I}$-saturated if it realizes all types with fewer than λ parameters that are finitely consistent in $\mathcal{N}$. When $\lambda = |\mathcal{N}|$ we simply say $\mathbb{I}$-saturated. The existence of $\mathbb{I}$-saturated models is obtained from the classical case just as for Theorem 9.

Theorem 10. Every model has an $\mathbb{I}$-elementary extension to a saturated model (possibly of inaccessible cardinality).

6. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{I}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained by replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{I}(M)$.

Formulas in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. Moreover the $\mathbb{I}$-atomic formulas in L are replaced with their negation and every connective is replaced with its dual i.e., $\vee, \wedge, \exists, \forall, \exists^{\mathsf{R}}, \forall^{\mathsf{R}}$ are replaced with $\wedge, \vee, \forall, \exists, \forall^{\mathsf{R}}, \exists^{\mathsf{R}}$ respectively.

It is clear that $\tilde{\varphi} \to \neg\varphi$. We say that $\tilde{\varphi}$ is a strong negation of φ.

Lemma 11.

1. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$.
2. For every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in R \smallsetminus O)$ is as required by the lemma.

Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of R, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = t \in C'$ is as required.

The lemma follows easily by induction. $\square$

7. The monster model

We denote by $\mathcal{U} = \langle U, R\rangle$ some large $\mathbb{I}$-saturated structure which we call the monster model. For convenience we assume that the cardinality of $\mathcal{U}$ is an inaccessible cardinal which we denote by κ. Below we say model for $\mathbb{I}$-elementary submodel of $\mathcal{U}$.

Let $A \subseteq U$ be a small set throughout this section. We define a topology on $U^{|x|} \times R^{|y|}$ which we call the $\mathbb{I}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x\,;y) \subseteq \mathbb{I}(A)$. This is a compact topology by the $\mathbb{I}$-compactness Theorem 9.

The following fact demostrate how $\mathbb{I}$-compactness applies. There are some subtle differences from the classical setting.

Fact 12. Let $p(x) \subseteq \mathbb{I}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{I}(U)$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. Claim (1) is immediate by saturation. Claim (2) follows from the first by Lemma 11. □

Proposition 13. For every $\varphi(x) \in \mathbb{I}(U)$

$$\bigwedge_{\varphi' > \varphi} \varphi'(x\,;y) \quad\leftrightarrow\quad \varphi(x\,;y)$$

Proof. We prove $\to$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifier is immediate. We consider case of the existential quantifiers of sort M. The case of existential quantifiers of sort R is identical. Assume inductively

ih. $$\bigwedge_{\varphi' > \varphi} \varphi'(x, z\,;y) \quad\to\quad \varphi(x, z\,;y)$$

We need to prove

$$\bigwedge_{\varphi' > \varphi} \exists z\, \varphi'(x, z\,;y) \quad\to\quad \exists z\, \varphi(x, z\,;y)$$

From (ih) we have

$$\exists z \bigwedge_{\varphi' > \varphi} \varphi'(x, z\,;y) \quad\to\quad \exists z\, \varphi(x, z\,;y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi' > \varphi} \exists z\, \varphi'(x, z\,;y) \quad\to\quad \exists z \bigwedge_{\varphi' > \varphi} \varphi'(x, z\,;y)$$

Replace x, y with some fix but arbitrary parameters, say a, α and assume the antecedent, that is, the truth of the theory $\{\exists z\, \varphi'(a, z\,;\alpha) \;:\; \varphi' > \varphi\}$. We need to prove the consistency of the type $\{\varphi'(a, z\,;\alpha) \;:\; \varphi' > \varphi\}$. By the saturation of $\mathfrak{U}$, finite concistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. □

When $A \subseteq U$, we write $S_{\mathbb{I}}(A)$ for the set of types

$$\mathbb{I}\text{-tp}(a/A) \quad=\quad \big\{\varphi(x) \;:\; \varphi(x) \in \mathbb{I}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of U. We write $S_{\mathbb{I},x}(A)$ when the tuple of variables x is fixed. The same notation applies also with $\mathbb{H}$ for $\mathbb{I}$.

The following corollary will be strengthen by Corollary 17 below.

Corollary 14. The types $p(x) \in S_{\mathbb{I}}(A)$ are maximally consistent subsets of $\mathbb{I}_x(A)$. That is, for every $\varphi(x) \in \mathbb{I}(A)$, either $\varphi(x) \in p$ or $p(x) \to \neg\varphi(x)$. The same holds with $\mathbb{H}$ for $\mathbb{I}$.

Proof. Let $p(x) = \mathbb{I}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 11 and Proposition 13 we obtain

$$\neg\varphi(x) \quad\to\quad \bigvee_{\tilde{\varphi} \perp \varphi} \tilde{\varphi}(x).$$

Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \to \neg\varphi(x)$ follows. □

The following will be useful below

Remark 15. For $p(x) \subseteq \mathbb{I}(U)$, we write $p'(x)$ for the type

$$p'(x) \quad = \quad \big\{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}.$$

Note that $p'(x)$ is equivalent to $p(x)$ by Proposition 13.

8. Elimination of quantifiers

We show that, if we allow some approximation, the quantifiers $\forall^{\mathsf{R}}$ and $\exists^{\mathsf{R}}$ can be eliminated.

Proposition 16. The monster model (or any saturated model, for that matter) is $\mathbb{H}$-homogeneous, that is, every $\mathbb{H}$-elementary map $f : U \to U$ of cardinality $< \kappa$ extends to an automorphism.

Proof. By Corollary 14, the inverse of an $\mathbb{H}$-elementary map $f : U \to U$ is $\mathbb{H}$-elementary. Then the usual proof by back-and-forth applies. □

Now, as promised, we strengthen Corollary 14. Let $A \subseteq U$ be a small set throughout this section.

Corollary 17. Let $p(x) \in S_{\mathbb{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathcal{L}_x(A)$. That is, for every $\varphi(x) \in \mathcal{L}(A)$, either $p(x) \to \varphi(x)$ or $p(x) \to \neg\varphi(x)$. Clearly, the same holds for $p'(x)$.

Proof. If $b \vDash p(x) = \mathbb{H}\text{-tp}(a/A)$ then there is an $\mathbb{H}$-elementary map $f \supseteq \mathrm{id}_A$ such that $f a = b$. As f exends to an automorphism and every automorphism is $\mathcal{L}$-elementary, the corollary follows. □

For $a, b \in U^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A, bearing in mind that formulas in $\mathbb{H}(A)$ or $\mathbb{I}(A)$ suffices to test the equivalence.

In the classical setting, from equivalence of types one can derive equivalence of formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence. Indeed, the next two propositions show that formulas in $\mathbb{I}(A)$ are approximated by formulas in $\mathbb{H}(A)$.

Proposition 18. Let $\varphi(x) \in \mathbb{I}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathbb{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 17 and Remark 15

$$\neg\varphi(x) \ \to \bigvee_{p'(x)\to\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathbb{H},x}(A)$. By Fact 12 and Lemma 11

$$\neg\varphi(x) \ \to \bigvee_{\neg\tilde{\psi}(x)\to\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathbb{H}(A)$. Equivalently,

$$\varphi(x) \ \leftarrow \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 12, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathbb{H}(A)$ such that

$$\varphi'(x) \;\leftarrow\; \bigwedge_{i=1,\dots,n} \tilde{\psi}_i(x) \;\leftarrow\; \varphi(x)$$

which yields the interpolant required by the proposition. □

Proposition 19. Let $\varphi(x) \in \mathbb{I}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathbb{H}(A)$ and some $\psi' > \psi$.

Proof. Let $a \in U^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$. By Corollary 17, $p'(x) \to \neg\varphi(x)$. By compactness $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. □

9. The Tarski-Vaught test

The following proposition is not literally the Tarski-Vaught test. Infact, it only applies when the larger structure is the moster model.

Proposition 20. Let M be a subset of U. Then the following are equivalent

1. M is the domain of a model $\mathcal{M} = \langle M, R\rangle$;
2. for every formula $\varphi(x) \in \mathbb{H}(M)$
   $$\exists x\, \varphi(x) \;\Rightarrow\; \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathbb{I}(M)$
   $$\exists x\, \neg\varphi(x) \;\Rightarrow\; \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 11 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\forall x\, \tilde{\varphi}(x)$ hence, by (1), $\mathcal{M} \vDash \neg\forall x\, \varphi(x)$. Then $\mathcal{M} \vDash \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $\mathcal{M} \vDash \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathbb{I}(M)$ be such that $\exists x\, \neg\varphi(x)$. By Corollary 19, there are a consistent $\psi(x) \in \mathbb{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). By the classical Tarski-Vaught test $M \preceq U$. Let $\mathcal{M} = \langle M, R\rangle$ be the unique $\mathcal{L}$-structure that is a substructure of $\mathcal{U}$. Then $\varphi(a) \Leftrightarrow \mathcal{M} \vDash \varphi(a)$ holds for every $a \in M^{|x|}$ and for every $\mathbb{I}$-atomic formula $\varphi(x)$. Now, assume inductively

$$\mathcal{M} \vDash \varphi(a, b) \;\Rightarrow\; \varphi(a, b).$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$\mathcal{M} \vDash \forall y\, \varphi(a, y) \;\Rightarrow\; \forall y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned}
\neg\forall y\, \varphi(a, y) \;&\Rightarrow\; \exists y\, \neg\varphi(a, y) \\
&\Rightarrow\; \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow\; \mathcal{M} \vDash \neg\varphi(a, b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow\; \mathcal{M} \nvDash \forall y\, \varphi(a, y).
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists$, $\exists^{\mathsf{R}}$, and $\forall^{\mathsf{R}}$ is straightforward. $\square$

10. Completeness

Needs some deeper thinking

For $a, b \in U$ we write $a \sim_R b$ if for every parameter free term $t(x, y)$ of sort $\mathsf{M}^{1+|y|} \to \mathsf{R}$ it holds $t(a, y) = t(b, y)$. Note that universal quantification over the free variables is understood. We write $[a]_R$ for the class of equivalence of a.

Fact 21. If $\bar{a} = \langle a_i : i < \lambda\rangle$ and $\bar{b} = \langle b_i : i < \lambda\rangle$ are such that $a_i \sim_R b_i$ for every $i < \lambda$ then $t(\bar{a}, y) = t(\bar{b}, y)$ holds for every term $t(x, y)$ of sort $\mathsf{M}^{\lambda+|y|} \to \mathsf{R}$ with parameters in U.

Proof. By induction on λ we prove that $t(\bar{a}, y) = t(\bar{b}, y)$ holds for every parameter free term $t(x, y)$. This is sufficient to prove the fact. Assume that

$$t(\bar{a}, z, y) \quad = \quad t(\bar{b}, z, y)$$

holds for every term $t(\bar{x}, z, y)$. In particular for any a_λ

1. $$t(\bar{a}, a_\lambda, y) \quad = \quad t(\bar{b}, a_\lambda, y).$$

If $a_\lambda \sim_R b_\lambda$ then

$$t(\bar{x}, a_\lambda, y) \quad = \quad t(\bar{x}, b_\lambda, y)$$

and in particular

2. $$t(\bar{b}, a_\lambda, y) \quad = \quad t(\bar{b}, b_\lambda, y).$$

From (1) and (2) we obtain

$$t(\bar{a}, a_\lambda, y) \quad = \quad t(\bar{b}, b_\lambda, y).$$

For limit ordinals induction is trivial. $\square$

For $a \in U$ and $A \subseteq U$, let $p(x) = \mathbb{I}\text{-tp}(a/A)$. We say that a is a limit point of A if $p(x) \to a \sim_R x$.

Fact 22. Let $\mathcal{M}$ be a model that is λ-$\mathbb{I}$-saturated for some $\lambda > |\mathcal{L}|$. Then $M \cap [a]_R \neq \varnothing$ for every limit point a of M.

Proof. Let $p(x) = \mathbb{I}\text{-tp}(a/A)$. For every formula $\varphi(x)$ in the type $a \sim x$ and every $\varphi' > \varphi$ pick some $\psi(x) \in p$ such that $\psi(x) \to \varphi'(x)$. There are are at most $|\mathcal{L}|$ such formulas $\psi(x)$. By saturation they have a common solution b in M. Clearly $b \sim_R a$. $\square$

11. Metric spaces

We discuss a simple example. Let M, d is a metric space. Let L contain symbols for all continuous functions $M^n \to M$. The language $\mathcal{L}$ has also a symbol d for a function of sort $\mathsf{M}^2 \to \mathsf{R}$. Let $R = \mathbb{R} \cup \{\pm\infty\}$ be the compactification of $\mathbb{R}$. Let $\mathcal{M} = \langle M, R\rangle$ be the structure that interprets the symbols in $\mathcal{L}$ as natural.

Let $\mathcal{U}$ be a monster model such that $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{U}$. Clearly, d does not define a metric on $\mathcal{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathcal{U}$. Therefore the definition of Cauchy sequence and the corresponding notion of completeness make perfectly sense for ${}^*d$.

Fact 23. Let $\mathcal{U}$ be as above. Then for every model $\mathcal{N} = \langle N, R\rangle$ the following are equivalent

1. $a \in U$ is definable in the limit over N
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of N that converges to a.

Proof. First note that, by the continuity of the terms with respect to the metric, $a \sim_R b$ is equivalent to $d(a, b) = 0$.

($2 \Rightarrow 1$) Let $\langle \varepsilon_i : i \in \omega\rangle$ be a sequence in $[0, 1]$ that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_N a$. By the uniqueness of the limit $d(a, b) = 0$.

($1 \Rightarrow 2$) Let $p(x) = \mathbb{I}\text{-tp}(a/N)$. Assume that $p(x) \to a \sim_R x$. Then $p'(x) \to d(a, x) < 2^{-i}$ for all i. By compactness (see Fact 12) there are formulas $\varphi_i(x) \in p$ such that $\varphi'_i(x) \to d(a, x) < 2^{-i}$ for some $\varphi'_i > \varphi_i$. By $\mathbb{I}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega\rangle$ converges to a. □

The relation $x \sim_R y$ is defined by a parameter-free $\mathbb{I}$-type. We say that a formula $\varphi(x)$ is $\sim_R$-invariant if $x \sim_R y \to [\varphi(x) \leftrightarrow \varphi(y)]$.

12. Measure spaces

⚠ Garbage below this point

Let $\mathcal{B}$ be a σ-algebra of subsets of Ω. Let F be the set of $\mathcal{B}$-simple functions, i.e. linear combinations of indicator functions of sets in $\mathcal{B}$. Let M be the set of $\mathbb{R}$-valued $\mathcal{B}$-measures concentrated on finitely many points of Ω.

Let $\mathcal{M} = \langle M, F, R\rangle$, where $R = \mathbb{R} \cup \{\pm\infty\}$ is the compactification of $\mathbb{R}$. The sort M in the previous section is replaced here by two sorts which we call M and F. Here L is the language of the structure $\langle M, F\rangle$. As both F and M are lattice $\mathbb{R}$-algebras, we assume that L has symbols for these lattice algebras. There is also a function symbol of sort $\mathsf{F} \times \mathsf{M} \to \mathsf{M}$ that is interpreted as the natural action of functions on measures (via multiplication). Finally, in $\mathcal{L}$ there is a function symbol I of sort $\mathsf{M} \to \mathsf{R}$ that is interpreted as the integral over Ω.

Let $\mathcal{U} = \langle {}^*M, {}^*F, R\rangle$ be a monster model such that $\mathcal{M} \preceq^{\mathbb{I}} \mathcal{U}$.

Lemma 24. Let μ be a R-valued $\mathcal{B}$-measure. Then there is ${}^*\mu \in {}^*F$ such that

$$\mathrm{I}\big(f \cdot {}^*\mu\big) \quad = \quad \int_\Omega f \,\mathrm{d}\mu$$

for every $\mathcal{B}$-measurable R-valued functions f.

Proof. We can assume that both f and μ are non-negative. The general case follows by decomposition. Let $G_f \subseteq R$ be the following set

$$G_f \quad = \quad \left\{ \int_\Omega g \,\mathrm{d}\mu \ : \ 0 \leq g \leq f,\ g \in F \right\}.$$

If G_f is bounded, pick for every $\varepsilon \in \mathbb{R}^+$ some $\alpha_{\varepsilon,f} \in G_f$ such that

$$\int_\Omega f \,\mathrm{d}\mu \;\; \leq \;\; \alpha_{\varepsilon,f} + \varepsilon.$$

If G_f is unbounded, let $\alpha_{\varepsilon,f} = +\infty$. Let y be a variable of sort M. Let $p(y)$ be the $\mathbb{I}$-type that contains, for all f as in the lemma, the formulas

$$\mathrm{I}\big(f \cdot y\big) \;\; \geq \;\; \alpha \qquad \text{for all } \alpha \in G_f;$$

$$\mathrm{I}\big(f \cdot y\big) \;\; \leq \;\; \alpha_{\varepsilon,f} + \varepsilon \qquad \text{for all } \varepsilon \in \mathbb{R}^+$$

By basic measure theory, $p(y)$ is finitely consistent in $\mathcal{M}$. Any realization of $p(y)$ in $\mathcal{U}$ proves the lemma. □

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The following discrete version of the Radon-Nikodym theorem is completely trivial. For simplicity we state it for (nonnegative) $\mathcal{B}$-measures. As the Hahn decomposition theorem holds (trivially) in ${}^\circ\mathcal{M}$, this is no loss of generality.

Fact 25. [Discrete Radon-Nikodym] Let $\mu, \nu \in {}^\circ M$, where ${}^\circ M$ is the unit ball of ${}^\circ\mathcal{M}$, be non negative. For every $\varepsilon > 0$ there are $E \in \mathcal{A}$ and $f \in {}^\circ R$, where ${}^\circ R$ is the unit ball of ${}^\circ\mathcal{R}$, such that

1. $I(E, |\nu - \varepsilon^{-1} f\mu|) \; = \; 0$
2. $|I(\neg E, \mu)| \leq \varepsilon$.

We will use the following version of the fact above.

Fact 26. [Iterated discrete Radon-Nikodym] Let $\mu, \nu \in {}^\circ M$ be non negative. For every $n > 0$ there are $E_1, \dots, E_n \in \mathcal{A}$ and $f \in {}^\circ R$ such that

0. the E_i are pairwise disjoint;
1. $I\big(E_i,\ |\nu - 2^i f\mu|\big) \; = \; 0$
2. $|I(X,\ \mu)| \leq 2^{-i}$, where $X = \neg(E_1 \cup \dots \cup E_i)$;

Proof. Apply inductively the theorem above, with E_i for Ω, to find E_{i+1} and f_{i+1} such that

1_i. $I\big(E_i,\ |\nu - 2^i f_i\mu|\big) \; = \; 0$;

2_i. $|I(X,\ \mu)| \leq 2^{-i}$, where $X = \neg(E_1 \cup \dots \cup E_i)$;

3_i. $f_i\big[\neg E_i] = \{0\}$.

Finally, let $f = f_1 + \dots + f_n$. The third condition above (which is immediate to obtain) ensures that $f \in {}^\circ R$. □

We now consider a saturated extension of $\langle \mathcal{A}, {}^\circ\mathcal{R}, {}^\circ\mathcal{M}\rangle$ that we denote by $\langle {}^*\!\mathcal{A}, {}^*\mathcal{R}, {}^*\mathcal{M}\rangle$. The following fact is a direct consequence of saturation.

If $\mathcal{A}$ is a σ-algebra and μ is a σ-additive measure then the supremum and the infimum in the fact above are attained. We will use the following.

Corollary 27. Let $\mu \in \mathcal{M}$. Then for every $A \in {}^*\!\mathcal{A}$ there are $A_{\rm in}, A_{\rm ex} \in \mathcal{A}$ such that $A_{\rm in} \subseteq A \subseteq A_{\rm ex}$ and a measures ${}^*\mu$ such that ${}^*\mu_{\restriction\mathcal{A}} = \mu_{\restriction\mathcal{A}}$ and ${}^*\mu(A) = \mu(A_{\rm ex})$. A similar claim holds for $A_{\rm in}$.

Question 28. Let $\mu \in \mathcal{M}$. Let ${}^*\mu \in {}^*\mathcal{M}$ be as above. Let $f \in {}^*R$, the unit ball of ${}^*\mathcal{R}$, be given and assume f is non negative. We define the function ${}^\mu f : \Omega \to \mathbb{R}$ as follows (tentative)

$$ {}^\mu f(a) \ = \ \inf\Big\{\alpha \ : \ I(A, f\,{}^*\mu) \le \alpha\mu(A) \ : \ A \in \mathcal{A},\ a \in A\Big\} $$

Is it true that for every $A \in \mathcal{A}$

$$ \int_A {}^\mu f\,\mathrm{d}\mu \ = \ I(A, f\,{}^*\mu) \ \ ? $$

Theorem 29 (Radon-Nikodym). Let $\nu, \mu \in \mathcal{M}$ be such that $\nu \ll \mu$. Then there is an $f \in \mathcal{R}$ such that

$$ \int_X f\,\mathrm{d}\mu = \int_X \mathrm{d}\nu $$

for every $X \in \mathcal{A}$.

Proof. ?????????????????? $\square$