Continuous logic for the classical logician

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ABSTRACT. Let $\mathcal{L}$ be a first-order two-sorted language. In general, very little model theory is possible if we restrict to $\mathcal{L}$-structures that have the form $\langle M, I\rangle$ for some fixed structure I. But not everything is lost when I is a compact topological space. Continuous logic deals with this case. Rivisiting old ideas we present a back-to-basics approach to continuous logic.

1. Introduction

As a minimal motivating example, consider a real vector space M. This is are among the most simple strucures considered in model theory. Now, expand it by adding a norm. A norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two-sorted structure $\langle M, \mathbb{R}\rangle$. Now, we have two conflicting aspirations

i. to apply the basic tools of model theory (such as elementarity, compactness, and saturation);

ii. to stay within the realm of normed spaces (hence insist that $\mathbb{R}$ should remain $\mathbb{R}$ throughout).

These two requests are blatantly incompatible, something has to give. Different people have attemped different paths.

1. Nonstandard analysts happily embrace norms that take values in some ${}^*\mathbb{R} \succeq \mathbb{R}$. These normed spaces are nonstandard, but there are tricks to transfer results from nonstandard to standard normed spaces and vice versa. Model theorists, when confronted with similar problems, have used similar approaches (in more generality and with different notation). See [G] for a survey.

2. Henson and Iovino propose to stick to the standard notion of norm but restrict the notion of elementarity and saturation to a smaller class of formulas. They demostrate that most standard tools of model theory apply in their setting. The work of Henson and Iovino is focussed on Banach spaces and has hardly been applied to broader contexts. See [HI] for a survey.

3. Real valued logicians opt for an esoteric approach. They abandon classical true/false valued logic in favour of a logic valued in the interval $[0, 1]$. See [K] or [H] for a survey. Arguably, this esoteric flavor is what has helped the approach to its popularity. Unfortunately, this approach adds notational burden and unnatural restrictions. Our down-to-classical-logic approach ia an alternative.

We elaborate on the ideas of Henson and Iovino and generalize them to a larger class of structures. We also restrict the notions of elementarity and saturation but we use a larger class of formulas. We also borrow intuitions and results from nonstandard analisys to an extent that we could claim the title: *nonstandard analysis for the standard logician*.

2. A class of structures

Let I be some fixed first-order structure which is endowed with a Hausdorff compact[1] topology (in particular, a normal topology). The language L_{I} contains relation symbols for the compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \to I$. In particular, there is a constant for each

[1] The requirement of compactness seems to exclude the motivational example mentioned above. To include this and others similar examples we have to replace $\mathbb{R}$ with $\mathbb{R} \cup \{\pm\infty\}$.

element of I. According to the context, C and f denote either the symbols of L_{I} or their interpretation in the stucture I. The most common examples of structures I are the unit interval $[0,1]$, with the usual topology, and its homeomophic copies $\mathbb{R}^+ \cup \{0,\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language L_{H} which we call the language of the home sort.

Definition 1. Let $\mathcal{L}$ be a two sorted language that expands both L_{H} and L_{I}. By model we mean a two-sorted $\mathcal{L}$-structure of the form $\langle M, I\rangle$, where M ranges over all L_{H}-structures, while I is the structure we fixed above. We write H and I to denote the two sorts of $\mathcal{L}$. The new symbols allowed in $\mathcal{L}$ are function of sort $\mathsf{H}^n \to \mathrm{I}$.

Models are denoted by the domain of their home sort.

The notion of model will be (inessentially) restricted once the monster model is introduced in Section 7.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (i.e. when I is finite), they are not models in the sense of Definion 1. As a remedy, below we carve out a set of formulas $L_{\mathsf{H}} \subseteq \mathcal{J} \subseteq \mathcal{L}$, such that every model has an $\mathcal{J}$-elementary, $\mathcal{J}$-saturated extension that is also a model.

Warning: as usual, L_{I}, L_{H}, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The symbols $\mathcal{J}$ and $\mathcal{H}$ defined below denote only sets of $\mathcal{L}$-formulas.

Definition 2. Formulas in $\mathcal{J}$ are constructed inductively from two sorts of $\mathcal{J}$-atomic formulas

i. formulas of the form $t(x\,;\eta) \in C$, where $C \subseteq I^n$ is compact and $t(x\,;\eta)$ is a tuple of terms of sort $\mathsf{H}^{|x|} \times \mathrm{I}^{|\eta|} \to \mathrm{I}$;

ii. all formulas in L_{H}.

We require that $\mathcal{J}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathrm{I}}$, $\exists^{\mathrm{I}}$ of sort I. We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort I. Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

We write $\mathcal{H}$ for the set of formulas in $\mathcal{J}$ with only quantifiers of sort H (i.e. without quantifiers of sort I).

The class $\mathcal{H}$ is a generalization of the class of positive bounded formulas of Henson and Iovino. In fact, it applies to a larger variety of structures. But we also enlarge the class of Henson and Iovino as we include all L_{H}-formulas. This enlargment comes at a price: the failure of the Perturbation Lemma [HI, Proposition 5.15]. However, the Perturbation Lemma is partly recovered in Proposition 28.

We prove a Compactness Theorem for an even larger class formulas, $\mathcal{J}$. This class offers some advantages over $\mathcal{H}$ as it allows quantifiers of sort I. For instance, it is easy to see (cf. Example 3 for a hint) that $\mathcal{J}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive[2]). Somewhat surprisingly, we will see (cf. Propositions 22 and 23) that the formulas in $\mathcal{J}$ can be approximated by formulas in $\mathcal{H}$.

Example 3. Let $I = [0,1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort $\mathsf{H}^{|x|} \to \mathrm{I}$. Then there is a formula in $\mathcal{J}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall x\,\big[t(x) \dot- \tau \in \{0\}\big] \quad \wedge \quad \forall\varepsilon\Big[\varepsilon \in \{0\}\ \vee\ \exists x\,\big[\tau \dot- (t(x)+\varepsilon) \in \{0\}\big]\Big]$$

which, in a more legible form, becomes

[2] This unsubstantited claims should be interpreted as concjectures. A painstaking technical comparison with real valued logic is not in the scope of this indroductory exploratory paper.

$$\forall x\,\big[t(x) \le \tau\big] \quad\wedge\quad \forall \varepsilon > 0\,\exists x\,\big[\tau \le t(x) + \varepsilon\big].$$

We conclude this introduction with a question. Is it possible extend the Compactness Theorem 13 to a larger class of formulas? E.g. can we allow in $\mathcal{L}$ (and in $\mathcal{J}$) function symbols of sort $\mathsf{H}^n \times \mathsf{I}^m \to \mathsf{I}$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I.

3. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathcal{I}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained by replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C. Note that $>$ is a dense (pre)order of $\mathcal{I}(M)$. Formulas in (i) of Definition 2 do not occur under the scope of a negation therefore $\varphi \to \varphi'$ in all $\mathcal{L}$-structures.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. Moreover the $\mathcal{I}$-atomic formulas in L_{H} are replaced with their negation and each connective is replaced with its dual i.e., $\vee, \wedge, \exists, \forall$ are replaced with $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde{\varphi}$ is a strong negation of φ. It is clear that $\tilde{\varphi} \to \neg\varphi$.

Lemma 4. For all $\varphi \in \mathcal{I}(M)$

1. for every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$;
2. for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in I \setminus O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of I, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. □

For $p(x) \subseteq \mathcal{I}(A)$, we write

$$p'(x) \quad = \quad \big\{\varphi'(x) \ :\ \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}$$

in particular $\{\varphi(x)\}' = \big\{\varphi'(x) \ :\ \varphi' > \varphi\big\}$. The following is an interesting fact which we do not know if it holds for arbitrary I.

Proposition 5. Suppose $I = [0,1]$. Then for every $\varphi(x) \in \mathcal{I}$ there is a formula $\psi(x) \in \mathcal{I}$ such that $\{\varphi(x)\}' \leftrightarrow \psi(x)$ holds in every $\mathcal{L}$-structure.

Proof. ??? □

4. Morphisms

Let M and N be two models. We say that a partial map $f : M \to N$ is $\mathcal{I}$-elementary if for every $\varphi(x) \in \mathcal{I}$ and every $a \in (\mathrm{dom} f)^{|x|}$

1. $$M \vDash \varphi(a) \ \Rightarrow\ N \vDash \varphi(fa).$$

An $\mathcal{I}$-elementary map that is total is called an $\mathcal{I}$-elementary embedding. When the map $\mathrm{id}_M : M \to N$ is an $\mathcal{I}$-elementary embedding, we write $M \preceq^{\mathcal{I}} N$ and say that M is an $\mathcal{I}$-elementary submodel of N.

The definitions of $\mathcal{H}$-elementary map/embedding/submodel is obtained replacing $\mathcal{I}$ by $\mathcal{H}$.

As $\mathcal{I}$ and $\mathcal{H}$-elementary maps are in particular L_{H}-elementary, they are injective. However, their inverse need not be $\mathcal{I}$, respectively $\mathcal{H}$-elementary. In other words, the converse of the implication in (1) may not hold. We have choosen to work with the classical notion of satisfaction at the cost of this asymmetric notion of elementarity. This contrast with the approach of Henson and Iovino. They introduce the notion of *approximated satisfaction.* The approximated $\mathcal{H}$-morphisms we define below are the maps that preserve Henson-Iovino approximated satisfaction. These morphisms are invertible, cf. Fact 6.

We say that the map $f : M \to N$ is approximately $\mathcal{I}$-elementary if for every formula $\varphi(x) \in \mathcal{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$

$$M \vDash \{\varphi(a)\}' \;\;\Rightarrow\;\; N \vDash \{\varphi(fa)\}'.$$

We define approximately $\mathcal{H}$-elementary maps in the same manner. We leave to the reader to verify (using Lemma 4) that $\mathcal{I}$-elementarity implies its approximated version, similarly for $\mathcal{H}$-elementarity. We will see that with a slight amount of saturation also the converse holds (Proposition 15). Finally, under full saturation, all differences disappear as all these morphims becomes $\mathcal{L}$-elementarity maps (Corollary 21).

The following holds in general.

Fact 6. If $f : M \to N$ is approximately $\mathcal{I}$-elementary then

$$M \vDash \{\varphi(a)\}' \;\;\Leftrightarrow\;\; N \vDash \{\varphi(fa)\}'.$$

for every $\varphi(x) \in \mathcal{I}$, and every $a \in (\operatorname{dom} f)^{|x|}$. The same holds for approximately $\mathcal{H}$-elementary maps.

Proof. Only implication $\Leftarrow$ requires a proof. Assume the r.h.s. of the equivalence. Fix $\varphi' > \varphi$ and prove $M \vDash \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By Lemma 4 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \to \neg\tilde{\varphi} \to \varphi'$. Then $N \vDash \neg\tilde{\varphi}(fa)$ and therefore $M \vDash \neg\tilde{\varphi}(a)$. Then $M \vDash \varphi'(a)$. $\square$

Finally, we introduce the morphisms that corrisponds to the classical partial embeddings. We say that $f : M \to N$ is a partial $\mathcal{I}$-embedding if the implication in (1) holds for all $\mathcal{I}$-atomic formulas $\varphi(x)$. Note that this is the same as requiring that (1) holds for all formulas without quantifiers of sort H.

Fact 7. If $f : M \to N$ is a partial $\mathcal{I}$-embedding then

$$M \vDash \varphi(a) \;\;\Leftrightarrow\;\; N \vDash \varphi(fa)$$

for every $a \in (\operatorname{dom} f)^{|x|}$ and every $\mathcal{I}$-formulas $\varphi(x)$ without quantifiers of sort H.

Proof. The equivalence is trivial for formulas in L_{H} so we only consider $\mathcal{I}$-atomic formulas as in (i) of Definion 1. Implication $\Rightarrow$ holds by definition. Vice versa, if $M \vDash t(a) \notin C$ then, by normality, $M \vDash t(a) \in \tilde{C}$ for some compact $\tilde{C}$ disjoint of C. By the definition of $\mathcal{I}$-embedding, $N \vDash t(a) \in \tilde{C}$. Hence $N \vDash t(a) \notin C$. Induction is immediate. $\square$

5. The standard part

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 12 which in turn is required for the proof of the Compactness Theorem for $\mathcal{I}$. The reader willing to accept without proof Theorem 13 (Compactness Theorem for $\mathcal{I}$) may skip this section.

Here, instead of reinventing the wiel (i.e. going back to ultrafilters and ultapowers), we assume the classical compactness theorem and built on that. Let $\langle N, {}^*I\rangle$ be an $\mathcal{L}$-structure that extends $\mathcal{L}$-elementarily the model M. Let η be a free variable of sort I. For each $\beta \in I$, we define the type

$$\mathrm{m}_\beta(\eta) \;\;=\;\; \{\eta \in D \;:\; D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(\eta)$ in ${}^*I$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 8. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \vDash \mathrm{m}_\beta(\alpha)$.

Proof. Negate the existence of β. For every $\gamma \in I$ pick some compact neighborhood D_γ of γ, such that ${}^*I \vDash \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ, with $\gamma \in \Gamma$, cover I. By elementarity these D_γ also cover ${}^*I$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in I$ such that ${}^*I \vDash \mathrm{m}_\beta(\alpha)$. We write $\alpha \approx \alpha'$ if $\mathrm{st}(\alpha) = \mathrm{st}(\alpha')$.

Fact 9. For every $\alpha \in {}^*I$ and every compact $C \subseteq I$

$$ {}^*I \quad \vDash \quad \alpha \in C \ \rightarrow \ \mathrm{st}(\alpha) \in C. $$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*I \vDash \alpha \in D \subseteq \neg C$. □

Fact 10. For every $\alpha \in ({}^*I)^{|\alpha|}$ and every function symbol f of sort $\mathtt{I}^{|\alpha|} \rightarrow \mathtt{I}$

$$ {}^*I \quad \vDash \quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big). $$

Proof. By Fact 8 and the definition of st(-) it suffices to prove that ${}^*I \vDash f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*I \vDash \alpha \in f^{-1}[D]$ and, as $I \preceq {}^*I$ we obtain ${}^*I \vDash f(\alpha) \in D$. □

The standard part of $\langle N, {}^*I\rangle$ is the model $\langle N, I\rangle$ that interprets the symbols f of sort $\mathsf{H}^n \rightarrow \mathtt{I}$ as the functions

$$ f^N(a) \quad = \quad \mathrm{st}\big({}^*f(a)\big) \qquad\qquad \text{for all } a \in N^n, $$

where ${}^*f$ is the interpretation of f in $\langle N, {}^*I\rangle$. Symbols in L_H maintain the same interpretation.

Fact 11. With the notation as above. Let $t(x\,;\eta)$ be a term of sort $\mathsf{H}^{|x|} \times \mathtt{I}^{|\eta|} \rightarrow \mathtt{I}$. Then for every $a \in M^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$ t^N\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*t(a\,;\alpha)\big) $$

Proof. When a t is function symbol of sort $\mathsf{H}^{|x|} \rightarrow \mathtt{I}$, the claim holds by definition. When t is a function symbol of sort $\mathtt{I}^{|\eta|} \rightarrow \mathtt{I}$, the claim follows from Fact 10. Now, assume inductively that

$$ t_i^N\big(a\,;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*t_i(a\,;\alpha)\big) $$

holds for the terms $t_1(x\,;\eta),\ldots,t_n(x\,;\eta)$ and let $t = f(t_1,\ldots,t_n)$ for some function f of sort $\mathtt{I}^n \rightarrow \mathtt{I}$. Then the claim follows immediately from the induction hypothesis and Fact 10. □

Lemma 12. With the notation as above. For every $\varphi(x\,;\eta) \in \mathfrak{I}$, $a \in N^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$ \langle N, {}^*I\rangle \vDash \varphi(a\,;\alpha) \ \Rightarrow \ N \vDash \varphi\big(a\,;\mathrm{st}(\alpha)\big) $$

Proof. Suppose $\varphi(x\,;\eta)$ is $\mathfrak{I}$-atomic. If $\varphi(x\,;\eta)$ is a formula of L_H the claim is trivial. Otherwise $\varphi(x\,;\eta)$ has the form $t(x\,;\eta) \in C$. Assume that the tuple $t(x\,;\eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, {}^*I\rangle \vDash t(a\,;\alpha) \in C$. Then $\mathrm{st}\big({}^*t(a\,;\alpha)\big) \in C$ by Fact 9.

Therefore $t^N\big(a\,;\mathrm{st}(\alpha)\big) \in C$ follows from Fact 11. This proves the lemma for $\mathfrak{I}$-atomic formulas. Induction is immediate. $\square$

6. Compactness

It is convenient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\langle M, {}^*I\rangle \vDash T$ for some $\mathcal{L}$-structure. We say that T is consistent when $M \vDash T$ is for some model M.

Theorem 13 (Compactness Theorem for $\mathfrak{I}$). Let $T \subseteq \mathfrak{I}$ be finitely consistent. Then T is consistent.

Proof. Suppose T is finitely consistent (or finitely $\mathcal{L}$-consistent, for that matter). By the classical Compactness Theorem, there is an $\mathcal{L}$-structure $\langle M, {}^*I\rangle \vDash T$. Let $\langle M, I\rangle$ be its standard part as defined in Section 5. Then $M \vDash T$ by Lemma 12. $\square$

A model N is λ-$\mathfrak{I}$-saturated if it realizes all types $p(x\,;\eta) \subseteq \mathfrak{I}(N)$ with fewer than λ parameters that are finitely consistent in N. When $\lambda = |N|$ we simply say $\mathfrak{I}$-saturated. The existence of $\mathfrak{I}$-saturated models is obtained from the classical case just as for Theorem 13.

Theorem 14. Every model has an $\mathfrak{I}$-elementary extension to a saturated model (possibly of inaccessible cardinality).

The following proposition shows that a slight amount of saturation tames the $\mathfrak{I}$-formulas.

Proposition 15. Let N be an ω-$\mathfrak{I}$-saturated model. Then

$$\big\{\varphi(x)\big\}' \;\leftrightarrow\; \varphi(x)$$

holds in N for every formula $\varphi(x) \in \mathfrak{I}(N)$.

Proof. We prove $\rightarrow$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

ih. $$\{\varphi(x,z)\}' \;\rightarrow\; \varphi(x,z)$$

We need to prove

$$\{\exists z\,\varphi(x,z)\} \;\rightarrow\; \exists z\,\varphi(x,z)$$

From (ih) we have

$$\exists z\,\{\varphi(x,z)\} \;\rightarrow\; \exists z\,\varphi(x,z)$$

Therefore it suffices to prove

$$\{\exists z\,\varphi(x,z)\}' \;\rightarrow\; \exists z\,\{\varphi(x,z)\}'$$

Replace x with some parameters, say a and assume the antecedent, that is, the theory $\{\exists z\,\varphi'(a,z) : \varphi' > \varphi\}$ is true in N. We need to prove the consistency of the type $\{\varphi'(a,z) : \varphi' > \varphi\}$. By saturation, finite concistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly. $\square$

Remark 16. The proposition above will be frequently used in this form: if evaluated in an ω-$\mathfrak{I}$-saturated model, $p'(x)$ is equivalent to $p(x)$.

A consequence of Proposition 15 is that between ω-$\mathfrak{I}$-saturated models the approximated morphisms defined in Section 4 coincide with their unapproximated version. In particular, the following follows from Fact 6.

Corollary 17. Let M be an ω-$\mathfrak{I}$-saturated model. Let $f : M \to N$ be an $\mathfrak{I}$-elementary map. Then

$$M \vDash \varphi(a) \quad\Leftrightarrow\quad N \vDash \varphi(fa)$$

for every $\mathfrak{I}$-formula $\varphi(x)$ and every $a \in (\operatorname{dom} f)^{|x|}$.

7. The monster model

We denote by $\mathfrak{U}$ some large $\mathfrak{I}$-saturated model which we call the monster model. For convenience we assume that the cardinality of $\mathfrak{U}$ is an inaccessible cardinal. Truth is evaluated in $\mathfrak{U}$ unless otherwise is specified. Below we say model for $\mathfrak{I}$-elementary submodel of $\mathfrak{U}$. We stress once again that the truth of some $\varphi \in \mathfrak{I}(M)$ in a model M implies the truth of φ (in $\mathfrak{U}$) but not vice versa. If also the converse implication holds we say M a strong model. By Corollary 17, strong models are those that realize the equivalence in Proposition 15. Hence ω-$\mathfrak{I}$-saturated models are strong models. Also Cauchy complete models, which we introduce below, are strong models. However, models agree on the approximated truth, see Fact 6.

Let $A \subseteq \mathfrak{U}$ be a small set throughout this section. We define a topology on $\mathfrak{U}^{|x|}$ which we call the $\mathfrak{I}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x) \subseteq \mathfrak{I}(A)$. This is a compact topology by the Compactness Theorem for $\mathfrak{I}$ 13. The following fact demostrate how $\mathfrak{I}$-compactness applies in this context. There are some subtle differences from the classical setting.

Fact 18. Let $p(x) \subseteq \mathfrak{I}(A)$ be a type. Then for every $\varphi(x) \in \mathfrak{I}(\mathfrak{U})$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \neg\varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (1) is immediate by saturation; (2) follows from (1) by Lemma 4. □

When $A \subseteq \mathfrak{U}$, we write $S_{\mathfrak{I}}(A)$ for the set of types

$$\mathfrak{I}\text{-tp}(a/A) \quad=\quad \big\{\varphi(x) \;:\; \varphi(x) \in \mathfrak{I}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of $\mathfrak{U}$. We write $S_{\mathfrak{I},x}(A)$ when the tuple of variables x is fixed. The same notation applies also with $\mathfrak{H}$ for $\mathfrak{I}$.

The following proposition will be strengthen by Corollary 21 below.

Proposition 19. The types $p(x) \in S_{\mathfrak{I}}(A)$ are maximally consistent subsets of $\mathfrak{I}_x(A)$. That is, for every $\varphi(x) \in \mathfrak{I}(A)$, either $\varphi(x) \in p$ or $p(x) \to \neg\varphi(x)$. The same holds with $\mathfrak{H}$ for $\mathfrak{I}$.

Proof. Let $p(x) = \mathfrak{I}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 4 and Proposition 15 we obtain

$$\neg\varphi(x) \quad\to\quad \bigvee_{\tilde\varphi \perp \varphi} \tilde\varphi(x).$$

Hence $\tilde\varphi(a)$ holds for some $\tilde\varphi \perp \varphi$ and $p(x) \to \neg\varphi(x)$ follows. □

8. Approximate elimination of quantifiers of sort I

We show that the quantifiers ∀ and ∃ can be eliminated up to some approximation.

Proposition 20. The monster model (or any saturated model, for that matter) is $\mathcal{H}$-homogeneous, that is, every $\mathcal{H}$-elementary map $f : \mathcal{U} \to \mathcal{U}$ of cardinality $< |\mathcal{U}|$ extends to an automorphism.

Proof. By Proposition 19, the inverse of an $\mathcal{H}$-elementary map $f : \mathcal{U} \to \mathcal{U}$ is $\mathcal{H}$-elementary. Then the usual proof by back-and-forth applies. □

As promised, we strengthen Proposition 19. Let $A \subseteq \mathcal{U}$ be a small set throughout this section.

Corollary 21. Let $p(x) \in S_{\mathcal{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathcal{L}_x(A)$. That is, for every $\varphi(x) \in \mathcal{L}(A)$, either $p(x) \to \varphi(x)$ or $p(x) \to \neg\varphi(x)$. Clearly, the same holds for $p'(x)$.

Proof. If $b \models p(x) = \mathcal{H}\text{-tp}(a/A)$ then there is an $\mathcal{H}$-elementary map $f \supseteq \mathrm{id}_A$ such that $fa = b$. Then f exends to an automorphism. As every automorphism is $\mathcal{L}$-elementary, the corollary follows. □

For $a, b \in \mathcal{U}^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A, bearing in mind that formulas in $\mathcal{H}(A)$ or $\mathcal{I}(A)$, or their approximations, suffice to test the equivalence.

In the classical setting, from an equivalence between types one derives an equivalence between formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence. Next proposition show that formulas in $\mathcal{I}(A)$ are approximated by formulas in $\mathcal{H}(A)$.

Proposition 22. Let $\varphi(x) \in \mathcal{I}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathcal{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 21 and Remark 16

$$\neg\varphi(x) \;\to\; \bigvee_{p'(x)\to\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathcal{H},x}(A)$. By Fact 18 and Lemma 4

$$\neg\varphi(x) \;\to\; \bigvee_{\neg\tilde{\psi}(x)\to\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{H}(A)$. Equivalently,

$$\varphi(x) \;\leftarrow\; \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 18, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{H}(A)$ such that

$$\varphi'(x) \;\leftarrow\; \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \;\leftarrow\; \varphi(x)$$

which yields the interpolant required by the proposition. □

We also need an approximation restult for the negation of formulas in $\mathcal{I}(A)$.

Proposition 23. Let $\varphi(x) \in \mathcal{I}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathcal{H}(A)$ and some $\psi' > \psi$.

Proof. Let $a \in \mathcal{U}^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathcal{H}\text{-tp}(a/A)$. By Corollary 21, $p'(x) \to \neg\varphi(x)$. By compactness $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. □

9. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is not literally the Tarski-Vaught test because it only applies when the larger structure is the moster model.

Proposition 24. Let M be a subset of $\mathcal{U}$. Then the following are equivalent

1. M is a model;
2. for every formula $\varphi(x) \in \mathcal{H}(M)$
$$\exists x\, \varphi(x) \;\Rightarrow\; \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathcal{I}(M)$
$$\exists x\, \neg\varphi(x) \;\Rightarrow\; \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 4 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\forall x\, \tilde{\varphi}(x)$ hence, by (1), $M \vDash \neg\forall x\, \varphi(x)$. Then $M \vDash \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $M \vDash \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathcal{I}(M)$ be such that $\exists x\, \neg\varphi(x)$. By Corollary 23, there are a consistent $\psi(x) \in \mathcal{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). By the classical Tarski-Vaught test $M \preceq_{\mathsf{H}} \mathcal{U}$. Then M is the domain of an $\mathcal{L}$-substructure of $\mathcal{U}$. Then $\varphi(a) \Leftrightarrow M \vDash \varphi(a)$ holds for every $\mathcal{I}$-atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \vDash \varphi(a,b) \;\Rightarrow\; \varphi(a,b).$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$M \vDash \forall y\, \varphi(a,y) \;\Rightarrow\; \forall y\, \varphi(a,y).$$

Indeed,

$$\begin{aligned}
\neg\forall y\, \varphi(a,y) &\Rightarrow \exists y\, \neg\varphi(a,y) \\
&\Rightarrow \neg\varphi(a,b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow M \vDash \neg\varphi(a,b) && \text{for some } b \in M^{|y|} \\
&\Rightarrow M \nvDash \forall y\, \varphi(a,y).
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists^{\mathsf{H}}$, $\exists^{\mathsf{I}}$, and $\forall^{\mathsf{I}}$ is straightforward. □

Classically, the main application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all $\mathcal{L}$-structures. In particular every $A \subseteq U$ is contained in a model of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of $\mathcal{L}$ is eccessively large because of the aboundance of symbols in L_{I}.

We say that $\mathcal{L}$ is separable if there is a countable $\mathcal{H}_0 \subseteq \mathcal{H}$ such that for every $\varphi' > \varphi \in \mathcal{H}$ there is a $\varphi_0 \in \mathcal{H}_0$ such that $\varphi \to \varphi_0 \to \varphi'$ holds in every $\mathcal{L}$-structure.

Proposition 25. Assume that $\mathcal{L}$ is separable. Let $A \subseteq \mathcal{U}$ be countable. Then there is a countable model M containing A.

Proof. As in the classical proof of the Löwenheim-Skolem Theorem, we construct $M \subseteq \mathcal{U}$ that contain a witness for every consistent formula in $\mathcal{H}_0(M)$. By separability and (2) in Proposition 24, M is a model. □

We have defined separability as a property of $\mathcal{L}$. Arguably, it could have been introduced as a property of the theory of $\mathcal{U}$. The following proposition provides the examples that motivate the definition of separability (in these examples the property depends soley on $\mathcal{L}$).

Proposition 26. Assume that I is second countable, and that there are at most countably many symbols in $\mathcal{L} \smallsetminus L_{\mathrm{I}}$. Then $\mathcal{L}$ is separable.

Proof. ??? □

10. Cauchy complete models

For $t(x,z)$ a term of sort $\mathsf{H}^{|x|+|z|} \to \mathrm{I}$ we define the formula

$$x \sim_t y \quad = \quad \forall z \;\; t(x,z) = t(y,z).$$

We also define the type

$$x \sim_{\mathrm{I}} y \quad = \quad \Big\{ x \sim_t y \;\; : \;\; t(x,z) \text{ as above} \Big\}.$$

Fact 27. For any $a = \langle a_i : i < \lambda \rangle$ and $b = \langle b_i : i < \lambda \rangle$

$$a \sim_{\mathrm{I}} b \;\; \Leftrightarrow \;\; a_i \sim_{\mathrm{I}} b_i \quad \text{for every } i < \lambda.$$

Proof. Only implication $\Leftarrow$ requires a proof. Assume $a_i \sim_{\mathrm{I}} b_i$ for every i. Let $|a| = |b| = \lambda$ and assume inductively that

$$\forall y, z \;\; t(a,y,z) \;=\; t(b,y,z)$$

holds for every term $t(x,y,z)$, with $|x| = \lambda$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

1. $$\forall z \;\; t(a, a_\lambda, z) \;=\; t(b, a_\lambda, z).$$

As $a_\lambda \sim_{\mathrm{I}} b_\lambda$ then

$$\forall x, z \;\; t(x, a_\lambda, z) \;=\; t(x, b_\lambda, z)$$

and in particular

2. $$\forall z \;\; t(b, a_\lambda, z) \;=\; t(b, b_\lambda, z).$$

From (1) and (2) we obtain

$$\forall z \;\; t(a, a_\lambda, z) \;=\; t(b, b_\lambda, z).$$

For limit ordinals induction is trivial. □

Note that the approximations of the formula $x \sim_t y$ have the form

$$x \sim_{t,D} y \quad = \quad \forall z \;\; \langle t(x,z), t(y,z) \rangle \in D$$

for some compact neighborhood D of Δ, the diagonal of I^2. Hence we define

$$x \sim'_{\mathrm{I}} y \quad = \quad \Big\{ x \sim_{t,D} y \;\; : \;\; t(x,z) \text{ parameter-free term, } D \text{ compact neigborhood of } \Delta \Big\}.$$

If $t(x,z) = t_1(x,z), \dots, t_n(x,z)$ is a tuple of terms, by $x \sim_{t,D} y$ we denote the conjunction of the formulas $x \sim_{t_i,D} y$. As D ranges over the compact neighborhoods of Δ and $t(x,z)$ ranges over the finite tuples parameter-free terms. These formulas $x \sim_{t,D} y$ define a system of entougages on $\mathcal{U}^{|x|}$. We refer to this uniformity and the topology associated as the I-topology. Though not nedded in the sequel, it is worth mentioning that the I-topology on $\mathcal{U}^{|x|}$ coincides with the product of the I-topology on $\mathcal{U}$. This can be verified by an argument similar to what used in the proof of Fact 27.

We say that $p(x) \subseteq \mathcal{I}(\mathcal{U})$ is I-invariant if $x \sim_{\mathrm{I}} y \rightarrow [p(x) \leftrightarrow p(y)]$. It is easy to see that all formulas constructed without the use of (ii) of Definion 2 are I-invariant. The following proposition is an immediate consequence of compactness. It corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Proposition 28. The following are equivalent for every $\varphi(x) \in \mathcal{I}(\mathcal{U})$

1. $\varphi(x)$ is I-invariant;
2. for every $\varphi' > \varphi$ there are a tuple of therm t and a compact neighborhood of Δ such that

$$x \sim_{t,D} y \wedge \varphi(x) \quad \rightarrow \quad \varphi'(y).$$

We say that $p(x)$ is a Cauchy type if it is consistent and $p(x) \wedge p(y) \rightarrow x \sim_{\mathrm{I}} y$.

We say that a type $q(x)$ is finitely satisfiable in M if every conjunction of formulas in $q(x)$ has a solution in M. This definition is classical, but note that classically every type over M is finitely saisfiable in M. Not here, as we can only infer that $q'(x)$ is finitely satisfiable.

Lemma 29. The following are equivalent for every model M

1. $p(x) \subseteq \mathcal{I}(M)$ is an I-invariant Cauchy type;
2. $p(x) \leftrightarrow a \sim_{\mathrm{I}} x$ for some/any $a \vDash p(x)$;
3. $a \sim'_{\mathrm{I}} x$ is finitely satisfiable in M for some/any $a \vDash p(x)$.

Proof. (1⇒2) Let $a \vDash p(x)$. Substituting a for y in the definition of Cauchy type we obtain $p(x) \rightarrow a \sim_{\mathrm{I}} x$ Similarly, from the definition of I-invariance we obtain $a \sim_{\mathrm{I}} x \rightarrow p(x)$.

(2⇒3) By compactness, as $p'(x)$ is finitely satisfiable in M.

(3⇒1) Let $a_{t,D} \in M$ be a realization of $a \sim_{t,D} x$ as D ranges over the compact neighborhoods of the diagonal and $t(x, z)$ over the finite tuples parameter-free terms. Then, by the uniqueness of the limit, the type $p(x)$ cointaining the formulas $x \sim_{t,D} a_{t,D}$ is as required by the lemma. □

We say that M is Cauchy complete if every I-invariant Cauchy type $p(x) \subseteq \mathcal{I}(M)$ is realized in M. Note that if $p(x)$ is Cauchy type then any consistent type $q(x)$ containing $p(x)$ is also Cauchy. In particular the type $p(x) \cup \mathcal{I}\text{-Th}(\mathcal{U}/M)$ is Cauchy. The following trivial consequence of this fact that is worth noticing

Remark 30. Cauchy complete models are strong models. I.e., $M \vDash \varphi$ for every true $\varphi \in \mathcal{I}(M)$.

Fact 31. Let M be a model that is λ-$\mathcal{I}$-saturated for some $\lambda > |\mathcal{L}|$. Then M is Cauchy complete.

Proof. Let $p(x) \subseteq \mathcal{I}(M)$ be an I-invariant Cauchy type. Pick any $a \vDash p(x)$. By Lemma 29, $p(x) \leftrightarrow a \sim_{\mathrm{I}} x$. For every formula $\varphi(x)$ in the type $a \sim_{\mathrm{I}} x$ and every $\varphi' > \varphi$ pick some $\psi(x) \in p$ such that $\psi(x) \rightarrow \varphi'(x)$. There are $|\mathcal{L}|$ many formulas in $a \sim_{\mathrm{I}} x$. Then by saturation the formulas $\psi(x)$ we picked above have a common solution b in M. Then $b \sim_{\mathrm{I}} a$ and, by invariance, $b \vDash p(x)$. □

It is clear from the proof that when $\mathcal{L}$ is separable it suffices to require $\lambda > \omega$.

11. Example: metric spaces

We discuss a simple example. Let I be $\mathbb{R}^+ \cup \{0, \infty\}$, with the topology that makes it homeomorphic to the unit interval. Let M, d is a metric space. Let L_{H} contain symbols for functions $M^n \rightarrow M$ that are uniformly continuous. Let $\mathcal{L}$ contain a symbol for d and possibly for some funtions of $M^n \to I$ that are uniformly continuous w.r.t. the metric d.

Let $\langle M, I\rangle$ be the structure that interprets the symbols of the language as natural.

Let $\mathcal{U}$ be a monster model that is an $\mathcal{I}$ elementary extension of M. Clearly, d does not define a metric on $\mathcal{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathcal{U}$. Therefore the notion of convergent sequence makes perfectly sense in $\mathcal{U}$. As all functions have been required to be uniformly continuous, it is immediate that $x \sim_{\mathrm{I}} y$ is equivalent to $d(x, y) = 0$.

Fact 32. Let $\mathcal{U}$ be as above. Then for every model N the following are equivalent for every $a \in \mathcal{U}$

1. $p(x) = \mathcal{I}\text{-tp}(a/N)$ is a Cauchy type;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of N that converges to a.

Proof. (2⇒1) Let $\langle \varepsilon_i : i \in \omega\rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then the formulas $d(a_i, x) \leq \varepsilon_i$ are in $p(x)$. Then every element realizing $p(x)$ is at distance 0 from a. Therefore $p(x) \to a \sim_{\mathrm{I}} x$.

(1⇒2) As $p(x)$ is Cauchy type, $p(x) \to a \sim_{\mathrm{I}} x$. Then $p'(x) \to d(a, x) < 2^{-i}$ for all i. By compactness (see Fact 18) there are formulas $\varphi_i(x) \in p$ such that $\varphi_i'(x) \to d(a, x) < 2^{-i}$ for some $\varphi_i' > \varphi_i$. By $\mathcal{I}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega\rangle$ converges to a. □

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⚠ Garbage after this point

12.

13. Separability

Let $t(x, y)$ and D be a term of sort $\mathsf{H}^{|x|+|z|} \to \mathrm{I}$, respectively a compact neighborhood of the diagonal. For $\gamma, \xi \in \mathrm{Aut}(\mathcal{U})$ and a a finite tuple of elements of M we write $\gamma \sim_{a,t,C} \xi$ for $\gamma a \sim_{t,D} \xi a$. This defines a system of entourages on $\mathrm{Aut}(\mathcal{U})$.

By $\mathrm{Aut}(\mathcal{U}/\{M\})$ we denote the set of automorphisms of $\mathcal{U}$ that fix M setwise. When $\mathrm{Aut}(\mathcal{U}/\{M\})$ is dense in $\mathrm{Aut}(\mathcal{U})$ with respect to the topology above, we say that $\mathcal{M}$ is approximatively ω-homogeneous. Note that this is the classical notion of homogeneity only with the discrete topology replaced by the I-topology.

We write $\mathcal{J}_\forall$ for the set of $\mathcal{J}$-formulas where the quantifier $\exists^{\mathsf{H}}$ do not occur. Let T_0 be an $\mathcal{J}_\forall$-theory with countable models (L_{H} is countable). Let $\mathcal{C}$ be a category whose objects, which we call $\mathcal{C}$-models, are countable models of T_0. For ease of speaking we identify these models with the domain of their sort H. The morphisms of $\mathcal{C}$, which we call $\mathcal{C}$-morphisms, are finite maps between $\mathcal{C}$-models.

c1. the identity map $\mathrm{id}_A : M \to M$ is a $\mathcal{C}$-morphism, for any $A \subseteq M$;

c2. $\mathcal{C}$-morphisms are partial $\mathcal{J}$-embeddings;

c3. the inverse of a $\mathcal{C}$-morphism is a $\mathcal{C}$-morphism;

c3. if M is a $\mathcal{C}$-model and $f : M \to N$ and $\mathcal{J}$-elementary map, then N is a $\mathcal{C}$-model and f is a $\mathcal{C}$-morphism.

We write $M \leq N$ if the identity map $\mathrm{id}_M : M \to N$ is a $\mathcal{C}$-morphism. We also assume that $\mathcal{C}$ is connected (c4) and that it has the amalgamation property (c5).

c4. the empty map between two $\mathcal{C}$-models is a $\mathcal{C}$-morphism;

c5. for every $\mathcal{C}$-morphism $k : M \to N$ and every $b \in M$ there is a $\mathcal{C}$-model $N \geq M$ and a $\mathcal{C}$-morphism $h : M \to N'$ that exiends k and is defined on b.