

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted structure. Ideally, we would like to have that an elementary extension of a normed space maintain the usual notion of real numbers¹ but this is not possible if we insist to maintain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 Definition. Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L' -structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L -structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

- i. $R^n \rightarrow R$;
- ii. $M^n \rightarrow M$;
- iii. $M^n \rightarrow R$.

The interpretation of symbols of sort $R^n \rightarrow R$ is required to be continuous, the interpretation of symbols of sort $M^n \rightarrow R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n .

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set \mathbb{L} of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an \mathbb{L} -elementary extension that is \mathbb{L} -saturated in the same class.

¹Non standard standard analysts have no problem in expanding \mathbb{R} . In these notes, for a change, we insist as \mathbb{R} should remain \mathbb{R} throughout.

For convenience we assume that the functions of sort $M^n \rightarrow M$ and the relation of sort M^n are all in L . It is also convenient to assume that L' contains names for all continuous bounded functions $R^n \rightarrow R$.

2 Definition. Formulas in \mathbb{L} are constructed inductively from the following two sets of formulas

- i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact² and $t(x, y)$ is a n -tuple of terms of sort $M^{|x|} \times R^{|y|} \rightarrow R$;
- ii. all formulas in L .

We require that \mathbb{L} is closed under the Boolean connectives \wedge, \vee ; the quantifiers \forall, \exists of sort M ; and the quantifiers \forall^C, \exists^C , by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

2. HENSON-IOVINO APPROXIMATIONS

For $\varphi, \varphi' \in \mathbb{L}$ (free variables are hidden) we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C . If no such atomic formulas occurs in φ , then $\varphi > \varphi$. We also have that $(t \in C) < (t \in C)$ when C is clopen.

Note that this is a dense (pre)order of \mathbb{L} .

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \rightarrow \varphi'$. Vice versa, if φ' for every $\varphi' > \varphi$, then φ .

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . Let Y be a topological space. If $f : I \rightarrow Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ . Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

The lemma below is required for the proof of Łoś Theorem in the next section. But first a preliminary fact.

3 Fact. Assume the following data

- $t(x; y)$, a term of sort $M^{|x|} \times R^{|y|} \rightarrow R$;
- $\langle a_i : i \in I \rangle$, a sequence of elements $a_i \in M^{|x|}$;
- $\langle \alpha_i : i \in I \rangle$, a sequence of elements $\alpha_i \in C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I .

²We confuse the relation symbols in L' of sort R^m with their interpretation and write $t \in C$ for $C(t)$

Then

$$\# \quad F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i)$$

Proof. By induction on the nesting of functions of sort $R^n \rightarrow R$. In the basis case the t is either a term of sort $M^{|x|} \rightarrow R$ or a variable of sort R . In both cases (#) is trivial. Assume inductively that the fact holds for the n -tuple $t(x, y)$. By the continuity of the functions of sort $R^n \rightarrow R$ and the induction hypothesis, (#) holds for $f(t(x, y))$. \square

4 Lemma. Assume the following data

- $\varphi(x; y) \in \mathbb{L}$;
- $\langle a_i : i \in I \rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I \rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I .

Then the following are equivalent

- 1 $\{i \in I : \mathcal{M} \models \varphi(a_i; \alpha)\} \in F$.
- 2 $\{i \in I : \mathcal{M} \models \varphi'(a_i; \alpha_i)\} \in F$ for every $\varphi' > \varphi$.

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L , it does not depend on α , and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume $\{i : t^{\mathcal{M}}(a_i; \alpha) \in C\} \in F$. Then $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \in C$. By Fact 3, $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i) \in C$. Then $\{i : t^{\mathcal{M}}(a_i; \alpha_i) \in D\} \in F$ for every D that is a compact neighborhood of C .

Vice versa, if $\{i : t^{\mathcal{M}}(a_i; \alpha_i) \in D\} \in F$ for every D that is a compact neighborhood of C , then $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \in C$, and (1) follows by Fact 3.

This completes the proof of the base case of the induction.

Induction clear for the connectives \vee, \wedge . To deal with the universal quantifier of sort M we assume inductively that for every $\varphi' > \varphi$

$$\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F.$$

We prove

$$1. \quad \{i : \mathcal{M} \models \forall y \varphi(a_i, y; \alpha)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \forall y \varphi'(a_i, y; \alpha_i)\} \in F$$

Negate the r.h.s. of the implication. Then there is a sequence $\langle b_i : i \in I \rangle$ such that (using that F is an ultrafilter)

$$\{i : \mathcal{M} \not\models \varphi'(a_i, b_i; \alpha_i)\} \in F$$

Assume the l.h.s. of (1) and reason for a contradiction. Then $\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F$. A contradiction that proves the implication (\Rightarrow) in (1). The converse implication is immediate.

Induction for the existential quantifier of sort M is similar (dual) to the proof for the universal quantifier.

Induction for the quantifier \forall^B is also virtually identical, but we repeat the argument for convenience. Assume inductively that for every $\varphi' > \varphi$ and every sequence $\langle \beta_i : i \in I \rangle$ in B such that $\beta = F\text{-}\lim_i \beta_i$

$$\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F.$$

We prove

$$2. \quad \{i : \mathcal{M} \models \forall^C y \varphi(a_i, ; \alpha, y)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \forall^C y \varphi'(a_i; \alpha_i, y)\} \in F$$

Negate the r.h.s. of the implication. Then there is a sequence $\langle \beta_i : i \in I \rangle$ such that (again using that F is an ultrafilter)

$$\{i : \mathcal{M} \not\models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$$

As B is compact, $F\text{-}\lim_i \beta_i$ exists, say it equals $\beta \in B$. Assume the l.h.s. of the implication and reason for a contradiction. Then $\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$. A contradiction that proves the implication (\Rightarrow) in (2). The converse implication is immediate.

Induction for the quantifier \exists^B is proved by the dual version of argument above. \square

3. ULTRAPRODUCTS

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that for each function symbol f of sort $M^n \rightarrow R$ there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two simplifications: (1) we only consider ultratowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F . The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

5 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \rightarrow M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I \rangle$.
3. The interpretation of functions of sort $R^n \rightarrow R$ remains unchanged.

4. If f is a function of sort $M^n \rightarrow R$ then

$$f^{\mathcal{N}}(\hat{a}) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$

5. If r is a relation symbol of sort M^n then

$$\mathcal{N} \models r(\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i)\} \in F.$$

6. The interpretation of functions of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the uniformity assumption above.

6 Fact. The structure \mathcal{N} satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

7 Fact. If $t(x)$ is a term of type $M^{|x|} \rightarrow M$ then

$$t^{\mathcal{N}}(\hat{a}) = \langle t^{\mathcal{M}}(\hat{a}i) : i \in I \rangle.$$

□

We also have that

8 Fact. If $t(x; y)$ has sort $M^{|x|} \times R^{|y|} \rightarrow R$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Proof. By induction. If t is a function symbol then x or z do not occur in t . If x does not occur in t the claim is trivial. If y does not occur in t the claim holds by definition. Finally, induction is clear by the continuity of the functions of sort $R^n \rightarrow R$. □

Finally, we prove

9 Proposition (Łoś Theorem). Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x; y) \in L$ then the theorem reduces to the classical Łoś Theorem. Then, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. By local compactness of R there is a compact neighborhood of C , such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$. By Fact 8 and the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives \vee and \wedge is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

$$1. \quad \mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F$. Choose \hat{b} such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$. If for a contradiction $\mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha)$, then in particular $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ a contradiction. This completes the proof of (1).

Induction for the existential quantifier of sort M is similar (dual) to the proof above.

Finally, to deal with the quantifier \forall^C we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

$$3. \quad \mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}; \alpha, \beta)$ for some β . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha, \beta_i)\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y)$, then $\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta)$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \alpha, \beta)\} \in F$ for every $\varphi'' > \varphi$. Fix φ'' such that $\varphi' > \varphi'' > \varphi$ and obtain a contradiction from Lemma 4. This completes the proof of (3).

Induction for the quantifier \exists^C is again dual to the proof above. □

4. \mathbb{L} -ELEMENTARITY

(Work in progress)

Let $\mathcal{M} = \langle M, R \rangle$ and $\mathcal{N} = \langle N, R \rangle$ be two structures. We say that $f : M \rightarrow N$, a partial map, is an \mathbb{L} -elementary map if

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(a) \quad \text{for every } \varphi(x) \in \mathbb{L} \text{ and every } a \in (\text{dom } f)^{|x|}$$

The following fact is proved as in the classical case using that $\mathcal{M} \models \varphi'$ for every $\varphi' > \varphi$ if and only if $\mathcal{M} \models \varphi$.

10 Fact. If \mathcal{N} is an ultrapower of \mathcal{M} then there is an \mathbb{L} -elementary embedding of \mathcal{M} into \mathcal{N} .

For $A \subseteq M \cap N$, we say that \mathcal{M} and \mathcal{N} are \mathbb{L} -(elementary) equivalent over A and write $\mathcal{M} \equiv_A^{\mathbb{L}} \mathcal{N}$ if $\text{id}_A : M \rightarrow N$ is \mathbb{L} -elementary. We write $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \equiv_M^{\mathbb{L}} \mathcal{N}$. In words, we say that \mathcal{M} is an \mathbb{L} -(elementary) substructure of \mathcal{N} .

The following is proved as in the classical case.

11 Proposition (Tarski-Vaught Test). The following are equivalent

1. $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$
2. M is a subset of N and for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\mathcal{N} \models \exists x \varphi(x) \Rightarrow \mathcal{N} \models \varphi(a) \text{ for some } a \in M$$

□

5. \mathbb{L} -SATURATION

(Work in progress)



Let $p(x) \subseteq \mathbb{L}(M)$. We say that $p(x)$ is **finitely satisfied in \mathcal{M}** if $\mathcal{M} \models \exists x \varphi'(x)$ for every $\varphi' > \varphi$ where $\varphi(x)$ is any conjunction of formulas in $p(x)$.

The following definition is standard.

12 Definition. We say that \mathcal{M} is **\mathbb{L} -saturated** if for every $p(x)$ as in 1 and 2 below, $\mathcal{M} \models \exists x p(a)$.

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is finitely satisfied in \mathcal{M} .

Then proof that every model embeds \mathbb{L} -elementarily in an \mathbb{L} -saturated one proceeds as in the classical case. First, note that the classical ultrapower construction yields the the following lemma.

13 Lemma. Every model \mathcal{M} embeds \mathbb{L} -elementarily in a model \mathcal{N} that realizes all types as in the definition above.

Finally, using a sufficiently long \mathbb{L} -elementary chain we obtain the required saturated extension.

14 Corollary. Every model \mathcal{M} is an \mathbb{L} -elementary substructure of some saturated model \mathcal{N} (possibly of inaccessible cardinality).

We denote by \mathcal{U} some large \mathbb{L} -saturated structure which we call the **monster model**. The unit ball of \mathcal{U} is denoted by U . The cardinality of \mathcal{U} is an inaccessible cardinal that we denote by κ . Below we say **model** for \mathbb{L} -elementary substructure of \mathcal{U} .

6. COMPLETENESS

(Work in progress)

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \rightarrow R$.

15 Fact. If $\bar{a} = \langle a_i : i < \lambda \rangle$ and $\bar{b} = \langle b_i : i < \lambda \rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \rightarrow R$.

Proof. Assume inductively that the fact holds for λ and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. \square

16 Example (???). Let L be the language of \mathbb{R} -algebras expanded with two lattice operators \wedge, \vee . Let $\langle \Omega, \mathcal{B}, \text{Pr} \rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L . Let $R = \mathbb{R}$. Assume L' extends L with all the continuous functions $\mathbb{R}^n \rightarrow \mathbb{R}$ and for every $n \in \mathbb{N}$ the functions E_n that gives the expected value of $(X \wedge n) \vee -n$ for $n \in \mathbb{N}$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is **definable in the limit** over M if $a \equiv_M^{\mathbb{L}} x \rightarrow a \sim x$

17 Example. Assume that $R = \mathbb{R}$ and that L' contains a function of sort $M^2 \rightarrow R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \rightarrow R$ are continuous with respect to this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M , a model;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of M that converges to a .

Proof. (2 \Rightarrow 1) Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_M a$. By the uniqueness of the limit $d(a, b) = 0$.

(1 \Rightarrow 2) Assume that $a \equiv_M x \rightarrow a \sim x$. Then $a \equiv_M^{\mathbb{L}} x \rightarrow d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \text{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \rightarrow d(a, x) < 1/n$. By \mathbb{L} -elementarity there is an $a_n \in M$ such that $\varphi_n(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a . \square