CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A class of structures

1 Definition. In these notes we deal with 3-sorted[1] structures of the form $\mathcal{M} = \langle M, \check{M}, \mathbb{R}\rangle$. The language and its interpretation are subject to the following conditions.

1. Functions may only have one of the following sorts (for any $m, n \in \omega$)
   a. $\mathbb{R}^n \to \mathbb{R}$
   b. $M^n \times \check{M}^m \to$ any of M, $\check{M}$, or $\mathbb{R}$
2. There is a symbol for every (total) uniformly continuous function $\mathbb{R}^n \to \mathbb{R}$.
3. The (functions that interpret the) terms of sort $M^n \to \mathbb{R}$ have *bounded* range.
4. Every element of $\check{M}$ is the image of some term of sort $M^n \to \check{M}$.
5. There is only one predicate; it has sort $\mathbb{R}$ and is interpreted as $x \leq 0$.

We call M the unit ball of $\mathcal{M}$. The terminology is inspired by the example below.

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $M^n \times \mathbb{R}^m \to \mathbb{R}$ with parameters in some $A \subseteq M$. Up to equivalence, these terms have the form $f(t(x), y)$, where $t(x)$ is a tuple of terms of sort $M^{|x|} \to \mathbb{R}$, and f is a function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

3 Definition. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

i. $\mathbb{L}(A)$ contains the atomic formula $t \leq 0$ for every term $t \in \mathbb{T}(A)$.
ii. It is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort M.
iii. It is closed under the quantifier $\forall$ of sort $\mathbb{R}$ relativized to any definable subset of $\mathbb{R}$.

Note that in iii of the definition above, definability is intended with respect to the full first order language. In particular, if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$, also $\forall \varepsilon{>}0\ \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

4 Example (Metric spaces of finite diameter). Given a metric space M, d of finite diameter we define a structure $\mathcal{M} = \langle M, \check{M}, \mathbb{R}\rangle$ as follows. Set $M = \check{M}$ and let $\mathrm{id} : M \to \check{M}$ be the identical isomorphism (so, $\check{M}$ is irrelevant in this example). The function symbols are id and d. Note that 3 is satisfied because d is bounded..

[1] In some examples it may be more natural to use $(n+n'+1)$-sorted structures $\mathcal{M} = \langle M_1, \dots, M_n, \check{M}_1, \dots, \check{M}_{n'}, \mathbb{R}\rangle$. The generalization is straightforeward.

5 **Example (Banach spaces).** Given a Banach space V we define a structure $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$ as follows. Let $\breve{M} = V$ and let $M = \{a \in \breve{M} : \|a\| \leq 1\}$ be the closed unit ball of V. Besides the symbols mentioned above, $\mathbb{L}$ contains a function symbol for the natural embedding $\mathrm{id} : M \to \breve{M}$. It also contains a symbol for the norm $\|\text{-}\| : \breve{M} \to \mathbb{R}$. Finally, $\mathbb{L}$ contains the usual symbols of the language of vector spaces. These have sort $\breve{M}^n \to \breve{M}$, for the appropiate $n \in \{0, 1, 2\}$.

Terms of sort $M^{|x|} \to \mathbb{R}$ have the form $f(t(x))$ where f is a uniformly continuous function and

$$t(x) \quad = \quad \Big\| \sum_{i=1}^{|x|} \sum_{j=1}^{m} \lambda_{i,j} x_i \Big\|.$$

Therefore, as for all $a \in M^{|x|}$

$$t^{\mathcal{M}}(a) \quad \leq \quad \sum_{i=1}^{|x|} \sum_{j=1}^{m} |\lambda_{i,j}|,$$

condition 3 of Definition 1 is satisfied. Conditions 4 is immediate.

If $\varphi \in \mathbb{L}(\mathbb{A})$ and $\varepsilon > 0$ we write φ_ε for the formula obtained by replacing in φ the atomic formulas $t \leq 0$ with $t - \varepsilon \leq 0$. Note that $\varphi \to \varphi_\varepsilon$.

2. $\mathbb{L}$-ELEMENTARY RELATIONS

Let $\mathcal{M}$ and $\mathcal{N}$ be models. We say that $R \subseteq M \times N$ is an $\mathbb{L}$-elementary relation between $\mathcal{M}$ and $\mathcal{N}$ if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } a\ R\ b.$$

Recall that, when $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$ are tuples, $a\ R\ b$ stands for $a_i\ R\ b_i$ for every $i \in \{1, \dots, n\}$.

We define an equivalence relation $(\sim_{\mathcal{M}})$ on M as follows

1. $$a \sim_{\mathcal{M}} b \quad \Leftrightarrow \quad t^{\mathcal{M}}(a) = t^{\mathcal{M}}(b) \ \text{ for every } t(x) \in \mathbb{T}(M),$$

where $|x| = |a| = |b| = 1$.

6 **Lemma.** The relation $(\sim_{\mathcal{M}}) \subseteq M^2$ is an $\mathbb{L}$-elementary relation. Moreover, it is maximal among these, i.e. no $\mathbb{L}$-elementary relation properly contains $(\sim_{\mathcal{M}})$.

Proof. Assume $a \sim_{\mathcal{M}} b$, where $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$. Recall that this means that $a_i \sim_{\mathcal{M}} b_i$ for all $i \in \{1, \dots, n\}$. Let Δ denote the diagonal relation on M. Note that $a_i \sim_{\mathcal{M}} b_i$ is equivalent to saying that $\Delta \cup \{(a_i, b_i)\}$ is an $\mathbb{L}$-elementary relation. As $\mathbb{L}$-elementary relations are closed under composition $\Delta \cup \big\{(a_1, b_1), \dots, (a_n, b_n)\big\}$ is $\mathbb{L}$-elementary. It follows that for every $\varphi(x) \in \mathbb{L}$

2. $$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi(b).$$

The converse implication ($2 \Rightarrow a \sim_{\mathcal{M}} b$) is immediate. □

7 Lemma. Let $R \subseteq M \times N$ be total and surjective $\mathbb{L}$-elementary relation. Then there is a unique maximal $\mathbb{L}$-elementary relation containing R. This maximal $\mathbb{L}$-elementary relation is equal to both $(\sim_{\mathcal{M}})\, R$ and $R\, (\sim_{\mathcal{N}})$.

Proof. It is immediate to verify that $(\sim_{\mathcal{M}})\, R$ is an $\mathbb{L}$-elementary relation containing R. Let S be any maximal $\mathbb{L}$-elementary relation containing R. By maximality, $(\sim_{\mathcal{M}})\, S = S$. As S is a total relation $(\sim_{\mathcal{M}}) \subseteq SS^{-1}$. Therefore, by the lemma above, $(\sim_{\mathcal{M}}) = SS^{-1}$. As R is a surjective relation, $S \subseteq S\, S^{-1} R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}})\, R$. A similar argument proves that $S = R\, (\sim_{\mathcal{N}})$. □

We write $\mathrm{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, $\mathbb{L}$-elementary relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/{\sim_{\mathcal{M}}}$.

Thought in most situations one could dispense with $\mathcal{M}$ in favour of $\mathcal{M}/{\sim_{\mathcal{M}}}$, in concrete cases one has a better grip on the first than on the latter. Therefore below we insist in working with $\mathbb{L}$-elementary relations in place of $\mathbb{L}$-elementary maps.

3. Ultraproducts

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $f : I \to \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \to F} f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq \mathbb{R} \cup \{\pm\infty\}$ that is a neighborhood of λ. Such a λ is unique and, when F is an ultrafilter, it always exists. When f is bounded, $\lambda \in \mathbb{R}$.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle M_i, \check{M}_i, \mathbb{R} \rangle$, that are uniformly bounded, that is, the bounds in 3 of Definition 1 are the same for all $\mathcal{M}_i$.

Let F be an ultrafilter on I.

8 Definition. We define a structure $\mathcal{N} = \langle \check{N}, \check{N}, \mathbb{R} \rangle$ that we call the ultraproduct of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. N comprise the sequences $\hat{a} : I \to \bigcup_{i \in I} M_i$ such that $\hat{a}\, i \in M_i$.
2. $\check{N}$ comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a} i)$, where $t(x)$ is a term of sort $M^n \to \check{M}$.
3. If f is a function of sort $M^n \times \check{M}^m \to M$ then $f^{\mathcal{N}}(\hat{a}, t^{\mathcal{N}}(\hat{c}))$ is the sequence $f^{\mathcal{M}_i}(\hat{a} i, t^{\mathcal{M}_i}(\hat{c} i))$.
4. Similarly when f is of sort $M^n \times \check{M}^m \to \check{M}$.

5. If f is a function of sort $M^n \times \breve{M}^m \to \mathbb{R}$ then

$$f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big) \;=\; \lim_{i\to F} f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that $\mathcal{N}$ is an ultrapower of $\mathcal{M}$.

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models $\mathcal{M}_i$ have the same bounds.

The following fact is easily proved by induction on the syntax.

9 Fact. For every term $t(x)$ of sort $M^n \to \mathbb{R}$

$$t^{\mathcal{N}}\big(\hat{a}\big) \;=\; \lim_{i\to F} t^{\mathcal{M}_i}\big(\hat{a}i\big). \qquad \square$$

Finally, we prove

10 Proposition (Łŏś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x,y) \in \mathbb{L}(A)$. Let $\hat{a} \in N^{|x|}$. Then there is a function $u_{\_} : \mathbb{R}^+ \to F$ such that for every $\lambda \in \mathbb{R}^{|y|}$

1. $\mathcal{N} \vDash \varphi(\hat{a},\lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \vDash \varphi_\varepsilon(\hat{a}i,\lambda)\big\}$ for every $\varepsilon > 0$.
2. $\mathcal{N} \nvDash \varphi(\hat{a},\lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \nvDash \varphi_\varepsilon(\hat{a}i,\lambda)\big\}$ for some $\varepsilon > 0$.

Proof. Suppose that $\varphi(x,y)$ is atomic, say

$$\varphi(x,y) \;=\; f(t(x),\ y) \leq 0,$$

where t is a tuple of terms of sort $M^{|x|} \to \mathbb{R}$ and f is a uniformly continuous function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

Let $\beta' \in \mathbb{R}^{|t|}$ be such that $\lim\limits_{i\to F} t(\hat{a}\, i) = \beta'$.

Given $\varepsilon > 0$, let B_ε be a neighborhood of β' such that

$$\Big| f(\beta,\ \lambda) - f(\beta',\lambda)\Big| < \varepsilon \qquad \text{for every } \beta \in B_\varepsilon \text{ and every } \lambda \in \mathbb{R}^{|y|}.$$

Such a neighborhood exists by the uniform continuity of f. Finally, to obtain 1 and 2 above it suffices to define

$$u_\varepsilon \;=\; \big\{ i \;:\; t(\hat{a}i) \in B_\varepsilon \big\}.$$

The rest of the inductive proof is straightforeward and is left to the reader. $\square$