CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A class of structures

1 Definition. In these notes we deal with 3-sorted[1] structures of the form $\mathcal{M} = \langle \check{M}, M, \mathbb{R}\rangle$. The language and its interpretation are subject to the following conditions.

1. Functions may only have one of the following sorts (for any $m, n \in \omega$)
   a. $\mathbb{R}^n \to \mathbb{R}$
   b. $\check{M}^n \times M^m \to$ any of $\check{M},\ M,$ or $\mathbb{R}$
2. There is a symbol for very (total) continuous functions $\mathbb{R}^n \to \mathbb{R}$.
3. The (functions that interpret the) terms of sort $\check{M}^n \to \mathbb{R}$ have *bounded* range.
4. Every element of M is the image of some term of sort $\check{M}^n \to M$.
5. There is only one predicate; it has sort $\mathbb{R}$ and is interpreted as $x \leq 0$.

We call $\check{M}$ the unit ball of $\mathcal{M}$. The terminology is inspired by the example below.

The language is denoted by $\mathbb{L}$.

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $\check{M}^n \times \mathbb{R}^m \to \mathbb{R}$ and parameters in $A \subseteq \check{M}$. Up to equivalence, these terms have the form $f(t(x), y)$ where $t(x)$ is a tuple of terms of sort $\check{M}^{|x|} \to \mathbb{R}$, the variables y have sort $\mathbb{R}$, and f is a function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

3 Definition. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

i. $\mathbb{L}(A)$ contains the atomic formula $t \leq 0$ for every term $t \in \mathbb{T}(A)$.

ii. It is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort $\check{M}$.

iii. It is closed under the quantifier $\forall$ of sort $\mathbb{R}$ relativized to any definable subset of $\mathbb{R}$.

Note that in iii, definability, is with respect to the full first order language. In particular, if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$, also $\forall \varepsilon{>}0\ \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

4 Example (Banach spaces). Given a Banach space M we define a structure $\mathcal{M} = \langle \check{M}, M, \mathbb{R}\rangle$ as follows. Let $\check{M} = \{a \in M : \|a\| \leq 1\}$ be the closed unit ball of M. Besides the symbols mentioned above, $\mathbb{L}$ contains a function symbol for the natural embedding $\mathrm{id} : \check{M} \to M$. It also

[1] In some examples it may be more natural to use $(n + n' + 1)$-sorted structures $\mathcal{M} = \langle \check{M}_1, \ldots, \check{M}_n, M_1, \ldots, M_{n'}, \mathbb{R}\rangle$. The generalization is straightforward.

contains a symbol for the norm $\|\text{-}\| : M \to \mathbb{R}$. Finally, $\mathbb{L}$ contains the usual symbols of the language of vector spaces. These have sort $M^n \to M$, for the appropiate $n \in \{0, 1, 2\}$. Note that conditions 3 and 4 of Definition 1 are immediatly satisfied.

If $\varphi \in \mathbb{L}(\mathbb{A})$ and $\varepsilon > 0$ we write φ_ε for the formula obtained by replacing in φ the atomic formulas $t \leq 0$ with $t - \varepsilon \leq 0$. Note that $\varphi \to \varphi_\varepsilon$.

2. Ultraproducts

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $f : I \to \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \to F} f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq \mathbb{R} \cup \{\pm\infty\}$ that is a neighborhood of λ. Such a λ is unique and, when F is an ultrafilter, it always exists. When f is bounded, $\lambda \in \mathbb{R}$.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle \check{M}_i, M_i, \mathbb{R} \rangle$, that are uniformly bounded, that is, the bounds in 3 of Definition 1 are the same for all $\mathcal{M}_i$.

Let F be an ultrafilter on I.

5 **Definition.** We define a structure $\mathcal{N} = \langle \check{N}, N, \mathbb{R} \rangle$ that we call the ultraproduct of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. $\check{N}$ comprise the sequences $\hat{a} : I \to \bigcup_{i \in I} \check{M}_i$ such that $\hat{a}\, i \in \check{M}_i$.
2. N comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a}i)$, where $t(x)$ is a term of sort $\check{M}^n \to M$.
3. If f is a function of sort $\check{M}^n \times M^m \to \check{M}$ then $f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big)$ is the sequence $f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big)$.
4. Similarly when f is of sort $\check{M}^n \times M^m \to M$.
5. If f is a function of sort $\check{M}^n \times M^m \to \mathbb{R}$ then
$$f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big) \;=\; \lim_{i \to F} f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that $\mathcal{N}$ is an ultrapower of $\mathcal{M}$.

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models $\mathcal{M}_i$ have the same bounds.

The following fact is easily proved by induction on the syntax.

6 **Fact.** For every term of sort $\check{M}^n \times M^m \to \mathbb{R}$

$$f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big) \;=\; \lim_{i \to F} f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big).$$

Finally, we prove

7 **Proposition (Łǒś Theorem).** Let $\mathcal{N}$ be as above and let $\varphi(x, y) \in \mathbb{L}(A)$. Let $\hat{a} \in \check{N}^{|x|}$. Then there is a function $u_{-} : \mathbb{R}^{+} \to F$ such that for every $\lambda \in \mathbb{R}^{|y|}$

1. $\mathcal{N} \vDash \varphi(\hat{a}, \lambda) \;\Rightarrow\; u_{\varepsilon} \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \vDash \varphi_{\varepsilon}(\hat{a}i, \lambda) \big\}$ for every $\varepsilon > 0$.
2. $\mathcal{N} \nvDash \varphi(\hat{a}, \lambda) \;\Rightarrow\; u_{\varepsilon} \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \nvDash \varphi_{\varepsilon}(\hat{a}i, \lambda) \big\}$ for some $\varepsilon > 0$.

Proof. Without loss of generality, we can assume that Λ is compact. Suppose that $\varphi(x, y)$ is atomic, say

$$\varphi(x, y) \;=\; f(t(x),\ y) \leq 0,$$

where t is a tuple of terms of sort $\check{M}^{|x|} \to \mathbb{R}$ and f is a continuous function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

Let Λ_0 be the set of those $\lambda \in \Lambda$ such that

$$\mathcal{N} \;\vDash\; f(t(\hat{a}),\ \lambda) \leq 0$$

or, equivalently, such that

$$\mathcal{N} \;\vDash\; f(\beta',\ \lambda) \leq 0, \qquad \text{where } \beta' \in \mathbb{R}^{|t|} \text{ is such that } \lim_{i \to F} t(\hat{a}\, i).$$

As Λ_0 is compact, the function g defined below is continuous

$$\begin{aligned} g : \mathbb{R}^{|t|} \;&\to\; \mathbb{R} \\ \beta \;&\mapsto\; \sup_{\lambda \in \Lambda_0} f(\beta, \lambda). \end{aligned}$$

Given $\varepsilon > 0$, let $B_{\varepsilon} = g^{-1}(-\infty, \varepsilon]$ and $u = \{i \,:\, \mathcal{M}_i \vDash g(t(\hat{a}i)) \leq \varepsilon\}$. As B_{ε} is a neighborhood of β', the set u belongs to F and for every $\lambda \in \Lambda_0$ and $i \in u$

2. $$u_{\varepsilon} \;\subseteq\; \Big\{ i \,;\, \mathcal{M}_i \vDash f(t(\hat{a}i),\ \lambda) \leq \varepsilon \Big\}$$

$\square$