CONTINUOUS MODEL THEORY WITHOUT THE HASSLE

1. Formulas and their interpretation

The following definition extends that of classical first-order logic with clause 3.

1 Definition. A signature or language L consists of

1. two sets of symbols: L_{rel}, for relations, and L_{fun}, for functions;
2. an arity function $\mathrm{Ar} : L_{\mathrm{rel}} \cup L_{\mathrm{fun}} \to \omega$;
3. a function $n_{\_} : L_{\mathrm{at}} \to \mathbb{N}$;

The function in 3 above is called the bounding function. With L_{at} we denote the set of atomic formulas as defined below.

2 Definition. A structure or model $\mathcal{M}$ of signature L consists of

1. a set $\mathcal{M}$;
2. a set $M \subseteq \mathcal{M}$;
3. for every $f \in L_{\mathrm{fun}}$, a function $f^{\mathcal{M}} : \mathcal{M}^{\mathrm{Ar}(f)} \to \mathcal{M}$;
4. for every $r \in L_{\mathrm{rel}}$, a function $r^{\mathcal{M}} : \mathcal{M}^{\mathrm{Ar}(r)} \to \mathbb{R}$;

The set M is the unit ball of the model and is required to saisfy the axioms in Definition 5. It will be clear below that the unit ball determines, in a sense, the whole model. (E.g. in Definition 4 we define the interpretation only of formulas that have parameters in the unit ball.) Though, in principle, two models could coincide when restricted to the unit ball and yet be distinct overall, this will never be the case below.

The syntax and semantic of terms is as in classical first-order logic. We write L_{trm} for the set of terms in the language L. Atomic formulas are defined as in classical first-order logic: they have the form $r\,t$, where r is a relation symbol and t is a tuple of terms. The set of atomic formulas is denoted by L_{at}.

The set of all formulas is denoted as usual by L, the same symbol as the language. Formulas are defined inductively from the atomic formulas using the following connectives.

The propositional (or Riesz) connectives are those in $\{1, +, \wedge\} \cup \mathbb{R}$, where $+$ and $\wedge$ are binary connectives, the elements of $\mathbb{R}$ are unary connectives, 1 is a logical constant (i.e. a zero-ary connective). There is also the infimum quantifier $\bigwedge_x$.

3 Definition. The inductive definition of formula is as follows

i. atomic formulas are formulas

ii. 1 is a formula;

iii. $\varphi + \psi$ is a formula;

iv. $\varphi \wedge \psi$ is a formula;

v. $\alpha\varphi$ is a formula for every $\alpha \in \mathbb{R}$;

vi. $\bigwedge_x \varphi$ is a formula.

If $\alpha \in \mathbb{R}$, we may write α for the formula $\alpha 1$. We write $-\varphi$ for $(-1)\varphi$ and $\varphi - \psi$ for $\varphi + (-\psi)$ and we write $\varphi \vee \psi$ for $-(-\varphi \wedge -\psi)$. Also, φ^+ and φ^- stand for $0 \vee \varphi$ respectively $0 \vee (-\varphi)$, and we write $|\varphi|$ for $\varphi \vee (-\varphi)$. Finally, $\bigvee_x \varphi$ stands for $-\bigwedge_x -\varphi$.

If $A \subseteq M$ (that is, A is a subset of the unit ball), we write $L(A)$ for the language L expanded with a 0-ary function symbol for every element of a. As in classical first-order logic, the model M is canonically expanded to a structure of signature $L(A)$ by setting $a^{\mathcal{M}} = a$ for every $a \in A$.

Let $\mathcal{M}$ be a model and let t be be a tuple of closed terms with parameters in M. We write $t^{\mathcal{M}}$ for the element of $\mathcal{M}$ obtained interpreting function symbols and parameters in $\mathcal{M}$ just as in classical first-order logic.

4 Definition. If $\varphi \in L(M)$ is a sentence (i.e. a closed formula) we define $\varphi^{\mathcal{M}}$ inductively as follows

i. $(r\,t)^{\mathcal{M}} \;=\; r^{\mathcal{M}}\big(t^{\mathcal{M}}\big)$.

ii. $1^{\mathcal{M}} \;=\; 1$;

iii. $(\psi + \xi)^{\mathcal{M}} \;=\; \psi^{\mathcal{M}} + \xi^{\mathcal{M}}$;

iv. $(\alpha\varphi)^{\mathcal{M}} \;=\; \alpha\varphi^{\mathcal{M}}$;

v. $(\psi \wedge \xi)^{\mathcal{M}} \;=\; \min\big\{\psi^{\mathcal{M}},\ \xi^{\mathcal{M}}\big\}$;

vi. $\Big(\bigwedge_x \psi\Big)^{\mathcal{M}} \;=\; \inf\Big\{\psi[x/b]^{\mathcal{M}} \;:\; b \in M\Big\}$. ⚠

Note that in (vi) the infimum is taken as b ranges (only) over the unit ball. In Fact 6 below we prove that this infimum is indeed finite, hence that the definition is sound. Finally, we complete Definition 2 by stating the axioms that the unit ball has to satisfy.

5 Definition. The unit ball is required to satisfy the following axioms

1. $\mathcal{M} = \Big\{t^{\mathcal{M}} \;:\; t \in L_{\rm trm}(M) \text{ a closed term }\Big\}$;
2. $|\varphi(a)^{\mathcal{M}}| \leq n_{\varphi(x)}$ for every $a \in M^{|x|}$ and every $\varphi(x) \in L_{\rm at}$.

For $\varphi(x) \in L(M)$ we write $\varphi^{\mathcal{M}}$ for the function $\varphi^{\mathcal{M}} : M^{|x|} \to \mathbb{R}$ that maps $a \mapsto \varphi(a)^{\mathcal{M}}$.

A straightforeward proof by induction on the syntax yields the following.

6 Fact. For every formula $\psi(x)$ there is an $n \in \mathbb{N}$ such that $|\psi(a)^{\mathcal{M}}| \leq n$ for all $a \in M^{|x|}$. □

2. Examples

1. Firstly, we leave to the reader to check that classical first-order models are a (trivial) special cases of the models introduced above. Classical relations take values in $\{0, 1\}$ where 0 is interpreted as "true". The unit ball is the whole of $\mathcal{M}$. The function $n_{\_}$ is constantly 1. To

obtain the full strength of first-order logic one has to add equality as a predicate as it is absent from our logic.

2. Secondly, any BBHU-premodel is also a model as those in Section 1 if we let the unit ball coincide with $\mathcal{M}$. The language L contains a relation symbol $d(x, y)$ for a metric and, possibly, symbols for all other functions and relations of $\mathcal{M}$. As all relations of $\mathcal{M}$ (including the metric) take values in the interval $[0, 1]$, the function $n_{\text{-}}$ can be set to be the constant 1.

3. Now we consider an example that is non trivial and distant both from classical models and from BBHU-models: weighted graphs. The set L_{fun} is empty and L_{rel} contains only a binary relation r. We require that $r^{\mathcal{M}}(a, b) = r^{\mathcal{M}}(b, a) \in [0, 1]$. As unit ball take, again,the whole of $\mathcal{M}$. The function $n_{\text{-}}$ is constant 1.

4. Finally, and example where the unit ball is non trivial. Let L be the language of Banach spaces. Here we have only one symbol in L_{rel}, the symbol $\|\text{-}\|$ for the norm. The set L_{fun} is the same as for real vector spaces in classical logic. Let $\mathcal{M}$ be a Banach space and define a continuous model as follows. The interpretation of the symbols in $L_{\text{rel}} \cup L_{\text{fun}}$ is the natural one. The unit ball of is $M = \{a \in M : \|a\| \leq 1\}$. If $\varphi(x)$ is the atomic formulas

$$\varphi(x) \quad = \quad \Big\| \sum_{i=1}^{n} \lambda_i x \Big\|$$

then

$$n_{\varphi(x)} \quad = \quad \sum_{i=1}^{n} |\lambda_i|.$$

3. Conditions and types

For $\varphi, \psi \in L(M)$ some closed formulas, we write $M \vDash \varphi \leq \psi$ for $\varphi^{\mathcal{M}} \leq \psi^{\mathcal{M}}$. The meaning of $M \vDash \varphi = \psi$ and of $M \vDash \varphi < \psi$ is similar.

Expressions of the form $\varphi(x) \leq 0$ are called conditions. We write $M \vDash \exists x\, \varphi(x) \leq 0$ if there is an $a \in M^{|x|}$ such that $M \vDash \varphi(a) \leq 0$. Observe that $\varphi(x) = 0$ is equivalent to $|\varphi(x)| \leq 0$ and that $\varphi(x) \leq 0$ is equivalent to $\big(\varphi(x) \vee 0\big) = 0$. So, when convenient, we may use equalities to denote conditions.

The negation of a condition is called a co-conditions. So, $\varphi(x) \neq 0$, $\varphi(x) < 0$ or $\varphi(x) > 0$ are co-conditions.

It is easy to see that conditions are cosed under logical disjunction and logical conjunction.

$$M \vDash \varphi \vee \psi \leq 0 \quad \Leftrightarrow \quad M \vDash \varphi \leq 0 \text{ and } M \vDash \psi \leq 0$$

$$M \vDash \varphi \wedge \psi \leq 0 \quad \Leftrightarrow \quad M \vDash \varphi \leq 0 \ \text{ or } \ M \vDash \psi \leq 0$$

Condition are also closed under universal quanfication (over the unit ball).

$$M \vDash \bigvee_{x} \varphi(x) \leq 0 \quad \Leftrightarrow \quad M \vDash \forall x\, \varphi(x) \leq 0$$

Closure under existential quanfication is a more subtle point. It is not true in general but it can be ensured by a small amount of saturation (cf. Fact 17).

A set of conditions is called a type or, when x is the empty tuple, a theory. For $p(x) \subseteq L(M)$, we define

$$p(x) \leq 0 \quad = \quad \big\{\varphi(x) \leq 0 \quad : \quad \varphi(x) \in p\big\}.$$

Up to equivalence, all types have the form above.

Beware that $p(x)$ denotes just a set of formulas, the expression $p(x) \leq 0$ denotes a type.

We write $M \vDash p(a) \leq 0$ if $M \vDash \varphi(a) \leq 0$ for every $\varphi(x) \in p$. We write $M \vDash \exists x\, p(x) \leq 0$ if $M \vDash p(a) \leq 0$ for some $a \in M^{|x|}$.

4. Elementary relations

Let $\mathcal{M}$ and $\mathcal{N}$ be models with unit balls M and N, respectively. We say that $R \subseteq M \times N$ is an elementary relation between M and N if for every $\varphi(x) \in L$

$$\varphi^{\mathcal{M}}(a) \quad = \quad \varphi^{\mathcal{N}}(b) \qquad\qquad \text{for every } a, b \in M^{|x|}, N^{|x|} \text{ such that } aRb.$$

We define an equivalence relation $(\sim_M)$ on the unit ball of a given model M as follows

$$a \sim_M b \quad \Leftrightarrow \quad \varphi(a)^{\mathcal{M}} = \varphi^{\mathcal{M}}(b) \ \text{ for every } \varphi(x) \in L(M).$$

In ther words,

$$a \sim_M b \quad \Leftrightarrow \quad I_M \cup \{(a, b)\} \subseteq M^2 \ \text{ is an elementary relation.}$$

As elementary relations are closed under composition, if $a \sim_M b$ and $a' \sim_M b'$ then the relation $I_M \cup \big\{(a, b), (a', b')\big\} \subseteq M^2$ is also elementary. In general, for any $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ we have that

$$a_i \sim_M b_i \text{ for every } i \in (n] \quad \Leftrightarrow \quad \varphi(a)^{\mathcal{M}} = \varphi^{\mathcal{M}}(b) \ \text{ for every } \varphi(z) \in L(M).$$

From the consideration above we deduce the following.

7 **Lemma.** The relation $(\sim_M) \subseteq M^2$ is a maximal elementary relation. That is, no elementary relation contains $(\sim_M)$ properly.

Proof. The claim above the lemma proves that $(\sim_M)$ is elementary. By the definition of $(\sim_M)$, a pair $a \nsim_M b$ cannot belong to any elementary relation. Therefore $(\sim_M)$ is maximal. □

8 **Lemma.** Let $R \subseteq M \times N$ be total and surjective elementary relation. Then there is a unique maximal elementary relation containing R. This maximal elementary relation is equal to both $(\sim_M)\, R$ and $R\, (\sim_N)$.

Proof. It is immediate to verify that $(\sim_M)\,R$ is an elementary relation containing R. Let S be any maximal elementary relation containing R. By maximality, $(\sim_M)\,S = S$. As S is a total relation $(\sim_M) \subseteq SS^{-1}$. Therefore, by the lemma above, $(\sim_M) = SS^{-1}$. As R is a surjective relation, $S \subseteq S\,S^{-1}R$. Finally, by maximality, we conclude that $S = (\sim_M)\,R$. A similar argument proves that $S = R\,(\sim_N)$. □

We write $\mathrm{Aut}(M)$ for the set of maximal, total and surjective, elementary relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $M/{\sim_M}$.

5. Elementary substructures

An elementary embedding is an elementary relation that is functional, injective and total.

When $\varnothing$ is an elementary relation between M and N we say that these are elementarily equivalent and write $M \equiv N$. We write I_A for the diagonal relation on A. We write $M \equiv_A N$ if I_A is an elementary relation. In words, we say that M and N are elementary equivalent over A. Finally, we write $M \preceq N$ if $M \equiv_M N$ and $\mathcal{M}$ is a substructure of $\bar{N}$ in the classical sense. In wards, we say that M is an elementary substructure of N.

Let $A \subseteq M$. The limit A-topology on $M^{|x|}$ is the initial topology with respect to the functions interpret the formulas in $L(A)$. The sets $\{a \in M^{|x|} \;:\; \varphi^{\mathcal{M}}(a) \le 0\}$, as $\varphi(x)$ ranges over $L(A)$, form a base of closed sets for the limit A-topology. Note that the sets $\{a \in M^{|x|} \;:\; \varphi^{\mathcal{M}}(a) < 0\}$ are a base of open sets of the same topology.

The limit A-topology is a completely regular topology, almost by definition. Typically, it is not T_0.

9 **Proposition (Tarski-Vaught test).** Let $M \subseteq N$ and assume that $\mathcal{M}$ is a substructure of $\bar{N}$. Let x be a single variable. Then the following are equivalent:

1. $M \preceq N$;
2. M is dense in N w.r.t. the M-limit topology on N;
3. for every $\varphi(x) \in L(M)$, if $\varphi(N) < 0$ is non empty, then $N \vDash \varphi(b) < 0$ for some $b \in M^{|x|}$.

Proof. Equivalence 2⇔3 is tautological. Implication 1⇒2 is clear. To prove 3⇒1 we prove by induction on the syntax of $\varphi(z)$, where z is a finite tuple, that for every $\alpha \in \mathbb{R}$

$$M \vDash \varphi(a) < \alpha \quad\Leftrightarrow\quad N \vDash \varphi(a) < \alpha \qquad\qquad \text{for every } a \in M^{|z|}$$

We use the equivalence above as induction hypothesis. As M is a substructure of N, this equivalence holds for atomic $\varphi(z)$. We leave the case of the propositional connectives to the reader and consider only the case of the infimum quantifier.

$$N \vDash \bigwedge_x \varphi(x,a) < \alpha \quad\Leftrightarrow\quad \text{there is a } b \in N \text{ such that } N \vDash \varphi(b,a) < \alpha$$

a. $\Leftrightarrow$ there is a $b \in M^{|x|}$ such that $N \vDash \varphi(b, a) < \alpha$

b. $\Leftrightarrow$ $M \vDash \bigwedge_x \varphi(x) < \alpha.$

Implication $\Rightarrow$ in *a* uses the assumption 3 and the equivalence in *b* uses the induction hypothesis. □

10 **Theorem (Downward Löwenheim-Skolem).** For every $A \subseteq N$ there is a model $A \subseteq M \preceq N$ of cardinality $\leq |L(A)|$.

Proof. (Needs some checking.) For every set A there is a base for the A-topology that has cardinality $\leq |L(A)|$. A set M of the required cardinality that is dense in the M-topology is obtained as in the classical downward Löwenheim-Skolem theorem. □

6. Ultraproducts

Firstly, we recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $r : I \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we write

$$\lim_{i \uparrow F} r(i) = \lambda$$

if $r^{-1}[A] \in F$ for every $A \subseteq \mathbb{R}$ that is a neighborhood of λ. When r is bounded and F is an ultrafilter, such a λ always exists.

Let I be an infinite set. Let $\langle M_i : i \in I \rangle$ be a sequence of models. Let F be an ultrafilter on I.

11 **Definition.** Write N for the set of all functions $\hat{a} : I \to \bigcup_{i \in I} M_i$ such that $\hat{a}\, i \in M_i$. Such N is the unit ball of a structure N that we call the ultraproduct of $\langle M_i : i \in I \rangle$ and is defined as follows.

1. the elements of $\bar{N}$ are functions $\hat{c} : I \to \bigcup_{i \in I} \bar{M}_i$ such that $\hat{c}\, i = t(\hat{a}\, i)$ for some $\hat{a} \in M^{|x|}$ and some term $t(x) \in L_{\rm trm}$ — below we write $t(\hat{a})$ for $\hat{c}$;
2. if $f \in L_{\rm fun}$ and $r \in L_{\rm rel}$ and $t_1(\hat{a}), \ldots, t_n(\hat{a})$ are elements of $\bar{N}$ then
   a. $f^N\big(t_1(\hat{a}), \ldots, t_n(\hat{a})\big) = t(\hat{a})$ where t is the term obtained composing f with $t_1, \ldots, t_n$;
   b. $r^N\big(t_1(\hat{a}), \ldots, t_n(\hat{a})\big) = \lim_{i \uparrow F} r^{M_i}\big(t_1(\hat{a}\, i), \ldots, t_n(\hat{a}\, i)\big)$.

A usual, if we have that $M_i = M$ for all $i \in I$, we say that N is an ultrapower of M.

Note that the unit ball N satisfies the axioms in Definition 5.

The limit in 2b of Definition 11 exists because F is an ultrafilter and it is finite because all models M_i have the same bounding functions. In fact, the function $i \mapsto r^{M_i}\big(t_1(\hat{a}\, i), \ldots, t_n(\hat{a}\, i)\big)$ is bounded by $n_{\varphi(x)}$, where $\varphi(x) = r(t_1(x), \ldots, t_n(x))$.

12 **Proposition (Łǒś Theorem).** Let N be as above and let $\varphi(x) \in L$. Then for every $\hat{a} \in N^{|x|}$,

$$\varphi^{\mathcal{N}}\big(\hat{a}\big) \;=\; \lim_{i\uparrow F} \varphi^{M_i}\big(\hat{a}i\big).$$

Proof. This is just the usual run-of-the-mill proof. We proceed by induction on the syntax of $\varphi(x)$. If $\varphi(x)$ is atomic holds the equality above holds by definition. We spell out the proof for the infimum quantifier. Assume inductively that

$$\varphi^{\mathcal{N}}\big(\hat{a},\hat{b}\big) \;=\; \lim_{i\uparrow F} \varphi^{M_i}\big(\hat{a}i,\hat{b}i\big).$$

We want to prove that

$$\Big(\bigwedge_x \varphi\big(\hat{a},x\big)\Big)^{\mathcal{N}} \;=\; \lim_{i\uparrow F} \Big(\bigwedge_x \varphi\big(\hat{a}i,x\big)\Big)^{M_i}$$

By the interpretation of the infimum quantifier, this amounts to prove that

$$\inf\left\{\varphi\big(\hat{a},\hat{b}\big)^{\mathcal{N}} \;:\; \hat{b}\in(uN)\right\} \;=\; \lim_{i\uparrow F}\inf\left\{\varphi\big(\hat{a}i,b\big)^{M_i} \;:\; b\in(uM_i)\right\}$$

First we prove the $\leq$ inequality. Let α be an arbitrary positive real number.

Assume that for some $\hat{b}\in(uN)$ $\quad \varphi^{\mathcal{N}}\big(\hat{a},\hat{b}\big) \;<\; \alpha.$

By induction hypothesis $\quad \lim_{i\uparrow F} \varphi^{M_i}\big(\hat{a}i,\hat{b}i\big) \;<\; \alpha,$

then for some $u\in F$ $\quad \varphi^{U_i}\big(\hat{a}i,\hat{b}i\big) \;<\; \alpha$ for every $i\in u$.

Hence $\quad \inf\left\{\varphi^{M_i}\big(\hat{a}i,b\big) \;:\; b\in(uM_i)\right\} \;<\; \alpha$ for every $i\in u$.

Finally, $\quad \lim_{i\uparrow F}\inf\left\{\varphi^{M_i}\big(\hat{a}i,b\big) \;:\; b\in(uM_i)\right\} \;<\; \alpha.$

This proves the $\geq$ inequality. As for the $\leq$ inequality, assume that

$$\lim_{i\uparrow F}\inf\left\{\varphi\big(\hat{a}i,b\big)^{M_i} \;:\; b\in(uM_i)^{|x|}\right\} \;<\; \alpha.$$

Therefore, for some $u\in F$

$$\inf\left\{\varphi\big(\hat{a}i,b\big)^{M_i} \;:\; b\in(uM_i)^{|x|}\right\} \;<\; \alpha \quad \text{for every } i\in u.$$

$$\inf\left\{\varphi\big(\hat{a}i,b\big)^{M_i} \;:\; b\in(uM_i)^{|x|}\right\} \;<\; \alpha \quad \text{for every } i\in u.$$

Therefore, for some $\hat{b}\in(uN)^{|x|}$ $\quad \varphi^{M_i}\big(\hat{a}i,\hat{b}i\big) \;<\; \alpha$ for every $i\in u$.

Then $\quad \lim_{i\uparrow F} \varphi^{M_i}\big(\hat{a}i,\hat{b}i\big) \;<\; \alpha,$

and finally $\quad \varphi^{\mathcal{N}}\big(\hat{a},\hat{b}\big) \;<\; \alpha.$

This proves the desired inequality. □

The following corollary of Łǒś Theorem is identical to its classical counterpart.

13 **Corollary.** For every model M and every ultrafilter F on I, an infinite set, let N be corresponding ultrapower of M. Then there is an elementary embedding of M in N. □

We add the following easy remark for future reference.

14 **Remark.** Let $R \subseteq N^2$ be the set of those pairs $(\hat{a}, \hat{b})$ such that $\{i \in I : \hat{a}i = \hat{b}i\} \in F$. Then R is an elementary relation.

7. Compactness theorem

From Łŏś Theorem we obtain a form of compactness with the usual argument. A theory $T \leq 0$ is finitely consistent if for every φ, that is a disjunction of sentences in T, there is a model M such that $M \vDash \varphi \leq 0$. Note that it is implicit that all these models M have the same bounding fuction.

15 **Theorem (Compactness Theorem).** Every finitely consistent theory is consistent.

Proof. We construct a model N of $T \leq 0$. Let I be the set of closed formulas ξ such that $\xi \leq 0$ holds in some model M_ξ. For every condition $\varphi \leq 0$ define $X_\varphi \subseteq I$ as follows

$$X_\varphi \;=\; \Big\{\xi \in I \;:\; \xi \leq 0 \;\vdash\; \varphi \leq 0\Big\}$$

Note that $\varphi \leq 0$ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi \vee \psi} \;=\; X_\varphi \cap X_\psi$. Then, as T is finitely consistent, the set $B = \Big\{X_\varphi \;:\; \varphi \in T\Big\}$ has the finite intersection property. Extend B to an ultrafilter F on I. Define

$$N \;=\; \prod_{\xi \in I} M_\xi$$

Check that $N \vDash T \leq 0$. Let $\varphi \in T$. By Łŏś Theorem,

$$\varphi^{\mathcal{N}} \;=\; \lim_{\xi \uparrow F} \varphi^{M_\xi}.$$

Note that $X_\varphi \subseteq \Big\{\xi \;:\; \varphi^{M_\xi} \leq 0\Big\}$ and recall that $X_\varphi \in F$ whenever $\varphi \in T$. Therefore $\varphi^{\mathcal{N}} \leq 0$. □

8. Saturation

Let $p(x) \subseteq L(M)$. We say that $p(x) \leq 0$ is finitely satisfied if for every disjunction of formulas in $p(x)$, say $\psi(x)$, there is an $a \in M^{|x|}$ such that $M \vDash \psi(a) \leq 0$.

We say that M is λ-saturated if for every $p(x)$ as in 1 and 2 below, there is an $a \in M^{|x|}$ such that $M \vDash p(a) \leq 0$.

1. $p(x) \subseteq L(A)$ for some $A \subseteq M$ of cardinality $< \lambda$;
2. $p(x) \leq 0$ is finitely satisfied in M.

If λ is the cardinality of M we say that M is saturated.

16 Theorem (???). Let λ be an inaccessible cardinal larger than $|L|$ and $|M|$. Then M has a saturated elementary superstructure of cardinality M.

We conclude with a convenient property of ω-saturated models.

17 Fact. Let M is ω-saturated. Then for every $\varphi(x) \in L(M)$ the following are equivalent

1. $M \vDash \exists x\, \varphi(x) \leq 0$;
2. $M \vDash \bigwedge_x \varphi(x) \leq 0$

Proof. Only 2⇒1 requires a proof. Let $p(x) = \{\varphi(x) - \alpha : \alpha \in \mathbb{R}^+\}$. If 2, then $p(x) \leq 0$ is finitely satisfied in M. Hence 1 follows by saturation. □

Di qui in poi solo esperimenti selvaggi.

9. The monster model

Throughout the following we fix a saturated model $\mathcal{U}$ of cardinality κ, an inaccessible cardinal larger than $|L|$, where $|L|$ stands for max $\{|L_{\text{fun}}|, |L_{\text{rel}}|, 2^\omega\}$.

18 Fact. Let $R \subseteq \mathcal{U}^2$ be an elementary relation of cardinality $< \kappa$. Then there is a total and surjective elementary relation $S \subseteq \mathcal{U}^2$ containing R.

Proof. □

Let $a \in \mathcal{U}^{|x|}$. We write $p(x) = \text{tp}(a/A)$ for $p(x) = \{\psi(x) \in L(A) : \mathcal{U} \vDash \psi(a) \leq 0\}$.

19 Fact. Let $a \in \mathcal{U}^{|x|}$, where $|x| < \kappa$. Let $A \subseteq \mathcal{U}$ have cardinality $< \kappa$. Then

$$p(\mathcal{U}) \leq 0 \quad = \quad \{ha : h \in \text{Aut}(\mathcal{U}/A)\}$$

10. Random variables

We define a 3-sorted structure $\langle \mathcal{A}, \mathcal{R}, \mathcal{M} \rangle$.

Fix a nonempty set Ω.

The domain of the first sort consists of a Boolean algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. The language contains functions for the Boolean operations on $\mathcal{A}$.

The domain of the second sort $\mathcal{R}$ contains the functions $f : \Omega \to \mathbb{R}$ that are linear combinations of indicator functions of sets in $\mathcal{A}$. In the language we include the functions that make $\mathcal{R}$ an $\mathbb{R}$-algebra (sum and and multiplications are defined pointwise). We also include operations for the pointwise maximum and minimum of two functions.

The domain of the third sort, $\mathcal{M}$, contains signed measures $\mu : \mathcal{A} \to \mathbb{R}$ that are linear combinations of measures of the form δ_a, for $a \in \Omega$. These are defined as follows: $\delta_a(A)$ is 1 if $a \in A$ and 0 otherwise. The language is that of lattice vector spaces.

There is a function of sort $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ that maps (f, μ) to the measure $f\mu$ obtained multiplying in the natural way f and μ.

Finally, there is a predicate $\boldsymbol{I}$ of sort $\mathcal{A} \times \mathcal{M}$ such that $\boldsymbol{I}(A, \mu) = \mu(A)$.

The unit ball $\langle \mathcal{A}, \mathcal{R}, \mathcal{M} \rangle$ is $\langle \mathcal{A}, \boldsymbol{R}, \boldsymbol{M} \rangle$ where $\boldsymbol{R}$ and $\boldsymbol{M}$ are the functions, respectively measures that are ≤ 1 in absolute value.

***

The following finite version of the Radon-Nikodym theorem is completely trivial. For simplicity we state it for measures. As the Hahn decomposition theorem holds (trivially) in $\mathcal{M}$, this is no loss of generality.

20 Fact. [Discrete Radon-Nikodym] Let $\mu, \nu \in \boldsymbol{M}$, where $\boldsymbol{M}$ is the unit ball of $\mathcal{M}$, be non negative. For every $\varepsilon > 0$ there are $E \in \mathcal{A}$ and $f \in \boldsymbol{R}$, the unit ball of $\mathcal{R}$, such that

1. $\boldsymbol{I}(\neg E, |\nu - \varepsilon^{-1} f\mu|) \ = \ 0$
2. $|\boldsymbol{I}(E, \mu)| \leq \varepsilon$.

We will use the following version of the fact above.

21 Fact. [Iterated discrete Radon-Nikodym] Let $\mu, \nu \in \boldsymbol{M}$, where $\boldsymbol{M}$ is the unit ball of $\mathcal{M}$. For every $n > 0$ there are $E_1, \dots, E_n \in \mathcal{A}$ and $f \in \boldsymbol{R}$, such that

0. $E_n \subseteq E_{n-1} \subseteq \dots \dots \subseteq E_1 \subseteq E_0 = \Omega$
1. $\boldsymbol{I}(E_i \smallsetminus E_{i+1},\ |\nu - 2^{-(i+1)} f\mu|) \ = \ 0$
2. $|\boldsymbol{I}(E_{i+1},\ \mu)| \leq 2^{-(i+1)}$.

Proof. Apply inductively the theorem above, with E_i for Ω, to find E_{i+1} and f_{i+1} such that

1_i. $\boldsymbol{I}(X, |\nu - 2^{-(i+1)} f_{i+1}\mu|) \ = \ 0$ for every $X \subseteq E_i \smallsetminus E_{i+1}$;

2_i. $|\boldsymbol{I}(X, \mu)| \leq 2^{-(i+1)}$ for every $X \subseteq E_{i+1}$;

3_i. $f_{i+1}[E_i \smallsetminus E_{i+1}] = \{0\}$.

Note that the last requirement is immediate to satisfy. Finally, let $f = f_1 + \dots + f_n$. □

We now consider a saturated extension $\langle {}^*\!\mathcal{A}, {}^*\mathcal{R}, {}^*\mathcal{M} \rangle$. The following fact is a direct consequence of saturation.

22 Fact. Let μ be a bounded finitely additive measure on $\mathcal{A}$, not necessarily in $\mathcal{M}$. Then there is ${}^*\mu \in {}^*\mathcal{M}$ such that $\mu_{\restriction \mathcal{A}} = {}^*\mu_{\restriction \mathcal{A}}$.

Proof. It suffices to check that the following type is finitely consistent in $\mathcal{A}$ (read ν as a variable and $\mu(X)$ as a real number)

$$\big\{\nu(X) = \mu(X) \ : \ X \in \mathcal{A}\big\},$$

which immediate. □

We can improve on the fact above. In fact, we can also control the measure of ${}^*\mu(A)$ for any $A \in \mathcal{A}$.

23 **Fact.** Let μ be a bounded measure on $\mathcal{A}$, not necessarily in $\mathcal{M}$. Let $A \in {}^*\!\mathcal{A}$. Define

$$\begin{aligned} m_{\mathrm{i}} &= \sup\{\mu(X) : X \in \mathcal{A},\ X \subseteq A\} \\ m_{\mathrm{e}} &= \inf\{\mu(X) : X \in \mathcal{A},\ A \subseteq X\} \end{aligned}$$

Then for every $m_{\mathrm{i}} \leq r \leq m_{\mathrm{e}}$ there is a measure ${}^*\mu \in {}^*\mathcal{M}$ such that ${}^*\mu_{\restriction\mathcal{A}} = \mu_{\restriction\mathcal{A}}$ and ${}^*\mu(A) = r$.

Proof (sketch). Assume for simplicity that μ is bounded by 1. It suffices to check that the following type is finitely consistent (read ν as a variable and $\mu(X)$ as a real number).

$$\big\{\nu(X) = \mu(X) \ : \ X \in \mathcal{A}\big\} \ \cup \ \big\{\nu(A) = r\big\}.$$

In fact, any ${}^*\mu$ realizing this type is as required by the lemma.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be a finite Boolean algebra. We define a measure ν on the Boolean algebra generated by $A, X_1, \dots, X_n$ that satisfies the type above restricted to $\mathcal{A}' \cup \{A\}$. Let $X_1, \dots, X_n$ be the atoms of $\mathcal{A}'$. Assume $A \notin \mathcal{A}$, to avoid trivialities. Then there are some sets X_i such that both $X_i \cap A$ and $X_i \smallsetminus A$ are nonempty. Suppose these sets are $X_1, \dots, X_m$. For $i \leq m$ we define $\nu(X_i \cap A) = \varepsilon\, \mu(X)$ and $\nu(X_i \smallsetminus A) = (1-\varepsilon)\mu(X_i)$, where $0 \leq \varepsilon \leq 1$ is specified below. For $i > m$ let $\nu(X_i) = \mu(X_i)$. Note that $\mu(X_1 \cup \dots \cup X_m) \geq m_{\mathrm{e}} - m_{\mathrm{i}}$. Therefore with a suitable ε, we can obtain $\nu(A) = r$. □

If $\mathcal{A}$ is a σ-algebra and μ is a σ-additive measure then the suppremum and the infimum in the fact above is attained. We will use the following.

24 **Corollary.** If $\mathcal{A}$ is a σ-algebra and μ is a bounded σ-additive measure then for every $A \in {}^*\!\mathcal{A}$ there are $A_{\mathrm{i}}, A_{\mathrm{e}} \in \mathcal{A}$ such that $A_{\mathrm{i}} \subseteq A \subseteq A_{\mathrm{e}}$ and a measures ${}^*\mu$ such that ${}^*\mu_{\restriction\mathcal{A}} = \mu_{\restriction\mathcal{A}}$ and ${}^*\mu_{\mathrm{e}}(A) = \mu(A_{\mathrm{e}})$. A similar claim holds for A_{i}.

25 **Theorem (Radon-Nikodym).** Assume $\mathcal{A}$ is a σ-algebra and let ν and μ be bounded σ-additive measures. Then there is a random variable f such that

$$\int_X f\,\mathrm{d}\mu = \int_X \mathrm{d}\nu$$

for every $X \in \mathcal{A}$.