CONTINUOUS MODEL THEORY WITHOUT THE HASSLE

1. Formulas and their interpretation

The following definition extends that of classical first-order logic with clauses 3 and 4.

1 Definition. A signature or language L consists of

1. two sets of symbols: $L_{\rm rel}$, for relations, and $L_{\rm fun}$, for functions;
2. an arity function $\mathrm{Ar} : L_{\rm rel} \cup L_{\rm fun} \to \omega$;
3. a function $n_{\_} : L_{\rm at} \to \mathbb{N}$;

The function in 3 above is called the bounding function. The definition of $L_{\rm at}$ is given below.

2 Definition. A structure or model of signature L consists of

1. a set $\bar{M}$;
2. a set $M \subseteq \bar{M}$.
3. for every $f \in L_{\rm fun}$, a function $f^M : \bar{M}^{\mathrm{Ar}(f)} \to \bar{M}$;
4. for every $r \in L_{\rm rel}$, a function $r^M : \bar{M}^{\mathrm{Ar}(r)} \to \mathbb{R}$;

The set M is called the unit ball and is required to saisfy the axioms in Definition 3. It will be clear in Definition 5 that all our model theory plays inside the unit ball. Therefore we feel justified to denote the whole model by the unit ball (in principle different models could share the same unit ball this will never be the case in the following).

The syntax and semantic of terms is defined just as in classical first-order logic. We write $L_{\rm trm}$ for the set of terms in the language L. Atomic formulas are defined as in classical first-order logic. They have the form $r\,t$, where r is a relation symbol and t is a tuple of terms. The set of atomic formulas is denoted by $L_{\rm at}$.

Finally, we finish the definition of structure by giving the axiomatics of the unit ball and of the bounding function.

3 Definition. The unit ball is required to satisfy the following axioms

1. $\bar{M} = \big\{ t^M : t \in L_{\rm trm}(M) \text{ a closed term} \big\}$;
2. $|\varphi(a)^M| \leq n_{\varphi(x)}$ for every $a \in M^{|x|}$ and every $\varphi(x) \in L_{\rm at}$.

The set of all formulas is denoted as usual by L, the same symbol as the language. Formulas are defined inductively from the atomic one using the following connectives.

The propositional (or Riesz) connectives are those in $\{1,+,\wedge\}\cup\mathbb{R}$, where $+$ and $\wedge$ are binary connectives, the elements of $\mathbb{R}$ are unary connectives, 1 is a logical constant (i.e. a zero-ary connective). There is also the infimum quantifier $\bigwedge_x$.

4 **Definition.** The inductive definition of formula is as follows

i. atomic formulas are formulas
ii. 1 is a formula;
iii. $\varphi+\psi$ is a formula;
iv. $\varphi\wedge\psi$ is a formula;
v. $\alpha\varphi$ is a formula for every $\alpha\in\mathbb{R}$;
vi. $\bigwedge_x \varphi$ is a formula.

If $\alpha\in\mathbb{R}$, we may write α for the formula $\alpha 1$. We write $-\varphi$ for $(-1)\varphi$ and $\varphi-\psi$ for $\varphi+(-\psi)$ and we write $\varphi\vee\psi$ for $-(-\varphi\wedge-\psi)$. Also, φ^+ and φ^- stand for $0\vee\varphi$ respectively $0\vee(-\varphi)$, and we write $|\varphi|$ for $\varphi\vee(-\varphi)$. Finally, $\bigvee_x \varphi$ stands for $-\bigwedge_x -\varphi$.

If $A\subseteq M$, we write $L(A)$ for the language L expanded with a 0-ary function symbol for every element of a. As in classical first-order logic, M is canonically expanded to a structure of signature $L(A)$ by setting $a^M=a$ for every $a\in A$.

Let M be a model and let $t\in L_{\text{term}}(M)$ be a tuple of closed. We write t^M for the element of $\bar{M}$ obtained interpreting function symbols and parameters in M just as in classical first-order logic.

5 **Definition.** If $\varphi\in L(uM)$ is a sentence (i.e. a closed formula) we define φ^M inductively as follows

i. $(r\,t)^M \;=\; r^M\big(t^M\big)$.
ii. $1^M \;=\; 1$;
iii. $(\psi+\xi)^M \;=\; \psi^M+\xi^M$;
iv. $(\alpha\varphi)^M \;=\; \alpha\varphi^M$;
v. $(\psi\wedge\xi)^M \;=\; \min\big\{\psi^M,\,\xi^M\big\}$;
vi. $\big(\bigwedge_x \psi\big)^M \;=\; \inf\big\{\psi[x/b]^M \;:\; b\in M\big\}$.

In Fact 6 below we prove that the infimum in (vi) above is indeed finite, hence that the definition is sound.

For $\varphi(x)\in L(M)$ we write φ^M for the function $\varphi^M : M^{|x|}\to\mathbb{R}$ that maps $a\mapsto\varphi(a)^M$.

A straightforeward proof by induction on the syntax yields the following.

6 **Fact.** For every formula $\psi(x)$ there is an $n\in\mathbb{N}$ such that $|\psi(a)^M|\le n$ for all $a\in M^{|x|}$. □

2. Examples

1. Firstly, we leave to the reader to check that classical first-order models are a (trivial) special cases of the models introduced above. Classical relations take values in $\{0,1\}$ where 0 is interpreted as "true". The unit ball is the whole of $\bar{M}$. The function $n_{\_}$ is constantly 1. To obtain the full strength of first-order logic one has to add equality as a predicate as it is absent from our logic.

2. Secondly, any BBHU-premodel is also a model as those in Section 1 if we let the unit ball coincide with $\bar{M}$. The language L contains a relation symbol $d(x, y)$ for a metric and, possibly, symbols for all other functions and relations of M. As all relations of M (including the metric) take values in the interval $[0, 1]$, the function n_{-} can be set to be the constant 1.

3. Now we consider an example that is non trivial and distant both from classical models and from BBHU-models: weighted graphs. The set L_{fun} is empty and L_{rel} contains only a binary relation r. We require that $r^M(a, b) = r^M(b, a) \in [0, 1]$. As unit ball take, again,the whole of $\bar{M}$. The function n_{-} is constant 1.

4. Finally, and example where the unit ball is non trivial. Let L be the language of Banach spaces. Here we have only one symbol in L_{rel}, the symbol $\|\text{-}\|$ for the norm. The set L_{fun} is the same as for real vector spaces in classical logic. Let $\bar{M}$ be a Banach space and define a continuous model as follows. The interpretation of the symbols in $L_{\mathrm{rel}} \cup L_{\mathrm{fun}}$ is the natural one. The unit ball of is $M = \{a \in M : \|a\| \leq 1\}$. If $\varphi(x)$ is the atomic formulas

$$\varphi(x) \quad = \quad \Big\| \sum_{i=1}^{n} \lambda_i x \Big\|$$

then

$$n_{\varphi(x)} \quad = \quad \sum_{i=1}^{n} |\lambda_i|.$$

3. Conditions and types

For $\varphi, \psi \in L(M)$ some closed formulas, we write $M \vDash \varphi \leq \psi$ for $\varphi^M \leq \psi^M$. The meaning of $M \vDash \varphi = \psi$ and of $M \vDash \varphi < \psi$ is similar.

Expressions of the form $\varphi(x) \leq 0$ are called conditions. We write $M \vDash \exists x\, \varphi(x) \leq 0$ if $M \vDash \varphi(a) \leq 0$ for some $a \in M^{|x|}$. Observe that $\varphi(x) = 0$ is equivalent to $|\varphi(x)| \leq 0$ and that $\varphi(x) \leq 0$ is equivalent to $\big(\varphi(x) \vee 0\big) = 0$. So, when convenient we may use weak equalities to denote conditions.

The negation of a condition is called a co-conditions. So, $\varphi(x) \neq 0$, $\varphi(x) < 0$ or $\varphi(x) > 0$ are co-conditions.

It is easy to see that conditions are cosed under logical disjunction and logical conjunction.

$$M \vDash \varphi \vee \psi \leq 0 \;\Leftrightarrow\; M \vDash \varphi \leq 0 \text{ and } M \vDash \psi \leq 0$$

$$M \vDash \varphi \wedge \psi \leq 0 \;\Leftrightarrow\; M \vDash \varphi \leq 0 \;\text{ or }\; M \vDash \psi \leq 0$$

Condition are also closed under universal quanfication over the unit ball.

$$M \vDash \bigvee_{x} \varphi(x) \leq 0 \;\Leftrightarrow\; M \vDash \forall x \in u\; \varphi(x) \leq 0$$

Closure under existential quanfication is a more subtle point. It is not true in general but it can be ensured by a small amount of saturation (cf. Fact 16).

A set of conditions is called a type or, when x is the empty tuple, a theory. For $p(x) \subseteq L(M)$, we define

$$p(x) \leq 0 \quad = \quad \big\{\varphi(x) \leq 0 \quad : \quad \varphi(x) \in p\big\}.$$

Up to equivalence, all types have the form above.

Beware that $p(x)$ denotes just a set of formulas, the expression $p(x) \leq 0$ denotes a type.

We write $M \vDash p(a) \leq 0$ if $M \vDash \varphi(a) \leq 0$ for every $\varphi(x) \in p$. We write $M \vDash \exists x\, p(x) \leq 0$ if $M \vDash p(a) \leq 0$ for some $a \in M^{|x|}$.

4. Elementary relations

Let $A \subseteq uM$. The A-limit topology on $(uM)^{|x|}$ is the initial topology with respect to the functions interpret the formulas in $L(A)$. The sets $\big\{a \in (uM)^{|x|} \ :\ \varphi^M(a) \leq 0\big\}$, as $\varphi(x)$ ranges over $L(A)$, form a base of closed sets for the A-limit topology. Equivalently, we could take the sets $\big\{a \in (uM)^{|x|} \ :\ \varphi^M(a) < 0\big\}$ as a base of open sets.

The A-limit topology is a completely regular topology, almost by definition. Typically, it is not T_0.

We say that $R \subseteq uM \times uN$ is an elementary relation between M and N if for every $\varphi(x) \in L$

$$\varphi^M(a) \quad = \quad \varphi^N(b) \qquad\qquad \text{for every } a, b \in (uM)^{|x|}, (uN)^{|x|} \text{ such that } aRb.$$

An elementary embedding is an elementary relation that is functional, injective and total.

When $\varnothing$ is an elementary relation between M and N we say that these are elementarily equivalent and write $M \equiv N$. We write I_A for the diagonal relation on A. We write $M \equiv_A N$ if I_A is an elementary relation. In words, we say that M and N are elementary equivalent over A. Finally, we write $M \preceq N$ if $M \subseteq N$ and $M \equiv_M N$ and say that M is an elementary substructure of N. By $M \subseteq N$ we mean that M is a substructureof N in the classical sense.

7 **Proposition (Tarski-Vaught test).** Let N be a model and let $M \subseteq N$ be a substructure. Fix a tuple of variables x of finite length. Then the following are equivalent:

1. $M \preceq N$;
2. $(uM)^{|x|}$ is dense in $(uN)^{|x|}$ w.r.t. the uM-limit topology on $(uN)^{|x|}$;
3. for every $\varphi(x) \in L(M)$, if $\varphi(uN) < 0$ is non empty, then $N \vDash \varphi(b) < 0$ for some $b \in (uM)^{|x|}$.

Proof. Equivalence 2⇔3 is tautological. Implication 1⇒2 is clear. To prove 3⇒1 we prove by induction on the syntax of $\varphi(x)$ that for every $\alpha \in \mathbb{R}$

$$M \vDash \varphi(a) < \alpha \quad \Leftrightarrow \quad N \vDash \varphi(a) < \alpha \qquad\qquad \text{for every } a \in M^{|x|}$$

We use the equivalence above as induction hypothesis. As M is a substructure of N, this equivalence holds for atomic $\varphi(x)$. We leave the case of the propositional connectives to the reader and consider only the case of the infimum quantifier.

$$
\begin{array}{lrcl}
 & N \vDash \bigwedge_x \varphi(x,a) < \alpha & \Leftrightarrow & \text{there is a } b \in (uN) \text{ such that } N \vDash \varphi(b,a) < \alpha \\
a. & & \Leftrightarrow & \text{there is a } b \in (uM)^{|x|} \text{ such that } N \vDash \varphi(b,a) < \alpha) \\
b. & & \Leftrightarrow & M \vDash \bigwedge_x \varphi(x) < \alpha.
\end{array}
$$

Implication $\Rightarrow$ in a uses the assumption 3 and the equivalence in b uses the induction hypothesis. □

8 Remark. In the Tarski-Vaught test for classical logic it is not necessary to require that M is a substructure of N as this is a consequence of the test. In the continuous case this is necessary by the lack of a predicate for equality. □

9 Theorem (Downward Löwenheim-Skolem). For every $A \subseteq N$ there is a model $A \subseteq M \preceq N$ of cardinality $\leq |L(A)|$.

Proof. (Needs some checking.) For every set A there is a base for the A-topology that has cardinality $\leq |L(A)|$. A set M of the required cardinality that is dense in the M-topology is obtained as in the classical downward Löwenheim-Skolem theorem. □

5. Ultrapowers

Firstly, we recall some standard notation about limits. Let I be a non-empty set. Let F be a filter on I. If $r : I \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we write

$$\lim_{i \uparrow F} r(i) = \lambda$$

if $r^{-1}[A] \in F$ for every $A \subseteq \mathbb{R}$ that is a neighborhood of λ. When r is bounded and F is an ultrafilter, such a λ always exists.

Let I be an infinite set. Let $\langle M_i : i \in I \rangle$ be a sequence of models. Note that all models are required to have the same bounding function (as these are part of the language). Let F be an ultrafilter on I.

10 Definition. Write U for the set of all functions $\hat{a} : I \to \bigcup_{i \in I}(uM_i)$ such that $\hat{a}\, i \in (uM_i)$. We define the structure N which we call an ultraproduct of $\langle M_i : i \in I \rangle$. A usual, if $M_i = M$ for all $i \in I$, then we say that N is an ultrapower of M.

1. the elements of N are functions $\hat{c} : I \to \bigcup_{i \in I} M_i$ such that $\hat{c}\, i = t(\hat{a}\, i)$ for some $\hat{a} \in U^{|x|}$ and some term $t(x) \in L_{\mathrm{trm}}$ — below we write $t(\hat{a})$ for $\hat{c}$;
2. if $f \in L_{\mathrm{fun}}$ and $r \in L_{\mathrm{rel}}$ and $t_1(\hat{a}), \ldots, t_n(\hat{a})$ are elements of N then
   a. $f^N\big(t_1(\hat{a}), \ldots, t_n(\hat{a})\big) = t(\hat{a})$ where t is the term obtained composing f with $t_1, \ldots, t_n$;
   b. $r^N\big(t_1(\hat{a}), \ldots, t_n(\hat{a})\big) = \lim_{i \uparrow F} r^{M_i}\big(t_1(\hat{a}\, i), \ldots, t_n(\hat{a}\, i)\big)$;
   c. $uN = U$.

Note that uN satisfies the axioms in Definition 3.

The limit in 2b of Definition 10 exists because F is an ultrafilter and it is finite because all U_i have the same bounding functions. In fact, the function $i \mapsto r^{M_i}\big(t_1(\hat{a}\,i), \dots, t_n(\hat{a}\,i)\big)$ is bounded by $n_{\varphi(x)}$, where $\varphi(x) = r(t_1(x), \dots, t_n(x))$.

11 **Proposition (Łǒś Theorem).** Let N be as above and let $\varphi(x) \in L$. Then for every $\hat{a} \in N^{|x|}$,

$$\varphi^N\big(\hat{a}\big) \quad = \quad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i\big).$$

Proof. This is just the usual run-of-the-mill proof. We proceed by induction on the syntax of $\varphi(x)$. If $\varphi(x)$ is atomic holds the equality above holds by definition. We spell out the proof for the infimum quantifier. Assume inductively that

$$\varphi^N\big(\hat{a}, \hat{b}\big) \quad = \quad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big).$$

We want to prove that

$$\Big(\bigwedge_x \varphi\big(\hat{a}, x\big)\Big)^N \quad = \quad \lim_{i \uparrow F} \Big(\bigwedge_x \varphi\big(\hat{a}i, x\big)\Big)^{M_i}$$

By the interpretation of the infimum quantifier, this amounts to prove that

$$\inf\Big\{\varphi\big(\hat{a}, \hat{b}\big)^N \ :\ \hat{b} \in (uN)\Big\} \quad = \quad \lim_{i \uparrow F} \inf\Big\{\varphi\big(\hat{a}i, b\big)^{M_i} \ :\ b \in (uM_i)\Big\}$$

First we prove the $\leq$ inequality. Let α be an arbitrary positive real number.

Assume that for some $\hat{b} \in (uN)$ $\qquad \varphi^N\big(\hat{a}, \hat{b}\big) \ <\ \alpha.$

By induction hypothesis $\qquad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big) \ <\ \alpha,$

then for some $u \in F$ $\qquad \varphi^{U_i}\big(\hat{a}i, \hat{b}i\big) \ <\ \alpha \quad$ for every $i \in u$.

Hence $\qquad \inf\Big\{\varphi^{M_i}\big(\hat{a}i, b\big) \ :\ b \in (uM_i)\Big\} \ <\ \alpha \quad$ for every $i \in u$.

Finally, $\qquad \lim_{i \uparrow F} \inf\Big\{\varphi^{M_i}\big(\hat{a}i, b\big) \ :\ b \in (uM_i)\Big\} \ <\ \alpha.$

This proves the $\geq$ inequality. As for the $\leq$ inequality, assume that

$$\lim_{i \uparrow F} \inf\Big\{\varphi\big(\hat{a}i, b\big)^{M_i} \ :\ b \in (uM_i)^{|x|}\Big\} \ <\ \alpha.$$

Therefore, for some $u \in F$

$$\inf\Big\{\varphi\big(\hat{a}i, b\big)^{M_i} \ :\ b \in (uM_i)^{|x|}\Big\} \ <\ \alpha \quad \text{for every } i \in u.$$

$$\inf\Big\{\varphi\big(\hat{a}i, b\big)^{M_i} \ :\ b \in (uM_i)^{|x|}\Big\} \ <\ \alpha \quad \text{for every } i \in u.$$

Therefore, for some $\hat{b} \in (uN)^{|x|}$ $\qquad \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big) \ <\ \alpha \quad$ for every $i \in u$.

Then $\qquad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big) \ <\ \alpha,$

and finally
$$\varphi^N\left(\hat{a},\hat{b}\right) \;<\; \alpha.$$

This proves the desired inequality. □

The following corollary of Łŏś Theorem is identical to its classical counterpart.

12 Corollary. For every model M and every ultrafilter F on I, an infinite set, let N be corresponding ultrapower of M. Then there is an elementary embedding of M in N. □

We add the following easy remark for future reference.

13 Remark. Let $R \subseteq N^2$ be the set of those pairs $(\hat{a},\hat{b})$ such that $\{i \in I \;:\; \hat{a}i = \hat{b}i\} \in F$. Then R is an elementary relation.

6. Compactness theorem

From Łŏś Theorem we obtain a form of compactness with the usual argument.

A theory $T \leq 0$ is finitely consistent if for every φ, that is a disjunction of sentences in T, there is a model M such that $M \vDash \varphi \leq 0$. Note that it is implicit that all these models M have the same bounding fuction.

14 Theorem (Compactness Theorem). Every finitely consistent theory is consistent.

Proof. We construct a model N of $T \leq 0$. Let I be the set of closed formulas ξ such that $\xi \leq 0$ holds in some model M_ξ. For every condition $\varphi \leq 0$ define $X_\varphi \subseteq I$ as follows
$$X_\varphi \;=\; \Big\{\xi \in I \;:\; \xi \leq 0 \;\vdash\; \varphi \leq 0\Big\}$$
Note that $\varphi \leq 0$ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi\vee\psi} \;=\; X_\varphi \cap X_\psi$. Then, as T is finitely consistent, the set $B = \Big\{X_\varphi \;:\; \varphi \in T\Big\}$ has the finite intersection property. Extend B to an ultrafilter F on I. Define
$$N \;=\; \prod_{\xi\in I} M_\xi$$
Check that $N \vDash T \leq 0$. Let $\varphi \in T$. By Łŏś Theorem,
$$\varphi^N \;=\; \lim_{\xi\uparrow F} \varphi^{M_\xi}.$$
Note that $X_\varphi \subseteq \Big\{\xi \;:\; \varphi^{M_\xi} \leq 0\Big\}$ and recall that $X_\varphi \in F$ whenever $\varphi \in T$. Therefore $\varphi^N \leq 0$. □

7. Saturation

Let $p(x) \subseteq L(M)$. We say that $p(x) \leq 0$ is finitely satisfied (in uM) if for every disjunction of formulas in $p(x)$, say $\psi(x)$, there is an $a \in (uM)^{|x|}$ such that $M \vDash \psi(a) \leq 0$.

We say that M is λ-saturated if for every $p(x) \subseteq L(A)$ as in 1 and 2 below, there is an $a \in (uM)^{|x|}$ such that $M \vDash p(a) \leq 0$.

1. $A \subseteq M$ has cardinality $< \lambda$;
2. $p(x) \leq 0$ is finitely satisfied in uM.

If λ is the cardinality of M we say that M is saturated.

15 Theorem (???). Let λ be an inaccessible cardinal larger than $|L|$ and $|M|$. Then M has a saturated elementary superstructure of cardinality M.

We conclude with an interesting property of ω-saturated models.

16 Fact. Let M is ω-saturated. Then for every $\varphi(x) \in L(M)$ the following are equivalent

1. $M \vDash \exists x \in u\, \varphi(x) \leq 0$;
2. $M \vDash \bigwedge\limits_{x} \varphi(x) \leq 0$

Proof. Only 2⇒1 requires a proof. Let $p(x) = \{\varphi(x) - \alpha \,:\, \alpha \in \mathbb{R}^+\}$. If 2, then $p(x) \leq 0$ is finitely satisfied in uM. Hence 1 follows by saturation. □

Di qui in poi solo esperimenti selvaggi.

8. Total and surjective elementary relations

Let $R \subseteq M \times N$. Define

$$\langle R\rangle \quad = \quad \Big\{\big(t(a), t(b)\big) \quad : \quad t(x) \in L_{\mathrm{trm}} \quad |x| < \omega \quad a, b \in M^{|x|} \quad aRb\Big\}$$

Clearly, if R is elementary, so is $\langle R\rangle$. We say that R is bounded if

$$R \quad = \quad \big\langle R \,\cap\, (uM \times uN)\big\rangle$$

We write $\mathrm{Tser}(M)$ for the set of elementary relations $R \subseteq M^2$ that are total and surjective. We write $\mathrm{Tser}(M/A)$ the set of relations in $\mathrm{Tser}(M)$ that contain I_A.

9. The monster model

Throughout the following we fix a saturated model $\mathfrak{U}$ of cardinality κ, an inaccessible cardinal larger than $|L|$, where $|L|$ stands for max $\big\{|L_{\mathrm{fun}}|, |L_{\mathrm{rel}}|, 2^{\omega}\big\}$.

17 Fact. Let $R \subseteq \mathfrak{U}^2$ be an elementary relation of cardinality $< \kappa$. Then there is an S such that $R \subseteq S \in \mathrm{Tser}(\mathfrak{U})$.

Proof. □

Let $a \in ({}^u\mathcal{U})^{|x|}$. We write $p(x) = \operatorname{tp}(a/A)$ for $p(x) = \{\psi(x) \in L(A) : \mathcal{U} \vDash \psi(a) \leq 0\}$.

18 Fact. Let $a \in ({}^u\mathcal{U})^{|x|}$, where $|x| < \kappa$. Let $A \subseteq \mathcal{U}$ have cardinality $< \kappa$. Then

$$p(\mathcal{U}) \leq 0 \quad = \quad \{ha : h \in \operatorname{Aut}(\mathcal{U}/A)\}$$