

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A CLASS OF STRUCTURES

1 Definition. In these notes we deal with 3-sorted¹ structures of the form $\mathcal{M} = \langle \check{M}, M, \mathbb{R} \rangle$. The language and its interpretation are subject to the following conditions.

1. Functions may only have one of the following sorts (for any $m, n \in \omega$)
 - a. $\mathbb{R}^n \rightarrow \mathbb{R}$
 - b. $\check{M}^n \times M^m \rightarrow \text{any of } \check{M}, M, \text{ or } \mathbb{R}$
2. There is a symbol for every (total) uniformly continuous functions $\mathbb{R}^n \rightarrow \mathbb{R}$.
3. The (functions that interpret the) terms of sort $\check{M}^n \rightarrow \mathbb{R}$ have *bounded* range.
4. Every element of M is the image of some term of sort $\check{M}^n \rightarrow M$.
5. There is only one predicate; it has sort \mathbb{R} and is interpreted as $x \leq 0$.

We call \check{M} the **unit ball** of \mathcal{M} . The terminology is inspired by the example below.

The language is denoted by \mathbb{L} .

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $\check{M}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ with parameters in some $A \subseteq \check{M}$. Up to equivalence, these terms have the form $f(t(x), y)$, where $t(x)$ is a tuple of terms of sort $\check{M}^{|x|} \rightarrow \mathbb{R}$, and f is a function $\mathbb{R}^{|x|+|y|} \rightarrow \mathbb{R}$.

3 Definition. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

- i. $\mathbb{L}(A)$ contains the atomic formula $t \leq 0$ for every term $t \in \mathbb{T}(A)$.
- ii. It is closed under the Boolean connectives \wedge, \vee , and the quantifiers \forall, \exists of sort \check{M} .
- iii. It is closed under the quantifier \forall of sort \mathbb{R} relativized to any definable subset of \mathbb{R} .

Note that in iii of the definition above, definability is intended with respect to the full first order language. In particular, if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$, also $\forall \varepsilon > 0 \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

4 Example (Metric spaces of finite diameter). Given a metric space M, d of finite diameter we define a structure $\mathcal{M} = \langle \check{M}, M, \mathbb{R} \rangle$ as follows. Set $\check{M} = M$ and let $\text{id} : \check{M} \rightarrow M$ be the identical isomorphism. The function symbols are id and d . Note that 3 is satisfied because d is bounded.

¹In some examples it may be more natural to use $(n + n' + 1)$ -sorted structures $\mathcal{M} = \langle \check{M}_1, \dots, \check{M}_n, M_1, \dots, M_{n'}, \mathbb{R} \rangle$. The generalization is straightforward.

5 Example (Banach spaces). Given a Banach space M we define a structure $\mathcal{M} = \langle \check{M}, M, \mathbb{R} \rangle$ as follows. Let $\check{M} = \{a \in M : \|a\| \leq 1\}$ be the closed unit ball of M . Besides the symbols mentioned above, \mathbb{L} contains a function symbol for the natural embedding $\text{id} : \check{M} \rightarrow M$. It also contains a symbol for the norm $\|\cdot\| : M \rightarrow \mathbb{R}$. Finally, \mathbb{L} contains the usual symbols of the language of vector spaces. These have sort $M^n \rightarrow M$, for the appropriate $n \in \{0, 1, 2\}$. Note that conditions 3 and 4 of Definition 1 are immediatly satisfied.

If $\varphi \in \mathbb{L}(\mathbb{A})$ and $\varepsilon > 0$ we write φ_ε for the formula obtained by replacing in φ the atomic formulas $t \leq 0$ with $t - \varepsilon \leq 0$. Note that $\varphi \rightarrow \varphi_\varepsilon$.

2. ULTRAPRODUCTS

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I . If $f : I \rightarrow \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \rightarrow F} f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq \mathbb{R} \cup \{\pm\infty\}$ that is a neighborhood of λ . Such a λ is unique and, when F is an ultrafilter, it always exists. When f is bounded, $\lambda \in \mathbb{R}$.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle \check{M}_i, M_i, \mathbb{R} \rangle$, that are **uniformly bounded**, that is, the bounds in 3 of Definition 1 are the same for all \mathcal{M}_i .

Let F be an ultrafilter on I .

6 Definition. We define a structure $\mathcal{N} = \langle \check{N}, N, \mathbb{R} \rangle$ that we call the **ultraproduct** of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. \check{N} comprise the sequences $\hat{a} : I \rightarrow \bigcup_{i \in I} \check{M}_i$ such that $\hat{a} i \in \check{M}_i$.
2. N comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a}i)$, where $t(x)$ is a term of sort $\check{M}^n \rightarrow M$.
3. If f is a function of sort $\check{M}^n \times M^m \rightarrow \check{M}$ then $f^{\mathcal{N}}(\hat{a}, t^{\mathcal{N}}(\hat{c}))$ is the sequence $f^{\mathcal{M}_i}(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i))$.
4. Similarly when f is of sort $\check{M}^n \times M^m \rightarrow M$.
5. If f is a function of sort $\check{M}^n \times M^m \rightarrow \mathbb{R}$ then

$$f^{\mathcal{N}}(\hat{a}, t^{\mathcal{N}}(\hat{c})) = \lim_{i \rightarrow F} f^{\mathcal{M}_i}(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that \mathcal{N} is an **ultrapower** of \mathcal{M} .

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models \mathcal{M}_i have the same bounds.

The following fact is easily proved by induction on the syntax.

7 Fact. For every term $t(x)$ of sort $\check{M}^n \rightarrow \mathbb{R}$

$$t^{\mathcal{N}}(\hat{a}) = \lim_{i \rightarrow F} t^{\mathcal{M}_i}(\hat{a}i).$$

□

Finally, we prove

8 Proposition (Łoś Theorem). Let \mathcal{N} be as above and let $\varphi(x, y) \in \mathbb{L}(A)$. Let $\hat{a} \in \check{N}^{|x|}$. Then there is a function $u_- : \mathbb{R}^+ \rightarrow F$ such that for every $\lambda \in \mathbb{R}^{|y|}$

1. $\mathcal{N} \models \varphi(\hat{a}, \lambda) \Rightarrow u_\varepsilon \subseteq \{i \in I : \mathcal{M}_i \models \varphi_\varepsilon(\hat{a}i, \lambda)\}$ for every $\varepsilon > 0$.
2. $\mathcal{N} \not\models \varphi(\hat{a}, \lambda) \Rightarrow u_\varepsilon \subseteq \{i \in I : \mathcal{M}_i \not\models \varphi_\varepsilon(\hat{a}i, \lambda)\}$ for some $\varepsilon > 0$.

Proof. Suppose that $\varphi(x, y)$ is atomic, say

$$\varphi(x, y) = f(t(x), y) \leq 0,$$

where t is a tuple of terms of sort $\check{M}^{|x|} \rightarrow \mathbb{R}$ and f is a uniformly continuous function $\mathbb{R}^{|t|+|y|} \rightarrow \mathbb{R}$.

Let $\beta' \in \mathbb{R}^{|t|}$ be such that $\lim_{i \rightarrow F} t(\hat{a}i) = \beta'$.

Given $\varepsilon > 0$, let B_ε be a neighborhood of β' such that

$$\left| f(\beta, \lambda) - f(\beta', \lambda) \right| < \varepsilon \quad \text{for every } \beta \in B_\varepsilon \text{ and every } \lambda \in \mathbb{R}^{|y|}.$$

Such a neighborhood exists by the uniform continuity of f . Finally, to obtain 1 and 2 above it suffices to define

$$u_\varepsilon = \{i : t(\hat{a}i) \in B_\varepsilon\}.$$

The rest of the inductive proof is straightforward and is left to the reader. □