

# CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

C. L. C. L. POLYMATH

## 1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted structure. Ideally, we would like that elementary extensions of normed spaces maintain the usual notion of real numbers<sup>1</sup>. Unfortunately, this is not possible if we insist to maintain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

**1 Definition.** Let  $L$  be a one-sorted (first-order) language. Let  $L' \supseteq L$  be a two-sorted language. We consider the class of  $L'$ -structures of the form  $\mathcal{M} = \langle M, R \rangle$ , where  $M$  ranges over  $L$ -structures, while  $R$  is fixed.

We assume that  $R$  is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

- i.  $R^n \rightarrow R$ ;
- ii.  $M^n \rightarrow M$ ;
- iii.  $M^n \rightarrow R$ .

The interpretation of symbols of sort  $R^n \rightarrow R$  is required to be continuous, the interpretation of symbols of sort  $M^n \rightarrow R$  is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts  $M^n$  and  $R^n$ .

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set  $\mathbb{L}$  of formulas,  $L \subseteq \mathbb{L} \subseteq L'$ , such that every model as in Definition 1 has an  $\mathbb{L}$ -elementary extension that is  $\mathbb{L}$ -saturated in the same class.

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<sup>1</sup>Non standard standard analysts have no problem in expanding  $\mathbb{R}$ . In these notes, for a change, we insist as  $\mathbb{R}$  (or, in general  $R$ ) should remain  $\mathbb{R}$  throughout.

For convenience we assume that the functions of sort  $M^n \rightarrow M$  and the relations of sort  $M^n$  are all in  $L$ . It is also convenient to assume that  $L'$  contains names for all continuous bounded functions  $R^n \rightarrow R$ .

**2 Definition.** Formulas in  $\mathbb{L}$  are constructed inductively from the following two sets of formulas

- i. formulas of the form  $t(x; y) \in C$ , where  $C \subseteq R^n$  is compact<sup>2</sup> and  $t(x, y)$  is a  $n$ -tuple of terms of sort  $M^{|x|} \times R^{|y|} \rightarrow R$ ;
- ii. all formulas in  $L$ .

We require that  $\mathbb{L}$  is closed under the Boolean connectives  $\wedge, \vee$ ; the quantifiers  $\forall, \exists$  of sort  $M$ ; and the quantifiers  $\forall^C, \exists^C$ , by which we mean the quantifiers of sort  $R$  restricted to some (any) compact set  $C \subseteq R^m$ .

## 2. HENSON-IOVINO APPROXIMATIONS

For  $\varphi, \varphi' \in \mathbb{L}(M)$  (free variables are hidden) we write  $\varphi' > \varphi$  if  $\varphi'$  can be obtained replacing each atomic formula  $t \in C$  occurring in  $\varphi$  by  $t \in D$  for some compact neighborhood  $D$  of  $C$ . If no such atomic formulas occurs in  $\varphi$ , then  $\varphi > \varphi$ . By the normality of  $R$  we also have  $\varphi > \varphi$  when  $\varphi = (t \in C)$  for some clopen set  $C$ .

Note that  $>$  is a dense (pre)order of  $\mathbb{L}(M)$ .

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that  $\varphi \rightarrow \varphi'$ . In fact, we have the following.

**3 Proposition (fake theorem).** For every  $\varphi(x) \in \mathbb{L}(M)$

$$\varphi(x) \leftrightarrow \bigwedge_{\varphi' > \varphi} \varphi'(x)$$

*Proof.* The claim is clear for atomic formulas. Induction for conjunction and the universal quantifiers is immediate.

Consider disjunction. Let  $\varphi = \varphi_i \vee \varphi_1$ . Assume inductively

$$\varphi_i(x) \leftrightarrow \bigwedge_{\varphi'_i > \varphi_i} \varphi'_i(x)$$

Then

$$\varphi(x) \leftrightarrow \bigwedge_{\varphi'_1 > \varphi_1} \varphi'_1(x) \vee \bigwedge_{\varphi'_2 > \varphi_2} \varphi'_2(x)$$

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<sup>2</sup>We confuse the relation symbols in  $L'$  of sort  $R^m$  with their interpretation and write  $t \in C$  for  $C(t)$

$$\begin{aligned} &\leftrightarrow \bigwedge_{\substack{\varphi'_1 > \varphi_1 \\ \varphi'_2 > \varphi_2}} \varphi'_1(x) \vee \varphi'_2(x) \\ &\leftrightarrow \bigwedge_{\varphi' > \varphi} \varphi'(x) \end{aligned}$$

Now consider the existential quantifiers of sort  $M$ .... □

**4 Lemma.** (*R compact*) For every  $\varphi' > \varphi$  there is a formula  $\psi \in \mathbb{L}(M)$  such that  $\varphi \rightarrow \neg\psi \rightarrow \varphi'$ .

*Proof.* By induction. If  $\varphi \in L$  the claim is obvious. Suppose  $\varphi$  is of the form  $t \in C$ . Then  $\varphi'$  is  $t \in D$  for some compact neighborhood of  $C$ . Let  $O$  be an open set such that  $C \subseteq O \subseteq D$ . Then  $\psi = (t \in R \setminus O)$  is as required.

Induction is easy (conjunctions and disjunctions in  $\varphi$  occur swapped in  $\psi$ ; similarly universal and existential quantifiers). □

We recall the standard definition of  $F$ -limits. Let  $I$  be a non-empty set. Let  $F$  be a filter on  $I$ . Let  $Y$  be a topological space. If  $f : I \rightarrow Y$  and  $\lambda \in Y$  we write

$$F\text{-}\lim_i f(i) = \lambda$$

if  $f^{-1}[A] \in F$  for every  $A \subseteq Y$  that is a neighborhood of  $\lambda$ . Such a  $\lambda$  is unique if  $Y$  is Hausdorff. When  $F$  is an ultrafilter and, in addition,  $Y$  is compact the limit always exists.

The lemma below is required for the proof of Łoś Theorem in the next section. But first a preliminary fact.

**5 Fact.** Assume the following data

- $t(x; y)$ , a term of sort  $M^{|x|} \times R^{|y|} \rightarrow R$ ;
- $\langle a_i : i \in I \rangle$ , a sequence of elements  $a_i \in M^{|x|}$ ;
- $\langle \alpha_i : i \in I \rangle$ , a sequence of elements of  $C \subseteq R^{|y|}$ , a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$ , for some ultrafilter  $F$  on  $I$ .

Then

$$\# \quad F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i)$$

*Proof.* By induction on the nesting of functions of sort  $R^n \rightarrow R$ . In the basis case  $t$  is either a term of sort  $M^{|x|} \rightarrow R$  or a variable of sort  $R$ . In both cases (#) is trivial. Assume inductively that the fact holds for the  $n$ -tuple  $t(x, y)$ . By the continuity of the functions of sort  $R^n \rightarrow R$  and the induction hypothesis, (#) holds for  $f(t(x, y))$ . □

**6 Lemma.** Assume the following data

- $\varphi(x; y) \in \mathbb{L}$ ;
- $\langle a_i : i \in I \rangle$ , a sequence of elements of  $M^{|x|}$ ;
- $\langle \alpha_i : i \in I \rangle$ , a sequence of elements of  $C \subseteq R^{|y|}$ , a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$ , for some ultrafilter  $F$  on  $I$ .

Then the following are equivalent

- i.  $\{i \in I : \mathcal{M} \models \varphi'(a_i; \alpha)\} \in F$  for every  $\varphi' > \varphi$ ;
- ii.  $\{i \in I : \mathcal{M} \models \varphi'(a_i; \alpha_i)\} \in F$  for every  $\varphi' > \varphi$ .

*Proof.* By induction on the syntax. When  $\varphi(x; y)$  is in  $L$ , it does not depend on  $\alpha$ , and the lemma is trivial. Suppose that  $\varphi(x; y)$  is as in (i) of Definition 2, say it is the formula  $t(x; y) \in C$ .

Assume  $\{i : t(a_i; \alpha) \in C'\} \in F$  for every  $C'$  that is a compact neighborhood of  $C$ . Let  $C''$  be a neighborhood of  $C$ . Pick  $C'$  such that  $C''$  be a neighborhood of  $C'$ . Then  $F\text{-}\lim_i t(a_i; \alpha) \in C'$  by normality of  $R$ . By Fact 5,  $F\text{-}\lim_i t(a_i; \alpha_i) \in C'$ . Then  $\{i : t(a_i; \alpha_i) \in C''\} \in F$ .

The converse implication is similar. This completes the proof of the base case of the induction.

Induction clear for the connectives  $\vee, \wedge$ . To deal with the universal quantifier of sort  $M$  we assume inductively that

$$\{i : \varphi'(a_i, b_i; \alpha)\} \in F \text{ for all } \varphi' > \varphi \Leftrightarrow \{i : \varphi'(a_i, b_i; \alpha_i)\} \in F \text{ for all } \varphi' > \varphi.$$

We prove

$$1. \{i : \forall y \varphi'(a_i, y; \alpha)\} \in F \text{ for all } \varphi' > \varphi \Leftrightarrow \{i : \forall y \varphi'(a_i, y; \alpha_i)\} \in F \text{ for all } \varphi' > \varphi.$$

Negate the r.h.s. of the equivalence. Then there are  $\varphi' > \varphi$  and a sequence  $\langle b_i : i \in I \rangle$  such that  $\{i : \neg \varphi'(a_i, b_i; \alpha_i)\} \in F$ . Pick  $\varphi''$  such that  $\varphi' > \varphi'' > \varphi$ . Assume also the l.h.s. of (1) and reason for a contradiction. Then  $\{i : \varphi''(a_i, b_i; \alpha)\} \in F$  and, by induction hypothesis,  $\{i : \varphi'(a_i, b_i; \alpha_i)\} \in F$ . A contradiction that proves the implication ( $\Rightarrow$ ) in (1). The converse implication is similar.

To deal with the existential quantifier of sort  $M$  we prove

$$2. \{i : \exists y \varphi'(a_i, y; \alpha)\} \in F \text{ for all } \varphi' > \varphi \Leftrightarrow \{i : \exists y \varphi'(a_i, y; \alpha_i)\} \in F \text{ for all } \varphi' > \varphi.$$

Assume the r.h.s. of the implication. Fix  $\varphi' > \varphi$ . We prove that  $\{i : \exists y \varphi'(a_i, y; \alpha)\} \in F$ . Pick  $\varphi''$  such that  $\varphi' > \varphi'' > \varphi$ . Then there is a sequence  $\langle b_i : i \in I \rangle$  such that  $\{i : \varphi''(a_i, b_i; \alpha_i)\} \in F$ . By the induction hypothesis  $\{i : \exists y \varphi'(a_i, y; \alpha_i)\} \in F$ . As  $\varphi'$  is arbitrary, this proves the implication ( $\Leftarrow$ ) in (2). The converse implication is similar.

Induction for the quantifiers  $\forall^C$  and  $\exists^C$  is virtually identical, but we repeat the argument for convenience. Assume inductively that for every  $\varphi' > \varphi$  and every sequence  $\langle \beta_i : i \in I \rangle$  in  $C$  such that  $\beta = F\text{-}\lim_i \beta_i$

$$\{i : \varphi(a_i; \alpha, \beta)\} \in F \text{ for all } \varphi' > \varphi \Leftrightarrow \{i : \varphi'(a_i; \alpha_i, \beta_i)\} \in F \text{ for all } \varphi' > \varphi.$$

We prove

$$3. \quad \{i : \forall^C y \varphi(a_i; \alpha, y)\} \in F \text{ for all } \varphi' > \varphi \Leftrightarrow \{i : \forall^C y \varphi'(a_i; \alpha_i, y)\} \in F \text{ for all } \varphi' > \varphi.$$

Negate the r.h.s. of the implication. Then there are  $\varphi' > \varphi$  and a sequence  $\langle \beta_i : i \in I \rangle$  such that  $\{i : \neg \varphi'(a_i; \alpha_i, \beta_i)\} \in F$ . Pick  $\varphi''$  such that  $\varphi' > \varphi'' > \varphi$ . Assume also the l.h.s. of (3) and reason for a contradiction. Then  $\{i : \varphi''(a_i; \alpha, \beta_i)\} \in F$  and, by induction hypothesis,  $\{i : \varphi'(a_i; \alpha_i, \beta_i)\} \in F$ . A contradiction that proves the implication ( $\Rightarrow$ ) in (3). The converse implication is similar.

Finally we prove

$$4. \quad \{i : \exists^C y \varphi(a_i; \alpha, y)\} \in F \text{ for all } \varphi' > \varphi \Leftrightarrow \{i : \exists^C y \varphi'(a_i; \alpha_i, y)\} \in F \text{ for all } \varphi' > \varphi.$$

Assume the r.h.s. of the implication. Fix  $\varphi' > \varphi$ . We prove that  $\{i : \exists y \varphi'(a_i; \alpha, y)\} \in F$ . Pick  $\varphi''$  such that  $\varphi' > \varphi'' > \varphi$ . Then there is a sequence  $\langle \beta_i : i \in I \rangle$  such that  $\{i : \varphi''(a_i; \alpha_i, \beta_i)\} \in F$ . By the induction hypothesis  $\{i : \exists y \varphi'(a_i; \alpha_i, y)\} \in F$ . As  $\varphi'$  is arbitrary, this proves the implication ( $\Leftarrow$ ) in (4). The converse implication is similar.  $\square$

### 3. ULTRAPRODUCTS

Below we introduce a suitable notion of ultraproducts of some structures  $\langle \mathcal{M}_i : i \in I \rangle$ . We require that for each function symbol  $f$  of sort  $M^n \rightarrow R$

# there is a compact  $C \subseteq R$  that contains the range of all the functions  $f^{\mathcal{M}_i}$ .

To keep notation tidy, we make two simplifications: (1) we only consider ultrapowers; (2) we ignore formulas in  $L$  containing equality, so we can work with  $M^I$  in place of  $M^I/F$ . The generalization is straightforward and is left to the reader.

Let  $I$  be an infinite set. Let  $F$  be an ultrafilter on  $I$ . Let  $\mathcal{M} = \langle M, R \rangle$  be an  $L$ -structure.

**7 Definition.** We define a structure  $\mathcal{N} = \langle N, R \rangle$  that we call the **ultrapower** of  $\mathcal{M}$ .

1.  $N = M^I$  that is, it is the set of sequences  $\hat{a} : I \rightarrow M$ .
2. If  $f$  is a function of sort  $M^n \rightarrow M$  then  $f^{\mathcal{N}}(\hat{a})$  is the sequence  $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I \rangle$ .
3. The interpretation of functions of sort  $R^n \rightarrow R$  remains unchanged.
4. If  $f$  is a function of sort  $M^n \rightarrow R$  then

$$f^{\mathcal{N}}(\hat{a}) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$

5. If  $r$  is a relation symbol of sort  $M^n$  then

$$\mathcal{N} \models r(\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i)\} \in F.$$

6. The interpretation of relations of sort  $R^n$  remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the uniformity requirement (#) above.

**8 Fact.** The structure  $\mathcal{N}$  satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

**9 Fact.** If  $t(x)$  is a term of type  $M^{|x|} \rightarrow M$  then

$$t^{\mathcal{N}}(\hat{a}) = \langle t^{\mathcal{M}}(\hat{a}i) : i \in I \rangle.$$

□

We also have that

**10 Fact.** If  $t(x; y)$  has sort  $M^{|x|} \times R^{|y|} \rightarrow R$  then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

*Proof.* By induction. If  $t$  is a function symbol then  $x$  or  $z$  do not occur in  $t$ . If  $x$  does not occur in  $t$  the claim is trivial. If  $y$  does not occur in  $t$  the claim holds by definition. Finally, induction is clear by the continuity of the functions of sort  $R^n \rightarrow R$ . □

Finally, we prove

**11 Proposition (Łoś Theorem).** Let  $\mathcal{N}$  be as above and let  $\varphi(x) \in \mathbb{L}$ . Then for all  $\hat{a} \in N^{|x|}$

$$\mathcal{N} \models \varphi(\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i)\} \in F \text{ for every } \varphi' > \varphi.$$

*Proof.* By induction on the syntax. If  $\varphi(x) \in L$  then the theorem reduces to the classical Łoś Theorem. Then, suppose that  $\varphi(x)$  is as in (i) of Definition 2, say it is the formula  $t(x) \in C$ .

Assume  $\mathcal{N} \models t(\hat{a}) \in C$ . If  $D$  is a compact neighborhood of  $C$  then  $D$  is also a neighborhood of  $t^{\mathcal{N}}(\hat{a})$ . Hence, by the definition of  $F$ -limit,  $\{i : \mathcal{M} \models t(\hat{a}i) \in D\} \in F$ . Vice versa, assume  $\mathcal{N} \models t(\hat{a}) \notin C$ . By local compactness of  $R$  there is a compact neighborhood of  $C$ , such that  $t^{\mathcal{N}}(\hat{a}) \notin D$ . By Fact 10 and the definition of  $F$ -limit,  $\{i : \mathcal{M} \models t(\hat{a}i) \notin D\} \in F$ .

This completes the proof of the base case of the induction.

Induction for the connectives  $\vee$  and  $\wedge$  is clear. To deal with the quantifiers of sort  $M$  we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

$$1. \quad \mathcal{N} \models \forall y \varphi(\hat{a}, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y)\} \in F \text{ for every } \varphi' > \varphi.$$

( $\Leftarrow$ ) Assume  $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b})$  for some  $\hat{b}$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\} \notin F$  for some  $\varphi' > \varphi$ . A fortiori  $\{i : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y)\} \notin F$  as required.

( $\Rightarrow$ ) Assume that  $\{i : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i, y)\} \in F$ . Choose  $\hat{b}$  such that  $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i)\} \in F$ . If for a contradiction  $\mathcal{N} \models \forall y \varphi(\hat{a}, y)$ , then in particular  $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\} \in F$  a contradiction. This completes the proof of (1).

Now, we prove



$$2. \quad \mathcal{N} \models \exists y \varphi(\hat{a}, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y)\} \in F \text{ for every } \varphi' > \varphi.$$

( $\Rightarrow$ ) Assume that  $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$  for some  $\hat{b}$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\} \in F$  for every  $\varphi' > \varphi$ . A fortiori  $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y)\} \in F$  as required.

( $\Leftarrow$ ) Assume that  $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y)\} \in F$  for every  $\varphi' > \varphi$ . We prove that  $\mathcal{N} \models \exists y \varphi(\hat{a}, y)$  for every  $\varphi' > \varphi$ . Suppose not, for a contradiction, say  $\varphi'$  is a counterexample. Choose some  $\hat{b}$  such that  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\} \in F$ . By induction hypothesis  $\mathcal{N} \models \varphi'(\hat{a}, \hat{b})$ , a contradiction. This completes the proof of (2).

To deal with the quantifier  $\forall^C$  we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}; \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \beta)\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

$$3. \quad \mathcal{N} \models \forall^C y \varphi(\hat{a}; y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; y)\} \in F \text{ for every } \varphi' > \varphi.$$

( $\Leftarrow$ ) Assume  $\mathcal{N} \not\models \varphi(\hat{a}; \beta)$  for some  $\beta$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \beta)\} \notin F$  for some  $\varphi' > \varphi$ . A fortiori  $\{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; y)\} \notin F$  as required.

( $\Rightarrow$ ) Assume  $\{i : \mathcal{M} \not\models \forall^C y \varphi'(\hat{a}i; y)\} \in F$  for some  $\varphi' > \varphi$ . Choose some  $\beta_i \in C$  such that  $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \beta_i)\} \in F$  and let  $\beta = F\text{-}\lim_i \beta_i$ . If for a contradiction  $\mathcal{N} \models \forall^C y \varphi(\hat{a}; y)$ , then  $\mathcal{N} \models \varphi(\hat{a}; \beta)$  and, by induction hypothesis,  $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \beta)\} \in F$  for every  $\varphi'' > \varphi$ . By Lemma 6,  $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \beta_i)\} \in F$  for every  $\varphi'' > \varphi$ . In particular  $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \beta_i)\} \in F$ , a contradiction. This completes the proof of (3).

Finally, for the quantifier  $\exists^C$  we prove



$$4. \quad \mathcal{N} \models \exists^C y \varphi(\hat{a}; y) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists^C y \varphi'(\hat{a}i; y)\} \in F \text{ for every } \varphi' > \varphi.$$

( $\Rightarrow$ ) Assume  $\mathcal{N} \models \varphi(\hat{a}; \beta)$  for some  $\beta$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \beta)\} \in F$  for every  $\varphi' > \varphi$ . A fortiori  $\{i : \mathcal{M} \models \exists^C y \varphi'(\hat{a}i; y)\} \in F$  for every  $\varphi' > \varphi$ .

( $\Leftarrow$ ) Assume  $\{i : \mathcal{M} \models \exists^C y \varphi'(\hat{a}i; y)\} \in F$  for all  $\varphi' > \varphi$ . We prove that  $\mathcal{N} \models \exists y \varphi(\hat{a}, y)$  for every  $\varphi'' > \varphi$ . Suppose not, for a contradiction, say  $\varphi''$  is a counterexample. Pick  $\varphi'$  such that

$\varphi'' > \varphi' > \varphi$ . Choose some  $\beta_i \in C$  such that  $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \beta_i)\} \in F$  and let  $\beta = F\text{-}\lim_i \beta_i$ . By Lemma 6,  $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \beta)\} \in F$ . This completes the proof of (4).  $\square$

#### 4. $\mathbb{L}$ -ELEMENTRARITY

Let  $\mathcal{M} = \langle M, R \rangle$  and  $\mathcal{N} = \langle N, R \rangle$  be two structures. We say that  $f : M \rightarrow N$ , a partial map, is an  $\mathbb{L}$ -elementary map if

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(a) \quad \text{for every } \varphi(x) \in \mathbb{L} \text{ and every } a \in (\text{dom } f)^{|x|}$$

The following fact is proved as in the classical case using that  $\mathcal{M} \models \varphi$  holds if and only if  $\mathcal{M} \models \varphi'$  holds for every  $\varphi' > \varphi$ .

**12 Fact.** If  $\mathcal{N}$  is an ultrapower of  $\mathcal{M}$  then there is an  $\mathbb{L}$ -elementary embedding of  $\mathcal{M}$  into  $\mathcal{N}$ .

For  $A \subseteq M \cap N$ , we say that  $\mathcal{M}$  and  $\mathcal{N}$  are  $\mathbb{L}$ -(elementary) equivalent over  $A$  and write  $\mathcal{M} \equiv_A^{\mathbb{L}} \mathcal{N}$  if  $\text{id}_A : M \rightarrow N$  is  $\mathbb{L}$ -elementary. We write  $\mathcal{M} \leq_M^{\mathbb{L}} \mathcal{N}$  when  $\mathcal{M} \equiv_M^{\mathbb{L}} \mathcal{N}$ . In words, we say that  $\mathcal{M}$  is an  $\mathbb{L}$ -(elementary) substructure of  $\mathcal{N}$ .

#### 5. $\mathbb{L}$ -COMPACTNESS

A theory  $T \subseteq \mathbb{L}$  be a weakly finitely consistent if for every  $\varphi$ , conjunction of sentences in  $T$ , any  $\varphi' > \varphi$  is consistent. The following proposition follows from Łoś Theorem by the usual argument.

**13 Proposition (Compactness Theorem).** Let  $T \subseteq \mathbb{L}$  be an weakly finitely consistent theory. Then  $T$  is consistent.  $\square$

Let  $p(x) \subseteq \mathbb{L}(M)$ . We say that  $p(x)$  is weakly finitely satisfied in  $\mathcal{M}$  if  $\mathcal{M} \models \exists x \varphi'(x)$  for every  $\varphi' > \varphi$  where  $\varphi(x)$  is some conjunction of formulas in  $p(x)$ .

**14 Definition.** We say that  $\mathcal{M}$  is  $\mathbb{L}$ -saturated if for every  $p(x)$  as in 1 and 2 below,  $\mathcal{M} \models \exists x p(a)$ .

1.  $p(x) \subseteq \mathbb{L}(A)$  for some  $A \subseteq M$  of cardinality  $< |M|$  and  $|x| = 1$ ;
2.  $p(x)$  is weakly finitely satisfied in  $\mathcal{M}$ .

The existence of  $\mathbb{L}$ -saturated models is proved as in the classical case.

**15 Proposition.** Every model has an  $\mathbb{L}$ -elementary extension to a saturated model (possibly of inaccessible cardinality).  $\square$

We denote by  $\mathcal{U} = \langle U, R \rangle$  some large  $\mathbb{L}$ -saturated structure which we call the monster model. The cardinality of  $\mathcal{U}$  is an inaccessible cardinal that we denote by  $\kappa$ . Below we say model for  $\mathbb{L}$ -elementary substructure of  $\mathcal{U}$ .



**16 Fact.** For every  $\varphi(x), \psi(x) \in \mathbb{L}(U)$  such that  $\neg \exists x [\varphi(x) \wedge \psi(x)]$  there are  $\varphi' > \varphi$  and  $\psi' > \psi$  such that  $\neg \exists x [\varphi'(x) \wedge \psi'(x)]$ .

*Proof.* By Proposition 3 and saturation. □

## 6. THE TARSKI-VAUGHT $\mathbb{L}$ -TEST

The following lemma is a useful observation. It has its first application below, in the proof of our version of the Tarski-Vaught test.

**17 Lemma.** (*R compact*) Let  $A \subseteq N$ . Then the following are equivalent

- 1 for every formula  $\varphi(x) \in \mathbb{L}(A)$   
 $\mathcal{N} \models \exists x \neg \varphi(x) \Rightarrow$  there is an  $a \in A$  such that  $\mathcal{N} \models \neg \varphi(a)$ ;
2. for every formula  $\varphi(x) \in \mathbb{L}(A)$   
 $\mathcal{N} \models \exists x \varphi(x) \Rightarrow$  for every  $\varphi' > \varphi$  there is an  $a \in A$  such that  $\mathcal{N} \models \varphi'(a)$ .

The implication  $2 \Rightarrow 1$  does not require the assumption of compactness.

*Proof.* ( $2 \Rightarrow 1$ ) Assume  $\mathcal{N} \models \exists x \neg \varphi(x)$ . Pick any  $c \in N$  such that  $c \models \neg \varphi(x)$ . Let  $p(x) = \mathbb{L}\text{-tp}(c/A)$ . Without loss of generality we can assume that  $\mathcal{N}$  is saturated. As  $p(x) \cup \{\neg \varphi(x)\}$  is inconsistent there is a formula  $\psi(x) \in p(x)$  such that  $\psi'(x) \rightarrow \neg \varphi(x)$  for some  $\psi' > \psi$ . Then from (2) we obtain that  $\mathcal{N} \models \psi'(a)$  for some  $a \in A$ . Hence  $\mathcal{N} \models \neg \varphi(a)$  follows.

( $1 \Rightarrow 2$ ) follows immediately from Lemma 4. □

**18 Proposition (Tarski-Vaught  $\mathbb{L}$ -Test).** Let  $M$  be a subset of  $N$ . Then the following are equivalent

1.  $M$  is the domain of a structure  $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$ ;
2. for every formula  $\varphi(x) \in \mathbb{L}(M)$   
 $\mathcal{N} \models \exists x \varphi(x) \Rightarrow$  for every  $\varphi' > \varphi$  there is an  $a \in A$  such that  $\mathcal{N} \models \varphi'(a)$ .

*Proof.* Assume (2) of the lemma. Then, by the classical Tarski-Vaught test  $M \leq N$ . We also have that  $\mathcal{M}$  is an  $L'$ -substructure of  $\mathcal{N}$ . Therefore, from every formula  $\varphi(x)$  as in (i) and (ii) of Definition 2 we have

$$\mathcal{N} \models \varphi(a) \Leftrightarrow \mathcal{M} \models \varphi(a) \quad \text{for every } a \in M^{|x|}.$$

Assume inductively

$$\# \quad \mathcal{N} \models \varphi(a, b) \Leftrightarrow \mathcal{M} \models \varphi(a, b)$$

and prove

$$\mathcal{N} \models \exists y \varphi(a, y) \Leftrightarrow \mathcal{M} \models \exists y \varphi(a, y).$$

The implication  $\Leftarrow$  follows immediately from the induction hypothesis. For the converse note that

$$\begin{aligned} \mathcal{N} \models \exists y \varphi(a, y) &\Rightarrow \text{for every } \varphi' > \varphi \text{ there is an } b \in M^{|y|} \text{ such that } \mathcal{N} \models \varphi'(a, b) \\ &\Rightarrow \mathcal{M} \\ &\Rightarrow \mathcal{M} \models \exists y \varphi'(a, y) \text{ for every } \varphi' > \varphi \\ &\Rightarrow \mathcal{M} \models \exists y \varphi(a, y) \end{aligned}$$

Now, assuming (#) we prove

$$\mathcal{N} \models \forall y \varphi(a, y) \Leftrightarrow \mathcal{M} \models \forall y \varphi(a, y).$$

One implication,  $\Rightarrow$ , is again immediate. For the converse note that by Lemma 17

$$\begin{aligned} \mathcal{N} \not\models \forall y \varphi(a, y) &\Rightarrow \mathcal{N} \models \exists y \neg \varphi(a, y) \\ &\Rightarrow \mathcal{N} \models \neg \varphi(a, b) \text{ for some } b \in M^{|y|} \\ &\Rightarrow \mathcal{M} \\ &\Rightarrow \mathcal{M} \not\models \forall y \varphi(a, y) \end{aligned}$$

□

Finally, induction for the boolean connectives and the quantifiers of sort  $R$  is immediate.

## 7. COMPLETENESS

(Work in progress)

For  $a, b \in U$  we write  $a \sim b$  if  $t(a) = t(b)$  for every term  $t(x)$  with parameters in  $U$  of sort  $M \rightarrow R$ .

**19 Fact.** If  $\bar{a} = \langle a_i : i < \lambda \rangle$  and  $\bar{b} = \langle b_i : i < \lambda \rangle$  are such that  $a_i \sim b_i$  for every  $i < \lambda$  then  $t(\bar{a}) = t(\bar{b})$  for every term  $t(x)$  with parameters in  $U$  of sort  $M^\lambda \rightarrow R$ .

*Proof.* By induction on  $\lambda$ . Assume the fact and let  $a_\lambda \sim b_\lambda$ . Then  $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$ . Then the fact holds with  $\lambda + 1$  for  $\lambda$ . For limit ordinals induction is immediate. □

**20 Example (???).** Let  $L$  be the language of  $\mathbb{R}$ -algebras expanded with two lattice operators  $\wedge, \vee$ . Let  $\langle \Omega, \mathcal{B}, \text{Pr} \rangle$  be a probability space. Let  $M$  be the set the simple real valued random variables with the natural interpretation of the symbols in  $L$ . Let  $R = \mathbb{R}$ . Assume  $L'$  extends  $L$  with all the continuous functions  $\mathbb{R}^n \rightarrow \mathbb{R}$  and for every  $n \in \mathbb{N}$  the functions  $E_n$  that gives the expected value of  $(X \wedge n) \vee -n$  for  $n \in \mathbb{N}$ . Note that the cut-off enures that the range of these functions is bounded.

The relation  $a \sim b$  holds in these there cases

We say that  $a \in U$  is **definable in the limit** over  $M$  if  $a \equiv_M^{\mathbb{L}} x \rightarrow a \sim x$

**21 Example.** Assume that  $R = \mathbb{R}$  and that  $L'$  contains a function of sort  $M^2 \rightarrow R$  that is interpreted in a pseudometric. Assume that all terms of sort  $M^n \rightarrow R$  are continuous with respect to this pseudometric. It is easy to see that  $a \sim b$  if and only if  $d(a, b) = 0$ . We claim that the following are equivalent

1.  $a \in U$  is definable in the limit over  $M$ , a model;
2. there is a sequence  $\langle a_i : i \in \omega \rangle$  of elements of  $M$  that converges to  $a$ .

*Proof.* (2  $\Rightarrow$  1) Let  $\langle \varepsilon_i : i \in \omega \rangle$  be a sequence of reals that converges to 0 and such that  $d(a_i, a) \leq \varepsilon_i$  for every  $i \in \omega$ . Then  $d(a_i, b) \leq \varepsilon_i$  for every  $b \equiv_M a$ . By the uniqueness of the limit  $d(a, b) = 0$ .

(1  $\Rightarrow$  2) Assume that  $a \equiv_M x \rightarrow a \sim x$ . Then  $a \equiv_M^{\mathbb{L}} x \rightarrow d(a, x) < 1/n$  for every  $n > 0$ . By compactness there is a formula  $\varphi_n(x) \in \text{tp}_{\mathbb{L}}(a/M)$  such that  $\varphi_n(x) \rightarrow d(a, x) < 1/n$ . By  $\mathbb{L}$ -elementarity there is an  $a_n \in M$  such that  $\varphi_n(a_n)$ . As  $d(a, a_n) < 1/n$ , the sequence  $\langle a_n : n \in \omega \rangle$  converges to  $a$ .  $\square$