CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

C. L. C. L. POLYMATH

1. A class of structures

As a motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted struture. Ideally, we would like to have that an elementary extension of a normed space mantain the usal notion of real numbers[1] but this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L'-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

i. $R^n \to R$;

ii. $M^n \to M$;

iii. $M^n \to R$.

The interpretation of symbols of sort $R^n \to R$ is required to be continuous, the interpretation of symbols of sort $M^n \to R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated in the same class.

[1] Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ should remain $\mathbb{R}$ throughout.

For convenience we assume that the functions of sort $M^n \to M$ and the relation of sort M^n are all in L. It is also convenient to assume that L' contains names for all continuous bounded functions $R^n \to R$.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

i. formulas of the form $t(x;y) \in C$, where $C \subseteq R^n$ is compact[2] and $t(x,y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge, \vee$; the quantifiers $\forall, \exists$ of sort M; and the quantifier $\forall^C$, by which we mean a universal quantifier restricted to the compact set $C \subseteq R^m$.

2. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}$ (free variables are hidden) we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$.

Note that this is a dense (pre)order of $\mathbb{L}$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$. Vice versa, if φ' for every $\varphi' > \varphi$, then φ.

The lemma below is required for the proof of Łǒś Theorem in the next section. But first a preliminary fact.

3 Fact. Assume the following data

- $t(x;y)$, an n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;
- $\langle a_i : i \in I \rangle$, a sequence of elements $a_i \in M^{|x|}$;
- $U \subseteq R^n$ an open set.

Let F be an ultrafilter on I. Assume that for some $\alpha \in R^{|y|}$

$$\{i \in I : t^{\mathcal{M}}(a_i;\alpha) \in U\} \quad \in \quad F.$$

Then there is an open $V \ni \alpha$ such that for every $\alpha' \in V$

$$\{i \in I : t^{\mathcal{M}}(a_i;\alpha') \in U\} \quad \in \quad F.$$

[2] We confuse the relation symbols in L' of sort R^m with their interpretation and write $t \in C$ for $C(t)$

Proof. By induction on the syntax. If t is a function symbol f then x or y do not occur in t. If x does not occur in t the claim holds because $f^{\mathcal{M}} : R^{|y|} \to R$ is continuous. If y does not occur in t the claim is trivial. Finally, induction is clear by the continuity of the functions of sort $R^n \to R$. □

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

4 **Lemma.** Assume the following data

- $\varphi(x; y) \in \mathbb{L}$;
- $\langle a_i : i \in I \rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I \rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Assssume that

$$\{i \in I : \mathcal{M} \vDash \varphi(a_i; \alpha)\} \quad \in \quad F.$$

Then for every $\varphi' > \varphi$

$$\{i \in I : \mathcal{M} \vDash \varphi'(a_i; \alpha_i)\} \quad \in \quad F.$$

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume that $\{i : \mathcal{M} \vDash t(a_i; \alpha) \in C\} \in F$. Then $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \in C$. Let D be a compact neighborhood of C. By Fact 3, there is an open $V \ni \alpha$ such that $\{i : \mathcal{M} \vDash t(a_i; \alpha') \in D\} \in F$ for every $\alpha' \in V$. Then $\{i : \mathcal{M} \vDash t'(a_i; \alpha_i) \in D\} \in F$ follows and proves the base case of the induction.

Induction clear for the connectives $\vee$, $\wedge$ and for the existential quantifier of sort M. To deal with the universal quantifier of sort M we assume inductively that for every $\varphi' > \varphi$

$$\Big\{i : \mathcal{M} \vDash \varphi(a_i, b_i; \alpha)\Big\} \in F \;\;\Rightarrow\;\; \Big\{i : \mathcal{M} \vDash \varphi'(a_i, b_i; \alpha_i)\Big\} \in F.$$

We prove

1. $\{i : \mathcal{M} \vDash \forall y\, \varphi(a_i, y; \alpha)\} \in F \;\;\Rightarrow\;\; \Big\{i : \mathcal{M} \vDash \forall y\, \varphi'(a_i, y; \alpha_i)\Big\} \in F$

Negate the r.h.s. of the implication. Then there is a sequence $\langle b_i : i \in I \rangle$ such that (using that F is an ultrafilter)

$$\big\{i : \mathcal{M} \nvDash \varphi'(a_i, b_i\,; \alpha_i)\big\} \in F$$

Assume the l.h.s. of (1) and reason for a contradiction. Then $\big\{i : \mathcal{M} \vDash \varphi(a_i, b_i\,; \alpha)\big\} \in F$ and, by induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(a_i, b_i\,; \alpha_i)\big\} \in F$. A contradiction that proves (1).

Induction for the quantifier $\forall^B$ is virtually identical, but we repeat the argument for convenience. Assume inductively that for every $\varphi' > \varphi$ and every sequence $\langle \beta_i : i \in I\rangle$ in B such that $\beta = F\text{-}\lim_i \beta_i$

$$\big\{i : \mathcal{M} \vDash \varphi(a_i\,; \alpha, \beta)\big\} \in F \;\;\Rightarrow\;\; \big\{i : \mathcal{M} \vDash \varphi'(a_i\,; \alpha_i, \beta_i)\big\} \in F.$$

We prove

2. $\{i : \mathcal{M} \vDash \forall^C y\, \varphi(a_i, ; \alpha, y)\} \in F \;\;\Rightarrow\;\; \big\{i : \mathcal{M} \vDash \forall^C y\, \varphi'(a_i\,; \alpha_i, y)\big\} \in F$

Negate the r.h.s. of the implication. Then there is a sequence $\langle \beta_i : i \in I\rangle$ such that (again using that F is an ultrafilter)

$$\big\{i : \mathcal{M} \nvDash \varphi'(a_i\,; \alpha_i, \beta_i)\big\} \in F$$

As B is compact, $F\text{-}\lim_i \beta_i$ exists, say it equals $\beta \in B$. Assume the l.h.s. of the implication and reason for a contradiction. Then $\big\{i : \mathcal{M} \vDash \varphi(a_i\,; \alpha, \beta)\big\} \in F$ and, by induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(a_i\,; \alpha_i, \beta_i)\big\} \in F$. A contradiction that proves (2) and, with it, the lemma. □

3. Ultraproducts

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I\rangle$. We require that for each function symbol f of sort $M^n \to R$ there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two semplifications: (1) we only consider ultratpowers; (2) we do not consider formulas in L containing equality, so we need not take any quotient. The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

5 **Definition.** We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \to M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i\big) : i \in I\rangle$.
3. The interpretation of functions of sort $R^n \to R$ remains unchanged.
4. If f is a function of sort $M^n \to R$ then

$$f^{\mathcal{N}}\big(\hat{a}\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i\big).$$

5. If r is a relation symbol of sort M^n then

$$\mathcal{N} \models r(\hat{a}) \Leftrightarrow \big\{i \in I \ : \ \mathcal{M} \models r(\hat{a}i)\big\} \in F.$$

6. The interpretation of functions of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the uniformity assumption above.

6 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

7 Fact. If $t(x)$ is a term of type $M^{|x|} \to M$ then

$$t^{\mathcal{N}}\big(\hat{a}\big) \quad = \quad \langle t^{\mathcal{M}}\big(\hat{a}i\big) \ : \ i \in I\rangle. \qquad \Box$$

We also have that

8 Fact. If $t(x; y)$ has sort $M^{|x|} \times R^{|y|} \to R$ then

$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i; \alpha\big).$$

Proof. By induction. If t is a function symbol then x or z do not occor in t. If x does not occur in t the claim is trivial. If y does not occur in t the claim holds by definition. Finally, induction is is clear by the continuity of the functions of sort $R^n \to R$. $\Box$

Finally, we prove

9 Proposition (Łŏś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \ \Leftrightarrow \ \big\{i \in I \ : \ \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x; y) \in L$ then the theorem reduces to the classical Łŏś Theorem. Then, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F-limit, $\big\{i \ : \ \mathcal{M} \models t(\hat{a}i, \alpha) \in D\big\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. By local compactness of R there is a compact neighborhood of C, such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$. By Fact 8 and the definition of F-limit, $\big\{i \ : \ \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\big\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}\,; \alpha) \;\Leftrightarrow\; \big\{ i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,; \alpha) \big\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

1. $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y\,; \alpha) \;\Leftrightarrow\; \big\{ i \in I \;:\; \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y\,; \alpha) \big\} \in F$ for every $\varphi' > \varphi$.

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b}; \alpha)$ for some $\hat{b}$. By induction hypothesis, $\big\{ i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{ i : \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y; \alpha) \big\} \notin F$ as required.

($\Rightarrow$) Assume that $\big\{ i : \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y; \alpha) \big\} \in F$. Choose $\hat{b}$ such that $\big\{ i : \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \in F$. If for a contradiction $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y; \alpha)$, then in particular $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\big\{ i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \in F$ a contradiction. This completes the proof of (1).

Now we prove (the argument, which we repeat for convinience, is dual to the one above)

2. $\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y\,; \alpha) \;\Leftrightarrow\; \big\{ i \in I \;:\; \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y\,; \alpha) \big\} \in F$ for every $\varphi' > \varphi$.

($\Rightarrow$) Assume $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}; \alpha)$ for some $\hat{b}$. By induction hypothesis, $\big\{ i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \in F$ for every $\varphi' > \varphi$. A fortiori $\big\{ i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y\,; \alpha) \big\} \in F$ as required.

($\Leftarrow$) Assume that $\big\{ i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y; \alpha) \big\} \in F$ for every $\varphi' > \varphi$. Choose some $\hat{b}$ such that $\big\{ i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \in F$. Suppose for a contradiction that $\mathcal{N} \nvDash \exists y\, \varphi(\hat{a}, y\,; \alpha)$. Then, in particular, $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b}\,; \alpha)$. By induction hypothesis, $\big\{ i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,; \alpha) \big\} \notin F$ for some $\varphi' > \varphi$, a contradiction. This completes the proof of (2).

Finally, to deal with the quantifier $\forall^C$ we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}; \alpha, \beta) \;\Leftrightarrow\; \big\{ i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i; \alpha, \beta) \big\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

3. $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}; \alpha, y) \;\Leftrightarrow\; \big\{ i \in I \;:\; \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y) \big\} \in F$ for every $\varphi' > \varphi$.

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}; \alpha, \beta)$ for some β. By induction hypothesis, $\big\{ i : \mathcal{M} \vDash \varphi'(\hat{a}i; \alpha, \beta) \big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{ i : \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y) \big\} \notin F$ as required.

($\Rightarrow$) Assume that $\big\{ i : \mathcal{M} \nvDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y) \big\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\big\{ i : \mathcal{M} \nvDash \varphi'(\hat{a}i; \alpha, \beta_i) \big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}; \alpha, y)$, then $\mathcal{N} \vDash \varphi(\hat{a}; \alpha, \beta)$ and, by induction hypothesis, $\big\{ i : \mathcal{M} \vDash \varphi''(\hat{a}i; \alpha, \beta) \big\} \in F$ for every $\varphi'' > \varphi$. Fix φ'' such that $\varphi' > \varphi'' > \varphi$ and obtain a contradiction from Lemma 4. This completes the proof of (3) and, with it, of the theorem. □

4. $\mathbb{L}$-ELEMENTRARITY

Let $\mathcal{M} = \langle M, R \rangle$ and $\mathcal{N} = \langle N, R \rangle$ be two structures. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if

$$\mathcal{M} \vDash \varphi(a) \;\Rightarrow\; \mathcal{N} \vDash \varphi(a) \qquad \text{for every } \varphi(x) \in \mathbb{L} \text{ and every } a \in (\operatorname{dom} f)^{|x|}$$

The following fact is proved as in the classical case using that $\mathcal{M} \vDash \varphi'$ for every $\varphi' > \varphi$ if and only if $\mathcal{M} \vDash \varphi$.

10 Fact. If $\mathcal{N}$ is an ultrapower of $\mathcal{M}$ then there is an $\mathbb{L}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

For $A \subseteq M \cap N$, we say that $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{L}$-(elementary) equivalent over A and write $\mathcal{M} \equiv^{\mathbb{L}}_{A} \mathcal{N}$ if $\mathrm{id}_A : M \to N$ is $\mathbb{L}$-elementary. We write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \equiv^{\mathbb{L}}_{M} \mathcal{N}$. In words, we say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$.

The following is proved as in the classical case.

11 Proposition (Tarski-Vaught Test). The following are equivalent

1. $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$
2. M is a subset of N and for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\mathcal{N} \vDash \exists x\, \varphi(x) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(a) \;\text{ for some } a \in M \qquad \square$$

5. $\mathbb{L}$-SATURATED STRUCTURES

Let $p(x) \subseteq \mathbb{L}(M)$. We say that $p(x)$ is finitely satisfied in $\mathcal{M}$ if $\mathcal{M} \vDash \exists x\, \varphi'(x)$ for every $\varphi' > \varphi$ where $\varphi(x)$ is any conjunction of formulas in $p(x)$.

The following definition is standard.

12 Definition. We say that $\mathcal{M}$ is $\mathbb{L}$-saturated if for every $p(x)$ as in 1 and 2 below, $\mathcal{M} \vDash \exists x\, p(a)$.

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is finitely satisfied in $\mathcal{M}$.

Then proof that every model embeds $\mathbb{L}$-elementarily in an $\mathbb{L}$-saturated one proceeds as in the classical case. First, note that the classical ultrapower construction yieds the the following lemma.

13 Lemma. Every model $\mathcal{M}$ embeds $\mathbb{L}$-elementarily in a model $\mathcal{N}$ that realizes all types as in the definition above.

Finally, using a sufficiently long $\mathbb{L}$-elementary chain we obtain the required saturated extension.

14 Corollary. Every model $\mathcal{M}$ is an $\mathbb{L}$-elementary subtructure of some saturated model $\mathcal{N}$ (possibly of inaccessible cardinality).

We denote by $\mathcal{U}$ some large $\mathbb{L}$-saturated structure which we call the monster model. The unit ball of $\mathcal{U}$ is denoted by U. The cardinality of $\mathcal{U}$ is an inaccessible cardinal that we denote by κ. Below we say model for $\mathbb{L}$-elementariy substructure of $\mathcal{U}$.