CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

C. L. C. L. POLYMATH

1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted struture. Ideally, we would like that elementary extensions of normed spaces mantain the usal notion of real numbers[1]. Unfortunately, this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L'-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

i. $R^n \to R$;

ii. $M^n \to M$;

iii. $M^n \to R$.

The interpretation of symbols of sort $R^n \to R$ is required to be continuous, the interpretation of symbols of sort $M^n \to R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated in the same class.

[1] Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ (or, in general R) should remain $\mathbb{R}$ throughout.

For convenience we assume that the functions of sort $M^n \to M$ and the relations of sort M^n are all in L. It is also convenient to assume that L' contains names for all continuous bounded functions $R^n \to R$.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact[2] and $t(x, y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall$, $\exists$ of sort M; and the quantifiers $\forall^C$, $\exists^C$, by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

2. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C, by the normality of R..

Note that $<$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$. In fact, we have the following.

3 Proposition (fake theorem). For every $\varphi(x) \in \mathbb{L}(M)$

$$\varphi(x) \;\leftrightarrow\; \bigwedge_{\varphi' > \varphi} \varphi'(x)$$

Proof. The claim is clear for atomic formulas. Induction for conjunction and the universal quantifiers is immediate.

Consider disjunction. Let $\varphi = \varphi_i \vee \varphi_1$. Assume inductively

$$\varphi_i(x) \;\leftrightarrow\; \bigwedge_{\varphi_i' > \varphi_i} \varphi_i'(x)$$

Then

$$\varphi(x) \;\leftrightarrow\; \bigwedge_{\varphi_1' > \varphi_1} \varphi_1'(x) \;\vee\; \bigwedge_{\varphi_2' > \varphi_2} \varphi_2'(x)$$

[2]We confuse the relation symbols in L' of sort R^m with their interpretation and write $t \in C$ for $C(t)$

$$\leftrightarrow \bigwedge_{\substack{\varphi_1'>\varphi_1\\ \varphi_2'>\varphi_2}} \varphi_1'(x) \vee \varphi_2'(x)$$

$$\leftrightarrow \bigwedge_{\varphi'>\varphi} \varphi'(x)$$

Now consider the existential quantifiers of sort M.

□

4 Lemma. (R compact) For every $\varphi' > \varphi$ there is a formula $\psi \in \mathbb{L}(M)$ such that $\varphi \to \neg\psi \to \varphi'$.

Proof. By induction. If $\varphi \in L$ the claim is obious. Suppose φ is of the form $t \in C$. Then φ' is $t \in D$ for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq D$. Then $\psi = (t \in R \smallsetminus O)$ is as required.

Induction is easy (conjuctions and disjuctions in φ occur swapped in ψ; similarly universal and existential quantifiers). □

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The lemma below is required for the proof of Łǒś Theorem in the next section. But first a preliminary fact.

5 Fact. Assume the following data

- $t(x; y)$, a term of sort $M^{|x|} \times R^{|y|} \to R$;
- $\langle a_i : i \in I\rangle$, a sequence of elements $a_i \in M^{|x|}$;
- $\langle \alpha_i : i \in I\rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Then

$$F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i) \tag{\#}$$

Proof. By induction on the nesting of functions of sort $R^n \to R$. In the basis case t is either a term of sort $M^{|x|} \to R$ or a variable of sort R. In both cases (#) is trivial. Assume inductively that the fact holds for the n-tuple $t(x, y)$. By the continuity of the functions of sort $R^n \to R$ and the induction hypothesis, (#) holds for $f\big(t(x,y)\big)$. The step fron terms tu tule of term is clear becouse the topology on R^n is the product topology. □

6 Lemma. Assume the following data

- $\varphi(x;y) \in \mathbb{L}$;
- $\langle a_i : i \in I\rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I\rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Then the following are equivalent

1 $\quad \{i \in I : \mathcal{M} \vDash \varphi(a_i;\alpha)\} \ \in \ F$;

2 $\quad \{i \in I : \mathcal{M} \vDash \varphi'(a_i\,;\alpha_i)\} \ \in \ F$ for every $\varphi' > \varphi$.

Proof. By induction on the syntax. When $\varphi(x;y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x;y)$ is as in (i) of Definition 2, say it is the formula $t(x;y) \in C$.

Assume $\{i : t^{\mathcal{M}}(a_i;\alpha) \in C\} \in F$. Then $F\text{-}\lim_i t^{\mathcal{M}}(a_i;\alpha) \in C$ by normality of R. By Fact 5, $F\text{-}\lim_i t^{\mathcal{M}}(a_i;\alpha_i) \in C$. Then $\{i : t^{\mathcal{M}}(a_i\,;\alpha_i) \in D\} \in F$ for every D that is a compact neighborhood of C.

Vice versa, assime $\{i : t^{\mathcal{M}}(a_i\,;\alpha_i) \in D\} \in F$ for every D that is a compact neighborhood of C. Then, again by normality, $F\text{-}\lim_i t^{\mathcal{M}}(a_i;\alpha_i) \in C$, and (1) follows by Fact 5.

This completes the proof of the base case of the induction.

Induction clear for the connectives $\vee$, $\wedge$. To deal with the universal quantifier of sort M we assume inductively that

$$\big\{i : \mathcal{M} \vDash \varphi(a_i, b_i\,;\alpha)\big\} \in F \ \Leftrightarrow\ \big\{i : \mathcal{M} \vDash \varphi'(a_i, b_i\,;\alpha_i)\big\} \in F \quad \text{for every } \varphi' > \varphi.$$

We prove

1. $\quad \{i : \mathcal{M} \vDash \forall y\, \varphi(a_i, y\,;\alpha)\} \in F \ \Leftrightarrow\ \big\{i : \mathcal{M} \vDash \forall y\, \varphi'(a_i, y\,;\alpha_i)\big\} \in F \quad$ for every $\varphi' > \varphi$.

Negate the r.h.s. of the implication. Then there are a formula $\varphi' < \varphi$ and a sequence $\langle b_i : i \in I\rangle$ such that (using that F is an ultrafilter)

$$\big\{i : \mathcal{M} \nvDash \varphi'(a_i, b_i\,;\alpha_i)\big\} \in F$$

Assume the l.h.s. of (1) and reason for a contradiction. Then $\big\{i : \mathcal{M} \vDash \varphi(a_i, b_i\,;\alpha)\big\} \in F$ and, by induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(a_i, b_i\,;\alpha_i)\big\} \in F$. A contradiction that proves the implication ($\Rightarrow$) in (1). The converse implication is immediate.

Induction for the existential quantifier of sort M is similar (dual) to the proof for the universal quantifier.

Induction for the quantifier $\forall^C$ is also virtually identical, but we repeat the argument for convenience. Assume inductively that for every $\varphi' > \varphi$ and every sequence $\langle \beta_i : i \in I \rangle$ in C such that $\beta = F\text{-}\lim_i \beta_i$

$$\big\{ i : \mathcal{M} \vDash \varphi(a_i\,; \alpha, \beta) \big\} \in F \;\;\Leftrightarrow\;\; \big\{ i : \mathcal{M} \vDash \varphi'(a_i\,; \alpha_i, \beta_i) \big\} \in F.$$

We prove

2. $$\{ i : \mathcal{M} \vDash \forall^C y\, \varphi(a_i,; \alpha, y) \} \in F \;\;\Leftrightarrow\;\; \big\{ i : \mathcal{M} \vDash \forall^C y\, \varphi'(a_i\,; \alpha_i, y) \big\} \in F$$

Negate the r.h.s. of the implication. Then there is a sequence $\langle \beta_i : i \in I \rangle$ such that (again using that F is an ultrafilter)

$$\big\{ i : \mathcal{M} \nvDash \varphi'(a_i\,; \alpha_i, \beta_i) \big\} \in F$$

As C is compact, $F\text{-}\lim_i \beta_i$ exists, say it equals $\beta \in C$. Assume the l.h.s. of the implication and reason for a contradiction. Then $\big\{ i : \mathcal{M} \vDash \varphi(a_i\,; \alpha, \beta) \big\} \in F$ and, by induction hypothesis, $\big\{ i : \mathcal{M} \vDash \varphi'(a_i\,; \alpha_i, \beta_i) \big\} \in F$. A contradiction that proves the implication ($\Rightarrow$) in (2) . The converse implication is immediate.

Induction for the quantifier $\exists^C$ is proved by the dual version of argument above. □

3. Ultraproducts

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that for each function symbol f of sort $M^n \to R$

there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two semplifications: (1) we only consider ultratpowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F. The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R \rangle$ be an L-structure.

7 **Definition.** We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \to M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i\big) : i \in I \rangle$.
3. The interpretation of functions of sort $R^n \to R$ remains unchanged.
4. If f is a function of sort $M^n \to R$ then

$$f^{\mathcal{N}}\big(\hat{a}\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i\big).$$

5. If r is a relation symbol of sort M^n then
$$\mathcal{N} \vDash r(\hat{a}) \Leftrightarrow \big\{ i \in I \ : \ \mathcal{M} \vDash r(\hat{a}i) \big\} \in F.$$

6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultraproducts, it would not hold without the uniformity requirement (#) above.

8 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

9 Fact. If $t(x)$ is a term of type $M^{|x|} \to M$ then
$$t^{\mathcal{N}}\big(\hat{a}\big) \quad = \quad \langle t^{\mathcal{M}}\big(\hat{a}i\big) \ : \ i \in I \rangle. \qquad \square$$

We also have that

10 Fact. If $t(x; y)$ has sort $M^{|x|} \times R^{|y|} \to R$ then
$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i; \alpha\big).$$

Proof. By induction. If t is a function symbol then x or z do not occor in t. If x does not occur in t the claim is trivial. If y does not occur in t the claim holds by definition. Finally, induction is is clear by the continuity of the functions of sort $R^n \to R$. $\square$

Finally, we prove

11 Proposition (Łŏś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$
$$\mathcal{N} \vDash \varphi(\hat{a}; \alpha) \quad \Leftrightarrow \quad \big\{ i \in I \ : \ \mathcal{M} \vDash \varphi'(\hat{a}i; \alpha) \big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x; y) \in L$ then the theorem reduces to the classical Łŏś Theorem. Then, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume $\mathcal{N} \vDash t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F-limit, $\big\{ i \ : \ \mathcal{M} \vDash t(\hat{a}i, \alpha) \in D \big\} \in F$. Vice versa, assume $\mathcal{N} \vDash t(\hat{a}, \alpha) \notin C$. By local compactness of R there is a compact neighborhood of C, such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$. By Fact 10 and the definition of F-limit, $\big\{ i \ : \ \mathcal{M} \vDash t(\hat{a}i, \alpha) \notin D \big\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}\,; \alpha) \;\;\Leftrightarrow\;\; \big\{ i \in I \;\;:\;\; \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,; \alpha) \big\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

1. $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y\,; \alpha) \;\;\Leftrightarrow\;\; \big\{ i \in I \;\;:\;\; \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y\,; \alpha) \big\} \in F$ for every $\varphi' > \varphi$.

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b}; \alpha)$ for some $\hat{b}$. By induction hypothesis, $\big\{ i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{ i \,:\, \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y; \alpha) \big\} \notin F$ as required.

($\Rightarrow$) Assume that $\big\{ i \,:\, \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y; \alpha) \big\} \in F$. Choose $\hat{b}$ such that $\big\{ i \,:\, \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \in F$. If for a contradiction $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y; \alpha)$, then in particular $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\big\{ i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha) \big\} \in F$ a contradiction. This completes the proof of (1).

Induction for the existential quantifier of sort M is similar (dual) to the proof above.

Finally, to deal with the quantifier $\forall^C$ we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}; \alpha, \beta) \;\;\Leftrightarrow\;\; \big\{ i \in I \;\;:\;\; \mathcal{M} \vDash \varphi'(\hat{a}i; \alpha, \beta) \big\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

3. $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}; \alpha, y) \;\;\Leftrightarrow\;\; \big\{ i \in I \;\;:\;\; \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y) \big\} \in F$ for every $\varphi' > \varphi$.

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}; \alpha, \beta)$ for some β. By induction hypothesis, $\big\{ i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i; \alpha, \beta) \big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{ i \,:\, \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y) \big\} \notin F$ as required.

($\Rightarrow$) Assume that $\big\{ i \,:\, \mathcal{M} \nvDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y) \big\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\big\{ i \,:\, \mathcal{M} \nvDash \varphi'(\hat{a}i; \alpha, \beta_i) \big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}; \alpha, y)$, then $\mathcal{N} \vDash \varphi(\hat{a}; \alpha, \beta)$ and, by induction hypothesis, $\big\{ i \,:\, \mathcal{M} \vDash \varphi''(\hat{a}i; \alpha, \beta) \big\} \in F$ for every $\varphi'' > \varphi$. Fix φ'' such that $\varphi' > \varphi'' > \varphi$ and obtain a contradiction from Lemma 6. This completes the proof of (3).

Induction for the quantifier $\exists^C$ is again dual to the proof above. $\square$

4. $\mathbb{L}$-Elementrarity

Let $\mathcal{M} = \langle M, R \rangle$ and $\mathcal{N} = \langle N, R \rangle$ be two structures. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if

$$\mathcal{M} \vDash \varphi(a) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(a) \qquad\qquad \text{for every } \varphi(x) \in \mathbb{L} \text{ and every } a \in (\operatorname{dom} f)^{|x|}$$

The following fact is proved as in the classical case using that $\mathcal{M} \vDash \varphi$ holds if and only if $\mathcal{M} \vDash \varphi'$ holds for every $\varphi' > \varphi$.

12 Fact. If $\mathcal{N}$ is an ultrapower of $\mathcal{M}$ then there is an $\mathbb{L}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

For $A \subseteq M \cap N$, we say that $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{L}$-(elementary) equivalent over A and write $\mathcal{M} \equiv^{\mathbb{L}}_A \mathcal{N}$ if $\mathrm{id}_A : M \to N$ is $\mathbb{L}$-elementary. We write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \equiv^{\mathbb{L}}_M \mathcal{N}$. In words, we say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$.

5. $\mathbb{L}$-COMPACTNESS

A theory $T \subseteq \mathbb{L}$ be a weakly finitely consistent if for every φ, conjunction of sentences in T, any $\varphi' > \varphi$ is consistent. The following proposition follows from Łǒś Theorem by the usual argument.

13 **Proposition (Compactness Theorem).** Let $T \subseteq \mathbb{L}$ be an weakly finitely consistent theory. Then T is consistent. □

Let $p(x) \subseteq \mathbb{L}(M)$. We say that $p(x)$ is weakly finitely satisfied in $\mathcal{M}$ if $\mathcal{M} \vDash \exists x\, \varphi'(x)$ for every $\varphi' > \varphi$ where $\varphi(x)$ is some conjunction of formulas in $p(x)$.

14 **Definition.** We say that $\mathcal{M}$ is $\mathbb{L}$-saturated if for every $p(x)$ as in 1 and 2 below, $\mathcal{M} \vDash \exists x\, p(a)$.

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is weakly finitely satisfied in $\mathcal{M}$.

The existence of $\mathbb{L}$-saturated models is proved as in the classical case.

15 **Proposition.** Every model has an $\mathbb{L}$-elementary extension to a saturated model (possibly of inaccessible cardinality). □

We denote by $\mathcal{U} = \langle U, R\rangle$ some large $\mathbb{L}$-saturated structure which we call the monster model. The cardinality of $\mathcal{U}$ is an inaccessible cardinal that we denote by κ. Below we say model for $\mathbb{L}$-elementariy substructure of $\mathcal{U}$.

16 **Fact.** For every $\varphi(x), \psi(x) \in \mathbb{L}(U)$ such that $\neg\exists x\, [\varphi(x) \wedge \psi(x)]$ there are $\varphi' > \varphi$ and $\psi' > \psi$ such that $\neg\exists x\, [\varphi'(x) \wedge \psi'(x)]$.

Proof. By Proposition 3 and saturation. □

6. THE TARSKI-VAUGHT $\mathbb{L}$-TEST

The following lemma is a useful observation. It has its first application below, in the proof of our version of the Tarski-Vaught test.

17 **Lemma.** (R compact) Let $A \subseteq N$. Then the following are equivalent

1 for every formula $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{N} \vDash \exists x\, \neg\varphi(x) \quad\Rightarrow\quad \text{there is an } a \in A \text{ such that } \mathcal{N} \vDash \neg\varphi(a);$$

2. for every formula $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{N} \vDash \exists x\, \varphi(x) \quad\Rightarrow\quad \text{for every } \varphi' > \varphi \text{ there is an } a \in A \text{ such that } \mathcal{N} \vDash \varphi'(a).$$

The implication 2⇒1 does not require the assumption of compactness.

Proof. (2⇒1) Assume $\mathcal{N} \models \exists x\, \neg\varphi(x)$. Pick any $c \in N$ such that $c \models \neg\varphi(x)$. Let $p(x) = \mathbb{L}\text{-tp}(c/A)$. Without loss of generality we can assume that $\mathcal{N}$ is saturated. As $p(x) \cup \{\neg\varphi(x)\}$ is inconsistent there is is a formula $\psi(x) \in p(x)$ such that $\psi'(x) \rightarrow \neg\varphi(x)$ for some $\psi' > \psi$. Then from (2) we obtain that $\mathcal{N} \models \psi'(a)$ for some $a \in A$. Hence $\mathcal{N} \models \neg\varphi(a)$ follows.

(1⇒2) follows immediately from Lemma 4. □

18 Proposition (Tarski-Vaught $\mathbb{L}$-Test). Let M be a subset of N. Then the following are equivalent

1. M is the domain of a structure $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$;
2. for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\mathcal{N} \models \exists x\, \varphi(x) \;\Rightarrow\; \text{for every } \varphi' > \varphi \text{ there is an } a \in A \text{ such that } \mathcal{N} \models \varphi'(a).$$

Proof. Assume (2) of the lemma. Then, by the classical Tarski-Vaught test $M \preceq N$. We also have that $\mathcal{M}$ is an L'-substructure of $\mathcal{N}$. Therefore, form every formula $\varphi(x)$ as in (i) and (ii) of Definition 2 we have

$$\mathcal{N} \models \varphi(a) \;\Leftrightarrow\; \mathcal{M} \models \varphi(a) \qquad\qquad \text{for every } a \in M^{|x|}.$$

Assume inductively

$$\# \qquad \mathcal{N} \models \varphi(a,b) \;\Leftrightarrow\; \mathcal{M} \models \varphi(a,b)$$

and prove

$$\mathcal{N} \models \exists y\, \varphi(a,y) \;\Leftrightarrow\; \mathcal{M} \models \exists y\, \varphi(a,y).$$

The implication ⇐ follows immediately from the induction hypothesis. For the converse note that

$$\begin{aligned}
\mathcal{N} \models \exists y\, \varphi(a,y) \;&\Rightarrow\; \text{for every } \varphi' > \varphi \text{ there is an } b \in M^{|y|} \text{ such that } \mathcal{N} \models \varphi'(a,b)\\
&\Rightarrow\; \qquad\qquad\qquad\qquad\qquad\qquad \mathcal{M}\\
&\Rightarrow\; \mathcal{M} \models \exists y\, \varphi'(a,y) \text{ for every } \varphi' > \varphi\\
&\Rightarrow\; \mathcal{M} \models \exists y\, \varphi(a,y)
\end{aligned}$$

Now, assuming (#) we prove

$$\mathcal{N} \models \forall y\, \varphi(a,y) \;\Leftrightarrow\; \mathcal{M} \models \forall y\, \varphi(a,y).$$

One implication, ⇒, is again immediate. For the converse note that by Lemma 17

$$\begin{aligned}
\mathcal{N} \not\models \forall y\, \varphi(a,y) \;&\Rightarrow\; \mathcal{N} \models \exists y\, \neg\varphi(a,y)\\
&\Rightarrow\; \mathcal{N} \models \neg\varphi(a,b) \text{ for some } b \in M^{|y|}\\
&\Rightarrow\; \mathcal{M}
\end{aligned}$$

$$\Rightarrow \quad \mathcal{M} \nvDash \forall y\, \varphi(a, y) \qquad \square$$

Finally, induction for the boolean connectives and the quantifiers of sort R is immediate.

7. Completeness

(Work in progress)

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \to R$.

19 **Fact.** If $\bar{a} = \langle a_i : i < \lambda \rangle$ and $\bar{b} = \langle b_i : i < \lambda \rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \to R$.

Proof. By induction on λ. Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ. For limit ordinals induction is immediate. $\square$

20 **Example (???).** Let L be the language of $\mathbb{R}$-algebras expanded with two lattice operators $\wedge$, $\vee$. Let $\langle \Omega, \mathcal{B}, \Pr \rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L. Let $R = \mathbb{R}$. Assume L' estends L with all the continuous functions $\mathbb{R}^n \to \mathbb{R}$ and for every $n \in \mathbb{N}$ the functions E_n that gives the expected value of $(X \wedge n) \vee -n$ for $n \in \mathbb{N}$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is definable in the limit over M if $a \equiv^{\mathbb{L}}_M x \to a \sim x$

21 **Example.** Assume that $R = \mathbb{R}$ and that L' contains a function of sort $M^2 \to R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \to R$ are continuous with rispect to this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M, a model;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of M that converges to a.

Proof. $(2 \Rightarrow 1)$ Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_M a$. By the uniqueness of the limit $d(a, b) = 0$.

$(1 \Rightarrow 2)$ Assume that $a \equiv_M x \to a \sim x$. Then $a \equiv^{\mathbb{L}}_M x \to d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \mathrm{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \to d(a, x) < 1/n$. By $\mathbb{L}$-elementarity there is an $a_n \in M$ such that $\varphi(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a. $\square$