

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number, we may formalize this with a two sorted structure. Ideally, we would like to have that an elementary extension of a normed space maintain the usual notion of real numbers¹ but this is not possible with the usual notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 Definition. Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L' -structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L -structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols of sort $M^n \times R^m \rightarrow R$ are interpreted in functions such that

- i. $f^{\mathcal{M}} : M^n \times R^m \rightarrow R$ is bounded, i.e. its range is contained in a compact set;
- ii. $f^{\mathcal{M}} : M^n \times R^m \rightarrow R$ is continuous uniformly in M , this is made precise in Definition 3.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set \mathbb{L} of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an \mathbb{L} -elementary extension that is \mathbb{L} -saturated in the same class. Our choice of \mathbb{L} is not the most general. It is a compromise that avoids excessive complications.

The function symbols of L' fall in one of the two groups below according to their sort

- 1. $M^n \times R^m \rightarrow M$;
- 2. $M^n \times R^m \rightarrow R$.

¹Non standard standard analysts have no problem in expanding \mathbb{R} . In these notes, for a change, we insist as \mathbb{R} should remain \mathbb{R} throughout.

Below, \mathbb{T}_M denotes the set of the terms obtained composing the function symbols as in 1 above.

By \mathbb{T}_R we denote the set of the terms obtained composing the function symbols as in 2 above.

By $\mathbb{T}_{R \circ M}$ we denote the set of the terms obtained composing terms in \mathbb{T}_R with terms in \mathbb{T}_M .

Clearly, there are more terms in L' ; but we will restrict to the one above.

For definiteness, we assume that L' contains symbols for *all* compact sets $C \subseteq R^m$. It also contains symbols for *all* functions $R^m \rightarrow R$ that are bounded and continuous.

For convenience we assume that the functions of sort $M^n \rightarrow M$ and the relation of sort M^n are all in L .

2 Definition. Formulas in \mathbb{L} are constructed inductively from the following two sets of formulas

- i. formulas as $t(y; z) \in C$, where $C \subseteq R^n$ is compact² and $t(y; z)$ is a tuple of terms in $\mathbb{T}_{R \circ M}$;
- ii. all formulas in L .

We require that \mathbb{L} is closed under the Boolean connectives \wedge, \vee ; the quantifiers \forall, \exists of sort M ; and the quantifier \forall^C , by which we mean a universal quantifier that ranges over the compact set $C \subseteq R^m$.

2. TOPOLOGICAL TOOLS

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . Let Y be a topological space. If $f : I \rightarrow Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ . Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

By βI we denote the set of ultrafilters on I . This is a topological space with the topology generated by the clopen sets $[A] = \{F \in \beta I : A \in F\}$ as A ranges over the subsets of I . Let X, Y be topological spaces, and assume Y is Hausdorff and compact. For any $f : I \times X \rightarrow Y$, we define

$$\begin{aligned} \bar{f} : \beta I \times X &\rightarrow Y \\ \langle F, \alpha \rangle &\mapsto F\text{-}\lim_i f(i, \alpha) \end{aligned}$$

3 Definition. Let I be a pure set. We say that $f : I \times X \rightarrow Y$ is **continuous uniformly in I** if $\bar{f} : \beta I \times X \rightarrow Y$ is continuous.

The following propositions unravels the definition above into a more practicable form.

4 Proposition. Let X and Y be as above. The following are equivalent

²We confuse the relation symbols in L' of sort R^m with their interpretation and write $x \in C$ for $C(x)$

1. $f : I \times X \rightarrow Y$ is continuous uniformly in I .
2. For every $F \in \beta I$ and $\alpha \in X$, and for every $V \subseteq Y$ open neighborhood of $\bar{f}(F, \alpha)$ there is $U \subseteq X$ open neighborhood of α and an $A \in F$ such that $f(i, x) \in V$ for every $i \in A$ and every $x \in U$.

3. HENSON-IOVINO APPROXIMATIONS


For $\varphi, \varphi' \in \mathbb{L}$ (free variables are hidden) we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C . If no such atomic formulas occurs in φ , then $\varphi > \varphi$.

Note that this is a dense (pre)order of \mathbb{L} .

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \rightarrow \varphi'$. Vice versa, if φ' for every $\varphi' > \varphi$, then φ .

The following lemma is required for the proof of Łoś Theorem in the next section.

5 Lemma. Let $\alpha = F\text{-}\lim_i \alpha_i$, where $\langle \alpha_i : i \in I \rangle$ is a sequence in R and $F \in \beta I$. Let $\hat{\alpha} : I \rightarrow M$. Let $\varphi(x; y) \in \mathbb{L}$ be such that $\{i : \mathcal{M} \models \varphi(\hat{\alpha}i; \alpha)\} \in F$. Then $\{i : \mathcal{M} \models \varphi'(\hat{\alpha}i; \alpha_i)\} \in F$ for every $\varphi' > \varphi$.

 *Proof.* By induction on the syntax. When $\varphi(x; y)$ is in L , it does not depend on α , and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$, where $t(x; y) \in \mathbb{T}_{R \circ M}$.

By Definition 1 the map $f : i, \alpha \mapsto t^{\mathcal{M}}(\hat{\alpha}i, \alpha)$ is uniformly continuous in I . Now, assume that $\{i : \mathcal{M} \models t(\hat{\alpha}i; \alpha) \in C\} \in F$. Then $\bar{f}(F, \alpha) \in C$. Let D be a compact neighborhood of C . By Proposition 4, there is a set $A \in F$ and $V \ni \alpha$ such that $f(i; \alpha') \in V$ for every $i \in A$ and $\alpha' \in V$. Then $\{i : \mathcal{M} \models \varphi'(\hat{\alpha}i; \alpha_i)\} \in F$ follows. This completes the base case of the induction.

Induction clear for the connectives \vee, \wedge and for the existential quantifier of sort M . To deal with the universal quantifier of sort M we assume inductively that for every $\varphi' > \varphi$

$$\{i : \mathcal{M} \models \varphi(\hat{\alpha}i, \hat{\beta}i; \alpha)\} \in F \Rightarrow \{i : \mathcal{M} \models \varphi'(\hat{\alpha}i, \hat{\beta}i; \alpha_i)\} \in F.$$

We prove

$$1. \quad \{i : \mathcal{M} \models \forall y \varphi(\hat{\alpha}i, y; \alpha)\} \in F \Rightarrow \{i : \mathcal{M} \models \forall y \varphi'(\hat{\alpha}i, y; \alpha_i)\} \in F$$

Negate the r.h.s. of the implication. Using that F is an ultrafilter, there is a sequence $\hat{\beta}$ such that

$$\{i : \mathcal{M} \not\models \varphi'(\hat{\alpha}i, \hat{\beta}i; \alpha_i)\} \in F$$

Assume the l.h.s. of the implication, for a contradiction. Then $\{i : \mathcal{M} \models \varphi(\hat{\alpha}i, \hat{\beta}i; \alpha)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{\alpha}i, \hat{\beta}i; \alpha_i)\} \in F$. A contradiction that proves (1).

Induction for the quantifier \forall^C is similar. Assume inductively that for every $\varphi' > \varphi$

$$\{i : \mathcal{M} \models \varphi(\hat{a}i; \alpha, \beta)\} \in F \Rightarrow \{i : \mathcal{M} \models \varphi'(\hat{a}; \alpha_i, \beta_i)\} \in F.$$

We prove

$$2. \quad \{i : \mathcal{M} \models \forall^C y \varphi(\hat{a}i; \alpha, y)\} \in F \Rightarrow \{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha_i, y)\} \in F$$

Negate the r.h.s. of the implication. Using that F is an ultrafilter, there is a sequence $\beta_i \in C$ such that

$$\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha_i, \beta_i)\} \in F$$

As C is compact, $F\text{-}\lim_i \beta_i$ exists, say it equals $\beta \in C$. Assume, for a contradiction, that the l.h.s. of the implication holds. Then $\{i : \mathcal{M} \models \varphi(\hat{a}; \alpha, \beta)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \alpha_i, \beta_i)\} \in F$. A contradiction that proves (2) and, with it, the lemma. \square

4. ULTRAPOWERS

To keep notation tidy, we do not consider formulas containing equality, so we need not take any quotient. The generalization is straightforward.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

6 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \times R^m \rightarrow M$ then $f^{\mathcal{N}}(\hat{a}; \alpha)$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle$.
4. If f is a function of sort $M^n \times R^m \rightarrow R$ then for every $\alpha \in R^m$

$$f^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \alpha).$$

3. if r is a relation symbol of sort $M^n \times R^m$ then for every $\alpha \in R^m$

$$\mathcal{N} \models r(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i; \alpha)\} \in F.$$

The following is immediate.

7 Fact. The structure \mathcal{N} satisfies Definition 1. 

The following is easily proved by induction on the syntax

8 Fact. If $t(y; z) \in \mathbb{T}_M$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = \langle t^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle. \quad \square$$

We also have that

9 Fact. If $t(y; z) \in \mathbb{T}_R$ then

$$\#_0 \quad t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Proof. By induction. The fact holds by definition if t is a function symbol. Therefore, we assume $\#_0$ as induction hypothesis and prove

$$\#_1 \quad f^{\mathcal{N}}(\hat{a}; t^{\mathcal{N}}(\hat{a}; \alpha)) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; t^{\mathcal{M}}(\hat{a}i; \alpha)).$$

By induction hypothesis

$$\#_2 \quad f^{\mathcal{N}}(\hat{a}; t^{\mathcal{N}}(\hat{a}; \alpha)) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \lambda), \quad \text{where } \lambda = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Let σ be the limit in $\#_2$. Let $V \subseteq R$ be an open set containing σ . Note that we can identify F with an ultrafilter on $M^{|I|}$ concentrated on range of \hat{a} . Apply 2 of Proposition 4, to obtain $A \in F$ and an open set $U \subseteq R$ containing λ such that for every $i \in A$ and every $x \in U$

$$f^{\mathcal{M}}(\hat{a}i; x) \in V.$$

By induction hypothesis there is $B \in F$ be such that $t^{\mathcal{M}}(\hat{a}i; \alpha) \in U$ for every $i \in B$. Therefore

$$f^{\mathcal{M}}(\hat{a}i; t^{\mathcal{M}}(\hat{a}i; \alpha)) \in V$$

for every $i \in A \cap B$. As $A \cap B \in F$, this suffices to prove that σ is also the limit in $\#_1$. □

From the two facts above it immediately follows that

10 Fact. If $t(y; z) \in \mathbb{T}_{R \circ M}$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha). \quad \square$$

Finally, we prove Łoś Theorem.

11 Theorem. Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x; y) \in L$ then the theorem reduces to the classical Łoś Theorem. Then, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$, where $t(x; y) \in \mathbb{T}_{R \circ M}$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. Let D be a compact neighborhood of C such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$. By Fact 10 and the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives \vee and \wedge is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

$$1. \quad \mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F$. Choose \hat{b} such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$. If for a contradiction $\mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha)$, then in particular $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ a contradiction. This completes the proof of (1).

Now we prove (the argument, which we repeat for convinience, is dual to the one above)

$$2. \quad \mathcal{N} \models \exists y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Rightarrow) Assume $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ for every $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F$ as required.

(\Leftarrow) Assume that $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F$ for every $\varphi' > \varphi$. Choose some \hat{b} such that $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$. Suppose for a contradiction that $\mathcal{N} \not\models \exists y \varphi(\hat{a}, y; \alpha)$. Then, in particular, $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$ for some $\varphi' > \varphi$, a contradiction. This completes the proof of (2).

Finally, to deal with the quantifier \forall^C we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

$$3. \quad \mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}; \alpha, \beta)$ for some β . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha, \beta_i)\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y)$, then $\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta)$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \alpha, \beta)\} \in F$ for every $\varphi'' > \varphi$. Fix φ'' such that $\varphi' > \varphi'' > \varphi$ and obtain a contradiction from Lemma 5 This completes the proof of (3) and, with it, of the theorem. \square