CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A class of structures

As a motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number, we may formalize this with a two sorted struture. Ideally, we would like to have that an elementary extension of a normed space mantain the usal notion of real numbers[1] but this is not possible with the usual notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L'-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols of sort $M^n \times R^m \to R$ are interpreted in functions such that

i. $f^{\mathcal{M}} : M^n \times R^m \to R$ is bounded, i.e. its range is contained in a compact set;

ii. $f^{\mathcal{M}} : M^n \times R^m \to R$ is continuous uniformly in M, this is made precise in Definition 3.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated in the same class. Our choice of $\mathbb{L}$ is not the most general. It is a compromise that avoids eccessive complications.

The function symbols of L' fall in one of four groups accortding to their sort

0. $M^n \to M$ (we assume, for simplicity, that these are also in L);

1. $M^n \times R^m \to M$ with m positive;

2. $R^m \to R$ (we assume L' contains symbols for *all* continuous bounded functions of this sort);

[1] Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ should remain $\mathbb{R}$ throughout.

3. $M^n \times R^m \to R$ with m positive.

Below, $\mathbb{T}_M$ denotes the set of the terms obtained composing function symbols as in 0 or 1 above.

By $\mathbb{T}_R$ we denote the set of the terms obtained composing function symbols as in 2 or 3 above.

By $\mathbb{T}_{R\circ M}$ we denote the set of the terms obtained composing terms in $\mathbb{T}_R$ with terms in $\mathbb{T}_M$.

Clearly, there are many more terms in L'; but we will restrict to the one above.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

i. formulas as $t(y; z) \in C$, where $C \subseteq R^n$ is compact[2] and $t(y; z)$ is a tuple of terms in $\mathbb{T}_{R\circ M}$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall$, $\exists$ of sort M; and the quantifier $\forall^C$, by which we mean a universal quanfier that ranges over the compact set $C \subseteq R^m$.

2. Topological tools

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

By βI we denote the set of ultrafilters on I. This is a topological space with the topology generated by the clopen sets $[A] = \{F \in \beta I : A \in F\}$ as A ranges over the subsets of I. Let X, Y be topological spaces, and assume Y is Hausdorff and compact. For any $f : I \times X \to Y$, we define

$$\begin{array}{rccc} \bar{f} & : & \beta I \times X & \to & Y \\ & & \langle F, \alpha \rangle & \mapsto & F\text{-}\lim\limits_i f(i, \alpha) \end{array}$$

3 Definition. Let I be a pure set. We say that $f : I \times X \to Y$ is continuous uniformly in I if $\bar{f} : \beta I \times X \to Y$ is continuous.

The following propositions unravels the definition above into a more practicable form.

4 Proposition. Let X and Y be as above. The following are equivalent

1. $f : I \times X \to Y$ is continuous uniformly in I.

[2] We confuse the relation symbols in L' of sort R^m with their interpretation and write $x \in C$ for $C(x)$

2. For every $F \in \beta I$ and $\alpha \in X$, and for every $V \subseteq Y$ open neighborhood of $\bar{f}(F,\alpha)$ there is $U \subseteq X$ open neighborhoodof α and an $A \in F$ such that $f(i,x) \in V$ for every $i \in A$ and every $x \in U$.

3. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}$ (free variables are hidden) we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$.

Note that this is a dense (pre)order of $\mathbb{L}$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \rightarrow \varphi'$. Vice versa, if φ' for every $\varphi' > \varphi$, then φ.

The following lemma is required for the proof of Łǒś Theorem in the next section.

5 Lemma. Let $\alpha = F\text{-}\lim_i \alpha_i$ for some sequence $\langle \alpha_i : i \in I\rangle$ in R and some $F \in \beta I$. Let $\hat{a} : I \rightarrow M$. Let $\varphi(x;y) \in \mathbb{L}$ be such that $\{i : \mathcal{M} \vDash \varphi(\hat{a}i;\alpha)\} \in F$. Then $\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\alpha_i)\} \in F$ for every $\varphi' > \varphi$.

⚠ *Proof.* By induction on the syntax. When $\varphi(x;y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x;y)$ is as in (i) of Definition 2, say it is the formula $t(x;y) \in C$, where $t(x;y) \in \mathbb{T}_{R \circ M}$.

By Definition 1 the map $f : i, \alpha \mapsto t^{\mathcal{M}}(\hat{a}i, \alpha)$ is uniformly continuous in I. Now, assume that $\{i : \mathcal{M} \vDash t(\hat{a}i;\alpha) \in C\} \in F$. Then $\bar{f}(F,\alpha) \in C$. Let D be a compact neighborhood of C. By Proposition 4, there is a set $A \in F$ and $V \ni \alpha$ such that $f(i;\alpha') \in V$ for every $i \in A$ and $\alpha' \in V$. Then $\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\alpha_i)\} \in F$ follows. This completes the base case of the induction.

Induction clear for the connectives $\vee$, $\wedge$ and for the existential quantifier of sort M. To deal with the universal quantifier of sort M we assume inductively that for every $\varphi' > \varphi$

$$\left\{i : \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i\,;\alpha)\right\} \in F \;\Rightarrow\; \left\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,;\alpha_i)\right\} \in F.$$

We prove

1. $\{i : \mathcal{M} \vDash \forall y\, \varphi(\hat{a}i, y\,;\alpha)\} \in F \;\Rightarrow\; \left\{i : \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y\,;\alpha_i)\right\} \in F$

Negate the r.h.s. of the implication. Using that F is an ultrafilter, there is a sequence $\hat{b}$ such that

$$\left\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i\,;\alpha_i)\right\} \in F$$

Assume the l.h.s. of the implication, for a contradiction. Then $\left\{i : \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i\,;\alpha)\right\} \in F$ and, by induction hypothesis, $\left\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,;\alpha_i)\right\} \in F$. A contradiction that proves (1).

Induction for the quantifier $\forall^C$ is similar. Assume inductively that for every $\varphi' > \varphi$

$$\big\{i\,:\,\mathcal{M}\vDash\varphi(\hat{a}i\,;\alpha,\beta)\big\}\in F\ \ \Rightarrow\ \ \big\{i\,:\,\mathcal{M}\vDash\varphi'(\hat{a}\,;\alpha_i,\beta_i)\big\}\in F.$$

We prove

2. $\quad \{i\,:\,\mathcal{M}\vDash\forall^C y\,\varphi(\hat{a}i,;\alpha,y)\}\in F\ \ \Rightarrow\ \ \big\{i\,:\,\mathcal{M}\vDash\forall^C y\,\varphi'(\hat{a}i\,;\alpha_i,y)\big\}\in F$

Negate the r.h.s. of the implication. Using that F is an ultrafilter, there is a sequence $\beta_i\in C$ such that

$$\big\{i\,:\,\mathcal{M}\nvDash\varphi'(\hat{a}i\,;\alpha_i,\beta_i)\big\}\in F$$

As C is compact, $F\text{-}\lim_i\beta_i$ exists, say it equals $\beta\in C$. Assume, for a contradiction, that the l.h.s. of the implication holds. Then $\big\{i\,:\,\mathcal{M}\vDash\varphi(\hat{a}\,;\alpha,\beta)\big\}\in F$ and, by induction hypothesis, $\big\{i\,:\,\mathcal{M}\vDash\varphi'(\hat{a}i\,;\alpha_i,\beta_i)\big\}\in F$. A contradiction that proves (2) and, with it, the lemma. □

4. Ultrapowers

To keep notation tidy, we do not consider formulas containing equality, so we need not take any quotient. The generalization is straightforward.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M}=\langle M,R\rangle$ be an L-structure.

6 Definition. We define a structure $\mathcal{N}=\langle N,R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $\quad N=M^I$ that is, it is the set of sequences $\hat{a}\,:\,I\to M$.

2. $\quad$ If f is a function of sort $M^n\times R^m\to M$ then $f^{\mathcal{N}}(\hat{a};\alpha)$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\,:\,i\in I\rangle$.

4. $\quad$ If f is a function of sort $M^n\times R^m\to R$ then for every $\alpha\in R^m$

$$f^{\mathcal{N}}\big(\hat{a};\alpha\big)\ =\ F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

3. $\quad$ if r is a relation symbol of sort $M^n\times R^m$ then for every $\alpha\in R^m$

$$\mathcal{N}\vDash r(\hat{a};\alpha)\Leftrightarrow\big\{i\in I\ \ :\ \ \mathcal{M}\vDash r(\hat{a}i;\alpha)\big\}\in F.$$

The following is immediate.

7 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax

8 Fact. If $t(y;z)\in\mathbb{T}_M$ then

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big)\ \ =\ \ \langle t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\,:\,i\in I\rangle.$$ □

We also have that

9 Fact. If $t(y;z)\in\mathbb{T}_R$ then

$$\#_0 \qquad t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

Proof. By induction. The fact holds by definition if t is a function symbol. Therefore, we assume $\#_0$ as induction hypothesis and prove

$$\#_1 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\Big).$$

By induction hypothesis

$$\#_2 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;\lambda\Big), \qquad \text{where } \lambda = F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

Let σ be the limit in $\#_2$. Let $V \subseteq R$ be an open set containing σ. Note that we can identify F with an ultrafilter on $M^{|\hat{a}|}$ concentrated on range of $\hat{a}$. Apply 2 of Proposition 4, to obtain $A \in F$ and an open set $U \subseteq R$ containing λ such that for every $i \in A$ and every $x \in U$

$$f^{\mathcal{M}}\big(\hat{a}i;x\big) \;\in\; V.$$

By induction hypothesis there is $B \in F$ be such that $t^{\mathcal{M}}\big(\hat{a}i;\alpha\big) \in U$ for every $i \in B$. Therefore

$$f^{\mathcal{M}}\Big(\hat{a}i;t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\Big) \;\in\; V$$

for every $i \in A \cap B$. As $A \cap B \in F$, this sufficies to prove that σ is also the limit in $\#_1$. □

From the two facts above it immediately follows that

10 Fact. If $t(y;z) \in \mathbb{T}_{R\circ M}$ then

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$ □

Finally, we prove Łős Theorem.

11 Theorem. Let $\mathcal{N}$ be as above and let $\varphi(x;y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \vDash \varphi(\hat{a};\alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x;y) \in L$ then the theorem reduces to the classical Łős Theorem. Then, suppose that $\varphi(x;y)$ is as in (i) of Definition 2, say it is the formula $t(x;y) \in C$, where $t(x;y) \in \mathbb{T}_{R\circ M}$.

Assume $\mathcal{N} \vDash t(\hat{a},\alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a},\alpha)$. Hence, by the definition of F-limit, $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i,\alpha) \in D\big\} \in F$. Vice versa, assume $\mathcal{N} \vDash t(\hat{a},\alpha) \notin C$. Let D be a compact neighborhood of C such that $t^{\mathcal{N}}(\hat{a},\alpha) \notin D$. By Fact 10 and the definition of F-limit, $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i,\alpha) \notin D\big\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}\,; \alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,; \alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

1. $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y\,; \alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y\,; \alpha)\big\} \in F$ for every $\varphi' > \varphi$.

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b}; \alpha)$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha)\big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{i : \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y; \alpha)\big\} \notin F$ as required.

($\Rightarrow$) Assume that $\big\{i : \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y; \alpha)\big\} \in F$. Choose $\hat{b}$ such that $\big\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i; \alpha)\big\} \in F$. If for a contradiction $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y; \alpha)$, then in particular $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha)\big\} \in F$ a contradiction. This completes the proof of (1).

Now we prove (the argument, which we repeat for convinience, is dual to the one above)

2. $\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y\,; \alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y\,; \alpha)\big\} \in F$ for every $\varphi' > \varphi$.

($\Rightarrow$) Assume $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}; \alpha)$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha)\big\} \in F$ for every $\varphi' > \varphi$. A fortiori $\big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y\,; \alpha)\big\} \in F$ as required.

($\Leftarrow$) Assume that $\big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y; \alpha)\big\} \in F$ for every $\varphi' > \varphi$. Choose some $\hat{b}$ such that $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i; \alpha)\big\} \in F$. Suppose for a contradiction that $\mathcal{N} \nvDash \exists y\, \varphi(\hat{a}, y\,; \alpha)$. Then, in particular, $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b}\,; \alpha)$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,; \alpha)\big\} \notin F$ for some $\varphi' > \varphi$, a contradiction. This completes the proof of (2).

Finally, to deal with the quantifier $\forall^C$ we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}; \alpha, \beta) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i; \alpha, \beta)\big\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

3. $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}; \alpha, y) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y)\big\} \in F$ for every $\varphi' > \varphi$.

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}; \alpha, \beta)$ for some β. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i; \alpha, \beta)\big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{i : \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y)\big\} \notin F$ as required.

($\Rightarrow$) Assume that $\big\{i : \mathcal{M} \nvDash \forall^C y\, \varphi'(\hat{a}i; \alpha, y)\big\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\big\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i; \alpha, \beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}; \alpha, y)$, then $\mathcal{N} \vDash \varphi(\hat{a}; \alpha, \beta)$ and, by induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi''(\hat{a}i; \alpha, \beta)\big\} \in F$ for every $\varphi'' > \varphi$. Fix φ'' such that $\varphi' > \varphi'' > \varphi$ and obtain a contradiction from Lemma 5 This completes the proof of (3) and, with it, of the theorem. □