CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A class of structures

1 Definition. Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language.

We consider the class of L'-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a Hausdorff topology and that there is a relation symbol in L' for every compact $C \subseteq R$. We will write $x \in C$ for $C(x)$.

We require that the function symbols of sort $M^n \times R^m \to R$ are interpreted in functions such that

1. $f^{\mathcal{M}} : M^n \times R^n \to R$ is bounded, i.e. its range is contained in a compact set;
2. $f^{\mathcal{M}} : M^n \times R^n \to R$ is continuous uniformly in M, see Definition 3.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, we carve out a set of formulas $L \subseteq \mathbb{L} \subseteq L'$ such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension in the same class that is $\mathbb{L}$-saturated.

In a nutshell, formulas in $\mathbb{L}$ are required not to contain quantifiers of sort R. Moreover, function symbols in the second group below may only occur only inside compact relations and never under the scope of negation.

Function symbols are divided in two groups according to their sort

1. $M^n \times R^m \to M$
2. $M^n \times R^m \to R$

The ultrapower construction we introduce in the next section is completely classical w.r.t. functions in 1. On the other hand, functions in 2 are *critical* and need to be dealt with differently.

Below, $\mathbb{T}_M$ denotes the set of the terms obtained composing function symbols as in 1 above.

By $\mathbb{T}_R$ we denote the set of the terms obtained composing function symbols as in 2 above.

By $\mathbb{T}_R \circ \mathbb{T}_M$ we denote the set of the terms obtained composing terms in $\mathbb{T}_R$ with terms in $\mathbb{T}_M$. Clearly, there are many more terms in L' but for simplicity, we will restrict the discussion to these.

2 Definition. Formulas in $\mathbb{L}$ are defined inductively as follows

i. $\mathbb{L}$ contains formulas of the form $t(y;z) \in C$,where $C \subseteq R^n$ is compact and $t(y;z)$ is a tuple of terms in $\mathbb{T}_R \circ \mathbb{T}_M$;

ii. $\mathbb{L}$ contains all formulas without quantifiers of sort R and only terms in $\mathbb{T}_M$;

iii. $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort M;

iv. ??? it might be feasible to assume some form of closure under the quantifier $\forall$ of sort R.

Clearly, $L \subseteq \mathbb{L} \subseteq L'$.

2. Topological tools

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

By βI we denote the set of ultrafilters on I. This is a topological space with the topology generated by the clopen sets $[A] = \{F \in \beta I : A \in F\}$ as A ranges over the subsets of I.

3 **Definition.** Let I be a pure set. Let X, Y be topological spaces, and assume Y is compact.

We say that $f : I \times X \to Y$ is continuous uniformly in I if the function below is continuous

$$\begin{array}{rccl} \bar{f} & : & \beta I \times X & \to & Y \\ & & \langle F, \alpha \rangle & \mapsto & F\text{-}\lim_i f(i, \alpha) \end{array}$$

It is convenient to unravel the devinition above into a longer b.

4 **Proposition.** The following are equivalent

1. $f : I \times X \to Y$ is continuous uniformly in I.

2. For every $F \in \beta I$ and $\alpha \in X$, and for every $V \subseteq Y$ open neighborhood of $F\text{-}\lim_i f(i, \alpha)$ there is $U \subseteq X$ open neighborhoodof α and an $A \in F$ such that $f(i, x) \in V$ for every $i \in A$ and every $x \in U$.

3. Ultrapowers

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that each function symbol f of sort $M^n \times R^m \to R$ is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$. (Otherwise, Fact 6 below may fail.)

To keep notation tidy, we make two simplifications: (1) we only consider ultrapowers; (2) we do not consider formulas containing equality, so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

5 Definition. We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \times R^m \to M$ then $f^{\mathcal{N}}(\hat{a};\alpha)$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i;\alpha\big) : i \in I\rangle$.
4. If f is a function of sort $M^n \times R^m \to R$ then for every $\alpha \in R^m$

$$f^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

3. if r is a relation symbol of sort $M^n \times R^m$ then for every $\alpha \in R^m$

$$\mathcal{N} \vDash r(\hat{a};\alpha) \Leftrightarrow \big\{i \in I \;:\; \mathcal{M} \vDash r(\hat{a}i;\alpha)\big\} \in F.$$

The following is immediate.

6 Fact. The structure $\mathcal{N}$ satisfies condition 1 of Definition 1.

The following is easily proved by induction on the syntax

7 Fact. If $t(y;z) \in \mathbb{T}_M$ then

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; \langle t^{\mathcal{M}}\big(\hat{a}i;\alpha\big) : i \in I\rangle. \qquad \square$$

We also have that

8 Fact. If $t(y;z) \in \mathbb{T}_R$ then

$$\#_0 \qquad t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

Proof. By induction. The fact holds by definition if t is atomic. Therefore, assume $\#_0$ and prove

$$\#_1 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\Big).$$

By induction hypothesis

$$\#_2 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;\lambda\Big).$$

where

$$\lambda \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)$$

Let σ the limit in $\#_2$. Let $V \subseteq R$ be an open set containing σ. Apply the uniform continuity as in 2 of Proposition 4, to obtain $A \in F$ and an open set $U \subseteq R$ containing λ such that for every $i \in A$ and every $x \in U$

$$f^{\mathcal{M}}\big(\hat{a}i; x\big) \ \in \ V.$$

After possibly restricting A, we can assume that for every $i \in A$

$$t^{\mathcal{M}}\big(\hat{a}i; \alpha\big) \ \in \ U.$$

This sufficies to prove that σ is also the limit in #1. □

From the two facts above it immediately follows that

9 Fact. If $t(y; z) \in \mathbb{T}_R \circ \mathbb{T}_M$ then

$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \ = \ F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i; \alpha\big).$$ □

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C. Note that, if no such atomic formulas occurs in φ, then $\varphi > \varphi$.

Finally, we prove Łŏś Theorem.

10 Proposition. Assume $|I| \geq$ the weight of the topology on R. Let $\mathcal{N}$ be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \ \Leftrightarrow \ \big\{ i \in I \ : \ \mathcal{M} \models \varphi'(\hat{a}i; \alpha) \big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x; y)$ is as in (i) of Definition 2 then, after absorbing α into the language, the proposition reduces to the classical Łŏś Theorem.

Now, suppose that $\varphi(x; y)$ is as in (ii) of Definition 2, say it is the formula $t(x; y) \in C$, where $t(x; y)$ is a tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F-limit, $\big\{ i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D \big\} \in F$.

Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. Let D be a compact neighborhood of C such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$ (why should it exist ???). By Fact 9 and the definition of F-limit, $\big\{ i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D \big\} \in F$.

As for induction, only the existential quantifier requires some work. Inductively we assume that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}\,; \alpha) \ \Leftrightarrow \ \big\{ i \in I \ : \ \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i\,; \alpha) \big\} \in F \text{ for every } \varphi' > \varphi.$$

We need to prove

$$\mathcal{N} \models \exists y\, \varphi(\hat{a}, y\,; \alpha) \ \Leftrightarrow \ \big\{ i \in I \ : \ \mathcal{M} \models \exists y\, \varphi'(\hat{a}i, y\,; \alpha) \big\} \in F \text{ for every } \varphi' > \varphi.$$

The implication $\Rightarrow$ is clear. For the converse, assume the r.h.s. Assume for simplicity that $I = \omega$. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for for every $i < j$ and

$$\mathcal{M} \ \models \ \bigwedge_{i \in I} \varphi_i(x, y\,; \alpha) \ \to \ \varphi(x, y\,; \alpha).$$

For every $j \in I$ there is a sequence $\hat{b}_j$ such that

$$\left\{ i \in I \ : \ \mathcal{M} \vDash \varphi_j(\hat{a}i, \hat{b}_j i\,; \alpha) \right\} \in F.$$

As $\varphi_j(x, y\,; \alpha) \to \varphi_{j'}(x, y\,; \alpha)$ holds for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_i i \ : \ i \in I \rangle$ satisfies $\mathcal{N} \vDash \varphi(\hat{a}, \hat{d}\,; \alpha)$.

(...) $\square$