CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A class of structures

1 **Definition.** In these notes we deal with 3-sorted[1] structures of the form $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$. The language L and its interpretation are subject to the following conditions.

1. Functions only have one of the following sorts (for any $m, n \in \omega$)
   a. $\mathbb{R}^n \to \mathbb{R}$
   b. $M^n \times \breve{M}^m \to$ any of M, $\breve{M}$, or $\mathbb{R}$
2. There is a symbol for every (total) uniformly continuous function $\mathbb{R}^n \to \mathbb{R}$.
3. The interpretation of functions symbolsof sort $M^n \to \mathbb{R}$ have *bounded* range.
4. Every element of $\breve{M}$ is the image of some term of sort $M^n \to \breve{M}$.
5. There is only one unary predicate: $x \leq 0$, of sort $\mathbb{R}$.

Below, when the sort $\breve{M}$ is not displaied in the definitions, we assume that $M = \breve{M}$ and that the language contains the identity map $\mathrm{id} : M \to \breve{M}$. In other words, that $\breve{M}$ is redundant.

We call M the unit ball of $\mathcal{M}$: there is where the action takes place. The auxiliary sort $\breve{M}$ can be ignored at a first reading. Is is introduced as it makes the description of many important examples less contrived, e.g. Example 4 below.

2 **Definition.** We write $\mathbb{T}(A)$ for the set of terms of sort $M^n \times \mathbb{R}^m \to \mathbb{R}$ with parameters in some $A \subseteq M$. Up to equivalence, these terms have the form $f(t(x), y)$, where $t(x)$ is a tuple of terms of sort $M^{|x|} \to \mathbb{R}$, and f is a function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$. Note that the terms $\mathbb{T}(A)$ have bounded range.

Now define a subset of the set first order formulas with parameters in a set $A \subseteq M$. Note that we do not allow negation, nor quantification over $\breve{M}$. Also, equality among elements of M is excluded.

3 **Definition.** We define the set of formulas $\mathbb{L}(A)$ inductively

i. $\mathbb{L}(A)$ contains the atomic formula $t \leq 0$ for every term $t \in \mathbb{T}(A)$.
ii. It is closed under the Boolean connectives $\perp$, $\top$, $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort M.
iii. It is closed under the quantifier $\forall$ of sort $\mathbb{R}$ relativized to any definable subset of $\mathbb{R}$.

[1]In some examples it may be more natural to use $(n+n'+1)$-sorted structures $\mathcal{M} = \langle M_1, \ldots, M_n, \breve{M}_1, \ldots, \breve{M}_{n'}, \mathbb{R}\rangle$. The generalization is straightforward.

Note that in iii of the definition above, definability is intended with respect to $L(M)$, the full first order language of $\mathcal{M}$, which is strictly larger than $\mathbb{L}(A)$. In particular, if $\varphi(x,\varepsilon) \in \mathbb{L}(A)$, also $\forall\varepsilon{>}0\ \varphi(x,\varepsilon) \in \mathbb{L}(A)$.

4 **Example (Banach spaces).** Given a Banach space V we define a structure $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$ as follows. Let $\breve{M} = V$ and let $M = \{a \in \breve{M} : \|a\| \leq 1\}$ be the closed unit ball of V. Besides the symbols mentioned above, $\mathbb{L}$ contains a function symbol for the natural embedding $\mathrm{id} : M \to \breve{M}$. It also contains a symbol for the norm $\|\text{-}\| : \breve{M} \to \mathbb{R}$. Finally, $\mathbb{L}$ contains the usual symbols of the language of vector spaces. These have sort $\breve{M}^n \to \breve{M}$, for the appropiate $n \in \{0,1,2\}$.

Terms of sort $M^{|x|} \to \mathbb{R}$ have the form $f(t(x))$ where f is a uniformly continuous function and

$$t(x) \quad = \quad \Big\| \sum_{i=1}^{|x|} \lambda_i x_i \Big\|.$$

Therefore, as for all $a \in M^{|x|}$

$$t^{\mathcal{M}}(a) \quad \leq \quad \sum_{i=1}^{|x|} |\lambda_i|,$$

condition 3 of Definition 1 is satisfied. Conditions 4 is immediate.

5 **Example (Metric spaces of finite diameter).** Given a metric space (M, d) of finite diameter we define a structure $\mathcal{M} = \langle M, \mathbb{R}\rangle$ as follows. The only function symbols is d. Note that 3 is satisfied because d is bounded.

2. $\mathbb{L}$-RELATIONS

Equality between elements in the unit ball is not among the formulas that Definition 1 enumerates in $\mathbb{L}$. Logic without equality is not uncommon in mathematics. For instance, in the theory of integration equality beween functions is not significative notion. Of course, there is a quick remedy to this. One can work with equivalence classes of functions that coincide almost everywhere. In this way, equality regains it meaning. But this has some drawbacks. In fact, this makes impossible to speak of the value that a function takes at a point and there are situations when this is inconvenient.

In our general setting, for similar reasons, structures that contain irrelevat copies of equivalent objects may offer a better grasp than terse quotiented structures.

Without equality, relations, rather than partial maps, are the natural tool to discuss elementarity.

Let $\mathcal{M}$ and $\mathcal{N}$ be models. We say that $R \subseteq M \times N$ is an $\mathbb{L}$-(elementary) relation between $\mathcal{M}$ and $\mathcal{N}$ (or on $\mathcal{M}$ if the two coicide) if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } a\ R\ b.$$

Recall that, when $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ are tuples, $a\ R\ b$ stands for $a_i\ R\ b_i$ for every $i \in \{1, \ldots, n\}$.

We define an equivalence relation $(\sim_{\mathcal{M}})$ on M as follows

1. $$a \sim_{\mathcal{M}} b \quad \Leftrightarrow \quad \Big(\mathcal{M} \vDash \varphi(a) \leftrightarrow \varphi(b) \quad \text{for every } \varphi(x) \in \mathbb{L}(M)\Big),$$

where $|x| = |a| = |b| = 1$. Note that this relation would be trivial had we included in $\mathbb{L}(M)$ equality between elements of M.

The following proposition is easily proved by induction on the syntax.

6 **Proposition.** The following are equivalent for every $a, b \in M$.

1. $a \equiv_{\mathcal{M}} b$;
2. $\mathcal{M} \vDash t(a) = t(b)$ for every $t(x) \in \mathbb{T}(M)$, with $|x| = 1$. □

7 **Lemma.** The relation $(\sim_{\mathcal{M}}) \subseteq M^2$ is an $\mathbb{L}$-relation. Moreover, it is maximal among the $\mathbb{L}$-relations on $\mathcal{M}$, i.e. no $\mathbb{L}$-relation properly contains $(\sim_{\mathcal{M}})$.

Proof. Assume $a \sim_{\mathcal{M}} b$, where $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$. Recall that this means that $a_i \sim_{\mathcal{M}} b_i$ for all $i \in \{1, \ldots, n\}$. Let Δ denote the diagonal relation on M. Note that $a_i \sim_{\mathcal{M}} b_i$ is equivalent to saying that $\Delta \cup \{(a_i, b_i)\}$ is an $\mathbb{L}$-relation. As $\mathbb{L}$-relations are closed under composition $\Delta \cup \big\{(a_1, b_1), \ldots, (a_n, b_n)\big\}$ is $\mathbb{L}$-elementary. It follows that for every $\varphi(x) \in \mathbb{L}$

2. $$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{M} \vDash \varphi(b).$$

This proves that $(\sim_{\mathcal{M}})$ is an $\mathbb{L}$-relation. Finally, maximality is immediate. □

8 **Lemma.** Let $R \subseteq M \times N$ be total and surjective $\mathbb{L}$-relation. Then there is a unique maximal $\mathbb{L}$-relation containing R. This maximal $\mathbb{L}$-relation is equal to both $(\sim_{\mathcal{M}})\, R$ and $R\, (\sim_{\mathcal{N}})$, where justaposition of relations stands for composition.

Proof. It is immediate to verify that $(\sim_{\mathcal{M}})\, R$ is an $\mathbb{L}$-relation containing R. Let S be any maximal $\mathbb{L}$-relation containing R. By maximality, $(\sim_{\mathcal{M}})\, S = S$. As S is a total relation $(\sim_{\mathcal{M}}) \subseteq S S^{-1}$. Therefore, by the lemma above, $(\sim_{\mathcal{M}}) = S S^{-1}$. As R is a surjective relation, $S \subseteq S\, S^{-1} R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}})\, R$. A similar argument proves that $S = R\, (\sim_{\mathcal{N}})$. □

We write $\mathrm{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, $\mathbb{L}$-relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/{\sim_{\mathcal{M}}}$.

9 **Definition.** Let x be a single variable. Let $t(x) \in \mathbb{T}(M)^{<\omega}$ be a finte tuple of terms, say $t(x) = t_1(x), \ldots, t_n(x)$. We define a pseudometric $\mathrm{d}_t(\text{-}, \text{-})$ on M as follows

$$\mathrm{d}_t(a,b) \quad = \quad \max_{i=1,\ldots,n} \left\{ |t_i^{\mathcal{M}}(a) - t_i^{\mathcal{M}}(b)| \right\}$$

3. Elementary substructures

An $\mathbb{L}$-(elementary) embedding is an $\mathbb{L}$-relation that is functional, injective and total.

When $\varnothing$ is an $\mathbb{L}$-relation between $\mathcal{M}$ and $\mathcal{N}$ we say that $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{L}$-(elementary) equivalent and write $\mathcal{M} \equiv^{\mathbb{L}} \mathcal{N}$. When $A \subseteq M \cap N$, we write $\mathcal{M} \equiv_A \mathcal{N}$ if Δ_A, the diagonal relation on A, is an $\mathbb{L}$-relation. In words, we say that $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{L}$-equivalent over A. Finally, we write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \subseteq \mathcal{N}$, i.e. $\mathcal{M}$ is a substructure of $\mathcal{N}$, and $\mathcal{M} \equiv^{\mathbb{L}}_M \mathcal{N}$. In words, we say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$.

Let $A \subseteq M$. The $\mathbb{L}(A)$-topology on $M^{|x|}$ is the topology that has, as a base of closed sets, the $\mathbb{L}(A)$-definable sets.

The $\mathbb{L}(A)$-topology is not T_0. The following lemma asserts that to obtain a Hausdorff topology it suffices to quotient by $\equiv^{\mathbb{L}}_A$. This is the so-called the Kolmogorov quotient.

We need the following definitions that will be relevant in many of the proofs below. If $\varphi(x) \in L(A)$ and $\varepsilon > 0$ we write ${}^{\varepsilon}\varphi(x)$ for the formula obtained by replacing every occurence in φ of the atomic formula $t(x) \leq 0$ with $t(x) - \varepsilon \leq 0$ or, less pedantically, $t(x) \leq \varepsilon$. We write ${}^{\neg\varepsilon}\varphi(x)$ for the formula obtained by replacing $t(x) \leq 0$ with $\varepsilon \leq t(x)$.

Note that, if $\varphi(x) \in \mathbb{L}(A)$ then ${}^{\varepsilon}\varphi(x)$, ${}^{\neg\varepsilon}\varphi(x) \in \mathbb{L}(A)$. Moreover, the positivity of the formulas in $\mathbb{L}(A)$ ensures that

$$\varphi(x) \quad \to \quad {}^{\varepsilon}\varphi(x)$$

and

$${}^{\neg\varepsilon}\varphi(x) \quad \to \quad \neg\varphi(x)$$

The following fact is immediate.

10 Fact. For every $a \in M^{|x|}$ and very $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{M} \vDash \forall \varepsilon > 0 \;\; {}^{\varepsilon}\varphi(a) \quad \Leftrightarrow \quad \mathcal{M} \vDash \varphi(a)$$

$$\mathcal{M} \vDash \exists \varepsilon > 0 \; {}^{\neg\varepsilon}\varphi(a) \quad \Leftrightarrow \quad \mathcal{M} \vDash \neg\varphi(a)$$

□

11 Lemma. If $a \not\equiv^{\mathbb{L}}_A b$ are two elements of $M^{|x|}$ then there are two formulas $\varphi(x), \psi(x) \in \mathbb{L}(A)$ such that $\varphi(a)$ and $\psi(b)$ hold and $\varphi(M) \cap \psi(M) = \varnothing$.

Proof. Let $\varphi(x) \in \mathbb{L}(A)$ be such that $\varphi(a)$ and $\neg\varphi(b)$. By the fact above ${}^{\neg\varepsilon}\varphi(b)$ holds for some $\varepsilon > 0$. Then let $\psi(x) = {}^{\neg\varepsilon}\varphi(x)$. □

12 Proposition (Tarski-Vaught test). Let $\mathcal{M} \subseteq \mathcal{N}$. Let x be a single variable. Then the following are equivalent

1. $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$;
2. for every $\varphi(x) \in \mathbb{L}(M)$, if $\varphi(N) \neq \varnothing$, then $\mathcal{N} \vDash \varphi(b)$ for some $b \in M$.

Proof. Implication 1⇒2 is clear. As for 2⇒1, we prove by induction on the syntax of $\varphi(z)$, where z is a finite tuple, that

3. $$\mathcal{N} \vDash \varphi(a) \quad\Leftrightarrow\quad \mathcal{M} \vDash \varphi(a) \qquad\qquad \text{for every } a \in M^{|z|}$$

For $\varphi(z)$ atomic 3 follows from $\mathcal{M} \subseteq \mathcal{N}$. Induction for the connectives $\wedge$ and $\vee$ is straightforward. Induction for the existential quantifier is as in the classical case. We prove the case of the universal quantifier over the unit ball. Assume that 3 holds for the formula $\psi(z, y)$ and prove that it holds for the formula $\forall y\, \psi(z, y)$. The the implication ⇒ is immediate. To prove ⇐, note that

$$\begin{aligned}
\mathcal{N} \vDash \neg\forall y\, \psi(a, y) \quad &\Rightarrow \quad \mathcal{N} \vDash \forall y\, {}^{\neg\varepsilon}\psi(a, y) && \text{for some } \varepsilon > 0\\
&\Rightarrow \quad \mathcal{M} \vDash \forall y\, {}^{\neg\varepsilon}\psi(a, y) && \text{for some } \varepsilon > 0\\
&\Rightarrow \quad \mathcal{M} \vDash \neg\forall y\, \psi(a, y).
\end{aligned}$$

Induction for the universal quantifier of sort $\mathbb{R}$ is immediate. □

4. Ultraproducts

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $f : I \to \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \to F} f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq \mathbb{R} \cup \{\pm\infty\}$ that is a neighborhood of λ. Such a λ is unique and, when F is an ultrafilter, it always exists. When f is bounded, $\lambda \in \mathbb{R}$.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle M_i, \check{M}_i, \mathbb{R} \rangle$, that are uniformly bounded, that is, the bounds in 3 of Definition 1 are the same for all $\mathcal{M}_i$.

Let F be an ultrafilter on I.

13 **Definition.** We define a structure $\mathcal{N} = \langle N, \check{N}, \mathbb{R} \rangle$ that we call the ultraproduct of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. N comprise the sequences $\hat{a} : I \to \bigcup_{i \in I} M_i$ such that $\hat{a}\, i \in M_i$.
2. $\check{N}$ comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a} i)$, where $t(x)$ is a term of sort $M^n \to \check{M}$.
3. If f is a function of sort $M^n \times \check{M}^m \to M$ then $f^{\mathcal{N}}(\hat{a}, t^{\mathcal{N}}(\hat{c}))$ is the sequence $f^{\mathcal{M}_i}(\hat{a} i, t^{\mathcal{M}_i}(\hat{c} i))$.
4. Similarly when f is of sort $M^n \times \check{M}^m \to \check{M}$.

5. If f is a function of sort $M^n \times \check{M}^m \to \mathbb{R}$ then

$$f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big) \;=\; \lim_{i\to F} f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that $\mathcal{N}$ is an ultrapower of $\mathcal{M}$.

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models $\mathcal{M}_i$ have the same bounds.

The following fact is easily proved by induction on the syntax.

14 Fact. For every term $t(x)$ of sort $M^n \to \mathbb{R}$

$$t^{\mathcal{N}}\big(\hat{a}\big) \;=\; \lim_{i\to F} t^{\mathcal{M}_i}\big(\hat{a}i\big). \qquad \square$$

Finally, we prove

15 Proposition (Łǒś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x,y) \in \mathbb{L}(A)$. Let $\hat{a} \in N^{|x|}$. Then there is a function $u\_ : \mathbb{R}^+ \to F$ such that for every $\lambda \in \mathbb{R}^{|y|}$

1. $\mathcal{N} \models \varphi(\hat{a},\lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{i \in I \;:\; \mathcal{M}_i \models {}^{\varepsilon}\varphi(\hat{a}i,\lambda)\big\}$ for every $\varepsilon > 0$.
2. $\mathcal{N} \not\models \varphi(\hat{a},\lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{i \in I \;:\; \mathcal{M}_i \not\models {}^{\varepsilon}\varphi(\hat{a}i,\lambda)\big\}$ for some $\varepsilon > 0$.

Proof. Suppose that $\varphi(x,y)$ is atomic, say

$$\varphi(x,y) \;=\; f(t(x),\ y) \le 0,$$

where t is a tuple of terms of sort $M^{|x|} \to \mathbb{R}$ and f is a uniformly continuous function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

Let $\beta' \in \mathbb{R}^{|t|}$ be such that $\lim\limits_{i\to F}\ t(\hat{a}i) = \beta'$.

Given $\varepsilon > 0$, let B_ε be a neighborhood of β' such that

$$\Big|f(\beta,\ \lambda) - f(\beta',\lambda)\Big| < \varepsilon \qquad \text{for every } \beta \in B_\varepsilon \text{ and every } \lambda \in \mathbb{R}^{|y|}.$$

Such a neighborhood exists by the uniform continuity of f. Finally, to obtain 1 and 2 above it suffices to define

$$u_\varepsilon \;=\; \big\{i \;:\; t(\hat{a}i) \in B_\varepsilon\big\}.$$

The rest of the inductive proof is straightforward and is left to the reader. $\square$

5. Saturation

Let $p(x) \subseteq \mathbb{L}(M)$, where x has the sort of the unit ball. We say that $p(x)$ is finitely satisfied in $\mathcal{M}$ if for every conjunction of formulas in $p(x)$, say $\varphi(x)$, and for every $\varepsilon > 0$, there is an $a \in M^{|x|}$ such that $\mathcal{M} \models {}^{\varepsilon}\varphi(a)$.

The following definition is completely standard

16 Definition. We say that $\mathcal{M}$ is $\mathbb{L}$-saturated if for every $p(x)$ as in 1 and 2 below, there is an $a \in M^{|x|}$ such that $\mathcal{M} \vDash p(a)$

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is finitely satisfied in $\mathcal{M}$.

Now we prove that every model embeds $\mathbb{L}$-elementarily in an $\mathbb{L}$-saturated one. First we prove the following lemma.

17 Lemma. Every model $\mathcal{M}$ embeds $\mathbb{L}$-elementarily in a model $\mathcal{N}$ that realizes all types as in the definition above.

Proof. Consider the collection of types such that 1 and 2 above. Assume that each type has a distinct variable and let x be the concatenation of all these variables. We denote by $p(x)$ the union of all these types. Let I be the set of formulas $\xi(x)$ that are satisfied in $\mathcal{M}$. For every formula $\varphi(x)$ and every $\varepsilon > 0$ define $X_{\varphi,\varepsilon} \subseteq I$ as follows

$$X_{\varphi,\varepsilon} \;=\; \Big\{\xi(x) \in I \;:\; \xi(M) \;\subseteq\; {}^{\varepsilon}\varphi(M)\Big\}$$

Note that ${}^{\varepsilon}\varphi(x)$ is satisfied in $\mathcal{M}$ if and only if $X_{\varphi} \neq \varnothing$ if and only if ${}^{\varepsilon}\varphi(x) \in X_{\varphi,\varepsilon}$. Moreover

$$X_{\varphi\wedge\psi,\ \min\{\varepsilon,\delta\}} \;\subseteq\; X_{\varphi,\varepsilon} \cap X_{\psi,\delta}.$$

Then, as $p(x)$ is finitely consistent, the set

$$B \;=\; \big\{X_{\varphi,\varepsilon} \;:\; \varphi(x) \in p \text{ and } \varepsilon > 0\big\}$$

has the finite intersection property. Extend B to an ultrafilter F on I. Let $\mathcal{N}$ be the ultrapower of $\mathcal{M}$ over F. That is, the model with unit ball $N = M^I$ obtained as in Definition 13. By Łǒś Theorem, $\mathcal{M}$ embeds elementariy into $\mathcal{N}$. It remains to prove that $\mathcal{N}$ realizes $p(x)$.

For every formula $\xi(x) \in I$ choose some $a_{\xi} \in M^{|x|}$ such that $\xi(a_{\xi})$. Let $\hat{a} \in (M^{|x|})^I$ be the function that maps $\xi(x) \mapsto a_{\xi}$. We claim that $\mathcal{N} \vDash \varphi(\hat{a})$ for every $\varphi(x) \in p$. By Fact 10, it suffices to prove that $\mathcal{N} \vDash {}^{\varepsilon}\varphi(\hat{a})$ for every $\varepsilon > 0$. Suppose not, then by Łǒś Theorem (Proposition 15 with ε for λ), for some $\varepsilon' > 0$ the set

$$Y_{\varphi,\varepsilon+\varepsilon'} \;=\; \Big\{\xi(x) \in I \;:\; \mathcal{M} \nvDash {}^{\varepsilon+\varepsilon'}\varphi(a_{\xi})\Big\}$$

belongs to F. This is a contradiction because $B \subseteq F$ contains the set $X_{\varphi,\varepsilon+\varepsilon'} \subseteq Y_{\varphi,\varepsilon+\varepsilon'}$. □

18 Corollary. Every model $\mathcal{M}$ is an $\mathbb{L}$-elementary subtructure of some saturated model $\mathcal{N}$ (possibly of inaccessible cardinality).

We denote by $\mathcal{U}$ some large $\mathbb{L}$-saturated structure which we call the monster model. The unit ball of $\mathcal{U}$ is denoted by U. The cardinality of $\mathcal{U}$ is an inaccessible cardinal that we denote by κ. Below we say model for $\mathbb{L}$-elementariy substructure of $\mathcal{U}$.

19 **Lemma.** Let $p(x) \subseteq \mathbb{L}(A)$, where $A \subseteq U$ has small cardinality. Let $\varphi(x) \in \mathbb{L}(U)$ be such that $p(x) \to \varphi(x)$. Then for every $\varepsilon > 0$ there is a formula $\psi(x)$, conjunction of formulas in $p(x)$, such that $\psi(x) \to \neg^{\neg\varepsilon}\varphi(x)$.

Proof. Let $\varepsilon > 0$ be given. Suppose that $\psi(x) \wedge {}^{\neg\varepsilon}\varphi(x)$ is consistent for every $\psi(x)$ that is conjunction of formulas in $p(x)$. By saturation there is a realization of $p(x) \cup {}^{\neg\varepsilon}\varphi(x)$. By Fact 10, $p(x) \wedge \neg\varphi(x)$ is also consistent. □

6. Complete models

Let $\mathcal{M}$ be a model. Let $a \in U^{|x|}$. We say that $\mathcal{M}$ defines a in the limit if

$$p_{\restriction M}(x) \;\to\; p(x) := \mathrm{tp}_{\mathbb{L}}(a/U)$$

or, in other words, if $a' \equiv^{\mathbb{L}}_{M} a$ implies $a' \sim_{\mathcal{U}} a$.

Note that the definability of a tuple in the limit is equivalent to the definability of its components. This follows from the following obvious identity

$$\mathrm{tp}_{\mathbb{L}}(a, b/U) \;=\; \mathrm{tp}_{\mathbb{L}}(a/U) \,\cup\, \mathrm{tp}_{\mathbb{L}}(b/U).$$

Now we characterize definability in the limit in topological terms.

20 **Proposition.** The following are equivalent for every $a \in U$

1. $\mathcal{M}$ defines a in the limit;
2. there is a net of elements of M that converges to a.

Proof. Let $p(x) = \mathrm{tp}_{\mathbb{L}}(a/U)$.

(1⇒2) The set of metrics d_t in Definion 9 is a directed set under the relation of refinement. For every such metric d_t we have that

$$p_{\restriction M}(x) \;\to\; \mathrm{d}_t(x, a) \leq 0$$

By Lemma 19, for every $\varepsilon > 0$ there some consistent $\varphi(x) \in p$ such that

$$\varphi(x) \;\to\; \mathrm{d}_t(x, a) < \varepsilon$$

As $\varphi(x)$ is consistent in $\mathcal{M}$, there is some $b_{t,\varepsilon} \in M$ such that $\mathrm{d}_t(b_{t,\varepsilon}, a) < \varepsilon$. This defines a net of elements of M that converges to a.

(2⇒1) Let I be a directed set. Let $\langle b_i : i \in I\rangle$ be a net of elements of M that converges to a. Then, $c \sim_{\mathcal{U}} a$ for every $c \vDash p_{\restriction M}(x)$. Hence $p_{\restriction M}(x) \to p(x)$. □

21 **Definition.** A model $\mathcal{M}$ is complete if it contains, up to $(\sim_{\mathcal{U}})$-equivalence, all points it defines in the limit.

Let $\bar{M}$ be the set points that are defined in the limit by $\mathcal{M}$. We denote by $\bar{\mathcal{M}}$ the (unique) substructure of $\mathcal{U}$ with unit ball $\bar{M}$. We call $\bar{\mathcal{M}}$ the completion of $\mathcal{M}$.

We say that a model $\mathcal{N}$ is M-$\mathbb{L}$-saturated if it realizes all constistent types $p(x) \subseteq \mathbb{L}(M)$. The following proposition is easy to prove.

22 **Proposition.** $\bar{\mathcal{M}}$ is the intersection of all M-$\mathbb{L}$-saturated models.

23 **Proposition.** $\bar{\mathcal{M}}$ is complete.

Proof. Suppose $a \in U$ is defined in the limit by $\bar{\mathcal{M}}$, that is

$$p'(x, b) \quad \to \quad p(x) := \mathrm{tp}_{\mathbb{L}}(a/U)$$

for some $b \in \bar{M}^{|z|}$ and $p'(x, z) \subseteq \mathbb{L}(M)$ such that $p_{\restriction \bar{M}}(x) = p'(x, b)$. Let $q(z) = \mathrm{tp}_{\mathbb{L}}(b/U)$, then

$$q_{\restriction M}(z) \,\cup\, p'(x, z) \quad \to \quad p(x)$$

Finally, $a \in \bar{M}$ follows because

$$p_{\restriction M}(x) \quad \to \quad \exists z \left[q_{\restriction M}(z) \,\cup\, p'(x, z)\right].$$

□

24 **Question.** $\bar{\mathcal{M}} \preceq_{\mathbb{L}} \mathcal{U}$?

7. Homogeneity

Recall that κ, the cardinality of $\mathcal{U}$, is an inaccessible cardinal.

25 **Fact.** Let $R \subseteq \mathcal{U}^2$ be an $\mathbb{L}$-relation of cardinality $< \kappa$. Then there is a total and surjective $\mathbb{L}$-relation $S \subseteq \mathcal{U}^2$ containing R.

Proof. We apply the usual back-and-forth construction with a pinch of extra caution. Let a be an enumeration of the domain of R. Let $\bar{a} = \langle a_i : i < \lambda\rangle$ be an enumeration of all tuples of length $|a|$ such that aRa_i. As κ is inaccessible, $\lambda < \kappa$. Let $b \in U$. It suffices to prove that there is a c such that $R \cup \{\langle b, c\rangle\}$ is an $\mathbb{L}$-relation. Let $p(x, z) = \mathrm{tp}(b, a)$ and let

$$q(x, \bar{z}) \quad = \quad \bigcup_{i<\lambda} p(x, z_i).$$

We claim that $q(x, \bar{a})$ is a finitely consistent type. A finite conjunction of formulas in $q(x, \bar{a})$ has the form $\psi(x, a_{i_1}) \wedge \cdots \wedge \psi(x, a_{i_n})$. As $\psi(b, a)$ and $a_{i_1}, \dots, a_{i_n} \; \boldsymbol{R} \; a, \dots, a$, we conclude that the condition $\psi(x, a_{i_1}) \wedge \cdots \wedge \psi(x, a_{i_n})$ is satisfied. The existence of the required element c follows by saturation. □

26 **Corollary.** Let $A \subseteq \mathcal{U}$ have cardinality $< \kappa$. Let $p(x) = \mathrm{tp}_{\mathbb{L}}(a/A)$, where $a \in \mathcal{U}^{|x|}$ is a tuple of length $|x| < \kappa$. Then

$$p(\mathcal{U}) \;\; = \;\; \Big\{ b \, : \, b\boldsymbol{R}a, \;\; \boldsymbol{R} \in \mathrm{Aut}(\mathcal{U}/A) \Big\}$$