CONTINUOUS LOGIC FOR THE DISCRETE LOGICIAN

1. A class of structures

1 Definition. In this notes we deal with 3-sorted[1] structures of the form $\mathcal{M} = \langle \breve{M}, M, \mathbb{R}\rangle$. The language and its interpretation are subject to the following conditions.

1. The function may only have one of the following sorts (for any $m, n \in \omega$)
   a. $\mathbb{R}^n \to \mathbb{R}$
   b. $\breve{M}^n \times M^m \to$ any of $\breve{M}$, M, or $\mathbb{R}$
2. The functions of sort $\mathbb{R}^n \to \mathbb{R}$ are *uniformly continuous*.
3. The (functions that interpret the) terms of sort $\breve{M}^n \to \mathbb{R}$ have *bounded* range.
4. Every element of M is the image of some term of sort $\breve{M}^n \to M$.
5. The only relation symbol is the order relation $\leq$ on $\mathbb{R}$.

We call $\breve{M}$ the unit ball of $\mathcal{M}$. The terminology is inspired by the example below.

The language is denoted by $\mathbb{L}$. For simplicity we assume it contains *all* functions as in 2 above.

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $\breve{M}^n \times \mathbb{R}^m \to \mathbb{R}$ and parameters in some set $A \subseteq \breve{M}$. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

i. $\mathbb{L}(A)$ contains the atomic formula $t \leq s$ for every pair of terms $t, s \in \mathbb{T}(A)$.

ii. $\mathbb{L}(A)$ is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort $\breve{M}$.

iii. $\mathbb{L}(A)$ is closed under the quantifier $\forall$ of sort $\mathbb{R}$ possibly relativized to a definable subset of $\mathbb{R}$.

Note that iii asserts, in particular, that if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$ then also $\forall \varepsilon{>}0\ \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

3 Example (Banach spaces). Given a Banach space M we define a structure $\mathcal{M} = \langle \breve{M}, M, \mathbb{R}\rangle$ as follows. Let $\breve{M} = \{a \in M : \|a\| \leq 1\}$ be the closed unit ball of M. Besides the symbols mentioned above, $\mathbb{L}$ contains a function symbol for the natural embedding $\mathrm{id} : \breve{M} \to M$. It contains also a symbol for the norm $\|\text{-}\| : M \to \mathbb{R}$. Finally, $\mathbb{L}$ contains the usual symbols of the language of vector spaces. These have sort $M^n \to M$, for the appropiate $n \in \{0, 1, 2\}$.

Finally, note that condition 3 and 4 of Definition 1 are immediatly satisfied.

[1] In some examples it may be more natural to use $(n + n' + 1)$-sorted structures $\mathcal{M} = \langle \breve{M}_1, \ldots, \breve{M}_n, M_1, \ldots, M_{n'}, \mathbb{R}\rangle$. The generalization is straightforeward.

2. Ultraproducts

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $r : I \to \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \uparrow F} r(i) = \lambda$$

if $r^{-1}[A] \in F$ for every $A \subseteq \mathbb{R}$ that is a neighborhood of λ. When F is an ultrafilter, such a λ always exists.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle \breve{M}_i, M_i, \mathbb{R} \rangle$, that are uniformly bounded. That is, the bounds in 3 of Definition 1 are the same for all $\mathcal{M}_i$.

Let F be an ultrafilter on I.

4 **Definition.** We define a structure $\mathcal{N} = \langle \breve{N}, N, \mathbb{R} \rangle$ that we call the ultraproduct of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. $\breve{N}$ comprise the sequences $\hat{a} : I \to \bigcup_{i \in I} \breve{M}_i$ such that $\hat{a}\, i \in \breve{M}_i$.
2. N comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a}i)$, where $t(x)$ is a term of sort $\breve{M}^n \to M$.
3. If f is a function of sort $\breve{M}^n \times M^m \to \breve{M}$ then $f^{\mathcal{N}}(\hat{a}, t^{\mathcal{N}}(\hat{c}))$ is the sequence $f^{\mathcal{M}_i}(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i))$.
4. Similarly when f is of sort $\breve{M}^n \times M^m \to M$.
5. If f is a function of sort $\breve{M}^n \times M^m \to \mathbb{R}$ then
$$f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big) \;=\; \lim_{i \uparrow F} f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that $\mathcal{N}$ is an ultrapower of $\mathcal{M}$.

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models $\mathcal{M}_i$ have the same bounds.

5 **Proposition (Łoś Theorem).** Let $\mathcal{N}$ be as above and let $\varphi(x, y) \in \mathbb{L}(A)$. Then for every $\hat{a} \in \breve{N}^{|x|}$ and $\lambda \in \mathbb{R}^{|y|}$

$$\mathcal{N} \vDash \varphi(\hat{a}, \lambda) \;\Leftrightarrow\; \big\{ i \in I \;:\; \mathcal{M}_i \vDash \exists \alpha \in A\; \varphi(\hat{a}i, \alpha) \big\} \in F \text{ for every neighborhood } A \ni \lambda.$$