CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

C. L. C. L. POLYMATH

1. A CLASS OF STRUCTURES

1 **Definition.** Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L'-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

The function symbols of L' are classified in two groups

1. $M^n \times R^m \to M$
2. $M^n \times R^m \to R$

We require that function symbols of the second group are interpreted in functions such that

i. $f^{\mathcal{M}} : M^n \times R^m \to R$ is bounded, i.e. its range is contained in a compact set;

ii. $f^{\mathcal{M}} : M^n \times R^m \to R$ is continuous uniformly in M, as defined in Definition 3 below.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in this class. As a remedy, below we carve out a set of formulas $L \subseteq \mathbb{L} \subseteq L'$ such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension in the same class that is $\mathbb{L}$-saturated. Our choice of $\mathbb{L}$ is not the most general, it is a compromise that accomodates the relevant examples and avoids eccessive complications.

In a nutshell, we require that function symbols of the second group above do not occur under the scope of a negation. We also do not allow existential quantifiers of sort R.

The ultrapower construction in Section 3 is completely classical w.r.t. functions the first group. On the other hand, functions of the second group are *critical* and need to be dealt with differently.

Below, $\mathbb{T}_M$ denotes the set of the terms obtained composing function symbols as in 1 above.

By $\mathbb{T}_R$ we denote the set of the terms obtained composing function symbols as in 2 above.

By $\mathbb{T}_R{\circ}\mathbb{T}_M$ we denote the set of the terms obtained composing terms in $\mathbb{T}_R$ with terms in $\mathbb{T}_M$. Clearly, there are many more terms in L' but for simplicity, we will restrict to these.

2 **Definition.** Formulas in $\mathbb{L}$ are constructed inductively from two groups of formulas below

i. formulas as $t(y; z) \in C$, where $C \subseteq R^n$ is compact[1] and $t(y; z)$ is a tuple of terms in $\mathbb{T}_R \circ \mathbb{T}_M$;

ii. quantifier free formulas with only terms in $\mathbb{T}_M$.

Finally we require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$, the quantifiers $\forall$, $\exists$ of sort M; and the universal quantifier $\forall$ of sort R.

Clearly, $L \subseteq \mathbb{L} \subseteq L'$.

2. Topological tools

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

By βI we denote the set of ultrafilters on I. This is a topological space with the topology generated by the clopen sets $[A] = \{F \in \beta I : A \in F\}$ as A ranges over the subsets of I.

3 **Definition.** Let I be a pure set. Let X, Y be topological spaces, and assume Y is compact.

We say that $f : I \times X \to Y$ is continuous uniformly in I if the function below is continuous

$$\begin{array}{rcl} \bar{f} : \beta I \times X & \to & Y \\ \langle F, \alpha \rangle & \mapsto & F\text{-}\lim\limits_i f(i, \alpha) \end{array}$$

It is convenient to unravel the definition above in a more practicable form.

4 **Proposition.** The following are equivalent

1. $f : I \times X \to Y$ is continuous uniformly in I.
2. For every $F \in \beta I$ and $\alpha \in X$, and for every $V \subseteq Y$ open neighborhood of $F\text{-}\lim_i f(i, \alpha)$ there is $U \subseteq X$ open neighborhoodof α and an $A \in F$ such that $f(i, x) \in V$ for every $i \in A$ and every $x \in U$.

3. Ultraproducts

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that for each function symbol f of sort $M^n \times R^m \to R$ there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$. Note that, otherwise, Fact 6 below may fail.

[1] We confuse the relation symbols in L' of sort R^m with their interpretation and write $x \in C$ for $C(x)$

To keep notation tidy, we make two simplifications: (1) we only consider ultrapowers; (2) we do not consider formulas containing equality, so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

5 Definition. We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \times R^m \to M$ then $f^{\mathcal{N}}(\hat{a};\alpha)$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i;\alpha\big) : i \in I\rangle$.
4. If f is a function of sort $M^n \times R^m \to R$ then for every $\alpha \in R^m$

$$f^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

3. if r is a relation symbol of sort $M^n \times R^m$ then for every $\alpha \in R^m$

$$\mathcal{N} \vDash r(\hat{a};\alpha) \Leftrightarrow \big\{i \in I \;:\; \mathcal{M} \vDash r(\hat{a}i;\alpha)\big\} \in F.$$

The following is immediate.

6 Fact. The structure $\mathcal{N}$ satisfies condition 1 of Definition 1.

The following is easily proved by induction on the syntax

7 Fact. If $t(y;z) \in \mathbb{T}_M$ then

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; \langle t^{\mathcal{M}}\big(\hat{a}i;\alpha\big) : i \in I\rangle. \qquad \square$$

We also have that

8 Fact. If $t(y;z) \in \mathbb{T}_R$ then

$$\#_0 \qquad t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

Proof. By induction. The fact holds by definition if t is a function symbol. Therefore, we assume $\#_0$ as induction hypothesis and prove

$$\#_1 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\Big).$$

By induction hypothesis

$$\#_2 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;\lambda\Big), \qquad \text{where } \lambda = F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

Let σ be the limit in $\#_2$. Let $V \subseteq R$ be an open set containing σ. Apply 2 of Proposition 4, to obtain $A \in F$ and an open set $U \subseteq R$ containing λ such that for every $i \in A$ and every $x \in U$

$$f^{\mathcal{M}}\big(\hat{a}i;x\big) \;\in\; V.$$

After possibly restricting A, we can assume that for every $i \in A$

$$t^{\mathcal{M}}\big(\hat{a}i;\alpha\big) \quad \in \quad U.$$

This sufficies to prove that σ is also the limit in $\#_1$. □

From the two facts above it immediately follows that

9 Fact. If $t(y;z) \in \mathbb{T}_R{\circ}\mathbb{T}_M$ then

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$ □

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C. Note that, if no such atomic formulas occurs in φ, then $\varphi > \varphi$. Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$. Moreover

$$\bigwedge_{\varphi' > \varphi} \varphi' \quad \leftrightarrow \quad \varphi.$$

Finally, we prove Łŏś Theorem.

10 Proposition. Assume $|I| \geq$ the weight of the topology on R. Let $\mathcal{N}$ be as above and let $\varphi(x;y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \vDash \varphi(\hat{a};\alpha) \quad \Leftrightarrow \quad \big\{i \in I \ \ : \ \ \mathcal{M} \vDash \varphi'(\hat{a}i;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x;y)$ is as in (ii) of Definition 2 then the proposition reduces to the classical Łŏś Theorem.

Now, suppose that $\varphi(x;y)$ is as in (i) of Definition 2, say it is the formula $t(x;y) \in C$, where $t(x;y) \in \mathbb{T}_R{\circ}\mathbb{T}_M$.

Assume $\mathcal{N} \vDash t(\hat{a},\alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a},\alpha)$. Hence, by the definition of F-limit, $\big\{i \ : \ \mathcal{M} \vDash t(\hat{a}i,\alpha) \in D\big\} \in F$. Vice versa, assume $\mathcal{N} \vDash t(\hat{a},\alpha) \notin C$. Let D be a compact neighborhood of C such that $t^{\mathcal{N}}(\hat{a},\alpha) \notin D$. By Fact 9 and the definition of F-limit, $\big\{i \ : \ \mathcal{M} \vDash t(\hat{a}i,\alpha) \notin D\big\} \in F$.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the universal quantifier of sort R we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a};\alpha,\beta) \quad \Leftrightarrow \quad \big\{i \in I \ \ : \ \ \mathcal{M} \vDash \varphi'(\hat{a}i;\alpha,\beta)\big\} \in F \text{ for every } \varphi' > \varphi$$

and we prove that

$$\mathcal{N} \vDash \forall y\, \varphi(\hat{a};\alpha,y) \quad \Leftrightarrow \quad \big\{i \in I \ \ : \ \ \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i;\alpha,y)\big\} \in F \text{ for every } \varphi' > \varphi.$$

The implication $\Rightarrow$ is clear. For the converse assume $\mathcal{N} \nvDash \varphi(\hat{a};\alpha,\beta)$ for some β. By induction hypothesis, and the requirement that F is an ultrafilter, $\big\{i \in I \ \ : \ \ \mathcal{M} \nvDash \varphi'(\hat{a}i;\alpha,\beta)\big\} \in F$ for some $\varphi' > \varphi$. A fortiori $\big\{i \in I \ \ : \ \ \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i;\alpha,y)\big\} \in F$, hence the implication $\Leftarrow$ follows.

To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}\,; \alpha) \ \Leftrightarrow\ \big\{i \in I \ :\ \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i\,; \alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

We first we prove that

$$\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y\,; \alpha) \ \Leftrightarrow\ \big\{i \in I \ :\ \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y\,; \alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

This is identical to the case above. The implication $\Rightarrow$ is clear. For the converse assume $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b}\,; \alpha)$ for some $\hat{b}$. By induction hypothesis, $\big\{i \in I \ :\ \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i\,; \alpha)\big\} \in F$ for some $\varphi' > \varphi$. Hence $\big\{i \in I \ :\ \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y\,; \alpha)\big\} \notin F$ as required.

Finally we prove

$$\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y\,; \alpha) \ \Leftrightarrow\ \big\{i \in I \ :\ \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y\,; \alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Again, $\Rightarrow$ is clear. For the converse, assume the r.h.s. Assume for simplicity that $I = \omega$ and that R is second countable. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for for every $i < j$ and

$$\mathcal{M} \ \vDash \ \bigwedge_{i \in I} \varphi_i(x, y\,; \alpha) \ \to\ \varphi(x, y\,; \alpha).$$

For every $j \in I$ there is a sequence $\hat{b}_j$ such that

$$\big\{i \in I \ :\ \mathcal{M} \vDash \varphi_j(\hat{a}i, \hat{b}_j i\,; \alpha)\big\} \in F.$$

As $\varphi_j(x, y\,; \alpha) \to \varphi_{j'}(x, y\,; \alpha)$ holds for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_i i \ :\ i \in I \rangle$ satisfies $\mathcal{N} \vDash \varphi(\hat{a}, \hat{d}\,; \alpha)$. □