CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A class of structures

1 Definition. In these notes we deal with 3-sorted[1] structures of the form $\mathcal{M} = \langle \check{M}, M, \mathbb{R}\rangle$. The language and its interpretation are subject to the following conditions.

1. Functions may only have one of the following sorts (for any $m, n \in \omega$)
   a. $\mathbb{R}^n \to \mathbb{R}$
   b. $\check{M}^n \times M^m \to$ any of $\check{M}$, M, or $\mathbb{R}$
2. There is a symbol for every (total) uniformly continuous functions $\mathbb{R}^n \to \mathbb{R}$.
3. The (functions that interpret the) terms of sort $\check{M}^n \to \mathbb{R}$ have *bounded* range.
4. Every element of M is the image of some term of sort $\check{M}^n \to M$.
5. There is only one predicate; it has sort $\mathbb{R}$ and is interpreted as $x \leq 0$.

We call $\check{M}$ the unit ball of $\mathcal{M}$. The terminology is inspired by the example below.

The language is denoted by $\mathbb{L}$.

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $\check{M}^n \times \mathbb{R}^m \to \mathbb{R}$ with parameters in some $A \subseteq \check{M}$. Up to equivalence, these terms have the form $f(t(x), y)$, where $t(x)$ is a tuple of terms of sort $\check{M}^{|x|} \to \mathbb{R}$, and f is a function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

3 Definition. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

i. $\mathbb{L}(A)$ contains the atomic formula $t \leq 0$ for every term $t \in \mathbb{T}(A)$.

ii. It is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort $\check{M}$.

iii. It is closed under the quantifier $\forall$ of sort $\mathbb{R}$ relativized to any definable subset of $\mathbb{R}$.

Note that in iii of the definition above, definability is intended with respect to the full first order language. In particular, if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$, also $\forall \varepsilon{>}0\ \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

4 Example (Metric spaces of finite diameter). Given a metric space M, d of finite diameter we define a structure $\mathcal{M} = \langle \check{M}, M, \mathbb{R}\rangle$ as follows. Set $\check{M} = M$ and let $\mathrm{id} : \check{M} \to M$ be the identical isomorphism. The function symbols are id and d. Note that 3 is satisfied because d is bounded.

[1] In some examples it may be more natural to use $(n+n'+1)$-sorted structures $\mathcal{M} = \langle \check{M}_1, \dots, \check{M}_n, M_1, \dots, M_{n'}, \mathbb{R}\rangle$. The generalization is straightforeward.

5 **Example (Banach spaces).** Given a Banach space M we define a structure $\mathcal{M} = \langle \breve{M}, M, \mathbb{R}\rangle$ as follows. Let $\breve{M} = \{a \in M : \|a\| \leq 1\}$ be the closed unit ball of M. Besides the symbols mentioned above, $\mathbb{L}$ contains a function symbol for the natural embedding $\mathrm{id} : \breve{M} \to M$. It also contains a symbol for the norm $\|\text{-}\| : M \to \mathbb{R}$. Finally, $\mathbb{L}$ contains the usual symbols of the language of vector spaces. These have sort $M^n \to M$, for the appropiate $n \in \{0, 1, 2\}$. Note that conditions 3 and 4 of Definition 1 are immediatly satisfied.

If $\varphi \in \mathbb{L}(\mathbb{A})$ and $\varepsilon > 0$ we write φ_ε for the formula obtained by replacing in φ the atomic formulas $t \leq 0$ with $t - \varepsilon \leq 0$. Note that $\varphi \to \varphi_\varepsilon$.

2. Ĕlementary relations

Let $\mathcal{M}$ and $\mathcal{N}$ be models. We say that $R \subseteq \breve{M} \times \breve{N}$ is an ĕlementary relation between $\mathcal{M}$ and $\mathcal{N}$ if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } aRb.$$

Recall that, when $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ are tuples, aRb stands for $a_i R b_i$ for $i = 1, \ldots, n$.

We define an equivalence relation $(\sim_\mathcal{M})$ on $\breve{M}$ as follows

1. $$a \sim_\mathcal{M} b \quad \Leftrightarrow \quad t^\mathcal{M}(a) = t^\mathcal{M}(b) \ \text{ for every } t(x) \in \mathbb{T}(\breve{M}),$$

where $|x| = |a| = |b| = 1$.

6 **Lemma.** The relation $(\sim_\mathcal{M}) \subseteq \breve{M}^2$ is an ĕlementary relation. Among these, it is maximal, i.e. no elementary relation contains $(\sim_\mathcal{M})$ properly.

Proof. Assume $a \sim_\mathcal{M} b$, where $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$. Recall that this means that $a_i \sim_\mathcal{M} b_i$ for all $i \in \{1, \ldots, n\}$. Let Δ denote the diagonal relation on $\breve{M}$. Note that $a_i \sim_\mathcal{M} b_i$ is equivalent to saying that $\Delta \cup \{(a_i, b_i)\}$ is an ĕlementary relation. As ĕlementary relations are closed under composition $\Delta \cup \big\{(a_1, b_1), \ldots, (a_n, b_n)\big\}$ is ĕlementary. It follows that for every $\varphi(x) \in \mathbb{L}$

2. $$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi(b).$$

The converse implication ($2 \Rightarrow a \sim_\mathcal{M} b$) is immediate. □

7 **Lemma.** Let $R \subseteq \breve{M} \times \breve{N}$ be total and surjective elementary relation. Then there is a unique maximal elementary relation containing R. This maximal elementary relation is equal to both $(\sim_\mathcal{M})\, R$ and $R\, (\sim_\mathcal{N})$.

Proof. It is immediate to verify that $(\sim_\mathcal{M})\, R$ is an elementary relation containing R. Let S be any maximal elementary relation containing R. By maximality, $(\sim_\mathcal{M})\, S = S$. As S is a total relation $(\sim_\mathcal{M}) \subseteq SS^{-1}$. Therefore, by the lemma above, $(\sim_\mathcal{M}) = SS^{-1}$. As R is a surjective relation,

$S \subseteq S\,S^{-1}R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}})\,R$. A similar argument proves that $S = R\,(\sim_{\mathcal{N}})$. □

We write $\mathrm{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, elementary relations $R \subseteq \breve{M}^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/{\sim_{\mathcal{M}}}$.

Thought in most situations one could dispense with $\mathcal{M}$ in favour of $\mathcal{M}/{\sim_{\mathcal{M}}}$, in concrete cases one has a better grip on the first than on the latter. Therefore below we insist in working with elementary relations in place of elementary maps.

3. Ultraproducts

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $f : I \to \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \to F} f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq \mathbb{R} \cup \{\pm\infty\}$ that is a neighborhood of λ. Such a λ is unique and, when F is an ultrafilter, it always exists. When f is bounded, $\lambda \in \mathbb{R}$.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle \breve{M}_i, M_i, \mathbb{R} \rangle$, that are uniformly bounded, that is, the bounds in 3 of Definition 1 are the same for all $\mathcal{M}_i$.

Let F be an ultrafilter on I.

8 **Definition.** We define a structure $\mathcal{N} = \langle \breve{N}, N, \mathbb{R} \rangle$ that we call the ultraproduct of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. $\breve{N}$ comprise the sequences $\hat{a} : I \to \bigcup_{i \in I} \breve{M}_i$ such that $\hat{a}\,i \in \breve{M}_i$.
2. N comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a}i)$, where $t(x)$ is a term of sort $\breve{M}^n \to M$.
3. If f is a function of sort $\breve{M}^n \times M^m \to \breve{M}$ then $f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big)$ is the sequence $f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big)$.
4. Similarly when f is of sort $\breve{M}^n \times M^m \to M$.
5. If f is a function of sort $\breve{M}^n \times M^m \to \mathbb{R}$ then

$$f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big) \;=\; \lim_{i \to F} f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that $\mathcal{N}$ is an ultrapower of $\mathcal{M}$.

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models $\mathcal{M}_i$ have the same bounds.

The following fact is easily proved by induction on the syntax.

9 Fact. For every term $t(x)$ of sort $\breve{M}^n \to \mathbb{R}$

$$t^{\mathcal{N}}\big(\hat{a}\big) \quad = \quad \lim_{i \to F} t^{\mathcal{M}_i}\big(\hat{a}i\big). \qquad \square$$

Finally, we prove

10 Proposition (Łŏś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x, y) \in \mathbb{L}(A)$. Let $\hat{a} \in \breve{N}^{|x|}$. Then there is a function $u_{\_} : \mathbb{R}^+ \to F$ such that for every $\lambda \in \mathbb{R}^{|y|}$

1. $\mathcal{N} \vDash \varphi(\hat{a}, \lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \vDash \varphi_\varepsilon(\hat{a}i, \lambda) \big\}$ for every $\varepsilon > 0$.
2. $\mathcal{N} \nvDash \varphi(\hat{a}, \lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \nvDash \varphi_\varepsilon(\hat{a}i, \lambda) \big\}$ for some $\varepsilon > 0$.

Proof. Suppose that $\varphi(x, y)$ is atomic, say

$$\varphi(x, y) \quad = \quad f(t(x),\ y) \leq 0,$$

where t is a tuple of terms of sort $\breve{M}^{|x|} \to \mathbb{R}$ and f is a uniformly continuous function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

Let $\beta' \in \mathbb{R}^{|t|}$ be such that $\lim\limits_{i \to F}\ t(\hat{a}\, i) = \beta'$.

Given $\varepsilon > 0$, let B_ε be a neighborhood of β' such that

$$\Big| f(\beta,\ \lambda) - f(\beta', \lambda) \Big| < \varepsilon \qquad\qquad \text{for every } \beta \in B_\varepsilon \text{ and every } \lambda \in \mathbb{R}^{|y|}.$$

Such a neighborhood exists by the uniform continuity of f. Finally, to obtain 1 and 2 above it suffices to define

$$u_\varepsilon \quad = \quad \big\{ i \;:\; t(\hat{a}i) \in B_\varepsilon \big\}.$$

The rest of the inductive proof is straightforeward and is left to the reader. $\square$