CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A class of structures

1 Definition. In these notes we deal with 3-sorted[1] structures of the form $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$. The language and its interpretation are subject to the following conditions.

1. Functions only have one of the following sorts (for any $m, n \in \omega$)
   a. $\mathbb{R}^n \to \mathbb{R}$
   b. $M^n \times \breve{M}^m \to$ any of M, $\breve{M}$, or $\mathbb{R}$
2. There is a symbol for every (total) uniformly continuous function $\mathbb{R}^n \to \mathbb{R}$.
3. The (functions that interpret the) terms of sort $M^n \to \mathbb{R}$ have *bounded* range.
4. Every element of $\breve{M}$ is the image of some term of sort $M^n \to \breve{M}$.
5. There is only one unary predicate: $x \leq 0$, of sort $\mathbb{R}$.

We call M the unit ball of $\mathcal{M}$: here is where the action takes place. The auxiliary sort $\breve{M}$ can be ignored at a first reading. Is is introduced as it makes the description of many important examples less contrived, e.g. Example 5 below.

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $M^n \times \mathbb{R}^m \to \mathbb{R}$ with parameters in some $A \subseteq M$. Up to equivalence, these terms have the form $f(t(x), y)$, where $t(x)$ is a tuple of terms of sort $M^{|x|} \to \mathbb{R}$, and f is a function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

3 Definition. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

i. $\mathbb{L}(A)$ contains the atomic formula $t \leq 0$ for every term $t \in \mathbb{T}(A)$.
ii. It is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort M.
iii. It is closed under the quantifier $\forall$ of sort $\mathbb{R}$ relativized to any definable subset of $\mathbb{R}$.

Note that in iii of the definition above, definability is intended with respect to the full first order language. In particular, if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$, also $\forall \varepsilon{>}0\ \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

4 Example (Metric spaces of finite diameter). Given a metric space M, d of finite diameter we define a structure $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$ as follows. Set $M = \breve{M}$ and let $\mathrm{id} : M \to \breve{M}$ be the identical

[1] In some examples it may be more natural to use $(n+n'+1)$-sorted structures $\mathcal{M} = \langle M_1, \dots, M_n, \breve{M}_1, \dots, \breve{M}_{n'}, \mathbb{R}\rangle$. The generalization is straightforeward.

isomorphism (so, $\breve{M}$ is irrelevant in this example). The function symbols are id and d. Note that 3 is satisfied because d is bounded..

5 **Example (Banach spaces).** Given a Banach space V we define a structure $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$ as follows. Let $\breve{M} = V$ and let $M = \{a \in \breve{M} : \|a\| \leq 1\}$ be the closed unit ball of V. Besides the symbols mentioned above, $\mathbb{L}$ contains a function symbol for the natural embedding $\mathrm{id} : M \to \breve{M}$. It also contains a symbol for the norm $\|\text{-}\| : \breve{M} \to \mathbb{R}$. Finally, $\mathbb{L}$ contains the usual symbols of the language of vector spaces. These have sort $\breve{M}^n \to \breve{M}$, for the appropiate $n \in \{0, 1, 2\}$.

Terms of sort $M^{|x|} \to \mathbb{R}$ have the form $f(t(x))$ where f is a uniformly continuous function and

$$t(x) \quad = \quad \Big\| \sum_{i=1}^{|x|} \sum_{j=1}^{m} \lambda_{i,j} x_i \Big\|.$$

Therefore, as for all $a \in M^{|x|}$

$$t^{\mathcal{M}}(a) \quad \leq \quad \sum_{i=1}^{|x|} \sum_{j=1}^{m} |\lambda_{i,j}|,$$

condition 3 of Definition 1 is satisfied. Conditions 4 is immediate.

2. $\mathbb{L}$-RELATIONS

Equality between elements in the unit ball is not among the formulas in $\mathbb{L}$. For this reason, relations, rather than partial maps, are the natural tool to discuss elementarity.

Let $\mathcal{M}$ and $\mathcal{N}$ be models. We say that $R \subseteq M \times N$ is an $\mathbb{L}$-(elementary) relation between $\mathcal{M}$ and $\mathcal{N}$, or on $\mathcal{M}$ if the two coicide, if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } a\ R\ b.$$

Recall that, when $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$ are tuples, $a\ R\ b$ stands for $a_i\ R\ b_i$ for every $i \in \{1, \dots, n\}$.

We define an equivalence relation $(\sim_{\mathcal{M}})$ on M as follows

1. $$a \sim_{\mathcal{M}} b \quad \Leftrightarrow \quad \Big(\mathcal{M} \vDash \varphi(a) \leftrightarrow \varphi(b) \quad \text{for every } \varphi(x) \in \mathbb{L}(M)\Big),$$

where $|x| = |a| = |b| = 1$.

6 **Lemma.** The relation $(\sim_{\mathcal{M}}) \subseteq M^2$ is an $\mathbb{L}$-relation. Moreover, it is maximal among the $\mathbb{L}$-relations on $\mathcal{M}$, i.e. no $\mathbb{L}$-relation properly contains $(\sim_{\mathcal{M}})$.

Proof. Assume $a \sim_{\mathcal{M}} b$, where $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$. Recall that this means that $a_i \sim_{\mathcal{M}} b_i$ for all $i \in \{1, \dots, n\}$. Let Δ denote the diagonal relation on M. Note that $a_i \sim_{\mathcal{M}} b_i$ is equivalent to saying that $\Delta \cup \{(a_i, b_i)\}$ is an $\mathbb{L}$-relation. As $\mathbb{L}$-relations are closed under composition $\Delta \cup \big\{(a_1, b_1), \ldots, (a_n, b_n)\big\}$ is $\mathbb{L}$-elementary. It follows that for every $\varphi(x) \in \mathbb{L}$

2. $$\mathcal{M} \vDash \varphi(a) \quad \Leftrightarrow \quad \mathcal{M} \vDash \varphi(b).$$

This proves that $(\sim_{\mathcal{M}})$ is an $\mathbb{L}$-relation. Finally, maximality is immediate. □

7 **Lemma.** Let $R \subseteq M \times N$ be total and surjective $\mathbb{L}$-relation. Then there is a unique maximal $\mathbb{L}$-relation containing R. This maximal $\mathbb{L}$-relation is equal to both $(\sim_{\mathcal{M}})\, R$ and $R\,(\sim_{\mathcal{N}})$.

Proof. It is immediate to verify that $(\sim_{\mathcal{M}})\, R$ is an $\mathbb{L}$-relation containing R. Let S be any maximal $\mathbb{L}$-relation containing R. By maximality, $(\sim_{\mathcal{M}})\, S = S$. As S is a total relation $(\sim_{\mathcal{M}}) \subseteq S S^{-1}$. Therefore, by the lemma above, $(\sim_{\mathcal{M}}) = S S^{-1}$. As R is a surjective relation, $S \subseteq S\, S^{-1} R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}})\, R$. A similar argument proves that $S = R\,(\sim_{\mathcal{N}})$. □

We write $\mathrm{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, $\mathbb{L}$-relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/{\sim_{\mathcal{M}}}$.

Thought in most situations one could dispense with $\mathcal{M}$ in favour of $\mathcal{M}/{\sim_{\mathcal{M}}}$, in concrete cases one has a better grip on the first than on the latter. Therefore below we insist in working with $\mathbb{L}$-relations in place of $\mathbb{L}$-elementary maps.

3. Elementary substructures

An $\mathbb{L}$-(elementary) embedding is an $\mathbb{L}$-relation that is functional, injective and total.

When $\varnothing$ is an $\mathbb{L}$-relation between $\mathcal{M}$ and $\mathcal{N}$ we say that $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{L}$-(elementary) equivalent and write $\mathcal{M} \equiv^{\mathbb{L}} \mathcal{N}$. When $A \subseteq M \cap N$, we write $\mathcal{M} \equiv_A \mathcal{N}$ if Δ_A, the diagonal relation on A, is an $\mathbb{L}$-relation. In words, we say that $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{L}$-equivalent over A. Finally, we write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \subseteq \mathcal{N}$, i.e. $\mathcal{M}$ is a substructure of $\mathcal{N}$, and $\mathcal{M} \equiv^{\mathbb{L}}_M \mathcal{N}$. In words, we say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$.

Let $A \subseteq M$. The $\mathbb{L}(A)$-topology on $M^{|x|}$ is the topology that has, as a base of closed sets, the $\mathbb{L}(A)$-definable sets.

The $\mathbb{L}(A)$-topology is not T_0. The following lemma asserts that to obtain a Hausdorff topology it suffices to quotient by $\equiv^{\mathbb{L}}_A$. This is the so-called the Kolmogorov quotient.

We need the following definitions that will be relevant in many of the proofs below. If $\varphi(x) \in \mathbb{L}(A)$ and $\varepsilon > 0$ we write ${}^{\varepsilon}\varphi(x)$ for the formula obtained by replacing every occurence in φ of the atomic formula $t(x) \leq 0$ with $t(x) - \varepsilon \leq 0$ or, less petandically, $t(x) \leq \varepsilon$. We write ${}^{\neg\varepsilon}\varphi(x)$ for the formula obtained by replacing $t(x) \leq 0$ with $\varepsilon \leq t(x)$.

Note that ${}^{\varepsilon}\varphi(x),\ {}^{\neg\varepsilon}\varphi(x) \in \mathbb{L}(A)$. Moreover

$$\varphi(x) \quad\to\quad {}^{\varepsilon}\varphi(x)$$

and
$$ {}^{\neg\varepsilon}\varphi(x) \quad\to\quad \neg\varphi(x)$$

The following fact is immediate.

8 Fact. For every $a \in M^{|x|}$ and very $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{M} \models \forall\varepsilon > 0\ \ {}^{\varepsilon}\varphi(a) \quad\Leftrightarrow\quad \mathcal{M} \models \varphi(a)$$

$$\mathcal{M} \models \forall\varepsilon > 0\ {}^{\neg\varepsilon}\varphi(a) \quad\Leftrightarrow\quad \mathcal{M} \models \neg\varphi(a)$$

□

9 Lemma. If $a \not\equiv^{\mathbb{L}}_{A} b$ are two elements of $M^{|x|}$ then there are two formulas $\varphi(x), \psi(x) \in \mathbb{L}(A)$ such that $\varphi(a)$ and $\psi(b)$ hold and $\varphi(M) \cap \psi(M) = \varnothing$.

Proof. Let $\varphi(x) \in \mathbb{L}(A)$ be such that $\varphi(a)$ and $\neg\varphi(b)$. By the fact above ${}^{\neg\varepsilon}\varphi(b)$ holds for some $\varepsilon > 0$. Then let $\psi(x) = {}^{\neg\varepsilon}\varphi(x)$. □

10 Proposition (Tarski-Vaught test). Let $\mathcal{M} \subseteq \mathcal{N}$. Let x be a single variable. Then the following are equivalent

1. $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$;
2. for every $\varphi(x) \in \mathbb{L}(M)$, if $\varphi(N) \neq \varnothing$, then $\mathcal{N} \models \varphi(b)$ for some $b \in M$.

Proof. Implication 1⇒2 is clear. As for 2⇒1, we prove by induction on the syntax of $\varphi(z)$, where z is a finite tuple, that

3. $$\mathcal{N} \models \varphi(a) \quad\Leftrightarrow\quad \mathcal{M} \models \varphi(a) \qquad\qquad \text{for every } a \in M^{|z|}$$

For $\varphi(z)$ atomic 3 follows from $\mathcal{M} \subseteq \mathcal{N}$. Induction for the connectives $\wedge$ and $\vee$ is straightforeward. Induction for the existential quantifier is as in the classical case. We prove the case of the universal quantifier over the unit ball. Assume that 3 holds for the formula $\psi(z, y)$ and prove that it holds for the formula $\forall y\, \psi(z, y)$. The the implication $\Rightarrow$ is immediate. As for $\Leftarrow$,

$$\begin{aligned}
\mathcal{N} \models \neg\forall y\, \psi(a, y) \quad&\Rightarrow\quad \mathcal{N} \models \forall y\, {}^{\neg\varepsilon}\psi(a, y) && \text{for some } \varepsilon > 0\\
&\Rightarrow\quad \mathcal{M} \models \forall y\, {}^{\neg\varepsilon}\psi(a, y) && \text{for some } \varepsilon > 0\\
&\Rightarrow\quad \mathcal{M} \models \neg\forall y\, \psi(a, y).
\end{aligned}$$

Induction for the universal quantifier of sort $\mathbb{R}$ is immediate. □

4. Ultraproducts

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $f : I \to \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \to F} f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq \mathbb{R} \cup \{\pm\infty\}$ that is a neighborhood of λ. Such a λ is unique and, when F is an ultrafilter, it always exists. When f is bounded, $\lambda \in \mathbb{R}$.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle M_i, \breve{M}_i, \mathbb{R} \rangle$, that are uniformly bounded, that is, the bounds in 3 of Definition 1 are the same for all $\mathcal{M}_i$.

Let F be an ultrafilter on I.

11 **Definition.** We define a structure $\mathcal{N} = \langle N, \breve{N}, \mathbb{R} \rangle$ that we call the ultraproduct of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. N comprise the sequences $\hat{a} : I \to \bigcup_{i \in I} M_i$ such that $\hat{a}\, i \in M_i$.
2. $\breve{N}$ comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a}i)$, where $t(x)$ is a term of sort $M^n \to \breve{M}$.
3. If f is a function of sort $M^n \times \breve{M}^m \to M$ then $f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big)$ is the sequence $f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big)$.
4. Similarly when f is of sort $M^n \times \breve{M}^m \to \breve{M}$.
5. If f is a function of sort $M^n \times \breve{M}^m \to \mathbb{R}$ then
$$f^{\mathcal{N}}\big(\hat{a}, t^{\mathcal{N}}(\hat{c})\big) \;=\; \lim_{i \to F} f^{\mathcal{M}_i}\big(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)\big).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that $\mathcal{N}$ is an ultrapower of $\mathcal{M}$.

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models $\mathcal{M}_i$ have the same bounds.

The following fact is easily proved by induction on the syntax.

12 **Fact.** For every term $t(x)$ of sort $M^n \to \mathbb{R}$

$$t^{\mathcal{N}}\big(\hat{a}\big) \;=\; \lim_{i \to F} t^{\mathcal{M}_i}\big(\hat{a}i\big). \qquad \square$$

Finally, we prove

13 **Proposition (Łoś Theorem).** Let $\mathcal{N}$ be as above and let $\varphi(x, y) \in \mathbb{L}(A)$. Let $\hat{a} \in N^{|x|}$. Then there is a function $u_{\_} : \mathbb{R}^+ \to F$ such that for every $\lambda \in \mathbb{R}^{|y|}$

1. $\mathcal{N} \vDash \varphi(\hat{a}, \lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \vDash^{\varepsilon} \varphi(\hat{a}i, \lambda) \big\}$ for every $\varepsilon > 0$.
2. $\mathcal{N} \nvDash \varphi(\hat{a}, \lambda) \;\Rightarrow\; u_\varepsilon \subseteq \big\{ i \in I \;:\; \mathcal{M}_i \nvDash^{\varepsilon} \varphi(\hat{a}i, \lambda) \big\}$ for some $\varepsilon > 0$.

Proof. Suppose that $\varphi(x, y)$ is atomic, say

$$\varphi(x, y) \quad = \quad f(t(x),\ y) \leq 0,$$

where t is a tuple of terms of sort $M^{|x|} \to \mathbb{R}$ and f is a uniformly continuous function $\mathbb{R}^{|t|+|y|} \to \mathbb{R}$.

Let $\beta' \in \mathbb{R}^{|t|}$ be such that $\lim\limits_{i \to F} t(\hat{a}i) = \beta'$.

Given $\varepsilon > 0$, let B_ε be a neighborhood of β' such that

$$\Big| f(\beta,\ \lambda) - f(\beta', \lambda) \Big| < \varepsilon \qquad\qquad \text{for every } \beta \in B_\varepsilon \text{ and every } \lambda \in \mathbb{R}^{|y|}.$$

Such a neighborhood exists by the uniform continuity of f. Finally, to obtain 1 and 2 above it suffices to define

$$u_\varepsilon \quad = \quad \big\{ i \ :\ t(\hat{a}i) \in B_\varepsilon \big\}.$$

The rest of the inductive proof is straightforeward and is left to the reader. □

5. Saturation

Let $p(x) \subseteq \mathbb{L}(M)$, where x has the sort of the unit ball. We say that $p(x)$ is finitely satisfied in $\mathcal{M}$ if for every conjunction of formulas in $p(x)$, say $\varphi(x)$, and for every $\varepsilon > 0$, there is an $a \in M^{|x|}$ such that $\mathcal{M} \vDash {}^{\varepsilon}\varphi(a)$.

The following definition is completely standard

14 Definition. We say that $\mathcal{M}$ is $\mathbb{L}$-saturated if for every $p(x)$ as in 1 and 2 below, there is an $a \in M^{|x|}$ such that $\mathcal{M} \vDash p(a)$

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is finitely satisfied in $\mathcal{M}$.

Now we prove that every model embeds $\mathbb{L}$-elementarily in an $\mathbb{L}$-saturated one. First we prove the following lemma.

15 Lemma. Every model $\mathcal{M}$ embeds $\mathbb{L}$-elementarily in a model $\mathcal{N}$ that realizes all types as in the definition above.

Proof. Consider the collection of types such that 1 and 2 above. Assume that each type has a distinct variable and let x be the concatenation of all these variables. We denote by $p(x)$ the union of all these types. Let I be the set of formulas $\xi(x)$ that are satisfied in $\mathcal{M}$. For every formula $\varphi(x)$ and every $\varepsilon > 0$ define $X_{\varphi,\varepsilon} \subseteq I$ as follows

$$X_{\varphi,\varepsilon} \quad = \quad \Big\{ \xi(x) \in I \ :\ \xi(M) \ \subseteq\ {}^{\varepsilon}\varphi(M) \Big\}$$

Note that ${}^{\varepsilon}\varphi(x)$ is satisfied in $\mathcal{M}$ if and only if $X_\varphi \neq \varnothing$ if and only if ${}^{\varepsilon}\varphi(x) \in X_{\varphi,\varepsilon}$. Moreover

$$X_{\varphi\wedge\psi,\ \min\{\varepsilon,\delta\}} \quad\subseteq\quad X_{\varphi,\varepsilon}\cap X_{\psi,\delta}.$$

Then, as $p(x)$ is finitely consistent, the set

$$B \quad=\quad \big\{X_{\varphi,\varepsilon} \ :\ \varphi(x)\in p \text{ and } \varepsilon>0\big\}$$

has the finite intersection property. Extend B to an ultrafilter F on I. Let $\mathcal{N}$ be the ultrapower of $\mathcal{M}$ over F. That is, the model with unit ball $N = M^I$ obtained as in Definition 11.

For every formula $\xi(x)\in I$ choose some $a_\xi \in M^{|x|}$ such that $\xi(a_\xi)$. We are confusing $(M^I)^{|x|}$ with $(M^{|x|})^I$ as it simplifies notation. Let $\hat{a}\in(M^{|x|})^I$ be the function that maps $\xi(x)\mapsto a_\xi$. We claim that $\mathcal{N}\vDash\varphi(\hat{a})$ for every $\varphi(x)\in p$. By Lemma **??**, it suffices to prove that $\mathcal{N}\vDash{}^{\varepsilon}\varphi(\hat{a})$ for every $\varepsilon>0$. Suppose not, then by Łŏś Theorem (with ε for λ), for some $\varepsilon'>0$ the set

$$\Big\{\xi(x)\in I \quad:\quad \mathcal{M}\nvDash{}^{\varepsilon+\varepsilon'}\varphi(a_\xi)\Big\}$$

belongs to F. This is a contradiction because $B\subseteq F$ contains the set $X_{\varphi,\varepsilon+\varepsilon'}$. $\square$