CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted struture. Ideally, we would like that elementary extensions of normed spaces mantain the usal notion of real numbers[1]. Unfortunately, this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $\mathcal{L} \supseteq L$ be a two-sorted language. We consider the class of $\mathcal{L}$-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

i. $R^n \to R$;

ii. $M^n \to M$;

iii. $M^n \to R$.

The interpretation of symbols of sort $R^n \to R$ is required to be continuous, the interpretation of symbols of sort $M^n \to R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq \mathcal{L}$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated in the same class.

[1]Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ (or, in general R) should remain $\mathbb{R}$ throughout.

For convenience we assume that the functions of sort $M^n \to M$ and the relations of sort M^n are all in L. It is also convenient to assume that $\mathcal{L}$ contains names for all continuous bounded functions $R^n \to R$.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact[2] and $t(x, y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge, \vee$; the quantifiers $\forall, \exists$ of sort M; and the quantifiers $\forall^C, \exists^C$, by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

For later reference we remark the following which is an immediate consequence of the restrictions we imposed on the sorts of the function symbols.

3 Remark. Each component of the tuple $t(x\,; y)$ in the definition above is either of sort $M^{|x|} \to R$ or of sort $R^{|y|} \to R$. □

2. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. By the normality of R we also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The following ia technical lemma that is required in the proof of Łŏś Theorem in the next section.

4 Lemma. Assume the following data

· $\varphi(x; y) \in \mathbb{L}$;

[2]We confuse the relation symbols in $\mathcal{L}$ of sort R^m with their interpretation and write $t \in C$ for $C(t)$

- $\langle a_i : i \in I\rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I\rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Then the following are equivalent

$$\{i \in I : \mathcal{M} \vDash \varphi'(a_i; \alpha)\} \in F \qquad \text{for every } \varphi' > \varphi;$$
$$\{i \in I : \mathcal{M} \vDash \varphi'(a_i; \alpha_i)\} \in F \qquad \text{for every } \varphi' > \varphi.$$

Proof. We claim that the following implications hold for every $\varphi' > \varphi$

i. $$\big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha)\big\} \in F \;\Rightarrow\; \big\{i \in I : \mathcal{M} \vDash \varphi'(a_i\,; \alpha_i)\big\} \in F;$$

ii. $$\big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha_i)\big\} \in F \;\Rightarrow\; \big\{i \in I : \mathcal{M} \vDash \varphi'(a_i\,; \alpha)\big\} \in F.$$

The lemma follows from the claim by the density of $>$.

By induction on the syntax. We only proof (i), the proof of (ii) is essetially the same. When $\varphi(x; y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$. First, note that by Remark 3 we trivially have that

$$F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \;=\; F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i).$$

We prove (i). Assume $\{i : t(a_i; \alpha) \in C\} \in F$. Let $O \supseteq C$ be open. Then $F\text{-}\lim_i t(a_i; \alpha){\in}O$. By what noted above, $F\text{-}\lim_i t(a_i; \alpha_i) \in O$. As O is arbitrary, $\{i : t(a_i\,; \alpha_i) \in C'\} \in F$ for every compact neghborhod of C.

This proves the basis case of the induction.

Induction clear for the connectives $\vee$, $\wedge$. To deal with the universal quantifier of sort M we assume inductively that

ih. $$\big\{i \in I : \mathcal{M} \vDash \varphi(a_i, b_i\,; \alpha)\big\} \in F \;\Rightarrow\; \big\{i \in I : \mathcal{M} \vDash \varphi'(a_i, b_i\,; \alpha_i)\big\} \in F;$$

First we prove

i∀. $$\big\{i \in I : \mathcal{M} \vDash \forall y\, \varphi(a_i, y\,; \alpha)\big\} \in F \;\Rightarrow\; \big\{i \in I : \mathcal{M} \vDash \forall y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F.$$

Negate the consequence. Then there is a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \neg\varphi'(a_i, b_i\,; \alpha_i)\big\} \in F$. If for a contradiction the antecedent of (i∀) holds, from (ih) we would obtain $\big\{i : \varphi(a_i, b_i\,; \alpha)\big\} \in F$. A fotriori $\big\{i : \varphi'(a_i, b_i\,; \alpha)\big\} \in F$, which is a contradiction that proves (i∀).

To deal with the existential quantifier of sort M we prove

i∃. $$\big\{i \in I : \mathcal{M} \vDash \exists y\, \varphi(a_i, y\,; \alpha)\big\} \in F \;\Rightarrow\; \big\{i \in I : \mathcal{M} \vDash \exists y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F.$$

Assume the antecedent of the implication. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Then there is a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \varphi''(a_i, b_i\,; \alpha)\big\} \in F$. By (ih) of the induction hypothesis $\big\{i : \exists y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F$.

Induction for the quantifiers $\forall^C$ and $\exists^C$ is virtually identical. □

3. Ultraproducts

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I\rangle$. We require that for each function symbol f of sort $M^n \to R$

\# there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two semplifications: (1) we only consider ultratpowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F. The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

5 Definition. We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \to M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i\big) : i \in I\rangle$.
3. The interpretation of functions of sort $R^n \to R$ remains unchanged.
4. If f is a function of sort $M^n \to R$ then
$$f^{\mathcal{N}}\big(\hat{a}\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i\big).$$
5. If r is a relation symbol of sort M^n then
$$\mathcal{N} \vDash (\hat{a}) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash r(\hat{a}i)\big\} \in F.$$
6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultraproducts, it would not hold without the uniformity requirement (#) above.

6 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

7 Fact. If $t(x)$ is a term of type $M^{|x|} \to M$ then
$$t^{\mathcal{N}}\big(\hat{a}\big) \;=\; \langle t^{\mathcal{M}}\big(\hat{a}i\big) : i \in I\rangle.$$
□

By Remark 3 we also have that

8 Fact. For every tuple $t(x; y)$ of sort $M^{|x|} \times R^{|y|} \to R$

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big). \qquad \square$$

Finally, we prove

9 Proposition (Łŏś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x) \in \mathbb{L}$. Then for every $\hat{a} \in N^{|x|}$ the following are equivalent

- $\mathcal{N} \vDash \varphi'(\hat{a})$ for every $\varphi' > \varphi$;
- $\big\{i \in I \ : \ \mathcal{M} \vDash \varphi'(\hat{a}i)\big\} \in F$ for every $\varphi' > \varphi$.

Proof. We claim that the following implications hold for every $\varphi' > \varphi$

i. $\quad \big\{i \,:\, \mathcal{M} \vDash \varphi(\hat{a}i)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi'(\hat{a});$

ii. $\quad \mathcal{N} \vDash \varphi(\hat{a}) \;\Rightarrow\; \big\{i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i)\big\} \in F.$

By the density of $>$, the lemma follows from the claim.

The claim is proved by induction on the syntax. If $\varphi(x) \in L$ then the theorem reduces to the classical Łŏś Theorem. Then, suppose that $\varphi(x)$ is as in (i) of Definition 2, say it is the formula $t(x) \in C$.

(i) Assume $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i) \in C\big\} \in F$. Then $F\text{-}\lim_i t(\hat{a}i) \in C'$, where C' is any neighborhood of C. By Fact 8, $\mathcal{N} \vDash t(\hat{a}) \in C'$.

(ii) Assume $\mathcal{N} \vDash t(\hat{a}) \in C$. By Fact 8 and the definition of F-limit, $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i) \in C'\big\} \in F$ where C' is any neighborhood of C.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

ih. $\quad \big\{i \,:\, \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi'(\hat{a}, \hat{b});$

iih. $\quad \mathcal{N} \vDash \varphi(\hat{a}\,;\hat{b}) \;\Rightarrow\; \big\{i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i\,;\hat{b}i)\big\} \in F.$

First we prove

i$\forall$. $\quad \big\{i \,:\, \mathcal{M} \vDash \forall y\, \varphi(\hat{a}i, y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \forall y\, \varphi'(\hat{a}, y);$

ii$\forall$. $\quad \mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i \,:\, \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y)\big\} \in F.$

(i$\forall$) Assume $\mathcal{N} \nvDash \varphi'(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i \,:\, \mathcal{M} \nvDash \varphi(\hat{a}i, \hat{b}i)\big\} \in F$. A fortiori $\big\{i \,:\, \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y)\big\} \in F$ as required.

(ii$\forall$) Assume $\big\{i \,:\, \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y)\big\} \in F$. Pick $\hat{b}$ such that $\big\{i \,:\, \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. If for a contradiction $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y)$. Then in particular $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$. By induction hypothesis $\big\{i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$, a contradiction.

Now we prove

i∃. $\big\{i : \mathcal{M} \vDash \exists y\, \varphi(\hat{a}i, y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \exists y\, \varphi'(\hat{a}, y)$;

ii∃. $\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\big\} \in F$.

(i∃) Assume $\big\{i : \mathcal{M} \vDash \exists y\, \varphi(\hat{a}i, y)\big\} \in F$. Choose some $\hat{b}$ such that $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i)\big\} \in F$. By induction hypothesis $\mathcal{N} \vDash \varphi'(\hat{a}, \hat{b})$. Therefore $\mathcal{N} \vDash \exists y\, \varphi'(\hat{a}, y)$ as required.

(ii∃) Assume $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. A fortiori $\big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\big\} \in F$.

To deal with the quantifiers $\forall^C$ and $\exists^C$ we assume inductively

ih. $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,; \alpha)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi'(\hat{a}\,; \alpha)$;

iih. $\mathcal{N} \vDash \varphi(\hat{a}\,; \alpha) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,; \alpha)\big\} \in F$.

First we prove

i$\forall^C$. $\big\{i : \mathcal{M} \vDash \forall^C y\, \varphi(\hat{a}i\,; y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \forall^C y\, \varphi'(\hat{a}\,; y)$;

ii$\forall^C$. $\mathcal{N} \vDash \forall y^C \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i, y)\big\} \in F$.

(i$\forall^C$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}\,; \beta)$ for some β. By induction hypothesis, $\big\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i\,; \beta)\big\} \in F$. A fortiori $\big\{i : \mathcal{M} \nvDash \forall^C y\, \varphi'(\hat{a}i\,; y)\big\} \in F$ as required.

(ii$\forall^C$) Assume $\big\{i : \mathcal{M} \nvDash \forall^C y\, \varphi(\hat{a}i\,; y)\big\} \in F$. Pick $\beta_i \in C$ such that $\big\{i : \mathcal{M} \nvDash \varphi(\hat{a}i\,; \beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. By Lemma 4 $\big\{i : \mathcal{M} \nvDash \varphi''(\hat{a}i\,; \beta)\big\} \in F$ for some $\varphi'' > \varphi$. If for a contradiction $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}\,; y)$, then $\mathcal{N} \vDash \varphi(\hat{a}\,; \beta)$ and, by induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,; \beta)\big\} \in F$ for every $\varphi' > \varphi$, a contradiction.

Finally, we prove

i$\exists^C$. $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi(\hat{a}i\,; y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \exists^C y\, \varphi'(\hat{a}\,; y)$;

ii$\exists^C$. $\mathcal{N} \vDash \exists y^C \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i, y)\big\} \in F$.

(i$\exists^C$) Assume $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi(\hat{a}i\,; y)\big\} \in F$. Choose $\beta_i \in C$ such that $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,; \beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. By Lemma 4, $\big\{i : \mathcal{M} \vDash \varphi''(\hat{a}i\,; \beta)\big\} \in F$ for every $\varphi'' > \varphi$. Hence from the induction hypothesis and the density of $>$ we obtain $\mathcal{N} \vDash \varphi'(\hat{a}\,; \beta)$ and a fortiori $\mathcal{N} \vDash \exists^C y\, \varphi'(\hat{a}\,; y)$..

(ii$\exists^C$) Assume $\mathcal{N} \vDash \varphi(\hat{a}\,; \beta)$ for some β. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,; \beta)\big\} \in F$. Therefore $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i\,; y)\big\} \notin F$. $\square$