CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted struture. Ideally, we would like that elementary extensions of normed spaces mantain the usal notion of real numbers[1]. Unfortunately, this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $\mathcal{L} \supseteq L$ be a two-sorted language. We consider the class of $\mathcal{L}$-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

i. $R^n \to R$;

ii. $M^n \to M$;

iii. $M^n \to R$.

The interpretation of symbols of sort $R^n \to R$ is required to be continuous and bounded, the interpretation of symbols of sort $M^n \to R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq \mathcal{L}$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated in the same class.

[1]Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ (or, in general R) should remain itself throughout.

For convenience we assume that the functions of sort $M^n \to M$ and the relations of sort M^n are all in L. It is also convenient to assume that $\mathcal{L}$ contains names for all continuous bounded functions $R^n \to R$.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

i. formulas of the form $t(x;y) \in C$, where $C \subseteq R^n$ is compact[2] and $t(x,y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge, \vee$; the quantifiers $\forall, \exists$ of sort M; and the quantifiers $\forall^C, \exists^C$, by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

For later reference we remark the following which is an immediate consequence of the restrictions we imposed on the sorts of the function symbols.

3 Remark. Each component of the tuple $t(x\,;y)$ in the definition above is either of sort $M^{|x|} \to R$ or of sort $R^{|y|} \to R$. □

2. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained replacing each atomic formula $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. By the normality of R we also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in C'$ where C' is some compact set disjoint from C. Moreover the atomic formulas in L are replaced with their negation and every connective is replaced with its dual. I.e., $\vee, \wedge, \exists, \forall, \exists^C, \forall^C$ are replaced with $\wedge, \vee, \forall, \exists, \forall^C, \exists^C$ respectively.

It is clear that $\tilde{\varphi} \to \neg\varphi$ and that $\tilde{\varphi} \leftrightarrow \neg\varphi$ when $\varphi \in L$. We say that $\tilde{\varphi}$ is a strong negation of φ

4 Lemma. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$. Vice versa, for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi'' > \varphi$ such that $\varphi \to \varphi'' \to \neg\tilde{\varphi}$.

[2]We confuse the relation symbols in $\mathcal{L}$ of sort R^m with their interpretation and write $t \in C$ for $C(t)$

Proof. If $\varphi \in L$ the claim is obvious. Suppose φ is of the form $t \in C$. Then φ' is $t \in C'$ for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\psi = (t \in C' \smallsetminus O)$ is as required.

The lemma follows easily by induction. □

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The following is technical lemma that is required in the proof of Łőś Theorem in the next section.

5 Lemma. Assume the following data

- $\varphi(x; y) \in \mathbb{L}$;
- $\langle a_i : i \in I\rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I\rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Then the following implications hold

i. $\big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha)\big\} \in F \;\Rightarrow\; \big\{i \in I : \mathcal{M} \vDash \varphi'(a_i\,; \alpha_i)\big\} \in F$ for every $\varphi' > \varphi$;

ii. $\big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha_i)\big\} \in F \;\Rightarrow\; \big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha)\big\} \in F$.

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$. First, note that by Remark 3 we trivially have that

$$F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i).$$

(i) Assume $\{i : t(a_i; \alpha) \in C\} \in F$. Let $O \supseteq C$ be open. Then $F\text{-}\lim_i t(a_i; \alpha) \in O$. By what noted above, $F\text{-}\lim_i t(a_i; \alpha_i) \in O$. As O is arbitrary, $\{i : t(a_i\,; \alpha_i) \in C'\} \in F$ for every compact neghborhod of C.

(ii) Assume $\{i : t(a_i; \alpha) \notin C\} \in F$. Then $F\text{-}\lim_i t(a_i; \alpha) \notin C$. Hence $F\text{-}\lim_i t(a_i; \alpha_i) \notin C$ and $\{i : t(a_i; \alpha) \notin C\} \in F$ follows.

This proves the basis case of the induction.

Induction clear for the connectives $\vee$, $\wedge$. To deal with the universal quantifier of sort M we assume inductively that ih. $\big\{i\in I\ :\ \mathcal{M}\vDash\varphi(a_i,b_i\,;\alpha)\big\}\in F\ \Rightarrow\ \big\{i\in I\ :\ \mathcal{M}\vDash\varphi'(a_i,b_i\,;\alpha_i)\big\}\in F$;

iih. $\big\{i\in I\ :\ \mathcal{M}\vDash\varphi(a_i,b_i\,;\alpha_i)\big\}\in F\ \Rightarrow\ \big\{i\in I\ :\ \mathcal{M}\vDash\varphi(a_i,b_i\,;\alpha)\big\}\in F$;

We prove

i∀. $\big\{i\in I\ :\ \mathcal{M}\vDash\forall y\,\varphi(a_i,y\,;\alpha)\big\}\in F\ \Rightarrow\ \big\{i\in I\ :\ \mathcal{M}\vDash\forall y\,\varphi'(a_i,y\,;\alpha_i)\big\}\in F$.

ii∀. $\big\{i\in I\ :\ \mathcal{M}\vDash\forall y\,\varphi(a_i,y\,;\alpha_i)\big\}\in F\ \Rightarrow\ \big\{i\in I\ :\ \mathcal{M}\vDash\forall y\,\varphi(a_i,y\,;\alpha)\big\}\in F$.

(i∀) Negate the consequent and pick a sequence $\langle b_i\ :\ i\in I\rangle$ such that $\big\{i\ :\ \neg\varphi'(a_i,b_i\,;\alpha_i)\big\}\in F$. If for a contradiction the antecedent of (i∀) holds, from (ih) we would obtain $\big\{i\ :\ \varphi(a_i,b_i\,;\alpha)_i\big\}\in F$. A fotriori $\big\{i\ :\ \varphi'(a_i,b_i\,;\alpha_i)\big\}\in F$, which is a contradiction that proves (i∀).

(ii∀) Negate the consequent and pick a sequence $\langle b_i\ :\ i\in I\rangle$ such that $\big\{i\ :\ \neg\varphi(a_i,b_i\,;\alpha)\big\}\in F$. If for a contradiction the antecedent of (ii∀) holds, from (iih) we would obtain $\big\{i\ :\ \varphi(a_i,b_i\,;\alpha)\big\}\in F$, a contradiction that proves (ii∀).

To deal with the existential quantifier of sort M we prove

i∃. $\big\{i\in I\ :\ \mathcal{M}\vDash\exists y\,\varphi(a_i,y\,;\alpha)\big\}\in F\ \Rightarrow\ \big\{i\in I\ :\ \mathcal{M}\vDash\exists y\,\varphi'(a_i,y\,;\alpha_i)\big\}\in F$.

ii∃. $\big\{i\in I\ :\ \mathcal{M}\vDash\exists y\,\varphi(a_i,y\,;\alpha_i)\big\}\in F\ \Rightarrow\ \big\{i\in I\ :\ \mathcal{M}\vDash\exists y\,\varphi(a_i,y\,;\alpha)\big\}\in F$.

(i∃) Assume the antecedent. Pick φ'' such that $\varphi'>\varphi''>\varphi$. Then there is a sequence $\langle b_i\ :\ i\in I\rangle$ such that $\big\{i\ :\ \varphi''(a_i,b_i\,;\alpha)\big\}\in F$. By (ih) we obtain $\big\{i\ :\ \exists y\,\varphi'(a_i,y\,;\alpha_i)\big\}\in F$.

(ii∃) Assume the antecedent. Then there is a sequence $\langle b_i\ :\ i\in I\rangle$ such that $\big\{i\ :\ \varphi(a_i,b_i\,;\alpha_i)\big\}\in F$. By (iih) we obtain $\big\{i\ :\ \exists y\,\varphi(a_i,y\,;\alpha)\big\}\in F$.

Induction for the quantifiers $\forall^C$ and $\exists^C$ is virtually identical. □

3. Ultraproducts

Below we introduce a suitable notion of ultraproducts of some structures $\langle\mathcal{M}_i\ :\ i\in I\rangle$. We require that for each function symbol f of sort $M^n\to R$

\# there is a compact $C\subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two semplifications: (1) we only consider ultratpowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F. The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M}=\langle M,R\rangle$ be an L-structure.

6 **Definition.** We define a structure $\mathcal{N}=\langle N,R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N=M^I$ that is, it is the set of sequences $\hat{a}\ :\ I\to M$.
2. If f is a function of sort $M^n\to M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i\big)\ :\ i\in I\rangle$.

3. The interpretation of functions of sort $R^n \to R$ remains unchanged.

4. If f is a function of sort $M^n \to R$ then

$$f^{\mathcal{N}}\big(\hat{a}\big) \quad = \quad F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i\big).$$

5. If r is a relation symbol of sort M^n then

$$\mathcal{N} \vDash (\hat{a}) \quad \Leftrightarrow \quad \big\{i \in I \;\; : \;\; \mathcal{M} \vDash r(\hat{a}i)\big\} \in F.$$

6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultraproducts, it would not hold without the uniformity requirement (#) above.

7 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

8 Fact. If $t(x)$ is a term of type $M^{|x|} \to M$ then

$$t^{\mathcal{N}}\big(\hat{a}\big) \quad = \quad \langle t^{\mathcal{M}}\big(\hat{a}i\big) \; : \; i \in I\rangle. \qquad \square$$

By Remark 3 we also have that

9 Fact. For every tuple $t(x; y)$ of sort $M^{|x|} \times R^{|y|} \to R$

$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i; \alpha\big). \qquad \square$$

Finally, we prove an asymmetric version of Łŏś Theorem.

10 Proposition (Łŏś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x) \in \mathbb{L}$. Then for every $\hat{a} \in N^{|x|}$ and every $\varphi' > \varphi$

i. $$\big\{i \; : \; \mathcal{M} \vDash \varphi(\hat{a}i)\big\} \in F \quad \Rightarrow \quad \mathcal{N} \vDash \varphi(\hat{a});$$

ii. $$\mathcal{N} \vDash \varphi(\hat{a}) \quad \Rightarrow \quad \big\{i \; : \; \mathcal{M} \vDash \varphi'(\hat{a}i)\big\} \in F.$$

Proof. The claim is proved by induction on the syntax. If $\varphi(x) \in L$ then the theorem reduces to the classical Łŏś Theorem. Now, suppose instead that $\varphi(x)$ is as in (i) of Definition 2, say it is the formula $t(x) \in C$.

(i) Assume $\big\{i \; : \; \mathcal{M} \vDash t(\hat{a}i) \in C\big\} \in F$. Then $F\text{-}\lim_i t(\hat{a}i) \in C'$ for every neighborhood of C. By regularity, $\mathcal{N} \vDash t(\hat{a}) \in C$.

(ii) Assume $\mathcal{N} \vDash t(\hat{a}) \in C$. By the definition of F-limit, $\big\{i \; : \; \mathcal{M} \vDash t(\hat{a}i) \in C'\big\} \in F$ where C' is any neighborhood of C.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

ih. $\quad \big\{i : \mathcal{M} \models \varphi(\hat{a}i, \hat{b}i)\big\} \in F \;\Rightarrow\; \mathcal{N} \models \varphi(\hat{a}, \hat{b});$

iih. $\quad \mathcal{N} \models \varphi(\hat{a}\,; \hat{b}) \;\Rightarrow\; \big\{i : \mathcal{M} \models \varphi'(\hat{a}i\,; \hat{b}i)\big\} \in F.$

First we prove

i∀. $\quad \big\{i : \mathcal{M} \models \forall y\, \varphi(\hat{a}i, y)\big\} \in F \;\Rightarrow\; \mathcal{N} \models \forall y\, \varphi(\hat{a}, y);$

ii∀. $\quad \mathcal{N} \models \forall y\, \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \models \forall y\, \varphi'(\hat{a}i, y)\big\} \in F.$

(i∀) Assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \not\models \varphi(\hat{a}i, \hat{b}i)\big\} \in F$. A fortiori $\big\{i : \mathcal{M} \not\models \forall y\, \varphi(\hat{a}i, y)\big\} \in F$ as required.

(ii∀) Assume $\big\{i : \mathcal{M} \not\models \forall y\, \varphi'(\hat{a}i, y)\big\} \in F$. Pick $\hat{b}$ such that $\big\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. If for a contradiction $\mathcal{N} \models \forall y\, \varphi(\hat{a}, y)$. Then in particular $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$. By induction hypothesis $\big\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$, a contradiction.

Now we prove

i∃. $\quad \big\{i : \mathcal{M} \models \exists y\, \varphi(\hat{a}i, y)\big\} \in F \;\Rightarrow\; \mathcal{N} \models \exists y\, \varphi(\hat{a}, y);$

ii∃. $\quad \mathcal{N} \models \exists y\, \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \models \exists y\, \varphi'(\hat{a}i, y)\big\} \in F.$

(i∃) Assume $\big\{i : \mathcal{M} \models \exists y\, \varphi(\hat{a}i, y)\big\} \in F$. Choose some $\hat{b}$ such that $\big\{i : \mathcal{M} \models \varphi(\hat{a}i, \hat{b}i)\big\} \in F$. By induction hypothesis $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$. Therefore $\mathcal{N} \models \exists y\, \varphi(\hat{a}, y)$ as required.

(ii∃) Assume $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. A fortiori $\big\{i : \mathcal{M} \models \exists y\, \varphi'(\hat{a}i, y)\big\} \in F$.

To deal with the quantifiers $\forall^C$ and $\exists^C$ we assume inductively

ih. $\quad \big\{i : \mathcal{M} \models \varphi(\hat{a}i\,; \alpha)\big\} \in F \;\Rightarrow\; \mathcal{N} \models \varphi(\hat{a}\,; \alpha);$

iih. $\quad \mathcal{N} \models \varphi(\hat{a}\,; \alpha) \;\Rightarrow\; \big\{i : \mathcal{M} \models \varphi'(\hat{a}i\,; \alpha)\big\} \in F.$

First we prove

i$\forall^C$. $\quad \big\{i : \mathcal{M} \models \forall^C y\, \varphi(\hat{a}i\,; y)\big\} \in F \;\Rightarrow\; \mathcal{N} \models \forall^C y\, \varphi(\hat{a}\,; y);$

ii$\forall^C$. $\quad \mathcal{N} \models \forall y^C \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \models \forall^C y\, \varphi'(\hat{a}i, y)\big\} \in F.$

(i$\forall^C$) Assume $\mathcal{N} \not\models \varphi(\hat{a}\,; \beta)$ for some β. By induction hypothesis, $\big\{i : \mathcal{M} \not\models \varphi(\hat{a}i\,; \beta)\big\} \in F$. A fortiori $\big\{i : \mathcal{M} \not\models \forall^C y\, \varphi(\hat{a}i\,; y)\big\} \in F$ as required.

(ii$\forall^C$) Assume $\big\{i : \mathcal{M} \not\models \forall^C y\, \varphi'(\hat{a}i\,; y)\big\} \in F$. Pick $\beta_i \in C$ such that $\big\{i : \mathcal{M} \not\models \varphi'(\hat{a}i\,; \beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. Then from Lemma 5 we obtain $\big\{i : \mathcal{M} \not\models \varphi(\hat{a}i\,; \beta)\big\} \in F$. Finally, by (iih), we conclude $\mathcal{N} \not\models \forall y^C \varphi(\hat{a}, y)$.

Finally, we prove

i$\exists^C$. $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi(\hat{a}i\,; y)\big\} \in F \;\;\Rightarrow\;\; \mathcal{N} \vDash \exists^C y\, \varphi(\hat{a}\,; y)$;

ii$\exists^C$. $\mathcal{N} \vDash \exists y^C \varphi(\hat{a}, y) \;\;\Rightarrow\;\; \big\{i : \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i, y)\big\} \in F$.

(i$\exists^C$) Assume $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi(\hat{a}i\,; y)\big\} \in F$. Choose $\beta_i \in C$ such that $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,; \beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. By Lemma 5, $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,; \beta)\big\} \in F$.

(ii$\exists^C$) Assume $\mathcal{N} \vDash \varphi(\hat{a}\,; \beta)$ for some β. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,; \beta)\big\} \in F$. Therefore $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i\,; y)\big\} \in F$. □

4. $\mathbb{L}$-Elementrarity

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two structures. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \vDash \varphi(a) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(f a) \qquad\qquad \text{for every } a \in (\operatorname{dom} f)^{|x|}.$$

A total $\mathbb{L}$-elementary map is called an $\mathbb{L}$-(elementary) embedding. If the map $\mathrm{id}_M : M \hookrightarrow N$ is an $\mathbb{L}$-embedding, that is, if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \vDash \varphi(a) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(a) \qquad\qquad \text{for every } a \in M^{|x|},$$

we write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$. The following fact follows from Łŏś Theorem just as in the classical case.

11 Fact. If $\mathcal{N}$ is an ultrapower of $\mathcal{M}$ then there is an $\mathbb{L}$-embedding $\mathcal{M} \hookrightarrow \mathcal{N}$. □

12 Fact. If $f : M \to N$ is $\mathbb{L}$-elementary then for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\operatorname{dom} f)^{|x|}$

$$\mathcal{M} \vDash \varphi'(a) \text{ for every } \varphi' > \varphi \;\;\Leftrightarrow\;\; \mathcal{N} \vDash \varphi'(f a) \text{ for every } \varphi' > \varphi$$

Proof. Only the implication $\Leftarrow$ requires a proof. Suppose $\varphi' > \varphi$ is such that $\mathcal{M} \nvDash \varphi'(a)$. By Lemma 4 there is formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$ and a formula $\varphi'' > \varphi$ such that $\varphi \to \varphi'' \to \neg\tilde{\varphi}$. Then we have

$$\begin{aligned} \mathcal{M} \nvDash \varphi'(a) \;\;&\Rightarrow\;\; \mathcal{M} \vDash \tilde{\varphi}(a) \\ &\Rightarrow\;\; \mathcal{N} \vDash \tilde{\varphi}(f a) \\ &\Rightarrow\;\; \mathcal{N} \nvDash \varphi''(f a) \end{aligned}$$ □

We prove a version of Tarski-Vaught Test. A second equivalent version of this test will be proved in the sections below.

13 Proposition (Tarski-Vaught $\mathbb{L}$-Test). Let M be a subset of N. Then the following are equivalent

1. M is the domain of a structure $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$;

2. for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\mathcal{N} \vDash \exists x \neg\varphi(x) \;\;\Rightarrow\;\; \text{there is an } a \in M \text{ such that } \mathcal{N} \vDash \neg\varphi(a);$$

Proof. (1⇒2) Clear.

(2⇒1) Assume (2). Then, by the classical Tarski-Vaught test $M \preceq N$. We also have that $\mathcal{M}$ is an $\mathcal{L}$-substructure of $\mathcal{N}$. Therefore, form every formula $\varphi(x)$ as in (i) and (ii) of Definition 2 we have that $\mathcal{M} \vDash \varphi(a) \Leftrightarrow \mathcal{N} \vDash \varphi(a)$ for every $a \in M^{|x|}$

Now, assume inductively

$$\mathcal{M} \vDash \varphi(a, b) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(a, b)$$

Using (2) and the induction hypothesis we prove by contrapposition that

$$\mathcal{M} \vDash \forall y\, \varphi(a, y) \;\;\Rightarrow\;\; \mathcal{N} \vDash \forall y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned}
\mathcal{N} \nvDash \forall y\, \varphi(a, y) \;\;&\Rightarrow\;\; \mathcal{N} \vDash \exists y \neg\varphi(a, y)\\
&\Rightarrow\;\; \mathcal{N} \vDash \neg\varphi(a, b) \quad \text{for some } b \in M^{|y|}\\
&\Rightarrow\;\; \mathcal{M} \vDash \neg\varphi(a, b) \quad \text{for some } b \in M^{|y|}\\
&\Rightarrow\;\; \mathcal{M} \nvDash \forall y\, \varphi(a, y)
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists$, $\exists^C$, and $\forall^C$ is straightforward. □

5. $\mathbb{L}$-COMPACTNESS

We say that a teory is consistent if it has a model satisfying Definition 1. Also the notion of logical consequence is restricted to models in this class.

A theory $T \subseteq \mathbb{L}$ be a finitely uniformly consistent if for every φ, conjunction of sentences in T is consistent. Moreover the functions in the models witnessing this have the same bound.

The following proposition follows from Łŏś Theorem by the usual argument.

14 Proposition (Compactness Theorem). Let $T \subseteq \mathbb{L}$ be a finitely uniformly consistent theory. Then T is consistent. □

15 Proposition. Let $p(x) \subseteq \mathbb{L}(M)$ be a finitely consistent in $\mathcal{M}$. Then $\mathcal{N} \vDash \exists x\, p(x)$ for some $\mathcal{N}$ such that $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$. □

16 Definition. We say that $\mathcal{M}$ is $\mathbb{L}$-saturated if for every $p(x)$ as in 1 and 2 below, $\mathcal{M} \vDash \exists x\, p(a)$.

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;

2. $p(x)$ is finitely satisfied in $\mathcal{M}$.

The existence of $\mathbb{L}$-saturated models is proved as in the classical case.

17 Proposition. Every model has an $\mathbb{L}$-elementary extension to a saturated model (possibly of inaccessible cardinality). □

We denote by $\mathcal{U} = \langle U, R\rangle$ some large $\mathbb{L}$-saturated structure which we call the monster model. The cardinality of $\mathcal{U}$ is an inaccessible cardinal that we denote by κ. Below we say model for $\mathbb{L}$-elementariy substructure of $\mathcal{U}$.

18 Fact. Let $p(x) \subseteq \mathbb{L}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{L}(U)$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. The first claim is clear. The second follows from the first by Lemma 4. □

6. Completeness

Needs full rewriting

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \to R$.

19 Fact. If $\bar{a} = \langle a_i : i < \lambda\rangle$ and $\bar{b} = \langle b_i : i < \lambda\rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \to R$.

Proof. By induction on λ. Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ. For limit ordinals induction is immediate. □

20 Example (???). Let L be the language of $\mathbb{R}$-algebras expanded with two lattice operators $\wedge$, $\vee$. Let $\langle \Omega, \mathcal{B}, \Pr\rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L. Let $R = \mathbb{R}$. Assume $\mathcal{L}$ estends L with all the continuous functions $\mathbb{R}^n \to \mathbb{R}$ and for every $n \in \mathbb{N}$ the functions E_n that gives the expected value of $(X \wedge n) \vee -n$ for $n \in \mathbb{N}$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is definable in the limit over M if $a \equiv^{\mathbb{L}}_M x \to a \sim x$

21 Example. Assume that $R = \mathbb{R}$ and that $\mathcal{L}$ contains a function of sort $M^2 \to R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \to R$ are continuous with rispect to this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M, a model;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of M that converges to a.

Proof. (2 ⇒ 1) Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_M a$. By the uniqueness of the limit $d(a, b) = 0$.

(1 ⇒ 2) Assume that $a \equiv_M x \to a \sim x$. Then $a \equiv_M^{\mathbb{L}} x \to d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \mathrm{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \to d(a, x) < 1/n$. By $\mathbb{L}$-elementarity there is an $a_n \in M$ such that $\varphi(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a. □