

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A CLASS OF STRUCTURES

1 Definition. In these notes we deal with 3-sorted¹ structures of the form $\mathcal{M} = \langle M, \check{M}, \mathbb{R} \rangle$. The language and its interpretation are subject to the following conditions.

1. Functions only have one of the following sorts (for any $m, n \in \omega$)
 - a. $\mathbb{R}^n \rightarrow \mathbb{R}$
 - b. $M^n \times \check{M}^m \rightarrow \text{any of } M, \check{M}, \text{ or } \mathbb{R}$
2. There is a symbol for every (total) uniformly continuous function $\mathbb{R}^n \rightarrow \mathbb{R}$.
3. The (functions that interpret the) terms of sort $M^n \rightarrow \mathbb{R}$ have *bounded* range.
4. Every element of \check{M} is the image of some term of sort $M^n \rightarrow \check{M}$.
5. There is only one unary predicate: $x \leq 0$, of sort \mathbb{R} .

We call M the **unit ball** of \mathcal{M} : there is where the action takes place. The auxiliary sort \check{M} can be ignored at a first reading. It is introduced as it makes the description of many important examples less contrived, e.g. Example 5 below.

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $M^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ with parameters in some $A \subseteq M$. Up to equivalence, these terms have the form $f(t(x), y)$, where $t(x)$ is a tuple of terms of sort $M^{|x|} \rightarrow \mathbb{R}$, and f is a function $\mathbb{R}^{|t|+|y|} \rightarrow \mathbb{R}$.

3 Definition. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

- i. $\mathbb{L}(A)$ contains the atomic formula $t \leq 0$ for every term $t \in \mathbb{T}(A)$.
- ii. It is closed under the Boolean connectives \wedge, \vee , and the quantifiers \forall, \exists of sort M .
- iii. It is closed under the quantifier \forall of sort \mathbb{R} relativized to any definable subset of \mathbb{R} .

Note that in iii of the definition above, definability is intended with respect to the full first order language. In particular, if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$, also $\forall \varepsilon > 0 \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

4 Example (Metric spaces of finite diameter). Given a metric space (M, d) of finite diameter we define a structure $\mathcal{M} = \langle M, \check{M}, \mathbb{R} \rangle$ as follows. Set $\check{M} = M$ and let $\text{id} : M \rightarrow \check{M}$ be the identical

¹In some examples it may be more natural to use $(n + n' + 1)$ -sorted structures $\mathcal{M} = \langle M_1, \dots, M_n, \check{M}_1, \dots, \check{M}_{n'}, \mathbb{R} \rangle$. The generalization is straightforward.

isomorphism (hence \check{M} is pleonastic in this case). The function symbols are id and d . Note that 3 is satisfied because d is bounded.

5 Example (Banach spaces). Given a Banach space V we define a structure $\mathcal{M} = \langle M, \check{M}, \mathbb{R} \rangle$ as follows. Let $\check{M} = V$ and let $M = \{a \in \check{M} : \|a\| \leq 1\}$ be the closed unit ball of V . Besides the symbols mentioned above, \mathbb{L} contains a function symbol for the natural embedding $\text{id} : M \rightarrow \check{M}$. It also contains a symbol for the norm $\|\cdot\| : \check{M} \rightarrow \mathbb{R}$. Finally, \mathbb{L} contains the usual symbols of the language of vector spaces. These have sort $\check{M}^n \rightarrow \check{M}$, for the appropriate $n \in \{0, 1, 2\}$.

Terms of sort $M^{|x|} \rightarrow \mathbb{R}$ have the form $f(t(x))$ where f is a uniformly continuous function and

$$t(x) = \left\| \sum_{i=1}^{|x|} \sum_{j=1}^m \lambda_{i,j} x_i \right\|.$$

Therefore, as for all $a \in M^{|x|}$

$$t^{\mathcal{M}}(a) \leq \sum_{i=1}^{|x|} \sum_{j=1}^m |\lambda_{i,j}|,$$

condition 3 of Definition 1 is satisfied. Condition 4 is immediate.

2. \mathbb{L} -RELATIONS

Equality between elements in the unit ball is not among the formulas in \mathbb{L} . For this reason, relations, rather than partial maps, are the natural tool to discuss elementarity.

Let \mathcal{M} and \mathcal{N} be models. We say that $R \subseteq M \times N$ is an \mathbb{L} -(elementary) relation between \mathcal{M} and \mathcal{N} (or on \mathcal{M} if the two coincide) if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(b) \quad \text{for every } a \text{ and } b \text{ such that } a R b.$$

Recall that, when $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$ are tuples, $a R b$ stands for $a_i R b_i$ for every $i \in \{1, \dots, n\}$.

We define an equivalence relation $(\sim_{\mathcal{M}})$ on M as follows

$$1. \quad a \sim_{\mathcal{M}} b \Leftrightarrow \left(\mathcal{M} \models \varphi(a) \leftrightarrow \varphi(b) \text{ for every } \varphi(x) \in \mathbb{L}(M) \right),$$

where $|x| = |a| = |b| = 1$.

6 Lemma. The relation $(\sim_{\mathcal{M}}) \subseteq M^2$ is an \mathbb{L} -relation. Moreover, it is maximal among the \mathbb{L} -relations on \mathcal{M} , i.e. no \mathbb{L} -relation properly contains $(\sim_{\mathcal{M}})$.

Proof. Assume $a \sim_{\mathcal{M}} b$, where $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$. Recall that this means that $a_i \sim_{\mathcal{M}} b_i$ for all $i \in \{1, \dots, n\}$. Let Δ denote the diagonal relation on M . Note that $a_i \sim_{\mathcal{M}} b_i$ is

equivalent to saying that $\Delta \cup \{(a_i, b_i)\}$ is an \mathbb{L} -relation. As \mathbb{L} -relations are closed under composition $\Delta \cup \{(a_1, b_1), \dots, (a_n, b_n)\}$ is \mathbb{L} -elementary. It follows that for every $\varphi(x) \in \mathbb{L}$

$$2. \quad \mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{M} \models \varphi(b).$$

This proves that $(\sim_{\mathcal{M}})$ is an \mathbb{L} -relation. Finally, maximality is immediate. \square

7 Lemma. Let $R \subseteq M \times N$ be total and surjective \mathbb{L} -relation. Then there is a unique maximal \mathbb{L} -relation containing R . This maximal \mathbb{L} -relation is equal to both $(\sim_{\mathcal{M}}) R$ and $R(\sim_{\mathcal{N}})$.

Proof. It is immediate to verify that $(\sim_{\mathcal{M}}) R$ is an \mathbb{L} -relation containing R . Let S be any maximal \mathbb{L} -relation containing R . By maximality, $(\sim_{\mathcal{M}}) S = S$. As S is a total relation $(\sim_{\mathcal{M}}) \subseteq S S^{-1}$. Therefore, by the lemma above, $(\sim_{\mathcal{M}}) = S S^{-1}$. As R is a surjective relation, $S \subseteq S S^{-1} R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}}) R$. A similar argument proves that $S = R(\sim_{\mathcal{N}})$. \square

We write $\text{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, \mathbb{L} -relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/\sim_{\mathcal{M}}$.

Thought in most situations one could dispense with \mathcal{M} in favour of $\mathcal{M}/\sim_{\mathcal{M}}$, in concrete cases one has a better grip on the first than on the latter. Therefore below we insist in working with \mathbb{L} -relations in place of \mathbb{L} -elementary maps.

3. ELEMENTARY SUBSTRUCTURES

An \mathbb{L} -(elementary) embedding is an \mathbb{L} -relation that is functional, injective and total.

When \varnothing is an \mathbb{L} -relation between \mathcal{M} and \mathcal{N} we say that \mathcal{M} and \mathcal{N} are \mathbb{L} -(elementary) equivalent and write $\mathcal{M} \equiv^{\mathbb{L}} \mathcal{N}$. When $A \subseteq M \cap N$, we write $\mathcal{M} \equiv_A \mathcal{N}$ if Δ_A , the diagonal relation on A , is an \mathbb{L} -relation. In words, we say that \mathcal{M} and \mathcal{N} are \mathbb{L} -equivalent over A . Finally, we write $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \subseteq \mathcal{N}$, i.e. \mathcal{M} is a substructure of \mathcal{N} , and $\mathcal{M} \equiv_M^{\mathbb{L}} \mathcal{N}$. In words, we say that \mathcal{M} is an \mathbb{L} -(elementary) substructure of \mathcal{N} .

Let $A \subseteq M$. The $\mathbb{L}(A)$ -topology on $M^{|x|}$ is the topology that has, as a base of closed sets, the $\mathbb{L}(A)$ -definable sets.

The $\mathbb{L}(A)$ -topology is not T_0 . The following lemma asserts that to obtain a Hausdorff topology it suffices to quotient by $\equiv_A^{\mathbb{L}}$. This is the so-called the Kolmogorov quotient.

We need the following definitions that will be relevant in many of the proofs below. If $\varphi(x) \in \mathbb{L}(A)$ and $\varepsilon > 0$ we write ${}^{\varepsilon}\varphi(x)$ for the formula obtained by replacing every occurrence in φ of the atomic formula $t(x) \leq 0$ with $t(x) - \varepsilon \leq 0$ or, less petandically, $t(x) \leq \varepsilon$. We write ${}^{\neg\varepsilon}\varphi(x)$ for the formula obtained by replacing $t(x) \leq 0$ with $\varepsilon \leq t(x)$.

Note that ${}^\varepsilon\varphi(x), \neg^\varepsilon\varphi(x) \in \mathbb{L}(A)$. Moreover

$$\varphi(x) \rightarrow {}^\varepsilon\varphi(x)$$

and

$$\neg^\varepsilon\varphi(x) \rightarrow \neg\varphi(x)$$

The following fact is immediate.

8 Fact. For every $a \in M^{|x|}$ and very $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{M} \models \forall \varepsilon > 0 \quad {}^\varepsilon\varphi(a) \Leftrightarrow \mathcal{M} \models \varphi(a)$$

$$\mathcal{M} \models \forall \varepsilon > 0 \quad \neg^\varepsilon\varphi(a) \Leftrightarrow \mathcal{M} \models \neg\varphi(a) \quad \square$$

9 Lemma. If $a \not\equiv_A^{\mathbb{L}} b$ are two elements of $M^{|x|}$ then there are two formulas $\varphi(x), \psi(x) \in \mathbb{L}(A)$ such that $\varphi(a)$ and $\psi(b)$ hold and $\varphi(M) \cap \psi(M) = \emptyset$.

Proof. Let $\varphi(x) \in \mathbb{L}(A)$ be such that $\varphi(a)$ and $\neg\varphi(b)$. By the fact above $\neg^\varepsilon\varphi(b)$ holds for some $\varepsilon > 0$. Then let $\psi(x) = \neg^\varepsilon\varphi(x)$. \square

10 Proposition (Tarski-Vaught test). Let $\mathcal{M} \subseteq \mathcal{N}$. Let x be a single variable. Then the following are equivalent

1. $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$;
2. for every $\varphi(x) \in \mathbb{L}(M)$, if $\varphi(N) \neq \emptyset$, then $\mathcal{N} \models \varphi(b)$ for some $b \in M$.

Proof. Implication $1 \Rightarrow 2$ is clear. As for $2 \Rightarrow 1$, we prove by induction on the syntax of $\varphi(z)$, where z is a finite tuple, that

$$3. \quad \mathcal{N} \models \varphi(a) \Leftrightarrow \mathcal{M} \models \varphi(a) \quad \text{for every } a \in M^{|z|}$$

For $\varphi(z)$ atomic 3 follows from $\mathcal{M} \subseteq \mathcal{N}$. Induction for the connectives \wedge and \vee is straightforward. Induction for the existential quantifier is as in the classical case. We prove the case of the universal quantifier over the unit ball. Assume that 3 holds for the formula $\psi(z, y)$ and prove that it holds for the formula $\forall y \psi(z, y)$. The the implication \Rightarrow is immediate. As for \Leftarrow ,

$$\begin{aligned} \mathcal{N} \models \neg \forall y \psi(a, y) &\Rightarrow \mathcal{N} \models \forall y \neg^\varepsilon \psi(a, y) && \text{for some } \varepsilon > 0 \\ &\Rightarrow \mathcal{M} \models \forall y \neg^\varepsilon \psi(a, y) && \text{for some } \varepsilon > 0 \\ &\Rightarrow \mathcal{M} \models \neg \forall y \psi(a, y). \end{aligned}$$

Induction for the universal quantifier of sort \mathbb{R} is immediate. \square

4. ULTRAPRODUCTS

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I . If $f : I \rightarrow \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \rightarrow F} f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq \mathbb{R} \cup \{\pm\infty\}$ that is a neighborhood of λ . Such a λ is unique and, when F is an ultrafilter, it always exists. When f is bounded, $\lambda \in \mathbb{R}$.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle M_i, \check{M}_i, \mathbb{R} \rangle$, that are **uniformly bounded**, that is, the bounds in 3 of Definition 1 are the same for all \mathcal{M}_i .

Let F be an ultrafilter on I .

11 Definition. We define a structure $\mathcal{N} = \langle N, \check{N}, \mathbb{R} \rangle$ that we call the **ultraproduct** of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. N comprise the sequences $\hat{a} : I \rightarrow \bigcup_{i \in I} M_i$ such that $\hat{a} i \in M_i$.
2. \check{N} comprise the sequences $t^{\mathcal{N}}(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a}i)$, where $t(x)$ is a term of sort $M^n \rightarrow \check{M}$.
3. If f is a function of sort $M^n \times \check{M}^m \rightarrow M$ then $f^{\mathcal{N}}(\hat{a}, t^{\mathcal{N}}(\hat{c}))$ is the sequence $f^{\mathcal{M}_i}(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i))$.
4. Similarly when f is of sort $M^n \times \check{M}^m \rightarrow \check{M}$.
5. If f is a function of sort $M^n \times \check{M}^m \rightarrow \mathbb{R}$ then

$$f^{\mathcal{N}}(\hat{a}, t^{\mathcal{N}}(\hat{c})) = \lim_{i \rightarrow F} f^{\mathcal{M}_i}(\hat{a}i, t^{\mathcal{M}_i}(\hat{c}i)).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that \mathcal{N} is an **ultrapower** of \mathcal{M} .

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models \mathcal{M}_i have the same bounds.

The following fact is easily proved by induction on the syntax.

12 Fact. For every term $t(x)$ of sort $M^n \rightarrow \mathbb{R}$

$$t^{\mathcal{N}}(\hat{a}) = \lim_{i \rightarrow F} t^{\mathcal{M}_i}(\hat{a}i).$$

□

Finally, we prove

13 Proposition (Łoś Theorem). Let \mathcal{N} be as above and let $\varphi(x, y) \in \mathbb{L}(A)$. Let $\hat{a} \in N^{|x|}$. Then there is a function $u_- : \mathbb{R}^+ \rightarrow F$ such that for every $\lambda \in \mathbb{R}^{|y|}$

1. $\mathcal{N} \models \varphi(\hat{a}, \lambda) \Rightarrow u_- \subseteq \{i \in I : \mathcal{M}_i \models {}^\varepsilon \varphi(\hat{a}i, \lambda)\}$ for every $\varepsilon > 0$.
2. $\mathcal{N} \not\models \varphi(\hat{a}, \lambda) \Rightarrow u_- \subseteq \{i \in I : \mathcal{M}_i \not\models {}^\varepsilon \varphi(\hat{a}i, \lambda)\}$ for some $\varepsilon > 0$.

Proof. Suppose that $\varphi(x, y)$ is atomic, say

$$\varphi(x, y) = f(t(x), y) \leq 0,$$

where t is a tuple of terms of sort $M^{|x|} \rightarrow \mathbb{R}$ and f is a uniformly continuous function $\mathbb{R}^{|t|+|y|} \rightarrow \mathbb{R}$.

Let $\beta' \in \mathbb{R}^{|t|}$ be such that $\lim_{i \rightarrow F} t(\hat{a}i) = \beta'$.

Given $\varepsilon > 0$, let B_ε be a neighborhood of β' such that

$$\left| f(\beta, \lambda) - f(\beta', \lambda) \right| < \varepsilon \quad \text{for every } \beta \in B_\varepsilon \text{ and every } \lambda \in \mathbb{R}^{|y|}.$$

Such a neighborhood exists by the uniform continuity of f . Finally, to obtain 1 and 2 above it suffices to define

$$u_\varepsilon = \{i : t(\hat{a}i) \in B_\varepsilon\}.$$

The rest of the inductive proof is straightforward and is left to the reader. \square

5. SATURATION

Let $p(x) \subseteq \mathbb{L}(M)$, where x has the sort of the unit ball. We say that $p(x)$ is **finitely satisfied in \mathcal{M}** if for every conjunction of formulas in $p(x)$, say $\varphi(x)$, and for every $\varepsilon > 0$, there is an $a \in M^{|x|}$ such that $\mathcal{M} \models^\varepsilon \varphi(a)$.

The following definition is completely standard

14 Definition. We say that \mathcal{M} is **\mathbb{L} -saturated** if for every $p(x)$ as in 1 and 2 below, there is an $a \in M^{|x|}$ such that $\mathcal{M} \models p(a)$

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is finitely satisfied in \mathcal{M} .

Now we prove that every model embeds \mathbb{L} -elementarily in an \mathbb{L} -saturated one. First we prove the following lemma.

15 Lemma. Every model \mathcal{M} embeds \mathbb{L} -elementarily in a model \mathcal{N} that realizes all types as in the definition above.

Proof. Consider the collection of types such that 1 and 2 above. Assume that each type has a distinct variable and let x be the concatenation of all these variables. We denote by $p(x)$ the union of all these types. Let I be the set of formulas $\xi(x)$ that are satisfied in \mathcal{M} . For every formula $\varphi(x)$ and every $\varepsilon > 0$ define $X_{\varphi, \varepsilon} \subseteq I$ as follows

$$X_{\varphi, \varepsilon} = \left\{ \xi(x) \in I : \xi(M) \subseteq {}^\varepsilon \varphi(M) \right\}$$

Note that ${}^\varepsilon \varphi(x)$ is satisfied in \mathcal{M} if and only if $X_\varphi \neq \emptyset$ if and only if ${}^\varepsilon \varphi(x) \in X_{\varphi, \varepsilon}$. Moreover

$$X_{\varphi \wedge \psi, \min\{\varepsilon, \delta\}} \subseteq X_{\varphi, \varepsilon} \cap X_{\psi, \delta}.$$

Then, as $p(x)$ is finitely consistent, the set

$$B = \{X_{\varphi, \varepsilon} : \varphi(x) \in p \text{ and } \varepsilon > 0\}$$

has the finite intersection property. Extend B to an ultrafilter F on I . Let \mathcal{N} be the ultrapower of \mathcal{M} over F . That is, the model with unit ball $N = M^I$ obtained as in Definition 11. By Łoś Theorem, \mathcal{M} embeds elementarily into \mathcal{N} . It remains to prove that \mathcal{N} realizes $p(x)$.

For every formula $\xi(x) \in I$ choose some $a_\xi \in M^{|x|}$ such that $\xi(a_\xi)$. Let $\hat{a} \in (M^{|x|})^I$ be the function that maps $\xi(x) \mapsto a_\xi$. We claim that $\mathcal{N} \models \varphi(\hat{a})$ for every $\varphi(x) \in p$. By Fact 8, it suffices to prove that $\mathcal{N} \models {}^\varepsilon \varphi(\hat{a})$ for every $\varepsilon > 0$. Suppose not, then by Łoś Theorem (Proposition 13 with ε for λ), for some $\varepsilon' > 0$ the set

$$Y_{\varphi, \varepsilon + \varepsilon'} = \{\xi(x) \in I : \mathcal{M} \not\models {}^{\varepsilon + \varepsilon'} \varphi(a_\xi)\}$$

belongs to F . This is a contradiction because $B \subseteq F$ contains the set $X_{\varphi, \varepsilon + \varepsilon'} \subseteq Y_{\varphi, \varepsilon + \varepsilon'}$. \square

16 Corollary. Every model \mathcal{M} is an \mathbb{L} -elementary substructure of some saturated model \mathcal{N} (possibly of inaccessible cardinality).

6. COMPLETE MODELS

Fix a model \mathcal{M} with unit ball M . Let \mathcal{U} be a saturated extension of \mathcal{M} with unit ball U . Let $a \in U$. We say that \mathcal{M} **isolates a in the limit** if there are a finite set $A \subseteq M$ and an infinite set $A \subseteq B \subseteq M$ such that for every infinite $A \subseteq C \subseteq B$

$$p_{|C}(x) \vdash p(x) := \text{tp}_{\mathbb{L}}(a/U)$$

The following proposition is immediate consequence of the definition of the relation $(\sim_{\mathcal{U}})$, see Section 2.

17 Proposition. Suppose that $a' \equiv_M^{\mathbb{L}} a$ are isolated in the limit by \mathcal{M} . Then $a' \sim_{\mathcal{U}} a$.

Finally we define completeness which is a form of saturation whose strength lies between ω - \mathbb{L} -saturation and ω_1 - \mathbb{L} -saturation.

18 Definition. A model \mathcal{M} is complete if it contains all points isolated in the limit. The completion of \mathcal{M} is the minimal model \mathcal{M}' that contains all points isolated by \mathcal{M} . It is easy to verify that \mathcal{M}' is complete.

It is easy to verify that if \mathcal{M} is the metric space in Example 4, then the completeness of \mathcal{M} implies the Cauchy-completeness of (M, d) .