CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a minimal motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted struture. Ideally, we would like that elementary extensions of normed spaces mantain the usal notion of real numbers[1]. Unfortunately, this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

Our first definition restrict the class of structure we consider in these notes. Our choice is a compromise to avoid eccessive complications. It would be interesting to drop the restriction on the sort of the function symbols.

1 Definition. Let L be a one-sorted (first-order) language. Let $\mathcal{L} \supseteq L$ be a two-sorted language. We consider the class of $\mathcal{L}$-structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L-structures, while R is fixed. By model we intend a structure as those described in this definition.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

i. $R^n \to R$;

ii. $M^n \to M$;

iii. $M^n \to R$.

The interpretation of symbols of sort $R^n \to R$ is required to be continuous and bounded, the interpretation of symbols of sort $M^n \to R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n.

There might be models in the class described above that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq \mathcal{L}$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated and belongs to the same class.

For convenience we assume that the functions of sort $M^n \to M$ and the relations of sort M^n are all in L. It is also convenient to assume that $\mathcal{L}$ contains names for all continuous bounded functions

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[1] Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we are adamant that $\mathbb{R}$ (or, in general R) should remain itself throughout.

$R^n \to R$. There are relations symbol of sort R^n for all (and only the) compact subsets of R^n. Therefore, to descibe $\mathcal{L}$, we only need specify L and the function symbols of sort $M^n \to R$.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from the following sets of $\mathbb{L}$-atomic formulas

    i.   formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact[2] and $t(x, y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

    ii.   all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge, \vee$; the quantifiers $\forall, \exists$ of sort M; and the quantifiers $\forall^C, \exists^C$, by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

We write $\mathbb{H}$ for the set of formulas in $\mathbb{L}$ without quantifiers of sort R.

For later reference we remark the following which is an immediate consequence of the restrictions we imposed on the sorts of the function symbols.

3 Remark. Each component of the tuple $t(x; y)$ in the definition above is either of sort $M^{|x|} \to R$ or of sort $R^{|y|} \to R$. $\qquad\qquad\square$

Formulas in $\mathbb{H}$ are the immediate generalization of the positive bounded formulas of Henson and Iovino. Here we also introduce the larger class $\mathbb{L}$ because it offers some advantages. For instance, it is easy to see (cf. Example 4 for a hint) that $\mathbb{L}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive). Somewhat surprisingly, we will see (cf. Propositions 27 and 28) that the formulas in $\mathbb{L}$ can be very well approximated by formulas in $\mathbb{H}$.

4 Example. Let $R = [0, 1]$, the unit interval of $\mathbb{R}$. Let $t(x)$ be a term of sort $M^{|x|} \to R$. Then there is a formula in $\mathbb{L}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall x \left[t(x) \div \tau \in \{0\} \right] \quad \wedge \quad \forall \varepsilon \left[\varepsilon \in \{0\} \vee \exists x \left[(\tau \div t(x)) \div \varepsilon \in \{0\} \right] \right]$$

which, in a human readable form, becomes

$$\forall x \left[t(x) \leq \tau \right] \quad \wedge \quad \forall \varepsilon > 0 \, \exists x \left[\tau \leq t(x) + \varepsilon \right].$$
$\qquad\qquad\square$

2. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If such atomic formulas do not occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. Moreover the $\mathbb{L}$-atomic formulas in L are replaced with their negation and every connective is replaced with its dual. I.e., $\vee, \wedge, \exists, \forall, \exists^C, \forall^C$ are replaced with $\wedge, \vee, \forall, \exists, \forall^C, \exists^C$ respectively.

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[2]We confuse the relation symbols in $\mathcal{L}$ of sort R^m with their interpretation and write $t \in C$ for $C(t)$

It is clear that $\tilde{\varphi} \to \neg\varphi$. We say that $\tilde{\varphi}$ is a ▢strong negation▢ of φ.

5 Lemma. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$. Vice versa, for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Let T be a compact set containing tha range of t. Let φ' be $t \in C'$, for some compact neighborhod of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Note that $t \notin T \smallsetminus O$ is equivalent to $t \in O$ by the choice of T. Then $\tilde{\varphi} = (t \in T \smallsetminus O)$ is as required by the lemma.

Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the local compactness of R, there are C' and O, disjoint neighborhoods of C and $\tilde{C}$ respectively. We can also require that C' is compact. Then $\varphi' = t \in C'$ is as required.

The lemma follows easily by induction. ▢

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The following is technical lemma that is required in the proof of Łoś Theorem in the next section.

6 Lemma. Assume the following data

· $\varphi(x; y) \in \mathbb{L}$;

· $\langle a_i : i \in I \rangle$, a sequence of elements of $M^{|x|}$;

· $\langle \alpha_i : i \in I \rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;

· $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Then the following implications hold

i. $\{i \in I : \mathcal{M} \vDash \varphi(a_i; \alpha)\} \in F \;\Rightarrow\; \{i \in I : \mathcal{M} \vDash \varphi'(a_i; \alpha_i)\} \in F$ for every $\varphi' > \varphi$;

ii. $\{i \in I : \mathcal{M} \vDash \varphi(a_i; \alpha_i)\} \in F \;\Rightarrow\; \{i \in I : \mathcal{M} \vDash \varphi(a_i; \alpha)\} \in F$.

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$. First, note that by Remark 3 we trivially have that

$$F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \;=\; F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i).$$

(i) Assume $\{i : t(a_i; \alpha) \in C\} \in F$. Then $F\text{-}\lim_i t(a_i; \alpha) \in C$. By what noted above, $F\text{-}\lim_i t(a_i; \alpha_i) \in C$. Then $\{i : t(a_i; \alpha_i) \in C'\} \in F$ where C' is any compact neghborhod of C.

(ii) Assume $\{i : t(a_i; \alpha) \notin C\} \in F$. Then $F\text{-}\lim_i t(a_i; \alpha) \notin C$. Hence $F\text{-}\lim_i t(a_i; \alpha_i) \notin C$ and $\{i : t(a_i; \alpha) \notin C\} \in F$ follows.

This proves the basis case of the induction.

Induction clear for the connectives $\vee, \wedge$. To deal with the universal quantifier of sort M we assume inductively that

ih. $\{i \in I : \mathcal{M} \vDash \varphi(a_i, b_i ; \alpha)\} \in F \Rightarrow \{i \in I : \mathcal{M} \vDash \varphi'(a_i, b_i ; \alpha_i)\} \in F;$

iih. $\{i \in I : \mathcal{M} \vDash \varphi(a_i, b_i ; \alpha_i)\} \in F \Rightarrow \{i \in I : \mathcal{M} \vDash \varphi(a_i, b_i ; \alpha)\} \in F;$

We prove

i∀. $\{i \in I : \mathcal{M} \vDash \forall y \, \varphi(a_i, y ; \alpha)\} \in F \Rightarrow \{i \in I : \mathcal{M} \vDash \forall y \, \varphi'(a_i, y ; \alpha_i)\} \in F.$

ii∀. $\{i \in I : \mathcal{M} \vDash \forall y \, \varphi(a_i, y ; \alpha_i)\} \in F \Rightarrow \{i \in I : \mathcal{M} \vDash \forall y \, \varphi(a_i, y ; \alpha)\} \in F.$

(i∀) Negate the consequent and pick a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \neg\varphi'(a_i, b_i ; \alpha_i)\} \in F$. If for a contradiction the antecedent of (i∀) holds, from (ih) we would obtain $\{i : \varphi(a_i, b_i ; \alpha)_i\} \in F$. A fotriori $\{i : \varphi'(a_i, b_i ; \alpha_i)\} \in F$, which is a contradiction that proves (i∀).

(ii∀) Negate the consequent and pick a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \neg\varphi(a_i, b_i ; \alpha)\} \in F$. If for a contradiction the antecedent of (ii∀) holds, from (iih) we would obtain $\{i : \varphi(a_i, b_i ; \alpha)\} \in F$, a contradiction that proves (ii∀).

To deal with the existential quantifier of sort M we prove

i∃. $\{i \in I : \mathcal{M} \vDash \exists y \, \varphi(a_i, y ; \alpha)\} \in F \Rightarrow \{i \in I : \mathcal{M} \vDash \exists y \, \varphi'(a_i, y ; \alpha_i)\} \in F.$

ii∃. $\{i \in I : \mathcal{M} \vDash \exists y \, \varphi(a_i, y ; \alpha_i)\} \in F \Rightarrow \{i \in I : \mathcal{M} \vDash \exists y \, \varphi(a_i, y ; \alpha)\} \in F.$

(i∃) Assume the antecedent. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Then there is a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \varphi''(a_i, b_i ; \alpha)\} \in F$. By (ih) we obtain $\{i : \exists y \, \varphi'(a_i, y ; \alpha_i)\} \in F$.

(ii∃) Assume the antecedent. Then there is a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \varphi(a_i, b_i ; \alpha_i)\} \in F$. By (iih) we obtain $\{i : \exists y \, \varphi(a_i, y ; \alpha)\} \in F$.

Induction for the quantifiers $\forall^C$ and $\exists^C$ is virtually identical. $\qquad \square$

3. ULTRAPRODUCTS

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that the models occurring in the ultraproduct are uniformly bounded as defined below.

7 Definition. We say that a family of models $\{\mathcal{M}_i : i \in I\}$ is uniformly bounded if for every function symbol f there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two semplifications: (1) we only consider ultratpowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F. The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R \rangle$ be an L-structure.

8 Definition. We define a structure $\boxed{\mathcal{N} = \langle N, R \rangle}$ that we call the $\boxed{\text{ultrapower}}$ of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.

2. If f is a function of sort $M^n \to M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I \rangle$.

3. The interpretation of functions of sort $R^n \to R$ remains unchanged.

4. If f is a function of sort $M^n \to R$ then
$$f^{\mathcal{N}}(\hat{a}) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$

5. If r is a relation symbol of sort M^n then
$$\mathcal{N} \vDash (\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \vDash r(\hat{a}i)\} \in F.$$

6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the requirement of uniformity in Definition 7.

9 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

10 Fact. If $t(x)$ is a term of type $M^{|x|} \to M$ then

$$t^{\mathcal{N}}(\hat{a}) \;=\; \langle t^{\mathcal{M}}(\hat{a}i) : i \in I \rangle. \qquad \qquad \square$$

By Remark 3 we also have that

11 Fact. For every tuple $t(x; y)$ of sort $M^{|x|} \times R^{|y|} \to R$

$$t^{\mathcal{N}}(\hat{a}; \alpha) \;=\; F\text{-}\lim_{i} t^{\mathcal{M}}(\hat{a}i; \alpha). \qquad \qquad \square$$

Finally, we prove an asymmetric version of Łoś Theorem.

12 Proposition (Łoś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x) \in \mathbb{L}$. Then for every $\hat{a} \in N^{|x|}$ and every $\varphi' > \varphi$

i. $\qquad\qquad \{i : \mathcal{M} \vDash \varphi(\hat{a}i)\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi(\hat{a});$

ii. $\qquad\qquad\qquad \mathcal{N} \vDash \varphi(\hat{a}) \;\Rightarrow\; \{i : \mathcal{M} \vDash \varphi'(\hat{a}i)\} \in F.$

Proof. The claim is proved by induction on the syntax. If $\varphi(x) \in L$ then the theorem reduces to the classical Łoś Theorem. Now, suppose instead that $\varphi(x)$ is as in (i) of Definition 2, say it is the formula $t(x) \in C$.

(i) Assume $\{i : \mathcal{M} \vDash t(\hat{a}i) \in C\} \in F$. Then $F\text{-}\lim_i t(\hat{a}i) \in C$ by regularity.

(ii) Assume $\mathcal{N} \vDash t(\hat{a}) \in C$. By the definition of F-limit, $\{i : \mathcal{M} \vDash t(\hat{a}i) \in C'\} \in F$ where C' is any neighborhood of C.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

ih. $\qquad \{i : \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i)\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi(\hat{a}, \hat{b});$

iih. $\qquad\qquad \mathcal{N} \vDash \varphi(\hat{a}; \hat{b}) \;\Rightarrow\; \{i : \mathcal{M} \vDash \varphi'(\hat{a}i; \hat{b}i)\} \in F.$

First we prove

i∀. $\qquad \{i : \mathcal{M} \vDash \forall y\, \varphi(\hat{a}i, y)\} \in F \;\Rightarrow\; \mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y);$

ii∀. $\qquad\qquad \mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y) \;\Rightarrow\; \{i : \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y)\} \in F.$

(i∀) Assume $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\{i : \mathcal{M} \nvDash \varphi(\hat{a}i, \hat{b}i)\} \in F$. A fortiori $\{i : \mathcal{M} \nvDash \forall y\, \varphi(\hat{a}i, y)\} \in F$ as required.

(ii∀) Assume $\{i : \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y)\} \in F$. Pick $\hat{b}$ such that $\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i)\} \in F$. If for a contradiction $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y)$. Then in particular $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$. By induction hypothesis $\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\} \in F$, a contradiction.

Now we prove

i∃. $\left\{i : \mathcal{M} \vDash \exists y\, \varphi(\hat{a}i, y)\right\} \in F \;\Rightarrow\; \mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y);$

ii∃. $\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y) \;\Rightarrow\; \left\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\right\} \in F.$

(i∃) Assume $\left\{i : \mathcal{M} \vDash \exists y\, \varphi(\hat{a}i, y)\right\} \in F$. Choose some $\hat{b}$ such that $\left\{i : \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i)\right\} \in F$. By induction hypothesis $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$. Therefore $\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y)$ as required.

(ii∃) Assume $\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y)$, hence $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\left\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\right\} \in F$. A fortiori $\left\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\right\} \in F$.

To deal with the quantifiers $\forall^{C}$ and $\exists^{C}$ we assume inductively that for all $\varphi' > \varphi$ and all $\hat{a} \in N^{|x|}$

ih. $\left\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,;\alpha)\right\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi(\hat{a}\,;\alpha);$

iih. $\mathcal{N} \vDash \varphi(\hat{a}\,;\alpha) \;\Rightarrow\; \left\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\alpha)\right\} \in F.$

First we prove

i$\forall^{C}$. $\left\{i : \mathcal{M} \vDash \forall^{C} y\, \varphi(\hat{a}i\,;y)\right\} \in F \;\Rightarrow\; \mathcal{N} \vDash \forall^{C} y\, \varphi(\hat{a}\,;y);$

ii$\forall^{C}$. $\mathcal{N} \vDash \forall y^{C} \varphi(\hat{a}, y) \;\Rightarrow\; \left\{i : \mathcal{M} \vDash \forall^{C} y\, \varphi'(\hat{a}i, y)\right\} \in F.$

(i$\forall^{C}$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}\,;\beta)$ for some $\beta \in C$. By induction hypothesis, $\left\{i : \mathcal{M} \nvDash \varphi(\hat{a}i\,;\beta)\right\} \in F$. Therefore $\left\{i : \mathcal{M} \nvDash \forall^{C} y\, \varphi(\hat{a}i\,;y)\right\} \in F$ as required.

(ii$\forall^{C}$) Let $\varphi' > \varphi'' > \varphi$. Assume $\left\{i : \mathcal{M} \nvDash \forall^{C} y\, \varphi'(\hat{a}i\,;y)\right\} \in F$. Then there are some $\beta_i \in C$ such that $\left\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i\,;\beta_i)\right\} \in F$. Let $\beta = F\text{-}\lim_i \beta_i$. Then from Lemma 6 we obtain $\left\{i : \mathcal{M} \nvDash \varphi''(\hat{a}i\,;\beta)\right\} \in F$. Finally, by (iih), we conclude $\mathcal{N} \nvDash \forall y^{C} \varphi(\hat{a}, y)$.

Finally, we prove

i$\exists^{C}$. $\left\{i : \mathcal{M} \vDash \exists^{C} y\, \varphi(\hat{a}i\,;y)\right\} \in F \;\Rightarrow\; \mathcal{N} \vDash \exists^{C} y\, \varphi(\hat{a}\,;y);$

ii$\exists^{C}$. $\mathcal{N} \vDash \exists y^{C} \varphi(\hat{a}, y) \;\Rightarrow\; \left\{i : \mathcal{M} \vDash \exists^{C} y\, \varphi'(\hat{a}i, y)\right\} \in F.$

(i$\exists^{C}$) Assume $\left\{i : \mathcal{M} \vDash \exists^{C} y\, \varphi(\hat{a}i\,;y)\right\} \in F$. Choose $\beta_i \in C$ such that $\left\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,;\beta_i)\right\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. By Lemma 6, $\left\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,;\beta)\right\} \in F$.

(ii$\exists^{C}$) Assume $\mathcal{N} \vDash \varphi(\hat{a}\,;\beta)$ for some $\beta \in C$. By induction hypothesis, $\left\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\beta)\right\} \in F$. Therefore $\left\{i : \mathcal{M} \vDash \exists^{C} y\, \varphi'(\hat{a}i\,;y)\right\} \in F$. $\square$

4. ELEMENTARITY

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two structures. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\mathrm{dom}\, f)^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\Rightarrow\; \mathcal{N} \vDash \varphi(fa)$$

An $\mathbb{L}$-elementary map is in particular L-elementary and therefore it is injective. An $\mathbb{L}$-elementary map that is total is called an $\mathbb{L}$-(elementary) embedding. When the map $\mathrm{id}_M : M \hookrightarrow N$ is an $\mathbb{L}$-embedding, that is, if for every $\varphi(x) \in \mathbb{L}$ and every $a \in M^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\Rightarrow\; \mathcal{N} \vDash \varphi(a),$$

we write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$.

The following follows from Łoś Theorem just as in the classical case.

13 Fact. If $\mathcal{N}$ is an ultrapower of $\mathcal{M}$ then there is an $\mathbb{L}$-embedding $\mathcal{M} \hookrightarrow \mathcal{N}$. □

The definitions of $\boxed{\mathbb{H}\text{-elementary}}$ map/embedding/substructure are similar. Note that any total and surjective $\mathbb{H}$-elementary map is immediately an $\mathcal{L}$-isomorphism.

14 Proposition. Let $f : M \to N$ be an $\mathbb{L}$-elementary map. Then for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\mathrm{dom}\, f)^{|x|}$

$$\mathcal{M} \vDash \varphi'(a) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi'(fa)$$

for every $\varphi' > \varphi$. The same holds with $\mathbb{H}$ for $\mathbb{L}$

5. COMPACTNESS

A theory T is consistent if $\mathcal{M} \vDash T$ for some model $\mathcal{M}$ satisfying Definition 1. As in the classical case, a theory T is said to be finitely consistent if every conjunction of sentences in T is consistent. We further say that T is $\boxed{\text{uniformly}}$ finitely consistent if the finite consistency is witnessed by uniformly bounded models (see Definition 7). The following proposition follows from Łoś Theorem by the usual argument.

15 Proposition (Compactness Theorem). Let $T \subseteq \mathbb{L}$ be a uniformly finitely consistent theory. Then T is consistent. □

16 Proposition. Let $p(x) \subseteq \mathbb{L}(M)$ be a finitely consistent in $\mathcal{M}$. Then $\mathcal{N} \vDash \exists x\, p(x)$ for some $\mathcal{N}$ such that $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$. □

A model $\mathcal{N}$ is $\boxed{\mathbb{L}\text{-saturated}}$ if it realizes all types with fewer than $|\mathcal{N}|$ parameters that are finitely consistent in $\mathcal{N}$. The existence of $\mathbb{L}$-saturated models is proved as in the classical case.

17 Proposition. Every model has an $\mathbb{L}$-elementary extension to a saturated model (possibly of inaccessible cardinality). □

6. THE MONSTER MODEL

We denote by $\boxed{\mathcal{U} = \langle U, R \rangle}$ some large $\mathbb{L}$-saturated structure which we call the $\boxed{\text{monster model.}}$ The cardinality of $\mathcal{U}$ is an inaccessible cardinal that we denote by $\boxed{\kappa}$. Below we say $\boxed{\text{model}}$ for $\mathbb{L}$-elementary substructure of $\mathcal{U}$.

Let $A \subseteq U$ be a small set. As usual, we define a topology on $U^{|x|}$ which we call the $\mathbb{L}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x) \subseteq \mathbb{L}(A)$. This is a compact topology.

The $\mathbb{L}(A)$-topology on $U^{|x|} \times R^{|y|}$ is the product of the $\mathbb{L}(A)$-topology on $U^{|x|}$ and the topology on $R^{|y|}$.

18 Lemma. Let $S \subseteq R$ be a compact set. For every $\varphi(x\,;y) \in \mathbb{L}(A)$ the set $\varphi(U\,;S)$ is compact.

Proof. We can assume that S, or suitable power of it, contains all compact sets that occur in $\varphi(x\,;y)$ as predicate or as bounds of quantifiers.

□

19 Fact. Let $p(x) \subseteq \mathbb{L}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{L}(U)$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;

2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. The first claim is clear. The second follows from the first by Lemma 5. $\square$

20 Proposition. For every $\varphi(x) \in \mathbb{H}(U)$

$$\bigwedge_{\varphi' > \varphi} \varphi'(x) \quad \leftrightarrow \quad \varphi(x)$$

Proof. We prove $\to$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifier is immediate. Consider the existential quantifiers of sort U. Assume inductively

$$\bigwedge_{\varphi' > \varphi} \varphi'(x, y) \quad \to \quad \varphi(x, y)$$

then

$$\exists y \bigwedge_{\varphi' > \varphi} \varphi'(x, y) \quad \to \quad \exists y\, \varphi(x, y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi' > \varphi} \exists y\, \varphi'(x, y) \quad \to \quad \exists y \bigwedge_{\varphi' > \varphi} \varphi'(x, y)$$

Replace x with a parameter, say a and assume the antecedent. Note that $\{\exists y\, \varphi'(a, y) : \varphi' > \varphi\}$ implies the finite concistency of the type $\{\varphi'(a, y) : \varphi' > \varphi\}$. This because if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. Therefore $\exists y\, \{\varphi'(a, y) : \varphi' > \varphi\}$ follows by saturation. $\square$

When $A \subseteq U$, we write $S_{\mathbb{H}}(A)$ for the set of types

$$\mathbb{H}\text{-tp}(a/A) \quad = \quad \{\varphi(x) \;:\; \varphi(x) \in \mathbb{H}(A) \text{ such that } \varphi(a)\}$$

as a ranges over the tuples of elements of U. We write $S_{\mathbb{H},x}(A)$ when the tuple of variables x is fixed. The following corollary will be strengthen by Corollary 26 below.

21 Corollary. The types $p(x) \in S_{\mathbb{H}}(A)$ are complete. That is, either $\varphi(x) \in p$ or $p(x) \to \neg\varphi(x)$ for every $\varphi(x) \in \mathbb{H}(A)$.

Proof. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 5 and Proposition 20 we obtain

$$\neg\varphi(x) \quad \to \quad \bigvee_{\tilde\varphi \perp \varphi} \tilde\varphi(x).$$

Hence $\tilde\varphi(a)$ holds for some $\tilde\varphi \perp \varphi$ and $p(x) \to \neg\varphi(x)$ follows. $\square$

The following will be useful below

22 Remark. For $p(x) \subseteq \mathbb{L}(U)$, we write $p'(x)$ for the type

$$p'(x) \quad = \quad \{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\}.$$

Note that, by Proposition 20, if $p(x) \in S_{\mathbb{H}}(A)$ then $p'(x)$ is equivalent to $p(x)$. Therefore $p'(x)$ is also complete for formulas in $\mathbb{H}(A)$. □

23 Corollary. The inverse of an $\mathbb{H}$-elementary map $f : \mathcal{U} \to \mathcal{U}$ is $\mathbb{H}$-elementary. □

7. HOMOGENEITY

A model $\mathcal{M}$ is $\boxed{\mathbb{H}\text{-homogeneous}}$ if every $\mathbb{H}$-elementary map $f : \mathcal{M} \to \mathcal{M}$ of cardinality $< |\mathcal{M}|$ extends to an automorphism.

24 Proposition. $\mathcal{U}$ is $\mathbb{H}$-homogeneous.

Proof. By Corollary 23 the usual proof by back-and-forth applies. □

25 Corollary. For every $a, b \in U^{|x|}$, if $a \equiv^{\mathbb{H}} b$ then $a \equiv^{\mathbb{L}} b$. □

26 Corollary. Let $p(x) \in S_{\mathbb{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathbb{L}_x(A)$. Clearly, the same holds for $p'(x)$. □

27 Proposition. Let $\varphi(x) \in \mathbb{L}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathbb{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 26

$$\neg\varphi(x) \quad\leftrightarrow\quad \bigvee_{p'(x)\to\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathbb{H},x}(A)$. By Fact 19

$$\neg\varphi(x) \quad\leftrightarrow\quad \bigvee_{\psi(x)\to\neg\varphi(x)} \psi(x)$$

where $\psi'(x)$ is such that $\psi' > \psi$ for some $\psi(x) \in \mathbb{H}(A)$ such that $\psi(x) \to \neg\varphi(x)$. By compactness

$$\neg\varphi(x) \quad\leftrightarrow\quad \bigvee_{i=1}^{n} \psi'_i(x)$$

By Lemma 5 we can replace $\psi'_i(x)$ by $\neg\tilde{\psi}_i(x)$ for some $\tilde{\psi}_i \perp \psi_i$. The proposition follows. □

28 Proposition. Let $\varphi(x) \in \mathbb{L}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathbb{H}(A)$ and some $\psi' > \psi$. □

Proof. Let $a \in U^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$. By Corollary 23, $p'(x) \to \neg\varphi(x)$. By compactness $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. □

29 Proposition (Tarski-Vaught Test). Let M be a subset of U. Then the following are equivalent

1. M is the domain of a model;

2. for every formula $\varphi(x) \in \mathbb{H}(M)$
$$\exists x\, \varphi(x) \quad\Rightarrow\quad \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$

3.   for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\exists x \, \neg\varphi(x) \;\Rightarrow\; \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists x \, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 5 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\forall x \, \tilde{\varphi}(x)$ hence, by (1), $\mathcal{M} \vDash \neg\forall x \, \varphi(x)$. Then $\mathcal{M} \vDash \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $\mathcal{M} \vDash \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathbb{L}(M)$ be such that $\exists x \, \neg\varphi(x)$. Then, by Corollary 28, there are a consistent $\psi(x) \in \mathbb{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). Then, by the classical Tarski-Vaught test $\mathcal{M} \preceq \mathcal{U}$. We also have that $\mathcal{M}$ is an $\mathcal{L}$-substructure of $\mathcal{U}$. Therefore for every $a \in M^{|x|}$ and for every $\mathbb{L}$-atomic formula, $\varphi(a)$ if and only if $\mathcal{M} \vDash \varphi(a)$. Now, assume inductively

$$\mathcal{M} \vDash \varphi(a, b) \;\Rightarrow\; \varphi(a, b)$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$\mathcal{M} \vDash \forall y \, \varphi(a, y) \;\Rightarrow\; \forall y \, \varphi(a, y).$$

Indeed,

$$\neg\forall y \, \varphi(a, y) \;\Rightarrow\; \exists y \, \neg\varphi(a, y)$$
$$\Rightarrow\; \neg\varphi(a, b) \qquad \text{for some } b \in M^{|y|}$$
$$\Rightarrow\; \mathcal{M} \vDash \neg\varphi(a, b) \quad \text{for some } b \in M^{|y|}$$
$$\Rightarrow\; \mathcal{M} \nvDash \forall y \, \varphi(a, y)$$

Induction for the connectives $\vee$, $\wedge$, $\exists$, $\exists^C$, and $\forall^C$ is straightforward.   □

8. COMPLETENESS

Needs full rewriting

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \to R$.

30 Fact. If $\bar{a} = \langle a_i \,:\, i < \lambda \rangle$ and $\bar{b} = \langle b_i \,:\, i < \lambda \rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \to R$.

Proof. By induction on λ. Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ. For limit ordinals induction is immediate.   □

31 Example (???). Let L be the language of $\mathbb{R}$-algebras expanded with two lattice operators $\wedge$, $\vee$. Let $\langle \Omega, \mathcal{B}, \Pr \rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L. Let $R = \mathbb{R}$. Assume $\mathcal{L}$ contains a symbol for the functions E_n, for $n \in \mathbb{N}$, that give the expected value of $(X \wedge n) \vee -n$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is definable in the limit over M if $a \equiv_M^{\mathbb{L}} x \to a \sim x$

32 Example. Assume that $R = \mathbb{R}$ and that $\mathcal{L}$ contains a function of sort $M^2 \to R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \to R$ are continuous with rispect to

this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M, a model;

2.   there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of M that converges to a.

Proof. $(2 \Rightarrow 1)$ Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_M^{\mathbb{L}} a$. By the uniqueness of the limit $d(a, b) = 0$.

$(1 \Rightarrow 2)$ Assume that $a \equiv_M^{\mathbb{L}} x \rightarrow a \sim x$. Then $a \equiv_M^{\mathbb{L}} x \rightarrow d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \mathrm{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \rightarrow d(a, x) < 1/n$. By $\mathbb{L}$-elementarity there is an $a_n \in M$ such that $\varphi(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a. $\qquad \square$