

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted structure. Ideally, we would like that elementary extensions of normed spaces maintain the usual notion of real numbers¹. Unfortunately, this is not possible if we insist to maintain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 Definition. Let L be a one-sorted (first-order) language. Let $\mathcal{L} \supseteq L$ be a two-sorted language. We consider the class of \mathcal{L} -structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L -structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

- i. $R^n \rightarrow R$;
- ii. $M^n \rightarrow M$;
- iii. $M^n \rightarrow R$.

The interpretation of symbols of sort $R^n \rightarrow R$ is required to be continuous and bounded, the interpretation of symbols of sort $M^n \rightarrow R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n .

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set \mathbb{L} of formulas, $L \subseteq \mathbb{L} \subseteq \mathcal{L}$, such that every model as in Definition 1 has an \mathbb{L} -elementary extension that is \mathbb{L} -saturated in the same class.

¹Non standard standard analysts have no problem in expanding \mathbb{R} . In these notes, for a change, we insist as \mathbb{R} (or, in general R) should remain itself throughout.

For convenience we assume that the functions of sort $M^n \rightarrow M$ and the relations of sort M^n are all in L . It is also convenient to assume that \mathcal{L} contains names for all continuous bounded functions $R^n \rightarrow R$.

2 Definition. Formulas in \mathbb{L} are constructed inductively from the following two sets of formulas

- i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact² and $t(x, y)$ is a n -tuple of terms of sort $M^{|x|} \times R^{|y|} \rightarrow R$;
- ii. all formulas in L .

We require that \mathbb{L} is closed under the Boolean connectives \wedge, \vee ; the quantifiers \forall, \exists of sort M ; and the quantifiers \forall^C, \exists^C , by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

For later reference we remark the following which is an immediate consequence of the restrictions we imposed on the sorts of the function symbols.

3 Remark. Each component of the tuple $t(x; y)$ in the definition above is either of sort $M^{|x|} \rightarrow R$ or of sort $R^{|y|} \rightarrow R$. □

2. HENSON-IOVINO APPROXIMATIONS

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained replacing each atomic formula $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C . If no such atomic formulas occurs in φ , then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C .

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \rightarrow \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in C'$ where C' is some compact set disjoint from C . Moreover the atomic formulas in L are replaced with their negation and every connective is replaced with its dual. I.e., $\vee, \wedge, \exists, \forall, \exists^C, \forall^C$ are replaced with $\wedge, \vee, \forall, \exists, \forall^C, \exists^C$ respectively.

It is clear that $\tilde{\varphi} \rightarrow \neg\varphi$ and that $\tilde{\varphi} \leftrightarrow \neg\varphi$ when $\varphi \in L$. We say that $\tilde{\varphi}$ is a **strong negation** of φ

4 Lemma. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \rightarrow \neg\tilde{\varphi} \rightarrow \varphi'$. Vice versa, for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi'' > \varphi$ such that $\varphi \rightarrow \varphi'' \rightarrow \neg\tilde{\varphi}$.

²We confuse the relation symbols in \mathcal{L} of sort R^m with their interpretation and write $t \in C$ for $C(t)$

Proof. If $\varphi \in L$ the claim is obvious. Suppose φ is of the form $t \in C$. Then φ' is $t \in C'$ for some compact neighborhood of C . Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\psi = (t \in C' \setminus O)$ is as required.

The lemma follows easily by induction. \square

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . Let Y be a topological space. If $f : I \rightarrow Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ . Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The following is technical lemma that is required in the proof of Łoś Theorem in the next section.

5 Lemma. Assume the following data

- $\varphi(x; y) \in \mathbb{L}$;
- $\langle a_i : i \in I \rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I \rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I .

Then the following implications hold

- i. $\{i \in I : \mathcal{M} \models \varphi(a_i; \alpha)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \varphi'(a_i; \alpha_i)\} \in F$ for every $\varphi' > \varphi$;
- ii. $\{i \in I : \mathcal{M} \models \varphi(a_i; \alpha_i)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \varphi(a_i; \alpha)\} \in F$.

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L , it does not depend on α , and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$. First, note that by Remark 3 we trivially have that

$$F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i).$$

(i) Assume $\{i : t(a_i; \alpha) \in C\} \in F$. Then $F\text{-}\lim_i t(a_i; \alpha) \in C$. By what noted above, $F\text{-}\lim_i t(a_i; \alpha_i) \in C$. Then $\{i : t(a_i; \alpha_i) \in C'\} \in F$ where C' is any compact neighborhood of C .

(ii) Assume $\{i : t(a_i; \alpha) \notin C\} \in F$. Then $F\text{-}\lim_i t(a_i; \alpha) \notin C$. Hence $F\text{-}\lim_i t(a_i; \alpha_i) \notin C$ and $\{i : t(a_i; \alpha_i) \notin C\} \in F$ follows.

This proves the basis case of the induction.

Induction clear for the connectives \vee, \wedge . To deal with the universal quantifier of sort M we assume inductively that

- ih. $\{i \in I : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F$;

iih. $\{i \in I : \mathcal{M} \models \varphi(a_i, b_i; \alpha_i)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F;$

We prove

iV. $\{i \in I : \mathcal{M} \models \forall y \varphi(a_i, y; \alpha)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(a_i, y; \alpha_i)\} \in F.$

iiV. $\{i \in I : \mathcal{M} \models \forall y \varphi(a_i, y; \alpha_i)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \forall y \varphi(a_i, y; \alpha)\} \in F.$

(iV) Negate the consequent and pick a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \neg \varphi'(a_i, b_i; \alpha_i)\} \in F$. If for a contradiction the antecedent of (iV) holds, from (ih) we would obtain $\{i : \varphi(a_i, b_i; \alpha_i)\} \in F$. A fortiori $\{i : \varphi'(a_i, b_i; \alpha_i)\} \in F$, which is a contradiction that proves (iV).

(iiV) Negate the consequent and pick a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \neg \varphi(a_i, b_i; \alpha)\} \in F$. If for a contradiction the antecedent of (iiV) holds, from (iih) we would obtain $\{i : \varphi(a_i, b_i; \alpha)\} \in F$, a contradiction that proves (iiV).

To deal with the existential quantifier of sort M we prove

i∃. $\{i \in I : \mathcal{M} \models \exists y \varphi(a_i, y; \alpha)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(a_i, y; \alpha_i)\} \in F.$

ii∃. $\{i \in I : \mathcal{M} \models \exists y \varphi(a_i, y; \alpha_i)\} \in F \Rightarrow \{i \in I : \mathcal{M} \models \exists y \varphi(a_i, y; \alpha)\} \in F.$

(i∃) Assume the antecedent. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Then there is a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \varphi''(a_i, b_i; \alpha)\} \in F$. By (ih) we obtain $\{i : \exists y \varphi'(a_i, y; \alpha_i)\} \in F$.

(ii∃) Assume the antecedent. Then there is a sequence $\langle b_i : i \in I \rangle$ such that $\{i : \varphi(a_i, b_i; \alpha_i)\} \in F$. By (iih) we obtain $\{i : \exists y \varphi(a_i, y; \alpha)\} \in F$.

Induction for the quantifiers \forall^C and \exists^C is virtually identical. □

3. ULTRAPRODUCTS

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that for each function symbol f of sort $M^n \rightarrow R$

there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two simplifications: (1) we only consider ultrapowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F . The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

6 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \rightarrow M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I \rangle$.
3. The interpretation of functions of sort $R^n \rightarrow R$ remains unchanged.

4. If f is a function of sort $M^n \rightarrow R$ then

$$f^{\mathcal{N}}(\hat{a}) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$

5. If r is a relation symbol of sort M^n then

$$\mathcal{N} \models (\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i)\} \in F.$$

6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the uniformity requirement (#) above.

7 Fact. The structure \mathcal{N} satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

8 Fact. If $t(x)$ is a term of type $M^{|x|} \rightarrow M$ then

$$t^{\mathcal{N}}(\hat{a}) = \langle t^{\mathcal{M}}(\hat{a}i) : i \in I \rangle.$$

□

By Remark 3 we also have that

9 Fact. For every tuple $t(x; y)$ of sort $M^{|x|} \times R^{|y|} \rightarrow R$

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

□

Finally, we prove an asymmetric version of Łoś Theorem.

10 Proposition (Łoś Theorem). Let \mathcal{N} be as above and let $\varphi(x) \in \mathbb{L}$. Then for every $\hat{a} \in N^{|x|}$ and every $\varphi' > \varphi$

- i. $\{i : \mathcal{M} \models \varphi(\hat{a}i)\} \in F \Rightarrow \mathcal{N} \models \varphi(\hat{a});$
- ii. $\mathcal{N} \models \varphi(\hat{a}) \Rightarrow \{i : \mathcal{M} \models \varphi'(\hat{a}i)\} \in F.$

Proof. The claim is proved by induction on the syntax. If $\varphi(x) \in L$ then the theorem reduces to the classical Łoś Theorem. Now, suppose instead that $\varphi(x)$ is as in (i) of Definition 2, say it is the formula $t(x) \in C$.

(i) Assume $\{i : \mathcal{M} \models t(\hat{a}i) \in C\} \in F$. Then $F\text{-}\lim_i t(\hat{a}i) \in C$ by regularity.

(ii) Assume $\mathcal{N} \models t(\hat{a}) \in C$. By the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i) \in C'\} \in F$ where C' is any neighborhood of C .

This completes the proof of the base case of the induction.

Induction for the connectives \vee and \wedge is clear. To deal with the quantifiers of sort M we assume inductively that

- ih. $\{i : \mathcal{M} \models \varphi(\hat{a}i, \hat{b}i)\} \in F \Rightarrow \mathcal{N} \models \varphi(\hat{a}, \hat{b});$
 iih. $\mathcal{N} \models \varphi(\hat{a}; \hat{b}) \Rightarrow \{i : \mathcal{M} \models \varphi'(\hat{a}i; \hat{b}i)\} \in F.$

First we prove

- iV. $\{i : \mathcal{M} \models \forall y \varphi(\hat{a}i, y)\} \in F \Rightarrow \mathcal{N} \models \forall y \varphi(\hat{a}, y);$
 iiV. $\mathcal{N} \models \forall y \varphi(\hat{a}, y) \Rightarrow \{i : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y)\} \in F.$

(iV) Assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b})$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \not\models \varphi(\hat{a}i, \hat{b}i)\} \in F$. A fortiori $\{i : \mathcal{M} \not\models \forall y \varphi(\hat{a}i, y)\} \in F$ as required.

(iiV) Assume $\{i : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i, y)\} \in F$. Pick \hat{b} such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i)\} \in F$. If for a contradiction $\mathcal{N} \models \forall y \varphi(\hat{a}, y)$. Then in particular $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$. By induction hypothesis $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\} \in F$, a contradiction.

Now we prove

- iE. $\{i : \mathcal{M} \models \exists y \varphi(\hat{a}i, y)\} \in F \Rightarrow \mathcal{N} \models \exists y \varphi(\hat{a}, y);$
 iiE. $\mathcal{N} \models \exists y \varphi(\hat{a}, y) \Rightarrow \{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y)\} \in F.$

(iE) Assume $\{i : \mathcal{M} \models \exists y \varphi(\hat{a}i, y)\} \in F$. Choose some \hat{b} such that $\{i : \mathcal{M} \models \varphi(\hat{a}i, \hat{b}i)\} \in F$. By induction hypothesis $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$. Therefore $\mathcal{N} \models \exists y \varphi(\hat{a}, y)$ as required.

(iiE) Assume $\mathcal{N} \models \varphi(\hat{a}, \hat{b})$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i)\} \in F$. A fortiori $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y)\} \in F$.

To deal with the quantifiers \forall^C and \exists^C we assume inductively

- ih. $\{i : \mathcal{M} \models \varphi(\hat{a}i; \alpha)\} \in F \Rightarrow \mathcal{N} \models \varphi(\hat{a}; \alpha);$
 iih. $\mathcal{N} \models \varphi(\hat{a}; \alpha) \Rightarrow \{i : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F.$

First we prove

- iV^C. $\{i : \mathcal{M} \models \forall^C y \varphi(\hat{a}i; y)\} \in F \Rightarrow \mathcal{N} \models \forall^C y \varphi(\hat{a}; y);$
 iiV^C. $\mathcal{N} \models \forall^C y \varphi(\hat{a}, y) \Rightarrow \{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i, y)\} \in F.$

(iV^C) Assume $\mathcal{N} \not\models \varphi(\hat{a}; \beta)$ for some β . By induction hypothesis, $\{i : \mathcal{M} \not\models \varphi(\hat{a}i; \beta)\} \in F$. A fortiori $\{i : \mathcal{M} \not\models \forall^C y \varphi(\hat{a}i; y)\} \in F$ as required.

(iiV^C) Assume $\{i : \mathcal{M} \not\models \forall^C y \varphi'(\hat{a}i; y)\} \in F$. Pick $\beta_i \in C$ such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \beta_i)\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. Then from Lemma 5 we obtain $\{i : \mathcal{M} \not\models \varphi(\hat{a}i; \beta)\} \in F$. Finally, by (iih), we conclude $\mathcal{N} \not\models \forall^C y \varphi(\hat{a}, y)$.

Finally, we prove

- iE^C. $\{i : \mathcal{M} \models \exists^C y \varphi(\hat{a}i; y)\} \in F \Rightarrow \mathcal{N} \models \exists^C y \varphi(\hat{a}; y);$

$$\text{ii}\exists^C. \quad \mathcal{N} \models \exists y^C \varphi(\hat{a}, y) \Rightarrow \{i : \mathcal{M} \models \exists^C y \varphi'(\hat{a}i, y)\} \in F.$$

(i \exists^C) Assume $\{i : \mathcal{M} \models \exists^C y \varphi(\hat{a}i; y)\} \in F$. Choose $\beta_i \in C$ such that $\{i : \mathcal{M} \models \varphi(\hat{a}i; \beta_i)\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. By Lemma 5, $\{i : \mathcal{M} \models \varphi(\hat{a}i; \beta)\} \in F$.

(ii \exists^C) Assume $\mathcal{N} \models \varphi(\hat{a}; \beta)$ for some β . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \beta)\} \in F$. Therefore $\{i : \mathcal{M} \models \exists^C y \varphi'(\hat{a}i; y)\} \in F$. \square

4. \mathbb{L} -ELEMENTRARITY

Let $\mathcal{M} = \langle M, R \rangle$ and $\mathcal{N} = \langle N, R \rangle$ be two structures. We say that $f : M \rightarrow N$, a partial map, is an **\mathbb{L} -elementary map** if for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\text{dom } f)^{|x|}$

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(fa)$$

Note that an \mathbb{L} -elementary map is in particular L -elementary and therefore it is injective. A total \mathbb{L} -elementary map is called an **\mathbb{L} -(elementary) embedding**. When the map $\text{id}_M : M \hookrightarrow N$ is an \mathbb{L} -embedding, that is, if for every $\varphi(x) \in \mathbb{L}$ and every $a \in M^{|x|}$

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(a),$$

we write $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$ and say that \mathcal{M} is an **\mathbb{L} -(elementary) substructure** of \mathcal{N} . We say that the map $f : M \rightarrow N$ is an **approximatively \mathbb{L} -elementary** if for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\text{dom } f)^{|x|}$

$$\mathcal{M} \models \varphi'(a) \text{ for every } \varphi' > \varphi \Leftrightarrow \mathcal{N} \models \varphi'(fa) \text{ for every } \varphi' > \varphi$$

By the density of $>$ the equivalence above can be split into two distinct implications: for every $\varphi(x) \in \mathbb{L}$ every $\varphi' > \varphi$ and every $a \in (\text{dom } f)^{|x|}$

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi'(fa)$$

$$\mathcal{N} \models \varphi(fa) \Rightarrow \mathcal{M} \models \varphi'(a)$$

Let $A \subseteq M \cap N$. If the map $\text{id}_A : M \rightarrow N$ is approximatively \mathbb{L} -elementary, we write $\mathcal{M} \equiv^A \mathcal{N}$.

11 Fact. Every \mathbb{L} -elementary is approximatively \mathbb{L} -elementary

Proof. Only the implication \Leftarrow requires a proof. Suppose $\varphi' > \varphi$ is such that $\mathcal{M} \not\models \varphi'(a)$. By Lemma 4 there is formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \rightarrow \neg \tilde{\varphi} \rightarrow \varphi'$ and a formula $\varphi'' > \varphi$ such that $\varphi \rightarrow \varphi'' \rightarrow \neg \tilde{\varphi}$. Then we have

$$\mathcal{M} \not\models \varphi'(a) \Rightarrow \mathcal{M} \models \tilde{\varphi}(a)$$

$$\Rightarrow \mathcal{N} \models \tilde{\varphi}(fa)$$

$$\Rightarrow \mathcal{N} \not\models \varphi''(fa)$$

\square

The following fact follows from Łoś Theorem just as in the classical case.

12 Fact. If \mathcal{N} is an ultrapower of \mathcal{M} then there is an \mathbb{L} -embedding $\mathcal{M} \hookrightarrow \mathcal{N}$. □

We prove a version of Tarski-Vaught Test. A second equivalent version of this test will be proved in the sections below.

13 Proposition (Tarski-Vaught \mathbb{L} -Test). Let M be a subset of N . Then the following are equivalent

1. M is the domain of a structure $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$;

2. for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\mathcal{N} \models \exists x \neg \varphi(x) \Rightarrow \text{there is an } a \in M \text{ such that } \mathcal{N} \models \neg \varphi(a);$$

Proof. (1 \Rightarrow 2) Clear.

(2 \Rightarrow 1) Assume (2). Then, by the classical Tarski-Vaught test $M \leq N$. We also have that \mathcal{M} is an \mathcal{L} -substructure of \mathcal{N} . Therefore, from every formula $\varphi(x)$ as in (i) and (ii) of Definition 2 we have that $\mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(a)$ for every $a \in M^{|x|}$

Now, assume inductively

$$\mathcal{M} \models \varphi(a, b) \Rightarrow \mathcal{N} \models \varphi(a, b)$$

Using (2) and the induction hypothesis we prove by contraposition that

$$\mathcal{M} \models \forall y \varphi(a, y) \Rightarrow \mathcal{N} \models \forall y \varphi(a, y).$$

Indeed,

$$\begin{aligned} \mathcal{N} \not\models \forall y \varphi(a, y) &\Rightarrow \mathcal{N} \models \exists y \neg \varphi(a, y) \\ &\Rightarrow \mathcal{N} \models \neg \varphi(a, b) \quad \text{for some } b \in M^{|y|} \\ &\Rightarrow \mathcal{M} \models \neg \varphi(a, b) \quad \text{for some } b \in M^{|y|} \\ &\Rightarrow \mathcal{M} \not\models \forall y \varphi(a, y) \end{aligned}$$

Induction for the connectives $\vee, \wedge, \exists, \exists^C$, and \forall^C is straightforward. □

5. \mathbb{L} -SATURATION

We say that a theory is consistent if it has a model satisfying Definition 1. Also the notion of logical consequence is restricted to models in this class.

A theory $T \subseteq \mathbb{L}$ be a finitely uniformly consistent if for every φ , conjunction of sentences in T is consistent. Moreover the functions in the models witnessing this have the same bound.

The following proposition follows from Łoś Theorem by the usual argument.

14 Proposition (Compactness Theorem). Let $T \subseteq \mathbb{L}$ be a finitely uniformly consistent theory. Then T is consistent. \square

15 Proposition. Let $p(x) \subseteq \mathbb{L}(M)$ be a finitely consistent in \mathcal{M} . Then $\mathcal{N} \models \exists x p(x)$ for some \mathcal{N} such that $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$. \square

A model \mathcal{N} is \mathbb{L} -saturated if it realizes all types with fewer than $|\mathcal{N}|$ parameters that are finitely consistent in \mathcal{N} . The existence of \mathbb{L} -saturated models is proved as in the classical case.

16 Proposition. Every model has an \mathbb{L} -elementary extension to a saturated model (possibly of inaccessible cardinality). \square

We denote by $\mathcal{U} = \langle U, R \rangle$ some large \mathbb{L} -saturated structure which we call the **monster model**. The cardinality of \mathcal{U} is an inaccessible cardinal that we denote by κ . Below we say **model** for \mathbb{L} -elementary substructure of \mathcal{U} .

17 Fact. Let $p(x) \subseteq \mathbb{L}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{L}(U)$

1. if $p(x) \rightarrow \neg\varphi(x)$ then $\psi(x) \rightarrow \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \rightarrow \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \rightarrow \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. The first claim is clear. The second follows from the first by Lemma 4. \square

18 Proposition. For every $\varphi(x) \in \mathbb{L}(U)$ without existential quantifiers of sort R

$$\bigwedge_{\varphi' > \varphi} \varphi'(x) \leftrightarrow \varphi(x)$$

Proof. We prove \rightarrow , the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. Consider the existential quantifiers of sort U . Assume inductively

$$\bigwedge_{\varphi' > \varphi} \varphi'(x, y) \rightarrow \varphi(x, y)$$

then

$$\exists y \bigwedge_{\varphi' > \varphi} \varphi'(x, y) \rightarrow \exists y \varphi(x, y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi' > \varphi} \exists y \varphi'(x, y) \rightarrow \exists y \bigwedge_{\varphi' > \varphi} \varphi'(x, y)$$

Replace x with a parameter, say a and assume the antecedent. Note that $\{\exists y \varphi'(a, y) : \varphi' > \varphi\}$ implies the finite consistency of the type $\{\varphi'(a, y) : \varphi' > \varphi\}$. This because if $\varphi_1, \varphi_2 > \varphi$ then

$\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. Therefore $\exists y \{ \varphi'(a, y) : \varphi' > \varphi \}$ follows by saturation. \square

6. \mathbb{L} -HOMOGENEITY

7. COMPLETENESS

Needs full rewriting

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \rightarrow R$.

19 Fact. If $\bar{a} = \langle a_i : i < \lambda \rangle$ and $\bar{b} = \langle b_i : i < \lambda \rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \rightarrow R$.

Proof. By induction on λ . Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ . For limit ordinals induction is immediate. \square

20 Example (???). Let L be the language of \mathbb{R} -algebras expanded with two lattice operators \wedge, \vee . Let $\langle \Omega, \mathcal{B}, \text{Pr} \rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L . Let $R = \mathbb{R}$. Assume \mathcal{L} extends L with all the continuous functions $\mathbb{R}^n \rightarrow \mathbb{R}$ and for every $n \in \mathbb{N}$ the functions E_n that gives the expected value of $(X \wedge n) \vee -n$ for $n \in \mathbb{N}$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is **definable in the limit** over M if $a \equiv_M^{\mathbb{L}} x \rightarrow a \sim x$

21 Example. Assume that $R = \mathbb{R}$ and that \mathcal{L} contains a function of sort $M^2 \rightarrow R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \rightarrow R$ are continuous with respect to this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M , a model;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of M that converges to a .

Proof. $(2 \Rightarrow 1)$ Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_M a$. By the uniqueness of the limit $d(a, b) = 0$.

$(1 \Rightarrow 2)$ Assume that $a \equiv_M^{\mathbb{L}} x \rightarrow a \sim x$. Then $a \equiv_M^{\mathbb{L}} x \rightarrow d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \text{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \rightarrow d(a, x) < 1/n$. By \mathbb{L} -elementarity there is an $a_n \in M$ such that $\varphi_n(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a . \square