

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A CLASS OF STRUCTURES

1 Definition. As M ranges over the L -structures, we consider the class of L' -structures of the form $\mathcal{M} = \langle M, R \rangle$, where R is some fixed structure. We are given a Hausdorff topology on R and assume that there is a relation symbol in L' for every subset of R^n that is compact in the product topology. We require that the function symbols of sort $M^n \times R^m \rightarrow R$ are interpreted in functions such that

1. $f^{\mathcal{M}}(-; \alpha) : M^n \rightarrow R$ is bounded (i.e. its range is contained in a compact set) for every $\alpha \in R^m$.

2?? $f^{\mathcal{M}}(a; -) : R^m \rightarrow R$ is continuous uniformly in $a \in M^n$, that is (?),

$$\bigcap_{a \in M^n} f^{\mathcal{M}}(a; -)^{-1}(O) \text{ is open for every open } O \subseteq R.$$

There may be no saturated structure among those described in Definition 1. As a remedy, we restrict the notion of saturation to formulas in a set \mathbb{L} that we define below. In a nutshell, formulas in \mathbb{L} do not contain existential quantifiers of sort R . Moreover, we require that a few *critical* function symbols only occur only inside compact relations and never under the scope of negation.

Intuitively, the function symbols are divided in two groups according to their sort

1. $M^n \times R^m \rightarrow M$

2. $M^n \times R^m \rightarrow R$

The ultrapower construction we introduce in the next section is completely classical w.r.t. functions as in 1. On the other hand, the functions as in 2 are *critical* and need to be dealt differently.

Below, \mathbb{T} denotes the set of the terms of sort $M^n \times R^m \rightarrow R$. Definition 1 ensures that for all $t(x; y) \in \mathbb{T}$ the range of the functions $t^{\mathcal{M}}(-, \alpha) : M^n \rightarrow R$ is bounded.

2 Definition. Formulas in \mathbb{L} are defined inductively as follows

- i. \mathbb{L} contains all formulas without quantifiers of sort R and functions of sort $M^n \times R^m \rightarrow R$;
- ii. \mathbb{L} contains formulas of the form $t(y; z) \in C$ where $C \subseteq R^n$ is compact and $t(y; z) \in \mathbb{T}^n$;
- iii. \mathbb{L} is closed under the Boolean connectives \wedge, \vee , and the quantifiers \forall, \exists of sort M ;
- iv. \mathbb{L} is closed under the quantifier \forall of sort R .

Clearly, $L \subseteq \mathbb{L} \subseteq L'$.

2. ULTRAPOWERS

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . If $f : I \rightarrow R^m$ and $\lambda \in R^m$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq R^m$ that is a neighborhood of λ . Such a λ is always unique (by the property of filters). When F is an ultrafilter, and $\text{range } f$ is bounded (i.e. it is contained in a compact set), the limit always exists.

Below we introduce a suitable notion of ultrapower. To keep notation tidy, we make two simplifications: (1) we only consider ultraproducts; (2) we do not consider formulas containing equality, so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

3 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \times R^m \rightarrow M$ then $f^{\mathcal{N}}(\hat{a}; \alpha)$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle$.
4. If f is a function of sort $M^n \times R^m \rightarrow R$ then

$$f^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \alpha).$$

4 Fact. The structure \mathcal{N} satisfies the conditions of Definition 1. (Didin't check condition 2.)

The following fact is easily proved by induction on the syntax.

5 Fact. If only function symbols of sort $M^n \times R^m \rightarrow M$ occurs in $t(y; z)$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = \langle t^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle.$$

□

If true, the proof of the following fact is tedious.

6 Fact. For every term $t(y; z)$ of sort $M^n \times R^m \rightarrow R$

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Proof. The proof is by induction. Assume the claim is true for $t_M(y, z)$ a tuple of terms with values in M . Write λ_α for $t_M^{\mathcal{N}}(\hat{a}; \alpha)$. Let $t_R(y, z)$ be a tuple of terms with values in R . Let f be a function of sort $M^n \times R^m \rightarrow R$. We need to prove that

$$f^{\mathcal{N}}(t_M^{\mathcal{N}}(\hat{a}; \alpha); \lambda_\alpha) = F\text{-}\lim_i f^{\mathcal{M}}(t_M^{\mathcal{M}}(\hat{a}i; \alpha); t_R^{\mathcal{M}}(\hat{a}i; \alpha))$$

By Fact 5 we can replace $t_M^{\mathcal{N}}(\hat{a}; \alpha)$ by a sequence of elements of M . As we are reluctant to introduce a new letter, we absorb this sequence in the tuple \hat{a} . Therefore the claim to prove is

$$f^{\mathcal{N}}(\hat{a}; \lambda_\alpha) = F\text{-}\lim_i f(\hat{a}i; t_R^{\mathcal{M}}(\hat{a}i; \alpha)).$$

Write μ for the limit on the r.h.s. of the equality above. For every neighborhood $\mu \in S \subseteq R$ there is a $J \in F$ such that $f(\hat{a}i; t_R^{\mathcal{M}}(\hat{a}i; \alpha)) \in S$ for all $i \in J$ \square

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formulas $t \in C$ occurring in φ by $t \in D$ for some closed neighborhood of C . Note that, if such atomic formulas do not occur in φ , then $\varphi > \varphi$.

Finally, we prove Łoś Theorem

7 Proposition. Assume $|I| \geq$ the weight of the topology on R . Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|\mathbf{x}|}$ and all $\alpha \in R^{|\mathbf{y}|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x; y)$ is as in (i) of Definition 2 then, after absorbing α into the language, the proposition reduces to the classical Łoś Theorem.

Now, suppose that $\varphi(x; y)$ is as in (ii) of Definition 2, say it is the formula $t(x; y) \in C$, where $t(x; y)$ is a tuple of terms of sort $M^{|\mathbf{x}|} \times R^{|\mathbf{y}|} \rightarrow R$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$.

Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. We can assume to work inside a compact set containing C and $t(\hat{a}, \alpha)$. By regularity, $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$ for D a compact neighborhood of C . By the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$. Then $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \notin F$.

As for induction, only the existential quantifier require some work. Inductively we assume that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

We need to prove

$$\mathcal{N} \models \exists y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

The implication \Rightarrow is clear. For the converse, assume the r.h.s. Assume for simplicity that $I = \omega$. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for every $i < j$ and

$$\mathcal{M} \models \bigwedge_{i \in I} \varphi_i(x, y; \alpha) \rightarrow \varphi(x, y; \alpha).$$

For every $j \in I$ there is a sequence \hat{b}_j such that

$$\{i \in I : \mathcal{M} \models \varphi_j(\hat{a}i, \hat{b}_ji; \alpha)\} \in F.$$

As $\varphi_j(x, y; \alpha) \rightarrow \varphi_{j'}(x, y; \alpha)$ holds for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_i : i \in I \rangle$ satisfies $\mathcal{N} \models \varphi(\hat{a}, \hat{d}; \alpha)$. \square