

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

C. L. C. L. POLYMATH

1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number, we may formalize this with a two sorted structure. Ideally, we would like to have that an elementary extension of a normed space maintain the usual notion of real numbers¹ but this is not possible with the usual notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 Definition. Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L' -structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L -structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

- i. $R^n \rightarrow R$;
- ii. $M^n \rightarrow M$;
- iii. $M^n \rightarrow R$.

The interpretation of symbols of sort $R^n \rightarrow R$ is required to be continuous, the interpretation of symbols of sort $M^n \rightarrow R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n .

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set \mathbb{L} of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an \mathbb{L} -elementary extension that is \mathbb{L} -saturated in the same class.

¹Non standard standard analysts have no problem in expanding \mathbb{R} . In these notes, for a change, we insist as \mathbb{R} should remain \mathbb{R} throughout.

For convenience we assume that the functions of sort $M^n \rightarrow M$ and the relation of sort M^n are all in L .

2 Definition. Formulas in \mathbb{L} are constructed inductively from the following two sets of formulas

- i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact² and $t(x, y)$ is a n -tuple of terms of sort $M^{|x|} \times R^{|y|} \rightarrow R$;
- ii. all formulas in L .

We require that \mathbb{L} is closed under the Boolean connectives \wedge, \vee ; the quantifiers \forall, \exists of sort M ; and the quantifier \forall^C , by which we mean a universal quantifier restricted to the compact set $C \subseteq R^m$.

2. TOPOLOGICAL TOOLS

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . Let Y be a topological space. If $f : I \rightarrow Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ . Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

3 Fact. Consider the followin data

- $t(x; y)$, an n -tuple of terms of sort $M^{|x|} \times R^{|y|} \rightarrow R$;
- $\langle \mathcal{M}_i : i \in I \rangle$, a sequence of structures;
- $\langle a_i : i \in I \rangle$, a sequence of elements $a_i \in M_i^{|x|}$;
- $\langle U_i : i \in I \rangle$, a sequence of open sebsubsets of R^n .

Let F be an ultrafilter on I . Assume that for some $\alpha \in R^{|y|}$

$$\{i \in I : t^{\mathcal{M}_i}(a_i; \alpha) \in U_i\} \in F.$$

Then there is an open $V \ni \alpha$ such that for every $\alpha' \in V$

$$\{i \in I : t^{\mathcal{M}_i}(a_i; \alpha') \in U_i\} \in F.$$

Proof. By induction on the syntax. If t is a function symbol f then x or y do not occur in t . If x does not occur in t the claim holds because $f^{\mathcal{M}} : R^{|y|} \rightarrow R$ is continuous. If y does not occur in t the claim is trivial. Finally, induction is clear by the continuity of the functions of sort $R^n \rightarrow R$. \square

²We confuse the relation symbols in L' of sort R^m with their interpretation and write $t \in C$ for $C(t)$

3. HENSON-IOVINO APPROXIMATIONS

For $\varphi, \varphi' \in \mathbb{L}$ (free variables are hidden) we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C . If no such atomic formulas occurs in φ , then $\varphi > \varphi$.

Note that this is a dense (pre)order of \mathbb{L} .

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \rightarrow \varphi'$. Vice versa, if φ' for every $\varphi' > \varphi$, then φ .

The following lemma is required for the proof of Łoś Theorem in the next section.

4 Lemma. Consider the followin data

- $\varphi(x; y) \in \mathbb{L}$;
- $\langle \mathcal{M}_i : i \in I \rangle$, a sequence of structures;
- $\langle a_i : i \in I \rangle$, a sequence of elements $a_i \in M_i^{|x|}$;

Let F be an ultrafilter on I . Assume that for some $\alpha \in R^{|y|}$

$$\{i : \mathcal{M} \models \varphi(a_i; \alpha)\} \in F.$$

Then for every $\varphi' > \varphi$ there is open $V \ni \alpha$ such that for every $\alpha' \in V$

$$\{i : \mathcal{M} \models \varphi'(a_i; \alpha')\} \in F.$$

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L , it does not depend on α , and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume that $\{i : \mathcal{M} \models t(a_i; \alpha) \in C\} \in F$. Then $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \in C$. Let D be a compact neighborhood of C . By Fact 3, there is an open $V \ni \alpha$ such that $\{i : \mathcal{M} \models t(a_i; \alpha') \in D\} \in F$ for every $\alpha' \in V$. Then $\{i : \mathcal{M} \models t'(a_i; \alpha_i) \in D\} \in F$ follows. This completes the base case of the induction.

Induction clear for the connectives \vee, \wedge and for the existential quantifier of sort M . To deal with the universal quantifier of sort M we assume inductively that for every $\varphi' > \varphi$

$$\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F \Rightarrow \{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F.$$

We prove

$$1. \quad \{i : \mathcal{M} \models \forall y \varphi(a_i, y; \alpha)\} \in F \Rightarrow \{i : \mathcal{M} \models \forall y \varphi'(a_i, y; \alpha_i)\} \in F$$

Negate the r.h.s. of the implication. Using that F is an ultrafilter, there is a sequence $\langle b_i : i \in I \rangle$ such that

$$\{i : \mathcal{M} \not\models \varphi'(a_i, b_i; \alpha_i)\} \in F$$

Assume the l.h.s. of the implication, for a contradiction. Then $\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F$. A contradiction that proves (1).

Induction for the quantifier \forall^C is similar. Assume inductively that for every $\varphi' > \varphi$

$$\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F \Rightarrow \{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F.$$

We prove

$$2. \quad \{i : \mathcal{M} \models \forall^C y \varphi(a_i, ; \alpha, y)\} \in F \Rightarrow \{i : \mathcal{M} \models \forall^C y \varphi'(a_i; \alpha_i, y)\} \in F$$

Negate the r.h.s. of the implication. Using that F is an ultrafilter, there is a sequence $\beta_i \in C$ such that

$$\{i : \mathcal{M} \not\models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$$

As C is compact, $F\text{-}\lim_i \beta_i$ exists, say it equals $\beta \in C$. Assume, for a contradiction, that the l.h.s. of the implication holds. Then $\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$. A contradiction that proves (2) and, with it, the lemma. \square

4. ULTRAPRODUCTS

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that for each function symbol f of sort $M^n \rightarrow R$ there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$. Note that, otherwise, Fact 6 below may fail.

To keep notation tidy, we make two simplifications: (1) we only consider ultrapowers; (2) we do not consider formulas in L containing equality, so we need not take any quotient. The generalization is straightforward.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

5 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \rightarrow M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I \rangle$.
3. The interpretation of functions of sort $R^n \rightarrow R$ remains unchanged.
4. If f is a function of sort $M^n \rightarrow R$ then

$$f^{\mathcal{N}}(\hat{a}) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$

5. if r is a relation symbol of sort M^n then

$$\mathcal{N} \models r(\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i)\} \in F.$$

6. The interpretation of functions of sort R^n remains unchanged.

The following is immediate but it needs to be noted.

6 Fact. The structure \mathcal{N} satisfies Definition 1.

The following is easily proved by induction on the syntax

7 Fact. If $t(x)$ is a term of type $M^{|x|} \rightarrow M$

$$t^{\mathcal{N}}(\hat{a}) = \langle t^{\mathcal{M}}(\hat{a}i) : i \in I \rangle.$$

□

We also have that

8 Fact. If $t(x; y)$ has sort $M^{|x|} \times R^{|y|} \rightarrow R$

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Proof. By induction. If t is a function symbol then x or z do not occur in t . If x does not occur in t the claim is trivial. If y does not occur in t the claim holds by definition. Finally, induction is clear by the continuity of the functions of sort $R^n \rightarrow R$. □

Finally, we prove Łoś Theorem.

9 Theorem. Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x; y) \in L$ then the theorem reduces to the classical Łoś Theorem. Then, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. Let D be a compact neighborhood of C such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$. By Fact 8 and the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives \vee and \wedge is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

$$1. \quad \mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F$. Choose \hat{b} such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$. If for a contradiction $\mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha)$, then in particular $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ a contradiction. This completes the proof of (1).

Now we prove (the argument, which we repeat for convinience, is dual to the one above)

2. $\mathcal{N} \models \exists y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F$ for every $\varphi' > \varphi$.

(\Rightarrow) Assume $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ for every $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F$ as required.

(\Leftarrow) Assume that $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F$ for every $\varphi' > \varphi$. Choose some \hat{b} such that $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$. Suppose for a contradiction that $\mathcal{N} \not\models \exists y \varphi(\hat{a}, y; \alpha)$. Then, in particular, $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$ for some $\varphi' > \varphi$, a contradiction. This completes the proof of (2).

Finally, to deal with the quantifier \forall^C we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

3. $\mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F$ for every $\varphi' > \varphi$.

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}; \alpha, \beta)$ for some β . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha, \beta_i)\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y)$, then $\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta)$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \alpha, \beta)\} \in F$ for every $\varphi'' > \varphi$. Fix φ'' such that $\varphi' > \varphi'' > \varphi$ and obtain a contradiction from Lemma 4. This completes the proof of (3) and, with it, of the theorem. \square