

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A CLASS OF STRUCTURES

1 Definition. We will work with L -structures of the form $\mathcal{M} = \langle M, R \rangle$, where R is fixed. We are given, for each n , a set of definable subsets of R^n that form a base of closed sets of a compact Hausdorff topology.

(?) We require that the terms of sort $M^n \times R^m \rightarrow R$ are interpreted in functions $t^{\mathcal{M}}(a, -) : R^m \rightarrow R$ that are continuous uniformly in $a \in M^n$.

Caveat: occasionally, we work with topological spaces that are not T_0 . When we say that such a space is Hausdorff, we mean that its Kolmogoroff quotient is Hausdorff.

There may be no saturated structure among those described in Definition 1. As a remedy, we restrict the notion of saturation to formulas in \mathbb{L} , as defined below. In a nutshell, formulas in \mathbb{L} do not contain existential quantifiers of sort R . Moreover, there are some *critical* function symbols that may not occur under the scope of negation. Intuitively, the function symbols are divided in three classes according to their sort

1. $M^n \times R^m \rightarrow M$
2. $M^0 \times R^m \rightarrow R$
3. $M^{n+1} \times R^m \rightarrow R$

Functions in the latter group are critical and will be dealt with caution.

Below, \mathbb{T} denotes the set of the terms of sort $M^n \times R^m \rightarrow R$.

2 Definition. The set \mathbb{L} is a subset of L that is defined inductively as follows

- i. it contains all formulas without quantifiers of sort R and function of sort $M^{n+1} \times R^m \rightarrow R$;
- ii. it contains the formulas $t(y; z) \in C$ where $C \subseteq R^n$ is a basic closed set and $t(y; z) \in \mathbb{T}^n$;
- iii. it is closed under the Boolean connectives \wedge, \vee , and the quantifiers \forall, \exists of sort M ;
- iv. it is closed under the quantifier \forall of sort R .

2. ULTRAPOWERS

We recall some standard definitions about F -limits. Let I be a non-empty set. Let F be a filter on I . If $f : I \rightarrow R^m$ and $\lambda \in R^m$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq R^m$ that is a neighborhood of λ . Such a λ is unique and, when F is an ultrafilter, it always exists (because R^n is compact).

Below we introduce ultrapowers of a suitable form. To keep the notation as clean as possible we make two simplifications: (1) we discuss only ultraproducts; (2) we do not consider formulas containing equality among elements of M , so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

3 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \times R^m \rightarrow M$ then $f^{\mathcal{N}}(\hat{a}; \alpha)$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle$.
4. If f is a function of sort $M^n \times R^m \rightarrow R$ then

$$f^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \alpha).$$

The following fact is easily proved by induction on the syntax.

4 Fact. If no function symbol of sort $M^{n+1} \times R^m \rightarrow R$ occurs in $t(y; z)$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = \langle t^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle.$$

□

If true, the proof of the following fact is tedious.

5 Fact. For every term $t(y; z)$ of sort $M^n \times R^m \rightarrow R$

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

□

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' has been obtained replacing each atomic formulas $t \in C$ occurring in φ by $t \in D$ for some closed neighborhood of C . Note that, if such atomic formulas do not occur in φ , then $\varphi > \varphi$.

Finally, we prove

6 Proposition (Łoś Theorem). Assume $|I| \geq$ the weight of the topology on R . Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x; y)$ is as in (i) of Definition 2 then, after absorbing α into the language, the proposition reduces to the classical Łoś Theorem.

Now, suppose that $\varphi(x; y)$ is as in (ii) of Definition 2, say it is the formula $t(x; y) \in C$, where $t(x; y)$ is a tuple of terms of sort $M^{|x|} \times R^{|y|} \rightarrow R$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D a closed neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$.

Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. By normality, $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$ for D a closed neighborhood of C . By the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$. Then $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \notin F$.

As for induction, only the existential quantifier require some work. Assume for simplicity that $I = \omega$. inductively we assume that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

We need to prove

$$\mathcal{N} \models \exists y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

The implication \Rightarrow is clear. For the converse, assume the r.h.s. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for every $i < j$ and

$$\mathcal{M} \models \bigwedge_{i \in I} \varphi_i(x, y; \alpha) \rightarrow \varphi(x, y; \alpha).$$

For every $j \in I$ there is a sequence \hat{b}_j such that

$$\{i \in I : \mathcal{M} \models \varphi_j(\hat{a}i, \hat{b}_ji; \alpha)\} \in F.$$

As $\varphi_j(x, y; \alpha) \rightarrow \varphi_{j'}(x, y; \alpha)$ for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_ji : i \in I \rangle$ satisfies $\mathcal{N} \models \varphi(\hat{a}, \hat{d}; \alpha)$. \square