CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

C. L. C. L. POLYMATH

1. A class of structures

As a minimal motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted structure. Ideally, we would like that elementary extensions of normed spaces to remain a normed space (hence $\mathbb{R}$ shoud remain $\mathbb{R}$ throughout). Unfortunately, this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

Let R be some fixed first-order structure which is endowed with a Hausdorff compact topology. The language L_R contains relation symbols for the compact subsets $C \subseteq R^n$ and a function symbols for the continuous functions $f : R^n \to R$. According to the context, C and f denote either the symbols of L_R or their interpretation in R.

1 Definition. Let L be a one-sorted (first-order) language. By model we intend an $\mathcal{L}$-structure of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is the structure above. The language $\mathcal{L}$ expands both L and L_R but only with function symbols of sort $M^n \to R$.

Clearly, saturated $\mathcal{L}$-structures exit but, with the exeption of trivial cases (i.e. when R is finite), they are not models. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq \mathcal{L}$, such that every model has an $\mathbb{L}$-elementary, $\mathbb{L}$-saturated extension that is a model.

Warning: as usual L and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The set $\mathbb{L}$ defined below is only a set of first-order formulas.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from two sorts of $\mathbb{L}$-atomic formulas

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact (in the product topology) and $t(x, y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge, \vee$; the quantifiers $\forall, \exists$ of sort M; and the quantifiers of sort R which are denoted by $\forall^R, \exists^R$.

We write $\mathbb{H}$ for the set of formulas in $\mathbb{L}$ without quantifiers of sort R.

For later reference we remark the following which is an immediate consequence of the restrictions we imposed on the sorts of the function symbols.

3 Remark. Each component of the tuple $t(x\,; y)$ in the definition above is either of sort $M^{|x|} \to R$ or of sort $R^{|y|} \to R$. □

Formulas in $\mathbb{H}$ are a generalization of the positive bounded formulas of Henson and Iovino. Here we also introduce the larger class $\mathbb{L}$ because it offers some advantages. For instance, it is easy to see (cf. Example 4 for a hint) that $\mathbb{L}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive). Somewhat surprisingly, we will see (cf. Propositions 19 and 20) that the formulas in $\mathbb{L}$ can be very well approximated by formulas in $\mathbb{H}$.

4 Example. Let $R = [0,1]$, the unit interval of $\mathbb{R}$. Let $t(x)$ be a term of sort $M^{|x|} \to R$. Then there is a formula in $\mathbb{L}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall x\ \big[t(x) \dot{-} \tau \in \{0\}\big] \quad \wedge \quad \forall^R \varepsilon \Big[\varepsilon \in \{0\}\ \vee\ \exists x\ \big[\big(\tau \dot{-} t(x)\big) \dot{-} \varepsilon \in \{0\}\big]\Big]$$

which, in a human readible form, becomes

$$\forall x\ \big[t(x) \le \tau\big] \quad \wedge \quad \forall \varepsilon > 0\ \exists x\ \big[\tau \le t(x) + \varepsilon\big].$$

□

2. The standard part

Let $R \preceq {}^*R$. For $\alpha, \beta \in {}^*R$ we write define the equivalence relation $\alpha \sim \beta$ as follows

$$\alpha \sim \beta \quad \Leftrightarrow \quad {}^*R \vDash \alpha \in C \leftrightarrow \beta \in C \quad \text{for every compact } C \subseteq R.$$

By the compactness of R, every $\alpha \in {}^*R$ is equivalent to some element of R. By normality of R, this element is unique. We denote it by $\mathrm{st}(\alpha)$. It is easy to see that if $\alpha, \beta \in ({}^*R)^n$ then

$$\bigwedge_{i=1}^{n} \alpha_i \sim \beta_i \quad \Leftrightarrow \quad S \vDash \alpha \in C \leftrightarrow \beta \in C \quad \text{for every compact } C \subseteq R^n.$$

We write $\mathrm{st}(\alpha)$ for the tuple with components $\mathrm{st}(\alpha_i)$.

5 Lemma. Let ${}^*\mathcal{N} = \langle N, {}^*R\rangle$ be an $\mathcal{L}$-structure, and assume that $R \preceq {}^*R$. Let $\mathcal{N}$ denote the substructure $\langle N, R\rangle$. Then for every $\varphi(x\,;y) \in \mathbb{L}$, every $a \in N^{|x|}$ and every $\alpha \in ({}^*R)^{|y|}$

$${}^*\mathcal{N} \vDash \varphi(a\,;\alpha) \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi\big(a\,;\mathrm{st}(\alpha)\big)$$

Proof. Suppose $\varphi(x\,;y)$ is $\mathbb{L}$-atomic. If $\varphi(x\,;y)$ is a formula of L the claim is trivial. Otherwise $\varphi(x\,;y)$ has the form $t(x\,;y) \in C$. Assume ${}^*\mathcal{N} \vDash t(a\,;\alpha) \in C$. We prove that for every compact $C_i \subseteq R$ such that $C \subseteq C_1 \times \cdots \times C_{|y|}$

$$\mathcal{N} \quad \vDash \quad t_i(a\,;\alpha) \in C_i.$$

By the definition of product topology, C is the intersection of such product sets. Hence$\mathcal{N} \vDash t(a\,;\alpha) \in C$ follows.

By Remark 3, the terms $t_i(a\,;\alpha)$ have either the form $t_i(a)$ or $t_i(\alpha)$. We can ignore the first case because $\mathcal{N}$ is a substructure of ${}^*\mathcal{N}$ hence $t_i(a)$ has the same value in both structures. Then (1) is equivalent to

$${}^*\mathcal{N} \quad \vDash \quad \alpha \in \big(t_i^{\mathcal{N}}\big)^{-1}[C_i].$$

But $\alpha \in \big(t_i^{\mathcal{N}}\big)^{-1}[C_i]$ is equivalent, modulo $\mathrm{Th}(R)$, to $\alpha \in D$ for some compact $D \subseteq R$. Therefore

$${}^*\mathcal{N} \quad \vDash \quad \alpha \in D.$$

Finally this is equivalent to $\mathrm{st}(\alpha) \in D$.

This proves lemma for $\mathbb{L}$-atomic formulas. Induction is immediate. □

3. $\mathbb{L}$-ELEMENTARITY

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two models. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\operatorname{dom} f)^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(fa).$$

An $\mathbb{L}$-elementary map is in particular L-elementary and therefore it is injective. An $\mathbb{L}$-elementary map that is total is called an $\mathbb{L}$-(elementary) embedding. When the map $\mathrm{id}_M : M \hookrightarrow N$ is an $\mathbb{L}$-embedding, that is, if for every $\varphi(x) \in \mathbb{L}$ and every $a \in M^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(a),$$

we write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) submodel of $\mathcal{N}$.

The definitions of $\mathbb{H}$-elementary map/embedding/submodel are similar.

4. $\mathbb{L}$-COMPACTNESS

It is convinient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure $\mathcal{M}$. We say that T is consistent when $\mathcal{M}$ is required to be a model.

6 **Proposition ($\mathbb{L}$-compactness).** Let $T \subseteq \mathbb{L}$ be finitely consistent. Then T is consistent.

Proof. By the compactness theorem ${}^*\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure ${}^*\mathcal{M} = \langle M, {}^*R\rangle$. Let $\mathcal{M}$ be the model $\langle M, R\rangle$. Then $\mathcal{M} \vDash T$ by Lemma 5. □

7 **Corollary.** Let $p(x) \subseteq \mathbb{L}(M)$ be finitely consistent in $\mathcal{M}$, a model. Then $\mathcal{N} \vDash \exists x\, p(x)$ for some model $\mathcal{N}$ such that $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$. □

A model $\mathcal{N}$ is $\mathbb{L}$-saturated if it realizes all types with fewer than $|\mathcal{N}|$ parameters that are finitely consistent in $\mathcal{N}$. The existence of $\mathbb{L}$-saturated models is proved as in the classical case.

8 **Proposition.** Every model has an $\mathbb{L}$-elementary extension to a saturated model (possibly of inaccessible cardinality). □

5. HENSON-IOVINO APPROXIMATIONS

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If such atomic formulas do not occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. Moreover the $\mathbb{L}$-atomic formulas in L are replaced with their negation and every connective is replaced with its dual. I.e., $\vee, \wedge, \exists, \forall, \exists^C, \forall^C$ are replaced with $\wedge, \vee, \forall, \exists, \forall^C, \exists^C$ respectively.

It is clear that $\tilde{\varphi} \rightarrow \neg\varphi$. We say that $\tilde{\varphi}$ is a strong negation of φ.

9 Lemma. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \rightarrow \neg\tilde{\varphi} \rightarrow \varphi'$. Vice versa, for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \rightarrow \varphi' \rightarrow \neg\tilde{\varphi}$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in R \smallsetminus O)$ is as required by the lemma.

Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of R, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = t \in C'$ is as required.

The lemma follows easily by induction. □

6. The monster model

We denote by $\mathfrak{U} = \langle U, R\rangle$ some large $\mathbb{L}$-saturated structure which we call the monster model. The cardinality of $\mathfrak{U}$ is an inaccessible cardinal that we denote by κ. Below we say model for $\mathbb{L}$-elementary submodel of $\mathfrak{U}$.

Let $A \subseteq U$ be a small set. As usual, we define a topology on $U^{|x|}$ which we call the $\mathbb{L}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x) \subseteq \mathbb{L}(A)$. This is a compact topology.

The $\mathbb{L}(A)$-topology on $U^{|x|} \times R^{|y|}$ is the product of the $\mathbb{L}(A)$-topology on $U^{|x|}$ and the topology on $R^{|y|}$.

10 Lemma. For every $\varphi(x\,;y) \in \mathbb{L}(A)$ the set $\varphi(U\,;R)$ is compact.

Proof. We prove that every finitely intersecting family of closed subsets of $\varphi(U\,;R)$ have nonempty intersection. A family of closed subsets of $\varphi(U\,;R)$ has the form

□

11 Fact. Let $p(x) \subseteq \mathbb{L}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{L}(U)$

1. if $p(x) \rightarrow \neg\varphi(x)$ then $\psi(x) \rightarrow \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \rightarrow \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \rightarrow \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. The first claim is clear. The second follows from the first by Lemma 9. □

12 Proposition. For every $\varphi(x) \in \mathbb{H}(U)$

$$\bigwedge_{\varphi' > \varphi} \varphi'(x) \quad\leftrightarrow\quad \varphi(x)$$

Proof. We prove $\rightarrow$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifier is immediate. Consider the existential quantifiers of sort U. Assume inductively

$$\bigwedge_{\varphi' > \varphi} \varphi'(x, y) \quad\rightarrow\quad \varphi(x, y)$$

then

$$\exists y \bigwedge_{\varphi'>\varphi} \varphi'(x,y) \;\;\rightarrow\;\; \exists y\, \varphi(x,y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi'>\varphi} \exists y\, \varphi'(x,y) \;\;\rightarrow\;\; \exists y \bigwedge_{\varphi'>\varphi} \varphi'(x,y)$$

Replace x with a parameter, say a and assume the antecedent. Note that $\{\exists y\, \varphi'(a,y) : \varphi' > \varphi\}$ implies the finite concistency of the type $\{\varphi'(a,y) : \varphi' > \varphi\}$. This because if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. Therefore $\exists y\, \{\varphi'(a,y) : \varphi' > \varphi\}$ follows by saturation. □

When $A \subseteq U$, we write $S_{\mathbb{H}}(A)$ for the set of types

$$\mathbb{H}\text{-tp}(a/A) \;\;=\;\; \big\{\varphi(x) \;:\; \varphi(x) \in \mathbb{H}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of U. We write $S_{\mathbb{H},x}(A)$ when the tuple of variables x is fixed. The following corollary will be strengthen by Corollary 18 below.

13 Corollary. The types $p(x) \in S_{\mathbb{H}}(A)$ are complete. That is, either $\varphi(x) \in p$ or $p(x) \rightarrow \neg\varphi(x)$ for every $\varphi(x) \in \mathbb{H}(A)$.

Proof. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 9 and Proposition 12 we obtain

$$\neg\varphi(x) \;\;\rightarrow\;\; \bigvee_{\tilde\varphi \perp \varphi} \tilde\varphi(x).$$

Hence $\tilde\varphi(a)$ holds for some $\tilde\varphi \perp \varphi$ and $p(x) \rightarrow \neg\varphi(x)$ follows. □

The following will be useful below

14 Remark. For $p(x) \subseteq \mathbb{L}(U)$, we write $p'(x)$ for the type

$$p'(x) \;\;=\;\; \big\{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}.$$

Note that, by Proposition 12, if $p(x) \in S_{\mathbb{H}}(A)$ then $p'(x)$ is equivalent to $p(x)$. Therefore $p'(x)$ is also complete for formulas in $\mathbb{H}(A)$. □

15 Corollary. The inverse of an $\mathbb{H}$-elementary map $f : \mathcal{U} \rightarrow \mathcal{U}$ is $\mathbb{H}$-elementary. □

7. Homogeneity

A model $\mathcal{M}$ is $\mathbb{H}$-homogeneous if every $\mathbb{H}$-elementary map $f : \mathcal{M} \rightarrow \mathcal{M}$ of cardinality $< |\mathcal{M}|$ extends to an automorphism.

16 Proposition. $\mathcal{U}$ is $\mathbb{H}$-homogeneous.

Proof. By Corollary 15 the usual proof by back-and-forth applies. □

17 Corollary. For every $a, b \in U^{|x|}$, if $a \equiv^{\mathbb{H}} b$ then $a \equiv^{\mathbb{L}} b$. □

18 Corollary. Let $p(x) \in S_{\mathbb{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathbb{L}_x(A)$. Clearly, the same holds for $p'(x)$. □

19 Proposition. Let $\varphi(x) \in \mathbb{L}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathbb{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 18

$$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee_{p'(x) \to \neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathbb{H},x}(A)$. By Fact 11

$$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee_{\psi(x) \to \neg\varphi(x)} \psi(x)$$

where $\psi'(x)$ is such that $\psi' > \psi$ for some $\psi(x) \in \mathbb{H}(A)$ such that $\psi(x) \to \neg\varphi(x)$. By compactness

$$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \psi_i'(x)$$

By Lemma 9 we can replace $\psi_i'(x)$ by $\neg\tilde{\psi}_i(x)$ for some $\tilde{\psi}_i \perp \psi_i$. The proposition follows. □

20 Proposition. Let $\varphi(x) \in \mathbb{L}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathbb{H}(A)$ and some $\psi' > \psi$. □

Proof. Let $a \in U^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$. By Corollary 15, $p'(x) \to \neg\varphi(x)$. By compactness $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. □

21 Proposition (Tarski-Vaught Test). Let M be a subset of U. Then the following are equivalent

1. M is the domain of a model;
2. for every formula $\varphi(x) \in \mathbb{H}(M)$
$$\exists x\, \varphi(x) \quad \Rightarrow \quad \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathbb{L}(M)$
$$\exists x\, \neg\varphi(x) \quad \Rightarrow \quad \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 9 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\forall x\, \tilde{\varphi}(x)$ hence, by (1), $\mathcal{M} \models \neg\forall x\, \varphi(x)$. Then $\mathcal{M} \models \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $\mathcal{M} \models \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathbb{L}(M)$ be such that $\exists x\, \neg\varphi(x)$. Then, by Corollary 20, there are a consistent $\psi(x) \in \mathbb{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). Then, by the classical Tarski-Vaught test $M \preceq U$. We also have that $\mathcal{M}$ is an $\mathcal{L}$-submodel of $\mathcal{U}$. Therefore for every $a \in M^{|x|}$ and for every $\mathbb{L}$-atomic formula, $\varphi(a)$ if and only if $\mathcal{M} \models \varphi(a)$. Now, assume inductively

$$\mathcal{M} \models \varphi(a, b) \quad \Rightarrow \quad \varphi(a, b)$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$\mathcal{M} \models \forall y\, \varphi(a, y) \quad \Rightarrow \quad \forall y\, \varphi(a, y).$$

Indeed,

$$\neg\forall y\, \varphi(a, y) \quad \Rightarrow \quad \exists y\, \neg\varphi(a, y)$$

$$\begin{aligned}
&\Rightarrow\ \neg\varphi(a,b) && \text{for some } b\in M^{|y|}\\
&\Rightarrow\ \mathcal{M}\vDash\neg\varphi(a,b) && \text{for some } b\in M^{|y|}\\
&\Rightarrow\ \mathcal{M}\nvDash\forall y\,\varphi(a,y)
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists$, $\exists^C$, and $\forall^C$ is straightforward. □

8. Completeness

Needs full rewriting

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \to R$.

22 **Fact.** If $\bar{a} = \langle a_i : i < \lambda\rangle$ and $\bar{b} = \langle b_i : i < \lambda\rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \to R$.

Proof. By induction on λ. Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ. For limit ordinals induction is immediate. □

23 **Example (???).** Let L be the language of $\mathbb{R}$-algebras expanded with two lattice operators $\wedge$, $\vee$. Let $\langle \Omega, \mathcal{B}, \Pr\rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L. Let $R = \mathbb{R}$. Assume $\mathcal{L}$ contains a symbol for the functions E_n, for $n \in \mathbb{N}$, that give the expected value of $(X \wedge n) \vee -n$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is definable in the limit over M if $a \equiv^{\mathbb{L}}_M x \to a \sim x$

24 **Example.** Assume that $R = \mathbb{R}$ and that $\mathcal{L}$ contains a function of sort $M^2 \to R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \to R$ are continuous with rispect to this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M, a model;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of M that converges to a.

Proof. ($2 \Rightarrow 1$) Let $\langle \varepsilon_i : i \in \omega\rangle$ be a sequence that converges to 0 and such that $d(a_i, a) \le \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \le \varepsilon_i$ for every $b \equiv^{\mathbb{L}}_M a$. By the uniqueness of the limit $d(a, b) = 0$.

($1 \Rightarrow 2$) Assume that $a \equiv^{\mathbb{L}}_M x \to a \sim x$. Then $a \equiv^{\mathbb{L}}_M x \to d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \mathrm{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \to d(a, x) < 1/n$. By $\mathbb{L}$-elementarity there is an $a_n \in M$ such that $\varphi(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega\rangle$ converges to a. □