CONTINUOUS MODEL THEORY WITHOUT THE HASSLE

1. FORMULAS AND THEIR INTERPRETATION

The following definition does not differ from classical first-order logic but for the third clause (which you may ignore til the end of the section).

1 Definition. A signature or language L consists of the following data

1. two sets of symbols: L_{rel}, for relations, and L_{fun}, for functions;

2. an arity function $\mathrm{Ar} : L_{\mathrm{rel}} \cup L_{\mathrm{fun}} \to \omega$;

3. the functions $n_{\text{-}} : L_{\mathrm{at}} \to \mathbb{N}$ and $t_{\text{-}} : L \to L_{\mathrm{trm}}$.

The functions in 3 above are called bounding functions. The meaning of L, L_{at} and L_{trm} in 4 is clarified below.

2 Definition. A structure or model M of signature L consists of the following data

1. for every $f \in L_{\mathrm{fun}}$, a function $f^M : M^{\mathrm{Ar}(f)} \to M$;

2. for every $r \in L_{\mathrm{rel}}$, a function $r^M : M^{\mathrm{Ar}(r)} \to \mathbb{R}$;

3. a subset $U \subseteq M$.

The set U in 3 above is called the unit ball of M. Again, this can be ignored for the time being. The axioms for the unit ball are stated in Definition 5 where we will also clarify the role of the bounding functions.

The syntax and semantic of terms is defined just as in classical first-order logic. We write L_{trm} for the set of terms in the language L. Atomic formulas are defined as in classical first-order logic. They have the form $r\,t$, where r is a relation symbol and t is a tuple of terms. The set of atomic formulas is denoted by L_{at}. The set of all formulas is denoted as usual by L, the same symbol as the language. This is defined inductively from the atomic formulas using the following connectives.

The propositional (or Riesz) connectives are those in $\{1, +, \wedge\} \cup \mathbb{R}$, where $+$ and $\wedge$ are binary connectives, the elements of $\mathbb{R}$ are unary connectives, 1 is a logical constant (i.e. a zero-ary connective). There is also the infimum quantifier $\bigwedge\limits_{x}$.

3 Definition. The inductive definition of formula is as follows

i. atomic formulas are formulas

ii. 1 is a formula;

iii. $\varphi + \psi$ is a formula;

iv. $\varphi \wedge \psi$ is a formula;

v. $\alpha\varphi$ is a formula for every $\alpha \in \mathbb{R}$;

vi. $\bigwedge_x \varphi$ is a formula.

If $\alpha \in \mathbb{R}$, we may write $\boxed{\alpha}$ for the formula $\alpha 1$. We write $\boxed{-\varphi}$ for $(-1)\varphi$ and $\boxed{\varphi - \psi}$ for $\varphi + (-\psi)$ and we write $\boxed{\varphi \vee \psi}$ for $-(-\varphi \wedge -\psi)$. Also, $\boxed{\varphi^+}$ and $\boxed{\varphi^-}$ stand for $0 \vee \varphi$ respectively $0 \vee (-\varphi)$, and we write $\boxed{|\varphi|}$ for $\varphi \vee (-\varphi)$. Finally, $\boxed{\bigvee_x \varphi}$ stands for $-\bigwedge_x -\varphi$.

If $A \subseteq M$, we write $L(A)$ for the language L expanded with a 0-ary function symbol for every element of a. As in classical first-order logic, M is canonically expanded to a structure of signature $L(A)$ by setting $a^M = a$ for every $a \in A$.

Let M be a model and let t be a tuple of closed terms possibly with parameters from M. We write t^M for the element obtained interpreting function symbols and parameters in M just as in classical first-order logic.

4 Definition. If $\varphi \in L(M)$ is a sentence (i.e. a closed formula) we define φ^M inductively as follows

i. $(r\,t)^M = r^M(t^M)$.

ii. $1^M = 1$;

iii. $(\psi + \xi)^M = \psi^M + \xi^M$;

iv. $(\alpha\varphi)^M = \alpha\varphi^M$;

v. $(\psi \wedge \xi)^M = \min\{\psi^M, \xi^M\}$;

vi. $\left(\bigwedge_x \psi\right)^M = \inf\{\max\{0, \psi[x/b]^M\} : b \in M\}$.

Note that the cutoff in condition vi of the definition above is necessary because formulas are, in principle, unbounded.

For $\varphi(x) \in L(M)$ we write φ^M for the function $\varphi^M : M^{|x|} \to \mathbb{R}$ that maps $a \mapsto \varphi(a)^M$.

Finally, we complete the definition of structure by giving a meaning to the unit ball and to the bounding functions. However, the requirements below are only needed in Section 4 for the construction of ultrapower to work properly.

5 Definition. The unit ball U is required to satisfy the following axioms

1. $M = \{t^M : t \in L_{\mathrm{trm}}(U)$ a closed term $\}$;

2. $M \vDash |\varphi(a)| \le n_{\varphi(x)}$ for every $a \in U^{|x|}$ and every $\varphi(x) \in L_{\mathrm{at}}$;

3. for every formula $\psi(x, y) \in L$ and for every $a \in U$

$$M \vDash \exists y\, \psi(a, y) \le 0 \quad \Rightarrow \quad M \vDash \psi(a, t_{\psi(x,y)}(b)) \le 0 \text{ for some } b \in U^{|z|}$$

For clarity, we stress that the variables occurring in $t_{\psi(x,y)}(z)$ may be distinct from x, y.

2. EXAMPLES

1. Firstly, we leave to the reader to check that classical first order models are a (trivial) special cases of the models introduced above. Classical relations take values in $\{0, 1\}$ where 0 is interpreted as "true". The unit ball is the whole of M. The function n_- is constantly 1, and $t_{\psi(x,y)}$ is the term y. To obtain the full strength of firt order logic one has to add equality as a predicate which is abset from our language.

2. Secondly, we consider a class of examples that are non trivial (and distant from classical logic) but where unit ball and bounding functions are irrelevant.

   The sets L_{rel}, L_{fun} and the arity function $\mathrm{Ar}(\text{-})$ are arbitrary. The interpretation of the symbols is arbitrary but we require that the functions r^M, for all $r \in L$, range in the interval $[-1, 1]$. As unit ball take the whole of M. The function n_- is constant 1, and $t_{\psi(x,y)}$ is (trivially) the term y.

3. Any BBHU-premodel M is also a model as those in Section 1 if we take as unit ball the whole of M. The language L contains a relation symbol $d(x, y)$ for a metric and, possibly, symbols for all other functions and relations of M. As all relations of M (including the metric) take values in the interval $[0, 1]$, the function n_- can be set to be the constant 1.

4. Finally, let L be the language of Banach spaces. Here we have only one symbol in L_{rel}, the symbol $\|\text{-}\|$ for the norm. The set L_{fun} is the same as for real vector spaces in classical logic. Our model M is a Banach space. The interpretation of the symbols in $L_{\mathrm{rel}} \cup L_{\mathrm{fun}}$ is the natural one. The unit ball of M is $U = \{a \in M : \|a\| = 1\}$.

   (Need be completed)

3. CONDITIONS AND TYPES

For $\varphi, \psi \in L(M)$ some closed formulas, we write $\boxed{M \vDash \varphi \leq \psi}$ for $\varphi^M \leq \psi^M$. The meaning of $\boxed{M \vDash \varphi = \psi}$ and of $\boxed{M \vDash \varphi < \psi}$ is similar.

Equations of the form $\varphi(x) = 0$ are called $\boxed{\text{conditions}}$. We write $M \vDash \exists x\, \varphi(x) = 0$ if $M \vDash \varphi(a) = 0$ for some $a \in M^{|x|}$. Observe that $\varphi(x) = 0$ is equivalent to $|\varphi(x)| \leq 0$ and $\varphi(x) \leq 0$ is equivalent to $\big(\varphi(x) \vee 0\big) = 0$. So, when convenient we may use weak inequalities to denote conditions.

The negation of a condition is called a $\boxed{\text{co-conditions}}$. So, $\varphi(x) \neq 0$, $\varphi(x) < 0$ or $\varphi(x) > 0$ are co-conditions.

A set of conditions is called a $\boxed{\text{type}}$ or, when x is the empty tuple, a $\boxed{\text{theory.}}$ For $p(x) \subseteq L(M)$, we define

$$\boxed{p(x) \leq 0} \;=\; \big\{\varphi(x) \leq 0 \quad : \quad \varphi(x) \in p\big\}.$$

Up to equivalence, all type have the form above.

Beware that $p(x)$ denotes just a set of formulas, the expression $p(x) \leq 0$ denotes a type.

We write $M \vDash p(a) \leq 0$ if $M \vDash \varphi(a) \leq 0$ for every $\varphi(x) \in p$. We write $M \vDash \exists x\, p(x) \leq 0$ if $M \vDash p(a) \leq 0$ for some $a \in M^{|x|}$.

4. ULTRAPOWERS

Firstly, we recall some standard notation about limits. Let I be a non-empty set. Let F be a filter on I. If $r : I \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we write

$$\lim_{i \uparrow F} r(i) = \lambda$$

if $r^{-1}[A] \in F$ for every $A \subseteq \mathbb{R}$ that is a neighborhood of λ. When r is bounded and F is an ultrafilter, such a λ always exists.

Let I be an infinite set. Let $\langle M_i : i \in I \rangle$ be a sequence of models with unit balls U_i. Note that all models are required to have the same bounding functions (as these are part of the language). Let F be an ultrafilter on I.

6 Definition. Write U for the set of all functions $\hat{a} : I \to \bigcup_{i \in I} U_i$ such that $\hat{a}\, i \in U_i$. We define the structure N as follows

1. the elements of N are functions $\hat{c} : I \to \bigcup_{i \in I} M_i$ such that $\hat{c}\, i = t(\hat{a}\, i)$ for some $\hat{a} \in U^{|x|}$ and some term $t(x) \in L_{\mathrm{trm}}$ — below we write $t(\hat{a})$ for $\hat{c}$;

2. if $f \in L_{\mathrm{fun}}$ and $r \in L_{\mathrm{rel}}$ and $t_1(\hat{a}), \dots, t_n(\hat{a})$ are elements of N then

    a. $f^N\big(t_1(\hat{a}), \dots, t_n(\hat{a})\big) = t(\hat{a})$ where t is the term obtained composing f with $t_1, \dots, t_n$;

    b. $r^N\big(t_1(\hat{a}), \dots, t_n(\hat{a})\big) = \lim_{i \uparrow F} r^{M_i}\big(t_1(\hat{a}\, i), \dots, t_n(\hat{a}\, i)\big)$.

The limit in 2b of Definition 6 exists because F is an ultrafilter and it is finite because all U_i have the same bounding functions. In fact, the function $i \mapsto r^{M_i}\big(t_1(\hat{a}\, i), \dots, t_n(\hat{a}\, i)\big)$ is bounded by $n_{\varphi(x)}$, where $\varphi(x) = r(t_1(x), \dots, t_n(x))$.

7 Proposition (Łǒś Theorem). Let N be as above and let $\varphi(x) \in L$. Then for every $\hat{a} \in N^{|x|}$,

$$\varphi^N(\hat{a}) = \lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i).$$

Proof. If $\varphi(x)$ is atomic holds by definition. We proceed by induction on the syntax of $\varphi(x)$. We only spell out the proof for the infimum quantifier. Assume inductively that

$$\varphi^N(\hat{a}, \hat{b}) = \lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i, \hat{b}i)$$

We want to prove that

$$\left(\bigwedge_x \varphi(\hat{a}, x) \right)^N = \lim_{i \uparrow F} \left(\bigwedge_x \varphi(\hat{a}i, x) \right)^{M_i}$$

By the interpretation of the infimum quantifier, this amounts to prove that

$$\inf \left\{ \max \left\{ 0, \ \varphi(\hat{a}, \hat{b})^N \right\} \ : \ \hat{b} \in N \right\} \ = \ \lim_{i \uparrow F} \inf \left\{ \max \left\{ 0, \ \varphi(\hat{a}i, b)^{M_i} \right\} \ : \ b \in M_i \right\}$$

For simplicity, assume that $\varphi(\hat{a}, \hat{b})^N$ and $\varphi(\hat{a}i, b)^{M_i}$ are non negative. First we prove the $\leq$ inequality which is the run-of-the-mill half of the proof. Let r be an arbitrary positive real number.

Assume that for some $\hat{b}$
$$\varphi^N(\hat{a}, \hat{b}) \ < \ r.$$

By induction hypothesis
$$\lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i, \hat{b}i) \ < \ r,$$

then for some $u \in F$
$$\varphi^{M_i}(\hat{a}i, \hat{b}i) \ < \ r \quad \text{for every } i \in u.$$

Hence
$$\inf \left\{ \varphi^{M_i}(\hat{a}i, b) \ : \ b \in M_i \right\} \ < \ r \quad \text{for every } i \in u.$$

Finally,
$$\lim_{i \uparrow F} \inf \left\{ \varphi^{M_i}(\hat{a}i, b) \ : \ b \in M_i \right\} \ < \ r.$$

This proves the $\geq$ inequality. The proof of the $\leq$ inequality is where the bounding function t_- is required. Assume that

$$\lim_{i \uparrow F} \inf \left\{ \varphi(\hat{a}i, b)^{M_i} \ : \ b \in M_i^{|x|} \right\} \ < \ r.$$

Therefore, for some $u \in F$

$$\inf \left\{ \varphi(\hat{a}i, b)^{M_i} \ : \ b \in M_i^{|x|} \right\} \ < \ r \quad \text{for every } i \in u.$$

Without loss of generality we can assume that $\hat{a} \in U$. Hence, writing $t(z)$ for $t_{\varphi(x,y)}(z)$ we have

$$\inf \left\{ \varphi(\hat{a}i, t(c))^{M_i} \ : \ c \in U_i^{|z|} \right\} \ < \ r \quad \text{for every } i \in u.$$

Therefore, for some $\hat{c} \in U^{|z|}$
$$\varphi^{M_i}(\hat{a}i, t(\hat{c}i)) \ < \ r \quad \text{for every } i \in u.$$

Then
$$\lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i, t(\hat{b}i)) \ < \ r,$$

and finally
$$\varphi^N(\hat{a}, t(\hat{b})) \ < \ r.$$

This proves the desired inequality. $\qquad\square$

8 Fact. The model N defined above has unit ball U.

Proof. The claim about U and n_- follows immediately from the definition of N, see 2b above. As for t_-, the clain follows from Łŏś Theorem. $\qquad\square$

5. COMPACTNESS THEOREM

From Łŏś Theorem we obtain a form of compactness with the usual argument. Note however that, w.r.t. classical logic, here we add a requirement of uniformty.

A theory $T \leq 0$ is finitely consistent if for every φ, that is a disjunction of sentences in T, there is a model M such that $M \vDash \varphi \leq 0$. Note that require that all these models M have the same bounding fuctions.

9 Theorem (Compactness Theorem). Every finitely consistent theory is consistent.

Proof. Let n_-, t_- be some bounding function that witness the uniform finite consistency of T. Below by *model* we mean a model with those bounding functions.

We construct a model N of $T \leq 0$. Let I be the set of closed formulas ξ such that $\xi \leq 0$ holds in some model M_ξ. For every condition $\varphi \leq 0$ define $X_\varphi \subseteq I$ as follows

$$X_\varphi \;=\; \left\{ \xi \in I \;:\; \xi \leq 0 \vdash \varphi \leq 0 \right\}$$

Note that $\varphi \leq 0$ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi \vee \psi} = X_\varphi \cap X_\psi$. Then, as T is finitely consistent, the set $B = \left\{ X_\varphi \;:\; \varphi \in T \right\}$ has the finite intersection property. Extend B to an ultrafilter F on I. Define

$$N \;=\; \prod_{\xi \in I} M_\xi$$

Check that $N \vDash T \leq 0$. Let $\varphi \in T$. By Łǒś Theorem,

$$\varphi^N \;=\; \lim_{\xi \uparrow F} \varphi^{M_\xi}.$$

Note that $X_\varphi \subseteq \left\{ \xi \;:\; \varphi^{M_\xi} \leq 0 \right\}$ and recall that $X_\varphi \in F$ whenever $\varphi \in T$. Therefore $\varphi^N \leq 0$. $\square$

6. ELEMENTARY MAPS

The A-limit topology on $M^{|x|}$ is the initial topology with respect to the functions interpret the formulas in $L(A)$. The sets $\left\{ a \in M^{|x|} \;:\; \varphi^M(a) \leq 0 \right\}$, as $\varphi(x)$ ranges over $L(A)$, form a base of closed sets for the A-limit topology. Equivalently, we can take the sets $\left\{ a \in M^{|x|} \;:\; \varphi^M(a) < 0 \right\}$ as a base of open sets.

The A-limit topology is a completely regular topology, almost by definition. Typically, it is not T_0.

We say that the map $k \;:\; M \rightarrow N$ is an elementary map if

$$\varphi^M(ka) \;=\; \varphi^N(a) \qquad\qquad \text{for every } \varphi(x) \in L \text{ and every } a \in (\mathrm{dom} k)^{|x|}.$$

We write $M \equiv N$ when $\varnothing \;:\; M \rightarrow N$ is an elementary map and $M \equiv_A N$ if $\mathrm{id}_A \;:\; M \rightarrow N$ is an elementary map. In words, we say that M and N are elementary equivalent over A. An elementary embedding is an total elementary map. Finally, we write $M \preceq N$ for $M \equiv_M N$ and say that M is an elementary substructure of N.

The following corollary of Łǒś Theorem is identical to its classical counterpart.

10 Corollary. For every model M and every ultrafilter F on I, an infinite set, let N be corresponding ultrapower of N. Then there is an elementary embedding of M in N. $\square$

11 Proposition (Tarski-Vaught test). Let N be a model and and let $M \subseteq N$ be a substructure. Fix a tuple of variables x of finite length. Then the following are equivalent:

1. $M \preceq N$;

2. $M^{|x|}$ is dense in $N^{|x|}$ w.r.t. the limit M-topology on $N^{|x|}$;

3. for every $\varphi(x) \in L(M)$, if $\varphi(N) < 0$ is non empty, then $\varphi(b) < 0$ for some $b \in M^{|x|}$.

Proof. Equivalence 2$\Leftrightarrow$3 and implication 1$\Rightarrow$2 are clear. To prove 2$\Rightarrow$1 we prove by induction on the syntax of $\varphi(x)$ that for every $r \in \mathbb{R}$

$$M \vDash \varphi(a) < r \quad \Leftrightarrow \quad N \vDash \varphi(ka) < r \qquad\qquad \text{for every } a \in M^{|x|}$$

We use the equivalence above as induction hypothesis. As M is a substructure of N, this equivalence holds for atomic $\varphi(x)$. We leave the case of Riesz connectives to the reader and consider only the case of the infimum quantifier.

$$N \vDash \bigwedge_x \varphi(x) < r \quad \Leftrightarrow \quad \text{the set } \varphi(N) < r \text{ is non empty}$$

a. $\qquad\qquad\qquad\qquad \Leftrightarrow \quad$ there is $b \in M^{|x|}$ such that $N \vDash \varphi(b) < r$

b. $\qquad\qquad\qquad\qquad \Leftrightarrow \quad$ there is $b \in M^{|x|}$ such that $M \vDash \varphi(b) < r$

$$\qquad\qquad\qquad \Leftrightarrow \quad M \vDash \bigwedge_x \varphi(x) < r.$$

Implication $\Rightarrow$ in *a* uses 2 and the equivalence in *b* uses the induction hypothesis. $\qquad\square$

12 Remark. In the Tarski-Vaught test for classical logic it is not necessary to require that M is a substructure of N as this is a consequence of the test. In the continuous case this is necessary by the lack of equality. $\qquad\square$

13 Theorem (Downward Löwenheim-Skolem). For every $A \subseteq N$ there is a model $A \subseteq M \preceq N$ of cardinality $\leq |L(A)|$.

Proof. (Needs some checking.) For every set A there is a base for the A-topology that has cardinality $\leq |L(A)|$. A set M of the required cardinality that is dense in the M-topology is obtained as in the classical downward Löwenheim-Skolem theorem. $\qquad\square$

7. SATURATION

Throughout this section M is a model with unit ball U and bounding functions n_-, t_-.

Let $p(x) \subseteq L(M)$. We say that $p(x) \leq 0$ is finitely satisfied (in U) if for every formula $\psi(x)$ that is a disjunction of formulas in $p(x)$, there is an $a \in U^{|x|}$ such that $M \vDash \psi(a) \leq 0$.

We say that M is λ-saturated if for every $p(x) \subseteq L(A)$ as in 1 and 2 below, there is an $a \in U^{|x|}$ such that $M \vDash p(a) \leq 0$.

1. $A \subseteq M$ has cardinality $< \lambda$;

2. $p(x) \leq 0$ is finitely satisfied in U.

If λ is the cardinality of M we say that M is saturated.

14 Theorem (???). Let λ be an inaccessible cardinal larger than $|L|$ and $|M|$. Then M has a saturated elementary superstructure of cardinality M.

8. COMPLETENESS VS. SATURATION ¡ PLEASE EXPAND !

The A-limit uniformity on $M^{|x|}$ is the uniformity that has the following entourages

$$V_{\varphi(x),\varepsilon} = \left\{ (a,b) \in M^{|x|} \times M^{|x|} \quad : \quad \left| \varphi^M(a) - \varphi^M(b) \right| < \varepsilon \right\}$$

for $\varphi(x) \in L(A)$ and $\varepsilon \in \mathbb{R}^+$. It is the coarsest uniformity that makes all formulas in $L(A)$ uniformly continuous.

Let $p(x) \subseteq L(M)$ be a type. We say that $p(x) \leq 0$ is Cauchy (in the A-limit uniformity) if for every $\varphi(x) \in L(A)$ and every $\varepsilon \in \mathbb{R}^+$ there is a disjunction of formulas in $p(x)$, say $\psi(x)$, such that $\varnothing \neq \left(\psi(M) \leq 0 \right)^2 \subseteq V_{\varphi(x),\varepsilon}$.

15 Fact. Let $\lambda = L(A)$. If M is λ-saturated then and $p(x) \subseteq L(M)$ is Cauchy, then $p(x)$ is realized in M.

For every $\varphi(x) \in L(A)$ and $n \in \mathbb{N}$ let

Suppose $N \vDash p(a) \leq 0$ and let $\left(\psi(M) \leq 0 \right)^2 \subseteq V_{\varphi(x),\varepsilon}$