CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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These notes have no ambition of mathematcal novelty. In fact, most techniques applied here are at least half a century old. Still we believe that an important facet of mathematics progress consists of the simplificaytion and clarification of existent knowledge. We hope to contribute in this manner.

1. A CLASS OF STRUCTURES

Real vector spaces are among the most simple strucures considered in model theory. Unfortunately, in most situations they are not expressive enough. It is natural to expand them by adding a norm. The norm is a function that, given vector, outputs a real number. Unfortunately, this function messies up with our intuition of elementary extension. Ideally, we would like to extend the vector field while mantaining the usal notion of real number[1] but ths is impossible.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L'-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

Function symbols of L' falls in one of these two groups

1. $M^n \times R^m \to M$
2. $M^n \times R^m \to R$

We require that function symbols of the second group are interpreted in functions such that

i. $f^{\mathcal{M}} : M^n \times R^m \to R$ is bounded, i.e. its range is contained in a compact set;

ii. $f^{\mathcal{M}} : M^n \times R^m \to R$ is continuous uniformly in M, as defined in Definition 3 below.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated and is still same class. Our choice of $\mathbb{L}$ is not the most general. It is a compromise that avoids eccessive complications.

[1]Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ needs to remain $\mathbb{R}$ throughout.

In a nutshell, we require that function symbols of the second group above do not occur under the scope of a negation. We also do not allow existential quantifiers of sort R. In fact, the ultrapower construction that we present in Section 3 is completely classical w.r.t. functions the first group. On the other hand, functions of the second group are *critical* and need to be dealt with differently.

Below, $\mathbb{T}_M$ denotes the set of the terms obtained composing function symbols as in 1 above.

By $\mathbb{T}_R$ we denote the set of the terms obtained composing function symbols as in 2 above.

By $\mathbb{T}_R \circ \mathbb{T}_M$ we denote the set of the terms obtained composing terms in $\mathbb{T}_R$ with terms in $\mathbb{T}_M$.

Clearly, there are many more terms in L' but we will restrict to these.

2 **Definition.** Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

i. formulas as $t(y; z) \in C$, where $C \subseteq R^n$ is compact[2] and $t(y; z)$ is a tuple of terms in $\mathbb{T}_R \circ \mathbb{T}_M$;

ii. quantifier free formulas with only terms in $\mathbb{T}_M$.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$, the quantifiers $\forall$, $\exists$ of sort M; and the universal quantifier $\forall$ of sort R.

Clearly, $L \subseteq \mathbb{L} \subseteq L'$.

2. Topological tools

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

By βI we denote the set of ultrafilters on I. This is a topological space with the topology generated by the clopen sets $[A] = \{F \in \beta I : A \in F\}$ as A ranges over the subsets of I.

3 **Definition.** Let I be a pure set. Let X, Y be topological spaces, and assume Y is Hausdorff and compact. We say that $f : I \times X \to Y$ is continuous uniformly in I if the function below is continuous

$$\begin{array}{rcl} \bar{f} : \beta I \times X & \to & Y \\ \langle F, \alpha \rangle & \mapsto & F\text{-}\lim_i f(i, \alpha) \end{array}$$

The following propositions unravels the definition above into a more practicable form.

[2]We confuse the relation symbols in L' of sort R^m with their interpretation and write $x \in C$ for $C(x)$

4 Proposition. The following are equivalent

1. $f : I \times X \to Y$ is continuous uniformly in I.
2. For every $F \in \beta I$ and $\alpha \in X$, and for every $V \subseteq Y$ open neighborhood of $F\text{-}\lim_i f(i, \alpha)$ there is $U \subseteq X$ open neighborhoodof α and an $A \in F$ such that $f(i, x) \in V$ for every $i \in A$ and every $x \in U$.

5 Lemma (???)**.** Let $f : I \times X \to Y$ be continuous uniformly in I. Let $\langle y_i \in Y : i \in I\rangle$. For a given $F \in \beta I$ define $\alpha = F\text{-}\lim_i y_i$. Then $f(F, \alpha) = F\text{-}\lim_i f(i, y_i)$.

6 Lemma (???)**.** Let $f : I \times X \to Y$ be continuous uniformly in I. Let $g : I \times Y \to I$ be any function and let $\bar{g} : \beta I \times Y \to \beta I$ the unique continuous extension of g. Let $\langle y_i \in Y : i \in I\rangle$. For a given $F \in \beta I$ define $\alpha = F\text{-}\lim_i y_i$. Then $f(\bar{g}(F, \alpha), \alpha) = F\text{-}\lim_i f(g(i, y_i), y_i)$.

3. Ultrapowers

To keep notation tidy, we do not consider formulas containing equality, so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

7 Definition. We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \times R^m \to M$ then $f^{\mathcal{N}}(\hat{a}; \alpha)$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i; \alpha\big) : i \in I\rangle$.
4. If f is a function of sort $M^n \times R^m \to R$ then for every $\alpha \in R^m$
$$f^{\mathcal{N}}\big(\hat{a}; \alpha\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i; \alpha\big).$$
3. if r is a relation symbol of sort $M^n \times R^m$ then for every $\alpha \in R^m$
$$\mathcal{N} \vDash r(\hat{a}; \alpha) \Leftrightarrow \big\{i \in I \;:\; \mathcal{M} \vDash r(\hat{a}i; \alpha)\big\} \in F.$$

The following is immediate.

8 Fact. The structure $\mathcal{N}$ satisfies condition 1 of Definition 1.

The following is easily proved by induction on the syntax

9 Fact. If $t(y; z) \in \mathbb{T}_M$ then
$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \;=\; \langle t^{\mathcal{M}}\big(\hat{a}i; \alpha\big) : i \in I\rangle. \qquad \square$$

We also have that

10 Fact. If $t(y;z) \in \mathbb{T}_R$ then

$$\#_0 \qquad t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

Proof. By induction. The fact holds by definition if t is a function symbol. Therefore, we assume $\#_0$ as induction hypothesis and prove

$$\#_1 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\Big).$$

By induction hypothesis

$$\#_2 \qquad f^{\mathcal{N}}\Big(\hat{a};t^{\mathcal{N}}\big(\hat{a};\alpha\big)\Big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\Big(\hat{a}i;\lambda\Big), \qquad \text{where } \lambda = F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$

Let σ be the limit in $\#_2$. Let $V \subseteq R$ be an open set containing σ. Note that we can identify F with an ultrafilter on $M^{|\hat{a}|}$ concentrated on range of $\hat{a}$. Apply 2 of Proposition 4, to obtain $A \in F$ and an open set $U \subseteq R$ containing λ such that for every $i \in A$ and every $x \in U$

$$f^{\mathcal{M}}\big(\hat{a}i;x\big) \;\in\; V.$$

By induction hypothesis there is $B \in F$ be such that $t^{\mathcal{M}}\big(\hat{a}i;\alpha\big) \in U$ for every $i \in B$. Therefore

$$f^{\mathcal{M}}\Big(\hat{a}i;t^{\mathcal{M}}\big(\hat{a}i;\alpha\big)\Big) \;\in\; V$$

for every $i \in A \cap B$. As $A \cap B \in F$, this sufficies to prove that σ is also the limit in $\#_1$. □

From the two facts above it immediately follows that

11 Fact. If $t(y;z) \in \mathbb{T}_R \circ \mathbb{T}_M$ then

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big).$$ □

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C. Note that, if no such atomic formulas occurs in φ, then $\varphi > \varphi$. Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$. Moreover

$$\bigwedge_{\varphi' > \varphi} \varphi' \;\leftrightarrow\; \varphi.$$

Finally, we prove Łŏś Theorem.

12 Proposition. Assume $|I| \geq$ the weight of the topology on R. Let $\mathcal{N}$ be as above and let $\varphi(x;y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \vDash \varphi(\hat{a};\alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x;y)$ is as in (ii) of Definition 2 then the proposition reduces to the classical Łŏś Theorem.

Now, suppose that $\varphi(x;y)$ is as in (i) of Definition 2, say it is the formula $t(x;y) \in C$, where $t(x;y) \in \mathbb{T}_R \circ \mathbb{T}_M$.

Assume $\mathcal{N} \vDash t(\hat{a},\alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a},\alpha)$. Hence, by the definition of F-limit, $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i,\alpha) \in D\big\} \in F$. Vice versa, assume $\mathcal{N} \vDash t(\hat{a},\alpha) \notin C$. Let D be a compact neighborhood of C such that $t^{\mathcal{N}}(\hat{a},\alpha) \notin D$. By Fact 11 and the definition of F-limit, $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i,\alpha) \notin D\big\} \in F$.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the universal quantifier of sort R we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a};\alpha,\beta) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i;\alpha,\beta)\big\} \in F \text{ for every } \varphi' > \varphi$$

and we prove that

$$\mathcal{N} \vDash \forall y\, \varphi(\hat{a};\alpha,y) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i;\alpha,y)\big\} \in F \text{ for every } \varphi' > \varphi.$$

The implication $\Rightarrow$ is clear. For the converse assume $\mathcal{N} \nvDash \varphi(\hat{a};\alpha,\beta)$ for some β. By induction hypothesis, and the requirement that F is an ultrafilter, $\big\{i \in I \;:\; \mathcal{M} \nvDash \varphi'(\hat{a}i;\alpha,\beta)\big\} \in F$ for some $\varphi' > \varphi$. A fortiori $\big\{i \in I \;:\; \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i;\alpha,y)\big\} \in F$, hence the implication $\Leftarrow$ follows.

To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a},\hat{b}\,;\alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i,\hat{b}i\,;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

We first we prove that

$$\mathcal{N} \vDash \forall y\, \varphi(\hat{a},y\,;\alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i,y\,;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

This is identical to the case above. The implication $\Rightarrow$ is clear. For the converse assume $\mathcal{N} \nvDash \varphi(\hat{a},\hat{b}\,;\alpha)$ for some $\hat{b}$. By induction hypothesis, $\big\{i \in I \;:\; \mathcal{M} \nvDash \varphi'(\hat{a}i,\hat{b}i\,;\alpha)\big\} \in F$ for some $\varphi' > \varphi$. Hence $\big\{i \in I \;:\; \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i,y\,;\alpha)\big\} \notin F$ as required.

Finally we prove

$$\mathcal{N} \vDash \exists y\, \varphi(\hat{a},y\,;\alpha) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i,y\,;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Again, $\Rightarrow$ is clear. For the converse, assume the r.h.s. Assume for simplicity that $I = \omega$ and that R is second countable. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for for every $i < j$ and

$$\mathcal{M} \;\vDash\; \bigwedge_{i\in I} \varphi_i(x,y\,;\alpha) \;\rightarrow\; \varphi(x,y\,;\alpha).$$

For every $j \in I$ there is a sequence $\hat{b}_j$ such that

$$\big\{i \in I \;:\; \mathcal{M} \vDash \varphi_j(\hat{a}i,\hat{b}_j i\,;\alpha)\big\} \in F.$$

As $\varphi_j(x,y\,;\alpha) \rightarrow \varphi_{j'}(x,y\,;\alpha)$ holds for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_i i \;:\; i \in I\rangle$ satisfies $\mathcal{N} \vDash \varphi(\hat{a},\hat{d}\,;\alpha)$. $\square$