CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. INTRODUZIONE

As a minimal motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them by adding as norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two-sorted structure.

Now, we have two conflicting aspirations

i. to be able to apply the basic tools of model theory (such as elementarity and saturation);

ii. to remain in the realm of normed spaces (and insist that $\mathbb{R}$ should remain $\mathbb{R}$ throughout).

These two aspirations are blatantly incompatible, something has to give. Different groups made different choices.

1. Nonstandard analysts happily embraced norms that take values in some ${}^*\mathbb{R} \succeq \mathbb{R}$. They have a tookit of tricks to transfer the results from nonstandard to standard normed spaces and vice versa. Model theorists, when confronted with similar problems, have used a similar approach (in some more generality and with different notation).

2. Henson and Iovino stick to the standard notion of norm but restrict the notion of elementarity and saturation to a smaller class of formulas (which we denote by $\mathbb{H}$). They demostrate that many standard tools of model theory still apply in their setting. The work of Henson and Iovino is focussed on Banach spaces. It has hardly been applied to other contexts.

3. Real valued logicians have the most radical approach. They abandon classical $\{\text{true}, \text{false}\}$ valued logic altogether in favour of a logic valued in the interval $[0, 1]$. Unfortunately, this adds considerable notational burden, restricts the class of structures that can be considered, and (unnaturally) limits the expressive power of the logic. Though many have been attracted by the challange, this paper aims to offering an alternative to the rest.

We elaborate on the ideas of Henson and Iovino and generalize them to a larger class of structures. Just as Henson and Iovino, we restrict the notions of elementarity and saturation. Our class, $\mathbb{L}$, is larger than their $\mathbb{H}$. Though we prove that $\mathbb{L}$ has (in a sense) approximatively the same expressive power as $\mathbb{H}$ (cf. Propositions 17 and 18), it is evident that $\mathbb{L}$ is more convinent.

When possible we use methods and resusts of nonstandard analysis (i.e. classical model theory in disguise). This should help understanding the connection between approach (1) and (2) above.

2. A CLASS OF STRUCTURES

Let R be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language L_R contains relation symbols for the compact subsets $C \subseteq R^n$ and a function symbol for each continuous functions $f : R^n \to R$. According to the context, C and f denote either the symbols of L_R or their interpretation in R.

Definition 1. Let L be a one-sorted (first-order) language. By model we intend an $\mathcal{L}$-structure of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is the structure above. The language $\mathcal{L}$ is an expansion of both L and L_R. The new symbols allowed in $\mathcal{L}$ are function symbols of sort $M^n \to R$.

The scope of the notion of model above will be (inessentially) restricted once we introduce the monster model, in Section 7.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (i.e. when R is finite) they are not models. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq \mathcal{L}$, such that every model has an $\mathbb{L}$-elementary, $\mathbb{L}$-saturated extension that is a model.

Warning: as usual L and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. The set $\mathbb{L}$ defined below is only a set of first-order formulas.

Definition 2. Formulas in $\mathbb{L}$ are defined inductively from two sorts of $\mathbb{L}$-atomic formulas

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact (in the product topology) and $t(x, y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall$, $\exists$ of sort M; and the quantifiers of sort R which are denoted by $\forall^R$, $\exists^R$.

We write $\mathbb{H}$ for the set of formulas in $\mathbb{L}$ without quantifiers of sort R.

For later reference we remark the following which is an immediate consequence of the restrictions imposed by Definition 1 on the sorts of the function symbols in $\mathcal{L}$.

Remark 3. Each component of the tuple $t(x\,;y)$ in the definition above is either of sort $M^{|x|} \to R$ or of sort $R^{|y|} \to R$.

Formulas in $\mathbb{H}$ are a generalization of the positive bounded formulas of Henson and Iovino. Here we also introduce the larger class $\mathbb{L}$ because it offers some advantages. For instance, it is easy to see (cf. Example 4 for a hint) that $\mathbb{L}$ has at least the same expressive power as real valued logic (in fact, it is way more expressive). Somewhat surprisingly, we will see (cf. Propositions 17 and 18) that the formulas in $\mathbb{L}$ can be very well approximated by formulas in $\mathbb{H}$.

Example 4. Let $R = [0, 1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort $M^{|x|} \to R$. Then there is a formula in $\mathbb{L}$ that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall x\, \big[t(x) \dot{-} \tau \in \{0\}\big] \quad \wedge \quad \forall^R \varepsilon \Big[\varepsilon \in \{0\} \ \vee\ \exists x\, \big[\tau \dot{-} (t(x) + \varepsilon) \in \{0\}\big]\Big]$$

which, in a human readible form, becomes

$$\forall x\, \big[t(x) \leq \tau\big] \quad \wedge \quad \forall \varepsilon > 0\ \exists x\, \big[\tau \leq t(x) + \varepsilon\big].$$

3. The standard part

Let $R \preceq {}^*R$. Let $\alpha, \alpha' \in {}^*R$. For $\beta \in R$ we write $\alpha \approx \beta$ if ${}^*R \vDash \alpha \in D$ for every D compact neighborhood of β. We write $\alpha \approx \alpha'$ if $\alpha \approx \beta \approx \alpha'$ for some $\beta \in R$. From the following fact it follows that $\approx$ is an equivalence relation on ${}^*R$.

Fact 5. For every $\alpha \in {}^*R$ there is a unique $\beta \in R$ such that $\alpha \approx \beta$.

Proof. Negate the existence of β. For every $\gamma \in R$ pick some D_γ, compact neighborhood of γ, such that ${}^*R \vDash \alpha \notin D_\gamma$. By compactess there is some finite $\Gamma \subseteq R$ such that D_γ, with $\gamma \in \Gamma$, cover R. By elementarity these D_γ also cover ${}^*R$. A contradiction. The uniqueness of β follows from normality. □

We will denote it by $\mathrm{st}(\alpha)$ the unique $\beta \in R$ such that $\alpha \approx \beta$.

Fact 6. For every $\alpha \in {}^*R$ and every compact C

$$ {}^*R \quad \vDash \quad \alpha \in C \ \rightarrow \ \mathrm{st}(\alpha) \in C. $$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a set D, a compact neighborhood of $\mathrm{st}(\alpha)$, disjoint from C. Then ${}^*R \vDash \alpha \in D \subseteq \neg C$. □

Lemma 7. Let ${}^*\mathcal{N} = \langle N, {}^*R \rangle$ be an $\mathcal{L}$-structure, and assume that $R \preceq {}^*R$. Let $\mathcal{N}$ denote the substructure $\langle N, R \rangle$. Then for every $\varphi(x\,;y) \in \mathbb{L}$, every $a \in N^{|x|}$ and every $\alpha \in ({}^*R)^{|y|}$

$$ {}^*\mathcal{N} \vDash \varphi(a\,;\alpha) \quad \Rightarrow \quad \mathcal{N} \vDash \varphi\big(a\,;\mathrm{st}(\alpha)\big) $$

Proof. Suppose $\varphi(x\,;y)$ is $\mathbb{L}$-atomic. If $\varphi(x\,;y)$ is a formula of L the claim is trivial. Otherwise $\varphi(x\,;y)$ has the form $t(x\,;y) \in C$. Assume ${}^*\mathcal{N} \vDash t(a\,;\alpha) \in C$. We claim that for every compact sets $C_i \subseteq R$ such that $C \subseteq C_1 \times \cdots \times C_{|y|}$

$$ \mathcal{N} \quad \vDash \quad t_i(a\,;\alpha) \in C_i. $$

By the definition of product topology, C is the intersection of products as those above. Therefore $\mathcal{N} \vDash t(a\,;\alpha) \in C$ follows from the claim.

We prove the claim. By Remark 3, the terms $t_i(a\,;\alpha)$ have either the form $t_i(a)$ or $t_i(\alpha)$. We can ignore the first case because $\mathcal{N}$ is a substructure of ${}^*\mathcal{N}$ hence $t_i(a)$ has the same value in both structures. Then (1) is equivalent to

$$ {}^*\mathcal{N} \quad \vDash \quad \alpha \in \big(t_i^{\mathcal{N}}\big)^{-1}[C_i]. $$

But $\alpha \in \big(t_i^{\mathcal{N}}\big)^{-1}[C_i]$ is equivalent, modulo $\mathrm{Th}(R)$, to $\alpha \in D$ for some compact $D \subseteq R$. Therefore

$$ {}^*\mathcal{N} \quad \vDash \quad \alpha \in D. $$

Finally this is equivalent to $\mathrm{st}(\alpha) \in D$.

This proves, the claim and with it the lemma for $\mathbb{L}$-atomic formulas. Induction is immediate. □

4. $\mathbb{L}$-ELEMENTARITY

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two models. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\operatorname{dom} f)^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(fa).$$

An $\mathbb{L}$-elementary map is in particular L-elementary and therefore it is injective. An $\mathbb{L}$-elementary map that is total is called an $\mathbb{L}$-(elementary) embedding. When the map $\mathrm{id}_M : M \hookrightarrow N$ is an $\mathbb{L}$-embedding, that is, if for every $\varphi(x) \in \mathbb{L}$ and every $a \in M^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\;\Rightarrow\;\; \mathcal{N} \vDash \varphi(a),$$

we write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) submodel of $\mathcal{N}$.

The definitions of $\mathbb{H}$-elementary map/embedding/submodel are similar.

5. $\mathbb{L}$-COMPACTNESS

It is convinient to distinguish between consistency with respect to models and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent when $\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure $\mathcal{M}$. We say that T is consistent when $\mathcal{M}$ is required to be a model.

Theorem 8 ($\mathbb{L}$-compactness). Let $T \subseteq \mathbb{L}$ be finitely consistent. Then T is consistent.

Proof. Suppose T is finitely consistent (or finitely $\mathcal{L}$-consistent, for that matter). By the classical compactness theorem ${}^*\mathcal{M} \vDash T$ for some $\mathcal{L}$-structure ${}^*\mathcal{M} = \langle M, {}^*R\rangle$. Let $\mathcal{M}$ be the model $\langle M, R\rangle$. Then $\mathcal{M} \vDash T$ by Lemma 7. □

A model $\mathcal{N}$ is $\mathbb{L}$-saturated if it realizes all types with fewer than $|\mathcal{N}|$ parameters that are finitely consistent in $\mathcal{N}$. The existence of $\mathbb{L}$-saturated models is obtained from the classical case just as for theorem above.

Proposition 9. Every model has an $\mathbb{L}$-elementary extension to a saturated model (possibly of inaccessible cardinality).

6. HENSON-IOVINO APPROXIMATIONS

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained replacing each atomic formula of the form $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If such atomic formulas do not occur in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. Moreover the $\mathbb{L}$-atomic formulas in L are replaced with their negation and every connective is replaced with its dual. I.e., $\vee, \wedge, \exists, \forall, \exists^C, \forall^C$ are replaced with $\wedge, \vee, \forall, \exists, \forall^C, \exists^C$ respectively.

It is clear that $\tilde{\varphi} \to \neg\varphi$. We say that $\tilde{\varphi}$ is a strong negation of φ.

Lemma 10.

1. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$.
2. For every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in R \smallsetminus O)$ is as required by the lemma.

Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of R, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = t \in C'$ is as required.

The lemma follows easily by induction. □

7. The monster model

We denote by $\mathfrak{U} = \langle U, R \rangle$ some large $\mathbb{L}$-saturated structure which we call the monster model. For convenience we assume that the cardinality of $\mathfrak{U}$ is an inaccessible cardinal which we denote by κ. Below we say model for $\mathbb{L}$-elementary submodel of $\mathfrak{U}$.

Let $A \subseteq U$ be a small set. We define a topology on $U^{|x|} \times R^{|y|}$ which we call the $\mathbb{L}(A)$-topology. The closed sets of this topology are the sets defined by the types $p(x\,;y) \subseteq \mathbb{L}(A)$. This is a compact topology by the $\mathbb{L}$-compactness Theorem 8.

Fact 11. Let $p(x) \subseteq \mathbb{L}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{L}(U)$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. The first claim is clear. The second follows from the first by Lemma 10. □

Proposition 12. For every $\varphi(x) \in \mathbb{L}(U)$

$$\bigwedge_{\varphi' > \varphi} \varphi'(x\,;y) \quad \leftrightarrow \quad \varphi(x\,;y)$$

Proof. We prove $\to$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifier is immediate. We consider case of the existential quantifiers of sort M. The case of existential quantifiers of sort R is identical. Assume inductively

$$\text{ih.} \qquad \bigwedge_{\varphi' > \varphi} \varphi'(x, z\,;y) \quad \to \quad \varphi(x, z\,;y)$$

We need to prove

$$\bigwedge_{\varphi' > \varphi} \exists z\, \varphi'(x, z\,;y) \quad \to \quad \exists z\, \varphi(x, z\,;y)$$

From (ih) we have

$$\exists z \bigwedge_{\varphi'>\varphi} \varphi'(x,z\,;y) \;\rightarrow\; \exists z\, \varphi(x,z\,;y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi'>\varphi} \exists z\, \varphi'(x,z\,;y) \;\rightarrow\; \exists z \bigwedge_{\varphi'>\varphi} \varphi'(x,z\,;y)$$

Replace x, y with some fix but arbitrary parameters, say a, α and assume the antecedent, that is, the truth of the theory $\{\exists z\, \varphi'(a,z\,;\alpha) : \varphi' > \varphi\}$. We need to prove the consistency of the type $\{\varphi'(a,z\,;\alpha) : \varphi' > \varphi\}$. By the saturation of $\mathcal{U}$, finite concistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. □

When $A \subseteq U$, we write $S_{\mathbb{H}}(A)$ for the set of types

$$\mathbb{H}\text{-tp}(a/A) \;=\; \big\{\varphi(x) \;:\; \varphi(x) \in \mathbb{H}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples of elements of U. We write $S_{\mathbb{H},x}(A)$ when the tuple of variables x is fixed. The following corollary will be strengthen by Corollary 16 below.

Corollary 13. The types $p(x) \in S_{\mathbb{H}}(A)$ are complete. That is, either $\varphi(x) \in p$ or $p(x) \rightarrow \neg\varphi(x)$ for every $\varphi(x) \in \mathbb{H}(A)$.

Proof. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$ and suppose $\varphi(x) \notin p$. Then $\neg\varphi(a)$. From Lemma 10 and Proposition 12 we obtain

$$\neg\varphi(x) \;\rightarrow\; \bigvee_{\tilde{\varphi}\perp\varphi} \tilde{\varphi}(x).$$

Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \rightarrow \neg\varphi(x)$ follows. □

The following will be useful below

Remark 14. For $p(x) \subseteq \mathbb{L}(U)$, we write $p'(x)$ for the type

$$p'(x) \;=\; \big\{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}.$$

Note that $p'(x)$ is equivalent to $p(x)$ by Proposition 12.

8. Homogeneity

A model $\mathcal{M}$ is $\mathbb{H}$-homogeneous if every $\mathbb{H}$-elementary map $f : \mathcal{M} \rightarrow \mathcal{M}$ of cardinality $< |\mathcal{M}|$ extends to an automorphism. Clearly, every automorphism is $\mathcal{L}$-elementary.

Proposition 15. $\mathcal{U}$ is $\mathbb{H}$-homogeneous.

Proof. By Corollary 13, the inverse of an $\mathbb{H}$-elementary map $f : \mathcal{U} \rightarrow \mathcal{U}$ is $\mathbb{H}$-elementary. Then the usual proof by back-and-forth applies. □

As promised, we now strengthen Corollary 13.

Corollary 16. Let $p(x) \in S_{\mathbb{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathcal{L}_x(A)$. Clearly, the same holds for $p'(x)$.

Proof. If $b \vDash p(x) = \mathbb{H}\text{-tp}(a/A)$ then there is $\mathbb{H}$-elementary map $f$vsuch that $fa = b$. As f exends to an automorphism, $a \equiv^{\mathcal{L}} b$ and the corollary follows. $\square$

The next two propositions show that formulas in $\mathbb{L}(A)$ are approximated by formulas in $\mathbb{H}(A)$.

Proposition 17. Let $\varphi(x) \in \mathbb{L}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathbb{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 16 and Remark 14

$$\neg\varphi(x) \quad \to \quad \bigvee_{p'(x)\to\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over $S_{\mathbb{H},x}(A)$. By Fact 11 and Lemma 10

$$\neg\varphi(x) \quad \to \quad \bigvee_{\neg\tilde{\psi}(x)\to\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathbb{H}(A)$. Equivalently,

$$\varphi(x) \quad \leftarrow \quad \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 11, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathbb{H}(A)$ such that

$$\varphi'(x) \quad \leftarrow \quad \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \quad \leftarrow \quad \varphi(x)$$

which yields the interpolant required by the proposition. $\square$

Proposition 18. Let $\varphi(x) \in \mathbb{L}(A)$ be such that $\neg\varphi(x)$ is consistent. Then $\psi'(x) \to \neg\varphi(x)$ for some consistent $\psi(x) \in \mathbb{H}(A)$ and some $\psi' > \psi$.

Proof. Let $a \in U^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathbb{H}\text{-tp}(a/A)$. By Corollary **??**, $p'(x) \to \neg\varphi(x)$. By compactness $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in p(x)$. $\square$

Proposition 19 (Tarski-Vaught Test). Let M be a subset of U. Then the following are equivalent

1. M is the domain of a model;
2. for every formula $\varphi(x) \in \mathbb{H}(M)$
$$\exists x\, \varphi(x) \quad \Rightarrow \quad \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathbb{L}(M)$
$$\exists x\, \neg\varphi(x) \quad \Rightarrow \quad \text{there is an } a \in M \text{ such that } \neg\varphi(a).$$

Proof. (1⇒2) Assume $\exists x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 10 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\forall x\, \tilde{\varphi}(x)$ hence, by (1), $\mathcal{M} \vDash \neg\forall x\, \varphi(x)$. Then $\mathcal{M} \vDash \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $\mathcal{M} \vDash \varphi'(a)$ and $\varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathbb{L}(M)$ be such that $\exists x\, \neg\varphi(x)$. Then, by Corollary 18, there are a consistent $\psi(x) \in \mathbb{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). Then, by the classical Tarski-Vaught test $M \preceq U$. It is clear that $\mathcal{M}$ is an $\mathcal{L}$-subtructure of $\mathcal{U}$. Therefore for every $a \in M^{|x|}$ and for every $\mathbb{L}$-atomic formula, $\varphi(a)$ if and only if $\mathcal{M} \vDash \varphi(a)$. Now, assume inductively

$$\mathcal{M} \vDash \varphi(a, b) \;\Rightarrow\; \varphi(a, b)$$

Using (3) and the induction hypothesis we prove by contrapposition that

$$\mathcal{M} \vDash \forall y\, \varphi(a, y) \;\Rightarrow\; \forall y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned}
\neg\forall y\, \varphi(a, y) \;&\Rightarrow\; \exists y \neg\varphi(a, y) \\
&\Rightarrow\; \neg\varphi(a, b) \qquad \text{for some } b \in M^{|y|} \\
&\Rightarrow\; \mathcal{M} \vDash \neg\varphi(a, b) \qquad \text{for some } b \in M^{|y|} \\
&\Rightarrow\; \mathcal{M} \nvDash \forall y\, \varphi(a, y)
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists$, $\exists^R$, and $\forall^R$ is straightforward. □

9. Completeness

Needs full rewriting

For $a, b \in U$ we write $a \sim b$ if $\forall y\, [t(a, y) = t(b, y)]$ for every parameter free term $t(x, y)$ of $\mathcal{L}$ sort $M^{1+|y|} \to R$.

Fact 20 (obsoleto). If $\bar{a} = \langle a_i : i < \lambda\rangle$ and $\bar{b} = \langle b_i : i < \lambda\rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \to R$.

Proof. By induction on λ. Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ. For limit ordinals induction is immediate. □

Example 21. Let L be the language of $\mathbb{R}$-algebras expanded with the function symbols $\wedge$, $\vee$. Let $\langle \Omega, \mathcal{B}, \Pr\rangle$ be a probability space. Let M be the set the simple, real valued, random variables with the natural interpretation of the symbols in L. The symbols $\wedge$ and $\vee$ are interpreted as the pointwise minimum, respectively maximum, of two functions. Let $R = [0, 1]$. Assume $\mathcal{L}$ contains a symbol for the functions E_1 that gives the expected value of $(a \wedge 1) \vee -1$.

Let a, b are two random variables such that $-1 \leq a, b \leq 1$. The relation $a \sim b$ holds exactly when $a(\omega) = b(\omega)$ for almost every $\omega \in \Omega$.

We say that $a \in U$ is definable in the limit over M if $a \equiv^{\mathbb{L}}_M x \to a \sim x$

Fact 22. Assume that $R = [0, 1]$ and that $\mathcal{L}$ contains a function of sort $M^2 \to R$ which is interpreted in a pseudometric. Assume that all terms of sort $M^n \to R$ are continuous with rispect to this pseudometric. Then $a \sim b$ if and only if $d(a, b) = 0$. Moreover, we claim that the following are equivalent

1. $a \in U$ is definable in the limit over M, a model;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of M that converges to a.

Proof. (2 $\Rightarrow$ 1) Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv^{\mathbb{L}}_M a$. By the uniqueness of the limit $d(a, b) = 0$.

(1 $\Rightarrow$ 2) Assume that $a \equiv^{\mathbb{L}}_M x \to a \sim x$. Then $a \equiv^{\mathbb{L}}_M x \to d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \mathrm{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \to d(a, x) < 1/n$. By $\mathbb{L}$-elementarity there is an $a_n \in M$ such that $\varphi(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a. $\square$