CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted struture. Ideally, we would like that elementary extensions of normed spaces mantain the usal notion of real numbers[1]. Unfortunately, this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $\mathcal{L} \supseteq L$ be a two-sorted language. We consider the class of $\mathcal{L}$-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

i. $R^n \to R$;

ii. $M^n \to M$;

iii. $M^n \to R$.

The interpretation of symbols of sort $R^n \to R$ is required to be continuous and bounded, the interpretation of symbols of sort $M^n \to R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq \mathcal{L}$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated in the same class.

For convenience we assume that the functions of sort $M^n \to M$ and the relations of sort M^n are all in L. It is also convenient to assume that $\mathcal{L}$ contains names for all continuous bounded functions $R^n \to R$. There is a relation symbol for every compact subset of R^n and there are no other relations of sort R^n.

2 **Definition.** Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

[1] Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ (or, in general R) should remain itself throughout.

i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact[2] and $t(x, y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall$, $\exists$ of sort M; and the quantifiers $\forall^C$, $\exists^C$, by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

We write $\mathbb{H}$ for the set of formulas in $\mathbb{L}$ without quantifiers of sort R.

For later reference we remark the following which is an immediate consequence of the restrictions we imposed on the sorts of the function symbols.

3 Remark. Each component of the tuple $t(x\,; y)$ in the definition above is either of sort $M^{|x|} \to R$ or of sort $R^{|y|} \to R$. □

The class $\mathbb{H}$ is the immediate generalization of class of positive bounded formulas of Henson and Iovino. We introduced the larger class $\mathbb{L}$ because it offers some advantages. For instance, it is easy to see that $\mathbb{L}$ has at least the same expressive power of real valued logic (in fact, way more). However, we will see that the two classes of formulas are closer than one might expect.

2. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' is obtained replacing each atomic formula $t \in C$ occurring in φ with $t \in C'$ where C' is some compact neighborhood of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. We also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in C'$ where C' is some compact set disjoint from C. Moreover the atomic formulas in L are replaced with their negation and every connective is replaced with its dual. I.e., $\vee, \wedge, \exists, \forall, \exists^C, \forall^C$ are replaced with $\wedge, \vee, \forall, \exists, \forall^C, \exists^C$ respectively.

It is clear that $\tilde{\varphi} \to \neg\varphi$ and that $\tilde{\varphi} \leftrightarrow \neg\varphi$ when $\varphi \in L$. We say that $\tilde{\varphi}$ is a strong negation of φ

4 Lemma. For every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$. Vice versa, for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi'' > \varphi$ such that $\varphi \to \varphi'' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L$ the claims are obvious. Suppose φ is of the form $t \in C$. Then φ' is $t \in C'$ for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in C' \smallsetminus O)$ is as required.

Suppose instead that $\tilde{\varphi}$ is of the form $t \in C'$ for some compact C' disjoint from C. By the local compactness of R, there exist C'' and O, disjoint neighborhoods of C and C' respectively, where is C'' is compact. Then $\varphi'' = t \in C''$ is as required.

[2] We confuse the relation symbols in $\mathcal{L}$ of sort R^m with their interpretation and write $t \in C$ for $C(t)$

The lemma follows easily by induction. □

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The following is technical lemma that is required in the proof of Łǒś Theorem in the next section.

5 Lemma. Assume the following data

- $\varphi(x; y) \in \mathbb{L}$;
- $\langle a_i : i \in I\rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I\rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Then the following implications hold

i. $\big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha)\big\} \in F \;\;\Rightarrow\;\; \big\{i \in I : \mathcal{M} \vDash \varphi'(a_i\,; \alpha_i)\big\} \in F$ for every $\varphi' > \varphi$;

ii. $\big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha_i)\big\} \in F \;\;\Rightarrow\;\; \big\{i \in I : \mathcal{M} \vDash \varphi(a_i\,; \alpha)\big\} \in F$.

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$. First, note that by Remark 3 we trivially have that

$$F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i).$$

(i) Assume $\{i : t(a_i; \alpha) \in C\} \in F$. Then $F\text{-}\lim_i t(a_i; \alpha) \in C$. By what noted above, $F\text{-}\lim_i t(a_i; \alpha_i) \in C$. Then $\{i : t(a_i\,; \alpha_i) \in C'\} \in F$ where C' is any compact neghborhod of C.

(ii) Assume $\{i : t(a_i; \alpha) \notin C\} \in F$. Then $F\text{-}\lim_i t(a_i; \alpha) \notin C$. Hence $F\text{-}\lim_i t(a_i; \alpha_i) \notin C$ and $\{i : t(a_i; \alpha) \notin C\} \in F$ follows.

This proves the basis case of the induction.

Induction clear for the connectives $\vee$, $\wedge$. To deal with the universal quantifier of sort M we assume inductively that

ih. $\big\{i \in I : \mathcal{M} \vDash \varphi(a_i, b_i\,; \alpha)\big\} \in F \;\;\Rightarrow\;\; \big\{i \in I : \mathcal{M} \vDash \varphi'(a_i, b_i\,; \alpha_i)\big\} \in F$;

iih. $\big\{i \in I : \mathcal{M} \vDash \varphi(a_i, b_i\,; \alpha_i)\big\} \in F \;\;\Rightarrow\;\; \big\{i \in I : \mathcal{M} \vDash \varphi(a_i, b_i\,; \alpha)\big\} \in F$;

We prove

i∀. $\big\{i \in I : \mathcal{M} \vDash \forall y\, \varphi(a_i, y\,; \alpha)\big\} \in F \;\;\Rightarrow\;\; \big\{i \in I : \mathcal{M} \vDash \forall y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F$.

ii∀. $\big\{i \in I : \mathcal{M} \vDash \forall y\, \varphi(a_i, y\,; \alpha_i)\big\} \in F \;\;\Rightarrow\;\; \big\{i \in I : \mathcal{M} \vDash \forall y\, \varphi(a_i, y\,; \alpha)\big\} \in F$.

(i∀) Negate the consequent and pick a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \neg\varphi'(a_i, b_i\,; \alpha_i)\big\} \in F$. If for a contradiction the antecedent of (i∀) holds, from (ih) we would obtain $\big\{i : \varphi(a_i, b_i\,; \alpha)_i\big\} \in F$. A fotriori $\big\{i : \varphi'(a_i, b_i\,; \alpha_i)\big\} \in F$, which is a contradiction that proves (i∀).

(ii∀) Negate the consequent and pick a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \neg\varphi(a_i, b_i\,; \alpha)\big\} \in F$. If for a contradiction the antecedent of (ii∀) holds, from (iih) we would obtain $\big\{i : \varphi(a_i, b_i\,; \alpha)\big\} \in F$, a contradiction that proves (ii∀).

To deal with the existential quantifier of sort M we prove

i∃. $\big\{i \in I : \mathcal{M} \vDash \exists y\, \varphi(a_i, y\,; \alpha)\big\} \in F \quad\Rightarrow\quad \big\{i \in I : \mathcal{M} \vDash \exists y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F.$

ii∃. $\big\{i \in I : \mathcal{M} \vDash \exists y\, \varphi(a_i, y\,; \alpha_i)\big\} \in F \quad\Rightarrow\quad \big\{i \in I : \mathcal{M} \vDash \exists y\, \varphi(a_i, y\,; \alpha)\big\} \in F.$

(i∃) Assume the antecedent. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Then there is a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \varphi''(a_i, b_i\,; \alpha)\big\} \in F$. By (ih) we obtain $\big\{i : \exists y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F$.

(ii∃) Assume the antecedent. Then there is a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \varphi(a_i, b_i\,; \alpha_i)\big\} \in F$. By (iih) we obtain $\big\{i : \exists y\, \varphi(a_i, y\,; \alpha)\big\} \in F$.

Induction for the quantifiers $\forall^C$ and $\exists^C$ is virtually identical. □

3. Ultraproducts

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I\rangle$. We require that the models occurring in the ultraproduct are uniformly bounded as defined below.

6 Definition. We say that a family of models $\{\mathcal{M}_i : i \in I\}$ is uniformly bounded if there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two semplifications: (1) we only consider ultratpowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F. The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

7 Definition. We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \to M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I\rangle$.
3. The interpretation of functions of sort $R^n \to R$ remains unchanged.
4. If f is a function of sort $M^n \to R$ then
$$f^{\mathcal{N}}(\hat{a}) \;=\; F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$
5. If r is a relation symbol of sort M^n then
$$\mathcal{N} \vDash (\hat{a}) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash r(\hat{a}i)\big\} \in F.$$
6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultraproducts, it would not hold without the requirement of uniformity in Definition 6.

8 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

9 Fact. If $t(x)$ is a term of type $M^{|x|} \to M$ then
$$t^{\mathcal{N}}(\hat{a}) \;=\; \langle t^{\mathcal{M}}(\hat{a}i) : i \in I\rangle.$$
□

By Remark 3 we also have that

10 **Fact.** For every tuple $t(x;y)$ of sort $M^{|x|} \times R^{|y|} \to R$

$$t^{\mathcal{N}}\big(\hat{a};\alpha\big) \quad = \quad F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i;\alpha\big). \qquad \square$$

Finally, we prove an asymmetric version of Łŏś Theorem.

11 **Proposition (Łŏś Theorem).** Let $\mathcal{N}$ be as above and let $\varphi(x) \in \mathbb{L}$. Then for every $\hat{a} \in N^{|x|}$ and every $\varphi' > \varphi$

i. $\quad \big\{i : \mathcal{M} \vDash \varphi(\hat{a}i)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi(\hat{a});$

ii. $\quad \mathcal{N} \vDash \varphi(\hat{a}) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i)\big\} \in F.$

Proof. The claim is proved by induction on the syntax. If $\varphi(x) \in L$ then the theorem reduces to the classical Łŏś Theorem. Now, suppose instead that $\varphi(x)$ is as in (i) of Definition 2, say it is the formula $t(x) \in C$.

(i) Assume $\big\{i : \mathcal{M} \vDash t(\hat{a}i) \in C\big\} \in F$. Then $F\text{-}\lim_i t(\hat{a}i) \in C$ by regularity.

(ii) Assume $\mathcal{N} \vDash t(\hat{a}) \in C$. By the definition of F-limit, $\big\{i : \mathcal{M} \vDash t(\hat{a}i) \in C'\big\} \in F$ where C' is any neighborhood of C.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

ih. $\quad \big\{i : \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi(\hat{a}, \hat{b});$

iih. $\quad \mathcal{N} \vDash \varphi(\hat{a}\,; \hat{b}) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,; \hat{b}i)\big\} \in F.$

First we prove

i$\forall$. $\quad \big\{i : \mathcal{M} \vDash \forall y\, \varphi(\hat{a}i, y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y);$

ii$\forall$. $\quad \mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y)\big\} \in F.$

(i$\forall$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \nvDash \varphi(\hat{a}i, \hat{b}i)\big\} \in F$. A fortiori $\big\{i : \mathcal{M} \nvDash \forall y\, \varphi(\hat{a}i, y)\big\} \in F$ as required.

(ii$\forall$) Assume $\big\{i : \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y)\big\} \in F$. Pick $\hat{b}$ such that $\big\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. If for a contradiction $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y)$. Then in particular $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$. By induction hypothesis $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$, a contradiction.

Now we prove

i$\exists$. $\quad \big\{i : \mathcal{M} \vDash \exists y\, \varphi(\hat{a}i, y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y);$

ii$\exists$. $\quad \mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\big\} \in F.$

(i$\exists$) Assume $\big\{i : \mathcal{M} \vDash \exists y\, \varphi(\hat{a}i, y)\big\} \in F$. Choose some $\hat{b}$ such that $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i, \hat{b}i)\big\} \in F$. By induction hypothesis $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$. Therefore $\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y)$ as required.

(ii$\exists$) Assume $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. A fortiori $\big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\big\} \in F$.

To deal with the quantifiers $\forall^C$ and $\exists^C$ we assume inductively ih. $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,;\alpha)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \varphi(\hat{a}\,;\alpha)$;

iih. $\mathcal{N} \vDash \varphi(\hat{a}\,;\alpha) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\alpha)\big\} \in F$.

First we prove

i$\forall^C$. $\big\{i : \mathcal{M} \vDash \forall^C y\, \varphi(\hat{a}i\,;y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}\,;y)$;

ii$\forall^C$. $\mathcal{N} \vDash \forall y^C \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i, y)\big\} \in F$.

(i$\forall^C$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}\,;\beta)$ for some $\beta \in C$. By induction hypothesis, $\big\{i : \mathcal{M} \nvDash \varphi(\hat{a}i\,;\beta)\big\} \in F$. Therefore $\big\{i : \mathcal{M} \nvDash \forall^C y\, \varphi(\hat{a}i\,;y)\big\} \in F$ as required.

(ii$\forall^C$) Assume $\big\{i : \mathcal{M} \nvDash \forall^C y\, \varphi'(\hat{a}i\,;y)\big\} \in F$. Pick $\beta_i \in C$ such that $\big\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i\,;\beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. Then from Lemma 5 we obtain $\big\{i : \mathcal{M} \nvDash \varphi(\hat{a}i\,;\beta)\big\} \in F$. Finally, by (iih), we conclude $\mathcal{N} \nvDash \forall y^C \varphi(\hat{a}, y)$.

Finally, we prove

i$\exists^C$. $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi(\hat{a}i\,;y)\big\} \in F \;\Rightarrow\; \mathcal{N} \vDash \exists^C y\, \varphi(\hat{a}\,;y)$;

ii$\exists^C$. $\mathcal{N} \vDash \exists y^C \varphi(\hat{a}, y) \;\Rightarrow\; \big\{i : \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i, y)\big\} \in F$.

(i$\exists^C$) Assume $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi(\hat{a}i\,;y)\big\} \in F$. Choose $\beta_i \in C$ such that $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,;\beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. By Lemma 5, $\big\{i : \mathcal{M} \vDash \varphi(\hat{a}i\,;\beta)\big\} \in F$.

(ii$\exists^C$) Assume $\mathcal{N} \vDash \varphi(\hat{a}\,;\beta)$ for some $\beta \in C$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\beta)\big\} \in F$. Therefore $\big\{i : \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i\,;y)\big\} \in F$. □

4. Elementarity

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two structures. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if for every $\varphi(x) \in \mathbb{L}$ and every $a \in (\operatorname{dom} f)^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\Rightarrow\; \mathcal{N} \vDash \varphi(f a)$$

Note that an $\mathbb{L}$-elementary map is in particular L-elementary and therefore it is injective. An $\mathbb{L}$-elementary map that is total is called an $\mathbb{L}$-(elementary) embedding. When the map $\mathrm{id}_M : M \hookrightarrow N$ is an $\mathbb{L}$-embedding, that is, if for every $\varphi(x) \in \mathbb{L}$ and every $a \in M^{|x|}$

$$\mathcal{M} \vDash \varphi(a) \;\Rightarrow\; \mathcal{N} \vDash \varphi(a),$$

we write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ and say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$.

The same notation is used with $\mathbb{H}$ for $\mathbb{L}$.

The following fact follows from Łŏś Theorem just as in the classical case.

12 **Fact.** If $\mathcal{N}$ is an ultrapower of $\mathcal{M}$ then there is an $\mathbb{L}$-embedding $\mathcal{M} \hookrightarrow \mathcal{N}$. □

5. Compactness

We say that a teory is consistent if it has a model satisfying Definition 1. Also the notion of logical consequence is restricted to models in this class.

As in the classical case, a theory $T \subseteq \mathbb{L}$ is said to be finitely consistent if every φ, conjunction of sentences in T, is consistent. We further say that T is uniformly finitely consistent if the finite consistency is witnessed by uniformly bounded models (see Definition 6).

The following proposition follows from Łŏś Theorem by the usual argument.

13 Proposition (Compactness Theorem). Let $T \subseteq \mathbb{L}$ be a finitely uniformly consistent theory. Then T is consistent. □

14 Proposition. Let $p(x) \subseteq \mathbb{L}(M)$ be a finitely consistent in $\mathcal{M}$. Then $\mathcal{N} \vDash \exists x\, p(x)$ for some $\mathcal{N}$ such that $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$. □

A model $\mathcal{N}$ is $\mathbb{L}$-saturated if it realizes all types with fewer than $|\mathcal{N}|$ parameters that are finitely consistent in $\mathcal{N}$. The existence of $\mathbb{L}$-saturated models is proved as in the classical case.

15 Proposition. Every model has an $\mathbb{L}$-elementary extension to a saturated model (possibly of inaccessible cardinality). □

6. The monster model

We denote by $\mathcal{U} = \langle U, R\rangle$ some large $\mathbb{L}$-saturated structure which we call the monster model. The cardinality of $\mathcal{U}$ is an inaccessible cardinal that we denote by κ. Below we say model for $\mathbb{L}$-elementary substructure of $\mathcal{U}$.

16 Fact. Let $p(x) \subseteq \mathbb{L}(U)$ be a type of small cardinality. Then for every $\varphi(x) \in \mathbb{L}(U)$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. The first claim is clear. The second follows from the first by Lemma 4. □

17 Proposition. For every $\varphi(x) \in \mathbb{H}(U)$

$$\bigwedge_{\varphi' > \varphi} \varphi'(x) \;\leftrightarrow\; \varphi(x)$$

Proof. We prove $\to$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifier is immediate. Consider the existential quantifiers of sort U. Assume inductively

$$\bigwedge_{\varphi' > \varphi} \varphi'(x, y) \;\to\; \varphi(x, y)$$

then

$$\exists y \bigwedge_{\varphi' > \varphi} \varphi'(x, y) \;\to\; \exists y\, \varphi(x, y)$$

Therefore it suffices to prove

$$\bigwedge_{\varphi' > \varphi} \exists y\, \varphi'(x, y) \;\to\; \exists y \bigwedge_{\varphi' > \varphi} \varphi'(x, y)$$

Replace x with a parameter, say a and assume the antecedent. Note that $\{\exists y\, \varphi'(a, y) : \varphi' > \varphi\}$ implies the finite concistency of the type $\{\varphi'(a, y) : \varphi' > \varphi\}$. This because if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set. Therefore $\exists y\, \{\varphi'(a, y) : \varphi' > \varphi\}$ follows by saturation. □

The following are immediate consequences of the proposition above and Lemma 4.

18 Corollary. For every $a \in U^{|x|}$ and $\varphi(x) \in \mathbb{H}(U)$, if $\neg\varphi(a)$ then $\tilde{\varphi}(a)$ for some $\tilde{\varphi} \perp \varphi$. $\square$

When $A \subseteq U$, we write $S_{\mathbb{H},x}(A)$ for the set of types

$$\mathbb{H}\text{-tp}(a/A) \quad = \quad \big\{\varphi(x) \ : \ \varphi(x) \in \mathbb{H}(A) \text{ such that } \varphi(a)\big\}$$

as a ranges over the tuples in $U^{|x|}$. The following corollary will be strengthen by Corollary 23 below.

19 Corollary. The types $p(x) \in S_{\mathbb{H},x}(A)$ are complete. That is, for every $\varphi(x) \in \mathbb{H}(A)$, either $p(x) \vdash \varphi(x)$ or $p(x) \vdash \neg\varphi(x)$. $\square$

Proof. Follows from the proposition above and Lemma 4. $\square$

For $p(x) \subseteq \mathbb{L}(U)$, we write $p'(x)$ for the type

$$p'(x) \quad = \quad \big\{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}.$$

Note that, by the proposition above, if $p(x) \in S_{\mathbb{H}}(A)$ then $p'(x) \leftrightarrow p(x)$. Therefore $p'(x)$ is also complete.

20 Corollary. The inverse of an $\mathbb{H}$-elementary map $f : \mathcal{U} \to \mathcal{U}$ is $\mathbb{H}$-elementary. $\square$

7. Homogeneity

A model $\mathcal{M}$ is $\mathbb{H}$-homogeneous if every $\mathbb{H}$-elementary map $f : \mathcal{M} \to \mathcal{M}$ of cardinality $< |\mathcal{M}|$ extends to an automorphism.

21 Proposition. $\mathcal{U}$ is $\mathbb{H}$-homogeneous.

Proof. By Corollary 20 the usual proof by back-and-forth applies. $\square$

22 Corollary. For every $a, b \in U^{|x|}$, if $a \equiv^{\mathbb{H}} b$ then $a \equiv^{\mathbb{L}} b$.

23 Corollary. Let $p(x) \in S_{\mathbb{H}}(A)$. Then $p(x)$ is complete for formulas in $\mathbb{L}_x(A)$. Clearly, the same holds for $p'(x)$. $\square$

24 Proposition. Let $\varphi(x) \in \mathbb{L}(A)$. For every given $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathbb{H}(A)$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$.

Proof. By Corollary 23

$$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee_{p'(x)\to\neg\varphi(x)} \quad p'(x)$$

where $p(x)$ ranges over $S_{\mathbb{H},x}(A)$. By Fact 16

$$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee_{\psi(x)\to\neg\varphi(x)} \quad \psi(x)$$

where $\psi'(x)$ is such that $\psi' > \psi$ for some $\psi(x) \in \mathbb{H}(A)$ such that $\psi(x) \to \neg\varphi(x)$. By compactness

$$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \psi_i'(x)$$

By Lemma 4 we can replace $\psi_i'(x)$ by $\neg\tilde{\psi}_i(x)$ for some $\tilde{\psi}_i \perp \psi_i$. The proposition follows. □

25 Proposition. Let $\varphi(x) \in \mathbb{L}(A)$ be such that $\neg\varphi(x)$ is consistent. Then there is a consistent $\psi(x) \in \mathbb{H}(A)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. □

26 Proposition (Tarski-Vaught Test). Let M be a subset of U. Then the following are equivalent

1. M is the domain of a model;
2. for every formula $\varphi(x) \in \mathbb{H}(M)$
$$\mathcal{U} \vDash \exists x\, \varphi(x) \quad \Rightarrow \quad \text{for every } \varphi' > \varphi \text{ there is an } a \in M \text{ such that } \mathcal{U} \vDash \varphi'(a);$$
3. for every formula $\varphi(x) \in \mathbb{L}(M)$
$$\mathcal{U} \vDash \exists x \neg\varphi(x) \quad \Rightarrow \quad \text{there is an } a \in M \text{ such that } \mathcal{U} \vDash \neg\varphi(a).$$

Proof. (1⇒2) Assume $\mathcal{U} \vDash \exists x\, \varphi(x)$ and let $\varphi' > \varphi$ be given. By Lemma 4 there is some $\tilde{\varphi} \perp \varphi$ such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\mathcal{U} \nvDash \forall x\, \tilde{\varphi}(x)$ hence, by (1), $\mathcal{M} \nvDash \forall x\, \varphi(x)$. Then $\mathcal{M} \vDash \neg\tilde{\varphi}(a)$ for some $a \in M$. Hence $\mathcal{M} \vDash \varphi'(a)$ and $\mathcal{U} \vDash \varphi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\varphi(x) \in \mathbb{L}(M)$ be such that $\mathcal{U} \vDash \exists x \neg\varphi(x)$. Then, by Corollary 20, there are a consistent $\psi(x) \in \mathbb{H}(M)$ and some $\psi' > \psi$ such that $\psi'(x) \to \neg\varphi(x)$. Then (3) follows.

(3⇒1) Assume (3). Then, by the classical Tarski-Vaught test $M \preceq U$. We also have that $\mathcal{M}$ is an $\mathcal{L}$-substructure of $\mathcal{U}$. Therefore $\mathcal{M} \vDash \varphi(a) \Leftrightarrow \mathcal{U} \vDash \varphi(a)$ for every $a \in M^{|x|}$ and for every formula $\varphi(x)$ as in (i) and (ii) of Definition 2.

Now, assume inductively
$$\mathcal{M} \vDash \varphi(a, b) \quad \Rightarrow \quad \mathcal{U} \vDash \varphi(a, b)$$
Using (3) and the induction hypothesis we prove by contrapposition that
$$\mathcal{M} \vDash \forall y\, \varphi(a, y) \quad \Rightarrow \quad \mathcal{U} \vDash \forall y\, \varphi(a, y).$$
Indeed,
$$\begin{aligned}
\mathcal{U} \nvDash \forall y\, \varphi(a, y) \quad &\Rightarrow \quad \mathcal{U} \vDash \exists y \neg\varphi(a, y)\\
&\Rightarrow \quad \mathcal{U} \vDash \neg\varphi(a, b) \quad \text{for some } b \in M^{|y|}\\
&\Rightarrow \quad \mathcal{M} \vDash \neg\varphi(a, b) \quad \text{for some } b \in M^{|y|}\\
&\Rightarrow \quad \mathcal{M} \nvDash \forall y\, \varphi(a, y)
\end{aligned}$$
Induction for the connectives $\vee$, $\wedge$, $\exists$, $\exists^C$, and $\forall^C$ is straightforward. □