

CONTINUOUS MODEL THEORY WITHOUT THE HASSLE

1. FORMULAS AND THEIR INTERPRETATION

The following definition does not differ from classical first-order logic but for the third clause (which you may ignore til the end of the section).

1 Definition. A **signature** or **language** L consists of the following data

1. two sets of symbols: L_{rel} , for relations, and L_{fun} , for functions;
2. an **arity** function $\text{Ar} : L_{\text{rel}} \cup L_{\text{fun}} \rightarrow \omega$;
3. the functions $n_- : L_{\text{at}} \rightarrow \mathbb{N}$ and $t_- : L \rightarrow L_{\text{trm}}$.

The functions in 3 above are called **bounding functions**. The meaning of L , L_{at} and L_{trm} in 4 is clarified below.

2 Definition. A **structure** or **model** M of signature L consists of the following data

1. for every $f \in L_{\text{fun}}$, a function $f^M : M^{\text{Ar}(f)} \rightarrow M$;
2. for every $r \in L_{\text{rel}}$, a function $r^M : M^{\text{Ar}(r)} \rightarrow \mathbb{R}$;
3. a subset $U \subseteq M$.

The set U in 3 above is called the **unit ball** of M . Again, this can be ignored for the time being. The axioms for the unit ball are stated in Definition 5 where we will also clarify the role of the bounding functions.

The syntax and semantic of **terms** is defined just as in classical first-order logic. We write L_{trm} for the set of terms in the language L . Atomic formulas are defined as in classical first-order logic. They have the form $r t$, where r is a relation symbol and t is a tuple of terms. The set of atomic formulas is denoted by L_{at} . The set of all formulas is denoted as usual by L , the same symbol as the language. This is defined inductively from the atomic formulas using the following connectives.

The **propositional** (or **Riesz**) connectives are those in $\{1, +, \wedge\} \cup \mathbb{R}$, where $+$ and \wedge are binary connectives, the elements of \mathbb{R} are unary connectives, 1 is a logical constant (i.e. a zero-ary connective). There is also the infimum **quantifier** \bigwedge_x .

3 Definition. The inductive definition of formula is as follows

- i. atomic formulas are formulas
- ii. 1 is a formula;
- iii. $\varphi + \psi$ is a formula;
- iv. $\varphi \wedge \psi$ is a formula;
- v. $\alpha\varphi$ is a formula for every $\alpha \in \mathbb{R}$;
- vi. $\bigwedge_x \varphi$ is a formula.

If $\alpha \in \mathbb{R}$, we may write α for the formula $\alpha 1$. We write $-\varphi$ for $(-1)\varphi$ and $\varphi - \psi$ for $\varphi + (-\psi)$ and we write $\varphi \vee \psi$ for $-(-\varphi \wedge -\psi)$. Also, φ^+ and φ^- stand for $0 \vee \varphi$ respectively $0 \vee (-\varphi)$, and we write $|\varphi|$ for $\varphi \vee (-\varphi)$. Finally, $\bigvee_x \varphi$ stands for $-\bigwedge_x -\varphi$.

If $A \subseteq M$, we write $L(A)$ for the language L expanded with a 0-ary function symbol for every element of a . As in classical first-order logic, M is canonically expanded to a structure of signature $L(A)$ by setting $a^M = a$ for every $a \in A$.

Let M be a model and let t be a tuple of closed terms possibly with parameters from M . We write t^M for the element obtained interpreting function symbols and parameters in M just as in classical first-order logic.

4 Definition. If $\varphi \in L(M)$ is a sentence (i.e. a closed formula) we define φ^M inductively as follows

- i. $(rt)^M = r^M(t^M)$.
- ii. $1^M = 1$;
- iii. $(\psi + \xi)^M = \psi^M + \xi^M$;
- iv. $(\alpha\varphi)^M = \alpha\varphi^M$;
- v. $(\psi \wedge \xi)^M = \min\{\psi^M, \xi^M\}$;
- vi. $(\bigwedge_x \psi)^M = \inf\{\max\{0, \psi[x/b]^M\} : b \in M\}$.

Note that the cutoff in condition vi of the definition above is necessary because formulas are, in principle, unbounded.

For $\varphi(x) \in L(M)$ we write φ^M for the function $\varphi^M : M^{|x|} \rightarrow \mathbb{R}$ that maps $a \mapsto \varphi(a)^M$.

Finally, we complete the definition of structure by giving a meaning to the unit ball and to the bounding functions. However, the requirements below are only needed in Section 4 for the construction of ultrapower to work properly.

5 Definition. The unit ball U is required to satisfy the following axioms

- 1. $M = \{t^M : t \in L_{\text{trm}}(U) \text{ a closed term}\}$;
- 2. $M \models |\varphi(a)| \leq n_{\varphi(x)}$ for every $a \in U^{|x|}$ and every $\varphi(x) \in L_{\text{at}}$;
- 3. for every formula $\psi(x, y) \in L$ and for every $a \in U$

$$M \models \exists y \psi(a, y) \leq 0 \Rightarrow M \models \psi(a, t_{\psi(x,y)}(b)) \leq 0 \text{ for some } b \in U^{|z|}$$

For clarity, we stress that the variables occurring in $t_{\psi(x,y)}(z)$ may be distinct from x, y .

2. EXAMPLES

1. Firstly, we leave to the reader to check that classical first order models are a (trivial) special cases of the models introduced above. Classical relations take values in $\{0, 1\}$ where 0 is interpreted as “true”. The unit ball is the whole of M . The function n_- is constantly 1, and $t_{\psi(x,y)}$ is the term y .
2. Secondly, we consider a class of examples that are non trivial (and distant from classical logic) but where unit ball and bounding functions are irrelevant.

The sets L_{rel} , L_{fun} and the arity function $\text{Ar}(-)$ are arbitrary. The interpretation of the symbols is arbitrary but we require that the functions r^M , for all $r \in L$, range in the interval $[-1, 1]$. As unit ball take the whole of M . The function n_- is constant 1, and $t_{\psi(x,y)}$ is (trivially) the term y .

3. Any BBHU-premodel M is also a model as those in Section 1 if we take as unit ball the whole of M . The language L contains a relation symbol $d(x, y)$ for a metric and, possibly, symbols for all other functions and relations of M . As all relations of M (including the metric) take values in the interval $[0, 1]$, the function n_- can be set to be the constant 1. As $U = M$, we can set $t_{\psi(x,y)}$ to be the term y .

(The following requires some cleaning or could as well be ometted.)

In a BBHU-premodel functions are required to be uniformly continuous with a modulus of uniform continuity fixed by the language. We can reproduce this condition in our language by adding a for every modulus of uniform continuity $\delta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and every $\varepsilon \in \mathbb{R}^+$, a predicate r_ε such that

$$r_\varepsilon^M(x, y, z, w) = \begin{cases} \varepsilon \cdot d(z, w) & \text{if } d(x, y) \leq \delta(\varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

If we require $n_{r_\varepsilon(x,y,fx,fy)} = 1$, then the function interpreted in the models (as defined in Section 1) are uniformly continuous with modulo δ .

4. Finally, let L be the language of Banach spaces. Here we have only one symbol in L_{rel} , the symbol $\|\cdot\|$ for the norm. The set L_{fun} is the same as for real vector spaces in classical logic. Our model M is a Banach space. The interpretation of the symbols in $L_{\text{rel}} \cup L_{\text{fun}}$ is the natural one. The unit ball of M is $U = \{a \in M : \|a\| = 1\}$.

(Need be completed)

Let $\varphi(x)$ be an atomic formula. This has form $\|t(x)\|$ for some $t(x) \in L_{\text{trm}}$. Note that $\sup\{\|t(a)\| : a \in U^{|x|}\}$ does not depend on M . Therefore we can set it to be $n_{\varphi(x)}$

3. CONDITIONS AND TYPES

For $\varphi, \psi \in L(M)$ some closed formulas, we write $M \models \varphi \leq \psi$ for $\varphi^M \leq \psi^M$. The meaning of $M \models \varphi = \psi$ and of $M \models \varphi < \psi$ is similar.

Equations of the form $\varphi(x) = 0$ are called **conditions**. We write $M \models \exists x \varphi(x) = 0$ if $M \models \varphi(a) = 0$ for some $a \in M^{|x|}$. Observe that $\varphi(x) = 0$ is equivalent to $|\varphi(x)| \leq 0$ and $\varphi(x) \leq 0$ is equivalent to $(\varphi(x) \vee 0) = 0$. So, when convenient we may use weak inequalities to denote conditions.

The negation of a condition is called a **co-conditions**. So, $\varphi(x) \neq 0$, $\varphi(x) < 0$ or $\varphi(x) > 0$ are co-conditions.

A set of conditions is called a **type** or, when x is the empty tuple, a **theory**. For $p(x) \subseteq L(M)$, we define

$$p(x) \leq 0 = \{ \varphi(x) \leq 0 \quad : \quad \varphi(x) \in p \}.$$

Up to equivalence, all type have the form above.

Beware that $p(x)$ denotes just a set of formulas, the expression $p(x) \leq 0$ denotes a type.

We write $M \models p(a) \leq 0$ if $M \models \varphi(a) \leq 0$ for every $\varphi(x) \in p$. We write $M \models \exists x p(x) \leq 0$ if $M \models p(a) \leq 0$ for some $a \in M^{|x|}$.

4. ULTRAPOWERS

Firstly, we recall some standard notation about limits. Let I be a non-empty set. Let F be a filter on I . If $r : I \rightarrow \mathbb{R}$ and $\lambda \in \mathbb{R}$ we write

$$\lim_{i \uparrow F} r(i) = \lambda$$

if $r^{-1}[A] \in F$ for every $A \subseteq \mathbb{R}$ that is a neighborhood of λ . When r is bounded and F is an ultrafilter, such a λ always exists.

Let I be an infinite set. Let $\langle M_i : i \in I \rangle$ be a sequence of models with unit balls U_i . Note that all models are required to have the same bounding functions (as these are part of the language). Let F be an ultrafilter on I .

6 Definition. Write **U** for the set of all functions $\hat{a} : I \rightarrow \bigcup_{i \in I} U_i$ such that $\hat{a} i \in U_i$. We define the structure **N** as follows

1. the elements of **N** are functions $\hat{c} : I \rightarrow \bigcup_{i \in I} M_i$ such that $\hat{c} i = t(\hat{a} i)$ for some $\hat{a} \in U^{|x|}$ and some term $t(x) \in L_{\text{trm}}$ — below we write $t(\hat{a})$ for \hat{c} ;
2. if $f \in L_{\text{fun}}$ and $r \in L_{\text{rel}}$ and $t_1(\hat{a}), \dots, t_n(\hat{a})$ are elements of **N** then
 - a. $f^N(t_1(\hat{a}), \dots, t_n(\hat{a})) = t(\hat{a})$ where t is the term obtained composing f with t_1, \dots, t_n ;
 - b. $r^N(t_1(\hat{a}), \dots, t_n(\hat{a})) = \lim_{i \uparrow F} r^{M_i}(t_1(\hat{a} i), \dots, t_n(\hat{a} i))$.

The limit in 2b of Definition 6 exists because F is an ultrafilter and it is finite because all U_i have the same bounding functions. In fact, the function $i \mapsto r^{M_i}(t_1(\hat{a} i), \dots, t_n(\hat{a} i))$ is bounded by $n_{\varphi(x)}$, where $\varphi(x) = r(t_1(x), \dots, t_n(x))$.

7 Proposition (Łoś Theorem). Let **N** be as above and let $\varphi(x) \in L$. Then for every $\hat{a} \in N^{|x|}$,

$$\varphi^N(\hat{a}) = \lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i).$$

Proof. If $\varphi(x)$ is atomic holds by definition. We proceed by induction on the syntax of $\varphi(x)$. We only spell out the proof for the infimum quantifier. Assume inductively that

$$\varphi^N(\hat{a}, \hat{b}) = \lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i, \hat{b}i)$$

We want to prove that

$$\left(\bigwedge_x \varphi(\hat{a}, x) \right)^N = \lim_{i \uparrow F} \left(\bigwedge_x \varphi(\hat{a}i, x) \right)^{M_i}$$

By the interpretation of the infimum quantifier, this amounts to prove that

$$\inf \left\{ \max \{0, \varphi(\hat{a}, \hat{b})^N\} : \hat{b} \in N \right\} = \lim_{i \uparrow F} \inf \left\{ \max \{0, \varphi(\hat{a}i, b)^{M_i}\} : b \in M_i \right\}$$

For simplicity, assume that $\varphi(\hat{a}, \hat{b})^N$ and $\varphi(\hat{a}i, b)^{M_i}$ are non negative. First we prove the \leq inequality i.e., the run-of-the-mill half of the equality. Let r be an arbitrary positive real number.

$$\text{Assume that for some } \hat{b} \quad \varphi^N(\hat{a}, \hat{b}) < r.$$

$$\text{By induction hypothesis} \quad \lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i, \hat{b}i) < r,$$

$$\text{then for some } u \in F \quad \varphi^{M_i}(\hat{a}i, \hat{b}i) < r \quad \text{for every } i \in u.$$

$$\text{Hence} \quad \inf \left\{ \varphi^{M_i}(\hat{a}i, b) : b \in M_i \right\} < r \quad \text{for every } i \in u.$$

$$\text{Finally,} \quad \lim_{i \uparrow F} \inf \left\{ \varphi^{M_i}(\hat{a}i, b) : b \in M_i \right\} < r.$$

This proves the \geq inequality. The proof of the \leq inequality is where the bounding function t_- is required. Assume that

$$\lim_{i \uparrow F} \inf \left\{ \varphi(\hat{a}i, b)^{M_i} : b \in M_i^{|x|} \right\} < r.$$

Therefore, for some $u \in F$

$$\inf \left\{ \varphi(\hat{a}i, b)^{M_i} : b \in M_i^{|x|} \right\} < r \quad \text{for every } i \in u.$$

Without loss of generality we can assume that $\hat{a} \in U$. Hence, writing $t(z)$ for $t_{\varphi(x,y)}(z)$ we have

$$\inf \left\{ \varphi(\hat{a}i, t(c))^{M_i} : c \in U_i^{|z|} \right\} < r \quad \text{for every } i \in u.$$

$$\text{Therefore, for some } \hat{c} \in U^{|z|} \quad \varphi^{M_i}(\hat{a}i, t(\hat{c}i)) < r \quad \text{for every } i \in u.$$

$$\text{Then} \quad \lim_{i \uparrow F} \varphi^{M_i}(\hat{a}i, t(\hat{b}i)) < r,$$

$$\text{and finally} \quad \varphi^N(\hat{a}, t(\hat{b})) < r.$$

This proves the desired inequality. □

8 Fact. The model N defined above has unit ball U .

Proof. The claim about U and n_- follows immediately from the definition of N , see 2b above. As for t_- , the claim follows from Łoś Theorem. \square

5. COMPACTNESS THEOREM

From Łoś Theorem we obtain a form of compactness with the usual argument. Note however that, w.r.t. classical logic, here we add a requirement of uniformity.

A theory $T \leq 0$ is finitely consistent if for every φ , that is a disjunction of sentences in T , there is a model M such that $M \models \varphi \leq 0$. Note that require that all these models M have the same bounding functions.

9 Theorem (Compactness Theorem). Every finitely consistent theory is consistent.

Proof. Let n_-, t_- be some bounding function that witness the uniform finite consistency of T . Below by *model* we mean a model with those bounding functions.

We construct a model N of $T \leq 0$. Let I be the set of closed formulas ξ such that $\xi \leq 0$ holds in some model M_ξ . For every condition $\varphi \leq 0$ define $X_\varphi \subseteq I$ as follows

$$X_\varphi = \left\{ \xi \in I : \xi \leq 0 \vdash \varphi \leq 0 \right\}$$

Note that $\varphi \leq 0$ is consistent if and only if $X_\varphi \neq \emptyset$. Moreover $X_{\varphi \vee \psi} = X_\varphi \cap X_\psi$. Then, as T is finitely consistent, the set $B = \{X_\varphi : \varphi \in T\}$ has the finite intersection property. Extend B to an ultrafilter F on I . Define

$$N = \prod_{\xi \in I} M_\xi$$

Check that $N \models T \leq 0$. Let $\varphi \in T$. By Łoś Theorem,

$$\varphi^N = \lim_{\xi \uparrow F} \varphi^{M_\xi}.$$

Note that $X_\varphi \subseteq \{\xi : \varphi^{M_\xi} \leq 0\}$ and recall that $X_\varphi \in F$ whenever $\varphi \in T$. Therefore $\varphi^N \leq 0$. \square

6. ELEMENTARY MAPS

The **A-limit topology** on $M^{|x|}$ is the initial topology with respect to the functions interpret the formulas in $L(A)$. The sets $\{a \in M^{|x|} : \varphi^M(a) \leq 0\}$, as $\varphi(x)$ ranges over $L(A)$, form a base of closed sets for the A-limit topology. Equivalently, we can take the sets $\{a \in M^{|x|} : \varphi^M(a) < 0\}$ as a base of open sets.

The A-limit topology is a completely regular topology, almost by definition. Typically, it is not T_0 .

We say that the map $k : M \rightarrow N$ is an **elementary map** if

$$\varphi^M(ka) = \varphi^N(a) \quad \text{for every } \varphi(x) \in L \text{ and every } a \in (\text{dom}k)^{|x|}.$$

We write $M \equiv N$ when $\emptyset : M \rightarrow N$ is an elementary map and $M \equiv_A N$ if $\text{id}_A : M \rightarrow N$ is an elementary map. In words, we say that M and N are elementary equivalent over A . An elementary embedding is an total elementary map. Finally, we write $M \leq N$ for $M \equiv_M N$ and say that M is an elementary substructure of N .

The following corollary of Łoś Theorem is identical to its classical counterpart.

10 Corollary. For every model M and every ultrafilter F on I , an infinite set, let N be corresponding ultrapower of N . Then there is an elementary embedding of M in N . \square

11 Proposition (Tarski-Vaught test). Let N be a model and let $M \subseteq N$ be a substructure. Fix a tuple of variables x of finite length. Then the following are equivalent:

1. $M \leq N$;
2. $M^{|x|}$ is dense in $N^{|x|}$ w.r.t. the limit M -topology on $N^{|x|}$;
3. for every $\varphi(x) \in L(M)$, if $\varphi(N) < 0$ is non empty, then $\varphi(b) < 0$ for some $b \in M^{|x|}$.

Proof. Equivalence $2 \Leftrightarrow 3$ and implication $1 \Rightarrow 2$ are clear. To prove $2 \Rightarrow 1$ we prove by induction on the syntax of $\varphi(x)$ that for every $r \in \mathbb{R}$

$$M \models \varphi(a) < r \Leftrightarrow N \models \varphi(ka) < r \quad \text{for every } a \in M^{|x|}$$

We use the equivalence above as induction hypothesis. As M is a substructure of N , this equivalence holds for atomic $\varphi(x)$. We leave the case of Riesz connectives to the reader and consider only the case of the infimum quantifier.

$$\begin{aligned} N \models \bigwedge_x \varphi(x) < r &\Leftrightarrow \text{the set } \varphi(N) < r \text{ is non empty} \\ a. &\Leftrightarrow \text{there is } b \in M^{|x|} \text{ such that } N \models \varphi(b) < r \\ b. &\Leftrightarrow \text{there is } b \in M^{|x|} \text{ such that } M \models \varphi(b) < r \\ &\Leftrightarrow M \models \bigwedge_x \varphi(x) < \varepsilon. \end{aligned}$$

Implication \Rightarrow in a uses 2 and the equivalence in b uses the induction hypothesis. \square

12 Remark. In the Tarski-Vaught test for classical logic it is not necessary to require that M is a substructure of N as this is a consequence of the test. In the continuous case this is necessary by the lack of equality. \square

13 Theorem (Downward Löwenheim-Skolem). For every $A \subseteq N$ there is a model $A \subseteq M \leq N$ of cardinality $\leq |L(A)|$.

Proof. (Needs some checking.) For every set A there is a base for the A -topology that has cardinality $\leq |L(A)|$. A set M of the required cardinality that is dense in the M -topology is obtained as in the classical downward Löwenheim-Skolem theorem. \square

7. SATURATION

Throughout this section M is a model with unit ball U and bounding functions n_-, t_- .

Let $p(x) \subseteq L(M)$. We say that $p(x) \leq 0$ is **uniformly finitely satisfied in M** if for every formula $\psi(x)$ that is a disjunction of formulas in $p(x)$, there is an $a \in U^{|z|}$ such that $M \models \psi(t(a)) \leq 0$, where $t(z) \in L_{\text{trm}}$ is a term which only depends on $p(x)$.

We say that M is **λ -saturated** if for every $p(x) \subseteq L(A)$ as in 1 and 2 below, there is an $a \in M^{|x|}$ such that $M \models p(a) \leq 0$.

1. $A \subseteq M$ has cardinality $< \lambda$;
2. $p(x) \leq 0$ is uniformly finitely satisfied in M .

If λ is the cardinality of M we say that M is **saturated**.

14 Theorem (???). Let λ be an inaccessible cardinal larger than $|L|$ and $|M|$. Then M has a saturated elementary superstructure of cardinality M .

8. COMPLETENESS VS. SATURATION ; PLEASE EXPAND !

The A -limit uniformity on $M^{|x|}$ is the uniformity that has the following entourages

$$V_{\varphi(x), \varepsilon} = \left\{ (a, b) \in M^{|x|} \times M^{|x|} : \left| \varphi^M(a) - \varphi^M(b) \right| < \varepsilon \right\}$$

for $\varphi(x) \in L(A)$ and $\varepsilon \in \mathbb{R}^+$. It is the coarsest uniformity that makes all formulas in $L(A)$ uniformly continuous.

Let $p(x) \subseteq L(M)$ be a type. We say that $p(x) \leq 0$ is **Cauchy** (in the A -limit uniformity) if for every $\varphi(x) \in L(A)$ and every $\varepsilon \in \mathbb{R}^+$ there is a disjunction of formulas in $p(x)$, say $\psi(x)$, such that $\emptyset \neq (\psi(M) \leq 0)^2 \subseteq V_{\varphi(x), \varepsilon}$.

15 Fact. Let $\lambda = L(A)$. If M is λ -saturated then and $p(x) \subseteq L(M)$ is Cauchy, then $p(x)$ is realized in M .

For every $\varphi(x) \in L(A)$ and $n \in \mathbb{N}$ let

Suppose $N \models p(a) \leq 0$ and let $(\psi(M) \leq 0)^2 \subseteq V_{\varphi(x), \varepsilon}$