

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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These notes do not claim new mathematical results. In fact, the main ideas applied here originated in the mid 20th century (cf. Chang-Keisler, Gromov). Still, we believe that an important facet of mathematics progress consists of the simplification and clarification of existent knowledge. We hope to contribute in this manner.

1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them by adding a norm. The norm is a function that, given vector, outputs a real number, we may formalize this with a two sorted structure. Ideally, we would like to extend the vector space while maintaining the usual notion of real numbers¹ but this is incompatible with the usual notion of elementary extension. One way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas. We elaborate on this idea and generalize it to arbitrary structures. We also show that their approach applies to a larger class of formulas.

1 Definition. Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L' -structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L -structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

The function symbols of L' fall in one of two groups

1. $M^n \times R^m \rightarrow M$
2. $M^n \times R^m \rightarrow R$

We require that function symbols of the second group are interpreted in functions such that

- i. $f^{\mathcal{M}} : M^n \times R^m \rightarrow R$ is bounded, i.e. its range is contained in a compact set;
- ii. $f^{\mathcal{M}} : M^n \times R^m \rightarrow R$ is continuous uniformly in M , this is made precise in Definition 3.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set \mathbb{L} of formulas, $L \subseteq \mathbb{L} \subseteq L'$,

¹Non standard standard analysts have no problem in expanding \mathbb{R} . In these notes, for a change, we insist as \mathbb{R} should remain \mathbb{R} throughout.

such that every model as in Definition 1 has an \mathbb{L} -elementary extension that is \mathbb{L} -saturated in the same class. Our choice of \mathbb{L} is not the most general. It is a compromise that avoids excessive complications.

In a nutshell, we require that function symbols of the second group above do not occur under the scope of a negation. We also do not allow existential quantifiers of sort R . In fact, the ultrapower construction that we present in Section 3 is completely classical w.r.t. functions the first group. On the other hand, functions of the second group are *critical* and need to be dealt with differently.

Below, \mathbb{T}_M denotes the set of the terms obtained composing function symbols as in 1 above.

By \mathbb{T}_R we denote the set of the terms obtained composing function symbols as in 2 above.

By $\mathbb{T}_R \circ \mathbb{T}_M$ we denote the set of the terms obtained composing terms in \mathbb{T}_R with terms in \mathbb{T}_M .

Clearly, there are many more terms in L' but we will restrict to these.

2 Definition. Formulas in \mathbb{L} are constructed inductively from the following two sets of formulas

- i. formulas as $t(y; z) \in C$, where $C \subseteq R^n$ is compact² and $t(y; z)$ is a tuple of terms in $\mathbb{T}_R \circ \mathbb{T}_M$;
- ii. quantifier free formulas with only terms in \mathbb{T}_M .

We require that \mathbb{L} is closed under the Boolean connectives \wedge, \vee , the quantifiers \forall, \exists of sort M ; and the universal quantifier \forall of sort R .

Clearly, $L \subseteq \mathbb{L} \subseteq L'$.

2. TOPOLOGICAL TOOLS

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . Let Y be a topological space. If $f : I \rightarrow Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ . Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

By βI we denote the set of ultrafilters on I . This is a topological space with the topology generated by the clopen sets $[A] = \{F \in \beta I : A \in F\}$ as A ranges over the subsets of I .

3 Definition. Let I be a pure set. Let X, Y be topological spaces, and assume Y is Hausdorff and compact. We say that $f : I \times X \rightarrow Y$ is **continuous uniformly in I** if the function below is continuous

$$\tilde{f} : \beta I \times X \rightarrow Y$$

²We confuse the relation symbols in L' of sort R^m with their interpretation and write $x \in C$ for $C(x)$

$$\langle F, \alpha \rangle \mapsto F\text{-}\lim_i f(i, \alpha)$$

The following propositions unravels the definition above into a more practicable form.

4 Proposition. The following are equivalent

1. $f : I \times X \rightarrow Y$ is continuous uniformly in I .
2. For every $F \in \beta I$ and $\alpha \in X$, and for every $V \subseteq Y$ open neighborhood of $F\text{-}\lim_i f(i, \alpha)$ there is $U \subseteq X$ open neighborhood of α and an $A \in F$ such that $f(i, x) \in V$ for every $i \in A$ and every $x \in U$.

5 Lemma (???). Let $f : I \times X \rightarrow Y$ be continuous uniformly in I . Let $\langle y_i \in Y : i \in I \rangle$. For a given $F \in \beta I$ define $\alpha = F\text{-}\lim_i y_i$. Then $f(F, \alpha) = F\text{-}\lim_i f(i, y_i)$.

6 Lemma (???). Let $f : I \times X \rightarrow Y$ be continuous uniformly in I . Let $g : I \times Y \rightarrow I$ be any function and let $\bar{g} : \beta I \times Y \rightarrow \beta I$ the unique continuous extension of g . Let $\langle y_i \in Y : i \in I \rangle$. For a given $F \in \beta I$ define $\alpha = F\text{-}\lim_i y_i$. Then $f(\bar{g}(F, \alpha), \alpha) = F\text{-}\lim_i f(g(i, y_i), y_i)$.

3. ULTRAPOWERS

To keep notation tidy, we do not consider formulas containing equality, so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

7 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \times R^m \rightarrow M$ then $f^{\mathcal{N}}(\hat{a}; \alpha)$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle$.
4. If f is a function of sort $M^n \times R^m \rightarrow R$ then for every $\alpha \in R^m$

$$f^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \alpha).$$

3. if r is a relation symbol of sort $M^n \times R^m$ then for every $\alpha \in R^m$

$$\mathcal{N} \models r(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i; \alpha)\} \in F.$$

The following is immediate.

8 Fact. The structure \mathcal{N} satisfies condition 1 of Definition 1.

The following is easily proved by induction on the syntax

9 Fact. If $t(y; z) \in \mathbb{T}_M$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = \langle t^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle.$$

□

We also have that

10 Fact. If $t(y; z) \in \mathbb{T}_R$ then

$$\#_0 \quad t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Proof. By induction. The fact holds by definition if t is a function symbol. Therefore, we assume $\#_0$ as induction hypothesis and prove

$$\#_1 \quad f^{\mathcal{N}}(\hat{a}; t^{\mathcal{N}}(\hat{a}; \alpha)) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; t^{\mathcal{M}}(\hat{a}i; \alpha)).$$

By induction hypothesis

$$\#_2 \quad f^{\mathcal{N}}(\hat{a}; t^{\mathcal{N}}(\hat{a}; \alpha)) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \lambda), \quad \text{where } \lambda = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Let σ be the limit in $\#_2$. Let $V \subseteq R$ be an open set containing σ . Note that we can identify F with an ultrafilter on $M^{|\hat{a}|}$ concentrated on range of \hat{a} . Apply 2 of Proposition 4, to obtain $A \in F$ and an open set $U \subseteq R$ containing λ such that for every $i \in A$ and every $x \in U$

$$f^{\mathcal{M}}(\hat{a}i; x) \in V.$$

By induction hypothesis there is $B \in F$ be such that $t^{\mathcal{M}}(\hat{a}i; \alpha) \in U$ for every $i \in B$. Therefore

$$f^{\mathcal{M}}(\hat{a}i; t^{\mathcal{M}}(\hat{a}i; \alpha)) \in V$$

for every $i \in A \cap B$. As $A \cap B \in F$, this suffices to prove that σ is also the limit in $\#_1$. □

From the two facts above it immediately follows that

11 Fact. If $t(y; z) \in \mathbb{T}_R \circ \mathbb{T}_M$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

□

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C . Note that, if no such atomic formulas occurs in φ , then $\varphi > \varphi$. Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \rightarrow \varphi'$. Moreover

$$\bigwedge_{\varphi' > \varphi} \varphi' \leftrightarrow \varphi.$$

Finally, we prove Łoś Theorem.

12 Proposition. Assume $|I| \geq$ the weight of the topology on R . Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|\hat{x}|}$ and all $\alpha \in R^{|\hat{y}|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x; y)$ is as in (ii) of Definition 2 then the proposition reduces to the classical Łoś Theorem.

Now, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$, where $t(x; y) \in \mathbb{T}_R \circ \mathbb{T}_M$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. Let D be a compact neighborhood of C such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$. By Fact 11 and the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$.

Induction for the connectives \vee and \wedge is clear. To deal with the universal quantifier of sort R we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \in F \text{ for every } \varphi' > \varphi$$

and we prove that

$$\mathcal{N} \models \forall y \varphi(\hat{a}; \alpha, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i; \alpha, y)\} \in F \text{ for every } \varphi' > \varphi.$$

The implication \Rightarrow is clear. For the converse assume $\mathcal{N} \not\models \varphi(\hat{a}; \alpha, \beta)$ for some β . By induction hypothesis, and the requirement that F is an ultrafilter, $\{i \in I : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha, \beta)\} \in F$ for some $\varphi' > \varphi$. A fortiori $\{i \in I : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i; \alpha, y)\} \in F$, hence the implication \Leftarrow follows.

To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

We first we prove that

$$\mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

This is identical to the case above. The implication \Rightarrow is clear. For the converse assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i \in I : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ for some $\varphi' > \varphi$. Hence $\{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \notin F$ as required.

Finally we prove

$$\mathcal{N} \models \exists y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Again, \Rightarrow is clear. For the converse, assume the r.h.s. Assume for simplicity that $I = \omega$ and that R is second countable. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for every $i < j$ and

$$\mathcal{M} \models \bigwedge_{i \in I} \varphi_i(x, y; \alpha) \rightarrow \varphi(x, y; \alpha).$$

For every $j \in I$ there is a sequence \hat{b}_j such that

$$\{i \in I : \mathcal{M} \models \varphi_j(\hat{a}i, \hat{b}_ji; \alpha)\} \in F.$$

As $\varphi_j(x, y; \alpha) \rightarrow \varphi_{j'}(x, y; \alpha)$ holds for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_i : i \in I \rangle$ satisfies $\mathcal{N} \models \varphi(\hat{a}, \hat{d}; \alpha)$. \square