

# CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

C. L. C. L. POLYMATH

## 1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number, we may formalize this in a natural way by using a two sorted structure. Ideally, we would like to have that an elementary extension of a normed space maintain the usual notion of real numbers<sup>1</sup> but this is not possible with the usual notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

**1 Definition.** Let  $L$  be a one-sorted (first-order) language. Let  $L' \supseteq L$  be a two-sorted language. We consider the class of  $L'$ -structures of the form  $\mathcal{M} = \langle M, R \rangle$ , where  $M$  ranges over  $L$ -structures, while  $R$  is fixed.

We assume that  $R$  is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

- i.  $R^n \rightarrow R$ ;
- ii.  $M^n \rightarrow M$ ;
- iii.  $M^n \rightarrow R$ .

The interpretation of symbols of sort  $R^n \rightarrow R$  is required to be continuous, the interpretation of symbols of sort  $M^n \rightarrow R$  is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts  $M^n$  and  $R^n$ .

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set  $\mathbb{L}$  of formulas,  $L \subseteq \mathbb{L} \subseteq L'$ , such that every model as in Definition 1 has an  $\mathbb{L}$ -elementary extension that is  $\mathbb{L}$ -saturated in the same class.

---

<sup>1</sup>Non standard standard analysts have no problem in expanding  $\mathbb{R}$ . In these notes, for a change, we insist as  $\mathbb{R}$  should remain  $\mathbb{R}$  throughout.

For convenience we assume that the functions of sort  $M^n \rightarrow M$  and the relation of sort  $M^n$  are all in  $L$ . It is also convenient to assume that  $L'$  contains names for all continuous bounded functions  $R^n \rightarrow R$ .

**2 Definition.** Formulas in  $\mathbb{L}$  are constructed inductively from the following two sets of formulas

- i. formulas of the form  $t(x; y) \in C$ , where  $C \subseteq R^n$  is compact<sup>2</sup> and  $t(x, y)$  is a  $n$ -tuple of terms of sort  $M^{|x|} \times R^{|y|} \rightarrow R$ ;
- ii. all formulas in  $L$ .

We require that  $\mathbb{L}$  is closed under the Boolean connectives  $\wedge, \vee$ ; the quantifiers  $\forall, \exists$  of sort  $M$ ; and the quantifier  $\forall^C$ , by which we mean a universal quantifier restricted to the compact set  $C \subseteq R^m$ .

## 2. HENSON-IOVINO APPROXIMATIONS

For  $\varphi, \varphi' \in \mathbb{L}$  (free variables are hidden) we say that  $\varphi' > \varphi$  if  $\varphi'$  can be obtained replacing each atomic formula  $t \in C$  occurring in  $\varphi$  by  $t \in D$  for some compact neighborhood  $D$  of  $C$ . If no such atomic formulas occurs in  $\varphi$ , then  $\varphi > \varphi$ .

Note that this is a dense (pre)order of  $\mathbb{L}$ .

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that  $\varphi \rightarrow \varphi'$ . Vice versa, if  $\varphi'$  for every  $\varphi' > \varphi$ , then  $\varphi$ .

The lemma below is required for the proof of Łoś Theorem in the next section. But first a preliminary fact.

**3 Fact.** Assume the following data

- $t(x; y)$ , an  $n$ -tuple of terms of sort  $M^{|x|} \times R^{|y|} \rightarrow R$ ;
- $\langle a_i : i \in I \rangle$ , a sequence of elements  $a_i \in M^{|x|}$ ;
- $U \subseteq R^n$  an open set.

Let  $F$  be an ultrafilter on  $I$ . Assume that for some  $\alpha \in R^{|y|}$

$$\{i \in I : t^{\mathcal{M}}(a_i; \alpha) \in U\} \in F.$$

Then there is an open  $V \ni \alpha$  such that for every  $\alpha' \in V$

$$\{i \in I : t^{\mathcal{M}}(a_i; \alpha') \in U\} \in F.$$

---

<sup>2</sup>We confuse the relation symbols in  $L'$  of sort  $R^m$  with their interpretation and write  $t \in C$  for  $C(t)$

*Proof.* By induction on the syntax. If  $t$  is a function symbol  $f$  then  $x$  or  $y$  do not occur in  $t$ . If  $x$  does not occur in  $t$  the claim holds because  $f^{\mathcal{M}} : R^{|x|} \rightarrow R$  is continuous. If  $y$  does not occur in  $t$  the claim is trivial. Finally, induction is clear by the continuity of the functions of sort  $R^n \rightarrow R$ .  $\square$

We recall the standard definition of  $F$ -limits. Let  $I$  be a non-empty set. Let  $F$  be a filter on  $I$ . Let  $Y$  be a topological space. If  $f : I \rightarrow Y$  and  $\lambda \in Y$  we write

$$F\text{-}\lim_i f(i) = \lambda$$

if  $f^{-1}[A] \in F$  for every  $A \subseteq Y$  that is a neighborhood of  $\lambda$ . Such a  $\lambda$  is unique if  $Y$  is Hausdorff. When  $F$  is an ultrafilter, and  $Y$  is compact the limit always exists.

**4 Lemma.** Assume the following data

- $\varphi(x; y) \in \mathbb{L}$ ;
- $\langle a_i : i \in I \rangle$ , a sequence of elements of  $M^{|x|}$ ;
- $\langle \alpha_i : i \in I \rangle$ , a sequence of elements of  $C \subseteq R^{|y|}$ , a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$ , for some ultrafilter  $F$  on  $I$ .

Assssume that

$$\{i \in I : \mathcal{M} \models \varphi(a_i; \alpha)\} \in F.$$

Then for every  $\varphi' > \varphi$

$$\{i \in I : \mathcal{M} \models \varphi'(a_i; \alpha_i)\} \in F.$$

*Proof.* By induction on the syntax. When  $\varphi(x; y)$  is in  $L$ , it does not depend on  $\alpha$ , and the lemma is trivial. Suppose that  $\varphi(x; y)$  is as in (i) of Definition 2, say it is the formula  $t(x; y) \in C$ .

Assume that  $\{i : \mathcal{M} \models t(a_i; \alpha) \in C\} \in F$ . Then  $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \in C$ . Let  $D$  be a compact neighborhood of  $C$ . By Fact 3, there is an open  $V \ni \alpha$  such that  $\{i : \mathcal{M} \models t(a_i; \alpha') \in D\} \in F$  for every  $\alpha' \in V$ . Then  $\{i : \mathcal{M} \models t'(a_i; \alpha_i) \in D\} \in F$  follows and proves the base case of the induction.

Induction clear for the connectives  $\vee, \wedge$  and for the existential quantifier of sort  $M$ . To deal with the universal quantifier of sort  $M$  we assume inductively that for every  $\varphi' > \varphi$

$$\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F \Rightarrow \{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F.$$

We prove

$$1. \quad \{i : \mathcal{M} \models \forall y \varphi(a_i, y; \alpha)\} \in F \Rightarrow \{i : \mathcal{M} \models \forall y \varphi'(a_i, y; \alpha_i)\} \in F$$

Negate the r.h.s. of the implication. Then there is a sequence  $\langle b_i : i \in I \rangle$  such that (using that  $F$  is an ultrafilter)

$$\{i : \mathcal{M} \not\models \varphi'(a_i, b_i; \alpha_i)\} \in F$$

Assume the l.h.s. of (1) and reason for a contradiction. Then  $\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F$  and, by induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F$ . A contradiction that proves (1).

Induction for the quantifier  $\forall^B$  is virtually identical, but we repeat the argument for convenience. Assume inductively that for every  $\varphi' > \varphi$  and every sequence  $\langle \beta_i : i \in I \rangle$  in  $B$  such that  $\beta = F\text{-}\lim_i \beta_i$

$$\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F \Rightarrow \{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F.$$

We prove

$$2. \quad \{i : \mathcal{M} \models \forall^C y \varphi(a_i, ; \alpha, y)\} \in F \Rightarrow \{i : \mathcal{M} \models \forall^C y \varphi'(a_i; \alpha_i, y)\} \in F$$

Negate the r.h.s. of the implication. Then there is a sequence  $\langle \beta_i : i \in I \rangle$  such that (again using that  $F$  is an ultrafilter)

$$\{i : \mathcal{M} \not\models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$$

As  $B$  is compact,  $F\text{-}\lim_i \beta_i$  exists, say it equals  $\beta \in B$ . Assume the l.h.s. of the implication and reason for a contradiction. Then  $\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F$  and, by induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$ . A contradiction that proves (2) and, with it, the lemma.  $\square$

### 3. ULTRAPRODUCTS

Below we introduce a suitable notion of ultraproducts of some structures  $\langle \mathcal{M}_i : i \in I \rangle$ . We require that for each function symbol  $f$  of sort  $M^n \rightarrow R$  there is a compact  $C \subseteq R$  that contains the range of all the functions  $f^{\mathcal{M}_i}$ .

To keep notation tidy, we make two simplifications: (1) we only consider ultratowers; (2) we do not consider formulas in  $L$  containing equality, so we need not take any quotient. The generalization is straightforward and is left to the reader.

Let  $I$  be an infinite set. Let  $F$  be an ultrafilter on  $I$ . Let  $\mathcal{M} = \langle M, R \rangle$  be an  $L$ -structure.

**5 Definition.** We define a structure  $\mathcal{N} = \langle N, R \rangle$  that we call the **ultrapower** of  $\mathcal{M}$ .

1.  $N = M^I$  that is, it is the set of sequences  $\hat{a} : I \rightarrow M$ .
2. If  $f$  is a function of sort  $M^n \rightarrow M$  then  $f^{\mathcal{N}}(\hat{a})$  is the sequence  $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I \rangle$ .
3. The interpretation of functions of sort  $R^n \rightarrow R$  remains unchanged.
4. If  $f$  is a function of sort  $M^n \rightarrow R$  then

$$f^{\mathcal{N}}(\hat{a}) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$

5. If  $r$  is a relation symbol of sort  $M^n$  then

$$\mathcal{N} \models r(\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i)\} \in F.$$

6. The interpretation of functions of sort  $R^n$  remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the uniformity assumption above.

**6 Fact.** The structure  $\mathcal{N}$  satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

**7 Fact.** If  $t(x)$  is a term of type  $M^{|x|} \rightarrow M$  then

$$t^{\mathcal{N}}(\hat{a}) = \langle t^{\mathcal{M}}(\hat{a}i) : i \in I \rangle.$$

□

We also have that

**8 Fact.** If  $t(x; y)$  has sort  $M^{|x|} \times R^{|y|} \rightarrow R$  then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

*Proof.* By induction. If  $t$  is a function symbol then  $x$  or  $z$  do not occur in  $t$ . If  $x$  does not occur in  $t$  the claim is trivial. If  $y$  does not occur in  $t$  the claim holds by definition. Finally, induction is clear by the continuity of the functions of sort  $R^n \rightarrow R$ . □

Finally, we prove

**9 Proposition (Łoś Theorem).** Let  $\mathcal{N}$  be as above and let  $\varphi(x; y) \in \mathbb{L}$ . Then for all  $\hat{a} \in N^{|x|}$  and all  $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

*Proof.* By induction on the syntax. If  $\varphi(x; y) \in L$  then the theorem reduces to the classical Łoś Theorem. Then, suppose that  $\varphi(x; y)$  is as in (i) of Definition 2, say it is the formula  $t(x; y) \in C$ .

Assume  $\mathcal{N} \models t(\hat{a}, \alpha) \in C$ . If  $D$  is a compact neighborhood of  $C$  then  $D$  is also a neighborhood of  $t^{\mathcal{N}}(\hat{a}, \alpha)$ . Hence, by the definition of  $F$ -limit,  $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$ . Vice versa, assume  $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$ . By local compactness of  $R$  there is a compact neighborhood of  $C$ , such that  $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$ . By Fact 8 and the definition of  $F$ -limit,  $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$ .

This completes the proof of the base case of the induction.

Induction for the connectives  $\vee$  and  $\wedge$  is clear. To deal with the quantifiers of sort  $M$  we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

$$1. \quad \mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

( $\Leftarrow$ ) Assume  $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$  for some  $\hat{b}$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$  for some  $\varphi' > \varphi$ . A fortiori  $\{i : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \notin F$  as required.

( $\Rightarrow$ ) Assume that  $\{i : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F$ . Choose  $\hat{b}$  such that  $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ . If for a contradiction  $\mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha)$ , then in particular  $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$  a contradiction. This completes the proof of (1).

Now we prove (the argument, which we repeat for convinience, is dual to the one above)

$$2. \quad \mathcal{N} \models \exists y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

( $\Rightarrow$ ) Assume  $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$  for some  $\hat{b}$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$  for every  $\varphi' > \varphi$ . A fortiori  $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F$  as required.

( $\Leftarrow$ ) Assume that  $\{i : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F$  for every  $\varphi' > \varphi$ . Choose some  $\hat{b}$  such that  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ . Suppose for a contradiction that  $\mathcal{N} \not\models \exists y \varphi(\hat{a}, y; \alpha)$ . Then, in particular,  $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$  for some  $\varphi' > \varphi$ , a contradiction. This completes the proof of (2).

Finally, to deal with the quantifier  $\forall^C$  we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

$$3. \quad \mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F \text{ for every } \varphi' > \varphi.$$

( $\Leftarrow$ ) Assume  $\mathcal{N} \not\models \varphi(\hat{a}; \alpha, \beta)$  for some  $\beta$ . By induction hypothesis,  $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \notin F$  for some  $\varphi' > \varphi$ . A fortiori  $\{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \notin F$  as required.

( $\Rightarrow$ ) Assume that  $\{i : \mathcal{M} \not\models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F$  for some  $\varphi' > \varphi$ . Choose some  $\beta_i \in C$  such that  $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha, \beta_i)\} \in F$  and let  $\beta = F\text{-}\lim_i \beta_i$ . If for a contradiction  $\mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y)$ , then  $\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta)$  and, by induction hypothesis,  $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \alpha, \beta)\} \in F$  for every  $\varphi'' > \varphi$ . Fix  $\varphi''$  such that  $\varphi' > \varphi'' > \varphi$  and obtain a contradiction from Lemma 4. This completes the proof of (3) and, with it, of the theorem.  $\square$

#### 4. $\mathbb{L}$ -ELEMENTRARITY

Let  $\mathcal{M} = \langle M, R \rangle$  and  $\mathcal{N} = \langle N, R \rangle$  be two structures. We say that  $f : M \rightarrow N$ , a partial map, is an  **$\mathbb{L}$ -elementary map** if

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(a) \quad \text{for every } \varphi(x) \in \mathbb{L} \text{ and every } a \in (\text{dom } f)^{|x|}$$

The following fact is proved as in the classical case using that  $\mathcal{M} \models \varphi'$  for every  $\varphi' > \varphi$  if and only if  $\mathcal{M} \models \varphi$ .

**10 Fact.** If  $\mathcal{N}$  is an ultrapower of  $\mathcal{M}$  then there is an  $\mathbb{L}$ -elementary embedding of  $\mathcal{M}$  into  $\mathcal{N}$ .

For  $A \subseteq M \cap N$ , we say that  $\mathcal{M}$  and  $\mathcal{N}$  are  $\mathbb{L}$ -(elementary) equivalent over  $A$  and write  $\mathcal{M} \equiv_A^{\mathbb{L}} \mathcal{N}$  if  $\text{id}_A : M \rightarrow N$  is  $\mathbb{L}$ -elementary. We write  $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$  when  $\mathcal{M} \equiv_M^{\mathbb{L}} \mathcal{N}$ . In words, we say that  $\mathcal{M}$  is an  $\mathbb{L}$ -(elementary) substructure of  $\mathcal{N}$ .

The following is proved as in the classical case.

**11 Proposition (Tarski-Vaught Test).** The following are equivalent

1.  $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$
2.  $M$  is a subset of  $N$  and for every formula  $\varphi(x) \in \mathbb{L}(M)$ 

$$\mathcal{N} \models \exists x \varphi(x) \Rightarrow \mathcal{N} \models \varphi(a) \text{ for some } a \in M$$

□

## 5. $\mathbb{L}$ -SATURATED STRUCTURES



Let  $p(x) \subseteq \mathbb{L}(M)$ . We say that  $p(x)$  is **finitely satisfied in  $\mathcal{M}$**  if  $\mathcal{M} \models \exists x \varphi'(x)$  for every  $\varphi' > \varphi$  where  $\varphi(x)$  is any conjunction of formulas in  $p(x)$ .

The following definition is standard.

**12 Definition.** We say that  $\mathcal{M}$  is  **$\mathbb{L}$ -saturated** if for every  $p(x)$  as in 1 and 2 below,  $\mathcal{M} \models \exists x p(a)$ .

1.  $p(x) \subseteq \mathbb{L}(A)$  for some  $A \subseteq M$  of cardinality  $< |M|$  and  $|x| = 1$ ;
2.  $p(x)$  is finitely satisfied in  $\mathcal{M}$ .

Then proof that every model embeds  $\mathbb{L}$ -elementarily in an  $\mathbb{L}$ -saturated one proceeds as in the classical case. First, note that the classical ultrapower construction yields the the following lemma.

**13 Lemma.** Every model  $\mathcal{M}$  embeds  $\mathbb{L}$ -elementarily in a model  $\mathcal{N}$  that realizes all types as in the definition above.

Finally, using a sufficiently long  $\mathbb{L}$ -elementary chain we obtain the required saturated extension.

**14 Corollary.** Every model  $\mathcal{M}$  is an  $\mathbb{L}$ -elementary substructure of some saturated model  $\mathcal{N}$  (possibly of inaccessible cardinality).

We denote by  $\mathcal{U}$  some large  $\mathbb{L}$ -saturated structure which we call the **monster model**. The unit ball of  $\mathcal{U}$  is denoted by  $U$ . The cardinality of  $\mathcal{U}$  is an inaccessible cardinal that we denote by  $\kappa$ . Below we say **model** for  $\mathbb{L}$ -elementary substructure of  $\mathcal{U}$ .