CONTINUOUS MODEL THEORY WITHOUT THE HASSLE

1. Formulas and their interpretation

The following definition extends that of classical first-order logic with cluses 3 and 4.

1 Definition. A signature or language L consists of the following data

1. two sets of symbols: L_{rel}, for relations, and L_{fun}, for functions;
2. an arity function $\text{Ar} : L_{\text{rel}} \cup L_{\text{fun}} \to \omega$;
3. a function $n_{\_} : L_{\text{at}} \to \mathbb{N}$;
4. a distingished unary relation symbol u.

The function in 3 above is called the bounding function. The meaning of L_{at} clarified below.

2 Definition. A structure or model M of signature L consists of the following data

1. for every $f \in L_{\text{fun}}$, a function $f^M : M^{\text{Ar}(f)} \to M$;
2. for every $r \in L_{\text{rel}}$, a function $r^M : M^{\text{Ar}(r)} \to \mathbb{R}$;
3. a function $u^M : M \to \{0, 1\}$.

We write uM for the set $\{a \in M : u^M a = 0\}$. We call this the unit ball of M. It will not play any role till Definition 4.vi. The unit ball is required to saisfy two axioms which are postponed till Definition 5. These axioms also clarify the role of the bounding function.

The syntax and semantic of terms is defined just as in classical first-order logic. We write L_{trm} for the set of terms in the language L. Atomic formulas are defined as in classical first-order logic. They have the form $r\,t$, where r is a relation symbol and t is a tuple of terms. The set of atomic formulas is denoted by L_{at}. The set of all formulas is denoted as usual by L, the same symbol as the language. Formulas are defined inductively from the atomic one using the following connectives.

The propositional (or Riesz) connectives are those in $\{+, \cdot, \wedge\} \cup \mathbb{R}$, where $+$, $\cdot$, and $\wedge$ are binary connectives, the elements of $\mathbb{R}$ are logical constants (i.e. a zero-ary connectives). There is also the infimum quantifier $\bigwedge_x$.

3 Definition. The inductive definition of formula is as follows

i. atomic formulas are formulas
ii. α is a formula for every $\alpha \in \mathbb{R}$;
iii. $\varphi + \psi$ is a formula;
iv. $\varphi \cdot \psi$ is a formula;
v. $\varphi \wedge \psi$ is a formula;
vi. $\bigwedge\limits_x \varphi$ is a formula.

If $\alpha \in \mathbb{R}$, we may write α for the formula $\alpha 1$. We write $-\varphi$ for $(-1)\varphi$ and $\varphi - \psi$ for $\varphi + (-\psi)$ and we write $\varphi \vee \psi$ for $-(-\varphi \wedge -\psi)$. Also, φ^+ and φ^- stand for $0 \vee \varphi$ respectively $0 \vee (-\varphi)$, and we write $|\varphi|$ for $\varphi \vee (-\varphi)$. Finally, $\bigvee\limits_x \varphi$ stands for $-\bigwedge\limits_x -\varphi$.

If $A \subseteq M$, we write $L(A)$ for the language L expanded with a 0-ary function symbol for every element of a. As in classical first-order logic, M is canonically expanded to a structure of signature $L(A)$ by setting $a^M = a$ for every $a \in A$.

Let M be a model and let t be a tuple of closed terms possibly with parameters from M. We write t^M for the element obtained interpreting function symbols and parameters in M just as in classical first-order logic.

4 **Definition.** If $\varphi \in L(M)$ is a sentence (i.e. a closed formula) we define φ^M inductively as follows

i. $(r\,t)^M = r^M\big(t^M\big)$.
ii. $\alpha^M = \alpha$;
iii. $(\psi + \xi)^M = \psi^M + \xi^M$;
iv. $(\varphi \cdot \psi)^M = \varphi^M \cdot \psi^M$;
v. $(\psi \wedge \xi)^M = \min\big\{\psi^M,\ \xi^M\big\}$;
vi. $\Big(\bigwedge\limits_x \psi\Big)^M = \inf\big\{\psi[x/b]^M \,:\, b \in (uM)\big\}$.

In Fact 6 below we prove that the infimum in (vi) of the definition above is ineed finite, so that the definition is well-given.

For $\varphi(x) \in L(M)$ we write φ^M for the function $\varphi^M : M^{|x|} \to \mathbb{R}$ that maps $a \mapsto \varphi(a)^M$.

Finally, we complete the definition of structure by giving the axiomatics of the unit ball and of the bounding function.

5 **Definition.** The unit ball is required to satisfy the following axioms

1. $M = \big\{t^M \,:\, t \in L_{\mathrm{trm}}(uM) \text{ a closed term}\big\}$;
2. $|\varphi(a)^M| \leq n_{\varphi(x)}$ for every $a \in (uM)^{|x|}$ and every $\varphi(x) \in L_{\mathrm{at}}$.

A straightforeward proof by induction on the syntax yields the following.

6 **Fact.** For every formula $\psi(x)$ there is an $n \in \mathbb{N}$ such that $|\psi(a)^M| \leq n$ for all $a \in (uM)^{|x|}$. □

2. Examples

1. Firstly, we leave to the reader to check that classical first order models are a (trivial) special cases of the models introduced above. Classical relations take values in $\{0,1\}$ where 0 is interpreted as "true". The unit ball is the whole of M. The function $n_{\text{-}}$ is constantly 1. To

obtain the full strength of firt order logic one has to add equality as a predicate which is abset from our language.

2. Secondly, we consider a class of examples that are non trivial (and distant from classical logic) but where unit ball and bounding function are irrelevant.

   The sets $L_{\rm rel}$, $L_{\rm fun}$ and the arity function Ar(-) are arbitrary. The interpretation of the symbols is arbitrary but we require that the functions r^M, for all $r \in L$, range in the interval $[-1, 1]$. As unit ball take the whole of M. The function $n_{\text{-}}$ is constant 1.

3. Any BBHU-premodel M is also a model as those in Section 1 if we take as unit ball the whole of M. The language L contains a relation symbol $d(x, y)$ for a metric and, possibly, symbols for all other functions and relations of M. As all relations of M (including the metric) take values in the interval $[0, 1]$, the function $n_{\text{-}}$ can be set to be the constant 1.

4. Finally, let L be the language of Banach spaces. Here we have only one symbol in $L_{\rm rel}$, the symbol $\|\text{-}\|$ for the norm. The set $L_{\rm fun}$ is the same as for real vector spaces in classical logic. Our model M is a Banach space. The interpretation of the symbols in $L_{\rm rel} \cup L_{\rm fun}$ is the natural one. The unit ball of M is $uM = \{a \in M \ :\ \|a\| \leq 1\}$.

   (Need be completed)

3. Conditions and types

For $\varphi, \psi \in L(M)$ some closed formulas, we write $M \vDash \ \varphi \leq \psi$ for $\varphi^M \leq \psi^M$. The meaning of $M \vDash \ \varphi = \psi$ and of $M \vDash \ \varphi < \psi$ is similar.

Equations of the form $\varphi(x) = 0$ are called conditions. We write $M \vDash \exists x\, \varphi(x) = 0$ if $M \vDash \varphi(a) = 0$ for some $a \in M^{|x|}$. Observe that $\varphi(x) = 0$ is equivalent to $|\varphi(x)| \leq 0$ and $\varphi(x) \leq 0$ is equivalent to $\big(\varphi(x) \vee 0\big) = 0$. So, when convenient we may use weak inequalities to denote conditions.

The negation of a condition is called a co-conditions. So, $\varphi(x) \neq 0$, $\varphi(x) < 0$ or $\varphi(x) > 0$ are co-conditions.

7 **Remark.** It is easy to see that conditions and co-conditions are cosed under logical disjunction and logical conjunction. Condition are also closed under universal quanfication over the unit ball. Closure under existential quanfication is a more subtle point. It is not true in general but it can be ensured by a small amount of saturation (cf. Fact 16).

A set of conditions is called a type or, when x is the empty tuple, a theory. For $p(x) \subseteq L(M)$, we define

$$p(x) \leq 0 \quad = \quad \Big\{\varphi(x) \leq 0 \quad : \quad \varphi(x) \in p\Big\}.$$

Up to equivalence, all type have the form above.

Beware that $p(x)$ denotes just a set of formulas, the expression $p(x) \leq 0$ denotes a type.

We write $M \vDash p(a) \leq 0$ if $M \vDash \varphi(a) \leq 0$ for every $\varphi(x) \in p$. We write $M \vDash \exists x\, p(x) \leq 0$ if $M \vDash p(a) \leq 0$ for some $a \in M^{|x|}$.

4. Elementary maps

The A-limit topology on $M^{|x|}$ is the initial topology with respect to the functions interpret the formulas in $L(A)$. The sets $\{a \in M^{|x|} : \varphi^M(a) \leq 0\}$, as $\varphi(x)$ ranges over $L(A)$, form a base of closed sets for the A-limit topology. Equivalently, we can take the sets $\{a \in M^{|x|} : \varphi^M(a) < 0\}$ as a base of open sets.

The A-limit topology is a completely regular topology, almost by definition. Typically, it is not T_0.

We say that the map $k : M \to N$ is an elementary map if

$$\varphi^M(ka) \;=\; \varphi^N(a) \qquad \text{for every } \varphi(x) \in L \text{ and every } a \in (\mathrm{dom} k)^{|x|}.$$

An elementary embedding is an total elementary map. Note that we are slightly abusing the word embedding, in fact elementary maps need not be injective.

We write $M \equiv N$ when $\varnothing : M \to N$ is an elementary map. In word we say that M and N are elementarily equivalent. We write $M \equiv_A N$ if $\mathrm{id}_A : M \to N$ is an elementary map. In words, we say that M and N are elementary equivalent over A. Finally, we write $M \preceq N$ for $M \equiv_M N$ and say that M is an elementary substructure of N.

8 Proposition (Tarski-Vaught test). Let N be a model and and let $M \subseteq N$ be a substructure. Fix a tuple of variables x of finite length. Then the following are equivalent:

1. $M \preceq N$;
2. $M^{|x|}$ is dense in $N^{|x|}$ w.r.t. the limit M-topology on $N^{|x|}$;
3. for every $\varphi(x) \in L(M)$, if $\varphi(N) < 0$ is non empty, then $N \vDash \varphi(b) < 0$ for some $b \in M^{|x|}$.

Proof. Equivalence 2⇔3 is tautological. Implication 1⇒2 is clear. To prove 3⇒1 we prove by induction on the syntax of $\varphi(x)$ that for every $r \in \mathbb{R}^+$

$$M \vDash \varphi(a) < r \;\Leftrightarrow\; N \vDash \varphi(a) < r \qquad \text{for every } a \in M^{|x|}$$

(The restriction to positive r is no loss of generality.) We use the equivalence above as induction hypothesis. As M is a substructure of N, this equivalence holds for atomic $\varphi(x)$. We leave the case of Riesz connectives to the reader and consider only the case of the infimum quantifier.

$$\begin{aligned}
N \vDash \bigwedge_x \varphi(x,a) < r &\;\Leftrightarrow\; \text{there is a } b \in (uN) \text{ such that } N \vDash \varphi(b,a) < r\\
&\;\Leftrightarrow\; \text{there is a } b \in N^{|y|} \text{ such that } N \vDash (ub < 1 \text{ and } \varphi(b,a) < r)\\
a. \qquad &\;\Leftrightarrow\; \text{there is a } b \in M^{|x|} \text{ such that } N \vDash (ub < 1 \text{ and } \varphi(b,a) < r)\\
b. \qquad &\;\Leftrightarrow\; M \vDash \bigwedge_x \varphi(x) < r.
\end{aligned}$$

We are using that ($ub < 1$ and $\varphi(b,a) < r$) is equivalent to $(ub-1)\cdot\varphi(b,a) < (ub-1)\cdot r$. Implication $\Rightarrow$ in a uses the assumption 3 and the equivalence in b uses the induction hypothesis. □

9 **Remark.** In the Tarski-Vaught test for classical logic it is not necessary to require that M is a substructure of N as this is a consequence of the test. In the continuous case this is necessary by the lack of a predicate for equality. □

10 **Theorem (Downward Löwenheim-Skolem).** For every $A \subseteq N$ there is a model $A \subseteq M \preceq N$ of cardinality $\leq |L(A)|$.

Proof. (Needs some checking.) For every set A there is a base for the A-topology that has cardinality $\leq |L(A)|$. A set M of the required cardinality that is dense in the M-topology is obtained as in the classical downward Löwenheim-Skolem theorem. □

5. Ultrapowers

Firstly, we recall some standard notation about limits. Let I be a non-empty set. Let F be a filter on I. If $r : I \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ we write

$$\lim_{i \uparrow F} r(i) = \lambda$$

if $r^{-1}[A] \in F$ for every $A \subseteq \mathbb{R}$ that is a neighborhood of λ. When r is bounded and F is an ultrafilter, such a λ always exists.

Let I be an infinite set. Let $\langle M_i : i \in I\rangle$ be a sequence of models. Note that all models are required to have the same bounding function (as these are part of the language). Let F be an ultrafilter on I.

11 **Definition.** Write U for the set of all functions $\hat{a} : I \to \bigcup_{i\in I}(uM_i)$ such that $\hat{a}\, i \in (uM_i)$. We define the structure N which we call an ultraproduct of $\langle M_i : i \in I\rangle$. A usual, if $M_i = M$ for all $i \in I$, then we say that N is an ultrapower of M.

1. the elements of N are functions $\hat{c} : I \to \bigcup_{i\in I} M_i$ such that $\hat{c}\, i = t(\hat{a}\, i)$ for some $\hat{a} \in U^{|x|}$ and some term $t(x) \in L_{\rm trm}$ — below we write $t(\hat{a})$ for $\hat{c}$;
2. if $f \in L_{\rm fun}$ and $r \in L_{\rm rel}$ and $t_1(\hat{a}), \dots, t_n(\hat{a})$ are elements of N then
   a. $f^N\big(t_1(\hat{a}), \dots, t_n(\hat{a})\big) = t(\hat{a})$ where t is the term obtained composing f with $t_1, \dots, t_n$;
   b. $r^N\big(t_1(\hat{a}), \dots, t_n(\hat{a})\big) = \lim\limits_{i\uparrow F} r^{M_i}\big(t_1(\hat{a}\, i), \dots, t_n(\hat{a}\, i)\big)$;
   c. $uN = U$.

Note that uN satisfies the axioms in Definition 5.

The limit in 2b of Definition 11 exists because F is an ultrafilter and it is finite because all U_i have the same bounding functions. In fact, the function $i \mapsto r^{M_i}\big(t_1(\hat{a}\, i), \dots, t_n(\hat{a}\, i)\big)$ is bounded by $n_{\varphi(x)}$, where $\varphi(x) = r(t_1(x), \dots, t_n(x))$.

12 Proposition (Łŏś Theorem). Let N be as above and let $\varphi(x) \in L$. Then for every $\hat{a} \in N^{|x|}$,

$$\varphi^N\big(\hat{a}\big) \quad = \quad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i\big).$$

Proof. This is just the usual run-of-the-mill proof. We proceed by induction on the syntax of $\varphi(x)$. If $\varphi(x)$ is atomic holds the equality above holds by definition. We spell out the proof for the infimum quantifier. Assume inductively that

$$\varphi^N\big(\hat{a}, \hat{b}\big) \quad = \quad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big).$$

We want to prove that

$$\Big(\bigwedge_x \varphi\big(\hat{a}, x\big)\Big)^N \quad = \quad \lim_{i \uparrow F} \Big(\bigwedge_x \varphi\big(\hat{a}i, x\big)\Big)^{M_i}$$

By the interpretation of the infimum quantifier, this amounts to prove that

$$\inf\left\{\varphi\big(\hat{a}, \hat{b}\big)^N \ : \ \hat{b} \in (uN)\right\} \quad = \quad \lim_{i \uparrow F} \inf\left\{\varphi\big(\hat{a}i, b\big)^{M_i} \ : \ b \in (uM_i)\right\}$$

First we prove the $\leq$ inequality. Let r be an arbitrary positive real number.

Assume that for some $\hat{b} \in (uN)$ $\quad \varphi^N\big(\hat{a}, \hat{b}\big) \ < \ r.$

By induction hypothesis $\quad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big) \ < \ r,$

then for some $u \in F$ $\quad \varphi^{U_i}\big(\hat{a}i, \hat{b}i\big) \ < \ r$ for every $i \in u$.

Hence $\quad \inf\left\{\varphi^{M_i}\big(\hat{a}i, b\big) \ : \ b \in (uM_i)\right\} \ < \ r$ for every $i \in u$.

Finally, $\quad \lim_{i \uparrow F} \inf\left\{\varphi^{M_i}\big(\hat{a}i, b\big) \ : \ b \in (uM_i)\right\} \ < \ r.$

This proves the $\geq$ inequality. As for the $\leq$ inequality, assume that

$$\lim_{i \uparrow F} \inf\left\{\varphi\big(\hat{a}i, b\big)^{M_i} \ : \ b \in (uM_i)^{|x|}\right\} \quad < \quad r.$$

Therefore, for some $u \in F$

$$\inf\left\{\varphi\big(\hat{a}i, b\big)^{M_i} \ : \ b \in (uM_i)^{|x|}\right\} \quad < \quad r \quad \text{for every } i \in u.$$

$$\inf\left\{\varphi\big(\hat{a}i, b\big)^{M_i} \ : \ b \in (uM_i)^{|x|}\right\} \quad < \quad r \quad \text{for every } i \in u.$$

Therefore, for some $\hat{b} \in (uN)^{|x|}$ $\quad \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big) \ < \ r$ for every $i \in u$.

Then $\quad \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a}i, \hat{b}i\big) \ < \ r,$

and finally $\quad \varphi^N\big(\hat{a}, \hat{b}\big) \ < \ r.$

This proves the desired inequality. □

The following corollary of Łŏś Theorem is identical to its classical counterpart.

13 Corollary. For every model M and every ultrafilter F on I, an infinite set, let N be corresponding ultrapower of M. Then there is an elementary embedding of M in N. □

6. Compactness theorem

From Łǒś Theorem we obtain a form of compactness with the usual argument.

A theory $T \leq 0$ is finitely consistent if for every φ, that is a disjunction of sentences in T, there is a model M such that $M \vDash \varphi \leq 0$. Note that it is implicit that all these models M have the same bounding fuction.

14 Theorem (Compactness Theorem). Every finitely consistent theory is consistent.

Proof. We construct a model N of $T \leq 0$. Let I be the set of closed formulas ξ such that $\xi \leq 0$ holds in some model M_ξ. For every condition $\varphi \leq 0$ define $X_\varphi \subseteq I$ as follows

$$X_\varphi \quad = \quad \Big\{\xi \in I \ : \ \xi \leq 0 \vdash \ \varphi \leq 0\Big\}$$

Note that $\varphi \leq 0$ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi\vee\psi} \ = \ X_\varphi \cap X_\psi$. Then, as T is finitely consistent, the set $B = \big\{X_\varphi \ : \ \varphi \in T\big\}$ has the finite intersection property. Extend B to an ultrafilter F on I. Define

$$N \quad = \quad \prod_{\xi\in I} M_\xi$$

Check that $N \vDash T \leq 0$. Let $\varphi \in T$. By Łǒś Theorem,

$$\varphi^N \quad = \quad \lim_{\xi\uparrow F} \varphi^{M_\xi}.$$

Note that $X_\varphi \subseteq \big\{\xi \ : \ \varphi^{M_\xi} \leq 0\big\}$ and recall that $X_\varphi \in F$ whenever $\varphi \in T$. Therefore $\varphi^N \leq 0$. □

7. Saturation

Let $p(x) \subseteq L(M)$. We say that $p(x) \leq 0$ is finitely satisfied (in uM) if for every formula $\psi(x)$ that is a disjunction of formulas in $p(x)$, there is an $a \in (uM)^{|x|}$ such that $M \vDash \psi(a) \leq 0$.

We say that M is λ-saturated if for every $p(x) \subseteq L(A)$ as in 1 and 2 below, there is an $a \in (uM)^{|x|}$ such that $M \vDash p(a) \leq 0$.

1. $A \subseteq M$ has cardinality $< \lambda$;
2. $p(x) \leq 0$ is finitely satisfied in uM.

If λ is the cardinality of M we say that M is saturated.

15 Theorem (???). Let λ be an inaccessible cardinal larger than $|L|$ and $|M|$. Then M has a saturated elementary superstructure of cardinality M.

We conclude with an interesting property of ω-saturated models.

16 Fact. Let M is ω-saturated. Then for every $\varphi(x) \in L(M)$ the following are equivalent

1. there is an $a \in uM$ such that $M \vDash \varphi(a) \leq 0$;
2. $M \vDash \bigwedge_x \varphi(x) \leq 0$

Proof. Only 2⇒1 requires a proof. Let $p(x) = \{\varphi(x) - r : r \in \mathbb{R}^+\}$. If 2, then $p(x) \leq 0$ is finitely satisfied in uM. Hence 1 follows by saturation. □