

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A CLASS OF STRUCTURES

As a motivating example, consider real vector spaces. These are among the most simple structures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted structure. Ideally, we would like that elementary extensions of normed spaces maintain the usual notion of real numbers¹. Unfortunately, this is not possible if we insist to maintain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 Definition. Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L' -structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L -structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

- i. $R^n \rightarrow R$;
- ii. $M^n \rightarrow M$;
- iii. $M^n \rightarrow R$.

The interpretation of symbols of sort $R^n \rightarrow R$ is required to be continuous, the interpretation of symbols of sort $M^n \rightarrow R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n .

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set \mathbb{L} of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an \mathbb{L} -elementary extension that is \mathbb{L} -saturated in the same class.

¹Non standard standard analysts have no problem in expanding \mathbb{R} . In these notes, for a change, we insist as \mathbb{R} (or, in general R) should remain \mathbb{R} throughout.

For convenience we assume that the functions of sort $M^n \rightarrow M$ and the relations of sort M^n are all in L . It is also convenient to assume that L' contains names for all continuous bounded functions $R^n \rightarrow R$.

2 Definition. Formulas in \mathbb{L} are constructed inductively from the following two sets of formulas

- i. formulas of the form $t(x; y) \in C$, where $C \subseteq R^n$ is compact² and $t(x, y)$ is a n -tuple of terms of sort $M^{|x|} \times R^{|y|} \rightarrow R$;
- ii. all formulas in L .

We require that \mathbb{L} is closed under the Boolean connectives \wedge, \vee ; the quantifiers \forall, \exists of sort M ; and the quantifiers \forall^C, \exists^C , by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

2. HENSON-IOVINO APPROXIMATIONS

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C . If no such atomic formulas occurs in φ , then $\varphi > \varphi$. By the normality of R we also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C .

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \rightarrow \varphi'$. In fact, we have the following.

3 Proposition (fake theorem). For every $\varphi(x) \in \mathbb{L}(M)$

$$\varphi(x) \leftrightarrow \bigwedge_{\varphi' > \varphi} \varphi'(x)$$

Proof. The claim is clear for atomic formulas. Induction for conjunction and the universal quantifiers is immediate.

Consider disjunction. Let $\varphi = \varphi_i \vee \varphi_1$. Assume inductively

$$\varphi_i(x) \leftrightarrow \bigwedge_{\varphi'_i > \varphi_i} \varphi'_i(x)$$

Then

$$\varphi(x) \leftrightarrow \bigwedge_{\varphi'_1 > \varphi_1} \varphi'_1(x) \vee \bigwedge_{\varphi'_2 > \varphi_2} \varphi'_2(x)$$

²We confuse the relation symbols in L' of sort R^m with their interpretation and write $t \in C$ for $C(t)$

$$\begin{aligned} &\leftrightarrow \bigwedge_{\substack{\varphi'_1 > \varphi_1 \\ \varphi'_2 > \varphi_2}} \varphi'_1(x) \vee \varphi'_2(x) \\ &\leftrightarrow \bigwedge_{\varphi' > \varphi} \varphi'(x) \end{aligned}$$

Now consider the existential quantifiers of sort M □

4 Lemma. (R compact) For every $\varphi' > \varphi$ there is a formula $\psi \in \mathbb{L}(M)$ such that $\varphi \rightarrow \neg\psi \rightarrow \varphi'$.

Proof. By induction. If $\varphi \in L$ the claim is obvious. Suppose φ is of the form $t \in C$. Then φ' is $t \in D$ for some compact neighborhood of C . Let O be an open set such that $C \subseteq O \subseteq D$. Then $\psi = (t \in R \setminus O)$ is as required.

Induction is easy (conjunctions and disjunctions in φ occur swapped in ψ ; similarly universal and existential quantifiers). □

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . Let Y be a topological space. If $f : I \rightarrow Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ . Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The lemma below is required for the proof of Łoś Theorem in the next section. But first a preliminary fact.

5 Fact. Assume the following data

- $t(x; y)$, a term of sort $M^{|x|} \times R^{|y|} \rightarrow R$;
- $\langle a_i : i \in I \rangle$, a sequence of elements $a_i \in M^{|x|}$;
- $\langle \alpha_i : i \in I \rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I .

Then

$$\# \quad F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i)$$

Proof. By induction on the nesting of functions of sort $R^n \rightarrow R$. In the basis case t is either a term of sort $M^{|x|} \rightarrow R$ or a variable of sort R . In both cases (#) is trivial. Assume inductively that the fact holds for the n -tuple $t(x, y)$. By the continuity of the functions of sort $R^n \rightarrow R$ and the induction hypothesis, (#) holds for $f(t(x, y))$. □

6 Lemma. Assume the following data

- $\varphi(x; y) \in \mathbb{L}$;
- $\langle a_i : i \in I \rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I \rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I .

Then the following are equivalent

- 1 $\{i \in I : \mathcal{M} \models \varphi(a_i; \alpha)\} \in F$;
- 2 $\{i \in I : \mathcal{M} \models \varphi'(a_i; \alpha_i)\} \in F$ for every $\varphi' > \varphi$.

Proof. By induction on the syntax. When $\varphi(x; y)$ is in L , it does not depend on α , and the lemma is trivial. Suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume $\{i : t^{\mathcal{M}}(a_i; \alpha) \in C\} \in F$. Then $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha) \in C$ by normality of R . By Fact 5, $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i) \in C$. Then $\{i : t^{\mathcal{M}}(a_i; \alpha_i) \in D\} \in F$ for every D that is a compact neighborhood of C .

Vice versa, assume $\{i : t^{\mathcal{M}}(a_i; \alpha_i) \in D\} \in F$ for every D that is a compact neighborhood of C . Then, again by normality, $F\text{-}\lim_i t^{\mathcal{M}}(a_i; \alpha_i) \in C$, and (1) follows by Fact 5.

This completes the proof of the base case of the induction.

Induction clear for the connectives \vee, \wedge . To deal with the universal quantifier of sort M we assume inductively that

$$\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F \quad \text{for every } \varphi' > \varphi.$$

We prove

1. $\{i : \mathcal{M} \models \forall y \varphi(a_i, y; \alpha)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \forall y \varphi'(a_i, y; \alpha_i)\} \in F \quad \text{for every } \varphi' > \varphi.$

Negate the r.h.s. of the equivalence. Then there are a formula $\varphi' > \varphi$ and a sequence $\langle b_i : i \in I \rangle$ such that (using that F is an ultrafilter)

$$\{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F$$

Assume the l.h.s. of (1) and reason for a contradiction. Then $\{i : \mathcal{M} \models \varphi(a_i, b_i; \alpha)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F$. A contradiction that proves the implication (\Rightarrow) in (1). The converse implication is obtained by contraposition from the induction hypothesis.

To deal with the existential quantifier of sort M we prove

2. $\{i : \mathcal{M} \models \exists y \varphi(a_i, y; \alpha)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \exists y \varphi'(a_i, y; \alpha_i)\} \in F \quad \text{for every } \varphi' > \varphi.$

Assume the r.h.s. of the implication and fix $\varphi' > \varphi$. Then there is a sequence $\langle b_i : i \in I \rangle$ such that

$$\{i : \mathcal{M} \models \varphi'(a_i, b_i; \alpha_i)\} \in F$$

By the induction hypothesis we obtain that $\{i : \mathcal{M} \models \exists y \varphi'(a_i, y; \alpha_i)\} \in F$. As $\varphi' > \varphi$ is arbitrary the implication (\Rightarrow) in (2) follows from Proposition 3. The converse implication is easy.

Induction for the quantifiers \forall^C and \exists^C is virtually identical, but we repeat the argument for convenience. Assume inductively that for every $\varphi' > \varphi$ and every sequence $\langle \beta_i : i \in I \rangle$ in C such that $\beta = F\text{-}\lim_i \beta_i$

$$\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F \quad \text{for every } \varphi' > \varphi.$$

We prove

$$3. \quad \{i : \mathcal{M} \models \forall^C y \varphi(a_i; \alpha, y)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \forall^C y \varphi'(a_i; \alpha_i, y)\} \in F \quad \text{for every } \varphi' > \varphi.$$

Negate the r.h.s. of the implication. Then there is a sequence $\langle \beta_i : i \in I \rangle$ such that (again using that F is an ultrafilter)

$$\{i : \mathcal{M} \not\models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$$

As C is compact, $F\text{-}\lim_i \beta_i$ exists, say it equals $\beta \in C$. Assume the l.h.s. of the implication and reason for a contradiction. Then $\{i : \mathcal{M} \models \varphi(a_i; \alpha, \beta)\} \in F$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi'(a_i; \alpha_i, \beta_i)\} \in F$. A contradiction that proves the implication (\Rightarrow) in (3). The converse implication is proved similarly by contraposition..

Finally we prove

$$4. \quad \{i : \mathcal{M} \models \exists^C y \varphi(a_i; \alpha, y)\} \in F \Leftrightarrow \{i : \mathcal{M} \models \exists^C y \varphi'(a_i; \alpha_i, y)\} \in F \quad \text{for every } \varphi' > \varphi.$$

□

3. ULTRAPRODUCTS

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that for each function symbol f of sort $M^n \rightarrow R$

there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two simplifications: (1) we only consider ultratowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F . The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

7 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \rightarrow M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i) : i \in I \rangle$.

3. The interpretation of functions of sort $R^n \rightarrow R$ remains unchanged.
4. If f is a function of sort $M^n \rightarrow R$ then

$$f^{\mathcal{N}}(\hat{a}) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i).$$

5. If r is a relation symbol of sort M^n then

$$\mathcal{N} \models r(\hat{a}) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i)\} \in F.$$

6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the uniformity requirement (#) above.

8 Fact. The structure \mathcal{N} satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

9 Fact. If $t(x)$ is a term of type $M^{|x|} \rightarrow M$ then

$$t^{\mathcal{N}}(\hat{a}) = \langle t^{\mathcal{M}}(\hat{a}i) : i \in I \rangle.$$

□

We also have that

10 Fact. If $t(x; y)$ has sort $M^{|x|} \times R^{|y|} \rightarrow R$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Proof. By induction. If t is a function symbol then x or z do not occur in t . If x does not occur in t the claim is trivial. If y does not occur in t the claim holds by definition. Finally, induction is clear by the continuity of the functions of sort $R^n \rightarrow R$. □

Finally, we prove

11 Proposition (Łoś Theorem). Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x; y) \in L$ then the theorem reduces to the classical Łoś Theorem. Then, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. By local compactness of R there is a compact neighborhood of C , such that $t^{\mathcal{N}}(\hat{a}, \alpha) \notin D$. By Fact 10 and the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives \vee and \wedge is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

$$1. \quad \mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F$. Choose \hat{b} such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$. If for a contradiction $\mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha)$, then in particular $\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha)$. By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ a contradiction. This completes the proof of (1).

Induction for the existential quantifier of sort M is similar (dual) to the proof above.

Finally, to deal with the quantifier \forall^C we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

$$3. \quad \mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F \text{ for every } \varphi' > \varphi.$$

(\Leftarrow) Assume $\mathcal{N} \not\models \varphi(\hat{a}; \alpha, \beta)$ for some β . By induction hypothesis, $\{i : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\{i : \mathcal{M} \models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \notin F$ as required.

(\Rightarrow) Assume that $\{i : \mathcal{M} \not\models \forall^C y \varphi'(\hat{a}i; \alpha, y)\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\{i : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha, \beta_i)\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \models \forall^C y \varphi(\hat{a}; \alpha, y)$, then $\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta)$ and, by induction hypothesis, $\{i : \mathcal{M} \models \varphi''(\hat{a}i; \alpha, \beta)\} \in F$ for every $\varphi'' > \varphi$. Fix φ'' such that $\varphi' > \varphi'' > \varphi$ and obtain a contradiction from Lemma 6. This completes the proof of (3).

Induction for the quantifier \exists^C is again dual to the proof above. □

4. \mathbb{L} -ELEMENTRARITY

Let $\mathcal{M} = \langle M, R \rangle$ and $\mathcal{N} = \langle N, R \rangle$ be two structures. We say that $f : M \rightarrow N$, a partial map, is an \mathbb{L} -elementary map if

$$\mathcal{M} \models \varphi(a) \Rightarrow \mathcal{N} \models \varphi(a) \quad \text{for every } \varphi(x) \in \mathbb{L} \text{ and every } a \in (\text{dom } f)^{|x|}$$

The following fact is proved as in the classical case using that $\mathcal{M} \models \varphi$ holds if and only if $\mathcal{M} \models \varphi'$ holds for every $\varphi' > \varphi$.

12 Fact. If \mathcal{N} is an ultrapower of \mathcal{M} then there is an \mathbb{L} -elementary embedding of \mathcal{M} into \mathcal{N} .

For $A \subseteq M \cap N$, we say that \mathcal{M} and \mathcal{N} are \mathbb{L} -(elementary) equivalent over A and write $\mathcal{M} \equiv_A^{\mathbb{L}} \mathcal{N}$ if $\text{id}_A : M \rightarrow N$ is \mathbb{L} -elementary. We write $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \equiv_M^{\mathbb{L}} \mathcal{N}$. In words, we say that \mathcal{M} is an \mathbb{L} -(elementary) substructure of \mathcal{N} .

5. \mathbb{L} -COMPACTNESS

A theory $T \subseteq \mathbb{L}$ be a weakly finitely consistent if for every φ , conjunction of sentences in T , any $\varphi' > \varphi$ is consistent. The following proposition follows from Łoś Theorem by the usual argument.

13 Proposition (Compactness Theorem). Let $T \subseteq \mathbb{L}$ be an weakly finitely consistent theory. Then T is consistent. \square

Let $p(x) \subseteq \mathbb{L}(M)$. We say that $p(x)$ is weakly finitely satisfied in \mathcal{M} if $\mathcal{M} \models \exists x \varphi'(x)$ for every $\varphi' > \varphi$ where $\varphi(x)$ is some conjunction of formulas in $p(x)$.

14 Definition. We say that \mathcal{M} is \mathbb{L} -saturated if for every $p(x)$ as in 1 and 2 below, $\mathcal{M} \models \exists x p(a)$.

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is weakly finitely satisfied in \mathcal{M} .

The existence of \mathbb{L} -saturated models is proved as in the classical case.

15 Proposition. Every model has an \mathbb{L} -elementary extension to a saturated model (possibly of inaccessible cardinality). \square

We denote by $\mathcal{U} = \langle U, R \rangle$ some large \mathbb{L} -saturated structure which we call the monster model. The cardinality of \mathcal{U} is an inaccessible cardinal that we denote by κ . Below we say model for \mathbb{L} -elementary substructure of \mathcal{U} .

16 Fact. For every $\varphi(x), \psi(x) \in \mathbb{L}(U)$ such that $\neg \exists x [\varphi(x) \wedge \psi(x)]$ there are $\varphi' > \varphi$ and $\psi' > \psi$ such that $\neg \exists x [\varphi'(x) \wedge \psi'(x)]$.

Proof. By Proposition 3 and saturation. \square

6. THE TARSKI-VAUGHT \mathbb{L} -TEST

The following lemma is a useful observation. It has its first application below, in the proof of our version of the Tarski-Vaught test.

17 Lemma. (R compact) Let $A \subseteq N$. Then the following are equivalent

- 1 for every formula $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{N} \models \exists x \neg \varphi(x) \Rightarrow \text{there is an } a \in A \text{ such that } \mathcal{N} \models \neg \varphi(a);$$

2. for every formula $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{N} \models \exists x \varphi(x) \Rightarrow \text{for every } \varphi' > \varphi \text{ there is an } a \in A \text{ such that } \mathcal{N} \models \varphi'(a).$$

The implication $2 \Rightarrow 1$ does not require the assumption of compactness.

Proof. ($2 \Rightarrow 1$) Assume $\mathcal{N} \models \exists x \neg \varphi(x)$. Pick any $c \in N$ such that $c \models \neg \varphi(x)$. Let $p(x) = \mathbb{L}\text{-tp}(c/A)$. Without loss of generality we can assume that \mathcal{N} is saturated. As $p(x) \cup \{\neg \varphi(x)\}$ is inconsistent there is a formula $\psi(x) \in p(x)$ such that $\psi'(x) \rightarrow \neg \varphi(x)$ for some $\psi' > \psi$. Then from (2) we obtain that $\mathcal{N} \models \psi'(a)$ for some $a \in A$. Hence $\mathcal{N} \models \neg \varphi(a)$ follows.

($1 \Rightarrow 2$) follows immediately from Lemma 4. □

18 Proposition (Tarski-Vaught \mathbb{L} -Test). Let M be a subset of N . Then the following are equivalent

1. M is the domain of a structure $\mathcal{M} \leq^{\mathbb{L}} \mathcal{N}$;

2. for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\mathcal{N} \models \exists x \varphi(x) \Rightarrow \text{for every } \varphi' > \varphi \text{ there is an } a \in A \text{ such that } \mathcal{N} \models \varphi'(a).$$

Proof. Assume (2) of the lemma. Then, by the classical Tarski-Vaught test $M \leq N$. We also have that \mathcal{M} is an L' -substructure of \mathcal{N} . Therefore, from every formula $\varphi(x)$ as in (i) and (ii) of Definition 2 we have

$$\mathcal{N} \models \varphi(a) \Leftrightarrow \mathcal{M} \models \varphi(a) \quad \text{for every } a \in M^{|x|}.$$

Assume inductively

$$\# \quad \mathcal{N} \models \varphi(a, b) \Leftrightarrow \mathcal{M} \models \varphi(a, b)$$

and prove

$$\mathcal{N} \models \exists y \varphi(a, y) \Leftrightarrow \mathcal{M} \models \exists y \varphi(a, y).$$

The implication \Leftarrow follows immediately from the induction hypothesis. For the converse note that

$$\begin{aligned} \mathcal{N} \models \exists y \varphi(a, y) &\Rightarrow \text{for every } \varphi' > \varphi \text{ there is an } b \in M^{|y|} \text{ such that } \mathcal{N} \models \varphi'(a, b) \\ &\Rightarrow \mathcal{M} \\ &\Rightarrow \mathcal{M} \models \exists y \varphi'(a, y) \text{ for every } \varphi' > \varphi \\ &\Rightarrow \mathcal{M} \models \exists y \varphi(a, y) \end{aligned}$$

Now, assuming (#) we prove

$$\mathcal{N} \models \forall y \varphi(a, y) \Leftrightarrow \mathcal{M} \models \forall y \varphi(a, y).$$

One implication, \Rightarrow , is again immediate. For the converse note that by Lemma 17

$$\begin{aligned}
\mathcal{N} \not\models \forall y \varphi(a, y) &\Rightarrow \mathcal{N} \models \exists y \neg \varphi(a, y) \\
&\Rightarrow \mathcal{N} \models \neg \varphi(a, b) \text{ for some } b \in M^{|y|} \\
&\Rightarrow \mathcal{M} \\
&\Rightarrow \mathcal{M} \not\models \forall y \varphi(a, y)
\end{aligned}$$

□

Finally, induction for the boolean connectives and the quantifiers of sort R is immediate.

7. COMPLETENESS

(Work in progress)

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \rightarrow R$.

19 Fact. If $\bar{a} = \langle a_i : i < \lambda \rangle$ and $\bar{b} = \langle b_i : i < \lambda \rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \rightarrow R$.

Proof. By induction on λ . Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ . For limit ordinals induction is immediate. □

20 Example (???). Let L be the language of \mathbb{R} -algebras expanded with two lattice operators \wedge, \vee . Let $\langle \Omega, \mathcal{B}, \text{Pr} \rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L . Let $R = \mathbb{R}$. Assume L' extends L with all the continuous functions $\mathbb{R}^n \rightarrow \mathbb{R}$ and for every $n \in \mathbb{N}$ the functions E_n that gives the expected value of $(X \wedge n) \vee -n$ for $n \in \mathbb{N}$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is **definable in the limit** over M if $a \equiv_M^L x \rightarrow a \sim x$

21 Example. Assume that $R = \mathbb{R}$ and that L' contains a function of sort $M^2 \rightarrow R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \rightarrow R$ are continuous with respect to this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M , a model;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of M that converges to a .

Proof. (2 \Rightarrow 1) Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_M a$. By the uniqueness of the limit $d(a, b) = 0$.

(1 \Rightarrow 2) Assume that $a \equiv_M x \rightarrow a \sim x$. Then $a \equiv_M^{\mathbb{L}} x \rightarrow d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \text{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \rightarrow d(a, x) < 1/n$. By \mathbb{L} -elementarity there is an $a_n \in M$ such that $\varphi_n(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a . \square