CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A class of structures

1 Definition. We will work with L-structures of the form $\mathcal{M} = \langle M, R\rangle$, where R is fixed. Moreover, we are given for each n a Hausdorff topology on R^n. We assume that all compact sets of R^n are definable in L. We require that the functions symbols of sort $M^n \times R^m \to R$ are interpreted in functions such that

1? $f^{\mathcal{M}}(a, \text{-}) : R^m \to R$ is continuous uniformly in $a \in M^n$, that is,

$$\bigcap_{a\in M^n} f^{\mathcal{M}}(a, \text{-})^{-1}(O) \text{ is open for every open } O \subseteq R.$$

2? $f^{\mathcal{M}}(\text{-}, \alpha) : M^n \to R$ is bounded (i.e. its range is contained in a compact set) for every $\alpha \in R^m$.

Caveat: occasionally, we work with totopogical spaces that are not T_0. When we say that such a space is Hausdorff, we mean that its Kolmogorov quotient is Hausdorff.

There may be no saturated structure among those described in Definition 1. As a remedy, we restrict the notion of saturation to formulas in a set $\mathbb{L}$ that we define below. In a nutshell, formulas in $\mathbb{L}$ do not contain existential quantifiers of sort R. Moreover, we require that some *critical* function symbols only occour only inside compact relations and never under the scope of negation.

Intuitively, the function symbols are divided in three classes according to their sort

1. $M^n \times R^m \to M$
2. $M^0 \times R^m \to R$
3. $M^{n+1} \times R^m \to R$

Functions in the latter group are the *critical* one and will be dealt with caution.

Below, $\mathbb{T}$ denotes the set of the terms of sort $M^n \times R^m \to R$. Definition 1 ensures that for all $t(x; y) \in \mathbb{T}$ the range of the functions $t^{\mathcal{M}}(\text{-}, \alpha) : M^n \to R$ is bounded.

2 Definition. Formulas in $\mathbb{L}$ are defined inductively as follows

i. $\mathbb{L}$ contains all formulas without quantifiers of sort R and function of sort $M^{n+1} \times R^m \to R$;

ii. $\mathbb{L}$ contains formulas of the form $t(y; z) \in C$ where $C \subseteq R^n$ is compact and $t(y; z) \in \mathbb{T}^n$;

iii. $\mathbb{L}$ is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort M;

iv. $\mathbb{L}$ is closed under the quantifier $\forall$ of sort R.

2. ULTRAPOWERS

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. If $f : I \to R^m$ and $\lambda \in R^m$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq R^m$ that is a neighborhood of λ. Such a λ is always unique (by the property of filters). When F is an ultrafilter, and range$f$ is bounded (i.e. it is contained in a compact set), the limit always exists.

Below we introduce a suitable notion of ultrapower. To keep notation tidy, we make two simplifications: (1) we only consider ultraproducts; (2) we do not consider formulas containing equality, so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

3 Definition. We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \times R^m \to M$ then $f^{\mathcal{N}}(\hat{a}; \alpha)$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i; \alpha\big) : i \in I\rangle$.
4. If f is a function of sort $M^n \times R^m \to R$ then

$$f^{\mathcal{N}}\big(\hat{a}; \alpha\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i; \alpha\big).$$

4 Fact. The structure $\mathcal{N}$ satisfies the conditions of Definition 1.

The following fact is easily proved by induction on the syntax.

5 Fact. If no function symbol of sort $M^{n+1} \times R^m \to R$ occurs in $t(y; z)$ then

$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \;=\; \langle t^{\mathcal{M}}\big(\hat{a}i; \alpha\big) : i \in I\rangle. \qquad \square$$

If true, the proof of the following fact is tedious.

6 Fact. For every term $t(y; z)$ of sort $M^n \times R^m \to R$

$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i; \alpha\big). \qquad \square$$

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' has been obtained replacing each atomic formulas $t \in C$ occurring in φ by $t \in D$ for some closed neighborhood of C. Note that, if such atomic formulas do not occur in φ, then $\varphi > \varphi$.

Finally, we prove

7 Proposition (Łǒś Theorem). Assume $|I| \geq$ the weight of the topology on R. Let $\mathcal{N}$ be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$ and all $\alpha \in R^{|y|}$

$$\mathcal{N} \vDash \varphi(\hat{a};\alpha) \;\;\Leftrightarrow\;\; \big\{i \in I \;\;:\;\; \mathcal{M} \vDash \varphi'(\hat{a}i;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x;y)$ is as in (i) of Definition 2 then, after absorbing α into le language, the proposition reduces to the classical Łǒś Theorem.

Now, suppose that $\varphi(x;y)$ is as in (ii) of Definition 2, say it is the formula $t(x;y) \in C$, where $t(x;y)$ is a tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$.

Assume $\mathcal{N} \vDash t(\hat{a},\alpha) \in C$. If D a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a},\alpha)$. Hence, by the definition of F-limit, $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i,\alpha) \in D\big\} \in F$.

Vice versa, assume $\mathcal{N} \vDash t(\hat{a},\alpha) \notin C$. We can assume to work inside a compact set containing C and $t(\hat{a},\alpha)$. By regularity, $t^{\mathcal{N}}(\hat{a},\alpha) \notin D$ for D a compact neigboorhood of C. By the definition of F-limit, $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i,\alpha) \notin D\big\} \in F$. Then $\big\{i \,:\, \mathcal{M} \vDash t(\hat{a}i,\alpha) \in D\big\} \notin F$.

As for induction, only the existential quantifier require some work. Inductively we assume that

$$\mathcal{N} \vDash \varphi(\hat{a},\hat{b}\,;\alpha) \;\;\Leftrightarrow\;\; \big\{i \in I \;\;:\;\; \mathcal{M} \vDash \varphi'(\hat{a}i,\hat{b}i\,;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

We need to prove

$$\mathcal{N} \vDash \exists y\, \varphi(\hat{a},y\,;\alpha) \;\;\Leftrightarrow\;\; \big\{i \in I \;\;:\;\; \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i,y\,;\alpha)\big\} \in F \text{ for every } \varphi' > \varphi.$$

The implication $\Rightarrow$ is clear. For the converse, assume the r.h.s. Assume for simplicity that $I = \omega$. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for for every $i < j$ and

$$\mathcal{M} \;\;\vDash\;\; \bigwedge_{i \in I} \varphi_i(x,y\,;\alpha) \;\to\; \varphi(x,y\,;\alpha).$$

For every $j \in I$ there is a sequence $\hat{b}_j$ such that

$$\big\{i \in I \;\;:\;\; \mathcal{M} \vDash \varphi_j(\hat{a}i,\hat{b}_j i\,;\alpha)\big\} \in F.$$

As $\varphi_j(x,y\,;\alpha) \to \varphi_{j'}(x,y\,;\alpha)$ holds for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_i i \,:\, i \in I\rangle$ satisfies $\mathcal{N} \vDash \varphi(\hat{a},\hat{d}\,;\alpha)$. □