

CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

1. A CLASS OF STRUCTURES

1 Definition. Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L' -structures of the form $\mathcal{M} = \langle M, R \rangle$, where M ranges over L -structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

The relation symbols of L' are those of L plus symbols for every compact $C \subseteq R^m$. We write $x \in C$ for $C(x)$. In L' there are function symbols of any possible sort that we divide in to clades

1. $M^n \times R^m \rightarrow M$
2. $M^n \times R^m \rightarrow R$

We require that function symbols of the second clade are interpreted in functions such that

- i. $f^{\mathcal{M}} : M^n \times R^m \rightarrow R$ is bounded, i.e. its range is contained in a compact set;
- ii. $f^{\mathcal{M}} : M^n \times R^m \rightarrow R$ is continuous uniformly in M , as defined in Definition 3 below.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in this class. As a remedy, below we carve out a set of formulas $L \subseteq \mathbb{L} \subseteq L'$ such that every model as in Definition 1 has an \mathbb{L} -elementary extension in the same class that is \mathbb{L} -saturated. Our choice of \mathbb{L} is not the most general.

In a nutshell, we require that function symbols of the second clade above may not occur under the scope of a negation. We do not allow existential quantifiers of sort R .

The ultrapower construction in Section 3 is completely classical w.r.t. functions the first clade. On the other hand, functions of the second clade are *critical* and need to be dealt with differently.

Below, \mathbb{T}_M denotes the set of the terms obtained composing function symbols as in 1 above.

By \mathbb{T}_R we denote the set of the terms obtained composing function symbols as in 2 above.

By $\mathbb{T}_R \circ \mathbb{T}_M$ we denote the set of the terms obtained composing terms in \mathbb{T}_R with terms in \mathbb{T}_M . Clearly, there are many more terms in L' but for simplicity, we will restrict to these.

2 Definition. Formulas in \mathbb{L} are constricted inductively from

- i. formulas as $t(y; z) \in C$, where $C \subseteq R^n$ is compact and $t(y; z)$ is a tuple of terms in $\mathbb{T}_R \circ \mathbb{T}_M$;
- ii. formulas with only terms in \mathbb{T}_M and no quantifiers;

Finally we require that \mathbb{L} is closed under the Boolean connectives \wedge, \vee , the quantifiers \forall, \exists of sort M ; and the universal quantifier \forall of sort R .

Clearly, $L \subseteq \mathbb{L} \subseteq L'$.

2. TOPOLOGICAL TOOLS

We recall the standard definition of F -limits. Let I be a non-empty set. Let F be a filter on I . Let Y be a topological space. If $f : I \rightarrow Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ . Such a λ is unique if Y is Hausdorff. When F is an ultrafilter, and Y is compact the limit always exists.

By βI we denote the set of ultrafilters on I . This is a topological space with the topology generated by the clopen sets $[A] = \{F \in \beta I : A \in F\}$ as A ranges over the subsets of I .

3 Definition. Let I be a pure set. Let X, Y be topological spaces, and assume Y is compact.

We say that $f : I \times X \rightarrow Y$ is **continuous uniformly in I** if the function below is continuous

$$\begin{aligned} \bar{f} : \beta I \times X &\rightarrow Y \\ \langle F, \alpha \rangle &\mapsto F\text{-}\lim_i f(i, \alpha) \end{aligned}$$

It is convenient to unravel the definition above in a more practicable form.

4 Proposition. The following are equivalent

1. $f : I \times X \rightarrow Y$ is continuous uniformly in I .
2. For every $F \in \beta I$ and $\alpha \in X$, and for every $V \subseteq Y$ open neighborhood of $F\text{-}\lim_i f(i, \alpha)$ there is $U \subseteq X$ open neighborhood of α and an $A \in F$ such that $f(i, x) \in V$ for every $i \in A$ and every $x \in U$.

3. ULTRAPOWERS

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I \rangle$. We require that for each function symbol f of sort $M^n \times R^m \rightarrow R$ there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$. Note that, otherwise, Fact 6 below may fail.

To keep notation tidy, we make two simplifications: (1) we only consider ultrapowers; (2) we do not consider formulas containing equality, so we need not take any quotient.

Let I be an infinite set. Let F be an ultrafilter on I . Let $\mathcal{M} = \langle M, R \rangle$ be an L -structure.

5 Definition. We define a structure $\mathcal{N} = \langle N, R \rangle$ that we call the **ultrapower** of \mathcal{M} .

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \rightarrow M$.
2. If f is a function of sort $M^n \times R^m \rightarrow M$ then $f^{\mathcal{N}}(\hat{a}; \alpha)$ is the sequence $\langle f^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle$.
4. If f is a function of sort $M^n \times R^m \rightarrow R$ then for every $\alpha \in R^m$

$$f^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \alpha).$$
3. if r is a relation symbol of sort $M^n \times R^m$ then for every $\alpha \in R^m$

$$\mathcal{N} \models r(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models r(\hat{a}i; \alpha)\} \in F.$$

The following is immediate.

6 Fact. The structure \mathcal{N} satisfies condition 1 of Definition 1.

The following is easily proved by induction on the syntax

7 Fact. If $t(y; z) \in \mathbb{T}_M$ then

$$t^{\mathcal{N}}(\hat{a}; \alpha) = \langle t^{\mathcal{M}}(\hat{a}i; \alpha) : i \in I \rangle.$$

□

We also have that

8 Fact. If $t(y; z) \in \mathbb{T}_R$ then

$$\#_0 \quad t^{\mathcal{N}}(\hat{a}; \alpha) = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha).$$

Proof. By induction. The fact holds by definition if t is a function symbol. Therefore, we assume $\#_0$ as induction hypothesis and prove

$$\#_1 \quad f^{\mathcal{N}}(\hat{a}; t^{\mathcal{N}}(\hat{a}; \alpha)) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; t^{\mathcal{M}}(\hat{a}i; \alpha)).$$

By induction hypothesis

$$\#_2 \quad f^{\mathcal{N}}(\hat{a}; t^{\mathcal{N}}(\hat{a}; \alpha)) = F\text{-}\lim_i f^{\mathcal{M}}(\hat{a}i; \lambda).$$

where

$$\lambda = F\text{-}\lim_i t^{\mathcal{M}}(\hat{a}i; \alpha)$$

Let σ be the limit in $\#_2$. Let $V \subseteq R$ be an open set containing σ . Apply 2 of Proposition 4, to obtain $A \in F$ and an open set $U \subseteq R$ containing λ such that for every $i \in A$ and every $x \in U$

$$f^{\mathcal{M}}(\hat{a}i; x) \in V.$$

After possibly restricting A , we can assume that for every $i \in A$

$$t^{\mathcal{M}}(\hat{a}i; \alpha) \in U.$$

This suffices to prove that σ is also the limit in $\#_1$. \square

From the two facts above it immediately follows that

9 Fact. If $t(y; z) \in \mathbb{T}_R \circ \mathbb{T}_M$ then

$$t^N(\hat{a}; \alpha) = F\text{-}\lim_i t^M(\hat{a}i; \alpha). \quad \square$$

For $\varphi(x), \varphi'(x) \in \mathbb{L}(A)$ we say that $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C . Note that, if no such atomic formulas occurs in φ , then $\varphi > \varphi$.

Finally, we prove Łoś Theorem.

10 Proposition. Assume $|I| \geq$ the weight of the topology on R . Let \mathcal{N} be as above and let $\varphi(x; y) \in \mathbb{L}$. Then for all $\hat{a} \in N^{[x]}$ and all $\alpha \in R^{[y]}$

$$\mathcal{N} \models \varphi(\hat{a}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. First note that if $\varphi(x; y)$ is as in (ii) of Definition 2 then, if we absorb α into the language, the proposition reduces to the classical Łoś Theorem.

Now, suppose that $\varphi(x; y)$ is as in (i) of Definition 2, say it is the formula $t(x; y) \in C$, where $t(x; y) \in \mathbb{T}_R \circ \mathbb{T}_M$.

Assume $\mathcal{N} \models t(\hat{a}, \alpha) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^N(\hat{a}, \alpha)$. Hence, by the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \in D\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}, \alpha) \notin C$. Let D be a compact neighborhood of C such that $t^N(\hat{a}, \alpha) \notin D$. By Fact 9 and the definition of F -limit, $\{i : \mathcal{M} \models t(\hat{a}i, \alpha) \notin D\} \in F$.

As for induction, the connective \vee and \wedge are clear. To deal with the universal quantifier of sort R we assume that

$$\mathcal{N} \models \varphi(\hat{a}; \alpha, \beta) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i; \alpha, \beta)\} \in F \text{ for every } \varphi' > \varphi.$$

and we prove that

$$\mathcal{N} \models \forall y \varphi(\hat{a}; \alpha, y) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i; \alpha, y)\} \in F \text{ for every } \varphi' > \varphi.$$

The implication \Rightarrow is clear. For the converse assume $\mathcal{N} \not\models \varphi(\hat{a}; \alpha, \beta)$ for some β . By induction hypothesis, and the requirement that F to be an ultrafilter, $\{i \in I : \mathcal{M} \not\models \varphi'(\hat{a}i; \alpha, \beta)\} \in F$ for some $\varphi' > \varphi$. A fortiori $\{i \in I : \mathcal{M} \not\models \forall y \varphi'(\hat{a}i; \alpha, y)\} \in F$, hence the implication \Leftarrow follows.

To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \models \varphi(\hat{a}, \hat{b}; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove that

$$\mathcal{N} \models \forall y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

This is identical to the case above. The implication \Rightarrow is clear. For the converse assume $\mathcal{N} \not\models \varphi(\hat{a}, \hat{b}; \alpha)$ for some \hat{b} . By induction hypothesis, $\{i \in I : \mathcal{M} \not\models \varphi'(\hat{a}i, \hat{b}i; \alpha)\} \in F$ for some $\varphi' > \varphi$. Hence $\{i \in I : \mathcal{M} \models \forall y \varphi'(\hat{a}i, y; \alpha)\} \notin F$ as required.

Finally we prove

$$\mathcal{N} \models \exists y \varphi(\hat{a}, y; \alpha) \Leftrightarrow \{i \in I : \mathcal{M} \models \exists y \varphi'(\hat{a}i, y; \alpha)\} \in F \text{ for every } \varphi' > \varphi.$$

Again, \Rightarrow is clear. For the converse, assume the r.h.s.

Assume for simplicity that $I = \omega$. For every $i \in I$ pick a formula φ_i so that $\varphi_i > \varphi_j > \varphi$ for every $i < j$ and

$$\mathcal{M} \models \bigwedge_{i \in I} \varphi_i(x, y; \alpha) \rightarrow \varphi(x, y; \alpha).$$

For every $j \in I$ there is a sequence \hat{b}_j such that

$$\{i \in I : \mathcal{M} \models \varphi_j(\hat{a}i, \hat{b}_ji; \alpha)\} \in F.$$

As $\varphi_j(x, y; \alpha) \rightarrow \varphi_{j'}(x, y; \alpha)$ holds for $j > j'$, the diagonal sequence $\hat{d} = \langle \hat{b}_i i : i \in I \rangle$ satisfies $\mathcal{N} \models \varphi(\hat{a}, \hat{d}; \alpha)$.

□