CONTINUOUS LOGIC FOR THE DISCRETE LOGICIAN

1. A class of structures

1 Definition. In this notes we deal with 3-sorted[1] structures of the form $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$. The language and its interpretation are subject to the following conditions.

1. The function may only have one of the following sorts (for any $m, n \in \omega$)
   a. $\mathbb{R}^n \to \mathbb{R}$
   b. $M^n \times \breve{M}^m \to$ any of M, $\breve{M}$, or $\mathbb{R}$
2. The functions of sort $\mathbb{R}^n \to \mathbb{R}$ are *uniformly continuous*.
3. The (functions that interpret the) terms of sort $M^n \to \mathbb{R}$ have *bounded* range.
4. Every element of $\breve{M}$ is the image of some term of sort $M^n \to \breve{M}$.
5. The only relation symbol is the order relation $\leq$ on $\mathbb{R}$.

We call M the unit ball of $\mathcal{M}$. The terminology is inspired by the example below.

The language is denoted by $\mathbb{L}$. For simplicity we assume it contains *all* functions as in 2 above.

2 Definition. We write $\mathbb{T}(A)$ for the set of terms of sort $M^n \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{R}$ and parameters in some set $A \subseteq M$. We write $\mathbb{L}(A)$ for the set of formulas obtained inductively as follows

i. $\mathbb{L}(A)$ contains the atomic formula $t \leq s$ for every pair of terms $t, s \in \mathbb{T}(A)$.

ii. $\mathbb{L}(A)$ is closed under the Boolean connectives $\wedge$, $\vee$, and the quantifiers $\forall$, $\exists$ of sort M.

iii. $\mathbb{L}(A)$ is closed under the quantifier $\forall$ of sort $\mathbb{R}$ possibly relativized to a definable subset of $\mathbb{R}$.

E.g. iii asserts, in particular, that if $\varphi(x, \varepsilon) \in \mathbb{L}(A)$ then also $\forall \varepsilon{>}0\ \varphi(x, \varepsilon) \in \mathbb{L}(A)$.

3 Example (Banach spaces). Given a Banach space $\breve{M}$ we define a structure $\mathcal{M} = \langle M, \breve{M}, \mathbb{R}\rangle$ as follows. Let $M = \{a \in \breve{M} : \|a\| \leq 1\}$ be the closed unit ball of $\breve{M}$. Besides the symbols mentioned above, $\mathbb{L}$ contains a function symbol for the natural embedding $\mathrm{id} : M \to \breve{M}$. It contains also a symbol for the norm $\|\text{-}\| : \breve{M} \to \mathbb{R}$. Finally, $\mathbb{L}$ contains the usual symbols of the language of vector spaces. These have sort $\breve{M}^n \to \breve{M}$, for the appropiate $n \in \{0, 1, 2\}$.

Finally, note that condition 3 and 4 of Definition 1 are immediatly satisfied.

[1] In some examples it may be more natural to use $(n+n'+1)$-sorted structures $\mathcal{M} = \langle M_1, \dots, M_n, \breve{M}_1, \dots, \breve{M}_{n'}, \mathbb{R}\rangle$. The generalization is straightforeward.

2. Ultraproducts

We recall some standard definitions about limits. Let I be a non-empty set. Let F be a filter on I. If $r : I \to \mathbb{R}$ and $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ we write

$$\lim_{i \uparrow F} r(i) = \lambda$$

if $r^{-1}[A] \in F$ for every $A \subseteq \mathbb{R}$ that is a neighborhood of λ. When F is an ultrafilter, such a λ always exists.

Let I be an infinite set. Let $\langle \mathcal{M}_i : i \in I \rangle$ be a sequence of structures, say $\mathcal{M}_i = \langle M_i, \breve{M}_i, \mathbb{R} \rangle$, that are uniformly bounded. That is, the bounds in 3 of Definition 1 are the same for all $\mathcal{M}_i$.

Let F be an ultrafilter on I.

4 **Definition.** We define a structure $\mathcal{N} = \langle N, \breve{N}, \mathbb{R} \rangle$ that we call the ultraproduct of the models $\langle \mathcal{M}_i : i \in I \rangle$.

1. N is the set of the sequences $\hat{a} : I \to \bigcup_{i \in I} M_i$ such that $\hat{a}\, i \in M_i$.
2. $\breve{N}$ is the set of the sequences $t(\hat{a})$ of the form $t^{\mathcal{M}_i}(\hat{a} i)$, where $t(x)$ is a term of sort $M^n \to \breve{M}$.
3. If f is a function of sort $M^n \times \breve{M}^m \to M$ then $f^{\mathcal{N}}(\hat{a}, t^{\mathcal{M}_i}(\hat{c}))$ is the sequence $f^{\mathcal{M}_i}\big(\hat{a} i, t^{\mathcal{M}_i}(\hat{c} i)\big)$.
4. Similarly when f is of sort $M^n \times \breve{M}^m \to \breve{M}$.
5. If f is a function of sort $M^n \times \breve{M}^m \to \mathbb{R}$ then
$$f^{\mathcal{N}}\big(\hat{a}, t(\hat{c})\big) \;=\; \lim_{i \uparrow F} f^{\mathcal{M}_i}\big(\hat{a} i, t^{\mathcal{M}_i}(\hat{c} i)\big).$$

As usual, if $\mathcal{M}_i = \mathcal{M}$ for all $i \in I$, we say that $\mathcal{N}$ is an ultrapower of $\mathcal{M}$.

The limit in 5 of the definition above always exists because F is an ultrafilter. It is finite because all models $\mathcal{M}_i$ have the same bounds.

5 **Proposition (Łoś Theorem).** Let $\mathcal{N}$ be as above and let $\varphi(x) \in \mathbb{L}(A)$. Then for every $\hat{a} \in N^{|x|}$ and $\lambda \in \mathbb{R}$

$$\mathcal{N} \vDash \varphi(\hat{a}, \lambda) \;\Leftrightarrow\; \big\{ i \in I \;:\; \mathcal{M}_i \vDash \exists \alpha \in A\ \varphi(\hat{a} i, \alpha) \big\} \in F \text{ or every neighborhood } A \text{ of } \lambda.$$

Proof. We proceed by induction on the syntax of $\varphi(x)$. Assume $\varphi(\hat{a}, \lambda)$ is atomic, say $t(\hat{a}, \lambda) \leq s(\hat{a}, \lambda)$. We spell out the proof for the infimum quantifier. Assume inductively that

$$\varphi^{\mathcal{N}}\big(\hat{a}, \hat{b}\big) \;=\; \lim_{i \uparrow F} \varphi^{M_i}\big(\hat{a} i, \hat{b} i\big).$$

We want to prove that

$$\Big(\bigwedge_x \varphi\big(\hat{a}, x\big)\Big)^{\mathcal{N}} \;=\; \lim_{i \uparrow F} \Big(\bigwedge_x \varphi\big(\hat{a} i, x\big)\Big)^{M_i}$$

By the interpretation of the infimum quantifier, this amounts to prove that

$$\inf \left\{ \varphi\left(\hat{a}, \hat{b}\right)^{\mathcal{N}} \ : \ \hat{b} \in (uN) \right\} = \lim_{i \uparrow F} \inf \left\{ \varphi\left(\hat{a}i, b\right)^{M_i} \ : \ b \in (uM_i) \right\}$$

First we prove the $\leq$ inequality. Let α be an arbitrary positive real number.

Assume that for some $\hat{b} \in (uN)$ $\varphi^{\mathcal{N}}\left(\hat{a}, \hat{b}\right) < \alpha$.

By induction hypothesis $\lim\limits_{i \uparrow F} \varphi^{M_i}\left(\hat{a}i, \hat{b}i\right) < \alpha$,

then for some $u \in F$ $\varphi^{U_i}\left(\hat{a}i, \hat{b}i\right) < \alpha$ for every $i \in u$.

Hence $\inf \left\{ \varphi^{M_i}\left(\hat{a}i, b\right) : b \in (uM_i) \right\} < \alpha$ for every $i \in u$.

Finally, $\lim\limits_{i \uparrow F} \inf \left\{ \varphi^{M_i}\left(\hat{a}i, b\right) : b \in (uM_i) \right\} < \alpha$.

This proves the $\geq$ inequality. As for the $\leq$ inequality, assume that

$$\lim_{i \uparrow F} \inf \left\{ \varphi\left(\hat{a}i, b\right)^{M_i} \ : \ b \in (uM_i)^{|x|} \right\} < \alpha.$$

Therefore, for some $u \in F$

$$\inf \left\{ \varphi\left(\hat{a}i, b\right)^{M_i} \ : \ b \in (uM_i)^{|x|} \right\} < \alpha \quad \text{for every } i \in u.$$

$$\inf \left\{ \varphi\left(\hat{a}i, b\right)^{M_i} \ : \ b \in (uM_i)^{|x|} \right\} < \alpha \quad \text{for every } i \in u.$$

Therefore, for some $\hat{b} \in (uN)^{|x|}$ $\varphi^{M_i}\left(\hat{a}i, \hat{b}i\right) < \alpha$ for every $i \in u$.

Then $\lim\limits_{i \uparrow F} \varphi^{M_i}\left(\hat{a}i, \hat{b}i\right) < \alpha$,

and finally $\varphi^{\mathcal{N}}\left(\hat{a}, \hat{b}\right) < \alpha$.

This proves the desired inequality. □

The following corollary of Łǒś Theorem is identical to its classical counterpart.

6 **Corollary.** For every model $\mathcal{M}$ and every ultrafilter F on I, an infinite set, let $\mathcal{N}$ be corresponding ultrapower of $\mathcal{M}$. Then there is an elementary embedding of $\mathcal{M}$ in $\mathcal{N}$. □

Note also that, unlike the classical ultraproduct, here we do not quotient the structure obtained in Definition 4. However, we note the following fact which is an easy consequence of Łǒś Theorem.

7 **Fact.** Let $R \subseteq N^2$ be the set of those pairs $(\hat{a}, \hat{b})$ such that $\{i \in I \,:\, \hat{a}i = \hat{b}i\} \in F$. Then R is an elementary relation. □

3. Examples

1. Firstly, we leave to the reader to check that classical first-order models are a (trivial) special cases of the models introduced above. Classical relations take values in $\{0, 1\}$ where 0 is interpreted as "true". The unit ball is the whole of $\mathcal{M}$. The function $n_{\_}$ is constantly 1. To obtain the full strength of first-order logic one has to add equality as a predicate as it is absent from our logic.

2. Secondly, any BBHU-premodel is also a model as those in Section 1 if we let the unit ball coincide with $\mathcal{M}$. The language L contains a relation symbol $d(x, y)$ for a metric and, possibly, symbols for all other functions and relations of $\mathcal{M}$. As all relations of $\mathcal{M}$ (including the metric) take values in the interval $[0, 1]$, the function $n_{\_}$ can be set to be the constant 1.

3. Now we consider an example that is non trivial and distant both from classical models and from BBHU-models: weighted graphs. The set L_{fun} is empty and L_{rel} contains only a binary relation r. We require that $r^{\mathcal{M}}(a, b) = r^{\mathcal{M}}(b, a) \in [0, 1]$. As unit ball take, again,the whole of $\mathcal{M}$. The function $n_{\_}$ is constant 1.

4. Finally, and example where the unit ball is non trivial. Let L be the language of Banach spaces. Here we have only one symbol in L_{rel}, the symbol $\|\text{-}\|$ for the norm. The set L_{fun} is the same as for real vector spaces in classical logic. Let $\mathcal{M}$ be a Banach space and define a continuous model as follows. The interpretation of the symbols in $L_{\text{rel}} \cup L_{\text{fun}}$ is the natural one. The unit ball of is $M = \{a \in M \,:\, \|a\| \leq 1\}$. If $\varphi(x)$ is the atomic formulas
$$\varphi(x) \;=\; \Big\| \sum_{i=1}^{n} \lambda_i x \Big\|$$
then
$$n_{\varphi(x)} \;=\; \sum_{i=1}^{n} |\lambda_i|.$$

4. Conditions and types

For $\varphi, \psi \in L(M)$ some closed formulas, we write $\mathcal{M} \vDash \;\varphi \leq \psi$ for $\varphi^{\mathcal{M}} \leq \psi^{\mathcal{M}}$. The meaning of $\mathcal{M} \vDash \;\varphi = \psi$ and of $\mathcal{M} \vDash \;\varphi < \psi$ is similar.

Expressions of the form $\varphi(x) \leq 0$ are called conditions. We write $\mathcal{M} \vDash \exists x\, \varphi(x) \leq 0$ if there is an $a \in M^{|x|}$ such that $\mathcal{M} \vDash \varphi(a) \leq 0$. Observe that $\varphi(x) = 0$ is equivalent to $|\varphi(x)| \leq 0$ and that $\varphi(x) \leq 0$ is equivalent to $\big(\varphi(x) \vee 0\big) = 0$. So, when convenient, we may use equalities to denote conditions.

The negation of a condition is called a co-conditions. So, $\varphi(x) \neq 0$, $\varphi(x) < 0$ or $\varphi(x) > 0$ are co-conditions.

It is easy to see that conditions are cosed under logical disjunction and logical conjunction.

$$\mathcal{M} \vDash \varphi \vee \psi \leq 0 \;\;\Leftrightarrow\;\; \mathcal{M} \vDash \varphi \leq 0 \text{ and } \mathcal{M} \vDash \psi \leq 0$$

$$\mathcal{M} \vDash \varphi \wedge \psi \leq 0 \;\;\Leftrightarrow\;\; \mathcal{M} \vDash \varphi \leq 0 \;\text{ or }\; \mathcal{M} \vDash \psi \leq 0$$

Condition are also closed under universal quanfication (over the unit ball).

$$\mathcal{M} \vDash \bigvee_{x} \varphi(x) \leq 0 \;\;\Leftrightarrow\;\; \mathcal{M} \vDash \forall x\, \varphi(x) \leq 0$$

Closure under existential quanfication is a more subtle point. It is not true in general but it can be ensured by a small amount of saturation (cf. Fact 15).

A set of conditions is called a type or, when x is the empty tuple, a theory. For $p(x) \subseteq L(M)$, we define

$$p(x) \leq 0 \;\;=\;\; \big\{\varphi(x) \leq 0 \;\;:\;\; \varphi(x) \in p\big\}.$$

Up to equivalence, all types have the form above.

Beware that $p(x)$ denotes just a set of formulas, the expression $p(x) \leq 0$ denotes a type.

We write $\mathcal{M} \vDash p(a) \leq 0$ if $\mathcal{M} \vDash \varphi(a) \leq 0$ for every $\varphi(x) \in p$. We write $\mathcal{M} \vDash \exists x\, p(x) \leq 0$ if $\mathcal{M} \vDash p(a) \leq 0$ for some $a \in M^{|x|}$.

5. Elementary relations

Let $\mathcal{M}$ and $\mathcal{N}$ be models with unit balls M and N, respectively. We say that $R \subseteq M \times N$ is an elementary relation between M and N if for every $\varphi(x) \in L$

$$\varphi^{\mathcal{M}}(a) \;\;\Leftrightarrow\;\; \varphi^{\mathcal{N}}(b) \qquad\qquad \text{for every } a, b \in M^{|x|}, N^{|x|} \text{ such that } aRb.$$

Equivalently, if for every $\varphi(x) \in L$

$$\mathcal{M} \vDash \varphi(a) \leq 0 \;\;=\;\; \mathcal{N} \vDash \varphi(a) \leq 0 \qquad \text{for every } a, b \in M^{|x|}, N^{|x|} \text{ such that } aRb.$$

We define an equivalence relation $(\sim_M)$ on the unit ball M as follows

1. $$a \sim_M b \;\;\Leftrightarrow\;\; \varphi(a)^{\mathcal{M}} = \varphi^{\mathcal{M}}(b) \;\text{ for every } \varphi(x) \in L(M).$$

In ther words,

$$a \sim_M b \quad \Leftrightarrow \quad I_M \cup \{(a,b)\} \subseteq M^2 \text{ is an elementary relation.}$$

As elementary relations are closed under composition, if $a \sim_M b$ and $a' \sim_M b'$ then the relation $I_M \cup \{(a,b),(a',b')\} \subseteq M^2$ is also elementary. In general, for any $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$ we have that

2. $$a_i \sim_M b_i \text{ for all } i \quad \Leftrightarrow \quad \varphi(a)^{\mathcal{M}} = \varphi^{\mathcal{M}}(b) \text{ for every } \varphi(z) \in L(M).$$

From the consideration above we deduce the following.

8 **Lemma.** The relation $(\sim_M) \subseteq M^2$ is a maximal elementary relation. That is, no elementary relation on M contains $(\sim_M)$ properly.

Proof. By 2 above, $(\sim_M)$ is elementary. By 1, a pair $a \nsim_M b$ cannot belong to any elementary relation. Therefore $(\sim_M)$ is maximal. □

9 **Lemma.** Let $R \subseteq M \times N$ be total and surjective elementary relation. Then there is a unique maximal elementary relation containing R. This maximal elementary relation is equal to both $(\sim_M) R$ and $R (\sim_N)$.

Proof. It is immediate to verify that $(\sim_M) R$ is an elementary relation containing R. Let S be any maximal elementary relation containing R. By maximality, $(\sim_M) S = S$. As S is a total relation $(\sim_M) \subseteq SS^{-1}$. Therefore, by the lemma above, $(\sim_M) = SS^{-1}$. As R is a surjective relation, $S \subseteq S S^{-1} R$. Finally, by maximality, we conclude that $S = (\sim_M) R$. A similar argument proves that $S = R (\sim_N)$. □

We write $\mathrm{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, elementary relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/{\sim_M}$.

Thought in most situations one could dispense with $\mathcal{M}$ in favour of $\mathcal{M}/{\sim_M}$, in concrete cases one has a better grip on the first than on the latter. Therefore below we insist in working with elementary relations in place of elementary maps.

6. Elementary substructures

An elementary embedding is an elementary relation that is functional, injective and total.

When $\varnothing$ is an elementary relation between M and N, the unit balls of $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent and write $\mathcal{M} \equiv \mathcal{N}$. We write I_A for the diagonal relation on a set A. We write $\mathcal{M} \equiv_A \mathcal{N}$ if I_A is an elementary relation. In words, we say that $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent over A. Finally, we write $\mathcal{M} \preceq \mathcal{N}$ if $\mathcal{M} \equiv_M \mathcal{N}$ and $\mathcal{M} \subseteq \mathcal{N}$, that is $\mathcal{M}$ is a substructure of $\mathcal{N}$ in the classical sense. In words, we say that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

Let $A \subseteq M$. The limit A-topology on $M^{|x|}$ is the initial topology with respect to the functions that interpret the formulas in $L(A)$. The sets $\{a \in M^{|x|} : \varphi^{\mathcal{M}}(a) \leq 0\}$, as $\varphi(x)$ ranges over $L(A)$, form a base of closed sets for the limit A-topology. Note that the sets $\{a \in M^{|x|} : \varphi^{\mathcal{M}}(a) < 0\}$ are a base of open sets of the same topology.

The limit A-topology is a completely regular topology, almost by definition. Typically, it is not T_0, but its Kolmogorov quotient is Hausdorff.

10 Proposition (Tarski-Vaught test). Let $\mathcal{M} \subseteq \mathcal{M}$. Let x be a single variable. Then the following are equivalent

1. $\mathcal{M} \preceq \mathcal{N}$;
2. M is dense in N w.r.t. the M-limit topology on N;
3. for every $\varphi(x) \in L(M)$, if $\varphi(N) < 0$ is non empty, then $\mathcal{N} \vDash \varphi(b) < 0$ for some $b \in M$.

Proof. Equivalence 2⇔3 is tautological. Implication 1⇒2 is clear. To prove 3⇒1 we prove by induction on the syntax of $\varphi(z)$, where z is a finite tuple, that for every $\alpha \in \mathbb{R}$

$$\mathcal{M} \vDash \varphi(a) < \alpha \quad \Leftrightarrow \quad \mathcal{N} \vDash \varphi(a) < \alpha \qquad \text{for every } a \in M^{|z|}$$

We use the equivalence above as induction hypothesis. As M is a substructure of N, this equivalence holds for atomic $\varphi(z)$. We leave the case of the propositional connectives to the reader and consider only the case of the infimum quantifier.

$$\begin{array}{lrcl} & N \vDash \bigwedge_x \varphi(x,a) < \alpha & \Leftrightarrow & \text{there is a } b \in N \text{ such that } \mathcal{N} \vDash \varphi(b,a) < \alpha \\ a. & & \Leftrightarrow & \text{there is a } b \in M^{|x|} \text{ such that } \mathcal{N} \vDash \varphi(b,a) < \alpha \\ b. & & \Leftrightarrow & M \vDash \bigwedge_x \varphi(x) < \alpha. \end{array}$$

Implication ⇒ in a uses the assumption 3 and the equivalence in b uses the induction hypothesis. □

11 Theorem (Downward Löwenheim-Skolem). For every $A \subseteq N$ there is a model $A \subseteq M \preceq N$ of cardinality $\leq |L(A)|$.

Proof. (Needs some checking.) For every set A there is a base for the A-topology that has cardinality $\leq |L(A)|$. A set M of the required cardinality that is dense in the M-topology is obtained as in the classical downward Löwenheim-Skolem theorem. □

7. Saturation

From this section on, we assume the existence of unboundedly many inaccessible cardinals as this simplifies the exposition. We prove directly the existence of saturated extension (monster models), skipping the proof of the compactness theorem as this is not required in the following. (The compactness theorem could be proved along the same lines.)

Let $p(x) \subseteq L(M)$. We say that $p(x) \leq 0$ is finitely satisfied in $\mathcal{M}$ if for every disjunction of formulas in $p(x)$, say $\psi(x)$, there is an $a \in M^{|x|}$ such that $\mathcal{M} \vDash \psi(a) \leq 0$.

12 **Definition.** We say that $\mathcal{M}$ is saturated if for every $p(x)$ as in 1 and 2 below, there is an $a \in M^{|x|}$ such that $\mathcal{M} \vDash p(a) \leq 0$

1. $p(x) \subseteq L(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x) \leq 0$ is finitely satisfied in M.

From Łŏś Theorem we obtain that every model embeds elementarily in a saturated one. First we prove the following lemma.

13 **Lemma.** Every model $\mathcal{M}$ embeds elementarily in a model $\mathcal{N}$ that realizes all types as in 1 and 2.

Proof. Consider the collection of types such that 1 and 2 above. Assume that each type has its own set of variables and let x be the concatenation of all these variables. We denote by $p(x)$ the union of all these types. Let I be the set of formulas $\xi(x)$ such that $\xi(x) \leq 0$ is satisfied in M. For every condition $\varphi(x) \leq 0$ define $X_\varphi \subseteq I$ as follows

$$X_\varphi \;=\; \Big\{\xi(x) \in I \;:\; \xi(M) \leq 0 \;\subseteq\; \varphi(M) \leq 0\Big\}$$

Note that $\varphi(x) \leq 0$ is consistent if and only if $X_\varphi \neq \varnothing$ if and only if $\varphi(x) \in X_\varphi$. Moreover $X_{\varphi\vee\psi} \;=\; X_\varphi \cap X_\psi$. Then, as $p(x)$ is finitely consistent, the set $B = \big\{X_\varphi \;:\; \varphi(x) \in p\big\}$ has the finite intersection property. Extend B to an ultrafilter F on I. Let $\mathcal{N}$ be the ultrapower of $\mathcal{M}$ over F. That is the model with unit ball $N = M^I$ obtained as in Definition 4.

For every formula $\xi(x) \in I$ choose some $a_\xi \in M^{|x|}$ such that $\xi(a_\xi) \leq 0$. We may confuse $(M^I)^{|x|}$ with $(M^{|x|})^I$ as it simplifies notation. Let $\hat{a} \in (M^{|x|})^I$ be the function that maps $\xi(x) \mapsto a_\xi$. By Łŏś Theorem, for every formula $\varphi(x)$

$$\varphi^{\mathcal{N}}\big(\hat{a}\big) \;=\; \lim_{\xi\uparrow F} \varphi^M\big(a_\xi\big).$$

Therefore $\hat{a}$ realizes $p(x) \leq 0$ in $\mathcal{N}$. □

14 **Theorem.** Every model $\mathcal{M}$ embeds elementarily in a saturated model.

Proof. As usual, iterate the lemma to construct a chain of length λ, a sufficiently large inaccessible cardinal. □

We conclude with a convenient property of saturated models.

15 **Fact.** Let M is saturated. Then for every $\varphi(x) \in L(M)$ the following are equivalent

1. $M \vDash \exists x \big(\varphi(x) \leq 0\big)$;
2. $M \vDash \bigwedge_x \varphi(x) \ \leq \ 0$

Proof. Only 2⇒1 requires a proof. Let $p(x) = \{\varphi(x) - \alpha : \alpha \in \mathbb{R}^+\}$. If 2, then $p(x) \leq 0$ is finitely satisfied in M. Hence 1 follows by saturation. □

8. Homogeneity

Throughout the following we fix a saturated model $\mathfrak{U}$ of cardinality κ, an inaccessible cardinal larger than $|L|$, where $|L|$ stands for $\max\big\{|L_{\rm fun}|, |L_{\rm rel}|, 2^{\omega}\big\}$.

Let $a \in \mathfrak{U}^{|x|}$. We write $p(x) = {\rm tp}(a/A)$ for $p(x) = \big\{\psi(x) \in L(A) : \mathfrak{U} \vDash \psi(a) \leq 0\big\}$. We write ${\rm tp}(a)$ when $A = \varnothing$.

16 **Fact.** Let $R \subseteq \mathfrak{U}^2$ be an elementary relation of cardinality $< \kappa$. Then there is a total and surjective elementary relation $S \subseteq \mathfrak{U}^2$ containing R.

Proof. We apply the usual back-and-forth construction with a pinch of extra caution. Below we will make free use of Fact 15. Let a be an enumeration of the domain of R. Let $\bar{a} = \langle a_i : i < \lambda\rangle$ be an enumeration of all tuples of length $|a|$ such that aRa_i. As κ is inaccessible, $\lambda < \kappa$. Let $b \in \mathfrak{U}$. It suffices to prove that there is a c such that $R \cup \{\langle b, c\rangle\}$ is an elementary relation. Let $p(x, z) = {\rm tp}(b, a/A)$ and let

$$q(x, \bar{z}) \quad = \quad \bigcup_{i<\lambda} p(x, z_i).$$

We claim that $q(x, \bar{a}) \leq 0$ is finitely consistent. A finite conjunction of formulas in $q(x, \bar{a})$ has the form $\psi(x, a_{i_1}) \wedge \cdots \wedge \psi(x, a_{i_n})$. As $\psi(b, a) \leq 0$ and $a_{i_1}, \ldots, a_{i_n}\ R\ a, \ldots, a$, we conclude that the condition $\psi(x, a_{i_1}) \wedge \cdots \wedge \psi(x, a_{i_n}) \leq 0$ is satisfied. The existence of the required element c follows from saturation. □

17 **Corollary.** Let $a \in \mathfrak{U}^{|x|}$, where $|x| < \kappa$. Let $A \subseteq \mathfrak{U}$ have cardinality $< \kappa$. Then

$$p(\mathfrak{U}) \leq 0 \quad = \quad \big\{b : bRa,\ R \in {\rm Aut}(\mathfrak{U}/A)\big\}$$

Di qui in poi solo esperimenti selvaggi.

9. Random variables

Let $\mathcal{A}$ be a σ-algebra of subsets of Ω. Let $\mathcal{M}$ be the set of bounded σ-additive measures on $\mathcal{A}$. Let $\mathcal{R}$ be the set of functions $f : \Omega \to \mathbb{R}$ that are measurable w.r.t. all measures in $\mathcal{M}$.

We define a 3-sorted structure $\langle \mathcal{A}, {}^{\circ}\mathcal{R}, {}^{\circ}\mathcal{M} \rangle$.

The domain of the first sort is $\mathcal{A}$. The language contains functions for the Boolean operations.

The domain of the second sort ${}^{\circ}\mathcal{R}$ contains the functions $f \in \mathbb{R}$ that are linear combinations of indicator functions of sets in $\mathcal{A}$. In the language we include the functions that make ${}^{\circ}\mathcal{R}$ an $\mathbb{R}$-algebra (sum and and multiplications are defined pointwise). We also include operations for the pointwise maximum and minimum of two functions.

The domain of the third sort, ${}^{\circ}\mathcal{M}$, contains signed measures $\mu : \mathcal{A} \to \mathbb{R}$ that are linear combinations of measures of the form δ_a, for $a \in \Omega$. These are defined as follows: $\delta_a(A)$ is 1 if $a \in A$ and 0 otherwise. The language is that of lattice vector spaces.

There is a function of sort ${}^{\circ}\mathcal{R} \times {}^{\circ}\mathcal{M} \to {}^{\circ}\mathcal{M}$ that maps (f, μ) to the measure $f\mu$ obtained multiplying in the natural way f and μ.

Finally, there is a predicate I of sort $\mathcal{A} \times {}^{\circ}\mathcal{M}$ such that $I(A, \mu) = \mu(A)$.

The unit ball of $\langle \mathcal{A}, {}^{\circ}\mathcal{R}, {}^{\circ}\mathcal{M} \rangle$ is $\langle \mathcal{A}, {}^{\circ}\boldsymbol{R}, \boldsymbol{M} \rangle$ where ${}^{\circ}\boldsymbol{R}$ and ${}^{\circ}\boldsymbol{M}$ are the set of functions, respectively measures, that are ≤ 1 in absolute value.

The readers can easily convince themselfs that a bounding function exist. Its explicit definition is not required in the sequel.

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The following discrete version of the Radon-Nikodym theorem is completely trivial. For simplicity we state it for (nonnegative) measures. As the Hahn decomposition theorem holds (trivially) in ${}^{\circ}\mathcal{M}$, this is no loss of generality.

18 Fact. [Discrete Radon-Nikodym] Let $\mu, \nu \in {}^{\circ}\boldsymbol{M}$, where ${}^{\circ}\boldsymbol{M}$ is the unit ball of ${}^{\circ}\mathcal{M}$, be non negative. For every $\varepsilon > 0$ there are $E \in \mathcal{A}$ and $f \in {}^{\circ}\boldsymbol{R}$, where ${}^{\circ}\boldsymbol{R}$ is the unit ball of ${}^{\circ}\mathcal{R}$, such that

1. $I(E, |\nu - \varepsilon^{-1} f \mu|) = 0$
2. $|I(\neg E, \mu)| \leq \varepsilon.$

We will use the following version of the fact above.

19 Fact. [Iterated discrete Radon-Nikodym] Let $\mu, \nu \in {}^{\circ}\boldsymbol{M}$ be non negative. For every $n > 0$ there are $E_1, \ldots, E_n \in \mathcal{A}$ and $f \in {}^{\circ}\boldsymbol{R}$ such that

0. the E_i are pairwise disjoint;

1. $I\big(E_i,\ |\nu - 2^i f\mu|\big) \ = \ 0$
2. $|I(X,\ \mu)| \leq 2^{-i}$, where $X = \neg(E_1 \cup \cdots \cup E_i)$;

Proof. Apply inductively the theorem above, with E_i for Ω, to find E_{i+1} and f_{i+1} such that

1_i. $I\big(E_i,\ |\nu - 2^i f_i\mu|\big) \ = \ 0$;

2_i. $|I(X,\ \mu)| \leq 2^{-i}$, where $X = \neg(E_1 \cup \cdots \cup E_i)$;

3_i. $f_i\big[\neg E_i] = \{0\}$.

Finally, let $f \ = \ f_1 + \cdots + f_n$. The third condition above (which is immediate to obtain) ensures that $f \in {}^{\circ}\!R$. □

We now consider a saturated extension of $\langle \mathcal{A}, {}^{\circ}\mathcal{R}, {}^{\circ}\mathcal{M}\rangle$ that we denote by $\langle {}^{*}\!\mathcal{A}, {}^{*}\mathcal{R}, {}^{*}\mathcal{M}\rangle$. The following fact is a direct consequence of saturation.

20 Fact. Let $\mu \in \mathcal{M}$. Then there is ${}^{*}\!\mu \in {}^{*}\mathcal{M}$ such that $\mu_{\restriction\mathcal{A}} = {}^{*}\!\mu_{\restriction\mathcal{A}}$.

Proof. It suffices to check that the following type is finitely consistent in $\mathcal{A}$ (read ν as a variable and $\mu(X)$ as a real number)

$$\big\{\nu(X) = \mu(X)\ \ :\ \ X \in \mathcal{A}\big\},$$

which immediate. □

We can improve on the fact above. In fact, we can also control the value of ${}^{*}\!\mu(A)$ for any $A \in \mathcal{A}$.

21 Fact. Let μ be a bounded measure on $\mathcal{A}$, not necessarily in ${}^{\circ}\mathcal{M}$. Let $A \in {}^{*}\!\mathcal{A}$. Define

$$\begin{aligned} m_{\text{in}} &= \sup\{\mu(X) : X \in \mathcal{A},\ X \subseteq A\} \\ m_{\text{ex}} &= \inf\{\mu(X) : X \in \mathcal{A},\ A \subseteq X\} \end{aligned}$$

Then for every $m_{\text{in}} \leq r \leq m_{\text{ex}}$ there is a measure ${}^{*}\!\mu \in {}^{*}\mathcal{M}$ such that ${}^{*}\!\mu_{\restriction\mathcal{A}} = \mu_{\restriction\mathcal{A}}$ and ${}^{*}\!\mu(A) = r$.

Proof (sketch). Assume for simplicity that μ is bounded by 1. It suffices to check that the following type is finitely consistent (read ν as a variable and $\mu(X)$ as a real number).

$$\big\{\nu(X) = \mu(X)\ \ :\ \ X \in \mathcal{A}\big\}\ \ \cup\ \ \big\{\nu(A) = r\big\}.$$

In fact, any ${}^{*}\!\mu$ realizing this type is as required by the lemma.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be a finite Boolean algebra. We define a measure ν on the Boolean algebra generated by $A, X_1, \ldots, X_n$ that satisfies the type above restrected to $\mathcal{A}' \cup \{A\}$. Let $X_1, \ldots, X_n$ be the atoms of $\mathcal{A}'$. Assume $A \notin \mathcal{A}$, to avoid trivialities. Then there are some sets X_i such that both $X_i \cap A$ and $X_i \smallsetminus A$ are nonempty. Suppose these sets are $X_1, \ldots, X_m$. For $i \leq m$ we define $\nu(X_i \cap A) = \varepsilon\,\mu(X)$

and $\nu(X_i \smallsetminus A) = (1-\varepsilon)\mu(X_i)$, where $0 \leq \varepsilon \leq 1$ is specified below. For $i > m$ let $\nu(X_i) = \mu(X_i)$. Note that $\mu(X_1 \cup \cdots \cup X_m) \geq m_{\rm ex} - m_{\rm in}$. Therefore with a suitable ε, we can obtain $\nu(A) = r$. $\square$

If $\mathcal{A}$ is a σ-algebra and μ is a σ-additive measure then the supremum and the infimum in the fact above are attained. We will use the following.

22 **Corollary.** Let $\mu \in \mathcal{M}$. Then for every $A \in {}^*\!\mathcal{A}$ there are $A_{\rm in}, A_{\rm ex} \in \mathcal{A}$ such that $A_{\rm in} \subseteq A \subseteq A_{\rm ex}$ and a measures ${}^*\mu$ such that ${}^*\mu_{\restriction \mathcal{A}} = \mu_{\restriction \mathcal{A}}$ and ${}^*\mu(A) = \mu(A_{\rm ex})$. A similar claim holds for $A_{\rm in}$.