CONTINUOUS LOGIC FOR THE CLASSICAL LOGICIAN

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1. A class of structures

As a motivating example, consider real vector spaces. These are among the most simple strucures considered in model theory. Now expand them with a norm. The norm is a function that, given vector, outputs a real number. We may formalize this in a natural way by using a two sorted struture. Ideally, we would like that elementary extensions of normed spaces mantain the usal notion of real numbers[1]. Unfortunately, this is not possible if we insist to mantain the classical notion of elementary extension. A way out has been proposed by Henson and Iovino: restrict the notion of elementarity to a smaller class of formulas (which they call positive bounded formulas). We elaborate on this idea generalizing it to arbitrary structures and a wider class of formulas.

1 **Definition.** Let L be a one-sorted (first-order) language. Let $L' \supseteq L$ be a two-sorted language. We consider the class of L'-structures of the form $\mathcal{M} = \langle M, R\rangle$, where M ranges over L-structures, while R is fixed.

We assume that R is endowed with a locally compact Hausdorff topology.

We require that function symbols only have one of these sorts

i. $R^n \to R$;

ii. $M^n \to M$;

iii. $M^n \to R$.

The interpretation of symbols of sort $R^n \to R$ is required to be continuous, the interpretation of symbols of sort $M^n \to R$ is required to be bounded, i.e. the range is contained in a compact set.

We only allow relation symbols of sorts M^n and R^n.

There might be models in the class described in Definition 1 that do not have a saturated elementary extension in the same class. As a remedy, below we carve out a set $\mathbb{L}$ of formulas, $L \subseteq \mathbb{L} \subseteq L'$, such that every model as in Definition 1 has an $\mathbb{L}$-elementary extension that is $\mathbb{L}$-saturated in the same class.

[1] Non standard standard analists have no problem in expanding $\mathbb{R}$. In these notes, for a change, we insist as $\mathbb{R}$ (or, in general R) should remain $\mathbb{R}$ throughout.

For convenience we assume that the functions of sort $M^n \to M$ and the relations of sort M^n are all in L. It is also convenient to assume that L' contains names for all continuous bounded functions $R^n \to R$.

2 Definition. Formulas in $\mathbb{L}$ are constructed inductively from the following two sets of formulas

i. formulas of the form $t(x;y) \in C$, where $C \subseteq R^n$ is compact[2] and $t(x,y)$ is a n-tuple of terms of sort $M^{|x|} \times R^{|y|} \to R$;

ii. all formulas in L.

We require that $\mathbb{L}$ is closed under the Boolean connectives $\wedge, \vee$; the quantifiers $\forall, \exists$ of sort M; and the quantifiers $\forall^C, \exists^C$, by which we mean the quantifiers of sort R restricted to some (any) compact set $C \subseteq R^m$.

2. Henson-Iovino approximations

For $\varphi, \varphi' \in \mathbb{L}(M)$ (free variables are hidden) we write $\varphi' > \varphi$ if φ' can be obtained replacing each atomic formula $t \in C$ occurring in φ by $t \in D$ for some compact neighborhood D of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. By the normality of R we also have $\varphi > \varphi$ when $\varphi = (t \in C)$ for some clopen set C.

Note that $>$ is a dense (pre)order of $\mathbb{L}(M)$.

Formulas in as in (i) of Definition 2 do not occur under the scope of a negation, therefore we always have that $\varphi \to \varphi'$.

We recall the standard definition of F-limits. Let I be a non-empty set. Let F be a filter on I. Let Y be a topological space. If $f : I \to Y$ and $\lambda \in Y$ we write

$$F\text{-}\lim_i f(i) = \lambda$$

if $f^{-1}[A] \in F$ for every $A \subseteq Y$ that is a neighborhood of λ. Such a λ is unique if Y is Hausdorff. When F is an ultrafilter and, in addition, Y is compact the limit always exists.

The lemma below is required for the proof of Łőś Theorem in the next section. But first a preliminary fact.

3 Fact. Assume the following data

- $t(x;y)$, a term of sort $M^{|x|} \times R^{|y|} \to R$;
- $\langle a_i : i \in I\rangle$, a sequence of elements $a_i \in M^{|x|}$;
- $\langle \alpha_i : i \in I\rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

[2] We confuse the relation symbols in L' of sort R^m with their interpretation and write $t \in C$ for $C(t)$

Then

$$\# \qquad F\text{-}\lim_i t^{\mathcal{M}}(a_i;\alpha) \;=\; F\text{-}\lim_i t^{\mathcal{M}}(a_i;\alpha_i)$$

Proof. By induction on the nesting of functions of sort $R^n \to R$. In the basis case t is either a term of sort $M^{|x|} \to R$ or a variable of sort R. In both cases (#) is trivial. Assume inductively that the fact holds for the n-tuple $t(x,y)$. By the continuity of the functions of sort $R^n \to R$ and the induction hypothesis, (#) holds for $f\big(t(x,y)\big)$. □

4 Lemma. Assume the following data

- $\varphi(x;y) \in \mathbb{L}$;
- $\langle a_i : i \in I\rangle$, a sequence of elements of $M^{|x|}$;
- $\langle \alpha_i : i \in I\rangle$, a sequence of elements of $C \subseteq R^{|y|}$, a compact set;
- $\alpha = F\text{-}\lim_i \alpha_i$, for some ultrafilter F on I.

Then the following are equivalent

i. $$\{i \in I : \mathcal{M} \vDash \varphi'(a_i;\alpha)\} \;\in\; F \;\text{ for every } \varphi' > \varphi;$$

ii. $$\{i \in I : \mathcal{M} \vDash \varphi'(a_i;\alpha_i)\} \;\in\; F \;\text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. When $\varphi(x;y)$ is in L, it does not depend on α, and the lemma is trivial. Suppose that $\varphi(x;y)$ is as in (i) of Definition 2, say it is the formula $t(x;y) \in C$.

Assume $\{i : t(a_i;\alpha) \in C'\} \in F$ for every C' that is a compact neighborhood of C. Let C'' be a neighborhood of C. Pick C' such that C'' be a neighborhood of C'. Then $F\text{-}\lim_i t(a_i;\alpha) \in C'$ by normality of R. By Fact 3, $F\text{-}\lim_i t(a_i;\alpha_i) \in C'$. Then $\{i : t(a_i;\alpha_i) \in C''\} \in F$.

The converse implication is similar. This completes the proof of the base case of the induction.

Induction clear for the connectives $\vee$, $\wedge$. To deal with the universal quantifier of sort M we assume inductively that

$$\big\{i : \varphi'(a_i,b_i;\alpha)\big\} \in F \text{ for all } \varphi' > \varphi \;\Leftrightarrow\; \big\{i : \varphi'(a_i,b_i;\alpha_i)\big\} \in F \text{ for all } \varphi' > \varphi.$$

We prove

1. $$\big\{i : \forall y\, \varphi'(a_i,y;\alpha)\big\} \in F \text{ for all } \varphi' > \varphi \;\Leftrightarrow\; \big\{i : \forall y\, \varphi'(a_i,y;\alpha_i)\big\} \in F \text{ for all } \varphi' > \varphi.$$

Negate the r.h.s. of the equivalence. Then there are $\varphi' > \varphi$ and a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \neg\varphi'(a_i,b_i;\alpha_i)\big\} \in F$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Assume also the l.h.s. of (1) and reason for a contradiction. Then $\big\{i : \varphi''(a_i,b_i;\alpha)\big\} \in F$ and, by induction hypothesis, $\big\{i : \varphi'(a_i,b_i;\alpha_i)\big\} \in F$. A contradiction that proves the implication ($\Rightarrow$) in (1). The converse implication is similar.

To deal with the existential quantifier of sort M we prove

2. $\big\{i : \exists y\, \varphi'(a_i, y\,; \alpha)\big\} \in F$ for all $\varphi' > \varphi \;\Leftrightarrow\; \big\{i : \exists y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F$ for all $\varphi' > \varphi$.

Assume the r.h.s. of the implication. Fix $\varphi' > \varphi$. We prove that $\big\{i : \exists y\, \varphi'(a_i, y\,; \alpha)\big\} \in F$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Then there is a sequence $\langle b_i : i \in I\rangle$ such that $\big\{i : \varphi''(a_i, b_i\,; \alpha_i)\big\} \in F$. By the induction hypothesis $\big\{i : \exists y\, \varphi'(a_i, y\,; \alpha_i)\big\} \in F$. As φ' is arbitrary, this proves the implication ($\Leftarrow$) in (2). The converse implication is similar.

Induction for the quantifiers $\forall^C$ and $\exists^C$ is virtually identical, but we repeat the argument for convenience. Assume inductively that for every $\varphi' > \varphi$ and every sequence $\langle \beta_i : i \in I\rangle$ in C such that $\beta = F\text{-}\lim_i \beta_i$

$$\big\{i : \varphi(a_i\,; \alpha, \beta)\big\} \in F \text{ for all } \varphi' > \varphi \;\Leftrightarrow\; \big\{i : \varphi'(a_i\,; \alpha_i, \beta_i)\big\} \in F \text{ for all } \varphi' > \varphi.$$

We prove

3. $\{i : \forall^C y\, \varphi(a_i\,; \alpha, y)\} \in F$ for all $\varphi' > \varphi \;\Leftrightarrow\; \big\{i : \forall^C y\, \varphi'(a_i\,; \alpha_i, y)\big\} \in F$ for all $\varphi' > \varphi$.

Negate the r.h.s. of the implication. Then there are $\varphi' > \varphi$ and a sequence $\langle \beta_i : i \in I\rangle$ such that $\big\{i : \neg\varphi'(a_i\,; \alpha_i, \beta_i)\big\} \in F$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Assume also the l.h.s. of (3) and reason for a contradiction. Then $\big\{i : \varphi''(a_i\,; \alpha, \beta_i)\big\} \in F$ and, by induction hypothesis, $\big\{i : \varphi'(a_i\,; \alpha_i, \beta_i)\big\} \in F$. A contradiction that proves the implication ($\Rightarrow$) in (3). The converse implication is similar.

Finally we prove

4. $\{i : \exists^C y\, \varphi(a_i\,; \alpha, y)\} \in F$ for all $\varphi' > \varphi \;\Leftrightarrow\; \big\{i : \exists^C y\, \varphi'(a_i\,; \alpha_i, y)\big\} \in F$ for all $\varphi' > \varphi$.

Assume the r.h.s. of the implication. Fix $\varphi' > \varphi$. We prove that $\big\{i : \exists y\, \varphi'(a_i\,; \alpha, y)\big\} \in F$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. Then there is a sequence $\langle \beta_i : i \in I\rangle$ such that $\big\{i : \varphi''(a_i\,; \alpha_i, \beta_i)\big\} \in F$. By the induction hypothesis $\big\{i : \exists y\, \varphi'(a_i\,; \alpha_i, y)\big\} \in F$. As φ' is arbitrary, this proves the implication ($\Leftarrow$) in (4). The converse implication is similar. □

3. Ultraproducts

Below we introduce a suitable notion of ultraproducts of some structures $\langle \mathcal{M}_i : i \in I\rangle$. We require that for each function symbol f of sort $M^n \to R$

\# there is a compact $C \subseteq R$ that contains the range of all the functions $f^{\mathcal{M}_i}$.

To keep notation tidy, we make two semplifications: (1) we only consider ultratpowers; (2) we ignore formulas in L containing equality, so we can work with M^I in place of M^I/F. The generalization is straightforward and is left to the reader.

Let I be an infinite set. Let F be an ultrafilter on I. Let $\mathcal{M} = \langle M, R\rangle$ be an L-structure.

5 Definition. We define a structure $\mathcal{N} = \langle N, R\rangle$ that we call the ultrapower of $\mathcal{M}$.

1. $N = M^I$ that is, it is the set of sequences $\hat{a} : I \to M$.
2. If f is a function of sort $M^n \to M$ then $f^{\mathcal{N}}(\hat{a})$ is the sequence $\langle f^{\mathcal{M}}\big(\hat{a}i\big) : i \in I\rangle$.
3. The interpretation of functions of sort $R^n \to R$ remains unchanged.
4. If f is a function of sort $M^n \to R$ then
$$f^{\mathcal{N}}\big(\hat{a}\big) \;=\; F\text{-}\lim_i f^{\mathcal{M}}\big(\hat{a}i\big).$$
5. If r is a relation symbol of sort M^n then
$$\mathcal{N} \models r(\hat{a}) \Leftrightarrow \big\{i \in I \ : \ \mathcal{M} \models r(\hat{a}i)\big\} \in F.$$
6. The interpretation of relations of sort R^n remains unchanged.

The following is immediate but it needs to be noted. In fact, in the more general setting of ultra-products, it would not hold without the uniformity requirement (#) above.

6 Fact. The structure $\mathcal{N}$ satisfies Definition 1.

The following is easily proved by induction on the syntax as in the classical case

7 Fact. If $t(x)$ is a term of type $M^{|x|} \to M$ then
$$t^{\mathcal{N}}\big(\hat{a}\big) \;=\; \langle t^{\mathcal{M}}\big(\hat{a}i\big) : i \in I\rangle. \qquad \square$$

We also have that

8 Fact. If $t(x; y)$ has sort $M^{|x|} \times R^{|y|} \to R$ then
$$t^{\mathcal{N}}\big(\hat{a}; \alpha\big) \;=\; F\text{-}\lim_i t^{\mathcal{M}}\big(\hat{a}i; \alpha\big).$$

Proof. By induction. If t is a function symbol then x or z do not occor in t. If x does not occur in t the claim is trivial. If y does not occur in t the claim holds by definition. Finally, induction is is clear by the continuity of the functions of sort $R^n \to R$. $\square$

Finally, we prove

9 Proposition (Łŏś Theorem). Let $\mathcal{N}$ be as above and let $\varphi(x) \in \mathbb{L}$. Then for all $\hat{a} \in N^{|x|}$
$$\mathcal{N} \models \varphi(\hat{a}) \;\Leftrightarrow\; \big\{i \in I \ : \ \mathcal{M} \models \varphi'(\hat{a}i)\big\} \in F \text{ for every } \varphi' > \varphi.$$

Proof. By induction on the syntax. If $\varphi(x) \in L$ then the theorem reduces to the classical Łŏś Theorem. Then, suppose that $\varphi(x)$ is as in (i) of Definition 2, say it is the formula $t(x) \in C$.

Assume $\mathcal{N} \models t(\hat{a}) \in C$. If D is a compact neighborhood of C then D is also a neighborhood of $t^{\mathcal{N}}(\hat{a})$. Hence, by the definition of F-limit, $\big\{i : \mathcal{M} \models t(\hat{a}i) \in D\big\} \in F$. Vice versa, assume $\mathcal{N} \models t(\hat{a}) \notin C$.

By local compactness of R there is a compact neighborhood of C, such that $t^{\mathcal{N}}(\hat{a}) \notin D$. By Fact 8 and the definition of F-limit, $\big\{i : \mathcal{M} \vDash t(\hat{a}i) \notin D\big\} \in F$.

This completes the proof of the base case of the induction.

Induction for the connectives $\vee$ and $\wedge$ is clear. To deal with the quantifiers of sort M we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}, \hat{b}) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F \text{ for every } \varphi' > \varphi.$$

First we prove

1. $$\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y)\big\} \in F \text{ for every } \varphi' > \varphi.$$

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{i : \mathcal{M} \vDash \forall y\, \varphi'(\hat{a}i, y)\big\} \notin F$ as required.

($\Rightarrow$) Assume that $\big\{i : \mathcal{M} \nvDash \forall y\, \varphi'(\hat{a}i, y)\big\} \in F$. Choose $\hat{b}$ such that $\big\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. If for a contradiction $\mathcal{N} \vDash \forall y\, \varphi(\hat{a}, y)$, then in particular $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$ a contradiction. This completes the proof of (1).

Now, we prove

2. $$\mathcal{N} \vDash \exists y\, \varphi(\hat{a}, y) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\big\} \in F \text{ for every } \varphi' > \varphi.$$

($\Rightarrow$) Assume that $\mathcal{N} \vDash \varphi(\hat{a}, \hat{b})$ for some $\hat{b}$. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$ for every $\varphi' > \varphi$. A fortiori $\big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\big\} \in F$ as required.

($\Leftarrow$) Assume that $\big\{i : \mathcal{M} \vDash \exists y\, \varphi'(\hat{a}i, y)\big\} \in F$ for every $\varphi' > \varphi$. We prove that $\mathcal{N} \vDash \exists y\, \varphi'(\hat{a}, y)$ for every $\varphi' > \varphi$. Suppose not, for a contradiction, say φ' is a counterexample. Choose some $\hat{b}$ such that $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i, \hat{b}i)\big\} \in F$. By induction hypothesis $\mathcal{N} \vDash \varphi'(\hat{a}, \hat{b})$, a contradiction. This completes the proof of (2).

To deal with the quantifier $\forall^C$ we assume inductively that

$$\mathcal{N} \vDash \varphi(\hat{a}\,;\beta) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \varphi'(\hat{a}i\,;\beta)\big\} \in F \text{ for every } \varphi' > \varphi$$

and prove that

3. $$\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}\,; y) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i\,; y)\big\} \in F \text{ for every } \varphi' > \varphi.$$

($\Leftarrow$) Assume $\mathcal{N} \nvDash \varphi(\hat{a}\,;\beta)$ for some β. By induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\beta)\big\} \notin F$ for some $\varphi' > \varphi$. A fortiori $\big\{i : \mathcal{M} \vDash \forall^C y\, \varphi'(\hat{a}i\,; y)\big\} \notin F$ as required.

($\Rightarrow$) Assume $\big\{i : \mathcal{M} \nvDash \forall^C y\, \varphi'(\hat{a}i\,; y)\big\} \in F$ for some $\varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\big\{i : \mathcal{M} \nvDash \varphi'(\hat{a}i\,;\beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. If for a contradiction $\mathcal{N} \vDash \forall^C y\, \varphi(\hat{a}\,; y)$, then $\mathcal{N} \vDash \varphi(\hat{a}\,;\beta)$ and, by induction hypothesis, $\big\{i : \mathcal{M} \vDash \varphi''(\hat{a}i\,;\beta)\big\} \in F$ for every $\varphi'' > \varphi$. By Lemma 4, $\big\{i : \mathcal{M} \vDash \varphi''(\hat{a}i\,;\beta_i)\big\} \in F$ for every $\varphi'' > \varphi$. In particular $\big\{i : \mathcal{M} \vDash \varphi'(\hat{a}i\,;\beta_i)\big\} \in F$, a contradiction. This completes the proof of (3).

Finally, for the quantifier $\exists^C$ we prove

4. $\mathcal{N} \vDash \exists^C y\, \varphi(\hat{a}\,; y) \;\Leftrightarrow\; \big\{i \in I \;:\; \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i\,; y)\big\} \in F$ for every $\varphi' > \varphi$.

($\Rightarrow$) Assume $\mathcal{N} \vDash \varphi(\hat{a}\,; \beta)$ for some β. By induction hypothesis, $\big\{i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i\,; \beta)\big\} \in F$ for every $\varphi' > \varphi$. A fortiori $\big\{i \,:\, \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i\,; y)\big\} \notin F$ for every $\varphi' > \varphi$.

($\Leftarrow$) Assume $\big\{i \,:\, \mathcal{M} \vDash \exists^C y\, \varphi'(\hat{a}i\,; y)\big\} \in F$ for all $\varphi' > \varphi$. We prove that $\mathcal{N} \vDash \exists y\, \varphi'(\hat{a}, y)$ for every $\varphi'' > \varphi$. Suppose not, for a contradiction, say φ'' is a counterexample. Pick φ' such that $\varphi'' > \varphi' > \varphi$. Choose some $\beta_i \in C$ such that $\big\{i \,:\, \mathcal{M} \vDash \varphi'(\hat{a}i\,; \beta_i)\big\} \in F$ and let $\beta = F\text{-}\lim_i \beta_i$. By Lemma 4, $\big\{i \,:\, \mathcal{M} \vDash \varphi''(\hat{a}i\,; \beta)\big\} \in F$. This completes the proof of (4). □

4. $\mathbb{L}$-Elementrarity

Let $\mathcal{M} = \langle M, R\rangle$ and $\mathcal{N} = \langle N, R\rangle$ be two structures. We say that $f : M \to N$, a partial map, is an $\mathbb{L}$-elementary map if

$$\mathcal{M} \vDash \varphi(a) \;\Rightarrow\; \mathcal{N} \vDash \varphi(a) \qquad\qquad \text{for every } \varphi(x) \in \mathbb{L} \text{ and every } a \in (\operatorname{dom} f)^{|x|}$$

The following fact is proved as in the classical case using that $\mathcal{M} \vDash \varphi$ holds if and only if $\mathcal{M} \vDash \varphi'$ holds for every $\varphi' > \varphi$.

10 **Fact.** If $\mathcal{N}$ is an ultrapower of $\mathcal{M}$ then there is an $\mathbb{L}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

For $A \subseteq M \cap N$, we say that $\mathcal{M}$ and $\mathcal{N}$ are $\mathbb{L}$-(elementary) equivalent over A and write $\mathcal{M} \equiv^{\mathbb{L}}_A \mathcal{N}$ if $\mathrm{id}_A : M \to N$ is $\mathbb{L}$-elementary. We write $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$ when $\mathcal{M} \equiv^{\mathbb{L}}_M \mathcal{N}$. In words, we say that $\mathcal{M}$ is an $\mathbb{L}$-(elementary) substructure of $\mathcal{N}$.

5. $\mathbb{L}$-compactness

A theory $T \subseteq \mathbb{L}$ be a weakly finitely consistent if for every φ, conjunction of sentences in T, any $\varphi' > \varphi$ is consistent. The following proposition follows from Łŏś Theorem by the usual argument.

11 **Proposition (Compactness Theorem).** Let $T \subseteq \mathbb{L}$ be an weakly finitely consistent theory. Then T is consistent. □

Let $p(x) \subseteq \mathbb{L}(M)$. We say that $p(x)$ is weakly finitely satisfied in $\mathcal{M}$ if $\mathcal{M} \vDash \exists x\, \varphi'(x)$ for every $\varphi' > \varphi$ where $\varphi(x)$ is some conjunction of formulas in $p(x)$.

12 **Definition.** We say that $\mathcal{M}$ is $\mathbb{L}$-saturated if for every $p(x)$ as in 1 and 2 below, $\mathcal{M} \vDash \exists x\, p(a)$.

1. $p(x) \subseteq \mathbb{L}(A)$ for some $A \subseteq M$ of cardinality $< |M|$ and $|x| = 1$;
2. $p(x)$ is weakly finitely satisfied in $\mathcal{M}$.

The existence of $\mathbb{L}$-saturated models is proved as in the classical case.

13 Proposition. Every model has an $\mathbb{L}$-elementary extension to a saturated model (possibly of inaccessible cardinality). □

We denote by $\mathcal{U} = \langle U, R\rangle$ some large $\mathbb{L}$-saturated structure which we call the monster model. The cardinality of $\mathcal{U}$ is an inaccessible cardinal that we denote by κ. Below we say model for $\mathbb{L}$-elementariy substructure of $\mathcal{U}$.

14 Fact. For every $\varphi(x), \psi(x) \in \mathbb{L}(U)$ such that $\neg\exists x\,[\varphi(x) \wedge \psi(x)]$ there are $\varphi' > \varphi$ and $\psi' > \psi$ such that $\neg\exists x\,[\varphi'(x) \wedge \psi'(x)]$.

Proof. By Proposition **??** and saturation. □

6. The Tarski-Vaught $\mathbb{L}$-test

15 Lemma. (R compact) For every $\varphi' > \varphi$ there is a formula $\psi \in \mathbb{L}(M)$ such that $\varphi \to \neg\psi \to \varphi'$.

Proof. By induction. If $\varphi \in L$ the claim is obious. Suppose φ is of the form $t \in C$. Then φ' is $t \in D$ for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq D$. Then $\psi = (t \in R \smallsetminus O)$ is as required.

Induction is easy (conjuctions and disjuctions in φ occur swapped in ψ; similarly universal and existential quantifiers). □

The following lemma is a useful observation. It has its first application below, in the proof of our version of the Tarski-Vaught test.

16 Lemma. (R compact) Let $A \subseteq N$. Then the following are equivalent

1 for every formula $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{N} \vDash \exists x\,\neg\varphi(x) \quad\Rightarrow\quad \text{there is an } a \in A \text{ such that } \mathcal{N} \vDash \neg\varphi(a);$$

2. for every formula $\varphi(x) \in \mathbb{L}(A)$

$$\mathcal{N} \vDash \exists x\,\varphi(x) \quad\Rightarrow\quad \text{for every } \varphi' > \varphi \text{ there is an } a \in A \text{ such that } \mathcal{N} \vDash \varphi'(a).$$

The implication 2⇒1 does not require the assumption of compactness.

Proof. (2⇒1) Assume $\mathcal{N} \vDash \exists x\,\neg\varphi(x)$. Pick any $c \in N$ such that $c \vDash \neg\varphi(x)$. Let $p(x) = \mathbb{L}\text{-tp}(c/A)$. Without loss of generality we can assume that $\mathcal{N}$ is saturated. As $p(x) \cup \{\neg\varphi(x)\}$ is inconsistent there is is a formula $\psi(x) \in p(x)$ such that $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi$. Then from (2) we obtain that $\mathcal{N} \vDash \psi'(a)$ for some $a \in A$. Hence $\mathcal{N} \vDash \neg\varphi(a)$ follows.

(1⇒2) follows immediately from Lemma 15. □

17 Proposition (Tarski-Vaught $\mathbb{L}$-Test). Let M be a subset of N. Then the following are equivalent

1. M is the domain of a structure $\mathcal{M} \preceq^{\mathbb{L}} \mathcal{N}$;
2. for every formula $\varphi(x) \in \mathbb{L}(M)$

$$\mathcal{N} \vDash \exists x\, \varphi(x) \;\Rightarrow\; \text{for every } \varphi' > \varphi \text{ there is an } a \in A \text{ such that } \mathcal{N} \vDash \varphi'(a).$$

Proof. Assume (2) of the lemma. Then, by the classical Tarski-Vaught test $M \preceq N$. We also have that $\mathcal{M}$ is an L'-substructure of $\mathcal{N}$. Therefore, form every formula $\varphi(x)$ as in (i) and (ii) of Definition 2 we have

$$\mathcal{N} \vDash \varphi(a) \;\Leftrightarrow\; \mathcal{M} \vDash \varphi(a) \qquad\qquad \text{for every } a \in M^{|x|}.$$

Assume inductively

$$\# \qquad \mathcal{N} \vDash \varphi(a, b) \;\Leftrightarrow\; \mathcal{M} \vDash \varphi(a, b)$$

and prove

$$\mathcal{N} \vDash \exists y\, \varphi(a, y) \;\Leftrightarrow\; \mathcal{M} \vDash \exists y\, \varphi(a, y).$$

The implication $\Leftarrow$ follows immediately from the induction hypothesis. For the converse note that

$$\begin{aligned}
\mathcal{N} \vDash \exists y\, \varphi(a, y) \;&\Rightarrow\; \text{for every } \varphi' > \varphi \text{ there is an } b \in M^{|y|} \text{ such that } \mathcal{N} \vDash \varphi'(a, b)\\
&\Rightarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{M}\\
&\Rightarrow\; \mathcal{M} \vDash \exists y\, \varphi'(a, y) \text{ for every } \varphi' > \varphi\\
&\Rightarrow\; \mathcal{M} \vDash \exists y\, \varphi(a, y)
\end{aligned}$$

Now, assuming (#) we prove

$$\mathcal{N} \vDash \forall y\, \varphi(a, y) \;\Leftrightarrow\; \mathcal{M} \vDash \forall y\, \varphi(a, y).$$

One implication, $\Rightarrow$, is again immediate. For the converse note that by Lemma 16

$$\begin{aligned}
\mathcal{N} \nvDash \forall y\, \varphi(a, y) \;&\Rightarrow\; \mathcal{N} \vDash \exists y \neg\varphi(a, y)\\
&\Rightarrow\; \mathcal{N} \vDash \neg\varphi(a, b) \text{ for some } b \in M^{|y|}\\
&\Rightarrow\; \mathcal{M}\\
&\Rightarrow\; \mathcal{M} \nvDash \forall y\, \varphi(a, y)
\end{aligned}$$

□

Finally, induction for the boolean connectives and the quantifiers of sort R is immediate.

7. Completeness

(Work in progress)

For $a, b \in U$ we write $a \sim b$ if $t(a) = t(b)$ for every term $t(x)$ with parameters in U of sort $M \to R$.

18 Fact. If $\bar{a} = \langle a_i : i < \lambda \rangle$ and $\bar{b} = \langle b_i : i < \lambda \rangle$ are such that $a_i \sim b_i$ for every $i < \lambda$ then $t(\bar{a}) = t(\bar{b})$ for every term $t(x)$ with parameters in U of sort $M^\lambda \to R$.

Proof. By induction on λ. Assume the fact and let $a_\lambda \sim b_\lambda$. Then $t(\bar{a}, a_\lambda) = t(\bar{a}, b_\lambda) = t(\bar{b}, b_\lambda)$. Then the fact holds with $\lambda + 1$ for λ. For limit ordinals induction is immediate. □

19 Example (???). Let L be the language of $\mathbb{R}$-algebras expanded with two lattice operators $\wedge$, $\vee$. Let $\langle \Omega, \mathcal{B}, \Pr \rangle$ be a probability space. Let M be the set the simple real valued random variables with the natural interpretation of the symbols in L. Let $R = \mathbb{R}$. Assume L' estends L with all the continuous functions $\mathbb{R}^n \to \mathbb{R}$ and for every $n \in \mathbb{N}$ the functions E_n that gives the expected value of $(X \wedge n) \vee -n$ for $n \in \mathbb{N}$. Note that the cut-off enures that the range of these functions is bounded.

The relation $a \sim b$ holds in these there cases

We say that $a \in U$ is definable in the limit over M if $a \equiv_M^{\mathbb{L}} x \to a \sim x$

20 Example. Assume that $R = \mathbb{R}$ and that L' contains a function of sort $M^2 \to R$ that that is interpreted in a pseudometric. Assume that all terms of sort $M^n \to R$ are continuous with rispect to this pseudometric. It is easy to see that $a \sim b$ if and only if $d(a, b) = 0$. We claim that the following are equivalent

1. $a \in U$ is definable in the limit over M, a model;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ of elements of M that converges to a.

Proof. ($2 \Rightarrow 1$) Let $\langle \varepsilon_i : i \in \omega \rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then $d(a_i, b) \leq \varepsilon_i$ for every $b \equiv_M a$. By the uniqueness of the limit $d(a, b) = 0$.

($1 \Rightarrow 2$) Assume that $a \equiv_M x \to a \sim x$. Then $a \equiv_M^{\mathbb{L}} x \to d(a, x) < 1/n$ for every $n > 0$. By compactness there is a formula $\varphi_n(x) \in \mathrm{tp}_{\mathbb{L}}(a/M)$ such that $\varphi_n(x) \to d(a, x) < 1/n$. By $\mathbb{L}$-elementarity there is an $a_n \in M$ such that $\varphi(a_n)$. As $d(a, a_n) < 1/n$, the sequence $\langle a_n : n \in \omega \rangle$ converges to a. □