

Actions

A. Polymath

ABSTRACT.

1. Preliminaries

Let $\Delta \subseteq L_{xz}(\mathcal{U})$. Given any $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$, we define the structure $\mathcal{U}^\Delta = \langle \mathcal{X}; \mathcal{Z} \rangle$. This is a 2-sorted structure whose signature L^Δ contains relation symbols for every formula $\varphi(x; z) \in \Delta$. As there is little risk of confusion, these relations symbols are also denoted by $\varphi(x; z)$, and we identify L^Δ with a subset of L .

When \mathcal{X} and \mathcal{Z} are type definable and Δ is small, \mathcal{U}^Δ is a saturated structure. As compactness is not always required, these sets are arbitrary unless explicitly required.

In the first few sections we only work a subset of the quantifier-free fragment of L^Δ . Let $\Delta^B(\mathcal{Z})$ be the set of Boolean combinations of the sets $\varphi(\mathcal{X}; b)$, for $\varphi(x; z) \in \Delta$ and $b \in \mathcal{Z}$. The same symbol is also used to denote the collection of formulas defining these sets. A $\Delta^B(\mathcal{Z})$ -type is a collection of $\Delta^B(\mathcal{Z})$ -formulas. A $\Delta^B(\mathcal{Z})$ -type-definable set is one defined by a $\Delta^B(\mathcal{Z})$ -type of small cardinality.

We write $S_\Delta(\mathcal{Z})$ for the set of $\Delta^B(\mathcal{Z})$ -types that are maximally finitely consistent (in \mathcal{X}) – which we may conveniently identify with a sets of $\Delta(\mathcal{Z})$ and negated $\Delta(\mathcal{Z})$ -formulas. Note that the types in $S_\Delta(\mathcal{Z})$ are required to be finitely consistent \mathcal{X} – a piece of information that we will not display in the notation. When the types are required to be finitely consistent in some $\mathcal{Y} \subseteq \mathcal{X}$ we write $S_{\Delta, \mathcal{Y}}(\mathcal{Z})$.

When $p(x)$ a type (i.e. any set of formulas) and $\mathcal{D} \subseteq \mathcal{X}$ we write $p(x) \vdash x \in \mathcal{D}$ if the inclusion $\psi(\mathcal{X}) \subseteq \mathcal{D}$ holds for some $\psi(x)$ that is conjunctions of formulas in $p(x)$.

Let $G \leq \text{Aut}(\mathcal{U}^\Delta)$. We view G as a group acting on \mathcal{U}^Δ and write \cdot for such action. We write $g \cdot \mathcal{D}$ for the natural action of $g \in G$ on $\mathcal{D} \subseteq \mathcal{X}$. If $\Delta = \varphi(\mathcal{X}; b)$ then $g \cdot \mathcal{D} = \varphi(\mathcal{X}; g \cdot b)$. A formula is G -invariant if the set it defines is G -invariant (i.e. fixed by the action of G).

If $p(x)$ is a $\Delta^B(\mathcal{Z})$ -type, we write $g \cdot p(x)$ for the set of formulas of the form $\varphi(\mathcal{X}; g \cdot b)$ for $\varphi(\mathcal{X}; b) \in p$. We say that the type $p(x) \subseteq \Delta^B(\mathcal{Z})$ is G -invariant accordingly.

Let $\mathcal{Y} \subseteq \mathcal{X}$. We say that a set $\mathcal{D} \subseteq \mathcal{X}$ is **syndetic** under the action of G **relative** to \mathcal{Y} , or **G -syndetic** for short, if finitely many G -translates of \mathcal{D} cover \mathcal{Y} . Dually, we say that \mathcal{D} is **thick** under the action of G relative to \mathcal{Y} , or **G -thick** for short, if the intersection of any finitely many G -translates of \mathcal{D} is consistent with \mathcal{Y} .

We omit reference to \mathcal{Y} when this is \mathcal{X} or if it is made crystal clear by the context.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

Our terminology is taken from topological dynamics. In similar context syndetic sets are called *generic*, see eg. [N09]. In [CK] the authors write *quasi-non-dividing* for *thick* under the action of $\text{Aut}(\mathcal{U}/A)$.

Notation: for $\mathcal{D} \subseteq \mathcal{X}$ and $C \subseteq G$ we write $C \cdot \mathcal{D}$ for $\{h \cdot \mathcal{D} : h \in C\}$.

In this chapter many proofs require some juggling with negations as epitomized by the following fact which is proved by spelling out the definitions

1 Fact Work relative to some given $\mathcal{Y} \subseteq \mathcal{X}$. The following are equivalent

1. \mathcal{D} is not G -syndetic
2. $\neg \mathcal{D}$ is G -thick.

Define the following type

$$\Sigma_{G,\mathcal{Y}}(x) = \{\vartheta(x) \in \Delta^B(\mathcal{Z}) : \vartheta(x) \text{ is } G\text{-syndetic relative to } \mathcal{Y}\}.$$

We write $\Sigma_G(x)$ for $\Sigma_{G,\mathcal{X}}(x)$.

2 Theorem Work relative to some given $\mathcal{Y} \subseteq \mathcal{X}$. For every $p(x) \in S_{\Delta,\mathcal{Y}}(\mathcal{Z})$ the following are equivalent

1. $p(x)$ is G -invariant
2. $p(x) \vdash \Sigma_{G,\mathcal{Y}}(x)$
3. $p(x)$ is G -thick.

Proof. $1 \Rightarrow 2$. Let $\mathcal{D} \in \Delta^B(\mathcal{Z})$ be a G -syndetic. Pick $C \subseteq G$ be finite such that $\mathcal{Y} \subseteq \cup C \cdot \mathcal{D}$. Then $p(x)$ is finitely consistent with $x \in \cup C \cdot \mathcal{D}$. By completeness, $p(x) \vdash x \in h \cdot \mathcal{D}$ for some $h \in C$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

$2 \Rightarrow 3$. Let $\varphi(x) \in p$. As $\Sigma_{G,\mathcal{Y}}(x) \cup \{\varphi(x)\}$ is finitely consistent in \mathcal{Y} , it cannot be that $\neg \varphi(x)$ is G -syndetic. Then from Fact 1 we obtain that $\varphi(x)$ is G -thick.

$3 \Rightarrow 1$. Negate 1. Let $\mathcal{D} \in \Delta^B(\mathcal{Z})$ and $g \in G$ such that $p(x) \vdash x \in \mathcal{D}$ and $p(x) \not\vdash g \cdot \mathcal{D}$. By completeness $p(x) \vdash x \in (\mathcal{D} \cap \neg g \cdot \mathcal{D})$. Clearly $\mathcal{D} \cap \neg g \cdot \mathcal{D}$ is not G -thick as it is inconsistent with its g -translate. \square

The following theorem gives a necessary and sufficient condition for the existence of global G -invariant $\Delta^B(\mathcal{Z})$ -type. Ideally, we would like that every G -thick $\Delta^B(\mathcal{Z})$ -type extends to a global thick type. Unfortunately this is not true in general (it is a strong assumption, see Section ??).

A set \mathcal{D} is **G -wide** relative to \mathcal{Y} if every finite cover of $\mathcal{D} \cap \mathcal{Y}$ by $\Delta^B(\mathbb{Z})$ -sets contains a set that is G -thick relative to \mathcal{Y} . A type is G -wide if every conjunction of formulas in the type is G -wide.

In [?CK] a similar property is called *quasi-non-forking*. Our use of the term *wide* is consistent with [?Hr], though we apply it to a narrow context.

3 Theorem Work relative to some given $\mathcal{Y} \subseteq \mathcal{X}$. For every $\mathcal{D} \in \Delta^B(\mathbb{Z})$ the following are equivalent

1. $\Sigma_{G,\mathcal{Y}}(x) \cup \{x \in \mathcal{D}\}$ is finitely consistent in \mathcal{Y}
2. there is a G -thick type $p(x) \in S_{\Delta,\mathcal{Y}}(\mathbb{Z})$ that entails $x \in \mathcal{D}$
3. \mathcal{D} is G -wide.

Proof. $1 \Rightarrow 2$. By Theorem 2, it suffices to pick any $p(x) \in S_{\Delta,\mathcal{Y}}(\mathbb{Z})$ extending the type $\Sigma_{G,\mathcal{Y}}(x) \cup \{x \in \mathcal{D}\}$.

$2 \Rightarrow 1$. By Theorem 2.

$2 \Rightarrow 3$. Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be $\Delta^B(\mathbb{Z})$ -sets that cover \mathcal{D} . Pick $p(x)$ as in 2. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i . Therefore, \mathcal{C}_i is G -thick.

$3 \Rightarrow 2$. Let $p(x)$ be maximal among the $\Delta^B(\mathbb{Z})$ -types that are finitely satisfiable in $\mathcal{Y} \cap \mathcal{D}$ and are G -wide. We claim that $p(x)$ is a complete $\Delta^B(\mathbb{Z})$ -type. Suppose for a contradiction that $\theta(x), \neg\theta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\Delta^B(\mathbb{Z})$ -sets $\mathcal{C}_1, \dots, \mathcal{C}_n$ that cover both $\psi(\mathcal{X}) \cap \theta(\mathcal{X})$ and $\psi(\mathcal{X}) \setminus \theta(\mathcal{X})$ and such that no \mathcal{C}_i is G -thick. As $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\psi(\mathcal{X})$ this is a contradiction. It is only left to show that $p(x)$ is G -thick. This follows from completeness and Theorem 2. \square

2. Connected components

Let $G \leq \text{Aut}(\mathcal{U}^\Delta)$. Unfortunately, syndeticity is not preserved under intersection. In particular $\Sigma_G(x)$ is not a G -syndetic type, and it may even be inconsistent. Then following notion is relevant.

4 Definition

$$Q_G = \{q(x) \subseteq \Sigma_G(x) : q(x) \text{ maximally } G\text{-syndetic}\}.$$

In other words, the types in Q_G are maximal among the subtypes of $\Sigma_G(x)$ that are closed under conjunction.

It is easy to see that Q_G is closed under the action of G . We write $\text{Stab}(q)$ for the stabilizer of $q(x) \subseteq \Delta^B(\mathbb{Z})$ in G , that is, the subgroup $\{g \in G : g \cdot q(x) = q(x)\}$. We write $\text{Stab}(\mathcal{D})$ with a similar meaning. Finally we define

$$G^1 = \text{Stab}(Q_G) = \bigcap_{q \in Q_G} \text{Stab}(q).$$

It is easy to verify that $G^1 \trianglelefteq G$.

5 Proposition

$$G^1 = \{g \in G : \mathcal{D} \cap g \cdot \mathcal{D} \in \Sigma_G(x) \text{ whenever } \mathcal{D} \in \Sigma_G(x)\}.$$

Proof. \subseteq . Pick any $k \in G^1$ and $\mathcal{D} \in \Sigma_G(x)$. Let $q(x) \in Q_G$ be a type containing $x \in \mathcal{D}$. From the G^1 -invariance of $q(x)$ we obtain that $q(x) \vdash x \in k \cdot \mathcal{D}$. Then $q(x) \vdash x \in \mathcal{D} \cap k \cdot \mathcal{D}$, hence $\mathcal{D} \cap k \cdot \mathcal{D}$ is G -syndetic.

\supseteq . Pick any $g \notin G^1$. Then $q(x) \neq g \cdot q(x)$ for some $q(x) \in Q_G$. Let $\varphi(x) \in q$ such that $q(x) \not\vdash g \cdot \varphi(x)$. By maximality, $\psi(x) \wedge g \cdot \varphi(x)$ is not G -syndetic for some $\psi(x) \in q$. As $q(x)$ is closed under conjunction, we can assume $\varphi(x) = \psi(x)$, then g does not belong to the set on the r.h.s. \square

6 Theorem Any finite conjunction of formulas in $\Sigma_{G^1}(x)$ is G -syndetic. In particular $\Sigma_{G^1}(x)$ is finitely consistent.

Proof. Notice that from Proposition 5 it easily follows that for every $\mathcal{D} \in \Sigma_G(x)$ and every finite $F \subseteq G^1$ the set $\cap F \cdot \mathcal{D}$ is G -syndetic.

Let $\mathcal{D}_1, \dots, \mathcal{D}_n \in \Sigma_{G^1}(x)$. Assume inductively that $\mathcal{D}_1 \cap \dots \cap \mathcal{D}_{n-1}$ is G -syndetic. Let $F \subseteq G^1$ be such that $\cup F \cdot \mathcal{D}_n = \mathcal{X}$. Then

$$\begin{aligned} \cup F \cdot [\mathcal{D}_1 \cap \dots \cap \mathcal{D}_n] &\supseteq \cap F \cdot [\mathcal{D}_1 \cap \dots \cap \mathcal{D}_{n-1}] \cap \cup F \cdot \mathcal{D}_n \\ &= \cap F \cdot [\mathcal{D}_1 \cap \dots \cap \mathcal{D}_{n-1}]. \end{aligned}$$

This last set is G -syndetic by the inductive hypothesis and what remarked above. The G -syndeticity of $\mathcal{D}_1 \cap \dots \cap \mathcal{D}_n$ follows. \square

Unfortunately, we are unable to conclude that the intersection of G^1 -syndetic sets is G^1 -syndetic.

From Theorems 3 and 6 it follows that G^1 -syndetic sets are G^1 -wide. But we can do better. First, we remark a useful consequence of normality.

7 Remark Assume $H \trianglelefteq G$. For every $\mathcal{D} \subseteq \mathcal{X}$ and every $g \in G$

$$\mathcal{D} \text{ is } H\text{-foo} \iff g \cdot \mathcal{D} \text{ is } H\text{-foo},$$

where *foo* can be replaced by *syndetic*, *invariant*, *thick*, *wide*. In particular, the type $\Sigma_H(x)$ is G -invariant.

Recall that when $\Sigma_H(x)$ is finitely consistent then H -syndetic sets are H -wide, see Theorem 3. As it happens, under the assumption of normality, this can be strengthened as follows.

8 Proposition Assume $H \trianglelefteq G$ and that $\Sigma_H(x)$ is finitely consistent. Then every G -syndetic $\mathcal{D} \in \Delta^B(\mathcal{Z})$ is H -wide. In particular all types in Q_G are G^1 -wide.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be finitely consistent with $\Sigma_H(x)$. As \mathcal{D} is G -syndetic, by completeness $p(x) \vdash x \in g \cdot \mathcal{D}$ for some $g \in G$. Equivalently, $g^{-1} \cdot p(x) \vdash x \in \mathcal{D}$. As $p(x)$ is H -thick, by Remark 7 also $g^{-1} \cdot p(x)$ is H -thick. Then the proposition follows from Theorem 3. \square

When the context suggests it, $\Delta^B(\mathcal{Z})$ will denote the broader class that includes also formulas obtained substituting y for x and is closed under Boolean connectives. A $\Delta^B(\mathcal{Z})$ -definable equivalence relation is an equivalence relation defined by some formula $\varepsilon(x, y) \in \Delta^B(\mathcal{Z})$. We define $\Delta^B(\mathcal{Z})$ -type-definable equivalence relations similarly. Note that if $\varepsilon(x, y)$ is a $\Delta^B(\mathcal{Z})$ -definable, or type-definable, equivalence relation then $\varepsilon(x, a)$, for any $a \in \mathcal{X}$, is equivalent to a $\Delta^B(\mathcal{Z})$ -formula, respectively a $\Delta^B(\mathcal{Z})$ -type.

An equivalence relation is bounded if it has a small number of equivalence classes.

9 Definition Let Φ^0 be the collection of sets that are equivalence classes of a bounded G -invariant $\Delta^B(\mathcal{Z})$ -type-definable equivalence relation. Let Φ^{00} be the collection of sets that are equivalence classes of a bounded G -invariant $\Delta^B(\mathcal{Z})$ -type-definable equivalence relation. Define

$$G^0 = \text{Stab}(\Phi^0) = \bigcap \{ \text{Stab}(\mathcal{D}) : \mathcal{D} \in \Phi^0 \}$$

and

$$G^{00} = \text{Stab}(\Phi^{00}) = \bigcap \{ \text{Stab}(\mathcal{Y}) : \mathcal{Y} \in \Phi^{00} \}.$$

10 Proposition Let \mathcal{X} be type-definable. Assume that G acts transitively on \mathcal{X} . Then

$$G^1 \leq G^{00} \leq G^0.$$

Proof. The inclusion $G^{00} \leq G^0$ is trivial. Let $\mathcal{Y} \in \Phi^{00}$ and $\mathcal{Y} \subseteq \mathcal{D} \in \Delta^B(\mathcal{Z})$. Then a small number of G -translates of \mathcal{D} cover \mathcal{X} . Therefore, by compactness, \mathcal{D} is G -syndetic. Then the type defining \mathcal{Y} is G -syndetic. Then this type extends to some $q(x) \in Q_G$. By G^1 -invariance $q(x)$ entails also $k \cdot \mathcal{Y}$ for every $k \in G^1$. Then k is in the stabilizer of \mathcal{Y} . \square