Actions

ABSTRACT.

1. Preliminaries

Let $\Delta \subseteq L_{xz}(\mathcal{U})$. Given any $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$, we define the structure $\mathcal{U}^\Delta = \langle \mathcal{X};\mathcal{Z}\rangle$. This is a 2-sorted structure whose signature L^Δ contains relation symbols for every formula $\varphi(x;z) \in \Delta$. As there is little risk of confusion, these relations symbols are also denoted by $\varphi(x;z)$, and we identify L^Δ with a subset of L.

When $\mathcal{X}$ and $\mathcal{Z}$ are type definable and Δ is small, $\mathcal{U}^\Delta$ is a saturated structure. As compactness is not always required, these sets are arbitrary unless explicitly required.

In the first few sections we only work a subset of the quantifier-free fragment of L^Δ. Let $\Delta^{\mathsf{B}}(\mathcal{Z})$ be the set of Boolean combinations of the sets $\varphi(\mathcal{X};b)$, for $\varphi(x;z) \in \Delta$ and $b \in \mathcal{Z}$. The same symbol is also used to denote the collection of formulas defining these sets. A $\Delta^{\mathsf{B}}(\mathcal{Z})$-type is a collection of $\Delta^{\mathsf{B}}(\mathcal{Z})$-formulas. A $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable set is one defined by a $\Delta^{\mathsf{B}}(\mathcal{Z})$-type of small cardinality.

We write $S_\Delta(\mathcal{Z})$ for the set of $\Delta^{\mathsf{B}}(\mathcal{Z})$-types that are maximally finitely consistent (in $\mathcal{X}$). Sometimes these are conveniently identified with a sets of $\Delta(\mathcal{Z})$ and negated $\Delta(\mathcal{Z})$-formulas. Note that the types in $S_\Delta(\mathcal{Z})$ are required to be finitely consistent $\mathcal{X}$ – a piece of information that we will not display in the notation.

When $p(x)$ a type (i.e. any set of formulas) and $\mathcal{D} \subseteq \mathcal{X}$ we write $p(x) \vdash x \in \mathcal{D}$ if the inclusion $\psi(\mathcal{X}) \subseteq \mathcal{D}$ holds for some $\psi(x)$ that is conjunctions of formulas in $p(x)$.

Let $G \leq \mathrm{Aut}(\mathcal{U}^\Delta)$. We view G as a group acting on $\mathcal{U}^\Delta$ and write $\cdot$ for such action. We write $g\cdot\mathcal{D}$ for the natural action of $g \in G$ on $\mathcal{D} \subseteq \mathcal{X}$. If $\mathcal{D} = \varphi(\mathcal{X};b)$ then $g\cdot\mathcal{D} = \varphi(\mathcal{X};g\cdot b)$. A formula is G-invariant if the set it defines is G-invariant (i.e. fixed by the action of G).

If $p(x)$ is a $\Delta^{\mathsf{B}}(\mathcal{Z})$-type, we write $g\cdot p(x)$ for the set of formulas of the form $\varphi(\mathcal{X};g\cdot b)$ for $\varphi(\mathcal{X};b) \in p$. We say that the type $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ is G-invariant accordingly.

We say that a set $\mathcal{D} \subseteq \mathcal{X}$ is syndetic under the action of G, or G-syndetic for short, if finitely many G-translates of $\mathcal{D}$ cover $\mathcal{X}$. Dually, we say that $\mathcal{D}$ is thick under the action

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of G or G-thick, if the intersection of any finitely many G-translates of $\mathcal{D}$ is consistent (in $\mathcal{X}$).

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

Our terminology is taken from topological dynamics. In similars context syndetic sets are called *generic*, see eg. [Ne]. In [CK] the authors write *quasi-non-dividing* over A for *thick* under the action of $\mathrm{Aut}(\mathcal{U}/A)$.

Notation: for $\mathcal{D} \subseteq \mathcal{X}$ and $C \subseteq G$ we write $C{\cdot}\mathcal{D}$ for $\{h{\cdot}\mathcal{D} : h \in C\}$.

1 Remark Let $q(x)$ be a $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. We say that $\mathcal{D}$ is G-syndetic relative to $q(x)$ if $q(x)$ entails $\cup F{\cdot}\mathcal{D}$ for some finite $F \subseteq G$. Dually, $\mathcal{D}$ is G-syndetic relative to $q(x)$ if $G{\cdot}\mathcal{D}$ is finitely consistent with $q(x)$. The notions of *wide, strong syndetic,* and *weakly thick,* which we indroduce below relitivize likewise. All results below easily relativize to any type $q(x)$. We entrust the generalization to the reader (this is only required in Fact 22).

In this chapter many proofs require some juggling with negations as epitomized by the following fact which is proved by spelling out the definitions

2 Fact The following are equivalent

1. $\mathcal{D}$ is not G-syndetic
2. $\neg\mathcal{D}$ is G-thick.

Define the following type

$$\Sigma_G(x) \quad = \quad \big\{\vartheta(x) \in \Delta^{\mathsf{B}}(\mathcal{Z}) \ : \ \vartheta(x) \text{ is } G\text{-syndetic}\big\}.$$

3 Theorem For every $p(x) \in S_\Delta(\mathcal{Z})$ the following are equivalent

1. $p(x)$ is G-invariant
2. $p(x) \vdash \Sigma_G(x)$
3. $p(x)$ is G-thick.

Proof. 1⇒2. Let $\mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$ be a G-syndetic. Pick $C \subseteq G$ be finite such that $\mathcal{X} \subseteq \cup C{\cdot}\mathcal{D}$. Then $p(x)$ is finitely consistent with $x \in \cup C{\cdot}\mathcal{D}$. By completeness, $p(x) \vdash x \in h\mathcal{D}$ for some $h \in C$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

2⇒3. Let $\varphi(x) \in p$. As $\Sigma_G(x) \cup \{\varphi(x)\}$ is finitely consistent, it cannot be that $\neg\varphi(x)$ is G-syndetic. Then from Fact 2 we obtain that $\varphi(x)$ is G-thick.

3⇒1. Negate 1. Let $\mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$ and $g \in G$ such that $p(x) \vdash x \in \mathcal{D}$ and $p(x) \nvdash g{\cdot}\mathcal{D}$. By completeness $p(x) \vdash x \in (\mathcal{D} \cap \neg g{\cdot}\mathcal{D})$. Clearly $\mathcal{D} \cap \neg g{\cdot}\mathcal{D}$ is not G-thick as it is inconsistent with its g-translate. □

The following theorem gives a necessary and sufficient condition for the existence of global G-invariant $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Ideally, we would like that every G-thick $\Delta^{\mathsf{B}}(\mathcal{Z})$-type extends to a global thick type. Unfortunately this is not true in general (it is an assumption with important consequences, see Section 4).

A set $\mathcal{D}$ is G-wide if every finite cover of $\mathcal{D}$ by $\Delta^{\mathsf{B}}(\mathcal{Z})$-sets contains a set that is G-thick. A type is G-wide if every conjunction of formulas in the type is G-wide.

In [**CK**] a similar property is called *quasi-non-forking*. Our use of the term *wide* is consistent with [**Hr**], though we apply it to a narrow context.

4 Theorem For every $\mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$ the following are equivalent

1. $\Sigma_G(x) \cup \{x \in \mathcal{D}\}$ is finitely consistent
2. there is a G-thick type $p(x) \in S_\Delta(\mathcal{Z})$ that entails $x \in \mathcal{D}$
3. $\mathcal{D}$ is G-wide.

Proof. 1⇒2. By Theorem 3, it suffices to pick any $p(x) \in S_\Delta(\mathcal{Z})$ extending the type $\Sigma_G(x) \cup \{x \in \mathcal{D}\}$.

2⇒1. By Theorem 3.

2⇒3. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be $\Delta^{\mathsf{B}}(\mathcal{Z})$-sets that cover $\mathcal{D}$. Pick $p(x)$ as in 2. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i. Therefore, $\mathcal{C}_i$ is G-thick.

3⇒2. Let $p(x)$ be maximal among the $\Delta^{\mathsf{B}}(\mathcal{Z})$-types that are finitely consistent with $\mathcal{D}$ and are G-wide. We claim that $p(x)$ is a complete $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\Delta^{\mathsf{B}}(\mathcal{Z})$-sets $\mathcal{C}_1, \ldots, \mathcal{C}_n$ that cover both $\psi(\mathcal{X}) \cap \vartheta(\mathcal{X})$ and $\psi(\mathcal{X}) \smallsetminus \vartheta(\mathcal{X})$ and such that no $\mathcal{C}_i$ is G-thick. As $\mathcal{C}_1, \ldots, \mathcal{C}_n$ cover $\psi(\mathcal{X})$ this is a contradiction. It is only left to show that $p(x)$ is G-thick. This follows from completeness and Theorem 3. □

2. Connected components

Let $G \leq \mathrm{Aut}(\mathcal{U}^\Delta)$. Unfortunately, syndeticity is not preserved under intersection. In particular $\Sigma_G(x)$ is not a G-syndetic type, and it may even be inconsistent. Then following notion is relevant.

5 Definition

$$Q_G \quad = \quad \big\{q(x) \subseteq \Sigma_G(x) \,:\, q(x) \text{ maximally } G\text{-syndetic}\big\}.$$

In other words, the types in Q_G are maximal among the subtypes of $\Sigma_G(x)$ that are closed under conjunction.

It is easy to see that Q_G is closed under the action of G. We write $\mathrm{Stab}(q)$ for the stabilizer of $q(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ in G, that is, the subgroup $\{g \in G \,:\, g \cdot q(x) = q(x)\}$. We write $\mathrm{Stab}(\mathcal{D})$ with a similar meaning. Finally we define

$$G^1 \quad = \quad \mathrm{Stab}(Q_G) \quad = \quad \bigcap_{q\in Q_G} \mathrm{Stab}(q).$$

It is easy to verify that $G^1 \trianglelefteq G$.

6 Proposition

$$G^1 \quad = \quad \big\{g\in G \,:\, \mathcal{D}\cap g{\cdot}\mathcal{D}\in \Sigma_G(x) \text{ whenever } \mathcal{D}\in\Sigma_G(x)\big\}.$$

Proof. $\subseteq$. Pick any $k\in G^1$ and $\mathcal{D}\in\Sigma_G(x)$. Let $q(x)\in Q_G$ be a type containing $x\in\mathcal{D}$. From the G^1-invariance of $q(x)$ we obtain that $q(x)\vdash x\in k{\cdot}\mathcal{D}$. Then $q(x)\vdash x\in\mathcal{D}\cap k{\cdot}\mathcal{D}$, hence $\mathcal{D}\cap k{\cdot}\mathcal{D}$ is G-syndetic.

$\supseteq$. Pick any $g\notin G^1$. Then $q(x)\neq g{\cdot}q(x)$ for some $q(x)\in Q_G$. Let $\varphi(x)\in q$ such that $q(x)\nvdash g{\cdot}\varphi(x)$. By maximality, $\psi(x)\wedge g{\cdot}\varphi(x)$ is not G-syndetic for some $\psi(x)\in q$. As $q(x)$ is closed under conjunction, we can assume $\varphi(x)=\psi(x)$, then g does not belong to the set on the r.h.s. □

7 Definition

$$P_G \quad = \quad \big\{p(x)\subseteq\Sigma_G(x) \,:\, p(x) \text{ maximally finitely consistent}\big\}.$$

Along the same lines as above, we define

$$G^{11} \quad = \quad \mathrm{Stab}(P_G) \quad = \quad \bigcap_{p\in P_G} \mathrm{Stab}(p).$$

By the following proposition, $G^1\le G^{11}\trianglelefteq G$. Note that when $\Sigma_G(x)$ is finitely consistent, $\Sigma_G(x)$ is the only element of P_G and $G^{11}=G$.

8 Proposition

$$G^{11} \quad = \quad \big\{g\in G \,:\, \mathcal{D}\cap g{\cdot}\mathcal{D}\neq\varnothing \text{ for every } \mathcal{D}\neq\varnothing \text{ intersection of sets in } \Sigma_G(x)\big\}.$$

Proof. $\subseteq$. Pick $k\in G^{11}$ and an intersection of G-syndetic sets $\mathcal{D}$. Let $p(x)\in P_G$ be a type entailing $\mathcal{D}$. From the G^{11}-invariance of $p(x)$ we obtain that $p(x)\vdash x\in k{\cdot}\mathcal{D}$. Then $x\in\mathcal{D}\cap k{\cdot}\mathcal{D}$ is consistent.

$\supseteq$. Pick any $g\notin G^{11}$. Then $p(x)\nsubseteq g{\cdot}p(x)$ for some $p(x)\in P_G$. Then $\varphi(x)\wedge g{\cdot}\varphi(x)$ is inconsistent for some $\varphi(x)\in p$. But $\varphi(x)$ is a consistent conjunction of formulas in $\Sigma_G(x)$. Then g does not belong to the set on the r.h.s. □

9 Theorem Any finite conjunction of formulas in $\Sigma_{G^1}(x)$ is G-syndetic. In particular $\Sigma_{G^1}(x)$ is finitely consistent.

Unfortunately, we are unable to conclude that the intersection of G^1-syndetic sets is G^1-syndetic.

Proof. Notice that from Proposition 6 it easily follows that for every $\mathcal{D}\in\Sigma_G$ and every finite $F\subseteq G^1$ the set $\cap F{\cdot}\mathcal{D}$ is G-syndetic.

Let $\mathcal{D}_1,\ldots,\mathcal{D}_n \in \Sigma_{G^1}$. Assume inductively that $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}$ is G-syndetic. Let $F \subseteq G^1$ be such that $\cup F{\cdot}\mathcal{D}_n = \mathcal{X}$. Then

$$1. \qquad \begin{aligned} \cup F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n] &\supseteq \cap F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}] \cap \cup F{\cdot}\mathcal{D}_n \\ &= \cap F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}]. \end{aligned}$$

This last set is G-syndetic by the inductive hypothesis and what remarked above. The G-syndedicity of $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ follows. □

A similar argument shows the following.

10 Theorem $\Sigma_{G^{11}}(x)$ is finitely consistent.

Proof. Let $\mathcal{D}_1,\ldots,\mathcal{D}_n \in \Sigma_{G^{11}}$. Assume inductively that $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1} \neq \varnothing$. As the $\mathcal{D}_i$ are in particular G-syndetic, some $p(x) \in P_G$ entails this disjunction. Let $F \subseteq G^{11}$ be such that $\cup F{\cdot}\mathcal{D}_n = \mathcal{X}$. By G^{11}-invariance, $p(x)$ also entails $\cap F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}]$. Then by (1) in the above proof $\cup F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n] \neq \varnothing$. The consistency of $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ follows. □

From the above theorems and Theorems 4 it follows that G^1-syndetic sets are G^1-wide, similarly for G^{11}. But we can do better. First, we remark a useful consequence of normality.

11 Remark Assume $H \trianglelefteq G$. For every $\mathcal{D} \subseteq \mathcal{X}$ and every $g \in G$

$$\mathcal{D} \text{ is } H\text{-foo} \quad \Rightarrow \quad g{\cdot}\mathcal{D} \text{ is } H\text{-foo},$$

where *foo* can be replaced by *syndetic, invariant, thick, wide*. In particular, the type $\Sigma_H(x)$ is G-invariant.

Recall that when $\Sigma_H(x)$ is finitely consistent then H-syndetic sets are H-wide, see Theorem 4. As it happens, under the assumption of normality, this can be strengthened as follows.

12 Proposition Assume $H \trianglelefteq G$ and that $\Sigma_H(x)$ is finitely consistent. Then every G-syndetic $\mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$ is H-wide. In particular all types in Q_G are G^{11}-wide.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be finitely consistent with $\Sigma_H(x)$. As $\mathcal{D}$ is G-syndetic, by completeness $p(x) \vdash x \in g{\cdot}\mathcal{D}$ for some $g \in G$. Equivalently, $g^{-1}{\cdot}\, p(x) \vdash x \in \mathcal{D}$. As $p(x)$ is H-thick, by Remark 11 also $g^{-1}{\cdot}\, p(x)$ is H-thick. Then the proposition follows from Theorem 4. □

When the context suggests it, $\Delta^{\mathsf{B}}(\mathcal{Z})$ will denote the broader class that includes also formulas obtained substituting y for x and is closed under Boolean connectives. A $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable equivalence relation is an equivalence relation defined by some formula $\varepsilon(x,y) \in \Delta^{\mathsf{B}}(\mathcal{Z})$. We define $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable equivalence relations similarly. Note that if $\varepsilon(x,y)$ is a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable, or type-definable, equivalence relation then $\varepsilon(x,a)$, for any $a \in \mathcal{X}$, is equivalent to a $\Delta^{\mathsf{B}}(\mathcal{Z})$-formula, respectively a $\Delta^{\mathsf{B}}(\mathcal{Z})$-type.

An equivalence relation is bounded if it has a small number of equivalence classes.

13 Definition Let Φ^0 be the collection of sets that are equivalence classes of a bounded G-invariant $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable equivalence relation. Let Φ^{00} be the collection of sets that are equivalence classes of a bounded G-invariant $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable equivalence relation. Define

$$G^0 \quad = \quad \mathrm{Stab}(\Phi^0) \quad = \quad \bigcap\{\mathrm{Stab}(\mathcal{D}) \ : \ \mathcal{D} \in \Phi^0\}$$

and

$$G^{00} \quad = \quad \mathrm{Stab}(\Phi^{00}) \quad = \quad \bigcap\{\mathrm{Stab}(\mathcal{Y}) \ : \ \mathcal{Y} \in \Phi^{00}\}.$$

The subgroup G^{000} is defined similarly using the collection Φ^{000} of sets that are equivalence classes of some bounded G-invariant equivalence relations.

14 Proposition Let $\mathcal{X}$ be type-definable. Assume that G acts transitively on $\mathcal{X}$. Then

$$G^{11} \quad \leq \quad G^{00} \quad \leq \quad G^0.$$

Proof. The inclusion $G^{00} \leq G^0$ is trivial. Let $\mathcal{Y} \in \Phi^{00}$ and $\mathcal{Y} \subseteq \mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$. Then a small number of G-translates of $\mathcal{D}$ cover $\mathcal{X}$. Therefore, by compactness, $\mathcal{D}$ is G-syndetic. Then the type defining $\mathcal{Y}$ is G-syndetic. Then this type extends to some $p(x) \in P_G$. By G^{11}-invariance $p(x)$ entails also $k{\cdot}\mathcal{Y}$ for every $k \in G^{11}$. Then k is in the stabilizer of $\mathcal{Y}$. □

3. Strong syndeticity

A set $\mathcal{D}$ is strongly G-syndetic if for every finite $F \subseteq G$ the set $\cap F{\cdot}\mathcal{D}$ is G-syndetic (recall that $F{\cdot}\mathcal{D}$ stands for $\{h{\cdot}\mathcal{D} : h \in F\}$). Dually, we say that $\mathcal{D}$ is weakly G-thick if for some finite $F \subseteq G$ the set $\cup F{\cdot}\mathcal{D}$ is thick. Again, the same properties may be attributed to formulas and types when every conjunction of formulas in the type has the property.

⚠ In topological dynamic, strong syndedic sets are called *thickly syndetic* and weak thickness is called *piecewise syndetic*. Newelski in [Ne] says *weak generic* for weakly thick. These terminologies defy my intuition.

I belive most facts in this section are well-known.

15 Lemma Assume $\mathcal{D}$ is strongly G-syndetic and $\mathcal{C}$ is G-syndetic. Then $\mathcal{D} \cap \mathcal{C}$ is is G-syndetic.

Proof. Pick some finite $F \subseteq G$ such that $\mathcal{X} = \cup F{\cdot}\mathcal{C}$. Note that

$$\begin{aligned} \cup F{\cdot}[\mathcal{D} \cap \mathcal{C}] \quad &\supseteq \quad \big(\cap F{\cdot}\mathcal{D}\big) \cap \big(\cup F{\cdot}\mathcal{C}\big) \\ &= \quad \cap F{\cdot}\mathcal{D} \end{aligned}$$

As $\mathcal{D}$ is strongly G-syndetic, $\cap F{\cdot}\mathcal{D}$ is G-syndetic. Therefore $\cup F{\cdot}[\mathcal{D} \cap \mathcal{C}]$ is G-syndetic. The G-syndeticity of $\mathcal{D} \cap \mathcal{C}$ follows. □

16 Lemma The intersection of two strongly G-syndetic sets is strongly G-syndetic.

Proof. Let $\mathcal{D}$ and $\mathcal{C}$ be strongly G-syndetic and let $C \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap C{\cdot}(\mathcal{C}\cap\mathcal{D})$ is G-syndetic. Clearly $\mathcal{B} = \mathcal{C}'\cap\mathcal{D}'$, where $\mathcal{C}' = \cap C{\cdot}\mathcal{C}$ and $\mathcal{D}' = \cap C{\cdot}\mathcal{D}$. Note that $\mathcal{C}'$ and $\mathcal{D}'$ are both strongly G-syndetic. By the above lemma, $\mathcal{B}$ is G-syndetic. □

Define the following type

$$ {}^{\mathrm{s}}\Sigma_G(x) \quad = \quad \big\{\vartheta(x) \in \Delta^{\mathrm{B}}(\mathcal{Z}) \ : \ \vartheta(x) \text{ is strongly } G\text{-syndetic}\big\}. $$

Note that $\Sigma_{G^1}(x) \subseteq {}^{\mathrm{s}}\Sigma_G(x)$ by Theorem 9.

17 Corollary (with Alex Kruckman) The type ${}^{\mathrm{s}}\Sigma_G(x)$ is finitely consistent, strongly G-syndetic, and G-invariant. Moreover,

$$ {}^{\mathrm{s}}\Sigma_G(x) \quad = \quad \bigcap_{q\in Q_G} q(x), $$

where Q_G is as in Definition 5.

Proof. Strong G-syndeticity is an immediate consequence of Lemma 16. Finite consistency is a consequence of syndeticity. Finally, G-invariance is clear because any translate of a strongly G-syndetic formula is also strongly G-syndetic.

As for the final equality, we prove $\subseteq$ first. If some $q(x) \in Q_G$ does not contains $\mathcal{D}$, then it contains some $\mathcal{C}$ such that $\mathcal{C}\cap\mathcal{D}$ is not G-syndetic. Then $\mathcal{D}$ is not G-syndetic, by Lemma 15.

Conversely, let $\mathcal{D}$ a set in the r.h.s. Then any G-translation of $\mathcal{D}$ also belong to the r.h.s. As this is closed under conjunction $\cap F{\cdot}\mathcal{D}$ is G-syndetic for every finite $F \subseteq G$. □

18 Proposition For every $\mathcal{D} \in \Delta^{\mathrm{B}}(\mathcal{Z})$ the following are equivalent

1. ${}^{\mathrm{s}}\Sigma_G(x) \cup \{x \in \mathcal{D}\}$ is finitely consistent
2. $\mathcal{D}$ is weakly G-thick.

Proof. 1⇒2. If ${}^{\mathrm{s}}\Sigma_G(x) \cup \{x \in \mathcal{D}\}$ is finitely consistent, then $\neg\mathcal{D}$ is strongly G-syndetic. From Fact 2, we obtain that $\neg\mathcal{D}$ not being strongly G-syndetic is equivalent to $\mathcal{D}$ being weakly G-thick.

2⇒1. Suppose ${}^{\mathrm{s}}\Sigma_G(x) \vdash x \notin \mathcal{D}$. Then $\neg\mathcal{D}$ is strongly G-syndetic. From Fact 2, $\mathcal{D}$ is not weakly G-thick. □

The following theorem asserts that weak thickness is partition regular.

19 Theorem If $\mathcal{C}\cup\mathcal{B}$ is weakly G-thick then $\mathcal{B}$ or $\mathcal{C}$ is weakly G-thick.

Proof. As ${}^{\mathrm{s}}\Sigma_G(x)$ is closed under conjunction. If $x \in \mathcal{C}\cup\mathcal{B}$ is finitely consistent with ${}^{\mathrm{s}}\Sigma_G(x)$ then so is one of the two sets. Then the treorem follows from the above proposition. □

4. A tamer landscape

Under suitable assumptions some notions introduced in these notes coalesce, and we are left with a tamer landscape.

20 Theorem The following are equivalent

1. G-thick $\Delta^{\mathrm{B}}(\mathcal{Z})$-sets are G-wide
2. G-syndetic $\Delta^{\mathrm{B}}(\mathcal{Z})$-sets are closed under intersection
3. G-syndetic $\Delta^{\mathrm{B}}(\mathcal{Z})$-sets are strongly G-syndetic
4. weakly G-thick $\Delta^{\mathrm{B}}(\mathcal{Z})$-sets are G-thick.

Proof. Clearly 2⇔3⇔4.

1⇒2. Let $\mathcal{C}$ and $\mathcal{D}$ be G-syndetic $\Delta^{\mathrm{B}}(\mathcal{Z})$-sets. Suppose for a contradiction that $\mathcal{C}\cap\mathcal{D}$ is not G-syndetic. Then $\neg(\mathcal{C}\cap\mathcal{D})$ is G-thick. By (1) and Theorem 4 there is a G-invariant type $p(x)\in S_\Delta(\mathcal{Z})$ that is finitely consistent and such that $p(x)\vdash x\notin\mathcal{C}\cap\mathcal{D}$. By completeness either $p(x)\vdash x\notin\mathcal{C}$ or $p(x)\vdash x\notin\mathcal{D}$. But Theorem 3 says that $p(x)$ entails both $\mathcal{C}$ and $\mathcal{D}$, a contradiction.

4⇒1. By Theorem 19. □

It is convenient to prove one last characterization of the above phenomenon. This is used in the next section.

21 Proposition (with Alex Kruckman) The equivalent conditions in Theorem 20 are also equivalent to

5. $$\Sigma_G(x) \quad = \quad \bigcap\Big\{p(x)\in S_\Delta(\mathcal{Z})\ :\ p(x) \text{ is } G\text{-invariant}\Big\}$$

Note that $\subseteq$ is always true, therefore (5) amounts to claiming that the type on the r.h.s. is G-syndetic.

Proof. Assume (1) of Theorem 20. It suffices to prove $\supseteq$, because the converse inclusion is always true. Let $\mathcal{D}$ be non G-syndetic. Then $\neg\mathcal{D}$ is G-thick and therefore G-wide. Then some G-invariant global type $p(x)$ entails $\neg\mathcal{D}$. Therefore $x\in\mathcal{D}$ does not belong to the type on the r.h.s.

Vice versa, note that the type on the r.h.s. is closed under conjunction. Then (2) of Theorem 20 immediately follows from (5). □

Let $q(x)$ be a $\Delta^{\mathrm{B}}(\mathcal{Z})$-type. We say that $q(x)$ is G-stationary if it extends to a unique G-invariant $p(x)\in S_\Delta(\mathcal{Z})$.

The notions of relative syndeticity/thickness are defined in Remark 1. It is easy to verify that the above theorem relativize to any type $q(x)$.

The conditions in Theorem 20 together with stationarity, produce further simplification.

22 Fact Let $q(x)$ be G-stationary. Assume that the equivalent conditions in Theorem 20 hold relative to $q(x)$. Then the following are equivalent

1. $\mathcal{D}$ is G-syndetic relative to $q(x)$
2. $\mathcal{D}$ is G-wide/thick relative to $q(x)$.

Proof. Work relatively to $q(x)$.

1⇒2. Assume (1), then $\Sigma_G(x)$ is finitely consistent. Then G-syndetic $\Delta^{\mathsf{B}}(\mathcal{Z})$-sets are G-thick. Finally, they are G-wide by (1) in Theorem 20. (Stationarity is not required in this direction.)

2⇒1. If $\mathcal{D}$ was not G-syndetic then $\neg\mathcal{D}$ would be G-thick and therefore, by the assumption, G-wide. This contradicts G-stationarity. □

5. Example: random graph

In this section $\mathcal{U}$ is a saturated model of the of theory of the random graph. Let $\mathcal{X} = \mathcal{Z} = \mathcal{U}$ and let $G = \mathrm{Aut}(\mathcal{U})$. We prove that the equivalent conditions in Theorem 20 hold.

There are two G-invariant global types $p_i(x)$. The first says that x is adjacent to all vertices, the second says that x is adjacent to none. The type on the r.h.s. of (5) in Proposition 21 is equivalent to the disjunction of $p_1(x)$ and $p_2(x)$. This is in turn equivalent to the type containing the formulas

$$\varphi(x,a) \quad = \quad r(x,a_1) \leftrightarrow r(x,a_2) \leftrightarrow \cdots\cdots \leftrightarrow r(x,a_n)$$

for every tuple of vertices $a = \langle a_1,\dots,a_n\rangle$. Therefore we only need to prove that these formulas are G-syndetic.

23 Theorem (with Alex Kruckman) The formula $\varphi_n(x,a)$ is G-syndetic for all $n \geq 2$ and all $a = \langle a_1,\dots,a_n\rangle$.

Proof. Let $b^1,\dots,b^n$ be tuples of length n with disjoint ranges that have the same type as a (over $\varnothing$). Let C be the set of tuples $c = \langle c_1,\dots,c_n\rangle$ such that $c_i \in \mathrm{range}(b^i)$. Finally when we pick the b^i, we do so in such a way that every tuple $c \in C$ has the same type as a.

We claim that the following disjunction is a tautology

$$\bigvee_{c\in C} \varphi(x,c) \ \vee \ \bigvee_{i=1}^{n} \varphi(x,b^i)$$

Fix x. If $\varphi(x,b^i)$ holds for some i, then we are done. Otherwise, for every i we have some k such that $r(x,b^i_k)$ holds (as well as some h such that $r(x,b^i_h)$ does not). Let $c_i = b^i_k$ for some k such that $r(x,b^i_k)$ holds. Then $\varphi(x,c)$ holds. □

6. Questions

It would be interesting to separate G^1 from G^{00} and understand its relations with G^{000} as definitioned in Section 2. It would be interesting to find examples (outside stable theories) where the equivalent conditions in Theorem 20 hold with G^1 for G.

Let M^* be as in Section 4 of [CLPZ].

24 Question Let G be the automorphism group of M^*.

Check that G^{00} and G^{000} in Definition 13 are the group of Kim-Pillay, respectively Lascar, strong automorphisms. Describe G^1 and compare it with G^{00} and G^{000}.

25 Question Let G be the group that acts by rigid rotations on the coordinates of M^*. Compare G^1, G^{00}, and G^{000} in this case.

7. Definable actions

In this section $\mathcal{Z}$ is a type-definable group that acts on a type definable set $\mathcal{X}$. The group operations and the group action are assumed to be definable. We use the symbol $\cdot$ for both the group multiplication and the group action. Let $M \preceq \mathcal{U}$. In this section Δ contains formulas $\varphi(z;x)$ of the form $\psi(z^{-1}\cdot x)$ for some $\psi(x) \in L(M)$. Clearly, the sets $\varphi(\mathcal{Z};\mathcal{X})$ are $\mathcal{Z}$-invariant. We write 1 for the identity of $\mathcal{Z}$.

It is worth noticing that automorphisms of $\mathcal{U}^\Delta$ need not preserve the group operations nor the group action.

To each $h \in \mathcal{Z}$ we associate the L^Δ-automorphism $\langle g;a\rangle \mapsto \langle h\cdot g;h\cdot a\rangle$. Therefore $\mathcal{Z}$ is, up to isomorphism, a subgroup of $\mathrm{Aut}(\mathcal{U}^\Delta)$. Note that $g\cdot\varphi(h;\mathcal{X}) = \varphi(g\cdot h;\mathcal{X})$ for all $g,h \in \mathcal{Z}$ and $\varphi(z;x) \in \Delta$. Therefore, the orbit of $\varphi(\mathcal{Z};\mathcal{X})$ under the action of $\mathcal{Z}$ is $\{\varphi(g;\mathcal{X}) : g \in \mathcal{Z}\}$. It follows that the orbit of $\varphi(h;\mathcal{X})$ under the actions of $\mathrm{Aut}(\mathcal{U}^\Delta)$ coincides with the orbit under the action of $\mathcal{Z}$.

We write X and Z for $\mathcal{X}\cap M^x$ and $\mathcal{Z}\cap M^z$ respectively. Clearly $Z \le \mathrm{Aut}(\mathcal{U}^\Delta/\{Z\})$, the latter is the subgroup of $\mathrm{Aut}(\mathcal{U}^\Delta)$ that fixes Z setwise. By the observation above, when $h \in Z$ and $\varphi(z;x) \in \Delta$, the orbit of $\varphi(h;x)$ under the action of $\mathrm{Aut}(\mathcal{U}^\Delta/\{Z\})$ coincides with the orbit under the action of Z. Also note that, by how Δ is defined in this section, up to equivalence we $\Delta(\{1\})$, $\Delta(Z)$, and $\Delta^{\mathrm{B}}(Z)$ coincide with $L_x(M)$.

When $g \in \mathcal{Z}$ and $a \in \mathcal{X}$ we write $g \perp a$ if for every $\varphi(z;x) \in \Delta$ such that $\varphi(g;a)$ we have that $\varphi(Z;a) \neq \varnothing$. When $\mathcal{C} \subseteq \mathcal{Z}$ and $\mathcal{D} \subseteq \mathcal{X}$ we define

$$\mathcal{C}\circ\mathcal{D} \quad = \quad \Big\{g\cdot a \,:\, g \in \mathcal{C},\ a \in \mathcal{D},\ g \perp a\Big\}$$

If $G \le \mathrm{Aut}(\mathcal{U}^\Delta)$ and $a \in \mathcal{X}$, we write $G\cdot a$ for the orbit of a under the action of G. We abbreviate $\mathcal{C}\circ\big(\mathrm{Aut}(\mathcal{U}^\Delta/M)\cdot a\big)$ with $\mathcal{C}\circ a$.

If $\mathcal{A} \subseteq \mathcal{X}$ we write $\mathrm{cl}(\mathcal{A})$ for the closure of $\mathcal{A}$ in the topology generated by the subsets of $\mathcal{X}$ that are $L(M)$-definable. Explicitely,

1. $\mathrm{cl}(\mathcal{A}) \;=\; \bigcap\Big\{\varphi(\mathcal{X}) \,:\, \varphi(x) \in L^{\Delta}(M) \text{ such that } \mathcal{A} \subseteq \varphi(\mathcal{X})\Big\}$
2. $\phantom{\mathrm{cl}(\mathcal{A})} \;=\; \Big\{b \,:\, \varphi(\mathcal{A}) \neq \varnothing \text{ for every } \varphi(x) \in L(M) \text{ that is satisfied by } b\Big\}$

26 Fact For every $a \in \mathcal{X}$

$$\mathrm{cl}(Z{\cdot}a) \;=\; \mathcal{Z} \circ a$$

Proof. $\supseteq$. Let $b \in \mathcal{Z} \circ a$, say $b = g{\cdot}a$ for some $g \perp a$. Let $\varphi(x) \in L^{\Delta}(M)$ be such that $Z{\cdot}a \subseteq \varphi(\mathcal{X})$. Then $g \perp a$ implies $\varphi(g{\cdot}a)$. This proves that $b \in \mathrm{cl}(Z{\cdot}a)$.

$\subseteq$. Let $b \in \mathrm{cl}(Z{\cdot}a)$ be given and define $p(x) = L^{\Delta}\text{-tp}(b/M)$. It suffices to prove that $z \perp a$ is consistent with $p(z{\cdot}a)$. In fact, if $g \perp a$, then $g{\cdot}a \in \mathcal{Z} \circ a$. And, as $\mathcal{Z} \circ a$ is M-invariant, if $p(g{\cdot}a)$ then $b \in \mathcal{Z} \circ a$. Now, to prove concistency, assume for a contradiction that $z \perp a \rightarrow \neg\varphi(z{\cdot}a)$ for some $\varphi(x) \in p$. As any $m \in Z$ realizes $z \perp a$, no element of $Z{\cdot}a$ satisfies $\varphi(x)$. This is a contradiction by (2) above. □

Let $\mathcal{D} \subseteq \mathcal{X}$ be an M-definable set. As Z is small, $\mathcal{D}$ is Z-thick if and only if $\cap Z{\cdot}\mathcal{D}$ is consistent.

27 Fact Let $\mathcal{D} \subseteq \mathcal{X}$ be definable over M. Then the following are equivalent

1. $\mathcal{D}$ is Z-thick
2. $\mathcal{Z} \circ a \subseteq \mathcal{D}$ for some $a \in \mathcal{D}$.

Proof. 1⇒2. Pick $a \in \cap Z{\cdot}\mathcal{D}$. Suppose for a contradiction that $g{\cdot}a \notin \mathcal{D}$ for some $g \in \mathcal{Z}$ such that $g \perp_M a$. Then $m{\cdot}a \notin \mathcal{D}$ for some $m \in Z$. This contradicts the choice of a.

2⇒1. It suffices to note that $m \cdot \mathcal{Z} \circ a = \mathcal{Z} \circ a$ for every $m \in Z$. □

The following is the dual version of the above fact.

28 Fact Let $\mathcal{D} \subseteq \mathcal{X}$ be definable over M. Then the following are equivalent

1. $\mathcal{D}$ is Z-syndetic
2. $(\mathcal{Z} \circ a) \cap \mathcal{D} \neq \varnothing$ for every $a \in \mathcal{X}$.

We say that $\mathcal{C} \subseteq \mathcal{X}$ is $\mathcal{Z}\circ$-invariant if $\mathcal{Z} \circ \mathcal{C} \subseteq \mathcal{C}$. Note that, when $\mathcal{Z} = \mathcal{X}$, this says that $\mathcal{C}$ is a left ideal. The following fact is easy.

29 Fact Every minimal $\mathcal{Z}\circ$-invariant subset of $\mathcal{X}$ is of the form $\mathcal{Z} \circ a$ for some $a \in \mathcal{X}$ so, in particular, it is type-definable over M. We also have that $\mathcal{Z} \circ a = \mathcal{M} \circ a$ for $\mathcal{M}$ any left ideal of $\mathcal{Z}$.

We say that $a \in \mathcal{X}$ is minimal if $\mathcal{Z} \circ a$ is a minimal $\mathcal{Z}\circ$-invariant subset of $\mathcal{X}$.

30 Theorem Let $\mathcal{D} \subseteq \mathcal{X}$ be definable over M. Then the following are equivalent

1. $\mathcal{D}$ is weakly Z-thick
2. $a \subseteq \mathcal{D}$ for some minimal $a \in \mathcal{D}$.

Proof. 2⇒1. Let $b \in \mathcal{Z} \circ a$. Then $b \in \mathcal{Z} \circ \mathcal{D}$. Therefore $b \in m \cdot \mathcal{D}$ for some $m \in Z$. This proves that $\mathcal{Z} \circ a \subseteq \cup Z \cdot \mathcal{D}$. By compactness $\mathcal{Z} \circ a \subseteq \cup C \cdot \mathcal{D}$ for some finite $C \subseteq Z$. By Fact 27, $\mathcal{D}$ is weakly Z-thick.

1⇒2. Let $C \subseteq Z$ be a finite set such that $\cup C \cdot \mathcal{D}$ is Z-thick. By Fact 27, there is $a \in \cup C \cdot \mathcal{D}$ such that $\mathcal{Z} \circ a \subseteq \cup C \cdot \mathcal{D}$. Let $a' \in \mathcal{Z} \circ a$ be minimal. As $a' \in m \cdot \mathcal{D}$ for some $m \in C$, we conclude that $m^{-1} \cdot a' \in \mathcal{D}$. Then $m^{-1} \cdot a'$ is the minimal element required by (2). □

8. Notes and references

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