Actions

A. Polymath

ABSTRACT.

1. Preliminaries

Let $\Delta \subseteq L_{xz}(\mathcal{U})$. Given any $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$, we define the structure $\mathcal{U}^\Delta = \langle \mathcal{X}; \mathcal{Z} \rangle$. This is a 2-sorted structure whose signature L^Δ contains relation symbols for every formula $\varphi(x;z) \in \Delta$. As there is little risk of confusion, these relations symbols are also denoted by $\varphi(x;z)$, and we identify L^Δ with a subset of L.

When $\mathcal{X}$ and $\mathcal{Z}$ are type definable and Δ is small, $\mathcal{U}^\Delta$ is a saturated structure. As compactness is not always required, these sets are arbitrary unless explicitly required.

In the first few sections we only work a subset of the quantifier-free fragment of L^Δ. Let $\Delta^{\mathsf{B}}(\mathcal{Z})$ be the set of Boolean combinations of the sets $\varphi(\mathcal{X}; b)$, for $\varphi(x;z) \in \Delta$ and $b \in \mathcal{Z}$. The same symbol is also used to denote the collection of formulas defining these sets. A $\Delta^{\mathsf{B}}(\mathcal{Z})$-type is a collection of $\Delta^{\mathsf{B}}(\mathcal{Z})$-formulas. A $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable set is one defined by a $\Delta^{\mathsf{B}}(\mathcal{Z})$-type of small cardinality.

We write $S_\Delta(\mathcal{Z})$ for the set of $\Delta^{\mathsf{B}}(\mathcal{Z})$-types that are maximallyfinitely consistent (in $\mathcal{X}$) – which we may conveniently identify with a sets of $\Delta(\mathcal{Z})$ and negated $\Delta(\mathcal{Z})$-formulas. Note that the types in $S_\Delta(\mathcal{Z})$ are required to be finitely consistent $\mathcal{X}$ – a piece of information that we will not display in the notation. When the types are required to be finitely consistent in some $\mathcal{Y} \subseteq \mathcal{X}$ we write $S_{\Delta,\mathcal{Y}}(\mathcal{Z})$.

When $p(x)$ a type (i.e. any set of formulas) and $\mathcal{D} \subseteq \mathcal{X}$ we write $p(x) \vdash x \in \mathcal{D}$ if the inclusion $\psi(\mathcal{X}) \subseteq \mathcal{D}$ holds for some $\psi(x)$ that is conjunctions of formulas in $p(x)$.

Let $G \leq \mathrm{Aut}(\mathcal{U}^\Delta)$. We view G as a group acting on $\mathcal{U}^\Delta$ and write $\cdot$ for such action. We write $g\cdot\mathcal{D}$ for the natural action of $g \in G$ on $\mathcal{D} \subseteq \mathcal{X}$. If $\Delta = \varphi(\mathcal{X}; b)$ then $g\cdot\mathcal{D} = \varphi(\mathcal{X}; g\cdot b)$. A formula is G-invariant if the set it defines is G-invariant (i.e. fixed by the action of G).

If $p(x)$ is a $\Delta^{\mathsf{B}}(\mathcal{Z})$-type, we write $g\cdot p(x)$ for the set of formulas of the form $\varphi(\mathcal{X}; g\cdot b)$ for $\varphi(\mathcal{X}; b) \in p$. We say that the type $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ is G-invariant accordingly.

Let $\mathcal{Y} \subseteq \mathcal{X}$. We say that a set $\mathcal{D} \subseteq \mathcal{X}$ is syndetic under the action of G relative to $\mathcal{Y}$, or G-syndetic for short, if finitely many G-translates of $\mathcal{D}$ cover $\mathcal{Y}$. Dually, we say that $\mathcal{D}$ is thick under the action of G relative to $\mathcal{Y}$, or G-thick for short, if the intersection of any finitely many G-translates of $\mathcal{D}$ is consistent with $\mathcal{Y}$.

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We omit reference to $\mathcal{Y}$ when this is $\mathcal{X}$ or if it is made crystal clear by the context.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

Our terminology is taken from topological dynamics. In similars context syndetic sets are called *generic*, see eg. [?N09]. In [?CK] the authors write *quasi-non-dividing* for *thick* under the action of $\mathrm{Aut}(\mathcal{U}/A)$.

Notation: for $\mathcal{D} \subseteq \mathcal{X}$ and $C \subseteq G$ we write $C{\cdot}\mathcal{D}$ for $\{h{\cdot}\mathcal{D} : h \in C\}$.

In this chapter many proofs require some juggling with negations as epitomized by the following fact which is proved by spelling out the definitions

1 Fact Work relative to some given $\mathcal{Y} \subseteq \mathcal{X}$. The following are equivalent

1. $\mathcal{D}$ is not G-syndetic
2. $\neg\mathcal{D}$ is G-thick.

Define the following type

$$\Sigma_{G,\mathcal{Y}}(x) \quad = \quad \big\{\vartheta(x) \in \Delta^{\mathsf{B}}(\mathcal{Z}) \ : \ \vartheta(x) \text{ is } G\text{-syndetic relatve to } \mathcal{Y}\big\}.$$

We write $\Sigma_G(x)$ for $\Sigma_{G,\mathcal{X}}(x)$.

2 Theorem Work relative to some given $\mathcal{Y} \subseteq \mathcal{X}$. For every $p(x) \in S_{\Delta,\mathcal{Y}}(\mathcal{Z})$ the following are equivalent

1. $p(x)$ is G-invariant
2. $p(x) \vdash \Sigma_{G,\mathcal{Y}}(x)$
3. $p(x)$ is G-thick.

Proof. 1⇒2. Let $\mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$ be a G-syndetic. Pick $C \subseteq G$ be finite such that $\mathcal{Y} \subseteq \cup C{\cdot}\mathcal{D}$. Then $p(x)$ is finitely consistent with $x \in \cup C{\cdot}\mathcal{D}$. By completeness, $p(x) \vdash x \in h\mathcal{D}$ for some $h \in C$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

2⇒3. Let $\varphi(x) \in p$. As $\Sigma_{G,\mathcal{Y}}(x) \cup \{\varphi(x)\}$ is finitely consistent in $\mathcal{Y}$, it cannot be that $\neg\varphi(x)$ is G-syndetic. Then from Fact 1 we obtain that $\varphi(x)$ is G-thick.

3⇒1. Negate 1. Let $\mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$ and $g \in G$ such that $p(x) \vdash x \in \mathcal{D}$ and $p(x) \nvdash g{\cdot}\mathcal{D}$. By completeness $p(x) \vdash x \in (\mathcal{D} \cap \neg g{\cdot}\mathcal{D})$. Clearly $\mathcal{D} \cap \neg g{\cdot}\mathcal{D}$ is not G-thick as it is inconsistent with its g-translate. □

The following theorem gives a necessary and sufficient condition for the existence of global G-invariant $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Ideally, we would like that every G-thick $\Delta^{\mathsf{B}}(Z)$-type extends to a global thick type. Unfortunately this is not true in general (it is a strong assumption, see Section **??**).

A set $\mathcal{D}$ is G-wide relative to $\mathcal{Y}$ if every finite cover of $\mathcal{D}\cap\mathcal{Y}$ by $\Delta^{\mathsf{B}}(\mathcal{Z})$-sets contains a set that is G-thick relative to $\mathcal{Y}$. A type is G-wide if every conjunction of formulas in the type is G-wide.

In [?CK] a similar property is called *quasi-non-forking*. Our use of the term *wide* is consistent with [?Hr], though we apply it to a narrow context.

3 Theorem Work relative to some given $\mathcal{Y}\subseteq\mathcal{X}$. For every $\mathcal{D}\in\Delta^{\mathsf{B}}(\mathcal{Z})$ the following are equivalent

1. $\Sigma_{G,\mathcal{Y}}(x)\cup\{x\in\mathcal{D}\}$ is finitely consistent in $\mathcal{Y}$
2. there is a G-thick type $p(x)\in S_{\Delta,\mathcal{Y}}(\mathcal{Z})$ that entails $x\in\mathcal{D}$
3. $\mathcal{D}$ is G-wide.

Proof. 1⇒2. By Theorem 2, it suffices to pick any $p(x)\in S_{\Delta,\mathcal{Y}}(\mathcal{Z})$ extending the type $\Sigma_{G,\mathcal{Y}}(x)\cup\{x\in\mathcal{D}\}$.

2⇒1. By Theorem 2.

2⇒3. Let $\mathcal{C}_1,\dots,\mathcal{C}_n$ be $\Delta^{\mathsf{B}}(\mathcal{Z})$-sets that cover $\mathcal{D}$. Pick $p(x)$ as in 2. By completeness, $p(x)\vdash x\in\mathcal{C}_i$ for some i. Therefore, $\mathcal{C}_i$ is G-thick.

3⇒2. Let $p(x)$ be maximal among the $\Delta^{\mathsf{B}}(\mathcal{Z})$-types that are finitely satisfiable in $Y\cap\mathcal{D}$ and are G-wide. We claim that $p(x)$ is a complete $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Suppose for a contradiction that $\theta(x),\neg\theta(x)\notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\Delta^{\mathsf{B}}(\mathcal{Z})$-sets $\mathcal{C}_1,\dots,\mathcal{C}_n$ that cover both $\psi(\mathcal{X})\cap\theta(\mathcal{X})$ and $\psi(\mathcal{X})\smallsetminus\theta(\mathcal{X})$ and such that no $\mathcal{C}_i$ is G-thick. As $\mathcal{C}_1,\dots,\mathcal{C}_n$ cover $\psi(\mathcal{X})$ this is a contradiction. It is only left to show that $p(x)$ is G-thick. This follows from completeness and Theorem 2. □

2. Connected components

Let $G\leq\mathrm{Aut}(\mathcal{U}^{\Delta})$. Unfortunately, syndeticity is not preserved under intersection. In particular $\Sigma_G(x)$ is not a G-syndetic type, and it may even be inconsistent. Then following notion is relevant.

4 Definition

$$Q_G \quad=\quad \big\{q(x)\subseteq\Sigma_G(x)\ :\ q(x)\text{ maximally }G\text{-syndetic}\big\}.$$

In other words, the types in Q_G are maximal among the subtypes of $\Sigma_G(x)$ that are closed under conjunction.

It is easy to see that Q_G is closed under the action of G. We write $\mathrm{Stab}(q)$ for the stabilizer of $q(x)\subseteq\Delta^{\mathsf{B}}(\mathcal{Z})$ in G, that is, the subgroup $\{g\in G\ :\ g\cdot q(x)=q(x)\}$. We write $\mathrm{Stab}(\mathcal{D})$ with a similar meaning. Finally we define

$$G^1\quad=\quad\mathrm{Stab}(Q_G)\quad=\quad\bigcap_{q\in Q_G}\mathrm{Stab}(q).$$

It is easy to verify that $G^1\trianglelefteq G$.

5 Proposition

$$G^1 \quad = \quad \big\{g \in G \,:\, \mathcal{D} \cap g{\cdot}\mathcal{D} \in \Sigma_G(x) \text{ whenever } \mathcal{D} \in \Sigma_G(x)\big\}.$$

Proof. $\subseteq$. Pick any $k \in G^1$ and $\mathcal{D} \in \Sigma_G(x)$. Let $q(x) \in Q_G$ be a type containing $x \in \mathcal{D}$. From the G^1-invariance of $q(x)$ we obtain that $q(x) \vdash x \in k{\cdot}\mathcal{D}$. Then $q(x) \vdash x \in \mathcal{D} \cap k{\cdot}\mathcal{D}$, hence $\mathcal{D} \cap k{\cdot}\mathcal{D}$ is G-syndetic.

$\supseteq$. Pick any $g \notin G^1$. Then $q(x) \neq g{\cdot}q(x)$ for some $q(x) \in Q_G$. Let $\varphi(x) \in q$ such that $q(x) \nvdash g{\cdot}\varphi(x)$. By maximality, $\psi(x) \wedge g{\cdot}\varphi(x)$ is not G-syndetic for some $\psi(x) \in q$. As $q(x)$ is closed under conjunction, we can assume $\varphi(x) = \psi(x)$, then g does not belong to the set on the r.h.s. □

6 Theorem Any finite conjunction of formulas in $\Sigma_{G^1}(x)$ is G-syndetic. In particular $\Sigma_{G^1}(x)$ is finitely consistent.

Proof. Notice that from Proposition 8 it easily follows that for every $\mathcal{D} \in \Sigma_G(x)$ and every finite $F \subseteq G^1$ the set $\cap F{\cdot}\mathcal{D}$ is G-syndetic.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_n \in \Sigma_{G^1}(x)$. Assume inductively that $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}$ is G-syndetic. Let $F \subseteq G^1$ be such that $\cup F{\cdot}\mathcal{D}_n = \mathcal{X}$. Then

$$\begin{aligned} \cup F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n] \quad &\supseteq \quad \cap F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}] \;\cap\; \cup F{\cdot}\mathcal{D}_n \\ &= \quad \cap F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}]. \end{aligned}$$

This last set is G-syndetic by the inductive hypothesis and what remarked above. The G-syndeticity of $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ follows. □

Unfortunately, we are unable to conclude that the intersection of G^1-syndetic sets is G^1-syndetic.

7 Definition

$$P_G \quad = \quad \big\{p(x) \subseteq \Sigma_G(x) \,:\, p(x) \text{ maximally finitely consistent}\big\}.$$

Along the same lines as above, we define

$$G^{11} \quad = \quad \mathrm{Stab}(P_G) \quad = \quad \bigcap_{p \in P_G} \mathrm{Stab}(p).$$

By the following proposition, $G^1 \leq G^{11}$. We do not know if the inclusion is strict.

8 Proposition

$$G^{11} \quad = \quad \big\{g \in G \,:\, q(x) \cup g{\cdot}q(x) \text{ is finitely consistent for every } q(x) \in Q_G\big\}.$$

Proof. $\subseteq$. Pick any $k \in G^{11}$ and $q(x) \in Q_G$. Let $p(x) \in P_G$ be a type containing $q(x)$. From the G^{11}-invariance of $p(x)$ we obtain that $p(x) \vdash k{\cdot}q(x)$. Then $q(x) \cup k{\cdot}q(x)$ is finitely consistent.

$\supseteq$. Pick any $g \notin G^{11}$. Then $q(x) \cup g{\cdot}q(x)$ is not finitely consistent for some $q(x) \in Q_G$. Let $\varphi(x) \in q$ such that $\varphi(x) \wedge g{\cdot}\varphi(x)$ is inconsistent. but $\varphi(x) \in p$ then g does not belong to the set on the r.h.s. □

9 Theorem Any finite conjunction of formulas in $\Sigma_{G^1}(x)$ is G-syndetic. In particular $\Sigma_{G^1}(x)$ is finitely consistent.

Proof. Notice that from Proposition 8 it easily follows that for every $\mathcal{D} \in \Sigma_G(x)$ and every finite $F \subseteq G^1$ the set $\cap F{\cdot}\mathcal{D}$ is G-syndetic.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_n \in \Sigma_{G^1}(x)$. Assume inductively that $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}$ is G-syndetic. Let $F \subseteq G^1$ be such that $\cup F{\cdot}\mathcal{D}_n = \mathcal{X}$. Then

$$\begin{aligned} \cup F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n] \;\; &\supseteq \;\; \cap F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}] \;\cap\; \cup F{\cdot}\mathcal{D}_n \\ &= \;\; \cap F{\cdot}[\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_{n-1}]. \end{aligned}$$

This last set is G-syndetic by the inductive hypothesis and what remarked above. The G-syndeticity of $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ follows. □

From Theorems 3 and 9 it follows that G^1-syndetic sets are G^1-wide. But we can do better. First, we remark a useful consequence of normality.

10 Remark Assume $H \trianglelefteq G$. For every $\mathcal{D} \subseteq \mathcal{X}$ and every $g \in G$

$$\mathcal{D} \text{ is } H\text{-foo} \quad \Leftarrow \quad g{\cdot}\mathcal{D} \text{ is } H\text{-foo},$$

where *foo* can be replaced by *syndetic, invariant, thick, wide*. In particular, the type $\Sigma_H(x)$ is G-invariant.

Recall that when $\Sigma_H(x)$ is finitely consistent then H-syndetic sets are H-wide, see Theorem 3. As it happens, under the assumption of normality, this can be strengthened as follows.

11 Proposition Assume $H \trianglelefteq G$ and that $\Sigma_H(x)$ is finitely consistent. Then every G-syndetic $\mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$ is H-wide. In particular all types in Q_G are G^1-wide.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be finitely consistent with $\Sigma_H(x)$. As $\mathcal{D}$ is G-syndetic, by completeness $p(x) \vdash x \in g{\cdot}\mathcal{D}$ for some $g \in G$. Equivalently, $g^{-1}{\cdot}\, p(x) \vdash x \in \mathcal{D}$. As $p(x)$ is H-thick, by Remark 10 also $g^{-1}{\cdot}\, p(x)$ is H-thick. Then the proposition follows from Theorem 3. □

When the context suggests it, $\Delta^{\mathsf{B}}(\mathcal{Z})$ will denote the broader class that includes also formulas obtained substituting y for x and is closed under Boolean connectives. A $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable equivalence relation is an equivalence relation defined by some formula $\varepsilon(x,y) \in \Delta^{\mathsf{B}}(\mathcal{Z})$. We define $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable equivalence relations similarly. Note that if $\varepsilon(x,y)$ is a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable, or type-definable, equivalence relation then $\varepsilon(x,a)$, for any $a \in \mathcal{X}$, is equivalent to a $\Delta^{\mathsf{B}}(\mathcal{Z})$-formula, respectively a $\Delta^{\mathsf{B}}(\mathcal{Z})$-type.

An equivalence relation is bounded if it has a small number of equivalence classes.

12 Definition Let Φ^0 be the collection of sets that are equivalence classes of a bounded G-invariant $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable equivalence relation. Let Φ^{00} be the collection of sets that are equivalence classes of a bounded G-invariant $\Delta^{\mathsf{B}}(\mathcal{Z})$-type-definable equivalence relation. Define

$$G^0 \quad = \quad \mathrm{Stab}(\Phi^0) \quad = \quad \bigcap\{\mathrm{Stab}(\mathcal{D}) \,:\, \mathcal{D} \in \Phi^0\}$$

and

$$G^{00} \quad = \quad \mathrm{Stab}(\Phi^{00}) \quad = \quad \bigcap\{\mathrm{Stab}(\mathcal{Y}) \,:\, \mathcal{Y} \in \Phi^{00}\}.$$

13 Proposition Let $\mathcal{X}$ be type-definable. Assume that G acts transitively on $\mathcal{X}$. Then

$$G^{11} \quad \leq \quad G^{00} \quad \leq \quad G^0.$$

Proof. The inclusion $G^{00} \leq G^0$ is trivial. Let $\mathcal{Y} \in \Phi^{00}$ and $\mathcal{Y} \subseteq \mathcal{D} \in \Delta^{\mathsf{B}}(\mathcal{Z})$. Then a small number of G-translates of $\mathcal{D}$ cover $\mathcal{X}$. Therefore, by compactness, $\mathcal{D}$ is G-syndetic. Then the type defining $\mathcal{Y}$ is G-syndetic. Then this type extends to some $p(x) \in P_G$. By G^{11}-invariance $p(x)$ entails also $k{\cdot}\mathcal{Y}$ for every $k \in G^{11}$. Then k is in the stabilizer of $\mathcal{Y}$. □