Standard analysis

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ABSTRACT. Let $\mathcal{L}$ be a first-order two-sorted language. Let I be some fixed structure. A *standard* structure is an $\mathcal{L}$-structure of the form $\langle M, I\rangle$ where I is fixed. When I is a compact topological space (and $\mathcal{L}$ meets a few requirements) it is possible to adapt a significant part of model theory to the restricted class of standard strutures. This has been demostrated by Henson and Iovino for Banach spaces (see, e.g. [HI]) and has been generalized to arbitrary structures in [CLCL]. The starting point is to prove that every standard structure has a *positive* elementary extension that is standard and realizes all positive types that are finitely consistent. The main tool is the notion of approximation of a positive formula and of its negation. These have been introduced by Henson and Iovino.

We review and elaborate on the properties of positive formulas and their approximations. In parallel, we introduce *continuous* formulas which provide a better counterpart to Henson and Iovino theory of normed spaces and/or real-valued model theory. Within the setting of standard structures we discuss ω-categoricity.

1. Standard structures

We refer to the introduction of [CLCL] for motivations. Aside from that, this paper is reasonably self-contained. In the first part we revisit some results of [CLCL]. We rephrase some theorems and add a few remarks and examples[1]. Our notation and terminology diverges slightly, where necessary.

Let I be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language $\mathcal{L}_\mathsf{I}$ contains a relation symbol for each compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \to I$. In particular, there is a constant for each element of I. According to the context, C and f denote either the symbols of $\mathcal{L}_\mathsf{I}$ or their interpretation in the stucture I. Such a language $\mathcal{L}_\mathsf{I}$ is much larger than necessary but it is convenient because it uniquely associates a structure to a topological space. The notion of a *dense set of formulas* (Definition 8) helps to reduce the size of the language when required.

[1] We will omit Section 3 from the final version of this paper and refer to [CLCL] instead.

The most straightforward examples of structures I are the unit interval $[0,1]$ with the usual topology, and its homeomophic copies $\mathbb{R}^+ \cup \{0,\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language $\mathcal{L}_{\mathsf{H}}$ which we call the language of the home sort.

Definition 1. Let $\mathcal{L}$ be a two sorted language that expands both $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{I}}$.

A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, I\rangle$, where M is any $\mathcal{L}_{\mathsf{H}}$-structure, while I is the structure above. We write H and I to denote the two sorts of $\mathcal{L}$.

The language $\mathcal{L}$ has a relation symbol $r_\varphi(x)$ for every $\varphi(x) \in \mathcal{L}_{\mathsf{H}}$. All $\mathcal{L}$-structures are assumed to model $r_\varphi(x) \leftrightarrow \varphi(x)$. In other words, the Morleyzation of $\mathcal{L}_{\mathsf{H}}$ is assumed in the definition of $\mathcal{L}$-structure.

Finally and most relevantly, $\mathcal{L}$ contains arbitrarily many symbols for functions of sort $\mathsf{H}^n \to \mathsf{I}$.

Standard structures are denoted by the domain of their home sort.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (when I is finite), these are not standard. As a remedy, below we carve out a set of formulas $\mathcal{L}^{\mathrm{p}}$, the set of positive formulas, such that every model has an positive elementary saturated extension that is also standard.

As usual, $\mathcal{L}_{\mathsf{I}}$, $\mathcal{L}_{\mathsf{H}}$, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. We write $\mathcal{L}_x$ when when we restrict variables to x. Note that $\mathcal{L}$ has two types of atomic formulas

1. $r_\varphi(x)$ for some $\varphi(x) \in \mathcal{L}_{\mathsf{H}}$ (maybe more, but these suffices up to equivalence);
2. $\tau(x;\eta) \in C$, where $C \subseteq I^n$ is a compact set and $\tau(x;\eta)$ is a tuple of terms of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$ (up to equivalence, equality has also this form).

The latter are called continuous atomic formulas.

Definition 2. A formula in $\mathcal{L}$ is positive if it uses only the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{I}}$, $\exists^{\mathsf{I}}$ of sort I.

The set of positive formulas is denoted by $\mathcal{L}^{\mathrm{p}}$.

A continuous formula is a positive formula where only continuous atomic formulas occur. The set of continuous formulas is denoted by $\mathcal{L}^{\mathrm{c}}$.

We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort I. Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

Notation 3. Much of the theory below is developed in parallel in the positive and the continuous case. For the sake of conciseness, we write p/c for a placeholder that should be replaced by either of p or c.

The positive formulas of Henson and Iovino correspond to those formulas in $\mathcal{L}^{\mathrm{c}}$ that do not have quantifiers of sort I.

Allowing quantifiers of sort I comes almost for free. In Section 8 will see that the formulas in $\mathcal{L}^{\mathrm{c}}$ are approximated by formulas in $\mathcal{L}^{\mathrm{c}}$ without quantifiers of sort I.

Instead, working within $\mathcal{L}^{\mathrm{p}}$ comes at a price but it is convenient at it simplifies the comparision with classical model theory.

An important fact to note about $\mathcal{L}^{\mathrm{c}}$ is that it is a language without equality of sort H. The difference beween $\mathcal{L}^{\mathrm{p}}$ and $\mathcal{L}^{\mathrm{c}}$ is similar to the difference between the space of absolutely integrable functions (w.r.t. some measure μ) and the Lebesgue space $L^1(\mu)$. For the latter only equality almost everywhere is relevant. However, even when $L^1(\mu)$ is our focus of interest, it is often easier to argue about real valued functions. For a similar reason $\mathcal{L}^{\mathrm{p}}$ and $\mathcal{L}^{\mathrm{c}}$ are better studied in parallel.

The extra expressive power offered by quantifiers of sort I is also convenient. For instance, the example below (from [CLCL]) should convince the reader that $\mathcal{L}^{\mathrm{c}}$ has at least the same expressive power as real valued logic.

Example 4. Let $I = [0,1]$, the unit interval. Let $\tau(x)$ be a term of sort $\mathsf{H}^{|x|} \to \mathsf{I}$. Then there is a positive formula that says $\sup_x \tau(x) = \alpha$. Indeed, consider the continuous formula

$$\forall x\,\big[\tau(x) \dot{-} \alpha \in \{0\}\big] \quad \wedge \quad \forall \varepsilon \Big[\varepsilon \in \{0\} \ \vee\ \exists x\,\big[\alpha \dot{-} (\tau(x) + \varepsilon) \in \{0\}\big]\Big]$$

which, in a more legible form, becomes

$$\forall x\,\big[\tau(x) \leq \alpha\big] \quad \wedge \quad \forall \varepsilon > 0\ \exists x\,\big[\alpha \leq \tau(x) + \varepsilon\big].$$

The following example is inspired by [HPP].

Example 5. Let $\mathcal{U}$ be a saturated structure of signature $\mathcal{L}_{\mathsf{H}}$. The relevant part of $\mathcal{L}$ contains symbols for the functions $f : \mathcal{U}^n \to I$ that are continuous in the following sense: the inverse image of every compact $C \subseteq I$ is type-definable in $\mathcal{L}_{\mathsf{H}}(M)$ for every $M \preceq \mathcal{U}$.

The theory of normed spaces is one of the motivating examples for standard structures. The requirement for I to be compact seems to be a limitation, but it is not a real one. Essentially, functional analysis and model theory are only concerned with the unit ball of normed spaces. To formalize the unit ball of a normed space as a standard structure (where I is the unit interval) is straightforward but cumbersome. Alternatively one can view a normed space as a many-sorted structure: a sort for each ball of radius $n \in \mathbb{Z}^+$. Unfortunately, this also results in a bloated formalism.

A neater formalization is possible if one accepts that some positive elementary extensions (to be defined below) of a normed space could contain vectors of infinite norm hence not be themselves proper normed spaces. However these infinities are harmless if one restrics to balls of fine radius. In fact, inside any ball of finite radius these improper normed spaces are completely standard.

Example 6. Let $I = \mathbb{R} \cup \{\pm\infty\}$. Let $\mathcal{L}_{\mathsf{H}}$ be the language of real (or complex) vector spaces. The language $\mathcal{L}$ contains a function symbol $\|\text{-}\|$ of sort $\mathsf{H} \to \mathsf{I}$. Normed space are standard structures with the natural interpretation of the language. It is easy to verify that the unit ball of a standard structure is the unit ball of a normed space. Note that the same remains true if we add to the language any continuous (equivalently, bounded) operator or functional.

We conclude this introduction by advertising a question in [CLCL]. Is it possible to extend the Positive Compactness Theorem (Theorem 19) to a larger class of languages? E.g. can we allow in $\mathcal{L}$ (and in $\mathcal{L}^{\mathrm{p/c}}$) function symbols of sort $\mathsf{H}^n \times \mathsf{I}^m \to \mathsf{I}$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I.

2. Henson-Iovino approximations

The notion of approximation is a central tool in the work of Hension and Iovino on normed spaces. In [CLCL] the natural generalization to standard structures is presented. We recall the definition and the main properties.

For C, C' compact substes of I^n, we write $C' > C$ if C' is a neighborhood of C. For φ, φ' (free variables are hidden) positive formulas possibly with parameters we write $\varphi' > \varphi$ if φ' is obtained by replacing in φ each atomic formula of the form $t \in C$ with $t \in C'$, for some $C' > C$. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. We call φ' a weakening of φ. Note that $>$ is a dense transitive relation and that $\varphi \to \varphi'$ in every $\mathcal{L}$-structure.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. The atomic formulas in $\mathcal{L}_{\mathsf{H}}$ are replaced with their negation. Finally each connective is replaced by its dual i.e., $\vee, \wedge, \exists, \forall$ are replaced by $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde{\varphi}$ is a strong negation of φ. It is clear that $\tilde{\varphi} \to \neg\varphi$ in every $\mathcal{L}$-structure.

Notice that the weakening and the strong negation of a formula in $\mathcal{L}^{\mathrm{c}}$ is also in $\mathcal{L}^{\mathrm{c}}$.

Lemma 7. For all positive formulas φ

1. for every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \rightarrow \neg\tilde{\varphi} \rightarrow \varphi'$;
2. for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \rightarrow \varphi' \rightarrow \neg\tilde{\varphi}$.

Proof. If $\varphi \in \mathcal{L}_{\mathsf{H}}$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some $C' > C$. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in I \smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of I, there is $C' > C$ disjoint from $\tilde{C}$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. □

For every type $p(x)$, we write

$$p'(x) \quad = \quad \big\{\varphi'(x) \;:\; \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}$$

in particular $\{\varphi(x)\}' = \big\{\varphi'(x) \;:\; \varphi' > \varphi\big\}$.

Definition 8. A set of formulas $\mathcal{F} \subseteq \mathcal{L}$ is p/c-dense modulo T, a theory, if for every positive/continuous $\varphi(x)$ and every $\varphi' > \varphi$, there is $\psi(x) \in \mathcal{F}$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi'(x)$ holds in every standard structure that models T.

Example 9. By Lemma 7, the set of negations of positive/continuous formulas is p/c-dense.

The following fact is easily proved by induction.

Fact 10. Let $\mathcal{F}$ be a set of formulas closed under the connectives $\wedge$, $\vee$, $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$, $\forall^{\mathsf{I}}$, $\exists^{\mathsf{I}}$. Then $\mathcal{F}$ is dense if the condition above is satisfied when $\varphi(x)$ atomic.

Using the fact above, it is immediate to verify the following.

Example 11. Let $\mathcal{C}$ be a prebase of closed subsets of I. The set of formulas built inductively from atomic formulas of the form $\tau(x,\eta) \in C$ for $C \in \mathcal{C}$ is c-dense.

Example 12. Let $I = [0,1]$. The set of formulas built inductively from atomic formulas of the form $\tau = 0$ is c-dense. In fact, first note that by the example above we can restrict to formulas built inductively from atomic formulas of the form $\tau \in [\alpha,\beta]$. Then note that $\tau \in [\alpha,\beta]$ is equivalent to $\tau \dot{-} \beta \in \{0\} \;\wedge\; \alpha \dot{-} \tau \in \{0\}$.

In Section 8 we prove that the set of continuous formulas without quantifiers of sort I is c-dense.

3. The standard part[1]

Our goal is to prove a compactness theorem for positive theories that only requires standard structures. To bypass lengthy routine applications of ultrafilters, ultrapowers, and ultralimits, we assume the Classical Compactness Theorem and build on that.

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 17 which in turn is required for the proof of the Positive Compactness Theorem (Theorem 19). The reader willing to accept it without proof may skip this section.

Let $\langle N, {}^*I\rangle$ be an $\mathcal{L}$-structure that extends $\mathcal{L}$-elementarily the standard structure M. Let η be a free variable of sort I. For each $\beta \in I$, we define the type

$$\mathrm{m}_\beta(\eta) \quad = \quad \{\eta \in D \,:\, D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(\eta)$ in ${}^*I$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 13. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Proof. Negate the existence of β. For every $\gamma \in I$ pick some compact neighborhood D_γ of γ, such that ${}^*I \models \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ, with $\gamma \in \Gamma$, cover I. By elementarity, the interpretation in ${}^*I$ of these D_γ cover ${}^*I$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Fact 14. For every $\alpha \in {}^*I$ and every compact $C \subseteq I$

$${}^*I \quad \models \quad \alpha \in C \;\rightarrow\; \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*I \models \alpha \in D \subseteq \neg C$. □

Fact 15. For every $\alpha \in ({}^*I)^{|\alpha|}$ and every function symbol f of sort $\mathsf{I}^{|\alpha|} \to \mathsf{I}$

$${}^*I \quad \models \quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big).$$

Proof. By Fact 13 and the definition of st(-) it suffices to prove that ${}^*I \models f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

[1] We will omit this section from the final version of this paper and refer to [CLCL] instead.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*I \models \alpha \in f^{-1}[D]$ and, as $I \preceq {}^*I$ we obtain ${}^*I \models f(\alpha) \in D$. □

The standard part of $\langle N, {}^*I\rangle$ is the standard structure $\langle N, I\rangle$ that interprets the symbols f of sort $\mathsf{H}^n \to \mathsf{I}$ as the functions

$$f^N(a) \quad = \quad \mathrm{st}\big({}^*f(a)\big) \qquad\qquad \text{for all } a \in N^n,$$

where ${}^*f$ is the interpretation of f in $\langle N, {}^*I\rangle$. Symbols in $\mathcal{L}_{\mathsf{H}}$ maintain the same interpretation.

Fact 16. With the notation as above. Let $\tau(x;\eta)$ be a term of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$. Then for every $a \in M^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$\tau^N\big(a;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*\tau(a;\alpha)\big)$$

Proof. When a t is a function symbol of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{I}^{|\eta|} \to \mathsf{I}$, the claim follows from Fact 15. Now, assume inductively that

$$t_i^N\big(a;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*\,t_i(a;\alpha)\big)$$

holds for the terms $t_1(x;\eta),\ldots,t_n(x;\eta)$ and let $t = f(t_1,\ldots,t_n)$ for some function f of sort $\mathsf{I}^n \to \mathsf{I}$. Then the claim follows immediately from the induction hypothesis and Fact 15. □

Lemma 17. With the notation as above. For every $\varphi(x;\eta) \in \mathcal{L}^{\mathrm{p}}$, $a \in N^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$\langle N, {}^*I\rangle \models \varphi(a;\alpha) \quad \Rightarrow \quad N \models \varphi\big(a;\mathrm{st}(\alpha)\big)$$

Proof. Suppose $\varphi(x;\eta)$ is $\mathcal{L}^{\mathrm{p}}$-atomic. If $\varphi(x;\eta)$ is a formula of $\mathcal{L}_{\mathsf{H}}$ the claim is trivial. Otherwise $\varphi(x;\eta)$ has the form $\tau(x;\eta) \in C$. Assume that the tuple $\tau(x;\eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, {}^*I\rangle \models \tau(a;\alpha) \in C$. Then $\mathrm{st}\big({}^*\tau(a;\alpha)\big) \in C$ by Fact 14. Therefore $\tau^N\big(a;\mathrm{st}(\alpha)\big) \in C$ follows from Fact 16. This proves the lemma for atomic formulas. Induction is immediate. □

Corollary 18. Let M be a standard structure. Let $p(\eta) \subseteq \mathcal{L}^{\mathrm{p}}(M)$ be a type that does not contain existential quantifiers of sort H. Then, if $p(\eta)$ is finitely consistent in M, it is realized in M.

The corollary has also a direct proof. This goes through the observation that the formulas in $p(\eta)$ define compact subsets of I.

Proof. Let $\langle {}^*M, {}^*I\rangle$ be an $\mathcal{L}$-elementary saturated superstructure of $\langle M, I\rangle$. Pick some $\alpha \in {}^*I$ such that $\langle {}^*M, {}^*I\rangle \models p(\alpha)$. By the lemma above, ${}^*M \models p(\mathrm{st}(\alpha))$. Now observe that the truth of formulas without existential quantifiers of sort H is preserved by substructures. $\square$

The exclusion of existential quantifiers of sort H is necessary. For a counterexample take $M = I = [0,1]$. Assume that $\mathcal{L}$ contains a function symbol for the identity map $\iota : M \to I$. Let $p(\eta)$ contain the formulas $\exists x \left(x > 0 \wedge \iota x + \eta \in [0, 1/n]\right)$ for all positive integers n.

4. Compactness

It is convenient to distinguish between consistency with respect to standard structures and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent if $\langle M, {}^*I\rangle \models T$ for some $\mathcal{L}$-structure $\langle M, {}^*I\rangle$ (this is the classical notion of concistency). We say that T is standardly consistent if $M \models T$ is for some standard structure M. By Lemma 17 these two notions coincide if T is positive. Therefore we have the following.

Theorem 19 (Positive Compactness Theorem). Let T be a positive theory. Then, if T is finitely $\mathcal{L}$-consistent, it is also standardly consistent.

An $\mathcal{L}$-structure N is positively λ-saturated if it realizes all types $p(x;\eta) \subseteq \mathcal{L}^{\mathrm{p}}(N)$ with fewer than λ parameters that are finitely consistent in N. When $\lambda = |N|$ we simply say p-saturated. The existence of p-saturated standard structures is obtained from the classical case just as for Theorem 19.

We say that M is a p-elementary substructure of N if $M \subseteq N$ and

$$M \models \varphi(a) \quad \Rightarrow \quad N \models \varphi(a)$$

for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$ and every $a \in M^{|x|}$.

Theorem 20. Every standard structure has a p-elementary extension to a p-saturated standard structure (possibly of inaccessible cardinality).

Proof. Let $\langle {}^*M, {}^*I\rangle$ be an $\mathcal{L}$-saturated $\mathcal{L}$-elementary extension of the standard structure $\langle M, I\rangle$. Let $\langle {}^*M, I\rangle$ be its standard part as defined in Section 3. By Lemma 17, $\langle {}^*M, I\rangle$ realizes all finitely consistent positive types with fewer than $|{}^*M|$ parameters. $\square$

The following proposition shows that a slight amount of saturation tames the positive formulas.

Proposition 21. Let N be a positively ω-saturated standard structure. Then

$$\{\varphi(x;\eta)\}' \quad \leftrightarrow \quad \varphi(x;\eta)$$

holds in N for every formula $\varphi(x;\eta) \in \mathcal{L}^{\mathrm{p}}(N)$.

Proof. We prove $\rightarrow$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

ih. $$\{\varphi(x,z;\eta)\}' \quad \rightarrow \quad \varphi(x,z;\eta)$$

We need to prove

$$\{\exists z\,\varphi(x,z;\eta)\}' \quad \rightarrow \quad \exists z\,\varphi(x,z;\eta)$$

From (ih) we have

$$\exists z\,\{\varphi(x,z;\eta)\}' \quad \rightarrow \quad \exists z\,\varphi(x,z;\eta)$$

Therefore it suffices to prove

$$\{\exists z\,\varphi(x,z;\eta)\}' \quad \rightarrow \quad \exists z\,\{\varphi(x,z;\eta)\}'$$

Replace the variables $x;\eta$ with parameters, say $a;\alpha$, and assume that $N \models \exists z\,\varphi'(a,z,;\alpha)$ for every $\varphi' > \varphi$. We need to prove the consistency of the type $\{\varphi'(a,z,;\alpha) : \varphi' > \varphi\}$. By saturation, finite consistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly. □

Remark 22. By Corollary 18, when $\varphi(x;\eta)$ does not contains existential quantifiers of sort H, the proposition above does not require the assumption of saturation. In general, some saturation is necessary: consider the model presented after Corollary 18 and the formula $\exists x\,\big(x > 0 \wedge \iota(x) \in \{0\}\big)$. See [Paragraph after Corollary 5.16][HI] for a counterexample in $\mathcal{L}^{\mathrm{c}}$.

5. The monster model

We denote by $\mathcal{U}$ some large p-saturated standard structure which we call the positive monster model. Truth is evaluated in $\mathcal{U}$ unless otherwise is specified. We denote by T the positive theory of $\mathcal{U}$. The density of a set of formulas is understood modulo T. Below we say p/c-model for p/c-elementary substructure of $\mathcal{U}$. We stress that the truth of some $\varphi \in \mathcal{L}^{\mathrm{p/c}}(M)$ in a p/c-model M implies the truth of φ (in $\mathcal{U}$) but not vice versa. However, all p/c-models agree on the approximated truth.

Fact 23. The following are equivalent

1. M is a p/c-model;
2. $M \models \{\varphi(a)\}' \;\Leftrightarrow\; \varphi(a)$ for every $\varphi(x) \in \mathcal{L}^{\mathrm{p/c}}$ and every $a \in M^{|x|}$.

Proof. (1⇒2) Only the implication ⇐ requires a proof. Assume $\varphi(a)$. Let $\varphi' > \varphi$. We prove $M \models \varphi'(a)$. By Lemma 7 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \to \neg\tilde{\varphi} \to \varphi'$. Then $\neg\tilde{\varphi}(a)$ and therefore $M \models \neg\tilde{\varphi}(a)$. Then $M \models \varphi'(a)$.

(2⇒1) By Proposition 21. □

The following fact demostrates how positive compactness applies. There are some subtle differences from the classical setting. Let $A \subseteq \mathcal{U}$ be a small set throughout this section.

Fact 24. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ be a type. Then for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(\mathcal{U})$

i. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \neg\varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;

ii. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (i) is immediate by saturation; (ii) follows from (i) by Lemma 7. □

Fact 25. Let $\mathcal{F}$ be a p/c-dense set of positive/continuous formulas. Then $\mathcal{F}'$ is p/c-dense.

Proof. Let $\varphi' > \varphi$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$ and $\psi \in \mathcal{F}$ such that $\varphi(x) \to \psi(x) \to \varphi''(x)$. It suffices to prove that $\psi'(x) \to \varphi'(x)$ for some $\psi' > \psi$. By Proposition 21, $\psi(x) \leftrightarrow \{\psi(x)\}'$. Therefore, $\psi'(x) \to \varphi'(x)$ follows from Fact 24. □

We write p/c-tp(a/A) for the positive/continuous type of a over A, that is, the set of formulas $\{\varphi(x) \in \mathcal{L}^{\mathrm{p/c}}(A) \;:\; \varphi(a)\}$. In general, if $\mathcal{F}$ is any set of formulas, we write $\mathcal{F}$-tp(a/A) for the type $\{\varphi(x) \in \mathcal{F}(A) \;:\; \varphi(a)\}$. The undecorated symbol tp(a/A) denotes the $\mathcal{L}$-type.

Fact 26. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p/c}}(A)$. The following are equivalent

1. $p(x)$ is a maximally standardly consistent subsets of $\mathcal{L}^{\mathrm{p/c}}_x(A)$;
2. $p(x) = $ p/c-tp(a/A) for some $a \in \mathcal{U}^{|x|}$.

Proof. (1⇒2) If $p(x)$ is consistent then $p(x) \subseteq$ p/c-tp(a/A) for some $a \in \mathcal{U}^{|x|}$. By maximality $p(x) = $ p/c-tp(a/A).

(2⇒1) By Proposition 21 and Lemma 7

$$\varphi(x) \quad \leftrightarrow \quad \bigwedge_{\tilde{\varphi}\perp\varphi} \neg\tilde{\varphi}(x).$$

Suppose $\varphi(x) \in \mathcal{L}^{\mathrm{p/c}}(A) \smallsetminus p$. Then $\neg\varphi(a)$. Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \rightarrow \neg\varphi(x)$ follows. □

The fact above extends as follows.

Fact 27. Let $p(x) \subseteq \mathcal{F}(A)$, where $\mathcal{F}$ is a p/c-dense set of positive/continuous formulas. Then the following are equivalent

1. $p(x)$ is a maximally standardly consistent subset of $\mathcal{F}_x(A)$;
2. $p(x) = \mathcal{F}\text{-tp}(a/A)$ for some $a \in \mathfrak{U}^{|x|}$;
3. $p(x) \leftrightarrow \text{p/c-tp}(a/A)$ for some $a \in \mathfrak{U}^{|x|}$.

Proof. (1⇒2) As above.

(2⇒3) As $q(x) \leftrightarrow q'(x)$, then $q(x) \leftrightarrow p(x) = \mathcal{F}\text{-tp}(a/A)$.

(3⇒1) Immediate. □

Proposition 28. Let $\mathcal{F}$ be a p/c-dense set of positive/continuous formulas. Then for every formula $\varphi(x) \in \mathcal{L}^{\mathrm{p/c}}$

i. $$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee\big\{\ \psi(x)\ :\ \psi'(x) \rightarrow \neg\varphi(x) \text{ and } \psi' > \psi \text{ for some } \psi(x) \in \mathcal{F}\big\};$$

ii. $$\neg\varphi(x) \quad \leftrightarrow \quad \bigvee\big\{\neg\psi(x)\ :\ \neg\psi(x) \rightarrow \neg\varphi(x) \text{ and } \psi(x) \in \mathcal{F}\big\}.$$

Proof. (i) Only → requires a proof. Let $a \in \mathfrak{U}^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \text{p/c-tp}(a)$. By Fact 26, $p(x) \rightarrow \neg\varphi(x)$. As $\mathcal{F}$ is p/c-dense, $p(x) \leftrightarrow q(x) = \mathcal{F}\text{-tp}(a)$. Then $q'(x) \rightarrow \neg\varphi(x)$ hence, by compactness, $\psi'(x) \rightarrow \neg\varphi(x)$ for some $\psi(x) \in \mathcal{F}$.

(ii) By density

$$\varphi(x) \quad \leftrightarrow \quad \bigwedge\big\{\psi(x) \in \mathcal{F}\ :\ \varphi(x) \rightarrow \psi(x)\big\}.$$

Negating both sides of the equivalence we obtain (ii). □

6. Cauchy completions

For $\tau(x,z) = \tau_1(x,z),\ldots,\tau_n(x,z)$ a tuple of terms of sort $\mathsf{H}^{|x|+|z|} \rightarrow \mathsf{I}$ we define the formula

$$x \sim_\tau y \quad = \quad \bigwedge_{i=1}^{n} \forall z\ \ \tau_i(x,z) = \tau_i(y,z),$$

where the expression $\alpha = \beta$ is shorthand for $\langle\alpha,\beta\rangle \in \Delta$, where Δ is the diagonal of I^2. We also define the type

$$x \sim y \quad = \quad \Big\{x \sim_\tau y \,:\, \tau(x,z) \text{ as above}\Big\}.$$

The following fact will be used below without mention.

Fact 29. For any $a = \langle a_i : i < \lambda\rangle$ and $b = \langle b_i : i < \lambda\rangle$

$$a \sim b \quad \Leftrightarrow \quad a_i \sim b_i \quad \text{for every } i < \lambda.$$

Proof. By induction. Only implication $\Leftarrow$ requires a proof For limit ordinals induction is trivial. Assume $a_i \sim b_i$ for every $i \le \lambda$. Let $|a| = |b| = \lambda$ and assume inductively that (note that it suffices to consider tuples τ of arity 1)

$$\forall y, z \;\; \tau(a,y,z) \quad = \quad \tau(b,y,z)$$

holds for every term $\tau(x,y,z)$, with $|x| = \lambda$ and $|y| = 1$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

1. $$\forall z \;\; \tau(a,a_\lambda,z) \quad = \quad \tau(b,a_\lambda,z).$$

As $a_\lambda \sim b_\lambda$ then

$$\forall x, z \;\; \tau(x,a_\lambda,z) \quad = \quad \tau(x,b_\lambda,z)$$

and in particular

2. $$\forall z \;\; \tau(b,a_\lambda,z) \quad = \quad \tau(b,b_\lambda,z).$$

From (1) and (2) we obtain

$$\forall z \;\; \tau(a,a_\lambda,z) \quad = \quad \tau(b,b_\lambda,z).$$ □

Note that the approximations of the formula $x \sim_\tau y$ have the form

$$x \sim_{\tau,D} y \quad = \quad \bigwedge_{i=1}^{n} \forall z \;\; \langle\tau_i(x,z), \tau_i(y,z)\rangle \in D$$

for some compact neighborhood D of Δ. We write $\mathcal{E}$ for the set containing the pairs τ, D as above. The formulas $x \sim_\varepsilon y$, as ε ranges over $\mathcal{E}$, form a base for a system of entougages on $\mathcal{U}^{|x|}$. We refer to this uniformity and the topology associated as the topology induced by I. Though not needed in the sequel, it is worth mentioning that the topology induced by I on $\mathcal{U}^{|x|}$ coincides with the product of the topology induced by I on $\mathcal{U}$. This can be verified by an argument similar to the proof of Fact 29.

Fact 30. For every $\varphi(x) \in \mathcal{L}^{\mathrm{c}}$

$$x \sim y \quad \rightarrow \quad \varphi(x) \leftrightarrow \varphi(y)$$

Proof. By induction. □

We define

$$x \sim' y \quad = \quad \{x \sim_\varepsilon y : \varepsilon \in \mathcal{E}\}.$$

The following corollary corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Corollary 31. For every $\varphi(x) \in \mathcal{L}^{\mathrm{c}}$, every $\varphi' > \varphi$, and every $\tilde{\varphi} \perp \varphi$ there an $\varepsilon \in \mathcal{E}$ such that

i. $x \sim_\varepsilon y \wedge \varphi(y) \quad \to \quad \varphi'(x)$

ii. $x \sim_\varepsilon y \wedge \tilde{\varphi}(y) \quad \to \quad \neg\varphi(x).$

Proof. As $x \sim' y \cup \{\varphi(x)\} \to \varphi(y)$ by the fact above, (i) follows from Fact 24. Similarly, we obtain (ii) from $x \sim' y \cup \{\tilde{\varphi}(x)\} \to \neg\varphi(y)$. □

We say what a type $q(x)$ is finitely satisfiable in A if every conjunction of formulas in $q(x)$ has a solution in $A^{|x|}$. This definition coincides with the classical one, but in our context, the notion is less roboust. We may happen that $p(x)$ is finitely satisfiable while $q(x) \leftrightarrow p(x)$ is not. In particular if M is a p-model and $q(x) = \text{p-tp}(a/M)$ then $q'(x)$ is always finitely satisfiable while $q(x)$ need not.

A set $A \subseteq \mathcal{U}$ is Cauchy complete if it contains all those $a \in \mathcal{U}$ such that $a \sim' x$ is finitely satisfied in A. Note that Cauchy complete sets are in particular closed under $\sim$-equivalence. The Cauchy completion of A is the set

$$\mathrm{Ccl}(A) \quad = \quad \{a : a \sim' x \text{ is finitely satisfied in } A\}.$$

Fact 32. $\mathrm{Ccl}(A)$ is Cauchy complete.

Proof. Suppose that $x \sim' a$ is finitely satisfied in $\mathrm{Ccl}(A)$. Let $\varepsilon \in \mathcal{E}$ be given. We prove that $a \sim_\varepsilon x$ is satisfied in A. Let $\eta \in \mathcal{E}$ be such that $x \sim_\eta y \sim_\eta z \to x \sim_\varepsilon z$. There is some $b \in \mathrm{Ccl}(A)$ such that $b \sim_\eta a$. There is some $c \in A$ such that $c \sim_\eta b$. Then c satisfies $x \sim_\varepsilon a$, as required. □

We say that $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(\mathcal{U})$ is a Cauchy type if it is consistent and $p(x) \wedge p(y) \to x \sim y$.

Fact 33. Let M be a c-model and let $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(M)$ be a Cauchy type. Then all realizations of $p(x)$ belong to $\mathrm{Ccl}(M)$.

Proof. If $p(x)$ is a Cauchy type, then $p(x) \to a \sim x$ for some $a \models p(x)$. As M is a c-model, $p'(x)$ is finitely satisfied in M. Then also $a \sim' x$ is finitely satisfied. Hence $a \in \mathrm{Ccl}(M)$. □

7. Morphims

Here we only discuss the continuous case and refer to [CLCL] for the positive case. The definition of c-elementary maps is delicate due to the lack of equality of sort H. A possible approach is to work in the quotient $\mathcal{U}/\sim$. A second possibility is to replace functions with relations. The two options differs only in the notation. Here we go for the second.

Let R be a binary relation on $\mathcal{U}$. If $a = \langle a_i : i < \lambda\rangle$ and $b = \langle b_i : i < \lambda\rangle$ we write $\langle a, b\rangle \in R$ to abbreviate: $\langle a_i, b_i\rangle \in R$ for all $i < \lambda$. Let $\mathcal{F}$ be a set of formulas. We say that R preserves the truth of $\mathcal{F}$-formulas if

$$\varphi(a) \quad\rightarrow\quad \varphi(b) \qquad\qquad \text{for every } \varphi(x) \in \mathcal{F} \text{ and every } \langle a, b\rangle \in R. \tag{\#}$$

We will consider two extreem cases for $\mathcal{F}$: the set of all continuous formulas and the set of atomic continuous formlas. We donote the latter by $\mathcal{L}^{\mathrm{c}}_{\mathrm{at}}$. Relations that preserve the truth of $\mathcal{L}^{\mathrm{c}}$-formulas are called c-elementary.

By the following fact, if R is c-elementary, also R^{-1} is c-elementary.

Fact 34. Let $\mathcal{F}$ be either $\mathcal{L}^{\mathrm{c}}$ or $\mathcal{L}^{\mathrm{c}}_{\mathrm{at}}$. Let R be a relation that preserves the truth of $\mathcal{F}$-formulas. Then the converse implication in (#) holds.

Proof. $\mathcal{F} = \mathcal{L}^{\mathrm{c}}$. Assume $\neg\varphi(a)$ and apply Proposition 28 to infer that $\psi(a)$ holds for some $\psi(x) \in \mathcal{L}^{\mathrm{c}}$ that implies $\neg\varphi(x)$.

$\mathcal{F} = \mathcal{L}^{\mathrm{c}}_{\mathrm{at}}$. Assume $\tau(a) \notin C$ then $\tau \in \tilde{C}$ for some compact $\tilde{C}$ disjoint of C. Then $\tau(b) \notin C$ follows. □

We say that the relation R is reduced if there is no $R' \subset (\sim) \circ R \circ (\sim)$ such that $R \subset (\sim) \circ R' \circ (\sim)$.

Fact 35. Every relation R contains a reduced relation R' such that $(\sim) \circ R \circ (\sim) = (\sim) \circ R' \circ (\sim)$.

Proof. Let $\langle R_i : i < \lambda\rangle$ be maximal decreasing chain of relations $R_i \subseteq R \subseteq (\sim) \circ R_i \circ (\sim)$. Let R' the intersection of the chain. By maximality R' is reduced and $(\sim) \circ R \circ (\sim) = (\sim) \circ R' \circ (\sim)$ is clear. □

Fact 36. Let $\mathcal{F} = \mathcal{L}^{\mathrm{c}}$. Let R be a reduced relation that preserves of the truth of $\mathcal{F}$-formulas. Then R is (the graph of) an injective map $f : \mathrm{dom}(R) \to \mathrm{range}(R)$.

The same is holds when $\mathcal{F} = \mathcal{L}^{\mathrm{c}}_{\mathrm{at}}$ if $\mathrm{dom}(R)$ and $\mathrm{range}(R)$ are c-models.

Proof. $\mathcal{F} = \mathcal{L}^{\mathrm{c}}$. Let $a \in \mathrm{dom}R$ and assume that $\langle a, b\rangle, \langle a, b'\rangle \in R$. To prove functionality suppose for a contradiction that $b \neq b'$. As R is reduced, $b \not\sim b'$. Then $b \not\sim_\varepsilon b'$ for some

$\varepsilon \in \mathcal{E}$. By Proposition 28 b, b' satisfies a continuous formula $\psi(x, x')$ that implies $x \not\sim_\varepsilon x'$. Then a, a also satisfies $\psi(x, x')$, a contradiction. This proves functionality, for injectivity apply the same argument to R^{-1}.

$\mathcal{F} = \mathcal{L}^{\mathrm{c}}_{\mathrm{at}}$. Reason as above to obtain $b \not\sim_\varepsilon b'$, say $\varepsilon = \langle \tau, D \rangle$. Then the formula

$$\neg\varphi(b, b', z) \quad = \quad \neg \bigwedge_{i=1}^{n} \langle \tau_i(b, z), \tau_i(b', z) \rangle \in D.$$

is consistent. By the Tarski-Vaught test (Theorem 43) $\neg\varphi(b, b', z)$ has a solution $c \in \mathrm{range}(R)^{|z|}$. Then $\neg\varphi(a, a, d)$ for some $d \in \mathrm{dom}(R)^{|z|}$ which is a contradiction. □

Let $f \subseteq R$ be a reduced c-elementary relation such that $R \subseteq (\sim) \circ f \circ (\sim)$. Then f is a function defined in exacly one representative a of each $(\sim)$-equivalence class $[a]$ that intersects $\mathrm{dom}(R)$.

Fact 37. Let $\mathcal{F}$ be either $\mathcal{L}^{\mathrm{c}}$ or $\mathcal{L}^{\mathrm{c}}_{\mathrm{at}}$. The following are equivalent

1. R preserves the truth of $\mathcal{F}$-formulas;
2. $\mathrm{Ccl}(R)$ preserves the truth of $\mathcal{F}$-formulas.

Proof. We prove the nontrivial implication, 1⇒2. Let $\varphi(x) \in \mathcal{F}$ and $\varphi'' > \varphi' > \varphi$ be given. Let $\langle a, b \rangle \in \mathrm{Ccl}(R)$ and assume $\varphi(a)$. Let $\varepsilon \in \mathcal{E}$ be such that

1. $x \sim_\varepsilon y \wedge \varphi\ (x) \quad \to \quad \varphi'(y)$
2. $x \sim_\varepsilon y \wedge \varphi'(x) \quad \to \quad \varphi''(y)$

By the definition of Cauchy completion, there are $a', b' \sim_\varepsilon a, b$ such that $\langle a', b' \rangle \in R$. Then from (1) we infer that $\varphi'(a')$. Note that $\varphi'(x) \in \mathcal{F}$, then $\varphi'(b')$. Then $\varphi''(b)$ follows from (2). As $\varphi'' > \varphi$ is arbitrary, the fact follows from Proposition 21. □

Definition 38. Let M and N be c-models. An c-isomorphism between M and M is a relation R that preserves the truth of $\mathcal{L}^{\mathrm{c}}_{\mathrm{at}}$-formulas and $M \subseteq \mathrm{dom}(R)$, $N \subseteq \mathrm{range}(R)$.

The term isomorphism may sound inappropriate but it is justified by the following fact.

Fact 39. Let R be a c-isomorphism between M and N, two c-models. Let $f \subseteq \mathrm{Ccl}(R)$ be a reduced relation such that $\mathrm{Ccl}(R) = (\sim) \circ f \circ (\sim)$. Then $f : \mathrm{Ccl}(M) \to \mathrm{Ccl}(N)$ is a bijective map such that $\varphi(a) \leftrightarrow \varphi(fa)$ for all $\varphi(x) \in \mathcal{L}^{\mathrm{c}}$ and $a \in \mathrm{Ccl}(M)^{|x|}$.

Proof. From Fact 36 we obtain that $f : \mathrm{Ccl}(M) \to \mathrm{Ccl}(N)$ is bijective map. To prove $\varphi(a) \leftrightarrow \varphi(fa)$ we reason by induction as in the classical case. □

8. Elimination of quantifiers of sort l

We write $\mathcal{L}^{\mathrm{p/c}}_{\mathsf{lqf}}$ for the set of positive/continuous formulas without quantifiers of sort l. In [CLCL] it is proved that $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$ is a p-dense set. Here we adapt the argument to continuous formulas.

We ask the reader to verify that Fact 36 holds, with the same proof, when R preserves the truth of formulas in $\mathcal{L}^{\mathrm{c}}_{\mathsf{lqf}}$.

Proposition 40. Every relation of small cardinality R that preserves the truth of formulas in $\mathcal{L}^{\mathrm{c}}_{\mathsf{lqf}}$ extends to c-automorphisms of $\mathcal{U}$.

Proof. We construct by back-and-forth a sequence $f_i : \mathcal{U} \to \mathcal{U}$ of reduced relations (i.e. injective maps) that preserve the truth formulas in $\mathcal{L}^{\mathrm{c}}_{\mathsf{lqf}}$. Finally we set

$$R \quad = \quad \bigcup_{i \in |\mathcal{U}|} (\sim) \circ f_i \circ (\sim).$$

We will ensure that $\mathcal{U} = \mathrm{dom}(R) = \mathrm{range}(R)$. Then R is a c-automorphism of $\mathcal{U}$ by Remark 39.

Let i be even. Let a be an enumeration of $\mathrm{dom}(f_n)$. Let $b \in \mathcal{U}$. Let $p(x, y) = \text{c-tp}(a, b)$. By c-elementarity the type $p(f_i a, y)$ is finitely satisfied. Let $c \in \mathcal{U}$ realize $p(f_i a, y)$. Let f_{i+1} be a reduced relation such that $f_{i+1} \subseteq f_i \cup \{\langle b, c\rangle\} \subseteq (\sim) \circ f_{i+1} \circ (\sim)$.

When i is odd we proceed similarly with f_i^{-1}. Limit stages are obvious. □

Corollary 41. Let $a \in \mathcal{U}^{|x|}$. Let $p(x) = \mathcal{L}^{\mathrm{c}}_{\mathsf{lqf}}\text{-tp}(a)$ and $q(x) = \text{c-tp}(a)$. Then $p(x) \leftrightarrow q(x)$.

Proof. Only $\to$ requires a proof. If a contains entries $a_i \sim a_j$, replace a_j with a_i. Note that the tuple a' obtained in this manner has the same c-type of a.

Let $b \models p(x)$ and let b' obtained with the same procedure as a'.

Then the function that maps $a' \mapsto b'$ preserves the truth of $\mathcal{L}^{\mathrm{c}}_{\mathsf{lqf}}$-formulas. Then f extends to an c-automorphism. As every c-automorphism is c-elementary and the corollary follows. □

Proposition 42. The set $\mathcal{L}^{\mathrm{c}}_{\mathsf{lqf}}$ is c-dense modulo T.

Proof. Let $\varphi(x)$ be a positive formula. We need to prove that for every $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$. By Corollary 41 and Proposition 21

$$\neg\varphi(x) \quad\rightarrow \quad \bigvee_{p'(x)\rightarrow\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over the maximally consistent $\mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$-types. By Fact 24 and Lemma 7

$$\neg\varphi(x) \quad\rightarrow \quad \bigvee_{\neg\tilde{\psi}(x)\rightarrow\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$. Equivalently,

$$\varphi(x) \quad\leftarrow \quad \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 24, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$ such that

$$\varphi'(x) \quad\leftarrow \quad \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \quad\leftarrow\quad \varphi(x)$$

which yields the interpolant required by the proposition. □

9. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is our version of the Tarski-Vaught test.

Theorem 43. Let M be a subset of $\mathcal{U}$. Let $\mathcal{F}$ be a p-dense set of positive formulas. Then the following are equivalent

1. M is a p-model;
2. for every formula $\psi(x) \in \mathcal{F}(M)$

$$\exists x\,\psi(x) \quad\Rightarrow\quad \text{for every } \psi' > \psi \text{ there is an } a \in M \text{ such that } \psi'(a);$$

3. for every formula $\psi(x) \in \mathcal{F}(M)$

$$\exists x\,\neg\psi(x) \quad\Rightarrow\quad \text{there is an } a \in M \text{ such that } \neg\psi(a).$$

If $\mathcal{F}$ is a c-dense set of continuous formulas then (2) and (3) above are equivalent to

1′. Ccl(M) is a c-model.

Moreover, if M is a substructure, then (1′), (2) and (3) are also equivalent to

1″. M is a c-model.

Proof. (1⇒2) Assume $\exists x\,\psi(x)$ and let $\psi' > \psi$ be given. By Lemma 7 there is some $\tilde{\psi} \perp \psi$ such that $\psi(x) \rightarrow \neg\tilde{\psi}(x) \rightarrow \psi'(x)$. Then $\neg\forall x\,\tilde{\psi}(x)$ hence, by (1), $M \models \neg\forall x\,\tilde{\psi}(x)$. Then $M \models \neg\tilde{\psi}(a)$ for some $a \in M$. Hence $M \models \psi'(a)$ and $\psi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\psi(x) \in \mathcal{F}(M)$ be such that $\exists x\,\neg\psi(x)$. By Proposition 28.i, there are a consistent $\varphi(x) \in \mathcal{F}(M)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \rightarrow \neg\psi(x)$. Then (3) follows.

(3⇒2) Let $\psi' > \psi$ for some $\psi(x) \in \mathcal{F}(M)$. Let $\tilde{\psi} \perp \psi$ such that $\psi(x) \to \neg\tilde{\psi}(x) \to \psi'(x)$. By Proposition 28.ii, $\neg\varphi(x) \to \neg\tilde{\psi}(x)$ for some $\varphi(x) \in \mathcal{F}(M)$ such that $\neg\varphi(x)$ is consistent. Then (2) follows from (3).

(2⇒1) Assume (2). By the classical Tarski-Vaught test $M \preceq_{\mathsf{H}} \mathcal{U}$. Then M is the domain of a substructure of $\mathcal{U}$. Then $M \models \varphi(a) \;\Rightarrow\; \varphi(a)$ holds for every atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \models \varphi(a, b) \quad \Rightarrow \quad \varphi(a, b).$$

Using (2) and the induction hypothesis we prove that

$$M \models \exists y\, \varphi(a, y) \quad \Rightarrow \quad \forall y\, \varphi(a, y).$$

Indeed, for any $\varphi' > \varphi$,

$$\begin{aligned}
M \models \exists y\, \varphi(a, y) \quad &\Rightarrow \quad M \models \exists y\, \psi(a, y) \text{for some } \psi \in \mathcal{F} \text{ such that } \varphi(a, y) \to \psi(a, y) \to \varphi'(a, y)\\
&\Rightarrow \quad M \models \psi(a, b) \text{for some } b \in M \text{ by (2)}\\
&\Rightarrow \quad M \models \varphi'(a, b)\\
&\Rightarrow \quad \varphi'(a, b) \text{by induction hypothesis}\\
&\Rightarrow \quad \exists y\, \varphi'(a, y).
\end{aligned}$$

As $\varphi' > \varphi$ is arbitrary, $\exists y\, \varphi(a, y)$ follows from Proposition 21.

Induction for the connectives $\vee$, $\wedge$, $\forall^{\mathsf{H}}$, $\exists^{\mathsf{I}}$, and $\forall^{\mathsf{I}}$ is straightforward.

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(1′⇒2) Let $\psi' > \psi'' > \psi$. Reasoning as in the proof of (1⇒2) we obtain that $\mathrm{Ccl}(M) \models \psi''(a)$ for some $a \in \mathrm{Ccl}(M)$. By Corollary 31, $a \sim_{\varepsilon} x \to \psi'(x)$ for some ε. As $a \sim' x$ is finitely satisflied in M, it follows that $\psi'(c)$ for some $c \in M$.

(2⇔3) The proof above applies verbatim when $\mathcal{F}$ is a c-dense set of continuous formulas.

(2⇒1′) Assume (2). We claim that $\mathrm{Ccl}(M)$ is a substructure of $\mathcal{U}$. Let $a \in M^n$ and let f be a function symbol of sort $\mathsf{H}^n \to \mathsf{H}$. We prove that $f a \in M$. We show that $f a \sim' x$ is finitely satisfied in M. Consider the formula $f a \sim_{\varepsilon} x$ where ε is the pair τ, D. By Lemma 7, there is a formula in $\tilde{\varphi}(x) \in \mathcal{L}^{\mathrm{p}}(M)$ such that

$$f a \sim_{\tau} x \quad \to \quad \neg\tilde{\varphi}(x) \quad \to \quad f a \sim_{\varepsilon} x$$

By Fact 25.ii there is a consistent formula $\neg\psi(x)$, for some $\psi(x) \in \mathcal{F}(M)$, that implies $f a \sim_{\varepsilon} x$. Then, by (3), $f a \sim_{\varepsilon} x$ is satisfied in M. This proves our claim.

Now, we claim that (2) holds also for every $\psi(x) \in \mathcal{F}\big(\mathrm{Ccl}(M)\big)$. Let $\psi(x, z) \in \mathcal{F}(M)$ and $\psi' > \psi$ be given. Let $b \in \mathrm{Ccl}(M)^{|z|}$. Suppose that $\exists x\, \psi(x, b)$ and let ε be such that $z \sim_{\varepsilon} b \to \exists x\, \psi''(x, z)$ where $\psi' > \psi'' > \psi$. By Corollary 31, we can also assume that $z \sim_{\varepsilon} b \wedge$

$\psi''(x,z) \to \psi'(x,b)$. Let $b' \in M^{|z|}$ be such that $b' \sim_\varepsilon b$. By (2) there is an $a \in M^{|x|}$ such that $\psi''(a,b')$. Then $\psi'(a,b)$ follows. This proves the second claim.

By the two claims above, the inductive argument in the proof in (2⇒1) applies to prove that $\mathrm{Ccl}(M)$ is a c-model.

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(1″⇒2) By the same argument as in (1⇒2).

(2⇒1″) As M is a substructure by assumption, the inductive argument in the proof in (2⇒1) applies. □

Remark 44. The theorem above shows in particular that for every substructure M the following are equivalent

1. M is a c-model;
2. $\mathrm{Ccl}(M)$ is a c-model.

Classically, the first application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all $\mathcal{L}$-structures. In particular every $A \subseteq \mathfrak{U}$ is contained in a standard structure of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of $\mathcal{L}$ is eccessively large because of the aboundance of symbols in $\mathcal{L}_{\mathsf{I}}$.

We say that $\mathcal{L}^{\mathrm{p/c}}$ is separable if there is a countable p/c-dense set $\mathcal{F}$ of positive/continuous formulas.

Proposition 45. Let $\mathcal{L}^{\mathrm{p/c}}$ be separable. Let A be a countable set. Then there is a countable p/c-model M containing A.

Proof. Let $\mathcal{F}$ be a countable p/c-dense set of positive/continuous formulas. As in the classical proof of the Löwenheim-Skolem Theorem, we construct a countable $M \subseteq \mathfrak{U}$ that contains a witness of every consistent formula $\neg\psi(x)$ for $\psi(x) \in \mathcal{F}(M)$. In continuous case we also ensure that M is a substructure. Then the proposition follows from Theorem 43. □

Proposition 46. Assume that $\mathcal{L}$ has at most countably many symbols of sort $\mathsf{H}^n \to \mathsf{I}$. Then, if I is a second countable (the topology has a countable base), then $\mathcal{L}^{\mathrm{c}}$ is separable modulo $\mathcal{L}$. Moreover if $\mathcal{L}_{\mathsf{H}}$ is countable also then $\mathcal{L}^{\mathrm{p}}$ is separable.

Proof. Fix a countable filter of compact neighborhoods of the diagonal of I^2. □

10. Continuous omitting types

In this section we present a version of the omitting type theorem. We are interested in omitting types in Cauchy complete models. We only consider the continuous case.

A type $p(x)$ is isolated by $\varphi(x)$, a consistent formula, if $\varphi(x) \to p(x)$. If $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{L}^{\mathrm{c}}(A)$ we say that it is c-isolated by A. We omit the reference to A when this is clear from the context (e.g., when $p(x)$ is presented as a type over A).

By Theorem 43, $p(x)$ is c-isolated by A, then $p(x)$ is realized in every c-model containing A.

Fact 47. Let $\mathcal{F}$ be a c-dense set of continuous formulas. Then the following are equivalent

1. $p(x)$ is c-isolated by A;
2. $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{F}(A)$;
3. $p(x)$ is isolated by some $\varphi'(x)$ such that $\varphi' > \varphi$ for some consistent $\varphi(x) \in \mathcal{F}(A)$.

Proof. (1⇒2) By Proposition 28.ii.

(1⇒3) By Proposition 28.i.

(3⇒1) Let $\varphi' > \varphi$ be as in (3). Let $\tilde{\varphi} \perp \varphi$ be such that $\varphi(x) \to \neg\tilde{\varphi}(x) \to \varphi'(x)$. Then $\neg\tilde{\varphi}(x)$ isolates $p(x)$. □

We say that the type $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(\mathcal{U})$ is approximately c-isolated by A if for every $\varepsilon \in \mathcal{E}$ there is a formula $\varphi_\varepsilon(x) \in \mathcal{L}^{\mathrm{c}}(A)$ such that $\neg\varphi_\varepsilon(x)$ is consistent with $p(x)$ and isolates $\exists x' \sim_\varepsilon x\; p(x')$.

Notice the requirement that the formula that isolates $\exists x' \sim_\varepsilon x\; p(x')$ is consistent with $p(x)$. This is used in the proof of Lemma 53.

Assume for the rest of this section that $\mathcal{L}^{\mathrm{c}}$ is separable. Then there is a sequence $\langle \varepsilon_n : n \in \omega \rangle$ such that for every $\varepsilon \in \mathcal{E}$ there is an n such that $x \sim_{\varepsilon_n} y \;\to\; x \sim_\varepsilon y$. We abbreviate $x \sim_{\varepsilon_n} y$ by $x \sim_n y$.

Fact 48. Let $\mathcal{L}^{\mathrm{c}}$ be separable. Let $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ be maximally consistent type that is approximately c-isolated. Then this is witnessed by formulas $\psi(x) \in \mathcal{L}^{\mathrm{c}}(A)$ such that $\psi_n(x) \to \psi_{n+1}(x)$. Finally, if $a_n \models \neg\psi_n(x)$ the the type $p(x) \cup \big\{a_n \sim_n x \;:\; n \in \omega\big\}$ is consistent.

Proof. Let $\psi_n(x) = \varphi_0(x) \vee \cdots \vee \varphi_n(x)$. Clearly $\neg\psi_n(x) \;\to\; x \sim_n y$, then we only need verify that $\neg\psi_n(x)$ is consistent with $p(x)$. But this is the case because by maximality $p(x) \to \neg\varphi_n(x)$. The second claim follows immediately. □

Fact 49. Let $\mathcal{L}^{\mathrm{c}}$ be separable. Let $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ be maximally consistent and approximately c-isolated by A. Let M be a c-model containing A. Then $p(x)$ is realized in $\mathrm{Ccl}(M)$.

Proof. If $a_n \in M$ is as in the fact above, then any solution of the type $p(x) \cup \{a_n \sim_n x : n \in \omega\}$ belongs to $\mathrm{Ccl}(M)$. □

In words, the proof above shows that if $p(x)$ is approximately c-isolated, then in every c-model contains a sequence of elements that converges, in the topology induced by I, to a realization of $p(x)$.

Fact 50. Let $\mathcal{L}^{\mathrm{c}}$ be separable. Let M be a c-model containing A. Then any $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ that is realized in $\mathrm{Ccl}(M)$ is approximately c-isolated by M.

Proof. Let $b \in \mathrm{Ccl}(M)^{|x|}$ realize $p(x)$. Let $\varepsilon \in \mathcal{E}$ be given. By Fact 47 it suffices to find a continuous formula $\varphi(x)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \to \exists x' \sim_\varepsilon x\; p(x')$.

Let $\eta' \in \mathcal{E}$ be such that $x \sim_{\eta'} y \sim_{\eta'} z \;\to\; x \sim_\varepsilon z$. Pick some $a \in M^{|x|}$ that satisfies $x \sim_{\eta'} b$. Then $x \sim_{\eta'} a \;\to \exists\, x' \sim_\varepsilon x\; p(x')$. Finally, note that $(\sim_{\eta'}) > (\sim_\eta)$ for some $\eta \in \mathcal{E}$. □

Lemma 51. Let $\mathcal{L}^{\mathrm{c}}(A)$ be separable. Let $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$, be non approximately c-isolated. Then every consistent formula $\neg\psi(z)$, with $\psi(z) \in \mathcal{L}^{\mathrm{c}}(A)$, has a solution a such that A, a does not c-isolate $p(x)$.

Proof. Let $\varepsilon \in \mathcal{E}$ witness that $p(x)$ is not approximately c-isolated by A. We construct a sequence of $\mathcal{L}^{\mathrm{c}}(A)$-formulas $\langle \gamma_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\gamma_i(z) : i < \omega\}$ is the required solution of $\neg\psi(z)$, witnessed by the same ε.

Let $\langle \xi_i(x,z) : i < \omega \rangle$ enumerate a countable c-dense subset of $\mathcal{L}^{\mathrm{c}}_{x,z}(A)$. Let $\gamma_0(z)$ be a consistent continuous formulas such that $\gamma'_0(z) \to \neg\psi(z)$ for some $\gamma'_0 > \gamma_0$. This exists by Proposition 28. Now we define $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$. Let $\gamma_i(z)$ and $\gamma'_i > \gamma_i$ be given. Pick $\tilde\gamma \perp \gamma_i$ such that $\gamma_i(z) \to \neg\tilde\gamma(z) \to \gamma'_i(z)$.

1. If $\neg\xi_i(x,z) \wedge \gamma_i(z)$ is inconsistent with $p(x)$, let $\gamma_{i+1}(z) = \gamma_i(z)$ and $\gamma'_{i+1}(z) = \gamma'_i(z)$.
2. Otherwise, pick $\varphi(x) \in p$ such that (#) below is consistent

   (#) $$\neg\tilde\gamma(z) \;\wedge\; \exists x \big[\neg\xi_i(x,z) \wedge \neg\exists x' \sim_\varepsilon x\; \varphi(x')\big].$$

   Finally, let $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$ consistent continuous formulas such that $\gamma'_{i+1}(z)$ implies (#). Such formulas exist by Proposition 28.

Let $a \models \{\gamma_i(z) : i < \omega\}$. We claim that that A, a does not c-isolate $p(x)$, witnessed by ε. Otherwise $\neg\xi_i(x, a) \to \exists x' \sim_\varepsilon x\ p(x')$ for some $\neg\xi_i(x, a)$ consistent with $p(x)$. This contradicts $a \models \gamma_{i+1}(z)$.

Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in (2).

Suppose for a contradiction that $\neg\xi_i(x, z) \wedge \gamma_i(z)$ is consistent with $p(x)$ while (#) is inconsistent for all formulas $\varphi(x) \in p$. This immediately implies that

$$\exists z \left[\neg\xi_i(x, z) \wedge \neg\tilde{\gamma}(z)\right] \quad \to \quad \exists x' \sim_\varepsilon x\ p(x').$$

By Fact 47, this yelds the desired contradiction. □

Theorem 52 (Continuous Omitting Types). Let $\mathcal{L}^{\mathrm{c}}$ be separable. Let A be countable. Assume also that $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ is not approximately c-isolated. Then there is a c-model M containing A such that $\mathrm{Ccl}(M)$ omits $p(x)$.

Proof. By assumption $\exists y \sim_\varepsilon x\ p(y)$ is not approximately c-isolated for some $\varepsilon \in \mathcal{E}$. Just as in the classical case, we appy Lemma 51 and the Tarski-Vaught test (Theorem 43) to obtain a countable c-model M that does not isolate $\exists y \sim_\varepsilon x\ p(y)$. By Fact 50, $\mathrm{Ccl}(M)$ omits $p(x)$. □

11. Continuous ω-categoricity

In the discussion of countable categoricity we assume that the underlying theory is complete. This assumption is not necessary (it is rather a consequence of categoricity), but it allows to work inside a monster model and simplify the notation.

Let M be a c-model. We say that M is atomic if for every finite tuple a of elements of M the type c-tp(a) is approximately c-isolated.

Lemma 53. Let $\mathcal{L}^{\mathrm{c}}$ be separable. Let M and N be two atomic c-models. Let k be a finite c-elementary relation between $\mathrm{Ccl}(M)$ and $\mathrm{Ccl}(N)$. Then for every $c \in M$ there is $d \in \mathrm{Ccl}(N)$ such that $k \cup \{\langle c, d\rangle\}$ is a c-elementary relation.

Proof. We can assume that k is reduced, i.e. it is an injective map $\mathrm{Ccl}(M) \to \mathrm{Ccl}(N)$. Let a enumerate dom$k$. Let $p(x, y) = \text{c-tp}(a, c)$, it suffices to prove that $p(x, y)$ is realized by some $d \in \mathrm{Ccl}(N)$. By Fact 48 there are some formulas $\psi_n(x, y) \in \mathcal{L}^{\mathrm{c}}$ such that

$$\neg\psi_n(x, y) \quad \to \quad \exists x', y' \sim_n x, y\ \ p(x', y')$$

and $\psi_n(x, y) \to \psi_{n+1}(x, y)$. As $\neg\psi_n(x, y)$ is consistent with $p(x, y)$ then $a, c \models \neg\psi_n(x, y)$. Then by c-elementarity, $\neg\varphi(ka, y)$ is consistent. Let $d_n \in \mathrm{Ccl}(M)$ be such that $ka, d_n \models$

$\neg\psi_n(x,y)$. Let b,d be a realization of the type $\{ka, d_n \sim_n x, y : n \in \omega\} \cup p(x,y)$. As $ka \sim b$, also ka, d realizes $p(x,y)$. As $d \in \mathrm{Ccl}(N)$, the proof is complete. □

Fact 54. Let M and N be countable atomic c-models. Then $\mathrm{Ccl}(M)$ and $\mathrm{Ccl}(N)$ are c-isomorphic.

Proof. We construct by back-and-forth a sequence $f_n : \mathrm{Ccl}(M) \to \mathrm{Ccl}(N)$ of finite c-elementary reduced relations (i.e. injective maps) and let

$$R \quad = \quad \bigcup_{n\in\omega} (\sim) \circ f_n \circ (\sim).$$

We will ensure that $M \subseteq \mathrm{dom}(R)$ and $N \subseteq \mathrm{range}(R)$, hence that R is a c-isomorphism between M and N.

Let n be even. Let a be an enumeration of $\mathrm{dom}(f_n)$. Let $c \in M$. Let $p(x,y) = \text{c-tp}(a,b)$. As M is atomic by Lemma 53, for some $d \in \mathrm{Ccl}(N)$ the relation $f_n \cup \{\langle c,d\rangle\}$ is c-elementary. Let f_{n+1} be a reduced relation such that $f_{n+1} \subseteq f_n \cup \{\langle a,c\rangle\} \subseteq (\sim) \circ f_{n+1} \circ (\sim)$.

When n is odd we proceed similarly with f_n^{-1}. □

Let $\mathcal{L}^{\mathrm{c}}$ be separable. We say that T is c-ω-categorical if any two countable c-models are c-isomorphic, see Definition 38. The following analogue of Ryll-Nardzewski's Theorem follows by the classical argument.

Theorem 55. Assume that $\mathcal{L}^{\mathrm{c}}$ is separable. Then the following are equivalent

1. T is c-ω-categorical;
2. every consistent type $p(x) \subseteq \mathcal{L}^{\mathrm{c}}$, for x any finite tuple of variables, is approximately c-isolated.

Proof. (1⇒2) If $p(x)$ is consistent it is realized in some countable c-model M. If $p(x)$ is not approximately c-isolated, by the continuous omitting type theorem there is a c-model N such that $\mathrm{Ccl}(N)$ omits $p(x)$. Then $\mathrm{Ccl}(M)$ and $\mathrm{Ccl}(N)$ are not c-isomorphic, hence T is not c-ω-categorical.

(2⇒1) As, every c-model is atomic, c-ω-categoricity follows from Fact 54. □

12. Example: Lebesgue spaces

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