Standard analysis

C. L. C. L. Polymath

ABSTRACT. Let $\mathcal{L}$ be a first-order two-sorted language. Let I be some fixed structure. A *standard* structure is an $\mathcal{L}$-structure of the form $\langle M, I\rangle$ where I is fixed. We ask if there is a nontrivial model theory of standard structures. When I is a compact topological space (and the topology is coded into $\mathcal{L}$) the answer is YES. Indeed, a significant part of classical model theory can be adapted, with a few caveats, to this restricted class of standard structures. E.g., we prove that every standard structure has a *positive* elementary extension that is standard and realizes all positive types that are finitely consistent. We show that types realized in this positive monster model are equivalent to positive types.

This approach offer a slick alternative to both nonstandard analysis and to real-valued model theory. It is based on [**CLCL**] and, in turn, inspired by the Henson-Iovino model theory of Banach spaces [**HI**].

1. Introduction

We refer to the introduction of [**CLCL**] for motivations. Aside from that, this paper is reasonably self-contained. In the first part we biefly revisit the main results of [**CLCL**]. We rephrase some theorem and add a few remarks and examples[1]. Our notation and terminology diverges slightly.

In the second part of this paper we discuss: criteria for elimination of quantifers; omitting types; ω-categoricity; stability; and indiscernibles.

The model theory of standard structures is based on the notions of positive formulas and of approximations. These evolved from the Henson-Iovino model theory of Banach spaces [**HI**]. Note that though the model theory of standard structures is an example of positive model theory there are important differences. For instance, we show that the negation of a positive formula is equivalent to an infinite disjunction of positive formulas. We also prove that these negations can be *approximated* arbitrarily well by positive formulas. This makes the model theory of standard structures closer to classical model theory than one may expect.

2. Standard structures

Let I be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language L_{I} contains relation symbols for the compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \to I$. In particular, there is a constant for each element of I. According to the context, C and f denote either the symbols of L_{I} or their interpretation in the stucture I. Such a language L_{I} is much larger than necessary but it is convenient because it uniquely associates a structure to a topological space. The notion of dense set of formulas (Definition 7) helps to reduce the size of the language when required.

The most straightforward examples of structures I are the unit interval $[0,1]$ with the usual topology, and its homeomophic copies $\mathbb{R}^+ \cup \{0,\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language L_{H} which we call the language of the home sort.

[1] In the final version we may drop a few sections that do not add new insight to [**CLCL**]

Definition 1. Let $\mathcal{L}$ be a two sorted language that expands both L_{H} and L_{I}. A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, I\rangle$, where M is any L_{H}-structure, while I is the structure we fixed above. We write H and I to denote the two sorts of $\mathcal{L}$.

The language $\mathcal{L}$ adds a relation symbol for every L_{H}-definable subset of M, in other words, $\mathcal{L}$ performs the Morleyzation of L_{H}.

Finally and most relevantly, $\mathcal{L}$ contains arbitrarily many function symbols of sort $\mathsf{H}^n \to \mathsf{I}$.

Standard structures are denoted by the domain of their home sort.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (when I is finite), they are not standard. As a remedy, below we carve out a set of formulas $\mathcal{L}^{\mathrm{p}}$, the set of positive formulas, such that every model has an positive elementary saturated extension that is also standard.

As usual, L_{I}, L_{H}, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. When we restrict to formulas with a fix tuple of free variables we write $\mathcal{L}_x$.

Definition 2. A formula in $\mathcal{L}$ is positive if it uses only the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{I}}$, $\exists^{\mathsf{I}}$ of sort I.

The set of positive formulas is denoted by $\mathcal{L}^{\mathrm{p}}$.

We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort I. Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

Note that $\mathcal{L}$ has two types of atomic formulas: those of sort H (that is, those equivalent to formulas in L_{H}) and those of the form $t(x;\eta) \in C$, where $C \subseteq I^n$ is compact and $t(x;\eta)$ is a tuple of terms of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$.

Henson and Iovino have a more stringent notion of positive formulas (that they define in the language of normed spaces). Their positive formulas exclude formulas in L_{H} and quantifiers of sort I. In our setting L_{H}-formulas are positive (up to equivalence) which comes at a small price: the failure of the Perturbation Lemma [HI, Proposition 5.15]. However, we will recover part of it in Proposition 39. Instead, somewhat surprisingly, allowing quantifiers of sort I is almost for free. We will see (cf. Propositions 27) that the formulas in $\mathcal{L}^{\mathrm{p}}$ are approximated by formulas without quantifiers of sort I.

The extra expressive power offered by quantifiers of sort I is convenient. For instance, it is easy to see (cf. Example 3) that $\mathcal{L}^{\mathrm{p}}$ has at least the same expressive power as real valued logic

Example 3. Let $I = [0,1]$, the unit interval in $\mathbb{R}$. Let $t(x)$ be a term of sort $\mathsf{H}^{|x|} \to \mathsf{I}$. Then there is a positive formula that says $\sup_x t(x) = \tau$. Indeed, consider the formula

$$\forall x\, \big[t(x) \dot{-} \tau \in \{0\}\big] \quad \wedge \quad \forall \varepsilon \Big[\varepsilon \in \{0\} \;\vee\; \exists x\, \big[\tau \dot{-} (t(x) + \varepsilon) \in \{0\}\big]\Big]$$

which, in a more legible form, becomes

$$\forall x\, \big[t(x) \le \tau\big] \quad \wedge \quad \forall \varepsilon > 0\, \exists x\, \big[\tau \le t(x) + \varepsilon\big].$$

The following example is inspired by [HPP].

Example 4. Let $\bar{M}$ be a saturated structure of signature L_{H}. The relevant part of $\mathcal{L}$ contains symbols for the functions $f : \bar{M}^n \to I$ that are continuous in the following sense: the inverse image of every compact $C \subseteq I$ is type-definable in $L_{\mathsf{H}}(M)$ for every $M \preceq \bar{M}$. It is easy to verify that $\langle \bar{M}, I\rangle$ is positively saturated, see Section 6.

Normed spaces is one of the motivating examples. Essentially, functional analysis and model theory are only interested in the unit ball of normed spaces. To formalize the unit ball of a normed space as a standard structure (where I is the unit interval) is straightforward but pretty ugly. Alternatively one can view a normed space as a many-sorted structure: a sort for each ball of radius $n \in \mathbb{Z}^+$. Unfortunately, this also results into a bloated formalism.

A neater formalization is possible if one accepts that some potitive elementary extensions (to be defined below) of a normed space could contain vectors of infinite norm hence not be themselves normed spaces. These infinities are harmless if one is only interested in the unit ball. In fact, inside any ball of finite radius these nonstandard normed spaces are completely standard.

Example 5. Let $I = \mathbb{R} \cup \{\pm\infty\}$. Let L_H be the language of real (or complex) vector spaces. The language $\mathcal{L}$ contains a function symbol $\|\text{-}\|$ of sort $\mathsf{H} \to \mathsf{I}$. Normed space are standard structures with the natural interpretation of the language. It is easy to verify that the unit ball of a standard structure is the unit ball of a normed space. Note that the same remains true if we add to the language any continuous (equivalently, bounded) operator or functional.

We conclude this introduction advertising a question of [CLCL]. Is it possible to extend the Positive Compactness Theorem (Theorem 19) to a larger class of languages? E.g. can we allow in $\mathcal{L}$ (and in $\mathcal{L}^\mathrm{p}$) function symbols of sort $\mathsf{H}^n \times \mathsf{I}^m \to \mathsf{I}$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I.

3. Henson-Iovino approximations

For φ, φ' (free variables are hidden) positive formulas possibly with parameters we write $\varphi' > \varphi$ if φ' is obtained by replacing in φ each atomic formula of the form $t \in C$ with $t \in C'$, where C' is some compact neighborhood of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. We call φ' a weakening of φ. Note that $>$ is a dense transitive relation and that $\varphi \to \varphi'$ in every $\mathcal{L}$-structure.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. The atomic formulas in L_H are replaced with their negation. Finally each connective is replaced by its dual i.e., $\vee, \wedge, \exists, \forall$ are replaced by $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde{\varphi}$ is a strong negation of φ. It is clear that $\tilde{\varphi} \to \neg\varphi$ in every $\mathcal{L}$-structure.

Lemma 6. For all positive formulas φ

1. for every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$;
2. for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in L_\mathsf{H}$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in I \smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of I, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. □

For every type $p(x)$, we write

$$p'(x) \quad = \quad \big\{\varphi'(x) \;:\; \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}$$

in particular $\{\varphi(x)\}' = \big\{\varphi'(x) \;:\; \varphi' > \varphi\big\}$.

Definition 7. A set of formulas $\mathcal{F} \subseteq \mathcal{L}$ is dense modulo T, a theory, if for every positive $\varphi(x)$ and every $\varphi' > \varphi$, there is $\psi(x) \in \mathcal{F}$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$ holds in every standard structure that models T. If T is empty, we say that $\mathcal{F}$ is dense modulo $\mathcal{L}$.

Example 8. By Lemma 6, the set of negations of positive formulas is dense modulo $\mathcal{L}$.

Example 9 (???). Read $\alpha = \beta$ as synonimous of $\langle \alpha, \beta \rangle \in \Delta$, where Δ is the diagonal of I^2, a compact set. It is easy to see that the set of formulas build inductively from the atomic formulas of sort H and $t(x) = s(x)$ is dense modulo $\mathcal{L}$.

Example 10. Let $\mathcal{H}$ be the set of positive formulas without quantifiers of sort I. In Section 8 we will prove that this set is dense.

4. Morphisms

Let M and N be two standard structures. We say that a partial map $f : M \to N$ is p-elementary if for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$ and every $a \in (\operatorname{dom} f)^{|x|}$

1. $$M \models \varphi(a) \quad \Rightarrow \quad N \models \varphi(fa).$$

A p-elementary map that is total is called an p-elementary embedding. When the map $\mathrm{id}_M : M \to N$ is an p-elementary embedding, we write $M \preceq^{\mathrm{p}} N$ and say that M is an p-elementary substructure of N.

As p-elementary maps are in particular L_{H}-elementary, they are injective. However, their inverse need not be p-elementary. In other words, the converse of the implication in (1) may not hold. Hence the following notion of elementarity which is more robust. We say that the map $f : M \to N$ is an approximate p-elementary if for every formula $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$, and every $a \in (\operatorname{dom} f)^{|x|}$

2. $$M \models \varphi(a) \quad \Rightarrow \quad N \models \{\varphi(fa)\}'.$$

One convenient feature of approximate p-elementarity is that it suffices verify (2) for any/some dense set of formulas. It is clear that p-elementarity implies its approximate variant. We will see that with a slight amount of saturation also the converse holds (Proposition 21). Finally, under full saturation, all differences disappear as these morphims becomes $\mathcal{L}$-elementarity maps (Corollary 26).

The following holds in general.

Fact 11. If $f : M \to N$ is approximate p-elementary then

$$M \models \{\varphi(a)\}' \quad \Leftrightarrow \quad N \models \{\varphi(fa)\}'.$$

for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$, and every $a \in (\operatorname{dom} f)^{|x|}$.

Proof. $\Rightarrow$ Fix $\varphi' > \varphi$ and assume $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By the definition of approximate elementarity, $N \models \varphi''(fa)$ and $N \models \varphi'(fa)$ follows.

$\Leftarrow$ Assume the r.h.s. of the equivalence. Fix $\varphi' > \varphi$ and prove $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By Lemma 6 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \to \neg\tilde{\varphi} \to \varphi'$. Then $N \models \neg\tilde{\varphi}(fa)$ and therefore $M \models \neg\tilde{\varphi}(a)$. Then $M \models \varphi'(a)$. □

Finally, we consider the notion of partial embeddings. Though the definition seems weaker, we prove that the notion is competely classical. We say that the map $f : M \to N$ is a partial embedding if the implication in (1) holds for all atomic formulas $\varphi(x)$. Note that this is the same as requiring that (1) holds for all formulas without quantifiers of sort H.

Fact 12. If $f : M \to N$ is a partial embedding then

$$M \models \varphi(a) \quad\Leftrightarrow\quad N \models \varphi(fa)$$

for every $a \in (\operatorname{dom} f)^{|x|}$ and every formula $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$ without quantifiers of sort H.

Proof. The equivalence is trivial for formulas in L_{H} so we only consider atomic formulas of the form $t \in C$. Implication $\Rightarrow$ holds by definition. Vice versa, if $M \models t(a) \notin C$ then, by normality, $M \models t(a) \in \tilde{C}$ for some compact $\tilde{C}$ disjoint of C. By the definition of partial embedding, $N \models t(a) \in \tilde{C}$. Hence $N \models t(a) \notin C$. Induction is immediate. □

5. The standard part

Our goal is to prove a compactness theorem for positive theories that only requires standard structures. To bypass lengthy routine applications of ultrafilters, ultrapowers, and ultralimits, we assume the Classical Compactness Theorem and built on that.

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 17 which in turn is required for the proof of the Positive Compactness Theorem (Theorem 19). The reader willing to accept it without proof may skip this section.

Let $\langle N, {}^*I\rangle$ be an $\mathcal{L}$-structure that extends $\mathcal{L}$-elementarily the standard structure M. Let η be a free variable of sort I. For each $\beta \in I$, we define the type

$$\mathrm{m}_\beta(\eta) \quad=\quad \{\eta \in D \,:\, D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(\eta)$ in ${}^*I$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 13. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Proof. Negate the existence of β. For every $\gamma \in I$ pick some compact neighborhood D_γ of γ, such that ${}^*I \models \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ, with $\gamma \in \Gamma$, cover I. By elementarity, the interpretation in ${}^*I$ of these D_γ cover ${}^*I$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Fact 14. For every $\alpha \in {}^*I$ and every compact $C \subseteq I$

$${}^*I \quad\models\quad \alpha \in C \;\to\; \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*I \models \alpha \in D \subseteq \neg C$. □

Fact 15. For every $\alpha \in ({}^*I)^{|\alpha|}$ and every function symbol f of sort $\mathsf{I}^{|\alpha|} \to \mathsf{I}$

$${}^*I \quad\models\quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big).$$

Proof. By Fact 13 and the definition of st(-) it suffices to prove that ${}^*I \models f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*I \models \alpha \in f^{-1}[D]$ and, as $I \preceq {}^*I$ we obtain ${}^*I \models f(\alpha) \in D$. □

The standard part of $\langle N, {}^*I\rangle$ is the standard structure $\langle N, I\rangle$ that interprets the symbols f of sort $\mathsf{H}^n \to \mathsf{I}$ as the functions

$$f^N(a) \quad = \quad \mathrm{st}\big({}^*f(a)\big) \qquad\qquad \text{for all } a \in N^n,$$

where ${}^*f$ is the interpretation of f in $\langle N, {}^*I\rangle$. Symbols in L_H maintain the same interpretation.

Fact 16. With the notation as above. Let $t(x;\eta)$ be a term of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$. Then for every $a \in M^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$t^N\big(a;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*t(a;\alpha)\big)$$

Proof. When a t is a function symbol of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{I}^{|\eta|} \to \mathsf{I}$, the claim follows from Fact 15. Now, assume inductively that

$$t_i^N\big(a;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*\,t_i(a;\alpha)\big)$$

holds for the terms $t_1(x;\eta),\dots,t_n(x;\eta)$ and let $t = f(t_1,\dots,t_n)$ for some function f of sort $\mathsf{I}^n \to \mathsf{I}$. Then the claim follows immediately from the induction hypothesis and Fact 15. □

Lemma 17. With the notation as above. For every $\varphi(x;\eta) \in \mathcal{L}^{\mathrm{p}}$, $a \in N^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$\langle N, {}^*I\rangle \models \varphi(a;\alpha) \;\;\Rightarrow\;\; N \models \varphi\big(a;\mathrm{st}(\alpha)\big)$$

Proof. Suppose $\varphi(x;\eta)$ is $\mathcal{L}^{\mathrm{p}}$-atomic. If $\varphi(x;\eta)$ is a formula of L_H the claim is trivial. Otherwise $\varphi(x;\eta)$ has the form $t(x;\eta) \in C$. Assume that the tuple $t(x;\eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, {}^*I\rangle \models t(a;\alpha) \in C$. Then $\mathrm{st}\big({}^*\,t(a;\alpha)\big) \in C$ by Fact 14. Therefore $t^N\big(a;\mathrm{st}(\alpha)\big) \in C$ follows from Fact 16. This proves the lemma for atomic formulas. Induction is immediate. □

Corollary 18. Let M be a standard structure. Let $p(\eta) \subseteq \mathcal{L}^{\mathrm{p}}(M)$ be a type that does not contain existential quantifiers of sort H. Then, if $p(\eta)$ is finitely consistent in M, it is realized in M.

The corollary has also a direct proof. This goes through the observation that the formulas in $p(\eta)$ define compact subsets of I.

Proof. Let $\langle {}^*M, {}^*I\rangle$ be an $\mathcal{L}$-elementary saturated superstructure of $\langle M, I\rangle$. Then $\langle {}^*M, {}^*I\rangle \models p(\alpha)$ for some $\alpha \in^* I$. By the lemma above, ${}^*M \models p(\mathrm{st}(\alpha))$. Now observe that the truth of formulas without existential quantifiers of sort H is preserved by substructures. □

The exclusion of existential quantifiers of sort H is necessary. For a counterexample take $M = I = [0,1]$. Assume that $\mathcal{L}$ contains a function symbol for the identity map $\iota : M \to I$. Let $p(\eta)$ contain the formulas $\exists x\big(x > 0 \wedge \iota x + \eta \in [0, 1/n]\big)$ for all positive integers n.

6. Compactness

It is convenient to distinguish between consistency with respect to standard structures and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent if $\langle M, {}^*I\rangle \models T$ for some

$\mathcal{L}$-structure $\langle M, {}^*I\rangle$. We say that T is st-consistent if $M \models T$ is for some standard structure M. By Lemma 17 these two notions coincide if T is positive. Therefore we have the following.

Theorem 19 (Positive Compactness Theorem). Let T be a positive theory that is finitely $\mathcal{L}$-consistent. Then T is st-consistent.

An $\mathcal{L}$-structure N is positively λ-saturated if it realizes all types $p(x;\eta) \subseteq \mathcal{L}^{\mathrm{p}}(N)$ with fewer than λ parameters that are finitely consistent in N. When $\lambda = |N|$ we simply say p-saturated. The existence of p-saturated standard structures is obtained from the classical case just as for Theorem 19.

Theorem 20. Every standard structure has an p-elementary extension to a p-saturated standard structure (possibly of inaccessible cardinality).

Proof. Let $\langle M, {}^*I\rangle$ be an $\mathcal{L}$-saturated $\mathcal{L}$-structure. Let $\langle M, I\rangle$ be its standard part as defined in Section 5. By Lemma 17, $\langle M, I\rangle$ realizes all finitely consistent positive types with fewer than $|M|$ parameters. □

The following proposition shows that a slight amount of saturation tames the positive formulas.

Proposition 21. Let N be a positively ω-saturated standard structure. Then

$$\big\{\varphi(x;\eta)\big\}' \;\leftrightarrow\; \varphi(x;\eta)$$

holds in N for every formula $\varphi(x;\alpha) \in \mathcal{L}^{\mathrm{p}}(N)$.

Proof. We prove $\rightarrow$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

ih. $$\{\varphi(x,z;\eta)\}' \;\rightarrow\; \varphi(x,z;\eta)$$

We need to prove

$$\{\exists z\,\varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\,\varphi(x,z;\eta)$$

From (ih) we have

$$\exists z\,\{\varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\,\varphi(x,z;\eta)$$

Therefore it suffices to prove

$$\{\exists z\,\varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\,\{\varphi(x,z;\eta)\}'$$

Replace the variables $x;\eta$ with parameters, say $a;\alpha$, and assume that $N \models \exists z\,\varphi'(a,z,;\alpha)$ for every $\varphi' > \varphi$. We need to prove the consistency of the type $\{\varphi'(a,z,;\alpha) : \varphi' > \varphi\}$. By saturation, finite consistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly. □

By Corollary 18, when $\varphi(x;\eta)$ does not contains existential quantifiers of sort H, the proposition above does not require the assumption of saturation. In general, some saturation is necessary: consider the model presented after Corollary 18 and the formula $\exists x\big(x > 0 \;\wedge\; \iota(x) \in \{0\}\big)$.

A consequence of Proposition 21 is that the approximate morphisms defined in Section 4 coincide with their unapproximate version when the codomain is positively ω-saturated.

7. The monster model

We denote by $\mathfrak{U}$ some large p-saturated standard structure which we call the monster model. Truth is evaluated in $\mathfrak{U}$ unless otherwise is specified. We denote by T the theory of $\mathfrak{U}$. The density of a set of formulas is understood modulo T. Below we say model for p-elementary substructure of $\mathfrak{U}$. We stress once again that the truth of some $\varphi \in \mathcal{L}^{\mathrm{p}}(M)$ in a model M implies the truth of φ (in $\mathfrak{U}$) but not vice versa. If also the converse implication holds we say M a strong model. However, all models agree on the approximated truth, see Fact 11,

$$M \models \{\varphi\}' \quad \Leftrightarrow \quad \{\varphi\}'$$

Strong models are those that realize the equivalence in Proposition 21, e.g. positively ω-saturated models are strong models.

The following fact demostrates how positive compactness applies. There are some subtle differences from the classical setting. Let $A \subseteq \mathfrak{U}$ be a small set throughout this section.

Fact 22. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ be a type. Then for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(\mathfrak{U})$

1. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \neg\varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
2. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (1) is immediate by saturation; (2) follows from (1) by Lemma 6. □

We write $S^{\mathrm{p}}_x(A)$ for the set of maximally consistent subsets of $\mathcal{L}^{\mathrm{p}}_x(A)$. We define p-tp$(a/A)$, the positive type of a over A, to be the set of formulas $\{\varphi(x) \in \mathcal{L}^{\mathrm{p}}(A) \ :\ \varphi(a)\}$. In general, if $\mathcal{F}$ is any set of formulas, we write $\mathcal{F}$-tp(a/A), for the type $\{\varphi(x) \in \mathcal{F}(A) \ :\ \varphi(a)\}$. The undecorated symbol tp(a/A) denotes the $\mathcal{L}$-type.

Fact 23. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$. The following are equivalent

1. $p(x)$ is a maximally consistent subsets of $\mathcal{L}^{\mathrm{p}}_x(A)$;
2. $p(x) = $ p-tp(a/A) for some $a \in \mathfrak{U}^{|x|}$.

Proof. (1⇒2) By the saturation of $\mathfrak{U}$, there is an $a \in \mathfrak{U}^{|x|}$ such that $p(x) \subseteq$ p-tp(a/A). By maximality $p(x) = $ p-tp(a/A).

(2⇒1) From Lemma 6 and Proposition 21 we obtain

$$\neg\varphi(x) \quad \to \quad \bigvee_{\tilde{\varphi} \perp \varphi} \tilde{\varphi}(x).$$

Suppose $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(A) \smallsetminus p$. Then $\neg\varphi(a)$. Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \to \neg\varphi(x)$ follows. □

Let $\mathcal{H}$ be the set of positive formulas without quantifiers of sort I. Note that in the proof above we could replace $\mathcal{L}^{\mathrm{p}}$ with any class of positive formulas closed under strong negation. Therefore we can also claim the following.

Fact 24. Let $p(x) \subseteq \mathcal{H}(A)$. The following are equivalent

1. $p(x)$ is a maximally consistent subset of $\mathcal{H}_x(A)$;
2. $p(x) = \mathcal{H}$-tp(a/A) for some $a \in \mathfrak{U}^{|x|}$.

8. Elimination of quantifiers of sort I

We show that the quantifiers of sort I can be eliminated up to some approximation. Precisely, we prove that the set formulas $\mathcal{H}$ defined above is dense (modulo T, the theory of $\mathcal{U}$), see Definion 7.

We say that partial map $f : \mathcal{U} \to \mathcal{U}$ is $\mathcal{H}$-elementary if it preserves the truth of formulas in $\mathcal{H}$.

Proposition 25. Every $\mathcal{H}$-elementary map of small cardinality extend to an automorphism of $\mathcal{U}$.

Proof. By Fact 24, the inverse of an $\mathcal{H}$-elementary map is also $\mathcal{H}$-elementary. Then we can extend the map by back-and-forth as usual. □

Let $A \subseteq \mathcal{U}$ be a small set throughout this section. We apply the proposition above to obtain a strengthening of Fact 24. For $a, b \in \mathcal{U}^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A.

Corollary 26. Let $p(x) = \mathcal{H}\text{-tp}(a/A)$ and $q(x) = \text{tp}(a/A)$. Then $p(x) \leftrightarrow q(x)$.

Proof. Only $\rightarrow$ requires a proof. If $b \models p(x)$ then there is an $\mathcal{H}$-elementary map $f \supseteq \text{id}_A$ such that $fa = b$. Then f extends to an automorphism. As every automorphism is $\mathcal{L}$-elementary, $a \equiv_A b$, and the corollary follows. □

The corollary says that $\mathcal{U}$ does not distinguish between $\mathcal{L}$, $\mathcal{L}^{\text{p}}$, and $\mathcal{H}$ types. In the classical setting, from an equivalence between types one derives an equivalence between formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence.

Proposition 27. The set $\mathcal{H}$ is dense (modulo the theory of $\mathcal{U}$).

Proof. Let $\varphi(x)$ be a positive formula. We need to prove that for every $\varphi' > \varphi$ there is some $\psi(x) \in \mathcal{H}$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$. By Corollary 26 and Proposition 21

$$\neg\varphi(x) \quad\rightarrow \bigvee_{p'(x)\rightarrow\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over the maximally consistent $\mathcal{H}$-types. By Fact 22 and Lemma 6

$$\neg\varphi(x) \quad\rightarrow \bigvee_{\neg\tilde{\psi}(x)\rightarrow\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{H}$. Equivalently,

$$\varphi(x) \quad\leftarrow \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 22, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{H}$ such that

$$\varphi'(x) \quad\leftarrow \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \quad\leftarrow\quad \varphi(x)$$

which yields the interpolant required by the proposition. □

Proposition 28. Let $\mathcal{F}$ be a dense set of positive formulas. Then for every $\varphi(x) \in \mathcal{L}^{\text{p}}$

$$\neg\varphi(x) \quad\leftrightarrow\quad \bigvee\big\{\psi(x) \in \mathcal{F} \;:\; \psi'(x) \rightarrow \neg\varphi(x) \text{ for some } \psi' > \psi\big\}.$$

Proof. Only $\rightarrow$ requires a proof. We will make repeated use of the equivalence $p(x) \leftrightarrow p'(x)$ that holds for all $p(x) \subseteq \mathcal{L}^{\text{p}}$. Let $a \in \mathcal{U}^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \mathcal{H}\text{-tp}(a)$. By Corollary 26,

$p(x) \to \neg\varphi(x)$. As $\mathcal{F}$ is dense, $p'(x) \leftrightarrow q(x) = \mathcal{F}\text{-tp}(a)$. Then $q'(x) \to \neg\varphi(x)$ and, by compactness, $\psi'(x) \to \neg\varphi(x)$ for some $\psi' > \psi \in \mathcal{F}$. □

We will need a similar result for larger class formulas. We write Σ^{p} for the set of formulas of the form $\exists y\, \varphi(x, y)$ where $\varphi(x, y)$ is positive Boolean combination of formulas such as $\neg\psi(x, y)$ and $\psi'(x, y)$ for $\psi(x, y)$ positive and $\psi' > \psi$. The following follows easily from the proposition above.

Proposition 29. For every $\varphi(x) \in \Sigma^{\mathrm{p}}$

$$\varphi(x) \quad \leftrightarrow \quad \bigvee \{\psi(x) \in \mathcal{L}^{\mathrm{p}} \,:\, \psi'(x) \to \varphi(x) \text{ for some } \psi' > \psi\}.$$

9. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is our version of the Tarski-Vaught test (which applies when the larger model is sufficiently saturated).

Proposition 30. Let M be a *subset* of $\mathcal{U}$. Let $\mathcal{F}$ be a dense set of positive formulas. Then the following are equivalent

1. M is a model;
2. for every formula $\psi(x) \in \mathcal{F}(M)$
$$\exists x\, \psi(x) \quad \Rightarrow \quad \text{for every } \psi' > \psi \text{ there is an } a \in M \text{ such that } \psi'(a);$$
3. for every formula $\psi(x) \in \mathcal{F}(M)$
$$\exists x \neg\psi(x) \quad \Rightarrow \quad \text{there is an } a \in M \text{ such that } \neg\psi(a).$$

Proof. (1⇒2) Assume $\exists x\, \psi(x)$ and let $\psi' > \psi$ be given. By Lemma 6 there is some $\tilde{\psi} \perp \psi$ such that $\psi(x) \to \neg\tilde{\psi}(x) \to \psi'(x)$. Then $\neg\forall x\, \tilde{\psi}(x)$ hence, by (1), $M \models \neg\forall x\, \tilde{\psi}(x)$. Then $M \models \neg\tilde{\psi}(a)$ for some $a \in M$. Hence $M \models \psi'(a)$ and $\psi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\psi(x) \in \mathcal{F}(M)$ be such that $\exists x \neg\psi(x)$. By Proposition 31, there are a consistent $\varphi(x) \in \mathcal{F}(M)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \to \neg\psi(x)$. Then (3) follows.

(2⇒1) Assume (2). By the classical Tarski-Vaught test $M \preceq_{\mathrm{H}} \mathcal{U}$. Then M is the domain of an $\mathcal{L}$-substructure of $\mathcal{U}$. Then $\varphi(a) \;\Leftrightarrow\; M \models \varphi(a)$ holds for every atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \models \varphi(a, b) \quad \Rightarrow \quad \varphi(a, b).$$

Using (2) and the induction hypothesis we prove by contrapposition that

$$M \models \forall y\, \varphi(a, y) \quad \Rightarrow \quad \forall y\, \varphi(a, y).$$

Indeed,

$$\begin{aligned}
\neg\forall y\, \varphi(a, y) \quad &\Rightarrow \quad \exists y \neg\varphi(a, y) \\
&\Rightarrow \quad \exists y \neg\varphi'(a, y) && \text{for some } \varphi' > \varphi \\
&\Rightarrow \quad \exists y \neg\psi(a, y) && \text{for some } \psi \in \mathcal{F} \text{ such that } \varphi(a, y) \to \psi(a, y) \to \varphi'(a, y) \\
&\Rightarrow \quad \neg\psi(a, b) && \text{for some } b \in M \text{ by (2)} \\
&\Rightarrow \quad \neg\varphi(a, b) \\
&\Rightarrow \quad M \models \neg\varphi(a, b) && \text{by induction hypothesis} \\
&\Rightarrow \quad M \not\models \forall y\, \varphi(a, y).
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists^{\mathsf{H}}$, $\exists^{\mathsf{I}}$, and $\forall^{\mathsf{I}}$ is straightforward. □

Classically, the main application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all $\mathcal{L}$-structures. In particular every $A \subseteq \mathcal{U}$ is contained in a standard structure of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of $\mathcal{L}$ is eccessively large because of the aboundance of symbols in L_{I}.

We say that $\mathcal{L}$ is separable modulo T if there is a countable set of positive formulas that is dense modulo T.

Proposition 31. Assume that $\mathcal{L}$ is separable modulo the theory of $\mathcal{U}$. Let $A \subseteq \mathcal{U}$ be countable. Then there is a countable model M containing A. (The size of a model is the size of the domain of its home sort.)

Proof. As in the classical proof of the Löwenheim-Skolem Theorem, we construct $M \subseteq \mathcal{U}$ that contains a witness for every consistent formula in $\mathcal{F}(M)$. By separability and Proposition 30, M is a model. □

Proposition 32. Assume that L_{H} is countable and that $\mathcal{L}$ has at most countably many symbols of sort $\mathsf{H}^n \to \mathsf{I}$. Then, if I is a second countable (the topology has a countable base), $\mathcal{L}$ is separable modulo $\mathcal{L}$.

Proof. Fix a countable filter of compact neighborhoods of the diagonal of I^2. □

10. Omitting types and countable categoricity

A type $p(x)$ is isolated by $\varphi(x)$, a consistent formula, if $\varphi(x) \to p(x)$. If $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(A)$ we say that it is p-isolated over A. Note that, by Propsition 30, if $p(x)$ is p-isolated over A, then $p(x)$ is realized in every model containing A.

Proposition 33. Let $\mathcal{F}$ be a dense set of positive formulas. Then the following are equivalent

1. $p(x)$ is p-isolated over A;
2. $p(x)$ is isolated by $\neg\psi(x)$ for some $\psi(x) \in \mathcal{F}(A)$;
3. $p(x)$ is isolated by some $\psi'(x)$ such that $\psi' > \psi$ for some consistent $\psi(x) \in \mathcal{F}(A)$;
4. $p(x)$ is isolated by some formula $\varphi(x) \in \Sigma^{\mathrm{p}}(A)$.

Proof. 1⇔2 For the nontrivial direction, note that by density

$$\varphi(x) \leftrightarrow \bigwedge\{\psi(x) \in \mathcal{F} \,:\, \varphi(x) \to \psi(x)\}$$

therefore, negating both sides of the equivalence,

$$\neg\varphi(x) \leftrightarrow \bigvee\{\neg\psi(x) \,:\, \psi(x) \in \mathcal{F} \text{ and } \neg\psi(x) \to \neg\varphi(x)\}.$$

If $\neg\varphi(x)$ isolates $p(x)$ then any consistent formula in the disjunction on th l.h.s. also isolates $p(x)$.

1⇔3 By Proposition 28 and Fact 6.

3⇔4 By Proposition 29. □

Lemma 34. Let $\mathcal{L}$ be separable and let A be countable. Assume that $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ is not p-isolated over A. Then every consistent formula $\neg\psi(z)$, with $\psi(z) \in \mathcal{L}^{\mathrm{p}}(A)$, has a solution a such that $p(x)$ is not isolated over A, a.

Proof. We construct a sequence of $\mathcal{L}^{\mathrm{p}}(A)$-formulas $\langle \gamma_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\gamma_i(z) : i < \omega\}$ is the required solution of $\neg\psi(z)$.

Let $\langle \xi_i(x;z) : i < \omega \rangle$ enumerate a countable dense subset of $\mathcal{L}^{\mathrm{p}}_{x;z}(A)$. Let $\gamma_0(z)$ be a consistent positive formula such that $\gamma'_0(z) \to \neg\psi(z)$ for some $\gamma'_0 > \gamma_0$, which exists by Proposition 28. Define $\gamma_{i+1}(z)$ and $\gamma'_{i+1}(z)$ inductively as follows.

1. If $\neg\xi_i(x;z) \wedge \gamma'_i(z)$ is inconsistent, let $\gamma_{i+1}(z) = \gamma_i(z)$ and $\gamma'_{i+1}(z) = \gamma'_i(z)$;
2. otherwise, pick $\varphi(x) \in p$ such that (#) below is consistent

   (#) $$\gamma'_i(z) \;\wedge\; \exists x\big[\neg\xi_i(x;z) \wedge \neg\varphi(x)\big].$$

   Finally, let $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$ consistent positive formulas such that $\gamma'_{i+1}(z)$ implies (#) –this exists by Proposition 29.

Let $a \models \{\gamma_i(z) : i < \omega\}$. We claim that that A, a does not p-isolate $p(x)$. Otherwise, by density, $\neg\xi(x;a) \to p(x)$ for some $i < \omega$, where $\neg\xi_i(x;a)$ is consistent. This contradicts $a \models \gamma_{i+1}(z)$.

Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in (2).

Suppose for a contradiction that $\neg\xi_i(x;z) \wedge \gamma'_i(z)$ is consistent while (#) is inconsistent for all formulas $\varphi(x) \in p$, that is,

$$\neg\xi_i(x;z) \wedge \gamma'_i(z) \;\to\; \varphi(x).$$

This immediately implies that

$$\exists z\big[\neg\xi_i(x;z) \wedge \gamma'_i(z)\big] \;\to\; p(x).$$

Then $p(x)$ is isolated by a formula in $\Sigma^{\mathrm{p}}(A)$. By Proposition 33, this is a contradiction. □

Theorem 35 (Positive Omitting Types). Assume that $\mathcal{L}$ is separable and that A is countable. Assume also that $p(x) \subseteq \mathcal{L}(A)$ is not p-isolated. Then there is a model M containing A that omits $p(x)$.

Proof. As in the classical proof, appy the lemma above and the Tarski-Vaught test (Proposition 30) to obtain a countable model M that does not isolate $p(x)$. For models, isolating and realizing are equivalent. □

Assume that $\mathcal{L}$ is separable modulo T. A theory T is positively ω-categorical if any two countable models $M, N \models T$ are isomorphic. The following analogue of Ryll-Nardzewski's Theorem follows as usual.

Theorem 36. Assume that $\mathcal{L}$ is separable modulo T. The following are equivalent

1. T is positively ω-categorical;
2. every positive type $p(x) \subseteq \mathcal{L}^{\mathrm{p}}$, for x any finite tuple of variables, is p-isolated over $\varnothing$.

The classical argument proves also the following.

Proposition 37. Fix a finite tuple of variables x. The following are equivalent

1. every positive type $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ is p-isolated over A;
2. there are finitely many complete types.

Proof. (1⇒2) Let $p_i(x) = \text{p-tp}(a_i/A)$, for $i < \lambda$, be an enumeration without repetitions of all positive types. Let $\neg\varphi_i(x)$ be the negative formulas that isolate the $p_i(x)$. As these are complete types, $\neg\varphi_i(x) \leftrightarrow p_i(x)$. If λ is infinite then the positive type $\{\varphi_i(x) : i < \lambda\}$ is consistent. Any realization of this type yield a contradiction.

(2⇒1) Let $p_i(x) = \text{p-tp}(a_i/A)$, for $i < n$, be an enumeration without repetitions of all positive types. Pick a formula $\varphi_i(x) \in p_i \smallsetminus \bigcup\big\{p_j : j \in n \smallsetminus \{i\}\big\}$. Then the formula $\neg\bigvee\big\{\varphi_i(x) : j \in n \smallsetminus \{i\} p\big\}$ isolates $p_i(x)$. □

11. Cauchy complete models

For $t(x,z)$ a term of sort $\mathsf{H}^{|x|+|z|} \to \mathsf{I}$ we define the formula

$$x \sim_t y \quad = \quad \forall z \;\; t(x,z) = t(y,z).$$

We also define the type

$$x \sim_{\mathsf{I}} y \quad = \quad \Big\{x \sim_t y \; : \; t(x,z) \text{ as above}\Big\}.$$

Fact 38. For any $a = \langle a_i : i < \lambda\rangle$ and $b = \langle b_i : i < \lambda\rangle$

$$a \sim_{\mathsf{I}} b \;\Leftrightarrow\; a_i \sim_{\mathsf{I}} b_i \quad \text{for every } i < \lambda.$$

Proof. Only implication ⇐ requires a proof. Assume $a_i \sim_{\mathsf{I}} b_i$ for every i. Let $|a| = |b| = \lambda$ and assume inductively that

$$\forall y, z \;\; t(a,y,z) \;\; = \;\; t(b,y,z)$$

holds for every term $t(x,y,z)$, with $|x| = \lambda$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

1. $$\forall z \;\; t(a,a_\lambda,z) \;\; = \;\; t(b,a_\lambda,z).$$

As $a_\lambda \sim_{\mathsf{I}} b_\lambda$ then

$$\forall x, z \;\; t(x,a_\lambda,z) \;\; = \;\; t(x,b_\lambda,z)$$

and in particular

2. $$\forall z \;\; t(b,a_\lambda,z) \;\; = \;\; t(b,b_\lambda,z).$$

From (1) and (2) we obtain

$$\forall z \;\; t(a,a_\lambda,z) \;\; = \;\; t(b,b_\lambda,z).$$

For limit ordinals induction is trivial. □

Note that the approximations of the formula $x \sim_t y$ have the form

$$x \sim_{t,D} y \quad = \quad \forall z \;\; \langle t(x,z), t(y,z)\rangle \in D$$

for some compact neighborhood D of Δ, the diagonal of I^2. Hence we define

$$x \sim'_{\mathsf{I}} y \quad = \quad \Big\{x \sim_{t,D} y \; : \; t(x,z) \text{ parameter-free term, } D \text{ compact neigborhood of } \Delta\Big\}.$$

If $t(x,z) = t_1(x,z), \ldots, t_n(x,z)$ is a tuple of terms, by $x \sim_{t,D} y$ we denote the conjunction of the formulas $x \sim_{t_i,D} y$. As D ranges over the compact neighborhoods of Δ and $t(x,z)$ ranges over the finite tuples parameter-free terms. These formulas $x \sim_{t,D} y$ define a system of entougages on $\mathcal{U}^{|x|}$. We refer to this uniformity and the topology associated as the I-topology. Though not needed in the sequel, it is worth mentioning that the I-topology on $\mathcal{U}^{|x|}$ coincides with the product of the I-topology on $\mathcal{U}$. This can be verified by an argument similar to the proof of Fact 38.

We say that $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(\mathcal{U})$ is I-invariant if $x \sim_{\mathsf{I}} y \to [p(x) \leftrightarrow p(y)]$. It is clear that all formulas that do not contain atomic formulas of sort H are invariant. The following proposition is an immediate consequence of compactness. It corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Proposition 39. The following are equivalent for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(\mathcal{U})$

1. $\varphi(x)$ is I-invariant;
2. for every $\varphi' > \varphi$ there are a tuple of therm t and a compact neighborhood of Δ such that
$$x \sim_{t,D} y \wedge \varphi(x) \;\to\; \varphi'(y).$$

We say that $p(x)$ is a Cauchy type if it is consistent and $p(x) \wedge p(y) \to x \sim_{\mathsf{I}} y$.

We say that a type $q(x)$ is finitely satisfiable in M if every conjunction of formulas in $q(x)$ has a solution in M. This definition is classical but note that, though classically every type over M is finitely saisfiable in M, here, unless M is sufficiently saturated, we can only infer that $q'(x)$ is finitely satisfiable.

Lemma 40. The following are equivalent for every model M

1. $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(M)$ is an I-invariant Cauchy type;
2. $p(x) \leftrightarrow a \sim_{\mathsf{I}} x$ for some/any $a \models p(x)$;
3. $a \sim'_{\mathsf{I}} x$ is finitely satisfiable in M for some/any $a \models p(x)$.

Proof. (1⇒2) Let $a \models p(x)$. Substituting a for y in the definition of Cauchy type we obtain $p(x) \to a \sim_{\mathsf{I}} x$ Similarly, from the definition of I-invariance we obtain $a \sim_{\mathsf{I}} x \to p(x)$.

(2⇒3) By compactness, as $p'(x)$ is finitely satisfiable in M.

(3⇒1) Let $a_{t,D} \in M$ be a realization of $a \sim_{t,D} x$ as D ranges over the compact neighborhoods of the diagonal and $t(x,z)$ over the finite tuples parameter-free terms. Then, by the uniqueness of the limit (up to $\sim_I$-equivalence), the type $p(x)$ cointaining the formulas $x \sim_{t,D} a_{t,D}$ is as required by the lemma. □

We say that a set $A \subseteq \mathcal{U}$ is Cauchy complete if every I-invariant Cauchy type $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ is realized by some $c \in A^{|x|}$.

Fact 41. Let M be a model that is positively λ-saturated for some $\lambda > |\mathcal{L}|$. Then M is Cauchy complete.

Proof. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(M)$ be an I-invariant Cauchy type. Pick any $a \models p(x)$. By Lemma 40, $p(x) \leftrightarrow a \sim_{\mathsf{I}} x$. For every formula $\varphi(x)$ in the type $a \sim_{\mathsf{I}} x$ and every $\varphi' > \varphi$ pick some $\psi(x) \in p$ such that $\psi(x) \to \varphi'(x)$. There are $|\mathcal{L}|$ many formulas in $a \sim_{\mathsf{I}} x$. Then by saturation the formulas $\psi(x)$ we picked above have a common solution b in M. Then $b \sim_{\mathsf{I}} a$ and, by invariance, $b \models p(x)$. □

It is clear from the proof that when $\mathcal{L}$ is separable it suffices to require $\lambda > \omega$.

12. Example: metric spaces

We discuss a simple example. Let I be $\mathbb{R}^+ \cup \{0, \infty\}$, with the topology that makes it homeomorphic to the unit interval. Let M, d is a metric space. Let L_{H} contain symbols for functions $M^n \to M$ that are uniformly continuous. Let $\mathcal{L}$ contain a symbol for d and possibly for some funtions of $M^n \to I$ that are uniformly continuous w.r.t. the metric d.

Let $\langle M, I\rangle$ be the structure that interprets the symbols of the language as natural.

Let $\mathcal{U}$ be a monster model that is an $\mathcal{L}^{\mathsf{p}}$ elementary extension of M. Clearly, d does not define a metric on $\mathcal{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathcal{U}$. Therefore the notion of convergent sequence makes perfectly sense in $\mathcal{U}$. As all functions have been required to be uniformly continuous, it is immediate that $x \sim_{\mathsf{I}} y$ is equivalent to $d(x, y) = 0$.

Fact 42. Let $\mathcal{U}$ be as above. Then for every model N and $a \in \mathcal{U}$ the following are equivalent

1. $p(x) = \text{p-tp}(a/N)$ is a Cauchy type;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of N that converges to a.

Proof. (2⇒1) Let $\langle \varepsilon_i : i \in \omega\rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \le \varepsilon_i$ for every $i \in \omega$. Then the formulas $d(a_i, x) \le \varepsilon_i$ are in $p(x)$. Then every element realizing $p(x)$ is at distance 0 from a. Therefore $p(x) \to a \sim_{\mathsf{I}} x$.

(1⇒2) As $p(x)$ is Cauchy type, $p(x) \to a \sim_{\mathsf{I}} x$. Then $p'(x) \to d(a, x) < 2^{-i}$ for all i. By compactness (see Fact 22) there are formulas $\varphi_i(x) \in p$ such that $\varphi_i'(x) \to d(a, x) < 2^{-i}$ for some $\varphi_i' > \varphi_i$. By $\mathcal{L}^{\mathsf{p}}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega\rangle$ converges to a. □

References

[G] Isaac Goldbring, *Lecture notes on nonstandard analysis*, UCLA Summer School in Logic 2012, available at www.math.uci.edu/~isaac/NSA%20notes.pdf.

[H] Bradd Hart, *An Introduction To Continuous Model Theory* (2023), arXiv:2303.03969

[Hr] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.

[HPP] Ehud Hrushovski, Ya'acov Peterzil, and Anand Pillay, *Groups, measures, and the NIP*, J. Amer. Math. Soc. **21** (2008), no. 2, 563–596.

[HI] C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 1–110.

[CLCL] C.L.C.L. Polymath, *Continuous Logic for the Classical Logician* (202?). T.a.

Danger after this point

13. Elimination of quantifiers

Let T be a positive theory. We say that T has (approximate, positive) elimination of quantifiers if the quantifier-free positive formulas are dense modulo T. Equivalently, if every complete type is equivalent to a quantifier-free positive types.

The following fact is routinely proved by back-and-forth.

Fact 43. The following are equivalent

1. T has elimination of quantifiers;
2. every finite partial embedding $k : M \to N$ beween models of T is an approximate p-elementary map;
3. for every finite partial embedding $k : M \to N$ between positively ω-saturated models of T, and for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is also a partial embedding.

14. Pseudofinite randomizations

Let T be a complete first-order theory of signature L with an infinite model. Let $\mathcal{U}$ be a model of T. Let Ω be a finite set. We denote by $\mathcal{U}^\Omega$ the set of functions $\Omega \to \mathcal{U}$. The elements of $\mathcal{U}^\Omega$ are denoted by letters decorated with a circonflexe accent $\hat{a}, \hat{b}$, etc. The value of $\hat{a}$ at $\omega \in \Omega$ is denoted by a_ω. We now define models of the form $\langle \mathcal{U}^\Omega, I\rangle$, where I is the real interval $[0,1]$ with the usual topology.

The language L_{H} is empty. The language $\mathcal{L}$ contains functions of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, one for each formula $\varphi(x) \in L$. These are denoted by $\Pr\varphi(\hat{x})$. Variables of sort H are decorated with a circonflexe accent. The interpretation of the function symbol $\Pr\varphi(\hat{x})$ is

$$\Pr\big(\varphi(\hat{a})\big) \quad = \quad \frac{\#[\![\varphi(\hat{a})]\!]}{\#\Omega},$$

where by # denotes the finite cardinality, and

$$[\![\varphi(\hat{a})]\!] \quad = \quad \big\{\omega \in \Omega \;:\; \mathcal{U} \models \varphi(a_\omega)\big\}.$$

The pseudofinite randomization of T is the positive theory

$$T^{\mathrm{pfr}} \quad = \quad \Big\{\varphi \in \mathcal{L}^{\mathrm{p}} \;:\; \langle \mathcal{U}^\Omega, I\rangle \models \varphi \text{ for all sufficiently large finite set } \Omega\Big\}$$

In this section we prove that T^{pfr} is a compete theory with elimination of quantifiers. We prove the lemma that is required for the back-and-forth.

Lemma 44. Let $k : M \to N$ be a finite partial embedding between positively ω-saturated models of T^{pfr}. Then for every $\hat{b} \in M$ and there is a $\hat{c} \in N$ such that $k \cup \{\langle \hat{b}, \hat{c}\rangle\} : M \to N$ is a partial embedding.

Proof. Let $\hat{a}$ be an enumeration of $\operatorname{dom} k$. Let $p(\hat{x}, \hat{z})$ be set of formulas of the form $\gamma = \Pr\varphi(\hat{x}, \hat{z})$, for some $\gamma \in I$, that hold in $M, \hat{b}, \hat{a}$. Note that $p(\hat{x}, \hat{z})$ implies the atomic type of $\hat{b}, \hat{a}$ in M. Therefore it suffices to show that $p(\hat{x}, k\hat{a})$ is finitely consistent in N.

Pick a finite set of these formulas, say $\gamma_i = \Pr\varphi_i(\hat{x}, \hat{z})$, for $i < n$. Me may assume that the fomulas $\varphi_i(x, z)$ define a partition of $\mathcal{U}^{|x,z|}$. By the saturation of N, it suffices to prove that for every $\varepsilon > 0$ there is a $\hat{c} \in N$ such that

$$\gamma_i \quad =_\varepsilon \quad \Pr\varphi_i(\hat{c}, k\hat{a}) \qquad\qquad (\alpha =_\varepsilon \beta \text{ is a shorthand for } |\alpha - \beta| \leq \varepsilon)$$

is true in N for all $i < n$.

Some preliminary work is required. For each $J \subseteq n$ define the following L-formulas

$$\xi_J(z) \quad = \quad \bigwedge_{i\in J} \exists x\,\varphi_i(x,z) \wedge \bigwedge_{i\in n\smallsetminus J} \neg\exists x\,\varphi_i(x,z)$$

These formulas $\xi_J(z)$ partition $\mathcal{U}^{|z|}$. For $J \subseteq n$ let α_J be such that

2. $$\alpha_J \quad = \quad \Pr\big(\xi_J(\hat z)\big)$$

is in $p(\hat x, \hat z)$. Let $\mathcal{J} = \{J \subseteq n \,:\, \alpha_J \neq 0\}$. When $J \in \mathcal{J}$, there are some $\beta_{i,J}$ such that the formulas

3. $$\beta_{i,J} \quad = \quad \Pr\Big(\varphi_i(\hat x, \hat z) \,\Big|\, \xi_J(\hat z)\Big).$$

are are among the consequences of $p(\hat x, \hat z)$. As the formulas $\varphi_i(x,z)$ define a partition, the $\beta_{i,J}$ add up to 1 for any fixed J. Note that $\beta_{i,J} = 0$ for $i \notin J$. Clearly, have that

4. $$\gamma_i \quad = \quad \sum_{J\in\mathcal{J}} \beta_{i,J}\alpha_J.$$

It is plain that for $i \in J$ the following is a consequence of $p(\hat x, \hat z)$

5. $$1 \quad = \quad \Pr\Big(\exists x\,\varphi_i(x, \hat z) \,\Big|\, \xi_J(\hat z)\Big)$$

Now we prove that (2) & (5) $\Rightarrow$ (1) holds in $\mathcal{U}^\Omega$ when Ω large enough (larger than n/ε suffices). Strictly speaking, this implication is not a positive formula, but the reader can easily verify that a suitable approximation of (2) and (5) suffices.

Let $\hat a' \in \mathcal{U}^\Omega$ satisfy (2) and (5) for every $i \in J \subseteq n$. We define $\hat c' \in \mathcal{U}^\Omega$ such that

$$\mathcal{U}^\Omega \quad \models \quad \beta_{i,J} =_\varepsilon \Pr\Big(\varphi_i(\hat c', \hat a') \,\Big|\, \xi_J(\hat a')\Big).$$

We may define $\hat c'$ separately in each event $[\![\xi_J(\hat a')]\!]$. Partition $[\![\xi_J(\hat a')]\!]$ into events $E_i \subseteq \Omega$, for $i \in J$, such that $\#E_i/\#\Omega =_\varepsilon \beta_{i,J}$. Then define c'_ω for $\omega \in E_i$ to be any witness of $\exists x\,\varphi_i(x, a_\omega)$. By (5), we can always find such a witness.

Finally, by (4), we deduce

$$\mathcal{U}^\Omega \quad \models \quad \gamma_i =_\varepsilon \Pr\big(\varphi_i(\hat c', \hat a')\big).$$

As $k\hat a$ satisfy (2) and (5), we conclude that some $\hat c \in N$ satisfy (1). □

Now, from Fact 43 we obtain that $T^{\rm pfr}$ has elimination of quantifiers. Completeness follows.

Corollary 45. The theory $T^{\rm pfr}$ is compete and has elimination of quantifiers.

15. Stability

A partitioned formula $\varphi(x;z) \in \mathcal{L}^{\rm p}$ is stable if there is no sequence $\langle a_i\,;b_i \,:\, i<\omega\rangle$ and no $\tilde\varphi \perp \varphi$ such that for every $i, j < \omega$

$$i<j \quad \Rightarrow \quad \varphi(a_i\,;b_j)$$

$$i>j \quad \Rightarrow \quad \tilde\varphi(a_i\,;b_j)$$

Using the terminology of [Hr], $\varphi(x;z)$ is stable if it is stably separated from all $\tilde\varphi \perp \varphi$.

Note that by compactness if $\varphi(x;z)$ and $\tilde\varphi \perp \varphi$ are stably separated then there is a maximal length m of a sequence $\langle a_i\,;b_i \,:\, i<m\rangle$ such that (1) and (2) above.