

Standard analysis

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ABSTRACT. Let \mathcal{L} be a first-order two-sorted language. Let I be some fixed structure. A *standard structure* is an \mathcal{L} -structure of the form $\langle M, I \rangle$ where I is fixed. When I is a compact topological space (and \mathcal{L} meets a few requirements) it is possible to adapt a significant part of model theory to the restricted class of standard structures. This has been demonstrated by Henson and Iovino for Banach spaces (see, e.g. [HI]) and has been generalized to arbitrary structures in [CLCL]. The starting point is to prove that every standard structure has a *positive* elementary extension that is standard and realizes all positive types that are finitely consistent. The main tool is the notion of approximation of a positive formula and of its negation. These have been introduced by Henson and Iovino.

We review and elaborate on the properties of positive formulas and their approximations. In parallel, we introduce *continuous* formulas which are convenient to discuss examples and provide a better counterpart to Henson and Iovino theory and/or real-valued model theory. Finally, we discuss criteria for elimination of quantifiers, omitting types, ω -categoricity, stability, and indiscernibles within the setting of standard structures.

1. Standard structures

We refer to the introduction of [CLCL] for motivations. Aside from that, this paper is reasonably self-contained. In the first part we revisit some results of [CLCL]. We rephrase some theorems and add a few remarks and examples¹. Our notation and terminology diverges slightly, where necessary.

Let I be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language \mathcal{L}_I contains a relation symbol for each compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \rightarrow I$. In particular, there is a constant for each element of I . According to the context, C and f denote either the symbols of \mathcal{L}_I or their interpretation in the structure I . Such a language \mathcal{L}_I is much larger than necessary but it is convenient because it uniquely associates a structure to a topological space. The notion of a *dense set of formulas* (Definition 8) helps to reduce the size of the language when required.

The most straightforward examples of structures I are the unit interval $[0, 1]$ with the usual topology, and its homeomorphic copies $\mathbb{R}^+ \cup \{0, \infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language \mathcal{L}_H which we call the language of the **home sort**.

Definition 1. Let \mathcal{L} be a two sorted language that expands both \mathcal{L}_H and \mathcal{L}_I . A **standard structure** is a two-sorted \mathcal{L} -structure of the form $\langle M, I \rangle$, where M is any \mathcal{L}_H -structure, while I is the structure above. We write **H** and **I** to denote the two sorts of \mathcal{L} .

The language \mathcal{L} has a relation symbol $r_\varphi(x)$ for every $\varphi(x) \in \mathcal{L}_H$. All \mathcal{L} -structures are assumed to model $r_\varphi(x) \leftrightarrow \varphi(x)$. In other words, the Morleyzation of \mathcal{L}_H is assumed in the definition of \mathcal{L} -structure.

Finally and most relevantly, \mathcal{L} contains arbitrarily many function symbols of sort $H^n \rightarrow I$.

Standard structures are denoted by the domain of their home sort.

¹We will omit Section 3 from the final version of this paper and refer to [CLCL] instead.

Clearly, saturated \mathcal{L} -structures exist but, with the exception of trivial cases (when I is finite), these are not standard. As a remedy, below we carve out a set of formulas \mathcal{L}^P , the set of positive formulas, such that every model has an positive elementary saturated extension that is also standard.

As usual, \mathcal{L}_I , \mathcal{L}_H , and \mathcal{L} denote both first-order languages and the corresponding set of formulas. We write \mathcal{L}_x when we restrict variables to x . Note that \mathcal{L} has three types of atomic formulas

0. $t(x) = s(x)$ where $t(x)$ and $s(x)$ are terms of sort $H^{|x|} \rightarrow H$;
1. $r_\varphi(x)$ for some $\varphi(x) \in \mathcal{L}_H$ (maybe more, but these suffices up to equivalence);
2. $\tau(x; \eta) \in C$, where $C \subseteq I^n$ is compact and $\tau(x; \eta)$ is a tuple of terms of sort $H^{|x|} \times I^{|\eta|} \rightarrow I$.

The latter are called **continuous atomic** formulas.

Definition 2. A formula in \mathcal{L} is **positive** if it uses only the Boolean connectives \wedge, \vee ; the quantifiers \forall^H, \exists^H of sort H ; and the quantifiers \forall^I, \exists^I of sort I .

The set of positive formulas is denoted by \mathcal{L}^P .

A **continuous** formula is a positive formula where only continuous atomic formulas occur. The set of continuous formulas is denoted by \mathcal{L}^C .

We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort I . Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

Notation 3. When we state facts that holds both for \mathcal{L}^P and \mathcal{L}^C we write $\mathcal{L}^{P/C}$ for short.

The positive formulas of Henson and Iovino correspond to those formulas in \mathcal{L}^C that do not have quantifiers of sort I .

Working within \mathcal{L}^P simplifies the comparison with classical model theory but comes at a price. Instead, allowing quantifiers of sort I is almost for free. In Section 8 will see that the formulas in $\mathcal{L}^{P/C}$ are approximated by formulas in $\mathcal{L}^{P/C}$ without quantifiers of sort I .

The main fact to note about \mathcal{L}^C is that it is a language without equality of sort H . The difference between \mathcal{L}^P and \mathcal{L}^C is similar to the difference between the space of absolutely integrable functions (w.r.t. some measure μ) and the Lebesgue space $L^1(\mu)$. For the latter only equality almost everywhere is relevant. However, even when $L^1(\mu)$ is our focus of interest, it is often easier to argue about real valued functions. For a similar reason \mathcal{L}^P and \mathcal{L}^C are better studied in parallel.

The extra expressive power offered by quantifiers of sort I is convenient. For instance, it is easy to see (cf. Example 4) that \mathcal{L}^C has at least the same expressive power as real valued logic.

Example 4. Let $I = [0, 1]$, the unit interval. Let $\tau(x)$ be a term of sort $H^{|x|} \rightarrow I$. Then there is a positive formula that says $\sup_x \tau(x) = \alpha$. Indeed, consider the formula

$$x [\tau(x) \dot{-} \alpha \in \{0\}] \quad \wedge \quad \forall \varepsilon [\varepsilon \in \{0\} \vee \exists x [\alpha \dot{-} (\tau(x) + \varepsilon) \in \{0\}]]$$

which, in a more legible form, becomes

$$x [\tau(x) \leq \alpha] \quad \wedge \quad \forall \varepsilon > 0 \exists x [\alpha \leq \tau(x) + \varepsilon].$$

The following example is inspired by [HPP].

Example 5. Let \mathcal{U} be a saturated structure of signature \mathcal{L}_H . The relevant part of \mathcal{L} contains symbols for the functions $f : \mathcal{U}^n \rightarrow I$ that are continuous in the following sense: the inverse image of every compact $C \subseteq I$ is type-definable in $\mathcal{L}_H(M)$ for every $M \preceq \mathcal{U}$. It is easy to verify that $\langle \mathcal{U}, I \rangle$ is positively saturated as defined in Section 4.

The theory of normed spaces is one of the motivating examples for standard structures. The requirement for I to be compact seems to be a limitation, but it is not a real one. Essentially, functional analysis and model theory are only concerned with the unit ball of normed spaces. To formalize the unit ball of a normed space as a standard structure (where I is the unit interval) is straightforward but cumbersome. Alternatively one can view a normed space as a many-sorted structure: a sort for each ball of radius $n \in \mathbb{Z}^+$. Unfortunately, this also results in a bloated formalism.

A neater formalization is possible if one accepts that some positive elementary extensions (to be defined below) of a normed space could contain vectors of infinite norm hence not be themselves proper normed spaces. However these infinities are harmless if one restricts to balls of finite radius. In fact, inside any ball of finite radius these improper normed spaces are completely standard.

Example 6. Let $I = \mathbb{R} \cup \{\pm\infty\}$. Let \mathcal{L}_H be the language of real (or complex) vector spaces. The language \mathcal{L} contains a function symbol $\|\cdot\|$ of sort $H \rightarrow I$. Normed spaces are standard structures with the natural interpretation of the language. It is easy to verify that the unit ball of a standard structure is the unit ball of a normed space. Note that the same remains true if we add to the language any continuous (equivalently, bounded) operator or functional.

We conclude this introduction by advertising a question in [CLCL]. Is it possible to extend the Positive Compactness Theorem (Theorem 19) to a larger class of languages? E.g. can we allow in \mathcal{L} (and in $\mathcal{L}^{p/c}$) function symbols of sort $H^n \times I^m \rightarrow I$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I .

2. Henson-Iovino approximations

For C, C' compact subsets of I^n , we write $C' > C$ if C' is a neighborhood of C . For φ, φ' (free variables are hidden) positive formulas possibly with parameters we write $\varphi' > \varphi$ if φ' is obtained by replacing in φ each atomic formula of the form $t \in C$ with $t \in C'$, for some $C' > C$. If no such atomic formula occurs in φ , then $\varphi > \varphi$. We call φ' a **weakening** of φ . Note that $>$ is a dense transitive relation and that $\varphi \rightarrow \varphi'$ in every \mathcal{L} -structure.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where \tilde{C} is some compact set disjoint from C . The atomic formulas in \mathcal{L}_H are replaced with their negation. Finally each connective is replaced by its dual i.e., $\vee, \wedge, \exists, \forall$ are replaced by $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde{\varphi}$ is a **strong negation** of φ . It is clear that $\tilde{\varphi} \rightarrow \neg\varphi$ in every \mathcal{L} -structure.

Notice that the weakening and the strong negation of a formula in \mathcal{L}^c is also in \mathcal{L}^c .

Lemma 7. For all positive formulas φ

1. for every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \rightarrow \neg\tilde{\varphi} \rightarrow \varphi'$;
2. for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \rightarrow \varphi' \rightarrow \neg\tilde{\varphi}$.

Proof. If $\varphi \in \mathcal{L}_H$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some $C' > C$. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in I \setminus O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact \tilde{C} disjoint from C . By the normality

of I , there is $C' > C$ disjoint from \tilde{C} . Then $\varphi' = (t \in C')$ is as required. The lemma follows easily by induction. \square

For every type $p(x)$, we write

$$p'(x) = \{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\}$$

in particular $\{\varphi(x)\}' = \{\varphi'(x) : \varphi' > \varphi\}$.

Definition 8. A set of formulas $\mathcal{F} \subseteq \mathcal{L}$ is **p/c-dense** modulo T , a theory, if for every positive/continuous $\varphi(x)$ and every $\varphi' > \varphi$, there is $\psi(x) \in \mathcal{F}$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi'(x)$ holds in every standard structure that models T .

Example 9. By Lemma 7, the set of negations of positive/continuous formulas is p/c-dense.

Example 10. Let \mathcal{C} be a prebase of closed subsets of I . The set of formulas built inductively from atomic formulas of the form $\tau(x, \eta) \in C$ for $C \in \mathcal{C}$ is c-dense.

Example 11. Let $I = [0, 1]$. The set of formulas built inductively from atomic formulas of the form $\tau \in \{0\}$ is c-dense. In fact, as every compact subset of I is the intersection of finite unions of closed intervals, the formulas built using only atomic formulas of the form $\tau \in [\alpha, \beta]$ are c-dense. Finally note that $\tau \in [\alpha, \beta]$ is equivalent to $\tau \div \beta \in \{0\} \wedge \alpha \div \tau \in \{0\}$.

Example 12. In Section 8 we prove that the set of continuous formulas without quantifiers of sort l is c-dense.

3. The standard part¹

Our goal is to prove a compactness theorem for positive theories that only requires standard structures. To bypass lengthy routine applications of ultrafilters, ultrapowers, and ultralimits, we assume the Classical Compactness Theorem and build on that.

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 17 which in turn is required for the proof of the Positive Compactness Theorem (Theorem 19). The reader willing to accept it without proof may skip this section.

Let $\langle N, {}^*I \rangle$ be an \mathcal{L} -structure that extends \mathcal{L} -elementarily the standard structure M . Let η be a free variable of sort l . For each $\beta \in I$, we define the type

$$m_\beta(\eta) = \{\eta \in D : D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $m_\beta(\eta)$ in *I is known to nonstandard analysts as the monad of β . The following fact is well-known.

Fact 13. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \models m_\beta(\alpha)$.

Proof. Negate the existence of β . For every $\gamma \in I$ pick some compact neighborhood D_γ of γ , such that ${}^*I \models \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ , with $\gamma \in \Gamma$, cover I . By

¹We will omit this section from the final version of this paper and refer to [CLCL] instead.

elementarity, the interpretation in *I of these D_γ cover *I . A contradiction. The uniqueness of β follows from normality. \square

We denote by $\text{st}(\alpha)$ the unique $\beta \in I$ such that $^*I \models m_\beta(\alpha)$.

Fact 14. For every $\alpha \in ^*I$ and every compact $C \subseteq I$

$$^*I \models \alpha \in C \rightarrow \text{st}(\alpha) \in C.$$

Proof. Assume $\text{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\text{st}(\alpha)$. Then $^*I \models \alpha \in D \subseteq \neg C$. \square

Fact 15. For every $\alpha \in (^*I)^{|\alpha|}$ and every function symbol f of sort $|\alpha| \rightarrow I$

$$^*I \models \text{st}(f(\alpha)) = f(\text{st}(\alpha)).$$

Proof. By Fact 13 and the definition of $\text{st}(\cdot)$ it suffices to prove that $^*I \models f(\alpha) \in D$ for every compact neighborhood D of $f(\text{st}(\alpha))$.

Fix one such D . Then $\text{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\text{st}(\alpha)$. Therefore $^*I \models \alpha \in f^{-1}[D]$ and, as $I \leq ^*I$ we obtain $^*I \models f(\alpha) \in D$. \square

The **standard part** of $\langle N, ^*I \rangle$ is the standard structure $\langle N, I \rangle$ that interprets the symbols f of sort $H^n \rightarrow I$ as the functions

$$f^N(a) = \text{st}(^*f(a)) \quad \text{for all } a \in N^n,$$

where *f is the interpretation of f in $\langle N, ^*I \rangle$. Symbols in \mathcal{L}_H maintain the same interpretation.

Fact 16. With the notation as above. Let $\tau(x; \eta)$ be a term of sort $H^{|\alpha|} \times I^{|\eta|} \rightarrow I$. Then for every $a \in M^{|\alpha|}$ and $\alpha \in (^*I)^{|\eta|}$

$$\tau^N(a; \text{st}(\alpha)) = \text{st}(^*\tau(a; \alpha))$$

Proof. When t is a function symbol of sort $H^{|\alpha|} \rightarrow I$, the claim holds by definition. When t is a function symbol of sort $I^{|\eta|} \rightarrow I$, the claim follows from Fact 15. Now, assume inductively that

$$t_i^N(a; \text{st}(\alpha)) = \text{st}(^*t_i(a; \alpha))$$

holds for the terms $t_1(x; \eta), \dots, t_n(x; \eta)$ and let $t = f(t_1, \dots, t_n)$ for some function f of sort $I^n \rightarrow I$. Then the claim follows immediately from the induction hypothesis and Fact 15. \square

Lemma 17. With the notation as above. For every $\varphi(x; \eta) \in \mathcal{L}^P$, $a \in N^{|\alpha|}$ and $\alpha \in (^*I)^{|\eta|}$

$$\langle N, ^*I \rangle \models \varphi(a; \alpha) \Rightarrow N \models \varphi(a; \text{st}(\alpha))$$

Proof. Suppose $\varphi(x; \eta)$ is \mathcal{L}^P -atomic. If $\varphi(x; \eta)$ is a formula of \mathcal{L}_H the claim is trivial. Otherwise $\varphi(x; \eta)$ has the form $\tau(x; \eta) \in C$. Assume that the tuple $\tau(x; \eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, ^*I \rangle \models \tau(a; \alpha) \in C$. Then $\text{st}(^*\tau(a; \alpha)) \in C$ by Fact 14. Therefore $\tau^N(a; \text{st}(\alpha)) \in C$ follows from Fact 16. This proves the lemma for atomic formulas. Induction is immediate. \square

Corollary 18. Let M be a standard structure. Let $p(\eta) \subseteq \mathcal{L}^P(M)$ be a type that does not contain existential quantifiers of sort H . Then, if $p(\eta)$ is finitely consistent in M , it is realized in M .

The corollary has also a direct proof. This goes through the observation that the formulas in $p(\eta)$ define compact subsets of I .

Proof. Let $\langle {}^*M, {}^*I \rangle$ be an \mathcal{L} -elementary saturated superstructure of $\langle M, I \rangle$. Then $\langle {}^*M, {}^*I \rangle \models p(\alpha)$ for some $\alpha \in {}^*I$. By the lemma above, ${}^*M \models p(\text{st}(\alpha))$. Now observe that the truth of formulas without existential quantifiers of sort H is preserved by substructures. \square

The exclusion of existential quantifiers of sort H is necessary. For a counterexample take $M = I = [0, 1]$. Assume that \mathcal{L} contains a function symbol for the identity map $\iota : M \rightarrow I$. Let $p(\eta)$ contain the formulas $\exists x (x > 0 \wedge \iota x + \eta \in [0, 1/n])$ for all positive integers n .

4. Compactness

It is convenient to distinguish between consistency with respect to standard structures and consistency with respect to \mathcal{L} -structures. We say that a theory T is **\mathcal{L} -consistent** if $\langle M, {}^*I \rangle \models T$ for some \mathcal{L} -structure $\langle M, {}^*I \rangle$ (this is the classical notion of consistency). We say that T is **standardly consistent** if $M \models T$ is for some standard structure M . By Lemma 17 these two notions coincide if T is positive. Therefore we have the following.

Theorem 19 (Positive Compactness Theorem). Let T be a positive theory. Then, if T is finitely \mathcal{L} -consistent, it is also standardly consistent.

An \mathcal{L} -structure N is **positively λ -saturated** if it realizes all types $p(x; \eta) \subseteq \mathcal{L}^P(N)$ with fewer than λ parameters that are finitely consistent in N . When $\lambda = |N|$ we simply say **p-saturated**. The existence of p-saturated standard structures is obtained from the classical case just as for Theorem 19.

We say that M is a **p-elementary** substructure of N if $M \subseteq N$ and

$$M \models \varphi(a) \Rightarrow N \models \varphi(a)$$

for every $\varphi(x) \in \mathcal{L}^P$ and every $a \in M^{|x|}$.

Theorem 20. Every standard structure has a p-elementary extension to a p-saturated standard structure (possibly of inaccessible cardinality).

Proof. Let $\langle {}^*M, {}^*I \rangle$ be an \mathcal{L} -saturated \mathcal{L} -elementary extension of the standard structure $\langle M, I \rangle$. Let $\langle {}^*M, I \rangle$ be its standard part as defined in Section 3. By Lemma 17, $\langle {}^*M, I \rangle$ realizes all finitely consistent positive types with fewer than $|{}^*M|$ parameters. \square

The following proposition shows that a slight amount of saturation tames the positive formulas.

Proposition 21. Let N be a positively ω -saturated standard structure. Then

$$\{\varphi(x; \eta)\}' \leftrightarrow \varphi(x; \eta)$$

holds in N for every formula $\varphi(x; \eta) \in \mathcal{L}^P(N)$.

Proof. We prove \rightarrow , the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H . Assume inductively

$$\text{ih.} \quad \{\varphi(x, z; \eta)\}' \rightarrow \varphi(x, z; \eta)$$

We need to prove

$$\{\exists z \varphi(x, z; \eta)\}' \rightarrow \exists z \varphi(x, z; \eta)$$

From (ih) we have

$$\exists z \{\varphi(x, z; \eta)\}' \rightarrow \exists z \varphi(x, z; \eta)$$

Therefore it suffices to prove

$$\{\exists z \varphi(x, z; \eta)\}' \rightarrow \exists z \{\varphi(x, z; \eta)\}'$$

Replace the variables $x; \eta$ with parameters, say $a; \alpha$, and assume that $N \models \exists z \varphi'(a, z; \alpha)$ for every $\varphi' > \varphi$. We need to prove the consistency of the type $\{\varphi'(a, z; \alpha) : \varphi' > \varphi\}$. By saturation, finite consistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly. \square

Remark 22. By Corollary 18, when $\varphi(x; \eta)$ does not contains existential quantifiers of sort H, the proposition above does not require the assumption of saturation. In general, some saturation is necessary: consider the model presented after Corollary 18 and the formula $\exists x (x > 0 \wedge \iota(x) \in \{0\})$. See [Unnumbered paragraph after Corollary 5.16][HII] for a counterexample in \mathcal{L}^c .

5. The monster model

We denote by \mathcal{U} some large p-saturated standard structure which we call the **positive monster model**. Truth is evaluated in \mathcal{U} unless otherwise is specified. We denote by T the positive theory of \mathcal{U} . The density of a set of formulas is understood modulo T . Below we say **p/c-model** for p/c-elementary substructure of \mathcal{U} . We stress that the truth of some $\varphi \in \mathcal{L}^{p/c}(M)$ in a p/c-model M implies the truth of φ (in \mathcal{U}) but not vice versa. However, all p/c-models agree on the approximated truth.

Fact 23. The following are equivalent

1. M is a p/c-model;
2. $M \models \{\varphi(a)\}' \Leftrightarrow \varphi(a)$ for every $\varphi(x) \in \mathcal{L}^{p/c}$ and every $a \in M^{|x|}$.

Proof. (1 \Rightarrow 2) Only the implication \Leftarrow requires a proof. Assume $\varphi(a)$. Let $\varphi' > \varphi$. We prove $M \models \varphi'(a)$. By Lemma 7 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \rightarrow \neg \tilde{\varphi} \rightarrow \varphi'$. Then $\neg \tilde{\varphi}(a)$ and therefore $M \models \neg \tilde{\varphi}(a)$. Then $M \models \varphi'(a)$.

(2 \Rightarrow 1) By Proposition 21. \square

The following fact demonstrates how positive compactness applies. There are some subtle differences from the classical setting. Let $A \subseteq \mathcal{U}$ be a small set throughout this section.

Fact 24. Let $p(x) \subseteq \mathcal{L}^p(A)$ be a type. Then for every $\varphi(x) \in \mathcal{L}^p(\mathcal{U})$

- i. if $p(x) \rightarrow \neg \varphi(x)$ then $\psi(x) \rightarrow \neg \varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;
- ii. if $p(x) \rightarrow \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \rightarrow \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (i) is immediate by saturation; (ii) follows from (i) by Lemma 7. \square

Fact 25. Let \mathcal{F} be a p/c-dense set of positive/continuous formulas. Then \mathcal{F}' is p/c-dense.

Proof. Let $\varphi' > \varphi$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$ and $\psi \in \mathcal{F}$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi''(x)$. It suffices to prove that $\psi'(x) \rightarrow \varphi'(x)$ for some $\psi' > \psi$. By Proposition 21, $\psi(x) \leftrightarrow \{\psi(x)\}'$. Therefore, $\psi'(x) \rightarrow \varphi'(x)$ follows from Fact 24. \square

We write $\text{p/c-tp}(a/A)$ for the positive/continuous type of a over A , that is, the set of formulas $\{\varphi(x) \in \mathcal{L}^{\text{p/c}}(A) : \varphi(a)\}$. In general, if \mathcal{F} is any set of formulas, we write $\mathcal{F}\text{-tp}(a/A)$ for the type $\{\varphi(x) \in \mathcal{F}(A) : \varphi(a)\}$. The undecorated symbol $\text{tp}(a/A)$ denotes the \mathcal{L} -type.

Fact 26. Let $p(x) \subseteq \mathcal{L}^{\text{p/c}}(A)$. The following are equivalent

1. $p(x)$ is a maximally standardly consistent subsets of $\mathcal{L}_x^{\text{p/c}}(A)$;
2. $p(x) = \text{p/c-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$.

Proof. (1 \Rightarrow 2) If $p(x)$ is consistent then $p(x) \subseteq \text{p/c-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$. By maximality $p(x) = \text{p/c-tp}(a/A)$.

(2 \Rightarrow 1) By Proposition 21 and Lemma 7

$$\varphi(x) \leftrightarrow \bigwedge_{\tilde{\varphi} \perp \varphi} \neg \tilde{\varphi}(x).$$

Suppose $\varphi(x) \in \mathcal{L}^{\text{p/c}}(A) \setminus p$. Then $\neg \varphi(a)$. Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \rightarrow \neg \varphi(x)$ follows. \square

The fact above extends as follows.

Fact 27. Let $p(x) \subseteq \mathcal{F}(A)$, where \mathcal{F} is a p/c-dense set of positive/continuous formulas. Then the following are equivalent

1. $p(x)$ is a maximally standardly consistent subset of $\mathcal{F}_x(A)$;
2. $p(x) = \mathcal{F}\text{-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$;
3. $p(x) \leftrightarrow \text{p/c-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$.

Proof. (1 \Rightarrow 2) As above.

(2 \Rightarrow 3) As $q(x) \leftrightarrow q'(x)$, then $q(x) \leftrightarrow p(x) = \mathcal{F}\text{-tp}(a/A)$.

(3 \Rightarrow 1) Immediate. \square

Proposition 28. Let \mathcal{F} be a p/c-dense set of positive/continuous formulas. Then for every formula $\varphi(x) \in \mathcal{L}^{\text{p/c}}$

- i. $\neg \varphi(x) \leftrightarrow \bigvee \{ \psi(x) : \psi'(x) \rightarrow \neg \varphi(x) \text{ and } \psi' > \psi \text{ for some } \psi(x) \in \mathcal{F} \};$
- ii. $\neg \varphi(x) \leftrightarrow \bigvee \{ \neg \psi(x) : \neg \psi(x) \rightarrow \neg \varphi(x) \text{ and } \psi(x) \in \mathcal{F} \}.$

Proof. (i) Only \rightarrow requires a proof. Let $a \in \mathcal{U}^{|x|}$ be such that $\neg \varphi(a)$. Let $p(x) = \text{p/c-tp}(a)$. By Fact 26, $p(x) \rightarrow \neg \varphi(x)$. As \mathcal{F} is p/c-dense, $p(x) \leftrightarrow q(x) = \mathcal{F}\text{-tp}(a)$. Then $q'(x) \rightarrow \neg \varphi(x)$ hence, by compactness, $\psi'(x) \rightarrow \neg \varphi(x)$ for some $\psi(x) \in \mathcal{F}$.

(ii) By density

$$\varphi(x) \leftrightarrow \bigwedge \{ \psi(x) \in \mathcal{F} : \varphi(x) \rightarrow \psi(x) \}.$$

Negating both sides of the equivalence we obtain (ii). \square

6. Cauchy completions

For $\tau(x, z) = \tau_1(x, z), \dots, \tau_n(x, z)$ a tuple of terms of sort $H^{|x|+|z|} \rightarrow I$ we define the formula

$$x \sim_\tau y = \bigwedge_{i=1}^n \forall z \ \tau_i(x, z) = \tau_i(y, z),$$

where the expression $\alpha = \beta$ is shorthand for $\langle \alpha, \beta \rangle \in \Delta$, where Δ is the diagonal of I^2 . We also define the type

$$x \sim y = \{x \sim_\tau y : \tau(x, z) \text{ as above}\}.$$

The following fact will be used below without mention.

Fact 29. For any $a = \langle a_i : i < \lambda \rangle$ and $b = \langle b_i : i < \lambda \rangle$

$$a \sim b \Leftrightarrow a_i \sim b_i \text{ for every } i < \lambda.$$

Proof. By induction. Only implication \Leftarrow requires a proof. For limit ordinals induction is trivial. Assume $a_i \sim b_i$ for every $i \leq \lambda$. Let $|a| = |b| = \lambda$ and assume inductively that (note that it suffices to consider tuples τ of arity 1)

$$\forall y, z \ \tau(a, y, z) = \tau(b, y, z)$$

holds for every term $\tau(x, y, z)$, with $|x| = \lambda$ and $|y| = 1$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

$$1. \quad \forall z \ \tau(a, a_\lambda, z) = \tau(b, a_\lambda, z).$$

As $a_\lambda \sim b_\lambda$ then

$$\forall x, z \ \tau(x, a_\lambda, z) = \tau(x, b_\lambda, z)$$

and in particular

$$2. \quad \forall z \ \tau(b, a_\lambda, z) = \tau(b, b_\lambda, z).$$

From (1) and (2) we obtain

$$\forall z \ \tau(a, a_\lambda, z) = \tau(b, b_\lambda, z). \quad \square$$

Note that the approximations of the formula $x \sim_\tau y$ have the form

$$x \sim_{\tau, D} y = \bigwedge_{i=1}^n \forall z \ \langle \tau_i(x, z), \tau_i(y, z) \rangle \in D$$

for some compact neighborhood D of Δ . We write \mathcal{E} for the set containing the pairs τ, D as above. The formulas $x \sim_\varepsilon y$ form a prebase for a system of entougages on $\mathcal{U}^{|x|}$. We refer to this uniformity and the topology associated as the topology induced by I . Though not needed in the sequel, it is worth mentioning that the topology induced by I on $\mathcal{U}^{|x|}$ coincides with the product of the topology induced by I on \mathcal{U} . This can be verified by an argument similar to the proof of Fact 29.

Fact 30. For every $\varphi(x) \in \mathcal{L}^c$

$$x \sim y \rightarrow \varphi(x) \leftrightarrow \varphi(y)$$

Proof. By induction. □

We define

$$x \sim' y = \{x \sim_\varepsilon y : \varepsilon \in \mathcal{E}\}.$$

The following corollary corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Corollary 31. For every $\varphi(x) \in \mathcal{L}^c$, every $\varphi' > \varphi$, and every $\tilde{\varphi} \perp \varphi$ there an $\varepsilon \in \mathcal{E}$ such that

- i. $x \sim_\varepsilon y \wedge \varphi(y) \rightarrow \varphi'(x)$
- ii. $x \sim_\varepsilon y \wedge \tilde{\varphi}(y) \rightarrow \neg\varphi(x).$

Proof. As $x \sim' y \cup \{\varphi(x)\} \rightarrow \varphi(y)$ by the fact above, (i) follows from Fact 24. Similarly, we obtain (ii) from $x \sim' y \cup \{\tilde{\varphi}(x)\} \rightarrow \neg\varphi(y)$. \square

We say what a type $q(x)$ is **finitely satisfiable** in A if every conjunction of formulas in $q(x)$ has a solution in $A^{|x|}$. This definition coincides with the classical one, but in our context, the notion is less robust. We may happen that $p(x)$ is finitely satisfiable while $q(x) \leftrightarrow p(x)$ is not. In particular if M is a p-model and $q(x) = \text{p-tp}(a/M)$ then $q'(x)$ is always finitely satisfiable while $q(x)$ need not.

We say that that a set $A \subseteq \mathcal{U}$ is **Cauchy complete** if it contains all those $a \in \mathcal{U}$ such that $a \sim' x$ is finitely satisfied in A . Note that Cauchy complete sets are in particular closed under \sim -equivalence. The **Cauchy completion** of A is the set

$$\text{Ccl}(A) = \{a : a \sim' x \text{ is finitely satisfied in } A\}.$$

Fact 32. $\text{Ccl}(A)$ is Cauchy complete.

Proof. Suppose that $x \sim' a$ is finitely satisfied in $\text{Ccl}(A)$. Let $\varepsilon \in \mathcal{E}$ be given. We prove that $a \sim_\varepsilon x$ is satisfied in A . Let $\eta \in \mathcal{E}$ be such that $x \sim_\eta y \sim_\eta z \rightarrow x \sim_\varepsilon z$. There is some $b \in \text{Ccl}(A)$ such that $b \sim_\eta a$. There is some $c \in A$ such that $c \sim_\eta b$. Then c satisfies $x \sim_\varepsilon a$, as required. \square

We say that $p(x) \subseteq \mathcal{L}^c(\mathcal{U})$ is a **Cauchy type** if it is consistent and $p(x) \wedge p(y) \rightarrow x \sim y$.

Fact 33. Let M be a c-model and let $p(x) \subseteq \mathcal{L}^c(M)$ be a Cauchy type. Then all realizations of $p(x)$ belong to $\text{Ccl}(M)$.

Proof. If $p(x)$ is a Cauchy type, then $p(x) \rightarrow a \sim x$ for some $a \models p(x)$. As M is a c-model, $p'(x)$ is finitely satisfied in M . Then also $a \sim' x$ is finitely satisfied. Hence $a \in \text{Ccl}(M)$. \square

7. Morphisms

Here we only discuss the continuous case and refer to [CLCL] for the positive case. The definition of c-elementary maps is delicate due to the lack of equality of sort H. A possible approach is to work in the quotient \mathcal{U}/\sim . A second possibility is to replace functions with relations. The two options differs only in the notation. Here we go for the second.

Let R be a binary relation on \mathcal{U} . If $a = \langle a_i : i < \lambda \rangle$ and $b = \langle b_i : i < \lambda \rangle$ we write $\langle a, b \rangle \in R$ to abbreviate: $\langle a_i, b_i \rangle \in R$ for all $i < \lambda$. Let \mathcal{F} be a set of formulas. We say that R preserves the truth of \mathcal{F} -formulas if

$$\# \quad \varphi(a) \rightarrow \varphi(b) \quad \text{for every } \varphi(x) \in \mathcal{F} \text{ and every } \langle a, b \rangle \in R.$$

We will consider two extreme cases for \mathcal{F} : the set of all continuous formulas and the set of atomic continuous formulas. We denote the latter by $\mathcal{L}_{\text{at}}^c$. Relations that preserve the truth of \mathcal{L}^c -formulas are called **c-elementary**.

By the following fact, if R is c-elementary, also R^{-1} is c-elementary.

Fact 34. Let \mathcal{F} be either \mathcal{L}^c or $\mathcal{L}_{\text{at}}^c$. Let R be a relation that preserves the truth of \mathcal{F} -formulas. Then the converse implication in (#) holds.

Proof. $\mathcal{F} = \mathcal{L}^c$. Assume $\neg\varphi(a)$ and apply Proposition 28 to infer that $\psi(a)$ holds for some $\psi(x) \in \mathcal{L}^c$ that implies $\neg\varphi(x)$.

$\mathcal{F} = \mathcal{L}_{\text{at}}^c$. Assume $\tau(a) \notin C$ then $\tau \in \tilde{C}$ for some compact \tilde{C} disjoint of C . Then $\tau(b) \notin C$ follows. \square

We say that the relation R is **reduced** if there is no $R' \subset (\sim) \circ R \circ (\sim)$ such that $R \subset (\sim) \circ R' \circ (\sim)$.

Fact 35. Every relation R contains a reduced relation R' such that $(\sim) \circ R \circ (\sim) = (\sim) \circ R' \circ (\sim)$.

Proof. Let $\langle R_i : i < \lambda \rangle$ be maximal decreasing chain of relations $R_i \subseteq R \subseteq (\sim) \circ R_i \circ (\sim)$. Let R' the intersection of the chain. By maximality R' is reduced and $(\sim) \circ R \circ (\sim) = (\sim) \circ R' \circ (\sim)$ is clear. \square

Fact 36. Let $\mathcal{F} = \mathcal{L}^c$. Let R be a reduced relation that preserves of the truth of \mathcal{F} -formulas. Then R is (the graph of) an injective map $f : \text{dom}(R) \rightarrow \text{range}(R)$.

The same is holds when $\mathcal{F} = \mathcal{L}_{\text{at}}^c$ if $\text{dom}(R)$ and $\text{range}(R)$ are c-models.

Proof. $\mathcal{F} = \mathcal{L}^c$. Let $a \in \text{dom}R$ and assume that $\langle a, b \rangle, \langle a, b' \rangle \in R$. To prove functionality suppose for a contradiction that $b \neq b'$. As R is reduced, $b \not\sim_\varepsilon b'$. Then $b \not\sim_\varepsilon b'$ for some $\varepsilon \in \mathcal{E}$. By Proposition 28 b, b' satisfies a continuous formula $\psi(x, x')$ that implies $x \sim_\varepsilon x'$. Then a, a also satisfies $\psi(x, x')$, a contradiction. This proves functionality, for injectivity apply the same argument to R^{-1} .

$\mathcal{F} = \mathcal{L}_{\text{at}}^c$. Reason as above to obtain $b \not\sim_\varepsilon b'$, say $\varepsilon = \langle \tau, D \rangle$. Then the formula

$$\neg\varphi(b, b', z) = \neg \bigwedge_{i=1}^n \langle \tau_i(b, z), \tau_i(b', z) \rangle \in D.$$

is consistent. By the Tarski-Vaught test (Theorem 42) $\neg\varphi(b, b', z)$ has a solution $c \in \text{range}(R)^{|z|}$. Then $\neg\varphi(a, a, d)$ for some $d \in \text{dom}(R)^{|z|}$ which is a contradiction. \square

Let $f \subseteq R$ be a reduced c-elementary relation such that $R \subseteq (\sim) \circ f \circ (\sim)$. Then f is a function defined in exactly one representative a of each (\sim) -equivalence class $[a]$ that intersects $\text{dom}(R)$.

Fact 37. Let \mathcal{F} be either \mathcal{L}^c or $\mathcal{L}_{\text{at}}^c$. The following are equivalent

1. R preserves the truth of \mathcal{F} -formulas;
2. $\text{Ccl}(R)$ preserves the truth of \mathcal{F} -formulas.

Proof. We prove the nontrivial implication, $1 \Rightarrow 2$. Let $\varphi(x) \in \mathcal{F}$ and $\varphi'' > \varphi' > \varphi$ be given. Let $\langle a, b \rangle \in \text{Ccl}(R)$ and assume $\varphi(a)$. Let $\varepsilon \in \mathcal{E}$ be such that

1. $x \sim_\varepsilon y \wedge \varphi(x) \rightarrow \varphi'(y)$
2. $x \sim_\varepsilon y \wedge \varphi'(x) \rightarrow \varphi''(y)$

By the definition of Cauchy completion, there are $a', b' \sim_\varepsilon a, b$ such that $\langle a', b' \rangle \in R$. Then from (1) we infer that $\varphi'(a')$. Note that $\varphi'(x) \in \mathcal{F}$, then $\varphi'(b')$. Then $\varphi''(b)$ follows from (2). As $\varphi'' > \varphi$ is arbitrary, the fact follows from Proposition 21. \square

Let M and N be c-models. An **c-isomorphism** between M and M is a relation R that preserves the truth of $\mathcal{L}_{\text{at}}^c$ -formulas and $M \subseteq \text{dom}(R)$, $N \subseteq \text{range}(R)$. The term isomorphism may sound inappropriate but it is justified by the following fact.

Fact 38. Let R be a c-isomorphism between M and N , two c-models. Let $f \subseteq \text{Ccl}(R)$ be a reduced relation such that $\text{Ccl}(R) = (\sim) \circ f \circ (\sim)$. Then $f : \text{Ccl}(M) \rightarrow \text{Ccl}(N)$ is a bijective map such that $\varphi(a) \leftrightarrow \varphi(fa)$ for all $\varphi(x) \in \mathcal{L}^c$ and $a \in \text{Ccl}(M)^{|x|}$.

Proof. From Fact 36 we obtain that $f : \text{Ccl}(M) \rightarrow \text{Ccl}(N)$ is bijective map. To prove $\varphi(a) \leftrightarrow \varphi(fa)$ we reason by induction as in the classical case. \square

8. Elimination of quantifiers of sort I

We write $\mathcal{L}_{\text{lqf}}^{p/c}$ for the set of positive/continuous formulas without quantifiers of sort I. In [CLCL] it is proved that $\mathcal{L}_{\text{lqf}}^p$ is a p-dense set. Here we adapt the argument to continuous formulas.

We ask the reader to verify that Fact 36 holds, with the same proof, when R preserves the truth of formulas in $\mathcal{L}_{\text{lqf}}^c$.

Proposition 39. Every relation of small cardinality R that preserves the truth of formulas in $\mathcal{L}_{\text{lqf}}^c$ extends to c-automorphisms of \mathcal{U} .

Proof. We construct by back-and-forth a sequence $f_i : \mathcal{U} \rightarrow \mathcal{U}$ of reduced relations (i.e. injective maps) that preserve the truth formulas in $\mathcal{L}_{\text{lqf}}^c$. Finally we set

$$R = \bigcup_{i \in |\mathcal{U}|} (\sim) \circ f_i \circ (\sim).$$

We will ensure that $\mathcal{U} = \text{dom}(R) = \text{range}(R)$. Then R is a c-automorphism of \mathcal{U} by Remark 38.

Let i be even. Let a be an enumeration of $\text{dom}(f_n)$. Let $b \in \mathcal{U}$. Let $p(x, y) = \text{c-tp}(a, b)$. By c-elementarity the type $p(f_i a, y)$ is finitely satisfied. Let $c \in \mathcal{U}$ realize $p(f_i a, y)$. Let f_{i+1} be a reduced relation such that $f_{i+1} \subseteq f_i \cup \{ \langle b, c \rangle \} \subseteq (\sim) \circ f_{i+1} \circ (\sim)$.

When i is odd we proceed similarly with f_i^{-1} . Limit stages are obvious. \square

Corollary 40. Let $a \in \mathcal{U}^{|x|}$. Let $p(x) = \mathcal{L}_{\text{lqf}}^c\text{-tp}(a)$ and $q(x) = \text{c-tp}(a)$. Then $p(x) \leftrightarrow q(x)$.

Proof. Only \rightarrow requires a proof. If a contains entries $a_i \sim a_j$, replace a_j with a_i . Note that the tuple a' obtained in this manner has the same c-type of a .

Let $b \models p(x)$ and let b' obtained with the same procedure as a' .

Then the function that maps $a' \mapsto b'$ preserves the truth of $\mathcal{L}_{\text{lqf}}^c$ -formulas. Then f extends to an c-automorphism. As every c-automorphism is c-elementary and the corollary follows. \square

Proposition 41. The set $\mathcal{L}_{\text{lqf}}^c$ is c-dense modulo T .

Proof. Let $\varphi(x)$ be a positive formula. We need to prove that for every $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathcal{L}_{\text{lqf}}^p$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi'(x)$. By Corollary 40 and Proposition 21

$$\neg \varphi(x) \rightarrow \bigvee_{p'(x) \rightarrow \neg \varphi(x)} p'(x)$$

where $p(x)$ ranges over the maximally consistent $\mathcal{L}_{\text{lqf}}^p$ -types. By Fact 24 and Lemma 7

$$\neg \varphi(x) \rightarrow \bigvee_{\neg \tilde{\psi}(x) \rightarrow \neg \varphi(x)} \neg \tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{L}_{\text{qf}}^p$. Equivalently,

$$\varphi(x) \leftarrow \bigwedge_{\tilde{\psi}(x) \leftarrow \varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 24, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{L}_{\text{qf}}^p$ such that

$$\varphi'(x) \leftarrow \bigwedge_{i=1, \dots, n} \tilde{\psi}_i(x) \leftarrow \varphi(x)$$

which yields the interpolant required by the proposition. \square

9. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is our version of the Tarski-Vaught test.

Theorem 42. Let M be a subset of \mathcal{U} . Let \mathcal{F} be a p -dense set of positive formulas. Then the following are equivalent

1. M is a p -model;
2. for every formula $\psi(x) \in \mathcal{F}(M)$

$$\exists x \psi(x) \Rightarrow \text{for every } \psi' > \psi \text{ there is an } a \in M \text{ such that } \psi'(a);$$
3. for every formula $\psi(x) \in \mathcal{F}(M)$

$$\exists x \neg \psi(x) \Rightarrow \text{there is an } a \in M \text{ such that } \neg \psi(a).$$

If \mathcal{F} is a c -dense set of continuous formulas then (2) and (3) above are equivalent to

- 1'. $\text{Ccl}(M)$ is a c -model.

Moreover, if M is a substructure, then (1'), (2) and (3) are also equivalent to

- 1''. M is a c -model.

Proof. (1 \Rightarrow 2) Assume $\exists x \psi(x)$ and let $\psi' > \psi$ be given. By Lemma 7 there is some $\tilde{\psi} \perp \psi$ such that $\psi(x) \rightarrow \neg \tilde{\psi}(x) \rightarrow \psi'(x)$. Then $\neg \forall x \tilde{\psi}(x)$ hence, by (1), $M \models \neg \forall x \tilde{\psi}(x)$. Then $M \models \neg \tilde{\psi}(a)$ for some $a \in M$. Hence $M \models \psi'(a)$ and $\psi'(a)$ follows from (1).

(2 \Rightarrow 3) Assume (2) and let $\psi(x) \in \mathcal{F}(M)$ be such that $\exists x \neg \psi(x)$. By Proposition 28.i, there are a consistent $\varphi(x) \in \mathcal{F}(M)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \rightarrow \neg \psi(x)$. Then (3) follows.

(3 \Rightarrow 2) Let $\psi' > \psi$ for some $\psi(x) \in \mathcal{F}(M)$. Let $\tilde{\psi} \perp \psi$ such that $\psi(x) \rightarrow \neg \tilde{\psi}(x) \rightarrow \psi'(x)$. By Proposition 28.ii, $\neg \varphi(x) \rightarrow \neg \tilde{\psi}(x)$ for some $\varphi(x) \in \mathcal{F}(M)$ such that $\neg \varphi(x)$ is consistent. Then (2) follows from (3).

(2 \Rightarrow 1) Assume (2). By the classical Tarski-Vaught test $M \leq_H \mathcal{U}$. Then M is the domain of a substructure of \mathcal{U} . Then $M \models \varphi(a) \Rightarrow \varphi(a)$ holds for every atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \models \varphi(a, b) \Rightarrow \varphi(a, b).$$

Using (2) and the induction hypothesis we prove that

$$M \models \exists y \varphi(a, y) \Rightarrow \forall y \varphi(a, y).$$

Indeed, for any $\varphi' > \varphi$,

$$\begin{aligned} M \models \exists y \varphi(a, y) &\Rightarrow M \models \exists y \psi(a, y) \text{ for some } \psi \in \mathcal{F} \text{ such that } \varphi(a, y) \rightarrow \psi(a, y) \rightarrow \varphi'(a, y) \\ &\Rightarrow M \models \psi(a, b) \text{ for some } b \in M \text{ by (2)} \\ &\Rightarrow M \models \varphi'(a, b) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \varphi'(a, b) && \text{by induction hypothesis} \\ &\Rightarrow \exists y \varphi'(a, y). \end{aligned}$$

As $\varphi' > \varphi$ is arbitrary, $\exists y \varphi(a, y)$ follows from Proposition 21.

Induction for the connectives \vee , \wedge , \forall^H , \exists , and \forall^I is straightforward.

(1' \Rightarrow 2) Let $\psi' > \psi'' > \psi$. Reasoning as in the proof of (1 \Rightarrow 2) we obtain that $\text{Ccl}(M) \models \psi''(a)$ for some $a \in \text{Ccl}(M)$. By Corollary 31, $a \sim_\varepsilon x \rightarrow \psi'(x)$ for some ε . As $a \sim' x$ is finitely satisfied in M , it follows that $\psi'(c)$ for some $c \in M$.

(2 \Leftrightarrow 3) The proof above applies verbatim when \mathcal{F} is a c-dense set of continuous formulas.

(2 \Rightarrow 1') Assume (2). We claim that $\text{Ccl}(M)$ is a substructure of \mathcal{U} . Let $a \in M^n$ and let f be a function symbol of sort $H^n \rightarrow H$. We prove that $fa \in M$. We show that $fa \sim' x$ is finitely satisfied in M . Consider the formula $fa \sim_\varepsilon x$ where ε is the pair τ, D . By Lemma 7, there is a formula in $\tilde{\varphi}(x) \in \mathcal{L}^p(M)$ such that

$$fa \sim_\tau x \rightarrow \neg \tilde{\varphi}(x) \rightarrow fa \sim_\varepsilon x$$

By Fact 25.ii there is a consistent formula $\neg\psi(x)$, for some $\psi(x) \in \mathcal{F}(M)$, that implies $fa \sim_\varepsilon x$. Then, by (3), $fa \sim_\varepsilon x$ is satisfied in M . This proves our claim.

Now, we claim that (2) holds also for every $\psi(x) \in \mathcal{F}(\text{Ccl}(M))$. Let $\psi(x, z) \in \mathcal{F}(M)$ and $\psi' > \psi$ be given. Let $b \in \text{Ccl}(M)^{|z|}$. Suppose that $\exists x \psi(x, b)$ and let ε be such that $z \sim_\varepsilon b \rightarrow \exists x \psi''(x, z)$ where $\psi' > \psi'' > \psi$. By Corollary 31, we can also assume that $z \sim_\varepsilon b \wedge \psi''(x, z) \rightarrow \psi'(x, b)$. Let $b' \in M^{|z|}$ be such that $b' \sim_\varepsilon b$. By (2) there is an $a \in M^{|x|}$ such that $\psi''(a, b')$. Then $\psi'(a, b)$ follows. This proves the second claim.

By the two claims above, the inductive argument in the proof in (2 \Rightarrow 1) applies to prove that $\text{Ccl}(M)$ is a c-model.

(1'' \Rightarrow 2) By the same argument as in (1 \Rightarrow 2).

(2 \Rightarrow 1'') As M is a substructure by assumption, the inductive argument in the proof in (2 \Rightarrow 1) applies. \square

Remark 43. The theorem above shows in particular that for every substructure M the following are equivalent

1. M is a c-model;
2. $\text{Ccl}(M)$ is a c-model.

Classically, the first application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all \mathcal{L} -structures. In particular every $A \subseteq \mathcal{U}$ is contained in a standard structure of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of \mathcal{L} is excessively large because of the abundance of symbols in \mathcal{L}_1 .

We say that $\mathcal{L}^{p/c}$ is **separable** if there is a countable p/c-dense set \mathcal{F} of positive/continuous formulas.

Proposition 44. Let $\mathcal{L}^{p/c}$ be separable. Let A be a countable set. Then there is a countable p/c-model M containing A .

Proof. Let \mathcal{F} be a countable p/c-dense set of positive/continuous formulas. As in the classical proof of the Löwenheim-Skolem Theorem, we construct a countable $M \subseteq \mathcal{U}$ that contains a witness of every consistent formula $\neg\psi(x)$ for $\psi(x) \in \mathcal{F}(M)$. In continuous case we also ensure that M is a substructure. Then the proposition follows from Theorem 42. \square

Proposition 45. Assume that \mathcal{L} has at most countably many symbols of sort $H^n \rightarrow I$. Then, if I is a second countable (the topology has a countable base), then \mathcal{L}^c is separable modulo \mathcal{L} . Moreover if \mathcal{L}_H is countable also then \mathcal{L}^p is separable.

Proof. Fix a countable filter of compact neighborhoods of the diagonal of I^2 . \square

10. Positive omitting types

In this section we prove a version of the omitting types theorem.

A type $p(x)$ is isolated by $\varphi(x)$, a consistent formula, if $\varphi(x) \rightarrow p(x)$. If $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{L}^{p/c}(A)$ we say that it is **p/c-isolated** by A . We omit reference to A when this is clear from the context (e.g. when A is the set of parameters of $p(x)$).

Note that, by Theorem 42, $p(x)$ is p/c-isolated by A , then $p(x)$ is realized in every p/c-model containing A . We also have that if $p(x)$ is realized in M , a p-model, then $p(x)$ is p-isolated by M . In fact, if $b \models p(x)$ then $\neg(b \neq x)$ isolates $p(x)$ where $b \neq x$ is positive by Morleyzation.

The continuous case will be considered in the next section. In this case a weaker notion of isolation is appropriate.

Fact 46. Let \mathcal{F} be a p/c-dense set of positive/continuous formulas. Then the following are equivalent

1. $p(x)$ is p/c-isolated by A ;
2. $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{F}(A)$;
3. $p(x)$ is isolated by some $\varphi'(x)$ such that $\varphi' > \varphi$ for some consistent $\varphi(x) \in \mathcal{F}(A)$.

Proof. (1 \Rightarrow 2) By Proposition 28.ii.

(1 \Rightarrow 3) By Proposition 28.i.

(3 \Rightarrow 1) Let $\varphi' > \varphi$ be as in (3). Let $\tilde{\varphi} \perp \varphi$ be such that $\varphi(x) \rightarrow \neg\tilde{\varphi}(x) \rightarrow \varphi'(x)$. Then $\neg\tilde{\varphi}(x)$ isolates $p(x)$. \square

Lemma 47. Let $\mathcal{L}^{p/c}(A)$ be separable. Let $p(x) \subseteq \mathcal{L}^{p/c}(A)$, be non p/c-isolated by A . Then every consistent formula $\neg\psi(z)$, with $\psi(z) \in \mathcal{L}^{p/c}(A)$, has a solution a such that A, a does not p/c-isolate $p(x)$.

Proof. We construct a sequence of $\mathcal{L}^{p/c}(A)$ -formulas $\langle \gamma_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\gamma_i(z) : i < \omega\}$ is the required solution of $\neg\psi(z)$.

Let $\langle \xi_i(x, z) : i < \omega \rangle$ enumerate a countable p/c-dense subset of $\mathcal{L}_{x,z}^{p/c}(A)$. Let $\gamma_0(z)$ be a consistent positive/continuous formulas such that $\gamma'_0(z) \rightarrow \neg\psi(z)$ for some $\gamma'_0 > \gamma_0$. This exists by Proposition 28. Now we define $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$. Let $\gamma_i(z)$ and $\gamma'_i > \gamma_i$ be given. Pick $\tilde{\gamma} \perp \gamma_i$ such that $\gamma_i(z) \rightarrow \neg\tilde{\gamma}(z) \rightarrow \gamma'_i(z)$.

1. If $\neg\xi_i(x, z) \wedge \gamma_i(z)$ is inconsistent, let $\gamma_{i+1} = \gamma_i$ and $\gamma'_{i+1} = \gamma'_i$.

2. Otherwise, pick $\varphi(x) \in p$ such that (#) below is consistent

$$(\#) \quad \neg \tilde{\gamma}(z) \wedge \exists x [\neg \xi_i(x, z) \wedge \neg \varphi(x)].$$

Finally, let $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$ consistent positive/continuous formulas such that $\gamma'_{i+1}(z)$ implies (#). Such formulas exist by Proposition 28.

Let $a \models \{\gamma_i(z) : i < \omega\}$. We claim that that A, a does not p/c-isolate $p(x)$. Otherwise, by Fact 46, $\neg \xi_i(x, a) \rightarrow p(x)$ for some consistent $\neg \xi_i(x, a)$. This contradicts $a \models \gamma_{i+1}(z)$.

Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in (2).

Suppose for a contradiction that $\neg \xi_i(x, z) \wedge \gamma_i(z)$ is consistent while (#) is inconsistent for all formulas $\varphi(x) \in p$. This immediately implies that

$$\exists z [\neg \xi_i(x, z) \wedge \neg \tilde{\gamma}(z)] \rightarrow p(x).$$

By Fact 46, $p(x)$ is p/c-isolated, a contradiction. \square

From the lemma we easily obtain an omitting types theorem for positive types that is very close to the classical one.

Theorem 48 (Positive Omitting Types). Let \mathcal{L}^p be separable. Let A be countable. Assume also that $p(x) \subseteq \mathcal{L}^p(A)$ is not p-isolated. Then there is a p-model M containing A that omits $p(x)$.

Proof. As in the classical proof, apply the lemma above and the Tarski-Vaught test (Theorem 42) to obtain a countable p-model M that does not isolate $p(x)$. For p-models, p-isolating a type is equivalent to realizing it. \square

11. Continuous omitting types

We say that the type $p(x) \subseteq \mathcal{L}^c(\mathcal{U})$ is **approximately c-isolated** by A if for every $\varepsilon \in \mathcal{E}$ there is a formula $\varphi_\varepsilon(x) \in \mathcal{L}^c(A)$ such that $\neg \varphi_\varepsilon(x)$ is consistent with $p(x)$ and isolates $\exists x' \sim_\varepsilon x p(x')$.

Notice the requirement that the formula that isolates $\exists x' \sim_\varepsilon x p(x')$ is consistent with $p(x)$. This is used in the proof of Lemma 54.

Assume for the rest of this section that \mathcal{L}^c is separable. Then there is a sequence $\langle \varepsilon_n : n \in \omega \rangle$ such that for every $\varepsilon \in \mathcal{E}$ there is an n such that $x \sim_{\varepsilon_n} y \rightarrow x \sim_\varepsilon y$. We abbreviate $x \sim_{\varepsilon_n} y$ by $x \sim_n y$.

Fact 49. Let \mathcal{L}^c be separable. Let $p(x) \subseteq \mathcal{L}^c(A)$ be maximally consistent type that is approximately c-isolated. Let $\varphi_n(x) \in \mathcal{L}^c(A)$ be such that $\neg \varphi_n(x)$ is consistent with $p(x)$ and isolates $\exists x' \sim_n x p(x')$. We can assume that $\varphi_n(x) \rightarrow \varphi_{n+1}(x)$.

Proof. Let $\psi_n(x) = \varphi_0(x) \vee \dots \vee \varphi_n(x)$. Clearly $\neg \psi_n(x) \rightarrow x \sim_\varepsilon y$, then we only need verify that $\neg \psi_n(x)$ is consistent with $p(x)$. But this is the case because by maximality $p(x) \rightarrow \neg \varphi_n(x)$. \square

Fact 50. Let \mathcal{L}^c be separable. Let $p(x) \subseteq \mathcal{L}^c(A)$ be maximally consistent and approximately c-isolated by A . Let M be a c-model containing A . Then $p(x)$ is realized in $\text{Ccl}(M)$.

Proof. Let $a_n \in M$ be a solution of $\neg \varphi_n(x)$, where the $\varphi_n(x)$ are as in the fact above. Then the type

$$p(x) \cup \{a_n \sim_n x : n \in \omega\}$$

is consistent. Its realizations belong to $\text{Ccl}(M)$. \square

In words, the proof above shows that if $p(x)$ is approximately c-isolated, then in every c-model contains a sequence of elements that converges, in the topology induced by I , to a realization of $p(x)$.

Fact 51. Let \mathcal{L}^c be separable. If $p(x) \subseteq \mathcal{L}^c(A)$ is realized in $\text{Ccl}(M)$ for some c-model M containing A , then $p(x)$ is approximately c-isolated by M .

Proof. Let $b \in \text{Ccl}(M)^{|x|}$ realize $p(x)$. Let $\varepsilon \in \mathcal{E}$ be given. By Fact 46 it suffices to find a continuous formula $\varphi(x)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \rightarrow \exists x' \sim_\varepsilon x p(x')$.

Let $\eta' \in \mathcal{E}$ be such that $x \sim_{\eta'} y \sim_{\eta'} z \rightarrow x \sim_\varepsilon z$. Pick some $a \in M^{|x|}$ that satisfies $x \sim_{\eta'} b$. Then $x \sim_{\eta'} a \rightarrow \exists x' \sim_\varepsilon x p(x')$. Finally, note that $(\sim_{\eta'}) > (\sim_\eta)$ for some $\eta \in \mathcal{E}$. \square

Lemma 52. Let $\mathcal{L}^c(A)$ be separable. Let $p(x) \subseteq \mathcal{L}^c(A)$, be non approximately c-isolated. Then every consistent formula $\neg\psi(z)$, with $\psi(z) \in \mathcal{L}^c(A)$, has a solution a such that A, a does not c-isolate $p(x)$.

Proof. The proof is essentially that of Lemma 47. We go over it again to check that the stronger requirement of consistency does not invalidate the argument.

Let $\varepsilon \in \mathcal{E}$ witness that $p(x)$ is not approximately c-isolated by A . We construct a sequence of $\mathcal{L}^c(A)$ -formulas $\langle \gamma_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\gamma_i(z) : i < \omega\}$ is the required solution of $\neg\psi(z)$, witnessed by the same ε .

Let $\langle \xi_i(x, z) : i < \omega \rangle$ enumerate a countable c-dense subset of $\mathcal{L}_{x,z}^c(A)$. Let $\gamma_0(z)$ be a consistent continuous formulas such that $\gamma'_0(z) \rightarrow \neg\psi(z)$ for some $\gamma'_0 > \gamma_0$. This exists by Proposition 28. Now we define $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$. Let $\gamma_i(z)$ and $\gamma'_i > \gamma_i$ be given. Pick $\tilde{\gamma} \perp \gamma_i$ such that $\gamma_i(z) \rightarrow \neg\tilde{\gamma}(z) \rightarrow \gamma'_i(z)$.

1. If $\neg\xi_i(x, z) \wedge \gamma_i(z)$ is inconsistent with $p(x)$, let $\gamma_{i+1}(z) = \gamma_i(z)$ and $\gamma'_{i+1}(z) = \gamma'_i(z)$.

2. Otherwise, pick $\varphi(x) \in p$ such that (#) below is consistent

$$(\#) \quad \neg\tilde{\gamma}(z) \wedge \exists x [\neg\xi_i(x, z) \wedge \neg\exists x' \sim_\varepsilon x \varphi(x')].$$

Finally, let $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$ consistent continuous formulas such that $\gamma'_{i+1}(z)$ implies (#). Such formulas exist by Proposition 28.

Let $a \models \{\gamma_i(z) : i < \omega\}$. We claim that that A, a does not c-isolate $p(x)$, witnessed by ε . Otherwise $\neg\xi_i(x, a) \rightarrow \exists x' \sim_\varepsilon x p(x')$ for some $\neg\xi_i(x, a)$ consistent with $p(x)$. This contradicts $a \models \gamma_{i+1}(z)$.

Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in (2).

Suppose for a contradiction that $\neg\xi_i(x, z) \wedge \gamma_i(z)$ is consistent with $p(x)$ while (#) is inconsistent for all formulas $\varphi(x) \in p$. This immediately implies that

$$\exists z [\neg\xi_i(x, z) \wedge \neg\tilde{\gamma}(z)] \rightarrow \exists x' \sim_\varepsilon x p(x').$$

By Fact 46, this yields the desired contradiction. \square

Theorem 53 (Continuous Omitting Types). Let \mathcal{L}^c be separable. Let A be countable. Assume also that $p(x) \subseteq \mathcal{L}^c(A)$ is not approximately c-isolated. Then there is a c-model M containing A such that $\text{Ccl}(M)$ omits $p(x)$.

Proof. By assumption $\exists y \sim_\varepsilon x p(y)$ is not approximately c-isolated for some $\varepsilon \in \mathcal{E}$. Just as in the classical case, we apply Lemma 47 and the Tarski-Vaught test (Theorem 42) to obtain a countable c-model M that does not isolate $\exists y \sim_\varepsilon x p(y)$. By Fact 51, $\text{Ccl}(M)$ omits $p(x)$. \square

12. Continuous ω -categoricity

In the discussion of countable categoricity we assume that the underlying theory is complete. This assumption is not necessary (it is rather a consequence of categoricity), but it allows to work inside a monster model and simplify the notation.

Let M be a c-model. We say that M is **atomic** if for every finite tuple a of elements of M the type $\text{c-tp}(a)$ is approximately c-isolated.

Lemma 54. Let \mathcal{L}^c be separable. Let M and N be two atomic c-models. Let k be a finite c-elementary relation between $\text{Ccl}(M)$ and $\text{Ccl}(N)$. Then for every $c \in M$ there is $d \in \text{Ccl}(N)$ such that $k \cup \{\langle c, d \rangle\}$ is a c-elementary relation.

Proof. We can assume that k is reduced, i.e. it is an injective map $\text{Ccl}(M) \rightarrow \text{Ccl}(N)$. Let a enumerate $\text{dom} k$. Let $p(x, y) = \text{c-tp}(a, c)$, it suffices to prove that $p(x, y)$ is realized by some $d \in \text{Ccl}(N)$. Let

$$\neg \varphi_n(x, y) \rightarrow \exists x', y' \sim_n x, y p(x', y').$$

As $\neg \varphi_n(x, y)$ is consistent with $p(x, y)$ then $a, c \models \neg \varphi_n(x, y)$. Then by c-elementarity, $\neg \varphi(ka, y)$ is consistent. Let $d_n \in \text{Ccl}(M)$ be such that $ka, d_n \models \neg \varphi_n(x, y)$. Let b, d be a realization of the type $\{ka, d_n \sim_n x, y : n \in \omega\} \cup p(x, y)$. As $ka \sim b$, also ka, d realizes $p(x, y)$. As $d \in \text{Ccl}(N)$, the proof is complete \square

Fact 55. Let M and N be countable atomic c-models. Then $\text{Ccl}(M)$ and $\text{Ccl}(N)$ are c-isomorphic.

Proof. We construct by back-and-forth a sequence $f_n : \text{Ccl}(M) \rightarrow \text{Ccl}(N)$ of finite c-elementary reduced relations (i.e. injective maps) and let

$$R = \bigcup_{n \in \omega} (\sim) \circ f_n \circ (\sim).$$

We will ensure that $M \subseteq \text{dom}(R)$ and $N \subseteq \text{range}(R)$. By Fact ?? this suffices to prove the fact.

Let n be even. Let a be an enumeration of $\text{dom}(f_n)$. Let $c \in M$. Let $p(x, y) = \text{c-tp}(a, c)$. As M is atomic by Lemma 54, for some $d \in \text{Ccl}(N)$ the relation $f_n \cup \{\langle c, d \rangle\}$ is c-elementary. Let f_{n+1} be a reduced relation such that $f_{n+1} \subseteq f_n \cup \{\langle a, c \rangle\} \subseteq (\sim) \circ f_{n+1} \circ (\sim)$.

When n is odd we proceed similarly with f_n^{-1} . \square

Let \mathcal{L}^c be separable. We say that T is **c- ω -categorical** if $\text{Ccl}(M)$ and $\text{Ccl}(N)$ are c-isomorphic for any two countable c-models M, N . The following analogue of Ryll-Nardzewski's Theorem follows by the classical argument.

Theorem 56. Assume that \mathcal{L}^c is separable. Then the following are equivalent

1. T is c- ω -categorical;
2. every consistent type $p(x) \subseteq \mathcal{L}^c$, for x any finite tuple of variables, is approximately c-isolated.

Proof. (1 \Rightarrow 2) If $p(x)$ is consistent it is realized in some countable c-model M . If $p(x)$ is not approximately c-isolated, by the continuous omitting type theorem there is a c-model N such that $\text{Ccl}(N)$ omits $p(x)$. Then $\text{Ccl}(M)$ and $\text{Ccl}(N)$ are not c-isomorphic, hence T is not c- ω -categorical. (2 \Rightarrow 1) As, every c-model is atomic, c- ω -categoricity follows from Fact 55. \square

13. Example: Lebesgue spaces

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Danger after this point

14. Elimination of quantifiers

Let T be a positive theory. We say that T has (approximate, positive) elimination of quantifiers if the quantifier-free positive formulas are p-dense modulo T . Equivalently, if every complete type is equivalent to a quantifier-free positive types.

The following fact is routinely proved by back-and-forth.

Fact 57. The following are equivalent

1. T has elimination of quantifiers;
2. every finite partial embedding $k : M \rightarrow N$ between p-models of T is an approximate p-elementary map;
3. for every finite partial embedding $k : M \rightarrow N$ between ω -saturated p-models of T , and for every $b \in M$ there is a $c \in N$ such that $k \cup \{ \langle b, c \rangle \} : M \rightarrow N$ is also a partial embedding.

15. Pseudofinite randomizations

Let T be a complete first-order theory of signature L with an infinite model. Let \mathcal{U} be a model of T . Let Ω be a finite set. We denote by \mathcal{U}^Ω the set of functions $\Omega \rightarrow \mathcal{U}$. The elements of \mathcal{U}^Ω are denoted

by letters decorated with a circonflexe accent \hat{a}, \hat{b} , etc. The value of \hat{a} at $\omega \in \Omega$ is denoted by a_ω . We now define models of the form $\langle \mathcal{U}^\Omega, I \rangle$, where I is the real interval $[0, 1]$ with the usual topology.

The language \mathcal{L}_H is empty. The language \mathcal{L} contains functions of sort $H^{|\mathbf{x}|} \rightarrow I$, one for each formula $\varphi(x) \in L$. These are denoted by $\text{Pr } \varphi(\hat{x})$. Variables of sort H are decorated with a circonflexe accent. The interpretation of the function symbol $\text{Pr } \varphi(\hat{x})$ is

$$\text{Pr}(\varphi(\hat{a})) = \frac{\#\llbracket \varphi(\hat{a}) \rrbracket}{\#\Omega},$$

where by $\#$ denotes the finite cardinality, and

$$\llbracket \varphi(\hat{a}) \rrbracket = \{\omega \in \Omega : \mathcal{U} \models \varphi(a_\omega)\}.$$

The pseudofinite randomization of T is the positive theory

$$T^{\text{pfr}} = \left\{ \varphi \in \mathcal{L}^p : \langle \mathcal{U}^\Omega, I \rangle \models \varphi \text{ for all sufficiently large finite set } \Omega \right\}$$

In this section we prove that T^{pfr} is a complete theory with elimination of quantifiers. We prove the lemma that is required for the back-and-forth.

Lemma 58. Let $k : M \rightarrow N$ be a finite partial embedding between positively ω -saturated models of T^{pfr} . Then for every $\hat{b} \in M$ and there is a $\hat{c} \in N$ such that $k \cup \{\langle \hat{b}, \hat{c} \rangle\} : M \rightarrow N$ is a partial embedding.

Proof. Let \hat{a} be an enumeration of $\text{dom } k$. Let $p(\hat{x}, \hat{z})$ be set of formulas of the form $\gamma = \text{Pr } \varphi(\hat{x}, \hat{z})$, for some $\gamma \in I$, that hold in M, \hat{b}, \hat{a} . Note that $p(\hat{x}, \hat{z})$ implies the atomic type of \hat{b}, \hat{a} in M . Therefore it suffices to show that $p(\hat{x}, k\hat{a})$ is finitely consistent in N .

Pick a finite set of these formulas, say $\gamma_i = \text{Pr } \varphi_i(\hat{x}, \hat{z})$, for $i < n$. We may assume that the formulas $\varphi_i(x, z)$ define a partition of $\mathcal{U}^{|\mathbf{x}, \mathbf{z}|}$. By the saturation of N , it suffices to prove that for every $\varepsilon > 0$ there is a $\hat{c} \in N$ such that

$$\gamma_i =_\varepsilon \text{Pr } \varphi_i(\hat{c}, k\hat{a}) \quad (\alpha =_\varepsilon \beta \text{ is a shorthand for } |\alpha - \beta| \leq \varepsilon)$$

is true in N for all $i < n$.

Some preliminary work is required. For each $J \subseteq n$ define the following L -formulas

$$\xi_J(z) = \bigwedge_{i \in J} \exists x \varphi_i(x, z) \wedge \bigwedge_{i \in n \setminus J} \neg \exists x \varphi_i(x, z)$$

These formulas $\xi_J(z)$ partition $\mathcal{U}^{|\mathbf{z}|}$. For $J \subseteq n$ let α_J be such that

$$2. \quad \alpha_J = \text{Pr}(\xi_J(\hat{z}))$$

is in $p(\hat{x}, \hat{z})$. Let $\mathcal{J} = \{J \subseteq n : \alpha_J \neq 0\}$. When $J \in \mathcal{J}$, there are some $\beta_{i,J}$ such that the formulas

$$3. \quad \beta_{i,J} = \text{Pr}(\varphi_i(\hat{x}, \hat{z}) \mid \xi_J(\hat{z})).$$

are among the consequences of $p(\hat{x}, \hat{z})$. As the formulas $\varphi_i(x, z)$ define a partition, the $\beta_{i,J}$ add up to 1 for any fixed J . Note that $\beta_{i,J} = 0$ for $i \notin J$. Clearly, have that

$$4. \quad \gamma_i = \sum_{J \in \mathcal{J}} \beta_{i,J} \alpha_J.$$

It is plain that for $i \in J$ the following is a consequence of $p(\hat{x}, \hat{z})$

$$5. \quad 1 = \text{Pr}(\exists x \varphi_i(x, \hat{z}) \mid \xi_J(\hat{z}))$$

Now we prove that (2) & (5) \Rightarrow (1) holds in \mathcal{U}^Ω when Ω large enough (larger than n/ε suffices). Strictly speaking, this implication is not a positive formula, but the reader can easily verify that a suitable approximation of (2) and (5) suffices.

Let $\hat{a}' \in \mathcal{U}^\Omega$ satisfy (2) and (5) for every $i \in J \subseteq n$. We define $\hat{c}' \in \mathcal{U}^\Omega$ such that

$$\mathcal{U}^\Omega \models \beta_{i,J} =_\varepsilon \Pr\left(\varphi_i(\hat{c}', \hat{a}') \mid \xi_J(\hat{a}')\right).$$

We may define \hat{c}' separately in each event $\llbracket \xi_J(\hat{a}') \rrbracket$. Partition $\llbracket \xi_J(\hat{a}') \rrbracket$ into events $E_i \subseteq \Omega$, for $i \in J$, such that $\#E_i / \#\Omega =_\varepsilon \beta_{i,J}$. Then define c'_ω for $\omega \in E_i$ to be any witness of $\exists x \varphi_i(x, a_\omega)$. By (5), we can always find such a witness.

Finally, by (4), we deduce

$$\mathcal{U}^\Omega \models \gamma_i =_\varepsilon \Pr(\varphi_i(\hat{c}', \hat{a}')).$$

As $k\hat{a}$ satisfy (2) and (5), we conclude that some $\hat{c} \in N$ satisfy (1). \square

Now, from Fact 57 we obtain that T^{pfr} has elimination of quantifiers. Completeness follows.

Corollary 59. The theory T^{pfr} is complete and has elimination of quantifiers.

16. Stability

A partitioned formula $\varphi(x; z) \in \mathcal{L}^P$ is stable if there is no sequence $\langle a_i; b_i : i < \omega \rangle$ and no $\tilde{\varphi} \perp \varphi$ such that for every $i, j < \omega$

$$i < j \Rightarrow \varphi(a_i; b_j)$$

$$i > j \Rightarrow \tilde{\varphi}(a_i; b_j)$$

Using the terminology of [Hr], $\varphi(x; z)$ is stable if it is stably separated from all $\tilde{\varphi} \perp \varphi$.

Note that by compactness if $\varphi(x; z)$ and $\tilde{\varphi} \perp \varphi$ are stably separated then there is a maximal length m of a sequence $\langle a_i; b_i : i < m \rangle$ such that (1) and (2) above.

17. Elementary relations

Let M and N be models. We say that $R \subseteq M \times N$ is an **p-elementary relation** between M and N if for every $\varphi(x) \in \mathcal{L}$

$$M \models \varphi(a) \Rightarrow N \models \varphi(b) \quad \text{for every } a \text{ and } b \text{ such that } a R b.$$

Note that, when $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$ are tuples, $a R b$ stands for $a_i R b_i$ for every $i \in \{1, \dots, n\}$.

Fact 60. Let $R \subseteq \mathcal{U}^2$ be an p-elementary relation of cardinality $< \kappa$. Then there is a total and surjective p-elementary relation $S \subseteq \mathcal{U}^2$ containing R .

Proof. We apply the usual back-and-forth construction with a pinch of extra caution. Let a be an enumeration of the domain of R . Let $\bar{a} = \langle a_i : i < \lambda \rangle$ be an enumeration of all tuples of length $|a|$ such that $a R a_i$. As κ is inaccessible, $\lambda < \kappa$. Let $b \in U$. It suffices to prove that there is a c such that $R \cup \{\langle b, c \rangle\}$ is an \mathbb{L} -relation. Let $p(x, z) = \text{tp}(b, a)$ and let

$$q(x, \bar{z}) = \bigcup_{i < \lambda} p(x, z_i).$$

We claim that $q(x, \bar{a})$ is a finitely consistent type. A finite conjunction of formulas in $q(x, \bar{a})$ has the form $\psi(x, a_{i_1}) \wedge \dots \wedge \psi(x, a_{i_n})$. As $\psi(b, a)$ and $a_{i_1}, \dots, a_{i_n} R a, \dots, a$, we conclude that the condition $\psi(x, a_{i_1}) \wedge \dots \wedge \psi(x, a_{i_n})$ is satisfied. The existence of the required element c follows by saturation. \square

Corollary 61. Let $A \subseteq \mathcal{U}$ have cardinality $< \kappa$. Let $p(x) = \text{tp}_{\mathbb{L}}(a/A)$, where $a \in \mathcal{U}^{|x|}$ is a tuple of length $|x| < \kappa$. Then

$$p(\mathcal{U}) = \{b : bRa, R \in \text{Aut}(\mathcal{U}/A)\}$$

Let \mathcal{M} and \mathcal{N} be models. We say that $R \subseteq M \times N$ is an \mathbb{L} -(elementary) relation between \mathcal{M} and \mathcal{N} (or on \mathcal{M} if the two coincide) if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{N} \models \varphi(b) \quad \text{for every } a \text{ and } b \text{ such that } aRb.$$

Recall that, when $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$ are tuples, aRb stands for $a_i R b_i$ for every $i \in \{1, \dots, n\}$.

We define an equivalence relation $(\sim_{\mathcal{M}})$ on M as follows

$$1. \quad a \sim_{\mathcal{M}} b \Leftrightarrow \left(\mathcal{M} \models \varphi(a) \leftrightarrow \varphi(b) \text{ for every } \varphi(x) \in \mathbb{L}(M) \right),$$

where $|x| = |a| = |b| = 1$. Note that this relation would be trivial had we included in $\mathbb{L}(M)$ equality between elements of M .

The following proposition is easily proved by induction on the syntax.

Proposition 62. The following are equivalent for every $a, b \in M$.

1. $a \equiv_{\mathcal{M}} b$;
2. $\mathcal{M} \models \tau(a) = \tau(b)$ for every $\tau(x) \in \mathbb{T}(M)$, with $|x| = 1$.

Lemma 63. The relation $(\sim_{\mathcal{M}}) \subseteq M^2$ is an \mathbb{L} -relation. Moreover, it is maximal among the \mathbb{L} -relations on \mathcal{M} , i.e. no \mathbb{L} -relation properly contains $(\sim_{\mathcal{M}})$.

Proof. Assume $a \sim_{\mathcal{M}} b$, where $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$. Recall that this means that $a_i \sim_{\mathcal{M}} b_i$ for all $i \in \{1, \dots, n\}$. Let Δ denote the diagonal relation on M . Note that $a_i \sim_{\mathcal{M}} b_i$ is equivalent to saying that $\Delta \cup \{(a_i, b_i)\}$ is an \mathbb{L} -relation. As \mathbb{L} -relations are closed under composition $\Delta \cup \{(a_1, b_1), \dots, (a_n, b_n)\}$ is \mathbb{L} -elementary. It follows that for every $\varphi(x) \in \mathbb{L}$

$$2. \quad \mathcal{M} \models \varphi(a) \Leftrightarrow \mathcal{M} \models \varphi(b).$$

This proves that $(\sim_{\mathcal{M}})$ is an \mathbb{L} -relation. Finally, maximality is immediate. \square

Lemma 64. Let $R \subseteq M \times N$ be total and surjective \mathbb{L} -relation. Then there is a unique maximal \mathbb{L} -relation containing R . This maximal \mathbb{L} -relation is equal to both $(\sim_{\mathcal{M}})R$ and $R(\sim_{\mathcal{N}})$, where juxtaposition of relations stands for composition.

Proof. It is immediate to verify that $(\sim_{\mathcal{M}})R$ is an \mathbb{L} -relation containing R . Let S be any maximal \mathbb{L} -relation containing R . By maximality, $(\sim_{\mathcal{M}})S = S$. As S is a total relation $(\sim_{\mathcal{M}}) \subseteq SS^{-1}$. Therefore, by the lemma above, $(\sim_{\mathcal{M}}) = SS^{-1}$. As R is a surjective relation, $S \subseteq SS^{-1}R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}})R$. A similar argument proves that $S = R(\sim_{\mathcal{N}})$. \square

We write $\text{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, \mathbb{L} -relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/\sim_{\mathcal{M}}$.