

Standard analysis

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ABSTRACT. Let \mathcal{L} be a first-order two-sorted language. Let X be some fixed structure. A *standard* structure is an \mathcal{L} -structure of the form $\langle M, X \rangle$, where M is arbitrary. When X is a compact topological space (and \mathcal{L} meets a few additional requirements) it is possible to adapt a significant part of model theory to the restricted class of standard structures. This has been shown by Henson and Iovino for Banach spaces (see, e.g. [HI] and references cited therein) and has been generalized to a larger class of structures in [CLCL]. We prove that their approach applies to an even larger class (which includes e.g. distal G -flows).

The starting point is to prove that every standard structure has a *positive* elementary extension that is standard and realizes all positive types that are finitely consistent. A second important step is to prove that (in a sufficiently saturated structure) the negation of a positive formula is equivalent to an infinite (but small) disjunction of positive formulas. The main tool is the notion of approximation of a positive formula and of its negation that have been introduced by Henson and Iovino.

We review and elaborate on the properties of positive formulas and their approximations. In parallel, we introduce *continuous* formulas which provide a better counterpart to Henson and Iovino theory of normed spaces and the real-valued model theory of metric spaces of [BBHU]. To demonstrate this setting in action we discuss ω -categoricity and (local) stability.

1. Standard structures

In order to make the paper reasonably self contained, in the first few sections we revisit and extend some results of [CLCL]. Some theorems are rephrased and we add a few remarks and examples. Where necessary our notation and terminology slightly diverges.

Let X be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language \mathcal{L}_X contains a relation symbol for each compact subsets $C \subseteq X^n$ and a function symbol for each continuous functions $f : X^n \rightarrow X$. In particular, there is a constant for each element of X . According to the context, C and f denote either the symbols of \mathcal{L}_X or their interpretation in the structure X . Such a language \mathcal{L}_X is much larger than necessary but it is convenient because it uniquely associates a structure to a topological space. The notion of a *dense set of formulas* (Definition 7) helps to reduce the size of the language when required.

We need the following definition. Let A be a pure set. We say that $f : A \times X^m \rightarrow X$ is **equicontinuous** if for every ε there is a δ such that $(f(a, \alpha), f(a, \alpha')) \in \varepsilon$ for every $a \in A$ and every $(\alpha, \alpha') \in \delta$, where ε and δ range over the neighborhoods of the diagonal of $X^m \times X^m$. In words we say that, as a ranges over A , the functions $f(a, -)$ have the same modulus of uniform continuity.

We introduce the the notion of distality. The reader may skip this at a first reading and assume equicontinuity instead. We say that $\beta, \beta' \in X$ are f -proximal if for every δ there are $a \in A$, $\alpha, \alpha' \in X^m$ such that $(\alpha, \alpha'), (\beta, f(a, \alpha)), (\beta', f(a, \alpha')) \in \delta$. We say that f is **distal** if any two f -proximal points are equal. It is easy to see that every equicontinuous function is distal.

The most straightforward examples of structures X are the unit interval $[0, 1]$ with the usual topology, and its homeomorphic copies $\mathbb{R}^+ \cup \{0, \infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language \mathcal{L}_H which we call the language of the **home sort**.

Definition 1. Let \mathcal{L} be a two sorted language that expands both \mathcal{L}_H and \mathcal{L}_X .

A **standard structure** is a two-sorted \mathcal{L} -structure of the form $\langle M, X \rangle$, where M is any \mathcal{L}_H -structure. We write **H** and **X** to denote the two sorts of \mathcal{L} .

The language \mathcal{L} has a relation symbol $r_\varphi(x)$ for every $\varphi(x) \in \mathcal{L}_H$. All \mathcal{L} -structures are assumed to model $r_\varphi(x) \leftrightarrow \varphi(x)$. In other words, the Morleyzation of \mathcal{L}_H is assumed in the definition of \mathcal{L} -structure.

Finally and most relevantly, \mathcal{L} contains symbols for distal functions that have sort $H^n \times X^m \rightarrow X$.

Standard structures are denoted by the domain of their home sort.

Clearly, saturated \mathcal{L} -structures exist but, with the exception of trivial cases (when X is finite), these are not standard. As a remedy, below we carve out a set of formulas \mathcal{F}^p , the set of positive formulas, such that every model has an positive elementary saturated extension that is also standard.

As usual, \mathcal{L}_X , \mathcal{L}_H , and \mathcal{L} denote both first-order languages and the corresponding set of formulas. We write \mathcal{L}_x when when we restrict variables to x . Note that \mathcal{L} has two types of atomic formulas

1. $r_\varphi(x)$ for some $\varphi(x) \in \mathcal{L}_H$ (maybe more, but these suffices up to equivalence);
2. $\tau(x; \eta) \in C$, where $C \subseteq X^n$ is a compact set and $\tau(x; \eta)$ is a tuple of terms of sort $H^{|x|} \times X^{|\eta|} \rightarrow X$ (up to equivalence, equality has also this form).

Definition 2. A formula in \mathcal{L} is **positive** if it uses only the Boolean connectives \wedge, \vee ; the quantifiers \forall^H, \exists^H of sort H; and the quantifiers \forall^X, \exists^X of sort X. The set of positive formulas is denoted by \mathcal{F}^p .

A **continuous** formula is a positive formula where only atomic formulas as in (2) occur. The set of continuous formulas is denoted by \mathcal{F}^c .

We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort X. Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

Notation 3. Much of the theory below is developed in parallel in the positive and the continuous case. For the sake of conciseness, we write **p/c** as a placeholder that can be replaced for either one of **p** or **c**.

The positive formulas of Henson and Iovino correspond to those formulas in \mathcal{F}^c that do not have quantifiers of sort X. They are also not interested in functions symbols of sort $H^n \times X^m \rightarrow X$ with n and m both positive.

Allowing quantifiers of sort X comes almost for free. In Section 8 will see that the formulas in \mathcal{F}^c are approximated by formulas in \mathcal{F}^c without quantifiers of sort X. Instead, working within \mathcal{F}^p comes at a price but it is convenient as it simplifies the comparison with classical model theory.

An important fact to note about \mathcal{F}^c is that it is a language without equality of sort H. The difference between \mathcal{F}^p and \mathcal{F}^c is similar to the difference between the space of absolutely integrable functions (w.r.t. some measure μ) and the Lebesgue space $L^1(\mu)$. For the latter only equality almost everywhere is relevant. However, even when $L^1(\mu)$ is our focus of interest, it is often easier to argue about real valued functions. For a similar reason \mathcal{F}^p and \mathcal{F}^c are better studied in parallel.

The extra expressive power offered by quantifiers of sort X is convenient. For instance, the example below (from [CLCL]) should convince the reader that \mathcal{F}^c has at least the same expressive power as real valued logic, see [BBHU] and [K].

Example 4. Let $X = [0, 1]$, the unit interval. Let $\tau(x)$ be a term of sort $H^{|x|} \rightarrow X$. Then there is a positive formula that says $\sup_x \tau(x) = \alpha$. Indeed, consider the continuous formula

$$\forall x [\tau(x) \div \alpha \in \{0\}] \quad \wedge \quad \forall \varepsilon [\varepsilon \in \{0\} \vee \exists x [\alpha \div (\tau(x) + \varepsilon) \in \{0\}]]$$

which, in a more legible form, becomes

$$\forall x [\tau(x) \leq \alpha] \quad \wedge \quad \forall \varepsilon > 0 \exists x [\alpha \leq \tau(x) + \varepsilon].$$

The theory of normed spaces is one of the motivating examples for standard structures. The requirement for X to be compact seems a limitation, but it is not a real one. Essentially, functional analysis and model theory are only concerned with the unit ball of normed spaces.

To formalize the unit ball of a normed space as a standard structure (where X is the unit interval) is straightforward in theory but cumbersome in practice. Alternatively, one could view a normed space as a many-sorted structure: a sort for each ball of radius $n \in \mathbb{Z}^+$. Unfortunately, this also results in a bloated formalism.

A neater formalization is possible if one accepts that some positive elementary extensions (to be defined below) of a normed space could contain vectors of infinite norm hence not be themselves proper normed spaces. However these infinities are harmless if one restricts to balls of finite radius. In fact, inside any ball of finite radius these improper normed spaces are completely standard.

Example 5. Let $X = \mathbb{R} \cup \{\pm\infty\}$. Let \mathcal{L}_H be the language of real (or complex) vector spaces. The language \mathcal{L} contains a function symbol $\|\cdot\|$ of sort $H \rightarrow X$. Normed space are standard structures with the natural interpretation of the language. It is easy to verify that the unit ball of a standard structure is the unit ball of a normed space. Note that the same remains true if we add to the language any continuous (equivalently, bounded) operator or functional.

2. Henson-Iovino approximations

The notion of approximation is a central tool in the work of Henson and Iovino on the model theory of normed spaces, see e.g. [HI]. In [CLCL] the natural generalization to standard structures is presented. We recall the definitions and the main properties.

For C, C' compact subsets of X^n , we write $C' > C$ if C' is a neighborhood of C . For φ, φ' (free variables are hidden) positive formulas possibly with parameters we write $\varphi' > \varphi$ if φ' is obtained by replacing in φ each atomic formula of the form $t \in C$ with $t \in C'$, for some $C' > C$. If no such atomic formulas occurs in φ , then $\varphi > \varphi$. We call φ' a **weakening** of φ . Note that $>$ is a dense transitive relation and that $\varphi \rightarrow \varphi'$ in every \mathcal{L} -structure. For every type $p(x)$, we write

$$p'(x) = \{\varphi'(x) : \varphi' > \varphi \text{ for some } \varphi(x) \in p\}$$

in particular $\{\varphi(x)\}' = \{\varphi'(x) : \varphi' > \varphi\}$.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where \tilde{C} is some compact set disjoint from C . The atomic formulas in \mathcal{L}_H are replaced with their negation. Finally each connective is replaced by its dual i.e.,

$\vee, \wedge, \exists, \forall$ are replaced by $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde{\varphi}$ is a **strong negation** of φ . It is clear that $\tilde{\varphi} \rightarrow \neg\varphi$ in every \mathcal{L} -structure.

Notice that the weakening and the strong negation of a formula in \mathcal{F}^c is also in \mathcal{F}^c .

Lemma 6. For all positive formulas φ

1. for every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \rightarrow \neg\tilde{\varphi} \rightarrow \varphi'$;
2. for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \rightarrow \varphi' \rightarrow \neg\tilde{\varphi}$.

Proof. If $\varphi \in \mathcal{L}_H$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some $C' > C$. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in X \smallfrown O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact \tilde{C} disjoint from C . By the normality of X , there is $C' > C$ disjoint from \tilde{C} . Then $\varphi' = (t \in C')$ is as required. The lemma follows easily by induction. \square

Definition 7. A set of formulas $\mathcal{F} \subseteq \mathcal{L}$ is **p/c-dense** modulo T , a theory, if for every positive/continuous $\varphi(x)$ and every $\varphi' > \varphi$, there is a formula $\psi(x) \in \mathcal{F}$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi'(x)$ holds in every standard structure that models T .

Example 8. By Lemma 6, the set of negations of positive/continuous formulas is p/c-dense.

The following fact is easily proved by induction.

Fact 9. Let \mathcal{F} be a set of formulas closed under the connectives $\wedge, \vee, \forall^H, \exists^H, \forall^X, \exists^X$. Then \mathcal{F} is p/c-dense if the condition in Definition 7 is satisfied when $\varphi(x)$ atomic.

Using the fact, it is immediate to verify the following.

Example 10. Let \mathcal{C} be a prebase of closed subsets of X . The set of formulas built inductively from atomic formulas of the form $\tau(x, \eta) \in C$ for $C \in \mathcal{C}$ is c-dense.

Example 11. Let $X = [0, 1]$. The set of formulas built inductively from atomic formulas of the form $\tau = 0$ is c-dense. In fact, first note that by Example 10 we can restrict to formulas built inductively from atomic formulas of the form $\tau \in [\alpha, \beta]$. Then note that $\tau \in [\alpha, \beta]$ is equivalent to $\tau \div \beta \in \{0\} \wedge \alpha \div \tau \in \{0\}$.

In Section 8 we prove that the set of continuous formulas without quantifiers of sort X is c-dense.

3. The standard part of a structure

In this section we introduce the notion of standard structure. This allows to derive the compactness theorem for positive theories from its classical counterpart. We will be sketchy and refer the reader to [CLCL] for details (though we work with a larger \mathcal{L} , the structure of the argument is similar).

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 15 which in turn is required for the proof of the positive Compactness Theorem (Theorem 17). The reader willing to accept it without proof may skip this section.

Let $\langle N, {}^*X \rangle$ be an \mathcal{L} -structure that extends \mathcal{L} -elementarily the standard structure M . Let η be a free variable of sort X . For each $\beta \in X$, we define the type

$$m_\beta(\eta) = \{\eta \in D : D \text{ compact neighborhood of } \beta\}.$$

In nonstandard analysis, the set of the realizations of $m_\beta(\eta)$ in *X is called the monad of β .

The compactness of X ensures that for every $\alpha \in {}^*X$ there is a unique $\beta \in X$ such that ${}^*X \models m_\beta(\alpha)$. We denote by $\text{st}(\alpha)$ the unique $\beta \in X$ such that ${}^*X \models m_\beta(\alpha)$. The following fact is clear.

Fact 12. For every $\alpha \in {}^*X$ and every compact $C \subseteq X$

$${}^*X \models \alpha \in C \rightarrow \text{st}(\alpha) \in C.$$

We need also the following.

Lemma 13. For every $a \in N^n$, $\alpha \in ({}^*X)^m$ and every term t of the right sort

$$\langle N, {}^*X \rangle \models \text{st}(\tau(a, \alpha)) = \text{st}(\tau(a, \text{st}(\alpha))).$$

Proof. We prove the fact for function symbols; the general case follows by easy induction. We give two proofs the first assumes equicontinuity, the second, more general, assumes only distality. Let ε and δ range over the compact neighborhoods of the diagonal of $X^m \times X^m$. We know that for every ε there is a δ such that

$$M \models \forall x, \eta, \eta' [(\eta, \eta') \in \delta \rightarrow (f(x, \eta), f(x, \eta')) \in \varepsilon].$$

The same formula holds in $\langle N, {}^*X \rangle$. As ${}^*X \models (\alpha, \text{st}(\alpha)) \in \delta$ for every δ , we conclude that $(f(a, \alpha), f(a, \text{st}(\alpha))) \in \varepsilon$ for every ε . Therefore $f(a, \alpha)$ and $f(a, \text{st}(\alpha))$ have the same standard part.

Let $\beta = \text{st}(f(a, \alpha))$ and $\beta' = \text{st}(f(a, \alpha'))$, where $\alpha' = \text{st}(\alpha)$. The following

$$\langle N, *X \rangle \models \exists x \exists \eta \left[(\eta, \alpha') \in \delta \wedge (\beta, f(x, \eta)) \in \delta \wedge (\beta', f(x, \alpha')) \in \delta \right].$$

By elementarity there are $a \in M^n$ and $\alpha \in X^m$ the witness the existentials above. Then a, α, α' witness the proximality of β and β' . By distality $\beta = \beta'$. \square

The standard part of $\langle N, *X \rangle$ is the standard structure $\langle N, X \rangle$ that interprets the symbols f of sort $H^n \times X^m \rightarrow X$ as the functions

$$f^N(a, \alpha) = \text{st}(*f(a, \alpha)) \quad \text{for all } a \in N^n, \alpha \in X^{|\alpha|}.$$

where $*f$ is the interpretation of f in $\langle N, *X \rangle$. By the fact above, when $n = 0$ the definition agrees with the canonical interpretation of the functions in \mathcal{L}_X . Symbols in \mathcal{L}_H maintain the same interpretation.

Easy induction proves the following.

Fact 14. With the notation as above. For every $a \in N^n$ and $\alpha \in (*X)^m$

$$\tau^N(a; \text{st}(\alpha)) = \text{st}(*\tau(a; \alpha))$$

where τ^N and $*\tau$ are interpretations of τ in N , respectively $\langle N, *X \rangle$.

Now we move to positive formulas.

Lemma 15. With the notation as above. For every $\varphi(x; \eta) \in \mathcal{F}^p$, $a \in N^{|\alpha|}$, and $\alpha \in (*X)^{|\eta|}$

$$\langle N, *X \rangle \models \varphi(a; \alpha) \Rightarrow N \models \varphi(a; \text{st}(\alpha))$$

Proof. Suppose $\varphi(x; \eta)$ is \mathcal{F}^p -atomic. If $\varphi(x; \eta)$ is a formula of \mathcal{L}_H the claim is trivial. Otherwise $\varphi(x; \eta)$ has the form $\tau(x; \eta) \in C$. Assume that the tuple $\tau(x; \eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, *X \rangle \models \tau(a; \alpha) \in C$. Then $\text{st}(*\tau(a; \alpha)) \in C$ by Fact 12. Therefore $\tau^N(a; \text{st}(\alpha)) \in C$ follows from Fact 14. This proves the lemma for atomic formulas. Induction is immediate. \square

Corollary 16. Let M be a standard structure. Let $p(\eta) \subseteq \mathcal{F}^p(M)$ be a type that does not contain existential quantifiers of sort H . Then, if $p(\eta)$ is finitely consistent in M , it is realized in M .

The corollary has also a direct proof. This goes through the observation that the formulas in $p(\eta)$ define compact subsets of X .

Proof. Let $\langle *M, *X \rangle$ be an \mathcal{L} -elementary saturated superstructure of $\langle M, X \rangle$. Pick some $\alpha \in *X$ such that $\langle *M, *X \rangle \models p(\alpha)$. By Lemma 15, $*M \models p(\text{st}(\alpha))$. Now observe that the truth of formulas without the quantifier \exists^H are preserved by substructures. \square

The exclusion of existential quantifiers of sort H is necessary. For a counterexample take $M = X = [0, 1]$. Assume that \mathcal{L} contains a function symbol for the identity map $\iota : M \rightarrow X$. Let $p(\eta)$ contain the formulas $\exists x (x > 0 \wedge \iota x + \eta \in [0, 1/n])$ for all positive integers n .

4. Compactness

It is convenient to distinguish between consistency with respect to standard structures and consistency with respect to \mathcal{L} -structures. We say that a theory T is \mathcal{L} -consistent if $\langle M, *X \rangle \models T$ for some \mathcal{L} -structure $\langle M, *X \rangle$ (this is the classical notion of consistency). We say that T is **standardly consistent** if $M \models T$ for some standard structure M . By Lemma 15 these two notions coincide if T is positive. Therefore we have the following.

Theorem 17 (Positive Compactness Theorem). Let T be a positive theory. Then, if T is finitely \mathcal{L} -consistent, it is also standardly consistent.

An \mathcal{L} -structure N is **positively λ -saturated** if it realizes all types $p(x; \eta) \in \mathcal{F}^p(N)$ with fewer than λ parameters that are finitely consistent in N . When $\lambda = |N|$ we simply say **p-saturated**. The existence of p-saturated standard structures is obtained from the classical case just as for Theorem 17.

We say that M is a **p-elementary** substructure of N if $M \subseteq N$ and

$$M \models \varphi(a) \Rightarrow N \models \varphi(a)$$

for every $\varphi(x) \in \mathcal{F}^p$ and every $a \in M^{|x|}$.

Theorem 18. Every standard structure has a p-elementary extension to a p-saturated standard structure (possibly of inaccessible cardinality).

Proof. Let $\langle *M, *X \rangle$ be an \mathcal{L} -saturated \mathcal{L} -elementary extension of the standard structure $\langle M, X \rangle$. Let $\langle *M, X \rangle$ be its standard part as defined in Section 3. By Lemma 15, $\langle *M, X \rangle$ realizes all finitely consistent positive types with $< |*M|$ parameters. \square

The following proposition shows that a slight amount of saturation tames the positive formulas.

Proposition 19. Let N be a positively ω -saturated standard structure. Then

$$\{\varphi(x; \eta)\}' \leftrightarrow \varphi(x; \eta)$$

holds in N for every formula $\varphi(x; \eta) \in \mathcal{F}^p(N)$.

Proof. We prove \rightarrow , the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

$$\text{ih. } \{\varphi(x, z; \eta)\}' \rightarrow \varphi(x, z; \eta)$$

We need to prove

$$\{\exists z \varphi(x, z; \eta)\}' \rightarrow \exists z \varphi(x, z; \eta)$$

From (ih) we have

$$\exists z \{\varphi(x, z; \eta)\}' \rightarrow \exists z \varphi(x, z; \eta)$$

Therefore it suffices to prove

$$\{\exists z \varphi(x, z; \eta)\}' \rightarrow \exists z \{\varphi(x, z; \eta)\}'$$

Replace the variables $x; \eta$ with parameters, say $a; \alpha$, and assume that $N \models \exists z \varphi'(a, z; \alpha)$ for every $\varphi' > \varphi$. We need to prove the consistency of the type $\{\varphi'(a, z; \alpha) : \varphi' > \varphi\}$. By saturation, finite consistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort X we argue similarly. \square

Remark 20. By Corollary 16, when $\varphi(x; \eta)$ does not contains existential quantifiers of sort H, Proposition 19 does not require any assumption of saturation. In general, some saturation is necessary: consider the model presented after Corollary 16 and the formula $\exists x (x > 0 \wedge \iota(x) \in \{0\})$. See [Paragraph after Corollary 5.16][H1] for a counterexample in \mathcal{F}^c .

5. The monster model

We denote by \mathcal{U} some large p-saturated standard structure which we call the positive monster model. Truth is evaluated in \mathcal{U} unless otherwise is specified. We denote by T the positive theory of \mathcal{U} . The density of a set of formulas is understood modulo T . Below we say **p/c-model** for p/c-elementary substructure of \mathcal{U} . We stress that the truth of some $\varphi \in \mathcal{F}^{p/c}(M)$ in a p/c-model M implies the truth of φ (in \mathcal{U}) but not vice versa. However, all p/c-models agree on the approximated truth.

Fact 21. The following are equivalent

1. M is a p/c-model;
2. $M \models \{\varphi(a)\}' \Leftrightarrow \varphi(a)$ for every $\varphi(x) \in \mathcal{F}^{p/c}$ and every $a \in M^{|x|}$.

Proof. (1 \Rightarrow 2) Implication \Rightarrow follows from Proposition 19. For the converse, assume $\varphi(a)$. Let $\varphi' > \varphi$. We prove $M \models \varphi'(a)$. By Lemma 6 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \rightarrow \neg\tilde{\varphi} \rightarrow \varphi'$. Then $\neg\tilde{\varphi}(a)$ and therefore $M \models \neg\tilde{\varphi}(a)$. Then $M \models \varphi'(a)$.

(2 \Rightarrow 1) If $M \models \varphi(a)$ then $M \models \{\varphi(a)\}'$. Then $\varphi(a)$ follows from (2). \square

The following fact demonstrates how positive compactness applies. There are some subtle differences from the classical setting. Let $A \subseteq \mathcal{U}$ be a small set throughout this section.

Fact 22. Let $p(x) \subseteq \mathcal{F}^p(A)$ be a type. Then for every $\varphi(x) \in \mathcal{F}^p(\mathcal{U})$

- i. if $p(x) \rightarrow \neg\varphi(x)$ then for some conjunction $\psi(x)$ of formulas in $p(x)$

$$\psi(x) \rightarrow \neg\varphi(x);$$
- ii. if $p(x) \rightarrow \varphi(x)$ and $\varphi' > \varphi$ then for some conjunction $\psi(x)$ of formulas in $p(x)$

$$\psi(x) \rightarrow \varphi'(x).$$

Proof. (i) is immediate by saturation; (ii) follows from (i) by Lemma 6. \square

Fact 23. Let \mathcal{F} be a p/c-dense set of positive/continuous formulas. Then \mathcal{F}' is p/c-dense.

Proof. Let $\varphi' > \varphi$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$. By density, $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi''(x)$ for some $\psi(x) \in \mathcal{F}$. It suffices to prove that $\psi'(x) \rightarrow \varphi'(x)$ for some $\psi' > \psi$. By Proposition 19, $\psi(x) \leftrightarrow \{\psi(x)\}'$. Therefore, $\psi'(x) \rightarrow \varphi'(x)$ follows from Fact 22. \square

We write $\text{p/c-tp}(a/A)$ for the positive/continuous type of a over A , that is, the set of formulas $\{\varphi(x) \in \mathcal{F}^{p/c}(A) : \varphi(a)\}$. In general, if \mathcal{F} is any set of formulas, we write $\mathcal{F}\text{-tp}(a/A)$ for the type $\{\varphi(x) \in \mathcal{F}(A) : \varphi(a)\}$. The undecorated symbol $\text{tp}(a/A)$ denotes the \mathcal{L} -type.

Fact 24. Let $p(x) \subseteq \mathcal{F}^{p/c}(A)$. The following are equivalent

1. $p(x)$ is a subsets of $\mathcal{F}_x^{p/c}(A)$ that is maximally consistent in $\mathcal{U}^{|x|}$;
2. $p(x) = \text{p/c-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$.

Proof. (1 \Rightarrow 2) If $p(x)$ is consistent then $p(x) \subseteq \text{p/c-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$. By maximality $p(x) = \text{p/c-tp}(a/A)$.

(2 \Rightarrow 1) By Proposition 19 and Lemma 6

$$\varphi(x) \leftrightarrow \bigwedge_{\tilde{\varphi} \perp \varphi} \neg \tilde{\varphi}(x).$$

Suppose $\varphi(x) \in \mathcal{F}^{\text{p/c}}(A) \sim p$. Then $\neg \varphi(a)$. Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$. Therefore $p(x) \rightarrow \neg \varphi(x)$ follows. \square

The Fact 24 extends as follows.

Fact 25. Let $p(x) \subseteq \mathcal{F}(A)$, where \mathcal{F} is a p/c-dense set of positive/continuous formulas. Then the following are equivalent

1. $p(x)$ is a maximally standardly consistent subset of $\mathcal{F}_x(A)$;
2. $p(x) = \mathcal{F}\text{-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$;
3. $p(x) \leftrightarrow \text{p/c-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$.

Proof. (1 \Rightarrow 2) As above.

(2 \Rightarrow 3) As $q(x) \leftrightarrow q'(x)$, then $q(x) \leftrightarrow p(x) = \mathcal{F}\text{-tp}(a/A)$.

(3 \Rightarrow 1) Immediate. \square

Proposition 26. Let \mathcal{F} be a p/c-dense set of positive/continuous formulas. Then for every formula $\varphi(x) \in \mathcal{F}^{\text{p/c}}$

- i. $\neg \varphi(x) \leftrightarrow \bigvee \{ \psi(x) \in \mathcal{F} : \psi'(x) \rightarrow \neg \varphi(x) \text{ for some } \psi' > \psi \};$
- ii. $\neg \varphi(x) \leftrightarrow \bigvee \{ \neg \psi(x) : \neg \psi(x) \rightarrow \neg \varphi(x) \text{ and } \psi(x) \in \mathcal{F} \}.$

Proof. (i) Only \rightarrow requires a proof. Let $a \in \mathcal{U}^{|x|}$ be such that $\neg \varphi(a)$. Let $p(x) = \text{p/c-tp}(a)$. By Fact 24, $p(x) \rightarrow \neg \varphi(x)$. As \mathcal{F} is p/c-dense, $p(x) \leftrightarrow q(x) = \mathcal{F}\text{-tp}(a)$. Then $q'(x) \rightarrow \neg \varphi(x)$ hence, by compactness, $\psi'(x) \rightarrow \neg \varphi(x)$ for some $\psi(x) \in \mathcal{F}$.

(ii) By density

$$\varphi(x) \leftrightarrow \bigwedge \{ \psi(x) \in \mathcal{F} : \varphi(x) \rightarrow \psi(x) \}.$$

Negating both sides of the equivalence we obtain (ii). \square

6. Cauchy completions

We recall a few definitions from [CLCL]. For $\tau(x, z) = \tau_1(x, z), \dots, \tau_n(x, z)$ a tuple of terms of sort $H^{|x|+|z|} \rightarrow X$ we define the formula

$$x \sim_{\tau} y = \bigwedge_{i=1}^n \forall z \tau_i(x, z) = \tau_i(y, z),$$

where the expression $\alpha = \beta$ is shorthand for $\langle \alpha, \beta \rangle \in \Delta$, where Δ is the diagonal of X^2 .

We also define the type

$$x \sim y = \{x \sim_{\tau} y : \tau(x, z) \text{ as above}\}.$$

The following fact will be used below without mention.

Fact 27. For any $a = \langle a_i : i < \lambda \rangle$ and $b = \langle b_i : i < \lambda \rangle$

$$a \sim b \Leftrightarrow a_i \sim b_i \text{ for every } i < \lambda.$$

Proof. By induction. Only implication \Leftarrow requires a proof. For limit ordinals induction is trivial. Assume $a_i \sim b_i$ for every $i \leq \lambda$. Let $|a| = |b| = \lambda$ and assume inductively that (note that it suffices to consider tuples τ of arity 1)

$$\forall y, z \tau(a, y, z) = \tau(b, y, z)$$

holds for every term $\tau(x, y, z)$, with $|x| = \lambda$ and $|y| = 1$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_{λ}

$$1. \quad \forall z \tau(a, a_{\lambda}, z) = \tau(b, a_{\lambda}, z).$$

As $a_{\lambda} \sim b_{\lambda}$ then

$$\forall x, z \tau(x, a_{\lambda}, z) = \tau(x, b_{\lambda}, z)$$

and in particular

$$2. \quad \forall z \tau(b, a_{\lambda}, z) = \tau(b, b_{\lambda}, z).$$

From (1) and (2) we obtain

$$\forall z \tau(a, a_{\lambda}, z) = \tau(b, b_{\lambda}, z). \quad \square$$

Note that the approximations of the formula $x \sim_{\tau} y$ have the form

$$x \sim_{\tau, D} y = \bigwedge_{i=1}^n \forall z \langle \tau_i(x, z), \tau_i(y, z) \rangle \in D$$

for some compact neighborhood D of Δ . We write \mathcal{E} for the set containing the pairs τ, D as above. The formulas $x \sim_{\varepsilon} y$, as ε ranges over \mathcal{E} , form a base for a system of entougages on $\mathcal{U}^{[x]}$. We refer to this uniformity and the topology associated as the topology induced by X . Though not needed in the sequel, it is worth mentioning that the topology induced by X on $\mathcal{U}^{[x]}$ coincides with the product of the topology induced by X on \mathcal{U} . This can be verified by an argument similar to the proof of Fact 27. See [CLCL] for details.

Fact 28. For every $\varphi(x) \in \mathcal{F}^c$

$$x \sim y \rightarrow \varphi(x) \leftrightarrow \varphi(y)$$

Proof. By induction. □

We define

$$x \sim' y = \{x \sim_\varepsilon y : \varepsilon \in \mathcal{E}\}.$$

The following corollary corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Corollary 29. For every $\varphi(x) \in \mathcal{F}^c$ and every $\varphi' > \varphi$ there an $\varepsilon \in \mathcal{E}$ such that

$$\text{i. } x \sim_\varepsilon y \wedge \varphi(y) \rightarrow \varphi'(x)$$

Proof. As $x \sim' y \cup \{\varphi(x)\} \rightarrow \varphi(y)$ by Fact 28, (i) follows from Fact 22. □

We say what a type $q(x)$ is **finitely satisfiable** in A if every conjunction of formulas in $q(x)$ has a solution in $A^{|x|}$. This definition coincides with the classical one, but in our context, the notion is less robust. For instance, if M is a p-model and $q(x) = \text{p-tp}(a/M)$ then $q'(x)$ is always finitely satisfiable while $q(x)$ need not.

A set $A \subseteq \mathcal{U}$ is **Cauchy complete** if it contains all those $a \in \mathcal{U}$ such that $a \sim' x$ is finitely satisfied in A . Note that Cauchy complete sets are in particular closed under the equivalence (\sim). The **Cauchy completion** of A is the set

$$\text{Ccl}(A) = \{a : a \sim' x \text{ is finitely satisfied in } A\}.$$

Fact 30. $\text{Ccl}(A)$ is Cauchy complete.

Proof. Suppose that $x \sim' a$ is finitely satisfied in $\text{Ccl}(A)$. Let $\varepsilon \in \mathcal{E}$ be given. We prove that $a \sim_\varepsilon x$ is satisfied in A . Let $\eta \in \mathcal{E}$ be such that $x \sim_\eta y \sim_\eta z \rightarrow x \sim_\varepsilon z$. There is some $b \in \text{Ccl}(A)$ such that $b \sim_\eta a$. There is some $c \in A$ such that $c \sim_\eta b$. Then c satisfies $x \sim_\varepsilon a$, as required. □

We say that $p(x) \subseteq \mathcal{F}^c(\mathcal{U})$ is a **Cauchy type** if it is consistent and $p(x) \wedge p(y) \rightarrow x \sim y$.

Fact 31. Let M be a p/c-model and let $p(x) \subseteq \mathcal{F}^{p/c}(M)$ be a Cauchy type. Then all realizations of $p(x)$ belong to $\text{Ccl}(M)$.

Proof. If $p(x)$ is a Cauchy type, then $p(x) \rightarrow a \sim x$ for some $a \models p(x)$. As M is a c-model, $p'(x)$ is finitely satisfied in M . Then also $a \sim' x$ is finitely satisfied. Hence $a \in \text{Ccl}(M)$. □

7. Morphisms

The notion of p-elementarity is discussed in detail in [CLCL]. In this section we are mostly concerned with the continuous case. This is delicate due to the lack of equality of sort H . A possible approach is to work in the quotient \mathcal{U}/\sim . A second possibility is to replace functions with relations. The two options differs only in the notation. Here we go for the second.

For semplicity, we only discuss morphisms between saturated structures. We fix, besides \mathcal{U} , a second saturated structure \mathcal{V} .

Let $R \subseteq \mathcal{U} \times \mathcal{V}$ be a binary relation. If $a = \langle a_i : i < \lambda \rangle$ and $b = \langle b_i : i < \lambda \rangle$ we write $\langle a, b \rangle \in R$ to abbreviate: $\langle a_i, b_i \rangle \in R$ for all $i < \lambda$. Let \mathcal{F} be a set of formulas. We say that R preserves the truth of \mathcal{F} -formulas if

$$\# \quad \mathcal{U} \models \varphi(a) \Rightarrow \mathcal{V} \models \varphi(b) \quad \text{for every } \varphi(x) \in \mathcal{F} \text{ and every } \langle a, b \rangle \in R.$$

We will consider two extreem cases for \mathcal{F} : the set of all positive/continuous formulas and the set of atomic positive/continuous formulas. We denote the latter by $\mathcal{F}_{\text{at}}^{\text{p/c}}$. Relations that preserve the truth of $\mathcal{F}^{\text{p/c}}$ -formulas are called **p/c-elementary**. Note that p-elementary relations are (graphs of) injective functions and some of the facts below become trivial.

By the following fact, if R preserves the truth of \mathcal{F} -formulas so does R^{-1} .

Fact 32. Let \mathcal{F} be either $\mathcal{F}^{\text{p/c}}$ or $\mathcal{F}_{\text{at}}^{\text{p/c}}$. Let R be a relation that preserves the truth of \mathcal{F} -formulas. Then the converse implication in (#) holds.

Proof. $\mathcal{F} = \mathcal{F}^{\text{p/c}}$. Assume $\mathcal{U} \models \neg\varphi(a)$ and apply Proposition 26 to infer that $\mathcal{U} \models \psi(a)$ for some $\psi(x) \in \mathcal{F}^c$ that implies $\neg\varphi(x)$. Then $\mathcal{V} \models \neg\varphi(b)$ follows.

$\mathcal{F} = \mathcal{F}_{\text{at}}^{\text{p/c}}$. Assume $\mathcal{U} \models \tau(a) \notin C$ then $\mathcal{U} \models \tau(a) \in \tilde{C}$ for some compact \tilde{C} disjoint of C . Then $\mathcal{V} \models \tau(b) \notin C$ follows. \square

We denote by (\sim) the equivalence relation defined as in Section 6; we use the same symbol for the relation in \mathcal{U} and in \mathcal{V} . The context will disambiguate. We say that the relation R is **reduced** if there is no $R' \subset R$ such that $R \subseteq (\sim) \circ R' \circ (\sim)$.

Fact 33. For every R there is a reduced relation $R' \subseteq R$ such that

$$(\sim) \circ R \circ (\sim) = (\sim) \circ R' \circ (\sim).$$

Proof. Let $\langle R_i : i < \lambda \rangle$ be a decreasing chain of maximal length of relations such that $R_i \subseteq R \subseteq (\sim) \circ R_i \circ (\sim)$. Let R' the intersection of the chain. By maximality, R' is a reduced relation. Finally, $(\sim) \circ R \circ (\sim) = (\sim) \circ R' \circ (\sim)$ is clear. \square

Fact 34. Let $\mathcal{F} = \mathcal{F}^c$. Let R be a reduced relation that preserves of the truth of \mathcal{F} -formulas. Then R is (the graph of) an injective map $f : \text{dom}(R) \rightarrow \text{range}(R)$.

The same holds when $\mathcal{F} = \mathcal{F}_{\text{at}}^c$ if $\text{dom}(R)$ and $\text{range}(R)$ are c-models.

Proof. $\mathcal{F} = \mathcal{F}^c$. Let $a \in \text{dom}(R)$ and assume that $\langle a, b \rangle, \langle a, b' \rangle \in R$. To prove functionality suppose for a contradiction that $b \neq b'$. As R is reduced, $b \sim b'$. Then $b \sim_\varepsilon b'$ for some $\varepsilon \in \mathcal{E}$. By Proposition 26 b, b' satisfies a continuous formula $\psi(x, x')$ that implies $x \sim_\varepsilon x'$. Then a, a also satisfies $\psi(x, x')$, a contradiction. This proves functionality, for injectivity apply the same argument to R^{-1} .

$\mathcal{F} = \mathcal{F}_{\text{at}}^c$. Reason as above to obtain $b \sim_\varepsilon b'$, say $\varepsilon = \langle \tau, D \rangle$. Then the formula

$$\neg \varphi(b, b', z) = \neg \bigwedge_{i=1}^n \langle \tau_i(b, z), \tau_i(b', z) \rangle \in D.$$

is consistent. By the Tarski-Vaught test (Theorem 41) $\neg \varphi(b, b', z)$ has a solution, say c , in $\text{range}(R)^{|z|}$. Then $\neg \varphi(a, a, d)$ for some $d R c$ which is a contradiction. \square

Let $f \subseteq R$ be a reduced c-elementary relation such that $R \subseteq (\sim) \circ f \circ (\sim)$. Then f is a function defined in exactly one representative a of each (\sim) -equivalence class $[a]$ that intersects $\text{dom}(R)$.

Fact 35. Let \mathcal{F} be either \mathcal{F}^c or $\mathcal{F}_{\text{at}}^c$. The following are equivalent

1. R preserves the truth of \mathcal{F} -formulas;
2. $\text{Ccl}(R)$ preserves the truth of \mathcal{F} -formulas.

Proof. We prove the nontrivial implication, $1 \Rightarrow 2$. Let $\varphi(x) \in \mathcal{F}$ and $\varphi'' > \varphi' > \varphi$ be given. Let $\langle a, b \rangle \in \text{Ccl}(R)$ and assume $\varphi(a)$. Let $\varepsilon \in \mathcal{E}$ be such that

1. $x \sim_\varepsilon y \wedge \varphi(x) \rightarrow \varphi'(y)$
2. $x \sim_\varepsilon y \wedge \varphi'(x) \rightarrow \varphi''(y)$

By the definition of Cauchy completion, there are $a', b' \sim_\varepsilon a, b$ such that $\langle a', b' \rangle \in R$. Then from (1) we infer that $\varphi'(a')$. Note that $\varphi'(x) \in \mathcal{F}$, then $\varphi'(b')$. Then $\varphi''(b)$ follows from (2). As $\varphi'' > \varphi$ is arbitrary, the fact follows from Proposition 19. \square

Definition 36. Let M and N be p/c-elementary substructures of \mathcal{U} , respectively \mathcal{V} . An **p/c-isomorphism** between M and N is a relation R that preserves the truth of $\mathcal{F}_{\text{at}}^{\text{p/c}}$ -formulas and $M = \text{dom}(R)$, $N = \text{range}(R)$.

It is clear that p-isomorphisms coincide with \mathcal{L} -isomorphisms. Indeed, in this case R is a bijective function that preserve the truth of atomic and, by Fact 32, negated atomic

formulas. On the other hand, in the continuous case the term isomorphism may sound inappropriate. It is justified by the following fact.

Fact 37. Let R be a c-isomorphism between M and N . Let $f \subseteq \text{Ccl}(R)$ be a reduced relation such that $\text{Ccl}(R) = (\sim) \circ f \circ (\sim)$. Then $f : \text{Ccl}(M) \rightarrow \text{Ccl}(N)$ is an injective map such that

$$\mathcal{U} \models \varphi(a) \Leftrightarrow \mathcal{V} \models \varphi(fa) \quad \text{for all } \varphi(x) \in \mathcal{F}^c \text{ and } a \in \text{dom}(f)^{|x|}.$$

Proof. From Fact 34 we obtain that $f : \text{Ccl}(M) \rightarrow \text{Ccl}(N)$ is an injective map. To prove $\varphi(a) \leftrightarrow \varphi(fa)$ we reason by induction as in the classical case. \square

Note that, though $f : \text{Ccl}(M) \rightarrow \text{Ccl}(N)$ above is neither total nor surjective, $\text{dom}(f)$ and $\text{range}(f)$ intersect every (\sim) -equivalence class.

8. Elimination of quantifiers of sort X

We write $\mathcal{F}_{\text{Xqf}}^{\text{p/c}}$ for the set of positive/continuous formulas without quantifiers of sort X . In [CLCL] it is proved that $\mathcal{F}_{\text{Xqf}}^{\text{p}}$ is a p-dense set. Here we adapt the argument so that it applies also to the continuous case.

Proposition 38. Every relation of small cardinality R that preserves the truth of formulas in $\mathcal{F}_{\text{Xqf}}^{\text{p/c}}$ extends to p/c-automorphisms of \mathcal{U} .

Proof. We prove the proposition in the continuous case. A simplified version of the same argument applies to the positive case.

We construct by back-and-forth a sequence $f_i : \mathcal{U} \rightarrow \mathcal{U}$ of reduced relations (i.e. injective maps) that preserve the truth formulas in $\mathcal{F}_{\text{Xqf}}^c$. Finally we set

$$R = \bigcup_{i \in |\mathcal{U}|} (\sim) \circ f_i \circ (\sim).$$

We will ensure that $\mathcal{U} = \text{dom}(R) = \text{range}(R)$, hence that R is a c-automorphism of \mathcal{U} .

Let i be even. Let a be an enumeration of $\text{dom}(f_n)$. Let $b \in \mathcal{U}$. Let $p(x, y) = \text{c-tp}(a, b)$. By c-elementarity the type $p(f_i a, y)$ is finitely satisfied. Let $c \in \mathcal{U}$ realize $p(f_i a, y)$. Let f_{i+1} be a reduced relation such that $f_{i+1} \subseteq f_i \cup \{\langle b, c \rangle\} \subseteq (\sim) \circ f_{i+1} \circ (\sim)$.

When i is odd we proceed similarly with f_i^{-1} for f_i . Limit stages are obvious. \square

Corollary 39. Let $p(x)$ and $q(x)$ be as in (i) or (ii) below. Then $p(x) \leftrightarrow q(x)$.

- i. $p(x) = \mathcal{F}_{\text{Xqf}}^p\text{-tp}(a)$ and $q(x) = \text{tp}(a)$.
- ii. $p(x) = \mathcal{F}_{\text{Xqf}}^c\text{-tp}(a)$ and $q(x) = \text{c-tp}(a)$.

Proof. (i) By Proposition 38 as p-isomorphisms are \mathcal{L} -isomorphisms.

(ii) Only \rightarrow requires a proof. If a contains entries $a_i \sim a_j$, replace a_j with a_i . Note that the tuple a' obtained in this manner has the same c-type of a .

Let $b \models p(x)$ and let b' obtained with the same procedure as a' .

Then the function that maps $a' \mapsto b'$ preserves the truth of $\mathcal{F}_{\text{Xqf}}^c$ -formulas. Then f extends to an c-automorphism. As every c-automorphism is c-elementary and the corollary follows. \square

Proposition 40. The set $\mathcal{F}_{\text{Xqf}}^{p/c}$ is p/c-dense modulo T .

Proof. Let $\varphi(x)$ be a positive formula. We need to prove that for every $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathcal{F}_{\text{Xqf}}^{p/c}$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi'(x)$. By Corollary 39 and Proposition 19

$$\neg\varphi(x) \rightarrow \bigvee_{p'(x) \rightarrow \neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over the maximally consistent $\mathcal{F}_{\text{Xqf}}^{p/c}$ -types. By Fact 22 and Lemma 6

$$\neg\varphi(x) \rightarrow \bigvee_{\neg\tilde{\psi}(x) \rightarrow \neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{F}_{\text{Xqf}}^{p/c}$. Equivalently,

$$\varphi(x) \leftarrow \bigwedge_{\tilde{\psi}(x) \leftarrow \varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 22, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{F}_{\text{Xqf}}^p$ such that

$$\varphi'(x) \leftarrow \bigwedge_{i=1, \dots, n} \tilde{\psi}_i(x) \leftarrow \varphi(x)$$

which yields the interpolant required by the proposition. \square

9. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is our version of the Tarski-Vaught test.

Theorem 41. Let M be a subset of \mathcal{U} . Let \mathcal{F} be a p-dense set of positive formulas. Then the following are equivalent

1. M is a p-model;
2. for every formula $\psi(x) \in \mathcal{F}(M)$

$$\exists x \psi(x) \Rightarrow \text{for every } \psi' > \psi \text{ there is an } a \in M \text{ such that } \psi'(a);$$
3. for every formula $\psi(x) \in \mathcal{F}(M)$

$$\exists x \neg \psi(x) \Rightarrow \text{there is an } a \in M \text{ such that } \neg \psi(a).$$

If \mathcal{F} is a c-dense set of continuous formulas then (2) and (3) are equivalent to

- 1'. $\text{Ccl}(M)$ is a c-model.

Moreover, if M is a substructure, then (1'), (2) and (3) are equivalent to

- 1''. M is a c-model.

Proof. (1 \Rightarrow 2) Assume $\exists x \psi(x)$ and let $\psi' > \psi$ be given. By Lemma 6 there is some $\tilde{\psi} \perp \psi$ such that $\psi(x) \rightarrow \neg \tilde{\psi}(x) \rightarrow \psi'(x)$. Then $\neg \forall x \tilde{\psi}(x)$ hence, by (1), $M \models \neg \forall x \tilde{\psi}(x)$. Then $M \models \neg \tilde{\psi}(a)$ for some $a \in M$. Hence $M \models \psi'(a)$ and $\psi'(a)$ follows from (1).

(2 \Rightarrow 3) Assume (2) and let $\psi(x) \in \mathcal{F}(M)$ be such that $\exists x \neg \psi(x)$. By Proposition 26.i, there are a consistent $\varphi(x) \in \mathcal{F}(M)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \rightarrow \neg \psi(x)$. Then (3) follows.

(3 \Rightarrow 2) Let $\psi' > \psi$ for some $\psi(x) \in \mathcal{F}(M)$. Let $\tilde{\psi} \perp \psi$ such that $\psi(x) \rightarrow \neg \tilde{\psi}(x) \rightarrow \psi'(x)$. By Proposition 26.ii, $\neg \varphi(x) \rightarrow \neg \tilde{\psi}(x)$ for some $\varphi(x) \in \mathcal{F}(M)$ such that $\neg \varphi(x)$ is consistent. Then (2) follows from (3).

(2 \Rightarrow 1) Assume (2). By the classical Tarski-Vaught test $M \leq_{\text{H}} \mathcal{U}$. Then M is the domain of a substructure of \mathcal{U} . Then $M \models \varphi(a) \Rightarrow \varphi(a)$ holds for every atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \models \varphi(a, b) \Rightarrow \varphi(a, b).$$

Using (2) and the induction hypothesis we prove that

$$M \models \exists y \varphi(a, y) \Rightarrow \forall y \varphi(a, y).$$

Indeed, for any $\varphi' > \varphi$,

$$\begin{aligned} M \models \exists y \varphi(a, y) &\Rightarrow M \models \exists y \psi(a, y) && \text{for some } \psi \in \mathcal{F} \text{ such that } \varphi \rightarrow \psi \rightarrow \varphi' \\ &\Rightarrow M \models \psi(a, b) && \text{for some } b \in M \text{ by (2)} \\ &\Rightarrow M \models \varphi'(a, b) \\ &\Rightarrow \varphi'(a, b) && \text{by induction hypothesis} \end{aligned}$$

$$\Rightarrow \exists y \varphi'(a, y).$$

As $\varphi' > \varphi$ is arbitrary, $\exists y \varphi(a, y)$ follows from Proposition 19.

Induction for the connectives \vee , \wedge , \forall^H , \exists^X , and \forall^X is straightforward.

(1' \Rightarrow 2) Let $\psi' > \psi'' > \psi$. Reasoning as in the proof of (1 \Rightarrow 2) we obtain that $\text{Ccl}(M) \models \psi''(a)$ for some $a \in \text{Ccl}(M)$. By Corollary 29, $a \sim_\varepsilon x \rightarrow \psi'(x)$ for some ε . As $a \sim' x$ is finitely satisfied in M , it follows that $\psi'(c)$ for some $c \in M$.

(2 \Leftrightarrow 3) The proof above applies verbatim when \mathcal{F} is a c-dense set of continuous formulas.

(2 \Rightarrow 1') Assume (2). We claim that $\text{Ccl}(M)$ is a substructure of \mathcal{U} . Let $a \in M^n$ and let f be a function symbol of sort $H^n \rightarrow H$. We prove that $fa \in M$. We show that $fa \sim' x$ is finitely satisfied in M . Consider the formula $fa \sim_\varepsilon x$ where ε is the pair τ, D . By Lemma 6, there is a formula in $\tilde{\varphi}(x) \in \mathcal{F}^p(M)$ such that

$$fa \sim_\tau x \rightarrow \neg \tilde{\varphi}(x) \rightarrow fa \sim_\varepsilon x$$

By Fact 23.ii there is a consistent formula $\neg\psi(x)$, for some $\psi(x) \in \mathcal{F}(M)$, that implies $fa \sim_\varepsilon x$. Then, by (3), $fa \sim_\varepsilon x$ is satisfied in M . This proves our claim.

Now, we claim that (2) holds also for every $\psi(x) \in \mathcal{F}(\text{Ccl}(M))$. Let $\psi(x, z) \in \mathcal{F}(M)$ and $\psi' > \psi$ be given. Let $b \in \text{Ccl}(M)^{|z|}$. Suppose that $\exists x \psi(x, b)$ and let ε be such that $z \sim_\varepsilon b \rightarrow \exists x \psi''(x, z)$ where $\psi' > \psi'' > \psi$. By Corollary 29, we can also assume that $z \sim_\varepsilon b \wedge \psi''(x, z) \rightarrow \psi'(x, b)$. Let $b' \in M^{|z|}$ be such that $b' \sim_\varepsilon b$. By (2) there is an $a \in M^{|x|}$ such that $\psi''(a, b')$. Then $\psi'(a, b)$ follows. This proves the second claim.

By the two claims above, the inductive argument in the proof in (2 \Rightarrow 1) applies to prove that $\text{Ccl}(M)$ is a c-model.

(1'' \Rightarrow 2) By the same argument as in (1 \Rightarrow 2).

(2 \Rightarrow 1'') As M is a substructure by assumption, the inductive argument in the proof in (2 \Rightarrow 1) applies. \square

Remark 42. Theorem 41 shows in particular that for every substructure M the following are equivalent

1. M is a c-model;
2. $\text{Ccl}(M)$ is a c-model.

Classically, the first application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that by the classical downward Löwenheim-Skolem Theorem every $A \subseteq \mathcal{U}$ is contained in a standard structure of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of \mathcal{L} is excessively large because of the abundance of symbols in \mathcal{L}_X .

We say that $\mathcal{F}^{p/c}$ is **separable** if there is a countable p/c-dense set \mathcal{F} of positive/continuous formulas.

Proposition 43. Let $\mathcal{F}^{p/c}$ be separable. Let A be a countable set. Then there is a countable p/c-model M containing A .

Proof. Let \mathcal{F} be a countable p/c-dense set of positive/continuous formulas. As in the classical proof of the Löwenheim-Skolem Theorem, we construct a countable $M \subseteq \mathcal{U}$ that contains a witness of every consistent formula $\neg\psi(x)$ for $\psi(x) \in \mathcal{F}(M)$. In continuous case we also ensure that M is a substructure. Then the proposition follows from Theorem 41. \square

Proposition 44. If \mathcal{L} has at most countably many symbols of sort $H^n \rightarrow X$ and X is a second countable (the topology has a countable base), then \mathcal{F}^c is separable. Moreover if \mathcal{L}_H is countable then also \mathcal{F}^p is separable.

Proof. Fix a countable filter of compact neighborhoods of the diagonal of X^2 \square

10. Continuous omitting types

In this section we present a version of the omitting types theorem. We are interested in obtaining Cauchy complete models omitting a given type. We only consider the continuous case.

A type $p(x)$ is isolated by $\varphi(x)$, a consistent formula, if $\varphi(x) \rightarrow p(x)$. If $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{F}^{p/c}(A)$ we say that it is **p/c-isolated** by A . We omit the reference to A when this is clear from the context (e.g., when $p(x)$ is presented as a type over A). By Theorem 41, $p(x)$ is p/c-isolated by A , then $p(x)$ is realized in every p/c-model containing A .

Fact 45. Let \mathcal{F} be a p/c-dense set of positive/continuous formulas. Then the following are equivalent

1. $p(x)$ is p/c-isolated by A ;
2. $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{F}(A)$;
3. $p(x)$ is isolated by some $\varphi'(x)$ such that $\varphi' > \varphi$ for some consistent $\varphi(x) \in \mathcal{F}(A)$.

Proof. (1 \Rightarrow 2) By Proposition 26.ii.

(1 \Rightarrow 3) By Proposition 26.i.

(3 \Rightarrow 1) Let $\varphi' > \varphi$ be as in (3). Let $\tilde{\varphi} \perp \varphi$ be such that $\varphi(x) \rightarrow \neg\tilde{\varphi}(x) \rightarrow \varphi'(x)$. Then $\neg\tilde{\varphi}(x)$ isolates $p(x)$. \square

The following fact is stated without proof for the sake of comparison. It is a form of positive omitting type theorem not very distant from the classical one. Its countinuous analogue fails: consider the type $a \sim x$ which is realized in every model containing a but it in general not c-isolated.

Fact 46. Let \mathcal{F}^p be separable. Let $A \subseteq \mathcal{U}$ be a countable. Let $p(x) \subseteq \mathcal{F}^p(A)$. Then the following are equivalent:

1. $p(x)$ is p-isolated;
2. $p(x)$ is realized in every p-model containing A .

We need a weaker notion of isolation. This will also allow to extend the omitting types theorem to Cauchy complete models.

We say that the type $p(x) \subseteq \mathcal{F}^c(\mathcal{U})$ is **approximately c-isolated** by A if for every $\varepsilon \in \mathcal{E}$ there is a formula $\varphi_\varepsilon(x) \in \mathcal{F}^c(A)$ such that $\neg\varphi_\varepsilon(x)$ is consistent with $p(x)$ and isolates $\exists x' \sim_\varepsilon x \ p(x')$. Notice the requirement of consistency with $p(x)$. This is used in the proof of Lemma 52.

Assume for the rest of this section that \mathcal{F}^c is separable. Let $\langle \varepsilon_n : n \in \omega \rangle$ be such that for every $\varepsilon \in \mathcal{E}$ there is an n such that $x \sim_{\varepsilon_n} y \rightarrow x \sim_\varepsilon y$. We abbreviate $x \sim_{\varepsilon_n} y$ by $x \sim_n y$.

Fact 47. Let \mathcal{F}^c be separable. Let $p(x) \subseteq \mathcal{F}^c(A)$ be maximally consistent type that is approximately c-isolated. Then this can be witnessed by formulas $\psi_n(x) \in \mathcal{F}^c(A)$ such that $\psi_n(x) \rightarrow \psi_{n+1}(x)$.

Finally, if $a_n \models \neg\psi_n(x)$, the type $p(x) \cup \{a_n \sim_n x : n \in \omega\}$ is consistent.

Proof. Let $\psi_n(x) = \varphi_0(x) \vee \dots \vee \varphi_n(x)$. Clearly $\neg\psi_n(x) \rightarrow x \sim_n y$, then we only need verify that $\neg\psi_n(x)$ is consistent with $p(x)$. But this is the case because by maximality $p(x) \rightarrow \neg\varphi_n(x)$. The second claim follows immediately. \square

Fact 48. Let \mathcal{F}^c be separable. Let $p(x) \subseteq \mathcal{F}^c(A)$ be maximally consistent and approximately c-isolated by A . Let M be a c-model containing A . Then $p(x)$ is realized in $\text{Ccl}(M)$.

Proof. Let $a_n \in M$ be as in Fact 47. Then any solution of $p(x) \cup \{a_n \sim_n x : n \in \omega\}$ belongs to $\text{Ccl}(M)$. \square

In words, the proof above shows that if $p(x)$ is approximately c-isolated, then in every c-model contains a sequence of elements that converges, in the topology induced by X , to a realization of $p(x)$.

Fact 49. Let \mathcal{F}^c be separable. Let M be a c-model containing A . Then every type $p(x) \subseteq \mathcal{F}^c(A)$ that is realized in $\text{Ccl}(M)$ is approximately c-isolated by M .

Proof. Let $b \in \text{Ccl}(M)^{|x|}$ realize $p(x)$. Let $\varepsilon \in \mathcal{E}$ be given. By Fact 45 it suffices to find a formula $\varphi(x)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \rightarrow \exists x' \sim_\varepsilon x p(x')$.

Let $\eta' \in \mathcal{E}$ be such that $x \sim_{\eta'} y \sim_{\eta'} z \rightarrow x \sim_\varepsilon z$. Pick some $a \in M^{|x|}$ that satisfies $x \sim_{\eta'} b$. Then $x \sim_{\eta'} a \rightarrow \exists x' \sim_\varepsilon x p(x')$. Finally, note that $(\sim_{\eta'}) > (\sim_\varepsilon)$ for some $\eta \in \mathcal{E}$. \square

Lemma 50. Let $\mathcal{F}^c(A)$ be separable. Assume that $p(x) \subseteq \mathcal{F}^c(A)$ is not approximately c-isolated. Then every consistent formula $\neg\psi(z)$, with $\psi(z) \in \mathcal{F}^c(A)$, has a solution a such that A, a does not approximately c-isolate $p(x)$.

Proof. Let $\varepsilon \in \mathcal{E}$ witness that $p(x)$ is not approximately c-isolated by A . We construct a sequence of $\mathcal{F}^c(A)$ -formulas $\langle \gamma_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\gamma_i(z) : i < \omega\}$ is the required solution of $\neg\psi(z)$, witnessed by the same ε .

Let $\langle \xi_i(x, z) : i < \omega \rangle$ enumerate a countable c-dense subset of $\mathcal{F}_{x,z}^c(A)$. Let $\gamma_0(z)$ be a consistent continuous formulas such that $\gamma'_0(z) \rightarrow \neg\psi(z)$ for some $\gamma'_0 > \gamma_0$. This exists by Proposition 26. Now we define $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$. Let $\gamma_i(z)$ and $\gamma'_i > \gamma_i$ be given. Pick $\tilde{\gamma} \perp \gamma_i$ such that $\gamma_i(z) \rightarrow \neg\tilde{\gamma}(z) \rightarrow \gamma'_i(z)$.

1. If $\neg\xi_i(x, z) \wedge \gamma_i(z)$ is inconsistent with $p(x)$, let $\gamma_{i+1}(z) = \gamma_i(z)$ and $\gamma'_{i+1}(z) = \gamma'_i(z)$.
2. Otherwise, pick $\varphi(x) \in p$ such that (#) below is consistent

$$(\#) \quad \neg\tilde{\gamma}(z) \wedge \exists x [\neg\xi_i(x, z) \wedge \neg\exists x' \sim_\varepsilon x \varphi(x')].$$

Finally, let $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$ consistent continuous formulas such that $\gamma'_{i+1}(z)$ implies (#). Such formulas exist by Proposition 26.

Let $a \models \{\gamma_i(z) : i < \omega\}$. We claim that that A, a does not c-isolate $p(x)$, witnessed by ε . Otherwise $\neg \xi_i(x, a) \rightarrow \exists x' \sim_\varepsilon x p(x')$ for some $\neg \xi_i(x, a)$ consistent with $p(x)$. This contradicts $a \models \gamma_{i+1}(z)$.

Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in (2).

Suppose for a contradiction that $\neg \xi_i(x, z) \wedge \gamma_i(z)$ is consistent with $p(x)$ while (#) is inconsistent for all formulas $\varphi(x) \in p$. This immediately implies that

$$\exists z [\neg \xi_i(x, z) \wedge \neg \tilde{\gamma}(z)] \rightarrow \exists x' \sim_\varepsilon x p(x').$$

By Fact 45, this yields the desired contradiction. \square

Theorem 51 (Continuous Omitting Types). Let \mathcal{F}^c be separable. Let A be countable. Assume also that $p(x) \subseteq \mathcal{F}^c(A)$ is not approximately c-isolated. Then there is a c-model M containing A such that $\text{Ccl}(M)$ omits $p(x)$.

Proof. By assumption $\exists y \sim_\varepsilon x p(y)$ is not approximately c-isolated for some $\varepsilon \in \mathcal{E}$. Just as in the classical case, we apply Lemma 50 and the Tarski-Vaught test (Theorem 41) to obtain a countable c-model M that does not isolate $\exists y \sim_\varepsilon x p(y)$. By Fact 49, $\text{Ccl}(M)$ omits $p(x)$. \square

11. Continuous countable categoricity

Let \mathcal{F}^c be separable. We say that T is **c- ω -categorical** if any two countable c-models are c-isomorphic, see Definition 36. It is easy to see that c- ω -categorical theories are complete. This allows us to work inside a monster model and simplify the notation.

Let M be a c-model. We say that M is **atomic** if for every finite tuple a of elements of M the type $\text{c-tp}(a)$ is approximately c-isolated.

Lemma 52. Let \mathcal{F}^c be separable. Let M and N be two atomic c-models. Let k be a finite c-elementary relation between $\text{Ccl}(M)$ and $\text{Ccl}(N)$. Then for every $c \in M$ there is $d \in \text{Ccl}(N)$ such that $k \cup \{ \langle c, d \rangle \}$ is a c-elementary relation.

Proof. We can assume that k is reduced, i.e. it is an injective map $\text{Ccl}(M) \rightarrow \text{Ccl}(N)$. Let a enumerate $\text{dom}(k)$. Let $p(x, y) = \text{c-tp}(a, c)$, it suffices to prove that $p(x, y)$ is realized by some $d \in \text{Ccl}(N)$. By Fact 47 there are some formulas $\psi_n(x, y) \in \mathcal{F}^c$ such that

$$\neg\psi_n(x, y) \rightarrow \exists x', y' \sim_n x, y \ p(x', y')$$

and $\psi_n(x, y) \rightarrow \psi_{n+1}(x, y)$. As $\neg\psi_n(x, y)$ is consistent with $p(x, y)$ then $a, c \models \neg\psi_n(x, y)$. Then by c-elementarity, $\neg\varphi(ka, y)$ is consistent. Let $d_n \in \text{Ccl}(M)$ be such that $ka, d_n \models \neg\psi_n(x, y)$. Let b, d be a realization of the type $\{ka, d_n \sim_n x, y : n \in \omega\} \cup p(x, y)$. As $ka \sim b$, also ka, d realizes $p(x, y)$. As $d \in \text{Ccl}(N)$, the proof is complete. \square

Fact 53. Let M and N be countable atomic c-models. Then $\text{Ccl}(M)$ and $\text{Ccl}(N)$ are c-isomorphic.

Proof. We construct by back-and-forth a sequence $f_n : \text{Ccl}(M) \rightarrow \text{Ccl}(N)$ of finite c-elementary reduced relations (i.e. injective maps) and let

$$R = \bigcup_{n \in \omega} (\sim) \circ f_n \circ (\sim).$$

We will ensure that $M \subseteq \text{dom}(R)$ and $N \subseteq \text{range}(R)$, hence that R is a c-isomorphism between M and N .

Let n be even. Let a be an enumeration of $\text{dom}(f_n)$. Let $c \in M$. Let $p(x, y) = \text{c-tp}(a, b)$. As M is atomic, by Lemma 52 there is a $d \in \text{Ccl}(N)$ such that the relation $f_n \cup \{\langle c, d \rangle\}$ is c-elementary. Let f_{n+1} be a reduced relation such that $f_{n+1} \subseteq f_n \cup \{\langle a, c \rangle\} \subseteq (\sim) \circ f_{n+1} \circ (\sim)$. When n is odd we proceed similarly with f_n^{-1} . \square

The following analogue of Ryll-Nardzewski's Theorem follows by the classical argument.

Theorem 54. Let \mathcal{F}^c be separable. Then the following are equivalent

1. T is c- ω -categorical;
2. every consistent type $p(x) \subseteq \mathcal{F}^c$, for x any finite tuple of variables, is approximately c-isolated.

Proof. (1 \Rightarrow 2) If $p(x)$ is consistent it is realized in some countable c-model M . If $p(x)$ is not approximately c-isolated, by the continuous omitting type theorem there is a c-model N such that $\text{Ccl}(N)$ omits $p(x)$. Then $\text{Ccl}(M)$ and $\text{Ccl}(N)$ are not c-isomorphic, hence T is not c- ω -categorical.

(2 \Rightarrow 1) By (2), every c-model is atomic. Then c- ω -categoricity follows from Fact 53. \square

12. Indiscernibles

Let $I, <$ be an infinite linear order. A sequence $a = \langle a_i : i \in I \rangle$ is p/c-indiscernible if

$$\text{p/c-tp}(a_{\upharpoonright I_0}) = \text{p/c-tp}(a_{\upharpoonright I_1})$$

for every $I_0, I_1 \subseteq I$ such that $|I_0| = |I_1| < \omega$. Note that as maximal positive types are complete, positive indiscernibles are classical indiscernibles.

Let $x = \langle x_i : i < \omega \rangle$. Let $b = \langle b_i : i \in J \rangle$. The Ehrenfeucht-Mostowski p/c-type of a is the type

$$\text{EM-p/c-tp}(b) = \bigcup_{n < \omega} \left\{ \varphi(x_{\upharpoonright n}) \in \mathcal{F}^{p/c} : \varphi(b_{\upharpoonright I_0}) \text{ for every } I_0 \subseteq J, |I_0| = n \right\}.$$

The following classical results holds unmodified.

Theorem 55. Let $I, <$ and $J, <$ be infinite linear orders. Let $b = \langle b_i : i \in J \rangle$. Then there is a sequence of p/c-indiscernibles $a = \langle a_i : i \in I \rangle$ that realizes the Ehrenfeucht-Mostowski p/c-type as b .

Proof. The usual argument works but requires some additional attention.

If $\psi(z)$ is a formula and w is tuple of variables of the same length of z , we write $\psi(/w)$ for the formula obtained replacing z by w in $\psi(z)$.

Let $e(x) = \text{EM-p/c-tp}(b)$. Let $z = \langle z_i : i \in I \rangle$. Define

$$\begin{aligned} q(z) = & \bigcup_{n < \omega} \left\{ \varphi(/z_{\upharpoonright I_0}) : \varphi(x_{\upharpoonright n}) \in e, I_0 \subseteq I, |I_0| = n \right\} \cup \\ & \bigcup_{n < \omega} \left\{ \varphi(/z_{\upharpoonright I_0}) \leftrightarrow \varphi(/z_{\upharpoonright I_1}) : \varphi(x_{\upharpoonright n}) \in \mathcal{F}^{p/c}, I_i \subseteq I, |I_i| = n \right\}. \end{aligned}$$

By Ramsey's theorem q is finitely consistent (but note that positive saturation is not enough to infer the existence of a realization in \mathcal{U}). Let $p(z) \subseteq \mathcal{F}^{p/c}$ be maximal among the types that are finitely consistent with q .

We claim that any $a \models p(z)$ is as required by the theorem. Clearly a realized the EM-p/c type of b , hence we only have to prove that a is a sequence of indiscernibles. Let $a_{\upharpoonright I_0} \models \varphi(x_{\upharpoonright n})$ for some $\varphi(x_{\upharpoonright n}) \in \mathcal{F}^{p/c}$ and some $I_0 \subseteq I$ of cardinality n . We show that $a_{\upharpoonright I_1} \models \varphi(x_{\upharpoonright n})$ for every $I_1 \subseteq I$ of cardinality n . This is clear if $\varphi(/z_{\upharpoonright I_0})$ is in p , so assume it is not. Then $\{\varphi(/z_{\upharpoonright I_0})\}' \subseteq p$. Then also $\{\varphi(/z_{\upharpoonright I_1})\}' \subseteq p$. Therefore $a_{\upharpoonright I_1} \models \{\varphi(x_{\upharpoonright n})\}$ which suffices by saturation. \square

13. Stable formulas

The results in this section are completely classical. The formulas $\varphi(x; z)$ and $\tilde{\varphi}(x; z)$ in Theorem 58 are not required to be positive, in fact, saturation is not required in this section. We prove an enhanced version of a fundamental property of stable formulas. Namely, externally definable sets are internally definable when the defining formula is stable. The result is relevant to the subject of this paper when we take φ positive and $\tilde{\varphi} \perp \varphi$.

The following definition is from [Hr]. We say that the formulas $\varphi(x; z), \tilde{\varphi}(x; z) \in \mathcal{L}$ are **stably separated** if for some $n < \omega$ there is no sequence $\langle a_i; b_i : i < n \rangle$ such that

$$\text{i.} \quad i < j \Rightarrow \varphi(a_i; b_j) \wedge \tilde{\varphi}(a_j; b_i) \quad \text{for every } i, j < n.$$

Clearly, two stably separated formulas are mutually inconsistent. Note that $\varphi, \neg\varphi$ are stably separated exactly when φ is stable (in \mathcal{U}).

We say that the sets $\mathcal{D}, \tilde{\mathcal{D}} \subseteq \mathcal{U}^{[z]}$ are **finitely separated** by $\varphi(x; z), \tilde{\varphi}(x; z)$ if for every finite set $B \subseteq \mathcal{U}^{[z]}$ there is an $a \in \mathcal{U}^{[x]}$ such that

$$\begin{aligned} \text{ii.} \quad B \cap \mathcal{D} &\subseteq B \cap \varphi(a; \mathcal{U}); \\ \text{iii.} \quad B \cap \tilde{\mathcal{D}} &\subseteq B \cap \tilde{\varphi}(a; \mathcal{U}). \end{aligned}$$

Note that this equivalent to requiring that the type below is finitely consistent

$$p(x) = \{\varphi(x; b) : b \in \mathcal{D}\} \cup \{\tilde{\varphi}(x; b) : b \in \tilde{\mathcal{D}}\}$$

Therefore, when $\varphi, \tilde{\varphi}$ are positive, it follows that $\mathcal{D}, \tilde{\mathcal{D}}$ are **externally separated**. That is, there is a p-elementary extension ${}^*\mathcal{U}$ of \mathcal{U} and an ${}^*a \in {}^*\mathcal{U}^{[x]}$ such that $\mathcal{D} \subseteq \varphi({}^*a; {}^*\mathcal{U})$ and $\tilde{\mathcal{D}} \subseteq \tilde{\varphi}({}^*a; {}^*\mathcal{U})$.

Finally, we say that they are **honestly** finitely separated if we can also guarantee that

$$\begin{aligned} \text{iv.} \quad \mathcal{D} \cap \tilde{\varphi}(a; \mathcal{U}) &= \emptyset; \\ \text{v.} \quad \tilde{\mathcal{D}} \cap \varphi(a; \mathcal{U}) &= \emptyset. \end{aligned}$$

If only (iv) obtains, we say **half-honestly** finitely separated.

Lemma 56. Let $\varphi(x; z), \tilde{\varphi}(x; z)$ be stably separated. If $\mathcal{D}, \tilde{\mathcal{D}}$ are finitely separated by $\varphi, \tilde{\varphi}$ then $\mathcal{D}, \tilde{\mathcal{D}}$ are half-honestly finitely separated by $\psi, \tilde{\psi}$ where

$$\begin{aligned} \psi(x_0, \dots, x_m z) &= \varphi(x_0; z) \vee \dots \vee \varphi(x_m; z) \\ \tilde{\psi}(x_0, \dots, x_m z) &= \tilde{\varphi}(x_0; z) \wedge \dots \wedge \tilde{\varphi}(x_m; z) \end{aligned}$$

Proof. Let m be such that no sequence $\langle a_i; b_i : i \leq m \rangle$ satisfies (i'). Let $B \subseteq \mathcal{U}^{[z]}$ be finite. It suffices to prove that there are some a_0, \dots, a_m such that

$$\begin{aligned} B \cap \mathcal{D} &\subseteq \varphi(a_i; \mathcal{U}) && \text{for every } 0 \leq i \leq m; \\ B \cap \tilde{\mathcal{D}} &\subseteq \tilde{\varphi}(a_i; \mathcal{U}) && \text{for every } 0 \leq i \leq m; \\ \emptyset &= \mathcal{D} \cap \bigcap_{i=0}^m \tilde{\varphi}(a_i; \mathcal{U}). \end{aligned}$$

We define a_n and b_n by recursion. Suppose that a_0, \dots, a_{n-1} and $b_0, \dots, b_{n-1} \in \mathcal{D}$ have been defined. We first define a_n , then b_n . Choose $a_n \in \mathcal{U}^{[x]}$ such that

$$\{b_0, \dots, b_{n-1}\} \cup (B \cap \mathcal{D}) \subseteq \varphi(a_n; \mathcal{U});$$

$$B \cap \tilde{\mathcal{D}} \subseteq \tilde{\varphi}(a_n; \mathcal{U}).$$

This is possible by (ii) and (iii). Now, if possible, choose b_n such that

$$b_n \in \mathcal{D} \cap \bigcap_{i=0}^n \tilde{\varphi}(a_i; \mathcal{U}).$$

If this is not possible, then setting $a_i = a_m$ for $n < i \leq m$, we obtain the required a_0, \dots, a_m . We claim that at some $n \leq m$ the procedure above halts. In fact, otherwise we would contradict the choice of m as the sequence $\langle a_i; b_i : i \leq m \rangle$ satisfies (i) up to reversing the order. \square

Lemma 57. Let $\varphi(x; z), \tilde{\varphi}(x; z)$ be stably separated. If $\mathcal{D}, \tilde{\mathcal{D}}$ is half-honestly finitely separated by $\varphi, \tilde{\varphi}$ then for some m they are honestly finitely separated by $\psi, \tilde{\psi}$ where

$$\begin{aligned} \psi(x_0, \dots, x_m; z) &= \varphi(x_0; z) \wedge \dots \wedge \varphi(x_m; z) \\ \tilde{\psi}(x_0, \dots, x_m; z) &= \tilde{\varphi}(x_0; z) \vee \dots \vee \tilde{\varphi}(x_m; z) \end{aligned}$$

Proof. Let m be such that no sequence $\langle a_i; b_i : i \leq m \rangle$ satisfies (i'). Let $B \subseteq \mathcal{U}^{|z|}$ be finite. It suffices to prove that there are some a_0, \dots, a_m such that

$$\begin{aligned} B \cap \mathcal{D} &\subseteq \varphi(a_i; \mathcal{U}) && \text{for every } 0 \leq i \leq m; \\ B \cap \tilde{\mathcal{D}} &\subseteq \tilde{\varphi}(a_i; \mathcal{U}) && \text{for every } 0 \leq i \leq m; \\ \emptyset &= \mathcal{D} \cap \tilde{\varphi}(a_i; \mathcal{U}) && \text{for every } 0 \leq i \leq m; \\ \emptyset &= \tilde{\mathcal{D}} \cap \bigcap_{i=0}^m \varphi(a_i; \mathcal{U}). \end{aligned}$$

We define a_n and b_n by recursion. Suppose that a_0, \dots, a_{n-1} and $b_0, \dots, b_{n-1} \in \mathcal{D}$ have been defined. We first define a_n , then b_n . Choose $a_n \in \mathcal{U}^{|x|}$ such that

$$\begin{aligned} B \cap \mathcal{D} &\subseteq \varphi(a_n; \mathcal{U}); \\ \{b_0, \dots, b_{n-1}\} \cup (B \cap \tilde{\mathcal{D}}) &\subseteq \tilde{\varphi}(a_n; \mathcal{U}). \\ \emptyset &= \mathcal{D} \cap \tilde{\varphi}(a_i; \mathcal{U}). \end{aligned}$$

This is possible by (ii), (iii) and (iv). Now, if possible, choose b_n such that

$$b_n \in \tilde{\mathcal{D}} \cap \bigcap_{i=0}^n \varphi(a_i; \mathcal{U}).$$

If this is not possible, then setting $a_i = a_m$ for $n < i \leq m$, we obtain the required a_0, \dots, a_m . We claim that at some $n \leq m$ the procedure above halts. In fact, otherwise we would contradict the choice of m as the sequence $\langle a_i; b_i : i \leq m \rangle$ satisfies (i). \square

Theorem 58. Let $\varphi(x; z), \tilde{\varphi}(x; z)$ be stably separated. If $\mathcal{D}, \tilde{\mathcal{D}}$ is finitely separated by $\varphi, \tilde{\varphi}$ then for some m, n they are honestly finitely separated by $\psi, \tilde{\psi}$ where

$$\begin{aligned}\psi(\langle x_{i,j} \rangle_{i,j \leq m,n}; z) &= \bigwedge_{i=0}^m \bigvee_{j=0}^n \varphi(x_{i,j}; z) \\ \tilde{\psi}(\langle x_{i,j} \rangle_{i,j \leq m,n}; z) &= \bigvee_{i=0}^m \bigwedge_{j=0}^n \tilde{\varphi}(x_{i,j}; z)\end{aligned}$$

Proof. Apply Lemma 56 and 56 in succession. Note that the usual Ramsey's argument proves that the formulas $\psi, \tilde{\psi}$ in Lemma 56 are stably separated. Therefore they meet the assumption of Lemma 57. \square

Let ξ be a variable of sort X . Let $\varphi(x; z; \xi)$ be a formula such that $\forall x, z \exists^1 \xi \varphi(x; z; \xi)$. Write $f(x; z) = \xi$ for $\varphi(x; z; \xi)$. The following remark links the discussion of stability to the double limits interpretation in [BY].

Remark 59. The following are equivalent

1. $f(x; z) \in C$ and $f(x; z) \in \tilde{C}$ are stably separated for every disjoint compact sets C and \tilde{C} .
2. for every sequence $\langle a_i; b_i : i < \omega \rangle$

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} f(a_i; b_j) = \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} f(a_i; b_j)$$

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