Standard analysis

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ABSTRACT. Let $\mathcal{L}$ be a first-order two-sorted language. Let I be some fixed structure. A *standard* structure is an $\mathcal{L}$-structure of the form $\langle M, I\rangle$ where I is fixed. When I is a compact topological space (and $\mathcal{L}$ meets a few requirements) it is possible to adapt a significant part of model theory to the restricted class of standard strutures. This has been demostrated by Henson and Iovino for Banach spaces (see, e.g. [HI]) and has been generalized to arbitrary structures in [CLCL]. The starting point is to prove that every standard structure has a *positive* elementary extension that is standard and realizes all positive types that are finitely consistent. The main tool is the notion of approximation of a positive formula and of its negation. These have been introduced by Henson and Iovino.

We review and elaborate on the properties of positive formulas and their approximations. In parallel, we introduce *continuous* formulas which are convenient to discuss examples and provide a better counterpart to Henson and Iovino theory and/or real-valued model theory. Finally, we discuss criteria for elimination of quantifers, omitting types, ω-categoricity, stability, and indiscernibles within the setting of standard structures.

1. Standard structures

We refer to the introduction of [CLCL] for motivations. Aside from that, this paper is reasonably self-contained. In the first part we revisit some results of [CLCL]. We rephrase some theorems and add a few remarks and examples[1]. Our notation and terminology diverges slightly, only where necessary.

Let I be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language $\mathcal{L}_{\mathsf{I}}$ contains a relation symbol for each compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \to I$. In particular, there is a constant for each element of I. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{I}}$ or their interpretation in the stucture I. Such a language $\mathcal{L}_{\mathsf{I}}$ is much larger than necessary but it is convenient because it uniquely associates a structure to a topological space. The notion of a *dense set of formulas* (Definition 8) helps to reduce the size of the language when required.

The most straightforward examples of structures I are the unit interval $[0,1]$ with the usual topology, and its homeomophic copies $\mathbb{R}^+ \cup \{0,\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language $\mathcal{L}_{\mathsf{H}}$ which we call the language of the home sort.

Definition 1. Let $\mathcal{L}$ be a two sorted language that expands both $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{I}}$. A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, I\rangle$, where M is any $\mathcal{L}_{\mathsf{H}}$-structure, while I is the structure above. We write H and I to denote the two sorts of $\mathcal{L}$.

The language $\mathcal{L}$ adds a relation symbol $r_\varphi(x)$ for every $\varphi(x) \in \mathcal{L}_{\mathsf{H}}$. All $\mathcal{L}$-structures are assumed to model $r_\varphi(x) \leftrightarrow \varphi(x)$. In other words, the Morleyzation of $\mathcal{L}_{\mathsf{H}}$ is assumed in the definition of $\mathcal{L}$-structure.

Finally and most relevantly, $\mathcal{L}$ contains arbitrarily many function symbols of sort $\mathsf{H}^n \to \mathsf{I}$.

[1] We will omit Section 4 from the final version of this paper and refer to [CLCL] instead. We will also omit a few proofs from Sections 3, 5, and 7.

Standard structures are denoted by the domain of their home sort.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (when I is finite), they are not standard. As a remedy, below we carve out a set of formulas $\mathcal{L}^{\mathrm{p}}$, the set of positive formulas, such that every model has an positive elementary saturated extension that is also standard.

As usual, $\mathcal{L}_{\mathsf{I}}$, $\mathcal{L}_{\mathsf{H}}$, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. We write $\mathcal{L}_x$ when when we restrict variables to x. If $\varphi(x) \in \mathcal{L}$ contains only terms of sort $\mathsf{H}^n \to \mathsf{H}$ and relations of sort H^n we will improperly say that $\varphi(x) \in \mathcal{L}_{\mathsf{H}}$ (this is only correct up to equivalence).

Note that $\mathcal{L}$ has two types of atomic formulas:

1. atomic formulas in $\mathcal{L}_{\mathsf{H}}$ (hence, up to equivalence, all formulas of $\mathcal{L}_{\mathsf{H}}$);
2. formulas of the form $\tau(x;\eta) \in C$, where $C \subseteq I^n$ is compact and $\tau(x;\eta)$ is a tuple of terms of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$.

Definition 2. A formula in $\mathcal{L}$ is positive if it uses only the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{I}}$, $\exists^{\mathsf{I}}$ of sort I.

The set of positive formulas is denoted by $\mathcal{L}^{\mathrm{p}}$.

A continuous formula is a positive formula where only atomic formulas as in (2) above occur. The set of continuous formulas is denoted by $\mathcal{L}^{\mathrm{c}}$.

We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort I. Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

Notation 3. When we state facts that holds both for $\mathcal{L}^{\mathrm{p}}$ and $\mathcal{L}^{\mathrm{c}}$ we write $\mathcal{L}^{\mathrm{p/c}}$ for short.

The positive formulas of Henson and Iovino correspond to those formulas in $\mathcal{L}^{\mathrm{c}}$ that do not have quantifiers of sort I.

Working within $\mathcal{L}^{\mathrm{p}}$ simplifies the comparision with classical model theory but comes at a price. Instead, allowing quantifiers of sort I is almost for free. We will see (cf. Propositions 33 and 45) that the formulas in $\mathcal{L}^{\mathrm{p/c}}$ are approximated by formulas in $\mathcal{L}^{\mathrm{p/c}}$ without quantifiers of sort I.

The main fact to note about $\mathcal{L}^{\mathrm{c}}$ is that it is a language without equality of sort H. The difference beween $\mathcal{L}^{\mathrm{p}}$ and $\mathcal{L}^{\mathrm{c}}$ is similar to the difference between the space of absolutely integrable functions and the Lebesgue space L^1. For the latter only equality almost everywhere is relevant. However, even when L^1 is our focus of interest, it is often easier to argue about real valued functions. For a similar reason $\mathcal{L}^{\mathrm{p}}$ and $\mathcal{L}^{\mathrm{c}}$ are better studied in parallel.

The extra expressive power offered by quantifiers of sort I is convenient. For instance, it is easy to see (cf. Example 4) that $\mathcal{L}^{\mathrm{c}}$ has at least the same expressive power as real valued logic.

Example 4. Let $I = [0,1]$, the unit interval. Let $\tau(x)$ be a term of sort $\mathsf{H}^{|x|} \to \mathsf{I}$. Then there is a positive formula that says $\sup_x \tau(x) = \alpha$. Indeed, consider the formula

$$x\,\big[\tau(x) \dot{-} \alpha \in \{0\}\big] \quad \wedge \quad \forall \varepsilon \Big[\varepsilon \in \{0\} \,\vee\, \exists x\,\big[\alpha \dot{-} (\tau(x) + \varepsilon) \in \{0\}\big]\Big]$$

which, in a more legible form, becomes

$$x\,\big[\tau(x) \leq \alpha\big] \quad \wedge \quad \forall \varepsilon > 0\, \exists x\,\big[\alpha \leq \tau(x) + \varepsilon\big].$$

The following example is inspired by [HPP].

Example 5. Let $\mathcal{U}$ be a saturated structure of signature $\mathcal{L}_{\mathsf{H}}$. The relevant part of $\mathcal{L}$ contains symbols for the functions $f : \mathcal{U}^n \to I$ that are continuous in the following sense: the inverse image of every compact $C \subseteq I$ is type-definable in $\mathcal{L}_{\mathsf{H}}(M)$ for every $M \preceq \mathcal{U}$. It is easy to verify that $\langle \mathcal{U}, I \rangle$ is positively saturated as defined in Section 5.

The theory of normed spaces is one of the motivating examples for standard structures. The requirement for I to be compact seems to be a limitation, but is is not a real one. Essentially, functional analysis and model theory are only concerned with the unit ball of normed spaces. To formalize the unit ball of a normed space as a standard structure (where I is the unit interval) is straightforward but cumbersome. Alternatively one can view a normed space as a many-sorted structure: a sort for each ball of radius $n \in \mathbb{Z}^+$. Unfortunately, this also results in a bloated formalism.

A neater formalization is possible if one accepts that some positive elementary extensions (to be defined below) of a normed space could contain vectors of infinite norm hence not be themselves proper normed spaces. However these infinities are harmless if one restrics to balls of fine radius. In fact, inside any ball of finite radius these improper normed spaces are completely standard.

Example 6. Let $I = \mathbb{R} \cup \{\pm\infty\}$. Let $\mathcal{L}_{\mathsf{H}}$ be the language of real (or complex) vector spaces. The language $\mathcal{L}$ contains a function symbol $\|\text{-}\|$ of sort $\mathsf{H} \to \mathsf{I}$. Normed space are standard structures with the natural interpretation of the language. It is easy to verify that the unit ball of a standard structure is the unit ball of a normed space. Note that the same remains true if we add to the language any continuous (equivalently, bounded) operator or functional.

We conclude this introduction by advertising a question in [CLCL]. Is it possible to extend the Positive Compactness Theorem (Theorem 20) to a larger class of languages? E.g. can we allow in $\mathcal{L}$ (and in $\mathcal{L}^{\mathrm{p/c}}$) function symbols of sort $\mathsf{H}^n \times \mathsf{I}^m \to \mathsf{I}$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I.

2. Henson-Iovino approximations

For C, C' compact substes of I^n, we write $C' > C$ if C' is a neighborhood of C. For φ, φ' (free variables are hidden) positive formulas possibly with parameters we write $\varphi' > \varphi$ if φ' is obtained by replacing in φ each atomic formula of the form $t \in C$ with $t \in C'$, for some $C' > C$. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. We call φ' a weakening of φ. Note that $>$ is a dense transitive relation and that $\varphi \to \varphi'$ in every $\mathcal{L}$-structure.

We write $\tilde\varphi \perp \varphi$ when $\tilde\varphi$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde C$ where $\tilde C$ is some compact set disjoint from C. The atomic formulas in $\mathcal{L}_{\mathsf{H}}$ are replaced with their negation. Finally each connective is replaced by its dual i.e., $\vee, \wedge, \exists, \forall$ are replaced by $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde\varphi$ is a strong negation of φ. It is clear that $\tilde\varphi \to \neg\varphi$ in every $\mathcal{L}$-structure.

Lemma 7. For all positive formulas φ

1. for every $\varphi' > \varphi$ there is a formula $\tilde\varphi \perp \varphi$ such that $\varphi \to \neg\tilde\varphi \to \varphi'$;
2. for every $\tilde\varphi \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde\varphi$.

Proof. If $\varphi \in \mathcal{L}_{\mathsf{H}}$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some $C' > C$. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde\varphi = (t \in I \smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde\varphi$ is of the form $t \in \tilde C$ for some compact $\tilde C$ disjoint from C. By the normality of I, there is $C' > C$ disjoint from $\tilde C$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. □

For every type $p(x)$, we write

$$p'(x) \quad = \quad \{\varphi'(x) \,:\, \varphi' > \varphi \text{ for some } \varphi(x) \in p\}$$

in particular $\{\varphi(x)\}' = \{\varphi'(x) \,:\, \varphi' > \varphi\}$.

Definition 8. A set of formulas $\mathcal{F} \subseteq \mathcal{L}$ is p/c-dense modulo T, a theory, if for every positive/continuous $\varphi(x)$ and every $\varphi' > \varphi$, there is $\psi(x) \in \mathcal{F}$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$ holds in every standard structure that models T. If T is empty, we say p/c-dense modulo $\mathcal{L}$.

Example 9. By Lemma 7, the set of negations of positive/continuous formulas is p/c-dense modulo $\mathcal{L}$.

Example 10. Let $I = [0,1]$. The set of formulas built inductively from atomic formulas of the form $\tau \in \{0\}$, is c-dense modulo $\mathcal{L}$. In fact, as every compact subset of I is the intersection of finite unions of closed intervals, the formulas built using only atomic formulas of the form $\tau \in [\alpha,\beta]$ are c-dense. Finally note that $\tau \in [\alpha,\beta]$ is equivalent to $\tau \dot{-} \beta \in \{0\} \,\wedge\, \alpha \dot{-} \tau \in \{0\}$.

Example 11. In Section 7 we prove that the set of positive/continuous formulas without quantifiers of sort I is p/c-dense.

3. Morphisms

Let M and N be two standard structures. We say that a partial map $f : M \to N$ is p-elementary if for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$ and every $a \in (\operatorname{dom} f)^{|x|}$

1. $$M \models \varphi(a) \quad \Rightarrow \quad N \models \varphi(fa).$$

In words, we say that f preserve the truth of the positive formulas. A p-elementary map that is total is called a p-elementary embedding. When the identity map $\mathrm{id}_M : M \to N$ is an p-elementary embedding, we write $M \preceq^{\mathrm{p}} N$ and say that M is an p-elementary substructure of N.

The discussion of c-elementarity needs some extra care because $\mathcal{L}^{\mathrm{c}}$ does not contains equality in the home sort. Therefore we postpone it till Section 10.

On the other hand, as p-elementary maps are in particular $\mathcal{L}_{\mathsf{H}}$-elementary, they are injective. However, their inverse need not be p-elementary. In other words, the converse of the implication in (1) may not hold. Hence the following notion of elementarity which is more roboust. We say that the map $f : M \to N$ is an approximate p-elementary if for every formula $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$, and every $a \in (\operatorname{dom} f)^{|x|}$

2. $$M \models \varphi(a) \quad \Rightarrow \quad N \models \{\varphi(fa)\}'.$$

One convenient feature of approximate p-elementarity is that it suffices verify (2) for any/some p-dense set of formulas. It is clear that p-elementarity implies its approximate variant. We will see that with a slight amount of saturation also the converse holds (Proposition 22). Moreover, under full saturation, these morphims becomes $\mathcal{L}$-elementarity maps (Corollary 32).

The following holds in general.

Fact 12. If $f : M \to N$ is approximate p-elementary then

$$M \models \{\varphi(a)\}' \quad \Leftrightarrow \quad N \models \{\varphi(fa)\}'.$$

for every $\varphi(x) \in \mathcal{L}^{\mathrm{P}}$, and every $a \in (\operatorname{dom} f)^{|x|}$.

Proof. $\Rightarrow$ Fix $\varphi' > \varphi$ and assume $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By the definition of approximate elementarity, $N \models \varphi''(fa)$ and $N \models \varphi'(fa)$ follows.

$\Leftarrow$ Assume the r.h.s. of the equivalence. Fix $\varphi' > \varphi$ and prove $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By Lemma 7 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \to \neg\tilde{\varphi} \to \varphi'$. Then $N \models \neg\tilde{\varphi}(fa)$ and therefore $M \models \neg\tilde{\varphi}(a)$. Then $M \models \varphi'(a)$. □

Finally, we consider the notion of partial embeddings. Though the definition seems weaker, we show that the notion is competely classical. We say that the map $f : M \to N$ is a partial embedding if the implication in (1) holds for all atomic formulas $\varphi(x)$. Note that this is the same as requiring that (1) holds for all formulas without quantifiers of sort H.

Fact 13. If $f : M \to N$ is a partial embedding then

$$M \models \varphi(a) \quad \Leftrightarrow \quad N \models \varphi(fa)$$

for every $a \in (\operatorname{dom} f)^{|x|}$ and every formula $\varphi(x) \in \mathcal{L}^{\mathrm{P}}$ without quantifiers of sort H.

Proof. The equivalence is trivial for formulas in $\mathcal{L}_{\mathsf{H}}$ so we only consider atomic formulas of the form $t \in C$. Implication $\Rightarrow$ holds by definition. Vice versa, if $M \models \tau(a) \notin C$ then, by normality, $M \models \tau(a) \in \tilde{C}$ for some compact $\tilde{C}$ disjoint of C. By the definition of partial embedding, $N \models \tau(a) \in \tilde{C}$. Hence $N \models \tau(a) \notin C$. Induction is immediate. □

4. The standard part

Our goal is to prove a compactness theorem for positive theories that only requires standard structures. To bypass lengthy routine applications of ultrafilters, ultrapowers, and ultralimits, we assume the Classical Compactness Theorem and build on that.

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 18 which in turn is required for the proof of the Positive Compactness Theorem (Theorem 20). The reader willing to accept it without proof may skip this section.

Let $\langle N, {}^*I\rangle$ be an $\mathcal{L}$-structure that extends $\mathcal{L}$-elementarily the standard structure M. Let η be a free variable of sort I. For each $\beta \in I$, we define the type

$$\mathrm{m}_\beta(\eta) \quad = \quad \{\eta \in D \,:\, D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(\eta)$ in ${}^*I$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 14. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Proof. Negate the existence of β. For every $\gamma \in I$ pick some compact neighborhood D_γ of γ, such that ${}^*I \models \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ, with $\gamma \in \Gamma$, cover I. By elementarity, the interpretation in ${}^*I$ of these D_γ cover ${}^*I$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Fact 15. For every $\alpha \in {}^*I$ and every compact $C \subseteq I$

$$ {}^*I \models \alpha \in C \to \mathrm{st}(\alpha) \in C. $$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*I \models \alpha \in D \subseteq \neg C$. □

Fact 16. For every $\alpha \in ({}^*I)^{|\alpha|}$ and every function symbol f of sort $\mathsf{I}^{|\alpha|} \to \mathsf{I}$

$$ {}^*I \models \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big). $$

Proof. By Fact 14 and the definition of st(-) it suffices to prove that ${}^*I \models f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*I \models \alpha \in f^{-1}[D]$ and, as $I \preceq {}^*I$ we obtain ${}^*I \models f(\alpha) \in D$. □

The standard part of $\langle N, {}^*I\rangle$ is the standard structure $\langle N, I\rangle$ that interprets the symbols f of sort $\mathsf{H}^n \to \mathsf{I}$ as the functions

$$ f^N(a) = \mathrm{st}\big({}^*f(a)\big) \qquad \text{for all } a \in N^n, $$

where ${}^*f$ is the interpretation of f in $\langle N, {}^*I\rangle$. Symbols in $\mathcal{L}_\mathsf{H}$ maintain the same interpretation.

Fact 17. With the notation as above. Let $\tau(x;\eta)$ be a term of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$. Then for every $a \in M^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$ \tau^N\big(a;\mathrm{st}(\alpha)\big) = \mathrm{st}\big({}^*\tau(a;\alpha)\big) $$

Proof. When a t is a function symbol of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{I}^{|\eta|} \to \mathsf{I}$, the claim follows from Fact 16. Now, assume inductively that

$$ t_i^N\big(a;\mathrm{st}(\alpha)\big) = \mathrm{st}\big({}^*t_i(a;\alpha)\big) $$

holds for the terms $t_1(x;\eta),\ldots,t_n(x;\eta)$ and let $t = f(t_1,\ldots,t_n)$ for some function f of sort $\mathsf{I}^n \to \mathsf{I}$. Then the claim follows immediately from the induction hypothesis and Fact 16. □

Lemma 18. With the notation as above. For every $\varphi(x;\eta) \in \mathcal{L}^\mathrm{P}$, $a \in N^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$ \langle N, {}^*I\rangle \models \varphi(a;\alpha) \;\Rightarrow\; N \models \varphi\big(a;\mathrm{st}(\alpha)\big) $$

Proof. Suppose $\varphi(x;\eta)$ is $\mathcal{L}^\mathrm{P}$-atomic. If $\varphi(x;\eta)$ is a formula of $\mathcal{L}_\mathsf{H}$ the claim is trivial. Otherwise $\varphi(x;\eta)$ has the form $\tau(x;\eta) \in C$. Assume that the tuple $\tau(x;\eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, {}^*I\rangle \models \tau(a;\alpha) \in C$. Then $\mathrm{st}\big({}^*\tau(a;\alpha)\big) \in C$ by Fact 15. Therefore $\tau^N\big(a;\mathrm{st}(\alpha)\big) \in C$ follows from Fact 17. This proves the lemma for atomic formulas. Induction is immediate. □

Corollary 19. Let M be a standard structure. Let $p(\eta) \subseteq \mathcal{L}^\mathrm{P}(M)$ be a type that does not contain existential quantifiers of sort H. Then, if $p(\eta)$ is finitely consistent in M, it is realized in M.

The corollary has also a direct proof. This goes through the observation that the formulas in $p(\eta)$ define compact subsets of I.

Proof. Let $\langle {}^*M, {}^*I\rangle$ be an $\mathcal{L}$-elementary saturated superstructure of $\langle M, I\rangle$. Then $\langle {}^*M, {}^*I\rangle \models p(\alpha)$ for some $\alpha \in^* I$. By the lemma above, ${}^*M \models p(\mathrm{st}(\alpha))$. Now observe that the truth of formulas without existential quantifiers of sort H is preserved by substructures. □

The exclusion of existential quantifiers of sort H is necessary. For a counterexample take $M = I = [0,1]$. Assume that $\mathcal{L}$ contains a function symbol for the identity map $\iota : M \to I$. Let $p(\eta)$ contain the formulas $\exists x\big(x > 0 \wedge \iota x + \eta \in [0, 1/n]\big)$ for all positive integers n.

5. Compactness

It is convenient to distinguish between consistency with respect to standard structures and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent if $\langle M, {}^*I\rangle \models T$ for some $\mathcal{L}$-structure $\langle M, {}^*I\rangle$ (this is the classical notion of concistency). We say that T is standardly consistent if $M \models T$ is for some standard structure M. By Lemma 18 these two notions coincide if T is positive. Therefore we have the following.

Theorem 20 (Positive Compactness Theorem). Let T be a positive theory. Then, if T is finitely $\mathcal{L}$-consistent, it is also standardly consistent.

An $\mathcal{L}$-structure N is positively λ-saturated if it realizes all types $p(x;\eta) \subseteq \mathcal{L}^{\mathrm{p}}(N)$ with fewer than λ parameters that are finitely consistent in N. When $\lambda = |N|$ we simply say p-saturated. The existence of p-saturated standard structures is obtained from the classical case just as for Theorem 20.

Theorem 21. Every standard structure has a p-elementary extension to a p-saturated standard structure (possibly of inaccessible cardinality).

Proof. Let $\langle {}^*M, {}^*I\rangle$ be an $\mathcal{L}$-saturated $\mathcal{L}$-elementary extension of the standard structure $\langle M, I\rangle$. Let $\langle {}^*M, I\rangle$ be its standard part as defined in Section 4. By Lemma 18, $\langle {}^*M, I\rangle$ realizes all finitely consistent positive types with fewer than $|{}^*M|$ parameters. □

The following proposition shows that a slight amount of saturation tames the positive formulas.

Proposition 22. Let N be a positively ω-saturated standard structure. Then

$$\big\{\varphi(x;\eta)\big\}' \;\leftrightarrow\; \varphi(x;\eta)$$

holds in N for every formula $\varphi(x;\eta) \in \mathcal{L}^{\mathrm{p}}(N)$.

Proof. We prove $\to$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

ih. $$\{\varphi(x,z;\eta)\}' \;\to\; \varphi(x,z;\eta)$$

We need to prove

$$\{\exists z\, \varphi(x,z;\eta)\}' \;\to\; \exists z\, \varphi(x,z;\eta)$$

From (ih) we have

$$\exists z\, \{\varphi(x,z;\eta)\}' \;\to\; \exists z\, \varphi(x,z;\eta)$$

Therefore it suffices to prove

$$\{\exists z\, \varphi(x,z;\eta)\}' \;\to\; \exists z\, \{\varphi(x,z;\eta)\}'$$

Replace the variables $x;\eta$ with parameters, say $a;\alpha$, and assume that $N\models\exists z\,\varphi'(a,z,;\alpha)$ for every $\varphi'>\varphi$. We need to prove the consistency of the type $\{\varphi'(a,z,;\alpha):\varphi'>\varphi\}$. By saturation, finite consistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1,\varphi_2>\varphi$ then $\varphi_1\wedge\varphi_2>\varphi'$ for some $\varphi'>\varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly. $\square$

Remark 23. By Corollary 19, when $\varphi(x;\eta)$ does not contains existential quantifiers of sort H, the proposition above does not require the assumption of saturation. In general, some saturation is necessary: consider the model presented after Corollary 19 and the formula $\exists x\big(x>0\ \wedge\ \iota(x)\in\{0\}\big)$.

Remark 24. A consequence of Proposition 22 is that the approximate morphisms defined in Section 3 coincide with their unapproximate version when the codomain is positively ω-saturated. In particular they are invertible by Fact 12.

6. The monster model

We denote by $\mathfrak{U}$ some large p-saturated standard structure which we call the positive monster model. Truth is evaluated in $\mathfrak{U}$ unless otherwise is specified. We denote by T the positive theory of $\mathfrak{U}$. The density of a set of formulas is understood modulo T. Below we say p-model for p-elementary substructure of $\mathfrak{U}$. We stress once again that the truth of some $\varphi\in\mathcal{L}^{\mathrm{p}}(M)$ in a p-model M implies the truth of φ (in $\mathfrak{U}$) but not vice versa. However, all p-models agree on the approximated truth, see Fact 12 and Remark 24,

$$M\models\{\varphi\}'\quad\Leftrightarrow\quad\{\varphi\}'.$$

A substructure $M\subseteq\mathfrak{U}$ is a c-model if

c. $\qquad M\models\varphi(a)\quad\Rightarrow\quad\varphi(a)\qquad$ for all formulas $\varphi(x)\in\mathcal{L}^{\mathrm{c}}$ and $a\in M^{|x|}$.

Note that $\mathcal{L}^{\mathrm{c}}$ is closed under weakening and strong negation. Therefore the argument in the proof of Fact 12 proves that (c) is equivalent to

$$M\models\{\varphi(a)\}'\quad\Leftrightarrow\quad\{\varphi(a)\}'\qquad\text{for all formulas }\varphi(x)\in\mathcal{L}^{\mathrm{c}}\text{ and }a\in M^{|x|}.$$

The following fact demostrates how positive compactness applies. There are some subtle differences from the classical setting. Let $A\subseteq\mathfrak{U}$ be a small set throughout this section.

Fact 25. Let $p(x)\subseteq\mathcal{L}^{\mathrm{p}}(A)$ be a type. Then for every $\varphi(x)\in\mathcal{L}^{\mathrm{p}}(\mathfrak{U})$

i. if $p(x)\rightarrow\neg\varphi(x)$ then $\psi(x)\rightarrow\neg\varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;

ii. if $p(x)\rightarrow\varphi(x)$ and $\varphi'>\varphi$ then $\psi(x)\rightarrow\varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (i) is immediate by saturation; (ii) follows fromrom (i) by Lemma 7. $\square$

Fact 26. Let $\mathcal{F}$ be a p/c-dense set of positive formulas. Then $\mathcal{F}'$ is p/c-dense.

Proof. Let $\varphi'>\varphi$. Pick φ'' such that $\varphi'>\varphi''>\varphi$ and $\psi\in\mathcal{F}$ such that $\varphi(x)\rightarrow\psi(x)\rightarrow\varphi''(x)$. It suffices to prove that $\psi'(x)\rightarrow\varphi'(x)$ for some $\psi'>\psi$. By Proposition 22, $\psi(x)\leftrightarrow\{\psi(x)\}'$. Therefore, $\psi'(x)\rightarrow\varphi'(x)$ follows from Fact 25. $\square$

We write $S^{\rm p/c}_x(A)$ for the set of maximally consistent subsets of $\mathcal{L}^{\rm p/c}_x(A)$. We define $\text{p/c-tp}(a/A)$, the positive/continuous type of a over A, to be the set of formulas $\big\{\varphi(x) \in \mathcal{L}^{\rm p/c}(A) \ :\ \varphi(a)\big\}$.

In general, if $\mathcal{F}$ is any set of formulas, we write $\mathcal{F}\text{-tp}(a/A)$, for the type $\big\{\varphi(x) \in \mathcal{F}(A) \ :\ \varphi(a)\big\}$. The undecorated symbol $\text{tp}(a/A)$ denotes the $\mathcal{L}$-type.

Fact 27. Let $p(x) \subseteq \mathcal{L}^{\rm p/c}(A)$. The following are equivalent

1. $p(x)$ is a maximally consistent subsets of $\mathcal{L}^{\rm p/c}_x(A)$;
2. $p(x) = \text{p/c-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$.

Proof. (1⇒2) Then $p(x) \subseteq \text{p/c-tp}(a/A)$ and, by maximality $p(x) = \text{p/c-tp}(a/A)$.

(2⇒1) From Lemma 7 and Proposition 22 we obtain

$$\neg\varphi(x) \quad\rightarrow\quad \bigvee_{\tilde{\varphi}\perp\varphi} \tilde{\varphi}(x).$$

Suppose $\varphi(x) \in \mathcal{L}^{\rm p/c}(A) \smallsetminus p$. Then $\neg\varphi(a)$. Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \rightarrow \neg\varphi(x)$ follows. □

Note that in the proof above we could replace $\mathcal{L}^{\rm p}$ with any set of positive formulas closed under strong negation. Therefore we can also claim the following.

Fact 28. Let $p(x) \subseteq \mathcal{F}(A)$, where $\mathcal{F}$ is a set of positive formulas closed under strong negation. Then the following are equivalent

1. $p(x)$ is a maximally consistent subset of $\mathcal{F}_x(A)$;
2. $p(x) = \mathcal{F}\text{-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$.

Proposition 29. Let $\mathcal{F}$ be a p/c-dense set of positive/continuous formulas. Then for every formula $\varphi(x) \in \mathcal{L}^{\rm p/c}$

i. $$\neg\varphi(x) \quad\leftrightarrow\quad \bigvee\big\{\psi'(x) \ :\ \psi'(x) \rightarrow \neg\varphi(x) \text{ where } \psi' > \psi \text{ for some consistent } \psi(x) \in \mathcal{F}\big\};$$

ii. $$\neg\varphi(x) \quad\leftrightarrow\quad \bigvee\big\{\neg\psi(x) \ :\ \psi(x) \in \mathcal{F} \text{ and } \neg\psi(x) \rightarrow \neg\varphi(x)\big\}.$$

Proof. (i) Only $\rightarrow$ requires a proof. Let $a \in \mathcal{U}^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \text{p/c-tp}(a)$. By Fact 27, $p(x) \rightarrow \neg\varphi(x)$. As $\mathcal{F}$ is p/c-dense, $p'(x) \leftrightarrow q(x) = \mathcal{F}\text{-tp}(a)$. Therefore $p(x) \leftrightarrow q'(x)$. Then, by compactness, $\psi'(x) \rightarrow \neg\varphi(x)$ for some $\psi' > \psi \in q \subseteq \mathcal{F}$.

(ii) By density

$$\varphi(x) \quad\leftrightarrow\quad \bigwedge\big\{\psi(x) \in \mathcal{F} \ :\ \varphi(x) \rightarrow \psi(x)\big\}.$$

Negating both sides of the equivalence we obtain (ii). □

We will need a similar result for larger class formulas. We write $\Sigma^{\rm p/c}$ for the set of formulas of the form $\exists y\, \vartheta(x,y)$ where $\vartheta(x,y)$ is obtained by taking disjuctions and conjuctions of formulas of the form $\neg\psi(x,y)$ or $\psi'(x,y)$ for some $\psi(x,y) \in \mathcal{L}^{\rm p/c}$ and $\psi' > \psi$. The following follows easily from the proposition above by induction on the syntax of $\varphi(x)$.

Proposition 30. Let $\mathcal{F}$ be a p/c-dense set of positive/continuous formulas. Then, for every formula $\varphi(x) \in \Sigma^{\rm p/c}$

$$\varphi(x) \quad\leftrightarrow\quad \bigvee\big\{\psi(x) \in \mathcal{F} \ :\ \psi(x) \rightarrow \varphi(x)\big\}.$$

7. Elimination of quantifiers of sort l for positive formulas

We write $\mathcal{L}^{\mathrm{p/c}}_{\mathsf{lqf}}$ for the set of positive/continuous formulas without quantifiers of sort l.

In this section we show that $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$ is p-dense modulo T (the theory of $\mathcal{U}$). In plain words, this amounts to the elimination of the quantifiers of sort l up to some approximation. In Section 10 we prove the analogous result for $\mathcal{L}^{\mathrm{c}}_{\mathsf{lqf}}$.

A partial map is $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary if it preserves the truth of formulas in $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$.

Proposition 31. Every $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary map $f : \mathcal{U} \to \mathcal{U}$ of small cardinality extends to an automorphism of $\mathcal{U}$.

Proof. By Fact 27, the inverse of an $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary map is also $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary. Then we can extend the map by back-and-forth as usual. □

Let $A \subseteq \mathcal{U}$ be a small set throughout this section. We apply the proposition above to obtain a strengthening of Fact 28. For $a, b \in \mathcal{U}^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A.

Corollary 32. Let $p(x) = \mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}\text{-tp}(a/A)$ and $q(x) = \mathrm{tp}(a/A)$. Then $p(x) \leftrightarrow q(x)$.

Proof. Only $\to$ requires a proof. If $b \models p(x)$ then there is an $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary map $f \supseteq \mathrm{id}_A$ such that $fa = b$. Then f extends to an automorphism. As every automorphism is $\mathcal{L}$-elementary, $a \equiv_A b$, and the corollary follows. □

The corollary says that $\mathcal{U}$ does not distinguish between $\mathcal{L}$, $\mathcal{L}^{\mathrm{p}}$, and $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$ types. In the classical setting, from an equivalence between types one derives an equivalence between formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence.

Proposition 33. The set $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$ is p-dense modulo T.

Proof. Let $\varphi(x)$ be a positive formula. We need to prove that for every $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$. By Corollary 32 and Proposition 22

$$\neg\varphi(x) \quad \to \quad \bigvee_{p'(x) \to \neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over the maximally consistent $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-types. By Fact 25 and Lemma 7

$$\neg\varphi(x) \quad \to \quad \bigvee_{\neg\tilde{\psi}(x) \to \neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$. Equivalently,

$$\varphi(x) \quad \leftarrow \quad \bigwedge_{\tilde{\psi}(x) \leftarrow \varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 25, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$ such that

$$\varphi'(x) \quad \leftarrow \quad \bigwedge_{i=1,\dots,n} \tilde{\psi}_i(x) \quad \leftarrow \quad \varphi(x)$$

which yields the interpolant required by the proposition. □

8. Cauchy completions

For $\tau(x,z) = \tau_1(x,z),\ldots,\tau_n(x,z)$ a tuple of terms of sort $\mathsf{H}^{|x|+|z|} \to \mathsf{I}$ we define the formula

$$x \sim_\tau y \quad = \quad \bigwedge_{i=1}^{n} \forall z\ \tau_i(x,z) = \tau_i(y,z),$$

where the expression $\alpha = \beta$ is shorthand for $\langle\alpha,\beta\rangle \in \Delta$, where Δ is the diagonal of I^2. We also define the type

$$x \sim y \quad = \quad \Big\{x \sim_\tau y \ :\ \tau(x,z) \text{ as above}\Big\}.$$

The following fact will be used below without mention.

Fact 34. For any $a = \langle a_i : i < \lambda\rangle$ and $b = \langle b_i : i < \lambda\rangle$

$$a \sim b \quad \Leftrightarrow \quad a_i \sim b_i \quad \text{for every } i < \lambda.$$

Proof. By induction. Only implication $\Leftarrow$ requires a proof For limit ordinals induction is trivial. Assume $a_i \sim b_i$ for every $i \le \lambda$. Let $|a| = |b| = \lambda$ and assume inductively that (it suffices to consider tuples τ of arity 1)

$$\forall y,z \quad \tau(a,y,z) \quad = \quad \tau(b,y,z)$$

holds for every term $\tau(x,y,z)$, with $|x| = \lambda$ and $|y| = 1$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

1. $$\forall z \quad \tau(a,a_\lambda,z) \quad = \quad \tau(b,a_\lambda,z).$$

As $a_\lambda \sim b_\lambda$ then

$$\forall x,z \quad \tau(x,a_\lambda,z) \quad = \quad \tau(x,b_\lambda,z)$$

and in particular

2. $$\forall z \quad \tau(b,a_\lambda,z) \quad = \quad \tau(b,b_\lambda,z).$$

From (1) and (2) we obtain

$$\forall z \quad \tau(a,a_\lambda,z) \quad = \quad \tau(b,b_\lambda,z).$$ □

Note that the approximations of the formula $x \sim_\tau y$ have the form

$$x \sim_{\tau,D} y \quad = \quad \bigwedge_{i=1}^{n} \forall z\ \langle\tau_i(x,z),\tau_i(y,z)\rangle \in D$$

for some compact neighborhood D of Δ. Hence we define

$$x \sim' y \quad = \quad \Big\{x \sim_{\tau,D} y \ :\ \tau(x,z) \text{ tuple of terms, } D \text{ compact neigborhood of } \Delta\Big\}.$$

The formulas $x \sim_{\tau,D} y$, as τ, D range as above, form a prebase for a system of entougages on $\mathfrak{U}^{|x|}$. We refer to this uniformity and the topology associated as the I-topology. Though not needed in the sequel, it is worth mentioning that the I-topology on $\mathfrak{U}^{|x|}$ coincides with the product of the I-topology on $\mathfrak{U}$. This can be verified by an argument similar to the proof of Fact 34.

Fact 35. For every $\varphi(x) \in \mathcal{L}^{\mathrm{c}}$

$$x \sim y \quad \rightarrow \quad \varphi(x) \leftrightarrow \varphi(y)$$

Proof. By induction. □

The following corollary corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Corollary 36. For every $\varphi(x) \in \mathcal{L}^{\mathrm{c}}$, every $\varphi' > \varphi$, and every $\tilde{\varphi} \perp \varphi$ there is a tuple of terms τ and a compact neighborhood of the diagonal D such that

i. $\quad x \sim_{\tau,D} y \wedge \varphi(y) \quad \rightarrow \quad \varphi'(x)$

ii. $\quad x \sim_{\tau,D} y \wedge \tilde{\varphi}(y) \quad \rightarrow \quad \neg\varphi(x).$

Proof. As $x \sim y \cup \{\varphi(x)\} \rightarrow \varphi(y)$ by the fact above, (i) follows from Fact 25. Similarly, we obtain (ii) from $x \sim y \cup \{\tilde{\varphi}(x)\} \rightarrow \neg\varphi(y)$. □

We say what a type $q(x)$ is finitely satisfiable in A if every conjunction of formulas in $q(x)$ has a solution in $A^{|x|}$. This definition coincides with the classical one, but in our context, the notion is less roboust. We may happen that $p(x)$ is finitely satisfiable while $q(x) \leftrightarrow p(x)$ is not. In particular if M is a p-model and $q(x) = \text{p-tp}(a/M)$ then $q'(x)$ is always finitely satisfiable while $q(x)$ need not.

We say that that a set $A \subseteq \mathfrak{U}$ is Cauchy complete if it contains all those $a \in \mathfrak{U}$ such that $a \sim' x$ is finitely satisfied in A. Note that Cauchy complete sets are in particular closed under $\sim$-equivalence. The Cauchy completion of A is the set

$$\mathrm{Ccl}(A) \quad = \quad \{a : a \sim' x \text{ is finitely satisfied in } A\}.$$

Fact 37. $\mathrm{Ccl}(A)$ is Cauchy complete

Proof. Suppose that $a \sim' x$ is finitely satisfied in $\mathrm{Ccl}(A)$. Let τ, D be given. We prove that $a \sim_{\tau,D} x$ is satisfied in A. Let E be a compact neighborhood of the diagonal such that $E \circ E \subseteq D$. There is some $b \in \mathrm{Ccl}(A)$ such that $b \sim_{\tau,E} a$. There is some $c \in A$ such that $c \sim_{\tau,E} b$. Then $c \sim_{\tau,D} a$, as required. □

We say that $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(\mathfrak{U})$ is a Cauchy type if it is consistent and $p(x) \wedge p(y) \rightarrow x \sim y$.

Fact 38. Let M be a c-model and let $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(M)$ be a Cauchy type. Then all realizations of $p(x)$ belong to $\mathrm{Ccl}(M)$.

Proof. If $p(x)$ is a Cauchy type, then $p(x) \rightarrow a \sim x$ for some $a \models p(x)$. Thas M is a c-model, $p'(x)$ is finitely satisfied in M. Then also $a \sim' x$ is finitely satisfied. Hence $a \in \mathrm{Ccl}(M)$. □

9. Continuous morphims

The definition of c-elementary maps is delicate due to the lack of equality. A possible approach is to work in the quotient $\mathfrak{U}/{\sim}$. A second possibility is to replace functions with relations. The two options differs only in the notation. Here we go for the second.

Let M and N be c-models. Let $R \subseteq M \times N$ be a binary relation. If $a = \langle a_i : i < \lambda \rangle$ and $b = \langle b_i : i < \lambda \rangle$ we write $\langle a, b \rangle \in R$ to abbreviate: $\langle a_i, b_i \rangle \in R$ for all $i < \lambda$. We say that R is an c-elementary relation between M and N if

$$M \models \varphi(a) \quad \Rightarrow \quad N \models \varphi(b) \qquad \text{for every } \varphi(x) \in \mathcal{L}^{\mathrm{c}} \text{ and every } \langle a, b \rangle \in R.$$

Below, for simplicity we only consider c-elementary relations between $\mathfrak{U}$ and itself. Then, by Fact 12 and Proposition 22, the condition above is aquivalent to

$$\# \qquad \varphi(a) \quad \leftrightarrow \quad \varphi(b) \qquad \text{for every } \varphi(x) \in \mathcal{L}^{\mathrm{c}} \text{ and every } \langle a, b \rangle \in R.$$

Therefore if R is c-elementary, also R^{-1} is c-elementary.

In the next section we will also need the notion of $\mathcal{L}^{\mathrm{c}}_{\mathrm{lqf}}$-elementary relation between $\mathcal{U}$ and itself. This is defined similarly but (#) above is required to hold for formulas in $\mathcal{L}^{\mathrm{c}}_{\mathrm{lqf}}$. We ask the reader to verify that the content of this section applies verbatim to $\mathcal{L}^{\mathrm{c}}_{\mathrm{lqf}}$-elementary relation.

The relation $(\sim)$ is c-elementary by Fact 35. Then, if the relation R is c-elementary, also $(\sim)\circ R\circ(\sim)$ is c-elementary. We identify a c-elementary relation $R = (\sim)\circ R\circ(\sim)$ with the map

$$\begin{array}{rcccl} R_{/\sim} & : & \mathcal{U}/\sim & \to & \mathcal{U}/\sim \\ & & [a] & \mapsto & [b] \qquad\qquad \text{for some/any } b \text{ such that } \langle a,b\rangle \in R. \end{array}$$

It is easy to verify that the definition is well-given and that $R_{/\sim}$ is injective.

We say that R is reduced if there is no $R' \subset R \subseteq (\sim)\circ R'\circ(\sim)$. It is easy to verify that every relation R contains a reduced relation R' such that $R \subseteq (\sim)\circ R'\circ(\sim)$.

Fact 39. Let R be reduced. Then R is the graph of an injective map $f : \mathrm{dom}(R) \to \mathrm{range}(R)$.

Proof. Let $a \in \mathrm{dom}R$ and assume that $\langle a,b\rangle, \langle a,b'\rangle \in R$. To prove functionality suppose for a contradiction that $b \neq b'$. As R is reduced, $b \not\sim b'$. Then $a \not\sim a$, a contradiction. This prove functionality, for injectivity apply the same argument to R^{-1}. □

Assume $R = (\sim)\circ R\circ(\sim)$ for simplicity. Let f be a reduced relation such that $f \subseteq R = (\sim)\circ f \circ(\sim)$. Then f is a function defined in exacly one representative a of each $(\sim)$-equivalence class $[a] \subseteq \mathrm{dom}(R)$. Moreover, $f(a)$ the unique representative of $R_{/\sim}\big([a]\big)$ in the range of f.

Fact 40. If R is a c-elementary relation then $\mathrm{Ccl}(R)$ is also c-elementary. We also have that $\mathrm{Ccl}(R) = (\sim)\circ\mathrm{Ccl}(R)\circ(\sim)$.

Proof. Let $\varphi(x)$ and $\varphi'' > \varphi' > \varphi$ be given. Let $\langle a,b\rangle \in \mathrm{Ccl}(R)$ and assume $\varphi(a)$. Let τ, D be such that

1. $x \sim_{\tau,D} y \wedge \varphi(x) \;\to\; \varphi'(y)$
2. $x \sim_{\tau,D} y \wedge \varphi'(x) \;\to\; \varphi''(y)$

By the definition of Cauchy completion, there are $a', b' \sim_{\tau,D} a, b$ such that $\langle a', b'\rangle \in R$. Then from (1) we infer that $\varphi'(a')$. As R is c-elementary, $\varphi'(b')$. Then $\varphi''(b)$ follows from (2). As $\varphi'' > \varphi$ is arbitrary, the fact follows from Proposition 33.

The second claim is immediate. □

Let M and N be c-models. A c-elementary relation such that $\mathrm{dom}(R) = M$ and $\mathrm{range}(R) = N$ is called an approximate c-isomorphism between M and N. The modifier *approximate* is due because R is c-elementary between $\mathcal{U}$ and itself. We will only consider these c-isomorphisms, therefore we omit the modifier. A c-automosphism a c-isomorphism of a c-model in itself. The following is noted for future reference.

Remark 41. To be a c-automorphism of $\mathcal{U}$ the relation R only needs to satisfy (#) for atomic formulas of the form $\tau(x) \in C$.

Fact 42. Let R be a c-elementary relation. Let $M \subseteq \mathrm{dom}(R)$ and $N \subseteq \mathrm{range}(R)$ be c-models. Then there is a c-isomorphism between $\mathrm{Ccl}(M)$ and $\mathrm{Ccl}(N)$.

Proof. By Fact 40 we can assume that $\mathrm{Ccl}(M) \subseteq \mathrm{dom}(R)$ and $\mathrm{Ccl}(N) \subseteq \mathrm{range}(R)$, hence a restriction of R is the required c-isomorphism. □

10. Elimination of quantifiers of sort I for continuous formulas

We verify that the argument in Section 7 can be adapted to continuous formulas.

Proposition 43. All $\mathcal{L}^{\rm c}_{\rm Iqf}$-elementary maps of small cardinality extend to c-automorphisms of $\mathfrak{U}$.

Proof. We construct by back-and-forth a sequence $f_i : \mathfrak{U} \to \mathfrak{U}$ of finite c-elementary reduced relations (i.e. injective maps) and let

$$R \quad = \bigcup_{i\in|\mathfrak{U}|} (\sim)\circ f_i \circ (\sim).$$

We will ensure that $\mathfrak{U} = {\rm dom}(R) = {\rm range}(R)$. Then R is a c-automorphism of $\mathfrak{U}$ by Remark 41.

Let i be even. Let a be an enumeration of ${\rm dom}(f_n)$. Let $b \in \mathfrak{U}$. Let $p(x, y) = $ c-tp(a, b). By c-elementarity the type $p(f_i a, y)$ is finitely satisfied. Let $c \in \mathfrak{U}$ realize $p(f_i a, y)$. Let f_{i+1} be a reduced relation such that $f_{i+1} \subseteq f_i \cup \{\langle b, c\rangle\} \subseteq (\sim)\circ f_{i+1} \circ (\sim)$.

When i is odd we proceed similarly with f_i^{-1}. Limit stages are obvious. □

Corollary 44. Let $a \in \mathfrak{U}^{|x|}$. Let $p(x) = \mathcal{L}^{\rm c}_{\rm Iqf}$-tp$(a)$ and $q(x) = $ c-tp(a). Then $p(x) \leftrightarrow q(x)$.

Proof. Only $\to$ requires a proof. If a contains entries $a_i \sim a_j$, replace a_j with a_i. Note that the tuple a' obtained in this manner has the same c-type of a.

Let $b \models p(x)$ and let b' obtained with the same procedure as a'.

Then the function that maps $a' \mapsto b'$ preserves the truth of $\mathcal{L}^{\rm c}_{\rm Iqf}$-formulas. Then f extends to an c-automorphism. As every c-automorphism is c-elementary and the corollary follows. □

Proposition 45. The set $\mathcal{L}^{\rm c}_{\rm Iqf}$ is c-dense modulo T.

Proof. The proof of Proposition 33, where Corollary 44 replaces the reference to Corollary 32. □

11. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is our version of the Tarski-Vaught test.

Theorem 46. Let M be a subset of $\mathfrak{U}$. Let $\mathcal{F}$ be a p-dense set of positive formulas. Then the following are equivalent

1. M is a p-model;
2. for every formula $\psi(x) \in \mathcal{F}(M)$
$$\exists x\, \psi(x) \quad\Rightarrow\quad \text{for every } \psi' > \psi \text{ there is an } a \in M \text{ such that } \psi'(a);$$
3. for every formula $\psi(x) \in \mathcal{F}(M)$
$$\exists x\, \neg\psi(x) \quad\Rightarrow\quad \text{there is an } a \in M \text{ such that } \neg\psi(a).$$

If $\mathcal{F}$ is a c-dense set of continuous formulas then (2) and (3) above are equivalent to

1′. ${\rm Ccl}(M)$ is a c-model.

Moreover, if M is a substructure, then (1′), (2) and (3) are also equivalent to

$1''$. M is a c-model.

Proof. (1⇒2) Assume $\exists x\,\psi(x)$ and let $\psi' > \psi$ be given. By Lemma 7 there is some $\tilde\psi \perp \psi$ such that $\psi(x) \to \neg\tilde\psi(x) \to \psi'(x)$. Then $\neg\forall x\,\tilde\psi(x)$ hence, by (1), $M \models \neg\forall x\,\tilde\psi(x)$. Then $M \models \neg\tilde\psi(a)$ for some $a \in M$. Hence $M \models \psi'(a)$ and $\psi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\psi(x) \in \mathcal{F}(M)$ be such that $\exists x\,\neg\psi(x)$. By Proposition 29, there are a consistent $\varphi(x) \in \mathcal{F}(M)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \to \neg\psi(x)$. Then (3) follows.

(3⇒2) Let $\psi' > \psi$ for some $\psi(x) \in \mathcal{F}(M)$. Let $\tilde\psi \perp \psi$ such that $\psi(x) \to \neg\tilde\psi(x) \to \psi'(x)$. By Fact 29.ii, $\neg\varphi(x) \to \neg\tilde\psi(x)$ for some $\varphi(x) \in \mathcal{F}(M)$ such that $\neg\varphi(x)$ is consistent. Then (2) follows from (3).

(2⇒1) Assume (2). By the classical Tarski-Vaught test $M \preceq_{\mathsf{H}} \mathcal{U}$. Then M is the domain of a substructure of $\mathcal{U}$. Then $M \models \varphi(a) \;\Rightarrow\; \varphi(a)$ holds for every atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \models \varphi(a,b) \;\;\Rightarrow\;\; \varphi(a,b).$$

Using (2) and the induction hypothesis we prove that

$$M \models \exists y\,\varphi(a,y) \;\;\Rightarrow\;\; \forall y\,\varphi(a,y).$$

Indeed, for any $\varphi' > \varphi$,

$$\begin{aligned}
M \models \exists y\,\varphi(a,y) \;\;&\Rightarrow\;\; M \models \exists y\,\psi(a,y) && \text{for some } \psi \in \mathcal{F} \text{ such that } \varphi(a,y) \to \psi(a,y) \to \varphi'(a,y)\\
&\Rightarrow\;\; M \models \psi(a,b) && \text{for some } b \in M \text{ by (2)}\\
&\Rightarrow\;\; M \models \varphi'(a,b) &&\\
&\Rightarrow\;\; \varphi'(a,b) && \text{by induction hypothesis}\\
&\Rightarrow\;\; \exists y\,\varphi'(a,y). &&
\end{aligned}$$

As $\varphi' > \varphi$ is arbitrary, $\exists y\,\varphi(a,y)$ follows from Proposition 22.

Induction for the connectives $\vee$, $\wedge$, $\forall^{\mathsf{H}}$, $\exists^{\mathsf{I}}$, and $\forall^{\mathsf{I}}$ is straightforward.

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($1'$⇒2) Let $\psi' > \psi'' > \psi$. Reasoning as in the proof of (1⇒2) we obtain that $\mathrm{Ccl}(M) \models \psi''(a)$ for some $a \in \mathrm{Ccl}(M)$. By Corollary 36, $a \sim_{\tau,D} x \to \psi'(x)$ for some τ, D. As $a \sim' x$ is finitely satisflied in M, it follows that $\psi'(c)$ for some $c \in M$.

(2⇔3) The proof above applies verbatim when $\mathcal{F}$ is a c-dense set of continuous formulas.

(2⇒$1'$) Assume (2). We claim that $\mathrm{Ccl}(M)$ is a substructure of $\mathcal{U}$. Let $a \in M^n$ and let f be a function symbol of sort $\mathsf{H}^n \to \mathsf{H}$. We prove that $f a \in M$. We show that $f a \sim' x$ is finitely satisfied in M. Consider the formula $f a \sim_{\tau,D} x$. By Lemma 7, there is a formula in $\tilde\varphi(x) \in \mathcal{L}^{\mathrm{p}}(M)$ such that

$$f a \sim_{\tau,\Delta} x \;\;\to\;\; \neg\tilde\varphi(x) \;\;\to\;\; f a \sim_{\tau,D} x$$

By Fact 26.ii there is a consistent formula $\neg\psi(x)$, for some $\psi(x) \in \mathcal{F}(M)$, that implies $f a \sim_{\tau,D} x$. Then, by (3), $f a \sim_{\tau,D} x$ is satisfied in M. This proves our claim.

Now, we claim that (2) holds also for every $\psi(x) \in \mathcal{F}\big(\mathrm{Ccl}(M)\big)$. Let $\psi(x,z) \in \mathcal{F}(M)$ and $\psi' > \psi$ be given. Let $b \in \mathrm{Ccl}(M)^{|z|}$. Suppose that $\exists x\,\psi(x,b)$ and let τ, D be such that $z \sim_{\tau,D} b \to \exists x\,\psi''(x,z)$ where $\psi' > \psi'' > \psi$. By Corollary 36, we can also assume that $z \sim_{\tau,D} b \wedge \psi''(x,z) \to \psi'(x,b)$. Let $b' \in M^{|z|}$ be such that $b' \sim_{\tau,D} b$. By (2) there is an $a \in M^{|x|}$ such that $\psi''(a,b')$. Then $\psi'(a,b)$ follows. This proves the second claim.

By the two claims above, the inductive argument in the proof in (2⇒1) applies to prove that $\mathrm{Ccl}(M)$ is a c-model.

***

($1''\Rightarrow 2$) By the same argument as in ($1\Rightarrow 2$).

($2\Rightarrow 1''$) As M is a substructure by assumption, the inductive argument in the proof in ($2\Rightarrow 1$) applies. □

Remark 47. The theorem above shows in particular that for every substructure M the following are equivalent

1. M is a c-model;
2. $\mathrm{Ccl}(M)$ is a c-model.

Classically, the first application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all $\mathcal{L}$-structures. In particular every $A\subseteq\mathcal{U}$ is contained in a standard structure of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of $\mathcal{L}$ is eccessively large because of the aboundance of symbols in $\mathcal{L}_{\mathsf{I}}$.

If there is a countable p/c-dense set $\mathcal{F}$ of positive formulas, we say that $\mathcal{L}^{\mathrm{p/c}}$ is separable.

Proposition 48. Let $\mathcal{L}^{\mathrm{p/c}}$ be separable. Let A be a countable set. Then there is a countable p/c-model M containing A.

Proof. Let $\mathcal{F}$ be a countable p/c-dense set of positive/continuous formulas. As in the classical proof of the Löwenheim-Skolem Theorem, we construct a countable $M\subseteq\mathcal{U}$ that contains a witness of every consistent formula $\neg\psi(x)$ for $\psi(x)\in\mathcal{F}(M)$. In continuous case we need also to ensure that M is a substructure. Then the proposition follows from Theorem 46. □

Proposition 49. Assume that $\mathcal{L}$ has at most countably many symbols of sort $\mathsf{H}^n\to\mathsf{I}$. Then, if I is a second countable (the topology has a countable base), then $\mathcal{L}^{\mathrm{c}}$ is separable modulo $\mathcal{L}$. Moreover if $\mathcal{L}_{\mathsf{H}}$ is countable also then $\mathcal{L}^{\mathrm{p}}$ is separable.

Proof. Fix a countable filter of compact neighborhoods of the diagonal of I^2. □

12. Positive omitting types

In this section we prove a useful lemma about isolated positive/continuous types. Our goal is to prove in the sections a the continuous version of the omitting types theorem and the caracterization of *c-ω-categoricity.* These results have interesting examples in functional analysis (here we will only breafly discuss Hilbert spaces).

A type $p(x)$ is isolated by $\varphi(x)$, a consistent formula, if $\varphi(x)\to p(x)$. If $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x)\in\mathcal{L}^{\mathrm{p}}(A)$ we say that it is p-isolated by A. Similarly, we define c-isolated. Note that, by Theorem 46, $p(x)$ is p/c-isolated by A, then $p(x)$ is realized in every p/c-model containing A. In the prositive case we also have the following: if $p(x)$ is is realized in M, a p-model, then $p(x)$ is p-isolated by M. In fact, if $b\models p(x)$ then $\neg(b\neq x)$ isolates $p(x)$ where $b\neq x$ is positive by Morleyzation.

The continuous case will be considered in the next section. In this case a weaker notion of isolation is more appropriate, see Fact 54.

Fact 50. Let $\mathcal{F}$ be a p/c-dense set of positive formulas. Then the following are equivalent

1. $p(x)$ is p/c-isolated by A;
2. $p(x)$ is isolated by $\neg\psi(x)$ for some $\psi(x) \in \mathcal{F}(A)$;
3. $p(x)$ is isolated by some $\psi'(x)$ such that $\psi' > \psi$ for some consistent $\psi(x) \in \mathcal{F}(A)$;
4. $p(x)$ is isolated by some formula $\varphi(x) \in \Sigma^{\mathrm{p/c}}(A)$.

Proof. (1⇒2) By Proposition 29.ii.

(1⇒3) By Proposition 29.i.

(3⇒1) Let $\psi' > \psi$ be as in (3). Let $\tilde{\psi} \perp \psi$. Then $\neg\tilde{\psi}(x)$ isolates $p(x)$.

(3⇒4) Trivial.

(4⇒3) By Proposition 30 and Fact 26. □

Lemma 51. Let $\mathcal{L}^{\mathrm{p/c}}(A)$ be separable. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p/c}}(A)$, be non p/c-isolated by A. Then every consistent formula $\neg\psi(z)$, with $\psi(z) \in \mathcal{L}^{\mathrm{p/c}}(A)$, has a solution a such that A, a does not p/c-isolate $p(x)$.

Proof. We construct a sequence of $\mathcal{L}^{\mathrm{p/c}}(A)$-formulas $\langle \gamma'_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\gamma'_i(z) : i < \omega\}$ is the required solution of $\neg\psi(z)$.

Let $\langle \xi_i(x;z) : i < \omega \rangle$ enumerate a countable p/c-dense subset of $\mathcal{L}^{\mathrm{p/c}}_{x;z}(A)$. Let $\gamma_0(z)$ and $\gamma'_0 > \gamma_0$ be a consistent positive/continuous formulas such that $\gamma'_0(z) \to \neg\psi(z)$. These exist by Proposition 29 and Fact 26. Define $\gamma'_{i+1}(z)$ inductively as follows.

1. If $\neg\xi_i(x;z) \wedge \gamma'_i(z)$ is inconsistent, let $\gamma'_{i+1}(z) = \gamma'_i(z)$.
2. Otherwise, pick $\varphi(x) \in p$ such that (#) below is consistent

   (#) $$\gamma'_i(z) \ \wedge\ \exists x \big[\neg\xi_i(x;z) \wedge \neg\varphi(x)\big].$$

   Finally, let $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$ consistent positive/continuous formulas such that $\gamma'_{i+1}(z)$ implies (#). Such formulas exist by Propositions 30 and 29.

Let $a \models \{\gamma'_i(z) : i < \omega\}$. We claim that that A, a does not p/c-isolate $p(x)$. Otherwise, by Fact 50, $\neg\xi_i(x;a) \to p(x)$ for some consistent $\neg\xi_i(x;a)$. This contradicts $a \models \gamma'_{i+1}(z)$.

Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in (2).

Suppose for a contradiction that $\neg\xi_i(x;z) \wedge \gamma'_i(z)$ is consistent while (#) is inconsistent for all formulas $\varphi(x) \in p$, that is,

$$\neg\xi_i(x;z) \wedge \gamma'_i(z) \ \to\ \varphi(x).$$

This immediately implies that

$$\exists z \big[\neg\xi_i(x;z) \wedge \gamma'_i(z)\big] \ \to\ p(x).$$

Then $p(x)$ is isolated by a formula in $\Sigma^{\mathrm{p/c}}(A)$. By Fact 50, this is a contradiction. □

From the lemma we easily obtain an omitting types theorem for positive types that is very close to the classical one.

Theorem 52 (Positive Omitting Types). Let $\mathcal{L}^{\mathrm{p}}$ be separable. Let A be countable. Assume also that $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ is not p-isolated. Then there is a p-model M containing A that omits $p(x)$.

Proof. As in the classical proof, appy the lemma above and the Tarski-Vaught test (Theorem 46) to obtain a countable p-model M that does not isolate $p(x)$. For p-models, p-isolating a type is equivalent to realizing it. □

13. Continuous omitting types.

We say that $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(\mathfrak{U})$ is c-isolated in the limit by A if for every τ, D the type $\exists y \sim_{\tau,D} x\ p(y)$ is c-isolated by A.

Fact 53. Let $\mathcal{L}^{\mathrm{c}}$ be separable. If $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ is c-isolated in the limit by A, then $p(x)$ is realized in every Cauchy complete c-model containing A.

Proof. As $\mathcal{L}^{\mathrm{c}}$ is separable, there is a sequence $\langle \tau_n, D_n : n < \omega \rangle$ such that for every τ, D there is an $n < \omega$ such that $x \sim_{\tau_n, D_n} y \rightarrow x \sim_{\tau,D} y$. We write $x \sim_n y$ for $\sim_{\tau_n, D_n}$.

Let M be a Cauchy complete c-model containing A. Let $a_n \in M$ be such that $\exists x \sim_n a_n\ p(x)$. By compactness, $b \models p(x)$ for some $b \in \mathfrak{U}$ such that $b \sim_n a_n$ for every $n < \omega$. Then $b \sim' x$ is finitely satisfied in M and therefore $b \in \mathrm{Ccl}(M) = M$. □

In words, the proof above shows that if $p(x)$ is c-isolated in the limit by M, then there is a sequence of elements of M that converges inthe I-topology to a realization of $p(x)$.

Fact 54. If $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ is realized in $\mathrm{Ccl}(M)$ for some c-model M containing A, then $p(x)$ is c-isolated in the limit by M.

Proof. Let $b \in \mathrm{Ccl}(M)^{|x|}$ realize $p(x)$. Let τ, D be given. By Fact 50 it suffices to find a continuous formula $\varphi(x)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \rightarrow \exists y \sim_{\tau,D} x\ p(y)$. Let C be a neighborhood of the diagonal such that $C' \circ C' \subseteq D$ for some $C' > C$. Let $a \in M$ be such that $a \sim_{\tau,C} b$. Then we can take $x \sim_{\tau,C} a$ as $\varphi(x)$ and $x \sim_{\tau,C'} a$ as $\varphi'(x)$. □

Theorem 55 (Continuous Omitting Types). Let $\mathcal{L}^{\mathrm{c}}$ be separable. Let A be countable. Assume also that $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ is not c-isolated in the limit. Then there is a c-model M containing A such that $\mathrm{Ccl}(M)$ omits $p(x)$.

Proof. By assumption $\exists y \sim_{\tau,D} x\ p(y)$ is not c-isolated for some τ, D. Appy Lemma 51 and the Tarski-Vaught test (Theorem 46) to obtain a countable c-model M that does not isolate the type $\exists y \sim_{\tau,D} x\ p(y)$. Then M does not isolate $p(x)$ in the limit. By Fact 54, $\mathrm{Ccl}(M)$ omits $p(x)$. □

14. Continuous ω-categoricity.

In the discussion of countable categoricity we assume that the underlying theory is complete. This assumption is not necessary (it is rather a consequence of categoricity), but it allows to work inside a monster model and simplify the notation.

Let M be a c-model. Let $A \subseteq \mathrm{Ccl}(M)$. We say that M is c-atomic over A if the types c-tp(a/A), for any finite tuple a of elements of M, are c-isolated by A in the limit.

Fact 56. Let $\mathcal{L}^{\mathrm{c}}$ be separable. If M is c-atomic over A then it is c-atomic over A, a for every finite tuple a of elements of $\mathrm{Ccl}(M)$.

Proof. Let $b \in M^{|y|}$. Assume that $p(x, y) = \text{c-tp}(a, b/A)$ is c-isolated in the limit. We prove that $p(a, y) = \text{c-tp}(b/A, a)$ is c-isolated in the limit. Let $\neg\varphi_n(x, y)$ isolate $\exists x', y' \sim_n x, y\ p(x', y')$, where $\sim_n$ is as in the proof of Fact 53. Note that $\neg\varphi_n(a, b)$ holds for all n. Then $\neg\varphi_n(a, y)$ is consistent and $\neg\varphi_n(a, y) \rightarrow \exists y \sim_n y'\ p(a, y')$. □

Fact 57. Let M and N be countable c-atomic c-models. Then $\mathrm{Ccl}(M)$ and $\mathrm{Ccl}(N)$ are c-isomorphic.

Proof. We construct by back-and-forth a sequence $f_n : \mathrm{Ccl}(M) \rightarrow \mathrm{Ccl}(N)$ of finite c-elementary reduced relations (i.e. injective maps) and let

$$R \quad = \quad \bigcup_{n\in\omega} (\sim) \circ f_n \circ (\sim).$$

We will ensure that $M \subseteq \mathrm{dom}(R)$ and $N \subseteq \mathrm{range}(R)$. By Fact 42 this suffices to prove the fact.

Let n be even. Let a be an enumeration of $\mathrm{dom}(f_n)$. Let $b \in M$. Let $p(x, y) = \text{c-tp}(a, b)$. As M is atomic over a by Fact 56, the type $p(a, y)$ is c-isolated in the limit. Then also the type $p(f_n a, y)$ is c-isolated in the limit. Let $c \in \mathrm{Ccl}(N)$ realize $p(f_n a, y)$. Let f_{n+1} be a reduced relation such that $f_{n+1} \subseteq f_n \cup \{\langle a, c\rangle\} \subseteq (\sim) \circ f_{n+1} \circ (\sim)$.

When n is odd we proceed similarly with f_n^{-1}. □

Let $\mathcal{L}^{\mathrm{c}}$ be separable. We say that T is c-ω-categorical if $\mathrm{Ccl}(M)$ and $\mathrm{Ccl}(N)$ are c-isomorphic for any two countable c-models M, N. The following analogue of Ryll-Nardzewski's Theorem follows as usual.

Theorem 58. Assume that $\mathcal{L}^{\mathrm{c}}$ is separable. Then the following are equivalent

1. T is c-ω-categorical;
2. every consistent type $p(x) \subseteq \mathcal{L}^{\mathrm{c}}$, for x any finite tuple of variables, is c-isolated in the limit.

Proof. (1⇒2) This is the classica argument. If $p(x)$ is not c-isolated in the limit, by the compactness theorem and the continuous omitting type theorem we can find two countable c-models M and N such that M realize $p(x)$ and $\mathrm{Ccl}(N)$ omits $p(x)$. Then $\mathrm{Ccl}(M)$ and $\mathrm{Ccl}(N)$ are not c-isomorphic, hence T is not c-ω-categorical.

(2⇒1) As, every model is c-atomic, c-ω-categoricity follows from Fact 57 as in the classical case. □

The classical argument proves also the following.

Proposition 59. Fix a finite tuple of variables x. The following are equivalent

1. every consistent $p(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ is c-isolated in the limit;
2. for every given τ, D there there are at most $n_{\tau,D} < \omega$ many types $p_i(x) \subseteq \mathcal{L}^{\mathrm{c}}(A)$ such that $\exists y \sim_{\tau,D} x\ p_i(y)$ are pairwise inconsistent.

Proof. (1⇒2) Let τ, D be given. Let $p_i(x) \subseteq \mathcal{L}^{\mathrm{c}}$, for $i < \lambda$, be such that $\exists y \sim_{\tau,D} x\ p_i(y)$ are pairwise inconsistent. Suppose that λ is maximal and, for a contradiction, that it is infinite. Let $\neg\varphi_i(x)$, where $\varphi_i(x) \in \mathcal{L}^{\mathrm{c}}$, be the formulas that isolate $\exists y \sim_{\tau,D} x\ p_i(x)$. As these types are pairwise inconsistent, then the continuous type $q(x) = \{\varphi_i(x) : i < \lambda\}$ is consistent. The type $\exists y \sim_{\tau,D} x\ q(x)$ is inconsistent with all $\exists y \sim_{\tau,D} x\ p_i(x)$. This contradicts the maximality of λ.

(2⇒1?) Let τ, D be given and pick a compact neighborhood of the diagonal E such that $E\circ E\circ E \subseteq D$. Let $p_i(x) \subseteq \mathcal{L}^{\rm c}(A)$, for $i < n$, be such that $\exists y \sim_{\tau,E} x\; p_i(x)$ are pairwise inconsistent. By compactness there are some $\varphi_i(x) \in p_i$ such that $\exists y \sim_{\tau,E} x\; \varphi_i(x)$ are pairwise inconsistent. Assume $n = n_{\tau,E}$ is maximal. Let $F = E\circ E$. We claim that for every a

$$\bigvee_{i=1}^{n} \exists y \sim_{\tau,F} a\;\; \varphi_i(y)$$

In fact, if $q(x) = \text{c-tp}(a/A)$, where a does not satisfy the formula above, then $\exists y \sim_{\tau,E} x\; q(y)$ would be inconsistent with every $\exists y \sim_{\tau,E} x\; p_i(x)$. This contradicts the maximality of n.

Let $q(x) \subseteq \mathcal{L}^{\rm c}(A)$. We prove that $\exists y \sim_{\tau,D} x\; q(y)$ is c-isolated. By maximality $\exists y \sim_{\tau,E} x\; q(y)$ is consistent with, say, $\exists y \sim_{\tau,E} x\; \varphi_1(x)$. Without loss of generality, assume $q(x) \to \exists y \sim_{\tau,E} x\; \varphi_1(x)$. Note that

$$\exists y \sim_{\tau,E} x\, \varphi_1(x) \quad\to\quad \neg\bigvee_{i=2}^{n} \exists y \sim_{\tau,F} x\;\; \varphi_i(y)$$

We claim that

$$\neg\bigvee_{i=2}^{n} \exists y \sim_{\tau,F} x\;\; \varphi_i(y) \quad\to$$

Let

$$\mathcal{C} \quad=\quad \Big\{\langle C_1,\ldots,C_n\rangle \;:\; E \subseteq C_i \subseteq D \text{ compact}, \forall x \bigvee_{i=1}^{n} \exists y \sim_{\tau,C_i} x\;\; \varphi_i(y)\Big\}$$

, for $i < n$ be such that

1. $$\neg\bigvee_{i=1}^{n} \varphi_i(x) \quad\to\quad \exists y \sim_{\tau,D} x\, q(y)$$

Suppose not for a contradiction. Let a satisfy the antecedent but not the consequence of (1).

□

15. Example: metric spaces

We discuss a simple example. Let I be $\mathbb{R}^+ \cup \{0,\infty\}$, with the topology that makes it homeomorphic to the unit interval. Let M, d is a metric space. Let $\mathcal{L}_{\rm H}$ contain symbols for functions $M^n \to M$ that are uniformly continuous. Let $\mathcal{L}$ contain a symbol for d and possibly for some funtions of $M^n \to I$ that are uniformly continuous w.r.t. the metric d.

Let $\langle M, I\rangle$ be the structure that interprets the symbols of the language as natural.

Let $\mathfrak{U}$ be a positive monster model that is an $\mathcal{L}^{\rm p}$ elementary extension of M. Clearly, d does not define a metric on $\mathfrak{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathfrak{U}$. Therefore the notion of convergent sequence makes perfectly sense in $\mathfrak{U}$. As all functions have been required to be uniformly continuous, it is immediate that $x \sim_{\rm I} y$ is equivalent to $d(x, y) = 0$.

Fact 60. Let $\mathfrak{U}$ be as above. Then for every p-model N and $a \in \mathfrak{U}$ the following are equivalent

1. $p(x) = \text{p-tp}(a/N)$ is a Cauchy type;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of N that converges to a.

Proof. (2⇒1) Let $\langle \varepsilon_i : i \in \omega\rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \le \varepsilon_i$ for every $i \in \omega$. Then the formulas $d(a_i, x) \le \varepsilon_i$ are in $p(x)$. Then every element realizing $p(x)$ is at distance 0 from a. Therefore $p(x) \to a \sim x$.

(1⇒2) As $p(x)$ is Cauchy type, $p(x) \to a \sim x$. Then $p'(x) \to d(a,x) < 2^{-i}$ for all i. By compactness (see Fact 25) there are formulas $\varphi_i(x) \in p$ such that $\varphi_i'(x) \to d(a,x) < 2^{-i}$ for some $\varphi_i' > \varphi_i$. By $\mathcal{L}^{\mathrm{p}}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a,a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega\rangle$ converges to a. □

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16. Elimination of quantifiers

Let T be a positive theory. We say that T has (approximate, positive) elimination of quantifiers if the quantifier-free positive formulas are p-dense modulo T. Equivalently, if every complete type is equivalent to a quantifier-free positive types.

The following fact is routinely proved by back-and-forth.

Fact 61. The following are equivalent

1. T has elimination of quantifiers;
2. every finite partial embedding $k : M \to N$ beween p-models of T is an approximate p-elementary map;
3. for every finite partial embedding $k : M \to N$ between ω-saturated p-models of T, and for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b,c\rangle\} : M \to N$ is also a partial embedding.

17. Pseudofinite randomizations

Let T be a complete first-order theory of signature L with an infinite model. Let $\mathfrak{U}$ be a model of T. Let Ω be a finite set. We denote by $\mathfrak{U}^\Omega$ the set of functions $\Omega \to \mathfrak{U}$. The elements of $\mathfrak{U}^\Omega$ are denoted by letters decorated with a circonflexe accent $\hat{a}, \hat{b}$, etc. The value of $\hat{a}$ at $\omega \in \Omega$ is denoted by a_ω. We now define models of the form $\langle \mathfrak{U}^\Omega, I\rangle$, where I is the real interval $[0,1]$ with the usual topology.

The language $\mathcal{L}_{\mathsf{H}}$ is empty. The language $\mathcal{L}$ contains functions of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, one for each formula $\varphi(x) \in L$. These are denoted by $\Pr\varphi(\hat{x})$. Variables of sort H are decorated with a circonflexe accent. The interpretation of the function symbol $\Pr\varphi(\hat{x})$ is

$$\Pr\big(\varphi(\hat{a})\big) \quad = \quad \frac{\#[\![\varphi(\hat{a})]\!]}{\#\Omega},$$

where by # denotes the finite cardinality, and

$$[\![\varphi(\hat{a})]\!] \quad = \quad \big\{\omega \in \Omega \,:\, \mathcal{U} \models \varphi(a_\omega)\big\}.$$

The pseudofinite randomization of T is the positive theory

$$T^{\mathrm{pfr}} \quad = \quad \Big\{\varphi \in \mathcal{L}^{\mathrm{p}} \,:\, \langle \mathcal{U}^\Omega, I\rangle \models \varphi \text{ for all sufficiently large finite set } \Omega\Big\}$$

In this section we prove that T^{pfr} is a compete theory with elimination of quantifiers. We prove the lemma that is required for the back-and-forth.

Lemma 62. Let $k : M \to N$ be a finite partial embedding between positively ω-saturated models of T^{pfr}. Then for every $\hat{b} \in M$ and there is a $\hat{c} \in N$ such that $k \cup \{\langle \hat{b}, \hat{c}\rangle\} : M \to N$ is a partial embedding.

Proof. Let $\hat{a}$ be an enumeration of $\mathrm{dom}k$. Let $p(\hat{x}, \hat{z})$ be set of formulas of the form $\gamma = \Pr\varphi(\hat{x}, \hat{z})$, for some $\gamma \in I$, that hold in $M, \hat{b}, \hat{a}$. Note that $p(\hat{x}, \hat{z})$ implies the atomic type of $\hat{b}, \hat{a}$ in M. Therefore it suffices to show that $p(\hat{x}, k\hat{a})$ is finitely consistent in N.

Pick a finite set of these formulas, say $\gamma_i = \Pr\varphi_i(\hat{x}, \hat{z})$, for $i < n$. Me may assume that the fomulas $\varphi_i(x, z)$ define a partition of $\mathcal{U}^{|x,z|}$. By the saturation of N, it suffices to prove that for every $\varepsilon > 0$ there is a $\hat{c} \in N$ such that

$$\gamma_i \quad =_\varepsilon \quad \Pr\varphi_i(\hat{c}, k\hat{a}) \qquad\qquad (\alpha =_\varepsilon \beta \text{ is a shorthand for } |\alpha - \beta| \le \varepsilon)$$

is true in N for all $i < n$.

Some preliminary work is required. For each $J \subseteq n$ define the following L-formulas

$$\xi_J(z) \quad = \quad \bigwedge_{i \in J} \exists x\, \varphi_i(x, z) \wedge \bigwedge_{i \in n \smallsetminus J} \neg\exists x\, \varphi_i(x, z)$$

These formulas $\xi_J(z)$ partition $\mathcal{U}^{|z|}$. For $J \subseteq n$ let α_J be such that

2. $$\alpha_J \quad = \quad \Pr\big(\xi_J(\hat{z})\big)$$

is in $p(\hat{x}, \hat{z})$. Let $\mathfrak{J} = \{J \subseteq n \,:\, \alpha_J \neq 0\}$. When $J \in \mathfrak{J}$, there are some $\beta_{i,J}$ such that the formulas

3. $$\beta_{i,J} \quad = \quad \Pr\Big(\varphi_i(\hat{x}, \hat{z}) \,\Big|\, \xi_J(\hat{z})\Big).$$

are are among the consequences of $p(\hat{x}, \hat{z})$. As the formulas $\varphi_i(x, z)$ define a partition, the $\beta_{i,J}$ add up to 1 for any fixed J. Note that $\beta_{i,J} = 0$ for $i \notin J$. Clearly, have that

4. $$\gamma_i \quad = \quad \sum_{J \in \mathfrak{J}} \beta_{i,J}\alpha_J.$$

It is plain that for $i \in J$ the following is a consequence of $p(\hat{x}, \hat{z})$

5. $$1 \quad = \quad \Pr\Big(\exists x\, \varphi_i(x, \hat{z}) \,\Big|\, \xi_J(\hat{z})\Big)$$

Now we prove that (2) & (5) $\Rightarrow$ (1) holds in $\mathcal{U}^\Omega$ when Ω large enough (larger than n/ε suffices). Strictly speaking, this implication is not a positive formula, but the reader can easily verify that a suitable approximation of (2) and (5) suffices.

Let $\hat{a}' \in \mathcal{U}^\Omega$ satisfy (2) and (5) for every $i \in J \subseteq n$. We define $\hat{c}' \in \mathcal{U}^\Omega$ such that

$$\mathcal{U}^\Omega \quad \models \quad \beta_{i,J} =_\varepsilon \Pr\Big(\varphi_i(\hat{c}', \hat{a}') \,\Big|\, \xi_J(\hat{a}')\Big).$$

We may define $\hat{c}'$ separately in each event $[\![\xi_J(\hat{a}')]\!]$. Partition $[\![\xi_J(\hat{a}')]\!]$ into events $E_i \subseteq \Omega$, for $i \in J$, such that $\#E_i/\#\Omega =_\varepsilon \beta_{i,J}$. Then define c'_ω for $\omega \in E_i$ to be any witness of $\exists x\, \varphi_i(x, a_\omega)$. By (5), we can always find such a witness.

Finally, by (4), we deduce

$$\mathfrak{U}^\Omega \quad \models \quad \gamma_i =_\varepsilon \Pr\big(\varphi_i(\hat{c}', \hat{a}')\big).$$

As $k\hat{a}$ satisfy (2) and (5), we conclude that some $\hat{c} \in N$ satisfy (1). □

Now, from Fact 61 we obtain that T^{pfr} has elimination of quantifiers. Completeness follows.

Corollary 63. The theory T^{pfr} is compete and has elimination of quantifiers.

18. Stability

A partitioned formula $\varphi(x\,;z) \in \mathcal{L}^{\text{p}}$ is stable if there is no sequence $\langle a_i\,; b_i \,:\, i < \omega\rangle$ and no $\tilde{\varphi} \perp \varphi$ such that for every $i, j < \omega$

$$i < j \quad \Rightarrow \quad \varphi(a_i\,; b_j)$$

$$i > j \quad \Rightarrow \quad \tilde{\varphi}(a_i\,; b_j)$$

Using the terminology of [Hr], $\varphi(x\,;z)$ is stable if it is stably separated from all $\tilde{\varphi} \perp \varphi$.

Note that by compactness if $\varphi(x\,;z)$ and $\tilde{\varphi} \perp \varphi$ are stably separated then there is a maximal length m of a sequence $\langle a_i\,; b_i \,:\, i < m\rangle$ such that (1) and (2) above.

19. Elementary relations

Let M and N be models. We say that $R \subseteq M \times N$ is an p-elementary relation between M and N if for every $\varphi(x) \in \mathcal{L}$

$$M \models \varphi(a) \quad \Rightarrow \quad N \models \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } a\,R\,b.$$

Note that, when $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ are tuples, $a\,R\,b$ stands for $a_i\,R\,b_i$ for every $i \in \{1, \ldots, n\}$.

Fact 64. Let $R \subseteq \mathfrak{U}^2$ be an p-elementary relation of cardinality $< \kappa$. Then there is a total and surjective p-elemenrary relation $S \subseteq \mathfrak{U}^2$ containing R.

Proof. We apply the usual back-and-forth construction with a pinch of extra caution. Let a be an enumeration of the domain of R. Let $\bar{a} = \langle a_i : i < \lambda\rangle$ be an enumeration of all tuples of length $|a|$ such that aRa_i. As κ is inaccessible, $\lambda < \kappa$. Let $b \in U$. It suffices to prove that there is a c such that $R \cup \{\langle b, c\rangle\}$ is an $\mathbb{L}$-relation. Let $p(x, z) = \text{tp}(b, a)$ and let

$$q(x, \bar{z}) \quad = \quad \bigcup_{i<\lambda} p(x, z_i).$$

We claim that $q(x, \bar{a})$ is a finitely consistent type. A finite conjunction of formulas in $q(x, \bar{a})$ has the form $\psi(x, a_{i_1}) \wedge \cdots \wedge \psi(x, a_{i_n})$. As $\psi(b, a)$ and $a_{i_1}, \ldots, a_{i_n}\, R\; a, \ldots, a$, we conclude that the condition $\psi(x, a_{i_1}) \wedge \cdots \wedge \psi(x, a_{i_n})$ is satisfied. The existence of the required element c follows by saturation. □

20.

Corollary 65. Let $A \subseteq \mathcal{U}$ have cardinality $< \kappa$. Let $p(x) = \mathrm{tp}_{\mathbb{L}}(a/A)$, where $a \in \mathcal{U}^{|x|}$ is a tuple of length $|x| < \kappa$. Then

$$p(\mathcal{U}) \quad = \quad \big\{b : bRa,\ R \in \mathrm{Aut}(\mathcal{U}/A)\big\}$$

Let $\mathcal{M}$ and $\mathcal{N}$ be models. We say that $R \subseteq M \times N$ is an $\mathbb{L}$-(elementary) relation between $\mathcal{M}$ and $\mathcal{N}$ (or on $\mathcal{M}$ if the two coicide) if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \models \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \models \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } a\,R\,b.$$

Recall that, when $a = a_1,\ldots,a_n$ and $b = b_1,\ldots,b_n$ are tuples, $a\,R\,b$ stands for $a_i\,R\,b_i$ for every $i \in \{1,\ldots,n\}$.

We define an equivalence relation $(\sim_{\mathcal{M}})$ on M as follows

1. $$a \sim_{\mathcal{M}} b \quad \Leftrightarrow \quad \Big(\mathcal{M} \models \varphi(a) \leftrightarrow \varphi(b) \quad \text{for every } \varphi(x) \in \mathbb{L}(M)\Big),$$

where $|x| = |a| = |b| = 1$. Note that this relation would be trivial had we included in $\mathbb{L}(M)$ equality between elements of M.

The following proposition is easily proved by induction on the syntax.

Proposition 66. The following are equivalent for every $a, b \in M$.

1. $a \equiv_{\mathcal{M}} b$;
2. $\mathcal{M} \models \tau(a) = \tau(b)$ for every $\tau(x) \in \mathbb{T}(M)$, with $|x| = 1$.

Lemma 67. The relation $(\sim_{\mathcal{M}}) \subseteq M^2$ is an $\mathbb{L}$-relation. Moreover, it is maximal among the $\mathbb{L}$-relations on $\mathcal{M}$, i.e. no $\mathbb{L}$-relation properly contains $(\sim_{\mathcal{M}})$.

Proof. Assume $a \sim_{\mathcal{M}} b$, where $a = a_1,\ldots,a_n$ and $b = b_1,\ldots,b_n$. Recall that this means that $a_i \sim_{\mathcal{M}} b_i$ for all $i \in \{1,\ldots,n\}$. Let Δ denote the diagonal relation on M. Note that $a_i \sim_{\mathcal{M}} b_i$ is equivalent to saying that $\Delta \cup \{(a_i, b_i)\}$ is an $\mathbb{L}$-relation. As $\mathbb{L}$-relations are closed under composition $\Delta \cup \big\{(a_1,b_1),\ldots,(a_n,b_n)\big\}$ is $\mathbb{L}$-elementary. It follows that for every $\varphi(x) \in \mathbb{L}$

2. $$\mathcal{M} \models \varphi(a) \quad \Leftrightarrow \quad \mathcal{M} \models \varphi(b).$$

This proves that $(\sim_{\mathcal{M}})$ is an $\mathbb{L}$-relation. Finally, maximality is immediate. □

Lemma 68. Let $R \subseteq M \times N$ be total and surjective $\mathbb{L}$-relation. Then there is a unique maximal $\mathbb{L}$-relation containing R. This maximal $\mathbb{L}$-relation is equal to both $(\sim_{\mathcal{M}})\,R$ and $R\,(\sim_{\mathcal{N}})$, where justapposition of relations stands for composition.

Proof. It is immediate to verify that $(\sim_{\mathcal{M}})\,R$ is an $\mathbb{L}$-relation containing R. Let S be any maximal $\mathbb{L}$-relation containing R. By maximality, $(\sim_{\mathcal{M}})\,S = S$. As S is a total relation $(\sim_{\mathcal{M}}) \subseteq SS^{-1}$. Therefore, by the lemma above, $(\sim_{\mathcal{M}}) = SS^{-1}$. As R is a surjective relation, $S \subseteq SS^{-1}R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}})\,R$. A similar argument proves that $S = R\,(\sim_{\mathcal{N}})$. □

We write $\mathrm{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, $\mathbb{L}$-relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/{\sim_{\mathcal{M}}}$.