Standard analysis

C. L. C. L. Polymath

ABSTRACT. Let $\mathcal{L}$ be a first-order two-sorted language. Let I be some fixed structure. A *standard* structure is an $\mathcal{L}$-structure of the form $\langle M, I\rangle$ where I is fixed. When I is a compact topological space (and $\mathcal{L}$ meets a few requirements) it is possible to adapt a significant part of model theory to the restricted class of standard strutures. This has been demostrated by Henson and Iovino for Banach spaces (see, e.g. [HI]) and has been generalized to arbitrary structures in [CLCL]. The starting point is to prove that every standard structure has a *positive* elementary extension that is standard and realizes all positive types that are finitely consistent. The main tool is the notion of approximation of a positive formula and of its negation. These have been introduced by Henson and Iovino.

We review and elaborate on the properties of positive formulas. In parallel, we introduce *continuous* formulas which are convenient to discuss examples and provide a better counterpart to Henson and Iovino theory and/or real-valued model theory. Finally, we discuss criteria for elimination of quantifers, omitting types, ω-categoricity, stability, and indiscernibles within the setting of standard structures.

1. Standard structures

We refer to the introduction of [CLCL] for motivations. Aside from that, this paper is reasonably self-contained. In the first part we revisit some results of [CLCL]. We rephrase some theorems and add a few remarks and examples[1]. Our notation and terminology diverges slightly where needed.

Let I be some fixed first-order structure which is endowed with a Hausdorff compact topology (in particular, a normal topology). The language $\mathcal{L}_{\mathsf{I}}$ contains relation symbols for the compact subsets $C \subseteq I^n$ and a function symbol for each continuous functions $f : I^n \to I$. In particular, there is a constant for each element of I. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{I}}$ or their interpretation in the stucture I. Such a language $\mathcal{L}_{\mathsf{I}}$ is much larger than necessary but it is convenient because it uniquely associates a structure to a topological space. The notion of a *dense set of formulas* (Definition 7) helps to reduce the size of the language when required.

The most straightforward examples of structures I are the unit interval $[0,1]$ with the usual topology, and its homeomophic copies $\mathbb{R}^+ \cup \{0,\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$.

We also fix a first-order language $\mathcal{L}_{\mathsf{H}}$ which we call the language of the home sort.

Definition 1. Let $\mathcal{L}$ be a two sorted language that expands both $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{I}}$. A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, I\rangle$, where M is any $\mathcal{L}_{\mathsf{H}}$-structure, while I is the structure above. We write H and I to denote the two sorts of $\mathcal{L}$.

The language $\mathcal{L}$ adds a relation symbol $R_\varphi(x)$ for every $\varphi(x) \in \mathcal{L}_{\mathsf{H}}$. All $\mathcal{L}$-structures are assumed to model $R_\varphi(x) \leftrightarrow \varphi(x)$. In other words, the Morleyzation of $\mathcal{L}_{\mathsf{H}}$ is built in the definition of $\mathcal{L}$-structure.

Finally and most relevantly, $\mathcal{L}$ contains arbitrarily many function symbols of sort $\mathsf{H}^n \to \mathsf{I}$.

Standard structures are denoted by the domain of their home sort.

[1] In the final version of this paper we will omit Section 4 and refer to [CLCL] instead. We will also omit some proofs.

Clearly, saturated $\mathcal{L}$-structures exist but, with the exception of trivial cases (when I is finite), they are not standard. As a remedy, below we carve out a set of formulas $\mathcal{L}^{\mathrm{p}}$, the set of positive formulas, such that every model has an positive elementary saturated extension that is also standard.

As usual, $\mathcal{L}_{\mathsf{I}}$, $\mathcal{L}_{\mathsf{H}}$, and $\mathcal{L}$ denote both first-order languages and the corresponding set of formulas. We write $\mathcal{L}_x$ when when we restrict variables to x. If $\varphi(x) \in \mathcal{L}$ contains only terms of sort $\mathsf{H}^n \to \mathsf{H}$ and relations of sort H^n we will improperly say that $\varphi(x) \in \mathcal{L}_{\mathsf{H}}$ (which is only correct up to equivalence).

Note that $\mathcal{L}$ has two types of atomic formulas:

1. atomic formulas in $\mathcal{L}_{\mathsf{H}}$ (hence, up to equivalence, all formulas of $\mathcal{L}_{\mathsf{H}}$);
2. formulas of the form $\tau(x;\eta) \in C$, where $C \subseteq I^n$ is compact and $\tau(x;\eta)$ is a tuple of terms of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$.

Definition 2. A formula in $\mathcal{L}$ is positive if it uses only the Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{I}}$, $\exists^{\mathsf{I}}$ of sort I.

The set of positive formulas is denoted by $\mathcal{L}^{\mathrm{p}}$.

A continuous formula is a positive formula where only atomic formulas as in (2) above occur. The set of continuous formulas is denoted by $\mathcal{L}^{\mathrm{c}}$.

We will use Latin letters x, y, z for variables of sort H and Greek letters η, ε for variables of sort I. Therefore we can safely drop the superscript from quantifiers if they are followed by variables.

The positive formulas of Henson and Iovino correspond to those formulas in $\mathcal{L}^{\mathrm{c}}$ that do not have quantifiers of sort I.

Working within $\mathcal{L}^{\mathrm{p}}$ simplifies the comparision with classical model theory but comes at a price. Instead, allowing quantifiers of sort I comes almost for free. We will see (cf. Propositions 31) that the formulas in $\mathcal{L}^{\mathrm{p}}/\mathcal{L}^{\mathrm{c}}$ are approximated by formulas without quantifiers of sort I.

The extra expressive power offered by quantifiers of sort I is convenient. For instance, it is easy to see (cf. Example 3) that $\mathcal{L}^{\mathrm{c}}$ has at least the same expressive power as real valued logic

Example 3. Let $I = [0,1]$, the unit interval in $\mathbb{R}$. Let $\tau(x)$ be a term of sort $\mathsf{H}^{|x|} \to \mathsf{I}$. Then there is a positive formula that says $\sup_x \tau(x) = \alpha$. Indeed, consider the formula

$$\forall x\, \big[\tau(x) \dot{-} \alpha \in \{0\}\big] \quad \wedge \quad \forall \varepsilon \Big[\varepsilon \in \{0\} \,\vee\, \exists x\, \big[\alpha \dot{-} (\tau(x) + \varepsilon) \in \{0\}\big]\Big]$$

which, in a more legible form, becomes

$$\forall x\, \big[\tau(x) \leq \alpha\big] \quad \wedge \quad \forall \varepsilon > 0\, \exists x\, \big[\alpha \leq \tau(x) + \varepsilon\big].$$

The following example is inspired by [HPP].

Example 4. Let $\bar{M}$ be a saturated structure of signature $\mathcal{L}_{\mathsf{H}}$. The relevant part of $\mathcal{L}$ contains symbols for the functions $f : \bar{M}^n \to I$ that are continuous in the following sense: the inverse image of every compact $C \subseteq I$ is type-definable in $\mathcal{L}_{\mathsf{H}}(M)$ for every $M \preceq \bar{M}$. It is easy to verify that $\langle \bar{M}, I\rangle$ is positively saturated as defined in Section 5.

The theory of normed spaces is one of the motivating examples of standard structures. Essentially, functional analysis and model theory are only interested in the unit ball of normed spaces. To formalize the unit ball of a normed space as a standard structure (where I is the unit interval) is straightforward but cumbersome. Alternatively one can view a normed space as a many-sorted structure: a sort for each ball of radius $n \in \mathbb{Z}^+$. Unfortunately, this also results in a bloated formalism.

A neater formalization is possible if one accepts that some positive elementary extensions (to be defined below) of a normed space could contain vectors of infinite norm hence not be themselves proper normed spaces. However these infinities are harmless if one is only interested in the unit ball. In fact, inside any ball of finite radius these improper normed spaces are completely standard.

Example 5. Let $I = \mathbb{R} \cup \{\pm\infty\}$. Let $\mathcal{L}_{\mathsf{H}}$ be the language of real (or complex) vector spaces. The language $\mathcal{L}$ contains a function symbol $\|\text{-}\|$ of sort $\mathsf{H} \to \mathsf{I}$. Normed space are standard structures with the natural interpretation of the language. It is easy to verify that the unit ball of a standard structure is the unit ball of a normed space. Note that the same remains true if we add to the language any continuous (equivalently, bounded) operator or functional.

We conclude this introduction by advertising a question in [CLCL]. Is it possible to extend the Positive Compactness Theorem (Theorem 19) to a larger class of languages? E.g. can we allow in $\mathcal{L}$ (and in $\mathcal{L}^{\mathrm{p}}$) function symbols of sort $\mathsf{H}^n \times \mathsf{I}^m \to \mathsf{I}$ with both n and m positive? These would have natural interpretations, e.g. a group acting on a compact set I.

2. Henson-Iovino approximations

For φ, φ' (free variables are hidden) positive formulas possibly with parameters we write $\varphi' > \varphi$ if φ' is obtained by replacing in φ each atomic formula of the form $t \in C$ with $t \in C'$, where C' is some compact neighborhood of C. If no such atomic formulas occurs in φ, then $\varphi > \varphi$. We call φ' a weakening of φ. Note that $>$ is a dense transitive relation and that $\varphi \to \varphi'$ in every $\mathcal{L}$-structure.

We write $\tilde{\varphi} \perp \varphi$ when $\tilde{\varphi}$ is obtained by replacing each atomic formula $t \in C$ occurring in φ with $t \in \tilde{C}$ where $\tilde{C}$ is some compact set disjoint from C. The atomic formulas in $\mathcal{L}_{\mathsf{H}}$ are replaced with their negation. Finally each connective is replaced by its dual i.e., $\vee, \wedge, \exists, \forall$ are replaced by $\wedge, \vee, \forall, \exists$, respectively. We say that $\tilde{\varphi}$ is a strong negation of φ. It is clear that $\tilde{\varphi} \to \neg\varphi$ in every $\mathcal{L}$-structure.

Lemma 6. For all positive formulas φ

1. for every $\varphi' > \varphi$ there is a formula $\tilde{\varphi} \perp \varphi$ such that $\varphi \to \neg\tilde{\varphi} \to \varphi'$;
2. for every $\tilde{\varphi} \perp \varphi$ there is a formula $\varphi' > \varphi$ such that $\varphi \to \varphi' \to \neg\tilde{\varphi}$.

Proof. If $\varphi \in \mathcal{L}_{\mathsf{H}}$ the claims are obvious. Suppose φ is of the form $t \in C$. Let φ' be $t \in C'$, for some compact neighborhood of C. Let O be an open set such that $C \subseteq O \subseteq C'$. Then $\tilde{\varphi} = (t \in I \smallsetminus O)$ is as required by the lemma. Suppose instead that $\tilde{\varphi}$ is of the form $t \in \tilde{C}$ for some compact $\tilde{C}$ disjoint from C. By the normality of I, there is C', a compact neighborhood of C disjoint from $\tilde{C}$. Then $\varphi' = \big(t \in C'\big)$ is as required. The lemma follows easily by induction. □

For every type $p(x)$, we write

$$p'(x) \quad = \quad \big\{\varphi'(x) \;:\; \varphi' > \varphi \text{ for some } \varphi(x) \in p\big\}$$

in particular $\{\varphi(x)\}' = \big\{\varphi'(x) \;:\; \varphi' > \varphi\big\}$.

Definition 7. A set of formulas $\mathcal{F} \subseteq \mathcal{L}$ is p-dense modulo T, a theory, if for every positive $\varphi(x)$ and every $\varphi' > \varphi$, there is $\psi(x) \in \mathcal{F}$ such that $\varphi(x) \to \psi(x) \to \varphi'(x)$ holds in every standard structure that models T. If T is empty, we say that $\mathcal{F}$ is p-dense modulo $\mathcal{L}$.

We say c-dense if the property above is only required to hold for continuous formulas.

Example 8. By Lemma 6, the set of negations of positive formulas is p-dense modulo $\mathcal{L}$.

Example 9. Let $I = [0,1]$. The set of formulas built inductively from atomic formulas of the form $\tau \in \{0\}$, is c-dense modulo $\mathcal{L}$. In fact, as every compact subset of I is the intersection of finite unions of closed intervals, the set of formulas containing only atomic formulas of the form $\tau \in [\alpha,\beta]$ is c-dense. Finally note that $\tau \in [\alpha,\beta]$ is equivalent to $\tau \dot{-} \beta \in \{0\} \wedge \alpha \dot{-} \tau \in \{0\}$.

Example 10. In Section 7 we prove that the set of positive/continuous formulas without quantifiers of sort I is p-dense/c-dense.

3. Morphisms

Let M and N be two standard structures. We say that a partial map $f : M \to N$ is p-elementary if for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$ and every $a \in (\operatorname{dom} f)^{|x|}$

1. $$M \models \varphi(a) \quad\Rightarrow\quad N \models \varphi(fa).$$

In words, we say that f preserve the truth of the positive formulas. A p-elementary map that is total is called a p-elementary embedding. When the identity map $\mathrm{id}_M : M \to N$ is an p-elementary embedding, we write $M \preceq^{\mathrm{p}} N$ and say that M is an p-elementary substructure of N.

The discussion of c-elementarity needs some extra care because $\mathcal{L}^{\mathrm{c}}$ does not contains equality in the home sort. Therefore we postpone it till Section 9.

On the other hand, as p-elementary maps are in particular $\mathcal{L}_{\mathsf{H}}$-elementary, they are injective. However, their inverse need not be p-elementary. In other words, the converse of the implication in (1) may not hold. Hence the following notion of elementarity which is more robust. We say that the map $f : M \to N$ is an approximate p-elementary if for every formula $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$, and every $a \in (\operatorname{dom} f)^{|x|}$

2. $$M \models \varphi(a) \quad\Rightarrow\quad N \models \{\varphi(fa)\}'.$$

One convenient feature of approximate p-elementarity is that it suffices verify (2) for any/some p-dense set of formulas. It is clear that p-elementarity implies its approximate variant. We will see that with a slight amount of saturation also the converse holds (Proposition 21). Moreover, under full saturation, these morphims becomes $\mathcal{L}$-elementarity maps (Corollary 30).

The following holds in general.

Fact 11. If $f : M \to N$ is approximate p-elementary then
$$M \models \{\varphi(a)\}' \quad\Leftrightarrow\quad N \models \{\varphi(fa)\}'.$$
for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$, and every $a \in (\operatorname{dom} f)^{|x|}$.

Proof. $\Rightarrow$ Fix $\varphi' > \varphi$ and assume $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By the definition of approximate elementarity, $N \models \varphi''(fa)$ and $N \models \varphi'(fa)$ follows.

$\Leftarrow$ Assume the r.h.s. of the equivalence. Fix $\varphi' > \varphi$ and prove $M \models \varphi'(a)$. Let $\varphi' > \varphi'' > \varphi$. By Lemma 6 there is some $\tilde{\varphi} \perp \varphi''$ such that $\varphi'' \to \neg\tilde{\varphi} \to \varphi'$. Then $N \models \neg\tilde{\varphi}(fa)$ and therefore $M \models \neg\tilde{\varphi}(a)$. Then $M \models \varphi'(a)$. □

Finally, we consider the notion of partial embeddings. Though the definition seems weaker, we show that the notion is competely classical. We say that the map $f : M \to N$ is a partial embedding if the implication in (1) holds for all atomic formulas $\varphi(x)$. Note that this is the same as requiring that (1) holds for all formulas without quantifiers of sort H.

Fact 12. If $f : M \to N$ is a partial embedding then

$$M \models \varphi(a) \quad \Leftrightarrow \quad N \models \varphi(fa)$$

for every $a \in (\mathrm{dom} f)^{|x|}$ and every formula $\varphi(x) \in \mathcal{L}^{\mathrm{P}}$ without quantifiers of sort H.

Proof. The equivalence is trivial for formulas in $\mathcal{L}_{\mathsf{H}}$ so we only consider atomic formulas of the form $t \in C$. Implication $\Rightarrow$ holds by definition. Vice versa, if $M \models \tau(a) \notin C$ then, by normality, $M \models \tau(a) \in \tilde{C}$ for some compact $\tilde{C}$ disjoint of C. By the definition of partial embedding, $N \models \tau(a) \in \tilde{C}$. Hence $N \models \tau(a) \notin C$. Induction is immediate. □

4. The standard part

Our goal is to prove a compactness theorem for positive theories that only requires standard structures. To bypass lengthy routine applications of ultrafilters, ultrapowers, and ultralimits, we assume the Classical Compactness Theorem and build on that.

In this section we recall the notion of standard part of an element of the elementary extension of a compact Hausdorff topological space. Our goal is to prove Lemma 17 which in turn is required for the proof of the Positive Compactness Theorem (Theorem 19). The reader willing to accept it without proof may skip this section.

Let $\langle N, {}^*I\rangle$ be an $\mathcal{L}$-structure that extends $\mathcal{L}$-elementarily the standard structure M. Let η be a free variable of sort I. For each $\beta \in I$, we define the type

$$\mathrm{m}_\beta(\eta) \quad = \quad \{\eta \in D \;:\; D \text{ compact neighborhood of } \beta\}.$$

The set of the realizations of $\mathrm{m}_\beta(\eta)$ in ${}^*I$ is known to nonstandard analysts as the monad of β. The following fact is well-known.

Fact 13. For every $\alpha \in {}^*I$ there is a unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Proof. Negate the existence of β. For every $\gamma \in I$ pick some compact neighborhood D_γ of γ, such that ${}^*I \models \alpha \notin D_\gamma$. By compactness there is some finite $\Gamma \subseteq I$ such that D_γ, with $\gamma \in \Gamma$, cover I. By elementarity, the interpretation in ${}^*I$ of these D_γ cover ${}^*I$. A contradiction. The uniqueness of β follows from normality. □

We denote by $\mathrm{st}(\alpha)$ the unique $\beta \in I$ such that ${}^*I \models \mathrm{m}_\beta(\alpha)$.

Fact 14. For every $\alpha \in {}^*I$ and every compact $C \subseteq I$

$${}^*I \quad \models \quad \alpha \in C \;\rightarrow\; \mathrm{st}(\alpha) \in C.$$

Proof. Assume $\mathrm{st}(\alpha) \notin C$. By normality there is a compact set D disjoint from C that is a neighborhood of $\mathrm{st}(\alpha)$. Then ${}^*I \models \alpha \in D \subseteq \neg C$. □

Fact 15. For every $\alpha \in ({}^*I)^{|\alpha|}$ and every function symbol f of sort $\mathsf{I}^{|\alpha|} \to \mathsf{I}$

$${}^*I \quad \models \quad \mathrm{st}\big(f(\alpha)\big) = f\big(\mathrm{st}(\alpha)\big).$$

Proof. By Fact 13 and the definition of st(-) it suffices to prove that ${}^*I \models f(\alpha) \in D$ for every compact neighborhood D of $f\big(\mathrm{st}(\alpha)\big)$.

Fix one such D. Then $\mathrm{st}(\alpha) \in f^{-1}[D]$. By continuity $f^{-1}[D]$ is a compact neighborhood of $\mathrm{st}(\alpha)$. Therefore ${}^*I \models \alpha \in f^{-1}[D]$ and, as $I \preceq {}^*I$ we obtain ${}^*I \models f(\alpha) \in D$. □

The standard part of $\langle N, {}^*I\rangle$ is the standard structure $\langle N, I\rangle$ that interprets the symbols f of sort $\mathsf{H}^n \to \mathsf{I}$ as the functions

$$f^N(a) \quad = \quad \mathrm{st}\big({}^*f(a)\big) \qquad\qquad \text{for all } a \in N^n,$$

where ${}^*f$ is the interpretation of f in $\langle N, {}^*I\rangle$. Symbols in $\mathcal{L}_{\mathsf{H}}$ maintain the same interpretation.

Fact 16. With the notation as above. Let $\tau(x;\eta)$ be a term of sort $\mathsf{H}^{|x|} \times \mathsf{I}^{|\eta|} \to \mathsf{I}$. Then for every $a \in M^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$\tau^N\big(a;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*\tau(a;\alpha)\big)$$

Proof. When a t is a function symbol of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, the claim holds by definition. When t is a function symbol of sort $\mathsf{I}^{|\eta|} \to \mathsf{I}$, the claim follows from Fact 15. Now, assume inductively that

$$t_i^N\big(a;\mathrm{st}(\alpha)\big) \quad = \quad \mathrm{st}\big({}^*\,t_i(a;\alpha)\big)$$

holds for the terms $t_1(x;\eta),\ldots,t_n(x;\eta)$ and let $t = f(t_1,\ldots,t_n)$ for some function f of sort $\mathsf{I}^n \to \mathsf{I}$. Then the claim follows immediately from the induction hypothesis and Fact 15. □

Lemma 17. With the notation as above. For every $\varphi(x;\eta) \in \mathcal{L}^{\mathrm{p}}$, $a \in N^{|x|}$ and $\alpha \in ({}^*I)^{|\eta|}$

$$\langle N, {}^*I\rangle \models \varphi(a;\alpha) \quad\Rightarrow\quad N \models \varphi\big(a;\mathrm{st}(\alpha)\big)$$

Proof. Suppose $\varphi(x;\eta)$ is $\mathcal{L}^{\mathrm{p}}$-atomic. If $\varphi(x;\eta)$ is a formula of $\mathcal{L}_{\mathsf{H}}$ the claim is trivial. Otherwise $\varphi(x;\eta)$ has the form $\tau(x;\eta) \in C$. Assume that the tuple $\tau(x;\eta)$ consists of a single term. The general case follows easily from this special case. Assume that $\langle N, {}^*I\rangle \models \tau(a;\alpha) \in C$. Then $\mathrm{st}\big({}^*\tau(a;\alpha)\big) \in C$ by Fact 14. Therefore $\tau^N\big(a;\mathrm{st}(\alpha)\big) \in C$ follows from Fact 16. This proves the lemma for atomic formulas. Induction is immediate. □

Corollary 18. Let M be a standard structure. Let $p(\eta) \subseteq \mathcal{L}^{\mathrm{p}}(M)$ be a type that does not contain existential quantifiers of sort H. Then, if $p(\eta)$ is finitely consistent in M, it is realized in M.

The corollary has also a direct proof. This goes through the observation that the formulas in $p(\eta)$ define compact subsets of I.

Proof. Let $\langle {}^*M, {}^*I\rangle$ be an $\mathcal{L}$-elementary saturated superstructure of $\langle M, I\rangle$. Then $\langle {}^*M, {}^*I\rangle \models p(\alpha)$ for some $\alpha \in^* I$. By the lemma above, ${}^*M \models p(\mathrm{st}(\alpha))$. Now observe that the truth of formulas without existential quantifiers of sort H is preserved by substructures. □

The exclusion of existential quantifiers of sort H is necessary. For a counterexample take $M = I = [0,1]$. Assume that $\mathcal{L}$ contains a function symbol for the identity map $\iota : M \to I$. Let $p(\eta)$ contain the formulas $\exists x\big(x > 0 \wedge \iota x + \eta \in [0, 1/n]\big)$ for all positive integers n.

5. Compactness

It is convenient to distinguish between consistency with respect to standard structures and consistency with respect to $\mathcal{L}$-structures. We say that a theory T is $\mathcal{L}$-consistent if $\langle M, {}^*I\rangle \models T$ for some $\mathcal{L}$-structure $\langle M, {}^*I\rangle$ (this is the classical notion of concistency). We say that T is standardly consistent if $M \models T$ is for some standard structure M. By Lemma 17 these two notions coincide if T is positive. Therefore we have the following.

Theorem 19 (Positive Compactness Theorem). Let T be a positive theory. Then, if T is finitely $\mathcal{L}$-consistent, it is also standardly consistent.

An $\mathcal{L}$-structure N is positively λ-saturated if it realizes all types $p(x;\eta) \subseteq \mathcal{L}^{\mathrm{p}}(N)$ with fewer than λ parameters that are finitely consistent in N. When $\lambda = |N|$ we simply say p-saturated. The existence of p-saturated standard structures is obtained from the classical case just as for Theorem 19.

Theorem 20. Every standard structure has a p-elementary extension to a p-saturated standard structure (possibly of inaccessible cardinality).

Proof. Let $\langle {}^*M, {}^*I\rangle$ be an $\mathcal{L}$-saturated $\mathcal{L}$-elementary extension of the standard structure $\langle M, I\rangle$. Let $\langle {}^*M, I\rangle$ be its standard part as defined in Section 4. By Lemma 17, $\langle {}^*M, I\rangle$ realizes all finitely consistent positive types with fewer than $|{}^*M|$ parameters. □

The following proposition shows that a slight amount of saturation tames the positive formulas.

Proposition 21. Let N be a positively ω-saturated standard structure. Then

$$\{\varphi(x;\eta)\}' \;\leftrightarrow\; \varphi(x;\eta)$$

holds in N for every formula $\varphi(x;\eta) \in \mathcal{L}^{\mathrm{p}}(N)$.

Proof. We prove $\rightarrow$, the non trivial implication. The claim is clear for atomic formulas. Induction for conjunction, disjunction and the universal quantifiers is immediate. We consider case of the existential quantifiers of sort H. Assume inductively

ih. $$\{\varphi(x,z;\eta)\}' \;\rightarrow\; \varphi(x,z;\eta)$$

We need to prove

$$\{\exists z\, \varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\, \varphi(x,z;\eta)$$

From (ih) we have

$$\exists z\, \{\varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\, \varphi(x,z;\eta)$$

Therefore it suffices to prove

$$\{\exists z\, \varphi(x,z;\eta)\}' \;\rightarrow\; \exists z\, \{\varphi(x,z;\eta)\}'$$

Replace the variables $x;\eta$ with parameters, say $a;\alpha$, and assume that $N \models \exists z\, \varphi'(a,z,;\alpha)$ for every $\varphi' > \varphi$. We need to prove the consistency of the type $\{\varphi'(a,z,;\alpha) : \varphi' > \varphi\}$. By saturation, finite consistency suffices. This is clear if we show that the antecedent is closed under conjunction. Indeed it is easy to verify that if $\varphi_1, \varphi_2 > \varphi$ then $\varphi_1 \wedge \varphi_2 > \varphi'$ for some $\varphi' > \varphi$. In words, the set of approximations of φ is a directed set.

For existential quantifiers of sort I we argue similarly. □

By Corollary 18, when $\varphi(x;\eta)$ does not contains existential quantifiers of sort H, the proposition above does not require the assumption of saturation. In general, some saturation is necessary: consider the model presented after Corollary 18 and the formula $\exists x\big(x > 0 \;\wedge\; \iota(x) \in \{0\}\big)$.

Remark 22. A consequence of Proposition 21 is that the approximate morphisms defined in Section 3 coincide with their unapproximate version when the codomain is positively ω-saturated.

6. The monster model

We denote by $\mathcal{U}$ some large p-saturated standard structure which we call the positive monster model. Truth is evaluated in $\mathcal{U}$ unless otherwise is specified. We denote by T the positive theory of $\mathcal{U}$. The density of a set of formulas is understood modulo T. Below we say p-model for p-elementary substructure of $\mathcal{U}$. We stress once again that the truth of some $\varphi \in \mathcal{L}^{\mathrm{p}}(M)$ in a p-model M implies the truth of φ (in $\mathcal{U}$) but not vice versa. If also the converse implication holds we say M a strong p-model. However, all p-models agree on the approximated truth, see Fact 11 and Remark 22,

$$M \models \{\varphi\}' \quad \Leftrightarrow \quad \{\varphi\}'.$$

Strong p-models are those that realize the equivalence in Proposition 21, e.g. ω-saturated p-models are strong p-models.

The following fact demostrates how positive compactness applies. There are some subtle differences from the classical setting. Let $A \subseteq \mathcal{U}$ be a small set throughout this section.

Fact 23. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ be a type. Then for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(\mathcal{U})$

i. if $p(x) \to \neg\varphi(x)$ then $\psi(x) \to \neg\varphi(x)$ for some $\psi(x)$ conjunction of formulas in $p(x)$;

ii. if $p(x) \to \varphi(x)$ and $\varphi' > \varphi$ then $\psi(x) \to \varphi'(x)$ for some conjunction of formulas in $p(x)$.

Proof. (i) is immediate by saturation; (ii) follows fromrom (i) by Lemma 6. □

Fact 24. Let $\mathcal{F}$ be a p-dense/c-dense set of positive formulas. Then $\mathcal{F}'$ is p-dense/c-dense.

Proof. Let $\varphi' > \varphi$. Pick φ'' such that $\varphi' > \varphi'' > \varphi$ and $\psi \in \mathcal{F}$ such that $\varphi(x) \to \psi(x) \to \varphi''(x)$. It suffices to prove that $\psi'(x) \to \varphi'(x)$ for some $\psi' > \psi$. By Proposition 21, $\psi(x) \leftrightarrow \{\psi(x)\}'$. Therefore, $\psi'(x) \to \varphi'(x)$ follows from Fact 23. □

We write $S^{\mathrm{p}}_x(A)$ for the set of maximally consistent subsets of $\mathcal{L}^{\mathrm{p}}_x(A)$. We define p-tp$(a/A)$, the positive type of a over A, to be the set of formulas $\{\varphi(x) \in \mathcal{L}^{\mathrm{p}}(A) \ : \ \varphi(a)\}$. Similarly, we define $S^{\mathrm{c}}_x(A)$ and c-tp(a/A).

In general, if $\mathcal{F}$ is any set of formulas, we write $\mathcal{F}$-tp(a/A), for the type $\{\varphi(x) \in \mathcal{F}(A) \ : \ \varphi(a)\}$. The undecorated symbol tp(a/A) denotes the $\mathcal{L}$-type.

Fact 25. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$. The following are equivalent

1. $p(x)$ is a maximally consistent subsets of $\mathcal{L}^{\mathrm{p}}_x(A)$;
2. $p(x) =$ p-tp(a/A) for some $a \in \mathcal{U}^{|x|}$.

Proof. (1⇒2) Then $p(x) \subseteq$ p-tp(a/A) and, by maximality $p(x) =$ c-tp(a/A).

(2⇒1) From Lemma 6 and Proposition 21 we obtain

$$\neg\varphi(x) \ \to \ \bigvee_{\tilde{\varphi} \perp \varphi} \tilde{\varphi}(x).$$

Suppose $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(A) \smallsetminus p$. Then $\neg\varphi(a)$. Hence $\tilde{\varphi}(a)$ holds for some $\tilde{\varphi} \perp \varphi$ and $p(x) \rightarrow \neg\varphi(x)$ follows. □

Note that in the proof above we could replace $\mathcal{L}^{\mathrm{p}}$ with any set of positive formulas closed under strong negation. Therefore we can also claim the following.

Fact 26. Let $p(x) \subseteq \mathcal{F}(A)$, where $\mathcal{F}$ is a set of positive formulas closed under strong negation. Then the following are equivalent

1. $p(x)$ is a maximally consistent subset of $\mathcal{F}_x(A)$;
2. $p(x) = \mathcal{F}\text{-tp}(a/A)$ for some $a \in \mathcal{U}^{|x|}$.

Proposition 27. Let $\mathcal{F}$ be a p-dense set of positive formulas. Then for every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$

i. $$\neg\varphi(x) \leftrightarrow \bigvee\{\psi(x) \in \mathcal{F} : \psi(x) \rightarrow \neg\varphi(x)\}$$

ii. $$\neg\varphi(x) \leftrightarrow \bigvee\{\neg\psi(x) : \psi(x) \in \mathcal{F} \text{ and } \neg\psi(x) \rightarrow \neg\varphi(x)\}.$$

Proof. (i) Only $\rightarrow$ requires a proof. Let $a \in \mathcal{U}^{|x|}$ be such that $\neg\varphi(a)$. Let $p(x) = \text{p-tp}(a)$. By Fact 25, $p(x) \rightarrow \neg\varphi(x)$. As $\mathcal{F}$ is p-dense, $p(x) \leftrightarrow q(x) = \mathcal{F}\text{-tp}(a)$. Then, by compactness, $\psi(x) \rightarrow \neg\varphi(x)$ for some $\psi \in q \subseteq \mathcal{F}$.

(ii) By density

$$\varphi(x) \leftrightarrow \bigwedge\{\psi(x) \in \mathcal{F} : \varphi(x) \rightarrow \psi(x)\}.$$

Negating both sides of the equivalence we obtain (ii). □

We will need a similar result for larger class formulas. We write Σ^{p} for the set of formulas of the form $\exists y\, \vartheta(x,y)$ where $\vartheta(x,y)$ is a positive Boolean combination of formulas of the form $\neg\psi(x,y)$ or $\psi'(x,y)$ for some $\psi(x,y) \in \mathcal{L}^{\mathrm{p}}$ and $\psi' > \psi$. The following follows easily from the proposition above by induction on the syntax of $\varphi(x)$.

Proposition 28. Let $\mathcal{F}$ be a p-dense set of positive formulas. Then for every $\varphi(x) \in \Sigma^{\mathrm{p}}$

$$\varphi(x) \leftrightarrow \bigvee\{\psi(x) \in \mathcal{F} : \psi(x) \rightarrow \varphi(x)\}.$$

7. Elimination of quantifiers of sort I for positive formulas

We show that the quantifiers of sort I can be eliminated up to some approximation. Precisely, we prove that $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$, the set of positive formulas without quantifiers of sort I, is p-dense modulo T (the theory of $\mathcal{U}$).

A partial map is $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary if it preserves the truth of formulas in $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$.

Proposition 29. Every $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary map $f : \mathcal{U} \rightarrow \mathcal{U}$ of small cardinality extends to an automorphism of $\mathcal{U}$.

Proof. By Fact 25, the inverse of an $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary map is also $\mathcal{L}^{\mathrm{p}}_{\mathsf{lqf}}$-elementary. Then we can extend the map by back-and-forth as usual. □

Let $A \subseteq \mathcal{U}$ be a small set throughout this section. We apply the proposition above to obtain a strengthening of Fact 26. For $a, b \in \mathcal{U}^{|x|}$ we write $a \equiv_A b$ if a and b satisfy the same $\mathcal{L}$-formulas over A.

Corollary 30. Let $p(x) = \mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}\text{-tp}(a/A)$ and $q(x) = \mathrm{tp}(a/A)$. Then $p(x) \leftrightarrow q(x)$.

Proof. Only $\rightarrow$ requires a proof. If $b \models p(x)$ then there is an $\mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$-elementary map $f \supseteq \mathrm{id}_A$ such that $fa = b$. Then f extends to an automorphism. As every automorphism is $\mathcal{L}$-elementary, $a \equiv_A b$, and the corollary follows. □

The corollary says that $\mathcal{U}$ does not distinguish between $\mathcal{L}$, $\mathcal{L}^{\mathrm{p}}$, and $\mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$ types. In the classical setting, from an equivalence between types one derives an equivalence between formulas. Without negation, this is not true. Still, we can infer an approximate form of equivalence.

Proposition 31. The set $\mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$ is p-dense modulo T.

Proof. Let $\varphi(x)$ be a positive formula. We need to prove that for every $\varphi' > \varphi$ there is some formula $\psi(x) \in \mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$ such that $\varphi(x) \rightarrow \psi(x) \rightarrow \varphi'(x)$. By Corollary 30 and Proposition 21

$$\neg\varphi(x) \quad \rightarrow \quad \bigvee_{p'(x)\rightarrow\neg\varphi(x)} p'(x)$$

where $p(x)$ ranges over the maximally consistent $\mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$-types. By Fact 23 and Lemma 6

$$\neg\varphi(x) \quad \rightarrow \quad \bigvee_{\neg\tilde{\psi}(x)\rightarrow\neg\varphi(x)} \neg\tilde{\psi}(x),$$

where $\tilde{\psi}(x) \in \mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$. Equivalently,

$$\varphi(x) \quad \leftarrow \quad \bigwedge_{\tilde{\psi}(x)\leftarrow\varphi(x)} \tilde{\psi}(x).$$

By compactness, see Fact 23, for every $\varphi' > \varphi$ there are some finitely many $\tilde{\psi}_i(x) \in \mathcal{L}^{\mathrm{p}}_{\mathrm{lqf}}$ such that

$$\varphi'(x) \quad \leftarrow \quad \bigwedge_{i=1,\ldots,n} \tilde{\psi}_i(x) \quad \leftarrow \quad \varphi(x)$$

which yields the interpolant required by the proposition. □

8. Cauchy completeness and c-models

For $\tau(x,z)$ a term of sort $\mathsf{H}^{|x|+|z|} \rightarrow \mathsf{I}$ we define the formula

$$x \sim_\tau y \quad = \quad \forall z \;\; \tau(x,z) = \tau(y,z),$$

where the expression $\alpha = \beta$ is shorthand for $\langle\alpha,\beta\rangle \in \Delta$, where Δ is the diagonal of I^2. We also define the type

$$x \sim y \quad = \quad \Big\{x \sim_\tau y \;:\; \tau(x,z) \text{ as above}\Big\}.$$

Fact 32. For any $a = \langle a_i : i < \lambda\rangle$ and $b = \langle b_i : i < \lambda\rangle$

$$a \sim b \quad \Leftrightarrow \quad a_i \sim b_i \quad \text{for every } i < \lambda.$$

Proof. Only implication $\Leftarrow$ requires a proof. Assume $a_i \sim b_i$ for every i. Let $|a| = |b| = \lambda$ and assume inductively that

$$\forall y,z \quad \tau(a,y,z) \quad = \quad \tau(b,y,z)$$

holds for every term $\tau(x,y,z)$, with $|x| = \lambda$ (universal quantification over the free variables is understood throughout the proof). In particular for any a_λ

1. $$\forall z \quad \tau(a, a_\lambda, z) \;=\; \tau(b, a_\lambda, z).$$

As $a_\lambda \sim b_\lambda$ then

$$\forall x, z \quad \tau(x, a_\lambda, z) \;=\; \tau(x, b_\lambda, z)$$

and in particular

2. $$\forall z \quad \tau(b, a_\lambda, z) \;=\; \tau(b, b_\lambda, z).$$

From (1) and (2) we obtain

$$\forall z \quad \tau(a, a_\lambda, z) \;=\; \tau(b, b_\lambda, z).$$

For limit ordinals induction is trivial. □

Note that the approximations of the formula $x \sim_\tau y$ have the form

$$x \sim_{\tau,D} y \;=\; \forall z \;\langle \tau(x,z), \tau(y,z)\rangle \in D$$

for some compact neighborhood D of Δ. Hence we define

$$x \sim' y \;=\; \Big\{ x \sim_{\tau,D} y \;:\; \tau(x,z) \text{ as above, } D \text{ compact neigborhood of } \Delta \Big\}.$$

If $\tau(x,z) = \tau_1(x,z), \ldots, \tau_n(x,z)$ is a tuple of terms, by $x \sim_{\tau,D} y$ we denote the conjunction of the formulas $x \sim_{\tau_i,D} y$.

The formulas $x \sim_{\tau,D} y$, as τ ranges over the finite tuples parameter-free terms and D over the compact neighborhoods of the diagonal, form a base for a system of entougages on $\mathcal{U}^{|x|}$. We refer to this uniformity and the topology associated as the I-topology. Though not needed in the sequel, it is worth mentioning that the I-topology on $\mathcal{U}^{|x|}$ coincides with the product of the I-topology on $\mathcal{U}$. This can be verified by an argument similar to the proof of Fact 32.

Fact 33. For every $\varphi(x) \in \mathcal{L}^{\mathrm{p}}$ that does not contain subformulas of sort $\mathcal{L}_{\mathrm{H}}$

$$x \sim y \;\rightarrow\; \varphi(x,\eta) \leftrightarrow \varphi(y,\eta)$$

Proof. By induction on $\varphi(x,\eta)$. Suppose $\varphi(x,\eta)$ as the form $\tau(x,\eta) \in C$. Then clearly

$$x \sim_\tau y \;\rightarrow\; \varphi(x,\eta) \leftrightarrow \varphi(y,\eta)$$

Induction is easy. As an example, we spell out the proof for the quantifier $\exists^{\mathrm{H}}$. Assume inductively that

$$x, x' \sim y, y' \;\rightarrow\; \varphi(x,x',\eta) \leftrightarrow \varphi(y,y',\eta)$$

Then in paricular

$$x \sim y \;\rightarrow\; \varphi(x,z,\eta) \leftrightarrow \varphi(y,z,\eta).$$

Therefore

$$x \sim y \;\rightarrow\; \exists z\, \varphi(x,z,\eta) \leftrightarrow \exists z\, \varphi(y,z,\eta).$$ □

Following corollary corresponds to the Perturbation Lemma [HI, Proposition 5.15].

Corollary 34. For every $\varphi(x) \in \mathcal{L}^{\mathrm{c}}$, every $\varphi' > \varphi$, and every $\tilde{\varphi} \perp \varphi$ there is a tuple of terms τ and a compact neighborhood of the diagonal D such that

i. $$x \sim_{\tau,D} y \wedge \varphi(y) \;\rightarrow\; \varphi'(x)$$

ii. $x \sim_{\tau,D} y \wedge \tilde{\varphi}(y) \rightarrow \neg\varphi(x)$.

Proof. As $x \sim y \cup \{\varphi(x)\} \rightarrow \varphi(y)$ by the fact above, (i)) follows from Fact 23. Similarly, we obtain (ii) from $x \sim y \cup \{\tilde{\varphi}(x)\} \rightarrow \neg\varphi(y)$. □

We say that $p(x) \subseteq \mathcal{L}^{c}(\mathfrak{U})$ is a Cauchy type if it is consistent and $p(x) \wedge p(y) \rightarrow x \sim y$. Equivalenltly, $p(x)$ is a Cauchy type if $p(x) \leftrightarrow a \sim x$ for some/any $a \models p(x)$. We say that a set $A \subseteq \mathfrak{U}$ is Cauchy complete if $p(\mathfrak{U}) \subseteq A^{|x|}$ for every Cauchy type $p(x) \subseteq \mathcal{L}^{c}(A)$. Note that Cauchy complete sets are in particular closed under $\sim$-equivalence.

We say that $M \subseteq \mathfrak{U}$ is a c-model if it is Cauchy complete and

c. $M \models \varphi(a) \Rightarrow \varphi(a)$ for all formulas $\varphi(x) \in \mathcal{L}^{c}$ and $a \in M^{|x|}$.

Note that $\mathcal{L}^{c}$ is closed under weakening and strong negation. Therefore the argument in the proof of Fact 11 proves that (c) is equivalent to

ac. $M \models \{\varphi(a)\}' \Leftrightarrow \{\varphi(a)\}'$ for all formulas $\varphi(x) \in \mathcal{L}^{c}$ and $a \in M^{|x|}$.

We say what a type $q(x)$ is finitely satisfiable in M if every conjunction of formulas in $q(x)$ has a solution in M.

Fact 35. Let M satisfy (c) above. Then the following are equivalent

1. M is Cauchy complete (therefore it is a c-model);
2. $a \in M$ whenever $a \sim' x$ is finitely satisfied in M.

Proof. (1⇒2) Assume that $a \sim' x$ is finitely satisfied in M. Let $a_{\tau,D} \in M$ be a solution of $a \sim_{\tau,D} x$ where τ ranges over the finite tuples parameter-free terms and D ranges over the compact neighborhoods of the diagonal. Then, by the uniqueness of the limit (up to $\sim$-equivalence), the type $p(x)$ containing the formulas $x \sim_{\tau,D} a_{\tau,D}$ is a Cauchy type. By (1), $p(x)$ is realized in M. Then all its realizations belong to M.

(2⇒1) Let $p(x) \subseteq \mathcal{L}^{c}(M)$ be a Cauchy type. Then $p(x) \leftrightarrow a \sim x$ for some $a \models p(x)$. By (ac) above, $p'(x)$ is finitely satisfied in M. Then also $a \sim' x$ is finitely satisfied. Then $p(x)$ is realized in M. □

Fact 36. Let M be a c-model. Then M is a substructure of $\mathfrak{U}$.

Proof. Let $a \in M^{n}$ and let f be a function symbol of sort $\mathsf{H}^{n} \rightarrow \mathsf{H}$. We prove that $f a \in M$. By Fact 35, it suffices to show that $f a \sim' x$ is finitely satisfied in M. This is the case, in fact, $\{\exists x\, f a \sim_{\tau} x\}'$ holds in $\mathfrak{U}$ hence, by (ac), it holds also in M. □

9. Elimination of quantifiers of sort I for continuous formulas

We adapt the argument in the previous section to continuous formulas. But first we need some definitions.

Let $\mathfrak{U}/{\sim}$ be the quotient of $\mathfrak{U}$ by the equivalence relation defined in Section 8. The equivalence class of $a \in \mathfrak{U}$ is denoted by $[a]$. A c-elementary map is a (partial) map $f : \mathfrak{U}/{\sim} \rightarrow \mathfrak{U}/{\sim}$ such that $\varphi(a) \rightarrow \varphi(b)$ for every $\varphi(x) \in \mathcal{L}^{c}$ and a, b such that $f\big([a]\big) = [b]$.

The class $\mathcal{L}^{c}_{\mathsf{Iqf}}$ and the $\mathcal{L}^{c}_{\mathsf{Iqf}}$-elementary maps are defined in analogy to the respective positive variant. An automorphism of $\mathfrak{U}/{\sim}$ is a total and surjective $\mathcal{L}^{c}_{\mathsf{Iqf}}$-elementary map (which is also c-elementary).

For brevity we will often confuse $[a]$ with a.

Proposition 37. All $\mathcal{L}^{\mathsf{c}}_{\mathsf{lqf}}$-elementary maps of small cardinality extend to automorphisms of $\mathcal{U}/{\sim}$.

Proof. By Fact 26, the inverse of an $\mathcal{L}^{\mathsf{c}}_{\mathsf{lqf}}$-elementary map is also $\mathcal{L}^{\mathsf{c}}_{\mathsf{lqf}}$-elementary. Then we can extend the map by back-and-forth as usual........ □

10. The Tarski-Vaught test and the Löwenheim-Skolem theorem

The following proposition is our version of the Tarski-Vaught test.

Proposition 38. Let M be a subset of $\mathcal{U}$. Let $\mathcal{F}$ be a p-dense set of positive formulas. Then the following are equivalent

1. M is a p-model;
2. for every formula $\psi(x) \in \mathcal{F}(M)$
$$\exists x\, \psi(x) \;\;\Rightarrow\;\; \text{for every } \psi' > \psi \text{ there is an } a \in M \text{ such that } \psi'(a);$$
3. for every formula $\psi(x) \in \mathcal{F}(M)$
$$\exists x\, \neg\psi(x) \;\;\Rightarrow\;\; \text{there is an } a \in M \text{ such that } \neg\psi(a).$$

If M is Cauchy compelte and $\mathcal{F}$ is a c-dense set of continuous formulas then (2) and (3) above are equivalent to

1′. M is a c-model.

Proof. (1⇒2) Assume $\exists x\, \psi(x)$ and let $\psi' > \psi$ be given. By Lemma 6 there is some $\tilde{\psi} \perp \psi$ such that $\psi(x) \to \neg\tilde{\psi}(x) \to \psi'(x)$. Then $\neg\forall x\, \tilde{\psi}(x)$ hence, by (1), $M \models \neg\forall x\, \tilde{\psi}(x)$. Then $M \models \neg\tilde{\psi}(a)$ for some $a \in M$. Hence $M \models \psi'(a)$ and $\psi'(a)$ follows from (1).

(2⇒3) Assume (2) and let $\psi(x) \in \mathcal{F}(M)$ be such that $\exists x\, \neg\psi(x)$. By Proposition 39, there are a consistent $\varphi(x) \in \mathcal{F}(M)$ and some $\varphi' > \varphi$ such that $\varphi'(x) \to \neg\psi(x)$. Then (3) follows.

(2⇒1) Assume (2). By the classical Tarski-Vaught test $M \preceq_{\mathsf{H}} \mathcal{U}$. Then M is the domain of a substructure of $\mathcal{U}$. Then $\varphi(a) \Leftrightarrow M \models \varphi(a)$ holds for every atomic formula $\varphi(x)$ and for every $a \in M^{|x|}$. Now, assume inductively

$$M \models \varphi(a,b) \;\;\Rightarrow\;\; \varphi(a,b).$$

Using (2) and the induction hypothesis we prove by contrapposition that

$$M \models \forall y\, \varphi(a,y) \;\;\Rightarrow\;\; \forall y\, \varphi(a,y).$$

Indeed,

$$\begin{aligned}
\neg\forall y\, \varphi(a,y) \;\;&\Rightarrow\;\; \exists y\, \neg\varphi(a,y) \\
&\Rightarrow\;\; \exists y\, \neg\varphi'(a,y) && \text{for some } \varphi' > \varphi \\
&\Rightarrow\;\; \exists y\, \neg\psi(a,y) && \text{for some } \psi \in \mathcal{F} \text{ such that } \varphi(a,y) \to \psi(a,y) \to \varphi'(a,y) \\
&\Rightarrow\;\; \neg\psi(a,b) && \text{for some } b \in M \text{ by (2)} \\
&\Rightarrow\;\; \neg\varphi(a,b) \\
&\Rightarrow\;\; M \models \neg\varphi(a,b) && \text{by induction hypothesis} \\
&\Rightarrow\;\; M \not\models \forall y\, \varphi(a,y).
\end{aligned}$$

Induction for the connectives $\vee$, $\wedge$, $\exists^{\mathsf{H}}$, $\exists^{\mathsf{I}}$, and $\forall^{\mathsf{I}}$ is straightforward.

For the second part, note that the proof of (1⇒2) also proves (1′ ⇒2) while the proof of (3⇒1) proves (3⇒1′) if we can show that M is a substructure of $\mathcal{U}$. But this the case by Fact 36. □

Classically, the main application of the Tarski-Vaught test is in the proof of the downward Löwenheim-Skolem Theorem. Note that here the classical downward Löwenheim-Skolem Theorem holds in full for all $\mathcal{L}$-structures. In particular every $A \subseteq \mathcal{U}$ is contained in a standard structure of cardinality $|\mathcal{L}(A)|$. In this form the Löwenheim-Skolem Theorem is not very informative. In fact, the cardinality of $\mathcal{L}$ is eccessively large because of the aboundance of symbols in $\mathcal{L}_{\mathsf{I}}$.

We say that $\mathcal{L}^{\mathrm{p}}(A)/\mathcal{L}^{\mathrm{c}}(A)$ is separable if there is a countable subset $\mathcal{L}^{\mathrm{p}}(A)/\mathcal{L}^{\mathrm{c}}(A)$ that is p-dense/c-dense.

Proposition 39. Assume that $\mathcal{L}^{\mathrm{p}}(A)$ is separable modulo the theory of $\mathcal{U}$. Then there is a countable p-model M containing A. (The size of a p-model is the size of the domain of its home sort.)

Proof. As in the classical proof of the Löwenheim-Skolem Theorem, we construct $M \subseteq \mathcal{U}$ that contains a witness for every consistent formula in $\mathcal{F}(M)$. By separability and Proposition 38, M is a p-model. □

Proposition 40. Assume that $\mathcal{L}_{\mathsf{H}}$ is countable and that $\mathcal{L}$ has at most countably many symbols of sort $\mathsf{H}^n \to \mathsf{I}$. Then, if I is a second countable (the topology has a countable base), $\mathcal{L}$ is separable modulo $\mathcal{L}$.

Proof. Fix a countable filter of compact neighborhoods of the diagonal of I^2. □

11. Positive omitting types.

A type $p(x)$ is isolated by $\varphi(x)$, a consistent formula, if $\varphi(x) \to p(x)$. If $p(x)$ is isolated by $\neg\varphi(x)$ for some $\varphi(x) \in \mathcal{L}^{\mathrm{p}}(A)$ we say that it is p-isolated by A. Note that, by Propsition 38, if A p-isolates $p(x)$, then $p(x)$ is realized in every p-model containing A.

Fact 41. Let $\mathcal{F}$ be a p-dense set of positive formulas. Then the following are equivalent

1. $p(x)$ is p-isolated by A;
2. $p(x)$ is isolated by $\neg\psi(x)$ for some $\psi(x) \in \mathcal{F}(A)$;
3. $p(x)$ is isolated by some $\psi'(x)$ such that $\psi' > \psi$ for some consistent $\psi(x) \in \mathcal{F}(A)$;
4. $p(x)$ is isolated by some formula $\varphi(x) \in \Sigma^{\mathrm{p}}(A)$.

Proof. (1⇒2) By Proposition 27

(1⇒3) Recall that $\mathcal{F}'$ is p-dense by Fact 24. Apply Proposition 27 with $\mathcal{F}'$ for $\mathcal{F}$.

(3⇒4) Trivial.

(4⇒1) By Proposition 28. □

Lemma 42. Let $\mathcal{L}^{\mathrm{p}}(A)$ be separable. Let $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$, be non p-isolated by A. Then every consistent formula $\neg\psi(z)$, with $\psi(z) \in \mathcal{L}^{\mathrm{p}}(A)$, has a solution a such that A, a does not p-isolate $p(x)$.

Proof. We construct a sequence of $\mathcal{L}^{\mathrm{p}}(A)$-formulas $\langle\gamma'_i(z) : i < \omega\rangle$ such that any realization a of the type $\{\gamma'_i(z) : i < \omega\}$ is the required solution of $\neg\psi(z)$.

Let $\langle\xi_i(x;z) : i < \omega\rangle$ enumerate a countable p-dense subset of $\mathcal{L}^{\mathrm{p}}_{x;z}(A)$. Let $\gamma_0(z)$ and $\gamma'_0 > \gamma_0$ be a consistent positive formulas such that $\gamma'_0(z) \to \neg\psi(z)$. These exist by Proposition 27 and Fact 24. Define $\gamma'_{i+1}(z)$ inductively as follows.

1. If $\neg\xi_i(x;z) \wedge \gamma'_i(z)$ is inconsistent, let $\gamma'_{i+1}(z) = \gamma'_i(z)$.
2. Otherwise, pick $\varphi(x) \in p$ such that (#) below is consistent

(#) $$\gamma'_i(z) \;\wedge\; \exists x\big[\neg\xi_i(x;z) \wedge \neg\varphi(x)\big].$$

Finally, let $\gamma_{i+1}(z)$ and $\gamma'_{i+1} > \gamma_{i+1}$ consistent positive formulas such that $\gamma'_{i+1}(z)$ implies (#). Such a formula exists by Propositions 28 and 27.

Let $a \models \{\gamma'_i(z) : i < \omega\}$. We claim that that A, a does not p-isolate $p(x)$. Otherwise, by Fact 41, $\neg\xi_i(x;a) \to p(x)$ for some consistent $\neg\xi_i(x;a)$. This contradicts $a \models \gamma'_{i+1}(z)$.

Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in (2).

Suppose for a contradiction that $\neg\xi_i(x;z) \wedge \gamma'_i(z)$ is consistent while (#) is inconsistent for all formulas $\varphi(x) \in p$, that is,

$$\neg\xi_i(x;z) \wedge \gamma'_i(z) \;\to\; \varphi(x).$$

This immediately implies that

$$\exists z\big[\neg\xi_i(x;z) \wedge \gamma'_i(z)\big] \;\to\; p(x).$$

Then $p(x)$ is isolated by a formula in $\Sigma^{\mathrm{p}}(A)$. By Fact 41, this is a contradiction. □

Theorem 43 (Positive Omitting Types). Let $\mathcal{L}^{\mathrm{p}}(A)$ be separable. Assume also that $p(x) \subseteq \mathcal{L}(A)$ is not p-isolated. Then there is a p-model M containing A that omits $p(x)$.

Proof. As in the classical proof, appy the lemma above and the Tarski-Vaught test (Proposition 38) to obtain a countable p-model M that does not isolate $p(x)$. For p-models, isolating and realizing are equivalent. □

Assume that $\mathcal{L}$ is separable modulo T. A theory T is positively ω-categorical if any two countable p-models $M, N \models T$ are isomorphic. The following analogue of Ryll-Nardzewski's Theorem follows as usual.

Theorem 44. Assume that $\mathcal{L}$ is separable modulo T. The following are equivalent

1. T is positively ω-categorical;
2. every maximal positive type $p(x) \subseteq \mathcal{L}^{\mathrm{p}}$, for x any finite tuple of variables, is p-isolated over $\varnothing$.

The classical argument proves also the following.

Proposition 45. Fix a finite tuple of variables x. The following are equivalent

1. every positive type $p(x) \subseteq \mathcal{L}^{\mathrm{p}}(A)$ is p-isolated over A;
2. there are finitely many complete types.

Proof. (1⇒2) Let $p_i(x) = \text{p-tp}(a_i/A)$, for $i < \lambda$, be an enumeration without repetitions of all positive types. Let $\neg\varphi_i(x)$ be the negative formulas that isolate the $p_i(x)$. As these are complete types, $\neg\varphi_i(x) \leftrightarrow p_i(x)$. If λ is infinite then the positive type $\{\varphi_i(x) : i < \lambda\}$ is consistent. Any realization of this type yield a contradiction.

(2⇒1) Let $p_i(x) = \text{p-tp}(a_i/A)$, for $i < n$, be an enumeration without repetitions of all positive types. Pick a formula $\varphi_i(x) \in p_i \smallsetminus \bigcup\{p_j : j \in n \smallsetminus \{i\}\}$. Then the formula $\neg\bigvee\{\varphi_i(x) : j \in n \smallsetminus \{i\} p\}$ isolates $p_i(x)$. □

12. Example: metric spaces

We discuss a simple example. Let I be $\mathbb{R}^+ \cup \{0, \infty\}$, with the topology that makes it homeomorphic to the unit interval. Let M, d is a metric space. Let $\mathcal{L}_{\mathrm{H}}$ contain symbols for functions $M^n \to M$ that are uniformly continuous. Let $\mathcal{L}$ contain a symbol for d and possibly for some funtions of $M^n \to I$ that are uniformly continuous w.r.t. the metric d.

Let $\langle M, I\rangle$ be the structure that interprets the symbols of the language as natural.

Let $\mathcal{U}$ be a positive monster model that is an $\mathcal{L}^{\mathrm{p}}$ elementary extension of M. Clearly, d does not define a metric on $\mathcal{U}$ as there are pairs of elements at infinite distance. However, when restricted to a ball of finite radius, d defines a pseudometric on $\mathcal{U}$. Therefore the notion of convergent sequence makes perfectly sense in $\mathcal{U}$. As all functions have been required to be uniformly continuous, it is immediate that $x \sim_{\mathrm{I}} y$ is equivalent to $d(x, y) = 0$.

Fact 46. Let $\mathcal{U}$ be as above. Then for every p-model N and $a \in \mathcal{U}$ the following are equivalent

1. $p(x) = \text{p-tp}(a/N)$ is a Cauchy type;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ of elements of N that converges to a.

Proof. (2⇒1) Let $\langle \varepsilon_i : i \in \omega\rangle$ be a sequence of reals that converges to 0 and such that $d(a_i, a) \leq \varepsilon_i$ for every $i \in \omega$. Then the formulas $d(a_i, x) \leq \varepsilon_i$ are in $p(x)$. Then every element realizing $p(x)$ is at distance 0 from a. Therefore $p(x) \to a \sim x$.

(1⇒2) As $p(x)$ is Cauchy type, $p(x) \to a \sim x$. Then $p'(x) \to d(a, x) < 2^{-i}$ for all i. By compactness (see Fact 23) there are formulas $\varphi_i(x) \in p$ such that $\varphi'_i(x) \to d(a, x) < 2^{-i}$ for some $\varphi'_i > \varphi_i$. By $\mathcal{L}^{\mathrm{p}}$-elementarity there is an $a_i \in N$ such that $\varphi'(a_i)$. As $d(a, a_i) < 2^{-i}$ for all i, the sequence $\langle a_i : i \in \omega\rangle$ converges to a. □

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13. Elimination of quantifiers

Let T be a positive theory. We say that T has (approximate, positive) elimination of quantifiers if the quantifier-free positive formulas are p-dense modulo T. Equivalently, if every complete type is equivalent to a quantifier-free positive types.

The following fact is routinely proved by back-and-forth.

Fact 47. The following are equivalent

1. T has elimination of quantifiers;
2. every finite partial embedding $k : M \to N$ beween p-models of T is an approximate p-elementary map;
3. for every finite partial embedding $k : M \to N$ between ω-saturated p-models of T, and for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is also a partial embedding.

14. Pseudofinite randomizations

Let T be a complete first-order theory of signature L with an infinite model. Let $\mathcal{U}$ be a model of T. Let Ω be a finite set. We denote by $\mathcal{U}^{\Omega}$ the set of functions $\Omega \to \mathcal{U}$. The elements of $\mathcal{U}^{\Omega}$ are denoted by letters decorated with a circonflexe accent $\hat{a}, \hat{b}$, etc. The value of $\hat{a}$ at $\omega \in \Omega$ is denoted by a_{ω}. We now define models of the form $\langle \mathcal{U}^{\Omega}, I\rangle$, where I is the real interval $[0,1]$ with the usual topology.

The language $\mathcal{L}_{\mathsf{H}}$ is empty. The language $\mathcal{L}$ contains functions of sort $\mathsf{H}^{|x|} \to \mathsf{I}$, one for each formula $\varphi(x) \in L$. These are denoted by $\Pr\varphi(\hat{x})$. Variables of sort H are decorated with a circonflexe accent. The interpretation of the function symbol $\Pr\varphi(\hat{x})$ is

$$\Pr\big(\varphi(\hat{a})\big) \quad = \quad \frac{\#[\![\varphi(\hat{a})]\!]}{\#\Omega},$$

where by # denotes the finite cardinality, and

$$[\![\varphi(\hat{a})]\!] \quad = \quad \big\{\omega \in \Omega \,:\, \mathcal{U} \models \varphi(a_{\omega})\big\}.$$

The pseudofinite randomization of T is the positive theory

$$T^{\mathrm{pfr}} \quad = \quad \Big\{\varphi \in \mathcal{L}^{\mathrm{p}} \,:\, \langle \mathcal{U}^{\Omega}, I\rangle \models \varphi \text{ for all sufficiently large finite set } \Omega\Big\}$$

In this section we prove that T^{pfr} is a compete theory with elimination of quantifiers. We prove the lemma that is required for the back-and-forth.

Lemma 48. Let $k : M \to N$ be a finite partial embedding between positively ω-saturated models of T^{pfr}. Then for every $\hat{b} \in M$ and there is a $\hat{c} \in N$ such that $k \cup \{\langle \hat{b}, \hat{c}\rangle\} : M \to N$ is a partial embedding.

Proof. Let $\hat{a}$ be an enumeration of $\operatorname{dom} k$. Let $p(\hat{x}, \hat{z})$ be set of formulas of the form $\gamma = \Pr\varphi(\hat{x}, \hat{z})$, for some $\gamma \in I$, that hold in $M, \hat{b}, \hat{a}$. Note that $p(\hat{x}, \hat{z})$ implies the atomic type of $\hat{b}, \hat{a}$ in M. Therefore it suffices to show that $p(\hat{x}, k\hat{a})$ is finitely consistent in N.

Pick a finite set of these formulas, say $\gamma_i = \Pr\varphi_i(\hat{x}, \hat{z})$, for $i < n$. Me may assume that the fomulas $\varphi_i(x, z)$ define a partition of $\mathcal{U}^{|x,z|}$. By the saturation of N, it suffices to prove that for every $\varepsilon > 0$ there is a $\hat{c} \in N$ such that

$$\gamma_i \quad =_{\varepsilon} \quad \Pr\varphi_i(\hat{c}, k\hat{a}) \qquad\qquad (\alpha =_{\varepsilon} \beta \text{ is a shorthand for } |\alpha - \beta| \leq \varepsilon)$$

is true in N for all $i < n$.

Some preliminary work is required. For each $J \subseteq n$ define the following L-formulas

$$\xi_J(z) \quad = \quad \bigwedge_{i\in J} \exists x\, \varphi_i(x,z) \wedge \bigwedge_{i\in n\smallsetminus J} \neg\exists x\, \varphi_i(x,z)$$

These formulas $\xi_J(z)$ partition $\mathfrak{U}^{|z|}$. For $J \subseteq n$ let α_J be such that

2. $$\alpha_J \quad = \quad \Pr\big(\xi_J(\hat{z})\big)$$

is in $p(\hat{x},\hat{z})$. Let $\mathcal{J} = \{J \subseteq n \,:\, \alpha_J \neq 0\}$. When $J \in \mathcal{J}$, there are some $\beta_{i,J}$ such that the formulas

3. $$\beta_{i,J} \quad = \quad \Pr\Big(\varphi_i(\hat{x},\hat{z}) \;\Big|\; \xi_J(\hat{z})\Big).$$

are are among the consequences of $p(\hat{x},\hat{z})$. As the formulas $\varphi_i(x,z)$ define a partition, the $\beta_{i,J}$ add up to 1 for any fixed J. Note that $\beta_{i,J} = 0$ for $i \notin J$. Clearly, have that

4. $$\gamma_i \quad = \quad \sum_{J\in\mathcal{J}} \beta_{i,J}\alpha_J.$$

It is plain that for $i \in J$ the following is a consequence of $p(\hat{x},\hat{z})$

5. $$1 \quad = \quad \Pr\Big(\exists x\, \varphi_i(x,\hat{z}) \;\Big|\; \xi_J(\hat{z})\Big)$$

Now we prove that (2) & (5) $\Rightarrow$ (1) holds in $\mathfrak{U}^\Omega$ when Ω large enough (larger than n/ε suffices). Strictly speaking, this implication is not a positive formula, but the reader can easily verify that a suitable approximation of (2) and (5) suffices.

Let $\hat{a}' \in \mathfrak{U}^\Omega$ satisfy (2) and (5) for every $i \in J \subseteq n$. We define $\hat{c}' \in \mathfrak{U}^\Omega$ such that

$$\mathfrak{U}^\Omega \quad \models \quad \beta_{i,J} =_\varepsilon \Pr\Big(\varphi_i(\hat{c}',\hat{a}') \;\Big|\; \xi_J(\hat{a}')\Big).$$

We may define $\hat{c}'$ separately in each event $[\![\xi_J(\hat{a}')]\!]$. Partition $[\![\xi_J(\hat{a}')]\!]$ into events $E_i \subseteq \Omega$, for $i \in J$, such that $\#E_i/\#\Omega =_\varepsilon \beta_{i,J}$. Then define c'_ω for $\omega \in E_i$ to be any witness of $\exists x\, \varphi_i(x,a_\omega)$. By (5), we can always find such a witness.

Finally, by (4), we deduce

$$\mathfrak{U}^\Omega \quad \models \quad \gamma_i =_\varepsilon \Pr\big(\varphi_i(\hat{c}',\hat{a}')\big).$$

As $k\hat{a}$ satisfy (2) and (5), we conclude that some $\hat{c} \in N$ satisfy (1). □

Now, from Fact 47 we obtain that T^{pfr} has elimination of quantifiers. Completeness follows.

Corollary 49. The theory T^{pfr} is compete and has elimination of quantifiers.

15. Stability

A partitioned formula $\varphi(x\,;z) \in \mathcal{L}^{\mathrm{p}}$ is stable if there is no sequence $\langle a_i\,; b_i \,:\, i<\omega\rangle$ and no $\tilde{\varphi} \perp \varphi$ such that for every $i,j<\omega$

$$i<j \quad \Rightarrow \quad \varphi(a_i\,; b_j)$$
$$i>j \quad \Rightarrow \quad \tilde{\varphi}(a_i\,; b_j)$$

Using the terminology of [Hr], $\varphi(x\,;z)$ is stable if it is stably separated from all $\tilde{\varphi} \perp \varphi$.

Note that by compactness if $\varphi(x\,;z)$ and $\tilde{\varphi} \perp \varphi$ are stably separated then there is a maximal length m of a sequence $\langle a_i\,; b_i \,:\, i<m\rangle$ such that (1) and (2) above.

16. Elementary relations

Let M and N be models. We say that $R \subseteq M \times N$ is an p-elementary relation between M and N if for every $\varphi(x) \in \mathcal{L}$

$$M \models \varphi(a) \quad \Rightarrow \quad N \models \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } a\,R\,b.$$

Note that, when $a = a_1,\dots,a_n$ and $b = b_1,\dots,b_n$ are tuples, $a\,R\,b$ stands for $a_i\,R\,b_i$ for every $i \in \{1,\dots,n\}$.

Fact 50. Let $R \subseteq \mathcal{U}^2$ be an p-elementary relation of cardinality $< \kappa$. Then there is a total and surjective p-elemenrary relation $S \subseteq \mathcal{U}^2$ containing R.

Proof. We apply the usual back-and-forth construction with a pinch of extra caution. Let a be an enumeration of the domain of R. Let $\bar{a} = \langle a_i : i < \lambda \rangle$ be an enumeration of all tuples of length $|a|$ such that aRa_i. As κ is inaccessible, $\lambda < \kappa$. Let $b \in U$. It suffices to prove that there is a c such that $R \cup \{\langle b, c\rangle\}$ is an $\mathbb{L}$-relation. Let $p(x,z) = \text{tp}(b,a)$ and let

$$q(x,\bar{z}) \quad = \quad \bigcup_{i<\lambda} p(x,z_i).$$

We claim that $q(x,\bar{a})$ is a finitely consistent type. A finite conjunction of formulas in $q(x,\bar{a})$ has the form $\psi(x,a_{i_1}) \wedge \cdots \wedge \psi(x,a_{i_n})$. As $\psi(b,a)$ and $a_{i_1},\dots,a_{i_n}\,R\,a,\dots,a$, we conclude that the condition $\psi(x,a_{i_1}) \wedge \cdots \wedge \psi(x,a_{i_n})$ is satisfied. The existence of the required element c follows by saturation. □

17.

Corollary 51. Let $A \subseteq \mathcal{U}$ have cardinality $< \kappa$. Let $p(x) = \text{tp}_{\mathbb{L}}(a/A)$, where $a \in \mathcal{U}^{|x|}$ is a tuple of length $|x| < \kappa$. Then

$$p(\mathcal{U}) \quad = \quad \big\{b : bRa,\ R \in \text{Aut}(\mathcal{U}/A)\big\}$$

Let $\mathcal{M}$ and $\mathcal{N}$ be models. We say that $R \subseteq M \times N$ is an $\mathbb{L}$-(elementary) relation between $\mathcal{M}$ and $\mathcal{N}$ (or on $\mathcal{M}$ if the two coicide) if for every $\varphi(x) \in \mathbb{L}$

$$\mathcal{M} \models \varphi(a) \quad \Leftrightarrow \quad \mathcal{N} \models \varphi(b) \qquad\qquad \text{for every } a \text{ and } b \text{ such that } a\,R\,b.$$

Recall that, when $a = a_1,\dots,a_n$ and $b = b_1,\dots,b_n$ are tuples, $a\,R\,b$ stands for $a_i\,R\,b_i$ for every $i \in \{1,\dots,n\}$.

We define an equivalence relation $(\sim_{\mathcal{M}})$ on M as follows

1. $$a \sim_{\mathcal{M}} b \quad \Leftrightarrow \quad \Big(\mathcal{M} \models \varphi(a) \leftrightarrow \varphi(b) \quad \text{for every } \varphi(x) \in \mathbb{L}(M)\Big),$$

where $|x| = |a| = |b| = 1$. Note that this relation would be trivial had we included in $\mathbb{L}(M)$ equality between elements of M.

The following proposition is easily proved by induction on the syntax.

Proposition 52. The following are equivalent for every $a, b \in M$.

1. $a \equiv_{\mathcal{M}} b$;
2. $\mathcal{M} \models \tau(a) = \tau(b)$ for every $\tau(x) \in \mathbb{T}(M)$, with $|x| = 1$.

Lemma 53. The relation $(\sim_{\mathcal{M}}) \subseteq M^2$ is an $\mathbb{L}$-relation. Moreover, it is maximal among the $\mathbb{L}$-relations on $\mathcal{M}$, i.e. no $\mathbb{L}$-relation properly contains $(\sim_{\mathcal{M}})$.

Proof. Assume $a \sim_{\mathcal{M}} b$, where $a = a_1,\dots,a_n$ and $b = b_1,\dots,b_n$. Recall that this means that $a_i \sim_{\mathcal{M}} b_i$ for all $i \in \{1,\dots,n\}$. Let Δ denote the diagonal relation on M. Note that $a_i \sim_{\mathcal{M}} b_i$ is equivalent to saying that $\Delta \cup \{(a_i,b_i)\}$ is an $\mathbb{L}$-relation. As $\mathbb{L}$-relations are closed under composition $\Delta \cup \big\{(a_1,b_1),\dots,(a_n,b_n)\big\}$ is $\mathbb{L}$-elementary. It follows that for every $\varphi(x) \in \mathbb{L}$

2. $$\mathcal{M} \models \varphi(a) \quad \Leftrightarrow \quad \mathcal{M} \models \varphi(b).$$

This proves that $(\sim_{\mathcal{M}})$ is an $\mathbb{L}$-relation. Finally, maximality is immediate. □

Lemma 54. Let $R \subseteq M \times N$ be total and surjective $\mathbb{L}$-relation. Then there is a unique maximal $\mathbb{L}$-relation containing R. This maximal $\mathbb{L}$-relation is equal to both $(\sim_{\mathcal{M}})\, R$ and $R\, (\sim_{\mathcal{N}})$, where justaposition of relations stands for composition.

Proof. It is immediate to verify that $(\sim_{\mathcal{M}})\, R$ is an $\mathbb{L}$-relation containing R. Let S be any maximal $\mathbb{L}$-relation containing R. By maximality, $(\sim_{\mathcal{M}})\, S = S$. As S is a total relation $(\sim_{\mathcal{M}}) \subseteq SS^{-1}$. Therefore, by the lemma above, $(\sim_{\mathcal{M}}) = SS^{-1}$. As R is a surjective relation, $S \subseteq S\, S^{-1} R$. Finally, by maximality, we conclude that $S = (\sim_{\mathcal{M}})\, R$. A similar argument proves that $S = R\, (\sim_{\mathcal{N}})$. □

We write $\mathrm{Aut}(\mathcal{M})$ for the set of maximal, total and surjective, $\mathbb{L}$-relations $R \subseteq M^2$. The choice of the symbol Aut is motivated by the lemma above. In fact any such relation R induces a unique automorphism on the (properly defined) quotient structure $\mathcal{M}/\sim_{\mathcal{M}}$.