

# Local stability in structures with a standard sort

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ABSTRACT. Continuous logic [BBHU] has replaced Henson-Iovino logic [HI] as a formalism to study the model theory of continuous structures. One of the first articles on the subject, [BU], defines local stability within continuous logic - as noted by Henson, this escaped [HI]'s approach. Subsequently, this has been used as a major argument to advocate in favor of [BBHU]'s over [HI]'s approach.

Recently, a classical approach to continuous structures has been proposed in [ABBMZ] and [Z] that extend the class of structures falling under the scope of [HI] or [BBHU]. These articles introduce the notion of *structures with a standard sort*. We discuss local stability in this context. We examine three variants of order property which are *prima facie* non equivalent. For each variant we show that sets externally definable by stable formulas are definable in some appropriate sense.

## 1. Structures with a standard sort

Let  $S$  be a Hausdorff compact topological space. We associate to  $S$  a first order structure in a language  $\mathcal{L}_S$  that has a relation symbol for each compact subset  $C \subseteq S$  and a function symbol for each continuous function  $f : S^n \rightarrow S$ . Depending on the context,  $C$  and  $f$  denote either the symbols of  $\mathcal{L}_S$  or their interpretation in the structure  $S$ . Throughout these notes the letter  $C$  always denotes a compact subset of  $S$ , or a tuple of such sets.

Finally, note that we could allow in  $\mathcal{L}_S$  relation symbols for all compact subsets of  $S^n$ , for any  $n$ . But this would clutter the notation, therefore we prefer to leave the straightforward generalization to the reader.

We also fix an arbitrary first order language which we denote by  $\mathcal{L}_H$  and call the language of the home sort.

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**1 Definition** Let  $\mathcal{L}$  be a two sorted language. The two sorts are denoted by  $H$  and  $S$ . The language  $\mathcal{L}$  expands  $\mathcal{L}_H$  and  $\mathcal{L}_S$  with symbols of sort  $H^n \times S^m \rightarrow S$ . An  $\mathcal{L}$ -structure is a structure of signature  $\mathcal{L}$  that interprets these symbols with equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in  $H$ ).  
For a given  $S$  a **standard structure** is a two-sorted  $\mathcal{L}$ -structure of the form  $\langle M, S \rangle$ , where  $M$  is any structure of signature  $\mathcal{L}_H$ . Standard structures are denoted by the domain of their home sort.

We denote by  $\mathcal{F}$  the set of  $\mathcal{L}$ -formulas constructed inductively from atomic formulas of the form (i) and (ii) below using the Boolean connectives  $\wedge, \vee$ ; the quantifiers  $\forall^H, \exists^H$  of sort  $H$ ; and the quantifiers  $\forall^S, \exists^S$  of sort  $S$ .

We write  $x \in C$  for the predicate associated to a compact set  $C$ , and denote by  $|-|$  the length of a tuple.

**2 Definition** The atomic formulas of  $\mathcal{F}$  are

- i. atomic and negated atomic formulas of  $\mathcal{L}_H$
- ii. those of the form  $\tau \in C$ , where  $\tau$  is a term of sort  $H^n \times S^m \rightarrow S$ , and  $C$  a compact set.

We remark that a smaller fragment of  $\mathcal{F}$  may be of interest in some specific contexts. This fragment is obtained by excluding equalities and inequalities from (i). The discussion below applies to this smaller fragment as well.

Let  $M \subseteq N$  be standard structures. We say that  $M$  is an  **$\mathcal{F}$ -elementary** substructure of  $N$  if the latter models all  $\mathcal{F}(M)$ -sentences that are true in  $M$ .

Let  $x$  and  $\xi$  be variables of sort  $H$ , respectively  $S$ . A standard structure  $M$  is  **$\mathcal{F}$ -saturated** if it realizes every type  $p(x; \xi) \subseteq \mathcal{F}(A)$ , for any  $A \subseteq M$  of cardinality smaller than  $|M|$ , that is finitely consistent in  $M$ . The following theorem is proved in [ABBMZ] for signatures that do not contain symbols of sort  $H^n \times S^m \rightarrow S$  with  $n \cdot m > 0$ . A similar framework has been independently introduced in [CP] – only without quantifiers of sort  $H$  and with an approximate notion of satisfaction. The corresponding compactness theorem is proved there with a similar method. The setting in [CP] is a generalization of that used by Henson and Iovino for Banach spaces [HI]. By the elimination of quantifiers of sort  $S$ , proved in [ABBMZ, Proposition 3.6], the two approaches are equivalent – up to approximations.

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [ABBMZ] extends to the case  $n \cdot m > 0$ .

**3 Theorem (Compactness)** Every standard structure has an  $\mathcal{F}$ -saturated  $\mathcal{F}$ -elementary extension.

We introduce a new sort  $X$  with the purpose of conveniently describing classes of  $\mathcal{F}$ -formulas that only differ by the relation symbol  $C \subseteq S$  they contain. Let  $\mathcal{F}_X$  be defined as  $\mathcal{F}$  but replacing (ii) in Definition 2 by

- iii.  $\tau(x; \xi) \in X$ , where  $X$  is a variable of sort  $X$ .

Formulas in  $\mathcal{F}_X$  are denoted by  $\varphi(x; \xi; X)$ , where  $X = X_1, \dots, X_n$  is a tuple of variables of sort  $X$ . If  $C = C_1, \dots, C_n$  is a tuple of compact subsets, then  $\varphi(x; \xi; C)$ , is a formula in  $\mathcal{F}$ . This we call an **instance** of  $\varphi(x; \xi; X)$ . All formulas in  $\mathcal{F}$  are instances of formulas in  $\mathcal{F}_X$ .

**4 Definition** Let  $\varphi \in \mathcal{F}_X$  be a formula – possibly with some hidden free variables. The **pseudonegation** of  $\varphi \in \mathcal{F}_X$  is the formula obtained by replacing the atomic formulas in  $\mathcal{L}_H$  by their negation and the logical symbols  $\wedge, \vee, \forall, \exists$  by their respective dual  $\vee, \wedge, \exists, \forall$ . The atomic formulas of the form  $\tau \in X$  remain unchanged.

The pseudonegation of  $\varphi$  is denoted by  $\sim\varphi$ . Clearly, when no variable of sort  $X$  occurs in  $\varphi$ , we have  $\sim\varphi \leftrightarrow \neg\varphi$ .

Note that, unlike  $\neg\varphi$ , the pseudonegation  $\sim\varphi$  is again a formula in  $\mathcal{F}_X$ .

It is sometimes convenient to work with infinite conjunctions of formulas. In the following  $\sigma(x; z; X)$  denotes an  $\mathcal{F}_X$ -type. We write  $\sim\sigma(x; z; X)$  for the possibly infinite disjunction of the formulas  $\sim\varphi(x; z; X)$  for  $\varphi(x; z; X) \in \sigma$ .

When  $C$  and  $\tilde{C}$  are tuples we read  $C \subseteq C'$  and  $\tilde{C} \cap C = \emptyset$  componentwise. We say that  $C'$  is a neighborhood of  $C$  if every component of  $C'$  contains an open set containing the corresponding component of  $C$ .

**5 Fact** The following hold for every  $C \subseteq C'$  and  $\tilde{C} \cap C = \emptyset$ , every standard structure  $M$ , and every  $\mathcal{F}_X(M)$ -type  $\sigma(X)$

$$\begin{aligned} M &\models \sigma(C) \rightarrow \sigma(C') \\ M &\models \sim\sigma(\tilde{C}) \rightarrow \neg\sigma(C). \end{aligned}$$

**6 Fact** For every neighborhood  $C'$  of  $C$  there is  $\tilde{C}$  disjoint from  $C$  such that

$$M \models \varphi(C) \rightarrow \neg\sim\varphi(\tilde{C}) \rightarrow \varphi(C')$$

for every  $M$ , and every  $\mathcal{F}_X(M)$ -formula  $\varphi(X)$ . Conversely, for every  $\tilde{C} \cap C = \emptyset$  there is a neighborhood  $C'$  of  $C$  such that

$$M \models \varphi(C) \rightarrow \varphi(C') \rightarrow \neg\sim\varphi(\tilde{C})$$

for every  $M$ , and every  $\mathcal{F}_X(M)$ -formula  $\varphi(X)$ .

**Proof.** When  $\varphi(X)$  is the atomic formula  $\tau \in X$ , the first claim holds with  $\tilde{C} = S \setminus O$  where  $O$  is any open set  $C \subseteq O \subseteq C'$ . The second claim holds when  $C'$  is any neighborhood of  $C$  disjoint from  $\tilde{C}$ . Induction on the syntax of  $\varphi(X)$  proves the general case.  $\square$

The above fact has the following useful consequence.

**7 Fact** The following are equivalent for every standard structure  $M$ , every  $\mathcal{F}_X(M)$ -type  $\sigma(X)$ , and every  $C$

1.  $M \models \sigma(C) \leftarrow \bigwedge \{ \sigma(C') : C' \text{ neighborhood of } C \}$
2.  $M \models \neg \sigma(C) \rightarrow \bigvee \{ \sim \sigma(\tilde{C}) : \tilde{C} \cap C = \emptyset \}.$

(The converse implications are trivial – therefore not displayed.)

**8 Fact** The equivalent conditions in Fact 7 hold in all  $\mathcal{F}$ -saturated structures.

**Proof.** It suffices to prove the claim for formulas. We prove the first of the two implications by induction on the syntax of  $\varphi(X)$ . The existential quantifiers of sort H and S are the only cases that require attention. Assume inductively that

$$\bigwedge \{ \varphi(a; C') : C' \text{ neighborhood of } C \} \rightarrow \varphi(a; C)$$

holds for every  $\varphi(a; C)$ . Then induction for the existential quantifier follows from

$$\bigwedge \{ \exists x \varphi(x; C') : C' \text{ neighborhood of } C \} \rightarrow \exists x \bigwedge \{ \varphi(x; C') : C' \text{ neighborhood of } C \}$$

which is a consequence of saturation and the fact that if  $C', C''$  are compact neighbourhoods of  $C$  then  $\exists x \varphi(x; C' \cap C'')$  is satisfiable. The proof for the existential quantifier of sort S is similar.  $\square$

We say that  $M$  is  **$\mathcal{F}$ -maximal** if it models  $\varphi \in \mathcal{F}(M)$  whenever  $\varphi$  holds in some  $\mathcal{F}$ -elementary extensions of  $M$ . By Fact 8 and the following fact, all  $\mathcal{F}$ -saturated standard structures are  $\mathcal{F}$ -maximal.

**9 Fact** Let  $M$  be a standard structure. Then the following are equivalent

1.  $M$  is  $\mathcal{F}$ -maximal
2. the conditions in Fact 7 hold for every type (equivalently, for every formula).

**Proof.**  $1 \Rightarrow 2$ . If (2) in Fact 7 fails,  $\neg \varphi(C) \wedge \neg \sim \varphi(\tilde{C})$  holds in  $M$  for every  $\tilde{C} \cap C = \emptyset$ . Let  $\mathcal{U}$  be an  $\mathcal{F}$ -saturated  $\mathcal{F}$ -elementary extension of  $M$ . As  $M$  is  $\mathcal{F}$ -maximal,  $\neg \varphi(C) \wedge \neg \sim \varphi(\tilde{C})$  also holds in  $\mathcal{U}$ . Then (2) in Fact 7 fails in  $\mathcal{U}$ , contradicting Fact 8.

$2 \Rightarrow 1$ . Let  $N$  be any  $\mathcal{F}$ -elementary extension of  $M$ . Suppose  $N \models \varphi(C)$ . By (2), it suffices to prove that  $M \models \varphi(C')$  for every neighborhood  $C'$  of  $C$ . Suppose not. Then, by (2) in Fact 7,  $M \models \sim \varphi(\tilde{C})$  for some  $\tilde{C} \cap C' = \emptyset$ . Then  $N \models \sim \varphi(\tilde{C})$  by  $\mathcal{F}$ -elementarity. As  $\tilde{C} \cap C = \emptyset$ , this contradicts Fact 5.  $\square$

**10 Notation** In what follows we work in a fixed  $\mathcal{F}$ -saturated standard structure  $\mathcal{U}$ . We denote by  $\kappa$  the cardinality of  $\mathcal{U}$ . We assume that  $\kappa$  is a Ramsey cardinal larger than the cardinality of the language. However, the willing reader may check that we can do without large cardinals by careful application of the Erdős-Rado Theorem, see [TZ, Appendix C.3].

In the following  $\sigma(x; z; X)$  always denotes an  $\mathcal{F}_X(\mathcal{U})$ -type of small (i.e.  $< \kappa$ ) cardinality.

## 2. A duality in $K(S)$

Write  $K(S)$  for the set of compact subsets of  $S$ . In this section  $\mathcal{D}$  and  $\mathcal{C}$  range over subsets of  $K(S)$ . We define

$$\sim \mathcal{D} = \{ \tilde{C} : \tilde{C} \cap C \neq \emptyset \text{ for every } C \in \mathcal{D} \}.$$

Above we read  $\tilde{C} \cap C \neq \emptyset$  as the negation of  $\tilde{C} \cap C = \emptyset$ . That is, some components of the tuples  $\tilde{C}$  and  $C$  have nonempty intersection.

The following fact motivates the definition.

**11 Fact** Let  $\mathcal{D} = \{C : \varphi(C)\}$ . Then  $\sim \mathcal{D} = \{\tilde{C} : \sim \varphi(\tilde{C})\}$ .

**Proof.** We need to prove that

$$\sim \varphi(\tilde{C}) \Leftrightarrow \text{for every } C, \text{ if } C \cap \tilde{C} = \emptyset \text{ then } \neg \varphi(C).$$

The implication  $\Rightarrow$  follows immediately from Fact 5. To prove  $\Leftarrow$  assume the r.h.s. Then  $\neg \varphi(S \setminus O)$  holds for every open set  $O \supseteq \tilde{C}$ . From the second implication in Fact 7 we obtain  $\sim \varphi(\tilde{C}')$  for some  $\tilde{C} \subseteq \tilde{C}' \subseteq O$ . As  $O$  is arbitrary, we obtain  $\sim \varphi(\tilde{C})$  from the first implication in Fact 7.  $\square$

We prove some straightforward inclusions.

**12 Fact** For every  $\mathcal{D} \subseteq \mathcal{C}$  we have  $\sim \mathcal{C} \subseteq \sim \mathcal{D}$ . Moreover,  $\mathcal{D} \subseteq \sim \sim \mathcal{D}$ .

**Proof.** Let  $C \notin \sim \mathcal{D}$ . Then  $\tilde{C} \cap C = \emptyset$  for some  $\tilde{C} \in \mathcal{D}$ . As  $\tilde{C} \in \mathcal{C}$  we conclude that  $C \notin \sim \mathcal{C}$ . For the second claim, let  $C \notin \sim \sim \mathcal{D}$ . Then  $\tilde{C} \cap C = \emptyset$  for some  $\tilde{C} \in \sim \mathcal{D}$ . Then  $C \notin \mathcal{D}$  follows.  $\square$

**13 Fact** For every  $\mathcal{D}$  the following are equivalent

1.  $\mathcal{D} = \sim \sim \mathcal{D}$ .
2.  $\mathcal{D} = \sim \mathcal{C}$  for some  $\mathcal{C}$ .

**Proof.** Only  $2 \Rightarrow 1$  requires a proof. From Fact 12 we obtain  $\sim \sim \sim \mathcal{C} \subseteq \sim \mathcal{C}$ . Assume (2). Then  $\sim \sim \mathcal{D} \subseteq \mathcal{D}$  which, again by Fact 12, suffices to prove (1).  $\square$

We say that  $\mathcal{D}$  is **involutive** if  $\mathcal{D} = \sim \sim \mathcal{D}$ .

**14 Theorem** The following are equivalent

1.  $\mathcal{D}$  is involutive
2.  $C \in \mathcal{D} \Leftrightarrow C' \in \mathcal{D}$  for every neighborhood  $C'$  of  $C$ .

It is not difficult to see that (2) holds if and only if  $\mathcal{D}$  is closed in the product of Vietoris topologies and includes all the supersets of its elements.

**Proof.**  $2 \Rightarrow 1$ . By Fact 12 it suffices to prove  $\sim\sim\mathcal{D} \subseteq \mathcal{D}$ . Let  $C \in \sim\sim\mathcal{D}$ . Let  $O$  be a tuple of open sets containing  $C$  componentwise. Let  $\tilde{C}$  be the componentwise complement of  $O$ . Then  $\tilde{C} \notin \sim\mathcal{D}$ . As  $\sim\mathcal{D} = \{\tilde{C} : \tilde{C} \cap \tilde{C}' \neq \emptyset \text{ for every } \tilde{C}' \in \mathcal{D}\}$ , we have that  $\tilde{C} \in \mathcal{D}$  for some  $\tilde{C} \subseteq O$ . Assume (2). Then, by  $\Rightarrow$  every  $C' \supseteq O$  is in  $\mathcal{D}$ . As  $O$  is arbitrary,  $C \in \mathcal{D}$  by  $\Leftarrow$ .

$1 \Rightarrow 2$ . It suffices to show that (2) holds with  $\sim\mathcal{D}$  for  $\mathcal{D}$ . The implication  $\Rightarrow$  in (2) is obvious. To prove  $\Leftarrow$ , assume  $C \notin \sim\mathcal{D}$ . Then  $C \cap \tilde{C} = \emptyset$  for some  $\tilde{C} \in \mathcal{D}$ . Let  $C'$  be a neighborhood of  $C$  disjoint from  $\tilde{C}$ . Then the r.h.s. of the equivalence fails for  $C'$ .  $\square$

### 3. Stable formulas – the finitary case

There is more than one way to define stability in our context. In this paper we consider three variants and for each we prove a suitable version of the theorem that says that externally definable sets are definable. All three proofs follow the same line of reasoning – a finitary version of the classical argument as presented e.g. in [Z?].

In this section we consider the strongest possible requirement of stability. It is not the most interesting one, but it is very similar to the classical definition – the comparison may be useful though not strictly required for the sequel.

The following definition of stability is the classical one. We dub it *finitary* for the reason explained below.

**15 Definition** Let  $\varphi(x; z; X)$ , for  $x$  and  $z$  tuples of variables of sort  $H$ , be an  $\mathcal{F}_X$ -formula. Let  $C$  be a fixed tuple of compact sets. We say that  $\varphi(x; z; C)$  is **finitarily unstable** if for every  $m < \omega$  there is a sequence  $\langle a_i; b_i : i < m \rangle$  such that

1.  $\varphi(a_n; b_i; C) \wedge \neg\varphi(a_i; b_n; C)$  for every  $i < n < m$ .

A formula is **finitarily stable** if it is not finitarily unstable.

The following is an equivalent definition which only mentions formulas in  $\mathcal{F}$ . The reader may wish to compare it with Definition 20.

**16 Fact** The following are equivalent

1.  $\varphi(x; z; C)$  is finitarily stable
2. for some  $m$  there is no sequence  $\langle a_i; b_i : i < m \rangle$  such that for some  $\tilde{C} \cap C = \emptyset$ 

$$\varphi(a_n; b_i; C) \wedge \sim\varphi(a_i; b_n; \tilde{C}) \quad \text{for every } i < n < m.$$

**Proof.**  $1 \Rightarrow 2$ . By Fact 5.

$2 \Rightarrow 1$ . Negate (1). Let  $\langle a_i; b_i : i < m \rangle$  be such that (1) of Definition 15 holds. By Fact 7 there are  $\tilde{C}_{n,i} \cap C = \emptyset$  such that

$$\varphi(a_n; b_i; C) \wedge \sim \varphi(a_i; b_n; \tilde{C}_{n,i}) \quad \text{for every } i < n < m.$$

Then the union of the  $\tilde{C}_{n,i}$  is the set  $\tilde{C}$  required to negate (2).  $\square$

In a classical context, i.e. relying on  $\mathcal{L}$ -saturation, one can replace  $m$  in Definition 15 by  $\omega$ . But this may not be true if we only have  $\mathcal{F}$ -saturation. Therefore we dedicate a separate section to the non-finitary version of stability.

The following definitions are classical, i.e. they rely on the full language  $\mathcal{L}$ .

**17 Definition** For a given  $C$ , a **global  $\varphi(x; z; C)$ -type** is a maximally (finitely) consistent set of formulas of the form  $\varphi(x; b; C)$  or  $\neg \varphi(x; b; C)$  for some  $b \in \mathcal{U}^z$ .

Global  $\varphi(x; z; C)$ -types should not be confused with global  $\varphi(x; z; X)$ -types which are introduced in the following section.

In this section the symbol  $\mathcal{D}$  always denotes a subset of  $\mathcal{U}^z$ .

**18 Definition** We say that  $\mathcal{D}$  is **approximable** by  $\varphi(x; z; C)$  if for every finite  $B \subseteq \mathcal{U}^z$  there is an  $a \in \mathcal{U}^x$  such that

$$b \in \mathcal{D} \Leftrightarrow \varphi(a; b; C) \quad \text{for every } b \in B.$$

It goes without saying that these definitions describe two sides of the same coin. In fact, it is clear that  $\mathcal{D}$  is approximable by  $\varphi(x; z; C)$  if and only if

$$p(x) = \{\varphi(x; b; C) : b \in \mathcal{D}\} \cup \{\neg \varphi(x; b; C) : b \notin \mathcal{D}\}$$

is a global  $\varphi(x; z; C)$ -type.

The following theorem is often rephrased by saying that, when  $\varphi(x; z; C)$  is finitarily stable, all global  $\varphi(x; z; C)$ -types are definable. The proof of the classical case applies verbatim because no saturation is required. We only state it here for comparison with its non-finitary counterpart, Theorem 26, and its approximate counterpart, Theorem 31.

**19 Theorem** Let  $\varphi(x; z; C)$  be finitarily stable. Let  $\mathcal{D}$  be approximable by  $\varphi(x; z; C)$ . Then there are some  $\langle a_{i,j} : i < k, j < m \rangle$  such that for every  $b \in \mathcal{U}^z$

$$b \in \mathcal{D} \Leftrightarrow \bigvee_{i < k} \bigwedge_{j < m} \varphi(a_{i,j}; b; C).$$

#### 4. Stable formulas – the non-finitary case

We discuss a version of stability that seems more appropriate in our context. There are a few differences with the previous section. First, here the attribute *stable* refers to a

whole class of formulas – all instances of some  $\varphi(x; z; X)$ . Furthermore, the sequence witnessing the order property is required to be infinite. Finally, since we need to extend the definition of stability to types, sequences of length  $\kappa$  (the cardinality of  $\mathcal{U}$ , a Ramsey cardinal) are used.

In this section  $\mathcal{D}$  is always a subset of  $\mathcal{U}^z \times K(S)^{|X|}$ . We write  $\sim\mathcal{D}$  for the set obtained by applying the definition in Section 2 to all fibers of  $\mathcal{D}$ . That is, if  $\mathcal{D}_b$  is the  $b$ -fiber  $\{C : \langle b; C \rangle \in \mathcal{D}\}$ , we define

$$\sim\mathcal{D} = \bigcup_{b \in \mathcal{U}^z} \{b\} \times (\sim\mathcal{D}_b).$$

We say that  $\mathcal{D}$  is involutive if all its fibers are.

**20 Definition** Let  $x$  and  $z$  be tuples of variables of sort  $H$ . Recall that  $\sigma(x; z; X)$  always denotes a small  $\mathcal{F}_X(\mathcal{U})$ -type.

We say that  $\sigma(x; z; X)$  is **unstable** if there are a sequence  $\langle a_i; b_i : i < \kappa \rangle$ , some  $C$ , and some  $\tilde{C} \cap C = \emptyset$  such that

$$\sigma(a_n; b_i; C) \wedge \sim\sigma(a_i; b_n; \tilde{C}) \quad \text{for every } i < n < \kappa \text{ or for every } n < i < \kappa.$$

A type is **stable** if it is not unstable.

By saturation, when  $\sigma(x; z; X)$  is a formula we can replace  $\kappa$  with  $\omega$  in the previous definition

**21 Definition** Let  $\varphi(x; z; X)$  be an  $\mathcal{F}_X$ -formula. A **global  $\varphi(x; z; X)$ -type** is a maximally (finitely) consistent set of formulas of the form  $\varphi(x; b; C)$  or  $\sim\varphi(x; b; C)$  for some  $b \in \mathcal{U}^z$  and some  $C$ .

It is convenient to have a different characterization of global types, therefore the following definition.

**22 Definition** We say that  $\mathcal{D}$  is **approximable** by  $\sigma(x; z; X)$  if for every small set  $B \subseteq \mathcal{U}^z$  there is an  $a \in \mathcal{U}^x$  such that

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \sigma(a; b; C)$  and
2.  $\langle b, C \rangle \in \mathcal{D} \Leftarrow \sigma(a; b; C)$  for every  $b \in B$  and every  $C$ .

When  $\sigma(x; z; X)$  is a formula and  $\mathcal{D}$  is involutive (2) in the above definition can be rephrased in a more convenient form.

**23 Remark** If  $\mathcal{D}$  is involutive, the following are equivalent

2.  $\langle b, C \rangle \in \mathcal{D} \Leftarrow \varphi(a; b; C)$
- 2'  $\langle b, C \rangle \in \sim\mathcal{D} \Rightarrow \sim\varphi(a; b; C).$



**Proof.** Let  $\mathcal{C} = \{\langle b, C \rangle : \varphi(a, b, C)\}$ . This is also an involutive set by Fact 11. Then (2) says  $\mathcal{C} \subseteq \mathcal{D}$ . By Fact 12 this is equivalent to  $\sim \mathcal{D} \subseteq \sim \mathcal{C}$  which in turn is equivalent to (2'), again by Fact 11.  $\square$

Under the assumptions of the remark (i.e. when  $\mathcal{D}$  is involutive and  $\sigma(x; z; X)$  is a formula), it is easy to verify by saturation that in Definition 22 the requirement *for every small*  $B$  can be replaced by *for every finite*  $B$  (indeed, one can verify that the proof of the following fact also works when  $B$  is required to be finite).

**24 Fact** Let  $\varphi(x; z; X)$  be an  $\mathcal{F}_X$ -formula. For every involutive  $\mathcal{D}$  the following are equivalent

1.  $\mathcal{D}$  is approximable by  $\varphi(x; z; X)$
2. the following is a global  $\varphi(x; z; X)$ -type

$$p(x) = \{ \varphi(x; b; C) : \langle b, C \rangle \in \mathcal{D} \} \cup \{ \sim \varphi(x; b; C) : \langle b, C \rangle \in \sim \mathcal{D} \}.$$

**Proof.**  $2 \Rightarrow 1$ . Let  $B$  be a given small subset of  $\mathcal{U}^z$ . The consistency of  $p(x) \upharpoonright B$  provides some  $a$  such that

$$\begin{aligned} \langle b, C \rangle \in \mathcal{D} &\Rightarrow \varphi(a; b; C) \quad \text{and} \\ \langle b, C \rangle \in \sim \mathcal{D} &\Rightarrow \sim \varphi(a; b; C) \end{aligned} \quad \text{for every } b \in B \text{ and every } C.$$

By Remark 23, these yield (1) and (2) in Definition 22.

$1 \Rightarrow 2$ . First we prove that any  $p(x)$  defined as in (2) is maximal whenever it is consistent – in fact, this only depends on  $\mathcal{D}$  being involutive. We need to consider two cases. First, assume that  $\varphi(x; b; C)$  is consistent with  $p(x)$ . Then consistency implies that  $\langle b, \tilde{C} \rangle \notin \sim \mathcal{D}$  for every  $\tilde{C} \cap C = \emptyset$ . But this implies that  $\langle b, C \rangle \in \sim \sim \mathcal{D} = \mathcal{D}$ , therefore  $\varphi(x; b; C) \in p$ . The second case is when  $\sim \varphi(x; b; C)$  is consistent with  $p(x)$ . This implies that  $\langle b, \tilde{C} \rangle \notin \mathcal{D}$  for every  $\tilde{C} \cap C = \emptyset$ . Then  $\langle b, C \rangle \in \sim \mathcal{D}$ , therefore  $\sim \varphi(x; b; C) \in p$ .

Finally, to prove consistency, let  $B$  be a given small set. Let  $a$  be as in Definition 22. Then  $a \models p(x) \upharpoonright B$  by Remark 23.  $\square$

The proof of the main theorem of this section requires the following notion of approximation. The definition is inspired by the classical notion of approximation from below in [Z15] where it is used as rephrasing of the notion of *honest definability* in [CS]. Here this notion plays a purely technical role, but we present it as potentially interesting in its own right.

**25 Definition** We say that  $\mathcal{D}$  is **approximable** by  $\sigma(x; z; X)$  **from below** if for every small set  $B \subseteq \mathcal{U}^z$  there is an  $a \in \mathcal{U}^x$  such that for every  $C$

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \sigma(a; b; C)$  for every  $b \in B$  and
2.  $\langle b, C \rangle \in \mathcal{D} \Leftarrow \sigma(a; b; C)$  for every  $b \in \mathcal{U}^z$ .

**26 Theorem** Let  $\sigma(x; z; X)$  be stable. Assume that  $\mathcal{D}$  is approximable by  $\sigma(x; z; X)$ . Then there are some  $\langle a_{i,j} : i, j < \lambda \rangle$  such that for every  $b \in \mathcal{U}^z$  and every  $C$

$$\langle b, C \rangle \in \mathcal{D} \Leftrightarrow \bigvee_{i < \lambda} \bigwedge_{j < \lambda} \sigma(a_{i,j}; b; C).$$

**Proof.** The theorem is an immediate consequence of the following three lemmas.  $\square$

**27 Lemma** Let  $\sigma(x; z; X)$  be stable. Assume that  $\mathcal{D}$  is approximable by  $\sigma(x; z; X)$  from below. Then there are some  $\langle a_i : i < \lambda \rangle$  such that for every  $b \in \mathcal{U}^z$  and every  $C$

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \bigvee_{i < \lambda} \sigma(a_i; b; C)$
2.  $\langle b, C \rangle \in \mathcal{D} \Leftarrow \bigvee_{i < \lambda} \sigma(a_i; b; C).$

**Proof.** We define recursively the required parameters  $a_i$  together with some auxiliary parameters  $b_i$ . The element  $a_n$  is chosen so that

3.  $\langle b, C \rangle \in \mathcal{D} \Leftarrow \sigma(a_n; b; C)$  for every  $b \in \mathcal{U}^z$  and every  $C$

and

4.  $\langle b_i, C \rangle \in \mathcal{D} \Rightarrow \sigma(a_n; b_i; C)$  for every  $i < n$  and every  $C$ .

This is possible because  $\mathcal{D}$  is approximated from below. Note that (3) immediately guarantees (2). Now, assume (1) fails for  $\lambda = n + 1$ , and choose  $b_n$  and  $C_n$  witnessing this. Then, by Fact 7, for some  $\tilde{C}_{i,n} \cap C_n = \emptyset$

5.  $\langle b_n, C_n \rangle \in \mathcal{D}$  and  $\bigwedge_{i \leq n} \sim \sigma(a_i; b_n; \tilde{C}_{i,n}).$

Suppose for a contradiction that the construction carries on for  $\kappa$  stages. Then, as (5) guarantees that  $\langle b_i, C_i \rangle \in \mathcal{D}$  for every  $i$ , from (4) we obtain  $\sigma(a_n; b_i; C_i)$  for every  $i < n$ . From (5) we also obtain  $\sim \sigma(a_i; b_n; \tilde{C}_{i,n})$ , for every  $i \leq n$ . Apply Ramsey with pairs  $\langle C, \tilde{C} \rangle$  of compact subsets of  $S$  as colors to obtain a subsequence  $\langle a'_i; b'_i : i < \kappa \rangle$  that contradicts Definition 20.  $\square$

**28 Lemma** Let  $\sigma(x; z; X)$  be stable. Assume that  $\mathcal{D}$  is approximable by  $\sigma(x; z; X)$ . Then for some  $\lambda < \kappa$  the type

$$\pi(\bar{x}; z; X) = \bigwedge_{i < \lambda} \sigma(x_i; z; X),$$

where the  $x_i$  are copies of  $x$  and  $\bar{x} = \langle x_i : i < \lambda \rangle$ , approximates  $\mathcal{D}$  from below.

**Proof.** Negate the claim and let  $B$  witness that  $\pi(\bar{x})$  does not approximate  $\mathcal{D}$  from below. Suppose that  $\langle a_i : i < n \rangle$  and  $\langle b_i : i < n \rangle$  have been defined. Choose  $a_n$  such that for every  $b \in B \cup \{b_i : i < n\}$  and every  $C$

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \sigma(a_n; b; C)$  and
2.  $\langle b, C \rangle \in \mathcal{D} \Leftarrow \sigma(a_n; b; C).$

Note that the latter implication is equivalent to: for every  $C$  there is some  $\tilde{C} \cap C = \emptyset$  such that

$$3. \quad \langle b, C \rangle \notin \mathcal{D} \Rightarrow \sim \sigma(a_n; b; \tilde{C}).$$

Now, as the lemma is assumed to fail, we can choose  $b_n$  and  $C_n$  such that

$$4. \quad \langle b_n, C_n \rangle \notin \mathcal{D} \quad \text{and} \quad \bigwedge_{i=0}^n \sigma(a_i; b_n; C_n).$$

Note that (4) ensures that  $\langle b_i, C_i \rangle \notin \mathcal{D}$  for every  $i$ . Then there is some  $\tilde{C}_{i,n}$  that witnesses (3) for  $\langle b_i, C_i \rangle \notin \mathcal{D}$ . We claim that the procedure has to stop after  $< \kappa$  steps. In fact, from (3) we obtain  $\sim \sigma(a_n; b_i; \tilde{C}_{i,n})$  for every  $i < n$ . On the other hand, by (4) we have that  $\sigma(a_i; b_n; C_n)$  for every  $i < n$ . Again, apply Ramsey with pairs  $\langle C, \tilde{C} \rangle$  of compact subsets of  $S$  as colors to obtain a subsequence  $\langle a'_i; b'_i : i < \kappa \rangle$  that contradicts Definition 20.  $\square$

**29 Lemma** If  $\sigma(x; z; X)$  is stable, then  $\pi(\bar{x}; z; X)$  in the previous lemma is stable.

**Proof.** Suppose  $\pi(\bar{x}; z; X)$  is unstable. Let  $\langle \bar{a}_i; b_i : i < \kappa \rangle$  be a sequence witnessing instability. Suppose  $\sim \pi(\bar{a}_i; b_n; \tilde{C})$  holds for every  $i < n < \kappa$  (the case  $n < i < \kappa$  is identical). For every pair  $i < n$  we can pick some  $j < \lambda$  such that  $\sim \sigma(a_{i,j}; b_n; \tilde{C})$ . Interpret  $j$  as a color. As  $\kappa$  is a Ramsey cardinal, every  $\lambda$ -coloring of a complete graph of size  $\kappa$  has a monochromatic subgraph of size  $\kappa$ . That is, there is some fixed  $j < \lambda$  such that  $\sim \sigma(a_{i,j}; b_n; \tilde{C})$  holds for every pair  $i < n$  restricted to range over a subsequence of length  $\kappa$ . Such a subsequence contradicts the stability of  $\sigma(x; z; X)$ .  $\square$

## 5. Yet another version of stability

In this section we present the notion of  $\varepsilon$ -stability and prove a version of Theorem 26 which, albeit approximate, only requires finite disjunctions and conjunctions.

In this section, the letter  $\varepsilon$  always denotes a compact symmetric neighborhood of the diagonal of  $S^2$ . If  $C$  is a subset of  $S$ , we write  $C^\varepsilon$  for the set of points that are  $\varepsilon$ -close to points in  $C$ . That is

$$C^\varepsilon = \{ \alpha \in S : \langle \xi, \alpha \rangle \in \varepsilon \text{ for some } \xi \in C \}.$$

When  $C$  is a tuple, the definition above applies componentwise.

**30 Definition** Let  $\varphi(x; z; X)$ , for  $x$  and  $z$  tuples of variables of sort  $H$ , be an  $\mathcal{F}_X$ -formula. We say that  $\varphi(x; z; X)$  is  $\varepsilon$ -unstable if for every  $m < \omega$  there is a sequence  $\langle a_i; b_i; C_i; \tilde{C}_i : i < m \rangle$  such that for every  $i < n < m$

$$\varphi(a_n; b_i; C_i) \wedge \sim \varphi(a_i; b_n; \tilde{C}_n) \quad \text{and} \quad C_i^\varepsilon \cap \tilde{C}_i = \emptyset.$$

A formula is  $\varepsilon$ -stable if it is not  $\varepsilon$ -unstable.

Note that the requirement of  $\varepsilon$ -stability is stronger the smaller the  $\varepsilon$ . It is also easy to see that  $\varepsilon$ -stable for every positive  $\varepsilon$  implies stable.

We state the main theorem of this section, which is proved along the same lines as Theorem 26.

**31 Theorem** Let  $\varphi(x; z; X)$  be  $\varepsilon$ -stable. Assume that  $\mathcal{D}$  is approximable by  $\varphi(x; z; X)$ . Then there are  $k, m < \omega$  and  $\langle a_{i,j} : i < k, j < m \rangle$  such that for every  $b \in \mathcal{U}^z$  and every  $C$

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \bigvee_{i < k} \bigwedge_{j < m} \varphi(a_{i,j}; b; C^\varepsilon)$
2.  $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \bigvee_{i < k} \bigwedge_{j < m} \varphi(a_{i,j}; b; C).$

**Proof.** The theorem is an immediate consequence of the following three lemmas.  $\square$

We need the following version of Definition 25.

**32 Definition** We say that  $\mathcal{D}$  is **approximable** by  $\varphi(x; z; X)$  **from  $\varepsilon$ -below** if for every finite  $B \subseteq \mathcal{U}^z$  there is an  $a \in \mathcal{U}^x$  such that for every  $C$

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \varphi(a; b; C)$  for every  $b \in B$  and
2.  $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \varphi(a; b; C)$  for every  $b \in \mathcal{U}^z$ .

**33 Lemma** Let  $\varphi(x; z; X)$  be  $\varepsilon$ -stable. Assume that  $\mathcal{D}$  is approximable by  $\varphi(x; z; X)$  from  $\varepsilon$ -below. Then there are  $k < \omega$  and  $\langle a_i : i < k \rangle$  such that for every  $b \in \mathcal{U}^z$  and every  $C$

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \bigvee_{i < k} \varphi(a_i; b; C^\varepsilon)$
2.  $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \bigvee_{i < k} \varphi(a_i; b; C).$

The proof is entirely similar to that of Lemma 27. It is nevertheless included to show the role of  $C^\varepsilon$ .

**Proof.** We define recursively the required parameters  $a_i$  together with some auxiliary parameters  $b_i$ . The element  $a_n$  is chosen so that

3.  $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \varphi(a_n; b; C)$  for every  $b \in \mathcal{U}^z$  and every  $C$

and

4.  $\langle b_i, C \rangle \in \mathcal{D} \Rightarrow \varphi(a_n; b_i; C)$  for every  $i < n$  and every  $C$ .

This is possible because  $\mathcal{D}$  is approximated from  $\varepsilon$ -below. Note that (3) immediately guarantees (2). Now, assume (1) fails for  $k = n + 1$ , and choose  $b_n$  and  $C_n$  witnessing this. Then, by Fact 7, for some  $\tilde{C}_{i,n} \cap C_n^\varepsilon = \emptyset$

5.  $\langle b_n, C_n \rangle \in \mathcal{D}$  and  $\bigwedge_{i \leq n} \sim \varphi(a_i; b_n; \tilde{C}_{i,n}).$

Let  $m$  witness the  $\varepsilon$ -stability of  $\varphi(x; z; X)$ . Suppose for a contradiction that the construction carries on for  $m$  stages. Then, as (5) guarantees that  $\langle b_i, C_i \rangle \in \mathcal{D}$  for every  $i$ , from (4)

we obtain  $\varphi(a_n; b_i; C_i)$  for every  $i < n < m$ . From (5) we also obtain, for every  $i < n < m$ , some  $\tilde{C}_{i,n} \cap C_n^\varepsilon = \emptyset$  such that  $\sim\varphi(a_i; b_n; \tilde{C}_{i,n})$ . By monotonicity,  $\sim\varphi(a_i; b_n; \tilde{C}_n)$  for  $\tilde{C}_n = \bigcup_{i < m} \tilde{C}_{i,n}$ . This contradicts our choice of  $m$ .  $\square$

**34 Lemma** Let  $\varphi(x; z; X)$  be  $\varepsilon$ -stable. Assume that  $\mathcal{D}$  is approximable by  $\varphi(x; z; X)$ . Let  $m$  be maximal such that a sequence as in Definition 30 exists for  $\varphi(x; z; X)$ . Write  $\bar{x}$  for  $\langle x_i : i < m \rangle$ , where the  $x_i$  are copies of  $x$ . Then the formula

$$\sigma(\bar{x}; z; X) = \bigwedge_{i < m} \varphi(x_i; z; X)$$

approximates  $\mathcal{D}$  from  $\varepsilon$ -below.

**Proof.** Negate the claim and let  $B$  witness that  $\sigma(\bar{x})$  does not approximate  $\mathcal{D}$  from  $\varepsilon$ -below. For  $n \leq m$  suppose that  $a_0, \dots, a_{n-1}$  and  $b_0, \dots, b_{n-1}$  have been defined. Choose  $a_n$  such that for every  $b \in B \cup \{b_0, \dots, b_{n-1}\}$  and every  $C$

1.  $\langle b, C \rangle \in \mathcal{D} \Rightarrow \varphi(a_n; b; C)$  and
2.  $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \varphi(a_n; b; C^\varepsilon)$ .

Note that the second implication is equivalent to: for every  $C$  there is some  $\tilde{C} \cap C^\varepsilon = \emptyset$  such that

3.  $\langle b, C^\varepsilon \rangle \notin \mathcal{D} \Rightarrow \sim\varphi(a_n; b; \tilde{C})$ .

Now, as the lemma is assumed to fail, we can choose  $b_n$  and  $C_n$  such that

4.  $\langle b_n, C_n^\varepsilon \rangle \notin \mathcal{D}$  and  $\bigwedge_{i=0}^n \varphi(a_i; b_n; C_n)$ .

Note that (4) ensures that  $\langle b_i, C_i^\varepsilon \rangle \notin \mathcal{D}$  for every  $i$ . Then there is some  $\tilde{C}_{i,n} \cap C_i^\varepsilon = \emptyset$  that witnesses (3) for  $\langle b_i, C_i^\varepsilon \rangle \notin \mathcal{D}$ . We claim that the procedure has to stop at some stage  $< m$ . Assume for a contradiction that we can continue till we have  $\langle a_i; b_i; C_i; \tilde{C}_i : i \leq m \rangle$ . From (3) we obtain  $\sim\varphi(a_n; b_i; \tilde{C}_{i,n})$  for every  $i < n \leq m$ . Therefore, by monotonicity,  $\sim\varphi(a_n; b_i; \tilde{C}_i)$  for  $\tilde{C}_i = \bigcup_{n \leq m} \tilde{C}_{i,n}$ . On the other hand, by (4) we have that  $\varphi(a_i; b_n; C_n)$  for every  $i < n \leq m$ . Therefore,  $\langle a_{m-i}; b_{m-i}; C_{m-i}; \tilde{C}_{m-i} : i \leq m \rangle$  contradicts the maximality of  $m$ .  $\square$

**35 Lemma** If  $\varphi(x; z; X)$  is  $\varepsilon$ -stable then  $\sigma(\bar{x}; z; X)$  in the previous lemma is  $\varepsilon$ -stable.

**Proof.** Let  $m$  be maximal such that a sequence as in Definition 30 exists for  $\varphi(x; z; X)$ . Let  $k$  be sufficiently large so that every  $m$ -coloring of a graph of size  $k$  has a monochromatic subgraph of size  $> m$ . Let  $\langle \bar{a}_i; b_i; C_i; \tilde{C}_i : i < k \rangle$  be a sequence witnessing the  $\varepsilon$ -instability of  $\sigma(\bar{x}; z; X)$ . Then for every pair  $i < n$  there is some  $j < m$  such that  $\sim\varphi(a_{i,j}; b_n; \tilde{C}_n)$ . By the choice of  $k$  there is a subsequence of length  $> m$  and a fixed  $j < m$  such that  $\sim\varphi(a_{i,j}; b_n; \tilde{C}_n)$  holds for every pair  $i < n$  in the subsequence. This contradicts the maximality of  $m$ .  $\square$

## 6. Stable definable functions

In this section we specialize the notions introduced in the previous sections to types  $\sigma(x; z; X)$  of the form  $\tau(x; z) \in X$ , where  $\tau(x; z)$  is an  $S$ -valued  $\mathcal{F}$ -type-definable function.

We say that  $\tau(x; z)$  is **stable** or  **$\varepsilon$ -stable** if so is the type  $\tau(x; z) \in X$ . The following fact is an interesting characterization of stability for functions which was first remarked in [B].

**36 Fact** Let  $S$  be a compact metric space. Let  $\tau(x; z)$  be a term in  $\mathcal{L}$ . Then the following are equivalent

1. the formula  $\tau(x; z) \in X$  is unstable
2. there is a sequence  $\langle a_i; b_i : i < \omega \rangle$  such that the two limits below exist and

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \tau(a_i; b_j) \neq \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \tau(a_i; b_j).$$

Terms are a specific kind of definable functions. When  $\tau(x; z)$  is a term, we have that  $\sim(\tau(x; z) \in X) = \tau(x; z) \in X$ , which is used throughout the proof below. We suspect that Fact 36 holds more generally when  $\tau(x; z)$  is an  $S$ -valued  $\mathcal{F}$ -type-definable function, but we have not been able to prove this.

**Proof.**  $1 \Rightarrow 2$ . Let  $C \cap \tilde{C} = \emptyset$  and  $\langle a_i; b_i : i < \omega \rangle$  be as given by (1). That is,  $\tau(a_i; b_j) \in C$  and  $\tau(a_j; b_i) \in \tilde{C}$  hold for every  $i < j < \omega$ . We can find a subsequence such that the two limits exist;  $C$  contains the limit on the left; and  $\tilde{C}$  contains the limit on the right – which therefore are distinct.

$2 \Rightarrow 1$ . Let  $C$  and  $\tilde{C}$  be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence  $\langle a_i; b_i : i < \omega \rangle$ .  $\square$

Let  $f : \mathcal{U}^z \rightarrow S$  be a function. We define

$$\mathcal{D}_f = \{ \langle b, C \rangle : f(b) \in C, b \in \mathcal{U}^z, C \in K(S) \}.$$

The following fact is immediate.

**37 Fact** The following are equivalent

1.  $\mathcal{D}_f$  is approximable by  $\tau(x; z) \in X$
2. for every small  $B \subseteq \mathcal{U}^z$  there is an  $a \in \mathcal{U}^x$  such that  $\tau(a; b) = f(b)$  for every  $b \in B$ .

We say that  $f$  is **approximable** by  $\tau(x; z)$  if the equivalent conditions above hold. The following is an easy consequence of Theorem 26 and Theorem 31.

**38 Theorem** Let  $S = [0, 1]$ . Let  $\tau(x; z)$  be a term and let  $f$  be approximable by  $\tau(x; z)$ .

1. If  $\tau(x; z)$  is stable, then there are  $\lambda$  and  $\langle a_{i,j} : i, j < \lambda \rangle$  such that for every  $b \in \mathcal{U}^z$

$$\sup_{i < \lambda} \inf_{j < \lambda} \tau(a_{i,j}; b) = f(b).$$

2. If  $\tau(x; z)$  is  $\varepsilon$ -stable there are  $m$  and  $\langle a_{i,j} : i, j < m \rangle$  such that for every  $b \in \mathcal{U}^z$

$$\left| f(b) - \max_{i < m} \min_{j < m} \tau(a_{i,j}; b) \right| \leq \varepsilon.$$

From the proof it is clear that the matrix  $a_{i,j}$  is such that  $\sup_i \inf_j$  can be replaced by  $\inf_i \sup_j$ . Similarly,  $\max_i \min_j$  can be replaced by  $\min_i \max_j$ .

**Proof.** We prove the first claim. Let  $\lambda$  and  $\langle a_{i,j} : i, j < \lambda \rangle$  be as in Theorem 26. Let  $C_i$  be the closure of the set  $\{\tau(a_{i,j}; b) : j < \lambda\}$ . By  $(\Leftarrow)$  in Theorem 26,  $f(b) \in C_i$  for every  $i < \lambda$ . Then  $f(b) \geq \inf_j \tau(a_{i,j}; b)$  for every  $i < \lambda$ . Therefore  $f(b) \geq \sup_i \inf_j \tau(a_{i,j}; b)$ .

Now, by  $(\Rightarrow)$  in Theorem 26 there is  $k < \lambda$  such that  $f(b) = \tau(a_{k,j}; b)$  for every  $j < \lambda$ . Therefore  $f(b) = \inf_j \tau(a_{k,j}; b)$ . Then  $f(b) \leq \sup_i \inf_j \tau(a_{i,j}; b)$ .

We prove the second claim. Let  $m$  and  $\langle a_{i,j} : i, j < m \rangle$  be as in Theorem 31 (without loss of generality  $k = m$ ). Then Theorem 31 gives

$$3. \quad \bigvee_{i < m} \bigwedge_{j < m} |\tau(a_{i,j}; b) - f(b)| \leq \varepsilon.$$

We claim that we also have

$$4. \quad \bigwedge_{i < m} \bigvee_{j < m} |\tau(a_{i,j}; b) - f(b)| \leq \varepsilon.$$

For  $i < m$ , let  $C_i = \{\tau(a_{i,j}; b) : j < m\}$ . Then

$$\bigwedge_{j < m} \tau(a_{i,j}; b) \in C_i,$$

hence by Theorem 31 (2) there is  $k$  such that

$$|\tau(a_{i,k}; b) - f(b)| \leq \varepsilon.$$

Then (4) follows.

Now, suppose (2) of the theorem fails for the given  $\varepsilon$ . Then for some  $b \in \mathcal{U}^z$  one of the following obtains

$$5. \quad f(b) - \min_{j < m} \tau(a_{i,j}; b) > \varepsilon \quad \text{for every } i < m$$

$$6. \quad \min_{j < m} \tau(a_{i,j}; b) - f(b) > \varepsilon \quad \text{for some } i < m.$$

But (5) contradicts (3), while (6) contradicts (4).  $\square$

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