Local stability in structures with a standard sort

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ABSTRACT. Continuous logic [BBHU] has replaced Henson-Iovino logic [HI], as a formalism to study model theory of continuous structures. One of the first articles on the subject, [BU], defines local stability within continuous logic - as noted by Henson, this escaped [HI]'s approach. Subsequently, this has been used as a major argument to advocate in favor of [BBHU]'s over [HI]'s approach.

Recently, a classical approach to continuous structures has been proposed in [AAVV] and [Z] that properly extend the class of structures that falls under the scope of [HI] or [BBHU]. These articles introduced the notion of structures with a standard sort. We discuss local stability in this context. We examine a few variants of the classical definition which are prima facie non equivalent. For each we prove that sets externally definable by a stable formulas are definable in some appropiate sense.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_{\mathsf{S}}$ that has a symbol for each compact subset $C \subseteq S$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{S}}$ or their interpretation in the structure S. Throughout these notes the letter C always denotes a compact subset of S, or a tuple of such sets.

Finally, note that we could allow in $\mathcal{L}_{\mathsf{S}}$ relation symbols for all compact subsets of S^n, for any n. But this would clutter the notation, therefore we prefer to entrust the straightforward generalization to the reader.

We also fix an arbitrary first-order language which we denote by $\mathcal{L}_{\mathsf{H}}$ and call the language of the home sort.

1 Definition Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{S}}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H).
A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_{\mathsf{H}}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

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We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S.

We write $x \in C$ for the predicate associated to a compact set C, and denote by $|\text{-}|$ the length of a tuple.

2 Definition We call atomic formulas those of the form

i. atomic and negated atomic formulas of $\mathcal{L}_{\mathsf{H}}$

ii. $\tau \in C$, where τ is a term of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$, and C a compact.

We remark that smaller fragment of $\mathcal{F}$ may be of interest in some specific contexts. This is obtained by excluding equalities and inequalities from (i) and requiring that C in (ii) is regular. The discussion below applies to this smaller fragment as well.

Let $M \subseteq N$ be standard structures. We say that M is an $\mathcal{F}$-elementary substructure of N if the latter models all $\mathcal{F}(M)$-sentences that are true in M.

Let x and ξ be variables of sort H, respectively S. A standard structure M is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M. The following theorem is proved in [AAVV] for signatures that do not contain symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n{\cdot}m > 0$. A similar framework has been independently introduced in [CP] – only without quantifiers of sort H and with an approximate notion of satisfaction. The corresponding compactness theorem is proved there with a similar method. The setting in [CP] is a generalization of that used by Henson and Iovino for Banach spaces [HI]. By the elimination of quantifiers of sort S, proved in [AAVV, Proposition 3.6], the two approaches are equivalent – up to approximations.

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [AAVV] extends to the case $n{\cdot}m > 0$.

3 Theorem (Compactness) Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

We introduce a new sort X with the scope of conveniently describing classes of $\mathcal{F}$-formulas that only differ by the sets/predicates $C \subseteq S$ they contain. Let $\mathcal{F}_{\mathsf{X}}$ be defined as $\mathcal{F}$ but replacing (ii) by

iii. $\tau(x;\xi) \in X$, where X is a variable of sort X.

Formulas in $\mathcal{F}_{\mathsf{X}}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1,\ldots,X_n$ be a tuple of variables of sort X. If $C = C_1,\ldots,C_n$ is a tuple of compact subsets, then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_{\mathsf{X}}$.

4 Definition Let $\varphi \in \mathcal{F}_{\mathsf{X}}$ be a formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_{\mathsf{X}}$ is the formula obtained by replacing the atomic formulas in $\mathcal{L}_{\mathsf{H}}$ by their negation and every connective $\wedge$, $\vee$, $\forall$, $\exists$ by their respective duals $\vee$, $\wedge$, $\exists$, $\forall$. The atomic formulas of the form $\tau \in X$ remain unchanged.

The pseudonegation of φ is denoted by $\sim\!\varphi$. Clearly, when X does not occur in φ, we have $\sim\!\varphi \leftrightarrow \neg\varphi$.

It is sometimes conveniet to work with infinite conjunctions of formulas. In the following $\sigma(x;z;X)$ denotes an $\mathcal{F}_{\mathsf{X}}$-type. We write $\sim\!\sigma(x;z;X)$ for the infinite disjunction of the formulas $\sim\!\varphi(x;z;X)$ for $\varphi(x;z;X) \in \sigma$.

The following fact is immediate and will be used without further mention.

5 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$, every standard structure M, and every $\mathcal{F}_{\mathsf{X}}(M)$-type $\sigma(X)$

$$\begin{array}{lcl} M & \models & \sigma(C) \rightarrow \sigma(C') \\ M & \models & \sim\!\sigma(\tilde{C}) \rightarrow \neg\sigma(C). \end{array}$$

6 Fact For every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from C such that

$$M \models \varphi(C) \rightarrow \neg\sim\!\varphi(\tilde{C}) \rightarrow \varphi(C')$$

for every M, and every $\mathcal{F}_{\mathsf{X}}(M)$-formula $\varphi(X)$. Conversely, for every $\tilde{C} \cap C = \varnothing$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \rightarrow \varphi(C') \rightarrow \neg\sim\!\varphi(\tilde{C}).$$

for every M, and every $\mathcal{F}_{\mathsf{X}}(M)$-formula $\varphi(X)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = C' \smallsetminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds with as C' any neighborhood of C disjoint from $\tilde{C}$. Induction on the syntax of $\varphi(X)$ proves the general case. □

The above fact has the following useful consequence.

7 Fact The following are equivalent for every standard structure M , every $\mathcal{F}_{\mathsf{X}}(M)$-type $\sigma(X)$, and every C

$$\begin{array}{lcl} M & \models & \sigma(C) \leftarrow \bigwedge\Big\{\sigma(C') \,:\, C' \text{ neighborhood of } C\Big\} \\ M & \models & \neg\sigma(C) \rightarrow \bigvee\Big\{\sim\!\sigma(\tilde{C}) \,:\, \tilde{C} \cap C = \varnothing\Big\}. \end{array}$$

(The converse implications are trivial – therefore not displayed.)

8 Fact The equivalent conditions in Fact 7 hold in all $\mathcal{F}$-saturated structures.

Proof. It suffices to prove the claim for formulas. We prove the first of the two implications by induction on the syntax of $\varphi(X)$. The existential quantifier of sort H is the only connective that requires attention. Assume inductively that

$$\bigwedge\Big\{\varphi(a;C')\ :\ C' \text{ neighborhood of } C\Big\}\ \ \rightarrow\ \ \varphi(a;C)$$

holds for every $\varphi(a;C)$. Then induction for the existential quantifier follows from

$$\bigwedge\Big\{\exists x\,\varphi(x;C')\ :\ C' \text{ neighborhood of } C\Big\}\ \ \rightarrow\ \ \exists x \bigwedge\Big\{\varphi(x;C')\ :\ C' \text{ neighborhood of } C\Big\}$$

which is a consequence of saturation. □

We say that M is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions. By Fact 8 and the following, $\mathcal{F}$-saturated standard structures are $\mathcal{F}$-maximal.

9 Fact Let M be a standard structure. Then the following are equivalent

1. M is $\mathcal{F}$-maximal
2. the conditions in Fact 7 hold for every type (equivalently, for every formula).

Proof. 1⇒2. If (2) in Fact 7 fails, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of M. As M is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds also in $\mathcal{U}$. Then (2) in Fact 7 fails in $\mathcal{U}$, contradicting Fact 8.

2⇒1. Let N be any $\mathcal{F}$-elementary extension of M. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C. Suppose not. Then, by (2) in Fact 7, $M \models {\sim}\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models {\sim}\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts Fact 8. □

10 Notation We denote by κ the cardinality of $\mathcal{U}$. We assume that κ is a Ramsey cardinal. However, the willing reader may check that we can do without large cardinals by carefull application of the Erdős-Rado Theorem, see [TZ, Appendix C.3].

In the following $\sigma(x;z;X)$ always denotes an $\mathcal{F}_{\mathsf{X}}(\mathcal{U})$-type of small (i.e. $<\kappa$) cardinality.

2. A duality in $K(S)$

Write $K(S)$ for the set of compact subsets of S. In this section $\mathcal{D}$ and $\mathcal{C}$ range over subsets of $K(S)^n$. We define

$${\sim}\mathcal{D} \quad = \quad \big\{\tilde{C}\ :\ \tilde{C} \cap C \neq \varnothing \text{ for every } C \in \mathcal{D}\big\}.$$

The following fact motivates the definition.

11 Fact Let $\mathcal{D} = \{C : \sigma(C)\}$. Then ${\sim}\mathcal{D} = \{\tilde{C} : {\sim}\sigma(\tilde{C})\}$.

Proof. We need to prove that

$${\sim}\sigma(\tilde{C}) \ \ \Leftrightarrow\ \ \text{for every } C,\ \text{ if } C \cap \tilde{C} = \varnothing \ \text{ then } \neg\sigma(C)$$

Implication $\Rightarrow$ follows immediately from Fact 5. To prove $\Leftarrow$ assume the r.h.s. Then $\neg\sigma(S \smallsetminus O)$ holds for every open set $O \supseteq \tilde{C}$. From the second implication in Fact 7 we obtain $\sim\sigma(\tilde{C}')$ for some $\tilde{C} \subseteq \tilde{C}' \subseteq O$. As O is arbitrary, we obtain $\sigma(\tilde{C})$ from the second implication in Fact 7. □

We prove a couple of straightforward inclusions.

12 Fact For every $\mathcal{D} \subseteq \mathcal{C}$ we have $\sim\mathcal{C} \subseteq \sim\mathcal{D}$. Moreover, $\mathcal{D} \subseteq \sim\sim\mathcal{D}$.

Proof. Let $C \notin \sim\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in \mathcal{D}$. As $\tilde{C} \in \mathcal{C}$ we conclude that $C \notin \sim\mathcal{C}$. For the second claim, let $C \notin \sim\sim\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in \sim\mathcal{D}$. Then $C \notin \mathcal{D}$ follows. □

13 Fact For every $\mathcal{D}$ the following are equivalent

1. $\mathcal{D} = \sim\sim\mathcal{D}$.
2. $\mathcal{D} = \sim\mathcal{C}$ for some $\mathcal{C}$.

Proof. Only 2⇒1 requires a proof. From Fact 12 we obtain $\sim\sim\sim\mathcal{C} \subseteq \sim\mathcal{C}$. Assume (2) then $\sim\sim\mathcal{D} \subseteq \mathcal{D}$ which, again by Fact 12, suffices to prove (1). □

We say that $\mathcal{D}$ is involutive if $\mathcal{D} = \sim\sim\mathcal{D}$. We say that $\mathcal{D}$ is closed if

$$C \in \mathcal{D} \quad \Leftrightarrow \quad C' \in \mathcal{D} \text{ for every neighborhood } C' \text{ of } C.$$

Note that we are requiring closure in two distintic contexts: topological, and Boolean (upward clousere by inclusion).

14 Theorem The following are equivalent

1. $\mathcal{D}$ is involutive
2. $\mathcal{D}$ is closed.

⚠ It is not difficult to see that (2) holds if and only if $\mathcal{D}$ is closed in the Vietoris topology and includes all the supersets of its elements.

Proof. 2⇒1. It suffices to prove $\sim\sim\mathcal{D} \subseteq \mathcal{D}$. Let $C \in \sim\sim\mathcal{D}$. Let $O \supseteq C$ be open. Then $S \smallsetminus O \notin \sim\mathcal{D}$. Then $\tilde{C} \in \mathcal{D}$ for some $\tilde{C} \subseteq O$. Assume (2). Then, by $\Rightarrow$ every $C' \supseteq O$ is in $\mathcal{D}$. As O is arbitrary, $C \in \mathcal{D}$ by $\Leftarrow$.

1⇒2. It suffices to show that (2) holds with $\sim\mathcal{D}$ for $\mathcal{D}$. Implication $\Rightarrow$ in the definition of closed is oivous. To prove $\Leftarrow$, assume $C \notin \sim\mathcal{D}$. Then $C \cap \tilde{C}$ for some $\tilde{C} \in \mathcal{D}$. Let C' be a neighboorhood of C disjoint of C. Then the r.h.s. of the equivalence fails for C'. □

3. Stable formulas – the finitary case

There is more than one way to define stability in our context. In this section we present the strongest possible requirement. It is not the most interesting one, but it is very similar to the classical definition – the comparision may be useful though not stricly required for the sequel.

The following is the classical definition of stability which we dub *finitary* for the reason explained below.

15 Definition We say that $\varphi(x\,;z\,;C)$ is finitarily stable if for some $m<\omega$ there is no sequence $\langle a_i\,;b_i \,:\, i<m\rangle$ such that for every $i<n$

$$\varphi(a_n\,;b_i\,;C) \quad \wedge \quad \neg\varphi(a_i\,;b_n\,;C).$$

From Fact 7 we easily obtain the following equivalent definition which only mentions formulas in $\mathcal{F}$. The reader may wish to compare it with Definition 20.

16 Fact The following are equivalent

1. $\varphi(x\,;z\,;C)$ is finitarily stable
2. for some m there is no sequence $\langle a_i\,;b_i\,;\tilde{C}_i \,:\, i<m\rangle$ such that for every $i<n$

$$\varphi(a_n\,;b_i\,;C) \quad \wedge \quad {\sim}\varphi(a_i\,;b_n\,;\tilde{C}_n) \quad \text{and} \quad C\cap\tilde{C}_i=\varnothing.$$

In a classical context, i.e. relying on $\mathcal{L}$-saturation, one can replace m in Definition 15 by ω. But this may not be true if only have $\mathcal{F}$-saturation. Therefore we dedicate a separate section to the non-finitary version of stability.

The following definitions are classical, i.e. they rely on the full language $\mathcal{L}$.

17 Definition A global $\varphi(x\,;z\,;C)$-type is a maximally (finitely) consistent set of formulas of the form $\varphi(x\,;b\,;C)$ or $\neg\varphi(x\,;b\,;C)$ for some $b\in\mathcal{U}^{|z|}$.

Global $\varphi(x\,;z\,;C)$-types should not be confused with global $\varphi(x\,;z\,;X)$-types which are introduced in the following section.

In this section the symbol $\mathcal{D}$ always denotes a subset of $\mathcal{U}^z$.

18 Definition We say that $\mathcal{D}$ is approximable by $\varphi(x\,;z\,;C)$ if for every finite $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that

$$b\in\mathcal{D} \quad \Leftrightarrow \quad \varphi(a\,;b\,;C) \qquad\qquad \text{for every } b\in B.$$

It goes without saying that these definitions describe two faces of the same coin. In fact, it is clear that $\mathcal{D}$ is approximable by $\varphi(x\,;z\,;C)$ if and only if

$$p(x) \quad = \quad \big\{\varphi(x\,;b\,;C) \,:\, b\in\mathcal{D}\big\} \ \cup\ \big\{\neg\varphi(x\,;b\,;C) \,:\, b\notin\mathcal{D}\big\}$$

is a global $\varphi(x;z;C)$-type.

The following theorem is often rephrased by saying that, when $\varphi(x;z;C)$ is finitarily stable, then all global $\varphi(x;z;C)$-types are definable. The proof of the classical case applies verbatim because no saturation is required. We state it here only for comparison with its non-finitary counterpart, see Theorem 25.

19 Theorem Let $\varphi(x;z;C)$ be finitarily stable. Let $\mathcal{D}$ be approximable by $\varphi(x;z;C)$. Then there are some $\langle a_{i,j} \,:\, i<k,\ j<m\rangle$ such that for every $b\in\mathcal{U}^z$

$$b\in\mathcal{D} \quad\Leftrightarrow\quad \bigvee_{i<k}\ \bigwedge_{j<m}\ \varphi(a_{i,j};b;C)$$

4. Stable formulas – the non-finitary case

We discuss a version of stability that seems more appropriate in this context. There are two main differences with the previous section. First, here the attribute *stable* refers to a whole class of formulas – all instances of some $\varphi(x;z;X)$. Second, the sequence witnessing the order property is required to be infinite. As we need to extend the definition of stability to types, sequences of length κ (a large cardinal) are used.

In this section $\mathcal{D}$ is always a subset of $\mathcal{U}^z\times K(S)^n$, where n has to be inferred from the context (tipically, $n=|X|$). We write $\sim\mathcal{D}$ for the set obtained by applying the definition in Section 2 to all fibers of $\mathcal{D}$. We say that $\mathcal{D}$ is involutive/closed if all its fibers are such.

20 Definition We say that $\sigma(x;z;X)$ is stable if there is no sequence $\langle a_i;b_i \,:\, i<\kappa\rangle$ such that for some C, some $\tilde{C}\cap C=\varnothing$

$$\sigma(a_n;b_i;C) \quad\wedge\quad \sim\sigma(a_i;b_n;\tilde{C}) \qquad\qquad \text{for every } i<n<\kappa$$

or for every $n<i<\kappa$.

Note that when $\sigma(x;z;X)$ is a formula we can replace κ with ω in the above definition. For types we can equivalently require that there is no sequence $\langle a_i;b_i \,:\, i<\omega\rangle$ such that for some C, some $\tilde{C}\cap C=\varnothing$, some formula $\varphi(x;z;X)\in\sigma$

$$\sigma(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;\tilde{C}) \qquad\qquad \text{for every } i<n<\kappa$$

or for every $n<i<\kappa$.

Note also that, by the pegeonhole principle, we can strengthen the requirement in Definition 20 by requiring that there is no $\langle a_i;b_i;C_i;\tilde{C}_i \,:\, i<\omega\rangle$ such that $\tilde{C}_i\cap C_i=\varnothing$ and

$$\sigma(a_n;b_i;C_i) \quad\wedge\quad \sim\varphi(a_i;b_n;\tilde{C}_n) \qquad\qquad \text{for every } i<n<\kappa.$$

21 Definition Let $\varphi(x;z;X)$ be a $\mathcal{F}_X$-formula. A global $\varphi(x;z;X)$-type is a maximally (finitely) consistent set of formulas of the form $\varphi(x;b;C)$ or $\sim\varphi(x;b;C)$ for some $b\in\mathcal{U}^{|z|}$ and some C.

It is convenient to have a different characterization of global types, therefore the following definition.

22 Definition We say that $\mathcal{D}$ is approximable by $\sigma(x;z;X)$ if for every small set $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that

1. $\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a;b;C)$ and
2. $\langle b, C\rangle \in \mathcal{D} \;\Leftarrow\; \varphi(a;b;C)$ for every $b \in B$ and every C.

In the above definition we quantify over all B of small cardinality. However, by compactness, the finite B are sufficient (this is used without further mention in Section **??**).

We remark that, by Facts 11 and 12, when $\mathcal{D}$ is involutive (2) can be rephrased as

$$\langle b, C\rangle \in {\sim}\mathcal{D} \;\Rightarrow\; {\sim}\sigma(a;b;C) \qquad \text{for every } b \in B \text{ and every } C.$$

23 Fact Let $\varphi(x;z;X)$ be a $\mathcal{F}_{\mathsf{X}}$-formula. For every involutive $\mathcal{D}$ the following are equivalent

1. $\mathcal{D}$ is approximable by $\varphi(x;z;X)$
2. the following is a global $\varphi(x;z;X)$-type

$$\begin{aligned} p(x) \;=\; & \big\{\; \varphi(x;b;C) : \langle b, C\rangle \in \mathcal{D}\big\} \;\cup \\ & \big\{{\sim}\varphi(x;b;C) : \langle b, C\rangle \in {\sim}\mathcal{D}\big\}. \end{aligned}$$

Proof. First we prove that $p(x)$ is maximal as soon as it is consistent – in fact, this only depends on $\mathcal{D}$ being involutive. We need to consider two cases. First, assume that $\varphi(x;b;C)$ is consistent with $p(x)$. Then consistency implies that $\langle b, \tilde{C}\rangle \notin {\sim}\mathcal{D}$ for every $\tilde{C} \cap C = \varnothing$. But this implies that $C \in {\sim}{\sim}\mathcal{D} = \mathcal{D}$, therefore $\varphi(x;b;C) \in p$. The second case is when ${\sim}\varphi(x;b;C)$ is consistent with $p(x)$. This implies that $\langle b, \tilde{C}\rangle \notin \mathcal{D}$ for every $\tilde{C} \cap C = \varnothing$. Then $\langle b, C\rangle \in {\sim}\mathcal{D}$, therefore ${\sim}\varphi(x;b;C) \in p$.

2⇒1. Let B be a given finite subset of $\mathcal{U}^z$. The consistency of $p(x) \restriction B$ provides some a such that for every finite B an a such that

$$\begin{aligned} \langle b, C\rangle \in \mathcal{D} \;&\Rightarrow\; \varphi(a;b;C) \quad \text{and} \\ \langle b, C\rangle \in {\sim}\mathcal{D} \;&\Rightarrow\; {\sim}\varphi(a;b;C) \qquad \text{for every } b \in B \text{ and every } C. \end{aligned}$$

By what remarked above, these yield (1) and (2) in Definition 22.

1⇒2. Let B be given. Let a be as in Definition 22. Then, by what remarked above, $a \models p(x) \restriction B$. □

The proof of the main theorem of this section requires the following notion of approximation. The definition is inspired by the classical notion of approximation from below in [Z15] where it is used as rephrasing of the notion of honest definability in [CS]. Here this notion plays a purely technical role, but we submit it as potentially interesing in itself.

24 Definition We say that $\mathcal{D}$ is approximable by $\sigma(x;z;X)$ from below if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that for every C

1. $\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \sigma(a;b;C)$ for every $b \in B$ and
2. $\langle b, C\rangle \in \mathcal{D} \;\Leftarrow\; \sigma(a;b;C)$ for every $b \in \mathcal{U}^z$.

25 Theorem Let $\sigma(x;z;X)$ be stable. Assume that $\mathcal{D}$ is approximable by $\sigma(x;z;X)$. Then there are some $\langle a_{i,j} : i<\lambda, j<\mu\rangle$ such that for every $b \in \mathcal{U}^z$ and every C

$$\langle b, C\rangle \in \mathcal{D} \;\Leftrightarrow\; \bigvee_{i<\lambda} \bigwedge_{j<\mu} \sigma(a_{i,j};b;C)$$

Proof. The theorem is an immediate consequence of the following three lemmas. □

26 Lemma Let $\sigma(x;z;X)$ be stable. Assume that $\mathcal{D}$ is approximable by $\sigma(x;z;X)$ from below. Then there is are some $\langle a_i : i<\lambda\rangle$ such that for every $b \in \mathcal{U}^z$ and every C

1. $\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \bigvee_{i<\lambda} \sigma(a_i;b;C)$
2. $\langle b, C\rangle \in \mathcal{D} \;\Leftarrow\; \bigvee_{i<\lambda} \sigma(a_i;b;C)$

Proof. We define recursively the required parameters a_i together with some auxiliary parameters b_i. The element a_n is choosen such that

3. $\langle b, C\rangle \in \mathcal{D} \;\Leftarrow\; \sigma(a_n;b;C)$ for every $b \in \mathcal{U}^z$ and every C

and

4. $\langle b_i, C\rangle \in \mathcal{D} \;\Rightarrow\; \sigma(a_n;b_i;C)$ for every $i<n$ and every C.

This is possible because $\mathcal{D}$ is approximated from below. Note that (3) immediately guarantees (2). Now, assume (1) fails for $\lambda = n$, and choose b_n and C_n witnessing this. Then, by Fact 7, for some $\tilde{C}_n \cap C_n = \varnothing$

5. $\langle b_n, C_n\rangle \in \mathcal{D} \;\;\&\;\; \bigwedge_{i<n} \sim\sigma(a_i;b_n;\tilde{C}_n)$

Suppose for a contradiction that the construction never ends. Then, as (5) guarantees that $\langle b_i, C_i\rangle \in \mathcal{D}$ for every i, from (4) we otain $\sigma(a_n;b_i;C_n)$ for every $i<n$. From (5) we also obtain $\sim\sigma(a_i;b_n;\tilde{C}_n)$, for every $i \le n$. This contradicts stability (as remarked after Definition 20). □

27 Lemma Let $\sigma(x;z;X)$ be stable. Then for some $\mu<\kappa$ the type

$$\pi(\bar{x};z;X) \;=\; \bigwedge_{i<\mu} \sigma(x_i;z;X),$$

where the x_i are copies of x and $\bar{x} = \langle x_i : i<\mu\rangle$, approximates $\mathcal{D}$ from below.

Proof. Negate the claim and let B witness that $\pi(\bar{x})$ does not approximate $\mathcal{D}$ from below. Suppose that $\langle a_i : i < n\rangle$ and $\langle b_i : i < n\rangle$ have been defined. Choose a_n such that for every $b \in B \cup \{b_i : i < n\}$ and every C

1. $\quad \langle b, C\rangle \in \mathcal{D} \ \Rightarrow\ \varphi(a_n ; b ; C) \quad$ and
2. $\quad \langle b, C\rangle \in \mathcal{D} \ \Leftarrow\ \varphi(a_n ; b ; C)$

Note that the latter implication is equivalent to: for every C there is some $\tilde{C} \cap C = \varnothing$ such that

3. $\quad \langle b, C\rangle \notin \mathcal{D} \ \Rightarrow\ \sim\varphi(a_n ; b ; \tilde{C})$.

Now, as the lemma is assumed to fail, we can choose b_n and C_n such that

4. $\quad \langle b_n, C_n\rangle \notin \mathcal{D} \ \ \& \ \ \bigwedge_{i=0}^{n} \varphi(a_i ; b_n ; C_n)$

Note that (4) ensure that $\langle b_i, C_i\rangle \notin \mathcal{D}$ for every i. The there is some $\tilde{C}_i$ that witnesses (3) for $\langle b_i, C_i\rangle \notin \mathcal{D}$. We claim that the procedure has to stop after $< \kappa$ steps. In fact, from (3) we obtain $\sim\varphi(a_n ; b_i ; \tilde{C}_i)$ for every $i < n$. On the other hand, by (4) we have that $\varphi(a_i ; b_n ; C_n)$ for every $i < n$. Therefore, $\langle a_i ; b_i ; C_i ; \tilde{C}_i \ :\ i < \kappa\rangle$ contradicts stability (as remarked after Definition 20). $\square$

28 Lemma If $\sigma(x ; z ; X)$ is stable then $\pi(\bar{x} ; z ; X)$ in the previous lemma is stable.

Proof. Suppose $\pi(\bar{x} ; z ; X)$ is unstable. Let $\langle \bar{a}_i ; b_i \ :\ i < k\rangle$ be a sequence witnessing instability. Then for every pair $i < n$ there is some $j < \mu$ such that $\sim\sigma(a_{j,n} ; b_i ; \tilde{C})$. Interprete such j as a color. As κ is a Ramsey cardinal, every μ-coloring of a complete graph of size κ has a monocromatic subgraph of size κ. That is, there is a $j < \mu$ such that $\sim\sigma(a_{j,n} ; b_i ; \tilde{C})$ obtains for κ many i. Therefore we can extract a subsequence that contradicts the stability of $\sigma(x ; z ; X)$. $\square$

5. Stable definable functions

In this section we specialize the notions introduced in the previous sections to types $\sigma(x ; z ; X)$ of the form $\tau(x ; z) \in X$ where $\tau(x ; z)$ is an S-valued $\mathcal{F}$-type definable function.

We say that $\tau(x ; z)$ is stable if so is the type $\tau(x ; z) \in X$. The following fact is a suggestive characterization of the stability of functions which has been first remarked in [B].

29 Fact Let $\tau(x ; z)$ be term in $\mathcal{L}$. Then the following are equivalent

1. the formula $\tau(x ; z) \in X$ is unstable
2. there is a sequence $\langle a_i ; b_i \ :\ i < \omega\rangle$ such that

$$\lim_{i\to\infty} \lim_{j\to\infty} \tau(a_i ; b_j) \ \neq\ \lim_{j\to\infty} \lim_{i\to\infty} \tau(a_i ; b_j)$$

Terms are a particular case of definable functions. When $\tau(x;z)$ is a term we have that $\sim\big(\tau(x;z)\in X\big) = \tau(x;z)\in X$ which is used throughout in the proof below. We suspect the fact holds in general though we have not been able to prove it.

⚠ **Proof.** 1⇒2. Let $C\cap\tilde{C}=\varnothing$ and $\langle a_i\,;b_i\;:\;i<\omega\rangle$ be as given by (1). That is, $\tau(a_i\,;b_j)\in C$ and $\tau(a_j\,;b_i)\in\tilde{C}$ hold for every $i<j<\omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and $\tilde{C}$ contains the limit on the right – which therefore are distinct.

2⇒1. Let C and $\tilde{C}$ be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i\,;b_i\;:\;i<\omega\rangle$. □

Let $f:\mathcal{U}^z\to S$ be a function. We define

$$\mathcal{D}_f \;\;=\;\; \big\{\langle b,C\rangle \;:\; f(b)\in C,\;\; b\in\mathcal{U}^z,\;\; C\in K(S)\big\}.$$

The following fact is immediate.

30 Fact The following are equivalent

1. $\mathcal{D}_f$ is approximable by $\tau(x;z)\in X$
2. for every small $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that $\tau(a;b)=f(b)$ for every $b\in B$.

We say that f is approximable by $\tau(x;z)$ is the equivalent conditions above apply. The following is an easy consequence if Theorem 25

31 Theorem Let $S=[0,1]$. Let $\tau(x;z)$ be stable. Let f be approximable by $\tau(x;z)$. Then there are some $\langle a_{i,j}\;:\;i<\lambda, j<\mu\rangle$ such that

$$f(b) \;\;=\;\; \sup_{i<\lambda}\,\inf_{j<\mu}\,\tau(a_{i,j}\,;b) \qquad\qquad \text{for every } b\in\mathcal{U}^z.$$

6. Yet another version of stability

In this section we present the notion of ε-stability and prove a version of Theorem 25 which, albeit approximate, only requires finite disjunction and congiunctions. Note that the requirement of ε-stability is stronger the smaller the ε and it always implies stability as defined in Defintion 20 – we do not know it coincides with it in the limit.

32 Definition We say that $\varphi(x;z;X)$ is ε-stable if there is a maximal m such that some sequence $\langle a_i\,;b_i\,;C_i\,;\tilde{C}_i\;:\;i<m\rangle$ is such that for every $i<n<m$

$$\varphi(a_n;b_i;C_i)\quad\wedge\quad\sim\varphi(a_i\,;b_n;\tilde{C}_n)\qquad\text{and}\qquad C_i^{\varepsilon}\cap\tilde{C}_i=\varnothing.$$

We state the main theorem of this section which is proved along the same lines as Theorem 25

33 Theorem Let $\varphi(x;z;X)$ be ε-stable. Assume that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$. Then there are some $\langle a_{i,j} \,:\, i<k,\ j<m\rangle$ such that for every $b \in \mathcal{U}^z$ and every C

$$\langle b,C\rangle \in \mathcal{D} \;\Rightarrow\; \bigvee_{i<k}\ \bigwedge_{j<m} \varphi(a_{i,j};b;C^{\varepsilon})$$

$$\langle b,C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \bigvee_{i<k}\ \bigwedge_{j<m} \varphi(a_{i,j};b;C)$$

Proof. The theorem is an immediate consequence of the following three lemmas. □

The proof of the theorem requires the following version of Definition 24

34 Definition We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ from ε-below if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that for every C

1. $\langle b,C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a;b;C)$ for every $b \in B$ and
2. $\langle b,C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \varphi(a;b;C)$ for every $b \in \mathcal{U}^z$.

35 Lemma Let $\varphi(x;z;X)$ be ε-stable. Assume that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ from ε-below. Then there is are some $\langle a_i \,:\, i<k\rangle$ such that for every $b \in \mathcal{U}^z$ and every C

1. $\langle b,C\rangle \in \mathcal{D} \;\Rightarrow\; \bigvee_{i<k} \varphi(a_i;b;C^{\varepsilon})$
2. $\langle b,C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \bigvee_{i<k} \varphi(a_i;b;C)$

Proof. We define recursively the required parameters a_i together with some auxiliary parameters b_i. The element a_n is choosen such that

3. $\langle b,C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \varphi(a_n;b;C)$ for every $b \in \mathcal{U}^z$ and every C

and

4. $\langle b_i,C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a_n;b_i;C)$ for every $i<n$ and every C.

This is possible because $\mathcal{D}$ is ε-approximated from below. Note that (3) immediately guarantees (2). Now, assume (1) fails for $k=n$, and choose b_n and C_n witnessing this. Then, by Fact 7, for some $\tilde{C}_n \cap C_n^{\varepsilon} = \varnothing$

5. $\langle b_n,C_n\rangle \in \mathcal{D} \;\;\&\;\; \bigwedge_{i<n} {\sim}\varphi(a_i;b_n;\tilde{C}_n)$

Suppose for a contradiction that the construction never ends. Then, as (5) guarantees that $\langle b_i,C_i\rangle \in \mathcal{D}$ for every i, from (4) we otain $\varphi(a_n;b_i;C_n)$ for every $i<n$. From (5) we also obtain ${\sim}\varphi(a_i;b_n;\tilde{C}_n)$, for every $i \le n$. This contradicts ε-stability. □

36 Lemma Let $\varphi(x;z;X)$ be ε-stable. Assume that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$. Let m be maximal so that a sequence as in (3) of Definition 32 exists. Let $\bar{x} = \langle x_i \ :\ i \leq m\rangle$ where the x_i are copies of x. Then the formula

$$\sigma(\bar{x};z;X) \quad = \quad \bigwedge_{i\leq m} \varphi(x_i;z;X)$$

approximate $\mathcal{D}$ from ε-below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x})$ does not approximate $\mathcal{D}$ from ε-below. Suppose that $a_0,\dots,a_{n-1}$ and $b_0,\dots,b_{n-1}$ have been defined. Choose a_n such that for every $b \in B\cup\{b_0,\dots,b_{n-1}\}$ and every C

1. $$\langle b, C\rangle \in \mathcal{D} \ \Rightarrow\ \varphi(a_n;b;C) \quad \text{and}$$
2. $$\langle b, C^{\varepsilon}\rangle \in \mathcal{D} \ \Leftarrow\ \varphi(a_n;b;C^{\varepsilon})$$

Note that the latter implication is equivalent to: for every C there is some $\tilde{C}\cap C^{\varepsilon} = \varnothing$ such that

3. $$\langle b, C^{\varepsilon}\rangle \notin \mathcal{D} \ \Rightarrow\ {\sim}\varphi(a_n;b;\tilde{C}).$$

Now, as the lemma is assumed to fail, we can choose b_n and C_n such that

4. $$\langle b_n, C_n^{\varepsilon}\rangle \notin \mathcal{D} \quad \& \quad \bigwedge_{i=0}^{n} \varphi(a_i;b_n;C_n)$$

Note that (4) ensure that $\langle b_i, C_i^{\varepsilon}\rangle \notin \mathcal{D}$ for every i. The there is some $\tilde{C}_i$ that witnesses (3) for $\langle b_i, C_i^{\varepsilon}\rangle \notin \mathcal{D}$. We claim that the procedure has to stop after $\leq m$ steps. In fact, from (3) we obtain ${\sim}\varphi(a_n;b_i;\tilde{C}_i)$ for every $i < n$. On the other hand, by (4) we have that $\varphi(a_i;b_n;C_n)$ for every $i < n$. Therefore, $\langle a_{m-i};b_{m-i};C_{m-i};\tilde{C}_{m-i} \ :\ i \leq m\rangle$ contradicts the maximality of m. □

37 Lemma If $\varphi(x;z;C)$ is ε-stable then $\sigma(\bar{x};z;C)$ in the previous lemma is ε-stable.

Proof. Let m be maximal such that a sequence as in (3) of Definition 32 exists. Let k be suffinciely large so that every m-coloring of a graph of size k has a monocromatic subgraph of size $> m$. Let $\langle \bar{a}_i;b_i;C_i;\tilde{C}_i \ :\ i < k\rangle$ be a sequence witnessing instability. Then for every pair $i < n$ there is some $j < m$ such that ${\sim}\varphi(a_{j,n};b_i;\tilde{C}_i)$. By the choice of k there is a $j < m$ such that ${\sim}\varphi(a_{j,n};b_i;\tilde{C}_i)$ obtains for $> m$ many i. Therefore we can extract a subsequence that contradicts the maximality of m. □

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