# Local stability in structures with a standard sort

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Abstract.

## 1. Structures with a standard sort

Let $S$ be some Hausdorff compact topological space. We associate to $S$ a first order structure in a language $\mathcal{L}_\mathsf{S}$ that has a symbol for each compact subset $C \subseteq S^n$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, $C$ and $f$ denote either the symbols of $\mathcal{L}_\mathsf{S}$ or their interpretation in the structure $S$. We write $x \in C$ for $C(x)$.

**1 Definition** Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by $\mathsf{H}$ and $\mathsf{S}$. The language $\mathcal{L}$ expands $\mathcal{L}_\mathsf{H}$ and $\mathcal{L}_\mathsf{S}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions. A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where $M$ is any structure of signature $\mathcal{L}_\mathsf{H}$ and $S$ is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^\mathsf{H}$, $\exists^\mathsf{H}$ of sort $\mathsf{H}$; and the quantifiers $\forall^\mathsf{S}$, $\exists^\mathsf{S}$ of sort $\mathsf{S}$.

**2 Definition** We call atomic formulas those of the form

i. atomic or negated atomic formulas of $\mathcal{L}_\mathsf{H}$;

ii. $\tau \in C$, where $C \subseteq S^n$ is a compact set and $\tau$ is a tuple of terms of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$.

**3 Remark** A smaller fragment of $\mathcal{F}$ may be of interest in some specific contexts. This is obtained by excluding equalities and inequalities from (i) and requiring that $C$ in (ii) is a regular compact set. The discussion below applies to this smaller fragment as well.

Let $M \subseteq N$ be standard structures. We say that $M$ is an $\mathcal{F}$-elementary substructure of $N$ if the latter models all sentences in $\mathcal{F}(M)$ true in $M$.

A standard structure $M$ is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in $M$. The following is

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proved in [AAVV] for signatures containing only symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n{\cdot}m = 0$. In [Z] it is observed that, under the assumption of equicontinuity, the proof extends to the case $n{\cdot}m \geq 0$.

**4 Theorem (Compactness)** Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

We introduce variables of some new sorts $\mathsf{X}_n$ with the sole scope of conveniently describing classes of $\mathcal{F}$-formulas that only differ by the sets/predicates $C \subseteq S^n$ that occur in the formula. To avoid cluttering the notation we write $\mathsf{X}$ for an unspecified tuple of sorts $\mathsf{X}_n$. Let $\mathcal{F}_\mathsf{X}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\xi) \in X$, where $X$ is a variable of sort $\mathsf{X}$.

Formulas in $\mathcal{F}_\mathsf{X}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1,\dots,X_k$ be a tuple of variables of sort $\mathsf{X}$. If $C = C_1,\dots,C_k$ is a tuple of compact subsets of $S$ then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_\mathsf{X}$.

**5 Definition** Let $\varphi \in \mathcal{F}_\mathsf{X}$ be formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_\mathsf{X}$ is the formula obtained by replacing in $\varphi$ the atomic formulas in $\mathcal{L}_\mathsf{H}$ by their negation and each connective $\wedge$, $\vee$, $\forall$, $\exists$ by its respective dual $\vee$, $\wedge$, $\exists$, $\forall$.
The pseudonegation of $\varphi$ is denoted by $\sim\!\varphi$.

The following fact will be used without further mention.

**6 Fact** Let $\varphi(X) \in \mathcal{F}_\mathsf{X}(M)$, where $M$ is a standard structure. Then the following holds in $M$ for every (tuples of) compact sets $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$

$$\begin{aligned} M &\models \varphi(C) \rightarrow \varphi(C') \\ M &\models \sim\!\varphi(C) \rightarrow \neg\varphi(\tilde{C}) \end{aligned}$$

Clearly, when $X$ does not occur in $\varphi$, we have $\sim\!\varphi \leftrightarrow \neg\varphi$.

**7 Fact** Let $\varphi(X) \in \mathcal{F}_\mathsf{X}(M)$, where $M$ is a standard structure. Then for every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from $C$ such that

$$M \models \varphi(C) \rightarrow \neg\sim\!\varphi(\tilde{C}) \rightarrow \varphi(C')$$

Conversely, for every $\tilde{C} \cap C = \varnothing$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \rightarrow \varphi(C') \rightarrow \neg\sim\!\varphi(\hat{C}).$$

**Proof.** When $\varphi(C)$ is the atomic formula of the form $\tau \in C$, the first claim holds with $\tilde{C} = C' \smallsetminus O$ for any open set $C \subseteq O \supseteq C'$. The second claim holds with as $C'$ any neighborhood of $C$ disjoint of $\tilde{C}$. Induction on the syntax of $\varphi(X)$ proves the general case. $\square$

The above fact has the following useful consequence.

**8 Fact** Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$, where $M$ is a standard structure. Then the following are equivalent for every tuple $C$ of compact subsets of $S$

$$\begin{aligned} M &\models \quad \varphi(C) \leftrightarrow \bigwedge\Big\{\varphi(C') : C' \text{ neighborhood of } C\Big\} \\ M &\models \neg\varphi(C) \leftrightarrow \bigvee\Big\{{\sim}\varphi(\tilde{C}) : \tilde{C} \cap C = \varnothing\Big\}. \end{aligned}$$

Note that the above fact also holds if we restrict the above $C'$ and $\tilde{C}$ to range over the subsets of some $C''$ neighboorhood of $C$.

**9 Fact** The equivalent conditions in Fact 8 hold in all $\mathcal{F}$-saturated structures.

**Proof.** By induction on the syntax of $\varphi(X)$. □

We say that astandard structure $M$ is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions. By the following fact $\mathcal{F}$-saturated structures are $\mathcal{F}$-maximal.

**10 Fact** Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Then the following are equivalent

1. $M$ is $\mathcal{F}$-maximal
2. the equivalent conditions in Fact 8 hold for every $\varphi(C)$.

**Proof.** 1⇒2. If (2) in Fact 8 fails, then $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds in $M$ for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of $M$. As $M$ is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ also holds in $\mathcal{U}$. Then (2) in Fact 8 also fails in $\mathcal{U}$, contradicting Fact 9.

2⇒1. Let $N$ be any $\mathcal{F}$-elementary extension of $M$. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every $C'$ neighborhood of $C$. Suppose not. Then, by (2) in Fact 8, $M \models {\sim}\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models {\sim}\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts the above fact. □

In what follows we fix a large saturated standard structure which we denote by $\mathcal{U}$. Unless otherwise specified, we work inside $\mathcal{U}$.

Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(\mathcal{U})$ be given. Define

$$A_\varphi \quad = \quad \{\alpha : \varphi(C) \text{ holds for every neighborhood } C \text{ of } \alpha\}$$

**11 Fact** Let $n$ be the total length of $X$. Then $A_\varphi \cup A_{\sim\varphi} = S^n$ and $A_\varphi \cap A_{\sim\varphi} = \varnothing$.

**Proof.** Negate the first claim. Let $\alpha \notin A_\varphi \cup A_{\sim\varphi}$. Then there is a neighborhood $C$ of $\alpha$ such that $\neg\varphi(C)$ and $\neg{\sim}\varphi(C)$.

By (2) in Fact 8, we have ${\sim}\,\varphi(\tilde{C})$ for some $\tilde{C} \cap C = \varnothing$. Thus, $\tilde{C} \cap A_\varphi = \varnothing$. □

## 2. Externally definable sets

Let $\varphi(x;z;X)$ be a formula in $\mathcal{F}_{\mathsf{X}}$ with the variables of sort $\mathsf{H}$ partitioned in two tuples $x;z$. A global $\varphi(x;z;X)$-type is a maximally consistent set of formulas that are instances of $\varphi(x;b;X)$ and/or $\sim\varphi(x;b;X)$ for some $b \in \mathcal{U}^{|z|}$.

Let $p(x)$ be a global $\varphi(x;z;X)$-type. We define

$$\mathcal{D}_{p,\varphi,C} \quad = \quad \{b \in \mathcal{U}^z \ : \ \varphi(x;b;C) \in p\}$$

**12 Fact** $$\neg\mathcal{D}_{p,\varphi,C} \quad = \quad \bigcup\Big\{\mathcal{D}_{p,\sim\varphi,\tilde{C}} \ : \ \tilde{C} \cap C = \varnothing\Big\}$$

**Proof.** $(\subseteq)$. Let $b \notin \mathcal{D}_{p,\varphi,C}$. Let $A \subseteq \mathcal{U}$ be such that every instance of $\varphi(x;b;X)$ and $\sim\varphi(x;b;X)$ that is consistent with $p(x)\restriction A$ is finitely consistent with $p(x)$. Finally pick $a \models p(x)\restriction A$. As $\neg\varphi(a,b,C)$, from Fact 10 we obtain that $\sim\varphi(a;b;\tilde{C})$ holds for some $\tilde{C} \cap C = \varnothing$. Then $\sim\varphi(x;b;\tilde{C})$ is consistent with $p(x)\restriction A$. By maximality of $p(x)$, we have $\sim\varphi(x;b;\tilde{C}) \in p$. Then $b \in \mathcal{D}_{p,\sim\varphi,\tilde{C}}$.

$(\supseteq)$. Similar. □

**13 Fact** Let $\varphi(x;z;X)$ and $C$ be given. Then for every $\mathcal{D} \subseteq \mathcal{U}^z$ the following are equivalent

1. $\mathcal{D} = \mathcal{D}_{p,\varphi,C}$ for some global $\varphi(x;z;X)$-type $p(x)$
2. for every finite set $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that
$$B \cap \mathcal{D} \quad = \quad B \cap \varphi(a;\mathcal{U}^x;C).$$

**Proof.** Let $B_0 = B \smallsetminus \mathcal{D}$ and $B_1 = B \cap \mathcal{D}$.

1⇒2. By the above fact, there is a $\tilde{C} \cap C = \varnothing$ such that $b \in \mathcal{D}_{p,\sim\varphi,\tilde{C}}$ form every $b \in B_0$. Let $a$ realize the type containing $\varphi(x;b;C)$ for every $b \in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$.

2⇒1. There is a set $\tilde{C} \cap C = \varnothing$ such that $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$. Let $p(x)$ be any global $\varphi(x;z;X)$-type containing $\varphi(x;b;C)$ for every $b \in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$. Such type exists as (2) guarantees consistency. □

Let $\mathcal{D} \subseteq \mathcal{U}^z \times S^{|X|}$. For $\alpha \in S^{|X|}$, we write $\mathcal{D}_\alpha$ for the set of all $b \in \mathcal{U}^z$ such that $\langle b, \alpha\rangle \in \mathcal{D}$. We say that $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$ if there is a global $\varphi(x;z;X)$-type $p(x)$ such that for every $\alpha$

$$\mathcal{D}_\alpha \quad = \quad \bigcap\{\mathcal{D}_{p,\varphi,C} \ : \ \alpha \in C^\circ\}$$

We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if for every $\alpha \in S^{|X|}$ and every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^{|x|}$ such that

$$B \cap \mathcal{D}_\alpha \quad = \quad B \cap \bigcap\{\varphi(a;\mathcal{U}^z;C) \ : \ \alpha \in C^\circ\}$$

**14 Fact** Let $\varphi(x;z;X)$ be given. Then, for every $\mathcal{D} \subseteq \mathcal{U}^z \times S^{|X|}$ the following are equivalent

1. $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$
2. $\mathcal{D}$ is approximated by $\varphi(x;z;X)$.

We say that $\mathcal{D}$ is approximable from below if for every $B \subseteq \mathcal{D}$ there is an $a \in \mathcal{U}^{|x|}$ such that

$$B \quad \subseteq \quad \varphi(a;\mathcal{U}^z;C) \quad \subseteq \quad \mathcal{D}.$$

## 3. Stable formulas

We say that $\varphi(x;z;X)$ is unstable if there are some compact sets $C \cap \tilde{C} = \varnothing$ and a sequence $\langle a_i;b_i \,:\, i<\omega\rangle$ such that for every $i<j<\omega$

$$\varphi(a_i;b_j;C) \quad \wedge \quad \sim\varphi(a_j;b_i;\tilde{C})$$

We say that $\varphi(x;z;X)$ is stable if it is not unstable.

The following fact is a characterization of stability in a context of particular interest. Let $\tau : \mathcal{U}^{x;z} \to S$ be a definable function. We say that $\tau(x;z)$ is stable if so is the formula $\tau(x;z) \in X$.

**15 Fact** Let $\tau : \mathcal{U}^{x;z} \to S$ be a definable function. Then the following are equivalent

1. the formula $\tau(x;z) \in X$ is unstable
2. for some sequence $\langle a_i;b_i \,:\, i<\omega\rangle$
$$\lim_{i\to\infty}\lim_{j\to\infty} \tau(a_i;b_j) \quad \neq \quad \lim_{j\to\infty}\lim_{i\to\infty} \tau(a_i;b_j)$$
3. for some sequence $\langle a_i;b_i \,:\, i<\omega\rangle$ and some neighborhood $\varepsilon$ of the diagonal of $S^2$
$$\big\langle \tau(a_i;b_j),\, \tau(a_j;b_i)\big\rangle \quad \notin \quad \varepsilon$$
for every $i<j<\omega$.

**Proof.** 2⇔3. Clear.

1⇒3. Let $C \cap \tilde{C} = \varnothing$ be such that $\tau(a_i;b_j) \in C$ and $\tau(a_j;b_i) \in \tilde{C}$ for every $i<j<\omega$ as given by (1). Then (3) holds for any $\varepsilon$ contained in the complement of $C \times \tilde{C}$.

2⇒1. Let $C$ and $\tilde{C}$ be two disjoint intervals centered in the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i;b_i \,:\, i<\omega\rangle$. □

**16 Fact** Let $S=[0,1]$. Let $\tau(x;z)$ be a stable definable function. Let $f:\mathcal{U}^{x;z}\to S$. Assume that for every finite $B\subseteq\mathcal{U}^z$ and every $\varepsilon>0$ there is an $a\in\mathcal{U}^x$ such that

$$\big|f(b)-\tau(a;b)\big| \quad\le\quad \varepsilon \qquad\qquad \text{for every } b\in B.$$

Then there are some $\langle a_{i,j} : 1\le i,j\le n\rangle$ such that

$$f(b) \quad=\quad \max_{i=1,\dots,n}\ \min_{j=1,\dots,n}\ \tau(a_{i,j};b) \qquad\qquad \text{for every } b\in\mathcal{U}^z.$$

**Proof.** □

**17 Lemma** Let $\varphi(x;z;X)$ be stable. Let $\mathcal{D}\subseteq\mathcal{U}^{|z|}$ be approximated by $\varphi(x;z;X)$ from below. Then

$$\mathcal{D}_{p,\varphi,C} \quad=\quad \bigcup_{i=1}^{n}\varphi(a_i;\mathcal{U}^z;C)$$

for some $\langle a_i : 1\le i\le n\rangle$.

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We define $\mathcal{D}_p\subseteq\mathcal{U}^{|z|}\times S^{|X|}$ as follows

$$\mathcal{D}_p \quad=\quad \Big\{\langle b,\alpha\rangle : \varphi(x;b;C)\in p \ \text{ for every } C \text{ neighborhood of } \alpha\Big\}.$$

**18 Fact**

$$\neg\mathcal{D}_p \quad=\quad \Big\{\langle b,\alpha\rangle : \sim\!\varphi(x;b;\tilde{C})\in p \ \text{ for every } \tilde{C}\not\ni\alpha\Big\}.$$

**Proof.** Let $a\models p(x)\restriction b$. Let $C$ be the intersection of all neighborhoods of $\alpha$ such that $\varphi(a;b;C')$. Assume $\langle b,\alpha\rangle\notin\mathcal{D}_p$, in other words $\varphi(a;b;C)$, then $C$ is a neighborhood of $\alpha$. By Fact 10 we have $\varphi(a;b;C)$. Let $\tilde{C}\subseteq C\smallsetminus\{\alpha\}$ be such that $C$ is a neighborhood of $\tilde{C}$. Then $\neg\varphi(a;b;\tilde{C})$. Therefor by the definition of $\mathcal{D}_p$.

Then we can find $\tilde{C}$ such that $\alpha\notin\tilde{C}$ and $C$ is a neighborhood of $\tilde{C}$. contains two dijoint compacts Then $\sim\!\varphi(x;b;\tilde{C}')\in p$ by the definition of $\mathcal{D}_p$.

Then $\sim\!\varphi(x;b;\tilde{C}')\in p$ for every neighborhood $\tilde{C}'$ of $\alpha$ disjoint from $C'$. Then for every neighborhood $C$ of $\alpha$, we have $\varphi(x;b;C)\in p$. □

We say that $\mathcal{D}$ is externally defined by $p(x)$ and $\varphi(x;z;X)$ if $\mathcal{D}_p\subseteq\mathcal{D}$ and $\tilde{\mathcal{D}}_p\subseteq\neg\mathcal{D}$.

We say that $\mathcal{D}\subseteq\mathcal{U}^{|z|}\times S^{|X|}$ is approximated by $\varphi(x;z;X)$ if for every finite $B\subseteq\mathcal{U}^{|z|}\times S^{|X|}$ there is an $a\in\mathcal{U}^{|x|}$ such that

$$\langle b,\alpha\rangle\in B\cap\mathcal{D} \quad\Rightarrow\quad \varphi(a;b;C) \text{ for every neighborhood } C \text{ of } \alpha$$

$$\langle b,\alpha\rangle\in B\smallsetminus\mathcal{D} \quad\Rightarrow\quad \sim\!\varphi(a;b;C) \text{ for every } C \text{ such that } \alpha\notin C$$

**19 Fact** For every $\mathcal{D} \subseteq \mathcal{U}^{|z|} \times S^{|X|}$, the following are equivalent

1. $\mathcal{D}$ is externally definable by some global $\varphi$-type
2. $\mathcal{D}$ is approximated by $\varphi(x\,;z\,;X)$.

**Proof.** (1)⇒(2). Let $B \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ be finite. For $\langle b, \alpha\rangle \in B \cap \mathcal{D}$ let $C_{\langle b,\alpha\rangle}$ be some neightborhood of $\alpha$. Let $C$ be the union of all these neighborhoods. For $\langle b, \alpha\rangle \in B \smallsetminus \mathcal{D}$ let $\tilde{C}_{\langle b,\alpha\rangle}$ be some neightborhood of $\alpha$. Let $\tilde{C}$ be the union of all these neighborhoods. By the finiteness of $B$, we can require that $C$ and $\tilde{C}$ are disjoint. Then (2) follows from the concistency of $p(x)$.

(2)⇒(1). Note that the condition of approximability asserts the finite concistency of the type $p(x)$ that is union of the following two sets of formulas

$$\Big\{\varphi(x\,;b\,;C)\ :\ \langle b, \alpha\rangle \in \mathcal{D} \text{ and } C \text{ neighborhood of } \alpha\Big\}$$

$$\Big\{{\sim}\varphi(x\,;b\,;C)\ :\ \langle b, \alpha\rangle \notin \mathcal{D} \text{ and } \alpha \notin C\Big\}$$

The inclusion $\mathcal{D} \subseteq \mathcal{D}_p$ is immediate. For the converse inclusion, let $\langle b, \alpha\rangle \notin \mathcal{D}$ and let $\tilde{C}$ be a neighborhood of $\alpha$ such that ${\sim}\varphi(x\,;b\,;\tilde{C})$. □

**4.**

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