

Local stability in structures with a standard sort

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ABSTRACT. Continuous logic [BBHU] has replaced Henson-Iovino logic [HI] as a formalism to study the model theory of continuous structures. One of the first articles on the subject, [BU], defines local stability within continuous logic - as noted by Henson, this escaped [HI]'s approach. Subsequently, this has been used as a major argument to advocate in favor of [BBHU]'s over [HI]'s approach.

Recently, a classical approach to continuous structures has been proposed in [AAVV] and [Z] that properly extend the class of structures that falls under the scope of [HI] or [BBHU]. These articles introduced the notion of structures with a standard sort. We discuss local stability in this context. We examine a few variants of the classical definition which are prima facie non equivalent. For each we prove that sets externally definable by stable formulas are definable in some appropriate sense.

1. Structures with a standard sort

Let S be a Hausdorff compact topological space. We associate to S a first order structure in a language \mathcal{L}_S that has a relation symbol for each compact subset $C \subseteq S$ and a function symbol for each continuous function $f : S^n \rightarrow S$. Depending on the context, C and f denote either the symbols of \mathcal{L}_S or their interpretation in the structure S . Throughout these notes the letter C always denotes a compact subset of S , or a tuple of such sets.

Finally, note that we could allow in \mathcal{L}_S relation symbols for all compact subsets of S^n , for any n . But this would clutter the notation, therefore we prefer to leave the straightforward generalization to the reader.

We also fix an arbitrary first order language which we denote by \mathcal{L}_H and call the language of the home sort.

1 Definition Let \mathcal{L} be a two sorted language. The two sorts are denoted by H and S . The language \mathcal{L} expands \mathcal{L}_H and \mathcal{L}_S with symbols of sort $H^n \times S^m \rightarrow S$. An \mathcal{L} -structure is a structure of signature \mathcal{L} that interprets these symbols with equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H). For a given S a **standard structure** is a two-sorted \mathcal{L} -structure of the form $\langle M, S \rangle$, where M is any structure of signature \mathcal{L}_H . Standard structures are denoted by the domain of their home sort.

We denote by \mathcal{F} the set of \mathcal{L} -formulas constructed inductively from atomic formulas of the form (i) and (ii) below using the Boolean connectives \wedge, \vee ; the quantifiers \forall^H, \exists^H of sort H ; and the quantifiers \forall^S, \exists^S of sort S .

We write $x \in C$ for the predicate associated to a compact set C , and denote by $|-|$ the length of a tuple.

2 Definition We call atomic formulas those of the form

- i. atomic and negated atomic formulas of \mathcal{L}_H
- ii. $\tau \in C$, where τ is a term of sort $H^n \times S^m \rightarrow S$, and C a compact set.

We remark that a smaller fragment of \mathcal{F} may be of interest in some specific contexts. This fragment is obtained by excluding equalities and inequalities from (i) and requiring that C in (ii) is regular. The discussion below applies to this smaller fragment as well.

Let $M \subseteq N$ be standard structures. We say that M is an \mathcal{F} -elementary substructure of N if the latter models all $\mathcal{F}(M)$ -sentences that are true in M .

Let x and ξ be variables of sort H , respectively S . A standard structure M is \mathcal{F} -saturated if it realizes every type $p(x; \xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M . The following theorem is proved in [AAVV] for signatures that do not contain symbols of sort $H^n \times S^m \rightarrow S$ with $n \cdot m > 0$. A similar framework has been independently introduced in [CP] – only without quantifiers of sort H and with an approximate notion of satisfaction. The corresponding compactness theorem is proved there with a similar method. The setting in [CP] is a generalization of that used by Henson and Iovino for Banach spaces [HI]. By the elimination of quantifiers of sort S , proved in [AAVV, Proposition 3.6], the two approaches are equivalent – up to approximations.

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [AAVV] extends to the case $n \cdot m > 0$.

3 Theorem (Compactness) Every standard structure has an \mathcal{F} -saturated \mathcal{F} -elementary extension.

We introduce a new sort X with the purpose of conveniently describing classes of \mathcal{F} -formulas that only differ by the relation symbol $C \subseteq S$ they contain. Let \mathcal{F}_X be defined as \mathcal{F} but replacing (ii) by

- iii. $\tau(x; \xi) \in X$, where X is a variable of sort X .

Formulas in \mathcal{F}_X are denoted by $\varphi(x; \xi; X)$, where $X = X_1, \dots, X_n$ is a tuple of variables of sort X . If $C = C_1, \dots, C_n$ is a tuple of compact subsets, then $\varphi(x; \xi; C)$ is a formula in \mathcal{F} . This we call an **instance** of $\varphi(x; \xi; X)$. All formulas in \mathcal{F} are instances of formulas in \mathcal{F}_X .

4 Definition Let $\varphi \in \mathcal{F}_X$ be a formula – possibly with some hidden free variables. The **pseudonegation** of $\varphi \in \mathcal{F}_X$ is the formula obtained by replacing the atomic formulas in \mathcal{L}_H by their negation and the logical symbols $\wedge, \vee, \forall, \exists$ by their respective dual $\vee, \wedge, \exists, \forall$. The atomic formulas of the form $\tau \in X$ remain unchanged.

The pseudonegation of φ is denoted by $\sim\varphi$. Clearly, when no variable of sort X occurs in φ , we have $\sim\varphi \leftrightarrow \neg\varphi$.

It is sometimes convenient to work with infinite conjunctions of formulas. In the following $\sigma(x; z; X)$ denotes an \mathcal{F}_X -type. We write $\sim\sigma(x; z; X)$ for the possibly infinite disjunction of the formulas $\sim\varphi(x; z; X)$ for $\varphi(x; z; X) \in \sigma$.

The following fact is immediate and will be used without further mention.

5 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \emptyset$, every standard structure M , and every $\mathcal{F}_X(M)$ -type $\sigma(X)$

$$\begin{aligned} M &\models \sigma(C) \rightarrow \sigma(C') \\ M &\models \sim\sigma(\tilde{C}) \rightarrow \neg\sigma(C). \end{aligned}$$

6 Fact For every $C \subseteq C'$ there is \tilde{C} disjoint from C such that

$$M \models \varphi(C) \rightarrow \neg\sim\varphi(\tilde{C}) \rightarrow \varphi(C')$$

for every M , and every $\mathcal{F}_X(M)$ -formula $\varphi(X)$. Conversely, for every $\tilde{C} \cap C = \emptyset$ there is a neighborhood C' of C such that

$$M \models \varphi(C) \rightarrow \varphi(C') \rightarrow \neg\sim\varphi(\tilde{C}).$$

for every M , and every $\mathcal{F}_X(M)$ -formula $\varphi(X)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = S \setminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds when C' is any neighborhood of C disjoint from \tilde{C} . Induction on the syntax of $\varphi(X)$ proves the general case. \square

The above fact has the following useful consequence.

7 Fact The following are equivalent for every standard structure M , every $\mathcal{F}_X(M)$ -type $\sigma(X)$, and every C

1. $M \models \sigma(C) \leftarrow \bigwedge \left\{ \sigma(C') : C' \text{ neighborhood of } C \right\}$
2. $M \models \neg\sigma(C) \rightarrow \bigvee \left\{ \sim\sigma(\tilde{C}) : \tilde{C} \cap C = \emptyset \right\}.$

(The converse implications are trivial – therefore not displayed.)

8 Fact The equivalent conditions in Fact 7 hold in all \mathcal{F} -saturated structures.

Proof. It suffices to prove the claim for formulas. We prove the first of the two implications by induction on the syntax of $\varphi(X)$. The existential quantifier of sort H is the only case that requires attention. Assume inductively that

$$\bigwedge \left\{ \varphi(a; C') : C' \text{ neighborhood of } C \right\} \rightarrow \varphi(a; C)$$

holds for every $\varphi(a; C)$. Then induction for the existential quantifier follows from

$$\bigwedge \left\{ \exists x \varphi(x; C') : C' \text{ neighborhood of } C \right\} \rightarrow \exists x \bigwedge \left\{ \varphi(x; C') : C' \text{ neighborhood of } C \right\}$$

which is a consequence of saturation. \square

We say that M is \mathcal{F} -maximal if it models $\varphi \in \mathcal{F}(M)$ whenever φ holds in some \mathcal{F} -elementary extensions of M . By Fact 8 and the following fact, all \mathcal{F} -saturated standard structures are \mathcal{F} -maximal.

9 Fact Let M be a standard structure. Then the following are equivalent

1. M is \mathcal{F} -maximal
2. the conditions in Fact 7 hold for every type (equivalently, for every formula).

Proof. $1 \Rightarrow 2$. If (2) in Fact 7 fails, $\neg\varphi(C) \wedge \neg\sim\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \emptyset$. Let \mathcal{U} be an \mathcal{F} -saturated \mathcal{F} -elementary extension of M . As M is \mathcal{F} -maximal, $\neg\varphi(C) \wedge \neg\sim\varphi(\tilde{C})$ also holds in \mathcal{U} . Then (2) in Fact 7 fails in \mathcal{U} , contradicting Fact 8.

$2 \Rightarrow 1$. Let N be any \mathcal{F} -elementary extension of M . Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every neighborhood C' of C . Suppose not. Then, by (2) in Fact 7, $M \models \sim\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \emptyset$. Then $N \models \sim\varphi(\tilde{C})$ by \mathcal{F} -elementarity. As $\tilde{C} \cap C = \emptyset$, this contradicts Fact 5. \square

10 Notation In what follows we work in a fixed \mathcal{F} -saturated standard structure \mathcal{U} . We denote by κ the cardinality of \mathcal{U} . We assume that κ is a Ramsey cardinal. However, the willing reader may check that we can do without large cardinals by careful application of the Erdős-Rado Theorem, see [TZ, Appendix C.3].

In the following $\sigma(x; z; X)$ always denotes an $\mathcal{F}_X(\mathcal{U})$ -type of small (i.e. $< \kappa$) cardinality.

2. A duality in $K(S)$

Write $K(S)$ for the set of compact subsets of S . In this section \mathcal{D} and \mathcal{C} range over subsets of $K(S)^n$. We define

$$\sim\mathcal{D} = \{ \tilde{C} : \tilde{C} \cap C \neq \emptyset \text{ for every } C \in \mathcal{D} \}.$$

The following fact motivates the definition.

11 Fact Let $\mathcal{D} = \{C : \varphi(C)\}$. Then $\sim\mathcal{D} = \{\tilde{C} : \sim\varphi(\tilde{C})\}$.

Proof. We need to prove that

$$\sim\varphi(\tilde{C}) \Leftrightarrow \text{for every } C, \text{ if } C \cap \tilde{C} = \emptyset \text{ then } \neg\varphi(C)$$

The implication \Rightarrow follows immediately from Fact 5. To prove \Leftarrow assume the r.h.s. Then $\neg\varphi(S \setminus O)$ holds for every open set $O \supseteq \tilde{C}$. From the second implication in Fact 7 we obtain $\sim\varphi(\tilde{C}')$ for some $\tilde{C} \subseteq \tilde{C}' \subseteq O$. As O is arbitrary, we obtain $\sim\varphi(\tilde{C})$ from the first implication in Fact 7. \square

We prove a couple of straightforward inclusions.

12 Fact For every $\mathcal{D} \subseteq \mathcal{C}$ we have $\sim\mathcal{C} \subseteq \sim\mathcal{D}$. Moreover, $\mathcal{D} \subseteq \sim\sim\mathcal{D}$.

Proof. Let $C \notin \sim\mathcal{D}$. Then $\tilde{C} \cap C = \emptyset$ for some $\tilde{C} \in \mathcal{D}$. As $\tilde{C} \in \mathcal{C}$ we conclude that $C \notin \sim\mathcal{C}$. For the second claim, let $C \notin \sim\sim\mathcal{D}$. Then $\tilde{C} \cap C = \emptyset$ for some $\tilde{C} \in \sim\mathcal{D}$. Then $C \notin \mathcal{D}$ follows. \square

13 Fact For every \mathcal{D} the following are equivalent


1. $\mathcal{D} = \sim\sim\mathcal{D}$.
2. $\mathcal{D} = \sim\mathcal{C}$ for some \mathcal{C} .

Proof. Only $2 \Rightarrow 1$ requires a proof. From Fact 12 we obtain $\sim\sim\sim\mathcal{C} \subseteq \sim\mathcal{C}$. Assume (2) then $\sim\sim\mathcal{D} \subseteq \mathcal{D}$ which, again by Fact 12, suffices to prove (1). \square

We say that \mathcal{D} is **involutive** if $\mathcal{D} = \sim\sim\mathcal{D}$.

14 Theorem The following are equivalent

1. \mathcal{D} is involutive
2. $C \in \mathcal{D} \Leftrightarrow C' \in \mathcal{D}$ for every neighborhood C' of C .

 It is not difficult to see that (2) holds if and only if \mathcal{D} is closed in the Vietoris topology and includes all the supersets of its elements.

Proof. $2 \Rightarrow 1$. By Fact 12 it suffices to prove $\sim\sim\mathcal{D} \subseteq \mathcal{D}$. Let $C \in \sim\sim\mathcal{D}$. Let $O \supseteq C$ be open. Then $S \setminus O \notin \sim\mathcal{D}$. Then $\tilde{C} \in \mathcal{D}$ for some $\tilde{C} \subseteq O$. Assume (2). Then, by \Rightarrow every $C' \supseteq O$ is in \mathcal{D} . As O is arbitrary, $C \in \mathcal{D}$ by \Leftarrow .

$1 \Rightarrow 2$. It suffices to show that (2) holds with $\sim\mathcal{D}$ for \mathcal{D} . The implication \Rightarrow in (2) is obvious. To prove \Leftarrow , assume $C \notin \sim\mathcal{D}$. Then $C \cap \tilde{C}$ for some $\tilde{C} \in \mathcal{D} = \emptyset$. Let C' be a neighborhood of C disjoint from \tilde{C} . Then the r.h.s. of the equivalence fails for C' . \square

3. Stable formulas – the finitary case

There is more than one way to define stability in our context. In this section we present the strongest possible requirement. It is not the most interesting one, but it is very similar to the classical definition – the comparison may be useful though not strictly required for the sequel.

The following is the classical definition of stability which we dub *finitary* for the reason explained below.

15 Definition We say that $\varphi(x; z; C)$ is **finitarily stable** if for some $m < \omega$ there is no sequence $\langle a_i; b_i : i < m \rangle$ such that

1. $\varphi(a_n; b_i; C) \wedge \neg \varphi(a_i; b_n; C)$ for every $i < n < m$.

The following is an equivalent definition which only mentions formulas in \mathcal{F} . The reader may wish to compare it with Definition 20.

16 Fact The following are equivalent

1. $\varphi(x; z; C)$ is finitarily stable
2. for some m there is no sequence $\langle a_i; b_i : i < m \rangle$ such that for some $\tilde{C} \cap C = \emptyset$

$$\varphi(a_n; b_i; C) \wedge \sim \varphi(a_i; b_n; \tilde{C}) \quad \text{for every } i < n < m.$$

Proof. $1 \Rightarrow 2$. By Fact 5.

$2 \Rightarrow 1$. Negate (2). Let $\langle a_i; b_i : i < m \rangle$ be such that (1) of Definition 15 holds. By Fact 7 there are $\tilde{C}_{n,i}$ such that

$$\varphi(a_n; b_i; C) \wedge \sim \varphi(a_i; b_n; \tilde{C}_{n,i}) \quad \text{for every } i < n < m.$$

Then the union of the $\tilde{C}_{n,i}$ is the set \tilde{C} required to negate (2). \square

In a classical context, i.e. relying on \mathcal{L} -saturation, one can replace m in Definition 15 by ω . But this may not be true if we only have \mathcal{F} -saturation. Therefore we dedicate a separate section to the non-finitary version of stability.

The following definitions are classical, i.e. they rely on the full language \mathcal{L} .

17 Definition For a given C , a **global $\varphi(x; z; C)$ -type** is a maximally (finitely) consistent set of formulas of the form $\varphi(x; b; C)$ or $\neg \varphi(x; b; C)$ for some $b \in \mathcal{U}^{|z|}$.

Global $\varphi(x; z; C)$ -types should not be confused with global $\varphi(x; z; X)$ -types which are introduced in the following section.

In this section the symbol \mathcal{D} always denotes a subset of \mathcal{U}^z .

18 Definition We say that \mathcal{D} is **approximable** by $\varphi(x; z; C)$ if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that

$$b \in \mathcal{D} \Leftrightarrow \varphi(a; b; C) \quad \text{for every } b \in B.$$

It goes without saying that these definitions describe two sides of the same coin. In fact, it is clear that \mathcal{D} is approximable by $\varphi(x; z; C)$ if and only if

$$p(x) = \{\varphi(x; b; C) : b \in \mathcal{D}\} \cup \{\neg \varphi(x; b; C) : b \notin \mathcal{D}\}$$

is a global $\varphi(x; z; C)$ -type.

The following theorem is often rephrased by saying that, when $\varphi(x; z; C)$ is finitarily stable, then all global $\varphi(x; z; C)$ -types are definable. The proof of the classical case applies verbatim because no saturation is required. We state it here only for comparison with its non-finitary counterpart, see Theorem 27 and its approximated counterpart, see Theorem 35.

19 Theorem Let $\varphi(x; z; C)$ be finitarily stable. Let \mathcal{D} be approximable by $\varphi(x; z; C)$. Then there are some $\langle a_{i,j} : i < k, j < m \rangle$ such that for every $b \in \mathcal{U}^z$

$$b \in \mathcal{D} \Leftrightarrow \bigvee_{i < k} \bigwedge_{j < m} \varphi(a_{i,j}; b; C)$$

4. Stable formulas – the non-finitary case

We discuss a version of stability that seems more appropriate in this context. There are two main differences with the previous section. First, here the attribute *stable* refers to a whole class of formulas – all instances of some $\varphi(x; z; X)$. Furthermore, the sequence witnessing the order property is required to be infinite. As we need to extend the definition of stability to types, sequences of length κ (a large cardinal) are used.

In this section \mathcal{D} is always a subset of $\mathcal{U}^z \times K(S)^n$, where n has to be inferred from the context (typically, $n = |X|$). We write $\sim\mathcal{D}$ for the set obtained by applying the definition in Section 2 to all fibers of \mathcal{D} . That is, if \mathcal{D}_b is the b -fiber $\{C : \langle b; C \rangle \in \mathcal{D}\}$, we define

$$\sim\mathcal{D} = \bigcup_{b \in \mathcal{U}^z} \{b\} \times (\sim\mathcal{D}_b)$$

We say that \mathcal{D} is involutive if all its fibers are.

20 Definition We say that $\sigma(x; z; X)$ is **stable** if there is no sequence $\langle a_i; b_i : i < \kappa \rangle$ such that for some C , some $\tilde{C} \cap C = \emptyset$

$$\sigma(a_n; b_i; C) \wedge \sim\sigma(a_i; b_n; \tilde{C}) \quad \text{for every } i < n < \kappa \text{ or for every } n < i < \kappa.$$

Note that when $\sigma(x; z; X)$ is a formula by compactness we can replace κ with ω in the above definition.

21 Remark In Definition 20 we can equivalently require the following. There is no sequence $\langle a_i; b_i : i < \omega \rangle$ such that for some C , some $\tilde{C} \cap C = \emptyset$, and some formula $\varphi(x; z; X) \in \sigma$

$$\sigma(a_n; b_i; C) \wedge \sim\varphi(a_i; b_n; \tilde{C}) \quad \text{for every } i < n < \kappa \text{ or for every } n < i < \kappa.$$

Proof. The conjunction in Definition 20 holds when $\sigma(a_n; b_i; C) \wedge \sim\varphi(a_i; b_n; \tilde{C})$ holds for some $\varphi(x; z; X) \in \sigma$. The formula $\varphi(x; z; X)$ depends on i and n but, since κ is a Ramsey cardinal, we can refine the sequence to obtain that the same $\varphi(x; z; X)$ for all $i < n < \kappa$.

Once we replace $\sim\sigma(x; z; X)$ by $\sim\varphi(x; z; X)$ compactness applies and the existence of a sequence of length ω is equivalent to the existence of a sequence of length κ . \square

Note also that, by the pigeonhole principle, we can strengthen the requirement in Definition 20 by requiring that there is no $\langle a_i; b_i; C_i; \tilde{C}_i : i < \omega \rangle$ such that $\tilde{C}_i \cap C_i = \emptyset$ and

$$\sigma(a_n; b_i; C_i) \wedge \sim\varphi(a_i; b_n; \tilde{C}_n) \quad \text{for every } i < n < \kappa \text{ or for every } n < i < \kappa.$$

22 Definition Let $\varphi(x; z; X)$ be a \mathcal{F}_X -formula. A **global $\varphi(x; z; X)$ -type** is a maximally (finitely) consistent set of formulas of the form $\varphi(x; b; C)$ or $\sim\varphi(x; b; C)$ for some $b \in \mathcal{U}^{|z|}$ and some C .

It is convenient to have a different characterization of global types, therefore the following definition.

23 Definition We say that \mathcal{D} is **approximable** by $\sigma(x; z; X)$ if for every small set $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that

1. $\langle b, C \rangle \in \mathcal{D} \Rightarrow \sigma(a; b; C)$ and
2. $\langle b, C \rangle \in \mathcal{D} \Leftarrow \sigma(a; b; C)$ for every $b \in B$ and every C .

When $\sigma(x; z; X)$ is a formula and \mathcal{D} is involutive (2) in the above definition can be rephrased in a more convenient form.

24 Remark If \mathcal{D} is involutive the following are equivalent

2. $\langle b, C \rangle \in \mathcal{D} \Leftarrow \varphi(a; b; C)$ for every $b \in B$ and every C
- 2' $\langle b, C \rangle \in \sim\mathcal{D} \Rightarrow \sim\varphi(a; b; C)$ for every $b \in B$ and every C .

Proof. Let $\mathcal{C} = \{C : \varphi(a, b, C)\}$. This is also an involutive set by Fact 11. Then (2) says $\mathcal{C} \subseteq \mathcal{D}$. By Fact 12 this is equivalent to $\sim\mathcal{D} \subseteq \sim\mathcal{C}$ which in turn is equivalent to (2'), again by Fact 11. \square

Under the assumption of the remark it is easy to verify by compactness that in Definition 18 the requirement *for every small B* can be replaced by *for every finite B*.

25 Fact Let $\varphi(x; z; X)$ be a \mathcal{F}_X -formula. For every involutive \mathcal{D} the following are equivalent

1. \mathcal{D} is approximable by $\varphi(x; z; X)$
2. the following is a global $\varphi(x; z; X)$ -type

$$p(x) = \{ \varphi(x; b; C) : \langle b, C \rangle \in \mathcal{D} \} \cup \{ \sim\varphi(x; b; C) : \langle b, C \rangle \in \sim\mathcal{D} \}.$$

Proof. First we prove that any $p(x)$ defined as in (2) is maximal as soon as it is consistent – in fact, this only depends on \mathcal{D} being involutive. We need to consider two

cases. First, assume that $\varphi(x; b; C)$ is consistent with $p(x)$. Then consistency implies that $\langle b, \tilde{C} \rangle \notin \sim \mathcal{D}$ for every $\tilde{C} \cap C = \emptyset$. But this implies that $C \in \sim \sim \mathcal{D} = \mathcal{D}$, therefore $\varphi(x; b; C) \in p$. The second case is when $\sim \varphi(x; b; C)$ is consistent with $p(x)$. This implies that $\langle b, \tilde{C} \rangle \notin \mathcal{D}$ for every $\tilde{C} \cap C = \emptyset$. Then $\langle b, C \rangle \in \sim \mathcal{D}$, therefore $\sim \varphi(x; b; C) \in p$.

2 \Rightarrow 1. Let B be a given small subset of \mathcal{U}^z . The consistency of $p(x) \upharpoonright B$ provides some a such that

$$\begin{aligned} \langle b, C \rangle \in \mathcal{D} &\Rightarrow \varphi(a; b; C) \quad \text{and} \\ \langle b, C \rangle \in \sim \mathcal{D} &\Rightarrow \sim \varphi(a; b; C) \end{aligned} \quad \text{for every } b \in B \text{ and every } C.$$

By Remark 24 above, these yield (1) and (2) in Definition 23.

1 \Rightarrow 2. Let B be a given small set. Let a be as in Definition 23. Then $a \models p(x) \upharpoonright B$ by Remark 24. \square

The proof of the main theorem of this section requires the following notion of approximation. The definition is inspired by the classical notion of approximation from below in [Z15] where it is used as rephrasing of the notion of honest definability in [CS]. Here this notion plays a purely technical role, but we submit it as potentially interesting in itself.

26 Definition We say that \mathcal{D} is **approximable** by $\sigma(x; z; X)$ **from below** if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that for every C

1. $\langle b, C \rangle \in \mathcal{D} \Rightarrow \sigma(a; b; C)$ for every $b \in B$ and
2. $\langle b, C \rangle \in \mathcal{D} \Leftarrow \sigma(a; b; C)$ for every $b \in \mathcal{U}^z$.

27 Theorem Let $\sigma(x; z; X)$ be stable. Assume that \mathcal{D} is approximable by $\sigma(x; z; X)$. Then there are some $\langle a_{i,j} : i < \lambda, j < \mu \rangle$ such that for every $b \in \mathcal{U}^z$ and every C

$$\langle b, C \rangle \in \mathcal{D} \Leftrightarrow \bigvee_{i < \lambda} \bigwedge_{j < \mu} \sigma(a_{i,j}; b; C)$$

Proof. The theorem is an immediate consequence of the following three lemmas. \square

28 Lemma Let $\sigma(x; z; X)$ be stable. Assume that \mathcal{D} is approximable by $\sigma(x; z; X)$ from below. Then there are some $\langle a_i : i < \lambda \rangle$ such that for every $b \in \mathcal{U}^z$ and every C

1. $\langle b, C \rangle \in \mathcal{D} \Rightarrow \bigvee_{i < \lambda} \sigma(a_i; b; C)$
2. $\langle b, C \rangle \in \mathcal{D} \Leftarrow \bigvee_{i < \lambda} \sigma(a_i; b; C)$

Proof. We define recursively the required parameters a_i together with some auxiliary parameters b_i . The element a_n is chosen such that

3. $\langle b, C \rangle \in \mathcal{D} \Leftarrow \sigma(a_n; b; C)$ for every $b \in \mathcal{U}^z$ and every C

and

$$4. \quad \langle b_i, C \rangle \in \mathcal{D} \Rightarrow \sigma(a_n; b_i; C) \quad \text{for every } i < n \text{ and every } C.$$

This is possible because \mathcal{D} is approximated from below. Note that (3) immediately guarantees (2). Now, assume (1) fails for $\lambda = n$, and choose b_n and C_n witnessing this. Then, by Fact 7, for some $\tilde{C}_n \cap C_n = \emptyset$

$$5. \quad \langle b_n, C_n \rangle \in \mathcal{D} \quad \& \quad \bigwedge_{i < n} \sim \sigma(a_i; b_n; \tilde{C}_n)$$

Suppose for a contradiction that the construction never ends. Then, as (5) guarantees that $\langle b_i, C_i \rangle \in \mathcal{D}$ for every i , from (4) we obtain $\sigma(a_n; b_i; C_n)$ for every $i < n$. From (5) we also obtain $\sim \sigma(a_i; b_n; \tilde{C}_n)$, for every $i \leq n$. This contradicts stability (as noted after Remark 21). \square

29 Lemma Let $\sigma(x; z; X)$ be stable. Then for some $\mu < \kappa$ the type

$$\pi(\bar{x}; z; X) = \bigwedge_{i < \mu} \sigma(x_i; z; X),$$

where the x_i are copies of x and $\bar{x} = \langle x_i : i < \mu \rangle$, approximates \mathcal{D} from below.

Proof. Negate the claim and let B witness that $\pi(\bar{x})$ does not approximate \mathcal{D} from below. Suppose that $\langle a_i : i < n \rangle$ and $\langle b_i : i < n \rangle$ have been defined. Choose a_n such that for every $b \in B \cup \{b_i : i < n\}$ and every C

$$\begin{aligned} 1. \quad & \langle b, C \rangle \in \mathcal{D} \Rightarrow \varphi(a_n; b; C) \quad \text{and} \\ 2. \quad & \langle b, C \rangle \in \mathcal{D} \Leftarrow \varphi(a_n; b; C) \end{aligned}$$

Note that the latter implication is equivalent to: for every C there is some $\tilde{C} \cap C = \emptyset$ such that

$$3. \quad \langle b, C \rangle \notin \mathcal{D} \Rightarrow \sim \varphi(a_n; b; \tilde{C}).$$

Now, as the lemma is assumed to fail, we can choose b_n and C_n such that

$$4. \quad \langle b_n, C_n \rangle \notin \mathcal{D} \quad \& \quad \bigwedge_{i=0}^n \varphi(a_i; b_n; C_n)$$

Note that (4) ensure that $\langle b_i, C_i \rangle \notin \mathcal{D}$ for every i . There is some \tilde{C}_i that witnesses (3) for $\langle b_i, C_i \rangle \notin \mathcal{D}$. We claim that the procedure has to stop after $< \kappa$ steps. In fact, from (3) we obtain $\sim \varphi(a_n; b_i; \tilde{C}_i)$ for every $i < n$. On the other hand, by (4) we have that $\varphi(a_i; b_n; C_n)$ for every $i < n$. Therefore, $\langle a_i; b_i; C_i; \tilde{C}_i : i < \kappa \rangle$ contradicts stability (as noted after Remark 21). \square

30 Lemma If $\sigma(x; z; X)$ is stable then $\pi(\bar{x}; z; X)$ in the previous lemma is stable.

Proof. Suppose $\pi(\bar{x}; z; X)$ is unstable. Let $\langle \bar{a}_i; b_i : i < \kappa \rangle$ be a sequence witnessing instability. Then for every pair $i < n$ there is some $j < \mu$ such that $\sim \sigma(a_{j,n}; b_i; \tilde{C})$. Interpret such j as a color. As κ is a Ramsey cardinal, every μ -coloring of a complete graph of size κ has a monochromatic subgraph of size κ . That is, there is a $j < \mu$ such

that $\sim\sigma(a_{j,n}; b_i; \tilde{C})$ obtains for κ many i . Therefore we can extract a subsequence that contradicts the stability of $\sigma(x; z; X)$. \square

5. Stable definable functions

In this section we specialize the notions introduced in the previous sections to types $\sigma(x; z; X)$ of the form $\tau(x; z) \in X$ where $\tau(x; z)$ is an S -valued \mathcal{F} -type definable function.

We say that $\tau(x; z)$ is **stable** if so is the type $\tau(x; z) \in X$. The following fact is a suggestive characterization of the stability of functions which has been first remarked in [B].

31 Fact Let $\tau(x; z)$ be term in \mathcal{L} . Then the following are equivalent

1. the formula $\tau(x; z) \in X$ is unstable
2. there is a sequence $\langle a_i; b_i : i < \omega \rangle$ such that

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \tau(a_i; b_j) \neq \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \tau(a_i; b_j)$$

Terms are a particular case of definable functions. When $\tau(x; z)$ is a term we have that $\sim(\tau(x; z) \in X) = \tau(x; z) \in X$ which is used throughout in the proof below. We suspect the fact holds in general though we have not been able to prove it.

Proof. $1 \Rightarrow 2$. Let $C \cap \tilde{C} = \emptyset$ and $\langle a_i; b_i : i < \omega \rangle$ be as given by (1). That is, $\tau(a_i; b_j) \in C$ and $\tau(a_j; b_i) \in \tilde{C}$ hold for every $i < j < \omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and \tilde{C} contains the limit on the right – which therefore are distinct.

$2 \Rightarrow 1$. Let C and \tilde{C} be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i; b_i : i < \omega \rangle$. \square

Let $f : \mathcal{U}^z \rightarrow S$ be a function. We define

$$\mathcal{D}_f = \{ \langle b, C \rangle : f(b) \in C, b \in \mathcal{U}^z, C \in K(S) \}.$$

The following fact is immediate.

32 Fact The following are equivalent

- 1 \mathcal{D}_f is approximable by $\tau(x; z) \in X$
- 2 for every small $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that $\tau(a; b) = f(b)$ for every $b \in B$.

We say that f is approximable by $\tau(x; z)$ if the equivalent conditions above apply. The following is an easy consequence if Theorem 27

33 Theorem Let $S = [0, 1]$. Let $\tau(x; z)$ be stable. Let f be approximable by $\tau(x; z)$. Then there are some $\langle a_{i,j} : i < \lambda, j < \mu \rangle$ such that

$$f(b) = \sup_{i < \lambda} \inf_{j < \mu} \tau(a_{i,j}; b) \quad \text{for every } b \in \mathcal{U}^z.$$

6. Yet another version of stability

In this section we present the notion of ε -stability and prove a version of Theorem 27 which, albeit approximate, only requires finite disjunctions and conjunctions. Note that the requirement of ε -stability is stronger the smaller the ε and it always implies stability as defined in Definition 20 – we do not know it coincides with it in the limit.

34 Definition We say that $\varphi(x; z; X)$ is ε -stable if there is a maximal m such that some sequence $\langle a_i; b_i; C_i; \tilde{C}_i : i < m \rangle$ is such that for every $i < n < m$

$$\varphi(a_n; b_i; C_i) \wedge \sim \varphi(a_i; b_n; \tilde{C}_n) \quad \text{and} \quad C_i^\varepsilon \cap \tilde{C}_i = \emptyset.$$

We state the main theorem of this section which is proved along the same lines as Theorem 27

35 Theorem Let $\varphi(x; z; X)$ be ε -stable. Assume that \mathcal{D} is approximable by $\varphi(x; z; X)$. Then there are some $\langle a_{i,j} : i < k, j < m \rangle$ such that for every $b \in \mathcal{U}^z$ and every C

$$\begin{aligned} \langle b, C \rangle \in \mathcal{D} &\Rightarrow \bigvee_{i < k} \bigwedge_{j < m} \varphi(a_{i,j}; b; C^\varepsilon) \\ \langle b, C^\varepsilon \rangle \in \mathcal{D} &\Leftarrow \bigvee_{i < k} \bigwedge_{j < m} \varphi(a_{i,j}; b; C) \end{aligned}$$

Proof. The theorem is an immediate consequence of the following three lemmas. \square

The proof of the theorem requires the following version of Definition 26

36 Definition We say that \mathcal{D} is approximable by $\varphi(x; z; X)$ from ε -below if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that for every C

1. $\langle b, C \rangle \in \mathcal{D} \Rightarrow \varphi(a; b; C)$ for every $b \in B$ and
2. $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \varphi(a; b; C)$ for every $b \in \mathcal{U}^z$.

37 Lemma Let $\varphi(x; z; X)$ be ε -stable. Assume that \mathcal{D} is approximable by $\varphi(x; z; X)$ from ε -below. Then there is some $\langle a_i : i < k \rangle$ such that for every $b \in \mathcal{U}^z$ and every C

1. $\langle b, C \rangle \in \mathcal{D} \Rightarrow \bigvee_{i < k} \varphi(a_i; b; C^\varepsilon)$
2. $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \bigvee_{i < k} \varphi(a_i; b; C)$

Proof. We define recursively the required parameters a_i together with some auxiliary parameters b_i . The element a_n is chosen such that

3. $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \varphi(a_n; b; C)$ for every $b \in \mathcal{U}^z$ and every C

and

4. $\langle b_i, C \rangle \in \mathcal{D} \Rightarrow \varphi(a_n; b_i; C)$ for every $i < n$ and every C .

This is possible because \mathcal{D} is ε -approximated from below. Note that (3) immediately guarantees (2). Now, assume (1) fails for $k = n$, and choose b_n and C_n witnessing this. Then, by Fact 7, for some $\tilde{C}_n \cap C_n^\varepsilon = \emptyset$

$$5. \quad \langle b_n, C_n \rangle \in \mathcal{D} \quad \& \quad \bigwedge_{i < n} \sim \varphi(a_i; b_n; \tilde{C}_n)$$

Suppose for a contradiction that the construction never ends. Then, as (5) guarantees that $\langle b_i, C_i \rangle \in \mathcal{D}$ for every i , from (4) we obtain $\varphi(a_n; b_i; C_n)$ for every $i < n$. From (5) we also obtain $\sim \varphi(a_i; b_n; \tilde{C}_n)$, for every $i \leq n$. This contradicts ε -stability. \square

38 Lemma Let $\varphi(x; z; X)$ be ε -stable. Assume that \mathcal{D} is approximable by $\varphi(x; z; X)$. Let m be maximal so that a sequence as in (3) of Definition 34 exists. Let $\bar{x} = \langle x_i : i \leq m \rangle$ where the x_i are copies of x . Then the formula

$$\sigma(\bar{x}; z; X) = \bigwedge_{i \leq m} \varphi(x_i; z; X)$$

approximate \mathcal{D} from ε -below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x})$ does not approximate \mathcal{D} from ε -below. Suppose that a_0, \dots, a_{n-1} and b_0, \dots, b_{n-1} have been defined. Choose a_n such that for every $b \in B \cup \{b_0, \dots, b_{n-1}\}$ and every C

1. $\langle b, C \rangle \in \mathcal{D} \Rightarrow \varphi(a_n; b; C)$ and
2. $\langle b, C^\varepsilon \rangle \in \mathcal{D} \Leftarrow \varphi(a_n; b; C^\varepsilon)$

Note that the latter implication is equivalent to: for every C there is some $\tilde{C} \cap C^\varepsilon = \emptyset$ such that

$$3. \quad \langle b, C^\varepsilon \rangle \notin \mathcal{D} \Rightarrow \sim \varphi(a_n; b; \tilde{C}).$$

Now, as the lemma is assumed to fail, we can choose b_n and C_n such that

$$4. \quad \langle b_n, C_n^\varepsilon \rangle \notin \mathcal{D} \quad \& \quad \bigwedge_{i=0}^n \varphi(a_i; b_n; C_n)$$

Note that (4) ensure that $\langle b_i, C_i^\varepsilon \rangle \notin \mathcal{D}$ for every i . There is some \tilde{C}_i that witnesses (3) for $\langle b_i, C_i^\varepsilon \rangle \notin \mathcal{D}$. We claim that the procedure has to stop after $\leq m$ steps. In fact, from (3) we obtain $\sim \varphi(a_n; b_i; \tilde{C}_i)$ for every $i < n$. On the other hand, by (4) we have that $\varphi(a_i; b_n; C_n)$ for every $i < n$. Therefore, $\langle a_{m-i}; b_{m-i}; C_{m-i}; \tilde{C}_{m-i} : i \leq m \rangle$ contradicts the maximality of m . \square

39 Lemma If $\varphi(x; z; C)$ is ε -stable then $\sigma(\bar{x}; z; C)$ in the previous lemma is ε -stable.

Proof. Let m be maximal such that a sequence as in (3) of Definition 34 exists. Let k be sufficiently large so that every m -coloring of a graph of size k has a monochromatic subgraph of size $> m$. Let $\langle \bar{a}_i; b_i; C_i; \tilde{C}_i : i < k \rangle$ be a sequence witnessing instability. Then for every pair $i < n$ there is some $j < m$ such that $\sim \varphi(a_{j,n}; b_i; \tilde{C}_i)$. By the choice of k there is a $j < m$ such that $\sim \varphi(a_{j,n}; b_i; \tilde{C}_i)$ obtains for $> m$ many i . Therefore we can extract a subsequence that contradicts the maximality of m . \square

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