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Local stability in structures with a standard sort

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Abstract.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_{\mathsf{S}}$ that has a symbol for each compact subset $C \subseteq S^n$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{S}}$ or their interpretation in the structure S.

We also fix an arbitrary first-order language which we denote by $\mathcal{L}_{\mathsf{H}}$ and call the language of the home sort.

1 Definition Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{S}}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H).
A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_{\mathsf{H}}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S.

We write $x \in C$ for the predicate associated to a compact set C, and denote by $|\text{-}|$ the length of a tuple.

2 Definition We call atomic formulas those of the form

i. atomic or negated atomic formulas of $\mathcal{L}_{\mathsf{H}}$;

ii. $\tau \in C$, where τ is a tuple of terms of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$, and $C \subseteq S^{|\tau|}$ is a compact set.

3 Remark A smaller fragment of $\mathcal{F}$ may be of interest in some specific contexts. This is obtained by excluding equalities and inequalities from (i) and requiring that C in (ii) is a regular compact set. The discussion below applies to this smaller fragment as well.

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Let $M \subseteq N$ be standard structures. We say that M is an $\mathcal{F}$-elementary substructure of N if the latter models all $\mathcal{F}(M)$-sentences that are true in M.

Let x and ξ be variables of sort H, respectively S. A standard structure M is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M. The following theorem is proved in [AAVV] for signatures that do not contain symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n{\cdot}m > 0$. A similar framework has been independently introduced in [PC] – only without quantifiers of sort H and with an approximate notion of satisfaction. By elimination of quantifiers of sort S, cf. [AAVV, Proposition 3.6], the two approaches are equivalent (up to approximations).

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [AAVV] extends to the case $n{\cdot}m \geq 0$.

4 Theorem (Compactness) Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

We introduce variables of some new sorts X_n with the sole scope of conveniently describing classes of $\mathcal{F}$-formulas that only differ by the sets/predicates $C \subseteq S^n$ that occur. To avoid cluttering the notation we write X to mean a suitable tuple of sorts $\mathsf{X}_{n_1},\dots,\mathsf{X}_{n_k}$. Let $\mathcal{F}_{\mathsf{X}}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\xi) \in X$, where X is a variable of sort X.

Formulas in $\mathcal{F}_{\mathsf{X}}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1,\dots,X_k$ be a tuple of sort X. If $C = C_1,\dots,C_k$ is a tuple of compact subsets of a suitable arity then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_{\mathsf{X}}$.

5 Definition Let $\varphi \in \mathcal{F}_{\mathsf{X}}$ be formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_{\mathsf{X}}$ is the formula obtained by replacing in φ the atomic formulas in $\mathcal{L}_{\mathsf{H}}$ by their negation and every connective $\wedge, \vee, \forall, \exists$ by their respective duals $\vee, \wedge, \exists, \forall$.
The pseudonegation of φ is denoted by $\sim\!\varphi$. Clearly, when X does not occur in φ, we have $\sim\!\varphi \leftrightarrow \neg\varphi$.

The following fact is immediate and will be used without further mention.

6 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$, every standard structure M, and every formula $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$

$$\begin{array}{lcl} M & \models & \varphi(C) \to \varphi(C') \\ M & \models & \sim\!\varphi(C) \to \neg\varphi(\tilde{C}). \end{array}$$

7 Fact For every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from C such that

$$M \models \varphi(C) \to \neg{\sim}\varphi(\tilde{C}) \to \varphi(C')$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Conversely, for every $\tilde{C} \cap C = \varnothing$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \to \varphi(C') \to \neg{\sim}\varphi(\tilde{C}).$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = C' \smallsetminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds with as C' any neighborhood of C disjoint from $\tilde{C}$. Induction on the syntax of $\varphi(X)$ proves the general case. □

The above fact has the following useful consequence.

8 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$, where M is a standard structure. Then the following are equivalent for every tuple C

$$M \models \varphi(C) \leftarrow \bigwedge\Big\{\varphi(C') : C' \text{ neighborhood of } C\Big\}$$

$$M \models \neg\varphi(C) \to \bigvee\Big\{{\sim}\varphi(\tilde{C}) : \tilde{C} \cap C = \varnothing\Big\}.$$

(The converse implications are trivial – therefore not displayed.)

Note that the fact also holds if we restrict the above C' and $\tilde{C}$ to range over the subsets of some C'' that is a neighboorhood of C.

9 Fact The equivalent conditions in Fact 8 hold in all $\mathcal{F}$-saturated structures.

Proof. By induction on the syntax of $\varphi(X)$. □

We say that M is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions. By the following fact $\mathcal{F}$-saturated structures are $\mathcal{F}$-maximal.

10 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Then the following are equivalent

1. M is $\mathcal{F}$-maximal
2. the equivalent conditions in Fact 8 hold for every $\varphi(C)$.

Proof. 1⇒2. If (2) in Fact 8 fails, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of M. As M is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds also in $\mathcal{U}$. Then (2) in Fact 8 fails in $\mathcal{U}$, contradicting Fact 9.

2⇒1. Let N be any $\mathcal{F}$-elementary extension of M. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C. Suppose not. Then, by (2) in Fact 8, $M \models {\sim}\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models {\sim}\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts Fact 9. □

In what follows we fix a large saturated standard structure which we denote by $\mathcal{U}$. Unless otherwise specified, we work inside $\mathcal{U}$. Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(\mathcal{U})$ be given. Define

$$C_\varphi \quad = \quad \big\{\alpha : \varphi(C) \text{ holds for every } C \text{ neighborhood of } \alpha\big\}$$

Then C_φ is a closed set. Note that either C_φ or $C_{\sim\varphi}$ is empty by Fact 6.

2. Stable formulas

Let $\varphi(x;z;X)$ be an infinite conjunction of formulas in $\mathcal{F}_{\mathsf{X}}$. The variables of sort H are partitioned in two tuples $x;z$ which may be infinite. We say that $\varphi(x;z;X)$ is unstable if there are some $C \cap \tilde{C} = \varnothing$ and a sequence $\langle a_i ; b_i \,:\, i < \omega\rangle$ such that for every $i < j < \omega$

$$\varphi(a_i;b_j;C) \quad \wedge \quad {\sim}\varphi(a_j;b_i;\tilde{C})$$

We say that $\varphi(x;z;X)$ is stable if it is not unstable.

The following fact is a characterization of stability in a context of particular interest. Let $\tau : \mathcal{U}^{x;z} \to S$ be a function definable by an infinite conjunction of formulas in $\mathcal{F}$. We say that $\tau(x;z)$ is stable if so is $\tau(x;z) \in X$.

11 Fact Let $\tau(x;z)$ be as above. Then the following are equivalent

1. the formula $\tau(x;z) \in X$ is unstable
2. for some sequence $\langle a_i ; b_i \,:\, i < \omega\rangle$

$$\lim_{i\to\infty}\lim_{j\to\infty} \tau(a_i;b_j) \quad \neq \quad \lim_{j\to\infty}\lim_{i\to\infty} \tau(a_i;b_j)$$

Proof. 1⇒2. Let $C \cap \tilde{C} = \varnothing$ and $\langle a_i ; b_i \,:\, i < \omega\rangle$ be as given by (1). That is, $\tau(a_i;b_j) \in C$ and $\tau(a_j;b_i) \in \tilde{C}$ hold for every $i < j < \omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and $\tilde{C}$ contains the limit on the right – which therefore are distinct.

2⇒1. Let C and $\tilde{C}$ be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i ; b_i \,:\, i < \omega\rangle$. □

Assume S is a metric space. Let $\varepsilon > 0$ We say that $f : \mathcal{U}^{x;z} \to S$ is ε-approximable by $\tau(x;z)$ if for every finite $B \subseteq \mathcal{U}^z$ and there is an $a \in \mathcal{U}^x$ such that

$$d\big(f(b)\,,\tau(a;b)\big) \quad \leq \quad \varepsilon \qquad\qquad \text{for every } b \in B.$$

In general (when S is not a metric space) then ε is a closed neighborhood of the diagonal of S^2 and the condition above becomes $\big\langle f(b)\,,\tau(a;b)\big\rangle \in \varepsilon$.

3. A case study

In this section we assume $S = [0,1]$. It is easy to see that when $\tau(x;z)$ has values in $[0,1]$ stability holds if there is no sequence $\langle a_i ; b_i \,:\, i < \omega\rangle$ such that for every $i < j < \omega$

$$\big|\tau(a_i;b_j) \; - \; \tau(a_j;b_i)\big| \quad \geq \quad \varepsilon.$$

The following theorem and its proof are well-known – though not literally in this form, cf. [BU, Proposition 7.6].

12 Theorem Let $S = [0,1]$. Let $\tau(x;z)$ be a stable definable function. Let $f : \mathcal{U}^{x;z} \to S$ be ε-approximable by $\tau(x;z)$. Then there are some $\langle a_{i,j} \;:\; i < n,\; j < \omega \rangle$ such that

$$\Big| f(b) - \max_{i=0,\ldots,n-1} \inf_{j<\omega} \tau(a_{i,j}; b) \Big| \;\leq\; 3\varepsilon \qquad \text{for every } b \in \mathcal{U}^z.$$

Proof. The theorem is an immediate consequence of the following three lemmas. □

We say that $f : \mathcal{U}^{x;z} \to [0,1]$ is ε-approximable from below if the parameter $a \in \mathcal{U}^x$ that witnesses approximability satisfies $\tau(a;b) \leq f(b) + \varepsilon$ for every $b \in \mathcal{U}^z$.

13 Lemma Assume that the f in the theorem is ε-approximable from below. Then there are some $\langle a_i \;\; i < n \rangle$ such that

$$\Big| f(b) - \max_{i=0,\ldots,n-1} \tau(a_i; b) \Big| \;\leq\; \varepsilon \qquad \text{for every } b \in \mathcal{U}^z.$$

Proof. Negate the lemma. We construct inductively a sequence $\langle a_i ; b_i \;:\; i < \omega \rangle$ that contradicts the stability of $\tau(x;z)$. Then, let b_n be such that (if $n = 0$ pick b_n arbitrarily)

1. $$\Big| f(b_n) - \max_{i=0,\ldots,n-1} \tau(a_i; b_n) \Big| \;>\; \varepsilon$$

By approximability from below, for some a_n we have

2. $$\tau(a_n; b_i) - f(b_i) \;\leq\; \varepsilon \qquad \text{for every } i \leq n$$

3. $$\tau(a_n; b) \;\leq\; f(b) + \varepsilon \qquad \text{for every } b \in \mathcal{U}^z.$$

We prove instability by showing that for every $i < n$

4. $$\tau(a_n; b_i) - \tau(a_i; b_n) \;>\; \varepsilon$$

Note that by virtue of (3)

$$\max_{i=0,\ldots,n-1} \tau(a_i; b_n) - f(b_n) \;\leq\; \varepsilon$$

Then (1) ensures that

1′. $$f(b_n) - \max_{i=0,\ldots,n-1} \tau(a_i; b_n) \;>\; \varepsilon$$

Using (1′) we obtain

$$\tau(a_n; b_i) - \tau(a_i; b_n) \;>\; \tau(a_n; b_i) - f(b_n) + \varepsilon$$

then by (2) we have

$$> \; f(b_i) - f(b_n) + 2\varepsilon$$

By refining the sequence we can assume that $|f(b_i) - f(b_n)| < \varepsilon$ for every $i, n \in \omega$. This yields (4). □

14 Lemma Under the assumptions of the theorem. Let $\bar{x} = \langle x_i : i < \omega \rangle$ where the x_i are copies of x. Then the function

$$\sigma(\bar{x}\,;z) \quad = \quad \inf_{i<\omega} \tau(x_i\,;z)$$

3ε-approximates f from below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x};z)$ does not ε-approximate f from below. We construct inductively a sequence $\langle a_i\,; b_i : i < \omega \rangle$ that contradicts the stability of $\tau(x;z)$. Then, let a_n be such that

1. $$\big|\tau(a_n\,;b) \;-\; f(b)\big| \quad \leq \quad \varepsilon \qquad\qquad \text{for every } b \in B \cup \{b_0, \ldots, b_n\}$$

then let b_n be such that

2. $$\min_{i=0,\ldots,n} \tau(a_i\,;b_n) \quad > \quad f(b_n) + 3\varepsilon.$$

We show that the sequence we constructed contradicts stability. In fact, by (2)

$$\tau(a_i\,;b_n) \;-\; \tau(a_n\,;b_i) \quad > \quad f(b_n) + 3\varepsilon - \tau(a_n\,;b_i).$$

Then, by (1)

$$\tau(a_i\,;b_n) \;-\; \tau(a_n\,;b_i) \quad > \quad f(b_n) + 3\varepsilon - f(b_i) - \varepsilon$$

As in the previous lemma, after refining the sequence we we obtain a contradiction. □

15 Lemma If $\tau(\bar{x};z)$ is stable, then $\sigma(\bar{x};z)$ in the previous lemma is stable.

Proof. It suffices to show that

$$\sigma(x_1, x_2\,;z) \quad = \quad \min_{i=1,2} \tau(x_i\,;z)$$

is stable. Suppose not. Pick $\langle a_{1,i}, a_{2,i}\,; b_i : i < \omega \rangle$ and $\varepsilon > 0$ such that for every $i < j < \omega$

$$\big|\sigma(a_{1,i}, a_{2,i}\,;b_j) - \sigma(a_{1,j}, a_{2,j}\,;b_i)\big| \quad \geq \quad \varepsilon$$

By Ramsey's theorem, we can refine the sequence so that one of the following holds for every $i < j$

$$\sigma(a_{1,i}, a_{2,i}\,;b_j) - \sigma(a_{1,j}, a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

$$-\sigma(a_{1,i}, a_{2,i}\,;b_j) + \sigma(a_{1,j}, a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

Assume the first – the second case is symmetric. Again by Ramsey's theorem, we can refine the sequence so that for every $i < j$ either $\tau(a_{1,i}\,;b_j) \leq \tau(a_{2,i}\,;b_j)$ or conversely $\tau(a_{1,i}\,;b_j) \geq \tau(a_{2,i}\,;b_j)$. In the first case we obtain

$$\tau(a_{1,i}\,;b_j) - \tau(a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

in the second case we obtain

$$\tau(a_{2,i}\,;b_j) - \tau(a_{1,j}\,;b_i) \quad \geq \quad \varepsilon.$$

In both cases we contradict the stability of $\tau(x;z)$. □

4. Tentative

We write K for a set of compact subsets of S^n. Let $\mathcal{D} \subseteq K$. We define

$$\sim\mathcal{D} \quad = \quad \big\{\tilde{C} \in K \ : \ \tilde{C} \cap C \neq \varnothing \text{ for every } C \in \mathcal{D}\big\}.$$

16 Fact If $\mathcal{D} = \{C : \varphi(C)\}$ then $\sim\mathcal{D} = \{\tilde{C} : \sim\varphi(\tilde{C})\}$

Proof. We need to prove that

$$\sim\varphi(\tilde{C}) \quad \Leftrightarrow \quad \text{for every } C, \ \ [C \cap \tilde{C} = \varnothing \rightarrow \neg\varphi(C)]$$

Implication $\Rightarrow$ follows immediately from Fact 6. To prove $\Leftarrow$ assume the r.h.s. Then $\neg\varphi(S \smallsetminus O)$ holds for every open set $O \supseteq \tilde{C}$. From the second implication in Fact 8 we obtain $\sim\varphi(\tilde{C}')$ for some $\tilde{C} \subseteq \tilde{C}' \subseteq O$. As O is arbitrary, we obtain $\varphi(\tilde{C})$ from the second implication in Fact 8. □

Note that we always have $\mathcal{D} \subseteq {\sim\sim}\mathcal{D}$. When $\mathcal{D} = {\sim\sim}\mathcal{D}$ we say that $\mathcal{D}$ is regular.

We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if for every finite $B \subseteq \mathcal{U}^z$ and every finitely many $C_i \in K(S)$, for $i = 1,\dots,n$, there is an $a \in \mathcal{U}^x$ such that

$$\begin{aligned} \langle b, C_i\rangle \in \mathcal{D} \quad &\rightarrow \quad \varphi(a;b;C_i) \quad \text{and} \\ \langle b, C_i\rangle \in \sim\mathcal{D} \quad &\rightarrow \quad \sim\varphi(a;b;C_i) \qquad \text{for all } b \in B \text{ and } i = 1,\dots,n. \end{aligned}$$

A global $\varphi(x;z;X)$-type is a maximally (finitely) consistent set of formulas that are instances of $\varphi(x;b;X)$ and/or $\sim\varphi(x;b;X)$ for some $b \in \mathcal{U}^{|z|}$.

The following is immediate.

17 Fact For every $\mathcal{D} \subseteq \mathcal{U}^z \times K(S)$ the following are equivalent

1. $\mathcal{D}$ is approximable by $\varphi(x;z;X)$
2. the following defines a (consistent) global $\varphi(x;z;X)$-type

$$p(x) \quad = \quad \big\{\varphi(x;b;C) \ : \ \langle b, C\rangle \in \mathcal{D}\big\} \cup \big\{\sim\varphi(x;b;C) \ : \ \langle b, C\rangle \in \sim\mathcal{D}\big\}.$$

Proof. 2$\Rightarrow$1. The finite consistency of $p(x)$ immediately implies that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$.

1$\Rightarrow$2. Finite consistency of $p(x)$ follows (again, immediately) from approximability. We prove $p(x)$ is maximal. We need to consider two cases. First, assume for a contradiction that $\varphi(x;b;C)$ is consistent with $p(x)$ but $\langle b, C\rangle \notin \mathcal{D}$. Then consistency implies that $\langle b, \tilde{C}\rangle \notin \sim\mathcal{D}$ for every $\tilde{C} \cap C = \varnothing$. But this contradicts the definition of $\sim\mathcal{D}$. The second case, for $\sim\varphi(x;b;C)$ is similar as ${\sim\sim}\mathcal{D} = \mathcal{D}$. □

5. Externally definable sets

global $\varphi(x;z;X)$-type is a maximally consistent set of formulas that are instances of $\varphi(x;b;X)$ and/or $\sim\varphi(x;b;X)$ for some $b \in \mathcal{U}^{|z|}$.

Let $p(x)$ be a global $\varphi(x;z;X)$-type. We define

$$\mathcal{D}_{p,\varphi,C} \quad = \quad \{b \in \mathcal{U}^z \ : \ \varphi(x;b;C) \in p\}$$

18 Fact $$\neg\mathcal{D}_{p,\varphi,C} \quad = \quad \bigcup\Big\{\mathcal{D}_{p,\sim\varphi,\tilde{C}} \ : \ \tilde{C} \cap C = \varnothing\Big\}$$

Proof. ($\subseteq$). Let $b \notin \mathcal{D}_{p,\varphi,C}$. Let $A \subseteq \mathcal{U}$ be such that every instance of $\varphi(x;b;X)$ and $\sim\varphi(x;b;X)$ that is consistent with $p(x) \restriction A$ is finitely consistent with $p(x)$. Finally pick $a \models p(x) \restriction A$. As $\neg\varphi(a,b,C)$, from Fact 10 we obtain that $\sim\varphi(a;b;\tilde{C})$ holds for some $\tilde{C} \cap C = \varnothing$. Then $\sim\varphi(x;b;\tilde{C})$ is consistent with $p(x) \restriction A$. By maximality of $p(x)$, we have $\sim\varphi(x;b;\tilde{C}) \in p$. Then $b \in \mathcal{D}_{p,\sim\varphi,\tilde{C}}$.

($\supseteq$). Similar. □

19 Fact Let $\varphi(x;z;X)$ and C be given. Then for every $\mathcal{D} \subseteq \mathcal{U}^z$ the following are equivalent

1. $\mathcal{D} = \mathcal{D}_{p,\varphi,C}$ for some global $\varphi(x;z;X)$-type $p(x)$
2. for every finite set $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that
$$B \cap \mathcal{D} \quad = \quad B \cap \varphi(a;\mathcal{U}^x;C).$$

Proof. Let $B_0 = B \smallsetminus \mathcal{D}$ and $B_1 = B \cap \mathcal{D}$.

1⇒2. By the above fact, there is a $\tilde{C} \cap C = \varnothing$ such that $b \in \mathcal{D}_{p,\sim\varphi,\tilde{C}}$ form every $b \in B_0$. Let a realize the type containing $\varphi(x;b;C)$ for every $b \in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$.

2⇒1. There is a set $\tilde{C} \cap C = \varnothing$ such that $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$. Let $p(x)$ be any global $\varphi(x;z;X)$-type containing $\varphi(x;b;C)$ for every $b \in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$. Such type exists as (2) guarantees consistency. □

Let $\mathcal{D} \subseteq \mathcal{U}^z \times S^{|X|}$. For $\alpha \in S^{|X|}$, we write $\mathcal{D}_\alpha$ for the set of all $b \in \mathcal{U}^z$ such that $\langle b, \alpha\rangle \in \mathcal{D}$. We say that $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$ if there is a global $\varphi(x;z;X)$-type $p(x)$ such that for every α

$$\mathcal{D}_\alpha \quad = \quad \bigcap\{\mathcal{D}_{p,\varphi,C} \ : \ \alpha \in C^\circ\}$$

We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if for every $\alpha \in S^{|X|}$ and every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^{|x|}$ such that

$$B \cap \mathcal{D}_\alpha \quad = \quad B \cap \bigcap\{\varphi(a;\mathcal{U}^z;C) \ : \ \alpha \in C^\circ\}$$

20 Fact Let $\varphi(x;z;X)$ be given. Then, for every $\mathcal{D} \subseteq \mathcal{U}^z \times S^{|X|}$ the following are equivalent

1. $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$
2. $\mathcal{D}$ is approximated by $\varphi(x;z;X)$.

We say that $\mathcal{D}$ is approximable from below if for every $B \subseteq \mathcal{D}$ there is an $a \in \mathcal{U}^{|x|}$ such that

$$B \ \subseteq\ \varphi(a;\mathcal{U}^z;C)\ \subseteq\ \mathcal{D}.$$

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We define $\mathcal{D}_p\subseteq\mathcal{U}^{|z|}\times S^{|X|}$ as follows

$$\mathcal{D}_p \;=\; \Big\{\langle b,\alpha\rangle\,:\,\varphi(x;b;C)\in p\ \text{ for every } C \text{ neighborhood of } \alpha\Big\}.$$

21 Fact $$\neg\mathcal{D}_p \;=\; \Big\{\langle b,\alpha\rangle\,:\,{\sim}\varphi(x;b;\tilde{C})\in p\ \text{ for every } \tilde{C}\not\ni\alpha\Big\}.$$

Proof. Let $a\models p(x)\restriction b$. Let C be the intersection of all neighborhoods of α such that $\varphi(a;b;C')$. Assume $\langle b,\alpha\rangle\notin\mathcal{D}_p$, in other words $\varphi(a;b;C)$, then C is a neighborhood of α. By Fact 10 we have $\varphi(a;b;C)$. Let $\tilde{C}\subseteq C\smallsetminus\{\alpha\}$ be such that C is a neighborhood of $\tilde{C}$. Then $\neg\varphi(a;b;\tilde{C})$. Therefor by the definition of $\mathcal{D}_p$.

Then we can find $\tilde{C}$ such that $\alpha\notin\tilde{C}$ and C is a neighborhood of $\tilde{C}$. contains two dijoint compacts Then ${\sim}\varphi(x;b;\tilde{C}')\in p$ by the definition of $\mathcal{D}_p$.

Then ${\sim}\varphi(x;b;\tilde{C}')\in p$ for every neighborhood $\tilde{C}'$ of α disjoint from C'. Then for every neighborhood C of α, we have $\varphi(x;b;C)\in p$. □

We say that $\mathcal{D}$ is externally defined by $p(x)$ and $\varphi(x;z;X)$ if $\mathcal{D}_p\subseteq\mathcal{D}$ and $\tilde{\mathcal{D}}_p\subseteq\neg\mathcal{D}$.

We say that $\mathcal{D}\subseteq\mathcal{U}^{|z|}\times S^{|X|}$ is approximated by $\varphi(x;z;X)$ if for every finite $B\subseteq\mathcal{U}^{|z|}\times S^{|X|}$ there is an $a\in\mathcal{U}^{|x|}$ such that

$$\langle b,\alpha\rangle\in B\cap\mathcal{D}\ \Rightarrow\ \varphi(a;b;C) \text{ for every neighborhood } C \text{ of } \alpha$$

$$\langle b,\alpha\rangle\in B\smallsetminus\mathcal{D}\ \Rightarrow\ {\sim}\varphi(a;b;C) \text{ for every } C \text{ such that } \alpha\notin C$$

22 Fact For every $\mathcal{D}\subseteq\mathcal{U}^{|z|}\times S^{|X|}$, the following are equivalent

1. $\mathcal{D}$ is externally definable by some global φ-type
2. $\mathcal{D}$ is approximated by $\varphi(x;z;X)$.

Proof. (1)⇒(2). Let $B\subseteq\mathcal{U}^{|z|}\times S^{|X|}$ be finite. For $\langle b,\alpha\rangle\in B\cap\mathcal{D}$ let $C_{\langle b,\alpha\rangle}$ be some neightborhood of α. Let C be the union of all these neighborhoods. For $\langle b,\alpha\rangle\in B\smallsetminus\mathcal{D}$ let $\tilde{C}_{\langle b,\alpha\rangle}$ be some neightborhood of α. Let $\tilde{C}$ be the union of all these neighborhoods. By the finiteness of B, we can require that C and $\tilde{C}$ are disjoint. Then (2) follows from the concistency of $p(x)$.

(2)⇒(1). Note that the condition of approximability asserts the finite concistency of the type $p(x)$ that is union of the following two sets of formulas

$$\Big\{\varphi(x;b;C)\,:\,\langle b,\alpha\rangle\in\mathcal{D} \text{ and } C \text{ neighborhood of } \alpha\Big\}$$

$$\Big\{{\sim}\varphi(x;b;C)\,:\,\langle b,\alpha\rangle\notin\mathcal{D} \text{ and } \alpha\notin C\Big\}$$

The inclusion $\mathcal{D} \subseteq \mathcal{D}_p$ is immediate. For the converse inclusion, let $\langle b, \alpha\rangle \notin \mathcal{D}$ and let $\tilde{C}$ be a neighborhood of α such that $\sim\varphi(x\,;b\,;\tilde{C})$. $\square$

6.

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