Local stability in structures with a standard sort

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ABSTRACT.

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_{\mathsf{S}}$ that has a symbol for each compact subsets $C \subseteq S^n$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{S}}$ or their interpretation in the structure S. Note that we write $x \in C$ for $C(x)$.

We also fix a first-order language $\mathcal{L}_{\mathsf{H}}$ which we call the language of the home sort.

Definition 1. Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{S}}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions.

A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_{\mathsf{H}}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from formulas of the form (i) ands (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S

Definition 2. We call atomic formulas those of the form

i. atomic or negated atomic formulas of $\mathcal{L}_{\mathsf{H}}$;

ii. $\tau \in C$, where $C \subseteq S$ is a compact set and τ is a term of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$.

In (ii) of the above definition we could replace C by a compact subset of S^n and τ with a tuple of terms – but this would complicate the notation and add nothing of relevance.

To convenienlty describe the substitution of a set/predicate $C \subseteq S$ occurring in a formula, we introduce variables of new sort X. Let $\mathcal{F}_{\mathsf{X}}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\eta) \in X$, where X is a variable of sort X.

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Formulas in $\mathcal{F}_\mathsf{X}$ are denoted by $\varphi(X)$, where $X = X_1, \ldots, X_k$ be a tuple of variables of the new sort. If $C = C_1, \ldots, C_k$ is a tuple of compact subsets of S then $\varphi(x\,\xi; C)$, is a formula in $\mathcal{F}$. All formulas in $\mathcal{F}$ are obtained as such istantiations a formulas in $\mathcal{F}_\mathsf{X}$.

Definition 3. The pseudonegation of $\varphi(X)$ is the formula obtained by replacing in $\varphi(X)$ the atomic formulas in $\mathcal{L}_\mathsf{H}$ by their negation and each connective $\wedge$, $\vee$, $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$, $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$by its respective dual $\vee$, $\wedge$, $\exists^{\mathsf{H}}$, $\forall^{\mathsf{H}}$, $\exists^{\mathsf{S}}$, $\forall^{\mathsf{S}}$.

The pseudonegation of $\varphi(X)$ is denoted by $\tilde{\varphi}(X)$.

Note that if $\varphi(X)$ is in $\mathcal{L}_\mathsf{H}$, so ξ and X are pleonastic, then $\tilde{\varphi}(X) \leftrightarrow \neg\varphi(X)$. If $\tilde{C}$ is (componentwise) disjoint of C then $\tilde{\varphi}(\tilde{C}) \leftarrow \neg\varphi(x\,\xi; C)$.

Let $\varphi(x\,;z\,;X)$ be a formula in $\mathcal{F}_\mathsf{X}$ with the tuples of variables of sort H partitioned in two $x\,;z$.

A global $\varphi(x\,;z\,;X)$-type is a consistent set of formulas $p(x) \subseteq \mathcal{F}(\mathcal{U})$ of the form $\varphi(x\,;b\,;C)$ and/or $\tilde{\varphi}(x\,;b\,;C)$ for some tuple C of compact subsets of S and some $b \in \mathcal{U}^{|z|}$.

Let $p(x)$ be a global $\varphi(x\,;z\,;X)$-type. We say that the set $\mathcal{D}_p \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ defined below is externally defined by $p(x)$

$$\mathcal{D}_p \quad = \quad \Big\{\langle b, \alpha\rangle \;:\; \varphi(x\,;b\,;C) \in p \text{ for every } C \text{ neighborhood of } \alpha\Big\}$$

We say that $\mathcal{D} \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ is approximated by $\varphi(x\,;z\,;X)$ if for every finite $B \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ there is a compact $C \subseteq S^{|X|}$ and some $a \in \mathcal{U}^{|x|}$ such that

$$\langle b, \alpha\rangle \in B \cap \mathcal{D} \quad \Rightarrow \quad \varphi(a\,;b\,;C) \text{ and } C \text{ is a neighborhood of } \alpha$$

$$\langle b, \alpha\rangle \in B \smallsetminus \mathcal{D} \quad \Rightarrow \quad \tilde{\varphi}(a\,;b\,;C) \text{ and } \neg\tilde{C} \text{ is a neighborhood of } \alpha$$

Fact 4. For every $\mathcal{D} \subseteq \mathcal{U}^{|z|} \times S^{|X|}$, the following are equivalent

1. $\mathcal{D}$ is externally definable by some $\varphi(x\,;z\,;X)$-type
2. $\mathcal{D}$ is approximated by $\varphi(x\,;z\,;X)$.

Proof. (1)⇒(2). Let $B \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ be finite. For $\langle b, \alpha\rangle \in B \cap \mathcal{D}$ let $C_{\langle b,\alpha\rangle}$ be some neightborhood of α. Let C be the union of all these neighborhoods. For $\langle b, \alpha\rangle \in B \smallsetminus \mathcal{D}$ let $\tilde{C}_{\langle b,\alpha\rangle}$ be some neightborhood of α. Let $\tilde{C}$ be the union of all these neighborhoods. By the finiteness of B, we can require that C and $\tilde{C}$ are disjoint. Then (2) follows from the concistency of $p(x)$.

(2)⇒(1). Note that the condition of approximability asserts the finite concistency of the type $p(x)$ that is union of the following two sets of formulas

$$\Big\{\varphi(x\,;b\,;C) \;:\; \langle b, \alpha\rangle \in \mathcal{D} \text{ and } C \text{ neighborhood of } \alpha\Big\}$$

$$\Big\{\tilde{\varphi}(x\,;b\,;\tilde{C}) \;:\; \langle b, \alpha\rangle \notin \mathcal{D} \text{ and } \tilde{C} \text{ neighborhood of } \alpha\Big\}$$

The inclusion $\mathcal{D} \subseteq \mathcal{D}_p$ is immediate. For the converse inclusion, let $\langle b, \alpha \rangle \notin \mathcal{D}$ and let $\tilde{C}$ be a neighborhood of α such that $\tilde{\varphi}(x; b; \tilde{C})$. $\square$

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References

[ABBMZ] Claudio Agostini, Stefano Baratella, Silvia Barbina, Luca Motto Ros, and Domenico Zambella, *Continuous logic in a classical setting* (2025). t.a. in Bull. Iranian Math. Soc.

[A] Josef Auslander, *Topological Dynamics*, Scholarpedia (2008), doi:10.4249/scholarpedia.3449.

[BY] Itaï Ben Yaacov, *Model theoretic stability and definability of types, after A. Grothendieck*, Bull. Symb. Log. **20** (2014), no. 4, 491–496.

[BBHU] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427.

[Hr] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.

[HI] C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 1–110.

[K] H. Jerome Keisler, *Model Theory for Real-valued Structures*, in Beyond First Order Model Theory (José Iovino, ed.), Vol. II, Chapman and Hall/CRC, 2023.