Local stability in structures with a standard sort

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ABSTRACT.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_{\mathsf{S}}$ that has a symbol for each compact subset $C \subseteq S^n$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{S}}$ or their interpretation in the structure S.

1 Definition Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{S}}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H).
A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_{\mathsf{H}}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S.

We write $x \in C$ for the predicate associated to a compact set C, and denote by $|\text{-}|$ the length of a tuple.

2 Definition We call atomic formulas those of the form

i. atomic or negated atomic formulas of $\mathcal{L}_{\mathsf{H}}$;
ii. $\tau \in C$, where τ is a tuple of terms of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$, and $C \subseteq S^{|\tau|}$ is a compact set.

3 Remark A smaller fragment of $\mathcal{F}$ may be of interest in some specific contexts. This is obtained by excluding equalities and inequalities from (i) and requiring that C in (ii) is a regular compact set. The discussion below applies to this smaller fragment as well.

Let $M \subseteq N$ be standard structures. We say that M is an $\mathcal{F}$-elementary substructure of N if the latter models all sentences in $\mathcal{F}(M)$ true in M.

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A standard structure M is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M. The following is proved in [AAVV] for signatures containing only symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n{\cdot}m = 0$. In [Z] it is observed that, under the assumption of equicontinuity, the proof extends to the case $n{\cdot}m \geq 0$.

4 Theorem (Compactness) Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

We introduce variables of some new sorts X_n with the sole scope of conveniently describing classes of $\mathcal{F}$-formulas that only differ by the sets/predicates $C \subseteq S^n$ that occur. To avoid cluttering the notation we write X to mean a suitable tuple of sorts $\mathsf{X}_{n_1}, \ldots, \mathsf{X}_{n_k}$. Let $\mathcal{F}_\mathsf{X}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\xi) \in X$, where X is a variable of sort X.

Formulas in $\mathcal{F}_\mathsf{X}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1, \ldots, X_k$ be a tuple of sort X. If $C = C_1, \ldots, C_k$ is a tuple of compact subsets of a suitable arity then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_\mathsf{X}$.

5 Definition Let $\varphi \in \mathcal{F}_\mathsf{X}$ be formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_\mathsf{X}$ is the formula obtained by replacing in φ the atomic formulas in $\mathcal{L}_\mathsf{H}$ by their negation and every connective $\wedge$, $\vee$, $\forall$, $\exists$ by their respective duals $\vee$, $\wedge$, $\exists$, $\forall$.
The pseudonegation of φ is denoted by ${\sim}\varphi$. Clearly, when X does not occur in φ, we have ${\sim}\varphi \leftrightarrow \neg\varphi$.

The following fact is immediate and will be used without further mention.

6 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$, every standard structure M, and every formula $\varphi(X) \in \mathcal{F}_\mathsf{X}(M)$

$$\begin{aligned} M &\models \varphi(C) \rightarrow \varphi(C') \\ M &\models {\sim}\varphi(C) \rightarrow \neg\varphi(\tilde{C}). \end{aligned}$$

7 Fact For every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from C such that

$$M \models \varphi(C) \rightarrow \neg{\sim}\varphi(\tilde{C}) \rightarrow \varphi(C')$$

for every M, and every $\varphi(X) \in \mathcal{F}_\mathsf{X}(M)$. Conversely, for every $\tilde{C} \cap C = \varnothing$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \rightarrow \varphi(C') \rightarrow \neg{\sim}\varphi(\tilde{C}).$$

for every M, and every $\varphi(X) \in \mathcal{F}_\mathsf{X}(M)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = C' \smallsetminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds with as C' any neighborhood of C disjoint from $\tilde{C}$. Induction on the syntax of $\varphi(X)$ proves the general case. $\square$

The above fact has the following useful consequence.

8 Fact Let $\varphi(X) \in \mathcal{F}_{\mathrm{X}}(M)$, where M is a standard structure. Then the following are equivalent for every tuple C

$$\begin{array}{rcl} M & \models & \varphi(C) \rightarrow \bigwedge\Big\{\varphi(C') \,:\, C' \text{ neighborhood of } C\Big\} \\ M & \models & \neg\varphi(C) \leftarrow \bigvee\Big\{{\sim}\varphi(\tilde{C}) \,:\, \tilde{C} \cap C = \varnothing\Big\}. \end{array}$$

(The converse of these implications are trivial, therefore not displayed.)

Note that the fact also holds if we restrict the above C' and $\tilde{C}$ to range over the subsets of some C'' that is a neighboorhood of C.

9 Fact The equivalent conditions in Fact 8 hold in all $\mathcal{F}$-saturated structures.

Proof. By induction on the syntax of $\varphi(X)$. $\square$

We say that M is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions. By the following fact $\mathcal{F}$-saturated structures are $\mathcal{F}$-maximal.

10 Fact Let $\varphi(X) \in \mathcal{F}_{\mathrm{X}}(M)$. Then the following are equivalent

1. M is $\mathcal{F}$-maximal
2. the equivalent conditions in Fact 8 hold for every $\varphi(C)$.

Proof. 1⇒2. If (2) in Fact 8 fails, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of M. As M is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds also in $\mathcal{U}$. Then (2) in Fact 8 fails in $\mathcal{U}$, contradicting Fact 9.

2⇒1. Let N be any $\mathcal{F}$-elementary extension of M. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C. Suppose not. Then, by (2) in Fact 8, $M \models {\sim}\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models {\sim}\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts Fact 9. $\square$

In what follows we fix a large saturated standard structure which we denote by $\mathcal{U}$. Unless otherwise specified, we work inside $\mathcal{U}$. Let $\varphi(X) \in \mathcal{F}_{\mathrm{X}}(\mathcal{U})$ be given. Define

$$C_\varphi \quad = \quad \big\{\alpha \,:\, \varphi(C) \text{ holds for every } C \text{ neighborhood of } \alpha\big\}$$

Then C_φ is a closed set. Note that either C_φ or $C_{\sim\varphi}$ is empty by Fact 6.

2. Stable formulas

Let $\varphi(x\,;z\,;X)$ be an infinite conjunction of formulas in $\mathcal{F}_{\mathsf{X}}$. The variables of sort H are partitioned in two tuples $x\,;z$ which may be infinite. We say that $\varphi(x\,;z\,;X)$ is unstable if there are some $C\cap\tilde{C}=\varnothing$ and a sequence $\langle a_i\,;b_i\ :\ i<\omega\rangle$ such that for every $i<j<\omega$

$$\varphi(a_i\,;b_j\,;C)\quad\wedge\quad\sim\varphi(a_j\,;b_i\,;\tilde{C})$$

We say that $\varphi(x\,;z\,;X)$ is stable if it is not unstable.

The following fact is a characterization of stability in a context of particular interest. Let $\tau:\mathcal{U}^{x;z}\to S$ be a function definable by an infinite conjunction of formulas in $\mathcal{F}$. We say that $\tau(x\,;z)$ is stable if so is $\tau(x\,;z)\in X$.

11 Fact Let $\tau(x\,;z)$ be as above. Then the following are equivalent

1. the formula $\tau(x\,;z)\in X$ is unstable
2. for some sequence $\langle a_i\,;b_i\ :\ i<\omega\rangle$

$$\lim_{i\to\infty}\lim_{j\to\infty}\tau(a_i\,;b_j)\quad\neq\quad\lim_{j\to\infty}\lim_{i\to\infty}\tau(a_i\,;b_j)$$

Proof. 1⇒2. Let $C\cap\tilde{C}=\varnothing$ and $\langle a_i\,;b_i\ :\ i<\omega\rangle$ be as given by (1). That is, $\tau(a_i\,;b_j)\in C$ and $\tau(a_j\,;b_i)\in\tilde{C}$ hold for every $i<j<\omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and $\tilde{C}$ contains the limit on the right – which therefore are distinct.

2⇒1. Let C and $\tilde{C}$ be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i\,;b_i\ :\ i<\omega\rangle$. □

Assume S is a metric space. Let $\varepsilon>0$ We say that $f:\mathcal{U}^{x;z}\to S$ is ε-approximable by $\tau(x\,;z)$ if for every finite $B\subseteq\mathcal{U}^z$ and there is an $a\in\mathcal{U}^x$ such that

$$d\big(f(b)\,,\tau(a\,;b)\big)\quad\leq\quad\varepsilon\qquad\qquad\text{for every } b\in B.$$

In general (when S is not a metric space) then ε is a closed neighborhood of the diagonal of S^2 and the condition above becomes $\big\langle f(b)\,,\tau(a\,;b)\big\rangle\in\varepsilon$.

3. A case study

In this section we assume $S=[0,1]$. It is easy to see that when $\tau(x\,;z)$ has values in $[0,1]$ stability holds if there is no sequence $\langle a_i\,;b_i\ :\ i<\omega\rangle$ such that for every $i<j<\omega$

$$\big|\tau(a_i\,;b_j)\ -\ \tau(a_j\,;b_i)\big|\quad\geq\quad\varepsilon.$$

The following theorem and its proof are well-known though not literally in this form.

12 Theorem Let $S=[0,1]$. Let $\tau(x\,;z)$ be a stable definable function. Let $f:\mathcal{U}^{x;z}\to S$ be ε-approximable by $\tau(x\,;z)$. Then there are some $\langle a_{i,j}\ :\ i<n,\ j<\omega\rangle$ such that

$$\Big|f(b)\ -\max_{i=0,\dots,n-1}\ \inf_{j<\omega}\ \tau(a_{i,j}\,;b)\Big|\quad\leq\quad\varepsilon\qquad\qquad\text{for every } b\in\mathcal{U}^z.$$

Proof. The theorem is an immediate consequence of the following three lemmas. □

We say that $f : \mathfrak{U}^{x;z} \to [0,1]$ is ε-approximable from below if the parameter $a \in \mathfrak{U}^x$ that witnesses approximability satisfies $\tau(a;b) \leq f(b) + \varepsilon$ for every $b \in \mathfrak{U}^z$.

13 Lemma Assume that the f in the theorem is ε-approximable from below. Then there are some $\langle a_i \;\; i < n\rangle$ such that

$$\Big|f(b) - \max_{i=0,\ldots,n-1} \tau(a_i;b)\Big| \;\leq\; \varepsilon \qquad \text{for every } b \in \mathfrak{U}^z.$$

Proof. Negate the lemma. We construct inductively a sequence $\langle a_i ; b_i \;:\; i < \omega\rangle$ that contradicts the stability of $\tau(x;z)$. Then, let b_n be such that (if $n = 0$ pick b_n arbitrarily)

1. $$\Big|f(b_n) - \max_{i=0,\ldots,n-1} \tau(a_i;b_n)\Big| \;>\; \varepsilon$$

By approximability from below, for some a_n we have

2. $$\tau(a_n;b_i) - f(b_i) \;\leq\; \varepsilon \qquad \text{for every } i \leq n$$
3. $$\tau(a_n;b) \;\leq\; f(b) + \varepsilon \qquad \text{for every } b \in \mathfrak{U}^z.$$

We prove instability by showing that for every $i < n$

4. $$\tau(a_n;b_i) - \tau(a_i;b_n) \;>\; \varepsilon$$

Note that by virtue of (3)

$$\max_{i=0,\ldots,n-1} \tau(a_i;b_n) - f(b_n) \;\leq\; \varepsilon$$

Then (1) ensures that

1′. $$f(b_n) - \max_{i=0,\ldots,n-1} \tau(a_i;b_n) \;>\; \varepsilon$$

Using (1′) we obtain

$$\tau(a_n;b_i) - \tau(a_i;b_n) \;>\; \tau(a_n;b_i) - f(b_n) + \varepsilon$$

then by (2) we have

$$> \; f(b_i) - f(b_n) + 2\varepsilon$$

By refining the sequence we can assume that $|f(b_i) - f(b_n)| < \varepsilon$ for every $i, n \in \omega$. This yields (4). □

14 Lemma Under the assumptions of the theorem. Let $\bar{x} = \langle x_i \;:\; i < \omega\rangle$ where the x_i are copies of x. Then the function

$$\sigma(\bar{x};z) \;=\; \inf_{i<\omega} \tau(x_i;z)$$

3ε-approximates f from below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x};z)$ does not ε-approximate f from below. We construct inductively a sequence $\langle a_i ; b_i \;:\; i < \omega\rangle$ that contradicts the stability of $\tau(x;z)$. Then, let a_n be such that

1. $$\left|\tau(a_n;b)\ -\ f(b)\right| \ \leq\ \varepsilon \qquad\qquad \text{for every } b\in B\cup\{b_0,\dots,b_n\}$$

then let b_n be such that

2. $$\min_{i=0,\dots,n}\tau(a_i;b_n)\ >\ f(b_n)+3\varepsilon.$$

We show that the sequence we constructed contradicts stability. In fact, by (2)

$$\tau(a_i;b_n)\ -\ \tau(a_n;b_i)\ >\ f(b_n)+3\varepsilon-\tau(a_n;b_i).$$

Then, by (1)

$$\tau(a_i;b_n)\ -\ \tau(a_n;b_i)\ >\ f(b_n)+3\varepsilon-f(b_i)-\varepsilon$$

As in the previous lemma, after refining the sequence we we obtain a contradiction. □

15 Lemma Under the assumptions of the theorem. The function $\sigma(\bar{x};z)$ in the previous lemma is stable.

Proof. It suffices to show that

$$\sigma(x_1,x_2;z)\ =\ \min_{i=1,2}\tau(x_i;z)$$

is stable. Suppose not. Pick $\langle a_{1,i},a_{2,i};b_i\ :\ i<\omega\rangle$ and $\varepsilon>0$ such that for every $i<j<\omega$

$$\left|\sigma(a_{1,i},a_{2,i};b_j)-\sigma(a_{1,j},a_{2,j};b_i)\right|\ \geq\ \varepsilon$$

By Ramsey's theorem, we can refine the sequence so that one of the following holds for every $i<j$

$$\sigma(a_{1,i},a_{2,i};b_j)-\sigma(a_{1,j},a_{2,j};b_i)\ \geq\ \varepsilon$$

$$-\sigma(a_{1,i},a_{2,i};b_j)+\sigma(a_{1,j},a_{2,j};b_i)\ \geq\ \varepsilon$$

Assume the first – the second case is symmetric. Again by Ramsey's theorem, we can refine the sequence so that for every $i<j$ either $\tau(a_{1,i};b_j)\leq\tau(a_{2,i};b_j)$ or conversely $\tau(a_{1,i};b_j)\geq\tau(a_{2,i};b_j)$. In the first case we obtain

$$\tau(a_{1,i};b_j)-\tau(a_{2,j};b_i)\ \geq\ \varepsilon$$

in the second case we obtain

$$\tau(a_{2,i};b_j)-\tau(a_{1,j};b_i)\ \geq\ \varepsilon.$$

In both cases we contradict the stability of $\tau(x;z)$. □

4. Externally definable sets

global $\varphi(x;z;X)$-type is a maximally consistent set of formulas that are instances of $\varphi(x;b;X)$ and/or $\sim\varphi(x;b;X)$ for some $b\in\mathcal{U}^{|z|}$.

Let $p(x)$ be a global $\varphi(x;z;X)$-type. We define

$$\mathcal{D}_{p,\varphi,C}\ =\ \left\{b\in\mathcal{U}^z\ :\ \varphi(x;b;C)\in p\right\}$$

16 Fact
$$\neg\mathcal{D}_{p,\varphi,C} \quad = \quad \bigcup\Big\{\mathcal{D}_{p,\sim\varphi,\tilde{C}} \;:\; \tilde{C}\cap C=\varnothing\Big\}$$

Proof. ($\subseteq$). Let $b\notin\mathcal{D}_{p,\varphi,C}$. Let $A\subseteq\mathcal{U}$ be such that every instance of $\varphi(x;b;X)$ and $\sim\varphi(x;b;X)$ that is consistent with $p(x)\restriction A$ is finitely consistent with $p(x)$. Finally pick $a\models p(x)\restriction A$. As $\neg\varphi(a,b,C)$, from Fact 10 we obtain that $\sim\varphi(a;b;\tilde{C})$ holds for some $\tilde{C}\cap C=\varnothing$. Then $\sim\varphi(x;b;\tilde{C})$ is consistent with $p(x)\restriction A$. By maximality of $p(x)$, we have $\sim\varphi(x;b;\tilde{C})\in p$. Then $b\in\mathcal{D}_{p,\sim\varphi,\tilde{C}}$.

($\supseteq$). Similar. □

17 Fact Let $\varphi(x;z;X)$ and C be given. Then for every $\mathcal{D}\subseteq\mathcal{U}^z$ the following are equivalent

1. $\mathcal{D}=\mathcal{D}_{p,\varphi,C}$ for some global $\varphi(x;z;X)$-type $p(x)$
2. for every finite set $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that
$$B\,\cap\,\mathcal{D} \quad = \quad B\,\cap\,\varphi(a;\mathcal{U}^x;C).$$

Proof. Let $B_0=B\smallsetminus\mathcal{D}$ and $B_1=B\cap\mathcal{D}$.

1⇒2. By the above fact, there is a $\tilde{C}\cap C=\varnothing$ such that $b\in\mathcal{D}_{p,\sim\varphi,\tilde{C}}$ form every $b\in B_0$. Let a realize the type containing $\varphi(x;b;C)$ for every $b\in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b\in B_0$.

2⇒1. There is a set $\tilde{C}\cap C=\varnothing$ such that $\sim\varphi(x;b;\tilde{C})$ for every $b\in B_0$. Let $p(x)$ be any global $\varphi(x;z;X)$-type containing $\varphi(x;b;C)$ for every $b\in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b\in B_0$. Such type exists as (2) guarantees consistency. □

Let $\mathcal{D}\subseteq\mathcal{U}^z\times S^{|X|}$. For $\alpha\in S^{|X|}$, we write $\mathcal{D}_\alpha$ for the set of all $b\in\mathcal{U}^z$ such that $\langle b,\alpha\rangle\in\mathcal{D}$. We say that $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$ if there is a global $\varphi(x;z;X)$-type $p(x)$ such that for every α
$$\mathcal{D}_\alpha \quad = \quad \bigcap\{\mathcal{D}_{p,\varphi,C} \;:\; \alpha\in C^\circ\}$$
We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if for every $\alpha\in S^{|X|}$ and every finite $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^{|x|}$ such that
$$B\,\cap\,\mathcal{D}_\alpha \quad = \quad B\,\cap\bigcap\{\varphi(a;\mathcal{U}^z;C) \;:\; \alpha\in C^\circ\}$$

18 Fact Let $\varphi(x;z;X)$ be given. Then, for every $\mathcal{D}\subseteq\mathcal{U}^z\times S^{|X|}$ the following are equivalent

1. $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$
2. $\mathcal{D}$ is approximated by $\varphi(x;z;X)$.

We say that $\mathcal{D}$ is approximable from below if for every $B\subseteq\mathcal{D}$ there is an $a\in\mathcal{U}^{|x|}$ such that
$$B \quad\subseteq\quad \varphi(a;\mathcal{U}^z;C) \quad\subseteq\quad \mathcal{D}.$$

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We define $\mathcal{D}_p \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ as follows

$$\mathcal{D}_p \quad = \quad \Big\{ \langle b, \alpha \rangle \, : \, \varphi(x\,;b\,;C) \in p \;\text{ for every } C \text{ neighborhood of } \alpha \Big\}.$$

19 Fact

$$\neg\mathcal{D}_p \quad = \quad \Big\{ \langle b, \alpha \rangle \, : \, \sim\!\varphi(x\,;b\,;\tilde{C}) \in p \;\text{ for every } \tilde{C} \not\ni \alpha \Big\}.$$

Proof. Let $a \models p(x) \restriction b$. Let C be the intersection of all neighborhoods of α such that $\varphi(a\,;b\,;C')$. Assume $\langle b, \alpha \rangle \notin \mathcal{D}_p$, in other words $\varphi(a\,;b\,;C)$, then C is a neighborhood of α. By Fact 10 we have $\varphi(a\,;b\,;C)$. Let $\tilde{C} \subseteq C \smallsetminus \{\alpha\}$ be such that C is a neighborhood of $\tilde{C}$. Then $\neg\varphi(a\,;b\,;\tilde{C})$. Therefor by the definition of $\mathcal{D}_p$.

Then we can find $\tilde{C}$ such that $\alpha \notin \tilde{C}$ and C is a neighborhood of $\tilde{C}$. contains two dijoint compacts Then $\sim\!\varphi(x\,;b\,;\tilde{C}') \in p$ by the definition of $\mathcal{D}_p$.

Then $\sim\!\varphi(x\,;b\,;\tilde{C}') \in p$ for every neighborhood $\tilde{C}'$ of α disjoint from C'. Then for every neighborhood C of α, we have $\varphi(x\,;b\,;C) \in p$. □

We say that $\mathcal{D}$ is externally defined by $p(x)$ and $\varphi(x\,;z\,;X)$ if $\mathcal{D}_p \subseteq \mathcal{D}$ and $\tilde{\mathcal{D}}_p \subseteq \neg\mathcal{D}$.

We say that $\mathcal{D} \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ is approximated by $\varphi(x\,;z\,;X)$ if for every finite $B \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ there is an $a \in \mathcal{U}^{|x|}$ such that

$$\langle b, \alpha \rangle \in B \cap \mathcal{D} \;\;\Rightarrow\;\; \varphi(a\,;b\,;C) \text{ for every neighborhood } C \text{ of } \alpha$$

$$\langle b, \alpha \rangle \in B \smallsetminus \mathcal{D} \;\;\Rightarrow\;\; \sim\!\varphi(a\,;b\,;C) \text{ for every } C \text{ such that } \alpha \notin C$$

20 Fact For every $\mathcal{D} \subseteq \mathcal{U}^{|z|} \times S^{|X|}$, the following are equivalent

1. $\mathcal{D}$ is externally definable by some global φ-type
2. $\mathcal{D}$ is approximated by $\varphi(x\,;z\,;X)$.

Proof. (1)⇒(2). Let $B \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ be finite. For $\langle b, \alpha \rangle \in B \cap \mathcal{D}$ let $C_{\langle b,\alpha \rangle}$ be some neightborhood of α. Let C be the union of all these neighborhoods. For $\langle b, \alpha \rangle \in B \smallsetminus \mathcal{D}$ let $\tilde{C}_{\langle b,\alpha \rangle}$ be some neightborhood of α. Let $\tilde{C}$ be the union of all these neighborhoods. By the finiteness of B, we can require that C and $\tilde{C}$ are disjoint. Then (2) follows from the concistency of $p(x)$.

(2)⇒(1). Note that the condition of approximability asserts the finite concistency of the type $p(x)$ that is union of the following two sets of formulas

$$\Big\{ \varphi(x\,;b\,;C) \, : \, \langle b, \alpha \rangle \in \mathcal{D} \text{ and } C \text{ neighborhood of } \alpha \Big\}$$

$$\Big\{ \sim\!\varphi(x\,;b\,;C) \, : \, \langle b, \alpha \rangle \notin \mathcal{D} \text{ and } \alpha \notin C \Big\}$$

The inclusion $\mathcal{D} \subseteq \mathcal{D}_p$ is immediate. For the converse inclusion, let $\langle b, \alpha \rangle \notin \mathcal{D}$ and let $\tilde{C}$ be a neighborhood of α such that $\sim\!\varphi(x\,;b\,;\tilde{C})$. □

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References

[AAVV] Claudio Agostini, Stefano Baratella, Silvia Barbina, Luca Motto Ros, and Domenico Zambella, *Continuous logic in a classical setting* (2025). t.a. in Bull. Iranian Math. Soc.

[A] Josef Auslander, *Topological Dynamics*, Scholarpedia (2008), doi:10.4249/scholarpedia.3449.

[BY] Itaï Ben Yaacov, *Model theoretic stability and definability of types, after A. Grothendieck*, Bull. Symb. Log. **20** (2014), no. 4, 491–496.

[BBHU] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427.

[HI] C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 1–110.

[K] H. Jerome Keisler, *Model Theory for Real-valued Structures*, in Beyond First Order Model Theory (José Iovino, ed.), Vol. II, Chapman and Hall/CRC, 2023.

[Z] Domenico Zambella, *Standed analysis* (2023), arXiv:2311.15711