Local stability in structures with a standard sort

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Abstract.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_{\mathsf{S}}$ that has a symbol for each compact subset $C \subseteq S$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{S}}$ or their interpretation in the structure S. We write $x \in C$ for $C(x)$.

Finally, note that we could allow in $\mathcal{L}_{\mathsf{S}}$ relation symbols for all compact subsets of S^n, for any n. But this would clutter the notation adding very little to the theory.

1 Definition Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{S}}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions. A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_{\mathsf{H}}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S.

2 Definition We call atomic formulas those of the form

i. atomic or negated atomic formulas of $\mathcal{L}_{\mathsf{H}}$;

ii. $\tau \in C$, where $C \subseteq S$ is a compact set and τ is a term of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$.

3 Remark A smaller fragment of $\mathcal{F}$ may be of interest in some specific contexts. This is obtained by excluding equalities and inequalities from (i) and requiring that C in (ii) is a regular compact set. The discussion below applies to this smaller fragment as well.

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Let $M \subseteq N$ be standard structures. We say that M is an $\mathcal{F}$-elementary substructure of N if the latter models all sentences in $\mathcal{F}(M)$ true in M.

A standard structure M is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M. The following is proved in [AAVV] for signatures containing only symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n \cdot m = 0$. In [Z] it is observed that, under the assumption of equicontinuity, the proof extends to the case $n \cdot m \geq 0$.

4 Theorem (Compactness) Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

To conveniently describe classes of $\mathcal{F}$ that only differ by the sets/predicates $C \subseteq S$ that occur in the formula, we introduce variables of new sort X. These will be used as placeholders for compact subsets of S. Let $\mathcal{F}_{\mathsf{X}}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\xi) \in X$, where X is a variable of sort X.

Formulas in $\mathcal{F}_{\mathsf{X}}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1,\ldots,X_k$ be a tuple of variables of sort X. If $C = C_1,\ldots,C_k$ is a tuple of compact subsets of S then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_{\mathsf{X}}$.

5 Definition Let $\varphi \in \mathcal{F}_{\mathsf{X}}$ be formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_{\mathsf{X}}$ is the formula obtained by replacing in φ the atomic formulas in $\mathcal{L}_{\mathsf{H}}$ by their negation and each connective $\wedge$, $\vee$, $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$, $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ by its respective dual $\vee$, $\wedge$, $\exists^{\mathsf{H}}$, $\forall^{\mathsf{H}}$, $\exists^{\mathsf{S}}$, $\forall^{\mathsf{S}}$.
The pseudonegation of φ is denoted by $\sim\varphi$. Clearly, when X does not occur in φ, we have $\sim\varphi \leftrightarrow \neg\varphi$.

The following fact will be used without further mention.

6 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}$ have, possibly, hidden variables of sort H and/or S. Then for every (tuples of) compact sets $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$.

$$\begin{array}{rcl} \varphi(C) & \rightarrow & \varphi(C') \\ \sim\varphi(C) & \rightarrow & \neg\varphi(\tilde{C}) \end{array}$$

A standard structure M is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions.

7 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Then the following are equivalent

1. M is $\mathcal{F}$-maximal
2. for every tuple C of compact subsets of S
$$M \models \quad \varphi(C) \;\leftrightarrow\; \bigwedge\Big\{\varphi(C') \,:\, C' \text{ neighborhood of } C\Big\}$$
3. for every tuple C of compact subsets of S
$$M \models \quad \neg\varphi(C) \;\leftrightarrow\; \bigvee\Big\{{\sim}\varphi(\tilde{C}) \,:\, \tilde{C} \cap C = \varnothing\Big\}.$$

Moreover, all of the above hold when M is $\mathcal{F}$-saturated.

Proof. Arguing by induction on the syntax of $\varphi(X)$ we can prove that (2) holds when M is $\mathcal{F}$-saturated, see [AAVV].

2$\Leftrightarrow$3. By normality.

1$\Rightarrow$2. If (2) fails, then $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of M. As M is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ also holds in $\mathcal{U}$. Then (2) fails also in $\mathcal{U}$, a contradiction.

2$\Rightarrow$1. Let N be any $\mathcal{F}$-elementary extension of M. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C. Suppose not. Then, by (3), $M \models {\sim}\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models {\sim}\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts the above fact. □

2. Externally definable sets

In what follows we fix a large saturated standard structure which we denote by $\mathcal{U}$. Unless otherwise specified, we work inside $\mathcal{U}$.

Let $\varphi(x;z;X)$ be a formula in $\mathcal{F}_{\mathsf{X}}$ with the variables of sort H partitioned in two tuples $x;z$. A global $\varphi(x;z;X)$-type is a maximally consistent set of formulas that are instances of $\varphi(x;b;X)$ and/or ${\sim}\varphi(x;b;X)$ for some $b \in \mathcal{U}^{|z|}$.

Let $p(x)$ be a global $\varphi(x;z;X)$-type. We define
$$\mathcal{D}_{p,\varphi,C} \quad = \quad \big\{b \in \mathcal{U}^z \,:\, \varphi(x;b;C) \in p\big\}$$

8 Fact
$$\neg\mathcal{D}_{p,\varphi,C} \quad = \quad \bigcup\Big\{\mathcal{D}_{p,\sim\varphi,\tilde{C}} \,:\, \tilde{C} \cap C = \varnothing\Big\}$$

Proof. ($\subseteq$). Let $b \notin \mathcal{D}_{p,\varphi,C}$. Let $A \subseteq \mathcal{U}$ be such that every instance of $\varphi(x;b;X)$ and ${\sim}\varphi(x;b;X)$ that is consistent with $p(x) \restriction A$ is finitely consistent with $p(x)$. Finally pick $a \models p(x) \restriction A$. As $\neg\varphi(a,b,C)$, from Fact 7 we obtain that ${\sim}\varphi(a;b;\tilde{C})$ holds for some $\tilde{C} \cap C = \varnothing$. Then ${\sim}\varphi(x;b;\tilde{C})$ is consistent with $p(x) \restriction A$. By maximality of $p(x)$, we have ${\sim}\varphi(x;b;\tilde{C}) \in p$. Then $b \in \mathcal{D}_{p,\sim\varphi,\tilde{C}}$.

($\supseteq$). Similar. □

9 Fact Let $\varphi(x;z;X)$ and C be given. Then for every $\mathcal{D} \subseteq \mathcal{U}^z$ the following are equivalent

1. $\mathcal{D} = \mathcal{D}_{p,\varphi,C}$ for some global $\varphi(x;z;X)$-type $p(x)$
2. for every finite set $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that
$$B \cap \mathcal{D} \quad = \quad B \cap \varphi(a;\mathcal{U}^x;C).$$

Proof. Let $B_0 = B \smallsetminus \mathcal{D}$ and $B_1 = B \cap \mathcal{D}$.

1⇒2. By the above fact, there is a $\tilde{C} \cap C = \varnothing$ such that $b \in \mathcal{D}_{p,\sim\varphi,\tilde{C}}$ form every $b \in B_0$. Let a realize the type containing $\varphi(x;b;C)$ for every $b \in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$.

2⇒1. There is a set $\tilde{C} \cap C = \varnothing$ such that $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$. Let $p(x)$ be any global $\varphi(x;z;X)$-type containing $\varphi(x;b;C)$ for every $b \in B_1$ and $\sim\varphi(x;b;\tilde{C})$ for every $b \in B_0$. Such type exists as (2) guarantees consistency. □

Let $\mathcal{D} \subseteq \mathcal{U}^z \times S^{|X|}$. For $\alpha \in S^{|X|}$, we write $\mathcal{D}_\alpha$ for the set of all $b \in \mathcal{U}^z$ such that $\langle b, \alpha\rangle \in \mathcal{D}$. We say that $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$ if there is a global $\varphi(x;z;X)$-type $p(x)$ such that for every α
$$\mathcal{D}_\alpha \quad = \quad \bigcap\{\mathcal{D}_{p,\varphi,C} \;:\; \alpha \in C^\circ\}$$
We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if for every $\alpha \in S^{|X|}$ and every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^{|x|}$ such that
$$B \cap \mathcal{D}_\alpha \quad = \quad B \cap \bigcap\{\varphi(a;\mathcal{U}^z;C) \;:\; \alpha \in C^\circ\}$$

10 Fact Let $\varphi(x;z;X)$ be given. Then, for every $\mathcal{D} \subseteq \mathcal{U}^z \times S^{|X|}$ the following are equivalent

1. $\mathcal{D}$ is externally definable by $\varphi(x;z;X)$
2. $\mathcal{D}$ is approximated by $\varphi(x;z;X)$.

We say that $\mathcal{D}$ is approximable from below if for every $B \subseteq \mathcal{D}$ there is an $a \in \mathcal{U}^{|x|}$ such that
$$B \quad \subseteq \quad \varphi(a;\mathcal{U}^z;C) \quad \subseteq \quad \mathcal{D}.$$

3. Stable formulas

We say that $\varphi(x;z;X)$ is unstable if there are some compact sets $C \cap \tilde{C} = \varnothing$ and a sequence $\langle a_i;b_i \;:\; i < \omega\rangle$ such that for every $i < j < \omega$
$$\varphi(a_i;b_j;C) \quad \wedge \quad \sim\varphi(a_j;b_i;\tilde{C})$$
We say that $\varphi(x;z;X)$ is stable if it is not unstable.

The following fact is a characterization of stability in a context of particular interest. Let $\tau : \mathcal{U}^{x;z} \to S$ be a definable function. We say that $\tau(x;z)$ is stable if so is the formula $\tau(x;z) \in X$.

11 Fact Let $\tau : \mathcal{U}^{x;z} \to S$ be a definable function. Then the following are equivalent

1. the formula $\tau(x;z) \in X$ is unstable
2. for some sequence $\langle a_i ; b_i \ :\ i<\omega\rangle$
$$\lim_{i\to\infty}\lim_{j\to\infty} \tau(a_i;b_j) \quad \neq \quad \lim_{j\to\infty}\lim_{i\to\infty} \tau(a_i;b_j)$$
3. for some sequence $\langle a_i ; b_i \ :\ i<\omega\rangle$ and some neighborhood ε of the diagonal of S^2
$$\big\langle \tau(a_i;b_j),\ \tau(a_j;b_i)\big\rangle \quad \notin \quad \varepsilon$$
for every $i<j<\omega$.

Proof. 2⇔3. Clear.

1⇒3. Let $C \cap \tilde{C} = \varnothing$ be such that $\tau(a_i;b_j) \in C$ and $\tau(a_j;b_i) \in \tilde{C}$ for every $i<j<\omega$ as given by (1). Then (3) holds for any ε contained in the complement of $C \times \tilde{C}$.

2⇒1. Let C and $\tilde{C}$ be two disjoint intervals centered in the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i ; b_i \ :\ i<\omega\rangle$. □

12 Fact Let $S=[0,1]$. Let $\tau(x;z)$ be a stable definable function. Let $f : \mathcal{U}^{x;z} \to S$. Assume that for every finite $B \subseteq \mathcal{U}^z$ and every $\varepsilon > 0$ there is an $a \in \mathcal{U}^x$ such that
$$\big|f(b) - \tau(a;b)\big| \quad \le \quad \varepsilon \qquad\qquad \text{for every } b \in B.$$
Then there are some $\langle a_{i,j} \ :\ 1 \le i,j \le n\rangle$ such that
$$f(b) \quad = \quad \max_{i=1,\dots,n}\ \min_{j=1,\dots,n}\ \tau(a_{i,j};b) \qquad\qquad \text{for every } b \in \mathcal{U}^z.$$

Proof. □

13 Lemma Let $\varphi(x;z;X)$ be stable. Let $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ be approximated by $\varphi(x;z;X)$ from below. Then
$$\mathcal{D}_{p,\varphi,C} \quad = \quad \bigcup_{i=1}^{n} \varphi(a_i;\mathcal{U}^z;C)$$
for some $\langle a_i \ :\ 1\le i\le n\rangle$.

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We define $\mathcal{D}_p \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ as follows
$$\mathcal{D}_p \quad = \quad \Big\{\langle b,\alpha\rangle \ :\ \varphi(x;b;C) \in p \ \text{ for every } C \text{ neighborhood of } \alpha\Big\}.$$

14 Fact
$$\neg\mathcal{D}_p \quad = \quad \Big\{\langle b,\alpha\rangle \ :\ {\sim}\varphi(x;b;\tilde{C}) \in p \ \text{ for every } \tilde{C} \not\ni \alpha\Big\}.$$

Proof. Let $a \models p(x) \restriction b$. Let C be the intersection of all neighborhoods of α such that $\varphi(a;b;C')$. Assume $\langle b,\alpha\rangle \notin \mathcal{D}_p$, in other words $\varphi(a;b;C)$, then C is a neighborhood of α. By Fact 7 we have $\varphi(a;b;C)$. Let $\tilde{C} \subseteq C \smallsetminus \{\alpha\}$ be such that C is a neighborhood of $\tilde{C}$. Then $\neg\varphi(a;b;\tilde{C})$. Therefor by the definition of $\mathcal{D}_p$.

Then we can find $\tilde{C}$ such that $\alpha \notin \tilde{C}$ and C is a neighborhood of $\tilde{C}$. contains two dijoint compacts Then $\sim\varphi(x;b;\tilde{C}') \in p$ by the definition of $\mathcal{D}_p$.

Then $\sim\varphi(x;b;\tilde{C}') \in p$ for every neighborhood $\tilde{C}'$ of α disjoint from C'. Then for every neighborhood C of α, we have $\varphi(x;b;C) \in p$. □

We say that $\mathcal{D}$ is externally defined by $p(x)$ and $\varphi(x;z;X)$ if $\mathcal{D}_p \subseteq \mathcal{D}$ and $\tilde{\mathcal{D}}_p \subseteq \neg\mathcal{D}$.

We say that $\mathcal{D} \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ is approximated by $\varphi(x;z;X)$ if for every finite $B \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ there is an $a \in \mathcal{U}^{|x|}$ such that

$$\langle b,\alpha\rangle \in B \cap \mathcal{D} \quad \Rightarrow \quad \varphi(a;b;C) \text{ for every neighborhood } C \text{ of } \alpha$$

$$\langle b,\alpha\rangle \in B \smallsetminus \mathcal{D} \quad \Rightarrow \quad \sim\varphi(a;b;C) \text{ for every } C \text{ such that } \alpha \notin C$$

15 Fact For every $\mathcal{D} \subseteq \mathcal{U}^{|z|} \times S^{|X|}$, the following are equivalent

1. $\mathcal{D}$ is externally definable by some global φ-type
2. $\mathcal{D}$ is approximated by $\varphi(x;z;X)$.

Proof. (1)⇒(2). Let $B \subseteq \mathcal{U}^{|z|} \times S^{|X|}$ be finite. For $\langle b,\alpha\rangle \in B \cap \mathcal{D}$ let $C_{\langle b,\alpha\rangle}$ be some neightborhood of α. Let C be the union of all these neighborhoods. For $\langle b,\alpha\rangle \in B \smallsetminus \mathcal{D}$ let $\tilde{C}_{\langle b,\alpha\rangle}$ be some neightborhood of α. Let $\tilde{C}$ be the union of all these neighborhoods. By the finiteness of B, we can require that C and $\tilde{C}$ are disjoint. Then (2) follows from the concistency of $p(x)$.

(2)⇒(1). Note that the condition of approximability asserts the finite concistency of the type $p(x)$ that is union of the following two sets of formulas

$$\Big\{\varphi(x;b;C) \;:\; \langle b,\alpha\rangle \in \mathcal{D} \text{ and } C \text{ neighborhood of } \alpha\Big\}$$

$$\Big\{\sim\varphi(x;b;C) \;:\; \langle b,\alpha\rangle \notin \mathcal{D} \text{ and } \alpha \notin C\Big\}$$

The inclusion $\mathcal{D} \subseteq \mathcal{D}_p$ is immediate. For the converse inclusion, let $\langle b,\alpha\rangle \notin \mathcal{D}$ and let $\tilde{C}$ be a neighborhood of α such that $\sim\varphi(x;b;\tilde{C})$. □

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