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Local stability in structures with a standard sort

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Abstract.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_\mathsf{S}$ that has a symbol for each regular compact subset $C \subseteq S^n$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_\mathsf{S}$ or their interpretation in the structure S.

Throughout these notes the letter C always denotes a regular compact subset of S^n, or a tuple of such sets.

We also fix an arbitrary first-order language which we denote by $\mathcal{L}_\mathsf{H}$ and call the language of the home sort.

1 Definition Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_\mathsf{H}$ and $\mathcal{L}_\mathsf{S}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H).
A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_\mathsf{H}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S.

We write $x \in C$ for the predicate associated to a compact set C, and denote by $|\text{-}|$ the length of a tuple.

2 Definition We call atomic formulas those of the form

i. atomic and negated atomic formulas of $\mathcal{L}_\mathsf{H}$ but not equalities and inequalities

ii. $\tau \in C$, where τ is a k-tuple terms of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$, and $C \subseteq S^k$.

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Let $M \subseteq N$ be standard structures. We say that M is an $\mathcal{F}$-elementary substructure of N if the latter models all $\mathcal{F}(M)$-sentences that are true in M.

Let x and ξ be variables of sort H, respectively S. A standard structure M is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M. The following theorem is proved in [AAVV] for signatures that do not contain symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n{\cdot}m > 0$. A similar framework has been independently introduced in [PC] – only without quantifiers of sort H and with an approximate notion of satisfaction. The corresponding compactness theorem is proved there with a similar method. The setting in [PC] is a generalization of that used by Henson and Iovino for Banach spaces [HI]. By the elimination of quantifiers of sort S, proved in [AAVV, Proposition 3.6], the two approaches are equivalent – up to approximations.

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [AAVV] extends to the case $n{\cdot}m > 0$.

3 Theorem (Compactness) Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

We introduce some new sorts X_k with the sole scope of conveniently describing classes of $\mathcal{F}$-formulas that only differ by the sets/predicates $C \subseteq S^k$ they contain. Let $\mathcal{F}_{\mathsf{X}}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\xi) \in X$, where X is a variable of sort X_k.

Formulas in $\mathcal{F}_{\mathsf{X}}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1,\ldots,X_n$ be a tuple of variables of sort $\mathsf{X}_{k_1},\ldots,\mathsf{X}_{k_n}$. If $C = C_1,\ldots,C_n$ is a tuple of regular compact subsets of $S^{k_1},\ldots,S^{k_n}$, then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_{\mathsf{X}}$.

4 Definition Let $\varphi \in \mathcal{F}_{\mathsf{X}}$ be a formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_{\mathsf{X}}$ is the formula obtained by replacing in φ the atomic formulas in $\mathcal{L}_{\mathsf{H}}$ by their negation and every connective $\wedge$, $\vee$, $\forall$, $\exists$ by their respective duals $\vee$, $\wedge$, $\exists$, $\forall$.
The pseudonegation of φ is denoted by $\sim\!\varphi$. Clearly, when X does not occur in φ, we have $\sim\!\varphi \leftrightarrow \neg\varphi$.

The following fact is immediate and will be used without further mention.

5 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$, every standard structure M, and every formula $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$

$$\begin{array}{rcl} M & \models & \varphi(C) \to \varphi(C') \\ M & \models & \sim\!\varphi(\tilde{C}) \to \neg\varphi(C). \end{array}$$

6 Fact For every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from C such that

$$M \models \varphi(C) \to \neg{\sim}\varphi(\tilde{C}) \to \varphi(C')$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Conversely, for every $\tilde{C} \cap C = \varnothing$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \to \varphi(C') \to \neg{\sim}\varphi(\tilde{C}).$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = C' \smallsetminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds with as C' any neighborhood of C disjoint from $\tilde{C}$. Induction on the syntax of $\varphi(X)$ proves the general case. □

The above fact has the following useful consequence.

7 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$, where M is a standard structure. Then the following are equivalent for every tuple C

$$M \models \varphi(C) \leftarrow \bigwedge\Big\{\varphi(C') : C' \text{ neighborhood of } C\Big\}$$

$$M \models \neg\varphi(C) \to \bigvee\Big\{{\sim}\varphi(\tilde{C}) : \tilde{C} \cap C = \varnothing\Big\}.$$

(The converse implications are trivial – therefore not displayed.)

Note that the fact also holds if we restrict the above C' and $\tilde{C}$ to range over the subsets of some neighboorhood of C.

8 Fact The equivalent conditions in Fact 7 hold in all $\mathcal{F}$-saturated structures.

Proof. Prove the first of the two implications by induction on the syntax of $\varphi(X)$. The existential quantifier of sort H is the only connective that requires attention. Assume inductively that

$$\bigwedge\Big\{\varphi(a;C') : C' \text{ neighborhood of } C\Big\} \to \varphi(a;C)$$

holds for every $\varphi(a;C)$. Then induction for the existential quantifier follows from

$$\bigwedge\Big\{\exists x\, \varphi(x;C') : C' \text{ neighborhood of } C\Big\} \to \exists x \bigwedge\Big\{\varphi(x;C') : C' \text{ neighborhood of } C\Big\}$$

which is a consequence of saturation. □

We say that M is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions. By the following fact $\mathcal{F}$-saturated structures are $\mathcal{F}$-maximal.

9 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Then the following are equivalent

1. M is $\mathcal{F}$-maximal
2. the equivalent conditions in Fact 7 hold for every $\varphi(C)$.

Proof. 1⇒2. If (2) in Fact 7 fails, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of M. As M is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds also in $\mathcal{U}$. Then (2) in Fact 7 fails in $\mathcal{U}$, contradicting Fact 8.

2⇒1. Let N be any $\mathcal{F}$-elementary extension of M. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C. Suppose not. Then, by (2) in Fact 7, $M \models {\sim}\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models {\sim}\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts Fact 8. □

In what follows we fix a large saturated standard structure which we denote by $\mathcal{U}$.

2. A duality in $K(S)$

Write $K(S^n)$ for the set of compact subsets of S^n. In this section the variable X has sort $\mathsf{X}_{n_1}, \ldots, \mathsf{X}_{n_k}$ and $\mathcal{D}$ and $\mathcal{C}$ range over the subsets of $K(S^{n_1}) \times \cdots \times K(S^{n_k})$. We define

$$ {\sim}\mathcal{D} \quad = \quad \big\{\tilde{C} : \tilde{C} \cap C \neq \varnothing \text{ for every } C \in \mathcal{D}\big\}. $$

The following fact motivates the definition.

10 Fact For every $\varphi(X)$, if $\mathcal{D} = \{C : \varphi(C)\}$ then ${\sim}\mathcal{D} = \{\tilde{C} : {\sim}\varphi(\tilde{C})\}$

Proof. We need to prove that

$$ {\sim}\varphi(\tilde{C}) \quad \Leftrightarrow \quad \text{for every } C, \ \text{ if } C \cap \tilde{C} = \varnothing \ \text{ then } \neg\varphi(C) $$

Implication ⇒ follows immediately from Fact 5. To prove ⇐ assume the r.h.s. Then $\neg\varphi(S \smallsetminus O)$ holds for every open set $O \supseteq \tilde{C}$. From the second implication in Fact 7 we obtain ${\sim}\varphi(\tilde{C}')$ for some $\tilde{C} \subseteq \tilde{C}' \subseteq O$. As O is arbitrary, we obtain $\varphi(\tilde{C})$ from the second implication in Fact 7. □

We prove a couple of straightforward inclusions.

11 Fact For every $\mathcal{D} \subseteq \mathcal{C}$ we have ${\sim}\mathcal{C} \subseteq {\sim}\mathcal{D}$. Moreover, $\mathcal{D} \subseteq {\sim}{\sim}\mathcal{D}$.

Proof. Let $C \notin {\sim}\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in \mathcal{D}$. As $\tilde{C} \in \mathcal{C}$ we conclude that $C \notin {\sim}\mathcal{C}$. For the second claim, let $C \notin {\sim}{\sim}\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in {\sim}\mathcal{D}$. Then $C \notin \mathcal{D}$ follows. □

12 Fact For every $\mathcal{D}$ the following are equivalent

1. $\mathcal{D} = {\sim}{\sim}\mathcal{D}$.
2. $\mathcal{D} = {\sim}\mathcal{C}$ for some $\mathcal{C}$.

Proof. Only 2⇒1 requires a proof. From Fact 11 we obtain ${\sim}{\sim}{\sim}\mathcal{C} \subseteq {\sim}\mathcal{C}$. Assume (2) then ${\sim}{\sim}\mathcal{D} \subseteq \mathcal{D}$ which, again by Fact 11, suffices to prove (1). □

We say that $\mathcal{D}$ is involutive if $\mathcal{D} = {\sim}{\sim}\mathcal{D}$. We say that $\mathcal{D}$ is closed if

$$ C \in \mathcal{D} \quad \Leftrightarrow \quad C' \in \mathcal{D} \text{ for every neighborhood } C' \text{ of } C. $$

Note that we are requiring closure in two distintic contexts: topological, and Boolean (upward clousere by inclusion).

13 Theorem The following are equivalent

1. $\mathcal{D}$ is involutive
2. $\mathcal{D}$ is closed.

⚠ It is not difficult to see that (2) holds if and only if $\mathcal{D}$ is closed in the Vietoris topology and includes all the supersets of its elements.

Proof. 2⇒1. It suffices to prove $\sim\sim\mathcal{D} \subseteq \mathcal{D}$. Let $C \in \sim\sim\mathcal{D}$. Let $O \supseteq C$ be open. Then $S \smallsetminus O \notin \sim\mathcal{D}$. Then $\tilde{C} \in \mathcal{D}$ for some $\tilde{C} \subseteq O$. Assume (2). Then, by ⇒ every $C' \supseteq O$ is in $\mathcal{D}$. As O is arbitrary, $C \in \mathcal{D}$ by ⇐.

1⇒2. It suffices to show that (2) holds with $\sim\mathcal{D}$ for $\mathcal{D}$. Implication ⇒ in the definition of closed is ovious. To prove ⇐, assume $C \notin \sim\mathcal{D}$. Then $C \cap \tilde{C}$ for some $\tilde{C} \in \mathcal{D}$. Let C' be a neighboorhood of C disjoint of C. Then the r.h.s. of the equivalence fails for C'. □

3. Stable formulas – the finitary case

There is more than one way to define stability in our context. In this section we present the strongest possible definition. It is not the most interesting one, but it is very similar to the classical definition, so it is useful for comparison.

Let $\varphi(x;z;X)$ be a formulas in $\mathcal{F}_{\mathsf{X}}(\mathcal{U})$. The variables of sort H are partitioned in two tuples $x;z$ which may be infinite. The following is the classical definition of stability which we dub *finitary* for the reason explained below.

14 Definition We say that $\varphi(x;z;C)$ is finitarily stable if for some $m<\omega$ there is no sequence $\langle a_i;b_i \,:\, i<m\rangle$ such that for every $i<n$

$$\varphi(a_n;b_i;C) \quad\wedge\quad \neg\varphi(a_i;b_n;C).$$

From Fact 7 we easily obtain the following equivalent definition which only mentions formulas in $\mathcal{F}$. The reader may wish to compare it with Definition/Theorem 20.

15 Fact The following are equivalent

1. $\varphi(x;z;C)$ is finitarily stable
2. for some m there is no sequence $\langle a_i;b_i;C_i \,:\, i<m\rangle$ such that for every $i<n$

$$\varphi(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;C_n) \quad\text{and}\quad C\cap C_n=\varnothing.$$

In a classical context, i.e. reling on $\mathcal{L}$-saturation, we can replace m in Definition 14 by ω. But this may not be true if only have $\mathcal{F}$-saturation. Therefore we dedicate a separate section to the non finitary version of stability.

The following definitions are classical, i.e. they rely on the full language $\mathcal{L}$. Their $\mathcal{F}$-homologue we will be introduced later.

16 Definition A global $\varphi(x;z;C)$-type is a maximally (finitely) consistent set of formulas of the form $\varphi(x;b;C)$ or $\neg\varphi(x;b;C)$ for some $b \in \mathcal{U}^{|z|}$.

In this section the symbol $\mathcal{D}$ always denotes a subset of $\mathcal{U}^z$.

17 Definition We say that $\mathcal{D}$ is approximable by $\varphi(x;z;C)$ if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that

$$b \in \mathcal{D} \quad \Leftrightarrow \quad \varphi(a;b;C) \qquad \text{for every } b \in B.$$

It goes without saying that these definitions describe two faces of the same coin. In fact, it is clear that $\mathcal{D}$ is approximable by $\varphi(x;z;C)$ if and only if

$$p(x) \quad = \quad \big\{\varphi(x;b;C) \ : \ b \in \mathcal{D}\big\} \ \cup \ \big\{\neg\varphi(x;b;C) \ : \ b \notin \mathcal{D}\big\}$$

is a global $\varphi(x;z;C)$-type

The following theorem says that if $\varphi(x;z;C)$ is finitarily stable then global $\varphi(x;z;C)$-types are definable. The proof of the classical case applies verbatim because no saturation is required. We state it only for comparison with its non finitary counterpart, see Theorem 26.

18 Theorem Let $\varphi(x;z;C)$ be finitarily stable. Let $\mathcal{D}$ be approximable by $\varphi(x;z;C)$. Then there are some $\langle a_{i,j} \ : \ i<k,\ j<m\rangle$ such that for every $b \in \mathcal{U}^z$

$$b \in \mathcal{D} \quad \Leftrightarrow \quad \bigvee_{i<k} \bigwedge_{j<m} \varphi(a_{i,j};b;C)$$

4. Stable formulas – the non-finitary case

In this section $\mathcal{D}$ is always a subset of $\mathcal{U}^z \times K(S)$ or, more generally, a subset of $\mathcal{U}^z \times K(S^{n_1}) \times \cdots \times K(S^{n_k})$. We write $\sim\mathcal{D}$ for the set obtained by applying the definition in Section 2 to all fibers of $\mathcal{D}$. We say that $\mathcal{D}$ is involutive/closed if all its fibers are such.

The letters ε and δ always denote a compact neighborhood of the diagonal of S^2. By $\varepsilon+\delta$ we denote the composition $\varepsilon\circ\varepsilon$, where the two sets are viewed as relations.

If C is a compact subsets of S, we write C^ε for the set whose elements are ε-close to elements of C. This definition extends naturally to when $C = C_1,\ldots,C_k$ where each C_i is a compact subset of S^{n_i}.

19 Definition We say that $\varphi(x;z;X)$ is stable if there is no sequence $\langle a_i;b_i : i<\omega\rangle$ such that for some C, some $\tilde{C}\cap C=\varnothing$, and for every $i<n$

$$\varphi(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;\tilde{C})$$

20 Definition / Theorem We say that $\varphi(x;z;X)$ is ε-stable if the following equivalent conditions hold

1. there is no sequence $\langle a_i;b_i;C_i;\tilde{C}_i : i<\omega\rangle$ such that for every $i<n$
$$\varphi(a_n;b_i;C_i) \quad\wedge\quad \sim\varphi(a_i;b_n;\tilde{C}_n) \quad\text{and}\quad C_i^{\varepsilon}\cap\tilde{C}_i=\varnothing.$$
2. there is a maximal m such that some sequence $\langle a_i;b_i;C_i;\tilde{C}_i : i<m\rangle$ is such that for every $i<n$
$$\varphi(a_n;b_i;C_i) \quad\wedge\quad \sim\varphi(a_i;b_n;\tilde{C}_n) \quad\text{and}\quad C_i^{\varepsilon}\cap\tilde{C}_i=\varnothing.$$

Proof. 2⇒1. Any counter example to (1) immediately provides a counterexample to (2).

⚠ 1⇒2. ??? □

21 Theorem The following are equivalent

1. $\varphi(x;z;X)$ is stable.
2. $\varphi(x;z;X)$ is ε-stable for every ε.

Proof. 2⇒1. Given any counter example to (1), let ε be such that $C^{\varepsilon}\cap\tilde{C}=\varnothing$. To obtain a counter example to (2), set $C_i=C$ and $\tilde{C}_i=\tilde{C}$ for every $i<\omega$.

⚠ 1⇒2. Negate (2). Let $\langle a_i;b_i;C_i;\tilde{C}_i : i<\omega\rangle$ be a counter example to (1) in Definition 20. By the compactness of $K(S)$ in the Vietoris topology, there are C and $\tilde{C}$ such that for infinitely many $i<\omega$ we have

$$C_i\subseteq C^{\delta},\quad C\subseteq C_i^{\delta} \qquad\text{and}\qquad \tilde{C}_i\subseteq\tilde{C}^{\delta},\quad \tilde{C}\subseteq\tilde{C}_i^{\delta},$$

where δ is a neighborhood of the diagonal such that $2\delta\subseteq\varepsilon$ Then, by the definition of ε-stability, we have that for infinitely many $i<\omega$ the following holds Then $C^{\delta}\cap\tilde{C}^{\delta}\varnothing$, therefore a suitable subsequence of $\langle a_i;b_i : i<\omega\rangle$ yields a counter example to stability.
□

22 Definition A global $\varphi(x;z;X)$-type is a maximally (finitely) consistent set of formulas of the form $\varphi(x;b;C)$ or $\sim\varphi(x;b;C)$ for some $b\in\mathfrak{U}^{|z|}$ and some C.

It is convenient to have a different characterization of global types, therefore the following definition.

23 Definition We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if for every finite $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that

1. $\langle b,C\rangle\in\mathcal{D} \;\Rightarrow\; \varphi(a;b;C)$ and
2. $\langle b,C\rangle\in\mathcal{D} \;\Leftarrow\; \varphi(a;b;C)$ for every $b\in B$ and every C.

We remark that, by Facts 10 and 11, when $\mathcal{D}$ is involutive (2) can be rephrased as

$$\langle b,C\rangle\in{\sim}\mathcal{D} \;\Rightarrow\; {\sim}\varphi(a;b;C) \qquad \text{for every } b\in B \text{ and every } C.$$

24 Fact For every involutive $\mathcal{D}$ the following are equivalent

1. $\mathcal{D}$ is approximable by $\varphi(x;z;X)$
2. the following is a global $\varphi(x;z;X)$-type

$$\begin{aligned} p(x) \;=\; &\big\{\ \varphi(x;b;C)\ :\ \langle b,C\rangle\in\mathcal{D}\big\}\ \cup\\ &\big\{{\sim}\varphi(x;b;C)\ :\ \langle b,C\rangle\in{\sim}\mathcal{D}\big\}.\end{aligned}$$

Proof. First we prove that $p(x)$ is maximal as soon as it is consistent – in fact, this only depends on $\mathcal{D}$ being involutive. We need to consider two cases. First, assume that $\varphi(x;b;C)$ is consistent with $p(x)$. Then consistency implies that $\langle b,\tilde{C}\rangle\notin{\sim}\mathcal{D}$ for every $\tilde{C}\cap C=\varnothing$. But this implies that $C\in{\sim}{\sim}\mathcal{D}=\mathcal{D}$, therefore $\varphi(x;b;C)\in p$. The second case is when ${\sim}\varphi(x;b;C)$ is consistent with $p(x)$. This implies that $\langle b,\tilde{C}\rangle\notin\mathcal{D}$ for every $\tilde{C}\cap C=\varnothing$. Then $\langle b,C\rangle\in{\sim}\mathcal{D}$, therefore ${\sim}\varphi(x;b;C)\in p$.

2⇒1. Let B be a given finite subset of $\mathcal{U}^z$. The finite consistency of $p(x)$ provides some a such that for every finite B an a such that

$$\langle b,C\rangle\in\mathcal{D} \;\Rightarrow\; \varphi(a;b;C) \quad\text{and}$$

$$\langle b,C\rangle\in{\sim}\mathcal{D} \;\Rightarrow\; {\sim}\varphi(a;b;C) \qquad \text{for every } b\in B \text{ and every } C.$$

By what remarked above, these yield (1) and (2) in Definition 30.

1⇒2. Let B be given. Let a be as in Definition 30. Then, by what remarked above, $a\models p(x)\restriction B$. □

The proof of the main theorem of this section requires the following notion of approximation. The definition is inspired by the classical notion of approximation from below in [Z15] where it is used as rephrasing of the notion of honest definability in [CS13]. Here this notion plays a purely technical role, but we submit it is potentially interesing in itself.

25 Definition We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ from ε-below if for every finite $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that for every C

1. $\langle b,C\rangle\in\mathcal{D} \;\Rightarrow\; \varphi(a;b;C)$ for every $b\in B$ and
2. $\langle b,C^{\varepsilon}\rangle\in\mathcal{D} \;\Leftarrow\; \varphi(a;b;C)$ for every $b\in\mathcal{U}^z$.

26 Theorem Let $\varphi(x;z;X)$ be stable. Assume that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$. Then for every ε there are some $\langle a_{i,j} \;:\; i<k,\ j<m\rangle$ such that for every $b \in \mathcal{U}^z$ and every C

$$\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \bigvee_{i<k}\,\bigwedge_{j<m} \varphi(a_{i,j};b;C^{\varepsilon})$$

$$\langle b, C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \bigvee_{i<k}\,\bigwedge_{j<m} \varphi(a_{i,j};b;C)$$

Proof. The theorem is an immediate consequence of the following three lemmas. □

27 Lemma Let $\varphi(x;z;X)$ be stable. Assume that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ from ε-below. Then there is are some $\langle a_i \;:\; i<k\rangle$ such that for every $b \in \mathcal{U}^z$ and every C

1. $$\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \bigvee_{i<k} \varphi(a_i;b;C^{\varepsilon})$$
2. $$\langle b, C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \bigvee_{i<k} \varphi(a_i;b;C)$$

Proof. We define recursively the required parameters a_i together with some auxiliary parameters b_i. The element a_n is choosen such that

3. $$\langle b, C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \varphi(a_n;b;C) \qquad \text{for every } b \in \mathcal{U}^z \text{ and every } C$$

and

4. $$\langle b_i, C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a_n;b_i;C) \qquad \text{for every } i<n \text{ and every } C.$$

This is possible because $\mathcal{D}$ is ε-approximated from below. Note that (3) immediately guarantees (2). Now, assume (1) fails for $k=n$, and choose b_n and C_n witnessing this. Then, by Fact 7, for some $\tilde{C}_n \cap C_n^{\varepsilon} = \varnothing$

5. $$\langle b_n, C_n\rangle \in \mathcal{D} \;\;\&\;\; \bigwedge_{i<n} {\sim}\varphi(a_i;b_n;\tilde{C}_n)$$

Suppose for a contradiction that the construction never ends. Then, as (5) guarantees that $\langle b_i, C_i\rangle \in \mathcal{D}$ for every i, from (4) we otain $\varphi(a_n;b_i;C_n)$ for every $i<n$. From (5) we also obtain ${\sim}\varphi(a_i;b_n;\tilde{C}_n)$, for every $i \le n$. This contradicts stability. □

28 Lemma Let $\varphi(x;z;X)$ be ε-stable. Assume that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$. Let m be maximal so that a sequence as in (3) of Theorem 21 exists. Let $\bar{x} = \langle x_i \;:\; i \le m\rangle$ where the x_i are copies of x. Then the formula

$$\sigma(\bar{x};z;X) \;=\; \bigwedge_{i\le m} \varphi(x_i;z;X)$$

approximate $\mathcal{D}$ from ε-below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x})$ does not approximate $\mathcal{D}$ from ε-below. Suppose that $a_0,\dots,a_{n-1}$ and $b_0,\dots,b_{n-1}$ have been defined. Choose a_n such that for every $b \in B \cup \{b_0,\dots,b_{n-1}\}$ and every C

1. $$\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a_n;b;C) \quad \text{and}$$
2. $$\langle b, C^{\varepsilon}\rangle \in \mathcal{D} \;\Leftarrow\; \varphi(a_n;b;C^{\varepsilon})$$

Note that the latter implication is equivalent to: for every C there is some $\tilde{C} \cap C^{\varepsilon} = \varnothing$ such that

3. $$\langle b, C^{\varepsilon}\rangle \notin \mathcal{D} \;\;\Rightarrow\;\; {\sim}\varphi(a_n\,;b\,;\tilde{C}).$$

Now, as the lemma is assumed to fail, we can choose b_n and C_n such that

4. $$\langle b_n, C_n^{\varepsilon}\rangle \notin \mathcal{D} \;\;\&\;\; \bigwedge_{i=0}^{n} \varphi(a_i\,;b_n\,;C_n)$$

Note that (4) ensure that $\langle b_i, C_i^{\varepsilon}\rangle \notin \mathcal{D}$ for every i. The there is some $\tilde{C}_i$ that witnesses (3) for $\langle b_i, C_i^{\varepsilon}\rangle \notin \mathcal{D}$. We claim that the procedure has to stop after $\leq m$ steps. In fact, from (3) we obtain ${\sim}\varphi(a_n\,;b_i\,;\tilde{C}_i)$ for every $i < n$. On the other hand, by (4) we have that $\varphi(a_i\,;b_n\,;C_n)$ for every $i < n$. Therefore, $\langle a_{m-i}\,;b_{m-i}\,;C_{m-i}\,;\tilde{C}_{m-i} \;:\; i \leq m\rangle$ contradicts the maximality of m. □

29 Lemma If $\varphi(x;z;C)$ is stable then $\sigma(\bar{x};z;C)$ in the previous lemma is ε-stable.

Proof. Let ε be given and let m be maximal such that a sequence as in (3) of Theorem 21 exists. Let k be suffinciely large so that every m-coloring of a graph of size k has a monocromatic subgraph of size $> m$. Let $\langle \bar{a}_i\,;b_i\,;C_i\,;\tilde{C}_i \;:\; i < k\rangle$ be a sequence witnessing instability as in (3) of Theorem 21. Then for every pair $i < n$ there is some $j < m$ such that ${\sim}\varphi(a_{j,n}\,;b_i\,;\tilde{C}_i)$. By the choice of k there is a $j < m$ such that ${\sim}\varphi(a_{j,n}\,;b_i\,;\tilde{C}_i)$ obtains for $> m$ many i. Then we can extract a subsequence that contradicts the maximality of m. □

5. An approximate version of approximability

We ripropose Theorem 26 using using the weaker notion δ-approximability which is natural when discussing definable functions in the next section.

30 Definition We say that $\mathcal{D}$ is δ-approximable by $\varphi(x;z;X)$ if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that

1. $\langle b, C\rangle \in \mathcal{D} \;\;\Rightarrow\;\; \varphi(a;b;C^{\delta})$ and
2. $\langle b, C^{\delta}\rangle \in \mathcal{D} \;\;\Leftarrow\;\; \varphi(a;b;C)$ for every $b \in B$ and every C.

We say δ-approximable from from ε-below if we can also require that

2′. $\langle b, C^{\delta+\varepsilon}\rangle \in \mathcal{D} \;\;\Leftarrow\;\; \varphi(a;b;C)$ for every $b \in \mathcal{U}$ and every C.

It is evident that, when $\mathcal{D}$ is closed as defined in Section 2 then approximable is equivalent to δ-approximable for every δ.

Now we restate a variant of Theorem 26. The proof closely follows that in the previous section.

31 Theorem Let $\varphi(x;z;X)$ be stable. Assume that $\mathcal{D}$ is δ-approximable by $\varphi(x;z;X)$. Then for every ε there are some $\langle a_{i,j} \ :\ i<k,\ j<m\rangle$ such that for every $b\in\mathcal{U}^z$ and every C

$$\langle b,C\rangle\in\mathcal{D} \quad\Rightarrow\quad \bigvee_{i<k}\ \bigwedge_{j<m}\ \varphi(a_{i,j};b;C^{\delta+\varepsilon})$$

$$\langle b,C^{\delta+\varepsilon}\rangle\in\mathcal{D} \quad\Leftarrow\quad \bigvee_{i<k}\ \bigwedge_{j<m}\ \varphi(a_{i,j};b;C)$$

Proof. The theorem is an immediate consequence of the following two lemmas and Lemma 29 in the previous section. □

32 Lemma Let $\varphi(x;z;X)$ be stable. Assume that $\mathcal{D}$ is δ-approximable by $\varphi(x;z;X)$ from ε-below. Then there is are some $\langle a_i \ :\ i<k\rangle$ such that for every $b\in\mathcal{U}^z$ and every C

1. $$\langle b,C\rangle\in\mathcal{D} \quad\Rightarrow\quad \bigvee_{i<k}\ \varphi(a_i;b;C^{\delta+\varepsilon})$$
2. $$\langle b,C^{\delta+\varepsilon}\rangle\in\mathcal{D} \quad\Leftarrow\quad \bigvee_{i<k}\ \varphi(a_i;b;C)$$

Proof. We define recursively the required parameters a_i together with some auxiliary parameters b_i. The element a_n is choosen such that

3. $$\langle b,C^{\delta+\varepsilon}\rangle\in\mathcal{D} \quad\Leftarrow\quad \varphi(a_n;b;C) \qquad\qquad \text{for every } b\in\mathcal{U}^z \text{ and every } C$$

and

4. $$\langle b_i,C\rangle\in\mathcal{D} \quad\Rightarrow\quad \varphi(a_n;b_i;C^{\delta}) \qquad\qquad \text{for every } i<n \text{ and every } C.$$

This is possible because $\mathcal{D}$ is δ-approximated from ε-below. Note that (3) immediately guarantees (2). Now, assume (1) fails for $k=n$, and choose b_n and C_n witnessing this. Then, by Fact 7, for some $\tilde{C}_n\cap C_n^{\delta+\varepsilon}=\varnothing$

5. $$\langle b_n,C_n\rangle\in\mathcal{D} \quad\&\quad \bigwedge_{i<n}\ {\sim}\varphi(a_i;b_n;\tilde{C}_n)$$

Suppose for a contradiction that the construction never ends. Then, as (5) guarantees that $\langle b_i,C_i\rangle\in\mathcal{D}$ for every i, from (4) we otain $\varphi(a_n;b_i;C_n^{\delta})$ for every $i<n$. From (5) we also obtain ${\sim}\varphi(a_i;b_n;\tilde{C}_n)$, for every $i\le n$. This contradicts ε-stability. □

33 Lemma Let $\varphi(x;z;X)$ be ε-stable. Assume that $\mathcal{D}$ is δ-approximable by $\varphi(x;z;X)$. Let m be maximal so that a sequence as in (3) of Theorem 21 exists. Let $\bar{x}=\langle x_i \ :\ i\le m\rangle$ where the x_i are copies of x. Then the formula

$$\sigma(\bar{x};z;X) \quad=\quad \bigwedge_{i\le m}\ \varphi(x_i;z;X)$$

δ-approximate $\mathcal{D}$ from ε-below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x})$ does not δ-approximate $\mathcal{D}$ from ε-below. Suppose that $a_0,\ldots,a_{n-1}$ and $b_0,\ldots,b_{n-1}$ have been defined. Choose a_n such that for every $b\in B\cup\{b_0,\ldots,b_{n-1}\}$ and every C

1. $\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a_n; b; C^{\delta})$ and

2. $\langle b, C^{\varepsilon+\delta}\rangle \in \mathcal{D} \;\Leftarrow\; \varphi(a_n; b; C^{\varepsilon})$

Note that the latter implication is equivalent to: for every C there is some $\tilde{C} \cap C^{\varepsilon} = \varnothing$ such that

3. $\langle b, C^{\varepsilon+\delta}\rangle \notin \mathcal{D} \;\Rightarrow\; {\sim}\varphi(a_n; b; \tilde{C})$.

Now, as the lemma is assumed to fail, we can choose b_n and C_n such that

4. $\langle b_n, C_n^{\varepsilon+\delta}\rangle \notin \mathcal{D} \;\;\&\;\; \bigwedge_{i=0}^{n} \varphi(a_i; b_n; C_n)$

Note that (4) ensure that $\langle b_i, C_i^{\delta+\varepsilon}\rangle \notin \mathcal{D}$ for every i. The there is some $\tilde{C}_i$ that witnesses (3) for $\langle b_i, C_i^{\delta+\varepsilon}\rangle \notin \mathcal{D}$. We claim that the procedure has to stop after $\le m$ steps. In fact, from (3) we obtain ${\sim}\varphi(a_n; b_i; \tilde{C}_i)$ for every $i < n$. On the other hand, by (4) we have that $\varphi(a_i; b_n; C_n)$ for every $i < n$. Therefore, $\langle a_{m-i}; b_{m-i}; C_{m-i}; \tilde{C}_{m-i} \;:\; i \le m\rangle$ contradicts the maximality of m. □

6. Stable definable functions

In this section we specialize the notions introduced in the previous sections to formulas $\varphi(x; z; X)$ of the form $\tau(x; z) \in X$ where $\tau(x; z)$ is an S-valued $\mathcal{F}$-definable function. Notice that ${\sim}\big(\tau(x; z) \in X\big) = \tau(x; z) \in X$.

We say that $\tau(x; z)$ is stable if so is $\tau(x; z) \in X$. The following fact is a suggestive characterization of the stability of functions.

34 Fact Let $\tau(x; z)$ be as above. Then the following are equivalent

1. the formula $\tau(x; z) \in X$ is unstable
2. there is a sequence $\langle a_i; b_i \;:\; i < \omega\rangle$ such that
$$\lim_{i\to\infty}\lim_{j\to\infty} \tau(a_i; b_j) \;\;\neq\;\; \lim_{j\to\infty}\lim_{i\to\infty} \tau(a_i; b_j)$$
3. there is a sequence $\langle a_i; b_i \;:\; i < \omega\rangle$ and some ε such that for every $i < j < \omega$
$$\big\langle \tau(a_i; b_j),\, \tau(a_j; b_i)\big\rangle \;\;\notin\;\; \varepsilon.$$

Proof. 2⇔3 Clear.

1⇒2. Let $C \cap \tilde{C} = \varnothing$ and $\langle a_i; b_i \;:\; i < \omega\rangle$ be as given by (1). That is, $\tau(a_i; b_j) \in C$ and $\tau(a_j; b_i) \in \tilde{C}$ hold for every $i < j < \omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and $\tilde{C}$ contains the limit on the right – which therefore are distinct.

2⇒1. Let C and $\tilde{C}$ be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i; b_i \;:\; i < \omega\rangle$. □

Let $f : \mathcal{U}^z \to S$ be a function. We define

$$\mathcal{D}_f \;\;=\;\; \big\{\langle b, C\rangle \;:\; f(b) \in C,\;\; b \in \mathcal{U}^z,\;\; C \in K(S)\big\}$$

35 Fact The following are equivalent

1 $\mathcal{D}_f$ is approximable by $\tau(x;z) \in X$

2 for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that $\tau(a;b) = f(b)$ for every $b \in B$.

36 Fact (conjecture) The following are equivalent

1 $\mathcal{D}_f$ is δ-approximable by $\tau(x;z) \in X$

2 for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that $\langle \tau(a;b)\,;f(b)\rangle \in \delta$ for every $b \in B$.

References

[AAVV] Claudio Agostini, Stefano Baratella, Silvia Barbina, Luca Motto Ros, and Domenico Zambella, *Continuous logic in a classical setting* (2025). t.a. in Bull. Iranian Math. Soc.

[A] Josef Auslander, *Topological Dynamics*, Scholarpedia (2008), doi:10.4249/scholarpedia.3449.

[BY] Itaï Ben Yaacov, *Model theoretic stability and definability of types, after A. Grothendieck*, Bull. Symb. Log. **20** (2014), no. 4, 491–496.

[BU] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Trans. Amer. Math. Soc. **362** (2010), 5213–5259.

[BBHU] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427.

[CS13] Artem Chernikov and Pierre Simon, *Externally definable sets and dependent pairs*, Israel J. Math. **194** (2013), no. 1, 409–425, arXiv:1007.4468.

[HI] C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, Cambridge, 2002, pp. 1–110.

[K] H. Jerome Keisler, *Model Theory for Real-valued Structures*, in Beyond First Order Model Theory (José Iovino, ed.), Vol. II, Chapman and Hall/CRC, 2023.

[PC] Nicolas Chavarria and Anand Pillay, *On pp-elimination and stability in a continuous setting*, Ann. Pure Appl. Logic **174** (2023), 1–14.

[Z15] Domenico Zambella, *Elementary classes of finite VC-dimension*, Arch. Math. Logic (2015), 511–520.

[Z] ______, *Standard analysis* (2023), arXiv:2311.15711