Local stability in structures with a standard sort

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Abstract.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_{\mathsf{S}}$ that has a symbol for each regular compact subset $C \subseteq S$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{S}}$ or their interpretation in the structure S.

Throughout these notes the letter C always denotes a regular compact subset of S, or a tuple of such sets.

Finally, note that we could allow in $\mathcal{L}_{\mathsf{S}}$ relation symbols for all compact subsets of S^n, for any n. But this would clutter the notation adding very little to the theory. When needed, we will assume this generalization and leave the detail to the reader.

We also fix an arbitrary first-order language which we denote by $\mathcal{L}_{\mathsf{H}}$ and call the language of the home sort.

1 Definition Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{S}}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H).
A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_{\mathsf{H}}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S.

We write $x \in C$ for the predicate associated to a compact set C, and denote by $|\text{-}|$ the length of a tuple.

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2 Definition We call atomic formulas those of the form

i. atomic and negated atomic formulas of $\mathcal{L}_{\mathsf{H}}$ with exlusion of equalities and inequalities

ii. $\tau \in C$, where τ is a term of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$, and $C \subseteq S$ is a regular compact set.

Let $M \subseteq N$ be standard structures. We say that M is an $\mathcal{F}$-elementary substructure of N if the latter models all $\mathcal{F}(M)$-sentences that are true in M.

Let x and ξ be variables of sort H, respectively S. A standard structure M is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M. The following theorem is proved in [AAVV] for signatures that do not contain symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n{\cdot}m > 0$. A similar framework has been independently introduced in [PC] – only without quantifiers of sort H and with an approximate notion of satisfaction. The corresponding compactness theorem is proved there with a similar method. The setting in [PC] is a generalization of that used by Henson and Iovino for Banach spaces [HI]. By the elimination of quantifiers of sort S, proved in [AAVV, Proposition 3.6], the two approaches are equivalent – up to approximations.

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [AAVV] extends to the case $n{\cdot}m > 0$.

3 Theorem (Compactness) Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

We introduce variables of some new sorts X with the sole scope of conveniently describing classes of $\mathcal{F}$-formulas that only differ by the sets/predicates $C \subseteq S$ that occur. Let $\mathcal{F}_{\mathsf{X}}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\xi) \in X$, where X is a variable of sort X.

Formulas in $\mathcal{F}_{\mathsf{X}}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1,\ldots,X_k$ be a tuple of variables sort X. If $C = C_1,\ldots,C_k$ is a tuple of regular compact subsets of S then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_{\mathsf{X}}$.

4 Definition Let $\varphi \in \mathcal{F}_{\mathsf{X}}$ be a formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_{\mathsf{X}}$ is the formula obtained by replacing in φ the atomic formulas in $\mathcal{L}_{\mathsf{H}}$ by their negation and every connective $\wedge$, $\vee$, $\forall$, $\exists$ by their respective duals $\vee$, $\wedge$, $\exists$, $\forall$.
The pseudonegation of φ is denoted by $\sim\!\varphi$. Clearly, when X does not occur in φ, we have $\sim\!\varphi \leftrightarrow \neg\varphi$.

The following fact is immediate and will be used without further mention.

5 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$, every standard structure M, and every formula $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$

$$\begin{aligned} M &\models \quad \varphi(C) \to \quad \varphi(C') \\ M &\models \ \sim\!\varphi(\tilde{C}) \to \neg\varphi(C). \end{aligned}$$

6 Fact For every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from C such that

$$M \models \varphi(C) \to \neg \sim\!\varphi(\tilde{C}) \to \varphi(C')$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Conversely, for every $\tilde{C} \cap C = \varnothing$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \to \varphi(C') \to \neg \sim\!\varphi(\tilde{C}).$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = C' \smallsetminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds with as C' any neighborhood of C disjoint from $\tilde{C}$. Induction on the syntax of $\varphi(X)$ proves the general case. □

The above fact has the following useful consequence.

7 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$, where M is a standard structure. Then the following are equivalent for every tuple C

$$\begin{aligned} M &\models \quad \varphi(C) \leftarrow \bigwedge\Big\{\varphi(C') : C' \text{ neighborhood of } C\Big\} \\ M &\models \ \neg\varphi(C) \to \bigvee\Big\{\sim\!\varphi(\tilde{C}) : \tilde{C} \cap C = \varnothing\Big\}. \end{aligned}$$

(The converse implications are trivial – therefore not displayed.)

Note that the fact also holds if we restrict the above C' and $\tilde{C}$ to range over the subsets of some neighboorhood of C.

8 Fact The equivalent conditions in Fact 7 hold in all $\mathcal{F}$-saturated structures.

Proof. Prove the first of the two implications by induction on the syntax of $\varphi(X)$. The existential quantifier of sort H is the only connective that requires attention. Assume inductively that

$$\bigwedge\Big\{\varphi(a;C') : C' \text{ neighborhood of } C\Big\} \;\to\; \varphi(a;C)$$

holds for every $\varphi(a;C)$. Then induction for the existential quantifier follows from

$$\bigwedge\Big\{\exists x\,\varphi(x;C') : C' \text{ neighborhood of } C\Big\} \;\to\; \exists x \bigwedge\Big\{\varphi(x;C') : C' \text{ neighborhood of } C\Big\}$$

which is a consequence of saturation. □

We say that M is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions. By the following fact $\mathcal{F}$-saturated structures are $\mathcal{F}$-maximal.

9 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Then the following are equivalent

1. M is $\mathcal{F}$-maximal
2. the equivalent conditions in Fact 7 hold for every $\varphi(C)$.

Proof. 1⇒2. If (2) in Fact 7 fails, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of M. As M is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg{\sim}\varphi(\tilde{C})$ holds also in $\mathcal{U}$. Then (2) in Fact 7 fails in $\mathcal{U}$, contradicting Fact 8.

2⇒1. Let N be any $\mathcal{F}$-elementary extension of M. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C. Suppose not. Then, by (2) in Fact 7, $M \models {\sim}\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models {\sim}\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts Fact 8. □

In what follows we fix a large saturated standard structure which we denote by $\mathcal{U}$.

Let ε a neighborhood of the diagonal of S^2. For $a, b \in \mathcal{U}^n$ we write $a \sim_\varepsilon b$ if $\langle \tau(a), \tau(b) \rangle \in \varepsilon$ for every term $\tau(z)$ of sort $\mathsf{H}^n \to \mathsf{S}$.

2. A duality in $K(S)$

Write $K(S)$ for the set of compact subsets of S. In this section $\mathcal{D}$ and $\mathcal{C}$ range over the subsets of $\subseteq K(S)^n$, for some n. We define

$$\sim\mathcal{D} \quad = \quad \big\{\tilde{C} \,:\, \tilde{C} \cap C \neq \varnothing \text{ for every } C \in \mathcal{D}\big\}.$$

The following fact motivates the definition.

10 Fact For every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(\mathcal{U})$, if $\mathcal{D} = \{C : \varphi(C)\}$ then ${\sim}\mathcal{D} = \{\tilde{C} : {\sim}\varphi(\tilde{C})\}$

Proof. We need to prove that

$${\sim}\varphi(\tilde{C}) \quad \Leftrightarrow \quad \text{for every } C, \ \text{ if } C \cap \tilde{C} = \varnothing \ \text{ then } \neg\varphi(C)$$

Implication ⇒ follows immediately from Fact 5. To prove ⇐ assume the r.h.s. Then $\neg\varphi(S \smallsetminus O)$ holds for every open set $O \supseteq \tilde{C}$. From the second implication in Fact 7 we obtain ${\sim}\varphi(\tilde{C}')$ for some $\tilde{C} \subseteq \tilde{C}' \subseteq O$. As O is arbitrary, we obtain $\varphi(\tilde{C})$ from the second implication in Fact 7. □

We prove a couple of straightforward inclusions.

11 Fact For every $\mathcal{D} \subseteq \mathcal{C}$ we have ${\sim}\mathcal{C} \subseteq {\sim}\mathcal{D}$. Moreover, $\mathcal{D} \subseteq {\sim}{\sim}\mathcal{D}$.

Proof. Let $C \notin {\sim}\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in \mathcal{D}$. As $\tilde{C} \in \mathcal{C}$ we conclude that $C \notin {\sim}\mathcal{C}$. For the second claim, let $C \notin {\sim}{\sim}\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in {\sim}\mathcal{D}$. Then $C \notin \mathcal{D}$ follows. □

12 Fact For every $\mathcal{D}$ the following are equivalent

1. $\mathcal{D} = {\sim}{\sim}\mathcal{D}$.
2. $\mathcal{D} = {\sim}\mathcal{C}$ for some $\mathcal{C}$.

Proof. Only 2⇒1 requires a proof. From Fact 11 we obtain ${\sim}{\sim}{\sim}\mathcal{C} \subseteq {\sim}\mathcal{C}$. Assume (2) then ${\sim}{\sim}\mathcal{D} \subseteq \mathcal{D}$ which, again by Fact 11, suffices to prove (1). □

We say that $\mathcal{D}$ is involutive if $\mathcal{D} = {\sim}{\sim}\mathcal{D}$.

13 Theorem The following are equivalent

1. $\mathcal{D}$ is involutive
2. $C \in \mathcal{D} \quad \Leftrightarrow \quad C' \in \mathcal{D}$ for every neighborhood C' of C.

It is not difficult to see that (2) holds if and only if $\mathcal{D}$ is closed in the Vietoris topology and includes all the supersets of its elements.

Proof. 2⇒1. It suffices to prove ${\sim}{\sim}\mathcal{D} \subseteq \mathcal{D}$. Let $C \in {\sim}{\sim}\mathcal{D}$. Let $O \supseteq C$ be open. Then $S \smallsetminus O \notin {\sim}\mathcal{D}$. Then $\tilde{C} \in \mathcal{D}$ for some $\tilde{C} \subseteq O$. Assume (2). Then, by ⇒ every $C' \supseteq O$ is in $\mathcal{D}$. As O is arbitrary, $C \in \mathcal{D}$ by ⇐.

1⇒2. It suffices to show that (2) holds with ${\sim}\mathcal{D}$ for $\mathcal{D}$. Implication ⇒ is ovious. To prove ⇐, assume $C \notin {\sim}\mathcal{D}$. Then $C \cap \tilde{C}$ for some $\tilde{C} \in \mathcal{D}$. Let C' be a neighboorhood of C disjoint of C. Then the r.h.s. of the equivalence fails for C'. □

3. Attempt 1

Let $\varphi(x;z;X)$ be a formulas in $\mathcal{F}_{\mathsf{X}}(\mathcal{U})$. The variables of sort H are partitioned in two tuples $x;z$ which may be infinite. The following is the classical definition of stability which we dub *finitary* for the reason explained below.

14 Definition We say that $\varphi(x;z;C)$ is finitarily stable if for some $m < \omega$ there is no sequence $\langle a_i;b_i \ :\ i<m\rangle$ such that for every $i<n$

$$\varphi(a_n;b_i;C) \quad \wedge \quad \neg\varphi(a_i;b_n;C).$$

From Fact 7 we easily obtain the following equivalent definition which only mentions formulas in $\mathcal{F}$.

15 Fact The following are equivalent

1. $\varphi(x;z;C)$ is finitarily stable
2. for some m there is no sequence $\langle a_i;b_i;C_i \ :\ i<m\rangle$ such that for every $i<n$

$$\varphi(a_n;b_i;C) \quad \wedge \quad {\sim}\varphi(a_i;b_n;C_n) \quad \text{and} \quad C\cap C_n = \varnothing.$$

In a classical context, i.e. reling on $\mathcal{L}$-saturation, we can replace m in Definition 14 by ω. But this may not be true if only have $\mathcal{F}$-saturation. Therefore we intruduce the non finitary version of saturation as a separate notion. Namely, we say that $\varphi(x;z;C)$ is stable if the equivalent conditions in the following fact hold.

16 Theorem (conjecture) The following are equivalent

1. there is no sequence $\langle a_i;b_i : i<\omega\rangle$ such that for some $\tilde{C}\cap C=\varnothing$ and every $i<n$
$$\varphi(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;\tilde{C})$$
2. there is no sequence $\langle a_i;b_i;C_i : i<\omega\rangle$ such that for some neighboorhood C' of C and for every $i<n$
$$\varphi(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;C_n) \quad\text{and}\quad C'\cap C_n=\varnothing.$$
3. for every neighboorhood C' of C there is a maximal m such that some sequence $\langle a_i;b_i;C_i : i<m\rangle$ is such that for every $i<n$
$$\varphi(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;C_n) \quad\text{and}\quad C'\cap C_n=\varnothing.$$

The following two definition are classical, i.e. they rely on the full language $\mathcal{L}$. Their $\mathcal{F}$-homologue we will be introduced later.

17 Definition A global $\varphi(x;z;C)$-type is a maximally (finitely) consistent set of formulas of the form $\varphi(x;b;C)$ or $\neg\varphi(x;b;C)$ for some $b\in\mathcal{U}^{|z|}$.

In this section the symbol $\mathcal{D}$ always denotes a subset of $\mathcal{U}^z$.

18 Definition We say that $\mathcal{D}$ is approximable by $\varphi(x;z;C)$ if for every finite $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that
$$b\in\mathcal{D} \quad\Leftrightarrow\quad \varphi(a;b;C) \qquad\qquad \text{for every } b\in B.$$

It goes without saying that these definitions describe two faces of the same coin. In fact, it is clear that $\mathcal{D}$ is approximable by $\varphi(x;z;C)$ if and only if
$$p(x) \quad=\quad \big\{\varphi(x;b;C) \,:\, b\in\mathcal{D}\big\} \;\cup\; \big\{\neg\varphi(x;b;C) \,:\, b\notin\mathcal{D}\big\}$$
is a global $\varphi(x;z;C)$-type

The following theorem says that if $\varphi(x;z;C)$ is finitarily stable then global $\varphi(x;z;C)$-types are definable. The proof of the classical case applies verbatim because no saturation is required. We state it only for comparison with its non finitary counterpart, see Theorem 25.

19 Theorem Let $\varphi(x;z;C)$ be finitarily stable. Let $\mathcal{D}$ be approximable by $\varphi(x;z;C)$. Then there are some $\langle a_{i,j} \,:\, i<k,\ j<m\rangle$ such that for every $b\in\mathcal{U}^z$

$$b\in\mathcal{D} \quad\Leftrightarrow\quad \bigvee_{i<k}\ \bigwedge_{j<m}\ \varphi(a_{i,j};b;C)$$

The proof of the theorem fails for stable but non-finitary stable formulas. We can rescue part of the theorem we using a variant notion of approximation (hence loosing the correspondence to global types).

20 Definition Let C' be a neighborhood of C. We say that $\mathcal{D}$ is C'-approximable by $\varphi(x;z;C)$ if for every finite $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that

1. $b\in\mathcal{D} \ \Rightarrow\ \varphi(a;b;C)$ and
2. $\varphi(a;b;C') \ \Rightarrow\ b\in\mathcal{D}$ for every $b\in B$.

We say that the approximation is from below if (2) holds for every $b\in\mathcal{U}^z$.

Clearly if $\mathcal{D}$ is C'-approximable for every neighborhood of C, then it is approximable.

21 Lemma Let $\varphi(x;z;C)$ be stable. Let C' be a neighboorhood of C. Assme that $\mathcal{D}$ is C'-approximable by $\varphi(x;z;C)$. Then there are some $\langle a_{i,j} \,:\, i<k,\ j<m\rangle$ such that for every $b\in\mathcal{U}^z$

$$b\in\mathcal{D} \quad\Rightarrow\quad \bigvee_{i<k}\ \bigwedge_{j<m}\ \varphi(a_{i,j};b;C')$$

$$b\in\mathcal{D} \quad\Leftarrow\quad \bigvee_{i<k}\ \bigwedge_{j<m}\ \varphi(a_{i,j};b;C)$$

Note that for $\varphi(x;z;C)$ fixed, k and m depends on C'.

Proof. The theorem is an immediate consequence of the following three lemma. □

22 Lemma Under the assumptions of Lemma 21. If $\mathcal{D}$ is C'-approximable from below then there is a sequence $\langle a_i \,:\, i<k\rangle$ such that for every $b\in\mathcal{U}^z$

1. $b\in\mathcal{D} \ \Rightarrow\ \bigvee_{i<k} \varphi(a_i;b;C')$
2. $b\in\mathcal{D} \ \Leftarrow\ \bigvee_{i<k} \varphi(a_i;b;C)$

Proof. We define recursively the required parameters a_i together with some auxiliary parameters b_i. The element a_n is choosen such that

3. $b\in\mathcal{D} \ \Leftarrow\ \varphi(a_n;b;C)$ for every $b\in\mathcal{U}^z$

and

4. $b_i\in\mathcal{D} \ \Rightarrow\ \varphi(a_n;b_i;C)$ for every $i<n$.

This is possible because $\mathcal{D}$ is approximated from below. Note that (3) immediately guarantees (2). Now choose b_n and $C_n \cap C' = \varnothing$ such that

5. $\bigwedge_{i=0}^{n} \sim\varphi(a_i\,;b_n;C_n)$ and $b \in \mathcal{D}$

If such b_n and C_n do not exist, then

$$\sim\bigvee_{i=0}^{n} \varphi(a_i\,;b;\tilde{C}) \quad\Rightarrow\quad b \notin \mathcal{D} \qquad\qquad \text{for every } b \in \mathcal{U}^z \text{ and every } \tilde{C} \cap C' = \varnothing$$

which is equivalent to (1) and proves the lemma. Suppose for a contradiction that the construction never ends. Then, from (4) we otain $\varphi(a_n\,;b_i\,;C)$ for every $i < n$ while from (5) we obtain $\sim\varphi(a_i\,;b_n;C_n)$, for every $i \leq n$. This contradicts stability. □

23 Lemma Under the assumptions of Lemma 21. Let m be maximal so that a sequence as in (3) of Fact 15 exists – for C' as given. Let $\bar{x} = \langle x_i \;:\; i \leq m\rangle$ where the x_i are copies of x. Then the formula

$$\sigma(\bar{x}\,;z;C) \quad=\quad \bigwedge_{i\leq m} \varphi(x_i\,;z;C)$$

C'-weakly approximate $\mathcal{D}$ from below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x})$ does not approximate $\mathcal{D}$ from below. Suppose that $a_0,\ldots,a_{n-1}$ and $b_0,\ldots,b_{n-1} \notin \mathcal{D}$ have been defined. Choose a_n such that for every $b \in B \cup \{b_0,\ldots,b_{n-1}\}$

1. $b \in \mathcal{D} \quad\Rightarrow\quad \varphi(a_n\,;b;C)$ and
2. $\varphi(a_n\,;b;C') \quad\Rightarrow\quad b \in \mathcal{D}$

Note that the latter implication is equivalent to

3. $b \notin \mathcal{D} \quad\Rightarrow\quad \sim\varphi(a_n\,;b;\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$.

We write C_i for the $\tilde{C}$ that witnesses (2) for b_i. Now, as the lemma is assume to fail, we can choose b_n such that

$$\bigwedge_{i=0}^{n} \varphi(a_i\,;b_n;C) \text{ and } b_n \notin \mathcal{D}.$$

We claim that the sequence has to stop after $\leq m$ steps. Otherwise we contradict the maximality of m. In fact, from (3) and $b_i \notin \mathcal{D}$ we obtain $\sim\varphi(a_n\,;b_i\,;C_i)$ for every $i < n$. On the other hand, by (1) we have that $\varphi(a_n\,;b_i\,;C)$ for every $i < n$. □

24 Lemma If $\varphi(x;z;C)$ is stable then $\sigma(\bar{x};z;C)$ in the previous lemma is stable.

Proof. Let k be suffinciely large so that every m-coloring of a graph of size k has a monocromatic subgraph of size $> m$. Let $\langle \bar{a}_i\,;b_i\,;C_i \;:\; i < k\rangle$ be a sequence witnessing instability as in (3) of Fact 26. Then for every pair $i < n$ there is some $j < m$ such that $\sim\varphi(a_{j,n}\,;b_i\,;C_i)$. By the choice of k there is a $j < m$ such that $\sim\varphi(a_{j,n}\,;b_i\,;C_i)$ obtains for $> m$ many i. Then we can extract a subsequence that contradicts the maximality of m. □

25 Theorem (conjecture) Let $\varphi(x;z;C)$ be stable. Let $\mathcal{D}$ be approximable by $\varphi(x;z;C)$. Then there is $\pi(z;C)$, a small set of formulas of the form $\varphi(a;z;C)$, with $a \in \mathcal{U}^x$, such that

$$b \in \mathcal{D} \;\Leftrightarrow\; \pi(b;C) \qquad \text{for every } b \in \mathcal{U}^z.$$

4. Attempt 2

In this section $\mathcal{D}$ is always a subset of $\mathcal{U}^z \times K(S)^n$, where n has to be inferred from the context (typically, $n = |X|$). We write $\sim\mathcal{D}$ for the set obtained by applying the definition in Section 2 to all fibers of $\mathcal{D}$. We say that $\mathcal{D}$ is involutive if all its fibers are such.

We write ε for a compact neighborhood of the diagonal of $(S^n)^2$. We write C^ε for elements of S^n that are ε-close to C.

We say that $\varphi(x;z;C)$ is ε-stable if the equivalent conditions in the following fact hold.

26 Theorem (conjecture) The following are equivalent

1. there is no sequence $\langle a_i; b_i \,:\, i < \omega\rangle$ such that for some $\tilde{C} \cap C^\varepsilon = \varnothing$ and every $i < n$
$$\varphi(a_n; b_i; C) \;\wedge\; \sim\varphi(a_i; b_n; \tilde{C})$$
2. there is no sequence $\langle a_i; b_i; C_i \,:\, i < \omega\rangle$ such that for every $i < n$
$$\varphi(a_n; b_i; C) \;\wedge\; \sim\varphi(a_i; b_n; C_n) \quad \text{and} \quad C^\varepsilon \cap C_n = \varnothing.$$
3. there is a maximal m such that some sequence $\langle a_i; b_i; C_i \,:\, i < m\rangle$ is such that for every $i < n$
$$\varphi(a_n; b_i; C) \;\wedge\; \sim\varphi(a_i; b_n; C_n) \quad \text{and} \quad C^\varepsilon \cap C_n = \varnothing.$$

27 Definition A global $\varphi(x;z;X)$-type is a maximally (finitely) consistent set of formulas of the form $\varphi(x;b;C)$ or $\sim\varphi(x;b;C)$ for some $b \in \mathcal{U}^{|z|}$ and some C.

28 Definition We say that $\mathcal{D}$ is ε-approximable by $\varphi(x;z;X)$ if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that

1. $\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a;b;C)$ and
2. $\varphi(a;b;C^\varepsilon) \;\Rightarrow\; \langle b, C\rangle \in \mathcal{D}$ for every $b \in B$ and every C.

We say ε-approximable from below if the implication $\Leftarrow$ holds for every $b \in \mathcal{U}^z$. We say approximable for ε-approximable for every ε.

By Facts 10 and 11 when $\mathcal{D}$ is involutive (1) can be rephrased as

$$\sim\varphi(a;b;C) \;\Rightarrow\; \langle b, C\rangle \in \sim\mathcal{D} \qquad \text{for every } b \in B \text{ and every } C.$$

Analogously, (2) can be rephrased as

$$\langle b, C\rangle \in \sim\mathcal{D} \;\Rightarrow\; \sim\varphi(a;b;C^\varepsilon) \qquad \text{for every } b \in B \text{ and every } C.$$

29 Fact For every involutive $\mathcal{D}$ the following are equivalent

1. $\mathcal{D}$ is approximable by $\varphi(x;z;X)$
2. the following is a global $\varphi(x;z;X)$-type

$$\begin{aligned} p(x) &= \big\{\ \varphi(x;b;C) : \langle b,C\rangle \in \mathcal{D}\big\} \cup \\ &\quad \big\{{\sim}\varphi(x;b;C) : \langle b,C\rangle \in {\sim}\mathcal{D}\big\}. \end{aligned}$$

Proof. 2⇒1. Let B be a given finite subset of $\mathcal{U}^z$. The finite consistency of $p(x)$ provides some a such that for every finite B an a such that

$$\begin{aligned} \langle b,C\rangle \in \mathcal{D} &\Rightarrow \quad \varphi(a;b;C) \quad \text{and} \\ \langle b,C\rangle \in {\sim}\mathcal{D} &\Rightarrow {\sim}\varphi(a;b;C) \qquad\qquad \text{for every } b\in B \text{ and every } C. \end{aligned}$$

Therefore also

$$\langle b,C\rangle \in {\sim}\mathcal{D} \Rightarrow {\sim}\varphi(a;b;C^{\varepsilon}).$$

By what remarked above, these yield (1) and (2) in Definition 28.

1⇒2. Firstly, we prove finite consistency. Let B and $C_1,\dots,C_n$ be given. Let ε be small enough so that

$$\langle b,C_i^{\varepsilon}\rangle \in {\sim}\mathcal{D} \Rightarrow \langle b,C_i\rangle \in {\sim}\mathcal{D} \qquad\qquad \text{for every } b\in B \text{ and every } i<n.$$

Let a be given by Definition 28. Then

$$\begin{aligned} \langle b,C_i\rangle \notin \mathcal{D} &\Rightarrow C_i\cap C=\varnothing \text{ for some C} \\ \langle b,C_i\rangle \in \mathcal{D} &\Rightarrow \varphi(a;b;C_i) \qquad\qquad \text{for every } b\in B \text{ and every } i. \end{aligned}$$

Then the following holds for every ε

$$\langle b,C\rangle \in {\sim}\mathcal{D} \Rightarrow {\sim}\varphi(a;b;C_i^{\varepsilon})$$

As ε

$$\Rightarrow \varphi(a;b;C_i).$$

This implies that $\langle b,C_i\rangle \in \mathcal{D}$ for every $b\in B$ and every C_i.

Then for every $i<n$ and every $b\in B$ we have that

follows (again, immediately) from approximability. We prove $p(x)$ is maximal. We need to consider two cases. First, assume that $\varphi(x;b;C)$ is consistent with $p(x)$. Then consistency implies that $\langle b,\tilde{C}\rangle \notin {\sim}\mathcal{D}$ for every $\tilde{C}\cap C=\varnothing$. But this implies that $C\in{\sim}{\sim}\mathcal{D}=\mathcal{D}$, therefore $\varphi(x;b;C)\in p$. The second case is when ${\sim}\varphi(x;b;C)$ is consistent with $p(x)$. This implies that $\langle b,\tilde{C}\rangle\notin\mathcal{D}$ for every $\tilde{C}\cap C=\varnothing$. Then $\langle b,C\rangle\in{\sim}\mathcal{D}$, therefore ${\sim}\varphi(x;b;C)\in p$. □

5. Stable functions

Let $\varphi(x;z;X)$ be type-definable that is, definable by an infinite (small) conjunction of formulas in $\mathcal{F}_{\mathsf{X}}(\mathcal{U})$. The variables of sort H are partitioned in two tuples $x;z$ which may be infinite.

We say that $\varphi(x;z;X)$ is unstable if there are some $C\cap\tilde{C}=\varnothing$ and a sequence $\langle a_i;b_i : i<\omega\rangle$ such that for every $i<j<\omega$

$$\varphi(a_i;b_j;C) \quad\wedge\quad \sim\varphi(a_j;b_i;\tilde{C})$$

We say that $\varphi(x;z;X)$ is stable if it is not unstable.

In this section we consider formulas of a particular form which is of some interest. We deal with the general case in the next section.

Assume $S=[0,1]$ throughout this section.

Let $\tau:\mathcal{U}^{x;z}\to S$ be a function definable by an infinite conjunction of formulas in $\mathcal{F}$. We say that $\tau(x;z)$ is stable if so is $\tau(x;z)\in X$. The following fact is a suggestive characterization of the stability of functions.

30 Fact Let $\tau(x;z)$ be as above. Then the following are equivalent

1. the formula $\tau(x;z)\in X$ is unstable
2. there is a sequence $\langle a_i;b_i : i<\omega\rangle$ such that
$$\lim_{i\to\infty}\lim_{j\to\infty}\tau(a_i;b_j) \quad\neq\quad \lim_{j\to\infty}\lim_{i\to\infty}\tau(a_i;b_j)$$
3. there is a sequence $\langle a_i;b_i : i<\omega\rangle$ and some $\varepsilon>0$ such that for every $i<j<\omega$
$$\big|\tau(a_i;b_j)\;-\;\tau(a_j;b_i)\big| \quad\geq\quad \varepsilon.$$

Proof. 2⇔3 Clear.

1⇒2. Let $C\cap\tilde{C}=\varnothing$ and $\langle a_i;b_i : i<\omega\rangle$ be as given by (1). That is, $\tau(a_i;b_j)\in C$ and $\tau(a_j;b_i)\in\tilde{C}$ hold for every $i<j<\omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and $\tilde{C}$ contains the limit on the right – which therefore are distinct.

2⇒1. Let C and $\tilde{C}$ be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i;b_i : i<\omega\rangle$. □

We say that $f:\mathcal{U}^z\to S$ is ε-approximable by $\tau(x;z)$ if for every finite $B\subseteq\mathcal{U}^z$ and there is an $a\in\mathcal{U}^x$ such that

$$\big|f(b)\,,\tau(a;b)\big| \quad\leq\quad \varepsilon \qquad\qquad \text{for every } b\in B.$$

We say approximable for ε-approximable for every $\varepsilon>0$.

31 Theorem Let $S=[0,1]$. Let $\tau(x;z)$ be a stable type-definable function. Let $f:\mathcal{U}^z\to S$ be approximable by $\tau(x;z)$. Then f is type-definable.

Proof. The following three lemmas show that if $\tau(x;z)$ is ε-approximable, then there are some $\langle a_{i,j} \ :\ i<n,\ j<\omega\rangle$ such that

$$\Big|f(b) - \max_{i=0,\dots,n-1}\ \inf_{j<\omega}\ \tau(a_{i,j};b)\Big| \ \leq\ 3\varepsilon \qquad \text{for every } b\in\mathfrak{U}^z.$$

Then f is the limit of type-definable functions, hence type-definable. □

We say that $f:\mathfrak{U}^{x;z}\to[0,1]$ is ε-approximable from below if the parameter $a\in\mathfrak{U}^x$ that witnesses approximability satisfies $\tau(a;b)\leq f(b)+\varepsilon$ for every $b\in\mathfrak{U}^z$.

32 Lemma Assume that the f in the theorem is ε-approximable from below. Then there are some $\langle a_i\ \ i<n\rangle$ such that

$$\Big|f(b) - \max_{i=0,\dots,n-1}\ \tau(a_i;b)\Big| \ \leq\ \varepsilon \qquad \text{for every } b\in\mathfrak{U}^z.$$

Proof. Negate the lemma. We construct inductively a sequence $\langle a_i;b_i\ :\ i<\omega\rangle$ that contradicts the stability of $\tau(x;z)$. Then, let b_n be such that (if $n=0$ pick b_n arbitrarily)

1. $$\Big|f(b_n) - \max_{i=0,\dots,n-1}\ \tau(a_i;b_n)\Big| \ >\ \varepsilon$$

By approximability from below, for some a_n we have

2. $$\tau(a_n;b_i) - f(b_i) \ \leq\ \varepsilon \qquad \text{for every } i\leq n$$

3. $$\tau(a_n;b) \ \leq\ f(b)+\varepsilon \qquad \text{for every } b\in\mathfrak{U}^z.$$

We prove instability by showing that for every $i<n$

4. $$\tau(a_n;b_i) - \tau(a_i;b_n) \ >\ \varepsilon$$

Note that by virtue of (3)

$$\max_{i=0,\dots,n-1}\ \tau(a_i;b_n) - f(b_n) \ \leq\ \varepsilon$$

Then (1) ensures that

1′. $$f(b_n) - \max_{i=0,\dots,n-1}\ \tau(a_i;b_n) \ >\ \varepsilon$$

Using (1′) we obtain

$$\tau(a_n;b_i) - \tau(a_i;b_n) \ >\ \tau(a_n;b_i) - f(b_n)+\varepsilon$$

then from (2) we obtain

$$>\ f(b_i) - f(b_n)+2\varepsilon$$

By refining the sequence we can assume that $|f(b_i)-f(b_n)|<\varepsilon$ for every $i,n\in\omega$. This yields (4). □

33 Lemma Under the assumptions of the theorem. Let $\bar{x} = \langle x_i \ :\ i < \omega\rangle$ where the x_i are copies of x. Then the function

$$\sigma(\bar{x}\,;z) \quad = \quad \inf_{i<\omega} \tau(x_i\,;z)$$

3ε-approximates f from below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x};z)$ does not ε-approximate f from below. We construct inductively a sequence $\langle a_i\,;b_i \ :\ i < \omega\rangle$ that contradicts the stability of $\tau(x;z)$. Then, let a_n be such that

1. $$\big|\tau(a_n\,;b)\ -\ f(b)\big| \quad \leq \quad \varepsilon \qquad\qquad \text{for every } b \in B \cup \{b_0,\ldots,b_n\}$$

then let b_n be such that

2. $$\min_{i=0,\ldots,n} \tau(a_i\,;b_n) \quad > \quad f(b_n) + 3\varepsilon.$$

We show that the sequence we constructed contradicts stability. In fact, by (2)

$$\tau(a_i\,;b_n)\ -\ \tau(a_n\,;b_i) \quad > \quad f(b_n) + 3\varepsilon - \tau(a_n\,;b_i).$$

Then, by (1)

$$\tau(a_i\,;b_n)\ -\ \tau(a_n\,;b_i) \quad > \quad f(b_n) + 3\varepsilon - f(b_i) - \varepsilon$$

As in the previous lemma, after refining the sequence we we obtain a contradiction. □

34 Lemma If $\tau(\bar{x};z)$ is stable, then $\sigma(\bar{x};z)$ in the previous lemma is stable.

Proof. It suffices to show that

$$\sigma(x_1,x_2\,;z) \quad = \quad \min_{i=1,2} \tau(x_i\,;z)$$

is stable. Suppose not. Pick $\langle a_{1,i}, a_{2,i}\,;b_i \ :\ i < \omega\rangle$ and $\varepsilon > 0$ such that for every $i < j < \omega$

$$\big|\sigma(a_{1,i}, a_{2,i}\,;b_j) - \sigma(a_{1,j}, a_{2,j}\,;b_i)\big| \quad \geq \quad \varepsilon$$

By Ramsey's theorem, we can refine the sequence so that one of the following holds for every $i < j$

$$\sigma(a_{1,i}, a_{2,i}\,;b_j) - \sigma(a_{1,j}, a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

$$-\sigma(a_{1,i}, a_{2,i}\,;b_j) + \sigma(a_{1,j}, a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

Assume the first – the second case is symmetric. Again by Ramsey's theorem, we can refine the sequence so that for every $i < j$ either $\tau(a_{1,i}\,;b_j) \leq \tau(a_{2,i}\,;b_j)$ or conversely $\tau(a_{1,i}\,;b_j) \geq \tau(a_{2,i}\,;b_j)$. In the first case we obtain

$$\tau(a_{1,i}\,;b_j) - \tau(a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

in the second case we obtain

$$\tau(a_{2,i}\,;b_j) - \tau(a_{1,j}\,;b_i) \quad \geq \quad \varepsilon.$$

In both cases we contradict the stability of $\tau(x;z)$. □

6. Attempt 1

Let $\varphi(x;z;X)$ be type-definable that is, definable by an infinite (small) conjunction of formulas in $\mathcal{F}_{\mathsf{X}}(\mathcal{U})$. The variables of sort H are partitioned in two tuples $x;z$ which may be infinite.

We say that $\varphi(x;z;X)$ is unstable if there is a sequence $\langle a_i;b_i \ :\ i<\omega\rangle$ such that for some $C\cap\tilde{C}=\varnothing$ and every $i<n<\omega$

$$\varphi(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;\tilde{C})$$

We say that $\varphi(x;z;X)$ is stable if it is not unstable.

20 Theorem (conjecture) The following are equivalent

1. $\varphi(x;z;X)$ is stable
2. there is no sequence $\langle a_i;b_i;C_i \ :\ i<\omega\rangle$ such that for some C, some neighboorhood C' of C, and every $i<n<\omega$
$$\varphi(a_n;b_i;C) \quad\wedge\quad \sim\varphi(a_i;b_n;C_n)$$
and $C'\cap C_n=\varnothing$.

If $\mathcal{D}\subseteq\mathcal{U}^z\times K(S)$ then $\sim\mathcal{D}$ is obtained by applying the definition in Section 2 to all fibers of $\mathcal{D}$.

21 Theorem (conjecture) Let $\varphi(x;z;X)$ be stable. Let $\mathcal{D}\subseteq\mathcal{U}^z\times K(S)^{|X|}$ be involutive and approximable by $\varphi(x;z;X)$. Then there are some $\langle a_{i,j} \ :\ i\le n,\ j<\omega\rangle$ such that for every $b\in\mathcal{U}^z$

$$\langle b,C\rangle\in\mathcal{D} \quad\Leftrightarrow\quad \bigvee_{i\le n}\ \bigwedge_{j<\omega}\ \varphi(a_{i,j};b;C)$$

Proof. □

We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if from below if in the definition above the implication in (2) holds for every $b\in\mathcal{U}^z$ and $C\in K(S)$.

22 Lemma Under the assumptions of the theorem. If $\mathcal{D}$ is approximable from below then there are some $\langle a_i \ :\ i\le n\rangle$ such that for every $b\in\mathcal{U}^z$

$$\bigvee_{i\le n}\ \varphi(a_i;b;C) \quad\Leftrightarrow\quad \langle b,C\rangle\in\mathcal{D}$$

Proof. The sequences $a_0,\dots,a_n$ is defined recursively together with an auxiliary sequence $b_0,\dots,b_{n-1}$. The elements in the sequence $a_0,\dots,a_n$ are choosen to obtain

1. $$\bigvee_{i=0}^{n}\ \varphi(a_i;b;C) \quad\Rightarrow\quad \langle b,C\rangle\in\mathcal{D} \qquad\text{for every } b\in\mathcal{U}^z \text{ and every } C$$

and

2. $$\langle b_i,C\rangle\in\mathcal{D} \quad\Rightarrow\quad \varphi(a_n;b_i;C) \qquad\text{for every } i<n \text{ and every } C.$$

This is possible because $\mathcal{D}$ is approximated from below. Now choose b_n and C_n such that

3. $\displaystyle\bigwedge_{i=0}^{n} \sim\varphi(a_i\,;b_n\,;C_n)$ and $\langle b_n, C_n\rangle \notin \sim\mathcal{D}$

If such b_n and C_n do not exist, then

$$\sim\bigvee_{i=0}^{n} \varphi(a_i\,;b\,;C) \quad\Rightarrow\quad \langle b, C\rangle \in \sim\mathcal{D} \qquad\qquad \text{for every } b \in \mathcal{U}^z \text{ and every } C$$

which is equivalent to the converse of (1) and proves the lemma. Suppose for a contradiction that the construction never ends. Then, from (2) we otain $\varphi(a_n\,;b_i\,;C)$ for every $i < n$ and from (3) we obtain $\sim\varphi(a_i\,;b_n\,;C_n)$, again for every $i < n$. This contradicts stability. □

23 Lemma Under the assumptions of the theorem. Let $\bar{x} = \langle x_m \;:\; i < \omega\rangle$ where the x_i are copies of x and m is the maximal . Then the formula

$$\sigma(\bar{x}\,;z\,;X) \quad=\quad \bigwedge_{i<m} \varphi(x_i\,;z\,;X)$$

approximate $\mathcal{D}$ from below.

Proof. Negate the claim and let B and C witness that $\sigma(\bar{x})$ does not approximate $\mathcal{D}$ from below. Suppose that $a_0,\ldots,a_{n-1}$ and $b_0,\ldots,b_{n-1}$ have been defined. Choose a_n such that

1. $\langle b, C\rangle \in \mathcal{D} \;\Rightarrow\; \varphi(a_n\,;b\,;C)$
2. $\langle b, C\rangle \in \sim\mathcal{D} \;\Rightarrow\; \sim\varphi(a_n\,;b\,;C)$ for every $b \in B \cup \{b_0,\ldots,b_{n-1}\}$.

Now, as the lemma is assume to fail, we can choose b_n such that

3. $\displaystyle\bigwedge_{i=0}^{n} \varphi(a_i\,;b_n\,;C)$ and $\langle b_n, C\rangle \notin \mathcal{D}$.

Moreover, as $\mathcal{D}$ is involutive, there is $C_n \cap C = \varnothing$ such that $\langle b_n, C_n\rangle \in \sim\mathcal{D}$.

We claim that the sequence we obtain contradicts stability. In fact, the construction guarantees that $\langle b_i, C_i\rangle \in \sim\mathcal{D}$ for every i, then from (2) we obtain that $\sim\varphi(a_n\,;b_i\,;C_i)$ for every $i < n$. On the other hand, by (1) we have that $\varphi(a_n\,;b_i\,;C)$ for every $i < n$. □

24 Lemma If $\varphi(x\,;z\,;C)$ is stable then $\sigma(\bar{x}\,;z\,;X)$ in the previous lemma is stable.

Proof. Assume $\sigma(x_1, x_2\,;z\,;X)$ in unstable. Let $\langle a_{1,i}, a_{2,i}\,;b_i\,;C_i \;:\; i < \omega\rangle$ be a sequence witnessing instability. then for every $i < n < \omega$ either $\sim\varphi(a_{1,n}\,;b_i\,;C_i)$ or $\sim\varphi(a_{2,n}\,;b_i\,;C_i)$

□

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