Local stability in structures with a standard sort

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ABSTRACT.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language $\mathcal{L}_{\mathsf{S}}$ that has a symbol for each compact subset $C \subseteq S$ and a function symbol for each continuous functions $f : S^n \to S$. According to the context, C and f denote either the symbols of $\mathcal{L}_{\mathsf{S}}$ or their interpretation in the structure S.

Finally, note that we could allow in $\mathcal{L}_{\mathsf{S}}$ relation symbols for all compact subsets of S^n, for any n. But this would clutter the notation adding very little to the theory. When needed, we will assume this generalization and leave the detail to the reader.

We also fix an arbitrary first-order language which we denote by $\mathcal{L}_{\mathsf{H}}$ and call the language of the home sort.

1 Definition Let $\mathcal{L}$ be a two sorted language. The two sorts are denoted by H and S. The language $\mathcal{L}$ expands $\mathcal{L}_{\mathsf{H}}$ and $\mathcal{L}_{\mathsf{S}}$ with symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$. An $\mathcal{L}$-structure is a structure of signature $\mathcal{L}$ that interprets these symbols in equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H).
A standard structure is a two-sorted $\mathcal{L}$-structure of the form $\langle M, S\rangle$, where M is any structure of signature $\mathcal{L}_{\mathsf{H}}$ and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by $\mathcal{F}$ the set of $\mathcal{L}$-formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives $\wedge$, $\vee$; the quantifiers $\forall^{\mathsf{H}}$, $\exists^{\mathsf{H}}$ of sort H; and the quantifiers $\forall^{\mathsf{S}}$, $\exists^{\mathsf{S}}$ of sort S.

We write $x \in C$ for the predicate associated to a compact set C, and denote by $|\text{-}|$ the length of a tuple.

2 Definition We call atomic formulas those of the form

i. atomic or negated atomic formulas of $\mathcal{L}_{\mathsf{H}}$;

ii. $\tau \in C$, where τ is a term of sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$, and $C \subseteq S$ is a compact set.

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3 Remark A smaller fragment of $\mathcal{F}$ may be of interest in some specific contexts. This is obtained by excluding equalities and inequalities from (i) and requiring that C in (ii) is a regular compact set. The discussion below applies to this smaller fragment as well.

Let $M \subseteq N$ be standard structures. We say that M is an $\mathcal{F}$-elementary substructure of N if the latter models all $\mathcal{F}(M)$-sentences that are true in M.

Let x and ξ be variables of sort H, respectively S. A standard structure M is $\mathcal{F}$-saturated if it realizes every type $p(x;\xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M. The following theorem is proved in [AAVV] for signatures that do not contain symbols sort $\mathsf{H}^n \times \mathsf{S}^m \to \mathsf{S}$ with $n{\cdot}m > 0$. A similar framework has been independently introduced in [PC] – only without quantifiers of sort H and with an approximate notion of satisfaction. (The setting in [PC] is a generalization of that used by Henson and Iovino for Banach spaces [HI].) By the elimination of quantifiers of sort S, proved in [AAVV, Proposition 3.6], the two approaches are equivalent – up to approximations.

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [AAVV] extends to the case $n{\cdot}m > 0$.

4 Theorem (Compactness) Every standard structure has a $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension.

We introduce variables of some new sorts X with the sole scope of conveniently describing classes of $\mathcal{F}$-formulas that only differ by the sets/predicates $C \subseteq S$ that occur. Let $\mathcal{F}_{\mathsf{X}}$ be defined as $\mathcal{F}$ but replacing (ii) with

iii. $\tau(x;\xi) \in X$, where X is a variable of sort X.

Formulas in $\mathcal{F}_{\mathsf{X}}$ are denoted by $\varphi(x;\xi;X)$, where $X = X_1,\dots,X_k$ be a tuple of variables sort X. If $C = C_1,\dots,C_k$ is a tuple of compact subsets of S then $\varphi(x;\xi;C)$, is a formula in $\mathcal{F}$. This we call an instance of $\varphi(x;\xi;X)$. All formulas in $\mathcal{F}$ are instances of formulas in $\mathcal{F}_{\mathsf{X}}$.

5 Definition Let $\varphi \in \mathcal{F}_{\mathsf{X}}$ be a formula – possibly with some (hidden) free variables. The pseudonegation of $\varphi \in \mathcal{F}_{\mathsf{X}}$ is the formula obtained by replacing in φ the atomic formulas in $\mathcal{L}_{\mathsf{H}}$ by their negation and every connective $\wedge$, $\vee$, $\forall$, $\exists$ by their respective duals $\vee$, $\wedge$, $\exists$, $\forall$.
The pseudonegation of φ is denoted by ${\sim}\varphi$. Clearly, when X does not occur in φ, we have ${\sim}\varphi \leftrightarrow \neg\varphi$.

The following fact is immediate and will be used without further mention.

6 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \varnothing$, every standard structure M, and every formula $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$

$$\begin{array}{lcl} M & \models & \varphi(C) \rightarrow \varphi(C') \\ M & \models & {\sim}\varphi(\tilde{C}) \rightarrow \neg\varphi(C). \end{array}$$

7 Fact For every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from C such that

$$M \models \varphi(C) \rightarrow \neg{\sim}\varphi(\tilde{C}) \rightarrow \varphi(C')$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$. Conversely, for every $\tilde{C} \cap C = \varnothing$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \rightarrow \varphi(C') \rightarrow \neg{\sim}\varphi(\tilde{C}).$$

for every M, and every $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = C' \smallsetminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds with as C' any neighborhood of C disjoint from $\tilde{C}$. Induction on the syntax of $\varphi(X)$ proves the general case. □

The above fact has the following useful consequence.

8 Fact Let $\varphi(X) \in \mathcal{F}_{\mathsf{X}}(M)$, where M is a standard structure. Then the following are equivalent for every tuple C

$$\begin{array}{lcl} M & \models & \varphi(C) \leftarrow \bigwedge\Big\{\varphi(C') \,:\, C' \text{ neighborhood of } C\Big\} \\ M & \models & \neg\varphi(C) \rightarrow \bigvee\Big\{{\sim}\varphi(\tilde{C}) \,:\, \tilde{C} \cap C = \varnothing\Big\}. \end{array}$$

(The converse implications are trivial – therefore not displayed.)

Note that the fact also holds if we restrict the above C' and $\tilde{C}$ to range over the subsets of some C'' that is a neighboorhood of C.

9 Fact The equivalent conditions in Fact 8 hold in all $\mathcal{F}$-saturated structures.

Proof. Prove the first of the two implications by induction on the syntax of $\varphi(X)$. The existential quantifier of sort H is the only connective that requires attention. Assume inductively that

$$\bigwedge\Big\{\varphi(a;C') \,:\, C' \text{ neighborhood of } C\Big\} \;\rightarrow\; \varphi(a;C)$$

holds for every $\varphi(a;C)$. Then induction for the existential quantifier follows from

$$\bigwedge\Big\{\exists x\,\varphi(x;C') \,:\, C' \text{ neighborhood of } C\Big\} \;\rightarrow\; \exists x \bigwedge\Big\{\varphi(x;C') \,:\, C' \text{ neighborhood of } C\Big\}$$

which is a consequence of saturation. □

We say that M is $\mathcal{F}$-maximal if it models all $\mathcal{F}(M)$-sentences that hold in some of its $\mathcal{F}$-elementary extensions. By the following fact $\mathcal{F}$-saturated structures are $\mathcal{F}$-maximal.

10 Fact Let $\varphi(X) \in \mathcal{F}_X(M)$. Then the following are equivalent

1. M is $\mathcal{F}$-maximal
2. the equivalent conditions in Fact 8 hold for every $\varphi(C)$.

Proof. 1⇒2. If (2) in Fact 8 fails, $\neg\varphi(C) \wedge \neg\sim\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \varnothing$. Let $\mathcal{U}$ be an $\mathcal{F}$-saturated $\mathcal{F}$-elementary extension of M. As M is $\mathcal{F}$-maximal, $\neg\varphi(C) \wedge \neg\sim\varphi(\tilde{C})$ holds also in $\mathcal{U}$. Then (2) in Fact 8 fails in $\mathcal{U}$, contradicting Fact 9.

2⇒1. Let N be any $\mathcal{F}$-elementary extension of M. Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C. Suppose not. Then, by (2) in Fact 8, $M \models \sim\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \varnothing$. Then $N \models \sim\varphi(\tilde{C})$ by $\mathcal{F}$-elementarity. As $\tilde{C} \cap C = \varnothing$, this contradicts Fact 9. □

In what follows we fix a large saturated standard structure which we denote by $\mathcal{U}$.

2. Pseudocomplement in $K(S)$

Write $K(S)$ for the set of compact subsets of S. For $\mathcal{D} \subseteq K(S)$ we define

$$\sim\mathcal{D} \quad = \quad \big\{\tilde{C} \in K(S) \,:\, \tilde{C} \cap C \neq \varnothing \text{ for every } C \in \mathcal{D}\big\}.$$

The following fact motivates the definition.

11 Fact If $\mathcal{D} = \{C : \varphi(C)\}$ then $\sim\mathcal{D} = \{\tilde{C} : \sim\varphi(\tilde{C})\}$

Proof. We need to prove that

$$\sim\varphi(\tilde{C}) \quad \Leftrightarrow \quad \text{for every } C, \ \text{ if } C \cap \tilde{C} = \varnothing \ \text{ then } \neg\varphi(C)$$

Implication ⇒ follows immediately from Fact 6. To prove ⇐ assume the r.h.s. Then $\neg\varphi(S \smallsetminus O)$ holds for every open set $O \supseteq \tilde{C}$. From the second implication in Fact 8 we obtain $\sim\varphi(\tilde{C}')$ for some $\tilde{C} \subseteq \tilde{C}' \subseteq O$. As O is arbitrary, we obtain $\varphi(\tilde{C})$ from the second implication in Fact 8. □

We prove a couple of straightforward inclusions.

12 Fact For every $\mathcal{D} \subseteq \mathcal{C} \subseteq K(S)$ we have $\sim\mathcal{C} \subseteq \sim\mathcal{D}$. Moreover, $\mathcal{D} \subseteq \sim\sim\mathcal{D}$.

Proof. Let $C \notin \sim\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in \mathcal{D}$. As $\tilde{C} \in \mathcal{C}$ we conclude that $C \notin \sim\mathcal{C}$. For the second claim, let $C \notin \sim\sim\mathcal{D}$. Then $\tilde{C} \cap C = \varnothing$ for some $\tilde{C} \in \sim\mathcal{D}$. Then $C \notin \mathcal{D}$ follows. □

13 Fact For every $\mathcal{D} \subseteq K(S)$ the following are equivalent

1. $\mathcal{D} = \sim\sim\mathcal{D}$.
2. $\mathcal{D} = \sim\mathcal{C}$ for some $\mathcal{C} \subseteq K(S)$.

Proof. Only 2⇒1 requires a proof. From the two claims in the above fact we obtain $\sim\sim\sim\mathcal{C} \subseteq \sim\mathcal{C}$. Then form (2) we obtain $\sim\sim\mathcal{D} \subseteq \mathcal{D}$, which suffices to prove (1). □

We say that $\mathcal{D}$ is regular if

$$C \in \mathcal{D} \quad \Leftrightarrow \quad C' \in \mathcal{D} \text{ for every neighborhood } C' \text{ of } C.$$

It is not difficult to see that $\mathcal{D}$ is regular if and only if it is closed in the Vietoris topology and includes all the supersets of its elements.

14 Theorem The following are equivalent

1. $\mathcal{D}$ is regular
2. $\mathcal{D} = \sim\sim\mathcal{D}$.

Proof. 1⇒2. It suffices to prove $\sim\sim\mathcal{D} \subseteq \mathcal{D}$. Let $C \in \sim\sim\mathcal{D}$. Let $O \supseteq C$ be open. Then $S \smallsetminus O \notin \sim\mathcal{D}$. Then $\tilde{C} \in \mathcal{D}$ for some $\tilde{C} \subseteq O$. By regularity (⇒) every $C' \supseteq O$ is in $\mathcal{D}$. As O is arbitrary, $C \in \mathcal{D}$ by regularity (⇐).

2⇒1. It suffices to show that $\sim\mathcal{D}$ is regular. Regularity (⇒) is ovious. To prove regularity (⇐), assume $C \notin \sim\mathcal{D}$. Then $C \cap \tilde{C}$ for some $\tilde{C} \in \mathcal{D}$. Let C' be a neighboorhood of C disjoint of C. Then the r.h.s. of the above equivalence fails for C'. □

3. Stable functions and definabilitry

Let $\varphi(x;z;X)$ be type-definable that is, definable by an infinite (small) conjunction of formulas in $\mathcal{F}_{\mathsf{X}}(\mathcal{U})$. The variables of sort H are partitioned in two tuples $x;z$ which may be infinite. We say that $\varphi(x;z;X)$ is unstable if there are some $C\cap\tilde{C} = \varnothing$ and a sequence $\langle a_i;b_i \ :\ i<\omega\rangle$ such that for every $i<j<\omega$

$$\varphi(a_i;b_j;C) \quad \wedge \quad \sim\varphi(a_j;b_i;\tilde{C})$$

We say that $\varphi(x;z;X)$ is stable if it is not unstable.

In this section we consider formulas of a particular form which is of some interest. We deal with the general case in the next section.

Assume $S = [0,1]$ throughout this section.

Let $\tau : \mathcal{U}^{x;z} \to S$ be a function definable by an infinite conjunction of formulas in $\mathcal{F}$. We say that $\tau(x;z)$ is stable if so is $\tau(x;z) \in X$. The following fact is a suggestive characterization of the stability of functions.

15 Fact Let $\tau(x;z)$ be as above. Then the following are equivalent

1. the formula $\tau(x;z) \in X$ is unstable
2. there is a sequence $\langle a_i; b_i : i < \omega \rangle$ such that
$$\lim_{i\to\infty}\lim_{j\to\infty} \tau(a_i; b_j) \quad \neq \quad \lim_{j\to\infty}\lim_{i\to\infty} \tau(a_i; b_j)$$
3. there is a sequence $\langle a_i; b_i : i < \omega \rangle$ and some $\varepsilon > 0$ such that for every $i < j < \omega$
$$\big|\tau(a_i; b_j) - \tau(a_j; b_i)\big| \quad \geq \quad \varepsilon.$$

Proof. 2⇔3 Clear.

1⇒2. Let $C \cap \tilde{C} = \varnothing$ and $\langle a_i; b_i : i < \omega \rangle$ be as given by (1). That is, $\tau(a_i; b_j) \in C$ and $\tau(a_j; b_i) \in \tilde{C}$ hold for every $i < j < \omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and $\tilde{C}$ contains the limit on the right – which therefore are distinct.

2⇒1. Let C and $\tilde{C}$ be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i; b_i : i < \omega \rangle$. □

We say that $f : \mathcal{U}^z \to S$ is ε-approximable by $\tau(x;z)$ if for every finite $B \subseteq \mathcal{U}^z$ and there is an $a \in \mathcal{U}^x$ such that

$$\big|f(b)\,, \tau(a;b)\big| \quad \leq \quad \varepsilon \qquad\qquad \text{for every } b \in B.$$

We say approximable for ε-approximable for every $\varepsilon > 0$.

16 Theorem Let $S = [0,1]$. Let $\tau(x;z)$ be a stable type-definable function. Let $f : \mathcal{U}^z \to S$ be approximable by $\tau(x;z)$. Then f is type-definable.

Proof. The following three lemmas show that if $\tau(x;z)$ is ε-approximable, then there are some $\langle a_{i,j} : i < n,\ j < \omega \rangle$ such that

$$\Big|f(b) - \max_{i=0,\ldots,n-1}\ \inf_{j<\omega}\ \tau(a_{i,j}; b)\Big| \quad \leq \quad 3\varepsilon \qquad\qquad \text{for every } b \in \mathcal{U}^z.$$

Then f is the limit of type-definable functions, hence type-definable. □

We say that $f : \mathcal{U}^{x;z} \to [0,1]$ is ε-approximable from below if the parameter $a \in \mathcal{U}^x$ that witnesses approximability satisfies $\tau(a;b) \leq f(b) + \varepsilon$ for every $b \in \mathcal{U}^z$.

17 Lemma Assume that the f in the theorem is ε-approximable from below. Then there are some $\langle a_i \ \ i < n \rangle$ such that
$$\Big|f(b) - \max_{i=0,\ldots,n-1} \tau(a_i; b)\Big| \quad \leq \quad \varepsilon \qquad\qquad \text{for every } b \in \mathcal{U}^z.$$

Proof. Negate the lemma. We construct inductively a sequence $\langle a_i; b_i : i < \omega \rangle$ that contradicts the stability of $\tau(x;z)$. Then, let b_n be such that (if $n = 0$ pick b_n arbitrarily)

1. $$\Big|f(b_n) - \max_{i=0,\ldots,n-1} \tau(a_i; b_n)\Big| \quad > \quad \varepsilon$$

By approximability from below, for some a_n we have

2. $\tau(a_n;b_i) \;-\; f(b_i) \quad \le \quad \varepsilon$ for every $i \le n$

3. $\tau(a_n;b) \quad \le \quad f(b)+\varepsilon$ for every $b \in \mathfrak{U}^z$.

We prove instability by showing that for every $i < n$

4. $\tau(a_n;b_i) \;-\; \tau(a_i;b_n) \quad > \quad \varepsilon$

Note that by virtue of (3)

$$\max_{i=0,\ldots,n-1} \tau(a_i;b_n) - f(b_n) \quad \le \quad \varepsilon$$

Then (1) ensures that

1′. $f(b_n) \;-\; \max_{i=0,\ldots,n-1} \tau(a_i;b_n) \quad > \quad \varepsilon$

Using (1′) we obtain

$$\tau(a_n;b_i) - \tau(a_i;b_n) \quad > \quad \tau(a_n;b_i) - f(b_n) + \varepsilon$$

then from (2) we obtain

$$> \quad f(b_i) - f(b_n) + 2\varepsilon$$

By refining the sequence we can assume that $|f(b_i) - f(b_n)| < \varepsilon$ for every $i, n \in \omega$. This yields (4). □

18 Lemma Under the assumptions of the theorem. Let $\bar{x} = \langle x_i \;:\; i < \omega\rangle$ where the x_i are copies of x. Then the function

$$\sigma(\bar{x};z) \quad = \quad \inf_{i<\omega} \tau(x_i;z)$$

3ε-approximates f from below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x};z)$ does not ε-approximate f from below. We construct inductively a sequence $\langle a_i;b_i \;:\; i < \omega\rangle$ that contradicts the stability of $\tau(x;z)$. Then, let a_n be such that

1. $\big|\tau(a_n;b) \;-\; f(b)\big| \quad \le \quad \varepsilon$ for every $b \in B \cup \{b_0,\ldots,b_n\}$

then let b_n be such that

2. $\min_{i=0,\ldots,n} \tau(a_i;b_n) \quad > \quad f(b_n) + 3\varepsilon.$

We show that the sequence we constructed contradicts stability. In fact, by (2)

$$\tau(a_i;b_n) \;-\; \tau(a_n;b_i) \quad > \quad f(b_n) + 3\varepsilon - \tau(a_n;b_i).$$

Then, by (1)

$$\tau(a_i;b_n) \;-\; \tau(a_n;b_i) \quad > \quad f(b_n) + 3\varepsilon - f(b_i) - \varepsilon$$

As in the previous lemma, after refining the sequence we we obtain a contradiction. □

19 Lemma If $\tau(\bar{x};z)$ is stable, then $\sigma(\bar{x};z)$ in the previous lemma is stable.

Proof. It suffices to show that

$$\sigma(x_1,x_2\,;z) \quad = \quad \min_{i=1,2}\,\tau(x_i\,;z)$$

is stable. Suppose not. Pick $\langle a_{1,i},a_{2,i}\,;b_i\ :\ i<\omega\rangle$ and $\varepsilon>0$ such that for every $i<j<\omega$

$$\left|\sigma(a_{1,i},a_{2,i}\,;b_j)-\sigma(a_{1,j},a_{2,j}\,;b_i)\right| \quad \geq \quad \varepsilon$$

By Ramsey's theorem, we can refine the sequence so that one of the following holds for every $i<j$

$$\sigma(a_{1,i},a_{2,i}\,;b_j)-\sigma(a_{1,j},a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

$$-\sigma(a_{1,i},a_{2,i}\,;b_j)+\sigma(a_{1,j},a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

Assume the first – the second case is symmetric. Again by Ramsey's theorem, we can refine the sequence so that for every $i<j$ either $\tau(a_{1,i}\,;b_j)\leq\tau(a_{2,i}\,;b_j)$ or conversely $\tau(a_{1,i}\,;b_j)\geq\tau(a_{2,i}\,;b_j)$. In the first case we obtain

$$\tau(a_{1,i}\,;b_j)-\tau(a_{2,j}\,;b_i) \quad \geq \quad \varepsilon$$

in the second case we obtain

$$\tau(a_{2,i}\,;b_j)-\tau(a_{1,j}\,;b_i) \quad \geq \quad \varepsilon.$$

In both cases we contradict the stability of $\tau(x\,;z)$. □

4. Tentative

If $\mathcal{D}\subseteq\mathcal{U}^z\times K(S)$ then $\sim\mathcal{D}$ is obtained by applying the definition in Section 2 to all fibers of $\mathcal{D}$. We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if for every finite $B\subseteq\mathcal{U}^z$ there is an $a\in\mathcal{U}^x$ such that

$$\langle b,C\rangle\in\mathcal{D} \quad\Leftrightarrow\quad \varphi(a;b;C) \qquad\qquad \text{for all } b\in B \text{ and } C\in K(S).$$

By Facts 11 and 12 the equivalence above can be split into two imlications

$$\langle b,C\rangle\in\mathcal{D} \quad\Rightarrow\quad \varphi(a;b;C) \quad \text{and}$$

$$\langle b,C\rangle\in{\sim}\mathcal{D} \quad\Rightarrow\quad {\sim}\varphi(a;b;C) \qquad\qquad \text{for all } b\in B \text{ and } C\in K(S).$$

Or, symmetrically

$$\varphi(a;b;C) \quad\Rightarrow\quad \langle b,C\rangle\in\mathcal{D} \quad \text{and}$$

$${\sim}\varphi(a;b;C) \quad\Rightarrow\quad \langle b,C\rangle\in{\sim}\mathcal{D} \qquad\qquad \text{for all } b\in B \text{ and } C\in K(S).$$

A global $\varphi(x;z;X)$-type is a maximally (finitely) consistent set of formulas that are instances of $\varphi(x;b;X)$ and/or ${\sim}\varphi(x;b;X)$ for some $b\in\mathcal{U}^{|z|}$.

20 Fact For every regular $\mathcal{D}\subseteq\mathcal{U}^z\times K(S)$ the following are equivalent

1. $\mathcal{D}$ is approximable by $\varphi(x;z;X)$
2. the following is a (consistent) global $\varphi(x;z;X)$-type

$$\begin{aligned} p(x) \quad = \quad & \big\{\ \varphi(x;b;C)\ :\ \langle b,C\rangle\in\mathcal{D}\big\}\ \cup \\ & \big\{{\sim}\varphi(x;b;C)\ :\ \langle b,C\rangle\in{\sim}\mathcal{D}\big\}. \end{aligned}$$

Proof. 2⇒1. The finite consistency of $p(x)$ immediately implies that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$.

1⇒2. Finite consistency of $p(x)$ follows (again, immediately) from approximability. We prove $p(x)$ is maximal. We need to consider two cases. First, assume for a contradiction that $\varphi(x;b;C)$ is consistent with $p(x)$ but $\langle b,C\rangle \notin \mathcal{D}$. Then consistency implies that $\langle b,\tilde{C}\rangle \notin {\sim}\mathcal{D}$ for every $\tilde{C}\cap C=\varnothing$. But implies that $C \in {\sim}{\sim}\mathcal{D}=\mathcal{D}$, a contradiction. The second case is when ${\sim}\varphi(x;b;C)$ is consistent with $p(x)$ but $\langle b,C\rangle \notin {\sim}\mathcal{D}$. Then $\langle b,\tilde{C}\rangle \notin \mathcal{D}$ for every $\tilde{C}\cap C=\varnothing$. Then $\langle b,\tilde{C}\rangle \in {\sim}\mathcal{D}$, a contradiction. □

21 Theorem (conjecture) Let $\varphi(x;z;X)$ be stable. Let $\mathcal{D}\subseteq \mathcal{U}^z\times K(S)$ be approximable by $\varphi(x;z;X)$. Let $C\cap\tilde{C}=\varnothing$ be given. Then there are some $\langle a_{i,j} \;:\; i,j<n\rangle$ such that for every $b\in\mathcal{U}^z$

$$\langle b,C\rangle \in \mathcal{D} \;\Rightarrow\; \bigwedge_{i=0}^{n}\bigvee_{j=0}^{n}\varphi(a_{i,j};b;C)$$

$$\langle b,\tilde{C}\rangle \in {\sim}\mathcal{D} \;\Rightarrow\; \bigvee_{i=0}^{n}\bigwedge_{j=0}^{n}{\sim}\varphi(a_{i,j};b;\tilde{C})$$

Proof. □

We say that $\mathcal{D}$ is approximable by $\varphi(x;z;X)$ if from below if in the definition above the implication in (2) holds for every $b\in\mathcal{U}^z$ and $C\in K(S)$.

22 Lemma (conjecture) Let $C\cap\tilde{C}=\varnothing$ be given. If $\mathcal{D}$ is approximable from below then there are some $\langle a_i \;:\; i<n\rangle$ such that for every $b\in\mathcal{U}^z$

$$\langle b,\tilde{C}\rangle \in \mathcal{D} \;\Rightarrow\; \bigvee_{i=0}^{n}\varphi(a_i;b;C)$$

$$\langle b,\tilde{C}\rangle \in {\sim}\mathcal{D} \;\Rightarrow\; \bigwedge_{i=0}^{n}{\sim}\varphi(a_i;b;\tilde{C}).$$

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