

Local stability in structures with a standard sort

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ABSTRACT.

1. Structures with a standard sort

Let S be some Hausdorff compact topological space. We associate to S a first order structure in a language \mathcal{L}_S that has a symbol for each compact subset $C \subseteq S$ and a function symbol for each continuous functions $f : S^n \rightarrow S$. According to the context, C and f denote either the symbols of \mathcal{L}_S or their interpretation in the structure S .

Finally, note that we could allow in \mathcal{L}_S relation symbols for all compact subsets of S^n , for any n . But this would clutter the notation adding very little to the theory. When needed, we leave the generalization to the reader.

We also fix an arbitrary first-order language which we denote by \mathcal{L}_H and call the language of the home sort.

1 Definition Let \mathcal{L} be a two sorted language. The two sorts are denoted by H and S . The language \mathcal{L} expands \mathcal{L}_H and \mathcal{L}_S with symbols sort $H^n \times S^m \rightarrow S$. An \mathcal{L} -structure is a structure of signature \mathcal{L} that interprets these symbols in equicontinuous functions (i.e. uniformly continuous w.r.t. the variables in H). A **standard structure** is a two-sorted \mathcal{L} -structure of the form $\langle M, S \rangle$, where M is any structure of signature \mathcal{L}_H and S is fixed. Standard structures are denoted by the domain of their home sort.

We denote by \mathcal{F} the set of \mathcal{L} -formulas constructed inductively from atomic formulas of the form (i) and (ii) below using Boolean connectives \wedge, \vee ; the quantifiers \forall^H, \exists^H of sort H ; and the quantifiers \forall^S, \exists^S of sort S .

We write $x \in C$ for the predicate associated to a compact set C , and denote by $|-|$ the length of a tuple.

2 Definition We call atomic formulas those of the form

- i. atomic or negated atomic formulas of \mathcal{L}_H ;
- ii. $\tau \in C$, where τ is a term of sort $H^n \times S^m \rightarrow S$, and $C \subseteq S$ is a compact set.

3 Remark A smaller fragment of \mathcal{F} may be of interest in some specific contexts. This is obtained by excluding equalities and inequalities from (i) and requiring that C in (ii) is a regular compact set. The discussion below applies to this smaller fragment as well.

Let $M \subseteq N$ be standard structures. We say that M is an \mathcal{F} -elementary substructure of N if the latter models all $\mathcal{F}(M)$ -sentences that are true in M .

Let x and ξ be variables of sort H , respectively S . A standard structure M is \mathcal{F} -saturated if it realizes every type $p(x; \xi) \subseteq \mathcal{F}(A)$, for any $A \subseteq M$ of cardinality smaller than $|M|$, that is finitely consistent in M . The following theorem is proved in [AAVV] for signatures that do not contain symbols sort $H^n \times S^m \rightarrow S$ with $n \cdot m > 0$. A similar framework has been independently introduced in [PC] – only without quantifiers of sort H and with an approximate notion of satisfaction. By elimination of quantifiers of sort S , cf. [AAVV, Proposition 3.6], the two approaches are equivalent (up to approximations).

In [Z] it is observed that, under the assumption of equicontinuity, the proof in [AAVV] extends to the case $n \cdot m \geq 0$.

4 Theorem (Compactness) Every standard structure has a \mathcal{F} -saturated \mathcal{F} -elementary extension.

We introduce variables of some new sorts X with the sole scope of conveniently describing classes of \mathcal{F} -formulas that only differ by the sets/predicates $C \subseteq S$ that occur. Let \mathcal{F}_X be defined as \mathcal{F} but replacing (ii) with

iii. $\tau(x; \xi) \in X$, where X is a variable of sort X .

Formulas in \mathcal{F}_X are denoted by $\varphi(x; \xi; X)$, where $X = X_1, \dots, X_k$ be a tuple of variables sort X . If $C = C_1, \dots, C_k$ is a tuple of compact subsets of S then $\varphi(x; \xi; C)$, is a formula in \mathcal{F} . This we call an **instance** of $\varphi(x; \xi; X)$. All formulas in \mathcal{F} are instances of formulas in \mathcal{F}_X .

5 Definition Let $\varphi \in \mathcal{F}_X$ be formula – possibly with some (hidden) free variables. The **pseudonegation** of $\varphi \in \mathcal{F}_X$ is the formula obtained by replacing in φ the atomic formulas in \mathcal{L}_H by their negation and every connective $\wedge, \vee, \forall, \exists$ by their respective duals $\vee, \wedge, \exists, \forall$.

The pseudonegation of φ is denoted by $\sim\varphi$. Clearly, when X does not occur in φ , we have $\sim\varphi \leftrightarrow \neg\varphi$.

The following fact is immediate and will be used without further mention.

6 Fact The following hold for every $C \subseteq C'$ and $\tilde{C} \cap C = \emptyset$, every standard structure M , and every formula $\varphi(X) \in \mathcal{F}_X(M)$

$$\begin{aligned} M &\models \varphi(C) \rightarrow \varphi(C') \\ M &\models \sim\varphi(C) \rightarrow \neg\varphi(\tilde{C}). \end{aligned}$$

7 Fact For every $C \subseteq C'$ there is a $\tilde{C} \subseteq C'$ disjoint from C such that

$$M \models \varphi(C) \rightarrow \neg\sim\varphi(\tilde{C}) \rightarrow \varphi(C')$$

for every M , and every $\varphi(X) \in \mathcal{F}_X(M)$. Conversely, for every $\tilde{C} \cap C = \emptyset$ there is a $C' \supseteq C$ such that

$$M \models \varphi(C) \rightarrow \varphi(C') \rightarrow \neg\sim\varphi(\tilde{C}).$$

for every M , and every $\varphi(X) \in \mathcal{F}_X(M)$.

Proof. When $\varphi(X)$ is the atomic formula $\tau \in X$, the first claim holds with $\tilde{C} = C' \setminus O$ where O is any open set $C \subseteq O \subseteq C'$. The second claim holds with as C' any neighborhood of C disjoint from \tilde{C} . Induction on the syntax of $\varphi(X)$ proves the general case. \square

The above fact has the following useful consequence.

8 Fact Let $\varphi(X) \in \mathcal{F}_X(M)$, where M is a standard structure. Then the following are equivalent for every tuple C

$$\begin{aligned} M &\models \varphi(C) \leftarrow \bigwedge \left\{ \varphi(C') : C' \text{ neighborhood of } C \right\} \\ M &\models \neg\varphi(C) \rightarrow \bigvee \left\{ \sim\varphi(\tilde{C}) : \tilde{C} \cap C = \emptyset \right\}. \end{aligned}$$

(The converse implications are trivial – therefore not displayed.)

Note that the fact also holds if we restrict the above C' and \tilde{C} to range over the subsets of some C'' that is a neighborhood of C .

9 Fact The equivalent conditions in Fact 8 hold in all \mathcal{F} -saturated structures.

Proof. By induction on the syntax of $\varphi(X)$. \square

We say that M is **\mathcal{F} -maximal** if it models all $\mathcal{F}(M)$ -sentences that hold in some of its \mathcal{F} -elementary extensions. By the following fact \mathcal{F} -saturated structures are \mathcal{F} -maximal.

10 Fact Let $\varphi(X) \in \mathcal{F}_X(M)$. Then the following are equivalent

1. M is \mathcal{F} -maximal
2. the equivalent conditions in Fact 8 hold for every $\varphi(C)$.

Proof. $1 \Rightarrow 2$. If (2) in Fact 8 fails, $\neg\varphi(C) \wedge \neg\sim\varphi(\tilde{C})$ holds in M for every $\tilde{C} \cap C = \emptyset$. Let \mathcal{U} be an \mathcal{F} -saturated \mathcal{F} -elementary extension of M . As M is \mathcal{F} -maximal, $\neg\varphi(C) \wedge \neg\sim\varphi(\tilde{C})$ holds also in \mathcal{U} . Then (2) in Fact 8 fails in \mathcal{U} , contradicting Fact 9.

$2 \Rightarrow 1$. Let N be any \mathcal{F} -elementary extension of M . Suppose $N \models \varphi(C)$. By (2), it suffices to prove that $M \models \varphi(C')$ for every C' neighborhood of C . Suppose not. Then, by (2) in Fact 8, $M \models \sim\varphi(\tilde{C})$ for some $\tilde{C} \cap C' = \emptyset$. Then $N \models \sim\varphi(\tilde{C})$ by \mathcal{F} -elementarity. As $\tilde{C} \cap C = \emptyset$, this contradicts Fact 9. \square

In what follows we fix a large saturated standard structure which we denote by \mathcal{U} .

2. Stable functions and definability

Let $\varphi(x; z; X)$ be **type-definable** that is, definable by an infinite (small) conjunction of formulas in $\mathcal{F}_X(\mathcal{U})$. The variables of sort H are partitioned in two tuples $x; z$ which may be infinite. We say that $\varphi(x; z; X)$ is **unstable** if there are some $C \cap \tilde{C} = \emptyset$ and a sequence $\langle a_i; b_i : i < \omega \rangle$ such that for every $i < j < \omega$

$$\varphi(a_i; b_j; C) \wedge \sim\varphi(a_j; b_i; \tilde{C})$$

We say that $\varphi(x; z; X)$ is **stable** if it is not unstable.

In this section we consider formulas of particular form which is of some interest. We deal with the general case in the next section.

Assume $S = [0, 1]$ throughout this section.

Let $\tau : \mathcal{U}^{x;z} \rightarrow S$ be a function definable by an infinite conjunction of formulas in \mathcal{F} . We say that $\tau(x; z)$ is **stable** if so is $\tau(x; z) \in X$. The following fact is a suggestive characterization of the stability of functions.

11 Fact Let $\tau(x; z)$ be as above. Then the following are equivalent

1. the formula $\tau(x; z) \in X$ is unstable
2. there is a sequence $\langle a_i; b_i : i < \omega \rangle$ such that

$$\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \tau(a_i; b_j) \neq \lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} \tau(a_i; b_j)$$

3. there is a sequence $\langle a_i; b_i : i < \omega \rangle$ and some $\varepsilon > 0$ such that for every $i < j < \omega$

$$|\tau(a_i; b_j) - \tau(a_j; b_i)| \geq \varepsilon.$$

Proof. $2 \Leftrightarrow 3$ Clear.

$1 \Rightarrow 2$. Let $C \cap \tilde{C} = \emptyset$ and $\langle a_i; b_i : i < \omega \rangle$ be as given by (1). That is, $\tau(a_i; b_j) \in C$ and $\tau(a_j; b_i) \in \tilde{C}$ hold for every $i < j < \omega$. We can restrict to a subsequence such that the two limits exist; C contains the limit on the left; and \tilde{C} contains the limit on the right – which therefore are distinct.

$2 \Rightarrow 1$. Let C and \tilde{C} be disjoint neighborhoods of the two limits in (2). Then (1) is witnessed by a tail of the sequence $\langle a_i; b_i : i < \omega \rangle$. \square

We say that $f : \mathcal{U}^z \rightarrow S$ is **ε -approximable** by $\tau(x; z)$ if for every finite $B \subseteq \mathcal{U}^z$ and there is an $a \in \mathcal{U}^x$ such that

$$|f(b), \tau(a; b)| \leq \varepsilon \quad \text{for every } b \in B.$$

We say **approximable** for ε -approximable for every $\varepsilon > 0$.

12 Theorem Let $S = [0, 1]$. Let $\tau(x; z)$ be a stable type-definable function. Let $f : \mathcal{U}^z \rightarrow S$ be approximable by $\tau(x; z)$. Then f is type-definable.

Proof. The following three lemmas show that if $\tau(x; z)$ is ε -approximable, then there are some $\langle a_{i,j} : i < n, j < \omega \rangle$ such that

$$|f(b) - \max_{i=0,\dots,n-1} \inf_{j < \omega} \tau(a_{i,j}; b)| \leq 3\varepsilon \quad \text{for every } b \in \mathcal{U}^z.$$

Then f is the limit of type-definable functions, hence type-definable. \square

We say that $f : \mathcal{U}^{x,z} \rightarrow [0, 1]$ is ε -approximable **from below** if the parameter $a \in \mathcal{U}^x$ that witnesses approximability satisfies $\tau(a; b) \leq f(b) + \varepsilon$ for every $b \in \mathcal{U}^z$.

13 Lemma Assume that the f in the theorem is ε -approximable from below. Then there are some $\langle a_i : i < n \rangle$ such that

$$|f(b) - \max_{i=0,\dots,n-1} \tau(a_i; b)| \leq \varepsilon \quad \text{for every } b \in \mathcal{U}^z.$$

Proof. Negate the lemma. We construct inductively a sequence $\langle a_i; b_i : i < \omega \rangle$ that contradicts the stability of $\tau(x; z)$. Then, let b_n be such that (if $n = 0$ pick b_n arbitrarily)

$$1. \quad |f(b_n) - \max_{i=0,\dots,n-1} \tau(a_i; b_n)| > \varepsilon$$

By approximability from below, for some a_n we have

$$2. \quad \tau(a_n; b_i) - f(b_i) \leq \varepsilon \quad \text{for every } i \leq n$$

$$3. \quad \tau(a_n; b) \leq f(b) + \varepsilon \quad \text{for every } b \in \mathcal{U}^z.$$

We prove instability by showing that for every $i < n$

$$4. \quad \tau(a_n; b_i) - \tau(a_i; b_n) > \varepsilon$$

Note that by virtue of (3)

$$\max_{i=0,\dots,n-1} \tau(a_i; b_n) - f(b_n) \leq \varepsilon$$

Then (1) ensures that

$$1'. \quad f(b_n) - \max_{i=0,\dots,n-1} \tau(a_i; b_n) > \varepsilon$$

Using (1') we obtain

$$\tau(a_n; b_i) - \tau(a_i; b_n) > \tau(a_n; b_i) - f(b_n) + \varepsilon$$

then from (2) we obtain

$$> f(b_i) - f(b_n) + 2\varepsilon$$

By refining the sequence we can assume that $|f(b_i) - f(b_n)| < \varepsilon$ for every $i, n \in \omega$. This yields (4). \square

14 Lemma Under the assumptions of the theorem. Let $\bar{x} = \langle x_i : i < \omega \rangle$ where the x_i are copies of x . Then the function

$$\sigma(\bar{x}; z) = \inf_{i < \omega} \tau(x_i; z)$$

3ε -approximates f from below.

Proof. Negate the claim and let B witness that $\sigma(\bar{x}; z)$ does not ε -approximate f from below. We construct inductively a sequence $\langle a_i; b_i : i < \omega \rangle$ that contradicts the stability of $\tau(x; z)$. Then, let a_n be such that

$$1. \quad |\tau(a_n; b) - f(b)| \leq \varepsilon \quad \text{for every } b \in B \cup \{b_0, \dots, b_n\}$$

then let b_n be such that

$$2. \quad \min_{i=0, \dots, n} \tau(a_i; b_n) > f(b_n) + 3\varepsilon.$$

We show that the sequence we constructed contradicts stability. In fact, by (2)

$$\tau(a_i; b_n) - \tau(a_n; b_i) > f(b_n) + 3\varepsilon - \tau(a_n; b_i).$$

Then, by (1)

$$\tau(a_i; b_n) - \tau(a_n; b_i) > f(b_n) + 3\varepsilon - f(b_i) - \varepsilon$$

As in the previous lemma, after refining the sequence we we obtain a contradiction. \square

15 Lemma If $\tau(\bar{x}; z)$ is stable, then $\sigma(\bar{x}; z)$ in the previous lemma is stable.

Proof. It suffices to show that

$$\sigma(x_1, x_2; z) = \min_{i=1,2} \tau(x_i; z)$$

is stable. Suppose not. Pick $\langle a_{1,i}, a_{2,i}; b_i : i < \omega \rangle$ and $\varepsilon > 0$ such that for every $i < j < \omega$

$$|\sigma(a_{1,i}, a_{2,i}; b_j) - \sigma(a_{1,j}, a_{2,j}; b_i)| \geq \varepsilon$$

By Ramsey's theorem, we can refine the sequence so that one of the following holds for every $i < j$

$$\sigma(a_{1,i}, a_{2,i}; b_j) - \sigma(a_{1,j}, a_{2,j}; b_i) \geq \varepsilon$$

$$-\sigma(a_{1,i}, a_{2,i}; b_j) + \sigma(a_{1,j}, a_{2,j}; b_i) \geq \varepsilon$$

Assume the first – the second case is symmetric. Again by Ramsey's theorem, we can refine the sequence so that for every $i < j$ either $\tau(a_{1,i}; b_j) \leq \tau(a_{2,i}; b_j)$ or conversely $\tau(a_{1,i}; b_j) \geq \tau(a_{2,i}; b_j)$. In the first case we obtain

$$\tau(a_{1,i}; b_j) - \tau(a_{2,j}; b_i) \geq \varepsilon$$

in the second case we obtain

$$\tau(a_{2,i}; b_j) - \tau(a_{1,j}; b_i) \geq \varepsilon.$$

In both cases we contradict the stability of $\tau(x; z)$. \square

3. Tentative

We write $K(S)$ for the set of compact subsets of S^n , for some n which we leave implicit in the context. For $\mathcal{D} \subseteq K(S)$ we define

$$\sim\mathcal{D} = \{\tilde{C} \in K(S) : \tilde{C} \cap C \neq \emptyset \text{ for every } C \in \mathcal{D}\}.$$

16 Fact If $\mathcal{D} = \{C : \varphi(C)\}$ then $\sim\mathcal{D} = \{\tilde{C} : \sim\varphi(\tilde{C})\}$

Proof. We need to prove that

$$\sim\varphi(\tilde{C}) \Leftrightarrow \text{for every } C, \text{ if } C \cap \tilde{C} = \emptyset \text{ then } \neg\varphi(C)$$

Implication \Rightarrow follows immediately from Fact 6. To prove \Leftarrow assume the r.h.s. Then $\neg\varphi(S \setminus O)$ holds for every open set $O \supseteq \tilde{C}$. From the second implication in Fact 8 we obtain $\sim\varphi(\tilde{C}')$ for some $\tilde{C} \subseteq \tilde{C}' \subseteq O$. As O is arbitrary, we obtain $\varphi(\tilde{C})$ from the second implication in Fact 8. \square

If $\mathcal{D} \subseteq \mathcal{U}^z \times K(S)$ then $\sim\mathcal{D}$ is obtained by applying the definition above to all fibers of \mathcal{D} . We say that \mathcal{D} is **approximable** by $\varphi(x; z; X)$ if for every finite $B \subseteq \mathcal{U}^z$ there is an $a \in \mathcal{U}^x$ such that

$$\langle b, C \rangle \in \mathcal{D} \rightarrow \varphi(a; b; C) \quad \text{and}$$

$$\langle b, C \rangle \in \sim\mathcal{D} \rightarrow \sim\varphi(a; b; C) \quad \text{for all } b \in B \text{ and } C \in K(S).$$

A **global $\varphi(x; z; X)$ -type** is a maximally (finitely) consistent set of formulas that are instances of $\varphi(x; b; X)$ and/or $\sim\varphi(x; b; X)$ for some $b \in \mathcal{U}^{|z|}$.

Note that we always have $\mathcal{D} \subseteq \sim\sim\mathcal{D}$. When $\mathcal{D} = \sim\sim\mathcal{D}$ we say that \mathcal{D} is **regular**.

17 Fact For every regular $\mathcal{D} \subseteq \mathcal{U}^z \times K(S)$ the following are equivalent

1. \mathcal{D} is approximable by $\varphi(x; z; X)$
2. the following is a (consistent) global $\varphi(x; z; X)$ -type

$$p(x) = \{ \varphi(x; b; C) : \langle b, C \rangle \in \mathcal{D} \} \cup \{ \sim\varphi(x; b; C) : \langle b, C \rangle \in \sim\mathcal{D} \}.$$

Proof. $2 \Rightarrow 1$. The finite consistency of $p(x)$ immediately implies that \mathcal{D} is approximable by $\varphi(x; z; X)$.

$1 \Rightarrow 2$. Finite consistency of $p(x)$ follows (again, immediately) from approximability. We prove $p(x)$ is maximal. We need to consider two cases. First, assume for a contradiction that $\varphi(x; b; C)$ is consistent with $p(x)$ but $\langle b, C \rangle \notin \mathcal{D}$. Then consistency implies that $\langle b, \tilde{C} \rangle \notin \sim\mathcal{D}$ for every $\tilde{C} \cap C = \emptyset$. But implies that $C \in \sim\sim\mathcal{D} = \mathcal{D}$, a contradiction. The second case is when $\sim\varphi(x; b; C)$ is consistent with $p(x)$ but $\langle b, C \rangle \notin \sim\mathcal{D}$. Then $\langle b, \tilde{C} \rangle \notin \mathcal{D}$ for every $\tilde{C} \cap C = \emptyset$. Then $\langle b, \tilde{C} \rangle \in \sim\mathcal{D}$, a contradiction. \square

18 Theorem (conjecture) Let $\varphi(x; z; X)$ be stable. Let $\mathcal{D} \subseteq \mathcal{U}^z \times K(S)$ be approximable by $\varphi(x; z; X)$. Then there are some $\langle a_{i,j} : i, j < n \rangle$ such that

$$\langle b, C \rangle \in \mathcal{D} \Leftrightarrow \bigwedge_{i=0}^n \bigvee_{j=0}^n \varphi(a_{i,j}; b; C)$$

for every $b \in \mathcal{U}^z$ and $C \in K(S)$.

Proof. □

We say that \mathcal{D} is approximable by $\varphi(x; z; X)$ if **from below** if in the definition above the implication in (2) holds for every $b \in \mathcal{U}^z$ and $C \in K(S)$.

19 Lemma If \mathcal{D} is approximable from below then there are some $\langle a_i : i < n \rangle$ such that

$$\langle b, C \rangle \in \mathcal{D} \Leftrightarrow \bigvee_{i=0}^n \varphi(a_i; b; C)$$

for every $b \in \mathcal{U}^z$ and $C \in K(S)$.

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