A Crèche Course in Model Theory

Preface

These are the notes of a course that I have been giving for a few years in Amsterdam and many more in Turin. A few chapters are missing and will be added soon. I keep the most recent version in

https://github.com/domenicozambella/creche

Below I list a few peculiarities of the exposition.

▷ Fraïssé limits are presented in a slightly general setting which accommodates more examples (and I'll add a few more). This is used e.g. to discuss saturation.

▷ Quantifier elimination for ACF and Hilbert's Nullstellensatz are presented with more details than usual (which may annoy some experts but hopefully help some students).

▷ The proof of the Omitting Types Theorem uses a model theoretic construction (different from the standard syntactic proof).

▷ Imaginaries and the eq-expansion are introduced from the (equivalent) dual perspective: the canonical name of a definable set is the set itself.

▷ Lascar and Kim-Pillay strong types are introduced in a slightly unconventional way. I am grateful to Krzysztof Krupiński for helping me squaring the circle with KP-types.

▷ Ramsey Theorem is derived from the existence of coheir sequences. Admittedly, this is not the shortest proof, but it may be interesting. I plan to add a proof of Hindman's theorem along the same lines.

Chapter 1

Preliminaries and notation

This chapter introduces the syntax and semantic of first order logic. We assume that the reader is not completely unfamiliar with first order logic.

1 Structures

A **(first order) language L** (also called **signature**) is a triple that consists of

- a set L_{fun} whose elements are called **function symbols**.
- a set L_{rel} whose elements are called **relation symbols**.
- a function that assign to every $f \in L_{\text{fun}}$ and $r \in L_{\text{rel}}$ non negative integers n_f and n_r that we call **arity** of the function, respectively relation, symbol. In words we say that f is a **n_f-ary** function symbol, similarly for r. We may say **constant** for 0-ary function symbol.

Waring: it is custom to use the symbol L to denote both the language and the set of formulas (to be defined below) associated to it. With $|L|$ we denote he cardinality of $L_{\text{fun}} \cup L_{\text{rel}} \cup \omega$.

A **(first order) structure** M of signature L (for short **L-structure**) consists of

- A set that we call the **domain** or **support** and denote with the same symbol M used for the whole structure
- A function that assign to every $f \in L_{\text{fun}}$ a total map $f^M : M^{n_f} \to M$ and to every $r \in L_{\text{rel}}$ a relation $r^M \subseteq M^{n_r}$. We call f^M and r^M the **interpretation** of f, respectively r in M.

Recall that by definition $M^0 = \{\varnothing\}$. Therefore the interpretation of a constant c is a function that maps the unique element of M^0 into an element of M. We will identify c^M with $c^M(\varnothing)$.

We may use the word **model** as a synonym of structure. But beware that in some context the word is used to denote some particular sort of structures.

If M is an L-structure and $A \subseteq M$ is any subset we write $L(A)$ the language obtained adding to L_{fun} the elements of A as constants. In this context the elements of A are called **parameters**. There is a canonical expansion of M to an $L(A)$-structure by setting $a^M = a$ for every $a \in A$.

1.1 Example The **language of additive groups** consists of the following function symbols

- a constant (that is, a function symbol of arity 0): 0
- a unary function symbol (that is, of arity 1): $-$
- a binary function symbol (that is, of arity 2): $+$

In the **language of multiplicative groups** the three symbols above are replaced with 1, $^{-1}$, and $\cdot$ respectively. Any group is a structure in either of these two signatures with the obvious interpretation. Needless to say, not all structures with these signature are groups.

The **language of (unitary) rings** contains all function symbols above with exception of $^{-1}$. The **language of ordered rings** also contains the binary relation symbol $<$. $\square$

The following is a less straightforward example. The reason behind the choice of the language of vector spaces will be clear in Section 4 below.

1.2 Example Let K be a field. The **language of vector spaces over K**, that here we denote by L_K, extends that of additive groups with a unary function symbols k for every $k \in K$.

Recall that a vector space over K is an abelian group M together with an action of K over M, that is, a function $\mu : K \times M \to M$. To view a vector space over K as an L_K-structure we interpret the group symbols in the obvious way and each $k \in K$ as the function $\mu(k, \text{-})$. $\square$

The languages in the examples above, with the exception of that of ordered rings, are **functional languages**, i.e. $L_{\text{rel}} = \varnothing$. In the following we consider two important examples of **relational languages** where $L_{\text{fun}} = \varnothing$.

1.3 Example The **language of strict orders** contains only the binary relation symbol $<$. The **language of graphs** also contains only a binary relation symbol (there is no canonical choice). In combinatorics a graph is a set of unordered pairs. To associate to a graph a first order structure we identify identify sets of unordered pairs with symmetric irreflexive relations. $\square$

2 Tuples

Let A be any set. A **tuple of elements of A** is a function $a : \alpha \to A$ for some ordinal α. We call α the **length** of a and denote it by $|a|$. If a is surjective, we call a an **enumeration** of A. Occasionally, we may say **sequence** for tuple – a word which we use also when the domain is not necessarily an ordinal (though, typically, an ordered set).

We write a_i for the i-th element of a, that is, the element $a(i)$, where $i < \alpha$. We may present a tuple with the notation $a = \langle a_i : i < \alpha \rangle$; when α is finite, we may also write $a = a_0, \ldots, a_{\alpha-1}$.

The set of tuples of elements of a length α is denoted by A^α. The set of tuples of length $< \alpha$ is denoted by $A^{<\alpha}$. For instance, $A^{<\omega}$ is the set of all finite tuples of elements of A. When α is finite we confuse A^α with α-th Cartesian power of A. In particular A^1 is confused with A.

If $a, b \in A^{|\alpha|}$ and h is a function defined on A, we write $h(a) = b$ for $h(a_i) = b_i$. We often confuse the pair $\langle a, b \rangle$ with the tuple of pairs $\langle a_i, b_i \rangle$. The context will resolve the ambiguity.

Note that there is a unique tuple of length 0, we call it the **empty tuple**. In fact, for all A, even when A is empty, $\varnothing$ is the unique element of A^0.

We will often concatenate tuples. If a and b are tuples we will write $\boldsymbol{a\,b}$ or $\boldsymbol{a,b}$ indifferently.

3 Terms

Fix a infinite set V whose elements we call **variables**. We use the letters x, y, z, ecc. to denote variables or tuple of variables. We rarely refer to V explicitly and we always assume it is large enough for our needs.

We fix a signature L for the whole section.

1.4 Definition *A **term** is a finite sequence of elements of $L_{\text{fun}} \cup V$ that are obtained with the following inductive procedure*

0. *every variable, intended as a tuple of length 1, is a term;*
i. *If $f \in L_{\text{fun}}$ and t is a tuple obtained concatenating n_f terms, then $f\,t$ is a term. By $f\,t$ we understand the tuple obtained prefixing t by f.*

*We will say **L-term** when we with to specify the language L.* □

Note that any constant f, intended as a tuple of length 1, is a term (by i, the term f is obtained concatenating $n_f = 0$ terms and prefixing by f). Terms that do not contain variables are called **closed terms**.

The intended meaning of, for instance, the term $+ + x\,y\,z$ is $(x+y)+z$. The first expression uses an **prefix notation** the second the **infix notation**. When convenient we (informally) use the infix notation and add parenthesis to improve legibility and avoid ambiguity.

The following lemma shows that the prefix notation allows to write unambiguous terms without the use of parenthesis.

1.5 Lemma (unique legibility of terms) *Let a be a sequence of terms. Suppose a can be obtained both by the concatenation of the terms $t_1, \ldots, t_n$ or by the concatenation of the terms $s_1, \ldots, s_m$. Then $n = m$ and $s_i = t_i$.*

Proof By induction on $|a|$. If $|a| = 0$ than $n = m = 0$ and there is nothing to prove. Suppose the claim holds for tuples of length k and let $a = a_1, \ldots, a_{k+1}$. Then a_1 is the first element of both t_1 and s_1. If a_1 is a variable, say x, then t_1 and s_1 are the term x and $n = m = 1$. Otherwise a_1 is a function symbol, say f, then $t_1 = f\,\bar{t}$ and $s_1 = f\,\bar{s}$ where $\bar{t}$ and $\bar{s}$ are obtained concatenating he terms $t'_1, \ldots, t'_p$ and $s'_1, \ldots, s'_p$. Now apply the induction hypothesis to $a_2, \ldots, a_{k+1}$ and to the terms $t'_1, \ldots, t'_p, t_2, \ldots, t_n$ and $s'_1, \ldots, s'_p, s_2, \ldots, s_m$. □

If $x = x_1, \ldots, x_n$ is a tuple of distinct variables and $s = s_1, \ldots, s_n$ is a tuple of terms, we write $\boldsymbol{t[x/s]}$ for the sequence obtained replacing x by s coordinatewise. Proving that $t[x/s]$ is indeed a term is a tedious and can safely skip.

If t is a term and $x_1, \ldots, x_n$ are (tuples of) variables we write $\boldsymbol{t(x_1, \ldots, x_n)}$ to declare that the variables occurring in t are among those that occur in $x_1, \ldots, x_n$. When a term has been presented as $t(x, y)$, we write $\boldsymbol{t(s, y)}$ for $t[x/s]$.

Finally we define the interpretation o a term in a structure M. We begin with closed terms. These are interpreted in 0-ary functions, i.e. an elements of the structure.

1.6 Definition *Let t be a closed $L(M)$ term. We write* $\boldsymbol{t^M}$ *for the* ***interpretation of t****. This is defined by induction of the syntax of t*

i. if $t = f\,\bar{t}$, where $f \in L_{\text{fun}}$ and $\bar{t}$ is a tuple obtained concatenation the terms $t_1, \ldots, t_{n_f}$, then $t^M = f^M(t_1^M, \ldots, t_{n_f}^M)$.

Note that in i we have used Lemma 1.5 in an essential way. In fact this ensures that the sequence $\bar{t}$ uniquely determines the terms $t_1, \ldots, t_{n_f}$. □

The inductive definition above is based on the case $n_f = 0$, i.e. for f a constant (or a parameter). When $t = c$, a constant, $\bar{t}$ is the empty tuple, then $t^M = c^M(\varnothing)$ which we abreviate with c^M. In particular, if $t = a$, a parameter, then $t^M = a^M = a$.

Now we generalize the interpretation to all (non necessarily closed) terms. If $t(x)$ is a term then we define $\boldsymbol{t^M(x) : M^{|x|} \to M}$ to be the function that maps $a \mapsto t(a)^M$.

4 Substructures

1.7 Definition *Fix a signature L and let M and N be two L-structures. We say that M is a* ***substructure*** *of N, and write* $\boldsymbol{M \subseteq N}$*, if*

1. *the domain of M is a subset of the domain of N*
2. *$f^M = f^N \restriction M^{n_f}$ for every $f \in L_{\text{fun}}$*
3. *$r^M = r^N \cap M^{n_r}$ for every $f \in L_{\text{rel}}$.*

Note that when f is a constant 2 becomes $f^M = f^N$, in particular the substructures of N contains at least all constants of N.

If a set $A \subseteq N$ is such that

1. $f^N[A^{n_f}] \subseteq A$ for every $f \in L_{\text{fun}}$

then there is a unique substructure $M \subseteq N$ with domain A, namely, the structure with the following interpretation

2. $f^M = f^N \restriction M^{n_f}$ (which is a good definition by the assumption on A);
3. $r^M = r^N \cap M^{n_r}$.

It is usual to confuse subsets of N that satisfy 1 with the unuque substructure they support.

It is immediate to verify that the intersection of an arbitrary family of substructures of N is a substructure of N. Therefore, for any given $A \subseteq N$ we may define the **substructure of N generated A** as the intersection of all substructures of N that contain A. We write $\boldsymbol{\langle A \rangle_N}$. The following easy proposition $\langle A \rangle_N$ gives more concrete representation of $\langle A \rangle_N$

1.8 Lemma *The following hold for every $A \subseteq N$*

1 $\qquad \langle A \rangle_N \quad = \quad \Big\{ t^N \ : \ t \text{ a closed } L(A)\text{-term} \Big\}$

2 $\quad \langle A \rangle_N \;=\; \Big\{ t^N(a) \;:\; t(x) \text{ an } L\text{-term and } a \in A^{|x|} \Big\}$

3 $\quad \langle A \rangle_N \;=\; \bigcup_{n\in\omega} A_n, \quad \text{where} \quad A_0 \;=\; A$

$$A_{n+1} \;=\; A_n \,\cup\, \Big\{ f^N(a) \;:\; f \in L_{\mathrm{fun}},\; a \in A_n^{n_f} \Big\}. \qquad \square$$

1.9 Example Let L be the multiplicative (or additive) language of groups. Let N be a group, which we consider as an L-structure in the natural way. Then the substructures of N are exactly the subgroups of N and $\langle A \rangle_N$ is the group generated by $A \subseteq N$. A similar claim is true when L is the signature of vector spaces and N is a vector space. The choice of the language is more or less fixed if we want that the algebraic and the model theoretic notion of substructure coincide. $\square$

5 Formulas

Fix a language L and a set of variables V. A **formula** is a finite sequence of symbols in $L_{\mathrm{fun}} \cup L_{\mathrm{rel}} \cup V \cup \{\doteq, \perp, \neg, \vee, \exists\}$. The last set contains the logical symbols that are called respectively

$\doteq$	**equality**;	$\perp$	**contradiction**;	$\neg$	**negation**;
$\vee$	**disjunction**;	$\exists$	**existential quantifier**.		

Syntactically, $\doteq$ behaves like a binary relation symbol. So, for convenience set $n_{\doteq} = 2$. However $\doteq$ is considered as a logic symbol because its semantic is fixed (it is always interpreted in the diagonal).

The definition below uses the prefix notation which simplifies the proof of the unique legibility lemma. However, in practice we always we use the infix notation: $t \doteq s$, $\varphi \vee \psi$, etc.

1.10 Definition *A **formula** is any finite sequence is obtained with the following inductive procedure*

0. *if $r \in L_{\mathrm{rel}} \cup \{=\}$ and t is a tuple obtained concatenating n_r terms then $r\,t$ is a formula. Formulas of this form are called **atomic**;*

i. *if φ e ψ are formulas then the following are formulas: $\perp$, $\neg\,\varphi$, $\vee\,\varphi\,\psi$, and $\exists x\,\varphi$, for any $x \in V$.* $\square$

We use $\boldsymbol{L}$ to denote both the language and the set of formulas. We write $\boldsymbol{L}_{\mathbf{at}}$ for the set of atomic formulas and $\boldsymbol{L}_{\mathbf{qf}}$ for the set of **quantifier-free formulas** i.e., formulas where $\exists$ does not occur.

The proof of the following is similar to the analogous lemma for terms.

1.11 Lemma (unique legibility of formulas) *Let a be a sequence of formulas. Suppose a can be obtained both by the concatenation of the formulas $\varphi_1, \dots, \varphi_n$ or by the concatenation of the formulas $\psi_1, \dots, \psi_m$. Then $n = m$ and $\varphi_i = \psi_i$.* $\square$

A formula is **closed** if all its variables occur under the scope of a quantifier. Closed formulas are also called **sentences**. We will do without a formal definition of *occurs* *under the scope of a quantifier* which is too lengthy. An example suffices: all occurrences of x are under the scope a quantifiers in the formula $\exists x\, \varphi$. These occurrences are called **bonded**. The formula $x \doteq y \wedge \exists x\, \varphi$ has **free** (i.e., not bond) occurrences of x and y.

Let x is a tuple of variables and t is a tuple of terms such that $|x| = |t|$. We write $\varphi[x/t]$ for the formula obtained substituting t for all free occurrences of x, coordinatewise.

We write $\varphi(x)$ to declare that the free variables in the formula φ are all among those of the tuple x. In this case we write $\varphi(t)$ for $\varphi[x/t]$.

We will often use without explicit mention the following useful syntactic decomposition of formulas with parameters.

1.12 Lemma *For every formula $\varphi(x) \in L(A)$ there is a formula $\psi(x\,;z) \in L$ and a tuple of parameters $a \in A^{|z|}$ such that $\varphi(x) = \psi(x\,;a)$.* □

Just as a term $t(x)$ is a name for a function $t(x)^M : M^{|x|} \to M$, a formula $\varphi(x)$ is a name for a subset $\boldsymbol{\varphi(x)^M} \subseteq M^{|x|}$ which we call **the subset of $\boldsymbol{M}$ defined by $\boldsymbol{\varphi(x)}$**. It is also very common to write $\boldsymbol{\varphi(M)}$ for the set defined by $\varphi(x)$. In general sets of the form $\varphi(M)$ for some $\varphi(x) \in M$ are called **definable**.

1.13 Definition of truth *For every formua φ with variables amon thos of the tuple x we define $\boldsymbol{\varphi(x)^M}$ by induction as follows*

o1. $\quad (\doteq t\, s)(x)^M = \Big\{a \in M^{|x|} \;:\; t^M(a) = s^M(a)\Big\}$

o2. $\quad (r\, t_1 \ldots t_n)(x)^M = \Big\{a \in M^{|x|} \;:\; \langle t_1^M(a), \ldots, t_n^M(a)\rangle \in r^M\Big\}$

i0. $\quad \bot(x)^M = \varnothing$

i1. $\quad (\neg\xi)(x)^M = M^{|x|} \smallsetminus \xi(x)^M$

i2. $\quad (\vee\, \xi\, \psi)(x)^M = \xi(x)^M \cup \psi(x)^M$

i3. $\quad (\exists y\, \varphi)(x)^M = \bigcup\limits_{a \in M} (\varphi[y/a])(x)^M$

Condition i2 assumes that ξ and ψ are uniquely determined by $\vee\, \xi\, \psi$. This is a guaranteed by the unique legibility o formulas, Lemma 1.11. Analogously, o1 e o2 assume Lemma 1.5.

The case when x is the empty tuple is far from trivial. Note that $\varphi(\varnothing)^M$ is a subset of $M^0 = \{\varnothing\}$. Then there are two possibilities either $\{\varnothing\}$ or $\varnothing$. We wil read them as two **truth values**: **True** and **False**, respectively. If $\varphi^M = \{\varnothing\}$ we say that **$\boldsymbol{\varphi}$ is true in $\boldsymbol{M}$**, if $\varphi^M = \varnothing$., we say that **$\boldsymbol{\varphi}$ is false $\boldsymbol{M}$**. We write $\boldsymbol{M \vDash \varphi}$, respectively $\boldsymbol{M \nvDash \varphi}$. Or we may say that **$\boldsymbol{M}$ models $\boldsymbol{\varphi}$**, respectively **$\boldsymbol{M}$ does not model $\boldsymbol{\varphi}$**. It is immediate to verify at

$$\varphi(M) = \big\{a \in M^{|x|} : M \vDash \varphi(a)\big\}.$$

6 Yet more notation

The following logical connectives are defined as abbreviations:

$\top$	stands for	$\neg\bot$	**tautology**
$\varphi \wedge \psi$	stands for	$\neg[\neg\varphi \vee \neg\psi]$	**conjunction**
$\varphi \rightarrow \psi$	stands for	$\neg\varphi \vee \psi$	**implication**
$\varphi \leftrightarrow \psi$	stands for	$[\varphi \rightarrow \psi] \wedge [\psi \rightarrow \varphi]$	**bi-implication**
$\varphi \not\leftrightarrow \psi$	stands for	$\neg[\varphi \leftrightarrow \psi]$	**exclusive disjunction**
$\forall x \varphi$	stands for	$\neg\exists x \neg \varphi$	**universal quantifier**

We agree that $\rightarrow$ e $\leftrightarrow$ bind less than $\wedge$ e $\vee$. Unary connectives (quantifiers and negation) bind stronger then binary connectives. For example

$$\exists x\, \varphi \wedge \psi \;\rightarrow\; \neg\xi \vee \vartheta \quad \text{reads as} \quad \Big[[\exists x\, \varphi] \wedge \psi\Big] \;\rightarrow\; \Big[[\neg\xi] \vee \vartheta\Big]$$

We say that $\forall x\, \varphi(x)$ is the **universal closure** of $\varphi(x)$ and that $\exists x\, \varphi(x)$ is the **existential closure**. We say that $\varphi(x)$ **holds in M** when its universal closure is true in M. We say that $\varphi(x)$ is **consistent in M** when its existential closure is true in M.

The semantic of conjunction and disjunction is associative. Then for any finite set of formulas $\{\varphi_i : i \in I\}$ we can write without ambiguities

$$\bigwedge_{i\in I} \varphi_i \qquad\qquad \bigvee_{i\in I} \varphi_i$$

When $x = x_1, \ldots, x_n$ is a tuple of variables we write $\exists x \varphi$ or $\exists x_1, \ldots, x_n \varphi$ for $\exists x_1 \ldots \exists x_n\, \varphi$. With first order sentences we are able to say that $\varphi(M)$ has at least n elements (also, no more than, or exactly n). It is convenient to use the following abbreviations.

$$\exists^{\geq n} x\, \varphi(x) \quad \text{stands for} \quad \exists x_1, \ldots, x_n \Big[\bigwedge_{1\leq i\leq n} \varphi(x_i) \;\wedge \bigwedge_{1\leq i<j\leq n} x_i \dot{\neq} x_j\Big].$$

$$\exists^{\leq n} x\, \varphi(x) \quad \text{stands for} \quad \neg\exists^{\geq n+1} x\, \varphi(x)$$

$$\exists^{= n} x\, \varphi(x) \quad \text{stands for} \quad \exists^{\geq n} x\, \varphi(x) \;\wedge\; \exists^{\leq n} x\, \varphi(x)$$

1.14 Exercise Let M be an L-structure and let $\psi(x), \varphi(x, y) \in L$. Write a sentence true in M exactly when

a. $\psi(M) \in \{\varphi(a, M) : a \in M\}$;

b. $\{\varphi(a, M) : a \in M\}$ contains at least two sets;

c. $\{\varphi(a, M) : a \in M\}$ contains only sets that are pairwise disjoint. □

1.15 Exercise Let M be a structure in the signature of graphs as defined in Example 1.3 (but not necessarily a graph). Write a sentence φ such that,

a. $M \vDash \varphi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq A \times \neg A$.

Remark: φ assert an asymmetric version of the property below

b. $M \vDash \psi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq (A \times \neg A) \cup (\neg A \times A)$.

Assume M is a graph, what required in b is equivalent to saying that M is a *bipartite garaph*, or equivalently that it has *chromatic number* 2 i.e., we can color the vertices with 2 colors so that no two adjacent vertices share the same color. $\square$

Chapter 2

Theories and elementarity

1 Logical consequences

A **theory** is a set $T \subseteq L$ of sentences. We write $\boldsymbol{M \vDash T}$ if $M \vDash \varphi$ for every $\varphi \in T$. If $\varphi \in L$ is a sentence we write $\boldsymbol{T \vdash \varphi}$ when

$$M \vDash T \quad \Rightarrow \quad M \vDash \varphi \qquad \text{for every } M.$$

In words, we say that φ is a **logical consequence** of T or that φ **follows from** T. If S is a theory $\boldsymbol{T \vdash S}$ has a similar meaning. If $T \vdash S$ and $S \vdash T$ we say that T and S are **logically equivalent**. We may say that T **axiomatizes** S (or vice versa).

We say that a theory is **consistent** if it has a model. With the notation above, T is consistent if and only if $\boldsymbol{T \nvdash \bot}$.

The **closure of T under logical consequence** is the set $\mathrm{ccl}(T)$ which is defined as follows:

$$\mathbf{ccl}(\boldsymbol{T}) \quad = \quad \Big\{\varphi \in L \ : \ \text{sentence such that } T \vdash \varphi\Big\}$$

If T is a finite set, say $T = \{\varphi_1, \dots, \varphi_n\}$ we write $\mathrm{ccl}(\varphi_1, \dots, \varphi_n)$ for $\mathrm{ccl}(T)$. If $T = \mathrm{ccl}(T)$ we say that T is **closed under logical consequences**.

The **theory of M** is the set of sentences that hold in M and is denoted by $\mathbf{Th}(\boldsymbol{M})$. More generally, if $\mathcal{K}$ is a class of structures, $\mathbf{Th}(\mathcal{K})$ is the set of sentences that hold in every model in $\mathcal{K}$. That is

$$\mathrm{Th}(\mathcal{K}) \quad = \quad \bigcap_{M \in \mathcal{K}} \mathrm{Th}(M)$$

The class of all models of T is denoted by $\mathbf{Mod}(\boldsymbol{T})$. We say that $\mathcal{K}$ is **axiomatizable** if $\mathrm{Mod}(T) = \mathcal{K}$ for some theory T. If T is finite we say that $\mathcal{K}$ is **finitely axiomatizable**. To sum up

$$\begin{aligned}
\mathrm{Th}(M) \quad &= \quad \Big\{\varphi \ : \ M \vDash \varphi\Big\} \\
\mathrm{Th}(\mathcal{K}) \quad &= \quad \Big\{\varphi \ : \ M \vDash \varphi \text{ for all } M \in \mathcal{K}\Big\} \\
\mathrm{Mod}(T) \quad &= \quad \Big\{M \ : \ M \vDash T\Big\}
\end{aligned}$$

2.1 Example Let L be the language of multiplicative groups. Let T_{g} be the set containing the universal closure of following three formulas

ass. $\quad (x \cdot y) \cdot z \ = \ x \cdot (y \cdot z);$

inv. $\quad x \cdot x^{-1} \ = \ x^{-1} \cdot x \ = \ 1;$

neu. $\quad x \cdot 1 \ = \ 1 \cdot x \ = x.$

Then T_{g} axiomatizes the theory of groups, i.e. $\mathrm{Th}(\mathcal{K})$ for $\mathcal{K}$ the class of all groups. Let φ be the universal closure of the following formula can. $\quad z \cdot x \ = \ z \cdot y \ \rightarrow \ x \ = \ y.$

As φ formalizes the cancellation property then $T_{\mathrm{g}} \vdash \varphi$, that is, φ is a logical consequence of T_{g}. Now consider the sentence ψ which is the universal closure of

com. $\quad x \cdot y \ = \ y \cdot x.$

So, commutative groups model ψ and non commutative groups model $\neg\psi$. Hence neither $T_{\mathrm{g}} \vdash \psi$ nor $T_{\mathrm{g}} \vdash \neg\psi$. We say that T_{g} **does not decide** ψ. □

Note that even when T is a very concrete set, $\mathrm{ccl}(T)$ may be more difficult to grasp. In the example above T_{g} contains three sentences but $\mathrm{ccl}(T_{\mathrm{g}})$ is an infinite set containing sentences that code theorems of group theory yet to be proved.

2.2 Remark The following properties say that ccl is a finitary closure operator.

1. $T \subseteq \mathrm{ccl}(T)$ (extensive)
2. $\mathrm{ccl}(T) \ = \ \mathrm{ccl}\big(\mathrm{ccl}(T)\big)$ (increasing)
3. $T \subseteq S \ \ \Rightarrow \ \ \mathrm{ccl}(T) \subseteq \mathrm{ccl}(S)$ (idempotent)
4. $\mathrm{ccl}(T) \ = \ \bigcup\big\{\mathrm{ccl}(S) \ : \ S \text{ finite subset of } T\big\}.$ (finitary)

Properties 1-3 are easy to verify while 4 requires the compactness theorem. □

In the next example we list a few algebraic theories with straightforward axiomatization.

2.3 Example We write T_{ag} for the theory of abelian groups which contains the universal closure of following

a1. $(x+y)+z \ = \ y+(x+z);$

a2. $x+(-x) \ = \ 0;$

a3. $x+0 \ = \ x;$

a4. $x+y \ = \ y+x.$

The theory T_{r} of (unitary) rings extends T_{ag} with

a5. $(x \cdot y) \cdot z \ = \ x \cdot (y \cdot z);$

a6. $1 \cdot x \ = \ x \cdot 1 \ = \ x;$

a7. $(x+y) \cdot z \ = \ x \cdot z + y \cdot z;$

a8. $z \cdot (x+y) \ = \ z \cdot x + z \cdot y.$

The theory of commutative rings T_{cg} contains also com of examples 2.1. The theory of ordered rings T_{or} extends T_{cr} with

o1. $x < z \ \rightarrow \ x+y < z+y;$

o2. $0 < x \wedge 0 < z \ \rightarrow \ 0 < x \cdot z.$ □

The axiomatization of vector spaces is less straightforward.

2.4 Example Fix a field K. The language L_K extends the language of additive groups with a unary function for every element of K. The theory of vector fields over K extends T_{ag} with the following axioms (for all $h, k, l \in K$)

m1. $h\,(x+y) \;=\; h\,x+h\,y$

m2. $l\,x \;=\; h\,x+k\,x$, where $l \;=\; h+_K k$

m3. $l\,x \;=\; h\,(k\,x)$, where $l \;=\; h\cdot_K k$

m4. $0_K\,x \;=\; 0$

m5. $1_K\,x \;=\; x$

The symbols 0_K and 1_K denote the zero and the unit of K. The symbols $+_K$ and $\cdot_K$ denote the sum and the product in K. These are not part of L_K, they are symbols we use in the metalanguage. □

2.5 Example Recall from Example 1.3 that we represent a graph with a symmetric irreflexive relation. Therefore **theory of graphs** contains the following two axioms

1. $\neg r(x,x)$;
2. $r(x,y) \to r(y,x)$. □

Our last example is a trivial one.

2.6 Example Let L be the empty language The **theory of infinite sets** is axiomatized by the sentences $\exists^{\geq n}x\ (x=x)$ for all positive integer n. □

2.7 Exercise Prove that $\mathrm{ccl}(\varphi\vee\psi)=\mathrm{ccl}(\varphi)\cap\mathrm{ccl}(\psi)$. □

2.8 Exercise Prove that $T\cup\{\varphi\}\vdash\psi$ then $T\vdash\varphi\to\psi$. □

2.9 Exercise Prove that $\mathrm{Th}(\mathrm{Mod}(T))=\mathrm{ccl}(T)$. □

2 Elementary equivalence

The following is a fundamental notion in model theory.

2.10 Definition *We say that M and N are **elementarily equivalent** if*

ee. $N\vDash\varphi \;\Leftrightarrow\; M\vDash\varphi$, *for every* $\varphi\in L$.

In this case we write $\boldsymbol{M\equiv N}$. *More generally, we write* $\boldsymbol{M\equiv_A N}$ *and say that M and N are elementarily equivalent **over A** if the following hold*

a. $A\subseteq M\cap N$

ee'. equivalence ee above holds for every sentence $\varphi\in L(A)$. □

The case when A is the whole domain of M is particularly important.

2.11 Definition *When $M\equiv_M N$ we write $\boldsymbol{M\preceq N}$ and say that M is an **elementary substructure** of N.* □

In the definition above the use of the term *substructure* is appropriate by the following lemma.

2.12 Lemma *If M and N are such that $M \equiv_A N$ and A is the domain of a substructure of M then A is also the domain a substructure of N and the two substructures coincide.*

Proof Let f be a function symbol and ler r be a relation symbol. It suffices to prove that $f^M(a) = f^N(a)$ for every $a \in A^{n_f}$ and that $r^M \cap A^{n_r} = r^N \cap A^{n_r}$.

If $b \in A$ is such that $b = f^M a$ then $M \vDash fa = b$. So, from $M \equiv_A N$, we obtain $N \vDash fa = b$, hence $f^N a = b$. This proves $f^M(a) = f^N(a)$.

Now let $a \in A^{n_r}$ and suppose $a \in r^M$. Then $M \vDash ra$ and, by elementarity, $N \vDash ra$, hence $a \in r^N$. By symmetry $r^M \cap A^{n_r} = r^N \cap A^{n_r}$ follows. □

It is not easy to prove that two structures are elementary equivalent. A direct verification is unfeasible even for the most simple structures. It will take a few chapters before we are able to discuss concrete examples.

We generalize the definition of $\mathrm{Th}(M)$ to include parameters

$$\mathbf{Th}(\boldsymbol{M/A}) \quad = \quad \Big\{\varphi \ : \ \text{ sentence in } L(A) \text{ such that } \ M \vDash \varphi\Big\}.$$

The following proposition is immediate

2.13 Proposition *For every pair of structures M and N and every $A \subseteq N \cap N$ the following are equivalent*

a. $M \equiv_A N$;

b. $Th(M/A) = Th(N/A)$;

c. $N \vDash \varphi(a) \quad \Leftrightarrow \quad M \vDash \varphi(a) \qquad$ *for every $\varphi(x) \in L$ and every $a \in A^{|x|}$.*

d. $\varphi(M) \cap A^{|x|} \ = \ \varphi(N) \cap A^{|x|}$ *for every $\varphi(x) \in L$.* □

If we restate a and c of the proposition above when $A = M$ we obtain that the following are equivalent

a'. $M \preceq N$;

c'. $\varphi(M) \ = \ \varphi(N) \cap M^{|x|} \qquad$ for every $\varphi(x) \in L$.

Note that c' extends to all definable sets what Definition 1.7 requires for a few basic definable sets.

2.14 Example Let G be a group which we consider as a structure in the multiplicative language of groups. We show that if G is simple and $H \preceq G$ then also H is simple. Recall that G is simple if all its normal subgroups are trivial, equivalently, if for every $a \in G \smallsetminus \{1\}$ the set $\{gag^{-1} \ : \ g \in G\}$ generates the whole group G.

Assume H is not simple then there $a, b \in H$ be such that b is not product of elements of $\{hah^{-1} \ : \ h \in H\}$. hence for every n

$$H \ \vDash \ \neg\exists x_1, \ldots, x_n \ \ (b \ \ = \ \ x_1 a x_1^{-1} \cdots \ x_n a x_n^{-1})$$

By elementarity the same hold in G. Hence G is not simple. □

2.15 Exercise Let $A \subseteq M \cap N$. Prove that $M \equiv_A N$ if and only if $M \equiv_B N$ for every finite $B \subseteq A$. □

2.16 Exercise Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if

$\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$. $\square$

2.17 Exercise Let $M \preceq N$ and let $\varphi(x, y) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M. $\square$

2.18 Exercise Consider $\mathbb{Z}^n$ as a structure in the additive language of groups with the natural interpretation. Prove that $\mathbb{Z}^n \not\equiv \mathbb{Z}^m$ for every positive integers $n \neq m$. Hint: in $\mathbb{Z}^n$ there are at most $2^n - 1$ elements that are not congruent modulo 2. $\square$

3 Embeddings and isomorphisms

Here we prove that isomorphic structures are elementary equivalent and a few related results.

2.19 Definition *An **embedding** of M into N is an injective map $h : M \to N$ such that*

1. $a \in r^M \quad \Leftrightarrow \quad ha \in r^N$ *for every $r \in L_{\mathrm{rel}}$ and $a \in M^{n_r}$;*
2. $f^M(a) \quad = \quad f^N(h\,a)$ *for every $f \in L_{\mathrm{fun}}$ and $a \in M^{n_f}$.*

Note that when $c \in L_{\mathrm{fun}}$ is a constant then 2 reads $hc^M = c^N$. Therefore that $M \subseteq N$ if and only if $\mathrm{id}_M : M \to N$ is an embedding.

*An surjective embedding is an **isomorphism** or, when domain and codomain coincide, an **automorphism**.* $\square$

Condition 1 above and the assumption that h is injective can be summarized in the following

1'. $M \vDash r(a) \quad \Leftrightarrow \quad N \vDash r(ha)$ for every $r \in L_{\mathrm{rel}} \cup \{=\}$ and every $a \in M^{n_r}$.

Note also that, by straightforward induction on syntax, from 2 we obtain

2' $h\,t^M(a) \quad = \quad t^N(h\,a)$ for every term $t(x)$ and every $a \in M^{|x|}$.

Combining these two properties, and by straightforward induction on the syntax, we obtain

3. $M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(ha)$ for every $\varphi(x) \in L_{\mathrm{qf}}$ and every $a \in M^{|x|}$.

Recall that L_{qf} for the set of quantifier-free formulas. It is worth noting that when $M \subseteq N$ and $h = \mathrm{id}_M$ then 3 becomes

3' $M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(a)$ for every $\varphi(x) \in L_{\mathrm{qf}}$ and for every $a \in M^{|x|}$.

Finally we prove that first order truth is preserved under isomorphism. We say that a map $h : M \to N$ **fixes** $A \subseteq M$ (pointwise) if $\mathrm{id}_A \subseteq h$. An isomorphism that fixes A is also called an **A-isomorphism**.

2.20 Theorem *If $h : M \to N$ is an isomorphism then for every $\varphi(x) \in L$*

\# $\quad M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(ha)$ *for every $a \in M^{|x|}$*

In particular, if h is an A-isomorphism then $M \equiv_A N$.

Proof We proceed by induction of the syntax of $\varphi(x)$. When $\varphi(x)$ is atomic # holds by 3 above. Induction for the Boolean connectives is straightforward so we only need to consider the existential quantifier. Assume as induction hypothesis that

$$M \vDash \varphi(a,b) \quad \Leftrightarrow \quad N \vDash \varphi(ha,hb) \qquad \text{for every tupla } a \in M^{|x|} \text{ and } b \in M.$$

We prove that # holds for the formula $\exists y\, \varphi(x,y)$.

$$\begin{aligned} M \vDash \exists y\, \varphi(a,y) \quad &\Leftrightarrow \quad M \vDash \varphi(a,b) \quad \text{for some } b \in M \\ &\Leftrightarrow \quad N \vDash \varphi(ha,hb) \text{for some } b \in M \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad N \vDash \varphi(ha,c) \quad \text{for some } c \in N \qquad (\Leftarrow \text{by surjectivity}) \\ &\Leftrightarrow \quad N \vDash \exists y\, \varphi(ha,y). \end{aligned}$$

□

2.21 Corollary *If $h : M \to N$ is an isomorphism then $h\big[\varphi(M)\big] = \varphi(N)$ for every $\varphi(x) \in L$.* □

We can now give a few very simple examples of elementarily equivalent structures.

2.22 Example Let L be the language of strict orders. Consider intervals of $\mathbb{R}$ (or in $\mathbb{Q}$) as structures in the natural way. The intervals $[0,1]$ and $[0,2]$ are isomorphic, hence $[0,1] \equiv [0,2]$ follows from Theorem 2.20. Clearly, $[0,1]$ is a substructure of $[0,2]$. However $[0,1] \not\preceq [0,2]$, in fact the formula $\forall x\,(x{\le}1)$ holds in $[0,1]$ but is false in $[0,2]$.

The example above shows that $M \subseteq N$ and $M \equiv N$ does not imply $M \preceq M$.

Now we prove that $(0,1) \preceq (0,2)$. By Exercise 2.15 above, it suffices to verify that $(0,1) \equiv_B (0,2)$ for every finite $B \subseteq (0,1)$. This follows again by Theorem 2.20 as $(0,1)$ and $(0,2)$ are B-isomorphic for every finite $B \subseteq (0,1)$. □

For the sake of completeness we also give the definition of homomorphism.

2.23 Definition *A* ***homomorphism*** *is a total map $h : M \to N$ such that*

1. $a \in r^M \quad \Rightarrow \quad ha \in r^N \qquad$ *for every $r \in L_{\mathrm{rel}}$ and $a \in M^{n_r}$;*
2. $h\, f^M(a) \quad = \quad f^N(h\, a) \qquad$ *for every $f \in L_{\mathrm{fun}}$ and $a \in M^{n_f}$.* □

Note that only one implication is required in 1.

2.24 Exercise Prove that if $h : N \to N$ is an automorphism and $M \preceq N$ then $h[M] \preceq N$. □

2.25 Exercise Let L be the empty language. Let $A, D \subseteq M$. Prove that the following are equivalent

1. D is definable over M;
2. either D is finite and $A \subseteq A$ or $\neg D$ is finite and $\neg D \subseteq A$.

Hint: as L-structures are plain sets, every bijection $f : M \to M$ is an automorphism. □

2.26 Exercise Prove that if $\varphi(x)$ is an existential formula (i.e. of the form $\exists y\, \psi(x,y)$ for $\psi(x,y) \in L_{\mathrm{qf}}$) and $h : M \to M$ is an embedding then

$$M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha) \text{ for every } a \in M^{|x|}.$$

□

2.27 Exercise Assume that the language L contains only the symbol $+$. Consider $\mathbb{Z}$ as a structure of signature L with the usual interpretation of $+$. Prove that there is no existential formula that defines the set of even integers. Hint: use Exercise 2.26. □

2.28 Exercise Let N be the multiplicative group of $\mathbb{Q}$. Let M be the subgroup of those rational numbers that are of the form n/m for some odd integers m and n. Prove that $M \preceq N$. Hint: use the fundamental theorem of arithmetic and reason as in Example 2.22. □

4 Completeness

A theory T is **maximal consistent** if it is consistent and there is no consistent theory S such that $S \subset T$. Equivalently, T contains every sentence φ **consistent with** T, that is, such that $T \cup \{\varphi\}$ is consistent. Clearly a maximal consistent theory is closed under logical consequences.

A theory T is **complete** if ccl$T$ is maximal consistent. Concrete examples will be given in the next chapters as it is not easy to prove that a theory is complete.

2.29 Proposition *The following are equivalent*

a. *T is maximal consistent;*

b. *$T = Th(M)$ for some structure M;*

c. *T is consistent and $\varphi \in T$ or $\neg\varphi \in T$ for every sentence φ.*

Proof To prove a⇒b, assume that T is consistent. Then there is $M \vDash T$. Therefore $T \subseteq \mathrm{Th}(M)$. As T is maximal consistent $T = \mathrm{Th}(M)$. Implication b⇒c is immediate. As for c⇒a note that if $T \cup \{\varphi\}$ is consistent then $\neg\varphi \notin T$ therefore $\varphi \in T$ follows from c. □

The proof of the proposition below is is left as an exercise for the reader.

2.30 Proposition *The following are equivalent*

a. *T is complete;*

b. *there is a unique maximal consistent theory S such that $T \subseteq S$;*

c. *T is consistent and $T \vdash Th(M)$ for every $M \vDash T$;*

d. *T is consistent and $T \vdash \varphi$ o $T \vdash \neg\varphi$ for every sentence φ;*

e. *T is consistent and $M \equiv N$ for every pair of models of T.* □

2.31 Exercise Prove that the following are equivalent

a. T is complete;

b. for every sentence φ, o $T \vdash \varphi$ o $T \vdash \neg\varphi$ but not both.

By contrast prove that the following are *not* equivalent

a. T is maximal consistent;

b. for every sentence φ, o $\varphi \in T$ o $\neg\varphi \in T$ but not both.

Hint: consider the theory containing all sentences where the symbol $\neg$ occurs an even number of times. This theory is not consistent as it contains $\perp$. □

2.32 Exercise Prove that if T has exactly 2 maximal consistent extension T_1 and T_2 then there is a sentence φ such that $T, \varphi \vdash T_1$ and $T, \neg\varphi \vdash T_2$. □

5 The Tarski-Vaught test

There is no natural notion of *smallest* elementary substructure containing a set of parameters A. The downward Löwenheim-Skolem, which we prove in the next section, is the best result that holds in full generality. Given an arbitrary $A \subseteq N$ we shall construct a model $M \preceq N$ containing A that is small in the sense of cardinality. The construction selects one by one the elements of M that are required to realise the condition $M \preceq N$. Unfortunately, Definition 2.11 supposes full knowledge of the truth in M and it may not be applied during the construction. The following lemma comes to our rescue with a property equivalent to $M \preceq N$ that only mention the truth in N.

2.33 Lemma (Tarski-Vaught test) *For every $A \subseteq N$ the following are equivalent*

1. *A is the domain of a structure $M \preceq N$;*
2. *for every formula $\varphi(x) \in L(A)$, with $|x| = 1$,*
   $N \vDash \exists x\, \varphi(x) \quad \Rightarrow \quad N \vDash \varphi(b)$ for some $b \in A$.

Proof 1⇒2

$$\begin{aligned} N \vDash \exists x\, \varphi(x) &\Rightarrow M \vDash \exists x\, \varphi(x) \\ &\Rightarrow M \vDash \varphi(b) && \text{for some } b \in M \\ &\Rightarrow N \vDash \varphi(b) && \text{for some } b \in M \end{aligned}$$

2⇒1 we first check that A is the domain of a substructure of N, that is, $f^N a \in A$ for every function symbol f and every $a \in A^{n_f}$. This follows from 2 with $fa = x$ for $\varphi(x)$.

Write M for the substructure of N with domain A. By induction on the syntax we prove

$$M \vDash \varphi(a) \Leftrightarrow N \vDash \varphi(a) \qquad \text{for every } a \in M^{|x|}.$$

If $\varphi(x)$ is atomic the claim follows from $M \subseteq N$ and the remarks underneath Definition 2.19. The case of Boolean connectives is straightforward, so only the existential quantifier requires a proof

$$\begin{aligned} M \vDash \exists x\, \varphi(a, x) &\Leftrightarrow M \vDash \varphi(a, b) && \text{for some } b \in M \\ &\Leftrightarrow N \vDash \varphi(a, b) && \text{for some } b \in M \text{ by induction hypothesis} \\ &\Leftrightarrow N \vDash \exists x\, \varphi(a, x) && \Leftarrow \text{uses 2.} \end{aligned}$$

□

2.34 Exercise Consider $\mathbb{R}$ in the language of strict orders. Prove that $\mathbb{R} \smallsetminus \{0\} \preceq \mathbb{R}$. □

6 The downward Löwenheim-Skolem theorem

The main theorem of this section was proved by Löwenheim at the beginning of the last century. Skolem gave a simpler proof immediately afterwards. At the time, the result was perceived as paradoxical.

A few years earlier, the works of Zermelo and Fraenkel had convinced the mathematical community that the whole of set theory could be formalised in a first order language. The downward Löwenheim-Skolem theorem implies the existence of an infinite countable model M of set theory: this is the so-called **Skolem paradox**. The existence of M seems paradoxical because, in particular, a sentence that formalises the axiom of power set holds in M. Therefore M contains an element b which, in M, is the set of subsets of the natural numbers. But the set of elements of b is a subset of M, and therefore it is countable.

In fact, this is not a contradiction, because the expression 'all subsets of the natural numbers' does not have the same meaning in M as it has in the real world. The notion of cardinality, too, acquires a different meaning. In the language of set theory, there is a first order sentence that formalises the fact that b is uncountable: the sentence says that there is no bijection between b and the natural numbers. Therefore the bijection between the elements of b and the natural numbers (which exists in the real world) does not belong to M. The notion of equinumerosity has a different meaning in M and in the real world, but those who live in M cannot realise this.

2.35 Downward Löwenheim-Skolem Theorem *Let N be an infinite L-structure and let $A \subseteq N$. Then there is a structure M of cardinality $\leq |L(A)|$ such that $A \subseteq M \preceq N$.*

Proof Set $\lambda = |L(A)|$. Below we construct a chain $\langle A_i : i < \lambda \rangle$ of subsets of N. The chain begins at $A_0 = A$ and is continuous at limit ordinals. Finally we set $M = \bigcup_{i<\lambda} A_i$. All A_i will have cardinality $\leq \lambda$ so $|M| \leq \lambda$ follows.

Now we construct A_{i+1} given A_i. Assume as induction hypothesis that $|A_i| \leq \lambda$. Then $|L(A_i)| \leq \lambda$. For some fixed variable x let $\langle \varphi_k(x) : k < \lambda \rangle$ be an enumeration of the formulas in $L(A_i)$ that are consistent in N. For every k pick $a_k \in N$ such that $N \vDash \varphi_k(a_k)$. Define $A_{i+1} = A_i \cup \{a_k : k < \lambda\}$. Then $|A_{i+1}| \leq \lambda$ is clear.

Now use Tarski-Vaught test to prove $M \preceq N$. Suppose $\varphi(x) \in L(M)$ is consistent in N. As finitely many parameters occur in formulas, $\varphi(x) \in L(A_i)$ for some i. Then $\varphi(x)$ is among the formulas enumerated above and $A_{i+1} \subseteq M$ contains a solution of $\varphi(x)$. □

We will need to adapt the construction above to meet more requirements on the model M. To better control the elements that enters M it is convenient to add one element at the time (above we add λ elements at each stage). We need to enumerate formulas with care if we want to complete the construction by stage λ.

2.36 Second proof of the downward Löwenheim-Skolem Theorem From set theory we know there is a bijection $\pi : \lambda \to \lambda^2$ such that $\pi_1(i)$ and $\pi_2(i)$, the two components of $\pi(i)$, are both $\leq i$. Suppose we have defined the set A_j for very $j \leq i$. Let $\langle \varphi_k(x) : k < \lambda \rangle$ be an enumeration of the formulas in $L(A_{\pi_1(i)})$. Let b be a solution of the $\varphi_k(x)$ for $k = \pi_2(i)$ and define $A_{i+1} = A_i \cup \{b\}$.

We use Tarski-Vaught test to prove $M \preceq N$. Let $\varphi(x) \in L(M)$ be consistent in N. As π is a bijection, there is an i such that $\varphi(x)$ is the $\pi_2(i)$-th formula enumerated in $L(A_{\pi_1(i)})$. Hence a witness of $\varphi(x)$ is enumerated in M at stage i. □

2.37 Exercise Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove that there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Hint: adapt the construction used to prove the downward Löwenheim-Skolem Theorem. □

7 Elementary chains

An **elementary chain** is a chain $\langle M_i : i < \lambda \rangle$ of structures such that $M_i \preceq M_j$ for every $i < j < \lambda$. The **union** (or **limit**) of the chain is the structure with as domain the set $\bigcup_{i<\lambda} M_i$ and as relations and functions the union of the relations and functions of M_i. It is plain that all structures in the chain are substructures of the limit.

2.38 Lemma *Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of structures. Let N be the union of the chain. Then $M_i \preceq N$ for every i.*

Proof By induction on the syntax of $\varphi(x) \in L$ we prove

$$M_i \vDash \varphi(a) \;\Leftrightarrow\; N \vDash \varphi(a) \qquad \text{for every } i < \lambda \text{ and every } a \in M_i^{|x|}$$

As remarked in 3' of Section 3, the claim holds for quantifier-free formulas. Induction for Boolean connectives is straightforward so we only need to consider the existential quantifier

$$\begin{aligned} M_i \vDash \exists y\, \varphi(a,y) \;&\Rightarrow\; M_i \vDash \varphi(a,b) && \text{for some } b \in M_i. \\ &\Rightarrow\; N \vDash \varphi(a,b) && \text{for some } b \in M_i \subseteq N \end{aligned}$$

where the second implication follows from the induction hypothesis. Vice versa

$$N \vDash \exists y\, \varphi(a,y) \;\Rightarrow\; N \vDash \varphi(a,b) \qquad \text{for some } b \in N$$

As a is a finite tuple, $a \in M_j^{|x|}$ for some j. We can also assume that $b \in M_j$. Then we can apply the induction hypothesis to $\varphi(x,y)$

$$\begin{aligned} &\Rightarrow\; M_j \vDash \varphi(a,b) && \text{for some } b \in M_j \\ &\Rightarrow\; M_j \vDash \exists y\, \varphi(a,y) \\ &\Rightarrow\; M_i \vDash \exists y\, \varphi(a,y) \end{aligned}$$

The last implication holds because $M_i \preceq M_j$ or, in case $j < i$, because $M_j \preceq M_i$. □

2.39 Exercise Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of substructures of N. Let M be the union of the chain. Prove that $M \preceq N$. □

2.40 Exercise Give an alternative answer to Exercise 2.37 using the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two countable chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$. The required model is $K = \bigcup_{i \in \omega} K_i$. In fact it is easy to check that $K \cap M = \bigcup_{i \in \omega} M_i$. □

Chapter 3

Ultraproducts and compactness

1 Filters and ultrafilters

The matter in this section will be reconsidered in a more general setting in Chapter 4.

Let I be any set. A **filter on I** (or a **filter of $\mathcal{P}(I)$**), is a non empty set $F \subseteq \mathcal{P}(I)$ such that for evry $a, b \in \mathcal{P}(I)$:

f1 $\qquad a \in F \text{ and } a \subseteq b \quad \Rightarrow \quad b \in F$

f2 $\qquad a \in F \text{ and } b \in F \quad \Rightarrow \quad a \cap b \in F$

A filter F is **proper** if $F \neq \mathcal{P}(I)$, equivalently if $\varnothing \notin F$. Otherwise it is **improper**. By f1 above, F is proper if and only if $\varnothing \notin F$. A filter F is **principal** if $F = \{a \subseteq I \, : \, b \subseteq a\}$ for some set $b \subseteq I$. We say that $\{b\}$ **generates** F. When I is finite every filter F is principal generated by $\{\cap F\}$. Non principal filters on I exist as soon as I is infinite. In fact, if I is infinite it is easy to check that the following is filter

$$F \quad = \quad \Big\{ a \subseteq I \, : \, a \text{ is cofinite in } I \Big\}$$

where **cofinite (in I)** means that $I \smallsetminus a$ is finite. This is called the **Fréchet's filter on I**. Fréchet's filter is the minimal non principal filter.

3.1 Exercise Let I be infinite. Prove that Fréchet's filter on I contains all non principal filters on I. □

A proper filter F is **maximal** if there is no proper filter H such that $F \subset H$. A proper filter F is an **ultrafilter** if for every $a \subseteq I$:

$$a \notin F \quad \Rightarrow \quad \neg a \in F \qquad\qquad \text{where } \neg a \stackrel{\text{def}}{=} I \smallsetminus a$$

Below we prove that ultrafilters are exactly the maximal filters.

Let $B \subseteq \mathcal{P}(I)$ the **filter generated by B** is the intersection of all filters containing B. It is easy to check that the intersection of a family of filters is a filter, so the notion is well defined. The following easy proposition gives a workable characterization of the filter generated by a set.

3.2 Proposition *The filter generated by B is $\Big\{ a \subseteq I \, : \, \bigcap C \subseteq a$ for some finite $C \subseteq B \Big\}$.* □

We say that B has the **finite intersection property** if $\bigcap C \neq \varnothing$ for every finite $C \subseteq B$. The following proposition is immediate.

3.3 Proposition *The following are equivalent*

1. *the filter generated by B is non principal;*
2. *B has the finite intersection property.* □

A proper filter F is **prime** if for every $a, b \subseteq I$.

$$a \cup b \in F \quad \Rightarrow \quad a \in F \quad \text{or} \quad b \in F$$

Prime filters coincide with maximal filter. (However, in Chapter 4 we introduce more a general context where primeness is distinct from maximality.)

3.4 Proposition *For every filter F on I, the following are equivalent:*

1. *F is a maximal filter;*
2. *F is a prime filter;*
3. *F is an ultrafilter.*

Proof 1⇒2. Suppose $a, b \notin F$, where F is maximal. We claim that $a \cup b \notin F$. By maximality, there is a $c \in F$ such that $a \cap c = b \cap c = \varnothing$. Therefore $(a \cup b) \cap c = \varnothing$. Hence $a \cup b \notin F$.

2⇒3. It suffices to note that $I = a \cup \neg a \in F$.

3⇒1. If $a \notin F$, where F is an ultrafilter, then $\neg a \in F$ and no proper filter contains $F \cup \{a\}$. □

3.5 Proposition *Let $B \subseteq \mathcal{P}(I)$ have the finite intersection property, then B is contained in a maximal filter.*

Proof First prove that the union of a chain of subsets of $\mathcal{P}(I)$ with the finite intersection property has the finite intersection property. Let $\mathcal{B}$ be such a chain and suppose for a contradiction that $\bigcup \mathcal{B}$ does not have the finite intersection property. Fix a finite $C \subseteq \bigcap \mathcal{B}$ such that $\bigcap C = \varnothing$. As C is finite, $C \subseteq B$ for some $B \in \mathcal{B}$. Hence B does not have the finite intersection property. A contradiction.

Now apply Zorn lemma to obtain a $B \subseteq \mathcal{P}(I)$ which is maximal among the sets with the finite intersection property. It is immediate that B is a filter. □

3.6 Exercise Prove that all principal ultrafilters are generated by a singleton. □

2 Direct products

In this and the next section $\langle M_i : i \in I \rangle$ is a sequence of L-structures. (We are abusing of the word sequence, I is only a set.) The **direct product** of this sequence is a structure denoted by

$$\prod_{i \in I} M_i \qquad\qquad \text{below this is abbreviated by } N$$

and is defined by conditions 1-3 below. If $M_i = M$ for all $i \in I$ we say that N is a **direct power** of M and denote it by M^I.

The domain of N is the set of the functions

1. $$\hat{a} \quad : \quad I \;\to\; \bigcup_{i \in I} M_i$$
$$\hat{a} \quad : \quad i \;\mapsto\; \hat{a}i \;\in\; M_i$$

We confuse tuples of elements of N with tuple-valued functions, for instance, the tuple $\hat{a} = \langle \hat{a}_1 \ldots \hat{a}_n \rangle$ is confused with the function $\hat{a} : i \mapsto \hat{a}i = \langle \hat{a}_1 i, \ldots, \hat{a}_n i \rangle$. But at a first reading it might be convenient to pretend that all functions and relations are unary.

The interpretation of $f \in L_{\rm fun}$ is defined as follows

2. $$(f^N \hat{a})i \quad = \quad f^{M_i}(\hat{a}i) \qquad \text{for all } i \in I,$$

The interpretation of $r \in L_{\rm fun}$ is the product of the relations r^{M_i}, that is, we define

3. $$\hat{a} \in r^N \quad \Leftrightarrow \quad \hat{a}i \in r^{M_i} \qquad \text{for all } i \in I.$$

The following proposition is immediate.

3.7 Proposition *If only the connectives $\wedge$, $\forall$ and $\exists$ occur in $\varphi(x) \in L$ then for every $\hat{a} \in N^{|x|}$*

$\sharp$ $$N \vDash \varphi(\hat{a}) \quad \Leftrightarrow \quad M_i \vDash \varphi(\hat{a}i) \qquad \textit{for all } i \in I.$$

Proof By induction on syntax. First note that we can extend 2 to all terms $t(x)$

2'. $$(t^N \hat{a})i \quad = \quad t^{M_i}(\hat{a}i) \qquad \text{for all } i \in I.$$

Combining 3 and 2' we obtain that for every $r \in L_{\rm rel} \cup \{=\}$ and every L-term $t(x)$

3'. $$N \vDash rt\hat{a} \quad \Leftrightarrow \quad M_i \vDash rt\hat{a}i \qquad \text{for all } i \in I$$

This shows that $\sharp$ holds for $\varphi(x)$ atomic. Induction for the connectives $\wedge$, $\forall$ ed $\exists$ is immediate. □

A consequence of Proposition 3.7 is that the direct product of groups, rings and vector spaces are structures of the same sort. Notably, the product of fields is not a field.

3 Quotient structures

If E is an equivalence relation on N we write $[c]_E$ for the equivalence class of $c \in N$. We use the same symnol for the equivalenc erelation on M^n defined as follow: if $a = a_1, \dots, a_n$ and $b = b_1, \dots, b_n$ are n-tuples of elements of N then $\boldsymbol{a \; E \; b}$ means that $a_i \; E \; b_i$ holds for all i. By $[a]_E$ we mean both the equivalence class of $a \in M^n$ or the for the tuple of equivalence classes $[a_1]_E, \dots, [a_n]_E$. The context will disambiguate.

An equivalence relation E on a structure N is a **congruence** if for every $f \in L_{\rm fun}$

c1. $$a \; E \; b \quad \Rightarrow \quad f^N a \; E \; f^N b;$$

When E is a congruence on N we write $\boldsymbol{N/E}$ for the a structure that has as domain the set of E-equivalence classes in N and the following interpretation of $f \in L_{\rm fun}$ and $r \in L_{\rm rel}$:

c2. $$f^{N/E}[a]_E \quad = \quad \left[f^N a\right]_E;$$

c3. $$[a]_E \; \in \; r^{N/E} \quad \Leftrightarrow \quad [a]_E \cap r^N \neq \varnothing.$$

By c1 this interpretation is well defined. We call N/E the **quotient structure**. The reader will recognize it as a familiar notion by the following proposition (which is not required in the following and requires the notion of homomorphism, see Definition 2.23. Recall that the **kernel** of a total map $h : N \to M$ is the equivalence relation E such that

$$a \; E \; b \quad \Leftrightarrow \quad ha = hb$$

for every $a, b \in N$.

3.8 Proposition *Let $h : N \to M$ be a surjective homomorphism and let E be the kernel of h. Then there is an isomorphism k that makes the following diagram commute*

$$\begin{tikzcd} N \arrow[r, "h"] \arrow[d, "\pi"] & M \\ N/E \arrow[ur, "k"'] & \end{tikzcd}$$

where $\pi : a \mapsto [a]_E$ is the projection map. □

Quotients clutter the notation with brackets. To avoid the mess, we prefer to reason in N and tweak the satisfaction relation. We replace equality with E and adapt the interpretation of relation symbols to c3 above. The following definition makes this precise (an inductive definition is required for instance in the proof of Łoś Theorem). Warning: it is not standard.

3.9 Definition *For t_2, t_2 closed terms of $L(N)$ define*

1* $\quad N/E \stackrel{*}{\models} t_1 = t_2 \quad\Leftrightarrow\quad [t_1^N]_E = [t_2^N]_E$

For t a tuple of closed terms of $L(N)$ and $r \in L_{\text{rel}}$ a relation symbol

2* $\quad N/E \stackrel{*}{\models} rt \quad\Leftrightarrow\quad [t^N]_E \in r^{N/E}$

Finally the definition is extended to all sentences $\varphi \in L(N)$ by induction in the usual way

3* $\quad N/E \stackrel{*}{\models} \neg\varphi \quad\Leftrightarrow\quad$ *not* $N/E \stackrel{*}{\models} \varphi$

4* $\quad N/E \stackrel{*}{\models} \varphi \wedge \psi \quad\Leftrightarrow\quad N/E \stackrel{*}{\models} \varphi$ *and* $N/E \stackrel{*}{\models} \psi$

5* $\quad N/E \stackrel{*}{\models} \exists x\, \varphi(x) \quad\Leftrightarrow\quad N/E \stackrel{*}{\models} \varphi(\hat{a})$ *for some $a \in N$.* □

Now, by induction on the syntax of formulas one can prove $\stackrel{*}{\models}$ does what required. In particular, $N/E \stackrel{*}{\models} \varphi(a) \leftrightarrow \varphi(b)$ for every $a\ E\ b$.

3.10 Proposition *Let E be a congruence relation of N. Then the following are equivalent for every $\varphi(x) \in L$*

1. $\quad N/E \quad \stackrel{*}{\models} \quad \varphi(a)$;
2. $\quad N/E \quad \models \quad \varphi([a]_E)$. □

4 Ultraproducts

Assume the notation of the previous sections. In particular, $\langle M_i : i \in I\rangle$ is a sequence of structures and N is the direct product of this sequence.

Let F be a filter on I. We define the following congruence relation on N

$$\hat{a} \sim_F \hat{c} \quad\Leftrightarrow\quad \big\{\, i \in I \ :\ \hat{a}i = \hat{c}i \,\big\} \in F\,.$$

It is easy to verify that $\sim_F$ is indeed an equivalence relation: reflexivity and symmetry are evident, transitivity follows from f2 in Section 1. It is also clear that $\sim_F$ a congruence.

We now introduce an ad hoc notion quotient structure ***N/F*** by tweaking the definition in Section 3. Namely, we replace c3 by the following

c3'. $\quad [\hat{a}]_{\sim_F} \in r^{N/F} \quad \Leftrightarrow \quad \big\{ i \,:\, \hat{a}i \in r^{M_i} \big\} \in F$

We will not elaborate on the difference between c3 and c3'. We refer the interested reader to Exercise 3.14 and 3.15.

The structure N/F is called the **reduced product** of the structures $\langle M_i : i \in I \rangle$ or, when $M_i = M$ for all $i \in I$, the **reduced power** of M. When F is an ultrafilter we say **ultraproduct**, respectively **ultrapower**.

We the notation introduced in Definition 3.9 (see also Proposition 3.10) we are ready to state a fundamental theorem about ultraproducts.

3.11 Łoś Theorem *Let $\varphi(x) \in L$ and let F be an ultrafilter on I. Then for every $\hat{a} \in N^{|x|}$, the following are equivalent*

1. $N/F \overset{*}{\vDash} \varphi(\hat{a})$;
2. $\big\{ i \,:\, M_i \vDash \varphi(\hat{a}i) \big\} \in F$.

Proof We proceed by induction of the syntax of $\varphi(x)$. Suppose $\varphi(x)$ is of the form $rt(x)$ for some tuple of terms $t(x)$ and $r \in L_{\text{rel}} \cup \{=\}$. Then 1$\Leftrightarrow$2 follows easily from c3'.

We prove the inductive step for the connectives $\neg$, $\wedge$, and $\exists$. We begin with $\neg$. This is the only place in the proof where the assumption that F is an *ultra*filter is required. From the inductive hypothesis we obtain

$$N/F \overset{*}{\vDash} \neg\varphi(\hat{a}) \quad \Leftrightarrow \quad \big\{ i \,:\, M_i \vDash \varphi(\hat{a}i) \big\} \notin F$$

So, as F is an *ultra*filter

$$\Leftrightarrow \quad \big\{ i \,:\, M_i \vDash \neg\varphi(\hat{a}i) \big\} \in F.$$

Now consider $\wedge$. Assume as inductive hypothesis that the equivalence 1$\Leftrightarrow$2 holds for $\varphi(x)$ and $\psi(x)$, then

$$N/F \overset{*}{\vDash} \varphi(\hat{a}) \wedge \psi(\hat{a}) \quad \Leftrightarrow \quad \big\{ i \,:\, M_i \vDash \varphi(\hat{a}i) \big\} \in F \;\text{ and }\; \big\{ i \,:\, M_i \vDash \psi(\hat{a}i) \big\} \in F$$

As filters are closed under intersection we obtain

$$\Leftrightarrow \quad \big\{ i \,:\, M_i \vDash \varphi(\hat{a}i) \wedge \psi(\hat{a}i) \big\} \in F.$$

Finally consider $\exists y$. Assume as inductive hypothesis that the equivalence 1$\Leftrightarrow$2 holds for $\varphi(x, y)$, then

$$\begin{aligned} N/F \overset{*}{\vDash} \exists y\, \varphi(\hat{a}, y) \quad &\Leftrightarrow \quad N/F \overset{*}{\vDash} \varphi(\hat{a}, \hat{b}) && \text{for some } \hat{b} \in N \\ &\Leftrightarrow \quad \big\{ i \,:\, M_i \vDash \varphi(\hat{a}i, \hat{b}i) \big\} \in F && \text{for some } \hat{b} \in N. \end{aligned}$$

We claim this is equivalent to

$$\Leftrightarrow \quad \big\{ i \,:\, M_i \vDash \exists y\, \varphi(\hat{a}i, y) \big\} \in F.$$

In fact, direction $\Rightarrow$ is trivial and, as for $\Leftarrow$, we choose as $\hat{b}$ a sequence that picks a witness of $M_i \vDash \exists y\, \varphi(\hat{a}i, y)$ if it exists, some arbitrary element of M_i otherwise. □

Let ***a^I*** denote the element of M^I that has constant value a. The following is an immediate consequence of Łoś theorem.

3.12 Corollary *For every $a \in M$*

$$M^I/F \overset{*}{\vDash} \varphi(a^I) \quad \Leftrightarrow \quad M \vDash \varphi(a)\,.$$ □

Often one identifies M with its image under the map $h : a \mapsto [a^I]_F$ and say that M^I/F is an elementary extension of M.

3.13 Corollary *Every infinite structure has a proper elementary extension.*

Proof Let M be an infinite structure and let F be an *non principal* ultrafilter on ω. By the above remark, it suffices to show that $h[M]$ is a *proper* substructure of M^ω/F. As M is infinite, there is an injective function $\hat{d} \in M^\omega$. Then for every $a \in M$ the set $\{i : \hat{d}i = a\}$ is either empty or a singleton and, as F is non principal, it does not belong to F. So, by Łoš Theorem $M^\omega/F \;\overset{*}{\nvDash}\; \hat{d} \neq a^I$ for every $a \in M$, that is, $[\hat{d}]_F \notin h[M]$. □

3.14 Exercise Prove that c3 differs from 3c' only if $\{i \,:\, r^{M_i} = \varnothing\} \in F$.

3.15 Exercise Let o be some fresh element. Let M_i' is a superstructures of M_i with domain $M_i \cup \{o\}$, relations $r^{M_i'} = r^{M_i} \cup \{o\}^{n_r}$, and functions $f^{M_i'}$ just any extension of f^{M_i}. Denote by N' the direct product of the sequence $\langle M_i' : i \in I\rangle$. The element of N' that is identically o is denoted by o^I.

It is easy to verify that the $\sim_F$-equivalence classes that do not contain o^I form substructure of N'.

Let $N'/{\sim_F}$ be the quotient structure as defined in Section 3. Prove that N/F is isomorphic to the substructure of $N'/{\sim_F}$ that contains all equivalence classes but for the one of o^I. □

5 The compactness theorem

A theory is **finitely consistent** if all its finite subsets are consistent. The following theorem is the *fiat lux* of model theory.

3.16 Compactness Theorem *Every finitely consistent theory is consistent.*

Proof Let T be a finitely consistent theory. We claim that the structure N/F which we define below is a model of T. Let I be the set of consistent sentences I in the language L. For every $\xi \in I$ pick some $M_\xi \vDash \xi$. For any sentence $\varphi \in L$ we write X_φ for the following subset of I

$$X_\varphi \quad = \quad \Big\{\xi \in I \,:\, \xi \vdash \varphi\Big\}$$

Clearly φ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi\wedge\psi} \;=\; X_\varphi \cap X_\psi$. hence, as T is finitely consistent, the set $B = \big\{X_\varphi \,:\, \varphi \in T\big\}$ has the finite intersection property. Therefore B extends to an ultrafilter F on I. Define

$$N \quad = \quad \prod_{\xi \in I} M_\xi.$$

We claim that $N/F \vDash T$. By Łoš Theorem, for every sentence $\varphi \in L$

$$N/F \vDash \varphi \quad \Leftrightarrow \quad \Big\{\xi \,:\, M_\xi \vDash \varphi\Big\} \;\in\; F\,.$$

By the definition of F, for every $\varphi \in T$, the set $X_\varphi \subseteq \{\xi \,:\, M_\xi \vDash \varphi\}$ belongs to F.

Therefore $N/F \vDash T$, *et lux fuit*. $\square$

The compactness theorem can be formulated in the following apparently stronger way.

3.17 Corollary *If $T \vdash \varphi$ then there is some finite $S \subseteq T$ such that $S \vdash \varphi$.*

Proof Suppose $S \nvdash \varphi$ for every finite $S \subseteq T$. Then for every finite $S \subseteq T$ there is a model $M \vDash S \cup \{\neg\varphi\}$. In other words, $T \cup \{\neg\varphi\}$ is finitely consistent. By compactness $T \cup \{\neg\varphi\}$ hence $T \nvdash \varphi$. $\square$

3.18 Exercise Let $\Phi \subseteq L$ be a set of sentences and suppose that $\vdash \psi \leftrightarrow \bigvee \Phi$ for some sentence ψ. Prove that there is a finite $\Phi_0 \subseteq \Phi$ such that $\vdash \psi \leftrightarrow \bigvee \Phi_0$. $\square$

6 Finite axiomatizability

A theory T is **finitely axiomatizable** if $\text{ccl}(S) = \text{ccl}(T)$ for some finite S. The following theorem shows that we can restrict the search of S to the subsets of T.

3.19 Proposition *For every theory T the following are equivalent*

1. *T is finitely axiomatizable;*
2. *there a finite $S \subseteq T$ such that $S \vdash T$.*

Proof Only 1⇒2 requires a proof. If T is finitely axiomatizable, there is a sentence φ such that $\text{ccl}(\varphi) = \text{ccl}(T)$. Then $T \vdash \varphi \vdash T$. By Proposition 3.17 there is a finite $S \subseteq T$ such that $S \vdash \varphi$. Then also $S \vdash T$. $\square$

If L is empty, then every structure is a model. The **theory of infinite sets** is the set of sentences that hold in every infinite structure.

3.20 Example The theory of infinite sets is not finitely axiomatizable. Define

$$T_\infty \quad = \quad \{\exists^{\geq n} x\ (x = x)\} \ : \ n \in \omega\}$$

Every infinite set is a model of T_∞ and, vice versa, every model of T_∞ is is an infinite set. Then $\text{ccl}(T_\infty)$ is the theory of infinite sets. Suppose for a contradiction that T_∞ is finitely axiomatizable. By Proposition 3.19, $\exists^{\geq n} x(x = x) \vdash T_\infty$ for some n. Any set of cardinality $n + 1$ proves that this is not the case. $\square$

The following is a less trivial example. It proves a claim we made in Exercise 1.15.

3.21 Example Write T_{gph} for the theory of graphs, see Example 2.5. Let $\mathcal{K}$ be the following class of structures

$$\mathcal{K} = \Big\{M \vDash T_{\text{gph}} \ : \ r^M \subseteq (A \times \neg A) \cup (\neg A \times A) \text{ for some } A \subseteq M\Big\}$$

We prove that $\mathcal{K}$ is axiomatizable but not finitely axiomatizable, i.e. $\mathcal{K} = \text{Mod}(T)$ for some theory T, but T cannot be choosen finite.

A **path of length n** in M is a sequence $c_0, \ldots, c_n \in M$ such that $M \vDash r(c_i, c_{i+1})$ for every $0 \leq i < n$. A path is **closed** if $c_0 = c_n$. We claim that the following theory axiomatizes $\mathcal{K}$.

$$T \;=\; \left\{\neg\exists x_0,\ldots x_{2n+1}\left[\bigwedge_{i=0}^{2n} r(x_i,x_{i+1}) \;\wedge\; x_0 = x_{2n+1}\right] \;:\; n\in\omega\right\}$$

In words, T says that all closed paths have even length. Inclusion $\mathcal{K}\subseteq \mathrm{Mod}(T)$ is clear, we prove $\mathrm{Mod}(T)\subseteq\mathcal{K}$. Let $M\vDash T$ and let $A_o\subseteq M$ contain exactly one point for every connected component of M. Define

$$A \;=\; \left\{b \;:\; M\vDash \exists x_0,\ldots,x_{2n}\left[\bigwedge_{i=1}^{2n-1} r(x_i,x_{i+1})\wedge a{=}x_0\wedge x_{2n}{=}b\right],\; a\in A_o,\; n\in\omega\right\}$$

We claim that $r^M\subseteq (A\times\neg A)\cup(\neg A\times A)$, hence $M\in\mathcal{K}$. We need to verify that if $r(b,c)$ then neither $b,c\in A$ nor $b,c\in\neg A$. Suppose for a contradiction that $r(b,c)$ and $b,c\in A$ (the case $b,c\in\neg A$ is similar). As b and c belong to the same connected component, there are two paths $b_0,\ldots,b_{2n}$ and $c_0,\ldots,c_{2m}$ that connect $a=b_0=c_0\in A_o$ to $b=b_{2n}$ and $c=c_{2m}$. Then $a,b_1,\ldots,b_{2n},c_{2m},\ldots,c_1,a$ is a closed path of odd length. A contradiction.

We now prove that $\mathcal{K}$ is not finitely axiomatizable. By Proposition 3.19 it suffices to note that no finite $S\subseteq T$ axiomatizes $\mathcal{K}$. □

7 Realization of types and upward Löwenheim-Skolem

Recall that a **type** is a set of formulas. When we present types we usually declare the variables that may occur in it – we write $\boldsymbol{p(x)}$, $\boldsymbol{q(x)}$, etc. where x is a tuple of variables. Clearly, when x is the empty tuple, $p(x)$ is just a theory. We identify a finite types with the conjunction of the formulas contained in it.

We write $\boldsymbol{M\vDash p(a)}$ if $M\vDash\varphi(a)$ for every $\varphi(x)\in p$. We say that a is a **solution** or a **realization** of $p(x)$. An equivalent notation is $\boldsymbol{M, a\vDash p(x)}$ or, when M is clear from the context, $\boldsymbol{a\vDash p(x)}$. We say that $p(x)$ is **consistent in M** it has a solution in M. In this case we may write $\boldsymbol{M\vDash\exists x\, p(x)}$. We say that $p(x)$ is **consistent** tout court if it is consistent in some model.

We say that a type $p(x)$ is **finitely consistent** if all its finite subsets are consistent. If they are all consistent in the same model M, we say that $p(x)$ is **finitely consistent in M**. The following theorem shows that the latter notion, which is trivial for theories, is very interesting for types.

3.22 Compactness Theorem for types *Every finitely consistent type $p(x)\subseteq L$ is consistent. Moreover, if $p(x)\subseteq L(M)$ is finitely consistent in M then it is realised in some elementary extension of M.*

Proof Let L' be an expansion of L with a fresh constant symbols c, a tuple of constants of the same length as x. Then $p(c)$ is a finitely consistent theory in the language L'. By the compactness theorem there is an L'-structure $N'\vDash p(c)$. Let N be the reduced of N' to L. That is the L-structure with the same domain and the same interpretation of N' on the symbols of L. Then N realises $p(x)$.

As for the second claim, let a be an enumeration of M. We can assume that $p(x)$ has the form $p'(x,a)$ for some $p'(x,z)\in L$. Define

$$q(z) \;=\; \{\varphi(z) \;:\; M\vDash\varphi(a)\}$$

Clearly, $q(z)\cup p'(x,z)$ is finitely consistent (in M). By the first part of the proof there is a model N' such that $N' \vDash p'(a', b')$ for some a', b'. Let h be the function $\{\langle a, a' \rangle\}$. Then $h : M \to N'$ is an embedding and we can identify M with $h[M]$. The proposition is proved if we can show that $h[M] \preceq N'$. For any $\varphi(z) \in L$ we have

$$h[M] \vDash \varphi(ha) \quad \Leftrightarrow \quad M \vDash \varphi(a) \quad \Leftrightarrow \quad \varphi(z) \in q(z) \quad \Rightarrow \quad N' \vDash \varphi(a').$$

Then $h[M] \preceq N'$ follows because all sentences in $L(M)$ have the form $\varphi(a)$ for some $\varphi(z) \in L$. □

The following corollary is historically important.

3.23 Upward Löwenheim-Skolem Theorem *Every infinite structure has arbitrarily large elementary extensions.*

Proof Let $x = \langle x_i : i < \lambda \rangle$ be a tuple distinct variables, where λ is an arbitrary cardinal. The type $p(x) = \{x_i \neq x_j : i < j < \lambda\}$ is finitely consistent in every infinite structure and every structure that realises p has cardinality $\geq \lambda$. Hence the claim follows from Theorem 3.22. □

Chapter 4

Types and morphisms

In Section 1 and 2 we introduce distributive lattices and prime filters and prove Stone's representation theorem for distributive lattices. In Section 3 we discuss lattices that arise from sets of formulas and their prime filters (prime types). These sections are mainly required for the discussion of Hilbert's Nullstellensatz in Section 7.7 below.

There is a lot of notation in this chapter. The reader may skim through and return to it when we refer to it.

1 Semilattices and filters

A **preorder** is a set $\mathbb{P}$ with a transitive and symmetric relation which we usually denote by $\leq$. If $A, B \subseteq \mathbb{P}$ we write $A \leq B$ if $a \leq b$ for every $a \in A$ and $b \in B$. We write $a \leq B$ and $A \leq b$ for $\{a\} \leq B$ and $A \leq \{b\}$ respectively.

Quotienting a preorder by the equivalence relation

$$a \sim b \quad \Leftrightarrow \quad a \leq b \quad \text{and} \quad b \leq a$$

gives a **(partial) order**. We often do not distinguish between a preorder and the partial order associated to it. Preorders are very common; here we are interested in the one induced by the relation of logical consequence

$$\varphi \leq \psi \quad \Leftrightarrow \quad \varphi \vdash \psi,$$

or, more generally, by the relation of **logical consequence over a theory T**, that is

$$\varphi \leq \psi \quad \Leftrightarrow \quad T \cup \{\varphi\} \vdash \psi.$$

A partial order $\mathbb{P}$ is a **lower semilattice** if for each pair $a, b \in \mathbb{P}$ there is a maximal element c such that $c \leq \{a, b\}$. We call c the **meet** of a and b. The meet is unique and is denoted by $a \wedge b$. Dually, a partial order is an **upper semilattice** if for each pair of elements a and b there is a minimal element c such that $\{a, b\} \leq c$. This c is called the **joint** of a and b. The joint is unique and is denoted by $a \vee b$. A **lattice** is simultaneously a lower and an upper semilattice.

An element c such that $c \leq \mathbb{P}$ is called a **lower bound** or a **bottom**. An element such that $\mathbb{P} \leq c$ is called an **upper bound** or a **top**. Lower and upper bounds are unique and will be denoted by $\mathbf{0}$, respectively $\mathbf{1}$. Other symbols common in the literature are $\perp$, respectively $\top$. A semilattice is **bounded** if it has both an upper and a lower bound.

For the rest of this section we assume that $\mathbb{P}$ is a bounded lower semilattice.

The meet is associative and commutative

$$\begin{aligned} (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\ a \wedge b &= a \wedge b. \end{aligned}$$

Hence we may unambiguously write $a_1 \wedge \cdots \wedge a_n$. When $C \subseteq \mathbb{P}$ is finite, we write $\wedge C$ for the meet of all the elements of C. We agree that $\wedge\varnothing = 1$.

In an upper semilattice, the dual properties hold for the joint. We write $\vee C$ for the joint of all elements in C and we agree that $\vee\varnothing = 0$.

A **filter** of $\mathbb{P}$ is a non-empty set $F \subseteq \mathbb{P}$ that satisfies the following for all $a, b \in \mathbb{P}$

f1. $a \in F$ and $a \leq b \;\Rightarrow\; b \in F$

f2. $a, b \in F \;\Rightarrow\; a \wedge b \in F$.

We say that F is a **proper filter** if $F \neq \mathbb{P}$, equivalently if $0 \notin F$. We say that F is **principal** if $F = \{b \,:\, a \leq b\}$ for some $a \in \mathbb{P}$. More precisely, we say that F is the **principal filter generated by a**. A proper filter F is **maximal** if there is no filter H such that $F \subset H \subset \mathbb{P}$.

We say that a filter F is **maximal relative to c** if $c \notin F$ and $c \in H$ for every $H \supset F$. So, a filter F is maximal if it is maximal relative to 0. We say that F is **relatively maximal** when it is maximal relative to some c.

For $B \subseteq \mathbb{P}$ we define the **filter generated by B** to be the intersection of all the filters containing B. It is easy to verify that this is indeed a filter. When B is a finite set, the filter generated by B is the principal filter generated by $\wedge B$. In general we have the following.

4.1 Proposition *For every $B \subseteq \mathbb{P}$, the filter generated by B is the set*

$$\{a \,:\, \wedge C \leq a \text{ for some finite non-empty } C \subseteq B\}. \qquad \square$$

This has the following important consequence.

4.2 Proposition *Let $B \subseteq \mathbb{P}$ and let $c \in \mathbb{P}$. If $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$, then B is contained in a maximal filter relative to c.*

Proof Let $\mathcal{F}$ be the set of filters F such that $B \subseteq F$ and $c \notin F$. By Proposition 4.1, $\mathcal{F}$ is non-empty. It is immediate that $\mathcal{F}$ is closed under unions of arbitrary chains. Then, by Zorn's lemma, $\mathcal{F}$ has a maximal element. □

4.3 Exercise Prove that the following are equivalent

1. F is maximal relative to c;
2. for every $a \notin F$ there is $d \in F$ such that $d \wedge a \leq c$. □

4.4 Exercise (A generalization of the exercise above.) Let $B \subseteq \mathbb{P}$ and let $c \in \mathbb{P}$ be such that $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$. Prove that the following are equivalent

1. B is a maximal filter relative to c;
2. $a \notin B \;\Rightarrow\; b \wedge a \leq c$ for some $b \in B$. □

4.5 Exercise Let $F \subseteq \mathbb{P}$ be a non-principal filter. Is F always contained in a maximal non-principal filter? □

2 Distributive lattices and prime filters

Let $\mathbb{P}$ be a lattice. We say that $\mathbb{P}$ is **distributive** if for every $a, b, c \in \mathbb{P}$

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \end{aligned}$$

Throughout this section we assume that $\mathbb{P}$ is a bounded distributive lattice.

A proper filter F is **prime** if for every $a, b \in \mathbb{P}$

$$a \vee b \in F \quad \Rightarrow \quad a \in F \quad \text{or} \quad b \in F.$$

4.6 Proposition *Every relatively maximal filter of $\mathbb{P}$ is prime.*

Proof Let F be maximal relative to c and assume that $a \notin F$ and $b \notin F$. Then, by Exercise 4.3, there are $d_1, d_2 \in F$ such that $d_1 \wedge a \leq c$ and $d_2 \wedge b \leq c$. Let $d = d_1 \wedge d_2$. Then $d \wedge a \leq c$ and $d \wedge b \leq c$ and therefore $(d \wedge a) \vee (d \wedge b) \leq c$. Hence, by distributivity, $d \wedge (a \vee b) \leq c$. Then $a \vee b \notin F$. □

The **Stone space** of $\mathbb{P}$ is a topological space that we denote by $S(\mathbb{P})$. The points of $S(\mathbb{P})$ are the prime filters of $\mathbb{P}$. The closed sets of the **Stone topology** are arbitrary intersections of sets of the form

$$[a]_{\mathbb{P}} \quad = \quad \Big\{ F \ : \ \text{prime filter such that } a \in F \Big\}.$$

for $a \in \mathbb{P}$. In other words, the sets above form a base of closed sets of the Stone topology. Using 1 and 3 in the following proposition the reader can easily check that this is indeed a base for a topology.

4.7 Proposition *For every $a, b \in \mathbb{P}$ we have*

1. $[0]_{\mathbb{P}} = \varnothing$;
2. $[1]_{\mathbb{P}} = S(\mathbb{P})$;
3. $[a]_{\mathbb{P}} \cup [b]_{\mathbb{P}} = [a \vee b]_{\mathbb{P}}$;
4. $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = [a \wedge b]_{\mathbb{P}}$.

Proof The verification is immediate. Only 3 requires that the filters in $S(\mathbb{P})$ are prime. □

The closed subsets of $S(\mathbb{P})$ ordered by inclusion form a distributive lattice. The following is a representation theorem for distributive lattices.

4.8 Theorem *The map $a \mapsto [a]_{\mathbb{P}}$ is an embedding of $\mathbb{P}$ in the lattice of the closed subsets of $S(\mathbb{P})$. In particular*

1. $0 \mapsto \varnothing$;
2. $1 \mapsto S(\mathbb{P})$;
3. $a \vee b \mapsto [a]_{\mathbb{P}} \cup [b]_{\mathbb{P}}$;
4. $a \wedge b \mapsto [a]_{\mathbb{P}} \cap [b]_{\mathbb{P}}$.

Proof It is immediate that the map above preserves the order and Proposition 4.7 shows that it preserves the lattice operations. We prove that the map is injective. Let $a \neq b$, say $a \nleq b$. We claim that $[a]_{\mathbb{P}} \nsubseteq [b]_{\mathbb{P}}$. There is a filter F that contains a and is maximal relative to b. By Proposition 4.6 such an F is prime. Then $F \in [a]_{\mathbb{P}} \smallsetminus [b]_{\mathbb{P}}$. $\square$

4.9 Theorem *With the Stone topology, $S(\mathbb{P})$ is a compact space.*

Proof Let $\langle [a_i]_{\mathbb{P}} : i \in I \rangle$ be basic closed sets such that for every finite $J \subseteq I$

a. $$\bigcap_{i \in J} [a_i]_{\mathbb{P}} \neq \varnothing.$$

We claim that

b. $$\bigcap_{i \in I} [a_i]_{\mathbb{P}} \neq \varnothing.$$

By 4 of Proposition 4.7 and a we obtain that $\curlywedge C \nleq 0$ for every finite $C \subseteq \{a_i : i \in I\}$. By Proposition 4.2, there is a maximal (relative to 0) filter containing $\{a_i : i \in I\}$. By Proposition 4.6, such filter is prime and it belongs to the intersection in b. $\square$

Let $a, b \in \mathbb{P}$. If $a \curlywedge b = 0$ and $a \curlyvee b = 1$, we say that b is the **complement** of a (and vice versa). The complement of an element need not exist. If the complement exists it is unique, the complement of a is denoted by $\neg a$.

4.10 Lemma *Let $U \subseteq S(\mathbb{P})$ be a clopen set. Then $U = [a]_{\mathbb{P}}$ for some $a \in \mathbb{P}$, and $\neg a$ exists.*

Proof As both U and $S(\mathbb{P}) \smallsetminus U$ are closed, for some sets $A, B \subseteq \mathbb{P}$

$$\bigcap_{x \in A} [x]_{\mathbb{P}} = U$$

$$\bigcap_{y \in B} [y]_{\mathbb{P}} = S(\mathbb{P}) \smallsetminus U.$$

By compactness, that is Theorem 4.9, there are some finite $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap_{x \in A_0} [x]_{\mathbb{P}} \cap \bigcap_{y \in B_0} [y]_{\mathbb{P}} = \varnothing$$

Let $a = \curlywedge A_0$ and $b = \curlywedge B_0$. From claim 4 of Proposition 4.7 we obtain $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = \varnothing$. Therefore $U \subseteq [a]_{\mathbb{P}} \subseteq S(\mathbb{P}) \smallsetminus [b]_{\mathbb{P}} \subseteq U$. Hence $U = [a]_{\mathbb{P}}$ and $b = \neg a$. $\square$

A **Boolean algebra** is a bounded distributive lattice where every element has a complement. In a Boolean algebra, the sets $[a]_{\mathbb{P}}$ are clopen and they form also a base of open set of the topology of $S(\mathbb{P})$. A topology that has a base of clopen sets is called **zero-dimensional**. By the following proposition the Stone topology of a Boolean algebra is Hausdorff.

A proper filter of a Boolean algebra is an **ultrafilter** if either $a \in F$ or $\neg a \in F$ for every a.

4.11 Proposition *Let $\mathbb{P}$ be a Boolean algebra. Then the following are equivalent*

1. *F is maximal;*
2. *F is prime;*
3. *F is an ultrafilter.*

Proof Implication 2 ⇒ 3 is obtained observing that $a \curlyvee \neg a \in F$. The rest is immediate. $\square$

4.12 Exercise Prove that the stone topology on $S(\mathbb{P})$ has a base of open compact sets. $\square$

4.13 Exercise Suppose we had defined $S(\mathbb{P})$ as the set of relatively maximal filters. What could possibly go wrong? $\square$

3 Types as filters

This section we work with a fixed set of formulas $\boldsymbol{\Delta}$ all with free variables among those of some fixed tuple $\boldsymbol{x}$. In this section we do not display x in the notation. Subsets of Δ are called **Δ-types**.

We associate to Δ a bounded lattice $\mathbb{P}(\boldsymbol{\Delta})$. This is the closure under conjunction and disjunction of the formulas in $\Delta \cup \{\bot, \top\}$. The (pre)order relation in $\mathbb{P}(\Delta)$ is given by

$$\psi \leq \varphi \quad \Leftrightarrow \quad T \cup \{\psi\} \vdash \varphi,$$

for some fixed theory T. In this section, to lighten notation, we absorb T in the symbol $\vdash$. If p is a Δ-type we denote by $\langle p \rangle_{\mathbb{P}}$ the filter in $\mathbb{P}(\Delta)$ generated by p.

4.14 Lemma *For every Δ-type p*

$$\langle p \rangle_{\mathbb{P}} \quad = \quad \Big\{ \varphi \in \mathbb{P}(\Delta) \quad : \quad p \vdash \varphi \Big\}.$$

In particular p is consistent if and only if $\langle p \rangle_{\mathbb{P}}$ is a proper filter.

Proof Inclusion $\subseteq$ is clear. Inclusion $\supseteq$ is a consequence of the Compactness Theorem. In fact $p \vdash \varphi$ implies that $\psi \vdash \varphi$ for some formula ψ that is conjunction of formulas in p. Then $\varphi \in \langle p \rangle_{\mathbb{P}}$ follows from $\psi \in \langle p \rangle_{\mathbb{P}}$ and $\psi \leq \varphi$. $\square$

We say that $p \subseteq \Delta$ is a **principal Δ-type** if $\langle p \rangle_{\mathbb{P}}$ is a principal. The following lemma is an immediate consequence of the Compactness Theorem. Note that in 3 the formula φ is arbitrary, possibly not even in $\mathbb{P}(\Delta)$.

4.15 Lemma *For every Δ-type p the following are equivalent*

1. *p is principal;*
2. *$\psi \ \vdash \ p \ \vdash \ \psi$ for some formula ψ (here ψ is any formula, it need not be in Δ);*
3. *$\varphi \ \vdash \ p$ where φ is conjunction of formulas in p.* $\square$

Proof Implications 1$\Rightarrow$2 is immediate by Lemma 4.14. To prove 2$\Rightarrow$3 suppose $\psi \vdash p \vdash \psi$. Apply compactness to obtain a formula φ, conjunction of formulas in p, such that $\varphi \vdash \psi$. Implications 3$\Rightarrow$1is trivial. $\square$

4.16 Definition *We say that $p \subseteq \Delta$ is a* ***prime Δ-type*** *if $\langle p \rangle_{\mathbb{P}}$ is a prime filter. We say that p is a* ***complete Δ-type*** *if $\langle p \rangle_{\mathbb{P}}$ is a maximal filter.* $\square$

Though in general neither Δ nor $\mathbb{P}(\Delta)$ are closed under negation, Lemma 4.14 has the following consequence.

4.17 Proposition *For every consistent Δ-type p the following are equivalent*

1. *p is complete;*

2. *p is consistent and either $p \vdash \varphi$ or $p \vdash \neg\varphi$ for every formula $\varphi \in \Delta$.*

Proof 2⇒1. Assume 2. As p is consistent, $\langle p \rangle_{\mathbb{P}}$ is a proper filter. To prove that it is maximal suppose $\varphi \notin \langle p \rangle_{\mathbb{P}}$. Then $p \nvdash \varphi$ and from 2 it follows that $p \vdash \neg\varphi$. Hence no proper filter contains $p \cup \{\varphi\}$.

1⇒2. As $\langle p \rangle_{\mathbb{P}}$ is proper, p is consistent. Suppose $p \nvdash \varphi$. Then $\varphi \notin \langle p \rangle_{\mathbb{P}}$ and, as $\langle p \rangle_{\mathbb{P}}$ is maximal, $p \cup \{\varphi\}$ generates the improper filter. Then $p \cup \{\varphi\}$ is inconsistent. Hence $p \vdash \neg\varphi$. □

Given a model M and a tuple $c \in M^{|x|}$ the **Δ-type of c in M** is the sets

$$\Delta\text{-tp}_M(c) \quad = \quad \Big\{\varphi(x) \in \Delta \; : \; M \vDash \varphi(c)\Big\}$$

When the model M is clear from the context we omit the subscript. When x and c are the empty tuple, we write $\mathrm{Th}_\Delta(M)$ for $\Delta\text{-tp}_M(c)$.

4.18 Lemma *For every Δ-type p the following are equivalent*

1. *p is prime;*
2. *$p \cup \big\{\neg\varphi \; : \; \varphi \in \Delta$ such that $p \nvdash \varphi\big\}$ is consistent;*
3. *$\big\{\varphi \in \Delta : p \vdash \varphi\big\} = \Delta\text{-tp}_M(c)$ for some model M and some tuple $c \in M^{|x|}$.*

Proof Implications 2⇒3⇒1 are clear, we prove 1⇒2. By compactness if the type in 2 is inconsistent then there are finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Delta$ such that $p \nvdash \varphi_i$ and

$$p \quad \vdash \quad \bigvee_{i=1}^{n} \varphi_i.$$

hence p is not prime. □

The following corollary is immediate. When it comes to verifying that a given Δ-type is prime, it simplifies the proof.

4.19 Corollary *For every Δ-type p the following are equivalent*

1. *p is prime;*
2. $p \;\vdash\; \bigvee_{i=1}^{n} \varphi_i \quad \Rightarrow \quad p \vdash \varphi_i$ *for some* $i \leq n$, *for every n and every $\varphi_1, \ldots, \varphi_n \in \Delta$.* □

The set Δ in this section contains only formulas with variables among those of the tuple x. In the following it is convenient to consider sets Δ that are closed under substitution of variables with any other variable. Then we may write $\Delta_{\restriction x}$ for the set of formulas in Δ with variables among those of the tuple x. But when x is clear from the context Δ may be used for $\Delta_{\restriction x}$. The set of prime Δ-types is denoted by $S(\Delta)$, and we write $S_x(\Delta)$ for $S(\Delta_{\restriction x})$.

The most common Δ used in the sequel is the set of all formulas in $L(A)$ and the underlying theory is $\mathrm{Th}(M/A)$ for some given model M containing A. In this case we write $\mathrm{tp}_M(c/A)$ for $\Delta\text{-tp}_M(c)$ or, when A is empty, $\mathrm{tp}_M(c)$. The set $S_x(\Delta)$ is denoted by $S_x(A)$. The topology on $S_x(A)$ is generated by the clopen

$$[\varphi(x)] \quad = \quad \Big\{p \in S_x(A) \; : \; \varphi(x) \in p\Big\}.$$

Sometimes Δ is the set of all formulas of a given syntactic form. Then we use some more suggestive notation that we summarize below.

4.20 Notation The following are some of the most common Δ-types and Δ-theories

1. **at-tp**(c), $\quad$ **Th**$_{\mathrm{at}}(M)$ $\quad$ when $\Delta = L_{\mathrm{at}}$
2. **at**$^{\pm}$**-tp**(c), $\quad$ **Th**$_{\mathrm{at}^{\pm}}(M)$ $\quad$ when $\Delta = L_{\mathrm{at}^{\pm}}$
3. **qf-tp**(c), $\quad$ **Th**$_{\mathrm{qf}}(M)$ $\quad$ when $\Delta = L_{\mathrm{qf}}$.

Clearly, the types/theories in 2 and 3 are equivalent. Most used is the theory $\mathrm{Th}_{\mathrm{at}^{\pm}}(M/M)$ which is called the **diagram of M** and has a dedicated symbol: **Diag**(M). □

4.21 Remark Let $A \subseteq M \cap N$. The following are equivalent

1. $N \vDash \mathrm{Diag}\langle A\rangle_M$;
2. $\langle A\rangle_M$ is a substructure of N. □

4 Morphisms

First we set the meaning that the word **map** has in these notes.

4.22 Definition *A **map** consists a triple $f : M \to N$ where*

1. *M is a set (usually a structure) called the **domain of the map**;*
2. *N is a set (usually a structure) called the **codomain the map**;*
3. *f is a function with **domain of definition** $\mathrm{dom}\, f \subseteq M$ and **image** $\mathrm{img}\, f \subseteq N$.*

*By **cardinality of $f : M \to N$** we understand the cardinality of the function f.* □

If $\mathrm{dom}\, f = M$ we say that the map is **total**; if $\mathrm{img}\, f = N$ we say that it is **surjective**. The **composition** of two maps and the **inverse** of a map are defined in the obvious way.

4.23 Definition *Let Δ be a set of formulas with free variables in the tuple $x = \langle x_i : i < \lambda\rangle$. The map $h : M \to N$ is a **Δ-morphism** if it **preserves the truth** of all formulas in Δ. By this we mean that*

p. $M \vDash \varphi(a) \;\Rightarrow\; N \vDash \varphi(ha)$ $\qquad$ *for every $\varphi(x) \in \Delta$ and every $a \in (\mathrm{dom}\, h)^{|x|}$.*

Notation: if a is the tuple $\langle a_i : i < \lambda\rangle$ then ha is the tuple $\langle ha_i : i < \lambda\rangle$. □

When $\Delta = L$ we say **elementary map** for Δ-morphism. When $\Delta = L_{\mathrm{at}}$ we say **partial homomorphism** and when $\Delta = L_{\mathrm{at}^{\pm}}$ we say **partial embedding** or **partial isomorphism**. The reason for the latter name is explained in Remark 4.24.

It is immediate to verify that a partial embedding which is total is an embedding (so, there is not conflict with Definition 2.19 above). Similarly, a partial embedding which is a total and surjective is an isomorphism. The precise connection between partial and total homo/iso-morphisms is discussed in following remark.

4.24 Remark For every map $h : M \to N$ the following are equivalent

1. $h : M \to N$ is a partial isomorphism;
2. there is an isomorphism $k : \langle \operatorname{dom} h \rangle_M \to \langle \operatorname{img} h \rangle_N$ that extends h.

Moreover, k unique. The equivalence holds replacing isomorphism by homomorphism (in which case we in 2 we obtain an surjection). The extension k is obtained defining $k\big(t(a)\big) = t(ha)$.

A similar fact holds for partial homomorphims with epimorphism (surjective homomorphism) for isomorphism.

Unfortunately, there is no hope to generalize this fact to Δ-morphisms as, in general, there is no notion of generated Δ-elementary substructure. $\square$

We use Δ-morphisms to compare, locally, two structures (or two loci of the same structure). There are different ways to do this, in the proposition below we list a few synonymous expressions. But first some more notation. When x is a fixed tuple of variables, $a \in M^{|x|}$ and $b \in N^{|x|}$ we write

$$\boldsymbol{M, a \ \Rrightarrow_\Delta \ N, b} \qquad \text{if } M \vDash \varphi(a) \Rightarrow N \vDash \varphi(b) \text{ for every } \varphi(x) \in \Delta.$$

$$\boldsymbol{M, a \ \equiv_\Delta \ N, b} \qquad \text{if } M \vDash \varphi(a) \Leftrightarrow N \vDash \varphi(b) \text{ for every } \varphi(x) \in \Delta.$$

The following equivalences are immediate and will be used without explicit reference.

4.25 Proposition *For every given set of formulas Δ and every map $h : M \to N$ the following are equivalent*

1. *$h : M \to N$ is a Δ-morphism;*
2. *$M, a \Rrightarrow_\Delta N, ha$ for every $a \in (\operatorname{dom} h)^{|x|}$;*
3. *Δ-$tp_M(a) \subseteq \Delta$-$tp_N(ha)$ for every $a \in (\operatorname{dom} h)^{|x|}$;*
4. *$N, ha \vDash p(x)$ for every $a \in (\operatorname{dom} h)^{|x|}$ and $p(x) = \Delta$-$tp_M(a)$.* $\square$

4.26 Remark Condition p in Definition 4.23 apply to tuples x of any length, in particular to the empty tuple. In this case $\varphi(x)$ is a sentence, $a \in (\operatorname{dom} h)^0 = \{\varnothing\}$ is the empty tuple, and p asserts that $\mathrm{Th}_\Delta(M) \subseteq \mathrm{Th}_\Delta(N)$. When $h = \varnothing$ this is actually all what p says. In fact $\varnothing^{|x|} = \varnothing$ unless $|x| = 0$. Still, $\mathrm{Th}_\Delta(M) \subseteq \mathrm{Th}_\Delta(N)$ may be a non trivial requirement. $\square$

4.27 Definition *We call $Th_\Delta(M)$ the **Δ-characteristic** of M. We say that T **decides** the Δ-characteristic if $Th_\Delta(M) = Th_\Delta(N)$ for all $M, N \vDash T$. In other words, for any pair of models of T, the empty map $\varnothing : M \to N$ is a Δ-morphism. When $\Delta = L_{\mathrm{at}^\pm}$ we say simply **characteristic**, then T decides the characteristic $\langle \varnothing \rangle_M \simeq \langle \varnothing \rangle_N$ for all $M, N \vDash T$.* $\square$

We conclude this section with a couple of propositions that break Theorem 2.20 into parts. More interestingly, in Chapter 9 we shall prove a sort of converse of Propositions 4.29 and 4.30.

This is not immediately required in the following. There is a relation between some properties of Δ-morphisms closure of Δ logical connectives. When $\mathsf{C} \subseteq \{\forall, \exists, \neg, \vee, \wedge\}$ is a set of connectives, we write **$\mathsf{C}\Delta$** for the closure of Δ with respect to all connectives in C. We write **$\neg\Delta$** for the set containing the negation of the formulas in Δ. Warning: do not confuse $\neg\Delta$ with $\{\neg\}\Delta$.

It is clear that Δ-morphisms are $\{\wedge,\vee\}\Delta$-morphisms.

4.28 Proposition *For every given set of formulas Δ and every injective map $h : M \to N$ the following are equivalent*

a. *$h : M \to N$ is a $\neg\Delta$-morphism;*

b. *$h^{-1} : M \to N$ is a Δ-morphism.* □

4.29 Proposition *For every set of formulas Δ, every Δ-embedding $h : M \to N$ is a $\{\exists\}\Delta$-morphism.*

Proof Formulas in $\{\exists\}\Delta$ ave the form $\exists y\, \varphi(x,y)$ where y is a finite tuples of variables and $\varphi(x,y) \in \Delta$. For every tuple $a \in (\operatorname{dom} h)^{|x|}$ we have:

$$\begin{aligned}
M \vDash \exists y\, \varphi(a,y) \quad &\Rightarrow \quad M \vDash \varphi(a,b) \text{ per una tupla } b \in M^{|y|}\\
&\Rightarrow \quad N \vDash \varphi(ha,hb)\\
&\Rightarrow \quad N \vDash \varphi(ha,c) \text{ per una tupla } c \in N^{|y|}\\
&\Rightarrow \quad N \vDash \exists y\, \varphi(ha,y).
\end{aligned}$$

Note that the second implication requires the totality of $h : M \to N$ which guarantees that $\varphi(a,y)$ has a solution in $\operatorname{dom} h$. □

When $h : M \to N$ is injective the following is a corollary of Propositions 4.28 and 4.29. The general proof is the 'dual' of that of Propositions 4.29.

4.30 Proposition *For every set of formulas Δ, every surjective Δ-morphism $h : M \to N$ is a $\{\forall\}\Delta$-morphism.* □

Chapter 5

Some relational structures

In the first section we prove that theory of *dense linear orders without endpoints* is *ω-categorical*. That is, any two such countable orders are isomorphic. This is an easy classical result of Cantor. In this chapter we examine Cantor's construction (a so-called *back-and-forth* construction) in great detail. In the second section we apply the same technique to prove that the theory of the *random graph* is ω-categorical.

1 Dense linear orders

The **language of strict orders**, which in this section we denote by L, contains only a binary relation symbol $<$. A structure M of signature L is a **strict order** if it models (the universal closure of) the following formulas

1. $x \not< x$ — irreflexive;
2. $x < z < y \ \rightarrow \ x < y$ — transitive.

Note that the following is an immediate consequence of 1 and 2.

$x < y \ \rightarrow \ y \not< x$ — antisymmetric.

We say that the order is **total** or **linear** if

li. $x < y \ \vee \ y < x \ \vee \ x = y$ — linear or total.

An order is **dense** if

nt. $\exists x, y \ (x < y)$ — non trivial;

d. $x < y \ \rightarrow \ \exists z \ (x < z < y)$ — dense.

We need to require the existence of two comparable elements, then d implies that dense orders are in fact infinite. We say that the ordering has no **endpoints** if

e. $\exists y \ (x < y) \ \wedge \ \exists y \ (y < x)$ — without endpoints.

We denote by $\boldsymbol{T_{\mathrm{lo}}}$ the theory strict linear orders and by $\boldsymbol{T_{\mathrm{dlo}}}$ the theory of dense linear orders without endpoints. Clearly, these are consistent theories: $\mathbb{Q}$ with the usual ordering is a model of T_{dlo}.

We introduce some notation to improve readability of the proof of the following theorem. Let A and B be subsets of an ordered set. We write $A < B$ if $a < b$ for every $a \in A$ and $b \in B$. We write $a < B$ e $A < b$ for $\{a\} < B$, respectively $A < \{b\}$. Let $M \vDash T_{\mathrm{lo}}$. Then $M \vDash T_{\mathrm{dlo}}$ if and only if for every finite $A, B \subseteq M$ such that $A < B$ there is a c such that $A < c < B$. In fact axiom d is evident and axioms nt and e are obtained taking replacing A and/or B by the empty set.

Now we prove the first of a series of lemmas that we call **extension lemmas**. Recall that in the language of strict orders a map $k : M \to N$ is a partial isomorphism if

$$M \vDash a < b \Leftrightarrow N \vDash ka < kb \qquad \text{for every } a, b \in \operatorname{dom} k.$$

(When $M, N \vDash T_{\text{lo}}$ direction $\Rightarrow$ suffices.)

5.1 Lemma *Fix $M \vDash T_{\text{lo}}$ and $N \vDash T_{\text{dlo}}$. Let $k : M \to N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \to N$ that extends k and is defined in b.*

Proof Given a finite partial isomorphism $k : M \to N$ define

$$X = \{x \in \text{dom}(k) : x < b\};$$
$$Y = \{y \in \text{dom}(k) : b < y\}.$$

The sets X and Y are finite and partition $\text{dom}\, k$, and $X < Y$. As $k : M \to N$ is a partial isomorphism, $k[X] < k[Y]$. Then in N there is an element c such that $k[X] < c < k[Y]$. It is easy to check that setting $h = k \cup \{\langle b, c\rangle\}$ we obtain the required extension. □

The following is an equivalent version of Lemma 5.1.

5.2 Corollary *Let $M \vDash T_{\text{lo}}$ be countable and let $N \vDash T_{\text{dlo}}$. Let $k : M \to N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \to N$ that extends k.*

Proof Let $\langle a_i : i < \omega\rangle$ be an enumeration of M. Define by induction a chain of finite partial isomorphisms $h_i : M \to N$ such that $a_i \in \text{dom}\, h_{i+1}$. The construction starts with $h_0 = k$. At stage $i + 1$ we chose any finite partial isomorphism $h_{i+1} : M \to N$ that extends h_i and is defined in a_i. This is possible by Lemma 5.1. In the end we set

$$h = \bigcup_{i \in \omega} h_i.$$

It is immediate to verify that $h : M \to N$ is the required embedding. □

5.3 Exercise Prove that the extension Lemma 5.1 characterizes models of T_{dlo} among models of T_{lo}. That is, if N is a model of T_{lo} such that the conclusion of Lemma 5.1 holds, then $M \vDash T_{\text{dlo}}$. □

We are now ready to prove that any two countable models of T_{dlo} are isomorphic which is a classical result of Cantor's. Actually what we prove is slightly more general than that. In fact Cantor's theorem is obtained from the theorem below by setting $k = \varnothing$, which we are allowed to, because all models have the same empty characteristic (cfr. Remark 4.26).

5.4 Theorem *Every finite partial isomorphism $k : M \to N$ between models of T_{dlo} extends to an isomorphism $g : M \to N$.*

The following is the archetypal **back-and-forth** construction. It is important to note that it does not mention linear orders at all. It only uses the extension Lemma 5.1. In many other contexts, as soon as we have an extension lemma, we can apply verbatim the same construction (cfr. Theorem 6.6).

Proof Fix some enumerations of M and N, say $\langle a_i : i < \omega\rangle$, respectively $\langle b_i : i < \omega\rangle$. We define by induction a chain of finite partial isomorphisms $g_i : M \to N$ such that $a_i \in \text{dom}\, g_{i+1}$ and $b_i \in \text{img}\, g_{i+1}$. In the end we set

$$g = \bigcup_{i \in \omega} g_i$$

We begin by setting $g_0 = k$. The inductive step consists of two half-steps that we may call the *forth-step* and *back-step*. In the forth-step we define $g_{i+1/2}$ such that $a_i \in \operatorname{dom} g_{i+1/2}$. in the back-step to define g_{i+1} such that $b_i \in \operatorname{img} g_{i+1}$

By the extension lemma 5.1 there is a finite partial isomorphism $g_{i+1/2} : M \to N$ that extends g_i and is defined in a_i. Now apply the same lemma to extend $(g_{i+1/2})^{-1} : N \to M$ to a finite partial isomorphism $(g_{i+1})^{-1} : N \to M$ defined in b_i. □

Let λ be an infinite cardinal. We say that a theory is **λ-categorical** if any two models of T of cardinality λ are isomorphic. From Theorem 5.4, taking $k = \varnothing$, we obtain the following.

5.5 Corollary *The theory T_{dlo} is ω-categorical.* □

We also obtain that T_{dlo} is a complete theory. This is consequence of the following general fact.

5.6 Proposition *If T is λ-categorical, for some $\lambda \geq |L|$, then T is complete.*

Proof Let M and N be any two models of T_{dlo}. Applying the upward and/or downward Löwenheim-Skolem theorem, we may assume they both have cardinality λ. (Here we use that $\lambda \geq |L|$.) Hence $M \simeq N$ and in particular $M \equiv N$. □

5.7 Exercise Prove that T_{dlo} is not λ-categorical for any uncountable λ. □

5.8 Exercise Prove that $\mathbb{R}$, in the language of strict orders, is not isomorphic to $\mathbb{R} \smallsetminus \{0\}$. □

5.9 Exercise Let L be the language of strict orders augmented with countably many constants $\{c_i : i \in \omega\}$. Let T be the theory that extends T_{dlo} with the axioms $c_i < c_{i+1}$ for all i. Prove that T is complete. Find three non isomorphic models of this theory. For N a rightly chosen model of T, prove the statement in Lemma 5.1, with M any model of T. □

5.10 Exercise Show that in Theorem 5.6 the assumption $\lambda \geq |L|$ is necessary. (Hint. Let ν be an uncountable cardinal. The language contains only the ordinals $i < \nu$ as constants. The theory T says that there are infinitely many elements and either $i = 0$ for every $i < \nu$, or $i \neq j$ for every $i < j < \nu$. Prove that T is ω-categorical but incomplete.) □

2 Random graphs

Recall that the **language of graphs**, which in this section we denote by L, contains only a binary relation r. A **graph** structure of signature L such that

1. $\neg r(x, x)$ irreflexive;
2. $r(x, y) \to r(y, x)$ symmetric.

An element of a graph M is called a **vertex** or a **node**. An **arc** is an unordered pair of vertices $\{a, b\} \subseteq M$ such that $M \vDash r(a, b)$. In words we may say that a is adjacent to b.

A **random graph** is a graph that also satisfies the following axioms for every n

nt. $\exists x, y, z \ \ (x \neq y \neq z \neq x)$ non trivial;

r$_n$. $\displaystyle\bigwedge_{i,j=1}^{n} x_i \neq y_j \ \ \rightarrow \ \ \exists z \bigwedge_{i=1}^{n} \big[r(x_i, z) \wedge \neg r(z, y_i)\big]$ for every $n \in \mathbb{Z}^+$.

The theory of graphs is denoted by $\boldsymbol{T_{\rm gph}}$ and the theory of random graphs is denoted by $\boldsymbol{T_{\rm rg}}$. The scheme of axioms r$_n$ plays the same role of density in the previous section. It says that given two non empty disjoint sets A^+ and A^- of cardinality $\leq n$ there is a vertex z that is adjacent to all vertices in A^+ and to no vertex in A^-. Note that it might well be that $z \in A^-$, however $z \in A^+$ would contradict 1.

Some readers may doubt if $T_{\rm rg}$ is consistent.

5.11 Proposition *There exists a random graph.*

Proof The domain of N is the set of natural numbers. Let $N \vDash r(n, m)$ hold if the n-th prime number divides m or vice versa m-th prime number divides n. □

5.12 Proposition *Random graphs are infinite.*

Proof Suppose for a contradiction that M is a finite random graph. As M contains at least 3 elements, we can fix two distinct elements $a, b \in M$ and require a z adjacent to all elements of $M \smallsetminus \{a, b\}$ and no element of $\{a, b\}$. A remarked above, $z \in \{a, b\}$, then a and b are not adjacent. Now let w be adjacent to all vertices in $M \smallsetminus \{a\}$ but not to a. The only option is $w = b$, but this implies that a is adjacent to b. Contradiction. □

The following is the analogous of Lemma 5.1 for random graphs. Recall that in the language of graphs a map $k : M \to N$ is a partial isomorphism if

$$M \vDash r(a, b) \ \Leftrightarrow \ N \vDash r(ka, kb) \qquad \text{for every } a, b \in \operatorname{dom} k.$$

5.13 Lemma *Fix $M \vDash T_{\rm gph}$ and $N \vDash T_{\rm rg}$. Let $k : M \to N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \to N$ that extends k and is defined in b.*

Proof The schema of the proof is the same as in Lemma 5.1, so we use the same notation. Assume $b \notin \operatorname{dom} k$ and define

$$A^+ = \big\{x \in \operatorname{dom} k \ : \ M \vDash r(x, b)\big\} \quad \text{e} \quad A^- = \big\{y \in \operatorname{dom} k \ : \ M \vDash \neg r(y, b)\big\}.$$

These two sets are finite and disjoint, then so are $k[A^+]$ and $k[A^-]$. Then there is a c such that

$$\bigwedge_{a \in A^+} r(ka, c) \ \ \wedge \ \ \bigwedge_{a \in A^-} \neg r(ka, c).$$

As $k[A^+] \cup k[A^-] = \operatorname{img} k$, it is immediate to verify that $h = k \cup \big\{\langle b, c\rangle\big\}$ is the require extension. □

With the same proof of Corollary 5.2 obtain the following.

5.14 Corollary *Let $M \vDash T_{\rm gph}$ be countable and let $N \vDash T_{\rm rg}$. Let $k : M \to N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \to N$ that extends k.* □

The proof of Theorem 5.4 gives the following theorem and its corollary.

5.15 Theorem *Every finite partial isomorphism $k : M \to N$ between models of $T_{\rm rg}$ extends to an isomorphism $g : M \to N$.* $\square$

5.16 Corollary *The theory $T_{\rm rg}$ is ω-categorical (and therefore complete).* $\square$

5.17 Exercise Let $a, b, c \in N \vDash T_{\rm rg}$. Prove that $r(a, N) = r(b, N) \cap r(c, N)$ occurs only in the trivial case $a = b = c$. $\square$

5.18 Exercise Prove that for every $b \in N \vDash T_{\rm rg}$ the set $r(b, N)$ is a random graph. Assume N is countable. Is every random graph $M \subseteq N$ of the form $r(b, N)$ for some $b \in N$? $\square$

5.19 Exercise Let N be free union of two random graphs. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$, with N_1 and N_2 some countable random graphs. By $\sqcup$ we denote the disjoint union. Prove that N is not a random graph. Show that N_1 is not definable without parameters. Write a first order sentence $\psi(x, y)$ true if x and y belong to the same connected component of N. Axiomatize the class $\mathcal{K}$ of graphs that are free union of two random graphs. $\square$

5.20 Exercise Prove that $T_{\rm rg}$ is not λ-categorical for any uncountable λ. Hint. Prove that there is a random graph N of cardinality λ where every vertex is adjacent to $< \lambda$ vertices. Compare it with its complement graph (the graph that has arcs between pairs that are non adjacent in N). $\square$

5.21 Exercise Prove that $T_{\rm rg}$ is not finitely axiomatizable. Hint. Given a random graph N and a set P of cardinality $n + 1$ show that you can add arcs to $M \sqcup P$ and make it satisfy axiom r_n but not r_{n+1}. $\square$

5.22 Exercise Let $A \subseteq N \vDash T_{\rm rg}$ and let $\varphi(x) \in L(A)$, where $|x| = 1$. Prove that if $\varphi(N)$ is finite then $\varphi(N) \subseteq A$. $\square$

5.23 Exercise (Peter J. Cameron) Prove that for every countable graph M the following are equivalent

1. M is either the infinite complete graph, the infinite empty graph, or the random graph;
2. if $M_1, M_2 \subseteq M$ are such that $M_1 \sqcup M_2 = M$, then $M_1 \simeq M$ or $M_2 \simeq M$.

Hint. For 2⇒1 first show that if $r(a, M) = \varnothing$ for some $a \in M$ then the graph is null and if $\{b\} \cup r(b, M) = M$ for some b then the graph is complete. Also note that 2 implies that a similar property holds for any finite partition of M, not just bi-partitions. It follows that if 2 and there is a finite A such that $\bigwedge_{a \in A} r(a, M) = \varnothing$ then the graph is null and if there is a finite B such that $B \cup \bigvee_{b \in B} r(b, M) = M$ then the graph is complete.

Suppose M is not a random graph. Fix some finite, disjoint A and B such that no c satisfies both $r(A, c) = A$ and $r(B, c) = \varnothing$. Let $M_1 = \{c : r(A, c) \neq A\} \smallsetminus B$

and $M_2 = B \cup \{c : r(B,c) \neq \varnothing\} \setminus M_1$. This is a partition of M. Now note that $\bigwedge_{a \in A} r(a, M_1) = \varnothing$ and $\bigvee_{b \in B} r(b, M_2) = M_2$. $\square$

3 Notes and references

The following is a well-written accessible survey on the amazing model theoretic properties of the random graph.

[1] Peter J. Cameron, *The random graph* (2013), arXiv:1301.7544.

Chapter 6

Fraïssé limits

We introduce *Fraïssé limits*, aleas *homogeneous-universal* or *generic* structures, which here we call *rich* models, after Poizat. Rich models generalize the examples in Chapters 5 and the many more to come. We also prove *elimination of quantifiers* for the theories introduced in Chapters 5.

1 Rich models.

We now define *categories of models and partial morphisms*. These are example of concrete categories as intended in category theory. However, apart from the name, in what follows we dispense with all notions of category theory as they would make the exposition less basic than intended (without providing more technical instruments). We refer the interested reader to the references at the end of the chapter.

⚠ Warning: the terminology introduced in this section is not standard. In the literature many details are left implicit. This is generally safe when the category used is fixed. Here we prefer to be more explicit because, when discussing saturation and quantifier elimination, it helps to compare different categories.

A **category (of models and partial morphisms)** is a class $\mathcal{M}$ which is disjoint union of two classes: $\mathcal{M}_{\mathbf{ob}}$ and $\mathcal{M}_{\mathbf{ar}}$. The first is the class of **objects** and contains structures with a common signature L which we call **models**. The second is the class of **arrows** and contains (partial) maps between models which we call **morphisms**. We require that the identity maps are morphisms and that composition of two morphism is again morphism. This makes $\mathcal{M}$ a well-defined category.

For example, $\mathcal{M}$ could consist of all models of some theory T_0 and of all partial isomorphisms between these. Alternatively, as morphisms we could take elementary maps between models. At a first reading the reader may assume $\mathcal{M}$ is as in one of these two examples. In the general case we need to make some assumptions on $\mathcal{M}$.

6.1 Definition *For ease of reference we list together all properties required below*

c0. *there is a morphism between any two models M and N;*

c1. *the identity map $id_A : M \to M$ is a morphism, for any $A \subseteq M$;*

c2. *morphisms are invertible maps and the inverse of a morphism is a morphism;*

c3. *morphisms preserve the truth of $L_{\rm at}$-formulas;*

c4. *if M is a model, every elementary map $k : M \to N$ is morphism (and, N is a model);*

c5. *if $k_i : M \to N$ is a chain of morphisms, then $\bigcup_{i<\lambda} k_i : M \to N$ is a morphism.*

c6. *if $k' : M \to N$ is a morphism for every finite $k' \subseteq k$, then $k : M \to N$ is a morphism.* □

We say that a category is **connected** if c0 holds. Axiom c1 implies that any restriction of a morphism is a morphism. Given `c1`, we can rephrase `c0` by saying that $\varnothing : M \to N$ is a morphism for any pair of models M and N.

A consequence of `c4` is that the class of models is closed under elementary equivalence. In fact, if $N \equiv M$ then $\varnothing : M \to N$ is elementary, hence it is a morphism and, in particular, and N is a model.

We call `c6` the **finite character of morphisms**. Note that it implies `5`. We shall never use `6` in these notes, but most natural categories of models and partial morphisms have this property.

The following two definitions assume `c2`. The generalization to non injective morphisms is not straightforward (in fact, there are two generalizations: *projective* and *inductive*). These generalizations are not very common and will not be considered here.

6.2 Definition *Assume that $\mathcal{M}$ satisfies `c2` of Definition 6.1. We say that a model N is **λ-rich** if for every model M of cardinality $\leq \lambda$ and every morphism $k : M \to N$ of cardinality $< \lambda$ there is an embedding $h : M \hookrightarrow N$ that extends k. We say that N is **rich** tout court if it is λ-rich for $\lambda = |N|$. When $\mathcal{M}_{\rm ob} = \mathrm{Mod}(T_0)$ for some theory T_0 and $\mathcal{M}_{\rm ar}$ is clear from the context, we say **rich model of T_0**.* □

Rich models are also called *Fraïssé limits* or *homogeneous-universal* for a reason that will soon be clear; they are also called *generic*. Unfortunately these names are either too long or too generic, so we opt for the less common term *rich* proposed by Poizat.

The following two notions are closely connected with richness.

6.3 Definition *Assume that $\mathcal{M}$ satisfies `c2` of Definition 6.1. We say that a model N is **λ-universal** if for every model M of cardinality $\leq \lambda$ in the same connected component of N there is an embedding $k : M \hookrightarrow N$. We say that a model N is **λ-homogeneous** if every $k : N \to N$ of cardinality $< \lambda$ extends to a bijective morphism $h : N \hookrightarrow\!\!\!\!\!\to N$ (an automorphism when `c3` below holds).*

*Note that the larger $\mathcal{M}_{\rm ar}$, the stronger notion of homogeneity. When $\mathcal{M}_{\rm ar}$ contains all partial isomorphisms between models (the largest class of morphisms considered here), it is common to say **λ-ultrahomogeneous** for λ-homogeneous.*

*As above, when $\lambda = |N|$ we say **universal**, respectively **homogeneous** and **ultrahomogeneous**.* □

In Section 5.1 we implicitly used $\mathcal{M}_{\rm ob} = \mathrm{Mod}(T_{\rm lo})$ and partial isomorphisms as $\mathcal{M}_{\rm ar}$. In Section 5.2 we used $\mathcal{M}_{\rm ob} = \mathrm{Mod}(T_{\rm gph})$ and again partial isomorphisms as $\mathcal{M}_{\rm ar}$. Corollary 5.2 proves that every model of $T_{\rm dlo}$ is ω-rich. Corollary 5.14 claims the analogous fact for $T_{\rm rg}$.

In the following we fequently work under the following assumption (even when not all properties are strictly necessary).

6.4 Assumption *Assume $|L| \leq \lambda$ and suppose that $\mathcal{M}$ satisfies `c1`-`c5` of Definition 6.1*

The assumption on the cardinality L is only necessary to apply the downward Löwenheim-Skolem Theorem when required.

6.5 Proposition *(Assume 6.4) The following are equivalent*

1. *N is a λ-rich model;*
2. *for every morphism $k : M \to N$ of cardinality $< \lambda$ and every $b \in M$ there is a morphism $h : M \to N$ that extends k and is defined in b.*

Proof Closure under union of chains of morphisms, which is ensured by c5, immediately yields 2⇒1. As for implication 1⇒2 we only need to consider the case $\lambda < |M|$. By the downward Löwenheim-Skolem theorem there is an $M' \preceq M$ of cardinality λ containing b. Let $h : M' \hookrightarrow N$ be the embedding obtained from 1. By c4, the map $h : M \to N$ is a composition of morphisms, hence a morphism. □

The following theorem subsumes both Theorem 5.4 and Theorem 5.15.

6.6 Theorem *(Assume 6.4) Let M and N be two rich models of the same cardinality λ. Then every morphism $k : M \to N$ of cardinality $< \lambda$ extends to an isomorphism.*

Proof When $\lambda = \omega$, we can take the proof of Theorem 5.4 and replace *partial isomorphism* by *morphism* and the references to Lemma 5.1 by references to Proposition 6.5. As for uncountable λ, we only need to extend the construction through limit stages. By c5 we can simply take the union. □

6.7 Corollary *(Assume 6.4) Then all rich models of cardinality λ in the same connected component are isomorphic.* □

It is obvious that λ-rich models are λ-universals and, by Theorem 6.6, they are λ-homogeneous. These two notions are weaker than richness. For instance, when $\mathcal{M}$ is as in Section 5.2, the countable graph that has no arc is trivially ultrahomogeneous but it is not universal and a fortiori not rich. On the other hand if we add to a countable random graph an isolated point we obtain an universal graph which is not ultrahomogeneous. However, when taken together, these two properties are equivalent to richness.

6.8 Theorem *(Assume 6.4) The following are equivalent:*

1. *N is rich;*
2. *N is homogeneous and universal.*

Proof Implication 1⇒2 is clear as noted above, so we prove 2⇒1. Let $k : M \to N$ be a morphism of cardinality $< |N|$ and let $b \in M$. As N is universal, there is a morphism $f : M \to N$ defined on $\operatorname{dom} k \cup \{b\}$. By c2 the map $k \circ f^{-1} : N \to N$ is a morphism of cardinality $< |N|$. By homogeneity it has an extension to a bijective morphism $h : N \hookrightarrow\!\!\!\!\rightarrow N$. It is immediate that $h \circ f : M \to N$ is the required extension of k. □

6.9 Exercise Let N be the structure obtained by adding to a countable random graph an isolated point. Show that this graph is homogeneous if morphisms are elementary maps but not if morphisms are simply partial isomorphisms. □

Next theorem is one of the most important facts about λ-rich models

6.10 Theorem *(Assume 6.4) Then every morphism between λ-rich models is elementary. In particular, λ-rich models in the same connected component are elementary equivalent.*

Proof Let $k : M \to N$ be a morphism between rich models. It suffices to prove that every finite restriction of k is elementary. By c1, we may as well assume that k itself is finite. It suffices to construct $M' \preceq M$ and $N' \preceq N$ together with a morphism $h : M \to N$ that extends k and maps M' bijectively to N'. Then by c4 the map $h : M' \hookrightarrow\!\!\!\!\rightarrow N'$ is also a morphism because it is the composition of morphisms. Finally, by c3, it is an isomorphism, in particular an elementary map.

We construct simultaneously a chain $\langle A_i : i{<}\lambda \rangle$ of subsets of M, a chain $\langle B_i : i{<}\lambda \rangle$ of subsets of N, and a chain functions $\langle h_i : i{<}\lambda \rangle$ such that $h_i : M \to N$ are morphisms, $\operatorname{dom} h_i \subseteq A_i$ and $\operatorname{img} h_i \subseteq B_i$. In the end we set

$$M' \ = \ \bigcup_{i<\lambda} A_i, \qquad N' \ = \ \bigcup_{i<\lambda} B_i, \qquad h \ = \ \bigcup_{i<\lambda} h_i.$$

The chains start with $A_0 = \operatorname{dom} k$, $B_0 = \operatorname{img} k$ and $h_0 = k$. As usual at limit stages we take the union. Now we consider successor stages. Fix some enumerations of M and N. At stage $i+1$ we also fix some enumerations of A_i, B_i, $L_x(A_i)$, and $L_x(B_i)$, where $|x| = 1$. Let $\langle i_1, i_2 \rangle$ be the i-th pair of ordinals $< \lambda$. If the i_2-th formula in $L_x(A_{i_1})$ is consistent in M, pick any solution a. Also, if the i_2-th formula in $L_x(B_{i_1})$ is consistent in N, pick any solution b. Let a' be the i_2-th element of A_{i_1} and let b' be the i_2-th element of B_{i_1}.

Let $h_{i+1} : M \to N$ be an extension of h_i such that $\operatorname{dom} h_{i+1} = \operatorname{dom} h_i \cup \{a'\}$ and $\operatorname{img} h_{i+1} = \operatorname{img} h_i \cup \{b'\}$. Note that the latter requirement is met applying Proposition 6.5 to $h_i^{-1} : N \to M$. Define $A_{i+1} = A_i \cup \{a, h_{i+1}^{-1} b'\}$ and $B_{i+1} = B_i \cup \{b, h_{i+1} a'\}$.

The elements a and b, added to M' and N', ensure that $M' \preceq M$ and $N' \preceq N$, by the Tarski-Vaught test. In fact every formula in $L(M')$ belongs to $L(A_{i_1})$ for some i_1 and it occurs as i_2- formula in the enumeration that has been fixed along with the construction. Therefore at stage $i = \langle i_1, i_2 \rangle$ a witness is added to M'. The same happens for formulas in $L(N')$.

A similar argument shows that the elements a' and b', added to the domain and range of h, ensure that the $h : M' \hookrightarrow\!\!\!\!\rightarrow N'$ is both total and surjective. □

The **theory of rich models** of $\mathcal{M}$ is the set T_1 of sentences that hold in all rich models (or equivalently $|L|$-rich models), assuming they exist. By the theorem above, T_1 is complete as soon as $\mathcal{M}$ is connected.

6.11 Exercise Let L be the language of strict orders. Prove that $\mathbb{Q} \preceq \mathbb{R}$. □

Assume for simplicity that L is countable. Then all ω-rich models belong to $\operatorname{Mod}(T)$ for some complete theory T. It is interesting to ask if the converse is true: if T is the theory of the ω-rich models of $\mathcal{M}$, do all models in $\mathcal{M} \cap \operatorname{Mod}(T)$ are rich? The answer is affirmative when T is either T_{dlo} or T_{rg}, but this is not always the case, see Example 6.14 for an easy counterexample. An affirmative answer implies that the theory of rich models is ω-categorical. An interesting variant of this question is considered in Theorem 8.8.

For convenience of the reader, we instantiate Theorem 6.10 with the two categories used in Chapter 5.

6.12 Corollary *Partial isomorphisms between models of $T_{\rm dlo}$ and between models of $T_{\rm rg}$ are elementary maps.* □

When partial isomorphism between models of a given theory T coincide with elementary maps, it is always by a fundamental reason. Let us introduce some terminology. Let T be a consistent theory. We say that ***T* has (or admits) elimination of quantifiers** if for every formula $\varphi(x) \in L$ there is a quantifier-free formula $\psi(x) \in L_{\rm qf}$ such that

$$T \vdash \psi(x) \leftrightarrow \varphi(x).$$

We will discuss general criteria for elimination of quantifiers in Chapter 9, see for instance Corollary 9.10 and 9.11. Here we report without proof the following theorem.

6.13 Theorem *The following are equivalent*

1. *T has elimination of quantifiers;*
2. *every partial isomorphism between models T is an elementary map.* □

For the time being when saying that T *has quantifier elimination* we intend 2 of the theorem above.

In the next chapter we introduce important examples of ω-rich models that do not have an ω-categorical theory. These are algebraic structures (groups, fields etc.) hence more complex than pure relational structures. So, we conclude this section with an example of this phenomenon in almost trivial context.

6.14 Example Let L contain a unary predicate r_n for every positive integer n. The theory T_0 contains the axioms $\neg\exists x\,\big[r_n(x) \wedge r_m(x)\big]$ for $n \neq m$ and $\exists^{\leq n} x\, r_n(x)$ for every n. Work in the the category of models of T_0 and partial isomorphisms. Let T be the theory that extends T_0 with the axioms $\exists^{=n} x\, r_n(x)$ for every n. Let $q(x)$ be the type $\big\{\neg r_n(x) : n \in \omega\big\}$. There are models of T that do not realize $q(x)$, hence T is not ω-categorical. It is easy to verify that the following are equivalent.

1. N is ω-rich model;
2. $N \vDash T$ and $q(N)$ is infinite.

The reader may use Theorem 6.10 to prove that T is complete and has elimination of quantifiers. It is also easy to verify that every uncountable model of T is rich and consequently that T is uncountably categorical. □

6.15 Exercise Let T_0 and $\mathcal{M}$ be as in Example 6.14 except that we restrict the language to the relations $r_0, \ldots, r_n$ for a fixed n. Do ω-rich models of T_0 exist? Is their theory ω-categorical? What if we add to T_0 the axiom $r_0(x) \vee \cdots \vee r_n(x)$? □

6.16 Exercise The language contains only the binary relations $<$ and e. The theory T_0 says that $<$ is a strict linear order and that e is an equivalence relation. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? Is it ω-categorical? □

6.17 Exercise The language contains only two binary relations. The theory T_0 says that they are equivalence relations. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? Is it ω-categorical? □

6.18 Exercise In the language of graphs let T_0 say that there are no cycles (equivalently, there is at most one path between any two nodes). In combinatorics these graphs are called *forests*, their connected components, *trees*. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? Is it ω-categorical? □

2 Weaker notions of universality and homogeneity

We want to extend the equivalence in Theorem 6.8 to λ-rich models. For that we need to weaken the notions of λ-universality and λ-homogeneity.

6.19 Definition *We say that a structure N is* ***weakly λ-homogeneous*** *if for every $b \in N$ every morphism $k : N \to N$ of cardinality $< \lambda$ extends to one defined in b. The term* ***back-and-forth λ-homogeneous*** *is also used.* □

The following easy exercise on back-and-forth is required in the sequel.

6.20 Exercise Assume that $\mathcal{M}$ satisfies c0-c5 of Definition 6.1. Prove that any weakly λ-homogeneous structure of cardinality λ is homogeneous. □

6.21 Definition *We say that a structure N is* ***weakly λ-universal*** *if for every model M in the connected component of N and every $A \subseteq M$ of cardinality $< \lambda$ there is a morphism $k : M \to N$ such that $A \subseteq \operatorname{dom} k$.* □

6.22 Lemma *(Assume 6.4) Let N be a weakly λ-homogeneous model. Let $A \subseteq N$ have cardinality $\leq \lambda$ and let $k : N \to N$ be a morphism of cardinality $< \lambda$. Then there is a model $M \preceq N$ containing A and an automorphism $h : M \to M$ that extends k.*

Proof Similar to the proof of Theorem 6.10. We shall construct simultaneously a chain $\langle A_i : i < \lambda \rangle$ of subsets of N and a chain functions $\langle h_i : i < \lambda \rangle$, such that $h_i : N \to N$ are morphisms. In the end we will set

$$M \;=\; \bigcup_{i<\lambda} A_i \qquad\qquad \text{and} \qquad\qquad h \;=\; \bigcup_{i<\lambda} h_i$$

The chains start with $A_0 = A \cup \operatorname{dom} k \cup \operatorname{img} k$ and $h_0 = k$. As usual, at limit stages we take the union. Now we consider successor stages. At stage i we fix some enumerations of A_i and of $L_x(A_i)$, where $|x| = 1$. Let $\langle i_1, i_2 \rangle$ be the i-th pair of ordinals $< \lambda$. If the i_2-th formula in $L_x(A_{i_1})$ is consistent in N, let a be any of its solutions. Also let b be the i_2-th element of A_{i_1}. Let $h_{i+1} : N \to N$ be a minimal morphism that extends h_i and is such that $b \in \operatorname{dom} h_{i+1} \cap \operatorname{img} h_{i+1}$. Define $A_{i+1} = A_i \cup \{a,\ h_{i+1}b,\ h_{i+1}^{-1}b\}$. □

6.23 Theorem *(Assume 6.4) For every model N the following are equivalent*

1. *N is λ-rich;*
2. *N is weakly λ-universal and weakly λ-homogeneous.*

Proof Implication 1⇒2 is clear. To prove 2⇒1 we generalize the proof of Theorem 6.8. We assume 2 fix some morphism $k : M \to N$ of cardinality $< \lambda$ and let $b \in M$. By weak λ-universality there is a morphism $f : M \to N$ with domain of definition $\operatorname{dom} k \cup \{b\}$. The map $f \circ k^{-1} : N \to N$ has cardinality $< \lambda$ and, by Lemma 6.22, it has an extension to an automorphism $h : N' \to N'$ for some $N' \preceq N$ containing $\operatorname{img} k \cup \operatorname{img} f$. Then $h \circ f : M \to N$ extends k and is defined on b. □

3 The amalgamation property

We say that $\mathcal{M}$ has the **amalgamation property** if for every pair of morphisms $f_1 : M \to M_1$ and $f_2 : M \to M_2$ there are two of embeddings $g_1 : M_1 \hookrightarrow N$ and $g_2 : M_2 \hookrightarrow N$ such that $g_1 \circ f_1(a) = g_2 \circ f_2(a)$ for every a in the common domain of definition, $\operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$.

$$\begin{tikzcd} & M_1 \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "f_1"] \arrow[dr, "f_1"'] & & N \\ & M_2 \arrow[ur, hook, "g_2"'] & \end{tikzcd}$$

As we assume that morphisms are invertible, we may express the amalgamation property in a more concise form. Namely, for every morphism $k : M \to N$ there is morphism $g : M' \to N'$ such that the following diagram commutes

$$\begin{tikzcd} & N \arrow[dr, hook, "\mathrm{id}_N"] & \\ M \arrow[ur, "k"] \arrow[rr, hook, "g"'] & & N' \end{tikzcd}$$

It is convenient to use the following terminology. We write $M \leq N$ for $M \subseteq N$ and $\mathrm{id}_M : M \hookrightarrow N$ is a morphism. We say that $k : M \to N$ **extends to** $g : M' \to N'$ if $k \subseteq h$, $M \leq M'$, and $N \leq N'$.

6.24 Proposition *Assume c2, then the following are equivalent*

1. *$\mathcal{M}$ has the amalgamation property;*
2. *every morphism $k : M \to N$ extends to an embedding $g : M \hookrightarrow N'$.*

Proof 1⇒2 Given $k : M \to N$, the amalgamation property yields the following commutative diagram which can be simplified to the diagram at the right

$$\begin{tikzcd} & N \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "k"] \arrow[dr, "\mathrm{id}_M"'] & & N' \\ & M \arrow[ur, hook, "g_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & N \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "k"] \arrow[rr, hook, "g_2"'] & & N' \end{tikzcd}$$

Up to isomorphism we can assume $g_2 = \mathrm{id}_M$, i.e. that $N \leq N'$. Hence $g_2 : M \hookrightarrow N'$ is the required extension of $k : M \to N$.

2⇒1 Let $f_1 : M \to M_1$ and $f_2 : M \to M_2$ be given. Let $k = f_2 \circ f_1^{-1} : M_1 \to M_2$ and let $g : M_1 \hookrightarrow N'$ be the extension ensured by 2. Then we obtain

$$
\begin{tikzcd}
 & M_1 \arrow[dd, "f_2 \circ f_1^{-1}"] \arrow[dr, hook, "g"] & \\
M \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & N' \\
 & M_2 \arrow[ur, hook, "\mathrm{id}_{M_2}"'] &
\end{tikzcd}
$$

as required. $\square$

We say that $\langle M_i : i < \lambda \rangle$ is a **$\leq$ -chain** if $M_i \leq M_j$ for all $i < j < \lambda$. For the next theorem to hold we need the following property:

6.25 Definition *We say that $\mathcal{M}$ is **closed under union of $\leq$ -chains** if*

c7. if $\langle M_i : i < \lambda \rangle$ is a $\leq$ -chain, then $M_i \leq \bigcup_{j<\lambda} M_j$ for all $i < \lambda$. $\square$

The following is a general existence theorem for rich models. This general form requires large cardinalities. We leave to the reader to verify that if the number if finite morphisms is countable (up to isomorphism) then countable rich models exit.

6.26 Theorem *Assume 6.4. Assume further c7 and that $\mathcal{M}$ has the amalgamation property. Let λ be such that $|L| < \lambda = \lambda^{<\lambda}$. Then there is a rich model N of cardinality λ.*

Proof We construct N as union of a $\leq$ -chain of models $\langle N_i : i < \lambda \rangle$ such that $|N_i| = \lambda$. Let N_0 be any model of cardinality λ. At stage $i+1$, let $f : M \to N_i$ be the least morphism (in a well-ordering that we specify below) such that $|f| \leq |M| < \lambda$ and f has no extension to an embedding $f' : M \hookrightarrow N_i$. Apply the amalgamation property to obtain an total morphism $f' : M \to N'$ that extends $f : M \to N_i$. By the downword Löwenheim-Skolem Theorem we may assume $|N'| = \lambda$. Let $N_{i+1} = N'$. At limit stages take the union.

The well-ordering mentioned needs to be chosen so that in the end we forget nobody. So, first at each stage we well-order the isomorphism-type of the morphisms $f : M \to N_i$ such that $f \leq |M| < \lambda$. Then the required well-ordering is obtained by dovetailing all these well-orderings. The length of this enumeration is at most $\lambda^{<\lambda}$, which is λ by hypothesis.

We check that N is rich. Let $f : M \to N$ be a morphism and $|f| < |M| \leq \lambda$. As $|L| < \lambda$ we can approximate M with an elementary chain of structures of cardinality $< \lambda$. Hence we may as well assume that $|f| \leq |M| < \lambda$. The cofinality of λ is larger than $|f|$, hence $\operatorname{img} f \subseteq N_i$ for some $i < \lambda$. So $f : M \to N_i$ is a morphism and at some stage j we have ensured the existence of an embedding of $f' : M \hookrightarrow N_{j+1}$ that extends f. $\square$

6.27 Proposition *Let $\mathcal{M}$ consist of all structures of some fixed signature and the elementary maps between these. Then $\mathcal{M}$ has the amalgamation property.*

Proof Let $k : M \to N$ be an elementary map. Let a enumerate $\operatorname{dom} k$ and let b enumerate M. Let $p(x\,; z) = \operatorname{tp}_M(b\,; a)$. The type $p(x\,; a)$ is consisent in M, in particular, it is finitely consistent and, by elementarity, $p(x\,; ka)$ is finitely consistent in N. By the compactness theorem, there is $N' \succeq N$ such that $N' \vDash p(c\,; ka)$ for some $c \in N'^{|x|}$. Hence $g = \{\langle b, c\rangle\} : M \to N'$ is the required elementary map that extends $k : M \to N$. $\square$

4 Notes and references

[1] Michael Lieberman and Jirí Rosický, *Classification theory for accessible categories* (2014), arXiv:1404.2528.

Chapter 7

Some algebraic structures

The main result in this chapter is Corollary 7.24, the elimination of quantifiers in algebraically closed fields, which in algebra is called Chevalley's Theorem on constructible sets. From it we derive Hilbert's Nullstellensatz 7.24. Finally, we isolate the model theoretic properties of those types that correspond to the algebraic notions of prime and radical ideal of polynomials.

The first sections of this chapter are not a pre-requisite for Sections 4–7, at the cost of a few repetitions in the latter.

7.1 Notation Recall that when $A \subseteq M$ we denote by $\langle A\rangle_M$ the substructure of M generated by A. Then $\langle A\rangle_M \subseteq N$ is equivalent to $N \vDash \mathrm{Diag}\,\langle A\rangle_M$. The diagram of a structure has been defined at the end of Section 4.3.

In this chapter, whenever some $A \subseteq M \vDash T$ are fixed, by *model* we mean superstructures of $\langle A\rangle_M$ that models T. The notions of *logical consequence*, *consistency*, *completeness*, etc. are modified accordingly, and we write $\vdash$ for $T \cup \mathrm{Diag}\langle A\rangle_M \vdash$.

We say that a type $p(x)$ is **trivial** if $\vdash p(x)$. □

1 Abelian groups

In this section the language L is that of additive groups. The theory T_{ag} of **abelian groups** is axiomatized by the universal closure of the following axioms

a1 $(x+y)+z \;=\; y+(x+z)$;

a2 $x+(-x) \;=\; (-x)+x \;=\; 0$;

a3 $x+0 \;=\; 0+x \;=\; x$;

a4 $x+y \;=\; y+x$.

Let x be a tuple of variables of length α, an ordinal. We write $L_{\mathrm{ter},x}$ for the set of terms $t(x)$ with free variables among x. On this set we define the equivalence relation

$$t(x) \sim s(x) \quad = \quad T_{\mathrm{ag}} \vdash t(x) = s(x).$$

We define the group operations on $L_{\mathrm{ter},x}/\!\sim$ in the obvious way. We denote by $\mathbb{Z}^{\oplus\alpha}$ the set of tuples of integers of length α that are almost always 0. The group operations on $\mathbb{Z}^{\oplus\alpha}$ are defined coordinate-wise. The following immediate proposition implies in particular that $L_{\mathrm{ter},x}/\!\sim$ is isomorphic to $\mathbb{Z}^{\oplus\alpha}$.

7.2 Proposition *Let $A \subseteq M \vDash T_{\mathrm{ag}}$. Then for every formula $\varphi(x) \in L_{\mathrm{at}}(A)$ there are $n \in \mathbb{Z}^{\oplus\alpha}$ and $c \in \langle A\rangle_M$ such that*

$$\vdash \quad \varphi(x) \;\leftrightarrow\; \sum_{i<\alpha} n_i x_i \;=\; c.$$

where $n = \langle n_i : i < \alpha \rangle$ *and* $x = \langle x_i : i < \alpha \rangle$. □

Proof Up to equivalence over $T_{\rm ag}$ the formula $\varphi(x)$ has the form $s(x) = t(a)$ for some parameter-free terms $s(x)$ and $t(z)$. Over Diag $\langle A \rangle_M$, we can replace $t(a)$ with a single $c \in \langle A \rangle_M$ and write $s(x)$ as the linear combination shown above. □

7.3 Definition *Let $M \vDash T_{\rm ag}$. For $A \subseteq M$ and $c \in M$. We say that* ***c is independent from A*** *if $\langle A \rangle_M \cap \langle c \rangle_M = \{0\}$. Otherwise we say that* ***c is dependent from A****. The* ***rank*** *of M is the least cardinality of a subset $A \subseteq M$ such that all elements in M are dependent from A. We denote it by* **rank** $\boldsymbol{M}$. □

Note that when M is a vector space the condition $\langle A \rangle_M \cap \langle c \rangle_M = \{0\}$ is equivalent to saying that c is not a linear combination of vectors in A. Then rank M coincides with the dimension of M. In fact, what we do here for abelian groups could be easily generalized to D-modules, where D is any integral domain, and in particular to vector spaces.

The following proposition gives a convenient syntactic characterization of independence. An element c of a group is called **nilpotent**, or in abelian groups also a **torsion element**, if $c^n = 0$ for some n.

7.4 Proposition *Let $A \subseteq M \vDash T_{\rm ag}$. Let $c \in M$ be not a torsion element. Then the following are equivalent.*

1. $\langle A \rangle_M \cap \langle c \rangle_M = \{0\}$;
2. $p(x) =$ *at-tp*$_M(c/A)$ *is trivial (see Notation 4.20 and 7.1).*

Proof 1⇒2 By Proposition 7.2, a non trivial formula in $p(x)$ may be assumed to have the form $nx = a$ for some n and some a. If such a formula is satisfied by c then $a \in \langle A \rangle_M \cap \langle c \rangle_M$. As c is not a torsion element, $n = 0$ and the equation is trivial.

2⇒1 If $\langle A \rangle_M \cap \langle c \rangle_M \neq \{0\}$ then $nc = a$ for some $a \in \langle A \rangle_M \smallsetminus \{0\}$. Then c satisfies some non trivial equation $nx = a$. □

7.5 Remark Let $k : M \to N$ be a partial embedding and let a be an enumeration of dom k. We claim that $k \cup \{\langle b, c \rangle\} : M \to N$ is a partial embedding for every $b \in M$ and $c \in N$ that are independent from a, respectively ka. In fact, it suffices to check that $M, b, a \equiv_{\rm at} N, c, ka$. Suppose $\varphi(x\,;z) \in L_{\rm at}$ is such that $M \vDash \varphi(b\,;a)$. Then by independence $\varphi(x\,;a)$ is trivial, i.e.

$$T_{\rm ag} \cup \text{Diag}\langle a \rangle_M \quad \vdash \quad \varphi(x\,;a).$$

As $\langle a \rangle_M$ and $\langle ka \rangle_N$ are isomorphic structures

$$T_{\rm ag} \;\cup\; \text{Diag}\langle ka \rangle_N \quad \vdash \quad \varphi(x\,;ka).$$

Therefore $N \vDash \varphi(c\,;ka)$. The proof that $N \vDash \varphi(c\,;ka)$ implies $M \vDash \varphi(b\,;a)$ is similar. □

2 Torsion-free abelian groups

The theory of **torsion-free abelian groups** extends $T_{\rm ag}$ with the following axioms for all positive integers n

st $nx = 0 \rightarrow x = 0$.

We denote this theory by $\boldsymbol{T_{\mathrm{tfag}}}$. It is not difficult to see that in a torsion-free abelian group every equation of the form $nx = a$ has at most one solution.

7.6 Proposition *Let $M \vDash T_{\mathrm{tfag}}$ be uncountable. Then* $\operatorname{rank} M = |M|$.

Proof Let $A \subseteq M$ have cardinality $< |M|$. We claim that M contains some element that is independent from A. It suffices to show that the number of elements that are dependent from A is $< |M|$. If $c \in M$ is dependent from A then, by Proposition 7.7, it is a solution of some formula $L_{\mathrm{at}}(A)$. As there is no torsion, such a formula has at most one solution. Therefore the number of elements that are dependent from A is at most $|L_{\mathrm{at}}(A)|$, that is $\max\{|A|, \omega\}$. If M is uncountable the claim follows. □

7.7 Proposition *Let $A \subseteq M \vDash T_{\mathrm{tfag}}$. Let $p(x) = \mathrm{at}^{\pm}\text{-}tp(b/A)$, where $b \in M$. Then one of the following holds:*

1. *b is independent from A;*
2. *$M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\mathrm{at}}(A)$ such that $\vdash \varphi(x) \rightarrow p(x)$.*

Note the similarity with Example 6.14, where the independent type is $q(x)$ and the isolating formulas are the $r_i(x)$.

It is important to observe that the set A above may be infinite. This is essential to obtain Corollary 7.11, and it is one of the main differences between this example and the examples encountered in Chapter 5.

Proof If b is dependent from A, then b satisfies a non trivial atomic formula $\varphi(x)$ which we claim is the formula required in 2. It suffices to show that $\varphi(x)$ implies a complete $L_{\mathrm{at}^{\pm}}(A)$-type. Clearly this type must be $p(x)$. Let $\varphi(x)$ have the form $nx = a$ for some $n \in \mathbb{Z} \smallsetminus \{0\}$ and $a \in \langle A \rangle_M \smallsetminus \{0\}$. We show that for every $m \in \mathbb{Z}$ and every $c \in \langle A \rangle_M$ one of the following holds

a. $\vdash nx = a \rightarrow mx = c$;

b. $\vdash nx = a \rightarrow mx \neq c$.

Suppose not for a contradiction that neither a nor b holds and fix models N_1, N_2 and some $b_i \in N_i$ such that

a'. $N_1 \vDash nb_1 = a$ and $N_1 \vDash mb_1 \neq c$;

b'. $N_2 \vDash nb_2 = a$ and $N_2 \vDash mb_2 = c$.

From b' we obtain $N_2 \vDash ma = nc$. As N_1 is torsion-free, from a' we obtain $N_1 \vDash ma \neq nc$. But $ma = nc$ is a formula with parameters in $\langle A \rangle_M$, so it should have the same truth value in all superstructures of $\langle A \rangle_M$, a contradiction. □

3 Divisible abelian groups

The theory of **divisible abelian groups** extends T_{tfag} with the following axioms for all integers $n \neq 0$

div $y \neq 0 \rightarrow \exists x\, nx = y$.

We denote this theory by $\boldsymbol{T_{\mathrm{dag}}}$.

7.8 Proposition *Let $A \subseteq M \vDash T_{\text{tfag}}$ and let $\varphi(x) \in L_{\text{at}}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\, \varphi(x)$ for every model $N \vDash T_{\text{dag}}$.*

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A_M \rangle \subseteq M' \vDash T_{\text{tfag}}$.

The claim in the proposition holds more generally for all $\varphi(x) \in L_{\text{qf}}$ and also when x is a tuple of variables. This follows from Lemma 7.10, whose proof uses the proposition.

Proof We can assume that $\varphi(x)$ has the form $nx = a$ for some $n \in \mathbb{Z}$ and some $a \in \langle A \rangle_M$. If $n = 0$, then $a = 0$ since $\varphi(x)$ is consistent, and the claim is trivial. If $n \neq 0$ then by consistency $a \neq 0$, hence a solution exist in N by axiom div. □

7.9 Exercise Prove a converse of Proposition 7.8. Let $A \subseteq N \vDash T_{\text{ag}}$ and let x be a single variable. Prove that if $N \vDash \exists x\, \varphi(x)$ for every consistent $\varphi(x) \in L_{\text{at}}(A)$, then $N \vDash T_{\text{dag}}$. □

We are ready to prove that divisible abelian groups of infinite rank are ω-rich.

7.10 Lemma *Let $k : M \to N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\text{tfag}}$ and $N \vDash T_{\text{dag}}$ is a model of rank $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \{\langle b\,;c\rangle\} : M \to N$ is a partial isomorphism.*

Proof Let a be an enumeration of dom k and let $p(x\,;z) = \text{at}^{\pm}\text{-tp}(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is dependent from a, then Proposition 7.7 yields a formula $\varphi(x\,;z) \in L_{\text{at}}$ such that

i. $\vdash \varphi(x\,;a) \to p(x\,;a)$

ii. $\varphi(x\,;a)$ is consistent.

By isomorphism, i and ii hold with a replaced by ka. Then by Proposition 7.8 the formula $\varphi(x\,;ka)$ has a solution $c \in N$.

The second case, which has no analogue in Lemma 5.1, is when b is independent from a. Then by Remark 7.5 we may choose c to be any element of N independent from ka. Such an element exists because N has rank at least λ. □

7.11 Corollary *Every uncountable model of T_{dag} is rich in the category of models of T_{tfag} and partial isomorphisms. In particular T_{dag} is uncountably categorical, complete, and has quantifier elimination.*

Proof Any uncountable $N \vDash T_{\text{dag}}$ has rank $|N|$, therefore it is rich by Lemma 7.10; categoricity and completeness follow. As for quantifier elimination, let $k : M \to N$ is a partial isomorphism between models of T_{dag}. If $M \preceq M'$ and $M \preceq N'$ are elementary superstructures of uncountable cardinality then $k : M' \to N'$ is elementary by Theorem 6.10 and this suffices to conclude that $k : M \to N$ is elementary. □

7.12 Exercise Prove that every model of T_{dag} is ω-ultraomogeneous (indipendenlty of cardinality and rank). □

4 Commutative rings

In this section L is the **language of (unital) rings**. It contains two constants 0 and 1 the unary operation $-$ and two binary operations $+$ and $\cdot$. The theory of rings contains the following axioms

a1-a4 as for abelian groups

r1 $(x{\cdot}y){\cdot}z \;=\; y{\cdot}(x{\cdot}z)$,

r2 $1{\cdot}x \;=\; x{\cdot}1 \;=\; x$,

r3 $(x+y){\cdot}z \;=\; x{\cdot}z + y{\cdot}z$,

r4 $z{\cdot}(x+y) \;=\; z{\cdot}x + z{\cdot}y$.

All the rings we consider are **commutative**

c $x{\cdot}y \;=\; y{\cdot}x$.

We denote the theory of commutative rings by $\boldsymbol{T_{\mathrm{cr}}}$.

In what follows the theory $T_{\mathrm{cr}} \cup \mathrm{Diag}\langle A\rangle_M$ is implicit in the sense of Notation 7.1, so it is important to remember that $\mathrm{Diag}\langle A\rangle_M$ is not trivial even when $A = \varnothing$. In fact, $\mathrm{Diag}\langle\varnothing\rangle_M$ determines the characteristic of the models.

Let $A \subseteq M \vDash T_{\mathrm{cr}}$ and let x be a tuple of variables of length α, an ordinal. We write $L_{\mathrm{ter},x}(A)$ for the set of terms $t(x)$ with free variables among x and parameters in A. On this set we define the equivalence relation

$$\boldsymbol{t(x) \sim s(x)} \quad = \quad \vdash\; t(x) = s(x).$$

On $L_{\mathrm{ter},x}(A)/{\sim}$ we define the ring operations in the obvious way so that $L_{\mathrm{ter},x}(A)/{\sim}$ is a commutative ring. We denote by $A[x]$ the set of polynomials with variables among x and parameters in $\langle A\rangle_M$. The ring operations on $A[x]$ are defined as usual. The following proposition (which is clear, but tedious to prove) implies in particular that $L_{\mathrm{ter},x}(A)/{\sim}$ is isomorphic to $A[x]$. For simplicity we state it only for $|x| = 1$.

7.13 Proposition *Let $A \subseteq M \vDash T_{\mathrm{cr}}$ and let x be a single variable. Then for every formula $\varphi(x) \in L_{\mathrm{at}}(A)$ there is a unique $n < \omega$ and a unique tuple $\langle a_i : i \leq n\rangle$ of elements of $\langle A\rangle_M$ such that $a_n \neq 0$ and*

$$\vdash \quad \varphi(x) \;\leftrightarrow\; \sum_{i\leq n} a_i x^i \;=\; 0. \qquad \square$$

The integer n in the proposition above is called the **degree of $\boldsymbol{\varphi(x)}$**.

7.14 Definition *Let $A \subseteq M \vDash T_{\mathrm{cr}}$. We say that an element $b \in M$ is* ***transcendent over A*** *if the type $p(x) = $ at-tp$_M(b/A)$ is trivial (see Notation 4.20 and 7.1). Otherwise we say that b is* ***algebraic over A****. The* ***transcendence degree*** *of M is the least cardinality of a subset $A \subseteq M$ such that all elements of M are algebraic over A.* $\square$

7.15 Remark Remark 7.5 holds here with 'independent' replaced by 'transcendent' and T_{ag} replaced by T_{cr}. $\square$

5 Integral domains

Let $a \in M \vDash T$. We say that a is a **zero divisor** if $a\,b = 0$ for some $b \in M \smallsetminus \{0\}$. An **integral domain** is a commutative ring without zero divisors. The theorey of integral domains contains the axioms of commutative rings and the following

nt. $0 \neq 1$

id. $x \cdot y = 0 \ \rightarrow\ x = 0 \ \vee\ y = 0.$

We denote the theory of integral domains by $\boldsymbol{T_{\rm id}}$.

7.16 Proposition *Let $M \vDash T_{\rm id}$ be uncountable. Then M has transcendence degree $|M|$.*

Proof In an integral domain every polynomial has finitely many solutions and there are $|L(A)|$ polynomials over A. □

7.17 Proposition *Let $A \subseteq M \vDash T_{\rm id}$. For $b \in M$ let $p(x) = {\rm at}^{\pm}\text{-}tp(b/A)$. Then one of the following holds*

1. *b is transcendental over A;*
2. *$M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\rm at}(A)$ such that $\vdash\ \varphi(x) \rightarrow p(x)$.*

Note the similarity with Example 6.14, where the transcendental type is $q(x)$ and the isolating formulas are the $r_i(x)$.

As in Proposition 7.7, the set A may be infinite. This is essential to obtain Corollary 7.20.

Proof Suppose b is not transcendental, i.e. it satisfies a non trivial atomic formula. Let $\varphi(x) \in L_{\rm at}(A)$ be a non trivial formula with minimal degree such that $\varphi(b)$. We prove that $\varphi(x)$ implies a complete $L_{{\rm at}^{\pm}}(A)$-type. Clearly this type must be $p(x)$. We prove that for any $\xi(x) \in L_{\rm at}(A)$ one of the following holds

1. $\vdash\ \varphi(x) \rightarrow\ \ \xi(x)$
2. $\vdash\ \varphi(x) \rightarrow \neg\xi(x)$.

Let us write $a(x) = 0$ and $a'(x) = 0$ for the formulas $\varphi(x)$ and $\xi(x)$, respectively. Let n be the degree of the polynomial $a(x)$. If $\langle A\rangle_M$ is a field, choose a polynomial $d(x)$ of maximal degree such that

a. $d(x)t(x) = a(x)$

a'. $d(x)t'(x) = a'(x)$,

for some polynomials $t(x)$ and $t'(x)$.

If $\langle A\rangle_M$ is not a field, polynomials $d(x)$, $t(x)$ and $t'(x)$ as above exist with coefficients in the field of fractions of $\langle A\rangle_M$. Then a and a' hold up to a factor in $\langle A\rangle_M$ which we absorb in $a(x)$ and $a'(x)$.

From a we get $d(b) = 0$ or $t(b) = 0$. In the first case, as $a(x)$ has minimal degree, we conclude that $t(x)$ is constant. This implies that any zero of $a(x)$ is also a zero of $a'(x)$, that is, it implies 1.

Now suppose $t(b) = 0$. Then the minimality of the degree of $a(x)$ implies that $d(x) = d$, where d is a nonzero constant. If $\langle A\rangle_M$ is a field, apply Bézout's identity to obtain two polynomials $c(x)$ and $c'(x)$ such that $d = a(x)c(x) + a'(x)c'(x)$. Then $a(x)$ and $a'(x)$ have no common zeros, and 2 follows. If $\langle A\rangle_M$ is not a field, we use Bézout's identity in the field of fractions of $\langle A\rangle_M$ and, for some $d' \in \langle A\rangle_M \smallsetminus \{0\}$, obtain $d'd = a(x)c(x) + a'(x)c'(x)$. Then we reach the same conclusion. □

6 Algebraically closed fields

Let $a, b \in M \vDash T_{\mathrm{cr}}$. We say that b is the **inverse** of a if $a \cdot b = 1$. A field is a commutative ring where every non-zero element has an inverse. The **theory of fields** contains T_{id} and the axiom

f. $\exists y\ \big[x \neq 0\ \rightarrow\ x \cdot y = 1\big]$.

Fields are structures in the signature of rings: the language contains no symbol for the multiplicative inverse. So, substructures of fields are merely integral domains.

The theory of **algebraically closed field**, which we denote by T_{acf}, also contains the following axioms for every positive integer n

ac. $\exists x\ \big(x^n + z_{n-1}x^{n-1} + \cdots + z_1 x + z_0\ =\ 0\big)$

This not a complete theory because it does not decide the characteristic of the field. For a prime p, we define the theory T^p_{acf}, which contains T_{acf} and the axiom

ch$_p$. $1 + \ldots (p \text{ times}) \cdots + 1 = 0$.

The theory T^0_{acf} contains the negation of ch$_p$ for all p. Note that all models of T^p_{acf} have the same characteristic in the model theoretic sense defined in 4.27. In fact, any two fields M and N with the same (algebraic) characteristic have isomorphic prime subfield, that is, $\langle\varnothing\rangle_M \simeq \langle\varnothing\rangle_N$.

7.18 **Proposition** *Let $A \subseteq M \vDash T_{\mathrm{id}}$ and let $\varphi(x) \in L_{\mathrm{at}}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\, \varphi(x)$ for every model $N \vDash T_{\mathrm{acf}}$.*

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A_M\rangle \subseteq M' \vDash T_{\mathrm{id}}$.

The claim in the proposition holds more generally for all $\varphi(x) \in L_{\mathrm{qf}}$ when x is a tuple of variables. This follows from Lemma 7.19 whose proof uses the proposition.

Proof Up to equivalence $\varphi(x)$ has the form $a_n x^n + \cdots + a_1 x + a_0\ =\ 0$ for some $a_i \in \langle A\rangle_N$. Choose n minimal. If $n = 0$ then $a_0 = 0$ by the consistency of $\varphi(x)$ and the claim is trivial. Otherwise $a_n \neq 0$ and the claim follows from f and ac. □

7.19 **Lemma** *Let $k : M \to N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\mathrm{id}}$ and $N \vDash T_{\mathrm{acf}}$ has transcendence degree $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \big\{\langle b, c\rangle\big\} : M \to N$ is a partial isomorphism.*

The following is the proof of Lemma 7.10 which we have pasted here for the sake of the lazy reader.

Proof Let a be an enumeration of $\mathrm{dom}\, k$ and let $p(x\,;z)\ =\ \mathrm{at}^{\pm}\text{-}\mathrm{tp}_M(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is algebraic over a, then Proposition 7.17 yields a formula $\varphi(x\,;z) \in L_{\mathrm{at}}$ such that

i. $\varphi(x\,;a) \to p(x\,;a)$

ii. $\varphi(x\,;a)$ is consistent.

By isomorphism i and ii hold with a replaced by ka. Then by Proposition 7.18 the formula $\varphi(x\,;ka)$ has a solution in $c \in N$.

The second case, which has no analogue in Lemma 5.1, is when b is transcendent over a. Then by Remark 7.15 we may choose c to be any element of N transcendent over ka. This exists because N has transcendence degree $\geq \lambda$. $\square$

7.20 Corollary *Every uncountable model of $T^p_{\rm acf}$ is rich in the category of integral domains of characteristic p and partial isomorphisms. In particular $T^p_{\rm acf}$ is uncountably categorical, and it is complete. Moreover, $T_{\rm acf}$ has quantifier elimination.*

The proof requires the same argument as Corollary 7.11.

Proof Every uncountable $N \vDash T^p_{\rm acf}$ has transcendence degree $|N|$, therefore it is rich by Lemma 7.19. Categoricity and completeness follow.

For quantifier elimination, let $k : M \to N$ be a partial embedding between models of $T_{\rm acf}$. Then M and N have the same charactiristic, i.e. they are model of $T^p_{\rm acf}$. Let M' and N' be elementary superstructures of M and N respectively of sufficiently large cardinality. As M' and N' are rich, $k : M' \to N'$ is elementary by Theorem 6.10. Hence $k : M \to N$ is also elementary. $\square$

7.21 Exercise Prove that every model of $T_{\rm acf}$ is ω-ultrahomogeneous (indipendently of cardinality and transcendence degree). $\square$

7 Hilbert's Nullstellensatz

In this section we fix a tuple of variables $\boldsymbol{x}$ and a subset $\boldsymbol{A}$ of an integral domain. We denote by $\boldsymbol{\Delta(A)}$ the set of formulas of the form $t(x) = 0$ where $t(x)$ is a term with parameters in A. So $\Delta(A)$-types are (possibly infinite) systems of polynomial equations with coefficients in $\langle A\rangle_M$.

For convenience we define the closure of $p(x)$ under logical consequences as follows

$$\operatorname{ccl} p(x) \quad = \quad \Big\{t(x) = 0 \;:\; p(x) \,\vdash\, t(x){=}0\Big\}$$

Remember that we always work under the assumptions made in Notation 7.1. In particular, in this section we work over the theory $T_{\rm id} \cup {\rm Diag}\langle A\rangle_M$. In general, closure under logical consequences is an elusive notion. Hence Propositions 7.22 and 7.23 are useful because they give a model theoretical, respectively algebraic, characterisation of $\operatorname{ccl} p(x)$.

7.22 Proposition *Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x)$ be a $\Delta(A)$-type. Fix some N of sufficiently large cardinality such that $\langle A\rangle_M \subseteq N \vDash T_{\rm acf}$. Then*

$$\operatorname{ccl} p(x) \quad = \quad \Big\{t(x) = 0 \,:\, N \vDash \forall x \left[p(x) \to t(x){=}0\right]\Big\}.$$

It suffices to choose N such that $|A| < |N|$ and $|x| \leq |N|$; note that x has possibly infinite length. In Corollary 7.24 below we will considerably strengthen this proposition for finite x.

Proof Only the inclusion $\supseteq$ requires a proof. Fix some polynomial $t(x){=}0 \notin \operatorname{ccl} p(x)$. Then $p(x) \wedge t(x) \neq 0$ is consistent, so there is a model M' such that $M' \vDash p(a) \wedge t(a) \neq 0$ for some $a \in M'^{|x|}$. Then there is a partial isomorphism $h : M' \to N$ that extends id_A and is defined on a, provided N is large enough to accommodate a. This implies that $p(x) \wedge t(x) \neq 0$ has a solution in N. Hence $t(x)$ does not belong to the set on the r.h.s. □

Recall that $A[x]$ denotes the ring of polynomials with variables in x and coefficients in $\langle A \rangle_M$. We identify $\Delta(A)$ nd $A[x]$ in the obvious way. Consequently, a $\Delta(A)$-type $p(x)$ is identified with a set of polynomials $p \subseteq A[x]$. For $p \subseteq A[x]$ we write **rad p** for the **radical ideal generated by p**, that is, the intersection of all prime ideals containing p. When $p = \operatorname{rad} p$, we say that **p is a radical ideal**. Recall from algebra that if $p \subseteq A[x]$ is an ideal then

$$\operatorname{rad} p \;=\; \Big\{ t(x) \;:\; t^n(x) \in p \text{ for some positive integer } n \Big\},$$

which explains the name.

7.23 Proposition *Let $A \subseteq M \vDash T_{\mathrm{id}}$ and let $p(x)$ be a $\Delta(A)$-type. Then* $\operatorname{ccl} p(x) \simeq \operatorname{rad} p$.

The proposition holds with a similar proof for the broader class of rings without nilpotent elements. (Which does not come as a surprise.)

Proof ($\supseteq$) We claim that $\operatorname{ccl} p(x)$ is an ideal. In fact, for every pair of $L(A)$-terms $t(x)$ and $s(x)$

$$\begin{aligned} t(x) = 0 \;&\vdash\; s(x)t(x) = 0 \\ s(x) = t(x) = 0 \;&\vdash\; s(x) + t(x) = 0 \end{aligned}$$

Moreover, as integral domains do not have nilpotent elements

$$t^n(x) = 0 \;\vdash\; t(x) = 0$$

By # above, $\operatorname{ccl} p(x)$ is a radical ideal which proves $\operatorname{ccl} p(x) \supseteq \operatorname{rad} p$.

($\subseteq$) We fix some $t(x) \notin \operatorname{rad} p$ and prove that $p(x) \wedge t(x) \neq 0$ is consistent. Let q be some prime ideal containing p such that $t(x) \notin q$. As q is prime, the ring $A[x]/q$ is an integral domain. The polynomials that vanish in $A[x]/q$ at $x + q$ are exactly those in q. Hence $A[x]/q$ witnesses the consistency of $q(x) \wedge t(x) \neq 0$. □

When x is a finite tuple of variables, we can extend the validity of Proposition 7.22 to the case $A = M = N$.

7.24 Corollary (Hilbert's Nullstellensatz) *Let $N \vDash T_{\mathrm{acf}}$ and let $p(x)$, where $|x| < \omega$, be a $\Delta(N)$-type. Then*

$$\operatorname{rad} p \;\simeq\; \operatorname{ccl} p(x) \;=\; \Big\{ t(x) \;:\; N \vDash \forall x \,[p(x) \to t(x) = 0] \Big\}.$$

Proof Let N' be a large elementary extension of N. By Proposition 7.22, the claim holds for N'. By Hilbert's Basis Theorem, the ideal generated by p is finitely generated hence $p(x)$ is equivalent to a formula (cfr. Exercise 7.27). Therefore, by elementarity, the claim holds for N. □

Hilbert's Nullstellensatz comes in two variants. The one in Corollary 7.24 is sometimes referred to as the *strong* Nullstellensatz. The weaker variant is stated in Exercise 7.26.

We conclude this section by showing that the notions of primeness for types and ideals coincide (if we restrict to types closed under logical consequences).

7.25 Proposition *Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x)$ be a $\Delta(A)$-type closed under logical consequences. Then the following are equivalent*

1. *$p(x)$ is a prime $\Delta(A)$-type;*
2. *p is a prime ideal.*

Proof 1⇒2 Assume 1 and suppose that the polynomial $t(x) \cdot s(x)$ belongs to p. Clearly, over $T_{\rm id} \cup {\rm Diag}\langle A\rangle_M$ we have $\vdash\ t(x) \cdot s(x) = 0 \rightarrow t(x) = 0 \vee s(x) = 0$. As $p(x)$ is a prime $\Delta(A)$-type, $p(x) \vdash t(x) = 0$ or $p(x) \vdash s(x) = 0$. By Proposition 7.23, the type $p(x)$ is closed under logical consequences, therefore $t(x) \in p$ or $s(x) \in p$.

2⇒1 Assume p is a prime ideal and for some $t_i(x) = 0 \in \Delta(A)$

$$p(x) \quad \vdash \quad \bigvee_{i=1}^{n} t_i(x) = 0.$$

Then

$$p(x) \quad \vdash \quad \prod_{i=1}^{n} t_i(x) \ = \ 0$$

Since $p(x)$ is closed under logical consequences, and p is a prime ideal, $t_i(x) \in p$ for some i. Hence $p(x)$ contains the equation $t_i(x) = 0$. By Corollary 4.19 this suffices to prove that $p(x)$ is a prime $\Delta(A)$-type. □

7.26 Exercise Let $N \vDash T_{\rm acf}$ and let $p(x)$ be a $\Delta(N)$-type where $|x| < \omega$. Prove that the following are equivalent

1. p is a proper ideal;
2. $p(x)$ has a solution in N. □

7.27 Exercise Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x)$ be a $\Delta(A)$-type. Prove that the following are equivalent

1. $p(x)$ is a principal $\Delta(A)$-type;
2. the ideal generated by p is finitely generated. □

Chapter 8

Saturation and homogeneity

The first two section introduce saturation and homogeneity and Section 3 presents the notation we shall use in the following chapters when working inside a monster model.

1 Saturated structures

Recall that a type $p(x) \subseteq L(M)$ is **finitely consistent in M** if every $\varphi(x)$, conjunction of formulas in $p(x)$, has a solution in M. When $A \subseteq M$ we write $S_x(A)$ for the set of types that are complete and finitely consistent in M. We never display M in the notation as it will always be clear from the context. When A is empty it usual to write $S_x(T)$ for $S_x(A)$ where $T = \mathrm{Th}(M)$. We write $S(A)$ for the union of $S_x(A)$ as x ranges over all tuples of variables. Similarly for $S(T)$.

The following remark will be used in the sequel without explicit reference.

8.1 Remark Let $k : M \to N$ be an elementary map and let a be an enumeration of $\mathrm{dom}\, k$. If $p(x\,;a)$ is finitely consistent in M, then $p(x\,;ka)$ is finitely consistent in N. (We can drop *finitely* in the antecedent but not in the consequent.) □

8.2 Definition *Let x be a single variable and let λ be an infinite cardinal. We say that a structure N is* ***λ-saturated*** *if it realizes every type $p(x)$ such that*

1. $p(x) \subseteq L(A)$ for some $A \subseteq N$ of cardinality $< \lambda$;

2. $p(x)$ is finitely consistent in N.

We say that N is ***saturated*** *tout court if it is λ-saturated and $|N| = \lambda$.*

8.3 Exercise Suppose $|L| \leq \omega$ and let M be an infinite structure. Then for every non-principal ultrafilter F on ω the structure M^ω/F is a ω_1-saturated elementary superstructure of M.

Hint: the notation is as in Chapter 3. Let $|x| = 1$ and $|z| = \omega$. It suffices to consider the types form $p(x\,;\hat{c})$ where $p(x\,;z) = \{\varphi_i(x\,;z) : i < \omega\} \subseteq L$ and $\hat{c} \in (M^\omega)^{|z|}$. Without loss of generality we can also assume that $\varphi_{i+1}(x\,;z) \to \varphi_i(x\,;z)$, and that all formulas $\varphi_i(x\,;\hat{c})$ are consistent in M^ω.

Let $\langle X_i : i < \omega \rangle$ be a strictly decreasing chain of elements of the ultrafilter such that $X_{i+1} \subseteq \{j \;:\; M \vDash \exists x\; \varphi_i(x, \hat{c}j)\}$. Let $\hat{a} \in M^\omega$ be such that $\varphi_i(\hat{a}j, \hat{c}j)$ holds for every $j \in X_i \smallsetminus X_{i+1}$. Then $\hat{a}$ realizes $p(x)$. □

8.4 Theorem *Assume $|L| < \lambda$ where λ is be such that $\lambda^{<\lambda} = \lambda$. Then every structure M of cardinality λ has a saturated elementary extension of cardinality λ.*

Proof We construct an elementary chain $\langle M_i : i < \lambda \rangle$ of models of cardinality λ. The chain starts with M and is the union at limit stages. Given M_i we choose as

M_{i+1} any model of cardinality λ that realizes all types in $S_x(A)$ for all $A \subseteq M_i$ of cardinality $< \lambda$. The required M_{i+1} exists because there are $2^{|L(A)|} \leq \lambda$ types in $S_x(A)$ and there are $\lambda^{<\lambda} = \lambda$ sets A.

Let N be the union of the chain. We check that N is the required extension. Let $p(x) \in S(A)$ for some $A \subseteq N$ of cardinality $< \lambda$. As λ is a regular cardinal, $A \subseteq M_i$ for some $i < \lambda$. Then M_{i+1} realizes $p(x)$, and so does N, by elementarity. □

8.5 Theorem *Assume $|L| \leq \lambda$ and let N be an infinite structure. Let $\mathcal{M}$ be the category (see Section 6.1) that consists of models of a compete theory T and elementary maps between these. Then the following are equivalent*

1. *N is a λ-saturated structure;*
2. *N is a λ-rich model;*
3. *N realizes all types $p(x) \subseteq L(A)$, with $|x| \leq \lambda$ and $|A| < \lambda$, finitely consistent in N.*

Note that it is the completeness of T which makes the category $\mathcal{M}$ connected.

Proof 1⇒2. Let $k : M \to N$ be an elementary map of cardinality $< \lambda$. It suffices to show that for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is an elementary map. Let a be an enumeration of $\operatorname{dom} k$ and define $p(x\,;z) = \operatorname{tp}_M(b\,;a)$. The required c is any element of N such that $N \vDash p(c\,;ka)$. As $p(x\,;a)$ is finitely consistent in M then $p(x\,;ka)$ is finitely consistent in N. As $|a| < \lambda$, saturation yields the required c.

2⇒3. Let $p(x)$ be as in 3. By the compactness theorem $N \preceq K \vDash p(b)$ for some model K and $b \in K^{|x|}$. By the downward Löwenheim-Skolem theorem there is a model $A, b \subseteq M \preceq K$ of cardinality $\leq \lambda$. (Here we use $|L|, |A|, |x| \leq \lambda$.) By 2, there is an elementary embedding $h : M \hookrightarrow N$ that extends id_A. Finally, as $M \vDash p(b)$, elementarity yields $N \vDash p(hb)$.

3⇒1. Trivial. □

Two saturated structures of the same cardinality are isomorphic as soon as they are elementarily equivalent (i.e. as soon as $\varnothing : M \to N$ is an elementary map.) In fact, from Theorem 8.5 and 6.6 we obtain the following.

8.6 Corollary *Every elementary map $k : M \to N$ of cardinality $< \lambda$ between saturated models of the same cardinality λ extends to an isomorphism.* □

As it turns out, we already have many examples of saturated structures.

8.7 Corollary *The following models are ω-saturated*

1. *models of T_{dlo};*
2. *models of T_{rg};*
3. *models of T_{dag} with infinite rank;*
4. *models of T_{acf} with infinite degree of transcendence.*

Countable models of T_{dlo} and T_{rg} and uncountable models of T_{dag} or T_{acf} are saturated.

Proof By quantifier elimination embeddings coincide with elementary embeddings. Then saturation is proved applying Theorem 8.5 and the extension lemmas proved in Chapter 5 and 7. □

The following is a useful test for quantifier elimination.

8.8 Theorem *Assume $|L| \leq \lambda$. Consider the category that consists of models of some theory T_0 and partial isomorphism. Suppose λ-rich models exist and denote by T_1 their theory. Then the following are equivalent*

1. *every λ-saturated model of T_1 is λ-rich;*
2. *T_1 has elimination of quantifiers.*

Proof 2⇒1. Let $N \vDash T_1$ be λ-saturated. Fix a morphism $k : M \to N$ of cardinality $< \lambda$, some $b \in M$ and let $p(x\,;z) = \text{qf-tp}_M(b\,;a)$, where a enumerates $\operatorname{dom} k$. The type $p(x\,;ka)$ is realized in any λ-rich model N' that contains $\langle ka\rangle_N$. By 2, $N \equiv_{ka} N'$, so $p(x\,;ka)$ is finitely consistent in N. By saturation, it is realised by some $c \in N$. Then $k \cup \{\langle b, c\rangle\} : M \to N$ is the required extension.

1⇒2. Let $k : M \to N$ be a finite partial isomorphism between models of T. We claim that it is an elementary map. Let $M' \succeq M$ and $N' \succeq N$ are saturated models of equal cardinality. As these are rich, $k : M' \to N'$ extends to an isomorphism $h : M' \hookrightarrow\!\!\!\!\rightarrow N'$ and the claim follows. □

2 Homogeneous structures

Definition 6.3 introduces the notion of universal and homogeneous structures a the general context. When the morphisms of the underlying category are the elementary maps we refer to these notion as elementary homogeneity and elementary universality. Though one often omits to specify *elementary*. We repeat the definition in this particular case.

8.9 Definition *A structure N is* ***(elementarily) λ-universal*** *if every $M \equiv N$ of cardinality $\leq \lambda$ there is a elementary embedding $h : M \hookrightarrow N$. We say* ***universal*** *tout court if it is λ-universal and of cardinality λ.*

We say that N is ***(elementarily) λ-homogeneous*** *if every elementary map $k : N \to N$ of cardinality $< \lambda$ extends to an automorphism. We say that N is* ***homogeneous*** *tout court if it is λ-homogeneous and of cardinality λ.* □

As saturated structure are rich, the following theorem is an instance of Theorem 6.8.

8.10 Theorem *For every structure N of cardinality $\geq |L|$ the following are equivalent*

1. *N is saturated;*
2. *N is elementarily universal and homogeneous.* □

Given $A \subseteq N$ we denote by $\mathbf{Aut}(N/A)$ the group of ***A*-automorphisms** of N. That is the group of automorphisms that fix A point-wise. Let a be a tuple of elements of N. The **orbit of *a* over *A* in *N*** is the set

$$O_N(a/A) \quad = \quad \Big\{ fa \ : \ f \in \operatorname{Aut}(N/A) \Big\}$$

When the model N is clear from the context we omit the subscript.

Orbits are particularly interesting if taken in a homogeneous structure. The following proposition is immediate but its importance cannot be overestimated.

8.11 Proposition *Let N be a λ-homogeneous structure. Let $A \subseteq N$ have cardinality $< \lambda$ and $a \in N^{<\lambda}$. Then $O_N(a/A) = p(N)$, where $p(x) = \mathrm{tp}_N(a/A)$.* □

Finally, we want to extend the equivalence in Theorem 8.10 to λ-saturated structures. For this we only need to apply Theorem 6.23.

When the morphisms of the underlying category are the elementary maps, we say **weakly λ-saturated** for weakly λ-universal (cfr. Definition 6.21). This is not the standard definition (i.e. 2 of the proposition below) but the reader can easily verify that it is equivalent by reasoning as in the proof of theorem 8.5.

8.12 Proposition *The following are equivalent*

1. *N weakly λ-saturated;*
2. *N realizes every type $p(x) \subseteq L$, where $|x| < \lambda$, that is finitely consistent in N.* □

The following is an instance of Theorem 6.23 that the reader may prove directly as an exercise.

8.13 Corollary *Let $|L| \leq \lambda$. The following are equivalent*

1. *N is λ-saturated;*
2. *N is weakly λ-saturated and weakly λ-homogeneous.* □

8.14 Exercise Let M be an arbitrary structure. Prove that M has an ω-homogeneous elementary extension of the same cardinality. (There is no assumption on the cardinality of the language.) □

8.15 Exercise Let M and N be elementarily homogeneous structures of the same cardinality λ. Suppose that $M \vDash \exists x\, p(x) \Leftrightarrow N \vDash \exists x\, p(x)$ for every $p(x) \subseteq L$ such that $|x| < \lambda$. Prove that the two structures are isomorphic. □

8.16 Exercise Let L be a language that extends that of strict linear orders with the constants $\{c_i : i \in \omega\}$. Let T be the theory that extends T_{dlo} with the axioms $c_i < c_{i+1}$ for every $i \in \omega$. Prove that T has elimination of quantifiers and is complete (it can be deduced from what is known of T_{dlo}). Exhibit a countable saturated model and a countable model that is not homogeneous. □

3 The monster model

In this section we present some notation and terminology frequently adopted when dealing with a complete theory T. We fix a saturated structure $\mathcal{U}$ of cardinality larger than $|L|$. We assume $\mathcal{U}$ to be large enough that among its elementary substructures we can find any model of T we might be interested in. For this reason $\mathcal{U}$ is nicknamed the **monster model**. We denote by κ the cardinality of $\mathcal{U}$ when necessary we assume κ to be inaccessible.

Some terms acquire a slightly different meaning when working inside a monster model.

small/large — cardinalities smaller than κ are called small;

models — are elementary substructure of $\mathcal{U}$ of small cardinality, they are denoted by the letters M and N, and derived symbols;

parameters — are always in $\mathcal{U}$; the symbols A, B, C, etc. denote sets of parameters of small cardinality; calligraphic letters as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. are used for sets of arbitrary cardinality;

tuples — have length $< \kappa$ unless otherwise specified;

global types — are complete types over $\mathcal{U}$; the set of global types is denoted by $S(\mathcal{U})$;

formulas — have parameters in $\mathcal{U}$ unless otherwise specified;

definable sets — are sets of the form $\varphi(\mathcal{U})$ for some formula $\varphi(x) \in L(\mathcal{U})$; we may say A-definable if $\varphi(x) \in L(A)$;

type-definable sets — are sets of the form $p(\mathcal{U})$ for some $p(x) \subseteq L(A)$ where, as the symbol suggests, A has small cardinality;

types of tuples — are always evaluated in $\mathcal{U}$, we write $\mathrm{tp}(a/A)$ for $\mathrm{tp}_{\mathcal{U}}(a/A)$ and $a \equiv_A b$ for $\mathcal{U}, a \equiv_A \mathcal{U}, b$;

orbits — of tuples under the action of $\mathrm{Aut}(\mathcal{U}/A)$ are denoted by $\mathcal{O}(a/A)$.

Let x be a tuple of variables. For any fixed $A \subseteq \mathcal{U}$ we introduce a topology on $\mathcal{U}^{|x|}$ that we call the **topology induced by A** or, for short, **A-topology**. (This is non standard terminology, not to be confused with the *logic* A-topology in Section 13.4.) The closed sets of the A-topology are those of the form $p(\mathcal{U})$ where $p(x) \subseteq L(A)$ is any type over A.

For $\varphi(x) \in L(A)$ the sets of the form $\varphi(\mathcal{U})$ are clopen in this topology (and vice versa by Proposition 8.17). They form both a base of closed sets and base of open sets, which makes these topologies *zero-dimensional*. By saturation, the topology induced by A is compact. Actually, saturation is equivalent to the compactness of all these topologies as A ranges over the sets of small cardinality.

These topologies are never T_0 as any pair of tuples $a \equiv_A b$ have exactly the same neighborhoods. Such pairs always exist by cardinality reasons. However it is immediate that the topology induced on the quotient $\mathcal{U}^{|x|}/\equiv_A$ is Hausdorff (this is the so-called *Kolmogorov quotient*). Indeed, this quotient corresponds to $S_x(A)$ with the topology introduced in Section 4.3.

The following proposition is an immediate consequence of compactness. When $A = B$ it says that the topology induced by A is *normal*: any two closed sets are separated by open sets. It could be called **mutual normality** (not a standard name) because the two closed sets belong to different topologies and the separating sets are each found in the corresponding topology.

8.17 Proposition (mutual normality) *Let $p(x) \subseteq L(A)$ and $q(x) \subseteq L(B)$ be such that $p(x) \to \neg q(x)$. Then there are $\varphi(x)$, conjuction of formulas in $p(x)$, and $\psi(x)$, conjuction of formulas in $q(x)$, such that $\varphi(x) \to \neg\psi(x)$.*

Proof The assumptions say that $p(x) \wedge q(x)$ is inconsistent. Then the formulas $\varphi(x)$ and $\psi(x)$ exist by compactness (i.e. saturation). □

8.18 Remark There are many forms in which the proposition above can be applied. For instance, assuming for brevity that $p(x)$ and $q(x)$ are closed under conjunctions,

a. if $p(x) \leftrightarrow \neg q(x)$ then $p(x) \leftrightarrow \varphi(x)$ for some $\varphi(x) \in p(x)$;

b. if $p(x) \leftrightarrow \psi(x)$ for some $\psi(x) \in L(\mathcal{U})$ then $p(x) \leftrightarrow \varphi(x)$ for some $\varphi(x) \in p$;

d. if $p(x) \to \psi(x)$ for some $\psi(x) \in L(\mathcal{U})$ then $\varphi(x) \to \psi(x)$ for some $\varphi(x) \in p$;

c. if $p(x) \to \bigvee_{\psi \in \Psi} \psi(x)$, where $|\Psi| < \kappa$, then $p(x) \to \bigvee_{i=1}^{n} \psi_i(x)$ for some $\psi_i \in \Psi$. □

A definable set as the form $\varphi(\mathcal{U}\,;b)$ for some formula $\varphi(x\,;z) \in L$ and some $b \in \mathcal{U}^{|z|}$. If $f \in \mathrm{Aut}(\mathcal{U}/A)$ then

$$\begin{aligned} f[\varphi(\mathcal{U}\,;b)] &= \{fa \ :\ \varphi(a\,;b), \quad a \in \mathcal{U}^{|x|}\} \\ &= \{fa \ :\ \varphi(fa\,;fb),\ a \in \mathcal{U}^{|x|}\} \\ &= \varphi(\mathcal{U}\,;fb) \end{aligned}$$

Hence automorphisms act on definable sets in a very natural way. Their action of type-definable sets is similar.

We say that a set $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ is **invariant over A** if $f[\mathcal{D}] = \mathcal{D}$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$. Equivalently, if $\mathcal{O}(a/A) \subseteq \mathcal{D}$ for every $a \in \mathcal{D}$. By homogeneity this is equivalent to requiring that $q(x) \to x \in \mathcal{D}$ for every $q(x) = \mathrm{tp}(a/A)$ where $a \in \mathcal{D}$.

Proposition 8.21 below an important fact about invariant type-definable sets. It may clarify the proof to consider first the particular case of definable sets.

8.19 Proposition *Let $\varphi(x) \in L(\mathcal{U})$ then the following are equivalent*

1. *$\varphi(x)$ is equivalent to some formula $\psi(x) \in L(A)$;*
2. *$\varphi(\mathcal{U})$ is invariant over A.*

We give two proofs of this theorem as they are both instructive.

Proof 1⇒2 Obvious.

2⇒1 From 2 and homogeneity we obtain

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{q(x) \to \varphi(x)} q(x)$$

where $q(x)$ above range over all types in $S_x(A)$. By compactness, we can rewrite this equivalence

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{\vartheta(x) \to \varphi(x)} \vartheta(x)$$

where $\vartheta(x)$ ranges over all formulas in $L(A)$. This latter equivalence says that $\neg\varphi(x)$

is equivalent to a type over A. Again by compactness we obtain

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \vartheta_i(x)$$

for some formula $\vartheta_i(x) \in L(A)$. □

Second proof of Proposition 8.19 2⇒1 Let $\varphi(\mathcal{U}\,;b)$, where $\varphi(x\,;z) \in L$, be a formula invariant over A. Let $p(z) = \mathrm{tp}(b/A)$. As $f[\varphi(\mathcal{U}\,;b)] = \varphi(\mathcal{U}\,;fb)$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$, homogeneity and invariance yield

$$p(z) \quad \rightarrow \quad \forall x\, \big[\varphi(x\,;z) \leftrightarrow \varphi(x\,;b)\big]$$

By compactness there is a formula $\vartheta(z) \in p$ such that

$$\vartheta(z) \quad \rightarrow \quad \forall x\, \big[\varphi(x\,;z) \leftrightarrow \varphi(x\,;b)\big]$$

Hence $\varphi(\mathcal{U}\,;b)$ is defined by the formula $\exists z\, \big[\vartheta(z) \wedge \varphi(x\,;z)\big]$, which is a formula in $L(A)$ as required. □

8.20 Exercise Let $\varphi(x) \in L$. Prove that the following are equivalent

1. $\varphi(x)$ is equivalent to some $\psi(x) \in L_{\mathrm{qf}}$;
2. $\varphi(a) \leftrightarrow \varphi(fa)$ for every partial isomorphism $f : \mathcal{U} \rightarrow \mathcal{U}$ defined in a.

Use the result to prove Theorem 6.13 for T complete. □

8.21 Proposition *Let $p(x) \subseteq L(\mathcal{U})$ then the following are equivalent*

1. *$p(x)$ is equivalent to some type $q(x) \subseteq L(A)$;*
2. *$p(\mathcal{U})$ is invariant over A.*

We give two proofs of this theorem. The second one requires Proposition 8.22 below.

Proof 1⇒2 Obvious.

2⇒1 It suffices to show that for every formula $\psi(x) \in p(x)$ there is a formula $\varphi(x) \in L(A)$ such that $p(x) \rightarrow \varphi(x) \rightarrow \psi(x)$. Fix $\psi(x) \in p(x)$. By invariance, any $q(x) \in S(A)$ consistent with $p(x)$ implies $p(x)$, hence

$$p(x) \quad \rightarrow \quad \bigvee_{q(x) \rightarrow \psi(x)} q(x) \quad \rightarrow \quad \psi(x)$$

where $q(x)$ above range over all types in $S_x(A)$. By compactness we can rewrite this equivalence as follows

$$p(x) \quad \rightarrow \quad \bigvee_{\vartheta(x) \rightarrow \psi(x)} \vartheta(x) \quad \rightarrow \quad \psi(x)$$

where $\vartheta(x)$ ranges over all formulas in $L(A)$. Applying mutual normality (Proposition 8.17) to the first implication we obtain a finite number of formulas $\vartheta_i(x)$ such that

$$p(x) \quad \rightarrow \quad \bigvee_{i=1}^{n} \vartheta_i(x) \quad \rightarrow \quad \psi(x)$$

This completes the proof. □

The following easy proposition is very useful. Its proof is left to the reader. Note that it would not hold without saturation saturation: for instance, consider $\mathbb{R}$ as a structure in the language of strict orders and let $q(x,y) = \text{tp}(-1,0\,/\,A)$, where $A = \{1/n : 0 < n\}$. By quantifier elimination, $-1 \equiv_A 0$. But $0 \nvDash \exists y\, q(x,y)$.

8.22 Proposition *Let $p(x\,;z) \subseteq L(A)$. Then $\exists z\, p(x\,;z)$ is equivalent to a type over A, namely to the type $\big\{\exists z\, \varphi(x\,;z) \;:\; \varphi(x\,;z)$ conjunction of formulas in $p(x\,;z)\big\}$. The theorem holds also when x and z have length κ.* □

As an application we give a second proof of the proposition above.

Second proof of Proposition 8.21 2⇒1 Write $p(x)$ as the type $q(x\,;b)$ for some $q(x\,;z) \subseteq L$ and some $b \in \mathcal{U}^{|z|}$. Let $s(z) = \text{tp}(b/A)$. By invariance and homogeneity the types $p(x\,;fb)$ for $f \in \text{Aut}(\mathcal{U}/A)$ are all equivalent. Therefore

$$p(x) \;\leftrightarrow\; \bigvee_{f\in\text{Aut}(\mathcal{U}/A)} q(x\,;fb)$$

$$\begin{aligned} p(x) \;&\leftrightarrow\; \bigvee_{c\equiv_A b} q(x\,;c) \\ &\leftrightarrow\; \exists z\,\big[s(z) \wedge q(x\,;z)\big]. \end{aligned}$$

Hence, by Proposition 8.22, $p(x)$ is equivalent to a type over A. □

8.23 Exercise Let $p(x) \subseteq L(A)$, with $|x| < \omega$. Prove that if $p(\mathcal{U})$ is infinite then it has cardinality κ. Show that this may not be true if x is an infinite tuple. □

8.24 Exercise Let $\varphi(x,y) \in L(\mathcal{U})$. Prove that if the set $\big\{\varphi(a,\mathcal{U}) \;:\; a \in \mathcal{U}^{|x|}\big\}$ is infinite then it has cardinality κ. Does the claim remains true with a type $p(x,y) \subseteq L(A)$ for $\varphi(x,y)$? □

8.25 Exercise Let $\varphi(x,y) \in L(\mathcal{U})$. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i \in \omega\rangle$ such that $\varphi(\mathcal{U},a_i) \subset \varphi(\mathcal{U},a_{i+1})$ for every $i < \omega$;
2. there is a sequence $\langle a_i : i \in \omega\rangle$ such that $\varphi(\mathcal{U},a_{i+1}) \subset \varphi(\mathcal{U},a_i)$ for every $i < \omega$. □

Chapter 9

Preservation theorems

A few results dating from the 1950s describe the relationship between syntactic and semantic properties of first-order formulas. These theorems states that a certain syntactic class of formulas contains, up to logical equivalence, all first-order formulas preserved under a certain algebraic relationship between structures (the existence of a particular kind of homomorphism). Criteria for quantifier-elimination follow from these theorems, see for instance the frequently used back-and-forth method of Corollary 9.11.

1 Lyndon-Robinson Lemma

In this section T is a consistent theory without finite models and Δ is a set of formulas closed under renaming of variables. At a first reading the reader is encouraged to assume that Δ is $L_{\rm at}$.

If $\mathsf{C} \subseteq \{\forall, \exists, \neg, \vee, \wedge\}$ is a set of connectives, we write $\mathbf{C\Delta}$ for the closure of Δ with respect to all connectives in C.

Recall that **Δ-morphism** is a map $k : M \to N$ that preserves the truth of formulas in Δ. It is immediate that Δ-morphism are automatically $\{\wedge\vee\}\Delta$-morphisms. As Δ is closed under renaming of variables, $\{\exists\}\Delta$-morphism are $\{\exists\wedge\vee\}\Delta$-morphisms and similarly $\{\forall\}\Delta$-morphism are $\{\forall\wedge\vee\}\Delta$-morphisms.

Below we use the following proposition without further reference.

9.1 Proposition *Fix $M \vDash T$ and $b \in M^{|x|}$. Let $q(x) = \Delta\text{-}tp_M(b)$. Then for every $\varphi(x) \in L$ the following are equivalent*

1. *$N \vDash \varphi(kb)$ for every $k : M \to N \vDash T$ that is a Δ-morphism defined in b;*
2. *$T \vdash q(x) \to \varphi(x)$.*

Proof 2⇒1 Immediate.

1⇒2 Negate 2, then there are $N \vDash T$ and $c \in N^{|x|}$ such that $q(c) \wedge \neg\varphi(c)$. Therefore the map $k : M \to N$, where $k = \{\langle b, c\rangle\}$, contradicts 1. □

The following is sometimes referred to as the Lyndon-Robinson Lemma.

9.2 Lemma *For every $\varphi(x) \in L$ the following are equivalent*

1. *$\varphi(x)$ is equivalent over T to a formula in $\{\wedge\vee\}\Delta$;*
2. *$\varphi(x)$ is preserved by Δ-morphisms between models of T.*

Proof 1⇒2 Immediate.

2⇒1 We claim that from 2 it follows that

$$T \vdash \varphi(x) \leftrightarrow \bigvee \{\, p(x) \subseteq \Delta \;:\; T \vdash p(x) \to \varphi(x) \,\}$$

Direction $\leftarrow$ is clear. To verify $\rightarrow$, fix $M \vDash T$ and let $b \in M^{|x|}$ be such that $M \vDash \varphi(b)$. From 2 it follows that $\varphi(x)$ satisfy 1 of Proposition 9.1. Therefore $T \vdash q(x) \rightarrow \varphi(x)$ for $q(x) = \Delta\text{-tp}(b)$. Hence $q(x)$ is one of the types that occur in the disjuction in # which therefore is satisfied by b.

From # and compactness we obtain

$$T \quad \vdash \quad \varphi(x) \quad \leftrightarrow \quad \bigvee \{ \psi(x) \in \{\wedge\}\Delta \ : \ T \vdash \psi(x) \rightarrow \varphi(x) \}$$

Applying compactness again, we replace the infinite disjunction above with a finite one and prove 2. □

9.3 Proposition *Let N be λ-saturated and let $k : M \rightarrow N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \rightarrow N$ is a $\{\exists\}\Delta$-morphism;*
2. *for every $b \in M$ some $\{\exists\}\Delta$-morphism $h : M \rightarrow N$ defined in b extends k;*
3. *for every $b \in M^{\omega}$ some Δ-morphism $h : M \rightarrow N$ defined in b extends k.*

Proof 1⇒2 Let a enumerate $\operatorname{dom} k$. Define $p(x\,;z) = \{\exists\}\Delta\text{-tp}_M(b\,;a)$. By 1, $p(x\,;ka)$ is finitely consistent in N. By saturation there is a $c \in N$ that realizes $p(x\,;ka)$. Therefore, $h : M \rightarrow N$ where $h = k \cup \{\langle b, c\rangle\}$, witnesses 2.

2⇒3 Iterate ω-times the extension in 2.

3⇒1 Let a enumerate $\operatorname{dom} k$ and let $|z| = |a|$. Formulas in $\{\exists\}\Delta$ with free variables among z are of the form $\exists x\, \varphi(x\,;z)$ where $\varphi(x\,;z)$ is in Δ and x is some fixed tuple of length ω. Assume $M \vDash \exists x\, \varphi(x\,;a))$ and let b be such that $M \vDash \varphi(b\,;a)$. By 3, we can extend k to some Δ-morphism $h : M \rightarrow N$ defined in b. Then $N \vDash \varphi(hb\,;ha)$ and therefore $N \vDash \exists x\, \varphi(x\,;ka)$. □

Iterating the lemma above we obtain the following.

9.4 Corollary *Let N be λ-saturated and let $|M| \leq \lambda$. Let $k : M \rightarrow N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \rightarrow N$ is a $\{\exists\}\Delta$-morphism;*
2. *$k : M \rightarrow N$ extends to an $\{\exists\}\Delta$-embedding;*
3. *$k : M \rightarrow N$ extends to an Δ-embedding.* □

The following theorem is often paraphrased as follows: a formula is existential if and only if (its truth) is preserved under extensions of structures.

9.5 Theorem *For every $\varphi(x) \in L$ the following are equivalent*

1. *$\varphi(x)$ is equivalent over T to a formula in $\{\exists\wedge\vee\}\Delta$;*
2. *$\varphi(x)$ is preserved by Δ-embedding between models of T.*

Proof 1⇒2 Immediate.

2⇒1 Negate 1. By Lemma 9.2 there is a $\{\exists\}\Delta$-morphism $k : M \rightarrow N$ between models of T that does not preserve $\varphi(x)$. We can assume that N is λ-saturated for some sufficiently large λ. By Corollary 9.4 there is a Δ-embedding $h : M \hookrightarrow N$ that extends k and contradicts 2. □

A dual version of the results above is obtained replacing embeddings by epimorphisms (surjective homomorphisms) and $\{\exists\}$ by $\{\forall\}$. If Δ contains the formula $x = y$ and is closed under negation, then $k : M \to N$ is a Δ-morphism if and only is $k^{-1} : N \to M$ is a Δ-morphism. In this case the dual version follows from what proved above. Without these assumptions the results need a similar but independent proof.

9.6 Proposition *Let M be λ-saturated and let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \to N$ is a $\{\forall\}\Delta$-morphism;*
2. *for every $c \in N$ some $\{\forall\}\Delta$-morphism $h : M \to N$ extends k and $c \in \operatorname{img} h$;*
3. *for every $c \in N^{\omega}$ some Δ-morphism $h : M \to N$ extends k and $c \in (\operatorname{img} h)^{\omega}$.*

We write **$\neg\Delta$** for the set containing the negation of the formulas in Δ. Warning: do not confuse $\neg\Delta$ with $\{\neg\}\Delta$.

Proof Left as an exercise for the reader. Hint: to prove implication 1⇒2 define $p(x, y) = \neg\{\forall\}\Delta\text{-tp}_N(ka, c)$, where a is a tuple that enumerates $\operatorname{dom} k$. From 1 obtain that $p(a, y)$ is finitely consistent in M. Then proceed as in the proof of Proposition 9.3. □

9.7 Corollary *Let M be λ-saturated and let $|N| \leq \lambda$. Let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \to N$ is a $\{\forall\}\Delta$-morphism;*
2. *$k : M \to N$ extends to an $\{\forall\}\Delta$-epimorphism;*
3. *$k : M \to N$ extends to an Δ-epimorphism.* □

Finally we obtain the following.

9.8 Theorem *The following are equivalent*

1. *$\varphi(x)$ is equivalent to a formula in $\{\forall\}\Delta$;*
2. *every Δ-epimorphism between models of T preserves $\varphi(x)$.* □

2 Quantifier elimination by back-and-forth

We say that ***T* admits** (or **has**) **positive Δ-elimination of quantifiers** if for every formula $\varphi(x)$ in $\{\exists\forall\wedge\vee\}\Delta$ there is a formula $\psi(x)$ in $\{\wedge\vee\}\Delta$ such that

$$T \quad \vdash \quad \varphi(x) \leftrightarrow \psi(x).$$

When Δ is closed under negation the attribute *positive* becomes irrelevant and will be omitted. When Δ is $L_{\text{at}^{\pm}}$ or L_{qf}, we simply say that T admits elimination of quantifiers. This is by far the most common case.

Quantifier elimination is often used to prove that a theory is complete because it reduces it to something much simpler to prove. The following is an immediate consequence of the definition above with x replaced by the empty tuple.

9.9 Remark If T has elimination of quantifiers then the following are equivalent

1. T decides all quantifier free sentences;
2. T is complete.

Hence a theory with quantifier elimination is complete if it decides the characteristic, see Definition 4.26). □

The following is a consequence of Lemma 9.2.

9.10 Corollary *The following are equivalent*

1. *T has Δ-elimination of quantifiers;*
2. *every Δ-morphism between models of T is both a $\{\exists\}\Delta$ and a $\{\forall\}\Delta$-morphism.*

Proof 1⇒2 Immediate.

2⇒1 We prove by induction of syntax that Δ-morphism preserve the truth of all formulas in $\{\exists\forall\wedge\vee\}\Delta$, this suffices by Lemma 9.2. Induction for the connectives $\vee$ and $\wedge$ is trivial. So assume as induction hypothesis that the truth of $\varphi(x, y)$ is preserved. By Lemma 9.2 $\varphi(x, y)$ is equivalent to a formula in $\{\wedge\vee\}\Delta$, hence by 2 the truth of $\exists y\, \varphi(x, y)$ and $\forall y\, \varphi(x, y)$ is preserved. □

Condition 2 of the corollary above may be difficult to verify directly. The following corollary of Proposition 9.3 and 9.6 gives a back-and-forth condition with is easier to verify.

9.11 Corollary *Let $|L| \leq \lambda$. The following are equivalent*

1. *T has Δ-elimination of quantifiers;*
2. *for every finite Δ-morphism $k : M \to N$ between λ-saturated models of T*
   a. *for every $b \in M$ some Δ-morphism $h : M \to N$ extends k and $b \in \operatorname{dom} h$;*
   b. *for every $c \in N$ some Δ-morphism $h : M \to N$ extends k and $c \in \operatorname{img} h$.* □

Note that when Δ contains the formula $x = y$ and is closed under negation, then $k : M \to N$ is a Δ-morphism if and only if $k^{-1} : N \to M$ is a Δ-morphism. In this case a and b are equivalent.

9.12 Exercise Let T be a complete theory without finite models in a language that consists only of unary predicates. Prove that T has elimination of quantifiers. □

9.13 Exercise Let L be the language of strict orders. The theory T of **discrete linear orders** extends the theory of linear orders without endpoints (see Section 5.1) with the following two of axioms

dis↑. $\exists z \left[x < z \ \wedge\ \neg\exists y\ x < y < z\right]$;

dis↓. $\exists z \left[z < x \ \wedge\ \neg\exists y\ z < y < x\right]$.

Let Δ be the set of formulas that contains (all alphabetic variants of) the formulas $x <_n y := \exists^{\geq n} z\ (x{<}z{<}y)$ and their negations (read $<_0$ as $<$). Prove that T has Δ-elimination of quantifiers. Prove that the structure $\mathbb{Q} \times \mathbb{Z}$ ordered with the lexicographic order

$$(a_1, a_2) < (b_1, b_2) \quad \Leftrightarrow \quad a_1 < b_1 \text{ or } (a_1 = b_1 \text{ e } a_2 < b_2)$$

is a saturated model of T. □

9.14 Exercise Let T be a consistent theory. Suppose that all completions of T are of the form $T \cup S$ for some set S of quantifier-free sentences. Prove that, if all completion of T have elimination of quantifiers, so does T. Show that this fails when the completions of T have arbitrary complexity.

Note. Thought the claim follows immediately from Corollary 9.10, a direct proof is also instructive. Prove that for every formula $\varphi(x)$ there are some quantifier-free sentences σ_i and quantifier-free formulas $\psi_i(x)$ such that

$$\sigma_i \vdash \varphi(x) \leftrightarrow \psi_i(x), \qquad T \vdash \bigvee_{i=1}^{n} \sigma_i, \qquad \text{and} \qquad \sigma_i \vdash \neg\sigma_j \text{ for } i \neq j.$$

For a counter example consider the empty theory in the language with a single unary predicate. □

3 Model-completeness

We say that T is **model-complete** if every embedding $h : M \hookrightarrow N$ between models of T is an elementary embedding.

The notion was introduced by Abraham Robinson and has become standard. It is inspired by the fact that T is model-complete if and only if $T \cup \mathrm{Diag}(M)$ is a complete theory, in the language $L(M)$, for every $M \vDash T$.

To stress positivity in the next proposition, we generalize the definition as follows. We say that T is **Δ-model-complete** if every Δ-embedding $h : M \hookrightarrow N$ between models of T is a $\{\forall\exists\}\Delta$-embedding.

Model-completeness is equivalent to a form of elimination of quantifiers.

9.15 Proposition *The following are equivalent*

1. *T is Δ-model-complete;*
2. *T is has $\{\exists\}\Delta$-elimination of quantifiers.*

Proof 1⇒2 By 1, every formula $\{\forall\exists\wedge\vee\}\Delta$ is preserved by Δ-embeddings therefore, by Theorem 9.5, it is equivalent to a formula in $\{\exists\wedge\vee\}\Delta$.

2⇒1 Clear, because Δ-embeddings preserve formulas in $\{\exists\wedge\vee\}\Delta$. □

The theory of discrete linear orders defined in Exercise 9.13 is an example of a model-complete theory without elimination of quantifiers.

9.16 Exercise Prove that the theory of discrete linear orders is model-complete. □

The difference between quantifier elimination and model-completeness subtle. It boils down to models of T having or not the amalgamation property.

9.17 Proposition *Assume T is model-complete. Let $\mathcal{M}$ be the category that consists of models of T and partial isomorphisms. Then the following are equivalent*

1. *$\mathcal{M}$ has the amalgamation property;*

2. *T has elimination of quantifiers.*

Proof 1⇒2 By Proposition 6.24 every partial morphism $k : M \to N$ extends to an embedding $g : M \hookrightarrow N'$ which, by model-completeness, is an elementary embedding. Model-completeness also implies that $N \preceq N'$. Hence $k : M \to N$ is an elementary map. This proves 2.

2⇒1 If all morphisms are elementary maps, amalgamations follows from Proposition 6.27. □

Note however that the models of a model-complete theory T do have amalgamation when the proper notion of morphism is chosen.

Let $\mathcal{M}'$ be the category that consists of models of T and the maps $k : M \to N$ such that there is a partial isomorphism $h : M' \to N'$ with

1. $k \subseteq h;\quad M \preceq M';\quad N \preceq N'$;

2. $\operatorname{dom} h$ contains a substructure of M' that models T (equivalently $\operatorname{img} h$ and N').

It is clear that if T is model-complete then the morphisms of $\mathcal{M}'$ are exactly the elementary maps. In this case $\mathcal{M}'$ has amalgamation. Vice versa if $\mathcal{M}'$ has amalgamation, the theory of rich models is model-complete.

9.18 Exercise Prove that $\mathcal{M}'$ satisfies c6 of Definition 6.1, finite character of morphisms. □

Chapter 10

Geometry and dimension

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ larger than $|L|$. Notation and implicit assumptions are as presented in Section 8.3.

1 Algebraic and definable elements

Let $a \in \mathcal{U}$ and let $\mathcal{A} \subseteq \mathcal{U}$ be some set of parameters (of arbitrary cardinality). We say that a is **algebraic over $\mathcal{A}$** if $\varphi(a) \wedge \exists^{=k} x\, \varphi(x)$ holds for some formula $\varphi(x) \in L(\mathcal{A})$ and some positive integer k. In particular when $k = 1$ we say that a is **definable over $\mathcal{A}$**. We write **$\mathrm{acl}(\mathcal{A})$** for the **algebraic closure of $\mathcal{A}$**, that is, set of all elements algebraic over $\mathcal{A}$. If $\mathcal{A} = \mathrm{acl}(\mathcal{A})$, we say that $\mathcal{A}$ is **algebraically closed**. The **definable closure of $\mathcal{A}$** is defined similarly and is denoted by **$\mathrm{dcl}(\mathcal{A})$**.

A formula $\varphi(u) \in L(A)$ or a type $p(u) \subseteq L(A)$, where u is a finite tuple, with finitely many solutions are called an **algebraic formula**, respectively an **algebraic type**.

10.1 Proposition *For every $A \subseteq \mathcal{U}$ and every type $p(u) \subseteq L(A)$, where $|u| < \omega$, the following are equivalent*

1 $\exists^{\leq n} u\; p(u)$;

2 $\exists^{\leq n} u\; \varphi(u)$ *for some $\varphi(u)$ which is conjunction of formulas $p(u)$.*

Proof The non trivial implication is 1⇒2. Let $\{a_i, \dots, a_n\}$ be all the solutions of $p(u)$. Then

$$p(u) \;\leftrightarrow\; \bigvee_{i=1}^{n} a_i = u$$

Then 2 follows by compactness (cfr. Remark 8.18.b). □

10.2 Exercise Prove that Proposition 10.1 does not hold in general for infinite tuple u. □

10.3 Exercise For every $a \in \mathcal{U}^{|u|}$ and $A \subseteq \mathcal{U}$, the following are equivalent

1. a is solution of some algebraic formula $\varphi(u) \in L(A)$;
2. $a = a_1, \dots, a_n$ for some $a_1, \dots, a_n \in \mathrm{acl}(A)$.

Prove also that if a is definable then so are $a_1, \dots, a_n$, and vice versa. □

10.4 Theorem *For every $A \subseteq \mathcal{U}$ and every $a \in \mathcal{U}$ the following are equivalent*

1 $a \in \mathrm{dcl}(A)$;

2 $\mathcal{O}(a/A) = \{a\}$.

Proof Implication 1⇒2 is obvious. As for 2⇒1, recall that $\mathcal{O}(a/A)$ is the set of realizations of $\operatorname{tp}(a/A)$, then the theorem follows from Proposition 10.1. □

10.5 Theorem *For every $A \subseteq \mathcal{U}$ and every $a \in \mathcal{U}$ the following are equivalent*

1. $a \in \operatorname{acl}(A)$;
2. $\mathcal{O}(a/A)$ *is finite;*
3. a *belongs to every model containing* A.

Proof 1⇔2. This is proved as in Theorem 10.4.

1⇒3. Assume 1, then there is a formula $\varphi(x) \subseteq L(A)$ such that $\varphi(a) \wedge \exists^{=k} x\, \varphi(x)$ for some k. By elementarity $\exists^{=k} x\, \varphi(x)$ holds in every model M containing A. Again by elementarity, the k solutions of $\varphi(x)$ in M are solutions in $\mathcal{U}$, therefore a is one of these.

3⇒2. Assume $\mathcal{O}(a/A)$ is infinite then, by Exercise 8.23, it has cardinality κ. Hence, by cardinality reasons, $\mathcal{O}(a/A) \not\subseteq M$ for every model M containing A. Pick any $f \in \operatorname{Aut}(\mathcal{U}/A)$ such that $fa \notin M$. Then $a \notin f^{-1}[M]$, so $f^{-1}[M]$ is a models that contradicts 3. □

10.6 Corollary *For every $A \subseteq \mathcal{U}$ and every $a \in \mathcal{U}$ the following hold*

1. *Se* $a \in \operatorname{acl} A$ *then* $a \in \operatorname{acl} B$ *for some finite* $B \subseteq A$; ***finite character***
2. $A \subseteq \operatorname{acl} A$; ***extensivity***
3. *se* $A \subseteq B$ *allora* $\operatorname{acl} A \subseteq \operatorname{acl} B$; ***monotonicity***
4. $\operatorname{acl} A = \operatorname{acl}(\operatorname{acl} A)$; ***idempotency***
5. $\operatorname{acl} A = \displaystyle\bigcap_{A \subseteq M} M$.

Properties 1-4 say that the algebraic closure is a closure operator with finite character.

Proof Properties 1-3 are obvious, 4 follow from 5 which in turn follows from Theorem 10.5. □

10.7 Proposition *If $f \in Aut(\mathcal{U})$ then $f\big[\operatorname{acl}(A)\big] = \operatorname{acl}\big(f[A]\big)$ for every $A \subseteq \mathcal{U}$.*

Proof We prove $f\big[\operatorname{acl}(A)\big] \subseteq \operatorname{acl}\big(f[A]\big)$. Fix $a \in \operatorname{acl}(A)$ and let $\varphi(x\,;z) \in L$ and $b \in A^{|z|}$ be such that $\varphi(x\,;b)$ is algebraic formula satisfied by a. By elementarity, $\varphi(x\,;fb)$ is algebraic and satisfied by fa. Therefore fa is algebraic over $f[A]$, which proves the inclusion.

The converse inclusion is obtained by substituting f^{-1} for f and $f[A]$ for A. □

10.8 Exercise Let $\varphi(z) \in L(A)$ be a consistent formula. Prove that, if $a \in \operatorname{acl}(A, b)$ for every $b \vDash \varphi(z)$, then $a \in \operatorname{acl}(A)$. Prove the same claim with a type $p(z) \subseteq L(A)$ for $\varphi(z)$. □

10.9 Exercise Let $a \in \mathcal{U} \smallsetminus \operatorname{acl}\varnothing$. Prove that $\mathcal{U}$ is isomorphic to some $\mathcal{V} \preceq \mathcal{U}$ such that $a \notin \mathcal{V}$. Hint: let c be an enumeration of $\mathcal{U}$ and let $p(u) = \operatorname{tp}(c)$ prove that

$p(u) \cup \{u_i \neq a : i < |c|\}$ is realized in $\mathcal{U}$. □

10.10 Exercise Let C be a finite set. Prove that if $C \cap M \neq \varnothing$ for every model M containing A, then $C \cap \operatorname{acl}(A) \neq \varnothing$.

Hint: by induction of the cardinality of C. Suppose there is a $c \in C \smallsetminus \operatorname{acl}(A)$, then there is $\mathcal{V} \simeq \mathcal{U}$ such that $A \subseteq \mathcal{V} \preceq \mathcal{U}$ and $c \notin \mathcal{V}$, see Exercise 10.9. Apply the induction hypothesis to $C' = C \cap \mathcal{V}$ with $\mathcal{V}$ for $\mathcal{U}$. □

10.11 Exercise Prove that for every $A \subseteq N$ there is an M such that $\operatorname{acl} A = M \cap N$.

Hint: add the requirement $\operatorname{acl}(A_i) \cap N \subseteq \operatorname{acl}(A)$ to the construction used to prove the downward Löwenheim-Skolem theorem. You need to prove that every consistent $\varphi(x) \in L(A_i)$ has a solution a such that $\operatorname{acl}(A_i, a) \cap N \subseteq \operatorname{acl}(A)$. The required a has to realize the type

$$\{\varphi(x)\} \cup \Big\{\neg[\psi(b,x) \wedge \exists^{\leq n} y\, \psi(y,x)] \;:\; b \in N \smallsetminus \operatorname{acl}(A),\;\; \psi(y,x) \in L(A_i),\;\; n < \omega\Big\}$$

whose consistence need to be verified. □

10.12 Exercise Prove that for every $A \subseteq N$ there is an $f \in \operatorname{Aut}(\mathcal{U}/A)$ such that $\operatorname{acl} A = f[N] \cap N$. (This is a stronger version of the claim in Exercise 10.11.)

Hint: let $c = \langle c_i : i < \lambda \rangle$ be an enumeration of N. Let $p(x) = \operatorname{tp}(c/A)$. Consider the type

$$p(x) \cup \Big\{\neg[\psi(b,x) \wedge \exists^{\leq n} y\, \psi(y,x)] \;:\; b \in N \smallsetminus \operatorname{acl}(A),\;\; \psi(y,x) \in L_x(A),\;\; n < \omega\Big\}$$

Every $a \models p(x)$ enumerates a model A-isomorphic to N. The consistence of $p(x)$ needs to be verified. □

10.13 Exercise Let $\varphi(x) \in L(\mathcal{U})$ and fix an arbitrary set A. Prove that the following are equivalent

1. there is some model M containing A and such that $M \cap \varphi(\mathcal{U}) = \varnothing$;
2. there is no consistent formula $\psi(z_1, \dots, z_n) \in L(A)$ such that
$$\psi(z_1, \dots, z_n) \to \bigwedge_{i=1}^{n} \varphi(z_i).$$

Hint: let $c = \langle c_i : i < \lambda \rangle$ be an enumeration of $N^{|x|}$, where N is any model containing A. Let $p(z) = \operatorname{tp}(c/A)$. Prove that 2 implies the consistency of the type $p(z) \cup \{\neg\varphi(z_i) : i < \lambda\}$ and deduce the existence of the required M. □

2 Strongly minimal theories

Finite and cofinite sets are always (trivially) definable in every structure. We say that M is a **minimal structure** all its definable subsets of arity one are finite or cofinite. Unfortunately, this notion is not elementary, i.e. it is not a property of $\operatorname{Th}(M)$. For instance $\mathbb{N}$ with only the order relation in the language is a minimal structure but none of its elementary extensions is. Hence the following definition: we say that M is a **strongly minimal structure** if it is minimal and all its elementary extensions are minimal.

We say that T, a consistent theory without finite models, is a **strongly minimal theory** if for every formula $\varphi(x\,;z) \in L$, where x has arity one, there is an $n \in \omega$ tale che

$$T \;\vdash\; \forall z\left[\exists^{\leq n} x\, \varphi(x\,;z) \;\vee\; \exists^{\leq n} x\, \neg\varphi(x\,;z)\right].$$

We show that the semantic notions match with and the syntactic one.

10.14 Proposition *The following are equivalent*

1. *Th(M) is a strongly minimal theory;*
2. *M is a strongly minimal structure;*
3. *M has an elementary extension which is minimal and ω-saturated.*

Proof Implications 1⇒2⇒3 are immediate, we prove 3⇒1. Let $\varphi(x\,;z) \in L$ and fix an elementary extension N as required by 2. Let $p(z) \subseteq L$ be the following type

$$p(z) \;=\; \left\{\exists^{>n} x\, \varphi(x\,;z) \;\wedge\; \exists^{>n} x\, \neg\varphi(x\,;z) \;\;:\;\; n \in \omega\right\}.$$

As N is minimal, $N \not\models \exists z\, p(z)$. By saturation $p(z)$ is not finitely consistent in M. Hence, for some n

$$M \;\models\; \forall z\left[\exists^{\leq n} x\, \varphi(x\,;z) \;\vee\; \exists^{\leq n} x\, \neg\varphi(x\,;z)\right].$$

which proves that Th(M) is strongly minimal. □

By quantifier elimination, T_{acf} and T_{dag} are strongly minimal theories.

10.15 Exercise Let T be a complete theory without finite models. Prove that the following are equivalent

1. M is minimal;
2. $a \equiv_M b$ for every $a, b \in \mathfrak{U} \smallsetminus M$. □

3 Independence and dimension

Thoughout this section we assume that T is a complete strongly minimal theory.

When $a \notin \operatorname{acl} B$ we say that a is **algebraically independent from** B. We say that B is an **algebraically independent set** if every $a \in B$ is independent from $B \smallsetminus \{a\}$. Below we shall abbreviate $B \cup \{a\}$ by $\boldsymbol{B, a}$ and $B \smallsetminus \{a\}$ by $\boldsymbol{B \smallsetminus a}$.

The following is a pivotal property of independence that holds in strongly minimal structures. It is called **symmetry** or **exchange principle**. For every B and every pair of elements $a, b \in \mathfrak{U} \smallsetminus \operatorname{acl} B$

$$b \in \operatorname{acl}(B, a) \;\;\Leftrightarrow\;\; a \in \operatorname{acl}(B, b)$$

Note that when $T = T_{\mathrm{acf}}$ this principle is the so called **Steinitz exchange lemma**.

10.16 Theorem *Independence is symmetric.*

Proof Suppose $b \notin \operatorname{acl}(B, a)$ and $a \in \operatorname{acl}(B, b)$. We prove that $a \in \operatorname{acl} B$. Fix a formula $\varphi(x, y) \in L(B)$ such that $\varphi(x, b)$ witness $a \in \operatorname{acl}(B, b)$, i.e. for some n

$$\varphi(a, b) \;\;\wedge\;\; \exists^{\leq n} x\, \varphi(x, b).$$

As $b \notin \operatorname{acl}(B, a)$, the formula

$$\psi(a,y) \;=\; \varphi(a,y) \;\wedge\; \exists^{\leq n} x\, \varphi(x,y).$$

is not algebraic. Therefore, by strong minimality, $\psi(a,y)$ has cofinitely many solutions. Hence every model containing B contains a solution of $\psi(a,y)$. As a is algebraic in any of these solutions, a belongs to every model containing B. Therefore, $a \in \operatorname{acl} B$ by Theorem 10.5. □

We say that $B \subseteq C$ is a **base** of C if B is an independent set and $C \subseteq \operatorname{acl} B$. The following theorem proves that all bases have the same cardinality, which we call the **dimension** of C and denote by $\mathbf{dim}\, C$. First we need the following lemma.

10.17 Lemma *If B is an independent set and $a \notin \operatorname{acl} B$ then B, a is also an independent set.*

Proof Suppose B, a is not independent and that $a \notin \operatorname{acl} B$. Then $b \in \operatorname{acl}(B \smallsetminus b, a)$ for some $b \in B$. As $a, b \notin \operatorname{acl}(B \smallsetminus b)$, from symmetry we obtain $a \in \operatorname{acl}(B \smallsetminus b, b) = \operatorname{acl} B$. Hence B is not an independent set. □

10.18 Corollary *For every $B \subseteq C$ the following are equivalent*

1. *B is a base of C.*
2. *B is a maximally independent subset of C.* □

We may write $\mathbf{acl}(B/A)$ per $\operatorname{acl}(B \cup A)$. The notation is used to suggest to the reader that the role A in the argument is irrelevant and that it could be absorbed in the language. Independence and base **over A** are defined in the obvious way.

10.19 Theorem *Fix some arbitrary set C, then*

1 *every independent set $B \subseteq C$ can be extended to a base of C;*

2 *all bases of C have the same cardinality;*

3 *claims 1 and 2 hold over any set of parameter A.*

Proof By the finite character of algebraic closure, the independent set form an inductive class. Apply Zorn lemma to obtain a maximally independent subset of C containing B. By Corollary 10.18 this set is a base of C. This prove 1 and it is immediate that the argument can be generalized as required in 3.

As for 2, assume for a contradiction that $B_1, B_2 \subseteq C$ are two bases of C and that $|B_1| < |B_2|$. First consider the case when B_2 is infinite. For each $b \in B_1$ fix a set $D_b \subseteq B_2$ such that $b \in \operatorname{acl}(D_b)$. Let

$$D \;=\; \bigcup_{b \in B_1} D_b.$$

Then $B_1 \subseteq \operatorname{acl} D$ and $|D| < |B_2|$. By transitivity, $C \subseteq \operatorname{acl} D$ which contradicts the independence of B_2. Again the argument generalizes immediately as required in 3.

Now we suppose that B_2 is finite and prove directly the generalized version. Let n be the least integer such that, for some set of parameters A, and some B_1 and B_2 bases of C over A, we have $|B_1| < |B_2| = n+1$. Fix any $c \in B_2$. Then $B_2 \smallsetminus c$ is a base over A, c. Note that B_1 is not independent over A, c. In fact, let $D \subseteq B_1$ be minimal such that as $c \in \operatorname{acl}(D/A)$. Note that D cannot be empty and pick an arbitrary $b \in D$. By symmetry $b \notin \operatorname{acl}(D \smallsetminus b/A, c)$. Hence B_1 is not an independent set over

A, c. Let $D_1 \subset B_1$ be maximal independent over A, c. Then D_1 and $B_2 \setminus c$ are two bases over A, c such that $|D_1| < |B_2 \setminus c| = n$, which contradicts the minimality of n. □

10.20 Proposition *Let k be an elementary map. Then $k \cup \{\langle b, c\rangle\}$ is also an elementary map for every $b \notin \operatorname{acl}(\operatorname{dom} k)$ and $c \notin \operatorname{acl}(\operatorname{img} k)$.*

Proof Let a be an enumeration of $\operatorname{dom} k$. We need to show that $\varphi(b\,;a) \leftrightarrow \varphi(c\,;ka)$ holds for every $\varphi(x\,;z) \in L$. As k is elementary, the formulas $\varphi(x\,;a)$ and $\varphi(x\,;ka)$ are either both algebraic or both co-algebraic. As $b \notin \operatorname{acl}(a)$ and $c \notin \operatorname{acl}(ka)$, they are both false or both true respectively. So the proposition follows. □

10.21 Corollary *Every bijection between independent sets is an elementary map.* □

Finally we show that dimension classifies models of T.

10.22 Theorem *Models of T with the same dimension are isomorphic.*

Proof Let A e B be some bases of M and N respectively. By Corollary 10.21, any bijectiion between A and B is an elementary map. By Proposition 10.7, it extends to the required isomorphism between $\operatorname{acl} A = M$ and $\operatorname{acl} B = N$. □

10.23 Corollary *Strongly minimal theories are λ-categorical for every $\lambda > |L|$.*

Proof As $|\operatorname{acl} A| \leq |L(A)|$, all models of cardinality λ have the same dimension λ. □

10.24 Proposition *For every model N of cardinality $\geq |L|$ the following are equivalent*

1. *N is saturated;*
2. $\dim N = |N|$.

Proof 2⇒1. Assume 2 and let $k : M \to N$ be an elementary map of cardinality $< |N|$ and let $b \in M$. We want an extension of k defined in b. If $b \in \operatorname{acl}(\operatorname{dom} k)$ the required extension exists by Proposition 10.7. Otherwise, pick any $c \in N \setminus \operatorname{acl}(\operatorname{img} k)$, which exist as $|k| < \dim N = |N|$. By Proposition 10.20, $k \cup \{\langle b, c\rangle\}$ is the required extension.

1⇒2. If $B \subseteq N$ is a base of N the following type is not realized in N

$$p(x) \quad = \quad \Big\{\neg\varphi(x) \ : \ \varphi(x) \in L(B) \text{ is algebraic}\Big\}$$

Therefore, if N is saturated, $|B| = |N|$. □

10.25 Exercise Prove that every infinite algebraically closed set is a model. □

10.26 Exercise Prove that every model is homogeneous. □

10.27 Exercise Prove that if $\dim N = \dim M + 1$ then there is no model K such that $M \prec K \prec N$. □

Chapter 11

Countable models

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ larger than $|L|$. (Expect that most relevant theorems are proved under the assumption that L is countable.) Notation and implicit assumptions are as presented in Section 8.3.

1 The omitting types theorem

We say that a formula **$\varphi(x)$ isolates $p(x)$**, a type, if it is consistent and $\varphi(x) \to p(x)$. When Δ is a set of formulas, we say that **Δ isolates $p(x)$** if some formula in Δ does. In this chapter Δ is always a set of the form $L_x(A)$ and we say **A isolates $p(x)$** or every simply **$p(x)$ is isolated** when A is clear. We say that a model **M omits $p(x)$** if it does not realized it.

Observe that if $p(x) \subseteq L(M)$ then M realizes $p(x)$ if and only if M isolates $p(x)$. Therefore if A isolates $p(x)$, then every model containing A realizes $p(x)$. Below we prove that the converse also holds when L and A are countable. This a famous classical theorem which is called the *omitting types theorem*.

The construction of a model M omitting $p(x)$ proceeds by stages. Along the construction we preserve the property of being *isolated* since that of being *realized* is not well behaved in sets which are not models. In the limit, when a model is obtained, these two properties coincide.

The omitting types theorem require the following lemma which is an analogue of the Kuratowski-Ulam theorem (i.e., the so-called Fubini theorem for Baire category).

11.1 Lemma *Assume $L(A)$ is countable. Let $p(x) \subseteq L(A)$ and suppose that A does not isolate $p(x)$. Then every consistent formula $\psi(z) \in L(A)$ has a solution a such that A, a does not isolate $p(x)$.*

Proof We construct sequence of formulas $\langle \psi_i(z) : i < \omega \rangle$ such that any realization a of the type $q(z) = \{\psi_i(z) : i < \omega\}$ is as required by the lemma.

To begin with, we fix an enumeration $\langle \xi_i(x\,;z) : i < \omega \rangle$ of all formulas in $L_{x\,;z}(A)$ and set $\psi_0(z) = \psi(z)$. At stage $i+1$, if $\xi_i(x\,;z) \wedge \psi_i(z)$ in inconsistent, let $\psi_{i+1}(z) = \psi_i(z)$, otherwise let

$$\psi_{i+1}(z) \quad = \quad \psi_i(z) \quad \wedge \quad \exists x \left[\xi_i(x\,;z) \wedge \neg\varphi(x)\right]$$

for some formula $\varphi(x) \in p$ such that the resulting $\psi_{i+1}(z)$ is consistent. If possible, this guarantees that $\xi_i(x\,;a)$ for any $a \vDash q(z)$ does not isolate $p(x)$. The proof is complete if we can show that it is always possible to find a formula $\varphi(x)$ as required.

Suppose by way of contradiction that no formula makes $\psi_{i+1}(z)$ consistent, that is,

$$\xi_i(x\,;z) \wedge \psi_i(z) \quad \to \quad \varphi(x)$$

for every $\varphi(x) \in p$. This immediately imply that

$$\exists z\, \big[\xi_i(x\,;z) \wedge \psi_i(z)\big] \quad \rightarrow \quad p(x),$$

that is, $p(x)$ is isolated by a formula in $L_x(A)$, which contradicts our assumptions. $\square$

11.2 Theorem (Omitting types theorem) *Assume $L(A)$ is countable. Then for every consistent $p(x) \subseteq L(A)$ the following are equivalent*

1. *all models containing A realize $p(x)$;*
2. *$p(x)$ is isolated.*

The theorem owes its name to the contrapositive of implication 1⇒2.

Proof Implication 2⇒1 is clear, we prove 1⇒2. Assume A does not isolate $p(x)$. The model M is the union of a chain $\langle A_i : i < \omega \rangle$ of countable subsets of $\mathfrak{U}$ that starts with $A_0 = A$. Along the construction we require as inductive hypothesis that A_i does not isolate $p(x)$. At the end, M will not isolate $p(x)$ which for a model is equivalent to omitting.

We proceed as in the proof of the downward Löwenheim-Skolem theorem (precisely, we use the second proof 2.36). Lemma 11.1 ensure that we can satisfy any consistent formula in A_i and preserve the inductive hypothesis. $\square$

Gerald Sacks once famously remarked: *Any fool can realize a type but it take a model theorist to omit one.* (Though, as remarked above, the proof reminds more of set theory than model theory.)

2 Prime and atomic models

We say that ***M* is prime over *A*** if $A \subseteq M$ and for every N containing A there is an elementary embedding $h : M \to N$ that fixes A. When A is empty we simply say that M is **prime**.

There is no syntactic analogue of primeness; the notion that comes most close (and works pretty well when everything is countable) is atomicity. For $a \in \mathfrak{U}^{|x|}$ we say that ***a* is isolated over *A*** if the type $p(x) = \mathrm{tp}(a/A)$ is isolated. Note that this equivalent to claiming that a is an isolated point in $\mathfrak{U}^{|x|}$ w.r.t. the A-topology defined in Section 8.3. We say that ***M* is atomic over *A*** if $A \subseteq M$ every $a \in M^{<\omega}$ is isolated over A.

11.3 Proposition *Let b and a be finite tuples. Then the following are equivalent*

1. *A isolates b, a;*
2. *A, a isolates b and A isolates a.*

Proof Let $p(x,z) = \mathrm{tp}(b,a/A)$ and note that $p(x,a) = \mathrm{tp}(x/A,a)$ and $\exists x\, p(x,z) = \mathrm{tp}(a/A)$.

1⇒2 Let $\varphi(x,z) \in L(A)$ be such that $\varphi(x,z) \rightarrow p(x,z)$. It follows that $\varphi(x,a) \rightarrow p(x,a)$ and $\exists x\, \varphi(x,z) \rightarrow \exists x\, p(x,z)$. Therefore 2 holds.

2⇒1 Fix $\varphi(x\,;z), \psi(z) \in L(A)$ such that $\varphi(x\,;a)$ isolates $p(x\,;a)$ and $\psi(z)$ isolates $\exists x\, p(x\,;z)$. Let $\xi(x\,;z) \in p$ be arbitrary. As $\varphi(x\,;a) \rightarrow \xi(x\,;a)$, the formula

$\forall x\,[\varphi(x\,;z) \to \xi(x\,;z)]$ belongs to $\exists x\, p(x,z)$. Hence $\psi(z) \to \forall x\,[\varphi(x\,;z) \to \xi(x\,;z)]$. As this holds for all $\xi(x\,;z) \in p$, we conclude that $\psi(z) \wedge \varphi(x\,;z)$ isolates $p(x,z)$. □

The straightforward direction of the proposition above yields the following useful proposition.

11.4 Proposition *If M is atomic over A then M is atomic over A,a for every finite $a \in M^{<\omega}$.*

Proof Let $b \in M^{|x|}$ be a finite tuple. Then A isolates b,a hence A,a isolates b. □

11.5 Proposition *Let $k : M \to N$ be an elementary map and suppose that M is atomic over* $\operatorname{dom} k$. *Then for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b,c\rangle\} : M \to N$ is elementary.*

Proof Let $p(x\,;z) = \operatorname{tp}(b\,;a)$ where a is an enumeration of $\operatorname{dom} k$. Let $\varphi(x\,;z) \in L$ be such that $\varphi(x\,;a) \to p(x\,;a)$. Note that, by elementarity, $\varphi(x\,;ka) \to p(x\,;ka)$. Hence the required c is any solution of $\varphi(x\,;ka)$ in N. □

A limiting assumption in Proposition 11.4 is that a need to be finite. Therefore the following proposition is restricted to countable models.

11.6 Proposition *Any two is countable models atomic over A are isomorphic.*

Proof Easy, using Propositions 11.4 and 11.5 and back-and-forth. □

11.7 Proposition *Assume $L(A)$ is countable. Then for every model M the following are equivalent*

1. *M is countable and atomic over A;*
2. *M is prime over A.*

Proof 1⇒2 By Propositions 11.4 and 11.5.

2⇒1 Some countable model containing A exists, as M embeds in it, M has also to be countable. Now we prove that M is atomic over A. Suppose for a contradiction that there is some $b \in M^{<\omega}$ such that $p(x) = \operatorname{tp}(b/A)$ is not isolated. By the omitting types theorem there is a model N containing A that omits $p(x)$. Then there cannot be any A-elementary embedding of M into N. □

11.8 Proposition *Assume $L(A)$ is countable. Then the following are equivalent*

1. *there are models atomic over A;*
2. *for every $|z| < \omega$, every consistent $\varphi(z) \in L(A)$ has a solution that is isolated over A.*

Note that 2 says that in $\mathfrak{U}^{|z|}$ isolated points are dense w.r.t. the topology defined in Section 8.3.

Proof Implication 1⇒2 hold by elementarity. To prove 2⇒1 we construct by induction a sequence $\langle a_i : i < \omega\rangle$. Reasoning as in (the second proof of) the downword Löverheim-Skolem theorem ensures that $A \cup \{a_i : i < \omega\}$ is a model. To obtain an atomic model we require that $a_{\restriction i}$ is isolated over A.

Suppose $a_{\restriction i}$ has been defined and let $\varphi(z) \in L(A)$ be the formula that isolates $\operatorname{tp}(a_{\restriction i}/A)$. Let $\psi(x\,;z) \in L(A)$ be such that $\psi(x\,;a_{\restriction i})$ is consistent (we leave to the reader the details of the enumeration of such formulas). Then $\psi(x;z) \wedge \varphi(z)$ is also consistent and by assumption it has a solution $b;c$ that is isolated over A. As $a_{\restriction i} \equiv_A c$, there is an A-automorphism such that $fc = a_{\restriction i}$. Therefore $fb;a_{\restriction i}$ is a solution $\psi(x;z)$ that is also isolated over A. Then we can set $a_i = fb$. □

3 Countable categoricity

Here we present some important characterizations of ω-categoricity. The second property below can be stated in different equivalent ways; for convenience, these equivalents are considered in a separate proposition. For convenience we consider the following generalization: we say hat T is **ω-categorical over A** if any two countable models containing A are isomorphic.

11.9 **Theorem (Engeler, Ryll-Nardzewsky, and Svenonius Theorem)** *Assume $L(A)$ is countable. The following are equivalent:*

1. *T is ω-categorical over A;*
2. *every type $p(x) \subseteq L(A)$ with $|x| < \omega$ is isolated.*

By 3 of Proposition 11.10 below, no theory is ω-categorical over an infinite set. The reader can easily prove that that categoricity over some finite A is equivalent to categoricity over $\varnothing$ (see Exercise 11.11).

Proof The implication 1⇒2 is an immediate consequence of the omitting types theorem. In fact, if $p(x)$ is a non-isolated A-type, then there are two countable models M and N containing A such that M realizes $p(x) \subseteq L(A)$ while N omits it. Then M and N cannot be isomorphic over A. As for implication 2⇒1, observe that 2 implies that every countable model containing A is atomic over A. But, by Proposition 11.6, countable atomic models are unique up to isomorphism. □

11.10 **Proposition** *Fix a set A and a finite tuple of variables x. The following are equivalent*

1. *every A-type $p(x)$ is isolated;*
2. *$S_x(A)$ is finite;*
3. *$L_x(A)$ is finite up to equivalence;*
4. *in $\mathcal{U}^{|x|}$ there is a finite number of orbits under $Aut(\mathcal{U}/A)$.*

Proof To prove the implication 1⇒2 observe that $\mathcal{U}^{|x|}$ is the union of sets of the form $p(\mathcal{U})$ where $p \in S_x(A)$. If these types are isolated then $\mathcal{U}^{|x|}$ is the union of A-definable sets. By compactness this union has to be finite. To prove 2⇒1 let $p \in S_x(A)$. If $S_x(A)$ is finite, $\neg p(\mathcal{U})$ is the union of finitely many type definable sets. A finite union of type definable sets is type definable. So $\neg p(\mathcal{U})$ is type definable. Hence $p(\mathcal{U})$ is isolated. We prove implication 2⇒3 observe that each formula in $L_x(A)$ is equivalent to the disjunction of the types in $S_x(A)$ that contain this formula. If $S_x(A)$ is finite, $L_x(A)$ is finite up to equivalence. Implication 3⇒2 is clear and equivalence 2⇔4 follows from the characterization of orbits as type-definable sets. □

11.11 **Exercise** Prove that the following are equivalent for every finite set A

1. T is ω-categorical;
2. T is ω-categorical over A. □

11.12 Exercise Prove that the following are equivalent

1. T is ω-categorical;
2. there is countable model that is both saturated and atomic. □

11.13 Exercise Assume L is countable and that T is complete. Suppose that for every finite tuple x there is a model M that realizes only finitely many types in $S_x(T)$. Prove that T is ω-categorical.

4 Small theories

Let T be, as always in this chapter, a complete theory without finite models. We say that T is **small over A** if $S_x(A)$ is countable for every x of finite length. When A is empty, we simply say that T is **small**. The set A is only introduced for convenience, $A = \varnothing$ is the only interesting case.

11.14 Proposition *No T is small over an uncountable set and if T is small (over $\varnothing$) then it is small over any countable A.*

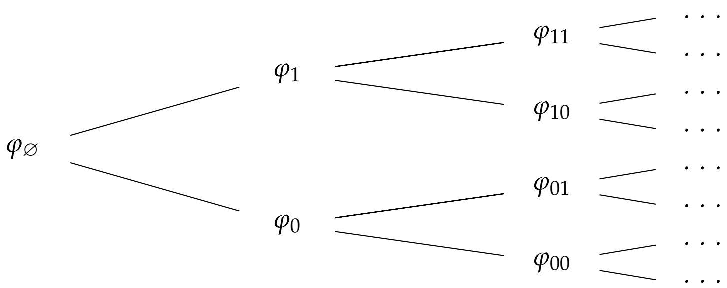
Proof Let a be an enumeration of A. As $S_x(A) = \{p(x\,;a) \ :\ p \in S_{x\,;z}(T)\}$ the proposition is immediate. □

Below we identify $S_x(A)$ with $\mathfrak{U}^{|x|}/{\equiv_A}$

11.15 Definition *Let Δ be a set of formulas (we mainly use $\Delta = L_x(A)$ in this section). A* ***binary tree of formulas in Δ*** *is a sequence $\langle \varphi_s \ :\ s \in 2^{<\omega}\rangle$ of formulas such that*

1. *for each $s \in 2^\omega$ the type $p_s \stackrel{def}{=} \{\varphi_{s\restriction n} \ :\ n < \omega\}$ is consistent;*
2. *$p_s \cup p_r$ is inconsistent for any two distinct $s, r \in 2^\omega$.*

We may depict a binary tree of formulas as follows



Where branches are consistent types and distinct branches are inconsistent. □

11.16 Proposition *Auppose $L(A)$ is countable. The following are equivalent*

1. *T is small over A;*
2. *there exists a countable saturated model containing A;*

3. there is no binary tree of formulas in $L_x(A)$ for any finite x.

Proof 1⇒2 There is a countable model M containing A that is weakly saturated (see Proposition 8.12). There is a countable homogeneous model N containing M (see Exercise 8.13). Clearly N is also weakly saturated. Then it is saturated by Corollary 8.13 .

2⇒3 Clear.

3⇒1 We prove the contrapositive: we assume that $\mathfrak{U}^{|x|}/{\equiv_A}$ is uncountable for some finite x and define a tree of formulas in $L_x(A)$ by induction. Let $\varphi_\varnothing(x) = \top$ and assume as induction hypothesis that $\varphi_s(\mathfrak{U})/{\equiv_A}$ is uncountable. This will guarantee the consistency of the branches. For convinience we also assume that $\varphi_s(x) \rightarrow \varphi_{s\restriction n}(x)$ for every $n < |s|$.

We claim that there is a formula $\psi(x) \in L(A)$ such that both sets $\varphi_s(\mathfrak{U}) \cap \psi(\mathfrak{U}) \,/\, {\equiv_A}$ and $\varphi_s(\mathfrak{U}) \cap \neg\psi(\mathfrak{U}) \,/\, {\equiv_A}$ are uncountable. If $\psi(x)$ is such a formula, we can define $\varphi_{s0}(x) = \varphi_s(x) \wedge \psi(x)$ and $\varphi_{s1}(x) = \varphi_s(x) \wedge \neg\psi(x)$ as this preserves the induction hypothesis.

First we prove that the following type is consistent

$$p(x) \quad = \quad \Big\{\psi(x) \in L(A) \;:\; \varphi_s(\mathfrak{U}) \cap \neg\psi(\mathfrak{U}) \,/\, {\equiv_A} \;\text{ is countable }\Big\}.$$

Suppose not, then $\neg\psi_1(x) \vee \dots \vee \neg\psi_n(x)$ holds for some $\psi_i(x) \in p$. Hence, as $\varphi_s(\mathfrak{U})/{\equiv_A}$ is uncountable, at least one of the sets $\varphi_s(\mathfrak{U}) \cap \neg\psi_i(\mathfrak{U}) \,/\, {\equiv_A}$ is uncountable, contradicting the definition of $p(x)$.

Finally note that, if the formula $\psi(x)$ required above does not exist, $p(x)$ is complete and, by the definition of $p(x)$ there are countably many types distinct from $p(x)$. This is a contradiction because we assumed $S_x(A)$ to be uncountable. □

11.17 Proposition *A small theory has countable atomic models over every countable set A.*

Proof We prove that every formula in $L_x(A)$, where $|x| < \omega$, has a solution isolated over A. Then it suffices to apply Proposition 11.8.

Suppose for a contradiction that $\varphi(x) \in L(A)$ is consistent but has no solution isolated over A. Then there is a formula $\psi(x) \in L(A)$ such that both $\varphi(x) \wedge \psi(x)$ and $\varphi(x) \wedge \neg\psi(x)$ are consistent, otherwise $\varphi(x)$ would imply a complete type and every solution of $\varphi(x)$ would be isolated. Fix such a $\psi(x)$. Clearly neither $\varphi(x) \wedge \psi(x)$ nor $\varphi(x) \wedge \neg\psi(x)$ have a solution isolated over A. This allows to construct a tree of formulas in $L_x(A)$ and prove that T is not small over A. □

11.18 Exercise Let $|x| = 1$. Prove that if $S_x(A)$ is countable for every finite set A, then T is small. □

11.19 Exercise (Vaught) Prove that no complete theory has exactly 2 countable models (assume L is countable – though it is not really necessary).

Hint: suppose T has exactly two countable models. Then T is small and there are a countable saturated model N and an atomic model $M \subseteq N$. As T is not ω-categorical, $M \not\simeq N$ and there is finite tuple a that is not isolated over $\varnothing$. Let K be an atomic model over a. Clearly $K \not\simeq M$ and, by Exercises 11.11 and 11.12, also $K \not\simeq N$. □

5 A toy version of a theorem of Zil'ber

As an application we prove that if T is ω-categorical and strongly minimal then it is not finitely axiomatizable.

We say that T has the **finite model property** if for every sentence $\varphi \in L$ there is a finite substructure $A \subseteq \mathfrak{U}$ such that

fmp $\qquad \mathfrak{U} \vDash \varphi \quad \Leftrightarrow \quad A \vDash \varphi$

The property is interesting because of the following proposition.

11.20 Proposition *If T has the finite model property then it is not finitely axiomatizable.*

Proof Assume fmp and suppose for a contadiction that there is a sentence $\varphi \in L$ such that $T \vdash \varphi \vdash T$. Then $A \vDash T$ for some finite structure A. But $T \vdash \exists^{>k} x\,(x = x)$ for every k. A contradiction. □

We need the following definition. We say that $C \subseteq \mathfrak{U}$ is an **homogeneous set** if for every pair of tuples $a, c \in C^{<\omega}$ such that $a \equiv c$ and for every $b \in C$ there is a $d \in C$ such that $a, b \equiv c, d$.

11.21 Lemma *Suppose L is countable. If T is ω-categorical and every finite set is contained in a finite homogeneous set, then T has the finite model property.*

Proof Assume for the moment that L is a relational language. We prove fmp also for formulas with parameters. We prove that for all n there is a finite structure $A \subseteq \mathfrak{U}$ where fmp holds for all sentences $\varphi \in L(A)$ such that

\# $\qquad$ number of parameters in φ + number of quantifiers in $\varphi \leq n$.

Fix n and pick some finite set A that is homogeneous and such that all types $p(z) \subseteq L$ with $|z| \leq n$ have a realization in $A^{|z|}$. Now we prove fmp by induction on the syntax of φ.

The claim is evident for atomic formulas and the induction step is immediate for boolean connectives. As for induction step for the existential quantifier, consider the formula $\exists x\, \varphi(x\,; c)$, where $c \in A^{<n}$ and $|x| = 1$. Implication $\Leftarrow$ of fmp follows immediately from the induction hypothesis and from the fact that, if $\exists x\, \varphi(x\,; c)$ satisfy #, also $\varphi(d\,; c)$ satisfies it. As for $\Rightarrow$, assume $\mathfrak{U} \vDash \exists x\, \varphi(x\,; c)$. Let $a, b \in A^{<\omega}$ be a solution of $\varphi(x\,; z)$ such that $a \equiv c$. Such a solution exists because all types with $\leq n$ variables are realized in A. By homogeneity there is a $d \in A$ such that $a, b \equiv c, d$ and therefore $A \vDash \varphi(d\,; c)$.

Finally consider the case when L contains functions symbols. Note that in fmp we may only require A to be a substructures w.r.t. symbols of L that actually occur in φ. Moreover, in # we can add the requirement that all functions symbols in φ have arity $< n$. Now note that if all types $p(z) \subseteq L$ with $|z| \leq n$ have a realization in $A^{|z|}$, then A is a substructure w.r.t. a fragment which only contains functions of arity $< n$. □

11.22 Proposition *If T is strongly minimal, then every algebraically closed set is homogeneous.*

Proof Let A be argebraically closed and fix some $a, c \in A^{<\omega}$ such that $a \equiv c$. Let b be an element of A. Suppose first the case $b \in \operatorname{acl} a$. Fix an $f \in \operatorname{Aut}(\mathfrak{U})$ such that $f(a) = c$. Then $d = fb$ is the required element, in fact $a, b \equiv c, d$ and $d \in \operatorname{acl} c \subseteq A$.

Now, suppose instead that $b \notin \operatorname{acl} a$. Then any $d \notin \operatorname{acl} c$ satisfies $a, b \equiv c, d$. Such a d exists in A otherwise $A = \operatorname{acl} c \neq \operatorname{acl} a$ which contradicts $a \equiv c$. □

From the propositions above we finally obtain the following.

11.23 Theorem *A theory which is ω-categorical and strongly minimal is not finitely axiomatizable.*

Proof If T is ω-categorical the algebraic closure of a finite set is finite. Therefore from Proposition 11.22 we infer that T satisfies the assumptions of Lemma 11.21. Hence T has the finite model property, so, by Proposition 11.20 it is not finitely axiomatizable. □

11.24 Exercise Assume L is countable and let T be strongly minimal. Prove that the following are equivalent

1. T is ω-categorical;
2. the algebraic closure of a finite set is finite. □

6 Notes and references

A famous theorem of Boris Zil'ber claims that Teorema 11.23 holds for any totally categorical theory. The same theorem has been proved independently by Cherlin, Harrington and Lachlan with a proof that uses the classification of finite simple groups. This theorem signs the birth of a subject known as *geometric stability theory* which studies in depth the geometric properties of model theories which we breafly hinted to in Chapter 10. The interested reader may consult Pillay's monograph on the subject.

[1] Anand Pillay, *Geometric stability theory*, Oxford Logic Guides, vol. 32, 1996.

Chapter 12

Definability and automorphisms

The description of first-order definability is simplified if we allow definable sets to be used as (second-order) parameters in formulas.

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ strictly larger than $|L|$. Notation and implicit assumptions are as presented in Section 8.3.

1 Many-sorted structures

A many-sorted language consists of three disjoint sets. Besides the usual L_{fun} and L_{rel}, we have a set L_{srt} of **sorts**. The language comprises also a **(many-sorted) arity function** that assigns to function and relations symbols r, f a tuple of sorts of finite positive length which we call **arity**.

A many sorted structure M consists of

1. a set M_s, for each $s \in L_{\text{srt}}$;
2. a function $f^M : M_{s_1} \times \cdots \times M_{s_n} \to M_{s_0}$, for each $f \in L_{\text{fun}}$ of arity $\langle s_0, \ldots, s_n \rangle$;
3. a relation $r^M \subseteq M_{s_0} \times \cdots \times M_{s_n}$, for each $r \in L_{\text{rel}}$ of arity $\langle s_0, \ldots, s_n \rangle$.

For every sort s we fix a sufficiently large a set of variables V_s. Now we define terms by induction. Along the way we assign a sort to each term t.

All variables are terms of the respective sort. If $t_1, \ldots, t_n$ are terms of sorts $s_1, \ldots, s_n$ and $f \in L_{\text{fun}}$ is of arity $\langle s_0, \ldots, s_n \rangle$ then $f\, t_1, \ldots, t_n$ is a term of sort s_0.

If $r \in L_{\text{rel}}$ has arity $\langle s_0, \ldots, s_n \rangle$ then $r\, t_0, \ldots, t_n$ is a formula. Also, $t_1 = t_2$ is formula for every pair of terms of equal sort. All other formulas are constructed by induction using the propositional connectives $\neg$ and $\vee$ and the quantifier $\exists x$ (or any other reasonable choice of logical connectives).

Truth of formulas is defined as for one-sorted languages up to one difference: we require that the witness of the quantifier $\exists x$ belongs to M_s, where s is the sort of the variable x.

Second order logic is arguably the most widely used example of many sorted structure. It may be formalized using a language with a sort n for every $n \in \omega$. The sort 0 is used for the first order elements; the sorts $n > 0$ is used for relations of arity n. For every $n > 0$ the language has a relation symbol $\in_n$ of arity $\langle 0^n, n \rangle$, where 0^n stands for n many 0s. There are also arbitrary many functions and relation symbols of sort $\langle 0^{n+1} \rangle$.

2 The eq-expansion

⚠ Warning: the structure $\mathcal{U}^{\text{eq}}$ and the theory T^{eq} defined below does not coincide with the standard one. As the difference is merely cosmetic, introducing a new notation would be an overkill and we prefer to abuse the existing terminology. In Section 6 below we compare our definition with the standard one. This will also explain the symbol used.

We recall the definition of partitioned formulas. A **partitioned formula** is a triple that consists of a formula and two tuples of variables. We use a semicolon, as in $\sigma(x\,;z)$, to separate the two tuples of variables. Below, z is always a placeholder for parameters and x is used for the definition of a set.

Given a language L we define a many-sorted language $\boldsymbol{L^{\text{eq}}}$ which has a sort for each (partitioned) formula $\sigma(x\,;z) \in L$ and a sort $\mathbf{0}$ which we call the **home sort** or the **first-order sort**. All other sorts are called **second-order sorts**. First of all, L^{eq} contains all relations and functions of L. The many-sorted arity of a relation r is $\langle 0^{n_r}\rangle$, a string of 0's of length the arity of r in L. Similarly, the many-sorted arity of a function f is $\langle 0^{1+n_f}\rangle$. Moreover L^{eq} contains a relation symbol $\in_{\sigma(x\,;z)}$ for each sort $\sigma(x\,;z)$. These have arity $\big\langle 0^{|x|}, \sigma(x\,;z)\big\rangle$. As there is no risk of ambiguity, in the sequel we write $\in$ without subscript.

The model $\boldsymbol{\mathcal{U}^{\text{eq}}}$ has $\mathcal{U}$ as domain for the home sort. The domain for the sort $\sigma(x\,;z)$ contains the definable sets $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ for some $b \in \mathcal{U}^{|z|}$. The symbols in L have the same interpretation as in the one-sorted case, and $\in$ is interpreted as set membership. We write $\boldsymbol{T^{\text{eq}}}$ for $\text{Th}(\mathcal{U}^{\text{eq}})$.

As usual $\boldsymbol{L^{\text{eq}}}$ also denotes the set of formulas constructed in this language and, if $A \subseteq \mathcal{U}^{\text{eq}}$, we write $\boldsymbol{L^{\text{eq}}(A)}$ for the language and the set of formulas that use elements of A as parameters. We write $\boldsymbol{L(A)}$ for the set of formulas in $L^{\text{eq}}(A)$ that contain no second-order variables (neither free nor quantified).

It is important to note right away that this expansion is a mild one: the definable subsets of the home sort of $\mathcal{U}^{\text{eq}}$ are the same as those of $\mathcal{U}$. In particular, iteration of the expansion would not yield anything new.

12.1 Proposition *Let $\varphi(u\,;\mathcal{X}) \in L^{\text{eq}}$ where u is a tuple of first-order variables and $\mathcal{X}$ a tuple of variables of sort $\sigma_i(x_i\,;z_i)$ for $i = 1,\dots,n$. Then there is a formula $\varphi'(u\,;z) \in L$ such that*

$$\forall \mathcal{X}, z\ \left[\bigwedge_{i=1}^{n} \mathcal{X}_i = \big\{x_i : \sigma_i(x_i\,;z_i)\big\} \rightarrow \forall u\ \big[\varphi(u\,;\mathcal{X})\ \leftrightarrow\ \varphi'(u\,;z)\big]\right]$$

where $\mathcal{X}$ stand for the tuple $\mathcal{X}_1,\dots,\mathcal{X}_n$ and z for the concatenation of $z_1,\dots,z_n$.

When $n = 0$ the proposition asserts that L^{eq}_u and L_u have the same expressive power.

Proof By induction on syntax. When φ is atomic $\varphi' = \varphi$ unless it is of the form $t \in \mathcal{X}$ for some tuple of terms t or it has the form $\mathcal{X}_1 = \mathcal{X}_2$. In the first case φ' is the formula $\sigma(t\,;z)$. In the second case it is the formula $\forall x\ \big[\sigma_1(x\,;z_1) \leftrightarrow \sigma_2(x\,;z_2)\big]$. In the latter case we assume that the variables x_1,x_2 in the sorts of $\mathcal{X}_1,\mathcal{X}_2$ have equal length otherwise we may replace $\mathcal{X}_1 = \mathcal{X}_2$ by $\perp$.

As for connectives they stay unchanged but for the quantifiers $\exists\mathcal{X}$ where $\mathcal{X}$ is a second-order variable, say of sort $\sigma(x\,;z)$. These quantifiers are replaced with $\exists z$. $\square$

Proposition 12.1 implies in particular that we can always eliminate second-order quantifiers. This is an obvious fact that may be obscured by the notation: we can replace $\exists \mathcal{X}$ by $\exists z$ if we substitute $\sigma(t\,;z)$ for $t \in \mathcal{X}$ in the quantified formula.

Proposition 12.1 should convince the reader that the move from $\mathcal{U}$ to $\mathcal{U}^{\mathrm{eq}}$ is almost a trivial one. Exercise 12.14 below should convince the reader that some trivial moves may be highly convenient.

12.2 Remark From Proposition 12.1 it follows that for every $A \subseteq \mathcal{U}^{\mathrm{eq}}$, there is a $B \subseteq \mathcal{U}$ such that $L(B)$ is at least as expressive as $L(A)$. The point of $\mathcal{U}^{\mathrm{eq}}$ is that there might not be any $B \subseteq \mathcal{U}$ such that $L(B)$ is exactly as expressive as $L^{\mathrm{eq}}(A)$. For instance, suppose L contains only an equivalence relation with infinitely many classes, all with infinite elements. Let $\mathcal{A}$ be an equivalence class. Then $\mathcal{A}$ is is definable in $L(B)$ if and only if $B \cap \mathcal{A} \neq \varnothing$. But no element of $\mathcal{A}$ is definable in $L^{\mathrm{eq}}(\{\mathcal{A}\})$. □

If $\mathcal{V} \preceq \mathcal{U}$ we write $\boldsymbol{\mathcal{V}^{\mathrm{eq}}}$ for the substructure of $\mathcal{U}^{\mathrm{eq}}$ that has $\mathcal{V}$ as domain of the home sort and the set of definable sets of the form $\sigma(\mathcal{U}\,;b)$ for some $b \in \mathcal{V}^{|z|}$ as domain of the sort $\sigma(x\,;z)$. The following proposition claims that elementary substructures of $\mathcal{U}^{\mathrm{eq}}$ are exactly those of the form $\mathcal{V}^{\mathrm{eq}}$ for some $\mathcal{V} \preceq \mathcal{U}$.

12.3 Proposition *The following are equivalent for every structure $\mathcal{V}^{\dagger}$ of signature L^{eq}*

1. $\mathcal{V}^{\dagger} \preceq \mathcal{U}^{\mathrm{eq}}$;
2. $\mathcal{V}^{\dagger} = \mathcal{V}^{\mathrm{eq}}$ *for some* $\mathcal{V} \preceq \mathcal{U}$.

Proof Implication 2⇒1 is a direct consequence of Proposition 12.1. We prove 1⇒2. Let $\mathcal{V}$ be the domain of the home sort of $\mathcal{V}^{\dagger}$. It is clear that $\mathcal{V} \preceq \mathcal{U}$. Let $\mathcal{A} \in \mathcal{V}^{\mathrm{eq}}$ have sort $\sigma(x\,;z)$, say $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ for some $b \in \mathcal{V}^{|z|}$. As $\exists^{=1}\mathcal{X}\,\forall x\,\big[x \in \mathcal{X} \leftrightarrow \sigma(x\,;b)\big]$ holds in $\mathcal{U}^{\mathrm{eq}}$, by elementarity it holds in $\mathcal{V}^{\dagger}$ and therefore $\mathcal{A} \in \mathcal{V}^{\dagger}$. This proves $\mathcal{V}^{\mathrm{eq}} \subseteq \mathcal{V}^{\dagger}$. A similar argument proves the converse inclusion. □

12.4 Proposition *Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$. Then every type $p(u\,;\mathcal{X}) \subseteq L^{\mathrm{eq}}(A)$ finitely consistent in $\mathcal{U}^{\mathrm{eq}}$ is realized in $\mathcal{U}^{\mathrm{eq}}$. That is, $\mathcal{U}^{\mathrm{eq}}$ is saturated.*

Proof By Remark 12.2, there is some $B \subseteq \mathcal{U}$ and some $q(u\,;\mathcal{X}) \subseteq L^{\mathrm{eq}}(B)$ equivalent to $p(u\,;\mathcal{X})$. This already proves the proposition when $\mathcal{X}$ is the empty tuple. Otherwise, let $q'(u\,;z)$ be obtained replacing every formula $\varphi(u\,;\mathcal{X})$ in $q(u\,;\mathcal{X})$ with with the formula $\varphi'(u\,;z)$ given by Proposition 12.1. Then $q'(u\,;z)$ is finitely consistent in $\mathcal{U}$. Assume for clarity of notation that $\mathcal{X}$ is a single variable of sort $\sigma(x\,;z)$. If $c\,;b \vDash q'(u\,;z)$ then $c\,;\sigma(\mathcal{U}\,;b) \vDash q(u\,;\mathcal{X})$. □

Automorphisms of a many-sorted structure are defined in the obvious way: sorts are preserved as well as functions and relations. Every automorphisms $f : \mathcal{U} \to \mathcal{U}$ extends to an automorphism $f : \mathcal{U}^{\mathrm{eq}} \to \mathcal{U}^{\mathrm{eq}}$ as follows. If $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ we define $f\mathcal{A} = \sigma(\mathcal{U}\,;fb) = f[\mathcal{A}]$, which clearly preserves the sort and the relation $\in$. Clearly, this extension is unique.

The homogeneity of $\mathcal{U}^{\mathrm{eq}}$ follows by back-and-forth as in the one-sorted case.

12.5 Proposition *Every elementary map $k : \mathcal{U}^{\mathrm{eq}} \to \mathcal{U}^{\mathrm{eq}}$ of cardinality $< \kappa$ extends to an automorphism of $\mathcal{U}^{\mathrm{eq}}$.* □

3 The definable closure in the eq-expansion

We may safely confuse automorphism of $\mathcal{U}$ with automorphisms of $\mathcal{U}^{\text{eq}}$. Let $A \subseteq \mathcal{U}^{\text{eq}}$ and let a be a tuple of elements of $\mathcal{U}^{\text{eq}}$. We denote by $\mathbf{Aut}(\mathcal{U}/A)$ the set of automorphisms (of $\mathcal{U}^{\text{eq}}$) that fix all elements of A. The symbol $\mathcal{O}(a/A)$ denotes the **orbit of a over A**. This has been defined in Section 2 and now we apply it to $\mathcal{U}^{\text{eq}}$

$$\mathcal{O}(a/A) \quad \stackrel{\text{def}}{=} \quad \{fa \ : \ f \in \operatorname{Aut}(\mathcal{U}/A)\}.$$

By homogeneity, $\mathcal{O}(a/A) = p(\mathcal{U}^{\text{eq}})$ where $p(v) = \operatorname{tp}(a/A)$. When $\mathcal{O}(a/A) = \{a\}$ we say that a is **invariant over A** or **A-invariant**, for short.

12.6 Definition *Let $A \subseteq \mathcal{U}^{\text{eq}}$ and $a \in \mathcal{U}^{\text{eq}}$. We say that **$a$ is definable over A** if $\varphi(a) \wedge \exists^{=1} v\, \varphi(v)$ holds for some formula $\varphi(v) \in L^{\text{eq}}(A)$. We write $\mathbf{dcl^{eq}(A)}$ for the set of those $a \in \mathcal{U}^{\text{eq}}$ that are definable over A. We write $\mathbf{dcl(A)}$ for $\operatorname{dcl}^{\text{eq}}(A) \cap \mathcal{U}$. This is the natural generalization of the notion introduced in Section 1.*

Saturation and homogeneity of $\mathcal{U}^{\text{eq}}$ allow us to prove the following proposition with virtually the same proof of Theorem 10.4

12.7 Theorem *For any $A \subseteq \mathcal{U}^{\text{eq}}$ and $a \in \mathcal{U}^{\text{eq}}$ the following are equivalent*

1. *a is invariant over A;*
2. *$a \in \operatorname{dcl}^{\text{eq}}(A)$.* □

Note that a tuple is invariant if and only it all its component are definable. Then in Theorem 12.7 we may replace a and/or A by a tuple.

Definition 12.6 treats first- and second-order elements of $\mathcal{U}^{\text{eq}}$ uniformly. However, for sets it can be rephrased in a more natural way.

12.8 Proposition *Let $A \subseteq \mathcal{U}^{\text{eq}}$ and let $\mathcal{A} \in \mathcal{U}^{\text{eq}}$ have sort $\sigma(x\,;z)$ then the following are equivalent*

1. *$\mathcal{A}$ is definable over A*
2. *$\mathcal{A} = \psi(\mathcal{U})$ for some $\psi(x) \in L(A)$*

Proof Implication 2⇒1 is clear because extensionality is implicit in the definition of $\mathcal{U}^{\text{eq}}$. We prove 1⇒2. Let $\varphi(\mathcal{X}) \in L^{\text{eq}}(A)$ be a formula $\mathcal{A}$ is the unique solution of. Then the required formula $\psi(x)$ is $\exists \mathcal{X}\, [x \in \mathcal{X} \wedge \varphi(\mathcal{X})]$. □

By the characterization above, Theorem 12.7 when applied to a definable set $\mathcal{A}$ gives an alternative proof of Proposition 8.19.

The formula $\psi(x)$ in Proposition 12.8 need not be related to the sort $\sigma(x\,;z)$. For example, consider the theory of a binary equivalence relation $e(x\,;z)$ with two infinite classes, let $\mathcal{A}$ be one of these classes and let $A \neq \varnothing$ be such that $A \cap \mathcal{A} = \varnothing$. Then $\mathcal{A}$ is definable over A though not by some formula of the form $e(x\,;b)$ for some $b \in A$. Things change radically if we replace A with a model.

12.9 Proposition *Let M be a model and let $\mathcal{A}$ be an element of sort $\sigma(x\,;z)$. Then the following are equivalent*

1. *$\mathcal{A} \in \operatorname{dcl}^{\text{eq}}(M)$;*

2. $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ *for some* $b \in M^{|z|}$.

In particular $M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$.

Proof Assume 1 and let $\psi(x) \in L(M)$ be such that $\mathcal{A} = \psi(\mathcal{U})$. Such a formula exists by Proposition 12.8. Then $\exists z\,\forall x\,\big[\psi(x) \leftrightarrow \sigma(x\,;z)\big]$ holds in $\mathcal{U}^{\mathrm{eq}}$. By elementarity it holds in M^{eq}, therefore $\exists z$ has a witness in M. This proves 1⇒2, the converse implication is obvious. □

4 The algebraic closure in the eq-expansion

Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and $a \in \mathcal{U}^{\mathrm{eq}}$. We say that ***a* is algebraic over *A*** if $\varphi(a) \wedge \exists^{=k} v\,\varphi(v)$ holds for some formula $\varphi(v) \in L^{\mathrm{eq}}(A)$ and some positive integer k. We write $\mathbf{acl^{eq}(A)}$ for the set of those $a \in \mathcal{U}^{\mathrm{eq}}$ that are algebraic over A. We write $\mathbf{acl(A)}$ for $\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathcal{U}$. This is the natural generalization of the notion introduced in Section 10.1.

The following proposition is proved with virtually the same proof of Theorem 10.5

12.10 Theorem *For every* $A \subseteq \mathcal{U}^{\mathrm{eq}}$ *and every* $a \in \mathcal{U}^{\mathrm{eq}}$ *the following are equivalent*

1. $\mathcal{O}(a/A)$ *is finite;*
2. $a \in \mathrm{acl}^{\mathrm{eq}}(A)$*;*
3. $a \in M^{\mathrm{eq}}$ *for every model such that* $A \subseteq M^{\mathrm{eq}}$. □

We say **finite equivalence relation** for an equivalence relation with finitely many classes. A **finite equivalence formula** or **type** is a formula, respectively a type, that defines a finite equivalence relation. Theorem 12.11 belows proves that sets algebraic over A are union of classes of a finite equivalence relations definable over A.

12.11 Theorem *Let* $A \subseteq \mathcal{U}^{\mathrm{eq}}$ *and let* $\mathcal{A} \in \mathcal{U}^{\mathrm{eq}}$ *be an element of sort* $\sigma(x\,;z)$. *Then the following are equivalent*

1. $\mathcal{A} \in \mathrm{acl}^{\mathrm{eq}}(A)$
2. *for some finite equivalence formula* $\varepsilon(x\,;y) \in L(A)$ *and some* $c_1, \ldots, c_n \in \mathcal{U}^{|y|}$

$$x \in \mathcal{A} \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \varepsilon(x\,;c_i).$$

Proof Let $\varphi(\mathcal{X}) \in L^{\mathrm{eq}}(A)$ be an algebraic formula that has $\mathcal{A}$ among its solutions and define

$$\varepsilon(x\,;y) \quad = \quad \forall \mathcal{X}\,\Big[\varphi(\mathcal{X}) \;\rightarrow\; \big[x \in \mathcal{X} \leftrightarrow y \in \mathcal{X}\big]\Big]$$

If $\varphi(\mathcal{X})$ has n, solutions, then $\varepsilon(x\,;y)$ has at most 2^n equivalence classes. Clearly, $\mathcal{A}$ is union of some these classes. □

We write $\boldsymbol{a \overset{\mathrm{Sh}}{\equiv}_A b}$ for $a \equiv_{\mathrm{acl}^{\mathrm{eq}} A} b$. In words we say that a and b have the same **Shelah strong-type over *A***.

12.12 Proposition *Let* $A \subseteq \mathcal{U}^{\mathrm{eq}}$ *and let* $a, b \in \mathcal{U}^{|x|}$. *Then the following are equivalent*

1. $a \overset{\mathrm{Sh}}{\equiv}_A b$*;*

2. $\varepsilon(a\,;b)$ *for every finite equivalence formula* $\varepsilon(x\,;y)\in L(A)$.

Proof Assume $\neg$2 and let $\varepsilon(x\,;y)$ witness this. Let $\mathcal{D}=\varepsilon(\mathfrak{U}\,;b)$, then $b\in\mathcal{D}$ and $a\notin\mathcal{D}$. As $\varepsilon(x\,;y)$ is an A-invariant finite equivalence formula, $\mathcal{D}\in\operatorname{acl}^{\mathrm{eq}}(A)$, and $\neg$1 follows. Vice versa, assume $\neg$1 and let $\varphi(x)\in L\big(\operatorname{acl}^{\mathrm{eq}}(A)\big)$ be such that $\varphi(a)\not\leftrightarrow\varphi(b)$. Let $\mathcal{D}=\varphi(\mathfrak{U})$, then $\mathcal{D}\in\operatorname{acl}^{\mathrm{eq}}(A)$. Therefore, by Proposition 12.11, the set $\mathcal{D}$ is union of equivalence classes of some finite equivalence formula $\varepsilon(x\,;y)\in L(A)$. Then $\neg\varepsilon(a\,;b)$ and $\neg$2 follows. □

12.13 Exercise Let $p(x)\subseteq L(A)$ and let $\varphi(x\,;y)\in L(A)$ be a formula that defines, when restricted to $p(\mathfrak{U})$, an equivalence relation with finitely many classes. Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x\,;y)$ on $p(\mathfrak{U})$. □

12.14 Exercise Let $A\subseteq\mathfrak{U}$, let $\mathcal{A}$ be a definable set with finite orbit over A. Without using $\mathfrak{U}^{\mathrm{eq}}$ prove that $\mathcal{A}$ is union of classes of a finite equivalence relation definable over A. □

12.15 Exercise Let T be strongly minimal and let $\varphi(x\,;y)\in L(A)$ with $|x|=|y|=1$. For arbitrary $c\in\mathfrak{U}$, prove that if the orbit of $\varphi(\mathfrak{U}\,;c)$ over A is finite, then $\varphi(\mathfrak{U}\,;c)$ is definable over $\operatorname{acl}A$.

Hint: you can use Theorem 12.11. □

5 Elimination of imaginaries

For the time being, we agree that *imaginary* is just another word for definable set. Though this is not formally correct (cfr. Section 6), it is morally true and helps to understand the terminology. The concept of elimination of imaginaries has been introduced by Poizat who also proved Theorem 12.22 below. A theory has elimination of imaginaries if for every $A\subseteq\mathfrak{U}^{\mathrm{eq}}$, there is a $B\subseteq\mathfrak{U}$ such that $L(B)$ and $L(A)$ have the same expressive power (i.e. they are the same up to to equivalence).

12.16 Definition *We say that* ***T has elimination of imaginaries*** *if for every definable set* $\mathcal{A}$ *there is a formula* $\varphi(x\,;z)\in L$ *such that*

ei $$\exists^{=1}z\;\forall x\;\Big[x\in\mathcal{A}\;\leftrightarrow\;\varphi(x\,;z)\Big]$$

We say that the witness of $\exists^{=1}z$ *in the formula above is a* ***canonical parameter*** *of* $\mathcal{A}$. *It is also called a a* ***canonical name*** *for* $\mathcal{A}$. *A set may have different canonical parameters for different formulas* $\varphi(x\,;z)$.

We say that T has ***weak*** *elimination of imaginaries if*

wei $$\exists^{=k}z\;\forall x\;\Big[x\in\mathcal{A}\;\leftrightarrow\;\varphi(x\,;z)\Big]$$

for some positive integer k. □

In the formulas above we allow z to be the empty string. In this case we read ei and wei omitting the quantifiers $\exists^{=1}z$, respectively $\exists^{=k}z$. Therefore $\varnothing$-definabile sets have all (at least) the empty string as a canonical parameter.

To show that the notions above are well-defined properties of a theory one needs to check that they are independent of our choice of monster model. We leave this to the reader as an exercise.

12.17 Exercise Let M be an arbitrary model. Prove that ei and wei hold (in $\mathcal{U}^{\rm eq}$) if and only if they hold in $M^{\rm eq}$. $\square$

We say that two tuples a and b of elements of $\mathcal{U}^{\rm eq}$ are **interdefinable** if $\operatorname{dcl}^{\rm eq}(a) = \operatorname{dcl}^{\rm eq}(b)$. By Theorem 12.7 this is equivalent to saying that $\operatorname{Aut}(\mathcal{U}/a) = \operatorname{Aut}(\mathcal{U}/b)$, that is, the automorphisms that fix a fix also b, and vice versa.

12.18 Theorem *The following are equivalent*

1. *T has weak elimination of imaginaries;*
2. *every definable set is interdefinable with a finite set;*
3. *every definable set $\mathcal{A}$ is definable over* $\operatorname{acl}\{\mathcal{A}\}$.

Proof 1⇒2 Assume 1 and let $\mathcal{B}$ be the set of solutions of the formula

$$\forall x\left[x \in \mathcal{A} \ \leftrightarrow\ \sigma(x\,;z)\right].$$

Hence $\mathcal{B}$ is finite and $\mathcal{B} \in \operatorname{dcl}^{\rm eq}\{\mathcal{A}\}$. We also have $\mathcal{A} \in \operatorname{dcl}^{\rm eq}\{\mathcal{B}\}$ because $\mathcal{A}$ is definibile by the formula $\exists z\left[z \in \mathcal{B} \wedge \sigma(x\,;z)\right]$. Therefore $\operatorname{dcl}^{\rm eq}\{\mathcal{A}\} = \operatorname{dcl}^{\rm eq}\{\mathcal{B}\}$.

2⇒3 Assume $\operatorname{dcl}^{\rm eq}\{\mathcal{A}\} = \operatorname{dcl}^{\rm eq}\{\mathcal{B}\}$ for some finite set $\mathcal{B}$. The elements of $\mathcal{B}$, say $b_1,\dots,b_n$, are the (finitely many) solutions of the formula $z \in \mathcal{B}$. Therefore $b_1,\dots,b_n \in \operatorname{acl}(\mathcal{B}) = \operatorname{acl}(\mathcal{A})$. Let $\varphi(\mathcal{X}\,;\mathcal{B})$ be a formula that has $\mathcal{A}$ as unique solution. Then $\exists \mathcal{X}\left[\mathcal{X} = \{b_1,\dots,b_n\} \wedge \varphi(\mathcal{X}\,;\mathcal{X})\right]$ is the formula that proves 3.

3⇒1 Assume 3. As wei holds trivially for all $\varnothing$-definable sets, we may assume $\mathcal{A} \neq \varnothing$. Let $\sigma(x\,;z)$ be such that

$$\forall x\left[x \in \mathcal{A} \ \leftrightarrow\ \sigma(x\,;b)\right].$$

for some tuple b of elements of $\operatorname{acl}\{\mathcal{A}\}$. Fix some algebraic formula $\delta(z\,;\mathcal{A})$ satisfied by b and write $\psi(z\,;\mathcal{X})$ for the formula

$$\forall x\left[x \in \mathcal{X} \ \leftrightarrow\ \sigma(x\,;z)\right] \ \wedge\ \delta(z\,;\mathcal{X}).$$

The following formula is clearly satisfied by b therefore, if we can prove that it has finitely many solutions, wei follows from Proposition 12.1

$$\sharp \qquad \forall x\left[x \in \mathcal{A} \ \leftrightarrow\ \sigma(x\,;z) \ \wedge\ \exists \mathcal{X}\, \psi(z\,;\mathcal{X})\right].$$

We check that any c that satisfies $\sharp$ also satisfies $\delta(z\,;\mathcal{A})$. As $\mathcal{A}$ is non empty, $\psi(c\,;\mathcal{A}')$ holds for some $\mathcal{A}'$. By $\sharp$ and the definition of $\psi(z\,;\mathcal{X})$ we obtain $\mathcal{A}' = \mathcal{A}$ and $\delta(c\,;\mathcal{A})$. $\square$

12.19 Theorem *The following are equivalent*

1. *T has elimination of imaginaries;*
2. *every definable set is interdefinable with a tuple of real elements;*

3. *every definable set* $\mathcal{A}$ *is definable over* $\mathrm{dcl}\{\mathcal{A}\}$.

Proof Implications 1⇒2⇒3 are immediate. Implication 3⇒1is identical to the homologous implication in Theorem 12.18, just substitute algebraic with definable. □

We now consider elimination of imaginaries in two concrete structures: algebraically closed fields and real closed fields. The following lemma is required in the proof of Theorem 12.21 below.

12.20 Lemma *The following is a sufficient condition for weak elimination of imaginaries*

$\sharp$ *for every* $A \subseteq \mathcal{U}^{\mathrm{eq}}$, *every consistent* $\varphi(z) \in L(A)$ *has a solution in* $\mathrm{acl}\, A$.

Proof Let $\mathcal{A} \in \mathcal{U}^{\mathrm{eq}}$ be a definable set and let $\sigma(x\,;z)$ be its sort. Then $\forall x\,\big[x \in \mathcal{A} \leftrightarrow \sigma(x\,;z)\big]$ is consistent and, by $\sharp$ it it has a solution in $\mathrm{acl}\{\mathcal{A}\}$. Hence weak elimination follows from Theorem 12.18. □

12.21 Theorem *Let* T *be a complete, strongly minimal theory. Then, if* $\mathrm{acl}\,\varnothing$ *is infinite,* T *has weak elimination of imaginaries.*

Proof If $\mathrm{acl}\,\varnothing$ is infinite, $\mathrm{acl}\,A$ is a model for every A (cfr. Exercise 10.25) so condition $\sharp$ of lemma 12.20 holds by elementarity and the theorem follows. □

12.22 Theorem *The theories* T^p_{acf} *have elimination of imaginaries.*

Proof By Theorem 12.21 we know that T^p_{acf} has weak elimination of imaginaries. Therefore, by Theorem 12.18 it suffices to prove that every finite set $\mathcal{A}$ is interdefinable with a tuple. Let $\mathcal{A} = \{a_1, \dots, a_n\}$ where each a_i is a tuple $a_{i,1}, \dots, a_{i,m}$ of elements of $\mathcal{U}$. Given $\mathcal{A}$ we define the term

$$t_{\mathcal{A}}(x\,;y) \quad = \quad \prod_{i=1}^{n}\Big(x - \sum_{k=1}^{m} a_{i,k}\, y_k\Big). \qquad \text{where } y = y_1, \dots, y_m.$$

Note that (the interpretation of) the term $t_{\mathcal{A}}(x\,;y)$ is independent on particular indexing of the set $\mathcal{A}$. So, any automorphism that fixes $\mathcal{A}$, fixes the $t_{\mathcal{A}}(x\,;y)$. Now rewrite $t_{\mathcal{A}}(x\,;y)$ as a sum of monomials and let c be the tuple of coefficients of these monomials. The tuple c uniquely determines $t_{\mathcal{A}}(x\,;y)$ and vice versa. Therefore every automorphism that fixes $\mathcal{A}$ fixes c and vice versa. Hence $\mathcal{A}$ and c are interdefinable. □

12.23 Exercise Let T have elimination of imaginaries and $\varphi(x\,;z) \in L(A)$. For arbitrary $c \in \mathcal{U}^{|z|}$, prove that if the orbit of $\varphi(\mathcal{U}\,;c)$ over A is finite, then $\varphi(\mathcal{U}\,;c)$ is definable over $\mathrm{acl}\,A$. □

6 Imaginaries: the true story

The point of the expansion to $\mathcal{U}^{\mathrm{eq}}$ is to add a canonical parameter for each definable set. In fact, in $\mathcal{U}^{\mathrm{eq}}$ every definable subset of $\mathcal{U}^{|z|}$ is the canonical parameter if itself. This allows to deal with theories without elimination of imaginaries in the most straightforward way.

The expansion to $\mathcal{U}^{\mathrm{eq}}$ that was originally introduced by Shelah (and still used everywhere else) is slightly different from the one introduced here. For a given set $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ Shelah considers the equivalence relation defined by the formula

$$\varepsilon(z\,;z') \stackrel{\text{def}}{=} \forall x\left[\sigma(x\,;z) \;\leftrightarrow\; \sigma(x\,;z')\right].$$

The equivalence class of b in the relation $\varepsilon(z\,;z')$ is what Shelah uses as canonical parameter of the set $\mathcal{A}$.

Shelah's $\mathcal{U}^{\rm eq}$ has a sort for each $\varnothing$-definable equivalence relation $\varepsilon(z\,;z')$. The domain of the sort $\varepsilon(z\,;z')$ contains the classes of the equivalence relation defined by $\varepsilon(z\,;z')$. These equivalence classes are called **imaginaries**. Shelah's $L^{\rm eq}$ contains functions that map tuples in the home sort to their equivalence class.

Tortuous routes are somewhat typical of Shelah. This did not hinder him from becoming one of the most productive mathematician of his century. Lesser minds may find it useful to take a simplified route.

7 Uniform elimination of imaginaries

Let us rephrase the definition of elimination of imaginaries: for every formula $\varphi(x\,;u) \in L$ and every tuple $c \in \mathcal{U}^{|u|}$ there is a formula $\sigma(x\,;z) \in L$ such that

$$\exists^{=1}z\;\forall x\left[\varphi(x\,;c) \;\leftrightarrow\; \sigma(x\,;z)\right].$$

A priori, the formula $\sigma(x\,;z)$ may depend on c in a very wild manner. We say that ***T* has uniform elimination of imaginaries** if for every $\varphi(x\,;u)$ there are a formula $\sigma(x\,;z)$ and a formula $\rho(z)$ such that

uei $$\forall u\;\exists^{=1}z\left[\rho(z)\;\wedge\;\forall x\;[\varphi(x\,;u) \;\leftrightarrow\; \sigma(x\,;z)]\right].$$

The role of the formula $\rho(z)$ above is may seem mysterious. It is clarified by the following propositions. In fact, uniform elimination of imaginaries is equivalent to a very natural property which in words says: every definable function is the kernel of a definable function. Recall that the kernel of the function f is the relation $fa = fb$.

Uniform elimination of imaginaries is convenient when dealing with interpretations of structure inside other structures. Let us consider a simple concrete example. Suppose $\mathcal{U}$ is a group and let $\mathcal{H}$ be a definable normal subgroup of $\mathcal{U}$. The elements of the quotient structure $\mathcal{U}/\mathcal{H}$ are equivalent classes of a definable equivalence relation. If there is uniform elimination of imaginaries we can identify $\mathcal{U}/\mathcal{H}$ with an actual definable subset of $\mathcal{G}$ (the range of the function f above). Moreover, the group operation of $\mathcal{G}$ are definable functions. As working in $\mathcal{U}/\mathcal{H}$ may be notationally cumbersome, $\mathcal{G}$ may offer a convenient alternative.

12.24 Proposition *The following are equivalent*

1. *T has uniform elimination of imaginaries;*
2. *for every $\varphi(x\,;u)$ such that $\forall u\;\exists x\;\varphi(x\,;u)$ there is a formula $\sigma(x\,;z)$ such that*

$$\forall u\;\exists^{=1}z\;\forall x\left[\varphi(x\,;u) \;\leftrightarrow\; \sigma(x\,;z)\right];$$

3. *for every equivalence formula $\varepsilon(x\,;u) \in L$ there is given by $\sigma(x\,;z) \in L$ such that*

$$\forall u\, \exists^{=1} z\, \forall x \left[\varepsilon(x\,;u) \;\leftrightarrow\; \sigma(x\,;z)\right].$$

Note that the definable function $f u = z$ mentioned above is defined by the formula

$$\vartheta(u\,;z) \;=\; \forall x \left[\varepsilon(x\,;u) \;\leftrightarrow\; \sigma(x\,;z)\right].$$

Proof 1⇒2. If $\forall u\, \exists x\, \varphi(x\,;u)$ we can rewrite uei as

$$\forall u\, \exists^{=1} z\, \forall x \left[\varphi(x\,;u) \;\leftrightarrow\; \rho(z) \wedge \sigma(x\,;z)\right].$$

2⇒3. Clear.

3⇒1. Apply 3 to the equivalence formula

$$\varepsilon(u\,;v) \;=\; \forall x \left[\varphi(x\,;u) \;\leftrightarrow\; \varphi(x\,;v)\right]$$

Let $\vartheta(u\,;z)$ be defined as above. The reader may check that uei holds substituting for $\delta(z)$ the formula $\exists u\; \vartheta(u\,;z)$ and and for $\sigma(x\,;z)$ the formula $\exists u\; [\varphi(x\,;u) \wedge \vartheta(u\,;z)]$. □

By the following theorem, uniformity comes almost for free.

12.25 Theorem *The following are equivalent*

1. *T has uniform elimination of imaginaries;*
2. $\operatorname{dcl}\varnothing$ *contains at least two elements and T has elimination of imaginaries.*

Proof 1⇒2. Let $\varphi(x, u)$ be the formula $u_1 = u_2$. From 1 we obtain a formula $\sigma(x\,;z) \in L$ such that

$$\forall u_1, u_2\, \exists^{=1} z \left[\rho(z) \;\wedge\; \forall x\, [u_1 = u_2 \;\leftrightarrow\; \sigma(x\,;z)]\right].$$

Therefore the formulas $\exists^{=1} z\, [\rho(z) \wedge \forall x\, \sigma(x\,;z)]$ and $\exists^{=1} z\, [\rho(z) \wedge \forall x \neg\sigma(x\,;z)]$ are both true. The witnesses of $\exists^{=1} z$ in these two formulas are two distinct elements of $\operatorname{dcl}\varnothing$.

2⇒1. Assume 2 and fix a formula formula $\varphi(x\,;u)$ such that $\varphi(x, a)$ is consistent for every $a \in \mathfrak{U}^{|u|}$. We prove 2 of Proposition 12.24.

Let $p(u)$ be the type that contains the formulas

$$\neg\, \exists^{=1} z\, \forall x \left[\varphi(x\,;u) \;\leftrightarrow\; \sigma(x;z)\right],$$

where $\sigma(x;z)$ ranges over all formulas in L. By elimination of imaginaries $p(u)$ is not consistent. Therefore, by compactness, there are some formulas $\sigma_i(x, z)$ such that

$$\sharp \qquad \forall u \bigvee_{i=0}^{n} \exists^{=1} z\, \forall x \left[\varphi(x\,;u) \;\leftrightarrow\; \sigma_i(x\,;z)\right].$$

To prove the theorem we need to move the disjunction in front of the $\sigma_i(x\,;z)$.

We can assume that if $\sigma_i(x\,;b) \leftrightarrow \sigma'_i(x\,;b')$ for some $\langle b, i\rangle \neq \langle b', i'\rangle$ then $\sigma_i(x\,;b)$ is inconsistent. Otherwise we can substitute the formula $\sigma_i(x\,;z)$ with

$$\sigma_i(x\,;z)\ \wedge\ \bigwedge_{j\leq i}\neg\exists y\neq b\,\forall x\,\Big[\sigma_j(x\,;y)\leftrightarrow\sigma_i(x\,;z)\Big].$$

As $\varphi(x,a)$ is consistent for every a, the substitution does not break the validity of $\sharp$.

Fix some distinct $\varnothing$-definable tuples $d_0,\ldots,d_n$ of the same length (these are easy to obtain from two $\varnothing$-definable elements). We claim that from $\sharp$ it follows that

$$\forall u\ \exists^{=1}z,y\ \forall x\left[\varphi(x\,;u)\ \leftrightarrow\ \bigvee_{i=1}^{n}\Big[\sigma_i(x\,;z)\ \wedge\ y=d_i\Big]\right].$$

(The tuple z,y plays the role of z.) We fix some a and check that the formula below has a unique solution

$$\flat\qquad\qquad \forall x\left[\varphi(x\,;a)\ \leftrightarrow\ \bigvee_{i=1}^{n}\Big[\sigma_i(x\,;z)\ \wedge\ y=d_i\Big]\right].$$

Existence follows immediately from $\sharp$. As for uniqueness, note that if b,d_i and $b',d_{i'}$ are two distinct solution of $\flat$ then $\sigma_i(x\,;b)\leftrightarrow\sigma_{i'}(x\,;b')$ for some $\langle b,i\rangle\neq\langle b',i'\rangle$. By what assumed on $\sigma_i(x\,;z)$, we obtain that $\varphi(x\,;a)$ is inconsistent. A contradiction which proves the theorem. $\square$

Chapter 13

Invariant sets

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ strictly larger than $|L|$. Notation and implicit assumptions are as presented in Section 8.3.

1 Invariant sets and types

Let $\mathcal{D} \subseteq \mathcal{U}^{|z|}$, where z is a tuple of length $< \kappa$. We say that $\mathcal{D}$ is a set **invariant over** $\boldsymbol{A}$, or $\boldsymbol{A}$**-invariant** if it is fixed (set-wise) by A-automorphisms. That is, $f[\mathcal{D}] = \mathcal{D}$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$ or, yet in other words,

is1. $\quad a \in \mathcal{D} \quad \leftrightarrow \quad fa \in \mathcal{D} \qquad$ for every $a \in \mathcal{U}^{|z|}$ and every $f \in \mathrm{Aut}(\mathcal{U}/A)$,

which, by homogeneity is equivalent to,

is2. $\quad a \in \mathcal{D} \quad \leftrightarrow \quad b \in \mathcal{D} \qquad$ for all $a, b \in \mathcal{U}^{|z|}$ such that $a \equiv_A b$.

which yield the following bound on the number of invariant sets

13.1 **Proposition** *Let $\lambda = |L_z(A)|$. There are at most 2^{2^λ} sets $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ that are invariant over A.*

Proof By is2, sets that are invariant over A are union of equivalence classes of the relation $\equiv_A$, that is, union of sets of the form $p(\mathcal{U})$ where $p(z) \in S(A)$. Then the number of A-invariant sets is $2^{|S_z(A)|}$. Clearly $|S_z(A)| \leq 2^\lambda$. □

We say that $\mathcal{D}$ is **invariant** tout court if it is invariant over some A. As we require κ to be inaccessible, there are exactly κ invariant sets.

In this chapter we work with Δ-types where Δ may be either $L(\mathcal{A})$ or the set of Boolean combinations of $\varphi(x\,;b)$ for $b \in \mathcal{A}^{|z|}$ and $\varphi(x\,;z) \in L$ some fixed formula. In the latter case a Δ-type is also called a $\boldsymbol{\varphi}$**-type**.

Typically $\mathcal{A}$ is either the whole of $\mathcal{U}$ or some small set $A \subseteq \mathcal{U}$. We denote by $\boldsymbol{S_\varphi(A)}$ the set of complete φ-types with parameters in A. Types in $\boldsymbol{S_\varphi(\mathcal{U})}$ are called **global** $\boldsymbol{\varphi}$**-types**.

Let $p(x) \subseteq L(\mathcal{U})$ be a consistent type. For every formula $\varphi(x\,;z) \in L$ we define

$$\mathcal{D}_{p,\varphi} \quad = \quad \Big\{ a \in \mathcal{U}^{|z|} \ : \ \varphi(x\,;a) \in p \Big\}$$

We can read the notation in two ways. Either the tuple z has infinite length and is the same for all formulas, or it is finite and depends on φ. This is possible because adding or erasing dummy variables to the second tuple of $\varphi(x\,;z)$ does not change $\mathcal{D}_{p,\varphi}$ in any relevant way, in particular invariance is preserved.

Let $p(x) \subseteq L(\mathcal{U})$ be a consistent type. We say that $p(x)$ is **invariant over** $\boldsymbol{A}$, or $\boldsymbol{A}$**-invariant**, if for every formula $\varphi(x\,;z) \in L$,

it1. $\quad \varphi(x\,;a) \in p \quad \Leftrightarrow \quad \varphi(x\,;fa) \in p \quad$ for every $a \in \mathcal{U}^{|z|}$ and every $f \in \mathrm{Aut}(\mathcal{U}/A)$.

A global φ-type can be safely identified with $\mathcal{D}_{p,\varphi}$ and a (complete) global type with the collection of all these sets. The notions of invariance coincide.

We say that the type $p(x) \subseteq L(\mathcal{U})$ **does not split over A** if

it2. $\quad a \equiv_A b \;\Rightarrow\; \Big(\varphi(x\,;a) \in p \Leftrightarrow \varphi(x\,;b) \in p\Big) \quad$ for all $a, b \in \mathcal{U}^{|z|}$

for every formula $\varphi(x\,;z) \in L$. For global types it2 is equivalent to

it2'. $\quad a \equiv_A b \;\Rightarrow\; \varphi(x\,;a) \leftrightarrow \varphi(x\,;b) \in p$

When p is a φ-type, it2 is equivalent to requiring it2' for $\varphi(x\,;z)$. By homogeneity, non splitting is equivalent to invariance.

The following is yet another important equivalent of invariance over A

it3. $\quad a \equiv_A b \;\Rightarrow\; a \equiv_{A,c} b \quad$ for all $a, b \in \mathcal{U}^{|z|}$ and for all $c \vDash p_{\restriction A,a,b}$.

This apply to global types $p(x) \in S(\mathcal{U})$ but not for φ-types.

2 Invariance from the dual perspective

⚠ The following terminology in not standard. We say that the set $\mathcal{B} \subseteq \mathcal{U}^{|x|}$ is **quasi-invariant over A** if for any finitely many automorphisms $f_1, \dots, f_n \in \mathrm{Aut}(\mathcal{U}/A)$ the sets $f_i[\mathcal{B}]$ have non-empty intersection.

We say that the type $p(x) \subseteq L(\mathcal{U})$ is **quasi-invariant over A** if $\varphi(\mathcal{U})$ is quasi-invariant over A for every $\varphi(x)$ conjunction of formulas in $p(x)$.

For global types quasi-invariance coincides with invariance.

13.2 Proposition *Let $p(x) \in S_\varphi(\mathcal{U})$ be a global φ-type, then the following are equivalent*

1. *$p(x)$ is invariant over A;*
2. *$p(x)$ is quasi-invariant over A.*

Proof 1⇒2. Assume $p(x)$ is invariant and let $\psi(x\,;b) \in p$ and some $b \in \mathcal{U}^{|z|}$. Then $\psi(x\,;fb) \in p$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$, so 2 follows from the finite consistency of $p(x)$.

2⇒1. Assume $p(x)$ is not invariant. Then there is $b \in \mathcal{U}^{|z|}$ such that $\varphi(x\,;b) \in p$ and $\varphi(x\,;fb) \notin p$ for some $f \in \mathrm{Aut}(\mathcal{U}/A)$. By completeness, $p(x)$ contains the formula $\varphi(x\,;b) \wedge \neg\varphi(x\,;fb)$ which clearly is not consistent with its f-translate. □

Quasi-invariance makes sense also for incomplete types. Unfortunately it is not true in general that every quasi-invariant type can be extended to a global one. In next section we consider a stronger notion, *coheirs*, which has this desirable property. In the next chapters we introduce a weaker notion, *non forking*, which also can be preserved by extensions.

3 Heirs and coheirs

The easiest way to obtain quasi-invariant types is via types that are finitely satisfiable. We say that a type **$p(x)$ is finitely satisfiable in A** if $\varphi(\mathcal{U}) \cap A^{|x|} \neq \varnothing$ for every $\varphi(x)$ that is a conjunction of formulas in p.

Note that by elementarity every type over a model is finitely satisfiable in the model, that is, if $p(x) \subseteq L(M)$ for some model M, then $p(x)$ is finitely satisfiable in M. For this reason the notion is mainly applied to models.

13.3 Proposition *Let $p(x) \in S_\varphi(\mathfrak{U})$ be a global φ-type finitely satisfiable in A. Then $p(x)$ is A-invariant.*

Proof By Proposition 13.2, because finitely satisfiable types are quasi-invariant. □

13.4 Proposition *Every type $q(x) \subseteq L(\mathfrak{U})$ finitely satisfiable in A has an extension to a global type finitely satisfiable in A.*

Proof Let $p(x) \subseteq L(\mathfrak{U})$ be maximal among the types containing $q(x)$ and finitely satisfiable in A. We prove that $p(x)$ is complete. If for a contradiction that $p(x)$ contains neither $\psi(x)$ nor $\neg\psi(x)$. Then neither $p(x) \cup \{\psi(x)\}$ nor $p(x) \cup \{\neg\psi(x)\}$ is finitely satisfiable in A. This contradicts the finite satisfiability of $p(x)$. □

If a global type $p(x) \in S(\mathfrak{U})$ is finitely satisfiable in M, we say that it is a **global coheir** of $p_{\restriction M}(x)$.

4 Morley sequences and indiscernibles

In the following α is some infinite ordinal $\leq \kappa$ and x is a tuple of variables of length $< \kappa$. Let $p(x) \in S(\mathfrak{U})$ be a global type invariant over A. We say that $c = \langle c_i : i < \alpha \rangle$ is a **Morley sequence of $p(x)$ over A** if for every $i < \alpha$

Ms. $$c_i \quad \models \quad p_{\restriction A, c_{\restriction i}}(x)$$

In particular, when $p(x)$ is finitely satisfiable in some model M, we say that c is a **coheirs sequence**. We may say that c is a Morley (or coheir) sequence without specifying a global type. Then we mean that c is a Morley (respectively coheir) sequence of some suitable global type.

The following is a convenient characterization of coheir sequences

13.5 Lemma *The following are equivalent*

1. *$c = \langle c_n : n < \alpha \rangle$ is a coheir sequence over A;*
2. *$c_{n+1} \equiv_{A, c_{\restriction n}} c_n$ and $\varphi(\mathfrak{U}) \cap A^{|x|} \neq \varnothing$ whenever $\varphi(x) \in L(A, c_{\restriction n})$ and $\varphi(c_n)$.*

Proof 1⇒2. Assume 1 and let $p(x) \in S(\mathfrak{U})$ be a global type finitely satisfiable in A and such that $c_i \models p_{\restriction A, c_{\restriction i}}(x)$. The requirement $c_{n+1} \equiv_{A, c_{\restriction n}} c_n$ is clear. Now, suppose $\varphi(c_n)$ for some $\varphi(x) \in L(A, c_{\restriction n})$. Then $\varphi(x)$ belongs to $p(x)$ hence $\varphi(\mathfrak{U}) \cap A^{|x|} \neq \varnothing$ because $p(x)$ is finitely satisfiable in A.

2⇒1. Assume 2 and define

$$q(x) \quad = \quad \{\varphi(x) \quad : \quad c_n \models \varphi(x) \in L(A, c_{\restriction n}) \ \text{ for some } \ n < \alpha\}.$$

Clearly c is a Morley sequence of any global type finitely satisfiable in A that extends $q(x)$. Hence it suffices to show that such type exists. So, we prove that $q(x)$ is finitely satisfiable in A and apply Proposition 13.4.

As 2 requires that every single formula in $q(x)$ is finitely satisfiable in A it suffices to show that $q(x)$ is closed under conjunctions. Let $\varphi_i(x) \in L(A, c_{\restriction n_i})$, for $i = 1, 2$, be two formulas in $q(x)$ satisfied by c_{n_i}. By the first requirement in 2, both formulas are satisfied by c_m where $m = \max\{n_1, n_2\}$. Hence their conjuction is in $q(x)$. □

Let $I, <_I$ be a linear order. We call a function $a : I \to \mathfrak{U}^{|x|}$ an ***I*-sequence**, or simply a **sequence** when I is clear. Often we introduce an I-sequence by writing $a = \langle a_i : i \in I \rangle$.

If $I_0 \subseteq I$ we call $a_{\restriction I_0}$ a **subsequence of *a***. Most relevant are the subsets $I_0 \subseteq I$ that are well-ordered by $<_I$ and, in particular, the finite ones. When I_0 has order-type α, an ordinal, we identify $a_{\restriction I_0}$ with a tuple of length α.

13.6 Definition *Let $I, <_I$ be an infinite linear order and let a be an I-sequence. We say that* ***a is a sequence of indiscernibles over A*** *or, a sequence of* ***A-indiscernibles****, if $a_{\restriction I_0} \equiv_A a_{\restriction I_1}$ for every $I_0, I_1 \in [I]^n$ and $n < \omega$. As usual,* $\mathbf{[I]^n}$ *is the set of subsets of I of cardinality n.* □

The condition above can be formulated in a number of clearly equivalent ways. We can require that that $\varphi(a_{i_0}, \ldots, a_{i_n}) \leftrightarrow \varphi(a_{j_0}, \ldots, a_{j_n})$ for every formula $\varphi(x_1, \ldots, x_n) \in L(A)$ and every pairs of tuples in I^n such that $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$. Or we can simply say that for every $i_0, \ldots, i_n \in I$ the type $\mathrm{tp}(a_{i_0}, \ldots, a_{i_n}/A)$ only depends on the order type of $i_0, \ldots, i_n$.

13.7 Proposition *Let $p(x) \in S(\mathfrak{U})$ be a global A-invariant type and let $c = \langle c_i : i < \alpha \rangle$ be a Morley sequence of $p(x)$ over A. Then c is a sequence of indiscernibles over A.*

Proof We prove by induction on $n < \omega$ that

$$\sharp \qquad c_{\restriction n} \equiv_A c_{\restriction I_0} \quad \text{for every } I_0 \subseteq \alpha \text{ of cardinality } n.$$

For $n = 0$ the claim is trivial, then we assume $\sharp$ above is true and prove that

$$c_{\restriction n}, c_n \equiv_A c_{\restriction I_0}, c_i \quad \text{for every } I_0 < i < \alpha.$$

As c is Morley sequence, $c_n \equiv_{A, c_{\restriction n}} c_i$ whenever $n < i$. Hence we can equivalently prove that

$$c_{\restriction n}, c_i \equiv_A c_{\restriction I_0}, c_i,$$

which is equivalent to

$$c_{\restriction n} \equiv_{A, c_i} c_{\restriction I_0}.$$

The latter holds by induction hypothesis $\sharp$ and the invariance of $p(x)$ as formulated in it3 of Section 13. □

5 From coheirs to Ramsey to indiscernibles

Let X be an infinite set an $n, k < \omega$. A total function $f : [X]^n \to k$ is also called a **coloring** of $[X]^n$ and is identified with a partition of $[X]^n$ into finitely k classes. A set $H \subseteq X$ such that f is constant on $[H]^n$ is called **monochromatic set**. In the literature it may be called a **homogeneous set**, but we already use this word with a different meaning. The following is a most famous theorem which here we prove it in unusual way.

13.8 Proposition (Ramsey Theorem) *For every infinite set X, every $n, k < \omega$ and every coloring $f : [X]^n \to k$ there is an infinite monochromatic set.*

Proof Let L be a language that contains k relation symbols $r_0, \ldots, r_{k-1}$ of arity n. Given $f : [X]^n \to k$ as above we define now a structure with domain X. The interpretation the relation symbols is

$$r_i^X \quad = \quad \Big\{a_0, \ldots, a_{n-1} \ : \ \{a_0, \ldots, a_{n-1}\} \in [X]^n \ \text{ and } \ f(\{a_0, \ldots, a_{n-1}\}) = i\Big\}.$$

Then

$$X \ \vDash \ \forall x_0, \ldots, x_{n-1} \left[\bigwedge_{0 \le i < j < n} x_i \ne x_j \ \ \to \ \ \bigvee_{i<k} r_i(x_0, \ldots, x_{n-1}) \right].$$

We may assume that X is an elementary substructure of some large saturated model $\mathcal{U}$. Pick any type $p(x) \in S(\mathcal{U})$ finitely satisfied in X but not realized in X and let $c = \langle c_i : i < \omega \rangle$ be a coheir sequence of $p(x)$. By # and indiscernibility, all tuples of n distinct elements of c satisfy the same relation, which we assume to be r_0. The theorem is proved if we can find in X a sequence $a = \langle a_i : i < \omega \rangle$ with the same property. (Note parenthetically that no element of c is in X.) We construct $a_{\restriction i}$ by induction on i as follows.

Assume as induction hypothesis that all subsequences of $a_{\restriction i}, c_{\restriction n}$ of length n satisfy r_0. Our goal is to find a_i such as the same property holds for $a_{\restriction i}, a_i, c_{\restriction n}$. By the indiscernibility of c, the property holds for $a_{\restriction i}, c_{\restriction n}, c_n$. And this can be written by a formula $\varphi(a_{\restriction i}, c_{\restriction n}, c_n)$. As c is a coheir sequence, by Lemma 13.5 we can find $a_i \in X$ such that $\varphi(a_{\restriction i}, c_{\restriction n}, a_i)$. So, as the order is irrelevant, $a_{\restriction i}, a_i, c_{\restriction n}$ satisfies the induction hypothesis. □

Let $I, <_I$ be an infinite linear order and let a be an I-sequence. Fix a sequence of distinct variables $x = \langle x_i : i < \omega \rangle$. We write $\boldsymbol{p(x) \ = \ \text{EM-tp}(a/A)}$ and say that **$p(x)$ is the Ehrenfeucht-Mostowski type of a over A** if $p(x)$ is the set of formulas $\varphi(x_{\restriction n}) \in L(A)$ such that $\varphi(a_{\restriction I_0})$ holds for every $I_0 \in [I]^n$. Note that if a is A-indiscernible then EM-tp(a/A) is a complete type, and vice versa. Moreover, if a and b are two A-indiscernible I-sequences with the same Ehrenfeucht-Mostowski type of a over A, then $a \equiv_A b$.

13.9 Theorem (Ehrenfeucht-Mostowski) *Let $I, <_I$ and $J, <_J$ be two infinite linear orders such that $|J| \le \kappa$. Then for every sequence $a = \langle a_i : i \in I \rangle$ there is an J-sequence of A-indiscernibles c such that EM-tp$(a/A) \subseteq$ EM-tp(c/A).*

Proof To begin with, we prove the theorem assuming that both I and J have order type ω. Then any realization of the following type yields a J-sequence of A-indiscernibles.

$$q(x) \quad \cup \quad \Big\{\varphi(x_{\restriction n}) \leftrightarrow \varphi(x_{\restriction J_0}) \ : \ \varphi(x_{\restriction n}) \in L(A), \ \ J_0 \in [\omega]^n, \ \ n < \omega\Big\}.$$

We will prove that any finite subset of the type above is realized by a subsequence of a. First note that any infinite subsequence of a realizes $q(x)$, so we only need to pay attention to the set on the right. We prove that for every $\varphi_0, \ldots, \varphi_{m-1} \in L_x(A)$ and every $n < \omega$ there is an infinite $H \subseteq I$ such that $a_{\restriction H}$

$$\Big\{\varphi_i(x_{\restriction n}) \leftrightarrow \varphi_i(x_{\restriction I_0}) \ : \ I_0 \in [\omega]^n \text{ and } i < m\Big\}.$$

Let $f : [I]^n \to \mathcal{P}(m)$ map $I_0 \in [I]^n$ to the set $\big\{i < m \ : \ \varphi_i(a_{\restriction I_0})\big\}$. By Ramsey theorem with 2^m colors there is some infinite monochromatic set $H \subseteq I$. As I has order type ω, so does H, and it is immediate that $a_{\restriction H}$ realizes #, as required to prove the theorem with I and J of order type ω.

Now we note that the argument above easily generalizes to arbitrary linear order I. Only, in this case we cannot guarantee that H has order type ω. However, as H is infinite, for every $k < \omega$ there is a any subset of H of size k that realizes the restriction of # to $n < k$ and $I_0 < k$. This suffices to prove the theorem with I arbitrary and J of order type ω.

Now we extend the result to arbitrary J of cardinality $\leq \kappa$. From the argument above, there is a sequence of indiscerbibles $\langle b_i : i < \omega \rangle$ such that EM-tp$(a/A) \subseteq$ EM-tp(b/A). Fix a J-sequence of variables $z = \langle z_i : i \in J \rangle$ and define

$$p(z) \quad = \quad \Big\{ \varphi(z_{\restriction J_0}) \in L(A) \ : \ \varphi(b_{\restriction n}) \ \text{ for some } J_0 \subseteq [J]^n, \ n < \omega \Big\}.$$

This type is clearly finitely consistent. Every c that realizes $q(z)$ is a J-sequence as required by the theorem. □

13.10 Proposition *Let $c = \langle c_i : i \in I \rangle$ be a sequence of A-indiscernibles. Then c is indiscernible over some model M containing A.*

Proof Fix an arbitrary model M containing A. By Theorem 13.9 there is an I-sequence of M-indiscernibles b such that EM-tp$(c/M) \subseteq$ EM-tp(b/M). As c is an A-indiscernible sequence $a \equiv_A b$, so some $h \in \mathrm{Aut}(\mathfrak{U}/A)$ takes b to c coordinatewise. Therefore $hb = c$ is indiscernible over $h[M]$. □

13.11 Exercise Let $a = \langle a_i : i \in I \rangle$ be an A-indiscernible sequence and let $I \subseteq J$ with $|J| \leq \kappa$. Then there is an A-indiscernible sequence $c = \langle c_i : i \in J \rangle$ such that $c_{\restriction I} = a$. □

13.12 Exercise Let $a \equiv_A b$ and let $q(x) = \mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. Prove that if $q(x)$ extends to a global A-invariant type $p(x) \in S(\mathfrak{U})$ then there is a sequence $c = \langle c_i : i < \omega \rangle$ such that a, c and b, c are both sequences of A-indiscernibles. □

Chapter 14

Lascar invariant sets

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ strictly larger than $|L|$. Notation and implicit assumptions are as presented in Section 8.3.

1 Expansions

In the next chapter, but marginally also in this chapter, we will find it convenient to expand the language L with a predicate for a given $\mathcal{D} \subseteq \mathcal{U}^{|z|}$. We denote by $\langle \mathcal{U}, \mathcal{D} \rangle$ the corresponding expansion of $\mathcal{U}$. Generally, we write $\boldsymbol{L(\mathcal{X})}$ for the expanded language but, if the intended interpretation of $\mathcal{X}$ only going to be $\mathcal{D}$, we may write $\boldsymbol{L(\mathcal{D})}$ and abbreviate $\langle \mathcal{U}\,; \mathcal{D} \rangle \vDash \varphi(\mathcal{X})$ as $\varphi(\mathcal{D})$.

If $\mathcal{C}, \mathcal{D} \subseteq \mathcal{U}^{|z|}$ we abbreviate $\langle \mathcal{U}, \mathcal{C} \rangle \equiv_A \langle \mathcal{U}, \mathcal{D} \rangle$ as $\mathcal{C} \equiv_A \mathcal{D}$. We also say that $\mathcal{D}$ is **saturated** if so is the model $\langle \mathcal{U}\,; \mathcal{D} \rangle$.

What said above is adapted to define the expansion $\boldsymbol{L(\mathcal{X}_i : i < \lambda)}$, where $\mathcal{X}_i$ are predicates of arity $|z_i|$. Again, when the sets $\mathcal{D}_i \subseteq \mathcal{U}^{|z_i|}$ are the only intended interpretation of $\mathcal{X}_i$, we may write $\boldsymbol{L(\mathcal{D}_i : i < \lambda)}$.

14.1 Remark The definitions above are straightforward when z finite tuple. When z is an infinite tuple the intuition stays the same but a more involved definition is required. In fact, first-order logic does not allow infinitary predicates. We think of $L(\mathcal{X})$ for a two sorted language. The *home sort*, denoted by 0, and the z*-sort*, denoted by z. The expansion $\langle \mathcal{U}, \mathcal{D} \rangle$ has domain $\mathcal{U}$ for the home sort, and $\mathcal{U}^{|z|}$ for the z-sort. Besides the symbols of L, there is a function symbol π_i for every $i < |z|$ which is interpreted as the projection to the i-coordinate. These functions have arity $\langle z, 0 \rangle$ (see Section 12.1 for the notation). There is also a predicate of sort $\langle z \rangle$ interpreted as $\mathcal{D}$. □

14.2 Proposition *If $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ is invariant over A then every A-indiscernible sequence is indiscernible in the language $L(A\,; \mathcal{D})$.*

Proof Let $c = \langle c_i : i \in I \rangle$ be an A-indiscernible sequence. For every $I_0, I_1 \in I^{[n]}$ there is an $f \in \operatorname{Aut}(\mathcal{U}/A)$ such that $fc_{\restriction I_0} = c_{\restriction I_1}$, Hence

$$\varphi(c_{\restriction I_0}\,; \mathcal{D}) \quad \leftrightarrow \quad \varphi(fc_{\restriction I_0}\,; f[\mathcal{D}]) \quad \leftrightarrow \quad \varphi(c_{\restriction I_1}\,; \mathcal{D}). \qquad \square$$

14.3 Exercise Prove that if $\mathcal{C} \subseteq \mathcal{U}^{|z|}$ is type-definable over B then $\equiv_{A,B}$ implies $\equiv_{A\,;\mathcal{C}}$. □

2 Lascar strong types

Let $\mathcal{D} \subseteq \mathcal{U}^{|z|}$, where z is a tuple of length $< \kappa$. The **orbit of $\mathcal{D}$ over A** is the set

$$o(\mathcal{D}/A) \stackrel{\text{def}}{=} \big\{ f[\mathcal{D}] \ : \ f \in \mathrm{Aut}(\mathcal{U}/A) \big\}$$

So, $\mathcal{D}$ is invariant over A when $o(\mathcal{D}/A) = \{\mathcal{D}\}$. We say that $\mathcal{D}$ is **Lascar invariant over** $\boldsymbol{A}$ if it is invariant over every model $M \supseteq A$. Recall that this means that if $a \equiv_M c$ for some model M containing A then $a \in \mathcal{D} \leftrightarrow c \in \mathcal{D}$.

14.4 Proposition *Let $\lambda = |L_z(A)|$. There are at most 2^{2^λ} sets $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ that are Lascar invariant over A.*

Proof Let N be a model containing A of cardinality $\leq \lambda$. Every set that is Lascar invariant over A is invariant over N. As $|L_z(N)| = \lambda$ the bound follows from Proposition 13.1. □

14.5 Theorem *For every $\mathcal{D}$ and every $A \subseteq M$ the following are equivalent*

1. *$\mathcal{D}$ is Lascar invariant over A;*
2. *every set in $o(\mathcal{D}/A)$ is M-invariant;*
3. *$o(\mathcal{D}/A)$ has cardinality $\leq 2^{2^{|L(A)|}}$;*
4. *$o(\mathcal{D}/A)$ has cardinality $< \kappa$;*
5. *$c_0 \in \mathcal{D} \leftrightarrow c_1 \in \mathcal{D}$ for every A-indiscernible sequence $\langle c_i : i < \omega \rangle$.*

Proof 1⇒2. This implication is clear because all sets in $o(\mathcal{D}/A)$ are Lascar invariant over A.

2⇒3. When $|M| \leq |L(A)|$ the implication follows from the bounds discussed in Section 13.1. We temporary add this assumption on M. Once the proof of the proposition is completed, is easily seen to be redundant.

3⇒4. This implication holds because κ is a strong limit cardinal.

4⇒5. Assume ¬5. Then we can find an A-indiscernible sequence $\langle c_i : i < \kappa \rangle$ such that $c_0 \in \mathcal{D} \not\leftrightarrow c_1 \in \mathcal{D}$. Define

$$E(u\,;v) \quad \Leftrightarrow \quad u \in \mathcal{C} \leftrightarrow v \in \mathcal{C} \quad \text{for every } \mathcal{C} \in o(\mathcal{D}/A).$$

Then $E(u\,;v)$ is an A-invariant equivalence relation. As $\neg E(c_0\,;c_1)$, by Proposition 14.2, indiscernibility over A implies that $\neg E(c_i\,,c_j)$ for every $i < j < \kappa$. Then $E(u\,;v)$ has κ equivalence classes. As κ is inaccessible, this implies ¬4.

5⇒1. Fix any $a \equiv_M b$. It suffices to prove that $a \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$. Let $p(z) \in S(\mathcal{U})$ be a global coheir of $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$. Let $c = \langle c_i : i < \omega \rangle$ be a Morley sequence of $p(z)$ over M, a, b. Then both a, c and b, c are A-indiscernible sequences. Therefore, from 5 we obtain $a \in \mathcal{D} \leftrightarrow c_0 \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$. □

For definable sets Lascar invariance reduces to definablity over the algebraic closure.

14.6 Corollary *For every definable set $\mathcal{D}$ the following are equivalent*

1. *$\mathcal{D}$ is Lascar invariant over A;*
2. *$\mathcal{D}$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$.* □

The following corollary easily follows from Theorem 14.5 and Proposition 14.2. As an exercise the reader may wish to prove it using Proposition 13.10.

14.7 Corollary *The following are equivalent*

1. *$\mathcal{D}$ is Lascar invariant over A;*
2. *every sequence of A-indiscernibles is $L(A\,;\mathcal{D})$-indiscernible.* □

Given a tuple $a \in \mathcal{U}^{|z|}$, we write $\mathcal{L}(a/A)$ for the intersection of all sets containing a that are Lascar invariant over A. Clearly $\mathcal{L}(a/A)$ is Lascar invariant over A. We say that a and b have the same **Lascar strong type over A** if $a \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$ for every set $\mathcal{D}$ that is Lascar invariant over A or, in other words, if $\mathcal{L}(a/A) = \mathcal{L}(b/A)$. The symbol used is $a \overset{\text{L}}{\equiv}_A b$.

Warning: the symbol $\mathcal{L}(a/A)$ is not standard. The relation $a \overset{\text{L}}{\equiv}_A b$ is sometime denoted by $aE_{L/A}b$ or by L-stp$(a/A) =$ L-stp(b/A).

14.8 Proposition *The relation $\overset{\text{L}}{\equiv}_A$ is the finest equivalence relation with $< \kappa$ classes that is invariant over A.* □

Proof Clearly $\overset{\text{L}}{\equiv}_A$ is an equivalence relation invariant over A. Each equivalence class is Lascar invariant over A, hence the number of equivalence classes is bounded by the number of Lascar invariant sets over A. To see that $\overset{\text{L}}{\equiv}_A$ is the finest of such equivalences. Suppose $\mathcal{D}$ is an equivalence class of an A-invariant equivalence relation with $< \kappa$ classes. Then $o(\mathcal{D}/A)$ has also cardinality $< \kappa$. Then $\mathcal{D}$ is Lascar invariant and as such is union of classes of the relation $\overset{\text{L}}{\equiv}_A$. □

Let $p(x) \in S(\mathcal{U})$ be global type; we say that p is **Lascar invariant over A** if the sets $\mathcal{D}_{p,\varphi}$ defined in Section 13 are all Lascar invariant over A.

14.9 Proposition *Let $p(x) \in S(\mathcal{U})$ be a global type. Then the following are equivalent*

1. *$p(x)$ is Lascar invariant over A;*
2. *every A-indiscernible sequence $c = \langle c_i : i < \omega\rangle$ is A, a-indiscernible for every $a \vDash p_{\restriction c}$.*

Here we either assume $|z| \geq \omega$ or in 2 we quantify over all $|z| < \omega$.

Proof 2⇒1 If $p(x)$ is not Lascar invariant over A then $c_0 \in \mathcal{D}_{p,\varphi} \not\leftrightarrow c_1 \in \mathcal{D}_{p,\varphi}$ for some A-indiscernible sequence $c = \langle c_i : i < \omega\rangle$ and some $\varphi(x\,;z) \in L$. Then $p(x)$ contains the formula $\varphi(x\,;c_0) \not\leftrightarrow \varphi(x\,;c_1)$. Hence, c is not indiscernible over any realization of $p(x)_{\restriction c_0,c_1}$

1⇒2 Assume 1 and fix an A-indiscernible sequence $c = \langle c_i : i < \omega\rangle$ and some $a \vDash p_{\restriction c}$. We need to prove that for every formula $\varphi(x\,;z') \in L$, where $z' = z_1,\dots,z_n$,

$$\varphi(a\,;c_0,\dots,c_{n-1}) \;\leftrightarrow\; \varphi(a\,;c_{i_1},\dots,c_{i_{n-1}}).$$

holds for every $i_0 < \dots < i_{n-1} < \omega$. Suppose not and let m be any integer larger than i_{n-1}. Then the following equivalences cannot both be true

$$\varphi(a\,;c_m,\dots,c_{m+n-1}) \;\leftrightarrow\; \varphi(a\,;c_0,\dots,c_{n-1});$$

$$\varphi(a\,;c_m,\dots,c_{m+n-1}) \;\leftrightarrow\; \varphi(a\,;c_{i_0},\dots,c_{i_{n-1}}).$$

If the first is false, define $c'_k = c_{km},\dots,c_{km+n-1}$ for all $k < \omega$. Otherwise, do this only for positive k and set $c'_0 = c_{i_0},\dots,c_{i_{n-1}}$. In either cases $\langle c'_k : k < \omega\rangle$ is a sequence of A-indiscernibles and $c'_0 \in \mathcal{D}_{p,\varphi} \not\leftrightarrow c'_1 \in \mathcal{D}_{p,\varphi}$. This contradicts 1. □

3 The Lascar graph and Newelski's theorem

Here we study Lascar strong types from a different viewpoint. The **Lascar graph over A** has an arc between all pairs $a, b \in \mathcal{U}^{|z|}$ such that $a \equiv_M b$ for some model M containing A. We write $\boldsymbol{d_A(a,b)}$ for the distance between a and b in the Lascar graph over A. Let us spell this out: $d_A(a,b) \leq n$ if there is a sequence $a_0, \ldots, a_n$ such that $a = a_0$, $b = a_n$, and $a_i \equiv_{M_i} a_{i+1}$ for some models M_i containing A. We write $\boldsymbol{d_A(a,b) < \infty}$ if a and b are in the same connected component of the Lascar graph over A.

14.10 Proposition *For every $a \in \mathcal{U}^{|z|}$*

$$\mathcal{L}(a/A) \quad = \quad \big\{c \; : \; d_A(a,c) < \infty\big\}.$$

Proof To prove inclusion $\supseteq$ it suffices to show that every Lascar A-invariant set containing a contains the set on the r.h.s. Let $\mathcal{D}$ be Lascar A-invariant, and let $b \in \mathcal{D}$. Then $\mathcal{D}$ contains also every c such that $b \equiv_M c$ for some model M containing A. That is, $\mathcal{D}$ contains every c such that $d_A(b,c) \leq 1$. It follows that $\mathcal{D}$ contains every c such that $d_A(a,c) < \infty$.

To prove inclusion $\subseteq$ we prove the set on the r.h.s. is Lascar A-invariant. Suppose the sequence $a_0, \ldots, a_n$, where $a_0 = a$ and $a_n = c$, witnesses $d_A(a,c) \leq n$ and suppose that $c \equiv_M b$ for some M containing A, then the sequence $a_0, \ldots, a_n, b$ witnesses $d_A(a,b) \leq n+1$. □

We write **Autf$(\mathcal{U}/A)$** for the subgroup of $\mathrm{Aut}(\mathcal{U}/A)$ that is generated by the automorphisms that fix point-wise some model M containing A. (The "f" in the symbol stands for *fort*, the French word for *strong*.) It is easy to verify that $\mathrm{Autf}(\mathcal{U}/A)$ is a normal subgroup of $\mathrm{Aut}(\mathcal{U}/A)$. The following is a corollary of Proposition 14.10.

14.11 Corollary *The following are equivalent*

1. $a \overset{\mathrm{L}}{\equiv}_A b$
2. $a = fb$ *for some* $f \in \mathrm{Autf}(\mathcal{U}/A)$. □

14.12 Exercise Prove that the equivalence relation $a \overset{\mathrm{L}}{\equiv}_A b$ is the transitive closure of the relation: there is a sequence $\langle c_i : i < \omega \rangle$ indiscernible over A such that $c_0 = a$ and $c_1 = b$. □

It may not be immediately obvious that the relation $d_A(z,y) \leq n$ is type-definable.

14.13 Proposition *For every $n < \omega$ there is a type $p_n(z,y) \subseteq L(A)$ equivalent to $d_A(z,y) \leq n$.*

Proof It suffices to prove the proposition with $n = 1$. Let $\lambda = |L(A)|$ and fix a tuple of distinct variables $w = \langle w_i : i < \lambda \rangle$, then $p_1(z,y) \;=\; \exists w\, p(w,z,y)$ where

$$p(w,z,y) \quad = \quad q(w) \;\cup\; \Big\{\varphi(z,w) \leftrightarrow \varphi(y,w) \;:\; \varphi(z,w) \in L(A)\Big\}$$

and $q(w) \subseteq L(A)$ is a consistent type with the property that all its realizations enumerate a model containing A.

It remains to verify that such a type exist. Let $\langle \psi_i(x, w_{\restriction i}) : i < \lambda \rangle$ be an enumeration of the formulas in $L_{x,w}(A)$, where x is a single variable. Let

$$q(w) \quad = \quad \Big\{\exists x\, \psi_i(x, w_{\restriction i}) \ \to \ \psi_i(w_i, w_{\restriction i}) \ : \ i < \lambda\Big\}.$$

Any realization of $q(w)$ satisfy the Tarski-Vaught test therefore it enumerates a model containing A. Vice versa it is clear that we can realize $q(w)$ in any model containing A. □

We conclude this section with a theorem of Ludomir Newelski.

The following notions apply generally to any group G acting on some set $\mathcal{X}$ and to any set $\mathcal{D} \subseteq \mathcal{X}$. Below we always have $G = \mathrm{Autf}(\mathcal{U}/A)$ and $\mathcal{X} = \mathcal{U}^{|z|}$. We say that $\mathcal{D}$ **is drifting** if for every finitely many $f_1, \dots, f_n \in G$ there is a $g \in G$ such that $g[\mathcal{D}]$ is disjoint from all the $f_i[\mathcal{D}]$. We say that $\mathcal{D}$ **is quasi-invariant** if for every finitely many $f_1, \dots, f_n \in G$ the sets $f_i[\mathcal{D}]$ have non-empty intersection. We say that a formula or a type is drifting or quasi-invariant if the set it defines is.

The union of drifting sets need not be drifting. However, by the following lemma it cannot be quasi-invariant.

14.14 Lemma *The union of finitely many drifting sets in not quasi-invariant.*

Proof It is convenient to prove an apparently more general claim. If $\mathcal{D}_1, \dots, \mathcal{D}_n$ are all drifting and $\mathcal{L}$ is such that for some finite $F \subseteq G$

$$\sharp \qquad \mathcal{L} \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{D}_1 \cup \dots \cup \mathcal{D}_n],$$

then $\mathcal{L}$ is not quasi-invariant. The claim is vacuously true for $n = 0$. Now, assume n is positive and that the claim holds for $n - 1$. Define $\mathcal{C} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_{n-1}$ and rewrite $\sharp$ as follows

$$\mathcal{L} \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{C}] \ \cup \ \bigcup_{h \in F} h[\mathcal{D}_n]$$

Since $\mathcal{D}_n$ is drifting there is a $g \in G$ such that $g[\mathcal{D}_n]$ is disjoint from $h[\mathcal{D}_n]$ for every $h \in F$, which implies that

$$\mathcal{L} \cap g[\mathcal{D}_n] \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{C}].$$

Hence for every h there holds

$$hg^{-1}[\mathcal{L}] \cap h[\mathcal{D}_n] \quad \subseteq \quad \bigcup_{f \in F} hg^{-1} f[\mathcal{C}]$$

So, from $\sharp$ we obtain

$$\mathcal{L} \cap \bigcap_{h \in F} hg^{-1}[\mathcal{L}] \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{C}] \ \cup \ \bigcup_{h \in F} \bigcup_{f \in F} hg^{-1} f[\mathcal{C}].$$

By the induction hypothesis, the set on the r.h.s. is not quasi-invariant. Hence neither is $\mathcal{L}$, proving the claim and with it the lemma. □

The following is a consequence of Baire's category theorem. We sketch a proof for the convenience of the reader.

14.15 Lemma *Let $p(x) \subseteq L(B)$ and $p_n(x) \subseteq L(A)$, for $n < \omega$, be consistent types such that*

1. $$p(x) \quad \to \quad \bigvee_{n < \omega} p_n(x)$$

Then there is an $n < \omega$ and a formula $\varphi(x) \in L(A)$ consistent with $p(x)$ such that

2. $\quad p(x) \wedge \varphi(x) \quad \rightarrow \quad p_n(x)$

Proof Negate 2 and choose inductively for every $n < \omega$ a formula $\psi_n(x) \in p_n(x)$ such that $p(x) \wedge \neg\psi_0(x) \wedge \cdots \wedge \neg\psi_n(x)$ is consistent. By compactness, this contradicts 1. $\square$

Finally we can prove Newelski's theorem.

14.16 Theorem (Newelski) *For every $a \in \mathcal{U}^{|z|}$ the following are equivalent*

1. *$\mathcal{L}(a/A)$ is type-definable;*
2. *$\mathcal{L}(a/A) \;=\; \{c \,:\, d_A(a,c) < n\}$ for some $n < \omega$.*

Proof Implications 2⇒1 holds by Proposition 14.13. We prove 1⇒2. Suppose $\mathcal{L}(a/A)$ is type-definable, say by the type $l(z)$. Let $p(z,y)$ be some consistent type (to be defined below) such that and $p(z,y) \rightarrow l(z) \wedge l(y)$. Then, in particular

$$p(z,y) \quad \rightarrow \quad \bigvee_{n<\omega} d_A(z,y) < n.$$

By Proposition 14.13 and Lemma 14.15, there is some $n < \omega$ and some $\varphi(z,y) \in L(A)$ consistent with $p(z,y)$ such that

$\sharp_1 \quad p(z,y) \wedge \varphi(z,y) \quad \rightarrow \quad d_A(z,y) < n.$

Below we define $p(z,y)$ so that for every $\psi(z,y) \in L(A)$

$\sharp_2 \quad p(z,a) \wedge \psi(z,a) \qquad$ is non-drifting whenever it is consistent.

Drifting and quasi-invariance are relative to the action of $\mathrm{Autf}(\mathcal{U}/A)$ on $\mathcal{U}^{|z|}$. Then, in particular, $p(z,a) \wedge \varphi(z,a)$ is non-drifting and the theorem follows. In fact, by non-drifting, there are some $a_0, \ldots, a_k \in \mathcal{L}(a/A)$ such that every set $p(\mathcal{U},c) \cap \varphi(\mathcal{U},c)$ for $c \in \mathcal{L}(a/A)$ intersects some $p(\mathcal{U},a_i) \cap \varphi(\mathcal{U},a_i)$. Let m be such that $d_A(a_i,a_j) \leq m$ for every $i,j \leq k$. From $\sharp_1$ we obtain that $d_A(a,c) \leq m + 2n$. As $c \in \mathcal{L}(a/A)$ is arbitrary, the theorem follows.

The required type $p(z,y)$ is union of a chain of types $p_\alpha(z,y)$ defined as follows

$$p_0(z,y) \;=\; l(z) \,\cup\, l(y);$$

$$\sharp_3 \qquad p_{\alpha+1}(z,y) \;=\; p_\alpha(z,y) \,\cup\, \Big\{\neg\psi(z,y) \;:\; p_\alpha(z,a) \wedge \psi(z,a) \text{ is drifting}\Big\};$$

$$p_\alpha(z,y) \;=\; \bigcup_{n<\alpha} p_n(z,y) \quad \text{for limit } \alpha.$$

Clearly, the chain stabilizes at some stage $\leq |L(A)|$ yielding a type which satisfies $\sharp_2$. So we only need to prove consistency. We prove that $p_\alpha(z,a)$ is quasi-invariant (so, in particular, consistent). Suppose that $p_n(z,a)$ is quasi-invariant for every $n < \alpha$ but, for a contradiction, $p_\alpha(z,a)$ is not. Then for some $f_1, \ldots, f_k \in \mathrm{Autf}(\mathcal{U}/A)$

$$p_\alpha(z,a) \;\cup\; \bigcup_{i=1}^{k} p_\alpha(z, f_i a)$$

is inconsistent. By compactness there is some $n < \alpha$ and some $\psi_i(z,y)$ as in $\sharp_3$ such that

$$p_n(z,a) \quad\to\quad \neg\bigwedge_{j=1}^{m}\bigwedge_{i=1}^{k}\neg\psi_j(z,f_ia)$$

As $p_n(z,a)$ is quasi-invariant, from Lemma 14.14 we obtain that $p_n(z,f_ia)\wedge\psi_j(z,f_ia)$ is non-drifting for some i,j. Clearly we can replace f_ia with a, then this contradicts the construction of $p_\alpha(z,y)$ and proves the theorem. □

14.17 Exercise Let $\mathcal{L}$ be quasi-invariant and let $\mathcal{D}$ be drifting, prove that $\mathcal{L}\smallsetminus\mathcal{D}$ is quasi-invariant. □

4 Kim-Pillay types

Given a tuple $a\in\mathcal{U}^{|z|}$, we write $\mathcal{K}(a/A)$ for the intersection of all type-definable sets containing a that are Lascar invariant over A. Or, more concisely, the intersection of all sets that are type-definable over a model containing A. We call $\mathcal{K}(a/A)$ the **Kim-Pillay strong type over A**. Clearly, $\mathcal{L}(a/A)\subseteq\mathcal{K}(a/A)\subseteq\mathcal{O}(a/A)$.

Clearly $\mathcal{K}(a/A)$ is Lascar invariant over A. It also easy to see that $\mathcal{K}(a/A)$ is type-definable. In fact, by invariance, we can assume that all types in the intersection above are over M, for any fixed model containing A. Hence $\mathcal{K}(a/A)$ is the minimal type-definable set containing a and closed under the relation $\overset{\text{L}}{\equiv}_A$. It follows that if $b\in\mathcal{K}(a/A)$ then $\mathcal{K}(b/A)\subseteq\mathcal{K}(a/A)$.

If $\mathcal{K}(a/A)=\mathcal{K}(b/A)$, we say that a and b have the same **Kim-Pillay strong type over A**. We abbreviate this by $a\overset{\text{KP}}{\equiv}_A b$. In other words, $a\overset{\text{KP}}{\equiv}_A b$ holds when $a\in\mathcal{D}\leftrightarrow b\in\mathcal{D}$ for every type-definable set $\mathcal{D}$ that is Lascar invariant over A.

⚠ Warning: the symbol $\mathcal{K}(a/A)$ is not standard. The symbol $a\overset{\text{KP}}{\equiv}_A b$ is not unusual, but others write KP-stp$(a/A)=$ KP-stp(b/A) or $a\ E_{\text{KP}/A}\ b$.

14.18 Proposition *Fix some $a\in\mathcal{U}^{|z|}$ and some $A\subseteq\mathcal{U}$. Then there is a type $e(z\,;w)\subseteq L(A)$ such that $\mathcal{K}(b/A)=e(\mathcal{U}\,;b)$ for all $b\in\mathcal{O}(a/A)$ and $e(z\,;w)$ defines an equivalence relation on $\mathcal{O}(a/A)$.*

Proof Notice that $\mathcal{K}(a/A)$ is type-definable over A,a. In fact, if $f\in\mathrm{Aut}(\mathcal{U}/A,a)$ and $\mathcal{D}$ is a set containing a that is type-definable and Lascar invariant over A, then so is $f[\mathcal{D}]$. Therefore $\mathcal{K}(a/A)$ is invariant over A,a. As $\mathcal{K}(a/A)$ is type-definable, invariance implies that it is type-definable over A,a. Let $e(z\,;w)\subseteq L(A)$ be such that $\mathcal{K}(a/A)=e(\mathcal{U}\,;a)$.

We prove that $\mathcal{K}(b/A)=e(\mathcal{U}\,;b)$ for all $b\in\mathcal{O}(a/A)$. Let $f\in\mathrm{Aut}(\mathcal{U}/A)$ be such that $fa=b$. If $\mathcal{D}$ is a type-definable Lascar invariant over A, then so is $f[\mathcal{D}]$. Therefore, f is a bijection between type-definable sets that are Lascar invariant over A and contain a and analogous sets containing b. Then $f[\mathcal{K}(a/A)]=\mathcal{K}(b/A)$ and $\mathcal{K}(b/A)=e(\mathcal{U}\,;b)$ follows.

We prove that $e(b\,;\mathcal{U})\cap\mathcal{O}(a/A)$ is Lascar invariant over A. Let $f\in\mathrm{Autf}(\mathcal{U}/A)$ and $c\in\mathcal{O}(a/A)$. Then $e(b\,;fc)$ is equivalent to $e(f^{-1}b\,;c)$ which in turn is equivalent to $e(b\,;c)$, by the invariance of $e(\mathcal{U}\,;c)$.

Finally we are ready to prove that $e(z\,;w)$ defines a symmetric relation on $\mathcal{O}(a/A)$. From what proved above, $e(b\,;\mathcal{U})\cap\mathcal{O}(a/A)$ is a type-definable Lascar invariant set containing b and therefore it contains $\mathcal{K}(b/A)$. We conclude that for all $b, c \in \mathcal{O}(a/A)$

$$e(c\,;b) \quad\leftrightarrow\quad c \in e(\mathcal{U}\,;b) \quad\leftrightarrow\quad c \in \mathcal{K}(b/A) \quad\rightarrow\quad e(b\,;c)$$

Reflexivity is clear; we prove transitivity. As remarked above, $\mathcal{K}(b/A) \subseteq \mathcal{K}(c/A)$, for all $b \in \mathcal{K}(c/A)$ or equivalently $e(b\,;c)$. Hence if $e(b\,;c)$ then

$$e(d\,;b) \quad\rightarrow\quad d \in \mathcal{K}(b/A) \quad\leftrightarrow\quad d \in \mathcal{K}(c/A) \quad\rightarrow\quad e(d\,;c).$$

Which completes the proof. □

14.19 Corollary *For every $a, b \in \mathcal{U}^{|z|}$ and $A \subseteq \mathcal{U}$ the following are equivalent*

1. $a \in \mathcal{K}(b/A)$;
2. $b \in \mathcal{K}(a/A)$;
3. $\mathcal{K}(a/A) = \mathcal{K}(b/A)$. □

The following useful lemma is the key ingredient in the proof of Theorem 14.21.

14.20 Lemma *Let $p(z) \subseteq L(A)$ and let $e(z\,;w) \subseteq L(A)$ define a bounded equivalence relation on $p(\mathcal{U})$. Then there is a type $e'(z\,;w) \subseteq L(A)$ which defines a bounded equivalence relation (on $\mathcal{U}$) and refines $e(z\,;w)$ on $p(\mathcal{U})$.*

Proof Let $\mathcal{C} = \langle \mathcal{C}_i : i < \lambda \rangle$ enumerate the partition of $p(\mathcal{U})$ induced by $e(z\,;w)$. Note that each $\mathcal{C}_i$ is type-definable over A, a for any $a \in \mathcal{C}_i$. If $x = \langle x_i : i < \lambda \rangle$, we write $x \in \mathcal{C}$ for the type that is the conjunction of $x_i \in \mathcal{C}_i$ for $i < \lambda$.

We claim that the required type is

$$e'(a\,;b) \quad\stackrel{\text{def}}{=}\quad \exists x' \in \mathcal{C}\; \exists x'' \in \mathcal{C} \quad a, x' \equiv_A b, x''$$

As the type above is invariant over A, we can assume that $e'(a\,;b)$ is type-definable over A. Moreover, it is clearly a reflexive and symmetric relation, so we only check it is transitive. Suppose $a, x' \equiv_A b, x''$ and $b, y' \equiv_A c, y''$ for some $x', x'', y', y'' \in \mathcal{C}$. Let z be such that $a, x', z \equiv_A b, x'', y'$ then $z \in \mathcal{C}$ and $a, z \equiv_A c, y''$.

The relation $e'(a\,;b)$ clearly refines the equivalence defined by $e(z\,;w)$ when restricted to to $p(\mathcal{U})$. To prove that it is bounded fix some $c \in \mathcal{C}$ and note that $e'(a\,;b)$ is refined by $a \equiv_{A,c} b$, which is bounded. □

Finally, we have the following.

14.21 Theorem *Denote by $e_A(z\,;w)$ be the finest bounded equivalence relation on $\mathcal{U}^{|z|}$ that is type-definable over A. Then for every $a, b \in \mathcal{U}^{|z|}$ the following are equivalent*

1. $a \stackrel{\text{KP}}{\equiv} b$
2. $e_A(a\,;b)$. □

Proof As $a \stackrel{\text{KP}}{\equiv} b$ is equivalent to $a \in \mathcal{K}(b/A)$, it suffices to prove that $e_A(\mathcal{U}\,;b) = \mathcal{K}(b/A)$ for every $b \in \mathcal{U}^{|z|}$.

$\subseteq$ The orbit of $e_A(\mathcal{U}\,;b)$ under $\mathrm{Aut}(\mathcal{U}/A)$ has cardinality $< \kappa$. Hence, by Theorem 14.5, it is Lascar invariant over A. As $\mathcal{K}(b/A)$ is the least of such sets, $\mathcal{K}(b/A) \subseteq e_A(\mathcal{U}\,;b)$.

$\supseteq$ Let $e(z\,;w)$ be the type-definable equivalence relation given by Proposition 14.18. The orbit of $\mathcal{K}(b/A)$ under $\mathrm{Aut}(\mathcal{U}/A)$ has cardinality $<\kappa$. Hence the equivalence relation that $e(z\,;w)$ defines on $\mathcal{O}(b/A)$ is bounded. By Lemma 14.20 there is be a type-definable bounded equivalence relation $e'(z\,;w) \subseteq L(A)$ that refines $e(z\,;w)$ on $\mathcal{O}(b/A)$. As $e'(z\,;w)$ is refined by $e_A(z\,;w)$, we obtain $e_A(\mathcal{U}\,;b) \subseteq e(\mathcal{U}\,;b) = \mathcal{K}(b/A)$. $\square$

The **logic A-topology**, or simply the **logic topology** when A is empty, is the topology on $\mathcal{U}^{|z|}$ whose closed sets are the type-definable Lascar A-invariant sets. It is clearly a compact topology. Its Kolmogorov quotient, that is $\mathcal{U}^{|z|}/\overset{\mathrm{KP}}{\equiv}$, is Hausdorff. In fact, by Corollary 14.19, the open sets $\neg\mathcal{K}(a/A)$ and $\neg\mathcal{K}(b/A)$ separate $b \overset{\mathrm{KP}}{\not\equiv} a$.

14.22 Exercise Let $\mathcal{S}(a/A)$ be the intersection of all definable sets that contain a and are Lascar invariant over A. Prove that $\mathcal{S}(a/A) = \{b \;:\; b \overset{\mathrm{Sh}}{\equiv}_A a\}$. $\square$

14.23 Exercise Prove that the clopen sets in the logic A-topology are definable over $\mathrm{acl}^{\mathrm{eq}} A$. $\square$

5 Notes and references

The original proof of Newelski theorem is rather long and complex. A simplified proof appears in Rodrigo Peláez's thesis [1, Section 3.3]. The proof here is a streamlined version the latter.

We are grateful to Volodya Shavrukov for fixing a wrong proof of Lemma 14.14.

[1] Rodrigo Peláez, *About the Lascar group*, PhD Thesis, Universitat de Barcelona, Departament de Lógica, História i Filosofia de la Ciéncia, 2008.

[2] Pierre Simon, *A guide to NIP theories*, Lecture Notes in Logic, vol. 44, Association for Symbolic Logic, Cambridge University Press, 2015.