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A Crèche Course in Model Theory

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Chapter 1

Preliminaries and notation

This chapter introduces the syntax and semantic of first order logic.

The definitions of terms and formulas we give in Section 1.3 and 1.5 are more formal than required here. Our main objective is to convince the reader that a rigorous definition of language and truth is possible. However, the actual details of such a definition are not relevant for our purposes.

1.1 Tuples

A sequence is a function $a : I \to A$ whose domain is a linear order $I, <_I$. We may use the notation $a = \langle a_i : i \in I \rangle$ for sequences. A tuple is a sequence whose domain is an ordinal, say α, then we write $a = \langle a_i : i < \alpha \rangle$. When α is finite, we may also write $a = a_0, \dots, a_{\alpha-1}$. The domain of the tuple a, the ordinal α, is denoted by $|a|$ and is called the length of a. In some contexts it may be convenient to confuse $\mathrm{rng}(a)$, the range of a, with a. If $A = \mathrm{rng}(a)$ we say that a is an enumeration of A.

If $J \subseteq I$ is a subset of the domain of the sequence $a = \langle a_i : i \in I \rangle$, we write $a_{\restriction J}$ for the restriction of a to J. When J is well ordered by $<_I$ we identify $a_{\restriction J}$ with a tuple. Sometimes (i.e. not always) we may overline tuples or sequences as mnemonic, as in $\bar{a}, \bar{c}$, etc. Still, restrictions are written as $a_{\restriction J}$, $c_{\restriction J}$ without the bar.

The set of tuples of elements of length α is denoted by A^α. The set of tuples of length $< \alpha$ is denoted by $A^{<\alpha}$. For instance, $A^{<\omega}$ is the set of all finite tuples of elements of A. When α is finite we do not distinguish between A^α and the α-th Cartesian power of A. In particular, we do not distinguish between A^1 and A.

When x is a tuple of variables, see Section 1.3, we will write write A^x for $A^{|x|}$. In fact, this notation is convenient when dealing with many-sorted structures, see Section 13.1.

If $a, b \in A^\alpha$ and h is a function defined on A, we write $h(a) = b$ for $h(a_i) = b_i$. We often do not distinguish between the pair $\langle a, b \rangle$ and the tuple of pairs $\langle a_i, b_i \rangle$. The context will resolve the ambiguity.

Note that there is a unique tuple of length 0, the empty set $\varnothing$, which in this context is called empty tuple. Recall that by definition $A^0 = \{\varnothing\}$ for every set A. Therefore, even when A is empty, A^0 contains the empty string.

We often concatenate tuples. If a and b are tuples, we write $a\,b$ for the concatenation or, when emphasis is required, $a^\frown b$.

1.2 Structures

Finally, we are really going to get started.

1.1 Definition A language L (also called signature) is a triple that consists of

1. a set L_{fun} whose elements are called function symbols
2. a set L_{rel} whose elements are called relation symbols
3. a function that assigns to every $f \in L_{\text{fun}}$, respectively $r \in L_{\text{rel}}$, non-negative integers n_f and n_r that we call arity of the function, respectively relation, symbol. We say that f is an n_f-ary function symbol, and similarly for r. A 0-ary function symbol is also called a constant.

Warning: it is customary to use the symbol L to denote both the language and the set of formulas associated to it (to be defined below). We denote by $|L|$ the cardinality of $L_{\text{fun}} \cup L_{\text{rel}} \cup \omega$. Note that, by definition, $|L|$ is always infinite.

1.2 Definition A structure M of signature L (for short L-structure) consists of

1. a set called the domain or support of the structure and is denoteed by the same symbol M used for the whole structure
2. a function that assigns to every $f \in L_{\text{fun}}$ a total map $f^M : M^{n_f} \to M$
3. a function that assigns to every $r \in L_{\text{rel}}$ a relation $r^M \subseteq M^{n_r}$.

We call f^M and r^M, the interpretation of f, respectively r, in M.

Recall that, by definition, $M^0 = \{\varnothing\}$. Therefore the interpretation of a constant c is a function that maps the unique element of M^0 to an element of M. We identify c^M with $c^M(\varnothing)$.

We may use the word model as a synonym for structure. But beware that, in some contexts, the word model is used to denote a particular kind of structure.

If M is an L-structure and $A \subseteq M$ is any subset, we write $L(A)$ for the language obtained by adding to L_{fun} the elements of A as constants. In this context, the elements of A are called parameters. There is a canonical expansion of M to an $L(A)$-structure that is obtained by setting $a^M = a$ for every $a \in A$.

1.3 Example The language of additive groups consists of the following function symbols:

1. a constant (that is, a function symbol of arity 0) $\ 0$
2. a unary function symbol (that is, of arity 1) $\ -$
3. a binary function symbol (that is, of arity 2) $\ +$.

In the language of multiplicative groups the three symbols above are replaced by 1, ${}^{-1}$, and $\cdot$ respectively. Any group is a structure in either of these two signatures with the obvious interpretation. Needless to say, not all structures with these signatures are groups.

The language of (unitary) rings contains all the symbols above except ${}^{-1}$. The language of ordered rings also contains the binary relation symbol $<$.

The following example is less straightforward. The reason for the choice of the language of vector spaces will become clear in Example 1.11 below.

1.4 Example Let F be a field. The language of vector spaces over F, which we denote by L_F, extends that of additive groups by a unary function symbol k for every $k \in F$.

Recall that a vector space over F is an abelian group M together with a function $\mu : F \times M \to M$ satisfying some well-known properties. To view a vector space over F as an L_F-structure, we interpret the group symbols in the obvious way and each $k \in F$ as the function $\mu(k, \text{-})$. See Example 2.6.

The languages in Examples 1.3 and 1.4, with the exception of that of ordered rings, are functional languages; that is, $L_{\text{rel}} = \varnothing$. The following example mention two important examples of relational languages, that is, languages where $L_{\text{fun}} = \varnothing$.

1.5 Example The language of strict orders contains a binary relation symbol, which is usually denoted by $<$. The language of graphs, contains a binary relation symbol $r(\text{-},\text{-})$. See Chapter 6.

1.3 Terms

Let V be an infinite set whose elements we call variables. We use the letters x, y, z, etc. to denote variables or tuples of variables. We rarely refer to V explicitly, and we always assume that V is large enough for our needs.

We fix a signature L for the whole section.

1.6 Definition A term is a finite sequence of elements of $L_{\text{fun}} \cup V$ that are obtained inductively as follows

0. every variable, intended as a tuple of length 1, is a term

i. if $f \in L_{\text{fun}}$ and $t_1, \dots, t_{n_f}$ are terms, then $f\, t_1, \dots, t_{n_f}$ is a term.

We say L-term when we need to specify the language L.

Note that any constant f, intended as a tuple of length 1, is a term (by i, the term f is obtained concatenating $n_f = 0$ terms and prefixing by f). Terms that do not contain variables are called closed terms.

The intended meaning of, for instance, the term $+ + x\, y\, z$ is $(x + y) + z$. The first expression uses prefix notation; the second uses infix notation. When convenient, we informally use infix notation and add parentheses to improve legibility and avoid ambiguity.

The following lemma shows that prefix notation allows to write terms unambiguously without using parentheses.

1.7 Lemma (Unique legibility of terms) Let a be a sequence of terms. Suppose a can be obtained both by concatenating the terms $t_1, \dots, t_n$ and by concatenating the terms $s_1, \dots, s_m$. Then $n = m$ and $s_i = t_i$.

Proof. By induction on $|a|$. If $|a| = 0$ than $n = m = 0$ and there is nothing to prove.

Suppose the claim holds for tuples of length k and let $a = a_1, \dots, a_{k+1}$. Then a_1 is the first element of both t_1 and s_1. If a_1 is a variable, say x, then t_1 and s_1 are the term x and $n = m = 1$. Otherwise a_1 is a function symbol, say f. Then $t_1 = f\,\bar{t}$ and $s_1 = f\,\bar{s}$, where $\bar{t}$ and $\bar{s}$ are obtained by concatenating the terms $t'_1, \dots, t'_p$ and $s'_1, \dots, s'_p$. Now apply the induction hypothesis to $a_2, \dots, a_{k+1}$ and to the terms $\bar{t}\, t_2 \dots t_n$ and $\bar{s}\, s_2 \dots s_m$. □

If $x = x_1, \dots, x_n$ is a tuple of distinct variables and $s = s_1, \dots, s_n$ is a tuple of terms, we write $t[x/s]$ for the sequence obtained by replacing x by s coordinatewise. Proving that $t[x/s]$ is indeed a term is a tedious task that can be safely skipped.

If t is a term and $x_1, \dots, x_n$ are (tuples of) variables, we write $t(x_1, \dots, x_n)$ to declare that the variables occurring in t are among those that occur in $x_1, \dots, x_n$. When a term has been presented as $t(x, y)$, we write $t(s, y)$ for $t[x/s]$.

Finally, we define the interpretation of a term in a structure M. We begin with closed terms. These are interpreted as 0-ary functions, i.e. as elements of the structure.

1.8 Definition Let t be a closed $L(M)$ term. The interpretation of t, denoted by t^M, is defined by induction on the syntax of t as follows

i. if $t = f\, t_1 \dots t_{n_f}$, where $f \in L_{\text{fun}}$, then $t^M = f^M(t_1^M, \dots, t_{n_f}^M)$.

Note that in i we have used Lemma 1.7 in an essential way. In fact this ensures that the sequence $t_1, \dots, t_{n_f}$ uniquely determines the terms $t_1^M, \dots, t_{n_f}^M$.

The inductive definition above is based on the case $n_f = 0$, that is, the case where f a constant, or a parameter. When $t = c$, a constant, $t_1 \dots t_{n_f}$ is the empty tuple, and so $t^M = c^M(\varnothing)$, which we abbreviate as c^M. In particular, if $t = a$, a parameter, then $t^M = a^M = a$.

Now we generalize the interpretation to all (not necessarily closed) terms. If $t(x)$ is a term, we define $t^M(x) : M^x \to M$ to be the function that maps a to $t(a)^M$.

1.4 Substructures

In the working practice, a *substructure* is a subset of a structure that is closed under the interpretation of the functions in the language. But there are a few cases when we need the following formal definition.

1.9 Definition Fix a signature L and let M and N be two L-structures. We say that M is a substructure of N, and write $M \subseteq N$, if

1. the domain of M is a subset of the domain of N
2. $f^M = f^N \restriction M^{n_f}$ for every $f \in L_{\text{fun}}$
3. $r^M = r^N \cap M^{n_r}$ for every $f \in L_{\text{rel}}$.

Note that when f is a constant 2 becomes $f^M = f^N$, in particular the substructures of N contains at least all the constants of N.

If a set $A \subseteq N$ is such that

1. $f^N[A^{n_f}] \subseteq A$ for every $f \in L_{\text{fun}}$

then there is a unique substructure $M \subseteq N$ with domain A, namely, the structure with the following interpretation

2. $f^M = f^N \restriction A^{n_f}$ (which is a good definition by assumption 1)
3. $r^M = r^N \cap A^{n_r}$.

It is usual to confuse subsets of N that satisfy 1 with the unique substructure they support.

It is immediate to verify that the intersection of an arbitrary family of substructures of N is a substructure of N. Therefore, for any given $A \subseteq N$ we may define the substructure of N generated A as the intersection of all substructures of N that contain A. We write $\langle A \rangle_N$. The following easy proposition gives more concrete representation of $\langle A \rangle_N$

1.10 Lemma The following hold for every $A \subseteq N$

1. $\langle A \rangle_N = \Big\{ t^N \ : \ t \text{ a closed } L(A)\text{-term} \Big\}$
2. $\langle A \rangle_N = \Big\{ t^N(a) \ : \ t(x) \text{ an } L\text{-term and } a \in A^x \Big\}$
3. $\langle A \rangle_N = \bigcup_{n \in \omega} A_n$, where $A_0 = A$

$$A_{n+1} = A_n \cup \Big\{ f^N(a) \ : \ f \in L_{\text{fun}},\ a \in A_n^{n_f} \Big\}.$$

1.11 Example Let L be the language of groups. Let N be a group, which we consider as an L-structure in the natural way. Then the substructures of N are exactly the subgroups of N and $\langle A \rangle_N$ is the group generated by $A \subseteq N$. A similar claim is true when L_F is the signature of vector spaces over some fixed field F. The choice of the language is more or less fixed if we want that the algebraic and the model theoretic notion of substructure coincide.

1.5 Formulas

Fix a language L and a set of variables V as in Section 1.3. A formula is a finite sequence of symbols in $L_{\text{fun}} \cup L_{\text{rel}} \cup V \cup \{\doteq, \perp, \neg, \vee, \exists\}$. The last set contains the logical symbols that are called respectively

$\doteq$	equality	$\perp$	contradiction	$\neg$	negation
$\vee$	disjunction	$\exists$	existential quantifier.		

Syntactically, $\doteq$ behaves like a binary relation symbol. So, for convenience set $n_{\doteq} = 2$. However $\doteq$ is considered as a logic symbol because its semantic is fixed (it is always interpreted in the diagonal).

The definition below uses the prefix notation which simplifies the proof of the unique legibility lemma. However, in practice we always we use the infix notation: $t \doteq s$, $\varphi \vee \psi$, etc.

1.12 Definition A formula is any finite sequence is obtained with the following inductive procedure

0. if $r \in L_{\mathrm{rel}} \cup \{\dot=\}$ and t is a tuple obtained concatenating n_r terms then $r\,t$ is a formula. Formulas of this form are called atomic
1. if φ e ψ are formulas then the following are formulas: $\bot$, $\neg\,\varphi$, $\vee\,\varphi\,\psi$, and $\exists x\,\varphi$, for any $x \in V$.

We use L to denote both the language and the set of formulas. We write L_{at} for the set of atomic formulas and L_{qf} for the set of quantifier-free formulas i.e. formulas where $\exists$ does not occur.

The proof of the following is similar to the analogous lemma for terms.

1.13 Lemma (Unique legibility of formulas) Let a be a sequence of formulas. Suppose a can be obtained both by the concatenation of the formulas $\varphi_1, \ldots, \varphi_n$ or by the concatenation of the formulas $\psi_1, \ldots, \psi_m$. Then $n = m$ and $\varphi_i = \psi_i$.

A formula is closed if all its variables occur under the scope of a quantifier. Closed formulas are also called sentences. We will do without a formal definition of *occurs under the scope of a quantifier* which is too lengthy. An example suffices: all occurrences of x are under the scope a quantifiers in the formula $\exists x\,\varphi$. These occurrences are called bounded. The formula $x \dot= y \wedge \exists x\,\varphi$ has free (i.e., not bound) occurrences of x and y.

Let x is a tuple of variables and t is a tuple of terms such that $|x| = |t|$. We write $\varphi[x/t]$ for the formula obtained substituting t for all free occurrences of x, coordinatewise.

We write $\varphi(x)$ to declare that the free variables in the formula φ are all among those of the tuple x. In this case we write $\varphi(t)$ for $\varphi[x/t]$.

We often use without explicit mention the following useful syntactic decomposition of formulas with parameters which we state without proof.

1.14 Lemma For every formula $\varphi(x) \in L(A)$ there is a formula $\psi(x\,;z) \in L$ and a tuple of parameters $a \in A^z$ such that $\varphi(x) = \psi(x\,;a)$.

Just as a term $t(x)$ is a name for a function $t(x)^M : M^x \to M$, a formula $\varphi(x)$ is a name for a subset $\varphi(x)^M \subseteq M^x$ which we call the subset of M defined by $\varphi(x)$. It is also very common to write $\varphi(M^x)$ for the set defined by $\varphi(x)$. In general sets of the form $\varphi(M^x)$ for some $\varphi(x) \in M$ are called definable.

1.15 Definition of truth For every formula φ with variables among those of the tuple x we define $\varphi(x)^M$ by induction as follows

o1. $(\doteq t\, s)(x)^M = \Big\{a \in M^x \ : \ t^M(a) = s^M(a)\Big\}$

o2. $(r\, t_1 \ldots t_n)(x)^M = \Big\{a \in M^x \ : \ \langle t_1^M(a), \ldots, t_n^M(a)\rangle \in r^M\Big\}$

i0. $\bot(x)^M = \varnothing$

i1. $(\neg\xi)(x)^M = M^x \smallsetminus \xi(x)^M$

i2. $(\vee\, \xi\, \psi)(x)^M = \xi(x)^M \cup \psi(x)^M$

i3. $(\exists y\, \varphi)(x)^M = \bigcup\limits_{a \in M} (\varphi[y/a])(x)^M.$

Condition i2 assumes that ξ and ψ are uniquely determined by $\vee\, \xi\, \psi$. This is a guaranteed by the unique legibility o formulas, Lemma 1.13. Analogously, o1 e o2 assume Lemma 1.7.

The case when x is the empty tuple is far from trivial. Note that $\varphi(\varnothing)^M$ is a subset of $M^0 = \{\varnothing\}$. Then there are two possibilities either $\{\varnothing\}$ or $\varnothing$. We will read them as two truth values: True and False, respectively. If $\varphi^M = \{\varnothing\}$ we say that φ is true in M, if $\varphi^M = \varnothing$, we say that φ is false M and we write $M \vDash \varphi$, respectively $M \nvDash \varphi$. In words we may say that M models φ, respectively M does not model φ. It is immediate to verify that

$$\varphi(M^x) = \big\{a \in M^x : M \vDash \varphi(a)\big\}.$$

Note that usually, we say *formula* when, strictly speaking, we mean *pair* that consists of a formula and a tuple of variables. Such pairs are interpreted in definable sets (cfr. Definition 1.15). In fact, if the tuple of variables were not given, the arity of the corresponding set is not determined.

In some contexts we also want to distinguish between two sorts of variables that play different roles. Some are placeholder for parameters, some are used to define a set. In the the first chapters this distinction is only a clue for the reader, in the last chapters it is an essential part of the definitions.

1.16 Definition A partitioned formula is a triple $\varphi(x\,;z)$ consisting of a formula and two tuples of variables such that the variables occurring in φ are all among x, z.

We use a semicolon to separate the two tuples of variables. Typically, z is the placeholder for parameters and x runs over the model.

1.6 Yet more notation

Now we abandon the prefix notation in favor of the infix notation. We also use the following logical connectives as abbreviations

$\top$ stands for $\neg\bot$ tautology

$\varphi \wedge \psi$	stands for	$\neg[\neg\varphi \vee \neg\psi]$	conjunction
$\varphi \rightarrow \psi$	stands for	$\neg\varphi \vee \psi$	implication
$\varphi \leftrightarrow \psi$	stands for	$[\varphi \rightarrow \psi] \wedge [\psi \rightarrow \varphi]$	bi-implication
$\varphi \nleftrightarrow \psi$	stands for	$\neg[\varphi \leftrightarrow \psi]$	exclusive disjunction
$\forall x\, \varphi$	stands for	$\neg\exists x \neg\, \varphi$	universal quantifier

We agree that $\rightarrow$ e $\leftrightarrow$ bind less than $\wedge$ e $\vee$. Unary connnectives (quantifiers and negation) bind stronger then binary connectives. For example

$$\exists x\, \varphi \wedge \psi \;\rightarrow\; \neg\xi \vee \vartheta \quad \text{reads as} \quad \Big[[\exists x\, \varphi] \wedge \psi\Big] \;\rightarrow\; \Big[[\neg\xi] \vee \vartheta\Big]$$

We say that $\forall x\, \varphi(x)$ and $\exists x\, \varphi(x)$ are the universal, respectively, existential closure of $\varphi(x)$. We say that $\varphi(x)$ holds in M when its universal closure is true in M. We say that $\varphi(x)$ is consistent in M when its existential closure is true in M.

The semantic of conjunction and disjunction is associative. Then for any finite set of formulas $\{\varphi_i : i \in I\}$ we can write without ambiguities

$$\bigwedge_{i\in I} \varphi_i \qquad\qquad \bigvee_{i\in I} \varphi_i$$

When $x = x_1, \ldots, x_n$ is a tuple of variables we write $\exists x\, \varphi$ or $\exists x_1, \ldots, x_n\, \varphi$ for $\exists x_1 \ldots \exists x_n\, \varphi$. There is a first order sentences that say that $\varphi(M^x)$ has at least n elements (also, no more than, or exactly n). It is convenient to use the following abbreviations.

$\exists^{\geq n} x\; \varphi(x)$	stands for	$\exists x_1, \ldots, x_n \Big[\bigwedge\limits_{1\leq i\leq n} \varphi(x_i) \;\wedge \bigwedge\limits_{1\leq i<j\leq n} x_i \dot{\neq} x_j \Big].$
$\exists^{\leq n} x\; \varphi(x)$	stands for	$\neg\exists^{\geq n+1} x\; \varphi(x)$
$\exists^{= n} x\; \varphi(x)$	stands for	$\exists^{\geq n} x\; \varphi(x) \;\wedge\; \exists^{\leq n} x\; \varphi(x)$

1.17 Exercise Let M be an L-structure and let $\psi(y), \varphi(x, y) \in L$. For each of the following conditions, write a sentence true in M exactly when

a. $\psi(M^y) \;\in\; \big\{\varphi(a, M^y) : a \in M^x\big\}$

b. $\big\{\varphi(a, M^y) : a \in M^x\big\}$ contains at least two sets

c. $\big\{\varphi(a, M^y) : a \in M^x\big\}$ contains only sets that are pairwise disjoint.

1.18 Exercise Let M be a structure in a signature that contains a symbol r for a binary relation. Write a sentence φ such that

a. $M \vDash \varphi$ if and only if there is $A \subseteq M$ such that $r^M \;\subseteq\; A \times \neg A$.

Remark: φ assert an asymmetric version of the property below

b. $M \vDash \psi$ if and only if there is $A \subseteq M$ such that $r^M \;\subseteq\; (A \times \neg A) \,\cup\, (\neg A \times A)$.

Assume M is a graph, what required in b is equivalent to saying that M is a *bipartite graph*, or equivalently that it has *chromatic number* 2 i.e., we can color the vertices with 2 colors so that no two adjacent vertices share the same color.

Chapter 2

Theories and elementarity

2.1 Logical consequences

A theory is a set $T \subseteq L$ of sentences. We write $M \vDash T$ when $M \vDash \varphi$ for every $\varphi \in T$. If $\varphi \in L$ is a sentence we write $T \vdash \varphi$ when

$$M \vDash T \quad \Rightarrow \quad M \vDash \varphi \qquad \text{for every } M.$$

In words, we say that φ is a logical consequence of T or that φ follows from T. If S is a theory $T \vdash S$ has a similar meaning. If $T \vdash S$ and $S \vdash T$ we say that T and S are logically equivalent. We may say that T axiomatizes S (or vice versa).

We say that a theory is consistent if it has a model. With the notation above, T is consistent if and only if $T \nvdash \perp$.

The closure of T under logical consequence is the set $\mathrm{ccl}(T)$ which is defined as follows:

$$\mathrm{ccl}(T) \quad = \quad \Big\{\varphi \in L \;:\; \text{sentence such that } T \vdash \varphi\Big\}$$

If T is a finite set, say $T = \{\varphi_1, \ldots, \varphi_n\}$ we write $\mathrm{ccl}(\varphi_1, \ldots, \varphi_n)$ for $\mathrm{ccl}(T)$. If $T = \mathrm{ccl}(T)$ we say that T is closed under logical consequences.

The theory of M is the set of sentences that hold in M and is denoted by $\mathrm{Th}(M)$. More generally, if $\mathcal{K}$ is a class of structures, $\mathrm{Th}(\mathcal{K})$ is the set of sentences that hold in every model in $\mathcal{K}$. That is

$$\mathrm{Th}(\mathcal{K}) \quad = \quad \bigcap_{M \in \mathcal{K}} \mathrm{Th}(M)$$

The class of all models of T is denoted by $\mathrm{Mod}(T)$. We say that $\mathcal{K}$ is axiomatizable if $\mathrm{Mod}(T) = \mathcal{K}$ for some theory T. If T is finite we say that $\mathcal{K}$ is finitely axiomatizable. To sum up

$$\begin{aligned}
\mathrm{Th}(M) \quad &= \quad \Big\{\varphi \;:\; M \vDash \varphi\Big\} \\
\mathrm{Th}(\mathcal{K}) \quad &= \quad \Big\{\varphi \;:\; M \vDash \varphi \text{ for all } M \in \mathcal{K}\Big\} \\
\mathrm{Mod}(T) \quad &= \quad \Big\{M \;:\; M \vDash T\Big\}
\end{aligned}$$

2.1 Example Let L be the language of multiplicative groups. Let T_{g} be the set containing the universal closure of following three formulas

1. $(x \cdot y) \cdot z \;=\; x \cdot (y \cdot z)$
2. $x \cdot x^{-1} \;=\; x^{-1} \cdot x \;=\; 1$
3. $x \cdot 1 \;=\; 1 \cdot x \;= x.$

Then T_{g} axiomatizes the theory of groups, i.e. $\mathrm{Th}(\mathcal{K})$ for $\mathcal{K}$ the class of all groups. Let φ be the universal closure of the following formula

$$z \cdot x \;=\; z \cdot y \;\rightarrow\; x \;=\; y.$$

As φ formalizes the cancellation property then $T_{\mathrm{g}} \vdash \varphi$, that is, φ is a logical consequence of T_{g}. Now consider the sentence ψ which is the universal closure of

4. $x \cdot y \;=\; y \cdot x.$

So, commutative groups model ψ and non commutative groups model $\neg\psi$. Hence neither $T_{\mathrm{g}} \vdash \psi$ nor $T_{\mathrm{g}} \vdash \neg\psi$. We say that T_{g} does not decide ψ.

Note that even when T is a very concrete set, $\mathrm{ccl}(T)$ may be more difficult to grasp. In the example above T_{g} contains three sentences but $\mathrm{ccl}(T_{\mathrm{g}})$ is an infinite set containing sentences that code theorems of group theory yet to be proved.

2.2 Remark The following properties say that ccl is a finitary closure operator.

1. $T \subseteq \mathrm{ccl}(T)$ (extensive)
2. $\mathrm{ccl}(T) \;=\; \mathrm{ccl}\big(\mathrm{ccl}(T)\big)$ (idempotent)
3. $T \subseteq S \;\;\Rightarrow\;\; \mathrm{ccl}(T) \subseteq \mathrm{ccl}(S)$ (increasing)
4. $\mathrm{ccl}(T) \;\;=\;\; \bigcup\big\{\mathrm{ccl}(S) \;:\; S \text{ finite subset of } T\big\}.$ (finitary)

Properties 1-3 are easy to verify while 4 requires the compactness theorem.

In the next example we list a few algebraic theories with straightforward axiomatization.

2.3 Example We write T_{ag} for the theory of abelian groups which contains the universal closure of following

a1. $(x+y)+z \;=\; y+(x+z)$

a2. $x+(-x) \;=\; 0$

a3. $x+0 \;=\; x$

a4. $x+y \;=\; y+x.$

2.4 Example The theory T_{r} of (unitary) rings extends T_{ag} with

a5. $(x \cdot y) \cdot z \;=\; x \cdot (y \cdot z)$

a6. $1 \cdot x \;=\; x \cdot 1 \;=\; x$

a7. $(x+y) \cdot z \;=\; x \cdot z + y \cdot z$

a8. $z \cdot (x+y) \;=\; z \cdot x + z \cdot y.$

The theory of commutative rings T_{cg} contains also com of examples 2.1.

2.5 Example The theory of ordered rings T_{or} extends T_{cr} with

o1. $x < z \ \rightarrow\ x + y < z + y$

o2. $0 < x \wedge 0 < z \ \rightarrow\ 0 < x \cdot z$.

2.6 Example The axiomatization of the theory of vector spaces is less straightforward. Fix a field F. The language L_F extends the language of additive groups with a unary function for every element of F. The theory of vector fields over F extends T_{ag} with the following axioms (for all $h, k, l \in F$)

m1. $h\,(x + y) \ =\ h\,x + h\,y$

m2. $l\,x \ =\ h\,x + k\,x$, where $l \ =\ h +_F k$

m3. $l\,x \ =\ h\,(k\,x)$, where $l \ =\ h \cdot_F k$

m4. $0_F\,x \ =\ 0$

m5. $1_F\,x \ =\ x$

The symbols 0_F and 1_F denote the zero and the unit of F. The symbols $+_F$ and $\cdot_F$ denote the sum and the product in F. These are not part of L_F, they are symbols we use in the metalanguage.

2.7 Example Recall from Example 1.5 that we represent a graph with a symmetric irreflexive relation. Therefore theory of graphs contains the following two axioms

1. $\neg r(x, x)$
2. $r(x, y) \rightarrow r(y, x)$.

Our last example is a trivial one.

2.8 Example Let L be the empty language The theory of infinite sets is axiomatized by the sentences $\exists^{\geq n} x\ (x = x)$ for all positive integer n.

2.9 Exercise Prove that $\mathrm{ccl}(\varphi \vee \psi) = \mathrm{ccl}(\varphi) \cap \mathrm{ccl}(\psi)$.

2.10 Exercise Prove that $\mathrm{Th}(\mathrm{Mod}(T)) = \mathrm{ccl}(T)$.

2.2 Elementary equivalence

The following is a fundamental notion in model theory.

2.11 Definition We say that M and N are elementarily equivalent if

ee. $N \vDash \varphi \ \Leftrightarrow\ M \vDash \varphi$, for every sentence $\varphi \in L$.

In this case we write $M \equiv N$. More generally, we write $M \equiv_A N$ and say that M and N are elementarily equivalent over A if the following hold

a. $A \subseteq M \cap N$

ee'. equivalence ee above holds for every sentence $\varphi \in L(A)$.

The case when A is the whole domain of M is particularly important.

2.12 Definition When $M \equiv_M N$ we say that M is an elementary substructure of N and write $M \preceq N$.

The following lemma shows that the use of the term *substructure* in the definition above is appropriate.

2.13 Lemma If M and N are such that $M \equiv_A N$ and A is the domain of a substructure of M then A is also the domain a substructure of N and the two substructures coincide.

Proof. Let f be a function symbol and ler r be a relation symbol. It suffices to prove that $f^M(a) = f^N(a)$ for every $a \in A^{n_f}$ and that $r^M \cap A^{n_r} = r^N \cap A^{n_r}$.

If $b \in A$ is such that $b = f^M a$ then $M \vDash fa = b$. So, from $M \equiv_A N$, we obtain $N \vDash fa = b$, hence $f^N a = b$. This proves $f^M(a) = f^N(a)$.

Now let $a \in A^{n_r}$ and suppose $a \in r^M$. Then $M \vDash ra$ and, by elementarity, $N \vDash ra$, hence $a \in r^N$. By symmetry $r^M \cap A^{n_r} = r^N \cap A^{n_r}$ follows. □

It is not easy to prove that two structures are elementary equivalent. A direct verification is unfeasible even for the most simple structures. It will take a few chapters before we are able to discuss concrete examples.

We generalize the definition of $\mathrm{Th}(M)$ to include parameters

$$\mathrm{Th}(M/A) = \Big\{\varphi \;:\; \text{sentence in } L(A) \text{ such that } M \vDash \varphi\Big\}.$$

The following proposition is immediate

2.14 Proposition For every pair of structures M and N and every $A \subseteq M \cap N$ the following are equivalent

a. $M \equiv_A N$

b. $\mathrm{Th}(M/A) = \mathrm{Th}(N/A)$

c. $M \vDash \varphi(a) \;\Leftrightarrow\; N \vDash \varphi(a)$ for every $\varphi(x) \in L$ and every $a \in A^x$.

d. $\varphi(M^x) \cap A^x = \varphi(N^x) \cap A^x$ for every $\varphi(x) \in L$.

If we restate a and c of the proposition above when $A = M$ we obtain that the following are equivalent

a'. $M \preceq N$

d'. $\varphi(M^x) = \varphi(N^x) \cap M^x$ for every $\varphi(x) \in L$.

Note that c' extends to all definable sets what Definition 1.9 requires for a few basic definable sets.

2.15 Example Let G be a group which we consider as a structure in the multiplicative language of groups. We show that if G is simple and $H \preceq G$ then also H is simple. Recall that G is simple if all its normal subgroups are trivial, equivalently, if for every $a \in G \smallsetminus \{1\}$ the set $\{gag^{-1} : g \in G\}$ generates the whole group G.

Assume H is not simple. Then there are $a, b \in H$ such that b is not the product of elements of $\{hah^{-1} : h \in H\}$. Then for every n

$$H \vDash \neg\exists x_1, \dots, x_n \ \left(b \ = \ x_1 a x_1^{-1} \cdots \ x_n a x_n^{-1}\right)$$

By elementarity the same hold in G. Hence G is not simple.

2.16 Exercise Let $A \subseteq M \cap N$. Prove that $M \equiv_A N$ if and only if $M \equiv_B N$ for every finite $B \subseteq A$.

2.17 Exercise Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M^x)$ is finite if and only if $\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M^x)$.

2.18 Exercise Let $M \preceq N$ and let $\varphi(x, z) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^x$. Prove that all these sets are definable over M.

2.19 Exercise Consider $\mathbb{Z}^n$ as a structure in the additive language of groups with the natural interpretation. Prove that $\mathbb{Z}^n \not\equiv \mathbb{Z}^m$ for every positive integers $n \neq m$. Hint: in $\mathbb{Z}^n$ there are at most 2^n elements that are not congruent modulo 2.

2.3 A nonstandard example

This section concerns an example that is useful to look at in some detail to get some familiarity with the notion of elementary substructure. The example is culturally interesting, because it formalises rigorously the notions of infinity and infinitesimal, which were used in Newton and Leibniz's time to develop real analysis. These notions were not well defined - in fact, they were inconsistent.

It was only in the mid 19th century that mathematicians of Weiertsrass' generation developed the notion of limit, thus providing rigorous grounds for the development of analysis.

Nonstandard analysis was developed by Abraham Robinson in the 1950s. Robinson found a way to formalise the ideas of infinity and infinitesimals through the concept of elementary extension.

The notation that follows will be used throughout this section.

The language we use contains

1. X, a relation symbol of arity n, for every $n \in \omega$ and every $X \subseteq \mathbb{R}^n$
2. f, a function symbol of arity n, for every $n \in \omega$ and every $f : \mathbb{R}^n \to \mathbb{R}$.

The standard model of real analysis is $\mathbb{R}$ with the natural interpretation of the symbols in our language, so we use the same symbol for an element of the language and its interpretation in $\mathbb{R}$. This is not an abuse of notation: in this case, elements and interpretations coincide.

Suppose $\mathbb{R}$ has a proper elementary extension ${}^*\mathbb{R}$. The existence of such an extension will be proved later. The interpretations of the symbols f and X in ${}^*\mathbb{R}$ will be denoted by ${}^*f$ and ${}^*X$, respectively. The elements of ${}^*\mathbb{R}$ are called hyperreals, the elements of $\mathbb{R}$ are called standard (hyper)reals, those in ${}^*\mathbb{R} \smallsetminus \mathbb{R}$ are called nonstandard (hyper)reals.

It is easy to verify that ${}^*\mathbb{R}$ is an ordered field. This is because the operations sum and product are in the language, and so is the order relation. The property of being an ordered field can be expressed via a set of sentences that are true in $\mathbb{R}$, and hence also in ${}^*\mathbb{R}$.

A hyperreal c is said to be infinitesimal if $|c| < \varepsilon$ for every positive standard ε. A hyperreal c is infinite if $k < |c|$ for every standard k; the hyperreal c if finite otherwise. Hence if c is infinite, then c^{-1} is infinitesimal. Clearly, all standard reals are finite, and 0 is the only infinitesimal standard real.

The proof of the following lemma is left to the reader.

2.20 Fact Infinitesimals are closed under sum, product and multiplication by standard reals.

We begin with an existence proof.

2.21 Lemma There are infinite nonstandard hyperreals and nonzero infinitesimals. Moreover, for every finite hyperreal c there is a unique standard b such that $b - c$ is infinitesimal.

Proof. If $c \in {}^*\mathbb{R}$ in infinite then c^{-1} is infinitesimal and if c is a nonzero infinitesimal, then c^{-1} is infinite. Then the first claim follows from the second one. Let c be finite. The set $\{a \in \mathbb{R} : c < a\}$ is a nonempty set of (standard) reals that is bounded from below. Let $b \in \mathbb{R}$ the least upper bound. We show that $b - c$ is a infinitesimal. Assume for a contradiction that $\varepsilon < |b - c|$ for some standard positive ε. Then $c < b - \varepsilon$, or $b + \varepsilon < c$, depending on whether $c < b$ or $b < c$. Both cases contradict that b is an infimum. This proves the existence of b.

To prove uniquenes, suppose that $b' \in \mathbb{R}$ and $b' - c$ is also infinitesimal. Then $b - b'$ is also infinitesimal. As 0 is the only standard infinitesimal $b = b'$. □

The existence of infinite non standard hyperreals shows that ${}^*\mathbb{R}$ is not archimedean: standard integers are not cofinal in ${}^*\mathbb{R}$. However, by elementarity, the nonstandard integers ${}^*\mathbb{N}$ are cofinal in ${}^*\mathbb{R}$: from the perspective of an inhabitant of ${}^*\mathbb{R}$, the latter is a normal archimedean field.

Dedekind completeness is another fundamental property of $\mathbb{R}$ that does not hold in ${}^*\mathbb{R}$. An ordered set is Dedekind complete if every subset that is bounded above has a least upper bound. Then ${}^*\mathbb{R}$ is not Dedekind complete, e.g. the set of infinitesimals does not have a least upper bound. However, it is easy to verify that all bounded definable subsets of ${}^*\mathbb{R}$ have a least upper bound. It follows that Dedekind completeness is *not* a first-order property.

We define the following equivalence relation on ${}^*\mathbb{R}$: we write $a \approx b$ if $|a - b|$ is infinitesimal. The fact that this is an equivalence relation follows from Fact 2.20. The equivalence class of c is called monad. By Lemma 2.21 if c is a finite hyperreal, there is a unique real in the monad of c.

Hyperreals that are not finite are said to be infinite. If c is finite, the unique standard real in the monad of c is called standard part of c and it is denoted by $\mathrm{st}(c)$.

In the following lemma, the expressions on the left can be formalised as first-order sentences, and therefore they hold in $\mathbb{R}$ if and only if they hold in ${}^*\mathbb{R}$.

2.22 Proposition For every $f : \mathbb{R} \to \mathbb{R}$, for every $a, l \in \mathbb{R}$ the following equivalences hold.

a. $\lim_{x \to +\infty} fx = +\infty \quad \Leftrightarrow \quad {}^*f(c)$ is positive and infinite for every infinite $c > 0$

b. $\lim_{x \to +\infty} fx = l \quad \Leftrightarrow \quad {}^*f(c) \approx l$ for every infinite $c > 0$

c. $\lim_{x \to a} fx = +\infty \quad \Leftrightarrow \quad {}^*f(c)$ is positive and infinite for every $c \approx a \neq c$

d. $\lim_{x \to a} fx = l \quad \Leftrightarrow \quad {}^*f(c) \approx l$ for every for every $c \approx a \neq c$.

Proof. We prove part d and we leave the other parts as an exercise.

$\Rightarrow$. The left-hand side of the equivalence d asserts

1 $$\mathbb{R} \models \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \, \Big[0 < |x - a| < \delta \;\; \rightarrow \;\; |fx - l| < \varepsilon \Big].$$

For consistency with standard notation in analysis, we use the Greek letters ε and δ as variables. The symbols $\dot{\varepsilon}$ e $\dot{\delta}$ denote parameters either in $\mathbb{R}$ or in ${}^*\mathbb{R}$.

Pick $c \neq a \approx c$ arbitrarily. We now check that ${}^*f(c) \approx l$, that is, $|{}^*fc - l| < \dot{\varepsilon}$ for every positive standard $\dot{\varepsilon}$. Pick $\dot{\varepsilon} \in \mathbb{R}^+$ arbitralily and let $\dot{\delta} \in \mathbb{R}$ be given by 1. By elementarity, we have

$${}^*\mathbb{R} \models \forall x \, \Big[0 < |x - a| < \dot{\delta} \;\; \rightarrow \;\; |fx - l| < \dot{\varepsilon} \Big].$$

In particular

$${}^*\mathbb{R} \models 0 < |c - a| < \dot{\delta} \;\; \rightarrow \;\; |fc - l| < \dot{\varepsilon}.$$

As $0 < |c - a| < \dot{\delta}$ certainly holds (because $\dot{\delta}$ is standard) $|{}^*fc - l| < \dot{\varepsilon}$ follows.

$\Leftarrow$. Assume that 1 is false, that is,

2 $$\mathbb{R} \models \exists \varepsilon > 0 \, \forall \delta > 0 \, \exists x \, \Big[0 < |x - a| < \delta \;\; \wedge \;\; \varepsilon \leq |fx - l| \Big].$$

We want to show that ${}^*f(c) \not\approx l$ for some $c \neq a$. Let $\dot{\varepsilon} \in \mathbb{R}$ witness 2. By elementarity we have

$${}^*\mathbb{R} \models \forall \delta > 0 \, \exists x \, \Big[0 < |x - a| < \delta \;\; \wedge \;\; \dot{\varepsilon} \leq |fx - l| \Big].$$

Let $\dot{\delta}$ be an arbitrary infinitesimal. Then

$${}^*\mathbb{R} \models \exists x \, \Big[0 < |x - a| < \dot{\delta} \;\; \wedge \;\; \dot{\varepsilon} \leq |fx - l| \Big].$$

Any c that witnesses the above formula is such that $c \approx a \neq c$ and, simultaneously, $\varepsilon \leq |{}^*fc - l|$. But $\dot{\varepsilon}$ is standard, hence ${}^*fc \not\approx l$. $\square$

The following corollary is immediate.

2.23 Corollary For every $f : \mathbb{R} \to \mathbb{R}$ the following are equivalent

a. f is continuous

b. ${}^*f(a) \approx {}^*f(c)$ for every pair of finite hyperreals such that $c \approx a$.

In Corollary 2.23, it is important to restrict c to *finite* hyperreals, otherwise we get a stronger property.

2.24 Proposition For every $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent

a. f is uniformly continuous

b. ${}^*f(a) \approx {}^*f(b)$ for every pair of hyperreals such that $a \approx b$.

Proof. a⇒b. Assume a, then

1 $$\mathbb{R} \quad \vDash \quad \forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \,\Big[|x-y| < \delta \quad \to \quad |fx - fy| < \varepsilon\Big].$$

Let $a \approx b$. We want to show that $|{}^*f(a) - {}^*f(b)| < \dot{\varepsilon}$ for every positive standard $\dot{\varepsilon}$. Given a standard positive $\dot{\varepsilon}$, let $\dot{\delta}$ be a standard real obtained from 1. Now elementarity gives

2 $$\mathbb{R} \quad \vDash \quad \forall x, y \,\Big[|x-y| < \dot{\delta} \quad \to \quad |fx - fy| < \dot{\varepsilon}\Big].$$

Hence, in particular,

$$^*\mathbb{R} \quad \vDash \quad |a-b| < \dot{\delta} \quad \to \quad |fa - fb| < \dot{\varepsilon}.$$

Since $a \approx b$ andd $\dot{\delta}$ is standard, $|a-b| < \dot{\delta}$ is true. Therefore $|{}^*f(a) - {}^*f(b)| < \dot{\varepsilon}$.

b⇒a. Negate a

3 $$\mathbb{R} \quad \vDash \quad \exists \varepsilon > 0\, \forall \delta > 0\, \exists x, y \,\Big[|x-y| < \delta \quad \wedge \quad \varepsilon \leq |fx - fy|\Big]$$

We want $a \approx b$ such that $\dot{\varepsilon} \leq |{}^*f(a) - {}^*f(b)|$ for some positive standard $\dot{\varepsilon}$. Let $\dot{\varepsilon}$ be a standard real that witnesses 3. Elementarity gives

$$^*\mathbb{R} \quad \vDash \quad \forall \delta > 0\, \exists x, y \,\Big[|x-y| < \delta \quad \wedge \quad \dot{\varepsilon} \leq |fx - fy|\Big].$$

Pick an arbitrary infinitesimal $\dot{\delta} > 0$ and let $a, b \in {}^*\mathbb{R}$ be given by the formula above

$$^*\mathbb{R} \quad \vDash \quad |a-b| < \delta \quad \wedge \quad \dot{\varepsilon} \leq |fa - fb|.$$

Since $\dot{\delta}$ is infinitesimal, we have $a \approx b$, as required.

The following proposition is an immediate consequence of Proposition 2.22.

2.25 Proposition For every unary function f and every standard a, the following are equivalent.

a. f is differentiable in a

b. $\operatorname{st}\left(\dfrac{f(a) - f(a+h)}{h}\right)$ exists and is constant for all infinitesimal $h \neq 0$.

2.26 Exercise Every definable (possibly with parameters) subset of $^*\mathbb{R}$ that is bounded from above has a least upper bound.

2.27 Exercise Prove that if the function $f : \mathbb{R} \to \mathbb{R}$ is injective, then ${}^*fa$ is nonstantard whenever a is nonstandard.

2.28 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is a finite set
2. ${}^*X = X$.

2.29 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is an open set in the usual topology on $\mathbb{R}$
2. $b \approx a \in X \quad \Rightarrow \quad b \in {}^*X$ for every $b \in {}^*\mathbb{R}$.

2.30 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is closed in the usual topology on $\mathbb{R}$
2. $a \in {}^*X \quad \Rightarrow \quad \operatorname{st} a \in X$ for every finite $a \in {}^*\mathbb{R}$.

2.31 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is bounded and closed in the usual topology on $\mathbb{R}$
2. for every $b \in {}^*X$ there is an $a \in X$ such that $a \approx b$.

2.32 Exercise For what sets $X \subseteq \mathbb{R}$ does the following hold for every $a, b \in {}^*\mathbb{R}$?

$b \approx a \in {}^*X \;\Rightarrow\; b \in {}^*X$.

2.33 Exercise Prove that $|\mathbb{R}| \;\le\; |{}^*\mathbb{Q}|$, that is, that the cardinality of ${}^*\mathbb{Q}$ is at least that of the continuum. (Hint: define an injective function $f : \mathbb{R} \to {}^*\mathbb{Q}$ by choosing a nonstandard rational in the monad of every standard real.)

2.34 Exercise Prove that every proper elementary extension of $\mathbb{Q}$ in the language of rings plus a relation symbol for every subset of $\mathbb{Q}$ has infinities and infinitesimals.

2.4 Embeddings and isomorphisms

Here we prove that isomorphic structures are elementarily equivalent and a few related results. The following definition generalizes the notion of substructure (and will be further generalized in Section 3.4).

2.35 Definition An embedding of M into N is an injective total map $h : M \hookrightarrow N$ such that

1. $a \in r^M \;\Leftrightarrow\; ha \in r^N$ for every $r \in L_{\text{rel}}$ and $a \in M^{n_r}$;
2. $h\, f^M(a) \;=\; f^N(h\, a)$ for every $f \in L_{\text{fun}}$ and $a \in M^{n_f}$.

Note that when $c \in L_{\text{fun}}$ is a constant 2 reads $h\, c^M = c^N$. Therefore $M \subseteq N$ if and only if $\mathrm{id}_M : M \to N$ is an embedding.

An surjective embedding is an isomorphism or, when domain and codomain coincide, an automorphism.

Condition 1 above and the assumption that h is injective can be summarized in the following

1′. $M \vDash r(a) \;\Leftrightarrow\; N \vDash r(ha)$ for every $r \in L_{\text{rel}} \cup \{\doteq\}$ and every $a \in M^{n_r}$.

Note also that, by straightforward induction on syntax, from 2 we obtain

2′ $h\, t^M(a) \;=\; t^N(h\, a)$ for every term $t(x)$ and every $a \in M^x$.

Combining 1′ and 2′ we obtain by straightforward induction on the syntax

3. $M \vDash \varphi(a) \;\Leftrightarrow\; N \vDash \varphi(ha)$ for every $\varphi(x) \in L_{\text{qf}}$ and every $a \in M^x$.

Recall that we write L_{qf} for the set of quantifier-free formulas. It is worth noting that when $M \subseteq N$ and $h = \mathrm{id}_M$ then 3 becomes

3′ $\quad M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(a) \qquad$ for every $\varphi(x) \in L_{\rm qf}$ and for every $a \in M^x$.

In words this is summarized by saying that the truth of quantifier-free formulas is preserved under sub- and superstructure.

Finally, we prove that first order truth is preserved under isomorphism. We say that a map $h : M \to N$ fixes $A \subseteq M$ (pointwise) if $\mathrm{id}_A \subseteq h$. An isomorphism that fixes A is also called an A-isomorphism.

2.36 Theorem If $h : M \to N$ is an isomorphism then for every $\varphi(x) \in L$

$$M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(ha) \text{ for every } a \in M^x$$

In particular, if h is an A-isomorphism then $M \equiv_A N$.

Proof. We proceed by induction on the syntax of $\varphi(x)$. When $\varphi(x)$ is atomic # holds by 3 above. Induction for the Boolean connectives is straightforward so we only need to consider the existential quantifier. Assume as induction hypothesis that

$$M \vDash \varphi(a,b) \quad \Leftrightarrow \quad N \vDash \varphi(ha,hb) \text{ for every } a \in M^x \text{ and } b \in M.$$

We prove that the equivalence in the theorem holds for the formula $\exists y\, \varphi(x,y)$.

$$\begin{aligned} M \vDash \exists y\, \varphi(a,y) \quad &\Leftrightarrow \quad M \vDash \varphi(a,b) \text{ for some } b \in M \\ &\Leftrightarrow \quad N \vDash \varphi(ha,hb) \text{ for some } b \in M \qquad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad N \vDash \varphi(ha,c) \text{for some } c \in N \qquad (\Leftarrow \text{ by surjectivity}) \\ &\Leftrightarrow \quad N \vDash \exists y\, \varphi(ha,y). \end{aligned}$$

□

2.37 Corollary If $h : M \to N$ is an isomorphism then for every $\varphi(x) \in L$.

$$h\big[\varphi(M^x)\big] \quad = \quad \varphi(N^x)$$

We can now give a few very simple examples of elementarily equivalent structures.

2.38 Example Let L be the language of strict orders. Consider the intervals of $\mathbb{R}$ as structures in the natural way. The intervals $[0,1]$ and $[0,2]$ are isomorphic, hence $[0,1] \equiv [0,2]$ follows from Theorem 2.36. Clearly, $[0,1]$ is a substructure of $[0,2]$. However $[0,1] \not\preceq [0,2]$, in fact the formula $\forall x\, (x \leq 1)$ holds in $[0,1]$ but is false in $[0,2]$. This shows that $M \subseteq N$ and $M \equiv N$ does not imply $M \preceq M$.

Now we prove that $(0,1) \preceq (0,2)$. By Exercise 2.16 it suffices to verify that for every finite $B \subseteq (0,1)$ we have $(0,1) \equiv_B (0,2)$. This follows again from Theorem 2.36 as $(0,1)$ and $(0,2)$ are B-isomorphic for every finite $B \subseteq (0,1)$.

For the sake of completeness we also give the definition of homomorphism.

2.39 Definition A homomorphism is a total map $h : M \to N$ such that

1. $a \in r^M \quad \Rightarrow \quad ha \in r^N$ for every $r \in L_{\rm rel}$ and $a \in M^{n_r}$;

2. $h\, f^M(a) \quad = \quad f^N(h\, a)$ for every $f \in L_{\rm fun}$ and $a \in M^{n_f}$.

Note that only one implication is required in 1.

2.40 Exercise Prove that if $h : N \to N$ is an automorphism and $M \preceq N$ then $h[M] \preceq N$.

2.41 Exercise Let L be the empty language. Let $A, D \subseteq M$. Prove that the following are equivalent

1. D is definable over A
2. either D is finite and $D \subseteq A$, or $\neg D$ is finite and $\neg D \subseteq A$.

Hint: as structures are plain sets, every bijection $f : M \to M$ is an automorphism.

2.42 Exercise Prove that if $\varphi(x)$ is an existential formula and $h : M \hookrightarrow N$ is an embedding then

$$M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha) \qquad\qquad \text{for every } a \in M^x.$$

Recall that existential formulas as those of the form $\exists y\, \psi(x, y)$ for $\psi(x, y) \in L_{\rm qf}$. Note that Theorem 10.7 proves that the property above characterizes existential formulas.

2.43 Exercise Let M be the model with domain $\mathbb{Z}$ in the language of additive groups which is interpreted in the natural way. Prove that there is no existential formula $\varphi(x)$ such that $\varphi(M^x)$ is the set of odd integers. Hint: use Exercise 2.42.

2.44 Exercise Let $\mathbb{Q}^+$ be the multiplicative group of positive rationals. Let M be the subgroup of the numbers of the form n/m for some odd integers m and n. Prove that $M \preceq \mathbb{Q}^+$. Hint: use the fundamental theorem of arithmetic and reason as in Example 2.38.

2.45 Exercise Prove that, in the language of strict orders, $\mathbb{R} \smallsetminus \{0\} \preceq \mathbb{R}$ and $\mathbb{R} \smallsetminus \{0\} \not\preceq \mathbb{R}$.

2.5 Quotient structure

The content of this section is mainly technical and only required later in the course. Its reading may be postponed.

If E is an equivalence relation on N we write $[c]_E$ for the equivalence class of $c \in N$. We use the same symbol for the equivalence relation on N^n defined as follow: if $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ are n-tuples of elements of N then $a\ E\ b$ means that $a_i\ E\ b_i$ holds for all i. It is easy to see that $b_1, \ldots, b_n \in [a, ..., a_n]_E$ if and only if $b_i \in [a_i]_E$ for all i. Therefore we use the notation $[a]_E$ for both the equivalence class of $a \in N^n$ and the tuple of equivalence classes $[a_1]_E, \ldots, [a_n]_E$.

2.46 Definition We say that the equivalence relation E on a structure N is a congruence if for every $f \in L_{\text{fun}}$

c1. $$a \mathrel{E} b \quad \Rightarrow \quad f^N a \mathrel{E} f^N b;$$

When E is a congruence on N we write N/E for the a structure that has as domain the set of E-equivalence classes in N and the following interpretation of $f \in L_{\text{fun}}$ and $r \in L_{\text{rel}}$:

c2. $$f^{N/E}[a]_E \quad = \quad \big[f^N a\big]_E\,;$$

c3. $$[a]_E \ \in \ r^{N/E} \quad \Leftrightarrow \quad [a]_E \cap r^N \neq \varnothing.$$

We call N/E the quotient structure.

By c1 the quotiont structure is well defined. The reader will recognize it as a familiar notion by the following proposition (which is not required in the following and requires the notion of homomorphism, see Definition 2.39. Recall that the kernel of a total map $h : N \to M$ is the equivalence relation E such that

$$a \mathrel{E} b \quad \Leftrightarrow \quad ha = hb$$

for every $a, b \in N$.

2.47 Proposition Let $h : N \to M$ be a surjective homomorphism and let E be the kernel of h. Then there is an isomorphism k that makes the following diagram commute

$$\begin{tikzcd}
N \arrow[r, "h"] \arrow[d, "\pi"] & M \\
N/E \arrow[ur, "k"] &
\end{tikzcd}$$

where $\pi : a \mapsto [a]_E$ is the projection map.

Quotients clutter the notation with brackets. To avoid the mess, we prefer to reason in N and tweak the satisfaction relation. Warning: though this is what everyone informally does, it is rarely spelled out so, feel free to reguard the following as a pedantry. In a nutshell, we are going to define the satisfaction relation $N/E \stackrel{*}{\models} \varphi(a)$ as an abbreviation of $N/E \models \varphi([a]_E)$.

Recall that in model theory, equality is not treated as a all other predicates. In fact, the interpretation of equality is fixed to always be the identity relation. In a few contexts is convinient to allow any congruence to interprete equality. This allows to work in N while thinking of N/E.

We define $N/E \stackrel{*}{\models}$ to be $N \models$ but with equality interpreted with E. The proposition below shows that this is the same thing as the regular truth in the quotient structure, $N/E \models$.

2.48 Definition For t_2, t_2 closed terms of $L(N)$ define

1* $\quad N/E \stackrel{*}{\models} t_1 = t_2 \quad \Leftrightarrow \quad t_1^N \, E \, t_2^N$

For t a tuple of closed terms of $L(N)$ and $r \in L_{\mathrm{rel}}$ a relation symbol

2* $\quad N/E \stackrel{*}{\models} r\,t \quad \Leftrightarrow \quad t^N \, E \, a \quad$ for some $a \in r^N$

Finally the definition is extended to all sentences $\varphi \in L(N)$ by induction in the usual way

3* $\quad N/E \stackrel{*}{\models} \neg\varphi \quad \Leftrightarrow \quad$ not $\; N/E \stackrel{*}{\models} \varphi$

4* $\quad N/E \stackrel{*}{\models} \varphi \wedge \psi \quad \Leftrightarrow \quad N/E \stackrel{*}{\models} \varphi$ and $N/E \stackrel{*}{\models} \psi$

5* $\quad N/E \stackrel{*}{\models} \exists x\, \varphi(x) \quad \Leftrightarrow \quad N/E \stackrel{*}{\models} \varphi(a) \quad$ for some $a \in N$.

Now, by induction on the syntax of formulas one can prove $\stackrel{*}{\models}$ does what required. In particular, $N/E \stackrel{*}{\models} \varphi(a) \leftrightarrow \varphi(b)$ for every $a \, E \, b$.

2.49 Proposition Let E be a congruence relation of N. Then the following are equivalent for every $\varphi(x) \in L$

1. $\quad N/E \stackrel{*}{\models} \varphi(a)$;
2. $\quad N/E \models \varphi([a]_E)$.

2.6 Completeness

A theory T is maximally consistent if it is consistent and there is no consistent theory S such that $T \subset S$. Equivalently, T contains every sentence φ consistent with T, that is, such that $T \cup \{\varphi\}$ is consistent. Clearly a maximally consistent theory is closed under logical consequences.

A theory T is complete if $\mathrm{ccl}T$ is maximally consistent. Concrete examples will be given in the next chapters as it is not easy to prove that a theory is complete.

2.50 Proposition The following are equivalent

a. T is maximally consistent
b. $T = \mathrm{Th}(M)$ for some structure M
c. T is consistent and $\varphi \in T$ or $\neg\varphi \in T$ for every sentence φ.

Proof. To prove a⇒b, assume that T is consistent. Then there is $M \models T$. Therefore $T \subseteq \mathrm{Th}(M)$. As T is maximally consistent $T = \mathrm{Th}(M)$. Implication b⇒c is immediate. As for c⇒a note that if $T \cup \{\varphi\}$ is consistent then $\neg\varphi \notin T$ therefore $\varphi \in T$ follows from c. □

The proof of the proposition below is left as an exercise for the reader.

2.51 Proposition The following are equivalent

a. T is complete
b. there is a unique maximally consistent theory S such that $T \subseteq S$
c. T is consistent and $T \vdash \mathrm{Th}(M)$ for every $M \vDash T$
d. T is consistent and either $T \vdash \varphi$ or $T \vdash \neg\varphi$ for every sentence φ
e. T is consistent and $M \equiv N$ for every pair of models of T.

2.52 Exercise Prove that the following are equivalent

a. T is complete
b. for every sentence φ, o $T \vdash \varphi$ o $T \vdash \neg\varphi$ but not both.

By contrast prove that the following are *not* equivalent

a. T is maximally consistent
b. for every sentence φ, o $\varphi \in T$ o $\neg\varphi \in T$ but not both.

2.53 Exercise Prove that if T has exactly 2 maximally consistent extension T_1 and T_2 then there is a sentence φ such that $T, \varphi \vdash T_1$ and $T, \neg\varphi \vdash T_2$. State and prove the generalization to finitely many maximally consistent extensions.

2.7 The Tarski-Vaught test

There is no natural notion of *smallest* elementary substructure containing a set of parameters A. The downward Löwenheim-Skolem, which we prove in the next section, is the best result that holds in full generality. Given an arbitrary $A \subseteq N$ we shall construct a model $M \preceq N$ containing A that is small in the sense of cardinality. The construction selects one by one the elements of M that are required to realise the condition $M \preceq N$. Unfortunately, Definition 2.12 supposes full knowledge of the truth in M and it may not be applied during the construction. The following lemma comes to our rescue with a property equivalent to $M \preceq N$ that only mention the truth in N.

2.54 Lemma (Tarski-Vaught test) For every $A \subseteq N$ the following are equivalent

1. A is the domain of a structure $M \preceq N$
2. for every formula $\varphi(x) \in L(A)$, with $|x| = 1$,
   $N \vDash \exists x\, \varphi(x) \quad \Rightarrow \quad N \vDash \varphi(b)$ for some $b \in A$.

Proof. 1⇒2.

$$\begin{aligned} N \vDash \exists x\, \varphi(x) \ &\Rightarrow\ M \vDash \exists x\, \varphi(x) && \\ &\Rightarrow\ M \vDash \varphi(b) && \text{for some } b \in M \\ &\Rightarrow\ N \vDash \varphi(b) && \text{for some } b \in A. \end{aligned}$$

2⇒1. Firstly, note that A is the domain of a substructure of N, that is, $f^N a \in A$ for every $f \in L_{\mathrm{fun}}$ and every $a \in A^{n_f}$. In fact, this follows from 2 with $fa = x$ for $\varphi(x)$.

Write M for the substructure of N with domain A. By induction on the syntax we prove that for every $\xi(x) \in L$

$$M \vDash \xi(a) \ \Leftrightarrow\ N \vDash \xi(a) \qquad \text{for every } a \in M^x.$$

If $\xi(x)$ is atomic the claim follows from $M \subseteq N$ and the remarks underneath Definition 2.35. The case of Boolean connectives is straightforward, so only the existential quantifier requires a proof. So, let $\xi(x)$ be the formula $\exists y\, \psi(x, y)$ and assume the induction hypothesis holds for $\psi(x, y)$

$$\begin{aligned} M \vDash \exists y\, \psi(a, y) \;\; &\Leftrightarrow \;\; M \vDash \psi(a, b) && \text{for some } b \in M \\ &\Leftrightarrow \;\; N \vDash \psi(a, b) && \text{for some } b \in M \\ &\Leftrightarrow \;\; N \vDash \exists y\, \psi(a, y). \end{aligned}$$

The second equivalence holds by induction hypothesis, in the last equivalence we use 2 for the implication $\Leftarrow$. □

2.8 Downward Löwenheim-Skolem

The main theorem of this section was proved by Löwenheim at the beginning of the last century. Skolem gave a simpler proof immediately afterwards. At the time, the result was perceived as paradoxical.

A few years earlier, Zermelo and Fraenkel provided a formalization of set theory in a first order language. The downward Löwenheim-Skolem theorem implies the existence of an infinite countable model M of set theory: this is the so-called Skolem paradox. The existence of M seems paradoxical because, in particular, a sentence that formalises the axiom of power set holds in M. Therefore M contains an element b which, in M, is the set of subsets of the natural numbers. But the set of elements of b is a subset of M, and therefore it is countable.

In fact, this is not a contradiction, because the expression *all subsets of the natural numbers* does not have the same meaning in M as it has in the real world. The notion of cardinality, too, acquires a different meaning. In the language of set theory, there is a first order sentence that formalises the fact that b is uncountable: the sentence says that there is no bijection between b and the natural numbers. Therefore the bijection between the elements of b and the natural numbers (which exists in the real world) does not belong to M. The notion of equinumerosity has a different meaning in M and in the real world, but those who live in M cannot realise this.

2.55 Theorem (Downward Löwenheim-Skolem) Let N be an infinite structure and fix some set $A \subseteq N$. Then there is a structure M of cardinality $\leq |L(A)|$ such that $A \subseteq M \preceq N$.

Proof. Set $\lambda = |L(A)|$. Below we construct a chain $\langle A_i : i < \omega\rangle$ of subsets of N. The chain begins at $A_0 = A$. Finally we set $M = \bigcup_{i<\omega} A_i$. All A_i will have cardinality $\leq \lambda$ so $|M| \leq \lambda$ follows.

Now we construct A_{i+1} given A_i. Assume as induction hypothesis that $|A_i| \leq \lambda$. Then $|L(A_i)| \leq \lambda$. For some fixed variable x let $\langle \varphi_k(x) \;:\; k < \lambda\rangle$ be an enumeration of the formulas in $L(A_i)$ that are consistent in N. For every k pick $a_k \in N$ such that $N \vDash \varphi_k(a_k)$. Define $A_{i+1} = A_i \cup \{a_k \;:\; k < \lambda\}$. Then $|A_{i+1}| \leq \lambda$ is clear.

We use the Tarski-Vaught test to prove $M \preceq N$. Suppose $\varphi(x) \in L(M)$ is consistent in N. As finitely many parameters occur in formulas, $\varphi(x) \in L(A_i)$ for some i. Then $\varphi(x)$ is among the formulas we enumerated at stage i and $A_{i+1} \subseteq M$ contains a solution of $\varphi(x)$. □

We will need to adapt the construction above to meet more requirements on the model M. To better control the elements that end up in M it is convenient to add one element at the time (above we add λ elements at each stage). We need to enumerate formulas with care if we want to complete the construction by stage λ.

Second proof of the downward Löwenheim-Skolem Theorem. Let $\pi : \lambda^2 \to \lambda$ be a bijection such that $j, k \leq \pi(j,k)$ for all $j, k < \lambda$. Suppose we have defined the sets A_j for every $j \leq i$ and let $\langle \varphi_{j,k}(x) \ : \ k < \lambda \rangle$ be an enumeration of the consistent formulas of $L(A_j)$. Let $j, k \leq i$ be such that $\pi(j,k) = i$. Let b be a solution of the formula $\varphi_{j,k}(x)$ and define $A_{i+1} = A_i \cup \{b\}$.

We use Tarski-Vaught test to prove $M \preceq N$. Let $\varphi(x) \in L(M)$ be consistent in N. Then $\varphi(x) \in L(A_j)$ for some j. Then $\varphi(x) = \varphi_{j,k}$ for some k. Hence a witness of $\varphi(x)$ is enumerated in M at stage $\pi(j,k) + 1$. □

2.56 Exercise Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Hint: adapt the construction used to prove the downward Löwenheim-Skolem Theorem.

2.57 Exercise The language contains only $<$, the relation of strict order. Prove that there is a countable ordinal α that is an elementary substructure of ω_1, the first uncountable ordinal.

2.9 Elementary chains

An elementary chain is a chain $\langle M_i : i < \lambda \rangle$ of structures such that $M_i \preceq M_j$ for every $i < j < \lambda$. The union (or limit) of the chain is the structure with as domain the set $\bigcup_{i<\lambda} M_i$ and as relations and functions the union of the relations and functions of M_i. It is plain that all structures in the chain are substructures of the limit.

2.58 Lemma Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of structures. Let N be the union of the chain. Then $M_i \preceq N$ for every i.

Proof. By induction on the syntax of $\varphi(x) \in L$ we prove

$$M_i \vDash \varphi(a) \ \Leftrightarrow \ N \vDash \varphi(a) \qquad \text{for every } i < \lambda \text{ and every } a \in M_i^x$$

As remarked in 3' of Section 2.4, the claim holds for quantifier-free formulas. Induction for Boolean connectives is straightforward so we only need to consider the existential quantifier

$$\begin{aligned} M_i \vDash \exists y\, \varphi(a,y) \ &\Rightarrow \ M_i \vDash \varphi(a,b) && \text{for some } b \in M_i. \\ &\Rightarrow \ N \vDash \varphi(a,b) && \text{for some } b \in M_i \subseteq N \end{aligned}$$

where the second implication follows from the induction hypothesis. Vice versa

$$N \vDash \exists y\, \varphi(a,y) \ \Rightarrow \ N \vDash \varphi(a,b) \qquad \text{for some } b \in N$$

Without loss of generality we can assume that $b \in M_j$ for some $j \geq i$ and obtain

$$\Rightarrow \ M_j \vDash \varphi(a,b) \qquad \text{for some } b \in M_j$$

Now apply the induction hypothesis to $\varphi(x,y)$ and M_j

$$\Rightarrow \quad M_j \vDash \exists y\, \varphi(a, y)$$
$$\Rightarrow \quad M_i \vDash \exists y\, \varphi(a, y)$$

where the last implication holds because $M_i \preceq M_j$. □

2.59 Exercise Let $\langle M_i : i \in \lambda \rangle$ be an chain of elementary substructures of N. Let M be the union of the chain. Prove that $M \preceq N$ and note that Lemma 2.58 is not required.

2.60 Exercise Give an alternative proof of Exercise 2.56 using the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two countable chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$. The required model is $K = \bigcup_{i \in \omega} K_i$. In fact it is easy to check that $K \cap M = \bigcup_{i \in \omega} M_i$.

Chapter 3

Types and morphisms

There is a lot of notation in this chapter. The reader may skim through and return to it when necessary. In Section 3.1 and 3.2 we introduce distributive lattices and prime filters and prove Stone's representation theorem for distributive lattices. In Section 3.3 we discuss lattices that arise from sets of formulas and their prime filters (prime types). These sections are only required for the discussion of Hilbert's Nullstellensatz, see Section 8.7 below.

3.1 Semilattices and filters

A preorder is a set $\mathbb{P}$ with a reflexive and transitive relation. We usually denote the preorder by $\leq$. If $A, B \subseteq \mathbb{P}$ we write $A \leq B$ if $a \leq b$ for every $a \in A$ and $b \in B$. We write $a \leq B$ and $A \leq b$ for $\{a\} \leq B$ and $A \leq \{b\}$ respectively.

Quotienting a preorder by the equivalence relation

$$a \sim b \quad \Leftrightarrow \quad a \leq b \quad \text{and} \quad b \leq a$$

gives a (partial) order. We often do not distinguish between a preorder and the partial order associated to it. Preorders are very common; here we are mainly interested in preorders induced by the relation of logical consequence

$$\varphi \leq \psi \quad \Leftrightarrow \quad \varphi \vdash \psi$$

or, more generally, induced by the relation of logical consequence modulo a theory T, that is,

$$\varphi \leq \psi \quad \Leftrightarrow \quad T \cup \{\varphi\} \vdash \psi.$$

A partial order $\mathbb{P}$ is a lower semilattice if for each pair $a, b \in \mathbb{P}$ there is a maximal element c such that $c \leq \{a, b\}$. We call c the meet of a and b. The meet is unique and is denoted by $a \wedge b$. Dually, a partial order is an upper semilattice if for each pair of elements a and b there is a minimal element c such that $\{a, b\} \leq c$. This c is called the join of a and b. The join is unique and is denoted by $a \vee b$. A lattice is simultaneously a lower and an upper semilattice.

For instance, the family of subsets of $\mathbb{R}$ that are finite union of left-open, right-closed intervals is an upper-semilattice w.r.t. the relation of inclusion. However, it is not a lattice.

An element c such that $c \leq \mathbb{P}$ is called a lower bound or a bottom. An element such that $\mathbb{P} \leq c$ is called an upper bound or a top. Lower and upper bounds are unique and will be denoted by 0, respectively 1. Other symbols common in the literature are $\perp$, respectively $\top$. A semilattice is bounded if it has both an upper and a lower bound.

For the rest of this section we assume that $\mathbb{P}$ is a bounded lower semilattice.

The meet is associative and commutative

$$\begin{aligned}(a \wedge b) \wedge c &= a \wedge (b \wedge c)\\ a \wedge b &= a \wedge b.\end{aligned}$$

Hence we may unambiguously write $a_1 \wedge \cdots \wedge a_n$. When $C \subseteq \mathbb{P}$ is finite, we write $\wedge C$ for the meet of all the elements of C. We agree that $\wedge\varnothing = 1$.

In an upper semilattice, the dual properties hold for the join. We write $\vee C$ for the join of all elements in C and we agree that $\vee\varnothing = 0$.

A filter of $\mathbb{P}$ is a non-empty set $F \subseteq \mathbb{P}$ that satisfies the following for all $a, b \in \mathbb{P}$

f1. $a \in F$ and $a \leq b \;\Rightarrow\; b \in F$

f2. $a, b \in F \;\Rightarrow\; a \wedge b \in F$.

We say that F is a proper filter if $F \neq \mathbb{P}$, equivalently if $0 \notin F$. We say that F is principal if $F = \{b : a \leq b\}$ for some $a \in \mathbb{P}$. More precisely, we say that F is the principal filter generated by a. A proper filter F is maximal if there is no filter H such that $F \subset H \subset \mathbb{P}$.

A set $B \subseteq \mathbb{P}$ has the finite intersection property if $\wedge C \neq \varnothing$ for every finite $C \subseteq B$. For $B \subseteq \mathbb{P}$ we define the filter generated by B to be the intersection of all the filters containing B. It is easy to verify that this is indeed a filter. When B is a finite set, the filter generated by B is the principal filter generated by $\wedge B$. In general we have the following.

3.1 Proposition For every $B \subseteq \mathbb{P}$, the filter generated by B is the set

$$\{a : \wedge C \leq a \text{ for some finite non-empty } C \subseteq B\}.$$

In particular B is contained in a proper filter if and only if it has the finite intersection property.

We say that a filter F is maximal relative to c if $c \notin F$ and $c \in H$ for every $H \supset F$. So, a filter F is maximal if it is maximal relative to 0. We say that F is relatively maximal when it is maximal relative to some c.

3.2 Proposition Let $B \subseteq \mathbb{P}$ and let $c \in \mathbb{P}$. If $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$, then B is contained in a maximal filter relative to c.

Proof. Let $\mathcal{F}$ be the set of filters F such that $B \subseteq F$ and $c \notin F$. By Proposition 3.1, $\mathcal{F}$ is non-empty. It is immediate that $\mathcal{F}$ is closed under unions of arbitrary chains. Then, by Zorn's lemma, $\mathcal{F}$ has a maximal element. □

3.3 Exercise Let $B \subseteq \mathbb{P}$. Let $c \in \mathbb{P}$ be such that $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$. Prove that the following are equivalent

1. B is a maximal filter relative to c
2. $a \notin B \;\Rightarrow\; b \wedge a \leq c$ for some $b \in B$.

3.4 Exercise Let $F \subseteq \mathbb{P}$ be a non-principal filter. Is F always contained in a maximal non-principal filter ?

3.2 Distributive lattices and prime filters

Let $\mathbb{P}$ be a lattice. We say that $\mathbb{P}$ is distributive if for every $a, b, c \in \mathbb{P}$

$$
\begin{aligned}
a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\
a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c)
\end{aligned}
$$

Throughout this section we assume that $\mathbb{P}$ is a bounded distributive lattice.

A proper filter F is prime if for every $a, b \in \mathbb{P}$

$$a \vee b \in F \quad \Rightarrow \quad a \in F \quad \text{or} \quad b \in F.$$

3.5 Proposition Every relatively maximal filter of $\mathbb{P}$ is prime.

Proof. Let F be maximal relative to c and assume that $a \notin F$ and $b \notin F$. Then, by Exercise 3.3, there are $d_1, d_2 \in F$ such that $d_1 \wedge a \leq c$ and $d_2 \wedge b \leq c$. Let $d = d_1 \wedge d_2$. Then $d \wedge a \leq c$ and $d \wedge b \leq c$ and therefore $(d \wedge a) \vee (d \wedge b) \leq c$. Hence, by distributivity, $d \wedge (a \vee b) \leq c$. Then $a \vee b \notin F$. □

The Stone space of $\mathbb{P}$ is a topological space that we denote by $S(\mathbb{P})$. The points of $S(\mathbb{P})$ are the prime filters of $\mathbb{P}$. The closed sets of the Stone topology are arbitrary intersections of sets of the form

$$[a]_{\mathbb{P}} \quad = \quad \Big\{ F \ : \ \text{prime filter such that } a \in F \Big\}.$$

for $a \in \mathbb{P}$. In other words, the sets above form a base of closed sets of the Stone topology. Using 1 and 3 in the following proposition the reader can easily check that this is indeed a base for a topology.

3.6 Proposition For every $a, b \in \mathbb{P}$ we have

1. $[0]_{\mathbb{P}} = \varnothing$
2. $[1]_{\mathbb{P}} = S(\mathbb{P})$
3. $[a]_{\mathbb{P}} \cup [b]_{\mathbb{P}} = [a \vee b]_{\mathbb{P}}$
4. $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = [a \wedge b]_{\mathbb{P}}$.

Proof. The verification is immediate. Only 3 requires that the filters in $S(\mathbb{P})$ are prime. □

The closed subsets of $S(\mathbb{P})$ ordered by inclusion form a distributive lattice. The following is a representation theorem for distributive lattices.

3.7 Theorem The map $a \mapsto [a]_{\mathbb{P}}$ is an embedding of $\mathbb{P}$ in the lattice of the closed subsets of $S(\mathbb{P})$. In particular

1. $0 \mapsto \varnothing$
2. $1 \mapsto S(\mathbb{P})$
3. $a \vee b \mapsto [a]_{\mathbb{P}} \cup [b]_{\mathbb{P}}$
4. $a \wedge b \mapsto [a]_{\mathbb{P}} \cap [b]_{\mathbb{P}}$.

Proof. It is immediate that the map above preserves the order and Proposition 3.6 shows that it preserves the lattice operations. We prove that the map is injective. Let $a \neq b$, say $a \nleq b$. We claim that $[a]_{\mathbb{P}} \nsubseteq [b]_{\mathbb{P}}$. There is a filter F that contains a and is maximal relative to b. By Proposition 3.5 such an F is prime. Then $F \in [a]_{\mathbb{P}} \smallsetminus [b]_{\mathbb{P}}$. □

3.8 Theorem With the Stone topology, $S(\mathbb{P})$ is a compact space.

Proof. Let $\langle [a_i]_{\mathbb{P}} : i \in I \rangle$ be basic closed sets such that for every finite $J \subseteq I$

a. $$\bigcap_{i \in J} [a_i]_{\mathbb{P}} \neq \varnothing.$$

We claim that

b. $$\bigcap_{i \in I} [a_i]_{\mathbb{P}} \neq \varnothing.$$

By 4 of Proposition 3.6 and a we obtain that $\wedge C \not\leq 0$ for every finite $C \subseteq \{a_i : i \in I\}$. By Proposition 3.2, there is a maximal (relative to 0) filter containing $\{a_i : i \in I\}$. By Proposition 3.5, such filter is prime and it belongs to the intersection in b. □

Let $a, b \in \mathbb{P}$. If $a \wedge b = 0$ and $a \vee b = 1$, we say that b is the complement of a (and vice versa). The complement of an element need not exist. If the complement exists it is unique, the complement of a is denoted by $\neg a$.

3.9 Lemma Let $U \subseteq S(\mathbb{P})$ be a clopen set. Then $U = [a]_{\mathbb{P}}$ for some $a \in \mathbb{P}$, and $\neg a$ exists.

Proof. As both U and $S(\mathbb{P}) \smallsetminus U$ are closed, for some sets $A, B \subseteq \mathbb{P}$

$$\bigcap_{x \in A} [x]_{\mathbb{P}} = U$$

$$\bigcap_{y \in B} [y]_{\mathbb{P}} = S(\mathbb{P}) \smallsetminus U.$$

By compactness, that is Theorem 3.8, there are some finite $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap_{x \in A_0} [x]_{\mathbb{P}} \cap \bigcap_{y \in B_0} [y]_{\mathbb{P}} = \varnothing$$

Let $a = \wedge A_0$ and $b = \wedge B_0$. From claim 4 of Proposition 3.6 we obtain $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = \varnothing$. Therefore $U \subseteq [a]_{\mathbb{P}} \subseteq S(\mathbb{P}) \smallsetminus [b]_{\mathbb{P}} \subseteq U$. Hence $U = [a]_{\mathbb{P}}$ and $b = \neg a$. □

A Boolean algebra is a bounded distributive lattice where every element has a complement. In a Boolean algebra, the sets $[a]_{\mathbb{P}}$ are clopen and they form also a base of open set of the topology of $S(\mathbb{P})$. A topology that has a base of clopen sets is called zero-dimensional. By the following proposition the Stone topology of a Boolean algebra is Hausdorff.

A proper filter of a Boolean algebra $\mathbb{P}$ is an ultrafilter if either $a \in F$ or $\neg a \in F$ for every $a \in \mathbb{P}$.

3.10 Proposition Let $\mathbb{P}$ be a Boolean algebra. Then the following are equivalent

1. F is maximal
2. F is prime
3. F is an ultrafilter.

Proof. For 2⇒3 observe that $a \vee \neg a \in F$. The rest is immediate. □

The most natural example of Boolean algebra is $\mathcal{P}(I)$ where I is any set. Union and intersection are the join, respectively meet of the algebra. We say filter on I for filter of $\mathcal{P}(I)$.

3.11 Exercise Let I be infinite. Let $F = \{a \subseteq I \ : \ |I \smallsetminus a| < \omega\}$. Then F is a filter on I which is called he Fréchet's filter. Prove that every non-principal ultrafilter on I contains Fréchet's filter.

3.12 Exercise Prove that the stone topology on $S(\mathbb{P})$ has a base of open compact sets.

3.13 Exercise Suppose we had defined $S(\mathbb{P})$ as the set of relatively maximal filters. What could possibly go wrong ?

3.3 Types as filters

In this section we work with a fixed set of formulas Δ all with free variables among those of some fixed tuple x. The tuple x is not displayed in the notation. Subsets of Δ are called Δ-types.

We associate to Δ a bounded lattice $\mathbb{P}(\Delta)$. This is the closure under conjunction and disjunction of the formulas in $\Delta \cup \{\bot, \top\}$. The (pre)order relation in $\mathbb{P}(\Delta)$ is given by

$$\psi \leq \varphi \quad \Leftrightarrow \quad T \cup \{\psi\} \ \vdash \ \varphi,$$

for some fixed theory T. In this section, to lighten notation, we absorb T in the symbol $\vdash$. If p is a Δ-type we denote by $\langle p \rangle_{\mathbb{P}}$ the filter in $\mathbb{P}(\Delta)$ generated by p.

A few propositions in this section requires the compactness theorem (for types), Theorem 5.7 which is only proved below.

3.14 Lemma For every Δ-type p

$$\langle p \rangle_{\mathbb{P}} \quad = \quad \Big\{ \varphi \in \mathbb{P}(\Delta) \quad : \quad p \vdash \varphi \Big\}.$$

In particular p is consistent if and only if $\langle p \rangle_{\mathbb{P}}$ is a proper filter.

Proof. Inclusion $\subseteq$ is clear. If $p \vdash \varphi$, by compactness, $\psi \vdash \varphi$ for some formula ψ that is conjunction of formulas in p. Then $\varphi \in \langle p \rangle_{\mathbb{P}}$ follows from $\psi \in \langle p \rangle_{\mathbb{P}}$ and $\psi \leq \varphi$. □

We say that $p \subseteq \Delta$ is a principal Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a principal. The following lemma is an immediate consequence of the Compactness Theorem. Note that in 3 the formula φ is arbitrary, possibly not even in $\mathbb{P}(\Delta)$.

3.15 Lemma For every Δ-type p the following are equivalent

1. p is principal
2. $\psi \ \vdash \ p \ \vdash \ \psi$ for some formula ψ (here ψ is any formula, it need not be in Δ)
3. $\varphi \ \vdash \ p$ where φ is conjunction of formulas in p.

Proof. Implications 1⇒2 is immediate by Lemma 3.14. To prove 2⇒3 suppose

$\psi \vdash p \vdash \psi$. Apply compactness to obtain a formula φ, conjunction of formulas in p, such that $\varphi \vdash \psi$. Implications 3⇒1is trivial. □

3.16 Definition We say that $p \subseteq \Delta$ is a prime Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a prime filter. We say that p is a complete Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a maximal filter.

Though in general neither Δ nor $\mathbb{P}(\Delta)$ are closed under negation, Lemma 3.14 has the following consequence.

3.17 Proposition For every consistent Δ-type p the following are equivalent

1. p is complete
2. p is consistent and either $p \vdash \varphi$ or $p \vdash \neg\varphi$ for every formula $\varphi \in \Delta$.

Proof. 2⇒1. Assume 2. As p is consistent, $\langle p \rangle_{\mathbb{P}}$ is a proper filter. To prove that it is maximal suppose $\varphi \notin \langle p \rangle_{\mathbb{P}}$. Then $p \nvdash \varphi$ and from 2 it follows that $p \vdash \neg\varphi$. Hence no proper filter contains $p \cup \{\varphi\}$.

1⇒2. As $\langle p \rangle_{\mathbb{P}}$ is proper, p is consistent. Suppose $p \nvdash \varphi$. Then $\varphi \notin \langle p \rangle_{\mathbb{P}}$ and, as $\langle p \rangle_{\mathbb{P}}$ is maximal, $p \cup \{\varphi\}$ generates the improper filter. Then $p \cup \{\varphi\}$ is inconsistent. Hence $p \vdash \neg\varphi$. □

Given a model M and a tuple $c \in M^x$ the Δ-type of c in M is the sets

$$\Delta\text{-tp}_M(c) \quad = \quad \Big\{ \varphi(x) \in \Delta \ : \ M \vDash \varphi(c) \Big\}$$

When the model M is clear from the context we omit the subscript. When x and c are the empty tuple, we write $\text{Th}_\Delta(M)$ for $\Delta\text{-tp}_M(c)$.

3.18 Lemma For every Δ-type p the following are equivalent

1. p is prime
2. $p \ \cup \ \big\{ \neg\varphi \ : \ \varphi \ \in \ \Delta \ \text{ such that } \ p \nvdash \varphi \big\}$ is consistent
3. $\big\{ \varphi \in \Delta : p \vdash \varphi \big\} \ = \ \Delta\text{-tp}_M(c)$ for some model M and some tuple $c \in M^x$.

Proof. Implications 2⇒3⇒1 are clear, we prove 1⇒2. By compactness if the type in 2 is inconsistent then there are finitely many formulas $\varphi_1, \dots, \varphi_n \in \Delta$ such that $p \nvdash \varphi_i$ and

$$p \ \vdash \ \bigvee_{i=1}^{n} \varphi_i.$$

hence p is not prime. □

The following corollary is immediate. It simplifies the task of verifying that a given type is prime.

3.19 Corollary For every Δ-type p the following are equivalent

1. p is prime
2. $p \ \vdash \ \bigvee_{i=1}^{n} \varphi_i \quad \Rightarrow \quad p \vdash \varphi_i$ for some $i \leq n$, $\quad$ for every n and every $\varphi_1, \dots, \varphi_n \in \Delta$.

The set Δ above contains only formulas with variables among those of the tuple x. In the following it is convenient to consider sets Δ that are closed under substitution of variables with any other variable. The set of prime Δ-types is denoted by $S(\Delta)$, and we write $S_x(\Delta)$ when we restrict to types in with variables among those of the tuple x.

The most common Δ used in the sequel is the set of all formulas in $L(A)$. Then the underlying theory is $\mathrm{Th}(M/A)$ for some given model M containing A. In this case we write $\mathrm{tp}_M(c/A)$ for Δ-$\mathrm{tp}_M(c)$ or, when A is empty, $\mathrm{tp}_M(c)$. The set $S_x(\Delta)$ is denoted by $S_x(A)$. The topology on $S_x(A)$ is generated by the clopen

$$[\varphi(x)] \quad = \quad \Big\{ p \in S_x(A) \ : \ \varphi(x) \in p \Big\}.$$

Frequently Δ is the set of all formulas of a given syntactic form. Then we use a more suggestive notation as summarized below.

3.20 Notation The following are some of the most common Δ-types and Δ-theories

1. $\text{at-tp}(c)$, $\quad \mathrm{Th}_{\mathrm{at}}(M)$ $\quad$ when $\Delta = L_{\mathrm{at}}$
2. $\text{at}^{\pm}\text{-tp}(c)$, $\quad \mathrm{Th}_{\mathrm{at}^{\pm}}(M)$ $\quad$ when $\Delta = L_{\mathrm{at}^{\pm}}$
3. $\text{qf-tp}(c)$, $\quad \mathrm{Th}_{\mathrm{qf}}(M)$ $\quad$ when $\Delta = L_{\mathrm{qf}}$.

The types/theories in 2 and 3 are logically equivalent. Most used is the theory $\mathrm{Th}_{\mathrm{at}^{\pm}}(M/M)$ which is called the diagram of M and has a dedicated symbol: $\mathrm{Diag}(M)$. The theory $\mathrm{Th}(M/M)$ is called the elementary diagram of M.

3.21 Remark Let $A \subseteq M \cap N$. The following are equivalent

1. $N \vDash \mathrm{Diag}\langle A \rangle_M$
2. $\langle A \rangle_M$ is a substructure of N.

3.4 Morphisms

First we set the meaning of the word map.

3.22 Definition A map consists a triple $f : M \to N$ where

1. M is a set (usually a structure) called the domain of the map
2. N is a set (usually a structure) called the codomain the map
3. f is a function with domain of definition $\mathrm{dom}\, f \subseteq M$ and image $\mathrm{rng}\, f \subseteq N$.

By cardinality of $f : M \to N$ we understand the cardinality of the function f.

If $\mathrm{dom}\, f = M$ we say that the map is total; if $\mathrm{rng}\, f = N$ we say that it is surjective. The composition of two maps and the inverse of a map are defined in the obvious way.

3.23 Definition Let Δ be a set of formulas. The map $h : M \to N$ is a Δ-morphism if it preserves the truth of all formulas in Δ. By this we mean that

p. $M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha)$ for every $\varphi(x) \in \Delta$ and every $a \in (\operatorname{dom} h)^x$.

When h is total we we say it is a Δ-embedding.

The following is easy remark is useful to simplify notation.

3.24 Remark When Δ is closed under substitution of free variables it is convenient to rewrite p using a fixed tuple a that enumerates $\operatorname{dom} h$ and a fixed tuple a valiables x of the same length. Then condition p can be rephrased as

p′. $M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha)$ for every $\varphi(x) \in \Delta$.

When $\Delta = L$ we say elementary map and elementary embedding for Δ-morphism, respectively Δ-embedding. When $\Delta = L_{\mathrm{at}}$ we say partial homomorphism and when $\Delta = L_{\mathrm{at}^{\pm}}$ we say either partial embedding or partial isomorphism. The reason for the latter name is explained in Remark 3.25. It is immediate to verify that a partial embedding which is total is an embedding as in Definition 2.35. Similarly, a partial embedding which is total and surjective is an isomorphism. The precise connection between partial and total homo/iso-morphisms is discussed in following remark.

3.25 Remark For every map $h : M \to N$ the following are equivalent

1. $h : M \to N$ is a partial isomorphism
2. there is an isomorphism $k : \langle \operatorname{dom} h \rangle_M \to \langle \operatorname{rng} h \rangle_N$ that extends h.

Moreover, this extension is unique. The equivalence holds replacing isomorphism by homomorphism (in this case, in 2 we obtain a surjection). The extension k is obtained defining $k\big(t(a)\big) = t(ha)$ for every term $t(x)$.

A similar fact holds for partial homomorphims if we replace isomorphism by epimorphism, i.e. surjective homomorphism.

We use Δ-morphisms to compare, locally, two structures. There are different ways to do this, in the proposition below we list a few synonymous expressions. But first some more notation. When x is a fixed tuple of variables, $a \in M^x$ and $b \in N^x$ we write

$$M, a \Rrightarrow_\Delta N, b \qquad \text{if } M \vDash \varphi(a) \Rightarrow N \vDash \varphi(b) \text{ for every } \varphi(x) \in \Delta.$$

$$M, a \equiv_\Delta N, b \qquad \text{if } M \vDash \varphi(a) \Leftrightarrow N \vDash \varphi(b) \text{ for every } \varphi(x) \in \Delta.$$

The following equivalences are immediate and will be used without explicit reference.

3.26 Proposition For every given set of formulas Δ and every map $h : M \to N$ the following are equivalent

1. $h : M \to N$ is a Δ-morphism
2. $M, a \Rrightarrow_\Delta N, ha$ for every $a \in (\operatorname{dom} h)^x$
3. $\Delta\text{-tp}_M(a) \subseteq \Delta\text{-tp}_N(ha)$ for every $a \in (\operatorname{dom} h)^x$
4. $N, ha \vDash p(x)$ for every $a \in (\operatorname{dom} h)^x$ and $p(x) = \Delta\text{-tp}_M(a)$.

As in Remark 3.24, one can equivalently require 2, 3, and 4 for some/any fixed tuple a that enumerates $\operatorname{dom} h$.

3.27 Remark Condition p in Definition 3.23 applies to tuples x of any length, in particular to the empty tuple. In this case $\varphi(x)$ is a sentence, $a \in (\operatorname{dom} h)^0 = \{\varnothing\}$ is the empty tuple, and p asserts that $\mathrm{Th}_\Delta(M) \subseteq \mathrm{Th}_\Delta(N)$. When $h = \varnothing$ this is actually all that p says. In fact $\varnothing^x = \varnothing$ unless $|x| = 0$. Still, $\mathrm{Th}_\Delta(M) \subseteq \mathrm{Th}_\Delta(N)$ may be a non trivial requirement.

3.28 Definition We call $\mathrm{Th}_\Delta(M)$ the Δ-theory of M. We say that the theory T is Δ-complete if $\mathrm{Th}_\Delta(M) = \mathrm{Th}_\Delta(N)$ for all $M, N \vDash T$. In other words, for any pair of models of T, the empty map $\varnothing : M \to N$ is a Δ-morphism. When T is $L_{\mathrm{at}^\pm}$-complete we say that T decides the characteristic of its models (by analogy with rings and fields). Note that when T decides the characteristic of its models, we have $\langle\varnothing\rangle_M \simeq \langle\varnothing\rangle_N$ for all $M, N \vDash T$.

We conclude this section with a couple of propositions that break Theorem 2.36 into parts. More interestingly, in Chapter 10 we shall prove a sort of converse of Propositions 3.30 and 3.31.

It is interesting to note that there is a relation between certain properties of Δ-morphisms and the closure of Δ under logical connectives. When $\mathrm{C} \subseteq \{\forall, \exists, \neg, \vee, \wedge\}$ is a set of connectives, we write $\mathrm{C}\Delta$ for the closure of Δ with respect to the connectives in C. We write $\neg\Delta$ for the set containing the negation of the formulas in Δ. Warning: do not confuse $\neg\Delta$ with $\{\neg\}\Delta$. Up to logical equivalence $\{\neg\}\Delta$ coincides with $\Delta \cup \neg\Delta$.

It is clear that Δ-morphisms are $\{\wedge, \vee\}\Delta$-morphisms.

3.29 Proposition For every given set of formulas Δ and every injective map $h : M \to N$ the following are equivalent

a. $h : M \to N$ is a $\neg\Delta$-morphism

b. $h^{-1} : M \to N$ is a Δ-morphism.

3.30 Proposition For every set of formulas Δ, every total Δ-morphism $h : M \to N$ is a $\{\exists\}\Delta$-morphism.

Proof. Formulas in $\{\exists\}\Delta$ have the form $\exists y\, \varphi(x, y)$ where y is a finite tuples of variables and $\varphi(x, y) \in \Delta$. For every tuple $a \in (\operatorname{dom} h)^x$ we have:

$$\begin{aligned} M \vDash \exists y\, \varphi(a, y) \quad &\Rightarrow \quad M \vDash \varphi(a, b) \text{ for } b \in M^y \\ &\Rightarrow \quad N \vDash \varphi(ha, hb) \\ &\Rightarrow \quad N \vDash \varphi(ha, c) \text{ for } c \in N^y \\ &\Rightarrow \quad N \vDash \exists y\, \varphi(ha, y). \end{aligned}$$

Note that the second implication requires the totality of $h : M \to N$ which guarantees that $\varphi(a, y)$ has a solution in $\operatorname{dom} h$. □

When $h : M \to N$ is injective the following proposition is a corollary of Propositions 3.29 and 3.30. The general proof is the dual version of the proof of Propositions 3.30.

3.31 Proposition For every set of formulas Δ, every surjective Δ-morphism $h : M \to N$ is a $\{\forall\}\Delta$-morphism.

3.32 Exercise Prove that if $h : M \to N$ is an elementary embedding $h[M]$ is an ememen-tary substructure of N and $h : M \to h[M]$ is an isomorphism.

Chapter 4

Ultraproducts

In these notes we only use ultraproducts to prove the compactness theorem. Since a syntactic proof of the compactness theorem is also given, this chapter is, strictly speaking, not required. However, the importance of ultraproducts transcends their application to model theory.

4.1 Direct products

In this and in the next section $\langle M_i : i \in I\rangle$ is a sequence of L-structures. (We are slightly abusing of the word sequence, since I is a just naked set.) The direct product of this sequence is a structure denoted by

$$N \quad = \quad \prod_{i\in I} M_i$$

and defined by conditions 1-3 below. If $M_i = M$ for all $i \in I$, we say that N is a direct power of M and denote it by M^I.

The domain of N is the set containing all functions

1. $$\hat{a} \quad : \quad I \;\to\; \bigcup_{i\in I} M_i$$ $$\hat{a} \quad : \quad i \;\mapsto\; \hat{a}i \;\in\; M_i$$

We do not distinguish between tuples of elements of N and tuple-valued functions. For instance, the tuple $\hat{a} = \langle \hat{a}_1 \ldots \hat{a}_n\rangle$ is identified with the function $\hat{a} : i \mapsto \hat{a}i = \langle \hat{a}_1 i, \ldots, \hat{a}_n i\rangle$. On a first reading of what follows, it may help to pretend that all functions and relations are unary.

The interpretation of $f \in L_{\text{fun}}$ is defined as follows:

2. $$(f^N\hat{a})i \quad = \quad f^{M_i}(\hat{a}i) \qquad\qquad \text{for all } i \in I.$$

The interpretation of $r \in L_{\text{rel}}$ is the product of the relations r^{M_i}, that is, we define

3. $$\hat{a} \in r^N \quad \Leftrightarrow \quad \hat{a}i \in r^{M_i} \qquad\qquad \text{for all } i \in I.$$

The following proposition is immediate.

4.1 Proposition If $\doteq, \wedge, \forall, \exists$ are the only logical symbol that occur in $\varphi(x) \in L$, then for every $\hat{a} \in N^x$

$\sharp$ $$N \vDash \varphi(\hat{a}) \quad \Leftrightarrow \quad M_i \vDash \varphi(\hat{a}i) \qquad\qquad \text{for all } i \in I.$$

Proof. By induction on syntax. First note that we can extend 2 to all terms $t(x)$ as follows:

2′. $$(t^N\hat{a})i \quad = \quad t^{M_i}(\hat{a}i) \qquad\qquad \text{for all } i \in I.$$

Combining 3 and 2′ gives that for every $r \in L_{\text{rel}} \cup \{=\}$ and every L-term $t(x)$

$$N \vDash rt\hat{a} \quad \Leftrightarrow \quad M_i \vDash rt\hat{a}i \qquad \text{for all } i \in I.$$

This shows that $\sharp$ holds for $\varphi(x)$ atomic. Induction for the connectives $\wedge$, $\forall$ ed $\exists$ is immediate. $\square$

A consequence of Proposition 4.1 is that a direct product of groups, rings or vector spaces is a structure of the same sort. However, a product of fields is not a field.

4.2 Łoś's Theorem

Let I be any set. By filter on I we mean a filter of the Boolean algebra $\mathcal{P}(I)$.

We (temporary) denote by N' the following structure. The domain and the interpretation of the function symbols is as in the structure N defined in 1, respectively 2, of the previous section. As interpretation of $r \in L_{\rm rel}$ we take

3′. $$\hat{a} \in r^{N'} \quad \Leftrightarrow \quad \big\{ i \in I \;:\; \hat{a}i \in r^{M_i} \big\} \in F\,.$$

Let F be a filter on I. We define the following congruence on N' (see Definition 2.46)

$$\hat{a} \sim_F \hat{c} \quad \Leftrightarrow \quad \big\{ i \in I \;:\; \hat{a}i = \hat{c}i \big\} \in F\,.$$

To check that $\sim_F$ is indeed a congruence, we first need to check that it is an equivalence relation. Reflexivity and symmetry are immediate, and transitivity follows from f2 in Section 3.1. Then we check that $\sim_F$ is compatible with the functions of L, that is, that c1 of Definition 2.46 is satisfied. This follows from f1 in Section 3.1.

For brevity, we write N'/F for $N'/{\sim_F}$ and $[\hat{a}]_F$ for $[\hat{a}]_{\sim_F}$. We write $N'/F \overset{*}{\vDash} \varphi(\hat{a})$ for $N'/F \vDash \varphi([\hat{a}]_F)$. A more formal interpretation of $\overset{*}{\vDash}$ is given in Definition 2.48.

The structure N'/F is called the reduced product of the structures $\langle M_i : i \in I\rangle$ or, when $M_i = M$ for all $i \in I$, the reduced power of M. When F is an ultrafilter we say ultraproduct, respectively ultrapower.

The difference between N and N' is highlighted in Exercise 4.6. For neater notation, below we write N for N'.

The following proposition is almost tautological. Note that it is a special case of Łoš's Theorem below (but it is does not require F to be an *ultra*filter).

4.2 Proposition Let $r \in L_{\rm rel}$. Let $t(x)$ be tuple of terms of length n_r, the arity of r. Then for every $\hat{a} \in N^x$ and every filter F the following are equivalent

1. $N/F \overset{*}{\vDash} r\,t(\hat{a})$
2. $\big\{i \;:\; M_i \vDash r\,t(\hat{a}i)\big\} \in F$.

Proof. 1⇒2. Assume 1. Then, by 2* of Definition 2.48, there is a $\hat{b} \sim t^N(\hat{a})$ such that $N \vDash r\,\hat{b}$. Therefore $\{i : M_i \vDash r\,\hat{b}i\}$ is in F. But $\hat{b} \sim t(\hat{a})$ means that $\big\{i \;:\; \hat{b}i = t(\hat{a}i)\big\}$ is in F, hence 2 follows.

2⇒1. Is clear. $\square$

4.3 Łoś's Theorem Let $\varphi(x) \in L$ and let F be an ultrafilter on I. Then for every $\hat{a} \in N^x$ the following are equivalent:

1. $N/F \overset{*}{\vDash} \varphi(\hat{a})$ (see Definition 2.48)
2. $\{ i : M_i \vDash \varphi(\hat{a}i) \} \in F.$

Proof. We proceed by induction on the syntax of $\varphi(x)$. If $\varphi(x)$ is equality, then equivalence holds by definition of $\sim$. If $\varphi(x)$ is of the form $r\,t(x)$ for some tuple of terms $t(x)$ and $r \in L_{\rm rel}$ then 1⇔2 is Proposition 4.2.

We prove the inductive step for the connectives $\neg$, $\wedge$, and the quantifier $\exists$. We begin with $\neg$. This is the only place in the proof where the assumption that F is an *ultra*filter is required. By the inductive hypothesis,

$$N/F \overset{*}{\vDash} \neg\varphi(\hat{a}) \quad\Leftrightarrow\quad \{ i : M_i \vDash \varphi(\hat{a}i) \} \notin F$$

So, as F is an *ultra*filter

$$\Leftrightarrow\quad \{ i : M_i \vDash \neg\varphi(\hat{a}i) \} \in F.$$

Now consider $\wedge$. Assume inductively that the equivalence 1⇔2 holds for $\varphi(x)$ and $\psi(x)$. Then

$$N/F \overset{*}{\vDash} \varphi(\hat{a}) \wedge \psi(\hat{a}) \quad\Leftrightarrow\quad \{i : M_i \vDash \varphi(\hat{a}i)\} \in F \text{ and } \{i : M_i \vDash \psi(\hat{a}i)\} \in F.$$

As filters are closed under intersection, we obtain

$$\Leftrightarrow\quad \{ i : M_i \vDash \varphi(\hat{a}i) \wedge \psi(\hat{a}i) \} \in F.$$

Finally, consider $\exists y$. Assume inductively that the equivalence 1⇔2 holds for $\varphi(x, y)$. Then

$$\begin{aligned} N/F \overset{*}{\vDash} \exists y\, \varphi(\hat{a}, y) \quad&\Leftrightarrow\quad N/F \overset{*}{\vDash} \varphi(\hat{a}, \hat{b}) && \text{for some } \hat{b} \in N \\ &\Leftrightarrow\quad \{ i : M_i \vDash \varphi(\hat{a}i, \hat{b}i) \} \in F && \text{for some } \hat{b} \in N. \end{aligned}$$

We claim this is equivalent to

$$\Leftrightarrow\quad \{ i : M_i \vDash \exists y\, \varphi(\hat{a}i, y) \} \in F.$$

The $\Rightarrow$ direction is trivial. For $\Leftarrow$, we choose as $\hat{b}$ a sequence that picks a witness of $M_i \vDash \exists y\, \varphi(\hat{a}i, y)$ if it exists, and some arbitrary element of M_i otherwise. □

Let a^I denote the element of M^I that has constant value a. The following is an immediate consequence of Łoś's theorem.

4.4 Corollary Let $N = M^I$. For every $a \in M$

$$N/F \overset{*}{\vDash} \varphi(a^I) \quad\Leftrightarrow\quad M \vDash \varphi(a).$$

We often identify M with its image under the embedding $h : a \mapsto [a^I]_F$, and say that M^I/F is an elementary extension of M.

The following corollary is an immediate consequence of the compactness theorem that we prove in the next chapter but here we give a direct proof.

4.5 Corollary Every infinite structure has a proper elementary extension.

Proof. Let M be an infinite structure and let F be a *non-principal* ultrafilter on ω. It suffices to show that $h[M]$, the image of the embedding defined above, is a *proper* substructure of M^ω/F. As M is infinite, there is an injective function $\hat{d} \in M^\omega$. Then for every $a \in M$ the set $\{i : \hat{d}\,i = a\}$ is either empty or a singleton and, as F is non principal, it does not belong to F. So, by Łoš Theorem, we have $M^\omega/F \overset{*}{\vDash} \hat{d} \neq a^I$ for every $a \in M$, that is, $[\hat{d}]_F \notin h[M]$. $\square$

4.6 Exercise Let N be the structure defined in the previous section. Prove that if $M_i = M$ for all $i \in I$ then $N'/F = N/F$. More generally, prove that if for every $r \in L_{\mathrm{rel}}$, either $r^{M_i} \neq \varnothing$ for all i or $r^{M_i} = \varnothing$ for all i then $N'/F = N/F$.

4.7 Exercise Consider $\mathbb{N}$ as a structure in the language of strict orders. Let F be a non-principal ultrafilter on ω. Prove that in $\mathbb{N}^\omega$ there is a sequence $\langle \hat{a}_i : i \in \omega \rangle$ such that $\mathbb{N}^\omega/F \overset{*}{\vDash} \hat{a}_{i+1} < \hat{a}_i$.

4.8 Exercise Let I be the set of integers $i > 1$. For $i \in I$, let $\mathbb{Z}_i$ denote the additive group of integers modulo i, and let N denote the product $\prod\limits_{i \in I} \mathbb{Z}_i$. Prove that, if F is a non-principal ultrafilter on I,

1. for some F, N/F does not contain any element of finite order
2. for some F, N/F has some elements of order 2
3. for all F, N/F contains an element of infinite order
4. for some F, $N/F \vDash \forall x\, \exists y\; my = x$ for every integer $m > 0$
5. for some F, N/F contains an element $\hat{a}$ such that $N/F \vDash \forall x\; mx \neq \hat{a}$ for every positive integer m.

Chapter 5

Compactness theorem(s)

We present two proofs of the compactness theorem. The first uses ultrapowers the second is syntactic. Finally, we generalize the theorem so that it applies to types.

5.1 Compactness via ultraproducts

A theory is finitely consistent if all its finite subsets are consistent. The following theorem is the *fiat lux* of model theory.

5.1 Compactness Theorem Every finitely consistent theory is consistent.

Proof. Let T be a finitely consistent theory. We claim that the structure N/F which we define below is a model of T. Let I be the set of consistent sentences $\xi \in L$. For every $\xi \in I$ pick some $M_\xi \vDash \xi$. For any sentence $\varphi \in L$ we define

$$X_\varphi \quad = \quad \Big\{\xi \in I \ : \ \xi \vdash \varphi\Big\}.$$

Clearly φ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi \wedge \psi} \; = \; X_\varphi \cap X_\psi$. Hence, as T is finitely consistent, the set $B = \big\{X_\varphi \ : \ \varphi \in T\big\}$ has the finite intersection property. Therefore B extends to an ultrafilter F on I. Define

$$N \quad = \quad \prod_{\xi \in I} M_\xi.$$

We claim that $N/F \vDash T$. By Łoš Theorem, for every sentence $\varphi \in L$

$$N/F \vDash \varphi \quad \Leftrightarrow \quad \Big\{\xi \ : \ M_\xi \vDash \varphi\Big\} \ \in \ F\,.$$

By the definition of F, for every $\varphi \in T$, the set $X_\varphi \subseteq \{\xi \ : \ M_\xi \vDash \varphi\}$ belongs to F. Therefore $N/F \vDash T$, *et lux fuit*. □

The compactness theorem can be formulated in the following apparently stronger way.

5.2 Corollary If $T \vdash \varphi$ then there is some finite $S \subseteq T$ such that $S \vdash \varphi$.

Proof. Suppose $S \nvdash \varphi$ for every finite $S \subseteq T$. Then for every finite $S \subseteq T$ there is a model $M \vDash S \cup \{\neg\varphi\}$. In other words, $T \cup \{\neg\varphi\}$ is finitely consistent. By compactness $T \cup \{\neg\varphi\}$ is consistent, hence $T \nvdash \varphi$. □

5.3 Exercise Let $\Phi \subseteq L$ be a set of sentences and suppose that $\vdash \psi \leftrightarrow \bigvee \Phi$ for some sentence ψ. Prove that there is a finite $\Phi_0 \subseteq \Phi$ such that $\vdash \psi \leftrightarrow \bigvee \Phi_0$.

5.4 Exercise Let $\mathcal{C}$ be a class of structures. Let $\mathrm{Th}(\mathcal{C}) = \{\varphi \mid M \vDash \varphi \text{ for all } M \in \mathcal{C}\}$ be the theory of $\mathcal{C}$. Prove the following are equivalent

1. $N \vDash \mathrm{Th}(\mathcal{C})$
2. N is elementarily equivalent to an ultraproduct of elements of $\mathcal{C}$.

5.2 Compactness for types

Recall that a type is a set of formulas. When we present types we usually declare the variables that may occur in it – we write $p(x)$, $q(x)$, etc. where x is a tuple of variables. When x is the empty tuple, $p(x)$ is just a theory. Often we identify a finite types with the (infinite) conjunction of the formulas contained in it.

We write $M \vDash p(a)$ if $M \vDash \varphi(a)$ for every $\varphi(x) \in p$. We say that a is a solution or a realization of $p(x)$. An equivalent notation is $M, a \vDash p(x)$ or, when M is clear from the context, $a \vDash p(x)$. We say that $p(x)$ is consistent in M if it has a solution in M. In this case we may write $M \vDash \exists x\, p(x)$. We say that $p(x)$ is consistent if it is realized in some model.

We say that a type $p(x) \subseteq L$ is finitely consistent if all its finite subsets are consistent. The following fact is an immediate consequence of the compactness theorem.

5.5 Fact Every finitely consistent type $p(x) \subseteq L$ is consistent.

Proof. Let L' be the expansion of L obtained by adding the fresh symbols c, a tuple of constants of the same length as x. Then $p(c)$ is a finitely consistent theory in the language L'. By the compactness theorem there is an L'-structure $N' \vDash p(c)$. Let N be the reduct of N' to L, that is, the L-structure with the same domain and the same interpretation as N' on the symbols of L. Note that, though the constants c are not in L, the elements of the tuple $c^{N'}$ remain in N. Then $N, c^{N'} \vDash p(x)$. □

If all finite subsets of a type $p(x) \subseteq L(M)$ are consistent in the very same model M, we say that $p(x)$ is finitely consistent in M. The following theorem shows that the notion of finite concistency *in a model*, which is trivial for theories, is instead very interesting for types. Before that, we recall an important particular case of Proposition 3.26.

5.6 Lemma (The elementary diagram method) Let M be an infinite model. Let c be an enumeration of M. Let $q(z) = \mathrm{tp}(c)$. Then, for every model N realizing $q(z)$, there is an elementary embedding $h : M \hookrightarrow N$.

We call $q(z)$ the elementary diagram of M though, stricticly speaking, the elementary diagram of M is theory $\mathrm{Th}(M/M) = q(c)$.

Proof. Let $N, a \vDash q(z)$. Let $h : M \hookrightarrow N$ map $c \mapsto a$. By Remark 3.24, this is an elementary embeddings. □

5.7 Compactness Theorem for types Let M be an infinite model. If $p(x) \subseteq L(M)$ is finitely consistent in M then it is realised in some elementary extension of M.

Proof. Let a be an enumeration of M. It is convenient to write $p(x)$ as $p(x\,;a)$ where $p(x\,;z) \subseteq L$. Let $q(z) = \mathrm{tp}_M(a)$. Clearly, $p(x\,;z) \cup q(z)$ is finitely consistent. Then, by Fact 5.5, it is realized in some model N' by some $c', a' \in (N')^{x,z}$. Let $h : M \to N'$ map $a \mapsto a'$. By Lemma 5.6, this is an elementary embedding. Note that $hc = c'$. By Exercise 3.32, $h[M]$ is an isomorphic copy of M and is an elementary substructure of N'. By construction $N', hc \vDash p(x\,;ha)$.

We may conclude that there is a structure N (an isomorphic copy of N') that extends elementarily M and realizes $p(x\,;a)$. □

The following corollary is historically important.

5.8 Upward Löwenheim-Skolem Theorem Every infinite structure has elementary extensions of arbitrarily large cardinality.

Proof. Let $x = \langle x_i : i < \lambda \rangle$ be a tuple of distinct variables, where λ is an arbitrary cardinal. The type $p(x) = \{x_i \neq x_j : i < j < \lambda\}$ is finitely consistent in every infinite structure and every structure that realises p has cardinality $\geq \lambda$. Hence the claim follows from Theorem 5.7. □

5.9 Exercise Prove that for every type $p(x) \subseteq L(M)$ the following are equivalent

1. $p(x)$ is finitely consistent in M
2. $p(x) \cup \mathrm{Th}(M/M)$ is finitely consistent.

5.3 Compactness via syntax

Here we prove the compactness theorem using the so-called Henkin method. We divide the proof in two steps. Firstly, we observe that when the language is rich enough to name witnesses of all existential statements of the theory, these witnesses (*Henkin constants*) form a canonical model. Secondly, we show that we can add the required Henkin constants to any finitely consistent theory.

5.10 Definition Fix a language L. Assume for simplicity that formulas use only the connectives $\wedge$, $\neg$ and $\exists$. We say that T is a Henkin theory if for all formulas φ and ψ

0. $\varphi \in T \Rightarrow \neg\varphi \notin T$
1. $\neg\neg\varphi \in T \Rightarrow \varphi \in T$
2. $\varphi \wedge \psi \in T \Rightarrow \varphi \in T$ and $\psi \in T$
3. $\neg(\varphi \wedge \psi) \in T \Rightarrow \neg\varphi \in T$ or $\neg\psi \in T$
4. $\exists x\, \varphi \in T \Rightarrow \varphi[x/a] \in T$ for some closed term a
5. $\neg\exists x\, \varphi \in T \Rightarrow \neg\varphi[x/a] \in T$ for all closed terms a.

Moreover, the following holds for all closed terms a, b, c

a. $a \doteq a \in T$
b. $a \doteq b \in T \Rightarrow b \doteq a \in T$
c. $a \doteq b,\ b \doteq c \in T \Rightarrow a \doteq c \in T$
d. $a \doteq b,\ \varphi[x/a] \in T \Rightarrow \varphi[x/b] \in T$.

Fix a theory T and let M be the structure that has as domain the set of closed terms. Define for every relation symbol r

$$r^M = \{\langle a_1, \ldots, a_n\rangle \in M^n : r(a_1, \ldots, a_n) \in T\},$$

where n is the arity of r. Define for every function symbol f

$$f^M = \{\langle t, a_1, \ldots, a_n\rangle \in M^{n+1} : t = fa_1 \ldots a_n\}.$$

where n is the arity of r. An easy proof by induction shows that $t^M = t$ for all closed terms t.

Finally, let E be the relation on M that holds when $a \doteq b \in T$.

5.11 Lemma The relation E is a congruence on M (as defined in Section 2.5).

Proof. Axioms a-c ensure that E is an equivalence. We claim that that E is a congruence. This is immediate for unary functions: apply e to the formula $fx \doteq fa$. In general the claim is easily proved by induction on the arity of f. □

Condition 0 is the only negative requirement of Definition 5.10 (it requires that T does *not* contain some formula). By condition 0, Henkin theories do not contain any blatant inconsistency. Surprisingly, this is all what is needed for the existence of a model.

5.12 Theorem If T is a Henkin theory then $M/E \vDash T$.

Proof. By induction on the complexity of the formula φ in T we prove that

1. $\quad \varphi \in T \Rightarrow M/E \overset{*}{\vDash} \varphi \quad$ for the notation cf. Definition 2.48
2. $\quad \neg\varphi \in T \Rightarrow M/E \overset{*}{\nvDash} \neg\varphi$

Induction is immediate by 1-5 of Definition 5.10. Hence we only need to verify the claim for atomic formulas. Consider first the formula $\varphi = (t_1 \doteq t_2)$ where the t_i are closed terms. By the definition of M, for every closed term t we have $t^M = t$ so claim 1 is clear. As for 2, suppose $M/E \overset{*}{\vDash} t_1 \doteq t_2$, that is $t_1 E t_2$. Then $t_1 \doteq t_2 \in T$ and $\neg t_1 \doteq t_2 \notin T$ follows from axiom 0.

Now assume $\varphi = rt$ for a relation r and a tuple of closed terms t. The argument is similar: 1 is immediate; to prove 2 suppose that $M/E \overset{*}{\vDash} rt$. Then $tEs \in r^M$ for some tuple of closed terms s. Then $rs \in T$, and by d $rt \in T$. Finally from 0 we obtain $\neg rt \notin T$. □

5.13 Proposition If every finite subset of T has a model then there is a Henkin theory T' containing T. The theory T' may be in an expanded language L'.

Proof. Set $\lambda = |L|$. Let $\langle c_i : i < \lambda\rangle$ be some constants not in L. Let L_i be the language with constants among $c_{\restriction i}$. Fix a variable x and an enumeration $\langle \varphi_i(x) : i < \lambda\rangle$ of the formulas in L_λ. Suppose that the enumeration is such that $\varphi_i(x) \in L_i$.

We now construct a sequence of finitely consistent L_i-theories T_i. If α is 0 or a limit ordinal we define

$$T_\alpha = T \cup \bigcup_{i<\alpha} T_i.$$

As for successor ordinals, let S_i be a maximally finitely consistent set of L_i-formulas containing T_i. (Here we use Zorn's lemma, but see the remark below.) It is immediate that S_i satisfies all requirements in Definition 5.10 but possibly for 4.

Now, if $\exists x\, \varphi_i(x) \in S_i$ set $T_{i+1} = S_i \cup \{\varphi[x/c_i]\}$. As c_i does not occur in S_i, it is evident that T_{i+1} is finitely consistent.

Recall that we assumed $\exists x\, \varphi_i(x) \in L_i$. Then, either $\exists x\, \varphi_i(x) \in T_{i+1}$ or it is not finitely consistent with T_{i+1}. Hence stage i settle requirement 4 in Definition 5.10 as far as $\varphi_i(x)$ is concerned.

At stage λ all possible counterexamples to 4 have been ruled out, then $T' = T_\lambda$ is the required Henkin theory. □

The following theorem is an immediate corollary of the proposition above.

5.14 Compactness Theorem If T is finitely consistent then T is consistent.

To keep the proof of Proposition 5.13 short, we applied Zorn's lemma. This is not strictly necessary. In fact, if we are given a finitely consistent theory T. We can extend T to a theory S that meets Definition 5.10, up to condition 4, by adding systematically all required formulas. The procedure is effective, hence Zorn's lemma is not required.

It is interesting to consider the case when T is finite. Assume also (though this is not really necessary) that the language contains finitely many symbols and no functions other then constants. Then the construction in Proposition 5.13 is an effective procedure that produces in ω steps a model of T. At each step T_n is finite and contains only subformulas of formulas in T or variant on these obtained by substituting constants for variables.

Now suppose instead that we start with an inconsistent T. The procedure above has to come to a halt at same (finite) stage because a model of T does not exist. When the procedure halts, we end up with a finite sequence of finite theories $T_0, \ldots, T_n$ where $T_0 = T$ and T_n contains some blatant inconsistency (i.e. φ and $\neg\varphi$). Many have interpreted $T_0, \ldots, T_n$ as a formal *proof* of the inconsistency of T.

All this has little or no interest to model theory. But it highlights a fascinating phenomenon. When we say that T is inconsistent, we say that no structure models T. This expression uses a (meta linguistic) universal quantifier that ranges over the class of all structures. Yet this is equivalent to an expression that merely asserts the existence of a finite sequence of finite theories.

Chapter 6

Some relational structures

In the first section we prove that theory of dense linear orders without endpoints is ω-categorical. That is, any two such countable orders are isomorphic. This is an easy classical result of Cantor. In this chapter we examine Cantor's construction (a so-called *back-and-forth* construction) in great detail. In the second section we apply the same technique to prove that the theory of the random graph is ω-categorical.

6.1 Dense linear orders

The language of strict orders, which in this section we denote by L, contains only a binary relation symbol $<$. A structure M of signature L is a strict order if it models (the universal closure of) the following formulas

1. $x \not< x$ — irreflexive;
2. $x < z < y \ \rightarrow\ x < y$ — transitive.

Note that the following is an immediate consequence of 1 and 2.

$x < y \ \rightarrow\ y \not< x$ — antisymmetric.

We say that the order is total or linear if

li. $x < y \ \vee\ y < x \ \vee\ x = y$ — linear or total.

An order is dense if

nt. $\exists x, y\ (x < y)$ — non trivial

d. $x < y \ \rightarrow\ \exists z\ (x < z < y)$ — dense.

We need to require the existence of two comparable elements, then d implies that dense orders are in fact infinite. We say that the ordering has no endpoints if

e. $\exists y\ (x < y)\ \wedge\ \exists y\ (y < x)$ — without endpoints.

We denote by $T_{\rm lo}$ the theory strict linear orders and by $T_{\rm dlo}$ the theory of dense linear orders without endpoints. Clearly, these are consistent theories: $\mathbb{Q}$ with the usual ordering is a model of $T_{\rm dlo}$.

We introduce some notation to improve readability of the proof of the following theorem. Let A and B be subsets of an ordered set. We write $A < B$ if $a < b$ for every $a \in A$ and $b \in B$. We write $a < B$ e $A < b$ for $\{a\} < B$, respectively $A < \{b\}$. Let $M \vDash T_{\rm lo}$. Then $M \vDash T_{\rm dlo}$ if and only if for every finite $A, B \subseteq M$ such that $A < B$ there is a c such that $A < c < B$. In fact axiom d is evident and axioms nt and e are obtained taking replacing A and/or B by the empty set.

Now we prove the first of a series of lemmas that we call extension lemmas. Recall that in the language of strict orders an injective map $k : M \to N$ is a partial isomorphism if

$$M \vDash a < b \Leftrightarrow N \vDash ka < kb \qquad \text{for every } a, b \in \operatorname{dom} k.$$

When $M, N \vDash T_{\rm lo}$ the direction $\Rightarrow$ suffices.

6.1 Lemma Fix $M \vDash T_{\mathrm{lo}}$ and $N \vDash T_{\mathrm{dlo}}$. Let $k : M \to N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \to N$ that extends k and is defined in b.

Proof. Given a finite partial isomorphism $k : M \to N$ define

$$\begin{aligned} A^- &= \{a \in \mathrm{dom}(k) \ : \ a < b\}; \\ A^+ &= \{a \in \mathrm{dom}(k) \ : \ b < a\}. \end{aligned}$$

The sets A^- and A^+ are finite and partition $\mathrm{dom}\, k$, and $A^- < A^+$. As $k : M \to N$ is a partial isomorphism, $k[A^-] < k[A^+]$. Then in N there is an element c such that $k[A^-] < c < k[A^+]$. It is easy to check that setting $h = k \cup \{\langle b, c\rangle\}$ gives the required extension. □

The following is an equivalent version of Lemma 6.1.

6.2 Corollary Let $M \vDash T_{\mathrm{lo}}$ be countable and let $N \vDash T_{\mathrm{dlo}}$. Let $k : M \to N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \hookrightarrow N$ that extends k.

Proof. Let $\langle a_i : i < \omega\rangle$ be an enumeration of M. Define by induction a chain of finite partial isomorphisms $h_i : M \to N$ such that $a_i \in \mathrm{dom}\, h_{i+1}$. The construction starts with $h_0 = k$. At stage $i+1$ we chose any finite partial isomorphism $h_{i+1} : M \to N$ that extends h_i and is defined in a_i. This is possible by Lemma 6.1. In the end we set

$$h = \bigcup_{i\in\omega} h_i.$$

It is immediate to verify that $h : M \hookrightarrow N$ is the required embedding. □

We are now ready to prove that any two countable models of T_{dlo} are isomorphic which is a classical result of Cantor's. Actually what we prove is slightly more general than that. In fact Cantor's theorem is obtained from the theorem below by setting $k = \varnothing$, which we are allowed to, because all models have the same empty characteristic (cfr. Remark 3.27).

6.3 Theorem Every finite partial isomorphism $k : M \to N$ between countable models of T_{dlo} extends to an isomorphism $g : M \xrightarrow{\sim} N$.

The following is the archetypal back-and-forth construction. It is important to note that it does not mention linear orders at all. It only uses the extension Lemma 6.1. The same construction can be applied in many other contexts where an extension lemma holds (cfr. Theorem 7.6).

Proof. Let $\langle a_i : i < \omega\rangle$ and $\langle b_i : i < \omega\rangle$ be enumerations of M and N respectively. We define by induction a chain of finite partial isomorphisms $g_i : M \to N$ such that $a_i \in \mathrm{dom}\, g_{i+1}$ and $b_i \in \mathrm{rng}\, g_{i+1}$. In the end we set

$$g = \bigcup_{i\in\omega} g_i$$

We begin by letting $g_0 = k$. The inductive step consists of two half-steps that we call the *forth step* and *back step*. In the forth step we define $g_{i+1/2}$ such that $a_i \in \operatorname{dom} g_{i+1/2}$. In the back step to define g_{i+1} such that $b_i \in \operatorname{rng} g_{i+1}$.

By the extension lemma 6.1 there is a finite partial isomorphism $g_{i+1/2} : M \to N$ that extends g_i and is defined in a_i. Now apply the same lemma to extend $\left(g_{i+1/2}\right)^{-1} : N \to M$ to a finite partial isomorphism $(g_{i+1})^{-1} : N \to M$ defined in b_i. □

Let λ be an infinite cardinal. We say that a theory is λ-categorical if any two models of T of cardinality λ are isomorphic. From Theorem 6.3, taking $k = \varnothing$, we obtain the following.

6.4 Corollary The theory T_{dlo} is ω-categorical.

We also obtain that T_{dlo} is a complete theory. This is consequence of the following general fact.

6.5 Proposition If T has no finite models and is λ-categorical for some $\lambda \geq |L|$, then T is complete.

Proof. Let M and N be any two models of T. Applying the upward and/or downward Löwenheim-Skolem theorem, we may assume they both have cardinality λ (here we use that M and N are both infinite and that $\lambda \geq |L|$). Hence $M \simeq N$ and in particular $M \equiv N$. □

6.6 Exercise Prove that the extension Lemma 6.1 characterizes models of T_{dlo} among models of T_{lo}. That is, if N is a model of T_{lo} such that the conclusion of Lemma 6.1 holds, then $M \vDash T_{\text{dlo}}$.

6.7 Exercise Prove that T_{dlo} is not λ-categorical for any uncountable λ.

6.8 Exercise Prove that, in the language of strict orders, $\mathbb{Q} \preceq \mathbb{R}$.

6.9 Exercise Let L be the language of strict orders expanded with countably many constants $\{c_i : i \in \omega\}$. Let T be the theory that extends T_{dlo} by the axioms $c_i < c_{i+1}$ for all i. Prove that T is complete. Find three non isomorphic countable models of this theory. For a suitably chosen model N of T, prove the statement in Lemma 6.1, where M any model of T.

6.10 Exercise Show that in Theorem 6.5 the assumption $\lambda \geq |L|$ is necessary. Hint: let ν be an uncountable cardinal. The language contains only the ordinals $i < \nu$ as constants. The theory T says that there are infinitely many elements and either $i = 0$ for every $i < \nu$, or $i \neq j$ for every $i < j < \nu$. Prove that T is incomplete. Prove that T has countable models and that these are all isomorphic.

6.2 Random graphs

Recall that the language of graphs, which in this section we denote by L, contains only a binary relation r. A graph structure of signature L such that

1. $\neg r(x,x)$ irreflexive
2. $r(x,y) \to r(y,x)$ symmetric.

An element of a graph M is called a vertex or a node. An edge is an unordered pair of vertices $\{a,b\} \subseteq M$ such that $M \vDash r(a,b)$. In words we may say that a is adjacent to b.

A random graph is a graph that also satisfies the following axioms for every n

nt. $\exists x, y \ \ (x \neq y)$ non trivial

r$_n$. $\displaystyle\bigwedge_{i,j=1}^{n} x_i \neq y_j \ \ \to \ \ \exists z \bigwedge_{i=1}^{n} \big[r(x_i,z) \wedge \neg r(z,y_i) \wedge z \neq y_i\big]$ for every $n \in \mathbb{Z}^+$.

The theory of graphs is denoted by $T_{\rm gph}$ and the theory of random graphs is denoted by $T_{\rm rg}$. The scheme of axioms r$_n$ plays the same role as density in the previous section. It says that given two disjoint sets A^+ and A^- of cardinality $\leq n$ there is a vertex z that is adjacent to all vertices in A^+ and to no vertex in A^-. We explicitly required that $z \notin A^-$, by 1 it is clear that $z \notin A^+$.

Strictly speaking, the axioms r$_n$ do not mention the cases when A^+ or A^- are empty. But as it is evident that random graphs are infinite, we can deal with them by adding redundant elements.

The following is the analogous of Lemma 6.1 for random graphs. Recall that in the language of graphs a map $k : M \to N$ is a partial isomorphism if it is injective and

$$M \vDash r(a,b) \ \Leftrightarrow \ N \vDash r(ka,kb) \qquad \text{for every } a,b \in \operatorname{dom} k.$$

6.11 Lemma Fix $M \vDash T_{\rm gph}$ and $N \vDash T_{\rm rg}$. Let $k : M \to N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \to N$ that extends k and is defined in b.

Proof. The structure of the proof is the same as in Lemma 6.1, so we use the same notation. Assume $b \notin \operatorname{dom} k$ and define

$$A^+ = \big\{x \in \operatorname{dom} k \ : \ M \vDash r(x,b)\big\} \quad \text{e} \quad A^- = \big\{y \in \operatorname{dom} k \ : \ M \vDash \neg r(y,b)\big\}.$$

These two sets are finite and disjoint, then so are $k[A^+]$ and $k[A^-]$. Then there is a $c \notin \operatorname{rng} k$ such that

$$\bigwedge_{a \in A^+} r(ka,c) \ \ \wedge \ \ \bigwedge_{a \in A^-} \neg r(ka,c).$$

As $k[A^+] \cup k[A^-] = \operatorname{rng} k$, it is immediate to verify that $h = k \cup \big\{\langle b,c\rangle\big\}$ is the required extension. □

Some readers may doubt that $T_{\rm rg}$ is consistent.

6.12 Proposition There exists a random graph.

Proof. The domain of is the set of natural numbers. Let $r(n,m)$ hold if the n-th prime number divides m or, conversely, the m-th prime number divides n. □

The same proof as that of Corollary 6.2 gives the following.

6.13 Corollary Let $M \vDash T_{\text{gph}}$ be countable and let $N \vDash T_{\text{rg}}$. Let $k : M \to N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \hookrightarrow N$ that extends k.

The proof of Theorem 6.3 gives the following theorem and its corollary.

6.14 Theorem Every finite partial isomorphism $k : M \to N$ between countable models of T_{rg} extends to an isomorphism $g : M \xrightarrow{\sim} N$.

6.15 Corollary The theory T_{rg} is ω-categorical (and therefore complete).

6.16 Exercise Let $a, b, c \in N \vDash T_{\text{rg}}$. Prove that $r(a, N) = r(b, N) \cap r(c, N)$ occurs only in the trivial case $a = b = c$.

6.17 Exercise Let $N \vDash T_{\text{rg}}$ prove that for every $b \in N$ the set $r(b, N)$ is a random graph. Is every random graph $M \subseteq N$ of the form $\varphi(N)$ for some $\varphi(x) \in L(N)$?

6.18 Exercise Let N be free union of two random graphs N_1 and N_2. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$, where $\sqcup$ denotes the disjoint union. Prove that N is not a random graph. Show that N_1 is not definable without parameters. Write a first order formula $\psi(x, y)$ true if x and y belong to the same connected component of N. Axiomatize the class of graphs that are free union of two random graphs.

6.19 Exercise Prove that T_{rg} is not λ-categorical for any uncountable λ. Hint: prove that there is a random graph N of cardinality λ where every vertex is adjacent to $< \lambda$ vertices. Compare it with its complement graph (the graph that has edges between pairs that are non adjacent in N).

6.20 Exercise Prove that T_{rg} is not finitely axiomatizable. Hint: given a random graph N and a set P of cardinality $n + 1$ show that you can add edges to $M \sqcup P$ and make it satisfy axiom r_n but not r_{n+1}.

6.21 Exercise (Peter J. Cameron) Prove that for every infinite countable graph M the following are equivalent

1. M is either random, complete, or empty (i.e. $r^M = M^2$ or $r^M = \varnothing$)
2. if $M_1, M_2 \subseteq M$ are such that $M_1 \sqcup M_2 = M$, then $M_1 \simeq M$ or $M_2 \simeq M$.

Hint: 2⇒1. Assume 2 and show that if $r(a, M) = \varnothing$ for some $a \in M$ then the graph is null; if $\{b\} \cup r(b, M) = M$ for some b, then the graph is complete. Clearly, 2 implies that any finite partition of M contains an element isomorphic to M. Then the claim above generalizes as follows: if there is a finite A such that $\bigcap_{a \in A} r(a, M) = \varnothing$ then M is the empty graph and if there is a finite B such that $B \cup \bigcup_{b \in B} r(b, M) = M$ then M is the complete graph.

Suppose M is not a random graph. Fix some finite, disjoint A and B such that no c satisfies both $r(A, c) = A$ and $r(B, c) = \varnothing$. Let $M_1 = \{c : r(A, c) \neq A\}$ and $M_2 = M \smallsetminus M_1$. Now note that

$$\bigcap_{a \in A} r(a, M_1) = \varnothing \qquad \text{and} \qquad \bigcup_{b \in B} r(b, M_2) = M_2.$$

6.3 Notes and references

We refer the reader to [1] for a well-written accessible survey on the amazing model theoretic properties of the random graph.

[1] Peter J. Cameron, *The random graph* (2013), arXiv:1301.7544. t.a. in The Mathematics of Paul Erdös III.

Chapter 7

Rich models

We introduce *Fraïssé limits*, also known as *homogeneous-universal* or *generic* structures, which here we call *rich* models, after Poizat. Rich models generalize the examples in Chapters 6 and the many more to come.

Elimination of quantifiers is briefly discussed at the end of Section 7.1. For the time being we identify quantifier elimination with the property that says that all partial isomorphisms are elementary maps. Proofs are easier with this notion in mind. The equivalence of this property with its syntactic counterpart is only proved in Chapter 10, when the reader is more familiar with arguments of compactness.

7.1 Models and morphisms

We now define *categories of models and partial morphisms*. These are example of concrete categories as intended in category theory. However, apart from the name, in what follows we dispense with all notions of category theory as they would make the exposition less basic than intended (without providing additional technical tools).

A category (of models and partial morphisms) is a class $\mathcal{M}$ which is disjoint union of two classes: $\mathcal{M}_{\text{ob}}$ and $\mathcal{M}_{\text{hom}}$. The first is the class of objects and contains structures with a common signature L which we call models. The second is the class of morphisms and contains (partial) maps between models. We require that the identity maps are morphisms and that composition of two morphism is again morphism. This makes $\mathcal{M}$ a well-defined category.

For example, $\mathcal{M}$ could consist of all models of some theory T_0 and of all partial isomorphisms between these. Alternatively, as morphisms we could take elementary maps between models. On a first reading the reader may assume $\mathcal{M}$ is as in one of the two examples above. In the general case we need to make some assumptions on $\mathcal{M}$.

7.1 Definition For ease of reference we list together all properties required below

c1. the (partial) identity map $\text{id}_A : M \to M$ is a morphism, for any $A \subseteq M$

c2. if $k' : M \to N$ is a morphism for every finite $k' \subseteq k$, then $k : M \to N$ is a morphism

c3. morphisms are invertible maps and the inverse of a morphism is a morphism

c4. morphisms preserve the truth of L_{at}-formulas

c5. if M is a model and $N \equiv M$, then also N is a model

c6. every elementary map between models is a morphism.

The connected component of a model M is the subclass of models N such that there is any morphism with domain M and codomain N (or vice versa, by c3). By axiom c1 the restriction of a morphism is a morphism, therefore M and N are in the same connected component if and only if the empty map $\varnothing : M \to N$ is a morphism. If the whole category $\mathcal{M}$ consists of one connected component we say that $\mathcal{M}$ is connected.

We call c2 the finite character of morphisms. Note that it implies the following

c7. if $k_i : M \to N$ is a chain of morphisms, then $\bigcup\limits_{i<\lambda} k_i : M \to N$ is a morphism.

Notably, the following two definitions require c3. The generalization to non injective morphisms is not straightforward (in fact, there are two generalizations: *projective* and *inductive*). These generalizations are not very common and will not be considered here.

7.2 Definition Assume that $\mathcal{M}$ satisfies c1-c3 of Definition 7.1. We say that a model N is λ-rich if for every model M, every $b \in M$ and every morphism $k : M \to N$ of cardinality $< \lambda$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is a morphism. We say that N is rich if it is λ-rich for $\lambda = |N|$. When $\mathcal{M}_{\rm ob} = {\rm Mod}(T_0)$ for some theory T_0 and $\mathcal{M}_{\rm hom}$ is clear from the context, we say rich model of T_0.

Rich models are also called *Fraïssé limits* or *homogeneous-universal* for a reason that will soon be clear; they are also called *generic*. Unfortunately these names are either too long or too generic, so we opt for the less common term *rich* that was proposed by Poizat.

The following two notions are closely connected with richness.

7.3 Definition Assume that $\mathcal{M}$ satisfies c1-c3 of Definition 7.1. We say that a model N is λ-universal if for every model M of cardinality $\leq \lambda$ in the same connected component as N there is an embedding $k : M \hookrightarrow N$. We say that a model N is λ-homogeneous if every $k : N \to N$ of cardinality $< \lambda$ extends to a bijective morphism $h : N \xrightarrow{\sim} N$ (an automorphism when c4 holds).

Note that the larger $\mathcal{M}_{\rm hom}$, the stronger notion of homogeneity. When $\mathcal{M}_{\rm hom}$ contains all partial isomorphisms between models (the largest class of morphisms considered here), it is common to say λ-ultrahomogeneous for λ-homogeneous.

As above, when $\lambda = |N|$ we say universal, homogeneous and ultrahomogeneous.

In Section 6.1 we implicitly used $\mathcal{M}_{\rm ob} = {\rm Mod}(T_{\rm lo})$ and partial isomorphisms as $\mathcal{M}_{\rm hom}$. In Section 6.2 we used $\mathcal{M}_{\rm ob} = {\rm Mod}(T_{\rm gph})$ and again partial isomorphisms as $\mathcal{M}_{\rm hom}$. Corollary 6.2 proves that every model of $T_{\rm dlo}$ is ω-rich. Corollary 6.13 claims the analogous fact for $T_{\rm rg}$.

In the following we frequently work under the following assumption (even when not all properties are strictly necessary).

7.4 Assumption Assume $|L| \leq \lambda$ and suppose that $\mathcal{M}$ satisfies c1-c6 of Definition 7.1

The assumption on the cardinality of L is necessary to apply the downward Löwenheim-Skolem Theorem when required.

7.5 Proposition (Assume 7.4) The following are equivalent

1. N is a λ-rich model
2. for every model M of cardinality $\leq \lambda$ and every morphism $k : M \to N$ of cardinality, say $< \lambda$ there is a embedding $h : M \hookrightarrow N$ that extends k.

Proof. Closure under union of chains of morphisms, which is ensured by c7, immediately yields 1⇒2. As for implication 2⇒1 we only need to consider the case $\lambda < |M|$. Let $k : M \to N$ be a morphism of cardinality $< \lambda$ and $b \in M$. By the downward Löwenheim-Skolem theorem there is an $M' \preceq M$ of cardinality λ containing $\operatorname{dom} k \cup \{b\}$. Let $h : M' \hookrightarrow N$ be the embedding obtained from 2. By c4, the map $h : M \to N$ is a composition of morphisms, hence a morphism. □

The following theorem subsumes both Theorem 6.3 and Theorem 6.14.

7.6 Theorem (Assume 7.4) Let M and N be two rich models of the same cardinality λ. Then every morphism $k : M \to N$ of cardinality $< \lambda$ extends to an isomorphism.

Proof. When $\lambda = \omega$, we can take the proof of Theorem 6.3 and replace *partial isomorphism* by *morphism* and the references to Lemma 6.1 by references to Proposition 7.5. As for uncountable λ, we only need to extend the construction through limit stages. By c7 we can simply take the union. □

7.7 Corollary (Assume 7.4) All rich models of cardinality λ in the same connected component are isomorphic.

It is obvious that rich models are universal. By Theorem 7.6, rich models they are homogeneous. These two notions are weaker than richness. For instance, when $\mathcal{M}$ is as in Section 6.2, the countable graph with no edge is trivially ultrahomogeneous but it is not universal and a fortiori not rich. On the other hand if we add to a countable random graph an isolated point we obtain a universal graph which is not ultrahomogeneous. However, when taken together, these two properties are equivalent to richness.

7.8 Theorem (Assume 7.4) The following are equivalent:

1. N is rich
2. N is homogeneous and universal.

Proof. Implication 1⇒2 is clear as noted above, so we prove 2⇒1. We use the characterization of richness given in Proposition 7.5. Let $k : M \to N$ be a morphism such that $|k| < |N|$ and $|M| \leq |N|$. As N is universal, there is a total morphism $f : M \hookrightarrow N$. By c3 the map $k \circ f^{-1} : N \to N$ is a morphism of cardinality $< |N|$. By homogeneity it has an extension to an automorphism $h : N \xrightarrow{\sim} N$. It is immediate that $h \circ f : M \hookrightarrow N$ is the required extension of k. □

A consequence of Theorem 7.6 is that morphisms between rich models of the same cardinality are elementary maps. However, the theorem gives no information when the models have different cardinality nor when they are merely λ-rich. This case is dealt with by next theorem, arguably the main result of this section.

To test the theorem below in a simple case we propose the following exercise.

7.9 Exercise Let N be the structure obtained by adding to a countable random graph an isolated point. Show that N is homogeneous if morphisms are elementary maps but it is not if morphisms are simply partial isomorphisms.

7.10 Exercise Every partial isomorphism $k : M \to N$ between models of T_{dlo} (or models of T_{rg}) is an elementary map. Hint: use downward Löwenheim-Skolem and Theorem 6.3.

7.2 The theory of rich models, and quantifier elimination

The theory of the rich models of $\mathcal{M}$ is the set T_1 of sentences that hold in all models that are λ-rich for some $\lambda \geq |L|$. The theorem below proves that T_1 is complete as soon as $\mathcal{M}$ is connected.

7.11 Theorem (Assume 7.4) Every morphism between λ-rich models is elementary. In particular, λ-rich models in the same connected component are elementarily equivalent.

Proof. Let $k : M \to N$ be a morphism between rich models. It suffices to prove that every finite restriction of k is elementary. By c2, we may as well assume that k itself is finite. It suffices to construct $M' \preceq M$ and $N' \preceq N$ together with a morphism $h : M \to N$ that extends k and maps M' bijectively to N'. Then by c6 the map $h : M' \rightarrow N'$ is the composition of morphisms, hence it is a morphism. Finally, by c4, it is an isomorphism, in particular an elementary map.

In general, richness is not preserved under elementary equivalence. Therefore M' and N' need to be constructed simultaneously with h. We define a chain of functions $\langle h_i : i{<}\lambda \rangle$ such that $h_i : M \to N$ are morphisms and in the end we set

$$h \;=\; \bigcup_{i<\lambda} h_i, \qquad M' \;=\; \operatorname{dom} h, \qquad N' \;=\; \operatorname{rng} h.$$

We interweave the usual back-and-forth-argument with the construction in the second proof of the Löwenheim-Skolem Theorem 2.55 in order to obtain $M' \preceq M$ and $N' \preceq N$.

The chains start with $h_0 = k$. At limit stages we take the union. Now assume we have h_i. Let $\varphi(x) \in L(\operatorname{dom} h_i)$ be some formula consistent in M and pick a solution $b \in M$. By λ-richness there is a $c \in N$ such that $h_i \cup \{\langle b, c\rangle\} : M \to N$ is a morphism. Let $h_{i+1/2} = h_i \cup \{\langle b, c\rangle\}$.

Finally, as in the proof of Theorem 6.3, we extend $h_{i+1/2}$ to obtain h_{i+1} by applying the same procedure with the roles of M and N inverted and $h^{-1}_{i+1/2}$ for h_i.

In the end we obtain $M' \preceq M$ if all formulas $\varphi(x) \in L(M')$ are considered. A similar consideration holds for N'. This is achieved using the same dovetail enumeration as in our second proof the downward Löwenheim-Skolem Theorem 2.55. □

Assume for simplicity that $\mathcal{M}$ is connected. Then all λ-rich models belong to

$\mathrm{Mod}(T_1)$ for some complete theory T_1. It is interesting to ask if the converse is true: if T_1 is the theory of the λ-rich models of $\mathcal{M}$, do all models in $\mathrm{Mod}(T_1)$ are rich?

7.12 Remark (Assume 7.4) Assume $\mathcal{M}$ is connected. Let T_1 be the theory of the rich models of $\mathcal{M}$. If every model of T_1 is λ-rich then T_1 is λ-categorical. (This is a conseguence of Corollary 7.7)

A very interesting interesting variant of the question asked above is considered in Theorem 9.7 where it is related to an important phenomenon that we now introduce. First, to have a concrete example at hand, we instantiate Theorem 7.11 with the two categories used in Chapter 6.

7.13 Corollary Partial isomorphisms between models of T_{dlo} and between models of T_{rg} are elementary maps.

When partial isomorphisms between models of a given theory T coincide with elementary maps, it is always by a fundamental reason. Let us introduce some terminology. Let T be a consistent theory. We say that T has (or admits) elimination of quantifiers if for every $\varphi(x) \in L$ there is a quantifier-free formula $\psi(x) \in L_{\mathrm{qf}}$ such that

$$T \vdash \psi(x) \leftrightarrow \varphi(x).$$

We will discuss general criteria for elimination of quantifiers in Chapter 10. Here we report without proof the following theorem.

7.14 Theorem The following are equivalent

1. T has elimination of quantifiers
2. every partial isomorphism between models of T is an elementary map.

⚠ This theorem will be proved only in Chapter 10, see Exercise 10.4 or Corollary 10.12. For the time being we do not need the syntactic version of elimination of quantifiers, so when saying that T *has quantifier elimination* we mean 2 of Theorem 7.14. For instance we rephrase Corollary 7.13 by saying that T_{dlo} and T_{rg} have elimination of quantifiers.

In the next chapter we introduce important examples of ω-rich models that do not have an ω-categorical theory. These are algebraic structures (groups, fields etc.) which are more complex than pure relational structures. So, we conclude this section with an example of this phenomenon in an almost trivial context.

7.15 Example Let L contain a unary predicate r_n for every positive integer n. The theory T_0 contains the axioms $\neg\exists x\, [r_n(x) \wedge r_m(x)]$ for $n \neq m$ and $\exists^{\leq n} x\, r_n(x)$ for every n. Work in the the category of models of T_0 and partial isomorphisms. Let T_1 be the theory that extends T_0 with the axioms $\exists^{=n} x\, r_n(x)$ for every n. Let $q(x)$ be the type $\{\neg r_n(x) : n \in \omega\}$. There are models of T_1 that do not realize $q(x)$, hence T_1 is not ω-categorical. It is easy to verify that the following are equivalent.

1. N is an ω-rich model
2. $N \vDash T_1$ and $q(N)$ is infinite.

The reader may use Theorem 7.11 and Compactness Theorem for Types 5.7 to prove that T_1 is complete and has elimination of quantifiers. It is also easy to verify that every uncountable model of T_1 is rich and consequently that T_1 is uncountably categorical.

7.16 Exercise Let T_0 and $\mathcal{M}$ be as in Example 7.15 except that we restrict the language to the relations $r_0, \ldots, r_n$ for a fixed n. Do ω-rich models of T_0 exist? If so, let T_1 be the set of sentences that hold in all rich model. Does T_1 has elimination of quantifiers? Is T_1 ω-categorical? Answer the questions above when we add to the language a constant 0. Answer the questions above when we drop the axioms $\neg\exists x\, [r_n(x) \wedge r_m(x)]$.

7.17 Exercise The language contains only the binary relations $<$ and e. The theory T_0 says that $<$ is a strict linear order and that e is an equivalence relation. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ?

7.18 Exercise The language contains only two binary relations. The theory T_0 says that they are equivalence relations. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ?

7.19 Exercise In the language of graphs let T_0 say that there are no cycles (equivalently, there is at most one path between any two nodes). In combinatorics these graphs are called *forests*, and their connected components are called *trees*. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ?

7.20 Exercise Assuming Theorem 7.14, prove that the following are equivalent

1. T has elimination of quantifiers
2. every *finite* partial isomorphism between models T is an elementary map.

7.21 Exercise A back-and-forth system between to models M and N is a nonempty set $\mathcal{P}$ of finite functions k such that

0. $k : M \to N$ is a partial embedding

1a. for every $b \in M$ there is $h \in \mathcal{P}$ such that $k \subseteq h$ and $b \in \operatorname{dom} h$

1b. for every $c \in N$ there is $h \in \mathcal{P}$ such that $k \subseteq h$ and $c \in \operatorname{rng} h$.

Prove that if L is countable and there is a back-and-forth system between M and N then there are $M' \preceq M$ and $N' \preceq N$ such that $M' \simeq N'$.

7.3 Weaker notions of universality and homogeneity

We want to extend the equivalence in Theorem 7.8 to λ-rich models. For that we need to weaken the notions of λ-homogeneity. This section is more technical and could be skipped at a first reading.

7.22 Definition We say that a structure N is weakly λ-homogeneous if for every $b \in N$ every morphism $k : N \to N$ of cardinality $< \lambda$ extends to one defined in b. The term back-and-forth λ-homogeneous is also used.

7.23 Lemma (Assume 7.4) Let N be a weakly λ-homogeneous model. Let $A \subseteq N$ have cardinality $\leq \lambda$ and let $k : N \to N$ be a morphism of cardinality $< \lambda$. Then there is a model $M \preceq N$ containing A and an automorphism $h : M \xrightarrow{\sim} M$ that extends k.

Proof. Similar to the proof of Theorem 7.11. We shall construct simultaneously a chain $\langle A_i : i < \lambda \rangle$ of subsets of N and a chain functions $\langle h_i : i < \lambda \rangle$, such that $h_i : N \to N$ are morphisms. In the end we will set

$$M = \bigcup_{i<\lambda} A_i \qquad \text{and} \qquad h = \bigcup_{i<\lambda} h_i$$

The chains start with $A_0 = A \cup \operatorname{dom} k \cup \operatorname{rng} k$ and $h_0 = k$. As usual, at limit stages we take the union. Now we consider successor stages. At stage i we fix some enumerations of A_i and of $L_x(A_i)$, where $|x| = 1$. Let $\langle i_1, i_2 \rangle$ be the i-th pair of ordinals $< \lambda$. If the i_2-th formula in $L_x(A_{i_1})$ is consistent in N, let a be any of its solutions. Also let b be the i_2-th element of A_{i_1}. Let $h_{i+1} : N \to N$ be a minimal morphism that extends h_i and is such that $b \in \operatorname{dom} h_{i+1} \cap \operatorname{rng} h_{i+1}$. Define $A_{i+1} = A_i \cup \{a,\ h_{i+1}b,\ h_{i+1}^{-1}b\}$. □

7.24 Theorem (Assume 7.4) For every model N the following are equivalent

1. N is λ-rich
2. N is λ-universal and weakly λ-homogeneous.

Proof. Implication 1⇒2 is clear. To prove 2⇒1 we generalize the proof of Theorem 7.8. We assume 2 fix some morphism $k : M \to N$ of cardinality $< \lambda$ and let $b \in M$. By λ-universality and the downward Löwenheim-Skolem theorem, there is a morphism $f : M \to N$ with domain of definition $\operatorname{dom} k \cup \{b\}$. The map $f \circ k^{-1} : N \to N$ has cardinality $< \lambda$ and, by Lemma 7.23, it has an extension to an automorphism $h : N' \xrightarrow{\sim} N'$ for some $N' \preceq N$ containing $\operatorname{rng} k \cup \operatorname{rng} f$. Then $h \circ f : M \to N$ extends k and is defined on b. □

7.25 Exercise (Assume 7.4) Prove that any weakly λ-homogeneous structure of cardinality λ is homogeneous.

7.4 The amalgamation property

In this section we discuss conditions that ensure the existence of rich models.

We say that $\mathcal{M}$ has the amalgamation property if for every pair of morphisms $f_1 : M \to M_1$ and $f_2 : M \to M_2$ there are two of embeddings $g_1 : M_1 \hookrightarrow N$ and

$g_2 : M_2 \hookrightarrow N$ such that $g_1 \circ f_1\,(a) = g_2 \circ f_2\,(a)$ for every a in the common domain of definition, $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$.

$$\begin{tikzcd} & M_1 \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "f_1"] \arrow[dr, "f_1"'] & & N \\ & M_2 \arrow[ur, hook, "g_2"'] & \end{tikzcd}$$

As we assume that morphisms are invertible, we may express the amalgamation property in a more concise form. It is convenient to use the following notation. We write $M \leq N$ if $M \subseteq N$ and $\mathrm{id}_M : M \hookrightarrow N$ is a morphism. We say that $k : M \to N$ extends to $g : M \to N'$ if $k \subseteq h$ and $N \leq N'$ namely, if the following diagram commutes

$$\begin{tikzcd} & N \arrow[dr, hook, "\mathrm{id}_N"] & \\ M \arrow[ur, "k"] \arrow[rr, hook, "g"'] & & N' \end{tikzcd}$$

7.26 Proposition Assume c3, then the following are equivalent

1. $\mathcal{M}$ has the amalgamation property
2. every morphism $k : M \to N$ extends to an embedding $g : M \hookrightarrow N'$.

Proof. 1⇒2 Given $k : M \to N$, the amalgamation property yields the following commutative diagram which can be simplified to the diagram at the right

$$\begin{tikzcd} & N \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "k"] \arrow[dr, "\mathrm{id}_M"'] & & N' \\ & M \arrow[ur, hook, "g_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & N \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "k"] \arrow[rr, hook, "g_2"'] & & N' \end{tikzcd}$$

Up to isomorphism we can assume $g_1 = \mathrm{id}_N$, i.e. that $N \leq N'$. Hence $g_2 : M \hookrightarrow N'$ is the required extension of $k : M \to N$.

2⇒1 Let $f_1 : M \to M_1$ and $f_2 : M \to M_2$ be given. Let $k = f_2 \circ f_1^{-1} : M_1 \to M_2$ and let $g : M_1 \hookrightarrow N$ be the extension ensured by 2. Then we obtain

$$\begin{tikzcd} & M_1 \arrow[dr, hook, "g"] \arrow[dd, "f_2 \circ f_1^{-1}"] & \\ M \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & N \\ & M_2 \arrow[ur, hook, "\mathrm{id}_{M_2}"'] & \end{tikzcd}$$

as required. □

We say that $\langle M_i : i < \lambda \rangle$ is a $\leq$-chain if $M_i \leq M_j$ for all $i < j < \lambda$. For the next theorem to hold we need the following property

7.27 Definition We say that $\mathcal{M}$ is closed under union of $\leq$-chains if

c8. if $\langle M_i : i < \lambda \rangle$ is a $\leq$-chain, then $M_i \leq \bigcup_{j<\lambda} M_j$ for all $i < \lambda$.

The following is a general existence theorem for rich models. This general form requires large cardinalities. We leave to the reader to verify that if the number if finite morphisms is countable (up to isomorphism) then countable rich models exit.

7.28 Theorem Assume 7.4. Assume further c8 and that $\mathcal{M}$ has the amalgamation property. Let λ be such that $|L| < \lambda = \lambda^{<\lambda}$. Then there is a rich model N of cardinality λ.

Proof. We construct N as union of a $\leq$-chain of models $\langle N_i : i < \lambda \rangle$ such that $|N_i| = \lambda$. Let N_0 be any model of cardinality λ. At stage $i+1$, let $f : M \to N_i$ be the least morphism (in a well-ordering that we specify below) such that $|f| \leq |M| < \lambda$ and f has no extension to an embedding $f' : M \hookrightarrow N_i$. Apply the amalgamation property to obtain an total morphism $f' : M \hookrightarrow N'$ that extends $f : M \to N_i$. By the downword Löwenheim-Skolem Theorem we may assume $|N'| = \lambda$. Let $N_{i+1} = N'$. At limit stages take the union.

The well-ordering mentioned needs to be chosen so that in the end we forget nobody. So, first at each stage we well-order the isomorphism-type of the morphisms $f : M \to N_i$ such that $f \leq |M| < \lambda$. Then the required well-ordering is obtained by dovetailing all these well-orderings. The length of this enumeration is at most $\lambda^{<\lambda}$, which is λ by hypothesis.

We check that N is rich. Let $f : M \to N$ be a morphism and $|f| < |M| \leq \lambda$. As $|L| < \lambda$ we can approximate M with an elementary chain of structures of cardinality $< \lambda$. Hence we may as well assume that $|f| \leq |M| < \lambda$. The cofinality of λ is larger than $|f|$, hence $\operatorname{rng} f \subseteq N_i$ for some $i < \lambda$. So $f : M \to N_i$ is a morphism and at some stage j we have ensured the existence of an embedding of $f' : M \hookrightarrow N_{j+1}$ that extends f. □

7.29 Proposition Let $\mathcal{M}$ consist of all structures of some fixed signature and the elementary maps between these. Then $\mathcal{M}$ has the amalgamation property.

Proof. Let $k : M \to N$ be an elementary map. Let a enumerate $\operatorname{dom} k$ and let b enumerate M. Let $p(x\,;z) = \operatorname{tp}_M(b\,;a)$. The type $p(x\,;a)$ is consisent in M, in particular, it is finitely consistent and, by elementarity, $p(x\,;ka)$ is finitely consistent in N. By the compactness theorem, there is $N' \succeq N$ such that $N' \vDash p(c\,;ka)$ for some $c \in N'^{x}$. Hence $g = \{\langle b, c \rangle\} : M \to N'$ is the required elementary map that extends $k : M \to N$. □

Chapter 8

Some algebraic structures

The main result in this chapter is the elimination of quantifiers in algebraically closed fields, Corollary 8.22. In commutative algebra this is called Chevalley's Theorem on constructible sets. From it we derive Hilbert's Nullstellensatz. First we state the theorem the model theoretic language, Theorem 8.27, then we tranlate it in the traditional algebraic setting, Theorem 8.31. To obtain the latter theorem we introduce the model theoretic objects that correspond to prime and radical ideals of polynomials.

The first sections of this chapter are not a pre-requisite for Sections 8.4–8.7, at the cost of a few repetitions in the latter.

8.1 Notation Recall that when $A \subseteq M$ we denote by $\langle A\rangle_M$ the substructure of M generated by A. Then $\langle A\rangle_M \subseteq N$ is equivalent to $N \vDash \mathrm{Diag}\,\langle A\rangle_M$. The diagram of a structure has been defined in Notation 3.20.

In this chapter, whenever some $A \subseteq M \vDash T$ are fixed, by *model* we mean superstructures of $\langle A\rangle_M$ that models T. The notions of *logical consequence*, *consistency*, *completeness*, etc. are modified accordingly, and we write $\vdash$ for $T \cup \mathrm{Diag}\langle A\rangle_M \vdash$.

We say that a type $p(x)$ is trivial if $\vdash p(x)$.

8.1 Abelian groups

The language L is that of additive groups. The theory T_{ag} of abelian groups is axiomatized by the universal closure of the usual axioms

a1 $(x+y)+z \;=\; y+(x+z)$

a2 $x+(-x) \;=\; (-x)+x \;=\; 0$

a3 $x+0 \;=\; 0+x \;=\; x$

a4 $x+y \;=\; y+x.$

Let x be a tuple of variables of length α, an ordinal. We write $L_{\mathrm{ter},x}$ for the set of terms $t(x)$ with free variables among x. On this set we define the equivalence relation

$$t(x) \sim s(x) \quad = \quad T_{\mathrm{ag}} \vdash t(x) = s(x).$$

We define the group operations on $L_{\mathrm{ter},x}/\!\sim$ in the obvious way. We denote by $\mathbb{Z}^{\oplus\alpha}$ the set of tuples of integers of length α that are almost always 0. The group operations on $\mathbb{Z}^{\oplus\alpha}$ are defined coordinate-wise. The following immediate proposition implies in particular that $L_{\mathrm{ter},x}/\!\sim$ is isomorphic to $\mathbb{Z}^{\oplus\alpha}$.

8.2 Proposition Let $A \subseteq M \vDash T_{\rm ag}$. Then for every formula $\varphi(x) \in L_{\rm at}(A)$ there are $n \in \mathbb{Z}^{\oplus\alpha}$ and $c \in \langle A\rangle_M$ such that

$$\vdash \quad \varphi(x) \quad \leftrightarrow \quad \sum_{i<\alpha} n_i x_i \;=\; c$$

where $n = \langle n_i : i < \alpha\rangle$ and $x = \langle x_i : i < \alpha\rangle$.

Proof. Up to equivalence over $T_{\rm ag}$ the formula $\varphi(x)$ has the form $s(x) = t(a)$ for some parameter-free terms $s(x)$ and $t(z)$. Over $\mathrm{Diag}\,\langle A\rangle_M$, we can replace $t(a)$ with a single $c \in \langle A\rangle_M$ and write $s(x)$ as the linear combination shown above. □

8.3 Definition Let $M \vDash T_{\rm ag}$. For $A \subseteq M$ and $c \in M$, we say that c is independent from A if $\langle A\rangle_M \cap \langle c\rangle_M = \{0\}$. Otherwise we say that c is dependent from A. The rank of M is the least cardinality of a subset $A \subseteq M$ such that all elements in M are dependent from A. We denote it by $\mathrm{rank}(M)$.

Note that when M is a vector space the condition $\langle A\rangle_M \cap \langle c\rangle_M = \{0\}$ is equivalent to saying that c is not a linear combination of vectors in A. Then $\mathrm{rank}\, M$ coincides with the dimension of M. In fact, what we do here for abelian groups could be easily generalized to D-modules, where D is any integral domain, and in particular to vector spaces. In practice, it is more convenient to use the following syntactic characterization of independence.

An element c of an abelian group is a torsion element if $nc = 0$ for some positive integer.

8.4 Proposition Let $A \subseteq M \vDash T_{\rm ag}$. Suppose that $c \in M$ is not a torsion element. Then the following are equivalent

1. c is independent from A
2. $p(x) = \text{at-tp}_M(c/A)$ is trivial (see Notation 8.1).

Proof. 1⇒2 By Proposition 8.2, formulas in $p(x)$ may be assumed to have the form $nx = a$ for some integer n and some $a \in \langle A\rangle_M$. As this formula is satisfied by c then $a \in \langle A\rangle_M \cap \langle c\rangle_M$. Hence $a = 0$. As c is not a torsion element, $n = 0$ and the equation is trivial.

2⇒1 If $\langle A\rangle_M \cap \langle c\rangle_M \neq \{0\}$ then $nc = a$ for some $a \in \langle A\rangle_M \smallsetminus \{0\}$ and some $n \in \mathbb{Z} \smallsetminus \{0\}$. Then c satisfies the equation $nx = a$. This equation is nontrivial (e.g., in M it is not satisfied by $2c$). □

8.5 Remark Let $k : M \to N$ be a partial embedding and let a be an enumeration of $\operatorname{dom} k$. We claim that $k \cup \{\langle b, c\rangle\} : M \to N$ is a partial embedding for every $b \in M$ and $c \in N$ that are independent from a, respectively ka. In fact, it suffices to check that $M, b, a \equiv_{\rm at} N, c, ka$. Suppose $\varphi(x\,;z) \in L_{\rm at}$ is such that $M \vDash \varphi(b\,;a)$. Then by independence $\varphi(x\,;a)$ is trivial, i.e.

$$T_{\rm ag} \cup \operatorname{Diag}\langle a\rangle_M \quad \vdash \quad \varphi(x\,;a).$$

As $\langle a\rangle_M$ and $\langle ka\rangle_N$ are isomorphic structures

$$T_{\rm ag} \cup \operatorname{Diag}\langle ka\rangle_N \quad \vdash \quad \varphi(x\,;ka).$$

Therefore $N \vDash \varphi(c\,;ka)$. This proves $M, b, a \Rrightarrow_{\rm at} N, c, ka$.

As the same assumptions apply to $k^{-1} : N \to M$, we also have $M, b, a \Lleftarrow_{\rm at} N, c, ka$.

8.2 Torsion-free abelian groups

The theory of torsion-free abelian groups extends $T_{\rm ag}$ with the following axioms for all positive integers n

tf $\quad nx = 0 \to x = 0.$

We denote this theory by $T_{\rm tfag}$. It is not difficult to see that in a torsion-free abelian group every equation of the form $nx = a$ has at most one solution.

8.6 Proposition Let $M \vDash T_{\rm tfag}$ be uncountable. Then $\operatorname{rank} M = |M|$.

Proof. Let $A \subseteq M$ have cardinality $< |M|$. We claim that M contains some element that is independent from A. It suffices to show that the number of elements that are dependent from A is $< |M|$. If $c \in M$ is dependent from A then, by Proposition 8.7, it is a solution of some formula $L_{\rm at}(A)$. As there is no torsion, such a formula has at most one solution. Therefore the number of elements that are dependent from A is at most $|L_{\rm at}(A)|$, that is $\max\{|A|, \omega\}$. If M is uncountable the claim follows. $\square$

8.7 Proposition Let $A \subseteq M \vDash T_{\rm tfag}$. Let $p(x) = \text{at}^{\pm}\text{-tp}(b/A)$, where $b \in M$. Then one of the following holds

1. b is independent from A
2. $M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\rm at}(A)$ such that $\vdash \varphi(x) \to p(x)$.

Note the similarity with Example 7.15, where the independent type is $q(x)$ and the isolating formulas are the $r_i(x)$.

It is important to observe that the set A above may be infinite. This is essential to obtain Corollary 8.11, and it is one of the main differences between this example and the examples encountered in Chapter 6.

Proof. If b is dependent from A, then b satisfies a non trivial atomic formula $\varphi(x)$ which we claim is the formula required in 2. It suffices to show that $\varphi(x)$ implies a complete $L_{\rm at^{\pm}}(A)$-type. Clearly this type must be $p(x)$. Let $\varphi(x)$ have the form $nx = a$ for some $n \in \mathbb{Z} \smallsetminus \{0\}$ and $a \in \langle A\rangle_M \smallsetminus \{0\}$. We show that for every $m \in \mathbb{Z}$ and every $c \in \langle A\rangle_M$ one of the following holds

a. $\vdash\ nx = a \to mx = c$

b. $\vdash\ nx = a \to mx \neq c$.

Suppose not for a contradiction that neither a nor b holds and fix models N_1, N_2 and some $b_i \in N_i$ such that

a′. $N_1 \vDash nb_1 = a$ and $N_1 \vDash mb_1 \neq c$

b′. $N_2 \vDash nb_2 = a$ and $N_2 \vDash mb_2 = c$.

From b′ we infer that $N_2 \vDash ma = nc$. As N_1 is torsion-free, from a' we infer that $N_1 \vDash ma \neq nc$. But $ma = nc$ is a formula with parameters in $\langle A\rangle_M$, so it should have the same truth value in all superstructures of $\langle A\rangle_M$, a contradiction. □

8.3 Divisible abelian groups

The theory of divisible abelian groups extends $T_{\rm tfag}$ with the following axioms for all integers $n \neq 0$

div $y \neq 0 \to \exists x\ nx = y$.

We denote this theory by $T_{\rm dag}$.

8.8 Proposition Let $A \subseteq M \vDash T_{\rm tfag}$ and let $\varphi(x) \in L_{\rm at}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\, \varphi(x)$ for every N such that $\langle A\rangle_M \subseteq N \vDash T_{\rm dag}$.

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A\rangle_M \subseteq M' \vDash T_{\rm tfag}$. The proposition holds more generally for all $\varphi(x) \in L_{\rm qf}$ and also when x is a tuple of variables. This follows from Lemma 8.10, whose proof uses the proposition.

Proof. We can assume that $\varphi(x)$ has the form $nx = a$ for some $n \in \mathbb{Z}$ and some $a \in \langle A\rangle_M$. If $n = 0$, then $a = 0$ since $\varphi(x)$ is consistent, and the claim is trivial. If $n \neq 0$ then by consistency $a \neq 0$, hence a solution exist in N by axiom div. □

8.9 Exercise Prove a converse of Proposition 8.8. Let $A \subseteq N \vDash T_{\rm ag}$ and let x be a single variable. Prove that if $N \vDash \exists x\, \varphi(x)$ for every consistent $\varphi(x) \in L_{\rm at}(A)$, then $N \vDash T_{\rm dag}$.

We are ready to prove that divisible abelian groups of infinite rank are ω-rich.

8.10 Lemma Let $k : M \to N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\rm tfag}$ and $N \vDash T_{\rm dag}$ is a model of rank $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \{\langle b\,;c\rangle\} : M \to N$ is a partial isomorphism.

Proof. Let a be an enumeration of $\operatorname{dom} k$ and let $p(x\,;z) = \text{at}^{\pm}\text{-tp}(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is dependent from a, then Proposition 8.7 yields a formula $\varphi(x\,;z) \in L_{\rm at}$ such that

i. $\vdash\ \varphi(x\,;a) \to p(x\,;a)$

ii. $\varphi(x\,;a)$ is consistent.

By isomorphism, i and ii hold with a replaced by ka. Then by Proposition 8.8 the formula $\varphi(x\,;ka)$ has a solution $c \in N$.

The second case, which has no analogue in Lemma 6.1, is when b is independent from a. Then by Remark 8.5 we may choose c to be any element of N independent from ka. Such an element exists because N has rank at least λ. □

Below a few important consequences of this lemma.

8.11 Corollary Work in the category of models of $T_{\rm tfag}$ with partial embeddings as morphisms. Then the following are equivalent

1. N is a λ-rich model
2. $N \vDash T_{\rm dag}$ and has rank $\geq \lambda$.

In particular every uncountable $N \vDash T_{\rm dag}$ is rich.

8.12 Corollary The theory $T_{\rm dag}$ is uncountably categorical, complete, and has quantifier elimination.

Proof. Categoricity and completeness ar immediate (the category presented in the corollary above is connected). As for quantifier elimination, let $k : M \to N$ is a partial isomorphism between models of $T_{\rm dag}$. If $M \preceq M'$ and $M \preceq N'$ are elementary superstructures of uncountable cardinality then $k : M' \to N'$ is elementary by Theorem 7.11 and this suffices to conclude that $k : M \to N$ is elementary. □

8.13 Exercise Prove that every model of $T_{\rm dag}$ is ω-ultrahomogeneous (independently of cardinality and rank).

8.4 Commutative rings

In this section L is the language of (unital) rings. It contains two constants 0 and 1 the unary operation $-$ and two binary operations $+$ and $\cdot$. The theory of rings contains the following axioms

a1-a4. as for abelian groups

r1. $(x\cdot y)\cdot z \;=\; y\cdot(x\cdot z)$

r2. $1\cdot x \;=\; x\cdot 1 \;=\; x$

r3. $(x+y)\cdot z \;=\; x\cdot z + y\cdot z$

r4. $z\cdot(x+y) \;=\; z\cdot x + z\cdot y$.

All the rings we consider are commutative

c. $x\cdot y \;=\; y\cdot x$.

We denote the theory of commutative rings by $T_{\rm cr}$.

In what follows the theory $T_{\rm cr} \cup {\rm Diag}\langle A\rangle_M$, for some M clear from the context, is implicit in the sense of Notation 8.1. So it is important to remember that ${\rm Diag}\langle A\rangle_M$ is not trivial even when $A = \varnothing$. In fact, ${\rm Diag}\langle\varnothing\rangle_M$ determines the characteristic of the models.

Let $A \subseteq M \vDash T_{\rm cr}$ and let x be a tuple of variables. We write $L_{{\rm ter},x}(A)$ for the set of terms $t(x)$ with free variables among x and parameters in A. On this set we define the equivalence relation

$$t(x) \sim s(x) \quad \Leftrightarrow \quad \vdash\ t(x) = s(x).$$

On $L_{\mathrm{ter},x}(A)/\!\sim$ we define the ring operations in the obvious way so that $L_{\mathrm{ter},x}(A)/\!\sim$ is a commutative ring. We denote by $A[x]$ the set of polynomials with variables among x and parameters in $\langle A\rangle_M$. The ring operations on $A[x]$ are defined as usual. The following proposition (which is clear, but tedious to prove) implies in particular that $L_{\mathrm{ter},x}(A)/\!\sim$ is isomorphic to $A[x]$. For simplicity we state it only for $|x| = 1$.

8.14 Proposition Let $A \subseteq M \vDash T_{\mathrm{cr}}$ and let x be a single variable. Then for every formula $\varphi(x) \in L_{\mathrm{at}}(A)$ there is a unique $n < \omega$ and a unique tuple $\langle a_i : i \leq n\rangle$ of elements of $\langle A\rangle_M$ such that $a_n \neq 0$ and

$$\vdash \quad \varphi(x) \ \leftrightarrow \ \sum_{i \leq n} a_i x^i \ = \ 0.$$

The integer n in the proposition above is called the degree of $\varphi(x)$.

8.15 Definition Let $A \subseteq M \vDash T_{\mathrm{cr}}$. An element $b \in M$ is transcendental over A if the type $p(x) = \text{at-tp}_M(b/A)$ is tr We say that aivial (see Notation 3.20 and 8.1). Otherwise we say that b is algebraic over A. The transcendence degree of M is the least cardinality of a subset $A \subseteq M$ such that all the elements of M are algebraic over A.

8.16 Remark Remark 8.5 holds here with 'independent' replaced by 'transcendental' and T_{ag} replaced by T_{cr}.

8.5 Integral domains

Let $a \in M \vDash T$. We say that a is a zero divisor if $a\,b = 0$ for some $b \in M \smallsetminus \{0\}$. An integral domain is a commutative ring without zero divisors. The theory of integral domains contains the axioms of commutative rings and the following

nt. $0 \neq 1$

id. $x{\cdot}y = 0 \ \rightarrow \ x = 0 \ \vee \ y = 0.$

We denote the theory of integral domains by T_{id}.

For a prime p, we define the theory T^p_{id}, which contains T_{id} and the axiom

ch$_p$. $1 + \cdots (p \text{ times}) \cdots + 1 = 0.$

The theory T^0_{id} contains the negation of ch$_p$ for all p. Note that all models of T^p_{id} have the same characteristic in the model theoretic sense defined in 3.28. In the remaining section we work in the category of models of T_{id} with partial embeddings as morphisms. This category consists of countably many connected components each containing all models of T^p_{id} for some p.

8.17 Proposition Let $M \vDash T_{\text{id}}$ be uncountable. Then M has transcendence degree $|M|$.

Proof. In an integral domain every polynomial has finitely many solutions and there are $|L(A)|$ polynomials over A. □

8.18 Proposition Let $A \subseteq M \vDash T_{\text{id}}$. For $b \in M$ let $p(x) = \text{at}^{\pm}\text{-tp}(b/A)$. Then one of the following holds

1. b is transcendental over A
2. $M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\text{at}}(A)$ such that $\vdash \varphi(x) \to p(x)$.

Note the similarity with Example 7.15, where the transcendental type is $q(x)$ and the isolating formulas are the $r_i(x)$.

As in Proposition 8.7, the set A may be infinite. This is essential to obtain Corollary 8.21.

Proof. Suppose b is not transcendental, i.e. it satisfies a non trivial atomic formula. Let $\varphi(x) \in L_{\text{at}}(A)$ be a non trivial formula with minimal degree such that $\varphi(b)$. We prove that $\varphi(x)$ implies a complete $L_{\text{at}^{\pm}}(A)$-type. Clearly this type must be $p(x)$. We prove that for any $\xi(x) \in L_{\text{at}}(A)$ one of the following holds

1. $\vdash \varphi(x) \to \xi(x)$
2. $\vdash \varphi(x) \to \neg\xi(x)$.

Let us write $a(x) = 0$ and $a'(x) = 0$ for the formulas $\varphi(x)$ and $\xi(x)$, respectively. If $\langle A\rangle_M$ is a field, choose a polynomial $d(x)$ of maximal degree such that for some polynomials $t(x)$ and $t'(x)$ the following hold

a. $d(x)\,t(x) = a(x)$
a'. $d(x)\,t'(x) = a'(x)$.

If $\langle A\rangle_M$ is not a field, polynomials $d(x)$, $t(x)$ and $t'(x)$ as above exist with coefficients in the field of fractions of $\langle A\rangle_M$. Then a and a' hold up to a factor in $\langle A\rangle_M$ which we absorb in $a(x)$ and $a'(x)$.

From a we get $d(b) = 0$ or $t(b) = 0$. In the first case, as $a(x)$ has minimal degree, we conclude that $t(x)$ is constant. This implies that any zero of $a(x)$ is also a zero of $a'(x)$, that is, it implies 1.

Now suppose $t(b) = 0$. Then the minimality of the degree of $a(x)$ implies that $d(x) = d$, where d is a nonzero constant. If $\langle A\rangle_M$ is a field, apply Bézout's identity to obtain two polynomials $c(x)$ and $c'(x)$ such that $d = a(x)c(x) + a'(x)c'(x)$. Then $a(x)$ and $a'(x)$ have no common zeros, and 2 follows. If $\langle A\rangle_M$ is not a field, we use Bézout's identity in the field of fractions of $\langle A\rangle_M$ and, for some $d' \in \langle A\rangle_M \smallsetminus \{0\}$, obtain $d'd = a(x)c(x) + a'(x)c'(x)$. Then we reach the same conclusion. □

8.6 Algebraically closed fields

Let $a, b \in M \vDash T_{\text{id}}$. We say that b is the inverse of a if $a{\cdot}b = 1$. A field is a commutative ring where every non-zero element has an inverse. The theory of fields contains T_{id} and the axiom

f. $\exists y\,\big[x \neq 0 \ \to\ x{\cdot}y = 1\big]$.

Fields are structures in the signature of rings: the language contains no symbol for the multiplicative inverse. So, substructures of fields are merely integral domains.

The theory of algebraically closed field, which we denote by $T_{\rm acf}$, also contains the following axioms for every positive integer n

ac. $\exists x\ \big(x^n + z_{n-1}x^{n-1} + \cdots + z_1 x + z_0\ =\ 0\big)$.

The theory $T^p_{\rm acf}$ is defined in analogy to $T^p_{\rm id}$ in the previous section.

8.19 Proposition Let $A \subseteq M \vDash T_{\rm id}$ and let $\varphi(x) \in L_{\rm at}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\, \varphi(x)$ for every model $N \vDash T_{\rm acf}$.

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A\rangle_M \subseteq M' \vDash T_{\rm id}$. The claim in the proposition holds more generally for all $\varphi(x) \in L_{\rm qf}$ when x is a tuple of variables. This follows from Lemma 8.20 whose proof uses the proposition.

Proof. Up to equivalence $\varphi(x)$ has the form $a_n x^n + \cdots + a_1 x + a_0\ = 0$ for some $a_i \in \langle A\rangle_N$. Choose n minimal. If $n = 0$ then $a_0 = 0$ by the consistency of $\varphi(x)$ and the claim is trivial. Otherwise $a_n \neq 0$ and the claim follows from f and ac. □

8.20 Lemma Let $k : M \to N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\rm id}$ and $N \vDash T_{\rm acf}$ has transcendence degree $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \big\{\langle b, c\rangle\big\} : M \to N$ is a partial isomorphism.

The following is the same proof given for Lemma 8.10 which we repeat here for convenience.

Proof. Let a be an enumeration of $\operatorname{dom} k$ and let $p(x\,;z)\ =\ \text{at}^{\pm}\text{-tp}_M(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is algebraic over a, then Proposition 8.18 yields a formula $\varphi(x\,;z) \in L_{\rm at}$ such that

i. $\quad \varphi(x\,;a) \to p(x\,;a)$

ii. $\quad \varphi(x\,;a)$ is consistent.

By isomorphism i and ii hold with a replaced by ka. Then by Proposition 8.19 the formula $\varphi(x\,;ka)$ has a solution in $c \in N$.

The second case, which has no analogue in Lemma 6.1, is when b is transcendental over a. Then by Remark 8.16 we may choose c to be any element of N transcendental over ka. This exists because N has transcendence degree $\geq \lambda$. □

Below a few important consequences of this lemma.

8.21 Corollary Work in the category of models of $T_{\rm id}$ with partial embeddings as morphisms. Then the following are equivalent

1. N is a λ-rich model
2. $N \vDash T_{\rm acf}$ and has transcendence degree $\geq \lambda$.

In particular every uncountable $N \vDash T_{\rm acf}$ is rich.

Proof. Implication 2 ⇒ 1 is an immediate consequence of Lemma 8.20.

In every connected component there is an $M \vDash T_{\rm acf}$ of cardinality λ and transcendence degree λ (by Proposition 8.17 when $\lambda > \omega$, by compactness for $\lambda = \omega$). As proved above, M is rich and therefore elementarily equivalent to any λ-rich model N in the same connected component. This proves 1 ⇒ 2. □

8.22 Corollary The theory $T_{\rm acf}$ has elimination of quantifiers.

Proof. Let $k : M \to N$ be a partial embedding between models of $T_{\rm acf}$. Let M' and N' be elementary superstructures of M and N respectively of sufficiently large cardinality. As M' and N' are rich, $k : M' \to N'$ is elementary by Theorem 7.11. Hence $k : M \to N$ is also elementary. □

8.23 Corollary The theories $T^p_{\rm acf}$ are complete and uncountably categorical (i.e. λ-categorical for every uncountable λ).

Proof. Two models of $T^p_{\rm acf}$ belong to the same connected component. Then, as every uncountable model of $T^p_{\rm acf}$ is rich, uncountable categoricity and completeness follow. □

8.24 Exercise Prove that every model of $T_{\rm acf}$ is ω-ultrahomogeneous (independently of cardinality and transcendence degree).

8.7 Hilbert's Nullstellensatz

Fix a tuple of variables x and a subset A of an integral domain M. In this section we are interested in formulas in $L_{\rm at}(A)$, hence we redefine the symbol of closure under logical consequence accordingly

$$\mathrm{ccl}\, p(x) \quad = \quad \Big\{\varphi(x) \in L_{\rm at}(A) \ : \ p(x) \vdash \varphi(x)\Big\}.$$

Recall that we work under the assumptions in Notation 8.1. In particular, in this section we work over the theory $T_{\rm id} \cup \mathrm{Diag}\langle A\rangle_M$, and the symbol $\vdash$ has to be interpreted accordingly.

In general, the notion of closure under logical consequence is elusive. In this respect Propositions 8.25 and 8.27 are useful, as they give a model theoretic characterization of $\mathrm{ccl}\, p(x)$. Corollary 8.30 gives an algebraic characterization.

8.25 Proposition Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x) \subseteq L_{\rm at}(A)$. Let N be of sufficiently large cardinality and such that $\langle A\rangle_M \subseteq N \vDash T_{\rm acf}$. Then

$$\mathrm{ccl}\, p(x) \quad = \quad \Big\{\varphi(x) \in L_{\rm at}(A) \ : \ N \vDash \forall x \, \big[p(x) \to \varphi(x)\big]\Big\}.$$

The cardinality of N is sufficiently large if $|L(A)| < |N|$ and $|x| \leq |N|$; note that here x has possibly infinite length.

Proof. Only the inclusion $\supseteq$ requires a proof. Suppose $\varphi(x) \in L_{\rm at}(A)$ is such that $p(x) \wedge \neg\varphi(x)$ is consistent. Then there is a model M_1 of cardinality $< |L(A)|$ and

$\leq |x|$ such that $M_1 \vDash p(a) \wedge \neg\varphi(a)$ for some $a \in M_1^x$. By Corollary 8.21, if N is large enough, there is a partial isomorphism $h : M_1 \to N$ that extends id_A and is defined on a. Therefore $N \vDash p(ha) \wedge \neg\varphi(ha)$. So $\varphi(x)$ does not belong to the set on the r.h.s. □

In the proposition above we could replace L_{at} by $L_{\mathrm{at}^{\pm}}$. But here we are interesed in a strenghening, the Nullstennensatz, that requires positive formulas.

Hilbert's Nullstellensatz extends the validity of the proposition above to the case $A = M = N$ (assuming x finite). While Proposition 8.25 could be generalized to other theories, the Nullstellensatz rests on an exquisitely algebraic phenomenon. In fact, model theory has no general tools to deal with large sets of parameters. Algebra comes to our aid with the following lemma. Which the reader may wish to compare with Hilbert's Basis Theorem.

8.26 Lemma Let $M \vDash T_{\mathrm{id}}$. Let $p(x) \subseteq L_{\mathrm{at}}(M)$, where x is finite, be closed under logical consequence. Then $\psi(x) \vdash p(x)$, for some $\psi(x)$ conjunction of formulas in $p(x)$.

Proof. By induction on the length of x. If x is the empty tuple, the lemma is trivial. Now, let x be a finite tuple and let y be a single variable. Let $p(x,y) \subseteq L_{\mathrm{at}}(M)$ be closed under logical consequence. Write $p(x)$ for the set of formulas $\varphi(x) \in L_{\mathrm{at}}(M)$ such that $p(x,y) \vdash \varphi(x)$. Assume as induction hypothesis that there is a conjunction of formulas in $p(x)$, say $\psi(x)$, such that $\psi(x) \vdash p(x)$. We prove the lemma with x, y for x.

Let $\psi(x,y) \in L_{\mathrm{at}}(M)$ have minimal degree (in y) among the formulas in $p(x,y)$ such that $\psi(x) \nvdash \psi(x,y)$. Let n be the degree of $\psi(x,y)$, which is non zero, because $\psi(x) \nvdash \psi(x,y)$. So we may write

$$\psi(x,y) \quad = \quad \Big(t_n(x) \cdot y^n + t'(x,y) = 0\Big),$$

where $t'(x,y)$ is a polynomial of degree $< n$. We claim that

1 $\qquad \psi(x) \wedge \psi(x,y) \quad \vdash \quad p(x,y)$

Suppose not for a contradiction. Pick among the formulas in $p(x,y)$ that are a counterexample to 1, one of minimal degree, say $\varphi(x,y)$. By the minimaly of n, the degree of $\varphi(x,y)$ is $n+i$ for some $i \geq 0$. Hence we may write

$$\varphi(x,y) \quad = \quad \Big(s_{n+i}(x) \cdot y^{n+i} + s'(x,y) = 0\Big),$$

for some polynomial $s'(x,y)$ of degree $< n+i$. From $p(x,y) \vdash \varphi(x,y)$ we obtain

$$p(x,y) \quad \vdash \quad \Big(s_{n+i}(x) \cdot t_n(x) \cdot y^{n+i} + t_n(x) \cdot s'(x,y) = 0\Big).$$

Now, using that $p(x,y) \vdash \psi(x,y)$

2. $\qquad p(x,y) \quad \vdash \quad \Big(-s_{n+i}(x) \cdot t'(x,y) \cdot y^i + t_n(x) \cdot s'(x,y) = 0\Big).$

But the polynomial in 2 has degree $< n+i$ and this contradicts the minimality of the degree of $\varphi(x,y)$. A contradicion that proves the proposition. □

8.27 Hilbert's Nullstellensatz (model theoretic version) Let $N \vDash T_{\text{acf}}$. For every type $p(x) \subseteq L_{\text{at}}(N)$, where x is a finite tuple, we have

$$\operatorname{ccl} p(x) \quad = \quad \Big\{\varphi(x) \in L_{\text{at}}(N) \;:\; N \vDash \forall x\, [p(x) \to \varphi(x)]\Big\}.$$

Proof. By Proposition 8.25, the equality above holds if we replace N by a sufficiently large elementay extension N', i.e.

$$= \quad \Big\{\varphi(x) \in L_{\text{at}}(N) \;:\; N' \vDash \forall x\, [p(x) \to \varphi(x)]\Big\}.$$

Since $p(x)$ and $\operatorname{ccl} p(x)$ are logically equivalent, we can replace one by the other. Then, by Lemma 8.26, we can replace $\operatorname{ccl} p(x)$ by an equivalent formula $\psi(x)$

$$= \quad \Big\{\varphi(x) \in L_{\text{at}}(N) \;:\; N' \vDash \forall x\, [\psi(x) \to \varphi(x)]\Big\}.$$

Now, by elementarity, we replace N back

$$= \quad \Big\{\varphi(x) \in L_{\text{at}}(N) \;:\; N \vDash \forall x\, [\psi(x) \to \varphi(x)]\Big\}.$$

Finally, we replace $p(x)$ back and obtain the desired equality. □

In the rest of this section, we show how to translate the model theoretic version of the Nullstellensatz into a more common algebraic formulation.

Let $A[x]$ be the ring of polynomials with variables in x and coefficients in $\langle A\rangle_M$. We identify $L_{\text{at}}(A)$ and $A[x]$ in the obvious way. A type $p(x) \subseteq L_{\text{at}}(A)$ is identified with a set $\mathcal{P}_p \subseteq A[x]$. Conversely, for any subset $\mathcal{P} \subseteq A[x]$ we write $p_{\mathcal{P}}(x)$ for the associated type.

We would like to characterize $\operatorname{ccl} p(x)$ in algebraic terms. The following is a preliminary result.

8.28 Proposition Let $A \subseteq M \vDash T_{\text{id}}$ and let $p(x) \subseteq L_{\text{at}}(A)$. The following are equivalent

1. $p(x)$ is a prime type
2. $\mathcal{P}_p$ is a prime ideal.

Proof. 1⇒2. Recall that, by our definition, a prime type is closed under logical consequence. Therefore to prove that $\mathcal{P}_p$ is an ideal, it suffices to note that the following entailments hold for every pair of polynomials $t(x)$ and $s(x)$

$$\begin{aligned} t(x) = 0 \;&\vdash\; s(x)t(x) = 0 \\ s(x) = t(x) = 0 \;&\vdash\; s(x) + t(x) = 0 \end{aligned}$$

Finally, to prove that $\mathcal{P}_p$ is prime, suppose that the polynomial $t(x) \cdot s(x)$ belongs to $\mathcal{P}_p$. As we are working over T_{id}

$$t(x) \cdot s(x) = 0 \;\vdash\; t(x) = 0 \vee s(x) = 0$$

If $p(x)$ is a prime type, $p(x) \vdash t(x) = 0$ or $p(x) \vdash s(x) = 0$. As $p(x)$ is closed under logical consequence, $t(x) \in \mathcal{P}_p$ or $s(x) \in \mathcal{P}_p$.

2⇒1. First, we prove that $p(x)$ is closed under logical consequence. Equivalently, we assume that $t(x) = 0 \notin p(x)$ and prove that $p(x) \wedge t(x) \neq 0$ is consistent. As $\mathcal{P}_p$ is a prime ideal, $A[x]/\mathcal{P}_p$ is integral domain. We denote by $x + \mathcal{P}_p$ the equivalence class of the polynomial $x \in A[x]$. Then $x + \mathcal{P}_p$ satisfies exactly the formulas in $p(x)$. Hence it witnesses the consistency of $p(x) \wedge t(x) \neq 0$ in $A[x]/\mathcal{P}_p$.

We now prove that $p(x)$ is prime. Let $t_i(x)$ be polynomials such that

$$p(x) \;\vdash\; \bigvee_{i=1}^{n} t_i(x) = 0.$$

Then

$$p(x) \;\vdash\; \prod_{i=1}^{n} t_i(x) \;=\; 0$$

Since $p(x)$ is closed under logical consequence, and $\mathcal{P}_p$ is a prime ideal, $t_i(x) \in \mathcal{P}_p$ for some i. Hence $p(x)$ contains the equation $t_i(x) = 0$. By Corollary 3.19 this suffices to prove that $p(x)$ is a prime type. □

8.29 Proposition Let $A \subseteq M \vDash T_{\text{id}}$ and let $p(x) \subseteq L_{\text{at}}(A)$. Then the following are equivalent for every $\varphi(x) \in L_{\text{at}}(A)$

1. $p(x) \vdash \varphi(x)$
2. $q(x) \vdash \varphi(x)$ for every prime type $q(x) \subseteq L_{\text{at}}(A)$ containing $p(x)$.

Proof. Only 2⇒1 requires a proof. Assume that $p(x) \nvdash \varphi(x)$. Then there is an N such that $\langle A \rangle_M \subseteq N \vDash T_{\text{id}}$ and $N \vDash p(a) \wedge \neg\varphi(a)$ for some $a \in N^x$. Let $q(x) =$ at-tp$_N(a/A)$. Then $q(x)$ is prime and $\varphi(x) \notin q(x)$. □

For any subset $\mathcal{P} \subseteq A[x]$ we write $\sqrt{\mathcal{P}}$ for the intersection of all prime ideals containing $\mathcal{P}$. This is called the radical ideal generated by $\mathcal{P}$. From Propositions 8.28 and 8.29 we obtain the following algebraic characterization of $\operatorname{ccl} p(x)$.

8.30 Corollary Let $A \subseteq M \vDash T_{\text{id}}$. Then $\mathcal{P}_{\operatorname{ccl} p(x)} = \sqrt{\mathcal{P}_p}$ for every $p(x) \subseteq L_{\text{at}}(A)$.

Equivalently, $\operatorname{ccl} p_{\mathcal{P}}(x) = p_{\sqrt{\mathcal{P}}}$ for every $\mathcal{P} \subseteq A[x]$.

We are now ready to translate the Nullstellensatz into a language more familiar to algebraists. Fix $N \vDash T_{\text{acf}}$. Let $\mathcal{P} \subseteq N[x]$. The algebraic variety associated to $\mathcal{P}$, often denoted by $V(\mathcal{P})$, is the set $p_{\mathcal{P}}(N)$. Viceversa, given $A \subseteq N$, let $\mathcal{I}(A)$ be the set of polynomials in $N[x]$ that vanish at all points in A^x. Clearly, $\mathcal{I}(A)$ is an ideal of $N[x]$.

8.31 Hilbert's Nullstellensatz (standard version) Let $N \vDash T_{\text{acf}}$. Let $\mathcal{P} \subseteq N[x]$ where x is a finite tuple. Then $\mathcal{I}\big(V(\mathcal{P})\big) = \sqrt{\mathcal{P}}$.

Proof. Note that $V(\mathcal{P}) = p_{\mathcal{P}}(N)$. Hence $p_{\mathcal{I}}(x)$, where $\mathcal{I} = \mathcal{I}(V(\mathcal{P}))$, is the set of formulas $\varphi(x) \in L_{\text{at}}(N)$ such that $V(\mathcal{P}) \subseteq \varphi(N)$. In other words, $p_{\mathcal{I}}(x)$ is the set containing those formulas such that $N \vDash \forall x[p_{\mathcal{P}}(x) \to \varphi(x)]$. Hence from the model theoretic version of the Nullstellensatz we obtain $p_{\mathcal{I}}(x) = \operatorname{ccl} p_{\mathcal{P}}(x)$, that in turn coincides with $p_{\mathcal{I}}(x) = p_{\sqrt{\mathcal{P}}}(x)$ by the corollary above. □

Chapter 9

Saturation and homogeneity

The first two sections introduce saturation and homogeneity and Section 9.3 presents the notation we shall use in the following chapters when working inside a monster model.

9.1 Saturated structures

Recall that a type $p(x) \subseteq L(M)$ is finitely consistent in M if every conjunction of formulas in $p(x)$ has a solution in M. When $A \subseteq M$ we write $S_x(A)$ for the set of types whose variables are in x which are complete and finitely consistent in M. We never display M in the notation as it will always be clear from the context. When A is empty it usual to write $S_x(T)$ for $S_x(A)$ where $T = \mathrm{Th}(M)$. We write $S(A)$ for the union of $S_x(A)$ as x ranges over all tuples of variables. Similarly for $S(T)$.

The following remark will be used in the sequel without explicit reference.

9.1 Remark Let $k : M \to N$ be an elementary map and let a be an enumeration of $\mathrm{dom}\, k$. Let $p(x\,;z) \subseteq L$. If $p(x\,;a)$ is finitely consistent in M, then $p(x\,;k\,a)$ is finitely consistent in N. (We can drop *finitely* in the antecedent but not in the consequent.)

We shall see that some theories have saturated models of small size. However, the existence of saturated models of arbitrary theories is problematic. The following theorem states the existence of a saturated model of cardinality λ whenever $|L| < \lambda = \lambda^{<\lambda}$. The existence of cardinals of this kind is independent of ZFC. If the generalized continuum hypothesis (GCH) holds, every successor cardinal is such that $\lambda = \lambda^{<\lambda}$. Without GCH, every inaccessible cardinal has this property. Both GCH and the existence of inaccessible cardinals are not generally accepted axioms.

Nevertheless, saturated models are widely used in model theory, without any worries about their existence. In fact, if consistency is an issue, they can be replaced by models that are both λ-saturated and λ-homogeneous (see next section) for some less problematic large cardinal λ. This is well known, so complications are commonly avoided by simply assuming that saturated models exist.

If T is the theory of the ring $\mathbb{Z}$, or any other sufficiently expressive theory, then one can prove that the cardinality λ of any saturated model of T is such that $\lambda = \lambda^{<\lambda}$. As the existence of cardinals with this property has to be assumed as an extra axiom, one could simply assume the existence of saturated models and skip the proof of the following theorem.

9.2 Theorem Assume $|L| < \lambda$ where λ is such that $\lambda^{<\lambda} = \lambda$. Then every structure M of cardinality $\leq \lambda$ has a saturated elementary extension of cardinality λ.

Proof. We can assume that M has cardinality λ. We construct an elementary chain $\langle M_i : i < \lambda\rangle$ of models of cardinality λ. The chain starts with M and is the union at limit stages. Given M_i we choose as M_{i+1} any model of cardinality λ that realizes all types $p(x) \subseteq L(A)$ for all $A \subseteq M_i$ of cardinality $< \lambda$. The required M_{i+1} exists because for any given A there are at most $2^{|L(A)|} \leq \lambda^{<\lambda} = \lambda$ such types and there are $2^{<\lambda} = \lambda$ sets A.

Let N be the union of the chain. We check that N is the required extension. Let $p(x) \subseteq L(A)$ for some $A \subseteq N$ of cardinality $< \lambda$. As $\lambda^{<\lambda} = \lambda$ implies in particular that λ is a regular cardinal, $A \subseteq M_i$ for some $i < \lambda$. Then M_{i+1} realizes $p(x)$, and so does N, by elementarity. $\square$

9.3 Remark The reader who did not read Section 7.1 may replace 2 in the theorem below with the following (and forget about $\mathcal{M}$).

2′ for every $b \in M$, every elementary map $k : M \to N$ of cardinality $< \lambda$ has an extension defined on b;

9.4 Theorem Assume $|L| \leq \lambda$ and let N be an infinite structure. Let $\mathcal{M}$ be the category (see Section 7.1) that consists of models of a complete theory T and elementary maps between these. Then the following are equivalent

1 N is a λ-saturated structure

2 N is a λ-rich model

3 N realizes all finitely consistent types $p(z) \subseteq L(A)$, with $|z| \leq \lambda$ and $|A| < \lambda$.

Note that it is the completeness of T which makes $\mathcal{M}$ connected.

Proof. 1⇒2. Let $k : M \to N$ be an elementary map of cardinality $< \lambda$. It suffices to show that for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is an elementary map. Let a be an enumeration of $\operatorname{dom} k$ and define $p(x\,;z) = \operatorname{tp}_M(b\,;a)$. As $p(x\,;a)$ is finitely consistent in M then $p(x\,;k\,a)$ is finitely consistent in N. The required c is any element of N such that $N \vDash p(c\,;k\,a)$. Such a c exists by saturation because $|a| < \lambda$.

2⇒3. Let $p(z)$ be as in 3. By the compactness theorem $N \preceq K \vDash p(a)$ for some model K and $a \in K^z$. By the downward Löwenheim-Skolem theorem there is a model $A, a \subseteq M \preceq K$ of cardinality $\leq \lambda$. (Here we use $|L|, |A|, |z| \leq \lambda$.) By 2, there is an elementary embedding $h : M \hookrightarrow N$ that extends id_A. (Here we use $|A| < \lambda$.) Finally, as $M \vDash p(a)$, elementarity yields $N \vDash p(h\,a)$.

3⇒1. Trivial. $\square$

Two saturated structures of the same cardinality are isomorphic as soon as they are elementarily equivalent (i.e. as soon as $\varnothing : M \to N$ is an elementary map.) In fact, from Theorems 9.4 and 7.6 we obtain the following (reference to Theorems 7.6 may be avoided with an easy back-and-forth construction).

9.5 Corollary Every elementary map $k : M \to N$ of cardinality $< \lambda$ between saturated models of the same cardinality λ extends to an isomorphism.

As it turns out, we already have many examples of saturated structures.

9.6 Corollary The following models are ω-saturated

1 models of T_{dlo}
2 models of T_{rg}
3 models of T_{dag} with infinite rank
4 models of T_{acf} with infinite degree of transcendence.

Countable models of T_{dlo} and T_{rg} and uncountable models of T_{dag} or T_{acf} are saturated.

Proof. By quantifier elimination embeddings coincide with elementary embeddings. Then saturation is proved applying Theorem 9.4 and the extension lemmas proved in Chapter 6 and 8. □

The following is a useful test for quantifier elimination.

9.7 Theorem Assume $|L| \leq \lambda$. Consider the category that consists of models of some theory T_0 and partial isomorphism. Suppose λ-rich models exist and denote by T_1 their theory. Then the following are equivalent

1. every λ-saturated model of T_1 is λ-rich
2. T_1 has elimination of quantifiers.

Proof. 2⇒1. Let $N \vDash T_1$ be λ-saturated. Fix a partial isomorphism $k : M \to N$ of cardinality $< \lambda$, some $b \in M$ and let $p(x\,;z) = \text{qf-tp}_M(b\,;a)$, where a enumerates $\operatorname{dom} k$. The type $p(x\,;k\,a)$ is realized in any λ-rich model N' that contains $\langle k\,a\rangle_N$. By 2, $N \equiv_{ka} N'$, so $p(x\,;k\,a)$ is finitely consistent in N. By saturation, it is realised by some $c \in N$. Then $k \cup \{\langle b, c\rangle\} : M \to N$ is the required extension.

1⇒2. Let $k : M \to N$ be a finite partial isomorphism between models of T. We claim that it is an elementary map. Let $M' \succeq M$ and $N' \succeq N$ are λ-saturated models. As these are λ-rich, $k : M' \to N'$ is elementary by Theorem 7.11. □

9.8 Definition Let x be a single variable and let λ be an infinite cardinal. We say that a structure N is λ-saturated if it realizes every type $p(x)$ such that

1. $p(x) \subseteq L(A)$ for some $A \subseteq N$ of cardinality $< \lambda$
2. $p(x)$ is finitely consistent in N.

We say that N is saturated if it is λ-saturated and $|N| = \lambda$.

9.9 Exercise Suppose $|L| \leq \omega$ and let M be an infinite structure. Then for every non-principal ultrafilter F on ω the structure M^ω/F is a ω_1-saturated elementary superstructure of M.

Hint: the notation is as in Chapter 4. Let $|x| = 1$ and $|z| = \omega$. It suffices to consider types of the form $p(x\,;\hat{c})$ where $p(x\,;z) = \{\varphi_i(x\,;z) : i < \omega\} \subseteq L$ and $\hat{c} \in (M^\omega)^z$. Without loss of generality we can also assume that $\varphi_{i+1}(x\,;z) \to \varphi_i(x\,;z)$, and that all formulas $\varphi_i(x\,;\hat{c})$ are consistent in M^ω.

Let $\langle X_i : i < \omega\rangle$ be a strictly decreasing chain of elements of the ultrafilter such that $X_{i+1} \subseteq \{j \ : \ M \vDash \exists x\ \varphi_i(x, \hat{c}j)\}$. Let $\hat{a} \in M^\omega$ be such that $\varphi_i(\hat{a}j, \hat{c}j)$ holds for every $j \in X_i \smallsetminus X_{i+1}$. Then $\hat{a}$ realizes $p(x)$.

9.10 Exercise Let $L = \{<\}$ and let N be a ω_1-saturated extension of $\mathbb{Q}$. Prove that there is an embedding $f : \mathbb{R} \to N$. Is it elementary? Is it an isomorphism?

9.11 Exercise Let $L = \{<\}$ and let N be a saturated extension of $\mathbb{Q}$. Prove that there $2^{|N|}$ Dedekind cuts of N.

9.2 Homogeneous structures

Definition 7.3 introduces the notions of universal and homogeneous structures in a general context. When the morphisms of the underlying category are the elementary maps, we refer to these notions as elementary homogeneity and elementary universality. However, one often omits to specify *elementary*. We repeat Definition 7.3 in this specific case.

9.12 Definition A structure N is (elementarily) λ-universal if every $M \equiv N$ of cardinality $\leq \lambda$ there is an elementary embedding $h : M \hookrightarrow N$. We say universal if it is λ-universal and of cardinality λ.

We say that N is (elementarily) λ-homogeneous if every elementary map $k : N \to N$ of cardinality $< \lambda$ extends to an automorphism. We say that N is homogeneous if it is λ-homogeneous and of cardinality λ.

As saturated structures are rich, the following theorem is an instance of Theorem 7.8.

9.13 Theorem For every structure N of cardinality $\geq |L|$ the following are equivalent

1. N is saturated
2. N is elementarily universal and homogeneous.

Given $A \subseteq N$ we denote by $\mathrm{Aut}(N/A)$ the group of A-automorphisms of N. That is the group of automorphisms that fix A point-wise. Let a be a tuple of elements of N. The orbit of a over A in N is the set

$$O_N(a/A) \quad = \quad \Big\{ fa \ : \ f \in \mathrm{Aut}(N/A) \Big\}.$$

When the model N is clear from the context we omit the subscript.

Orbits in a homogeneous structure are particularly interesting. The following proposition is immediate but its importance cannot be overestimated.

9.14 Proposition Let N be a λ-homogeneous structure. Let $A \subseteq N$ have cardinality $< \lambda$ and let $a \in N^{<\lambda}$. Then $O_N(a/A) = p(N)$, where $p(x) = \mathrm{tp}_N(a/A)$.

Finally, we want to extend the equivalence in Theorem 9.13 to λ-saturated structures. For this we only need to apply Theorem 7.24.

When the morphisms of the underlying category are the elementary maps, it is usual to replace the notion of λ-universal (cfr. Definition 7.3) with the following.

9.15 Definition We say that N is weakly λ-saturated if N realizes every type $p(x) \subseteq L$, where $|x| \leq \lambda$, that is finitely consistent in N.

In fact, by the Löwenheim-Skolem theorem the two notions are equivalent.

9.16 Proposition The following are equivalent

1. N weakly λ-saturated
2. N is λ-universal.

The following is an instance of Theorem 7.24 that the reader may prove directly as an exercise.

9.17 Corollary Let $|L| \leq \lambda$. The following are equivalent

1. N is λ-saturated
2. N is weakly λ-saturated and weakly λ-homogeneous.

9.18 Exercise Let M be an arbitrary structure of cardinality larger than $|L|$. Prove that M has an ω-homogeneous elementary extension of the same cardinality.

9.19 Exercise Let M and N be elementarily homogeneous structures of the same cardinality λ. Suppose that $M \vDash \exists x\, p(x) \Leftrightarrow N \vDash \exists x\, p(x)$ for every $p(x) \subseteq L$ such that $|x| < \lambda$ Prove that the two structures are isomorphic.

9.20 Exercise Let L be a language that extends that of strict linear orders with the constants $\{c_i : i \in \omega\}$. Let T be the theory that extends T_{dlo} with the axioms $c_i < c_{i+1}$ for every $i \in \omega$. Prove that T has elimination of quantifiers and is complete (it can be deduced from what is known of T_{dlo}). Exhibit a countable saturated model and a countable model that is not homogeneous.

9.3 The monster model

In this section we present some notation and terminology frequently adopted when dealing with a complete theory T. We fix a saturated structure $\mathcal{U}$ of cardinality larger than $|L|$. We assume $\mathcal{U}$ to be large enough that among its elementary substructures we can find any model of T we might be interested in. This structure is called the monster model. We denote by κ the cardinality of $\mathcal{U}$. When appropriate, we assume κ to be inaccessible.

Some terms acquire a slightly different meaning when working inside a monster model.

truth — we say that $\varphi(x)$ holds if $\mathcal{U} \vDash \forall x\, \varphi(x)$ and similarly for other expressions such as $p(x) \to \neg q(x)$ or $\exists y\, p(x, y)$ etc., which are neither first-order formulas nor types;

consistency — we say that $\varphi(x)$ is consistent if $\mathcal{U} \vDash \exists x\, \varphi(x)$; the consistency of a type $p(x)$ or expressions such as those above is defined similarly;

small/large	cardinalities smaller than κ are called small;
models	are elementary substructure of $\mathcal{U}$ of small cardinality they are denoted by the letters M and N;
parameters	are always in $\mathcal{U}$; the symbols A, B, C, etc. denote sets of parameters of small cardinality; calligraphic letters as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. are used for sets of arbitrary cardinality;
tuples	have length $< \kappa$ unless otherwise specified;
global types	are complete finitely consistent types over $\mathcal{U}$; the set of global types is denoted by $S(\mathcal{U})$;
formulas	have parameters in $\mathcal{U}$ unless otherwise specified;
definable sets	are sets of the form $\varphi(\mathcal{U})$ for some formula $\varphi(x) \in L(\mathcal{U})$; we may say A-definable if $\varphi(x) \in L(A)$;
type-definable sets	are sets of the form $p(\mathcal{U})$ for some $p(x) \subseteq L(A)$ where, as the symbol suggests, A has small cardinality;
types of tuples	are denoted by $\mathrm{tp}(a/A) = \mathrm{tp}_{\mathcal{U}}(a/A)$ and $a \equiv_A b$ abbreviates $\mathcal{U}, a \equiv_A \mathcal{U}, b$;
orbits of tuples	under the action of $\mathrm{Aut}(\mathcal{U}/A)$ are denoted by $\mathcal{O}(a/A)$.

Let x be a tuple of variables. For any fixed $A \subseteq \mathcal{U}$ we introduce a topology on $\mathcal{U}^{|x|}$ that we call the topology induced by A or, for short, A-topology. (This is non standard terminology, not to be confused with the *logic* A-topology in Section 16.5.) The closed sets of the A-topology are those of the form $p(\mathcal{U})$ where $p(x) \subseteq L(A)$ is a type over A.

For $\varphi(x) \in L(A)$ the sets of the form $\varphi(\mathcal{U})$ are clopen in this topology (and vice versa by Proposition 9.21). They form both a base of closed sets and base of open sets, which makes these topologies *zero-dimensional*. By saturation, the topology induced by A is compact. Actually, saturation is equivalent to the compactness of all these topologies as A ranges over the sets of small cardinality.

These topologies are never T_0 as any pair of tuples $a \equiv_A b$ have exactly the same neighborhoods. Such pairs always exist by cardinality reasons. However it is immediate that the topology induced on the quotient $\mathcal{U}^{|x|}/\equiv_A$ is Hausdorff (this is the so-called *Kolmogorov quotient*). Indeed, this quotient corresponds to $S_x(A)$ with the topology introduced in Section 3.3.

The following proposition is an immediate consequence of compactness. When $A = B$ it says that the topology induced by A is *normal*: any two closed sets are separated by open sets. It could be called mutual normality (not a standard name) because the two closed sets belong to different topologies and the separating sets are each found in the corresponding topology.

9.21 Proposition (mutual normality) Let $p(x) \subseteq L(A)$ and $q(x) \subseteq L(B)$ be such that $p(x) \rightarrow \neg q(x)$. Then there are a conjunction $\varphi(x)$ of formulas in $p(x)$ and a conjunction $\psi(x)$ of formulas in $q(x)$ such that $\varphi(x) \rightarrow \neg\psi(x)$.

Proof. The assumptions say that $p(x) \cup q(x)$ is inconsistent (i.e. not realized in $\mathcal{U}$). Then the formulas $\varphi(x)$ and $\psi(x)$ exist by compactness (i.e. saturation). □

9.22 Remark There are many forms in which the proposition above can be applied. For instance, assuming for brevity that $p(x)$ and $q(x)$ are closed under conjunctions,

a. if $p(x) \leftrightarrow \neg q(x)$ then $p(x) \leftrightarrow \varphi(x)$ for some $\varphi(x) \in p(x)$

b. if $p(x) \leftrightarrow \psi(x)$ for some $\psi(x) \in L(\mathcal{U})$ then $p(x) \leftrightarrow \varphi(x)$ for some $\varphi(x) \in p$

c. if $p(x) \rightarrow \psi(x)$ for some $\psi(x) \in L(\mathcal{U})$ then $\varphi(x) \rightarrow \psi(x)$ for some $\varphi(x) \in p$

d. if $p(x) \rightarrow \bigvee\limits_{\psi \in \Psi} \psi(x)$, where $|\Psi| < \kappa$, then $p(x) \rightarrow \bigvee\limits_{i=1}^{n} \psi_i(x)$ for some $\psi_i \in \Psi$.

9.23 Remark A definable set has the form $\varphi(\mathcal{U}^x; b)$ for some formula $\varphi(x; z) \in L$ and some $b \in \mathcal{U}^z$. If $f \in \mathrm{Aut}(\mathcal{U})$ then

$$\begin{aligned} f[\varphi(\mathcal{U}^x; b)] &= \{fa \ : \ \varphi(a; b), \quad a \in \mathcal{U}^{|x|}\} \\ &= \{fa \ : \ \varphi(fa; fb), \ a \in \mathcal{U}^{|x|}\} \\ &= \varphi(\mathcal{U}^x; fb). \end{aligned}$$

Hence automorphisms act on definable sets in a very natural way. Their action on type-definable sets is similar.

We say that a set $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ is invariant over A if $f[\mathcal{D}] = \mathcal{D}$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$ or, equivalently, if $\mathcal{O}(a/A) \subseteq \mathcal{D}$ for every $a \in \mathcal{D}$. By homogeneity this is equivalent to requiring that

$$q(x) \quad \rightarrow \quad x \in \mathcal{D}$$

for every $q(x) = \mathrm{tp}(a/A)$ and $a \in \mathcal{D}$.

Proposition 9.25 below is an important fact about invariant type-definable sets. It may clarify the proof to consider first the particular case of definable sets.

9.24 Proposition For every $\varphi(x) \in L(\mathcal{U})$ the following are equivalent

1. $\varphi(x)$ is equivalent to some formula $\psi(x) \in L(A)$
2. $\varphi(\mathcal{U}^x)$ is invariant over A.

We give two proofs of this theorem as they are both instructive.

Proof. 1⇒2 Obvious.

2⇒1 From 2 and homogeneity we obtain

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{q(x) \in Q} q(x)$$

where Q is the set of the types in $S_x(A)$ such that $q(x) \to \varphi(x)$. By compactness, we can rewrite this equivalence

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{\vartheta(x)\in\Theta} \vartheta(x)$$

where Θ is the set of the formulas in $L(A)$ such that $\vartheta(x) \to \varphi(x)$. The latter equivalence says that $\neg\varphi(x)$ is equivalent to a type over A. Again by compactness we obtain

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \vartheta_i(x)$$

for some formula $\vartheta_i(x) \in L(A)$. □

Second proof of Proposition 9.24. 2⇒1 Let $\varphi(\mathcal{U}^x;b)$, where $\varphi(x\,;z) \in L$, be a set invariant over A. Let $p(z) = \mathrm{tp}(b/A)$. As $f[\varphi(\mathcal{U}^x;b)] = \varphi(\mathcal{U}^x;fb)$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$, homogeneity and invariance yield

$$p(z) \quad \to \quad \forall x\,\big[\varphi(x\,;z) \leftrightarrow \varphi(x\,;b)\big].$$

By compactness there is a formula $\vartheta(z) \in p$ such that

$$\vartheta(z) \quad \to \quad \forall x\,\big[\varphi(x\,;z) \leftrightarrow \varphi(x\,;b)\big].$$

Hence $\varphi(\mathcal{U}^x;b)$ is defined by the formula $\exists z\,\big[\vartheta(z) \wedge \varphi(x\,;z)\big]$, which is a formula in $L(A)$ as required. □

9.25 Proposition Let $p(x) \subseteq L(B)$. Then the following are equivalent

1. $p(x)$ is equivalent to some type $q(x) \subseteq L(A)$
2. $p(\mathcal{U}^x)$ is invariant over A.

We give two proofs of this theorem. The second one requires Proposition 9.26 below.

Proof. 1⇒2 Obvious.

2⇒1 It suffices to show that for every formula $\psi(x) \in p(x)$ there is a formula $\vartheta(x) \in L(A)$ such that $p(x) \to \vartheta(x) \to \psi(x)$. Fix $\psi(x) \in p(x)$. By invariance, any $q(x) \in S(A)$ consistent with $p(x)$ implies $p(x)$, hence

$$p(x) \quad \to \quad \bigvee_{q(x)\to\psi(x)} q(x) \quad \to \quad \psi(x)$$

where $q(x)$ above range over all types in $S_x(A)$. By compactness we can rewrite this equivalence as follows

$$p(x) \quad \to \quad \bigvee_{\vartheta(x)\to\psi(x)} \vartheta(x) \quad \to \quad \psi(x)$$

where $\vartheta(x)$ ranges over all formulas in $L(A)$. Applying mutual normality (Proposition 9.21) to the first implication we obtain a finite number of formulas $\vartheta_i(x) \in L(A)$ such that

$$p(x) \quad \to \quad \bigvee_{i=1}^{n} \vartheta_i(x) \quad \to \quad \psi(x).$$ □

The following easy proposition is very useful.

9.26 Proposition Let $p(x\,;z) \subseteq L(A)$. There is a type $q(x) \subseteq L(A)$ such that

$$\mathfrak{U} \quad \vDash \quad \forall x\,\big[\exists z\, p(x\,;z) \;\leftrightarrow\; q(x)\big].$$

The theorem holds also when x and z have length κ.

Proof. It is easy to verify that the equivalence above holds with

$$q(x) \quad = \quad \big\{\exists z\, \varphi(x\,;z) \;:\; \varphi(x\,;z) \text{ conjunction of formulas in } p(x\,;z)\big\}. \qquad \square$$

Note that the proposition would not hold without saturation. For a counter example consider $\mathbb{R}$ as a structure in the language of strict orders and let $q(x,y) = \mathrm{tp}(0,1/A)$, where

$$A \quad = \quad \Big\{1 + \frac{1}{n+1} \;:\; n \in \omega\Big\}.$$

By quantifier elimination, $0 \equiv_A 1$ but $\mathbb{R}, 1 \nvDash \exists y\; q(x,y)$. However, in any sufficiently saturated elementary extension of $\mathbb{R}$, we have $1 \vDash \exists y\; q(x,y)$.

As an application we give a second proof of the proposition above.

Second proof of Proposition 9.25. 2$\Rightarrow$1 Write $p(x)$ as the type $q(x\,;b)$ for some $q(x\,;z) \subseteq L$ and some $b \in \mathfrak{U}^z$. Let $s(z) = \mathrm{tp}(b/A)$. By invariance and homogeneity the types $q(x\,;fb)$ for $f \in \mathrm{Aut}(\mathfrak{U}/A)$ are all equivalent. Therefore

$$\begin{aligned} p(x) \quad &\leftrightarrow \quad \bigvee_{f \in \mathrm{Aut}(\mathfrak{U}/A)} q(x\,;fb) \\ p(x) \quad &\leftrightarrow \quad \bigvee_{c \equiv_A b} q(x\,;c) \\ &\leftrightarrow \quad \exists z\,\big[s(z) \wedge q(x\,;z)\big]. \end{aligned}$$

Hence, by Proposition 9.26, $p(x)$ is equivalent to a type over A. $\square$

9.27 Exercise Let $\varphi(x) \in L$. Prove that the following are equivalent

1. $\varphi(x)$ is equivalent to some $\psi(x) \in L_{\mathrm{qf}}$
2. $\varphi(a) \leftrightarrow \varphi(fa)$ for every partial isomorphism $f : \mathfrak{U} \to \mathfrak{U}$ and $a \in (\mathrm{dom}\, f)^{|x|}$.

Use the result to prove Theorem 7.14 for T complete.

9.28 Exercise Prove that $\mathbb{R}, 1 \nvDash \exists y\; q(x,y)$, as claimed before Proposition 9.26.

9.29 Exercise Let $p(x) \subseteq L(A)$, with $|x| < \omega$. Prove that if $p(\mathfrak{U})$ is infinite then it has cardinality κ. Show that this is not true if x has length ω.

9.30 Exercise Let $\varphi(x,y) \in L(\mathfrak{U})$. Prove that if the set $\big\{\varphi(a,\mathfrak{U}) \;:\; a \in \mathfrak{U}^{|x|}\big\}$ is infinite then it has cardinality κ. Does the claim remain true with a type $p(x,y) \subseteq L(A)$ for $\varphi(x,y)$?

9.31 Exercise Let $\varphi(x\,;y) \in L(\mathfrak{U})$. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i \in \omega\rangle$ such that $\varphi(\mathfrak{U}\,;a_i) \subset \varphi(\mathfrak{U}\,;a_{i+1})$ for every $i < \omega$
2. there is a sequence $\langle a_i : i \in \omega\rangle$ such that $\varphi(\mathfrak{U}\,;a_{i+1}) \subset \varphi(\mathfrak{U}\,;a_i)$ for every $i < \omega$.

9.32 Exercise Let L be a countable language containing L_{gr} and assume $\mathfrak{U}$ is a group. Let $p(x) \subseteq L$ define a subgroup of $\mathfrak{U}$. Prove that there are some formulas $\varphi_n(x) \in L$ such that

1. $p(x) \leftrightarrow \{\varphi_n(x) : n < \omega\}$
2. $\varphi_{n+1}(x) \wedge \varphi_{n+1}(y) \rightarrow \varphi_n(x \cdot y)$.

9.33 Exercise Let $\varphi(x\,;z) \in L$ and $a \in \mathcal{U}^z$. Prove that the following are equivalent

1. the type $\{\varphi(x\,;fa) : f \in \mathrm{Aut}(\mathcal{U})\}$ is realized (in $\mathcal{U}$)
2. there is a formula $\psi(z) \in L$ such that $\exists x \forall z\, [\psi(z) \rightarrow \varphi(x\,;z)]$.

Chapter 10

Preservation theorems

In this chapter we present a few results dating from the 1950s that describe the relationship between syntactic and semantic properties of first-order formulas. These results characterize the classes of formulas that preserved under various sorts of morphisms. Criteria for quantifier-elimination follow from these theorems, see for instance the frequently used back-and-forth method of Corollary 10.13.

10.1 Lyndon-Robinson Lemma

We refer the reader to Exercise 9.27 for a simpler version of the main result in this section, the Lyndon-Robinson Lemma. In fact, under the additional assumption of completeness, Lemma 10.3 is essentially the same as the claim in Exercise 9.27.

However, we are interested in criteria for quantifier elimination, e.g. Corollary 10.12 below. We often need to prove quantifier elimination in order to prove completeness. Therefore any assumption of completeness would make criteria for quantifier elimination less applicable.

In this section T is a consistent theory without finite models and Δ is a set of formulas closed under renaming of variables. At a first reading the reader is encouraged to assume that Δ is L_{at}.

10.1 Definition If $\mathsf{C} \subseteq \{\forall, \exists, \neg, \vee, \wedge\}$ is a set of connectives, we write $\mathsf{C}\Delta$ for the closure of Δ with respect to all connectives in C. We may write $\Delta^{\pm}$ for $\{\neg\}\Delta$.

Recall that Δ-morphism is a map $k : M \to N$ that preserves the truth of formulas in Δ. It is immediate that Δ-morphism are automatically $\{\wedge\vee\}\Delta$-morphisms. Similarly $\{\exists\wedge\}\Delta$-morphism are $\{\exists\wedge\vee\}\Delta$-morphisms and similarly $\{\forall\vee\}\Delta$-morphism are $\{\forall\wedge\vee\}\Delta$-morphisms.

Below we use the following proposition without further reference.

10.2 Proposition Fix $M \vDash T$ and $b \in M^x$. Let $q(x) = \Delta\text{-tp}_M(b)$. Then for every $\varphi(x) \in L$ the following are equivalent

1. $N \vDash \varphi(kb)$ for every Δ-morphism $k : M \to N \vDash T$ that is defined in b

1′. $N \vDash \varphi(c)$ for every $N \vDash T$ such that $N, c \Rrightarrow_\Delta M, b$

2. $T \vdash q(x) \to \varphi(x)$.

Proof. 1⇔1′ In fact, the difference is just in the notation.

2⇒1 Immediate.

1⇒2 Negate 2, then there are $N \vDash T$ and $c \in N^x$ such that $N \vDash q(c) \wedge \neg\varphi(c)$. Therefore the map $k : M \to N$, where $k = \{\langle b, c\rangle\}$, contradicts 1. □

The following is sometimes referred to as the Lyndon-Robinson Lemma.

10.3 Lemma For every $\varphi(x) \in L$ the following are equivalent

1. $\varphi(x)$ is equivalent over T to a formula in $\{\wedge\vee\}\Delta$
2. $\varphi(x)$ is preserved by Δ-morphisms between models of T.

Proof. 1⇒2 Immediate.

2⇒1 We claim that 2 implies

$$\# \qquad T \quad \vdash \quad \varphi(x) \quad \leftrightarrow \quad \bigvee\{\, p(x) \subseteq \Delta \;:\; T \vdash p(x) \rightarrow \varphi(x) \,\}.$$

The implication $\leftarrow$ is clear. To verify the implication $\rightarrow$, let $M \vDash T$ and let $b \in M^x$ be such that $M \vDash \varphi(b)$. From 2 it follows that $\varphi(x)$ satisfies 1 of Proposition 10.2. Therefore $T \vdash q(x) \rightarrow \varphi(x)$ for $q(x) = \Delta\text{-tp}(b)$. Hence $q(x)$ is one of the types that occur in the disjuction in # which therefore is satisfied by b.

From # and compactness we obtain

$$T \quad \vdash \quad \varphi(x) \quad \leftrightarrow \quad \bigvee\{\, \psi(x) \in \{\wedge\}\Delta \;:\; T \vdash \psi(x) \rightarrow \varphi(x) \,\}.$$

Applying compactness again allows us to replace the infinite disjunction above with a finite one and prove 2. □

The following exercise is immediate but it is arguably the most important result of this chapter.

10.4 Exercise Prove Theorem 7.14, that is, that for every theory T the following are equivalent

1. T has elimination of quantifiers
2. every partial isomorphism between models of T is an elementary map.

In the rest of these section we ...

10.5 Proposition Let N be λ-saturated and let $k : M \rightarrow N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent

1. $k : M \rightarrow N$ is a $\{\exists\wedge\}\Delta$-morphism
2. for every $b \in M$ some $\{\exists\wedge\}\Delta$-morphism $h : M \rightarrow N$ defined in b extends k
3. for every $\bar{b} \in M^{\omega}$ some Δ-morphism $h : M \rightarrow N$ defined in $\bar{b}$ extends k.

Proof. 1⇒2 Let a enumerate $\operatorname{dom} k$. Define $p(x\,;z) = \{\exists\wedge\}\Delta\text{-tp}_M(b\,;a)$. By 1, $p(x\,;ka)$ is finitely consistent in N. By saturation there is a $c \in N$ that realizes $p(x\,;ka)$. Therefore, $h : M \rightarrow N$ where $h = k \cup \{\langle b, c\rangle\}$, witnesses 2.

2⇒3 Iterate ω-times the extension in 2.

3⇒1 Let a enumerate $\operatorname{dom} k$ and let $|z| = |a|$. Formulas in $\{\exists\wedge\}\Delta$ with free variables among z are of the form $\exists\bar{x}\,\varphi(\bar{x}\,;z)$ where $\varphi(\bar{x}\,;z)$ is in Δ and $\bar{x}$ is some fixed tuple of length ω. Assume $M \vDash \exists\bar{x}\,\varphi(\bar{x}\,;a)$ and let $\bar{b}$ be such that $M \vDash \varphi(\bar{b}\,;a)$. By 3, we can extend k to some Δ-morphism $h : M \rightarrow N$ defined in $\bar{b}$. Then $N \vDash \varphi(h\bar{b}\,;ha)$ and therefore $N \vDash \exists\bar{x}\,\varphi(\bar{x}\,;ka)$. □

Iterating the lemma above we obtain the following.

10.6 Corollary Let N be λ-saturated and let $|M| \leq \lambda$. Let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent

1. $k : M \to N$ is a $\{\exists\wedge\}\Delta$-morphism
2. $k : M \to N$ extends to an $\{\exists\wedge\}\Delta$-embedding
3. $k : M \to N$ extends to an Δ-embedding.

The following theorem is often paraphrased as follows: a formula is existential if and only if (its truth) is preserved under extensions of structures.

10.7 Theorem For every $\varphi(x) \in L$ the following are equivalent

1. $\varphi(x)$ is equivalent over T to a formula in $\{\exists\wedge\vee\}\Delta$;
2. $\varphi(x)$ is preserved by Δ-embedding between models of T.

Proof. 1$\Rightarrow$2 Immediate.

2$\Rightarrow$1 Negate 1. By the Lyndon-Robinson Lemma 10.3 there is a $\{\exists\wedge\}\Delta$-morphism $k : M \to N$ between models of T that does not preserve $\varphi(x)$. We can assume that N is λ-saturated for some sufficiently large λ. By Corollary 10.6 there is a Δ-embedding $h : M \hookrightarrow N$ that extends k and contradicts 2. $\square$

A dual version of the results above is obtained replacing embeddings by epimorphisms, i.e. surjective (partial) homomorphisms, and $\{\exists\wedge\}$ by $\{\forall\vee\}$. If Δ contains the formula $x = y$ and is closed under negation, then $k : M \to N$ is a Δ-morphism if and only is $k^{-1} : N \to M$ is a Δ-morphism. In this case the dual version follows from what proved above. Without these assumptions the results need a similar but independent proof.

10.8 Proposition Let M be λ-saturated and let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent

1. $k : M \to N$ is a $\{\forall\vee\}\Delta$-morphism
2. for every $c \in N$ some $\{\forall\vee\}\Delta$-morphism $h : M \to N$ extends k and $c \in \operatorname{rng} h$
3. for every $c \in N^\omega$ some Δ-morphism $h : M \to N$ extends k and $c \in (\operatorname{rng} h)^\omega$.

We write $\neg\Delta$ for the set containing the negation of the formulas in Δ. Warning: do not confuse $\neg\Delta$ with $\{\neg\}\Delta$.

Proof. Left as an exercise for the reader. Hint: to prove implication 1$\Rightarrow$2 define $p(x, y) = \neg\{\forall\vee\}\Delta\text{-tp}_N(ka, c)$, where a is a tuple that enumerates $\operatorname{dom} k$. From 1 obtain that $p(a, y)$ is finitely consistent in M. Then proceed as in the proof of Proposition 10.5. $\square$

10.9 Corollary Let M be λ-saturated and let $|N| \leq \lambda$. Let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent

1. $k : M \to N$ is a $\{\forall\vee\}\Delta$-morphism
2. $k : M \to N$ extends to an $\{\forall\vee\}\Delta$-epimorphism
3. $k : M \to N$ extends to an Δ-epimorphism.

Finally we obtain the following.

10.10 Theorem The following are equivalent

1. $\varphi(x)$ is equivalent to a formula in $\{\forall\wedge\vee\}\Delta$
2. every Δ-epimorphism between models of T preserves $\varphi(x)$.

10.2 Quantifier elimination by back-and-forth

We say that T admits (or has) positive Δ-elimination of quantifiers if for every formula $\varphi(x)$ in $\{\exists\forall\wedge\vee\}\Delta$ there is a formula $\psi(x)$ in $\{\wedge\vee\}\Delta$ such that

$$T \quad \vdash \quad \varphi(x) \leftrightarrow \psi(x).$$

When Δ is closed under negation the attribute *positive* becomes irrelevant and will be omitted. When Δ is $L_{\mathrm{at}^{\pm}}$ or L_{qf}, we simply say that T admits elimination of quantifiers. This is by far the most common case.

Quantifier elimination is often used to prove that a theory is complete because it reduces it to something much simpler to prove. The following is an immediate consequence of the definition above with x replaced by the empty tuple.

10.11 Remark If T has elimination of quantifiers then the following are equivalent

1. T decides all quantifier free sentences
2. T is complete.

Hence a theory with quantifier elimination is complete if it decides the characteristic, see Definition 3.27).

The following is a consequence of Lemma 10.3.

10.12 Corollary The following are equivalent

1. T has positive Δ-elimination of quantifiers
2. every Δ-morphism between models of T is an $\{\exists\wedge\}\Delta$ and a $\{\forall\vee\}\Delta$-morphism.

Proof. 1⇒2 Immediate.

2⇒1 We prove by induction of syntax that Δ-morphism preserve the truth of all formulas in $\{\exists\forall\wedge\vee\}\Delta$, this suffices by Lemma 10.3. Induction for the connectives $\vee$ and $\wedge$ is trivial. So assume as induction hypothesis that the truth of $\varphi(x, y)$ is preserved. By Lemma 10.3 $\varphi(x, y)$ is equivalent to a formula in $\{\wedge\vee\}\Delta$, hence by 2 the truth of $\exists y\, \varphi(x, y)$ and $\forall y\, \varphi(x, y)$ is preserved. □

Condition 2 of the corollary above may be difficult to verify directly. The following corollary of Proposition 10.5 and 10.8 gives a back-and-forth condition witch is easier to verify.

10.13 Corollary Let $|L| \leq \lambda$. The following are equivalent

1. T has positive Δ-elimination of quantifiers
2. for every finite Δ-morphism $k : M \to N$ between λ-saturated models of T
   a. for every $b \in M$ some Δ-morphism $h : M \to N$ extends k and $b \in \operatorname{dom} h$;
   b. for every $c \in N$ some Δ-morphism $h : M \to N$ extends k and $c \in \operatorname{rng} h$. $\square$

Note that when Δ contains the formula $x = y$ and is closed under negation, then $k : M \to N$ is a Δ-morphism if and only if $k^{-1} : N \to M$ is a Δ-morphism. In this case a and b are equivalent.

10.14 Exercise Let T be a complete theory without finite models in a language that consists only of unary predicates. Prove that T has elimination of quantifiers.

10.15 Exercise Consider $\mathbb{R}$ as a structure in the language of real vector spaces expanded with the usual order relation. Prove that $\mathrm{Th}(\mathbb{R})$ has elimination of quantifiers.

10.16 Exercise Let T be the theory of discrete linear orders, that is, T extends the theory of linear orders T_{lo} (see Section 6.1) with the following two of axioms

dis↑. $\exists z \left[x < z \ \wedge \ \neg\exists y \ x < y < z\right]$

dis↓. $\exists z \left[z < x \ \wedge \ \neg\exists y \ z < y < x\right]$.

Let Δ be the set of formulas that contains (all alphabetic variants of) the formulas $x <_n y := \exists^{\geq n} z \ (x{<}z{<}y)$ and their negations, for all positive integers n. Prove that the theory of discrete linear orders has Δ-elimination of quantifiers. Prove that the structure $\mathbb{Q} \times \mathbb{Z}$ ordered with the lexicographic order

$$(a_1, a_2) < (b_1, b_2) \quad \Leftrightarrow \quad a_1 < b_1 \ \text{ or } \ (a_1 = b_1 \ \text{ e } \ a_2 < b_2)$$

is a saturated model of T.

10.17 Exercise Let T be a consistent theory. Suppose that all completions of T are of the form $T \cup S$ for some set S of quantifier-free sentences. Prove that if all completions of T have elimination of quantifiers, so does T. Show that this fails when the completions of T have arbitrary complexity.

Note. Thought the claim follows immediately from Corollary 10.12, a direct proof by compactness is also instructive. Prove that for every formula $\varphi(x)$ there are some quantifier-free sentences σ_i and quantifier-free formulas $\psi_i(x)$ such that

$$\sigma_i \vdash \varphi(x) \leftrightarrow \psi_i(x), \qquad T \vdash \bigvee_{i=1}^{n} \sigma_i, \qquad \text{and} \qquad \sigma_i \vdash \neg\sigma_j \text{ for } i \neq j.$$

For a counter example consider the empty theory in the language with a single unary predicate.

10.3 Model-completeness

We say that T is model-complete if every embedding $h : M \hookrightarrow N$ between models of T is an elementary embedding. The terminology, introduced by Abraham Robinson, is inspired by the fact that T is model-complete if and only if $T \cup \mathrm{Diag}(M)$ is a complete theory, in the language $L(M)$, for every $M \vDash T$.

To stress positivity in the next proposition, we generalize the definition as follows.

We say that T is Δ-model-complete if every Δ-embedding $h : M \hookrightarrow N$ between models of T is a $\{\forall\exists \wedge \vee\}\Delta$-embedding.

Model-completeness is equivalent to a property akin to quantifier elimination.

10.18 Proposition The following are equivalent

1. T is Δ-model-complete
2. T has $\{\exists\wedge\}\Delta$-elimination of quantifiers.

Proof. 1⇒2 By 1, every formula $\{\forall\exists\wedge\vee\}\Delta$ is preserved by Δ-embeddings therefore, by Theorem 10.7, it is equivalent to a formula in $\{\exists\wedge\vee\}\Delta$.

2⇒1 Clear, because Δ-embeddings preserve formulas in $\{\exists\wedge\vee\}\Delta$. □

The theory of discrete linear orders defined in Exercise 10.16 is an example of a model-complete theory without elimination of quantifiers.

The difference between quantifier elimination and model-completeness subtle. It boils down to models of T having or not the amalgamation property.

10.19 Proposition Assume T is model-complete. Let $\mathcal{M}$ be the category that consists of models of T and partial isomorphisms. Then the following are equivalent

1. $\mathcal{M}$ has the amalgamation property
2. T has elimination of quantifiers.

Proof. 1⇒2 By Proposition 7.26 every partial morphism $k : M \to N$ extends to an embedding $g : M \hookrightarrow N'$ which, by model-completeness, is an elementary embedding. Model-completeness also implies that $N \preceq N'$. Hence $k : M \to N$ is an elementary map. This proves 2.

2⇒1 If all morphisms are elementary maps, amalgamations follows from Proposition 7.29. □

Note however that the models of a model-complete theory T do have amalgamation when the proper notion of morphism is chosen.

Let $\mathcal{M}'$ be the category that consists of models of T and the maps $k : M \to N$ such that there is a partial isomorphism $h : M' \to N'$ with

1. $k \subseteq h$; $M \preceq M'$; $N \preceq N'$;
2. $\operatorname{dom} h$ contains a substructure of M' that models T (equivalently $\operatorname{rng} h$ and N').

Moreover, we add as morphisms the maps that are obtained from those above by composition. It is clear that if T is model-complete then the morphisms of $\mathcal{M}'$ are exactly the elementary maps. In this case $\mathcal{M}'$ has amalgamation. Vice versa if $\mathcal{M}'$ has amalgamation, the theory of rich models is model-complete.

10.20 Exercise Prove that $\mathcal{M}'$ satisfies finite character of morphisms, c2 of Definition 7.1.

Chapter 11

Geometry and dimension

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ larger than $|L|$. The notation and implicit assumptions are as in Section 9.3.

11.1 Algebraic and definable elements

Let $a \in \mathcal{U}$ and let $\mathcal{A} \subseteq \mathcal{U}$ be some set of parameters (of arbitrary cardinality). We say that a is algebraic over $\mathcal{A}$ if $\varphi(a) \wedge \exists^{=k} x\, \varphi(x)$ holds for a formula $\varphi(x) \in L(\mathcal{A})$ and some positive integer k. In particular, when $k = 1$ we say that a is definable over $\mathcal{A}$. We write $\mathrm{acl}(\mathcal{A})$ for the algebraic closure of $\mathcal{A}$, that is, the set of all the elements that are algebraic over $\mathcal{A}$. If $\mathcal{A} = \mathrm{acl}(\mathcal{A})$, we say that $\mathcal{A}$ is algebraically closed. The definable closure of $\mathcal{A}$ is defined similarly and is denoted by $\mathrm{dcl}(\mathcal{A})$.

Let x be a finite tuple of variables. Formulas $\varphi(x) \in L(A)$, or types $p(x) \subseteq L(A)$, with finitely many solutions are called algebraic.

11.1 Proposition For every $A \subseteq \mathcal{U}$ and every type $p(x) \subseteq L(A)$, where x is finite, the following are equivalent

1. $\exists^{\leq n} x\, p(x)$
2. $\exists^{\leq n} x\, \varphi(x)$ for some $\varphi(x)$ which is a conjunction of formulas in $p(x)$.

Proof. Only 1⇒2 requires a proof. Let $\{a_1, \ldots, a_n\} = p(\mathcal{U})$. Then

$$p(x) \;\rightarrow\; \bigvee_{i=1}^{n} a_i = x$$

and 2 follows by compactness. □

11.2 Theorem For every $A \subseteq \mathcal{U}$ and every $a \in \mathcal{U}$ the following are equivalent

1. $a \in \mathrm{dcl} A$
2. $\mathcal{O}(a/A) = \{a\}$.

Proof. Only 2⇒1 requires a proof. Recall that $\mathcal{O}(a/A)$ coincides with the set of realizations of $\mathrm{tp}(a/A)$. Then the theorem follows from Proposition 11.1. □

11.3 Theorem For every $A \subseteq \mathcal{U}$ and every $a \in \mathcal{U}$ the following are equivalent

1. $a \in \mathrm{acl} A$
2. $\mathcal{O}(a/A)$ is finite
3. a belongs to every model containing A.

Proof. 1⇔2. This is proved as in Theorem 11.2.

1⇒3. Assume 1. Then there is a formula $\varphi(x) \subseteq L(A)$ such that $\varphi(a) \wedge \exists^{=k} x\, \varphi(x)$ for some k. By elementarity $\exists^{=k} x\, \varphi(x)$ holds in every model M containing A. Again by elementarity, the k solutions of $\varphi(x)$ in M are solutions in $\mathcal{U}$, therefore a is one of these.

3⇒2. Assume $\mathcal{O}(a/A)$ is infinite and fix any model M containing A. By Exercise 9.29, $\mathcal{O}(a/A)$ has cardinality κ, hence $\mathcal{O}(a/A) \nsubseteq M$. Pick any $f \in \mathrm{Aut}(\mathcal{U}/A)$ such that $fa \notin M$. Then $a \notin f^{-1}[M]$, so $f^{-1}[M]$ is a model that contradicts 3. □

11.4 Corollary For every $A \subseteq \mathcal{U}$ and every $a \in \mathcal{U}$

1. if $a \in \mathrm{acl}A$ then $a \in \mathrm{acl}B$ for some finite $B \subseteq A$ — finite character
2. $A \subseteq \mathrm{acl}A$ — extensivity
3. if $A \subseteq B$ then $\mathrm{acl}A \subseteq \mathrm{acl}B$ — monotonicity
4. $\mathrm{acl}A = \mathrm{acl}(\mathrm{acl}A)$ — idempotency
5. $\mathrm{acl}A = \bigcap_{A \subseteq M} M.$

Properties 1-4 say that acl(-) is a closure operator with finite character.

Proof. Properties 1-3 are obvious, 4 follows from 5 which in turn follows from Theorem 11.3. □

11.5 Proposition If $f \in \mathrm{Aut}(\mathcal{U})$ then $f\big[\mathrm{acl}A\big] = \mathrm{acl}\big(f[A]\big)$ for every $A \subseteq \mathcal{U}$.

Proof. We prove $f\big[\mathrm{acl}A\big] \subseteq \mathrm{acl}\big(f[A]\big)$. Fix $a \in \mathrm{acl}A$ and let $\varphi(x\,;z) \in L$ and $b \in A^z$ be such that $\varphi(x\,;b)$ is algebraic formula satisfied by a. By elementarity, $\varphi(x\,;fb)$ is algebraic and satisfied by fa. Therefore fa is algebraic over $f[A]$, which proves the inclusion.

The converse inclusion is obtained by substituting f^{-1} for f and $f[A]$ for A. □

11.6 Exercise For every $a \in \mathcal{U}^x$ and $A \subseteq \mathcal{U}$, the following are equivalent

1. a is solution of some algebraic formula $\varphi(x) \in L(A)$
2. $a = a_1, \ldots, a_n$ for some $a_1, \ldots, a_n \in \mathrm{acl}A$.

11.7 Exercise Let $\varphi(z) \in L(A)$ be a consistent formula. Prove that, if $a \in \mathrm{acl}(A, b)$ for every $b \vDash \varphi(z)$, then $a \in \mathrm{acl}A$. Prove the same claim with a type $p(z) \subseteq L(A)$ for $\varphi(z)$.

11.8 Exercise Let $a \in \mathcal{U} \smallsetminus \mathrm{acl}\varnothing$. Prove that $\mathcal{U}$ is isomorphic to some $\mathcal{V} \preceq \mathcal{U}$ such that $a \notin \mathcal{V}$. Hint: let $\bar{c}$ be an enumeration of $\mathcal{U}$ and let $p(\bar{u}) = \mathrm{tp}(\bar{c})$ prove that $p(\bar{u}) \cup \big\{u_i \neq a : i < |\bar{u}|\big\}$ is realized in $\mathcal{U}$ and that any realization yields the required substructure of $\mathcal{U}$.

11.9 Exercise Let C be a finite set. Prove that if $C \cap M \neq \varnothing$ for every model M containing A, then $C \cap \mathrm{acl}A \neq \varnothing$. Hint: by induction on the cardinality of C. Suppose there is a $c \in C \smallsetminus \mathrm{acl}A$, then there is $\mathcal{V} \simeq \mathcal{U}$ such that $A \subseteq \mathcal{V} \preceq \mathcal{U}$ and $c \notin \mathcal{V}$, see Exercise 11.8. Apply the induction hypothesis to $C' = C \cap \mathcal{V}$ with $\mathcal{V}$ for $\mathcal{U}$.

11.10 Exercise Prove that for every $A \subseteq N$ there is an M such that $\mathrm{acl}A = M \cap N$. Hint: add the requirement $\mathrm{acl}(A_i) \cap N \subseteq \mathrm{acl}A$ to the construction used to prove the downward Löwenheim-Skolem theorem. You need to prove that every consistent $\varphi(x) \in L(A_i)$ has a solution a such that $\mathrm{acl}(A_i, a) \cap N \subseteq \mathrm{acl}A$. The required a has to realize the type

$$\{\varphi(x)\} \cup \Big\{\neg\big[\psi(b,x) \wedge \exists^{\leq n} y\, \psi(y,x)\big] \;:\; b \in N \smallsetminus \mathrm{acl}A,\;\; \psi(y,x) \in L(A_i),\;\; n < \omega\Big\}$$

whose consistency need to be verified.

11.11 Exercise Prove that for every $A \subseteq N$ there is an automorphism $f \in \mathrm{Aut}(\mathcal{U}/A)$ such that $\mathrm{acl}A = f[N] \cap N$. (This is a stronger version of the claim in Exercise 11.10.) Hint: let $\bar{c}$ be an enumeration of N. Let $p(\bar{x}) = \mathrm{tp}(\bar{c}/A)$. Consider the type

$$p(x) \cup \Big\{\neg\big[\psi(b,\bar{x}) \wedge \exists^{\leq n} y\, \psi(y,\bar{x})\big] \;:\; b \in N \smallsetminus \mathrm{acl}A,\;\; \psi(y,x) \in L(A),\;\; n < \omega\Big\}$$

Any $\bar{a} \models p(\bar{x})$ enumerates a model A-isomorphic to N.

11.12 Exercise Let $\varphi(x) \in L(\mathcal{U})$ and fix an arbitrary set A. Prove that the following are equivalent

1. $\varphi(M^x) \neq \varnothing$ for every model M containing A
2. there is a consistent formula $\psi(z_1, \ldots, z_n) \in L(A)$ such that
$$\psi(z_1, \ldots, z_n) \to \bigvee_{i=1}^{n} \varphi(z_i).$$

Hint: let $\bar{c}$ be an enumeration of M^x, where M is any model containing A. Let $p(\bar{z}) = \mathrm{tp}(\bar{c}/A)$. Note that 1 implies that $p(\bar{z}) \cup \big\{\neg\varphi(z_i) \;:\; i < |\bar{z}|\big\}$ is inconsistent.

11.13 Exercise Let $T = T_{\mathrm{rg}}$. Prove that $\mathrm{acl}A = \mathrm{dcl}A$ for every set A.

11.14 Exercise Let $p(x) \in S(A)$. Let $Q = \big\{q(x) \in S(\mathrm{acl}A) \;:\; q(x) \supseteq p(x)\big\}$. Prove that $\mathrm{Aut}(\mathcal{U}/A)$ acts on Q transistively, i.e. any two types in Q are conjugated.

11.2 Strongly minimal theories

Finite and cofinite sets are always (trivially) definable in every structure. We say that M is a minimal structure if all its definable subsets of arity one are finite or cofinite. Unfortunately, this notion is not elementary, i.e. it is not a property of $\mathrm{Th}(M)$. For instance $\mathbb{N}$ with only the order relation in the language is a minimal structure but none of its elementary extensions is. Hence the following definition: we say that M is a strongly minimal structure if it is minimal and all its elementary extensions are minimal.

We say that T, a consistent theory without finite models, is strongly minimal if for every formula $\varphi(x\,;z) \in L$, where x has arity one, there is an $n \in \omega$ tale che

$$T \;\vdash\; \exists^{\leq n} x\; \varphi(x\,;z) \;\vee\; \exists^{\leq n} x\; \neg\varphi(x\,;z).$$

We show that the semantic notion matches the syntactic one.

11.15 Proposition The following are equivalent

1. $\mathrm{Th}(M)$ is a strongly minimal theory
2. M is a strongly minimal structure
3. M has an elementary extension which is minimal and ω-saturated.

Proof. Implications 1⇒2⇒3 are immediate, we prove 3⇒1. Let $\varphi(x\,;z) \in L$ and let N be the elementary extension given by 3. Let $p(z) \subseteq L$ be the following type

$$p(z) \;=\; \Big\{\exists^{>n}x\,\varphi(x\,;z) \wedge \exists^{>n}x\,\neg\varphi(x\,;z) \;\;:\;\; n\in\omega\Big\}.$$

As N is minimal, $N \nvDash \exists z\, p(z)$. By ω-saturation $p(z)$ is not finitely consistent in M. Hence, for some n

$$M \;\vDash\; \forall z\,\Big[\exists^{\leq n}x\,\varphi(x\,;z) \vee \exists^{\leq n}x\,\neg\varphi(x\,;z)\Big].$$

which proves that $\mathrm{Th}(M)$ is strongly minimal. □

By quantifier elimination, T_{acf} and T_{dag} are strongly minimal theories.

11.16 Exercise Let T be a complete theory without finite models. Prove that the following are equivalent

1. M is minimal
2. $a \equiv_M b$ for every $a, b \in \mathfrak{U} \smallsetminus M$.

11.3 Independence and dimension

Throughout this section we assume that T is a complete strongly minimal theory.

When $a \notin \mathrm{acl}B$ we say that a is algebraically independent from B. We say that B is an algebraically independent set if every $a \in B$ is independent from $B \smallsetminus \{a\}$. Below we shall abbreviate $B \cup \{a\}$ by B, a and $B \smallsetminus \{a\}$ by $B \smallsetminus a$.

The following is a pivotal property of independence that holds in strongly minimal structures. It is called symmetry or exchange principle. For every B and every pair of elements $a, b \in \mathfrak{U} \smallsetminus \mathrm{acl}B$

$$b \in \mathrm{acl}(B, a) \;\Leftrightarrow\; a \in \mathrm{acl}(B, b)$$

Note that when T is the theory of vector spaces (over any fixed field) this principle is the so called Steinitz exchange lemma.

11.17 Theorem (T strongly minimal.) Independence is symmetric. That is, if $a, b \notin \mathrm{acl}B$ then $b \in \mathrm{acl}(B, a) \Leftrightarrow a \in \mathrm{acl}(B, b)$

Proof. Suppose $b \notin \mathrm{acl}(B, a)$ and $a \in \mathrm{acl}(B, b)$. We prove that $a \in \mathrm{acl}B$. Fix a formula $\varphi(x, y) \in L(B)$ such that $\varphi(x, b)$ witnesses $a \in \mathrm{acl}(B, b)$, i.e. for some n

$$\varphi(a, b) \;\wedge\; \exists^{\leq n}x\,\varphi(x, b).$$

As $b \notin \mathrm{acl}(B, a)$, the formula

$$\psi(a, y) \;=\; \varphi(a, y) \;\wedge\; \exists^{\leq n}x\,\varphi(x, y).$$

is not algebraic. Therefore, by strong minimality, $\psi(a, y)$ has cofinitely many solutions. Hence every model containing B contains a solution of $\psi(a, y)$. As a is algebraic in any of these solutions, a belongs to every model containing B. Therefore, $a \in \mathrm{acl}B$ by Theorem 11.3. □

We say that $B \subseteq C$ is a basis of C if B is an independent set and $C \subseteq \mathrm{acl}B$. The following theorem proves that all bases have the same cardinality, which we call the dimension of C and denote by $\dim C$. First we need the following lemma.

11.18 Lemma (T strongly minimal.) If B is an independent set and $a \notin \operatorname{acl} B$ then B, a is also an independent set.

Proof. Suppose B, a is not independent and that $a \notin \operatorname{acl} B$. Then $b \in \operatorname{acl}(B \smallsetminus b, a)$ for some $b \in B$. As $a, b \notin \operatorname{acl}(B \smallsetminus b)$, from symmetry we obtain $a \in \operatorname{acl}(B \smallsetminus b, b) = \operatorname{acl} B$. Hence B is not an independent set. □

11.19 Corollary (T strongly minimal.) For every $B \subseteq C$ the following are equivalent

1. B is a basis of C
2. B is a maximally independent subset of C.

Finally we prove the main theorem about basis.

11.20 Theorem (T strongly minimal.) Fix some arbitrary set C. Then

1. every independent set $B \subseteq C$ can be extended to a basis of C
2. all bases of C have the same cardinality.

Proof. By the finite character of algebraic closure, the independent set form an inductive class. Apply Zorn lemma to obtain a maximally independent subset of C containing B. By Corollary 11.19 this set is a basis of C. This proves 1.

As for 2, assume for a contradiction that $A, B \subseteq C$ are two bases of C and that $|A| < |B|$. First consider the case when B is infinite. For each $a \in A$ fix a finite set $D_a \subseteq B$ such that $a \in \operatorname{acl}(D_a)$. Let

$$D \;=\; \bigcup_{a \in A} D_a.$$

Then $A \subseteq \operatorname{acl} D$ and $|D| < |B|$. By transitivity, $C \subseteq \operatorname{acl} D$ which contradicts the independence of B.

Now we suppose that $|A| < |B| = n < \omega$. Choose B such that $|B \smallsetminus A|$ is minimal among the bases of cardinality $\geq n$. As $|A| < |B|$, there is a $b \in B \smallsetminus A$. Let B' be a maximal indipendent subset of $(B \smallsetminus b), A$ containing $B \smallsetminus b$. By Corollary 11.19, B' is a base of $(B \smallsetminus b), A$, hence it is also a base of C. As $|B' \smallsetminus A| < |B \smallsetminus A|$, this is a contradiction. □

11.21 Proposition (T strongly minimal) Let k be an elementary map. Then $k \cup \{\langle b, c\rangle\}$ is also an elementary map for every $b \notin \operatorname{acl}(\operatorname{dom} k)$ and $c \notin \operatorname{acl}(\operatorname{rng} k)$.

Proof. Let a be an enumeration of $\operatorname{dom} k$. We need to show that $\varphi(b\,; a) \leftrightarrow \varphi(c\,; ka)$ holds for every $\varphi(x\,; z) \in L$. As k is elementary, the formulas $\varphi(x\,; a)$ and $\varphi(x\,; ka)$ are either both algebraic or both co-algebraic. As $b \notin \operatorname{acl}(a)$ and $c \notin \operatorname{acl}(ka)$, they are both false or both true respectively. So the proposition follows. □

11.22 Corollary (T strongly minimal.) Every bijection between independent sets is an elementary map.

Finally we show that dimension classifies models of T.

11.23 Theorem (T strongly minimal.) Models of T with the same dimension are isomorphic.

Proof. Let A e B be bases of M and N respectively. By Corollary 11.22, any bijection between A and B is an elementary map. By Proposition 11.5, it extends to the required isomorphism between $\text{acl}A = M$ and $\text{acl}B = N$. $\square$

11.24 Corollary (T strongly minimal.) Let $|L| < \lambda$. Then T is λ-categorical.

Proof. Let M have cardinality λ. Let $B \subseteq M$ be a base. Then $\lambda = |M| = |\text{acl}B| = |L(B)| = \max\{|L|, |B|\}$. If $|L| < \lambda$, then $\lambda = |B|$. Therefore all models of cardinality λ are isomorphic because they all have the same dimension λ. $\square$

11.25 Proposition (T strongly minimal.) For every model N of cardinality $\geq |L|$ the following are equivalent

1. N is saturated
2. $\dim N = |N|$.

Proof. 2⇒1. Assume 2 and let $k : M \to N$ be an elementary map of cardinality $< |N|$ and let $b \in M$. We want an extension of k defined in b. If $b \in \text{acl}(\text{dom}\, k)$ then the required extension exists by Proposition 11.5. Otherwise, we pick any element $c \in N \smallsetminus \text{acl}(\text{rng}\, k)$. Such an element exists as $|k| < \dim N = |N|$. Then $k \cup \big\{\langle b, c\rangle\big\}$ is the required extension by Proposition 11.21.

1⇒2. If $B \subseteq N$ is a basis of N the following type is not realized in N

$$p(x) \quad = \quad \Big\{\neg\varphi(x) \ : \ \varphi(x) \in L(B) \text{ is algebraic } \Big\}$$

Therefore, if N is saturated, $|B| = |N|$. $\square$

11.26 Exercise (T strongly minimal.) Prove that every infinite algebraically closed set is a model.

11.27 Exercise (T strongly minimal, L countable.) Prove that every model is homogeneous.

11.28 Exercise (T strongly minimal.) Prove that if $\dim N = \dim M + 1$ then there is no model K such that $M \prec K \prec N$.

Chapter 12

Countable models

In this chapter L is a fix signature, T a complete theory without finite models, and $\mathcal{U}$ is a saturated model of inaccessible cardinality κ larger than $|L|$. We make no blanket assumption on the cardinality of L, but the main theorems require L to be countable. The notation and implicit assumptions are as in Section 9.3.

12.1 The omitting types theorem

We say that the formula $\varphi(x)$ isolates the type $p(x)$ when $\varphi(x)$ is consistent and $\varphi(x) \rightarrow p(x)$. When Δ is a set of formulas, we say that Δ isolates $p(x)$ if some formula in Δ does. When $\Delta = L_x(A)$, we say that A isolates $p(x)$ or, when A is clear, that $p(x)$ is isolated. We say that a model M omits $p(x)$ if $p(x)$ is not realized in M.

Observe that if $p(x) \subseteq L(M)$ then M realizes $p(x)$ if and only if M isolates $p(x)$. Therefore if A isolates $p(x)$, then every model containing A realizes $p(x)$. Below we prove that the converse holds when L and A are countable. This is a famous classical theorem that is called the *omitting types theorem* because it is proved by constructing a model M that omits a given non-isolated type $p(x)$.

The core of the argument lies in the following lemma.

12.1 Lemma Assume $L(A)$ is countable. Let $p(x) \subseteq L(A)$ and suppose that A does not isolate $p(x)$. Then, if $\psi(z) \in L(A)$ is consistent, $\psi(z)$ has a solution a such that A, a does not isolate $p(x)$.

Proof. We construct a sequence of formulas $\langle \psi_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\psi_i(z) : i < \omega\}$ is the required solution of $\psi(z)$.

Let $\langle \xi_i(x\,; z) : i < \omega \rangle$ be an enumeration of $L_{x\,;z}(A)$. Set $\psi_0(z) = \psi(z)$ and define $\psi_{i+1}(z)$ inductively as follows:

1. if $\xi_i(x\,; z) \wedge \psi_i(z)$ is inconsistent, let $\psi_{i+1}(z) = \psi_i(z)$
2. otherwise, let $\psi_{i+1}(z) = \psi_i(z) \wedge \exists x\,[\xi_i(x\,; z) \wedge \neg\varphi(x)]$ for some/any $\varphi(x) \in p$ that makes $\psi_{i+1}(z)$ is consistent.

Note that 1 and 2 ensure that if $a \vDash \psi_{i+1}(z)$ then $\xi_i(x\,; a)$ does not isolate $p(x)$. Therefore the proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required in 2.

Suppose for a contradiction that no formula makes $\psi_{i+1}(z)$ consistent, that is,

$$\xi_i(x\,; z) \wedge \psi_i(z) \quad \rightarrow \quad \varphi(x)$$

for every $\varphi(x) \in p$. This immediately implies that

$$\exists z\,[\xi_i(x\,; z) \wedge \psi_i(z)] \quad \rightarrow \quad p(x),$$

that is, $p(x)$ is isolated by a formula in $L_x(A)$. This contradicts our assumption and proves the lemma. $\square$

12.2 Omitting Types Theorem Assume $L(A)$ is countable. Then for every consistent type $p(x) \subseteq L(A)$ the following are equivalent

1. all models containing A realize $p(x)$
2. A isolates $p(x)$.

Proof. The implication 2⇒1 is clear. We prove 1⇒2. Assume that A does not isolate $p(x)$. The model M is the union of a chain $\langle A_i : i < \omega \rangle$ of countable subsets of $\mathfrak{U}$ where $A_0 = A$. Along the construction we require that A_i does not isolate $p(x)$. At the end, M will not isolate $p(x)$. Since M is a model, this this is equivalent to M omitting $p(x)$.

We proceed as in the proof of the downward Löwenheim-Skolem theorem. Assume that A_i does not isolate $p(x)$. With the notation in the second proof of the Löwenheim-Skolem Theorem 2.55, at stage $i = \pi(j, k)$ apply Lemma 12.1 to find a solution a of $\varphi_k(x)$ such that $A_{i+1} = A_i, a$ does not isolates $p(x)$. $\square$

Gerald Sacks once famously remarked: *Any fool can realize a type but it takes a model theorist to omit one*. However, the diagonalization method in the proof of Lemma 12.1 lean towards descriptive set theory. (We invite the interested reader to compare this lemma with the Kuratowski-Ulam theorem.)

12.3 Example The following example shows that in the omitting types theorem we cannot drop the assumption that $L(A)$ is countable. Let F be the set of all bijections between two uncountable sets, X and Y. Let M be the model whose domain is domain the disjoint union of F, X and Y. The language has a ternary relation symbol for $f(x) = y$ and unary relation symbols for F, X, and Y. Let $\mathfrak{U}$ be a saturated elementary extension of M. Then $\mathfrak{U}$ is partitioned into three definable sets $\mathfrak{U}_F$, $\mathfrak{U}_X$ and $\mathfrak{U}_Y$. Each element of $\mathfrak{U}_F$ defines a bijection between $\mathfrak{U}_X$ and $\mathfrak{U}_Y$.

Note that for any two elements $a, b \in \mathfrak{U}_Y$, there is an automorphism of $\mathfrak{U}$ that fixes $\mathfrak{U}_X \cup \mathfrak{U}_Y \smallsetminus \{a, b\}$ and swaps a and b.

Now, let $Y_0 \subseteq \mathfrak{U}_Y$ be countable. Let $c \in \mathfrak{U}_Y \smallsetminus Y_0$ and let $p(y) = \mathrm{tp}(c/X, Y_0)$. We claim that $p(y)$ is realized in every model containing X, Y_0. In fact, by the remark above $p(\mathfrak{U}) = \mathfrak{U}_Y \smallsetminus Y_0$. But every model containing X also contains uncountably many elements of $\mathfrak{U}_Y$, hence it contains a conjugate of c which therefore realizes $p(y)$. We also claim that $p(y)$ is not isolated. Suppose for a contradiction there is a consistent formula $\varphi(y)$ such that $\varphi(y) \to p(y)$. Then $\varphi(y)$ has a solution in $\mathfrak{U}_Y \smallsetminus Y_0$. By the remark above, this implies that $\varphi(\mathfrak{U})$ is a cofinite subset of $\mathfrak{U}_Y$ and this contradicts $\varphi(\mathfrak{U}) \subseteq p(\mathfrak{U})$.

12.4 Exercise Let $p(x) \subseteq L(B)$ and $p_n(x) \subseteq L(A)$, for $n < \omega$, be such that

$$p(x) \quad \to \quad \bigvee_{n<\omega} p_n(x)$$

Prove that

$$p(x) \wedge \varphi(x) \quad \to \quad p_n(x)$$

for some $n < \omega$ and some formula $\varphi(x) \in L(A)$ consistent with $p(x)$.

12.5 Exercise Let $A \subseteq B$ and $p(x) \subseteq L(A)$. Suppose that $\operatorname{tp}(a/B)$ is isolated over B for every $a \vDash p(x)$. Prove that $p(x)$ is isolated over A.

12.2 Prime and atomic models

We say that M is prime over A if $A \subseteq M$ and for every N containing A there is an elementary embedding $h : M \to N$ that fixes A. When A is empty we simply say that M is prime.

There is no syntactic analogue of primeness. The closest notion, which works well for countable models in a countable language, is atomicity. For $a \in \mathcal{U}^x$ we say that a is isolated over A if the type $p(x) = \operatorname{tp}(a/A)$ is isolated. Note that this equivalent to claiming that a is an isolated point in $\mathcal{U}^x$ with respect to the A-topology defined in Section 9.3. We say that M is atomic over A if $A \subseteq M$ and every $a \in M^{<\omega}$ is isolated over A. When A is empty we say that M is atomic.

12.6 Proposition Let a and b be finite tuples. Then the following are equivalent

1. A isolates b, a
2. A, a isolates b and A isolates a.

Proof. Let $p(x,z) = \operatorname{tp}(b,a/A)$. Then $p(x,a) = \operatorname{tp}(b/A,a)$ and $\exists x\, p(x,z) \leftrightarrow \operatorname{tp}(a/A)$.

1⇒2. Let $\varphi(x,z) \in p$ be such that $\varphi(x,z) \to p(x,z)$. Then $\varphi(x,a) \to p(x,a)$ and $\exists x\, \varphi(x,z) \to \exists x\, p(x,z)$. Therefore 2 holds by the remark above.

2⇒1. Fix $\varphi(x\,;z), \psi(z) \in L(A)$ such that $\varphi(x\,;a)$ isolates $p(x\,;a)$ and $\psi(z)$ isolates $\exists x\, p(x\,;z)$. Let $\xi(x\,;z) \in p$ be arbitrary. As $\varphi(x\,;a) \to \xi(x\,;a)$, the formula $\forall x\, [\varphi(x\,;z) \to \xi(x\,;z)]$ belongs to $\operatorname{tp}(a/A)$ which, as noted above, coincides with $\exists x\, p(x,z)$. Hence $\psi(z) \to \forall x\, [\varphi(x\,;z) \to \xi(x\,;z)]$. As this holds for all $\xi(x\,;z) \in p$, we conclude that $\psi(z) \wedge \varphi(x\,;z)$ isolates $p(x,z)$. □

Implication 1⇒2 of the proposition above yields the following useful proposition.

12.7 Proposition If M is atomic over A then M is atomic over A, a for any $a \in M^{<\omega}$.

Proof. Let $b \in M^x$ be a finite tuple. Then A isolates b, a hence A, a isolates b. □

12.8 Proposition Let $k : M \to N$ be an elementary map and suppose that M is atomic over $\operatorname{dom} k$. Then for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is elementary.

Proof. Let $p(x\,;z) = \operatorname{tp}(b\,;a)$ where a is an enumeration of $\operatorname{dom} k$. Let $\varphi(x\,;z) \in L$ be such that $\varphi(x\,;a) \to p(x\,;a)$. Note that, by elementarity, $\varphi(x\,;ka) \to p(x\,;ka)$. Hence the required c is any solution of $\varphi(x\,;ka)$ in N. □

A limiting assumption in Proposition 12.7 is that a need to be finite. Therefore the following proposition is restricted to countable models.

12.9 Proposition Any two countable models atomic over A are isomorphic.

Proof. By Propositions 12.7 and 12.8 and an easy back-and-forth argument. □

12.10 Proposition Assume $L(A)$ is countable. Then for every model M the following are equivalent

1. M is countable and atomic over A
2. M is prime over A.

Proof. 1⇒2. By Propositions 12.7 and 12.8.

2⇒1. Some countable model containing A exists, as M embeds in it, M has also to be countable. Now we prove that M is atomic over A. Suppose for a contradiction that there is some $b \in M^{<\omega}$ such that $p(x) = \mathrm{tp}(b/A)$ is not isolated. By the omitting types theorem there is a model N containing A that omits $p(x)$. Then there cannot be any A-elementary embedding of M into N. □

12.11 Proposition Assume $L(A)$ is countable. Then the following are equivalent

1. there are models atomic over A
2. every consistent $\varphi(z) \in L(A)$ has a solution that is isolated over A.

Note that 2 says that in $\mathfrak{U}^z$ isolated points are dense w.r.t. the topology defined in Section 9.3.

Proof. 1⇒2. This holds by elementarity.

2⇒1. We construct by induction a sequence $\langle a_i : i < \omega \rangle$. Reasoning as in (the second proof of) the downward Löverheim-Skolem theorem we can easily ensure that $A \cup \{a_i : i < \omega\}$ is a model. To obtain an atomic model we require that $a_{\restriction i}$ is isolated over A.

Suppose $a_{\restriction i}$ has been defined and assume that some formula $\varphi(z) \in L(A)$ isolates $\mathrm{tp}(a_{\restriction i}/A)$. Let $\psi(x\,;z) \in L(A)$ be such that $\psi(x\,;a_{\restriction i})$ is consistent (we leave to the reader the details of the enumeration of such formulas). Then $\psi(x\,;z) \wedge \varphi(z)$ is also consistent and by assumption it has a solution $b\,;c$ that is isolated over A. As $a_{\restriction i} \equiv_A c$, there is an A-automorphism such that $fc = a_{\restriction i}$. Therefore $fb\,;a_{\restriction i}$ is a solution $\psi(x\,;z)$ that is also isolated over A. Then we can set $a_i = fb$. □

12.3 Countable categoricity

Here we present some important characterizations of ω-categoricity. The second property below can be stated in different equivalent ways; for convenience, these equivalents are considered in a separate proposition. For the time being we introduce the following generalization (which we will prove is completely unnecessary): we say hat T is ω-categorical over A if any two countable models containing A are isomorphic over A. We say ω-categorical for ω-categorical over $\varnothing$.

12.12 Theorem (Engeler, Ryll-Nardzewsky, and Svenonius) Assume $L(A)$ is countable. The following are equivalent:

1. T is ω-categorical over A
2. every type $p(x) \subseteq L(A)$ with $|x| < \omega$ is isolated.

The set A is introduced for convenience. By 3 of Proposition 12.13 below, no theory is ω-categorical over an infinite set, and categoricity over some finite A is equivalent to categoricity over $\varnothing$ (see Exercise 12.15).

Proof. 1⇒2. This is an immediate consequence of the omitting types theorem. In fact, if $p(x)$ is a non-isolated A-type, then there are two countable models M and N containing A such that M realizes $p(x) \subseteq L(A)$ while N omits it. Then M and N cannot be isomorphic over A.

2⇒1. Observe that 2 implies that every countable model containing A is atomic over A. But, by Proposition 12.9, countable atomic models are unique up to isomorphism. □

12.13 Proposition Fix a set A and a finite tuple of variables x. The following are equivalent

1. every A-type $p(x)$ is isolated
2. $S_x(A)$ is finite
3. $L_x(A)$ is finite up to equivalence
4. $\mathrm{Aut}(\mathcal{U}/A)$ induces finitely many orbits in $\mathcal{U}^x$.

Proof. To prove the implication 1⇒2 observe that $\mathcal{U}^x$ is the union of sets of the form $p(\mathcal{U})$ where $p \in S_x(A)$. If these types are isolated then $\mathcal{U}^x$ is the union of A-definable sets. By compactness this union has to be finite. To prove 2⇒1 let $p \in S_x(A)$. If $S_x(A)$ is finite, $\neg p(\mathcal{U})$ is the union of finitely many type definable sets. A finite union of type definable sets is type definable. So $\neg p(\mathcal{U})$ is type definable. Hence $p(\mathcal{U})$ is isolated. We prove implication 2⇒3 observe that each formula in $L_x(A)$ is equivalent to the disjunction of the types in $S_x(A)$ that contain this formula. If $S_x(A)$ is finite, $L_x(A)$ is finite up to equivalence. Implication 3⇒2 is clear and equivalence 2⇔4 follows from the characterization of orbits as type-definable sets. □

12.14 Exercise Prove that the following are equivalent

1. T is ω-categorical
2. for every M countable and x finite, $\mathrm{Aut}(M)$ induces finitely many orbits in M^x.

12.15 Exercise Prove that the following are equivalent for every finite set A

1. T is ω-categorical
2. T is ω-categorical over A.

12.16 Exercise Prove that the following are equivalent

1. T is ω-categorical
2. there is a countable model that is both saturated and atomic.

12.17 Exercise Assume L is countable and that T is complete. Suppose that for every finite tuple x there is a model M that realizes only finitely many types in $S_x(T)$. Prove that T is ω-categorical.

12.4 Small theories

Let T be, as always in this chapter, a complete theory without finite models. We say that T is small over A if $S_x(A)$ is countable for every x of finite length. When

A is empty, we simply say that T is small. The set A is introduced for convenience; in most application A is the empty set. A different term is used in another very interesting case: a theory which is small over every countable set A is said to be ω-stable. For this reason, the term 0-stable is sometimes used for small.

12.18 Proposition If T is small (over $\varnothing$) then it is small over any finite set A.

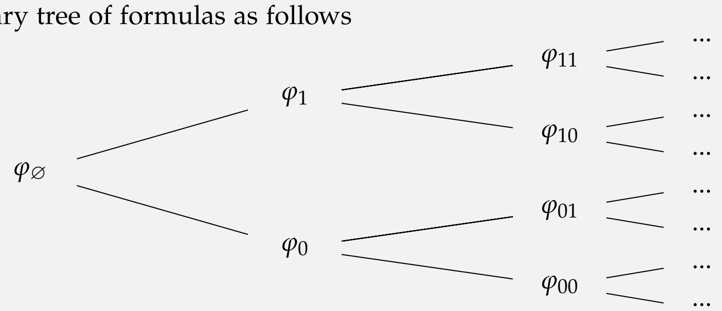
Proof. Let a be an enumeration of A. As $S_x(A) = \{p(x\,;a) \;:\; p \in S_{x;z}(T)\}$ the proposition is immediate. □

Below we identify $S_x(A)$ with $\mathcal{U}^x/{\equiv_A}$

12.19 Definition Let Δ be a set of formulas (we mainly use $\Delta = L_x(A)$ in this section). A binary tree of formulas in Δ is a sequence $\langle \varphi_s \;:\; s \in 2^{<\lambda}\rangle$ of formulas in $\Delta \cup \{\top\}$ such that

1. for each $s \in 2^\lambda$ the type $p_s \;=\; \{\varphi_r \;:\; r \subset s\}$ is consistent
2. $p_s \cup p_r$ is inconsistent for any two distinct $s, r \in 2^\lambda$.

(Condition 2 is usually obtained by taking $\varphi_{s0} \leftrightarrow \neg\varphi_{s1}$ for every s.) We call λ the height of the tree. If the height is not specified, we assume it is ω. We may depict a binary tree of formulas as follows



where branches are consistent types and distinct branches are inconsistent.

Let $S(\Delta)$ be denote the set of maximal consistent Δ-types.

12.20 Lemma Suppose Δ is countable and closed under negation. Then the following are equivalent

1. there is a binary tree of formulas in Δ
2. $|S(\Delta)| = 2^\omega$
3. $|S(\Delta)| > \omega$.

Proof. Since the implications 1⇒2⇒3 are clear, it suffices to prove 3⇒1. We assume that $S(\Delta)$ is uncountable and define a tree of formulas in Δ by induction. Begin with $\varphi_\varnothing = \top$. For $s \in 2^{<\omega}$ define

$$p_s \quad = \quad \{\varphi_r \;:\; r \subseteq s\}.$$

Assume inductively that p_s has uncountably many extensions in $S(\Delta)$. This will guarantee the consistency of the branches.

It suffices to show that there is a formula $\psi \in \Delta$ such that both $p_s \cup \{\psi\}$ and

$p_s \cup \{\neg\psi\}$ have uncountably many extensions in $S(\Delta)$. Then we define $\varphi_{s0} = \psi$ and $\varphi_{s1} = \neg\psi$.

Consider the following type that extends p_s

$$q \quad = \quad \Big\{\xi \in \Delta \ : \ p_s \cup \{\neg\xi\} \text{ has } \leq \omega \text{ extensions in } S(\Delta)\Big\}.$$

This type is consistent, otherwise $\neg\xi_1 \vee \cdots \vee \neg\xi_n$ would hold for some $\xi_i \in q$. This cannot happen, because p_s has uncountably many extensions in $S(\Delta)$, while by the definition of q each of $p_s \cup \{\neg\xi_i\}$ has countably many extensions.

If the formula ψ required above does not exist, q is complete, hence it belongs to $S(\Delta)$. Every type in $S(\Delta)$ that extends p_s and is distinct from q contains $p_s \cup \{\neg\xi\}$ for some $\xi \in q$. By the definition of q, there are countably many such types, so this contradicts the induction hypothesis. □

12.21 Proposition Suppose $L(A)$ is countable. The following are equivalent

1. T is small over A
2. there exists a countable saturated model containing A
3. there is no binary tree of formulas in $L_x(A)$ for any finite x.

Proof. 1⇒2. There is a countable model M containing A that is weakly saturated (see Proposition 9.16). There is a countable homogeneous model N containing M (see Exercise 9.17). Clearly N is also weakly saturated. Then it is saturated by Corollary 9.17.

For a direct argument: construct a countable chain of countable models M_i such that M_{i+1} realized all types in $S_x(A, B)$ for every finite $B \subseteq M_i$ and x. The union of the chain is the saturated model that proves 2.

2⇒3. Clear.

3⇒1. By Lemma 12.20. □

12.22 Proposition A small theory has countable atomic models over every countable set.

Proof. We prove that every formula in $L_x(A)$, where x is finite, has a solution isolated over A. Then it suffices to apply Proposition 12.11.

Suppose for a contradiction that $\varphi(x) \in L(A)$ is consistent but has no solution isolated over A. Then there is a formula $\psi(x) \in L(A)$ such that both $\varphi(x) \wedge \psi(x)$ and $\varphi(x) \wedge \neg\psi(x)$ are consistent, otherwise $\varphi(x)$ would imply a complete type and every solution of $\varphi(x)$ would be isolated. Fix such a $\psi(x)$. Clearly neither $\varphi(x) \wedge \psi(x)$ nor $\varphi(x) \wedge \neg\psi(x)$ have a solution isolated over A. This allows to construct a binary tree of formulas in $L_x(A)$ and prove that T is not small over A. □

12.23 Exercise Let $|x| = 1$. Prove that if $S_x(A)$ is countable for every finite set A, then T is small.

12.24 Exercise (Vaught) Prove that no complete theory has exactly 2 countable models (assume L is countable – though it is not really necessary).

Hint: suppose T has exactly two countable models. Then T is small and there are a countable saturated model N and an atomic model $M \subseteq N$. As T is not ω-categorical, $M \not\simeq N$ and there is finite tuple a that is not isolated over $\varnothing$. Let K

be an atomic model over a. Clearly $K \not\simeq M$ and, by Exercises 12.15 and 12.16, also $K \not\simeq N$.

12.5 A toy version of a theorem of Zil'ber

As an application we prove that if T is ω-categorical and strongly minimal then it is not finitely axiomatizable.

We say that T has the finite model property if for every sentence $\varphi \in L$ there is a finite substructure $A \subseteq \mathfrak{U}$ such that

fmp $\qquad \mathfrak{U} \vDash \varphi \quad \Leftrightarrow \quad A \vDash \varphi$

The property is interesting because of the following proposition.

12.25 Proposition If T has the finite model property then it is not finitely axiomatizable.

Proof. Assume fmp and suppose for a contadiction that there is a sentence $\varphi \in L$ such that $T \vdash \varphi \vdash T$. Then $A \vDash T$ for some finite structure A. But $T \vdash \exists^{>k} x\,(x = x)$ for every k. A contradiction. □

We need the following definition. We say that $C \subseteq \mathfrak{U}$ is a homogeneous set if for every pair of tuples $a, c \in C^{<\omega}$ such that $a \equiv c$ and for every $b \in C$ there is a $d \in C$ such that $a, b \equiv c, d$.

12.26 Lemma Suppose L is countable. If T is ω-categorical and every finite set is contained in a finite homogeneous substructure, then T has the finite model property.

Proof. We prove fmp also for formulas with parameters. We prove that for all n there is a finite structure $A \subseteq \mathfrak{U}$ where fmp holds for all sentences $\varphi \in L(A)$ such that

\# $\qquad$ number of parameters in φ $+$ number of quantifiers in φ $\leq n$.

Fix n and pick some finite substructure A that is homogeneous and such that all types $p(z) \subseteq L$ with $|z| \leq n$ have a realization in A^z. Now we prove fmp by induction on the syntax of φ.

The claim for atomic formulas is witnessed by any finite structure that contains the parameters of the formula. Such finite substructure exists in fact it suffices to take the algebraic closure which, in an ω-categorical theory, is finite (by 3 of Lemma 12.13). Induction for Boolean connectives is straightforward. As for induction step for the existential quantifier, consider the formula $\exists x\, \varphi(x\,;c)$, where $c \in A^{<n}$ and $|x| = 1$. Implication $\Leftarrow$ of fmp follows immediately from the induction hypothesis and from the fact that, if $\exists x\, \varphi(x\,;c)$ satisfy #, also $\varphi(d\,;c)$ satisfies it. As for $\Rightarrow$, assume $\mathfrak{U} \vDash \exists x\, \varphi(x\,;c)$. Let $a, b \in A^{<\omega}$ be a solution of $\varphi(x\,;z)$ such that $a \equiv c$. Such a solution exists because all types with $\leq n$ variables are realized in A. By homogeneity there is a $d \in A$ such that $a, b \equiv c, d$ and therefore $A \vDash \varphi(d\,;c)$. □

12.27 Proposition If T is strongly minimal, then every algebraically closed set is homogeneous.

Proof. Let A be algebraically closed and let $a, c \in A^{<\omega}$ be such that $a \equiv c$. Let b be an element of A. Suppose first that $b \in \operatorname{acl} a$. Let $f \in \operatorname{Aut}(\mathfrak{U})$ be such that $f(a) = c$ and $f[A] = \operatorname{acl} A$. Then $d = fb$ is the required element, in fact $a, b \equiv c, d$. Now, suppose instead that $b \notin \operatorname{acl} a$. Then any $d \notin \operatorname{acl} c$ satisfies $a, b \equiv c, d$. Such a d exists in A, otherwise $A = \operatorname{acl} c \neq \operatorname{acl} a$, which contradicts $a \equiv c$. □

From the propositions above we finally obtain the following.

12.28 Theorem A theory which is ω-categorical and strongly minimal is not finitely axiomatizable.

Proof. If T is ω-categorical the algebraic closure of a finite set is finite. Therefore from Proposition 12.27 we infer that T satisfies the assumptions of Lemma 12.26. Hence T has the finite model property, so, by Proposition 12.25 it is not finitely axiomatizable. □

12.29 Exercise Assume L is countable and let T be strongly minimal. Prove that the following are equivalent

1. T is ω-categorical
2. the algebraic closure of a finite set is finite.

Implication 1⇒2 does not require the strong minimality of T.

12.6 Notes and references

An uncountable, non-isolated, complete type that cannot be omitted was produced by Gebhard Fuhrken in 1962. Example 12.3 is inspired by a post by Alex Kruckman on StackExhange [1]. I am not aware of other expositions.

Boris Zil'ber famously proved that Theorem 12.28 holds for any totally categorical theory. The same theorem has been proved independently by Cherlin, Harrington and Lachlan. Their proof uses the classification of finite simple groups. This theorem marks the birth of a subject known as *geometric stability theory* which studies in depth the geometric properties which we briefly mentioned in Chapter 11. The interested reader may consult Pillay's monograph [2]. The material in Section 12.5 comes from [2, Section 2.6]

[1] Alex Kruckman, *Counterexample to the omitting types in uncountable language* (2017), available at https://math.stackexchange.com/q/2434851. URL accessed 2019-02-28.

[2] Anand Pillay, *Geometric stability theory*, Oxford Logic Guides, vol. 32, 1996.

Chapter 13

Imaginaries

The description of first-order definability is simplified if we allow definable sets to be used as second-order parameters in formulas. This leads to the theory of (elimination of) *imaginaries*. The technical reason that induced Shelah to introduce imaginaries will only be clear later, see Section 18.4, but the theory is of independent interest.

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ strictly larger than $|L|$. The notation and implicit assumptions are as in Section 9.3.

13.1 Many-sorted structures

A many-sorted language consists of three disjoint sets. Besides the usual $L_{\rm fun}$ and $L_{\rm rel}$, we have a set $L_{\rm srt}$ whose elements are called sorts. The language also includes a (many-sorted) arity function that assigns to function and relation symbols r, f a tuple of sorts of finite positive length which we call arity.

A many-sorted structure M consists of

1. a set M_s, for each $s \in L_{\rm srt}$
2. a function $f^M : M_{s_1} \times \cdots \times M_{s_n} \to M_{s_0}$, for each $f \in L_{\rm fun}$ of arity $\langle s_0, \ldots, s_n \rangle$
3. a relation $r^M \subseteq M_{s_0} \times \cdots \times M_{s_n}$, for each $r \in L_{\rm rel}$ of arity $\langle s_0, \ldots, s_n \rangle$.

For every sort s we fix a sufficiently large set of variables V_s. Now we define terms and their respective sorts by induction.

All variables are terms of their respective sort. If $t_1, \ldots, t_n$ are terms of sorts $s_1, \ldots, s_n$ and $f \in L_{\rm fun}$ is of arity $\langle s_0, \ldots, s_n \rangle$ then $f\, t_1, \ldots, t_n$ is a term of sort s_0.

Formulas are defined as follows. If $r \in L_{\rm rel}$ has arity $\langle s_0, \ldots, s_n \rangle$ then $r\, t_0, \ldots, t_n$ is a formula. Also, $t_1 = t_2$ is formula for every pair of terms of equal sort. All other formulas are constructed by induction using the propositional connectives $\neg$ and $\vee$ and the quantifier $\exists x$ (or any other reasonable choice of logical connectives).

Truth of formulas is defined as for one-sorted languages, except that here we require that the witness of the quantifier $\exists x$ belongs to M_s, where s is the sort of the variable x.

Models of second-order logic are arguably the most widely used examples of many-sorted structures. They may be described using a language with a sort n for every $n \in \omega$. The sort 0 is used for the first-order elements; the sort $n > 0$ is used for relations of arity n. For every $n > 0$ the language has a relation symbol $\in_n$ of arity $\langle 0^n, n \rangle$, where $0^n = 0 \overset{n \text{ times}}{\ldots\ldots} 0$. There are also arbitrarily many function and relation symbols of sort $\langle 0^n \rangle$ for any $n > 0$.

13.2 The eq-expansion

Warning: the structure $\mathcal{U}^{\mathrm{eq}}$ and the theory T^{eq} defined below do not coincide with the standard ones introduced by Shelah. As the difference is merely cosmetic, introducing new notation would be overkill and we prefer to abuse the existing terminology. In Section 13.7 below we compare our definition with the standard one.

Given a language L, we define a many-sorted language L^{eq} which has a sort for each partitioned formula $\sigma(x\,;z) \in L$ and a sort 0 which we call the home sort. (Partitioned formulas have been introduced in Definition 1.16.) For legibility, we pretend that all formulas σ depend on the same variables. So we assume that $x\,;z$ are infinite tuples. Hence, with the notation of the previous section $L_{\mathrm{srt}} = \{0\} \cup L_{x\,;z}$.

The home sort is also called first-order sort, the other sorts are called second-order.

Let $\mathcal{U}^{\mathrm{eq}}$ be the many-sorted structure that has $\mathcal{U}$ as domain of the home sort. The domain for the sort $\sigma(x\,;z)$ contains the definable sets $\mathcal{A} = \sigma(\mathcal{U}^x;b)$ as b ranges over $\mathcal{U}^z$.

The language of $\mathcal{U}^{\mathrm{eq}}$ is denoted by L^{eq}. It contains all the relations and functions of the first order language L with the same interpretation as in the one-sorted case. Moreover L^{eq} contains a relation symbol $\in_{\sigma(x\,;z)}$ for each sort $\sigma(x\,;z)$. These relation symbols have arity $\langle 0^{|x_\sigma|}, \sigma(x\,;z)\rangle$, where x_σ are the variables in x that actually occur in σ. These relations are interpreted as set membership. As there is no risk of ambiguity, in what follows we omit σ from the subscripts.

We write T^{eq} for $\mathrm{Th}(\mathcal{U}^{\mathrm{eq}})$. As usual L^{eq} also denotes the set of formulas constructed in this language and, if $A \subseteq \mathcal{U}^{\mathrm{eq}}$, we write $L^{\mathrm{eq}}(A)$ for the language and the set of formulas that use elements of A as parameters.

We write $L(A)$ for the set of formulas in $L^{\mathrm{eq}}(A)$ that contain no second-order variables, neither free nor quantified (when $A \subseteq \mathcal{U}^{\mathrm{eq}}$, it may contain second-order parameters).

We use the symbol $\mathcal{X}$ to denote a generic second-order variable.

It is important to note right away that this expansion of $\mathcal{U}$ is a mild one: the definable subsets of the home sort of $\mathcal{U}^{\mathrm{eq}}$ are the same as those of $\mathcal{U}$. In particular, iterating the expansion would not yield anything new.

13.1 Proposition Let $\bar{\mathcal{X}} = \mathcal{X}_1, \ldots, \mathcal{X}_n$ be a tuple of second order variables of sort $\sigma_i(x\,;z)$. Then for every formula $\varphi(x\,;\bar{\mathcal{X}}) \in L^{\mathrm{eq}}$ there is a formula $\varphi'(x\,;\bar{z}) \in L$, where $\bar{z} = z_1, \ldots, z_n$ is a tuple variables of length $|z|$, such that the following holds in $\mathcal{U}^{\mathrm{eq}}$

$$\varphi(x\,;\bar{\mathcal{A}}) \quad \leftrightarrow \quad \varphi'(x\,;\bar{b})$$

for every $\bar{\mathcal{A}} = \mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\bar{b} = b_1, \ldots, b_n$ such that $\mathcal{A}_i = \sigma(\mathcal{U}^x; b_i)$.

When $n = 0$ the proposition asserts that L^{eq}_x and L_x have the same expressive power.

Proof (sketch). By induction on syntax. When φ is atomic, we set $\varphi' = \varphi$ unless φ is of the form $t \in \mathcal{X}_i$ for some tuple of terms t or it has the form $\mathcal{X}_i = \mathcal{X}_j$. In the first case φ' is the formula $\sigma_i(t\,;z_i)$. In the second case it is the formula

$\forall x\,[\sigma_i(x\,;z_i) \leftrightarrow \sigma_j(x\,;z_j)]$.

The connectives stay unchanged except for the quantifiers $\exists\mathcal{X}$, where $\mathcal{X}$ is a second-order variable, say of sort $\sigma(x\,;z)$. These quantifiers are replaced by $\exists z$. □

Proposition 13.1 implies in particular that we can always replace $\exists\mathcal{X}$ by $\exists z$ if we substitute $\sigma(t\,;z)$ for $t \in \mathcal{X}$ in the quantified formula.

13.2 Remark Proposition 13.1 should convince the reader that the move from $\mathcal{U}$ to $\mathcal{U}^{\mathrm{eq}}$ is *almost* trivial. For instance, it implies that for every $A \subseteq \mathcal{U}^{\mathrm{eq}}$, there exists a $B \subseteq \mathcal{U}$ such that $L(B)$ is at least as expressive as $L(A)$. By this we mean that every formula in $L(A)$ is equivalent to some formula in $L(B)$. The set $B \subseteq \mathcal{U}$ contains the parameters that define the definable sets in $A \subseteq \mathcal{U}^{\mathrm{eq}}$. The point of $\mathcal{U}^{\mathrm{eq}}$ is that there might not be any $B \subseteq \mathcal{U}$ such that $L(B)$ is *exactly* as expressive as $L^{\mathrm{eq}}(A)$.

For instance, suppose L contains only a binary relation which is interpreted as an equivalence relation with infinitely many infinite classes. Let $\mathcal{A}$ be an equivalence class and let $A = \{\mathcal{A}\}$. Then for $\mathcal{A}$ is definable in $L(B)$ if and only if $B \cap \mathcal{A} \neq \varnothing$. But no element of $\mathcal{A}$ is definable in $L(A)$.

If $\mathcal{V} \preceq \mathcal{U}$ we write $\mathcal{V}^{\mathrm{eq}}$ for the substructure of $\mathcal{U}^{\mathrm{eq}}$ that has $\mathcal{V}$ as domain of the home sort and the set of definable sets of the form $\sigma(\mathcal{U}^x; b)$ for some $b \in \mathcal{V}^z$ as domain of the sort $\sigma(x\,;z)$. The following proposition claims that the elementary substructures of $\mathcal{U}^{\mathrm{eq}}$ are exactly those of the form $\mathcal{V}^{\mathrm{eq}}$ for some $\mathcal{V} \preceq \mathcal{U}$.

13.3 Proposition The following are equivalent for every structure $\mathcal{V}^\dagger$ of signature L^{eq}

1. $\mathcal{V}^\dagger \preceq \mathcal{U}^{\mathrm{eq}}$
2. $\mathcal{V}^\dagger = \mathcal{V}^{\mathrm{eq}}$ for some $\mathcal{V} \preceq \mathcal{U}$.

Proof. Implication 2⇒1 is a direct consequence of Proposition 13.1. We prove 1⇒2. Let $\mathcal{V}$ be the domain of the home sort of $\mathcal{V}^\dagger$. It is clear that $\mathcal{V} \preceq \mathcal{U}$. Let $\mathcal{A} \in \mathcal{V}^{\mathrm{eq}}$ have sort $\sigma(x\,;z)$, say $\mathcal{A} = \sigma(\mathcal{U}^x; b)$ for some $b \in \mathcal{V}^z$. As $\exists^{=1}\mathcal{X}\,\forall x\,[x \in \mathcal{X} \leftrightarrow \sigma(x\,;b)]$ holds in $\mathcal{U}^{\mathrm{eq}}$, by elementarity it holds in $\mathcal{V}^\dagger$ and therefore $\mathcal{A} \in \mathcal{V}^\dagger$. This proves $\mathcal{V}^{\mathrm{eq}} \subseteq \mathcal{V}^\dagger$. A similar argument proves the converse inclusion. Given $\mathcal{A} \in \mathcal{V}^\dagger$ of sort $\sigma(x\,;z)$, the formula $\forall x\,[x \in \mathcal{A} \leftrightarrow \sigma(x\,;b)]$ holds in $\mathcal{V}^\dagger$ for some b in the home sort. By elementarity, $\mathcal{A} = \sigma(\mathcal{U}^x; b)$ for some $b \in \mathcal{V}$. □

13.4 Proposition Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$. Then every type $p(u\,;\mathcal{X}) \subseteq L^{\mathrm{eq}}(A)$ that is finitely consistent in $\mathcal{U}^{\mathrm{eq}}$ is realized in $\mathcal{U}^{\mathrm{eq}}$. That is, $\mathcal{U}^{\mathrm{eq}}$ is saturated.

Proof. By Remark 13.2, there are some $B \subseteq \mathcal{U}$ and some $q(u\,;\mathcal{X}) \subseteq L^{\mathrm{eq}}(B)$ equivalent to $p(u\,;\mathcal{X})$. This already proves the proposition when $\mathcal{X}$ is the empty tuple. Otherwise, let $q'(u\,;z)$ be obtained by replacing every formula $\varphi(u\,;\mathcal{X})$ in $q(u\,;\mathcal{X})$ with the formula $\varphi'(u\,;z)$ given in Proposition 13.1. Then $q'(u\,;z)$ is finitely consistent in $\mathcal{U}$. Assume for clarity of notation that $\mathcal{X}$ is a single variable of sort $\sigma(x\,;z)$. If $c\,;b \vDash q'(u\,;z)$, then $c\,;\sigma(\mathcal{U}^x; b) \vDash q(u\,;\mathcal{X})$. □

Automorphisms of a many-sorted structure are defined in the obvious way: sorts are preserved and so are functions and relations. Every automorphism $f : \mathcal{U} \to \mathcal{U}$

extends to an automorphism $f : \mathcal{U}^{eq} \to \mathcal{U}^{eq}$ as follows. If $\mathcal{A} = \sigma(\mathcal{U}^x; b)$ we define $f\mathcal{A} = \sigma(\mathcal{U}^x; fb) = f[\mathcal{A}]$, which clearly preserves the sort and the relation $\in$. Clearly, this extension is unique.

The homogeneity of $\mathcal{U}^{eq}$ follows by back-and-forth as in the one-sorted case.

13.5 Proposition Every elementary map $k : \mathcal{U}^{eq} \to \mathcal{U}^{eq}$ of cardinality $< \kappa$ extends to an automorphism of $\mathcal{U}^{eq}$.

13.3 The eq-definable closure

We may safely identify automorphism of $\mathcal{U}$ with automorphisms of $\mathcal{U}^{eq}$. Let $A \subseteq \mathcal{U}^{eq}$ and let a be a tuple of elements of $\mathcal{U}^{eq}$. We denote by $\mathrm{Aut}(\mathcal{U}/A)$ the set of automorphisms (of $\mathcal{U}^{eq}$) that fix all elements of A. The symbol $\mathcal{O}(a/A)$ denotes the orbit of a over A. This has been defined in Section 9.2 and now we apply it to $\mathcal{U}^{eq}$

$$\mathcal{O}(a/A) \quad = \quad \{fa \ : \ f \in \mathrm{Aut}(\mathcal{U}/A)\}.$$

By homogeneity, $\mathcal{O}(a/A) = p(\mathcal{U}^{eq})$ where $p(v) = \mathrm{tp}(a/A)$. When $\mathcal{O}(a/A) = \{a\}$ we say that a is invariant over A or A-invariant, for short.

13.6 Definition Let $A \subseteq \mathcal{U}^{eq}$ and $a \in \mathcal{U}^{eq}$. When $\varphi(a) \wedge \exists^{=1} v\, \varphi(v)$ holds for some formula $\varphi(v) \in L^{eq}(A)$, we say that a is definable over A. We write $\mathrm{dcl}^{eq}(A)$ for the set of those $a \in \mathcal{U}^{eq}$ that are definable over A. We write $\mathrm{dcl}(A)$ for $\mathrm{dcl}^{eq}(A) \cap \mathcal{U}$. This is the natural generalization of the notion of definability introduced in Section 11.1.

The definition above treats first- and second-order elements of $\mathcal{U}^{eq}$ uniformly. The following propositions proves that when $a \in \mathcal{U}^{eq}$ is a definable set, the notion of definability coincides with the one usually applied to sets.

13.7 Proposition Let $A \subseteq \mathcal{U}^{eq}$ and let $\mathcal{A} \in \mathcal{U}^{eq}$ have sort $\sigma(x\,; z)$. Then the following are equivalent

1. $\mathcal{A} \in \mathrm{dcl}^{eq}(A)$
2. $\mathcal{A} = \psi(\mathcal{U}^x)$ for some $\psi(x) \in L(A)$.

Proof. Implication 2⇒1 is clear because extensionality is implicit in the definition of $\mathcal{U}^{eq}$. We prove 1⇒2. Let $\varphi(\mathcal{X}) \in L^{eq}(A)$ be a formula $\mathcal{A}$ is the unique solution of. Then 2 holds with $\exists \mathcal{X}\, \big[x \in \mathcal{X} \wedge \varphi(\mathcal{X})\big]$ for $\psi'(x)$. This $\psi'(x)$ is a formula in $L^{eq}(A)$. Proposition 13.1 yields the required formula $\psi(x) \in L(A)$. □

The saturation and homogeneity of $\mathcal{U}^{eq}$ allows us to prove the following proposition with virtually the same proof as for Theorem 11.2

13.8 Theorem For any $A \subseteq \mathcal{U}^{eq}$ and $a \in \mathcal{U}^{eq}$ the following are equivalent

1. a is invariant over A
2. $a \in \mathrm{dcl}^{eq}(A)$.

By Proposition 13.7, Theorem 13.8 when applied to a definable set $\mathcal{A}$ gives an alternative proof of Proposition 9.24.

We conclude this section with a remark about the canonicity of the definitions of sets. The formula $\psi(x)$ in Proposition 13.7 need not be the sort $\sigma(x\,;z)$. For example, consider the theory of a binary equivalence relation $e(x\,;z)$ with two infinite classes, let $\mathcal{A}$ be one of these classes and let $A \neq \varnothing$ be such that $A \cap \mathcal{A} = \varnothing$. Then $\mathcal{A}$ is definable over A though not by some formula of the form $e(x\,;b)$ for some $b \in A$. Things change if we replace A with a model.

13.9 Proposition Let M be a model and let $\mathcal{A}$ be an element of sort $\sigma(x\,;z)$. Then the following are equivalent

1. $\mathcal{A} \in \mathrm{dcl}^{\mathrm{eq}}(M)$
2. $\mathcal{A} = \sigma(\mathcal{U}^x; b)$ for some $b \in M^z$.

In particular $M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$.

Proof. Assume 1 and let $\psi(x) \in L(M)$ be such that $\mathcal{A} = \psi(\mathcal{U}^x)$. Such a formula exists by Proposition 13.7. Then $\exists z\, \forall x\, \big[\psi(x) \leftrightarrow \sigma(x\,;z)\big]$ holds in $\mathcal{U}$. By elementarity it holds in M, therefore $\exists z$ has a witness in M. This proves 1⇒2, the converse implication is obvious. □

13.4 The eq-algebraic closure

The following is the natural generalization of the notion introduced in Section 11.1.

13.10 Definition Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and $a \in \mathcal{U}^{\mathrm{eq}}$. We say that a is algebraic over A if a formula of the form $\varphi(a) \wedge \exists^{=k} v\, \varphi(v)$ holds for some $\varphi(v) \in L^{\mathrm{eq}}(A)$ and some positive integer k. We write $\mathrm{acl}^{\mathrm{eq}}(A)$ for the set of those $a \in \mathcal{U}^{\mathrm{eq}}$ that are algebraic over A. We write $\mathrm{acl}(A)$ for $\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathcal{U}$.

The following proposition is proved with virtually the same proof as Theorem 11.3

13.11 Theorem For every $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and every $a \in \mathcal{U}^{\mathrm{eq}}$ the following are equivalent

1. $\mathcal{O}(a/A)$ is finite
2. $a \in \mathrm{acl}^{\mathrm{eq}} A$
3. $a \in M^{\mathrm{eq}}$ for every model such that $A \subseteq M^{\mathrm{eq}}$.

It is worthwile to spell out the equivalence 2⇔3 of the theorem above when a is a definable set $\mathcal{A}$. Namely, $\mathcal{A} \in \mathrm{acl}^{\mathrm{eq}} A$ if and only if $\mathcal{A}$ is definable over every model containing A.

We say finite equivalence relation for an equivalence relation with finitely many classes. A finite equivalence formula or type is a formula, respectively a type, that defines a finite equivalence relation. Theorem 13.12 belows proves that sets algebraic over A are union of classes of a finite equivalence relations definable over A.

13.12 Theorem Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and let $\mathcal{A} \in \mathcal{U}^{\mathrm{eq}}$ be an element of sort $\sigma(x\,;z)$. Then the following are equivalent

1. $\mathcal{A} \in \mathrm{acl}^{\mathrm{eq}} A$
2. for some finite equivalence formula $\varepsilon(x\,;y) \in L(A)$ and some $c_1, \ldots, c_n \in \mathcal{U}^{|y|}$

$$x \in \mathcal{A} \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \varepsilon(x\,;c_i).$$

Proof. 2⇒1 If $\varepsilon(x\,;y)$ has m classes, then $\mathcal{O}(\mathcal{A}/A)$ contains at most $\binom{m}{n}$ sets.

1⇒2 Let $\varphi(\mathcal{X}) \in L^{\mathrm{eq}}(A)$ be an algebraic formula that has $\mathcal{A}$ among its solutions and define

$$\varepsilon(x\,;y) \quad = \quad \forall \mathcal{X} \Big[\varphi(\mathcal{X}) \ \rightarrow \ \big[x \in \mathcal{X} \leftrightarrow y \in \mathcal{X}\big]\Big]$$

If $\varphi(\mathcal{X})$ has n, solutions, then $\varepsilon(x\,;y)$ has at most 2^n equivalence classes. Clearly, $\mathcal{A}$ is union of some these classes. □

13.13 Definition We write $a \stackrel{\mathrm{Sh}}{\equiv}_A b$ when $\varepsilon(a\,;b)$ holds for every finite equivalence formula $\varepsilon(x\,;y) \in L(A)$. In words we say that a and b have the same Shelah strong-type over A.

By the following proposition, the Shelah strong type of a over A is $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}} A)$.

13.14 Proposition Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and let $a, b \in \mathcal{U}^x$. Then the following are equivalent

1. $a \stackrel{\mathrm{Sh}}{\equiv}_A b$
2. $a \equiv_{\mathrm{acl}^{\mathrm{eq}} A} b$.

Proof. 2⇒1 Assume ¬1 and let $\varepsilon(x\,;y) \in L(A)$ be a finite equivalence formula such that $\neg\varepsilon(a\,;b)$. Let $\mathcal{D} = \varepsilon(\mathcal{U}^x;b)$, then $b \in \mathcal{D}$ and $a \notin \mathcal{D}$. As $\varepsilon(x\,;y)$ is an A-invariant finite equivalence formula, $\mathcal{D} \in \mathrm{acl}^{\mathrm{eq}} A$, and ¬2 follows.

1⇒2 Assume ¬2 and let $\varphi(x) \in L\big(\mathrm{acl}^{\mathrm{eq}} A\big)$ be such that $\varphi(a) \not\leftrightarrow \varphi(b)$. Let $\mathcal{D} = \varphi(\mathcal{U}^x)$, then $\mathcal{D} \in \mathrm{acl}^{\mathrm{eq}} A$. Therefore, by Proposition 13.12, the set $\mathcal{D}$ is union of equivalence classes of some finite equivalence formula $\varepsilon(x\,;y) \in L(A)$. Then $\neg\varepsilon(a\,;b)$ and ¬1 follows. □

⚠ We write $\mathcal{S}(a/A)$ for the intersection of all definable sets that contain a and are algebraic over A. By the proposition above $\mathcal{S}(a/A) = \big\{b : b \stackrel{\mathrm{Sh}}{\equiv}_A a\big\} = \mathcal{O}(a/\mathrm{acl}^{\mathrm{eq}} A)$.

13.15 Exercise Let $p(x) \subseteq L(A)$ and let $\varphi(x\,;y) \in L(A)$ be a formula that defines, when restricted to $p(\mathcal{U}^x)$, an equivalence relation with finitely many classes. Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x\,;y)$ on $p(\mathcal{U}^x)$.

13.16 Exercise Let $A \subseteq \mathcal{U}$ and let $\mathcal{A}$ be a definable set with finite orbit over A. Without using the eq-expansion, prove that $\mathcal{A}$ is union of classes of a finite equivalence relation definable over A.

13.17 Exercise Let T be strongly minimal and let $\varphi(x\,;z) \in L(A)$ with $|x| = 1$. For arbitrary $b \in \mathcal{U}^z$, prove that if the orbit of $\varphi(\mathcal{U}^x;b)$ over A is finite, then $\varphi(\mathcal{U}^x;b)$ is definable over $\operatorname{acl} A$.

Hint: you can use Theorem 13.12.

13.5 Elimination of imaginaries

For the time being, we agree that *imaginary* is just another word for definable set. Though this is not formally correct (cfr. Section 13.7), it is morally true and helps to understand the terminology. The concept of elimination of imaginaries has been introduced by Poizat who also proved Theorem 13.27 below. A theory has elimination of imaginaries if for every $A \subseteq \mathcal{U}^{\text{eq}}$, there is a $B \subseteq \mathcal{U}$ such that $L(B)$ and $L(A)$ have the same expressive power (i.e. they are the same up to to equivalence).

13.18 Definition We say that T has elimination of imaginaries if for every definable set $\mathcal{A}$ there is a formula $\varphi(x\,;z) \in L$ such that

ei $$\exists^{=1} z\ \forall x \Big[x \in \mathcal{A} \ \leftrightarrow\ \varphi(x\,;z)\Big]$$

We say that the witness of $\exists^{=1}z$ in the formula above is a canonical parameter of $\mathcal{A}$ or a canonical name for $\mathcal{A}$. A set may have different canonical parameters for different formulas $\varphi(x\,;z)$.

We say that T has weak elimination of imaginaries if

wei $$\exists^{=k} z\ \forall x \Big[x \in \mathcal{A} \ \leftrightarrow\ \varphi(x\,;z)\Big]$$

for some positive integer k.

In the formulas above we allow z to be the empty string. In this case we read ei and wei omitting the quantifiers $\exists^{=1}z$, respectively $\exists^{=k}z$. Therefore $\varnothing$-definable sets have all (at least) the empty string as a canonical parameter.

To show that the notions above are well-defined properties of a theory one needs to check that they are independent of our choice of monster model. We leave this to the reader as an exercise.

We say that two tuples a and b of elements of $\mathcal{U}^{\text{eq}}$ are interdefinable if $\operatorname{dcl}^{\text{eq}}(a) = \operatorname{dcl}^{\text{eq}}(b)$. By Theorem 13.8 this is equivalent to saying that $\operatorname{Aut}(\mathcal{U}/a) = \operatorname{Aut}(\mathcal{U}/b)$, that is, the automorphisms that fix a fix also b, and vice versa.

13.19 Theorem The following are equivalent

1. T has weak elimination of imaginaries
2. every definable set is interdefinable with a finite set
3. every definable set $\mathcal{A} \in \operatorname{dcl}^{\text{eq}}(\operatorname{acl}\{\mathcal{A}\})$.

Proof. 1⇒2 Assume 1 and let $\mathcal{B}$ be the set of solutions of the formula

$$\forall x \Big[x \in \mathcal{A} \ \leftrightarrow\ \sigma(x\,;z)\Big].$$

Hence $\mathcal{B}$ is finite and $\mathcal{B} \in \mathrm{dcl}^{\mathrm{eq}}\{\mathcal{A}\}$. We also have $\mathcal{A} \in \mathrm{dcl}^{\mathrm{eq}}\{\mathcal{B}\}$ because $\mathcal{A}$ is definable by the formula $\exists z\,[z \in \mathcal{B} \wedge \sigma(x\,;z)]$. Therefore $\mathrm{dcl}^{\mathrm{eq}}\{\mathcal{A}\} = \mathrm{dcl}^{\mathrm{eq}}\{\mathcal{B}\}$.

2⇒3 Assume $\mathrm{dcl}^{\mathrm{eq}}\{\mathcal{A}\} = \mathrm{dcl}^{\mathrm{eq}}\{\mathcal{B}\}$ for some finite set $\mathcal{B}$. The elements of $\mathcal{B}$, say $b_1,\dots,b_n$, are the (finitely many) solutions of the formula $z \in \mathcal{B}$. Therefore $b_1,\dots,b_n \in \mathrm{acl}\{\mathcal{B}\} = \mathrm{acl}\{\mathcal{A}\}$. Let $\varphi(\mathcal{X}\,;\mathcal{B})$ be a formula that has $\mathcal{A}$ as unique solution. Then $\exists \mathcal{Y}\,[\mathcal{Y} = \{b_1,\dots,b_n\} \wedge \varphi(\mathcal{X}\,;\mathcal{Y})]$ is the formula that proves 3.

3⇒1 Assume 3. As wei holds trivially for all $\varnothing$-definable sets, we may assume $\mathcal{A} \neq \varnothing$. Let $\sigma(x\,;z)$ be such that

$$\forall x \Big[x \in \mathcal{A} \;\leftrightarrow\; \sigma(x\,;b) \Big].$$

for some tuple b of elements of $\mathrm{acl}\{\mathcal{A}\}$. Fix some algebraic formula $\delta(z\,;\mathcal{A})$ satisfied by b and write $\psi(z\,;\mathcal{X})$ for the formula

$$\forall x \Big[x \in \mathcal{X} \;\leftrightarrow\; \sigma(x\,;z) \Big] \;\wedge\; \delta(z\,;\mathcal{X}).$$

The formula $\sharp$ below is clearly satisfied by b therefore, if we can prove that it has finitely many solutions, wei follows from Proposition 13.1

$$\sharp \qquad \forall x \Big[x \in \mathcal{A} \;\leftrightarrow\; \sigma(x\,;z) \;\wedge\; \exists \mathcal{X}\, \psi(z\,;\mathcal{X}) \Big].$$

We check that any c that satisfies $\sharp$ also satisfies $\delta(z\,;\mathcal{A})$. As $\mathcal{A}$ is non empty, $\psi(c\,;\mathcal{A}')$ holds for some $\mathcal{A}'$. By $\sharp$ and the definition of $\psi(z\,;\mathcal{X})$ we obtain $\mathcal{A}' = \mathcal{A}$ and $\delta(c\,;\mathcal{A})$. □

There are a notions of elimination of imaginaries that are weaker than weak elimination. For instance, we say that T has geometric elimination of imaginaries if for every $A \subseteq \mathcal{U}$

$$\mathcal{A} \;\in\; \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}\{\mathcal{A}\})$$

This will not be applied in these notes, but see Exercises 13.22 and 13.23.

13.20 Theorem The following are equivalent

1. T has elimination of imaginaries
2. every definable set is interdefinable with a tuple of real elements
3. every definable set $\mathcal{A}$ is definable over $\mathrm{dcl}\{\mathcal{A}\}$.

Proof. Implications 1⇒2⇒3 are immediate. Implication 3⇒1 is identical to the homologous implication in Theorem 13.19, just substitute algebraic with definable. □

13.21 Exercise Let T have elimination of imaginaries and $\varphi(x\,;z) \in L(A)$. For arbitrary $c \in \mathcal{U}^z$, prove that if the orbit of $\varphi(\mathcal{U}^x;c)$ over A is finite, then $\varphi(\mathcal{U}^x;c)$ is definable over $\mathrm{acl}A$.

13.22 Exercise Prove that following are equivalent for every $A \subseteq \mathcal{U}$

1. $\mathrm{acl}^{\mathrm{eq}}A = \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}A)$ for every $A \subseteq \mathcal{U}$
2. $\mathrm{Aut}(\mathcal{U}/\mathrm{acl}^{\mathrm{eq}}A) = \mathrm{Aut}(\mathcal{U}/\mathrm{acl}A)$
3. $a \equiv_{\mathrm{acl}A} b \;\Leftrightarrow\; a \overset{\mathrm{Sh}}{\equiv}_A b$ for every $A \subseteq \mathcal{U}$ and $a,b \in \mathcal{U}^{<\omega}$.

13.23 Exercise Prove that the following are equivalent

1. T has weak elimination of imaginaries
2. T has geometric elimination of imaginaries and 1 of Exercise 13.22 holds.

13.24 Example Prove that the following are equivalent

1. T has weak elimination of imaginaries
2. for every $\mathcal{A} \in \mathcal{U}^{\mathrm{eq}}$ there exists the least algebraically closed set $A \subseteq \mathcal{U}$ such that $\mathcal{A}$ is definable over A.

13.6 Examples

We now consider elimination of imaginaries in two concrete theories that have been introduced in the previous chapters: algebraically closed fields and the random graph.

13.25 Lemma The following is a sufficient condition for weak elimination of imaginaries

$\sharp$ for every $A \subseteq \mathcal{U}^{\mathrm{eq}}$, every consistent $\varphi(z) \in L(A)$ has a solution in $\mathrm{acl}A$.

Proof. Let $\mathcal{A} \in \mathcal{U}^{\mathrm{eq}}$ be a definable set of sort $\sigma(x\,;z)$. Then $\forall x\, \big[x \in \mathcal{A} \leftrightarrow \sigma(x\,;z)\big]$ is consistent and, by $\sharp$ it has a solution in $\mathrm{acl}\{\mathcal{A}\}$. Hence weak elimination follows from Theorem 13.19. □

13.26 Theorem Let T be a complete, strongly minimal theory. Then, if $\mathrm{acl}\varnothing$ is infinite, T has weak elimination of imaginaries.

Proof. If $\mathrm{acl}\varnothing$ is infinite, $\mathrm{acl}A$ is a model for every A (cfr. Exercise 11.26) so condition $\sharp$ of lemma 13.25 holds by elementarity and the theorem follows. □

13.27 Theorem The theories T^p_{acf} have elimination of imaginaries.

Proof. By Theorem 13.26 we know that T^p_{acf} has weak elimination of imaginaries. Therefore, by Theorem 13.19 it suffices to prove that every finite set $\mathcal{A}$ is interdefinable with a tuple. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ where each a_i is a tuple $a_{i,1}, \ldots, a_{i,m}$ of elements of $\mathcal{U}$. Given $\mathcal{A}$ we define the term

$$t_{\mathcal{A}}(x\,;y) \quad = \quad \prod_{i=1}^{n} \bigg(x - \sum_{k=1}^{m} a_{i,k}\, y_k\bigg). \qquad \text{where } y = y_1, \ldots, y_m$$

Note that (the interpretation of) the term $t_{\mathcal{A}}(x\,;y)$ is independent on particular indexing of the set $\mathcal{A}$. So, any automorphism that fixes $\mathcal{A}$, fixes the $t_{\mathcal{A}}(x\,;y)$. Now rewrite $t_{\mathcal{A}}(x\,;y)$ as a sum of monomials and let c be the tuple of coefficients of these monomials. The tuple c uniquely determines $t_{\mathcal{A}}(x\,;y)$ and vice versa. Therefore every automorphism that fixes $\mathcal{A}$ fixes c and vice versa. Hence $\mathcal{A}$ and c are interdefinable. □

The rest of this section is dedicated to the proof that the theory of the random graph has weak elimination of quantifiers.

13.28 Lemma The following is a sufficient condition for weak elimination of imaginaries

♮ $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ is definable both over A and over B, then it is definable over $A \cap B$.

Proof. Let $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ be a definable set, and pick a finite set $A \subseteq \mathcal{U}$ of minimal size over which $\mathcal{D}$ is definable. Now, let f be an automorphism that fixes $\mathcal{D}$. Then $\mathcal{D}$ is definable over $A \cap f[A]$, so by minimality of $|A|$ we have $A = f[A]$. Now let a be an enumeration of A. By the above, a has a finite orbit under $\mathrm{Aut}(\mathcal{U}/\{\mathcal{D}\})$. Hence $A \subseteq \mathrm{acl}\{\mathcal{D}\}$. Therefore $\mathcal{D} \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}\{\mathcal{D}\})$ The lemma follows by applying Theorem 13.19. □

13.29 Lemma Let $\mathcal{U} \vDash T_{\mathrm{rg}}$. Let a and b be tuples such that $\mathrm{rng}\, a \cap A, \mathrm{rng}\, b \cap B \subseteq A \cap B$. Assume also that $a \equiv_{A\cap B} b$. Then there is a tuple c such that $a \equiv_A c \equiv_B b$.

Proof. For simplicity assume that $\mathrm{rng}\, a \cap A = \mathrm{rng}\, b \cap B = \varnothing$. The general case follows easily.

By induction on the length of a and b. The claim is trivial if this length is 0 so, assume they have length $n+1$ and that there are $c_0, \ldots, c_{n-1} \notin A, B, \mathrm{rng}\, a, \mathrm{rng}\, b$ such that $a_{\restriction n} \equiv_A c_0, \ldots, c_{n-1} \equiv_B b_{\restriction n}$.

We assume that $\mathrm{rng}\, a \cap \mathrm{rng}\, b = \varnothing$ otherwise we replace b with any tuple $b' \equiv_B b$ that is disjoint of $\mathrm{rng}\, a$.

Now, let $A' = A \cup \{a_0, \ldots, a_{n-1}\}$ and $B' = B \cup \{b_0, \ldots, b_{n-1}\}$. By the assumption above $A' \cap B' = A \cap B$.

Let c_n be a vertex the satisfies the following formulas

1a $$\bigwedge_{a' \in A'} \Big[r(x, a') \leftrightarrow r(a_n, a') \Big]$$

1b $$\bigwedge_{b' \in B'} \Big[r(x, b') \leftrightarrow r(b_n, b') \Big]$$

2a $$\bigwedge_{i<n} \Big[r(x, c_i) \leftrightarrow r(a_n, a_i) \Big]$$

2b $$\bigwedge_{i<n} \Big[r(x, c_i) \leftrightarrow r(b_n, b_i) \Big]$$

Note that 1a and 1b are mutually consistent by the assumption $a \equiv_{A\cap B} b$. By the same assumption 2a and 2b. As $c_0, \ldots, c_{n-1} \notin A, B, \mathrm{rng}\, a, \mathrm{rng}\, b$ the four formulas above are mutually consistent.

Clearly, any c_n not in $A, B, \mathrm{rng}\, a, \mathrm{rng}\, b$ proves the lemma. □

13.30 Lemma The theory T_{rg} has weak elimination of imaginaries.

Proof. We prove ♮ of Lemma 13.28. Suppose not and pick $a, b \in \mathcal{U}^{|x|}$ such that $a \equiv_{A\cap B} b$ and $a \in \mathcal{D} \nLeftrightarrow b \in \mathcal{D}$.

Note that $\mathcal{D}$ is definable over $f[A]$ for every $f \in \mathrm{Aut}(\mathcal{U}/B)$ and the same holds swapping A and B. Therefore we can replace A and B with suitable $f[A]$ and $g[B]$

that are disjoint of rng a and rng b, rispectively (for simplicity we are assuming that a and b do not contain elements in $A \cap B$). Clearly $f[A] \cap g[B] = A \cap B$.

By Lemma 13.29, there is c such that $a \equiv_{f[A]} c \equiv_{g[B]} b$. This contradicts the fact that $\mathcal{D}$ is definable both over $f[A]$ and over $g[B]$. □

13.7 Imaginaries: the true story

The point of the expansion to $\mathcal{U}^{\mathrm{eq}}$ is to add a canonical parameter for each definable set. In fact, in $\mathcal{U}^{\mathrm{eq}}$ every definable subset of $\mathcal{U}^{z}$ is the canonical parameter of itself. This allows us to deal with theories without elimination of imaginaries in the most straightforward way.

The expansion to $\mathcal{U}^{\mathrm{eq}}$ that was originally introduced by Shelah (and still used everywhere else) is slightly different from the one introduced here. For a given set $\mathcal{A} = \sigma(\mathcal{U}^{x}; b)$ Shelah considers the equivalence relation defined by the formula

$$\varepsilon(z\,;z') \quad = \quad \forall x\,\Big[\sigma(x\,;z) \;\leftrightarrow\; \sigma(x\,;z')\Big].$$

The equivalence class of b in the relation $\varepsilon(z\,;z')$ is what Shelah uses as canonical parameter of the set $\mathcal{A}$.

Shelah's $\mathcal{U}^{\mathrm{eq}}$ has a sort for each $\varnothing$-definable equivalence relation $\varepsilon(z\,;z')$. The domain of the sort $\varepsilon(z\,;z')$ contains the classes of the equivalence relation defined by $\varepsilon(z\,;z')$. These equivalence classes are called imaginaries. Shelah's L^{eq} contains functions that map tuples in the home sort to their equivalence class.

Chapter 14

Invariant sets

In this chapter, L is a signature, T is a complete theory without finite models, and $\mathcal{U}$ is a saturated model of inaccessible cardinality κ strictly larger than $|L|$. We use the same notation and make the same implicit assumptions as in Section 9.3.

14.1 Invariant sets and types

Let $\mathcal{D} \subseteq \mathcal{U}^z$, where z is a tuple of length $< \kappa$. We say that $\mathcal{D}$ is nvariant over A, or A-invariant for sort, if it is fixed setwise by all A-automorphisms. That is, $f[\mathcal{D}] = \mathcal{D}$ for every automorphism $f \in \mathrm{Aut}(\mathcal{U}/A)$ or, yet in other words, if

is1. $\quad a \in \mathcal{D} \;\leftrightarrow\; fa \in \mathcal{D} \qquad$ for every $a \in \mathcal{U}^z$ and every $f \in \mathrm{Aut}(\mathcal{U}/A)$,

which, by homogeneity, is equivalent to,

is2. $\quad a \in \mathcal{D} \;\leftrightarrow\; b \in \mathcal{D} \qquad$ for all $a, b \in \mathcal{U}^z$ such that $a \equiv_A b$.

The latter condition yield the following bound on the number of invariant sets.

14.1 Proposition Let $\lambda = |L_z(A)|$. There are at most 2^{2^λ} sets $\mathcal{D} \subseteq \mathcal{U}^z$ that are invariant over A.

Proof. By is2, sets that are invariant over A are unions of equivalence classes of the relation $\equiv_A$, that is, unions of sets of the form $p(\mathcal{U})$ where $p(z) \in S(A)$. Then the number of A-invariant sets is at most $2^{|S_z(A)|}$. Clearly $|S_z(A)| \leq 2^\lambda$. □

A formula is invariant if the set it defines is invariant.

The definition of invariant type is less straightforward. One is tempted to require that the formulas in the type are invariant – but this is too strong a demand. Alternatively, one could require that the set defined by the type is invariant – but this would be an empty requirement for large types that are not realized in $\mathcal{U}$.

First we introduce some notation. Let $\varphi(x\,;z) \in L$. A φ-formula is a formula the form $\vartheta(x\,;\bar{b})$ for some $\vartheta(x\,;\bar{z})$ that is a Boolean combination of the formulas $\varphi(x\,;z_i)$, where $\bar{z} = z_1, \ldots, z_n$, and some $\bar{b} \in \mathcal{U}^{\bar{z}}$. A φ-definable set is a set $\vartheta(\mathcal{U}^x;\bar{b})$ definable by a φ-formula. We write $L_\varphi(\mathcal{A})$ for the set of φ-formulas with parameters in $\mathcal{A}$.

We are mainly concerned in Δ-types where Δ is either $L(\mathcal{A})$ or $L_\varphi(\mathcal{A})$. In the latter case Δ-types are called φ-types. We denote by $S_\varphi(\mathcal{A})$ the set of complete φ-types with parameters in $\mathcal{A}$. Typically, $\mathcal{A}$ is either $\mathcal{U}$ or some small set $A \subseteq \mathcal{U}$. Types in $S_\varphi(\mathcal{U})$ are called global φ-types. Note that whenever convenient we can assume that complete φ-types only contain formulas of the form $\varphi(x\,;b)$ and $\neg\varphi(x\,;b)$.

Let $p(x) \subseteq L(\mathcal{U})$ be a consistent type. For every formula $\varphi(x\,;z) \in L$ we define

$$\mathcal{D}_{p,\varphi} \;=\; \Big\{a \in \mathcal{U}^z \;:\; \varphi(x\,;a) \in p\Big\}.$$

A global φ-type $p(x) \in S_\varphi(\mathcal{U})$ can be identified with the set $\mathcal{D}_{p,\varphi}$. Therefore a global type $p(x) \in S(\mathcal{U})$ can be identified with the collection of the sets $\mathcal{D}_{p,\varphi}$, where $\varphi(x\,;z)$ ranges over L.

We say that $p(x) \subseteq L(\mathcal{U})$ is invariant over A, or A-invariant for short, if for every formula $\varphi(x\,;z) \in L$

it1. $\varphi(x\,;a) \in p \;\Leftrightarrow\; \varphi(x\,;fa) \in p$ for every $a \in \mathcal{U}^z$ and every $f \in \mathrm{Aut}(\mathcal{U}/A)$.

Hence $p(x)$ is invariant exactly when all the sets $\mathcal{D}_{p,\varphi}$ are. That is, $p(x)$ is invariant over A if

it2. $a \equiv_A b \;\Rightarrow\; \Big(\varphi(x\,;a) \in p \Leftrightarrow \varphi(x\,;b) \in p\Big)$ for all $\varphi(x\,;z) \in L$ and $a, b \in \mathcal{U}^z$.

If $p(x)$ is a global φ-types types it2 is equivalent to

it2′. $a \equiv_A b \;\Rightarrow\; p(x) \vdash \varphi(x\,;a) \leftrightarrow \varphi(x\,;b)$ for all $a, b \in \mathcal{U}^z$.

Finally, note the following useful characterization of it2′

it3. $a \equiv_A b \;\Rightarrow\; \varphi(c\,;a) \leftrightarrow \varphi(c\,;b)$ for all $a, b \in \mathcal{U}^z$ and for all $c \vDash p_{\restriction A,a,b}$.

When $p(x) \in S(\mathcal{U})$, this can be rephrased as

it3′. $a \equiv_A b \;\Rightarrow\; a \equiv_{A,c} b$ for all $a, b \in \mathcal{U}^z$ and for all $c \vDash p_{\restriction A,a,b}$.

14.2 Heirs and coheirs

The easiest way to obtain types that are invariant over a model M is via types that are finitely satisfiable in M. We say that a type $p(x)$ is finitely satisfiable in A if every conjunction of formulas in $p(x)$ has a solution in A^x.

14.2 Proposition Every type $p(x) \subseteq L(\mathcal{U})$ that is finitely satisfiable in A is invariant over A.

Proof. Suppose not. By it3 there are $a \equiv_A b$ and $\varphi(x\,;z) \in L$ such that $p(x) \vdash \varphi(x\,;a) \not\leftrightarrow \varphi(x\,;b)$. Then the finite satisfiability of $p(x)$ contradicts $a \equiv_A b$. □

14.3 Proposition Every type $q(x) \subseteq L(\mathcal{U})$ that is finitely satisfiable in A has an extension to a global type that is finitely satisfiable in A.

Proof. Let $p(x) \subseteq L(\mathcal{U})$ be maximal among the types that contain $q(x)$ and are finitely satisfiable in A. We prove that $p(x)$ is complete. If for a contradiction $p(x)$ contains neither $\psi(x)$ nor $\neg\psi(x)$, then neither $p(x) \cup \{\psi(x)\}$ nor $p(x) \cup \{\neg\psi(x)\}$ are finitely satisfiable in A. This contradicts the finite satisfiability of $p(x)$. □

In most cases we work with types that are finitely satisfiable over a model. The reason is explained by the next proposition, which is clear by elementarity.

14.4 Proposition Every consistent type over a model is finitely satisfiable in that model, that is, whenever $p(x) \subseteq L(M)$ is consistent, $p(x)$ is finitely satisfiable in M.

14.5 Definition A type $p(x) \subseteq L(\mathfrak{U})$ that is finitely satisfiable in M is said to be a coheir of $p_{\restriction M}(x)$.

In many cases it is convenient to work with elements instead of types. We introduce the following notation to express that $\mathrm{tp}(a/M, b)$ is finitely satisfied in M.

14.6 Definition For every $a \in \mathfrak{U}^x$ and $b \in \mathfrak{U}^z$ we define

$$a \mathop{\smile\!\!\!\!\!\mid}_M b \quad \Leftrightarrow \quad \varphi(M^x, b) \neq \varnothing \text{ for all } \varphi(x\,;z) \in L(M) \text{ such that } \varphi(a\,;b).$$

We say that $\mathrm{tp}(a/M, b)$ is a coheir of $\mathrm{tp}(a/M)$ or, equivalently, that $\mathrm{tp}(b/M, a)$ is an heir of $\mathrm{tp}(b/M)$. We define the type

$$x \mathop{\smile\!\!\!\!\!\mid}_M b \quad = \quad \Big\{\neg\varphi(x\,;b) \;:\; \varphi(x\,;b) \in L(M) \text{ and } \varphi(M^x; b) = \varnothing\Big\}.$$

We will write $a \equiv_M x \mathop{\smile\!\!\!\!\!\mid}_M b$ for the union of the types $x \mathop{\smile\!\!\!\!\!\mid}_M b$ and $\mathrm{tp}(a/M)$.

The tuples a realizing $x \mathop{\smile\!\!\!\!\!\mid}_M b$ are exactly those such that $a \mathop{\smile\!\!\!\!\!\mid}_M b$. Note that $\mathrm{tp}(a/M, b)$ is a coheir of $\mathrm{tp}(a/M)$ according to Definition 14.5, so the terminology is consistent.

We may think of $a \mathop{\smile\!\!\!\!\!\mid}_M b$ as saying that a is independent from b over M. This is a strong notion of independence. In general it is not symmetric, that is $b \mathop{\smile\!\!\!\!\!\mid}_M a$ is not equivalent to $a \mathop{\smile\!\!\!\!\!\mid}_M b$. In Chaper 18 we will see that symmetry is equivalent to stability.

We will use the following easy lemma without explicit reference.

14.7 Lemma The following properties hold for all M, a, b, and c

1. $a \mathop{\smile\!\!\!\!\!\mid}_M b \;\Rightarrow\; fa \mathop{\smile\!\!\!\!\!\mid}_M fb$ for every $f \in \mathrm{Aut}(\mathfrak{U}/M)$ — *invariance*
2. $a \mathop{\smile\!\!\!\!\!\mid}_M b \;\Leftrightarrow\; a_0 \mathop{\smile\!\!\!\!\!\mid}_M b_0$ for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$ — *finite character*
3. $a \mathop{\smile\!\!\!\!\!\mid}_M c, b$ and $c \mathop{\smile\!\!\!\!\!\mid}_M b \;\Rightarrow\; a, c \mathop{\smile\!\!\!\!\!\mid}_M b$ — *transitivity*
4. $a \mathop{\smile\!\!\!\!\!\mid}_M b \;\Rightarrow$ there exists $a' \equiv_{M,b} a$ such that $a' \mathop{\smile\!\!\!\!\!\mid}_M b, c$ — *coheir extension*
5. $a \mathop{\smile\!\!\!\!\!\mid}_M b_1, b_2$ and $b_1 \equiv_M b_2 \;\Rightarrow\; b_1 \equiv_{M,a} b_2$ — *non-splitting*

Proof. Properties 1-3 follow immediately from Definition 14.6. We prove 4. Assume $a \mathop{\smile\!\!\!\!\!\mid}_M b$, that is, $\mathrm{tp}(a/M, b)$ is finitely satisfiable in M. By Proposition 14.3 $\mathrm{tp}(a/M, b)$ extends to a global type $p(x)$ that is finitely satisfiable in M. Then any $a' \models p_{\restriction M, b, c}(x)$ proves the lemma. The proof of 5 is left to the reader. □

14.8 Proposition Let $a \mathop{\smile\!\!\!\!\!\mid}_M b$ then there is $\mathcal{V} \preceq \mathfrak{U}$ that is isomorphic to $\mathfrak{U}$ over M, b and such that $a \mathop{\smile\!\!\!\!\!\mid}_M \mathcal{V}$.

Proof. Let c be an enumeration of $\mathfrak{U}$. Let $p(w, z) = \mathrm{tp}(c, b/M)$. We can take $\mathcal{V}$ to be the structure enumerated by the tuple that realizes symultaneously $p(w, b)$ and the following type

$$\Big\{\neg\varphi(a, w, b) \quad : \quad \varphi(x, w, z) \in L(M), \;\; p(w, b) \to \varphi(M^x, w, b) = \varnothing\Big\}.$$

We only need to prove consistency. If inconsistent, then

$$\vartheta(w,b) \quad\rightarrow\quad \bigvee_{i=1}^{n} \varphi_i(a,w,b)$$

for some $\vartheta(w,z) \in p$ and some $\varphi_i(x,w,z)$ such that $p(w,b) \rightarrow \varphi_i(M^x,w,b) = \varnothing$. We obtain a contradiction because, as $a \downfree_M b$, we can replace a by some $a' \in M^x$. $\square$

The type $a \equiv_M x \downfree_M b$ in Definition 14.6 is the intersection of all global coheirs of $\mathrm{tp}(a/M)$. Its consistency is guaranteed by the fact that M is a model (see Proposition 14.4). However, in general it need not be a complete type over M, b. In fact, its completeness of this type is a property with important consequences.

14.9 Definition We say that $\downfree_M$ is stationary if $a \equiv_M x \downfree_M b$ is a complete type over M, b for all finite tuples b and a. We say n-stationary if we limit the request to tuples a of length n.

An application of stationarity is given in Section 15.3. Stationarity is often ensured by the following property, which will receive due attention in Section 18.7.

14.10 Proposition Let x be a tuple of variables of length n. If for every $\varphi(x) \in L(\mathfrak{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(M^x) = \psi(M^x)$ then $\downfree_M$ is n-stationary.

Proof. Let $b \in \mathfrak{U}^z$ and $a_1, a_2 \in \mathfrak{U}^x$ be such that $a_i \downfree_M b$ and $a_1 \equiv_M a_2$. We claim that $a_1 \equiv_{M,b} a_2$. We need to prove that $\varphi(a_1\,;b) \leftrightarrow \varphi(a_2\,;b)$ for every $\varphi(x\,;z) \in L(M)$. Let $\psi(x) \in L(M)$ be such that $\varphi(\mathfrak{U}^x;b) = \psi(\mathfrak{U})$. From $a_i \downfree_M b$ we obtain that $\varphi(a_i\,;b) \leftrightarrow \psi(a_i)$. Finally, the claim follows because $a_1 \equiv_M a_2$. $\square$

14.11 Exercise Let T be strongly minimal. Let $a \in \mathfrak{U}$ and $b \in \mathfrak{U}^z$. Prove that $a \downfree_M b$ if and only if $a \in M$, $b \in M^z$ or $a \notin \mathrm{acl}(M,b)$.

14.3 Morley sequences and indiscernibles

In what follows α is some ordinal $\leq \kappa$, typically ω, and x is a tuple of variables of length $< \kappa$.

Let $p(x) \in S(\mathfrak{U})$ be a global type. We say that $\bar{c} = \langle c_i : i < \alpha \rangle$ is a Morley sequence of $p(x)$ over A if for every $i < \alpha$

Ms. $\qquad c_i \quad\models\quad p_{\restriction A, c_{\restriction i}}(x).$

When $p(x)$ is finitely satisfiable in A, we say that $\bar{c}$ is a coheir sequence of $p(x)$ over A.

When we say that $\bar{c}$ is a coheir sequence over A (with no explicit reference to a global type), we mean that *there is* a type $p(x) \in S(\mathfrak{U})$ that is finitely satisfiable in A such that $\bar{c}$ is a coheir sequence of $p(x)$.

The following is a convenient characterization of coheir sequences.

14.12 Lemma The following are equivalent

1. $\bar{c} = \langle c_n : n < \omega \rangle$ is a coheir sequence over M
2. $c_n \downfree_M c_{\restriction n}$ and $c_{n+1} \equiv_{M, c_{\restriction n}} c_n$ for every $n < \omega$.

Proof. 1⇒2. Assume 1 and let $p(x) \in S(\mathfrak{U})$ be a global type that is finitely satisfiable in M and such that $c_i \models p_{\restriction M, c_{\restriction i}}(x)$. The requirement $c_{n+1} \equiv_{M, c_{\restriction n}} c_n$ is clear. Now, suppose $\varphi(c_{n+1})$ for some $\varphi(x) \in L(M, c_{\restriction n+1})$. Then $\varphi(x)$ belongs to $p(x)$, so $\varphi(\mathfrak{U}^x) \cap M^x \neq \varnothing$ because $p(x)$ is finitely satisfiable in M. This proves $c_n \downarrow_M c_{\restriction n}$.

2⇒1. Let $q(x) = \{\varphi(x) \in L(M, \bar{c}) : \varphi(c_n)$ holds for cofinitely many $n\}$. We claim that $q(x)$ is finitely satisfiable in M. Let $\varphi(x\,;z) \in L(M)$ be such that $\varphi(x\,;c_{\restriction n}) \in q$. By the definition of $q(x)$, the formula $\varphi(c_m\,;c_{\restriction n})$ holds for all sufficiently large m. Hence, from 2 we infer $c_m \downarrow_M c_{\restriction n}$ and conclude that $\varphi(x\,;c_{\restriction n})$ is satisfied in M.

Let $p(x)$ be any global extension of $q(x)$ finitely saisfied in M. We prove that $\bar{c}$ is a Morley sequence of $p(x)$ over M. By 2 either $c_m \models p_{\restriction M, c_{\restriction n}}(x)$ for all $m \geq n$ or $c_m \not\models p_{\restriction M, c_{\restriction n}}(x)$ for all $m \geq n$. As $p(x)$ extends $q(x)$, the latter cannot occur. □

Let $(I, <_I)$ be a linear order. A function $\bar{a} : I \to \mathfrak{U}^x$ is said to be an *I*-sequence, or simply a sequence when I is clear. We will often introduce an I-sequence as $\bar{a} = \langle a_i : i \in I \rangle$.

If $I_0 \subseteq I$ we call $a_{\restriction I_0}$ a subsequence of $\bar{a}$. The subsets $I_0 \subseteq I$ that are well-ordered by $<_I$, in particular the finite ones, are especially relevant. When I_0 has order type α, an ordinal, we identify $a_{\restriction I_0}$ with a tuple of length α.

Recall that $I^{(n)}$ denotes that the set of *n*-subsets of I, i.e. the subsets of I of cardinality n. The notation

$$\binom{I}{n} \quad = \quad I^{(n)}$$

is also common.

14.13 Definition Let $(I, <_I)$ be an infinite linear order and let $\bar{a}$ be an I-sequence. We say that $\bar{a}$ is a sequence of indiscernibles over A or, an *A*-indiscernible sequence, if $a_{\restriction I_0} \equiv_A a_{\restriction I_1}$ for every $I_0, I_1 \in I^{(n)}$ and $n < \omega$.

The indiscernibility condition can be formulated in a number of equivalent ways. For example, we can require that, for every formula $\varphi(x_1, \ldots, x_n) \in L(A)$ and every pair of tuples in I^n such that $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$,

$$\varphi(a_{i_0}, \ldots, a_{i_n}) \quad \leftrightarrow \quad \varphi(a_{j_0}, \ldots, a_{j_n})$$

Alternatively, we can simply say that for all $i_0, \ldots, i_n \in I$ the type $\mathrm{tp}(a_{i_0}, \ldots, a_{i_n}/A)$ only depends on the order type of $i_0, \ldots, i_n$.

14.14 Proposition Let $p(x) \in S(\mathfrak{U})$ be a global A-invariant type and let $\bar{c} = \langle c_i : i < \alpha \rangle$ be a Morley sequence of $p(x)$ over A. Then $\bar{c}$ is an A-indiscernibles sequence.

Proof. We prove by induction on $n < \omega$ that

$\sharp \qquad c_{\restriction n} \equiv_A c_{\restriction I_0}$ for every $I_0 \subseteq \alpha$ of cardinality n.

For $n = 0$ the claim is trivial. We assume inductively that $\sharp$ above is true and prove that

$$c_{\restriction n}, c_n \equiv_A c_{\restriction I_0}, c_i \quad \text{for every } I_0 < i < \alpha.$$

As $\bar{c}$ is Morley sequence, $c_n \equiv_{A, c_{\restriction n}} c_i$ whenever $n < i$. Hence we can equivalently prove that

$$c_{\restriction n}, c_i \;\equiv_A\; c_{\restriction I_0}, c_i,$$

which is equivalent to

$$c_{\restriction n} \equiv_{A, c_i} c_{\restriction I_0}.$$

The latter holds by induction hypothesis $\sharp$ and the invariance of $p(x)$ as formulated in it3 of Section 14.1. $\square$

14.15 Exercise Let $\bar{a}$ be a sequence such that $a_{\restriction I_0} \equiv a_{\restriction I_1}$ for every I_0, I_1 such that $I_0 < I_1$. Prove that $\bar{a}$ is a sequence of indiscernibles.

Chapter 15

Ramsey theory

In Section 15.1 we prove Ramsey's theorem and in Section 15.2 we present its major application in model theory: the Ehrenfeucht-Mostowski construction of indiscernibles.

In the remaining sections we prove two important results of Ramsey theory. These results will not be used elsewhere in these notes. Our only purpose is to illustrate a concrete (relatively speaking) application of the notion of coheir.

15.1 Ramsey's theorem from coheir sequences

In this chapter we are interest in finite partitions. We may represent the partition of a set X into k subsets with a map $f : X \to [k]$. The elements of $[k] = \{1, \dots, k\}$ are also called colors, and the partition a coloring, or k-coloring, of X. We say that $Y \subseteq X$ is monochromatic if $f_{\restriction Y}$ is constant on Y.

Let M be an arbitrary infinite set. Fix $n, k < \omega$ and fix a coloring f of the set of all n-subsets of M, also known as the complete n-uniform hypergraph with vertex set M,

$$f \quad : \quad \binom{M}{n} \quad \to \quad [k].$$

We say that $H \subseteq M$ is a monochromatic subgraph if the subgraph induced by H is monochromatic. In combinatorics, monochromatic subgraphs are also called homogeneous sets.

The following is a very famous theorem which we prove here in an unusual (and overly convoluted) way. Our purpose is to illustrade the method of proof of Theorem 15.17. In the latter the model theoretic overhead really pays off.

15.1 Ramsey Theorem Let M be an infinite set and fix some positive integers n and k. Fix an arbitrary k-coloring of the (edges of the) complete n-uniform hypergraph with vertex set M. Then there is an infinite monochromatic subgraph $H \subseteq M$.

Proof. Let L be a language that contains k relation symbols $r_1, \dots, r_k$ of arity n. Given a k-coloring f we define a structure with domain M. The interpretation the relation symbols is

$$r_i^M \quad = \quad \Big\{ a \in M^n \ : \ |\operatorname{rng}(a)| = n \ \text{ and } \ f\big(\operatorname{rng}(a)\big) = i \Big\}.$$

We may assume that M is an elementary substructure of some large saturated model $\mathcal{U}$. Pick any type $p(x) \in S(\mathcal{U}^x)$ finitely satisfied in M but not realized in M and let $\bar{c} = \langle c_i : i < \omega \rangle$ be a coheir sequence of $p(x)$ over M.

There is a first-order sentence saying that the formulas $r_i(x_1, \dots, x_n)$ are a coloring of $M^{(n)}$. Then by elementarity the same holds in $\mathcal{U}$. By indiscernibility, all tuples of n distinct elements of $\bar{c}$ have the same color, say 1.

Incidentally, no element of $\bar{c}$ is in M. The theorem is proved if we can find in M a sequence $\bar{a} = \langle a_i : i < \omega\rangle$ with color 1. We construct $a_{\restriction i}$ by induction on i as follows.

Assume as induction hypothesis that the subsequences of length n of $a_{\restriction i}, c_{\restriction n}$ have all color 1. Our goal is to find $a_i \in M$ such that the same property holds for $a_{\restriction i}, a_i, c_{\restriction n}$. By the indiscernibility of $\bar{c}$, the property holds for $a_{\restriction i}, c_{\restriction n}, c_n$, and this can be written by a formula $\varphi(a_{\restriction i}, c_{\restriction n}, c_n)$. As $\bar{c}$ is a coheir sequence, by Lemma 14.12 we can find $a_i \in M$ such that $\varphi(a_{\restriction i}, c_{\restriction n}, a_i)$. So, as the order is irrelevant, $a_{\restriction i}, a_i, c_{\restriction n}$ satisfies the induction hypothesis. □

15.2 Exercise Let M be a graph with the property that for every finite $A \subseteq M$ there is a $c \in M$ such that $A \subseteq r(c, \mathcal{U})$. (This holds in particular when M is a random graph.) Prove that for every finite coloring of the edges of M, there is a subgraph that is complete and monochromatic.

15.3 Exercise The graph M is as in Exercise 15.2. A star in M is a subgraph whose edges all share a common vertex. We say that a coloring of the edges of M is locally finite if there is a k such that every star has at most k colors. Prove that for every locally finite coloring of the edges of M, there is an infinite monochromatic complete subgraph.

15.2 The Ehrenfeucht-Mostowski theorem

Let $I, <_I$ be an infinite linear order and let $\bar{a}$ be an I-sequence. Fix a tuple of distinct variables $\bar{x} = \langle x_i : i < \omega\rangle$. We write $p(\bar{x}) = \text{EM-tp}(\bar{a}/A)$ and say that $p(\bar{x})$ is the Ehrenfeucht-Mostowski type of $\bar{a}$ over A if

$$p(\bar{x}) \quad = \quad \Big\{\varphi(x_{\restriction n}) \in L(A) \ : \ \varphi(a_{\restriction I_0}) \ \text{ holds for every } \ I_0 \in \binom{I}{n}\Big\}.$$

Note that $\bar{x}$ is always of order-type ω, while $\bar{a}$ is an arbitrary infinite sequence. Clearly,

Note also that if $\bar{a}$ is A-indiscernible then $\text{EM-tp}(\bar{a}/A)$ is a complete type, and vice versa. So, if $\bar{a}$ and $\bar{c}$ are two A-indiscernible I-sequences with the same Ehrenfeucht-Mostowski type over A, then $\bar{a} \equiv_A \bar{c}$.

15.4 Ehrenfeucht-Mostowski Theorem Let $I, <_I$ and $J, <_J$ be two infinite linear orders such that $|J| \leq \kappa$. Then for every sequence $\bar{a} = \langle a_i : i \in I\rangle$ there is an J-sequence of A-indiscernibles $\bar{c}$ such that $\text{EM-tp}(\bar{a}/A) \subseteq \text{EM-tp}(\bar{c}/A)$.

Proof. We prove the theorem for $I = J = \omega$ and leave the general case to the reader. Let $q(\bar{x}) = \text{EM-tp}(\bar{a}/A)$. Any realization of the following type is a J-sequence of A-indiscernibles $\bar{c}$ as required by the theorem.

$$q(\bar{x}) \quad \cup \quad \Big\{\varphi(x_{\restriction I_0}) \leftrightarrow \varphi(x_{\restriction J_0}) \ : \ \varphi(x_{\restriction n}) \in L(A),\ I_0, J_0 \in \binom{\omega}{n},\ n < \omega\Big\}.$$

We will prove that any finite subset of the type above is realized by a finite subsequence of $\bar{a}$. First note that any finite subset of $q(\bar{x})$ is realized by any subsequence of $\bar{a}$ of the proper length by the definition of EM-type. Then we only need to pay attention to the set on the right.

We prove that for k and n arbitrary large and every $\varphi_1, \ldots, \varphi_k \in L_{x_{\restriction n}}(A)$ there is an infinite $H \subseteq \omega$ such that $a_{\restriction H}$ realizes

$$\# \qquad \Big\{\varphi_i(x_{\restriction I_0}) \leftrightarrow \varphi_i(x_{\restriction J_0}) \;:\; I_0, J_0 \in \binom{\omega}{n} \text{ and } i \in [k]\Big\}.$$

Consider the subsets of $[k]$ as colors and let f be the coloring of $\omega^{(n)}$ that maps I_0 to the set $\{i \;:\; \varphi_i(a_{\restriction I_0})\}$. By the Ramsey theorem, there is some infinite monochromatic set $H \subseteq \omega$. Hence

$$\Big\{\varphi_i(a_{\restriction I_0}) \leftrightarrow \varphi_i(a_{\restriction J_0}) \;:\; I_0, J_0 \in \binom{H}{n} \text{ and } i \in [k]\Big\}.$$

As H has order type ω, it is immediate that $a_{\restriction H}$ realizes #, as required to prove the theorem with $I = J = \omega$. □

15.5 Proposition Let $\bar{a} = \langle a_i \;:\; i \in I\rangle$ be a sequence of A-indiscernibles. Then $\bar{a}$ is indiscernible over some model M containing A.

Proof. Fix an model M containing A. By Theorem 15.4 there is an I-sequence of M-indiscernibles $\bar{c}$ such that $\text{EM-tp}(\bar{a}/M) \subseteq \text{EM-tp}(\bar{c}/M)$. As $\bar{a}$ is an A-indiscernible sequence $\bar{a} \equiv_A \bar{c}$. Therefore $h\bar{c} = \bar{a}$ for some some $h \in \text{Aut}(\mathcal{U}/A)$. Hence $\bar{a}$ is indiscernible over $h[M]$. □

15.6 Exercise Let $\bar{a} = \langle a_i \;:\; i \in I\rangle$ be an A-indiscernible sequence and let $J \supseteq I$ with $|J| \leq \kappa$. Then there is an A-indiscernible sequence $\bar{c} = \langle c_i : i \in J\rangle$ such that $c_{\restriction I} = \bar{a}$.

15.7 Exercise Let $p(x) \in S(\mathcal{U})$ be a global type invariant over A. Let $a, b \vDash p_{\restriction A}(x)$. Prove that there is a sequence $\bar{c} = \langle c_i : i < \omega\rangle$ such that $a, \bar{c}$ and $b, \bar{c}$ are both sequences of A-indiscernibles.

15.8 Exercise Let $\langle c_i \;:\; i < \omega\rangle$ be an indiscernible sequence. Prove that there is an indiscernible sequence $\langle d_i : i < \omega\rangle$ such that $d_0, d_1 = c_1, c_0$.

15.9 Exercise Let $\langle \mathcal{D}_i : i < \omega\rangle$ be an A-indiscernible sequence in $\mathcal{U}^{\text{eq}}$. Prove that there is an A-indiscernible sequence $\langle b_i \;:\; i < \omega\rangle$ and a formula $\varphi(x\,;z) \in L$ such that $\mathcal{D}_i = \varphi(\mathcal{U}\,;b_i)$.

15.3 Idempotent orbits in semigroups

In this and the following sections we focus on semigroups definable in a first-order structure. For a lighter notation, we assume that the semigroup operation $\cdot$ is among the symbols of L and that the whole monster model is a semigroup.

We fix a set of parameters A. For any two sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{U}$ we define

$$\mathcal{A} \cdot_A \mathcal{B} \quad = \quad \Big\{a \cdot b \;:\; a \in \mathcal{A},\ b \in \mathcal{B} \text{ and } \ a \mathop{\downfall}_A b\Big\}$$

In this and the next section we abbreviate $\mathcal{O}(a/A)$, the orbit of a under $\text{Aut}(\mathcal{U}/A)$, with a_A. We write $a \cdot_A \mathcal{B}$ for $\mathcal{O}(a/A) \cdot_A \mathcal{B}$. Similarly for $\mathcal{A} \cdot_A b$ and $a \cdot_A b$.

15.10 Proposition If $\mathcal{A}$ is type definable over A then so is $\mathcal{A} \cdot_A b$ for any b.

Proof. Let $p(y) = \text{tp}(b/A)$. Then $\mathcal{A} \cdot_A b$ is defined by the type $q(z)$, where

$$q(z) \quad = \quad \exists x, y\ \big[z = x \cdot y \ \wedge\ x \in \mathcal{A} \ \wedge\ y \mathop{\downfall}_A p\big]. \qquad \square$$

By the invariance of $\mathrel{\perp\!\!\!\!\!\perp}_A$, for every $f \in \mathrm{Aut}(\mathcal{U}/A)$ we have $f[\mathcal{A} \cdot_A \mathcal{B}] = f[\mathcal{A}] \cdot_A f[\mathcal{B}]$ Therefore, when $\mathcal{A}$ and $\mathcal{B}$ are invariant over A, also $\mathcal{A} \cdot_A \mathcal{B}$ is invariant over A. Below we mainly deal with invariant sets.

15.11 Proposition For every A-invariant sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

$$\mathcal{A} \cdot_A (\mathcal{B} \cdot_A \mathcal{C}) \quad \subseteq \quad (\mathcal{A} \cdot_A \mathcal{B}) \cdot_A \mathcal{C}$$

Proof. Let $a{\cdot}b{\cdot}c$ be an arbitrary element of the l.h.s. where $a \mathrel{\perp\!\!\!\!\!\perp}_A b{\cdot}c$ and $b \mathrel{\perp\!\!\!\!\!\perp}_A c$. By extension (Lemma 14.7), there exists a' such that $a \equiv_{A, b\cdot c} a' \mathrel{\perp\!\!\!\!\!\perp}_A b{\cdot}c, a, b, c$. By transitivity (again Lemma 14.7), $a'{\cdot}b \mathrel{\perp\!\!\!\!\!\perp}_A c$. Therefore $a'{\cdot}b{\cdot}c$ belongs to the r.h.s. Finally, as $a' \equiv_{A, b\cdot c} a$, also $a{\cdot}b{\cdot}c$ belongs to the r.h.s. by A-invariance. □

Let $\mathcal{A}$ be a non empty set. When $\mathcal{A} \cdot_A \mathcal{A} \subseteq \mathcal{A}$, we say that it is idempotent (over A).

15.12 Corollary Assume $\mathcal{B} \subseteq \mathcal{A}$ are both A-invariant. Then if $\mathcal{A}$ is idempotent, also $\mathcal{A} \cdot_A \mathcal{B}$ is idempotent.

Proof. We check that if $\mathcal{A}$ is idempotent so is $\mathcal{A} \cdot_A \mathcal{B}$

$$\begin{aligned}
(\mathcal{A} \cdot_A \mathcal{B}) \cdot_A (\mathcal{A} \cdot_A \mathcal{B}) \quad &\subseteq \quad \mathcal{A} \cdot_A (\mathcal{A} \cdot_A \mathcal{B}) && \text{because } \mathcal{A} \cdot_A \mathcal{B} \subseteq \mathcal{A} \\
&\subseteq \quad (\mathcal{A} \cdot_A \mathcal{A}) \cdot_A \mathcal{B} && \text{by the lemma above} \\
&\subseteq \quad \mathcal{A} \cdot_A \mathcal{B}
\end{aligned}$$

□

We show that, under the assumption of stationarity, the operation $\cdot_A$ is associative. The quotient map $\mathcal{U} \to \mathcal{U}/{\equiv_A}$ is almost a homeomorphism.

15.13 Proposition Assume $\mathrel{\perp\!\!\!\!\!\perp}_A$ is stationary, see Definition 14.9. Fix $a \mathrel{\perp\!\!\!\!\!\perp}_A b$ arbitrarily. Then $a'{\cdot}b' \equiv_A a{\cdot}b$ for every $a' \equiv_A a$ and $b' \equiv_A b$ such that $a' \mathrel{\perp\!\!\!\!\!\perp}_A b'$. Or, in other words,

$$(a{\cdot}b)_A \quad = \quad a \cdot_A b.$$

Proof. We prove two inclusions, only the second one requires stationarity.

$\subseteq$ As $a \mathrel{\perp\!\!\!\!\!\perp}_A b$ holds by hypothesis, $a{\cdot}b \in a \cdot_A b$. The inclusion follows by invariance.

$\supseteq$ By invariance it suffices to show that the l.h.s. contains $a \cdot_A \{b\}$. Let $a' \in a_A$ such that $a' \mathrel{\perp\!\!\!\!\!\perp}_A b$. We claim that $a'{\cdot}b \in (a{\cdot}b)_A$. Both a and a' satisfy $a \equiv_A x \mathrel{\perp\!\!\!\!\!\perp}_A b$. By stationarity, $a \equiv_{A,b} a'$. Hence $a{\cdot}b \equiv_A a'{\cdot}b$. □

15.14 Corollary (associativity) Let M be a model and assume $\mathrel{\perp\!\!\!\!\!\perp}_M$ is stationary. Then for every M-invariant sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

$$\mathcal{A} \cdot_M (\mathcal{B} \cdot_M \mathcal{C}) \quad = \quad (\mathcal{A} \cdot_M \mathcal{B}) \cdot_M \mathcal{C}$$

Proof. We can assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are M-orbits. Say of a, b, and c respectively. As we are working over a model, we can assume that $a \mathrel{\perp\!\!\!\!\!\perp}_M b{\cdot}c$ and $b \mathrel{\perp\!\!\!\!\!\perp}_M c$. By Proposition 15.13 the set on the l.h.s. equals $(a{\cdot}b{\cdot}c)_M$. By a similar argument the set on the r.h.s. equals $(a'\cdot b'\cdot c')_M$ for some elements a', b', and c'. Proposition 15.11 proves that inclusion $\subseteq$ holds in general. But inclusion between orbits amounts to equality. $\square$

15.15 Lemma Let M be a model and assume $\downfree_M$ is stationary. If $\mathcal{A}$ is minimal among the idempotent sets that are type-definable over M, then $\mathcal{A} = b_M$ for some (any) $b \in \mathcal{A}$.

Proof. Fix arbitrarily some $b \in \mathcal{A}$. By Corollary 15.12, the set $\mathcal{A} \cdot_M b$ is contained in $\mathcal{A}$, idempotent and type-definable over M by Proposition 15.10. Therefore by minimality $\mathcal{A} \cdot_M b = \mathcal{A}$. Let $\mathcal{A}' \subseteq \mathcal{A}$ contain those a such that $a \cdot_M b = b_M$. This set is non empty because $b \in \mathcal{A} \cdot_M b$. It is easy to verify that $\mathcal{A}'$ is type-definable over M, b. As it is clearly invariant over M, it is type-definable over M. By associativity it is idempotent. Hence, by minimality, $\mathcal{A}' = \mathcal{A}$. Then $b \in \mathcal{A}'$, which implies $b \cdot_M b = b_M$. That is, b has idempotent orbit. Finally, by minimality, $\mathcal{A} = b_M$. $\square$

15.16 Corollary Under the same assumptions of the lemma above, every type-definable idempotent set contains an element with an idempotent orbit.

15.4 Hindman's theorem

In this section we merge the theory of idempotents presented in Section 15.3 with the proof of Ramsey's theorem to obtain Hindman's theorem in a straightforward way.

Let $\bar{a}$ be a tuple of elements of $\mathcal{U}$ of length $\leq \omega$. We write $\operatorname{fp}\bar{a}$ for the set of finite products of elements of $\bar{a}$ taken in increasing order. Namely,

$$\operatorname{fp}\bar{a} \quad = \quad \Big\{ a_{i_0} \cdot \cdots \cdot a_{i_k} \;\; : \;\; i_0 < \cdots < i_k < |\bar{a}|, \;\; k < |\bar{a}| \Big\}.$$

Let $\lessdot$ be a relation on $\mathcal{U}$. Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{U}$. We say that $\mathcal{A}$ is $\lessdot$-covered by $\mathcal{C}$ if for every $a_1, \ldots, a_n \in \mathcal{A}$ there are infinitely many $c \in \mathcal{C}$ such that $a_i \lessdot c$ for all i. When $\mathcal{A} = \mathcal{C}$ we simply sat that $\mathcal{A}$ is $\lessdot$-covered. We say that $\mathcal{A}$ is $\lessdot$-closed if $a \lessdot b \lessdot c$ implies $a \lessdot b\cdot c$ for all $a, b, c \in \mathcal{A}$. A $\lessdot$-chain in $\mathcal{U}$ is a tuple $\bar{a} \in \mathcal{U}^{\leq\omega}$ such that $a_i \lessdot a_{i+1}$.

The requirements on $\lessdot$ are hardly restrictive. For example, on a free semigroup we can take the preorder relation given by the length of the words. Or, on any semigroup G, we could take the trivial relation G^2 –the theorem below would remain non trivial.

15.17 Hindman's Theorem Let $\lessdot$ be a relation on a semigroup G. Assume that G is $\lessdot$-closed and $\lessdot$-covered. Then for every finite coloring of G there is an infinite $\lessdot$-chain $\bar{a}$ such that $\operatorname{fp}\bar{a}$ is monochromatic.

Note that this implies that every commutative semigroup G has an infinite monochromatic subset closed under finite sums of distinct elements (order G arbitrarily).

Our proof follows closely the proof of Ramsey's theorem 15.1. The novelty is all in Lemma 15.15.

Proof. We interpret G as a structure in a language that extends the natural language of semigroups with a symbol for $\prec$ and one for each subset of G. Let $\mathcal{U}$ be a saturated elementary superstucture of G. As observed in Remark **??**, the language makes $\perp_G$ trivially stationary.

We write $\mathcal{G}'$ for the type-definable set $\{x : G \prec x\}$, which is non empty because G is $\prec$-covered. We claim that $\mathcal{G}'$ is idempotent. In fact, if $a, b \in \mathcal{G}'$ then, as $G \prec a, b$ and $a \perp_G b$, we must have that $a \prec b$. Therefore, from the $\prec$-closure of $\prec$ we infer $a \cdot b \in \mathcal{G}'$.

Let g be an element of $\mathcal{G}'$ with idempotent orbit as given by Corollary 15.16. Let $p(x) \in S(\mathcal{U})$ be a global coheir of $\operatorname{tp}(g/G)$. Let $\bar{g}$ a coheir sequence of $p(x)$, that is,

$$g_i \quad \models \quad p_{\restriction G, g_{\restriction i}}(x).$$

We write $\bar{g}_{\restriction i}$ for the tuple $g_{i-1}, \ldots, g_0$. By the idempotency of g_G and Proposition 15.13, $h \equiv_G g$ for all $h \in \operatorname{fp} \bar{g}_{\restriction i}$ and all i. It follows in particular that $\operatorname{fp} \bar{g}_{\restriction i}$ is monochromatic, say all its elements have color 1. Now, we use the sequence $\bar{g}$ to define $\bar{a} \in G^{\omega}$ such that all elements of $\operatorname{fp} \bar{a}$ have color 1.

Assume as induction hypothesis that we have $a_{\restriction i} \in G^i$ such that all elements of $\operatorname{fp}(a_{\restriction i}, g_0)$ have color 1. Our goal is to find a_i such that the same property holds for $\operatorname{fp}(a_{\restriction i+1}, g_0)$.

First we claim that from the induction hypothesis it follows that, for all j, all elements of $\operatorname{fp}(a_{\restriction i}, \bar{g}_{\restriction j})$ have color 1. In fact, the elements of $\operatorname{fp}(a_{\restriction i}, \bar{g}_{\restriction j})$ have the form $b \cdot h$ for some $b \in \operatorname{fp}(a_{\restriction i})$ and $h \in \operatorname{fp}(\bar{g}_{\restriction j})$. As $h \equiv_G g$, we conclude that $b \cdot h \equiv_G b \cdot g_0$, which proves the claim.

Let $\varphi(a_{\restriction i}, g_{i+1}, g_{\restriction i+1})$ say that all elements of $\operatorname{fp}(a_{\restriction i}, \bar{g}_{\restriction i+2})$ have color 1. As $\bar{g}$ is a coheir sequence we can find a_i such that $\varphi(a_{\restriction i}, a_i, g_{\restriction i+1})$. Hence all elements of $\operatorname{fp}(a_{\restriction i+1}, g_{\restriction i+1})$ have color 1. Therefore a_i is as required. □

15.5 The Hales-Jewett Theorem

The Hales-Jewett Theorem is a purely combinatorial statement that implies the van der Waerden Theorem.

We need a few definitions. We work with the same notation as Section 15.3. Let $\mathcal{A}$ be an idempotent set that is type-definable over A. We say that an element c is left-minimal (w.r.t. $\mathcal{A}$) if $c \in \mathcal{A} \cdot_A g$ for every $g \in \mathcal{A} \cdot_A c$.

15.18 Proposition Let $\mathcal{A}$ be idempotent and type-definable over A. Let a be arbitrary. Then $\mathcal{A} \cdot_A a$ contains a left-minimal element c. When $\mathcal{A} \cdot_A a \cap \mathcal{A}$ is non empty, we can also require that c has idempotent orbit.

Proof. If $b \in \mathcal{A} \cdot_A a$ then $\mathcal{A} \cdot_A b \subseteq \mathcal{A} \cdot_A a$. By compactness we obtain $c' \in \mathcal{A} \cdot_A a$ such that $\mathcal{A} \cdot_A b = \mathcal{A} \cdot_A c'$ for every $b \in \mathcal{A} \cdot_A c'$. Hence every $c \in \mathcal{A} \cdot_A c'$ is left-minimal. Note that if $b \in \mathcal{A}$ then by idempotency $\mathcal{A} \cdot_A b \subseteq \mathcal{A}$. Hence when $\mathcal{A} \cdot_A a$ and $\mathcal{A}$ have non empty intersection, we can also require that $c' \in \mathcal{A}$. Then $\mathcal{A} \cdot_A c'$ is idempotent. Therefore, by Corollary 15.16 there is some $c \in \mathcal{A} \cdot_A c'$ with idempotent orbit. □

15.19 Proposition Let M be a model and assume $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_M$ is stationary. Let $\mathcal{A}$ be idempotent and type-definable over M. Let c_M be idempotent and such that $c \cdot_M \mathcal{A}$, $\mathcal{A} \cdot_M c \subseteq \mathcal{A}$. Then

1. $c \cdot_M \mathcal{A} \cdot_M c$ contains some g with idempotent orbit
2. if moreover c is left-minimal, then $c \equiv_M g$ for every g as in 1.

Note, parenthetically, that the set in 1 may not be type-definable, therefore Corollary 15.16 does not apply directly and we need an indirect argument.

Proof. 1. From $c \cdot_M \mathcal{A} \subseteq \mathcal{A}$ we obtain that $\mathcal{A} \cdot_M c$ is idempotent. As it is also type-definable, by Corollary 15.16 it contains a b with idempotent orbit. Then $b \cdot_M c = b_M$, from which we obtain that $c \cdot_M b$ is idempotent and contained in $c \cdot_M \mathcal{A} \cdot_M c$.

2. From $g \in c \cdot_M \mathcal{A} \cdot_M c$ and the idempotency of c_M we obtain $g_M = c \cdot_M g$. As $g \in \mathcal{A} \cdot_M c$, from the left-minimality of c_M we obtain $c \in \mathcal{A} \cdot_M g$. Hence $c_M = c \cdot_M g$, by the idempotency of g_M. Therefore $c_M = g_M$, which proves 2. □

The following is a technical lemma that is required in the proof of the main theorem.

15.20 Proposition Let M be a model and assume $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_M$ is stationary. Let $\sigma : \mathfrak{U} \to \mathfrak{U}$ be a semigroup homomorphism definable over M. Then

1. $\sigma\big[a_M\big] = (\sigma\, a)_M$
2. $\sigma\big[a \cdot_M b\big] = \sigma\, a \cdot_M \sigma\, b$.

Proof. 1. As $a \equiv_M a'$ implies $\sigma\, a \equiv_M \sigma\, a'$, inclusion $\subseteq$ is clear. For the converse, note that the type $\exists y\, \big[\sigma\, y = x \wedge y \equiv_M a\big]$ is trivially realized by $\sigma\, a$. By invariance it is equivalent to a type over M. Therefore it is realized by all elements of $(\sigma\, a)_M$. Hence all elements of $(\sigma\, a)_M$ are the image of some element in a_M.

2. We have to prove the following equality

$$\Big\{\sigma(a'\cdot b')\ :\ a \equiv_M a' \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_M b' \equiv_M b\Big\} \quad = \quad \Big\{a'\cdot b'\ :\ \sigma\, a \equiv_M a' \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_M b' \equiv_M \sigma\, b\Big\}.$$

It suffices to prove one inclusion because by Proposition 15.13 both sides are orbits. We prove $\subseteq$. Note that $a' \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_M b'$ implies $\sigma\, a' \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_M \sigma\, b'$. Hence the set on l.h.s. is contained in the following

$$\Big\{\sigma(a'\cdot b')\ :\ \sigma\, a' \mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}_M \sigma\, b',\ a' \equiv_M a,\ b' \equiv_M b\Big\}.$$

which is in turn contained in the set on the r.h.s. □

Let $\mathfrak{U}$ be a semigroup. A nice subsemigroup of $\mathfrak{U}$ is a subsemigroup $\mathfrak{C}$ with the property that if $a \cdot b \in \mathfrak{C}$ then both $a, b \in \mathfrak{C}$.

15.21 Hales-Jewett Theorem (Koppelberg's version) Let G be an infinite semigroup and let $C \subset G$ be a nice subsemigroup. Let Σ be a finite set of retractions of G onto C, that is, homomorphisms $\sigma : G \to C$ such that $\sigma_{\restriction C} = \mathrm{id}_C$. Then, for every finite coloring of C, there is an $a \in G \smallsetminus C$ such that $\{\sigma\, a : \sigma \in \Sigma\}$ is monochromatic.

Proof. Let $G \preceq \mathfrak{U}$. Here $\mathfrak{U}$ is a monster model in a language that expands the natural one with a symbol for all subsets of G. As observed in Remark **??**, this makes

$\downfree_G$ stationary. Let $\mathcal{C}$ be the definable set such that $C = G \cap \mathcal{C}$. By elementarity, $\mathcal{C}$ is a nice subsemigroup of $\mathcal{U}$. The language contains also symbols for the retractions $\sigma : \mathcal{U} \to \mathcal{C}$.

By Proposition 15.18, there is a left-minimal $c \in \mathcal{C}$. As $\mathcal{C} \cdot_M c$ is clearly idempotent, we can further require that c has idempotent orbit.

By nicety, $\mathcal{U} \smallsetminus \mathcal{C}$ satisfy the assumptions of Proposition 15.19. Hence, by the first claim of that proposition, there is an idempotent $g \in c \cdot_G (\mathcal{U} \smallsetminus \mathcal{C}) \cdot_G c$. In particular, $g \in \mathcal{U} \smallsetminus \mathcal{C}$. Now apply the second claim of Proposition 15.20, with $\mathcal{C}$ for $\mathcal{A}$, to obtain $\sigma\, g \in c \cdot_G \mathcal{C} \cdot_G c$ for all $\sigma \in \Sigma$. As $\sigma\, g$ is also idempotent, we apply Proposition 15.19 to conclude that $\sigma\, g \equiv_G c$. In particular the set $\{\sigma\, g : \sigma \in \Sigma\}$ is monochromatic.

Though the element g above need not belong to $G \smallsetminus C$, by elementarity $G \smallsetminus C$ contains some a with the same property and this proves the theorem. □

Finally we show how the classical Hales-Jewett theorem follows from its abstract version.

15.22 Hales-Jewett Theorem (Classical version) Let C be an infinite semigroup. Fix a tuple of variables x and let $F \subseteq C^{|x|}$ be a finite set. Fix also a finite coloring of C. Then there is a non constant term $t(x)$ of the language of semigroups with parameters in C such that $\{t(a) : a \in F\}$ is monochromatic.

Proof. Let G be the set of terms $t(x)$ in the language of semigroups with parameters in C and free variables in x. Then G is a semigroup under the natural operation. For every $a \in C^{|x|}$ the map $\sigma_a : t(x) \mapsto t(a)$ is a retraction. Hence we can apply the theorem above. □

We conclude with a variant of Theorem 15.21 that applies to a broader class of semigroup homomorphisms. For Σ a set of maps $\sigma : G \to C$ and $c \in C$ we define

$$\Sigma^{-1}[c] \quad = \quad \bigcap_{\sigma \in \Sigma} \sigma^{-1}[c]$$

Clearly, when the maps in Σ are retractions, $\Sigma^{-1}[c]$ is non empty for all $c \in C$ because it contains at least c.

15.23 Hales-Jewett Theorem (Yet another variant) Let Σ be a finite set of homomorphisms $\sigma : G \to C$ between infinite semigroups such that $\Sigma^{-1}[c]$ is non empty for all $c \in C$. Then, for every finite coloring of C, there is a $g \in G$ such that the set $\{\sigma\, g \ : \ \sigma \in \Sigma\}$ is monochromatic.

Proof. Let $G * C$ be free product of the two semigroups. That is, $G * C$ contains finite sequences of elements of $G \cup C$, below called *words*, that alternate elements in G with elements in C. The product of two words is obtained concatenating them and, when it applies, replacing two contiguous elements of the same group by their product. Note that C is a nice subsemigroups of $G * C$.

Any homomorphism $\sigma : G \to C$ extends canonically to a retraction of $G * C$ to C. In fact, this extension is unique: the elements of G that occur in a word are replaced by their image under σ, finally the elements in the resulting sequence are multiplied. This extension is denoted by the same symbol σ.

Apply Theorem 15.21 to obtain some $w \in G * C$ such that $\{\sigma w : \sigma \in \Sigma\}$ is monochromatic. Suppose $w = c_0 \cdot g_0 \cdot \dots \cdot c_n \cdot g_n$ for some $g_i \in G$ and $c_i \in C$, where one or both of c_0 or g_n could be absent. Pick some $h_i \in \Sigma^{-1}[c_i]$ and let $g = h_0 \cdot g_0 \cdot \dots \cdot h_n \cdot g_n$. Then $\{\sigma g : \sigma \in \Sigma\}$ is monochromatic as required to complete the proof. □

15.6 Notes and references

The first application of the algebraic structure of βG (the Stone-Čech compactification of a semigroup G) to Ramsey Theory is the celebrated Galvin-Glazer proof of Hindman's theorem. Here we have used saturated models in place of Stone-Čech compactification. The idea to replace the semigroup βG has been pioneered by Ludomir Newelsi in the study of applications of topological dynamics to model theory.

The original proof of the Hales-Jewett Theorem by Alfred Hales and Robert Jewett is combinatorial. An alternative proof, also combinatorial, has been given by Sheron Shelah. Our proof is taken from [1].

[1] Eugenio Colla and Domenico Zambella, *Ramsey's coheirs*, J. Symb. Log. **87** (2022), no. 1, 377–391. arXiv:1901.04363.

Chapter 16

Lascar invariant sets

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ strictly larger than $|L|$. The notation and implicit assumptions are as in Section 9.3.

16.1 Expansions

This section is only marginally required in the present chapter so it can be postponed with minor consequences.

We will find it convenient to expand the language L with a predicate for a given $\mathcal{D} \subseteq \mathcal{U}^z$. We denote by $\langle \mathcal{U}, \mathcal{D} \rangle$ the corresponding expansion of $\mathcal{U}$. Generally, we write $L(\mathcal{X})$ for the expanded language but, when the intended interpretation of $\mathcal{X}$ is only going to be $\mathcal{D}$, we may write $L(\mathcal{D})$ and abbreviate $\langle \mathcal{U}\,;\mathcal{D} \rangle \vDash \varphi(\mathcal{X})$ as $\varphi(\mathcal{D})$.

16.1 Remark The definitions above are straightforward when z finite tuple. When z is an infinite tuple the intuition stays the same but a more involved definition is required. In fact, first-order logic does not allow infinitary predicates. We think of $L(\mathcal{X})$ for a two sorted language. The *home sort*, denoted by 0, and the z-*sort*, denoted by z. The expansion $\langle \mathcal{U}, \mathcal{D} \rangle$ has domain $\mathcal{U}$ for the home sort, and $\mathcal{U}^z$ for the z-sort. Besides the symbols of L, there is a function symbol π_i for every $i < |z|$ which is interpreted as the projection to the i-coordinate. These functions have sort $\langle z, 0 \rangle$ (see Section 13.1 for the notation). There is also a predicate of sort $\langle z \rangle$ interpreted as $\mathcal{D}$.

What said above is adapted to define the expansion $L(\mathcal{X}_i : i < \lambda)$, where $\mathcal{X}_i$ are predicates of sort z_i. Again, when the sets $\mathcal{D}_i \subseteq \mathcal{U}^{z_i}$ are the only intended interpretation of $\mathcal{X}_i$, we write $L(\mathcal{D}_i : i < \lambda)$.

16.2 Definition If $\mathcal{C}, \mathcal{D} \subseteq \mathcal{U}^z$ we abbreviate $\langle \mathcal{U}, \mathcal{C} \rangle \equiv_A \langle \mathcal{U}, \mathcal{D} \rangle$ as $\mathcal{C} \equiv_A \mathcal{D}$. We also say that $\mathcal{D}$ is saturated if so is the model $\langle \mathcal{U}\,;\mathcal{D} \rangle$.

16.3 Remark For every $\mathcal{D} \subseteq \mathcal{U}^z$ there is a saturated $\mathcal{C} \equiv \mathcal{D}$. In fact, it suffices to find a saturated model $\langle \mathcal{U}', \mathcal{D}' \rangle \equiv \langle \mathcal{U}, \mathcal{D} \rangle$ of cardinality κ. By saturation, there is an isomorphism $f : \mathcal{U}' \to \mathcal{U}$. Therefore $f[\mathcal{D}']$ is the required set $\mathcal{C}$.

16.4 Proposition If $\mathcal{D} \subseteq \mathcal{U}^z$ is invariant over A then every A-indiscernible sequence is indiscernible in the language $L(A\,;\mathcal{D})$.

Proof. Let $\bar{c} = \langle c_i : i \in I \rangle$ be an A-indiscernible sequence. For every $I_0, I_1 \in I^{[n]}$

there is an $f \in \mathrm{Aut}(\mathcal{U}/A)$ such that $fc_{\restriction I_0} = c_{\restriction I_1}$, Hence

$$\varphi(c_{\restriction I_0}\,;\mathcal{D}) \quad\leftrightarrow\quad \varphi(fc_{\restriction I_0}\,;f[\mathcal{D}]) \quad\leftrightarrow\quad \varphi(c_{\restriction I_1}\,;\mathcal{D}). \qquad \square$$

16.5 Exercise Prove that if $\mathcal{C} \subseteq \mathcal{U}^z$ is type-definable over B then $\equiv_{A,B}$ implies $\equiv_{A\,;\mathcal{C}}$.

16.6 Exercise Prove that if $\mathcal{D} \subseteq \mathcal{U}^z$ is saturated and invariant over A than it is definable over A.

16.2 Lascar strong types

Let $\mathcal{D} \subseteq \mathcal{U}^z$, where z is a tuple of length $< \kappa$. The orbit of $\mathcal{D}$ over A is the set

$$o(\mathcal{D}/A) \quad=\quad \Big\{ f[\mathcal{D}] \;:\; f \in \mathrm{Aut}(\mathcal{U}/A) \Big\}$$

So, $\mathcal{D}$ is invariant over A when $o(\mathcal{D}/A) = \{\mathcal{D}\}$. We say that $\mathcal{D}$ is Lascar invariant over A if it is invariant over every model $M \supseteq A$. Recall that this means that if $a \equiv_M c$ for some model M containing A then $a \in \mathcal{D} \leftrightarrow c \in \mathcal{D}$.

16.7 Proposition Let $\lambda = |L_z(A)|$. There are at most 2^{2^λ} sets $\mathcal{D} \subseteq \mathcal{U}^z$ that are Lascar invariant over A.

Proof. Let N be a model containing A of cardinality $\leq \lambda$. Every set that is Lascar invariant over A is invariant over N. As $|L_z(N)| = \lambda$ the bound follows from Proposition 14.1. $\square$

16.8 Theorem For every $\mathcal{D}$ and every $A \subseteq M$ the following are equivalent

1. $\mathcal{D}$ is Lascar invariant over A
2. every set in $o(\mathcal{D}/A)$ is M-invariant
3. $o(\mathcal{D}/A)$ has cardinality $\leq 2^{2^{|L(A)|}}$
4. $o(\mathcal{D}/A)$ has cardinality $< \kappa$
5. $c_0 \in \mathcal{D} \leftrightarrow c_1 \in \mathcal{D}$ for every A-indiscernible sequence $\langle c_i : i < \omega\rangle$.

Proof. 1⇒2. This implication is clear because all sets in $o(\mathcal{D}/A)$ are Lascar invariant over A.

2⇒3. When $|M| \leq |L(A)|$ the implication follows from the bounds discussed in Section 14.1. We temporary add this assumption on M. Once the proof of the proposition is completed, is easily seen to be redundant (by 4 and Proposition 16.7).

3⇒4. This implication holds because κ is a strong limit cardinal.

4⇒5. Assume ¬5. Then we can find an A-indiscernible sequence $\langle c_i : i < \kappa\rangle$ such that $c_0 \in \mathcal{D} \not\leftrightarrow c_1 \in \mathcal{D}$. Define

$$E(u\,;v) \quad\Leftrightarrow\quad u \in \mathcal{C} \leftrightarrow v \in \mathcal{C} \quad \text{for every } \mathcal{C} \in o(\mathcal{D}/A).$$

Then $E(u\,;v)$ is an A-invariant equivalence relation. As $\neg E(c_0\,;c_1)$, by Proposition 16.4, indiscernibility over A implies that $\neg E(c_i\,,c_j)$ for every $i < j < \kappa$. Then $E(u\,;v)$ has κ equivalence classes. As κ is inaccessible, this implies ¬4.

5⇒1. Fix any $a \equiv_N b$ where $A \subseteq N$. It suffices to prove that $a \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$. Let $p(z) \in S(\mathcal{U})$ be a global coheir of $\mathrm{tp}(a/N) = \mathrm{tp}(b/N)$. Let $\bar{c} = \langle c_i : i < \omega\rangle$ be a Morley sequence of $p(z)$ over N, a, b. Then both $a, \bar{c}$ and $b, \bar{c}$ are A-indiscernible sequences. Therefore, from 5 we obtain $a \in \mathcal{D} \leftrightarrow c_0 \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$. □

For definable sets Lascar invariance reduces to definablity over the algebraic closure.

16.9 Corollary For every definable set $\mathcal{D}$ the following are equivalent

1. $\mathcal{D}$ is Lascar invariant over A
2. $\mathcal{D}$ is definable over every model containing A
3. $\mathcal{D} \in \mathrm{acl}^{\mathrm{eq}}A$.

The following corollary easily follows from Theorem 16.8 and Proposition 16.4. As an exercise the reader may prove it using Proposition 15.5.

16.10 Corollary The following are equivalent

1. $\mathcal{D}$ is Lascar invariant over A
2. every sequence of A-indiscernibles is $L(A\,;\mathcal{D})$-indiscernible.

⚠ Given a tuple $a \in \mathcal{U}^z$, we write $\mathcal{L}(a/A)$ for the intersection of all sets containing a that are Lascar invariant over A. Clearly $\mathcal{L}(a/A)$ is Lascar invariant over A. The symbol $\mathcal{L}(a/A)$ is not standard.

16.11 Definition We write $a \overset{\mathrm{L}}{\equiv}_A b$ and say that a and b have the same Lascar strong type over A if the equivalence $a \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$ holds for every set $\mathcal{D}$ that is Lascar invariant over A or, in other words, if $\mathcal{L}(a/A) = \mathcal{L}(b/A)$. The notation $\text{L-stp}(a/A) = \text{L-stp}(b/A)$ and $aE_{L/A}b$ are also common in the the literature.

16.12 Proposition The relation $\overset{\mathrm{L}}{\equiv}_A$ is the finest equivalence relation with $< \kappa$ classes that is invariant over A.

Proof. Clearly $\overset{\mathrm{L}}{\equiv}_A$ is an equivalence relation invariant over A. Each equivalence class is Lascar invariant over A, hence the number of equivalence classes is bounded by the number of Lascar invariant sets over A. To see that $\overset{\mathrm{L}}{\equiv}_A$ is the finest of such equivalences. Suppose $\mathcal{D}$ is an equivalence class of an A-invariant equivalence relation with $< \kappa$ classes. Then $o(\mathcal{D}/A)$ has also cardinality $< \kappa$. Then $\mathcal{D}$ is Lascar invariant and as such it is union of classes of the relation $\overset{\mathrm{L}}{\equiv}_A$. □

Let $p(x) \in S(\mathcal{U})$ be global type. We say that p is Lascar invariant over A if the sets $\mathcal{D}_{p,\varphi}$, for $\varphi(x\,;z) \in L$, are all Lascar invariant over A. The sets $\mathcal{D}_{p,\varphi}$ are defined in Section 14.1.

16.13 Proposition Let $p(x) \in S(\mathcal{U})$. Then the following are equivalent

1. $p(x)$ is Lascar invariant over A
2. every A-indiscernible sequence $\bar{c}$ is indiscernible over A, a for every $a \models p_{\restriction A,\bar{c}}(x)$
3. every A-indiscernible sequence $\bar{c}$ is indiscernible over a for every $a \models p_{\restriction \bar{c}}(x)$.

For convenience the tuples c_i have length $|z| = \omega$.

Proof. 1⇒2. Assume 1 and fix an A-indiscernible sequence $\bar{c} = \langle c_i : i < \omega \rangle$ and some $a \models p_{\restriction \bar{c}}$. We need to prove that for every formula $\varphi(x\,;z_1,\dots,z_n) \in L(A)$

$$\varphi(a\,;c_{\restriction I_0}) \quad \leftrightarrow \quad \varphi(a\,;c_{I_1}).$$

holds for every $I_0, I_1 \subseteq \omega$ of cardinality n. Without loss of generality we can assume that $I_0 < I_1$. Define I_n for $n > 1$ arbitrarily such that $I_n < I_{n+1}$. Then the sequence $c'_n = c_{\restriction I_n}$ is a sequence of A-indiscernibles and $c'_0 \in \mathcal{D}_{p,\varphi} \nleftrightarrow c'_1 \in \mathcal{D}_{p,\varphi}$. By Theorem 16.8 this contradicts 1.

2⇒3 is trivial.

3⇒1. If $p(x)$ is not Lascar invariant over A then $c_0 \in \mathcal{D}_{p,\varphi} \nleftrightarrow c_1 \in \mathcal{D}_{p,\varphi}$ for some A-indiscernible sequence $\bar{c} = \langle c_i : i < \omega \rangle$ and some $\varphi(x\,;z) \in L$. Then $p(x)$ contains the formula $\varphi(x\,;c_0) \nleftrightarrow \varphi(x\,;c_1)$. Hence, $\bar{c}$ is not indiscernible over any realization of $p(x)_{\restriction c_0,c_1}$. □

16.3 Coheirs over sets

In this section we consider a property that implies the existence of global Lascar invariant types and has simple syntactic characterization. It is a natural generalization to a set of the notion of finite satisfiability over a model that we introduced in Section 14.2.

In Section 14.2 finite satisfiability over a model has been used to prove that every consistent type over a model has a global extension to a type that is invariant over that model. Ideally, we would like to prove here a similar existence property for Lascar invariance over a set. But this is not possible in general. In fact, in Example 16.17, we present a theory that has no global Lascar invariant type.

A global type $p(x) \in S(\mathfrak{U})$ that is finitely satisfiable in every model containing A, is called a global coheir of $p_{\restriction A}(x)$. Note that when A is a model we obtain the same notion defined in Section 14.2.

16.14 Proposition Let $\varphi(x\,;z) \in L$. Every type $p(x) \in S_\varphi(\mathfrak{U})$ that is finitely satisfiable in every $M \supseteq A$ is Lascar invariant over A.

Proof. Let $\langle c_i : i < \omega \rangle$ be an A-indiscernible sequence. Suppose for a contradiction that $c_0 \in \mathcal{D}_{p,\varphi} \nleftrightarrow c_1 \in \mathcal{D}_{p,\varphi}$ Then $\varphi(x\,;c_0) \nleftrightarrow \varphi(x\,;c_1) \in p$. Let $M \supseteq A$ be such that $\langle c_i : i < \omega \rangle$ is indiscernible over M. As $p(x)$ is finitely satisfiable in M, for some $a \in M^x$ we have $\varphi(a\,;c_0) \nleftrightarrow \varphi(a\,;c_1)$. This contradicts indiscernibility over M. □

⚠ We say that A is a coheir extension base if every type that is finitely satisfiable in every $M \supseteq A$ has an extension to a global type with the same property. The terminology is not standard. It is inspired by the analogy with extensions bases for forking, see e.g. [1], [2].

16.15 Proposition The following are equivalent

1. A is a coheir extension base
2. for every $\psi(x), \varphi(x) \in L(\mathfrak{U})$, if $\psi(x) \vee \varphi(x)$ is finitely statisfied in every $M \supseteq A$, then $\psi(x)$ or $\varphi(x)$ is finitely statisfied in every $M \supseteq A$.

Proof. 1⇒2. Clear.

2⇒1. Assume 2 and repeat the proof of Proposition 14.3. □

Trivially, every model is an extension base for coheirs. The following example shows that some theories may have no other extension bases.

16.16 Example Consider T_{dlo}. Suppose A is not a model. We show that A is not an extension base for coheirs. As A is not a model, there are $a < c$ in A such that either $A \cap (a,c) = \varnothing$ or $a \leq A$ or $A \leq c$. In the first case let $b \in (a,c)$. Then every model containing A intersects $(a,b) \cup (b,c)$, but there are both models that omit (a,b) and models that omit (b,c). In the second case let $b < a$ and consider $(-\infty, b) \cup (b,a)$. The third case is symmetric.

In Chapter 18 we will see that for some important class of theories, *stable theories*, all sets are coheir extension bases.

16.17 Example We present a theory without global types that are Lascar invariant over the empty set. Let T be the theory of $\mathbb{Q}$ with the cyclic order, i.e. with a ternary predicate $\text{cyc}(a,b,c)$ that holds if

$$(a < b < c) \ \vee\ (c < a < b) \ \vee\ (b < c < a).$$

The picture to keep in mind is that of a circle, where $\text{cyc}(a,b,c)$ holds if we encounter a,b,c in that order when going clockwise.

We will implicitly use that T has elimination of quantifiers. Let $p(x) \in S(\mathcal{U})$. We claim that $p(x)$ is not Lascar invariant over the empty set. Suppose it is, for a contradiction. We can find $a < b$ in $\mathcal{U}$ such that either $\text{cyc}(a,x,b)$ or $\text{cyc}(a,b,x)$ is in $p(x)$. We show that both these possibilities are contradictory.

1. If $p(x)$ contains $\text{cyc}(a,x,b)$, let $M < a$ be a model and pick c such that $b < c$. Then $a,b \equiv_M b,c$, hence $\text{cyc}(b,x,c)$ is also in $p(x)$. A contradiction.
2. If $p(x)$ contains $\text{cyc}(a,b,x)$, let $a < M < b$ and pick a' such that $a < a' < M$. As $a,a' \equiv_M b,a$, also $\text{cyc}(b,a,x)$ is in $p(x)$. A contradiction.

16.18 Exercise Let A be a coheir extension base. Prove that for every $b \in \mathcal{U}^z$ there is a structure $\mathcal{V} \preceq \mathcal{U}$ isomophic to $\mathcal{U}$ over A such that $\mathcal{V} \mathop{\downfree}_M b$ for every $A \subseteq M \preceq \mathcal{V}$.

16.19 Exercise Let A be a coheir extension base. Prove that for every $a \in \mathcal{U}^x$ there is a structure $\mathcal{V} \preceq \mathcal{U}$ isomophic to $\mathcal{U}$ over A such that $a \mathop{\downfree}_M \mathcal{V}$ for every $A \subseteq M \preceq \mathcal{V}$.

16.20 Exercise Let $\varphi(x) \in L(\mathcal{U})$ be satisfied in every model containing A. Let $M \supseteq A$. Prove that $\varphi(x)$ is contained in some global type finitely satisfied in M.

16.21 Exercise Let $\varphi(x\,;z) \in L$. Let $p(x) \in \varphi(\mathcal{U})$ be finitely satisfied in every $M \supseteq A$. Prove that if $\mathcal{D}_{p,\varphi}$ is saturated, see definition 16.2, then it is in $\text{acl}^{\text{eq}}A$.

16.4 The Lascar graph

Here we study Lascar strong types from a different viewpoint. The Lascar graph over A has an arc between all pairs $a,b \in \mathcal{U}^x$ such that $a \equiv_M b$ for some model

M containing A. We write $d_A(a,b)$ for the distance between a and b in the Lascar graph over A. Let us spell this out: $d_A(a,b) \leq n$ if there is a sequence $a_0,\dots,a_n$ such that $a = a_0$, $b = a_n$, and $a_i \equiv_{M_i} a_{i+1}$ for some models M_i containing A. We write $d_A(a,b) < \infty$ if a and b are in the same connected component of the Lascar graph over A.

16.22 Proposition For every $a \in \mathcal{U}^x$

$$\mathcal{L}(a/A) \quad = \quad \{c \ : \ d_A(a,c) < \infty\}.$$

Proof. To prove inclusion $\supseteq$ it suffices to show that every Lascar A-invariant set containing a contains the set on the r.h.s. Let $\mathcal{D}$ be Lascar A-invariant, and let $b \in \mathcal{D}$. Then $\mathcal{D}$ contains also every c such that $b \equiv_M c$ for some model M containing A. That is, $\mathcal{D}$ contains every c such that $d_A(b,c) \leq 1$. It follows that $\mathcal{D}$ contains every c such that $d_A(a,c) < \infty$.

To prove inclusion $\subseteq$ we prove the set on the r.h.s. is Lascar A-invariant. Suppose the sequence $a_0,\dots,a_n$, where $a_0 = a$ and $a_n = c$, witnesses $d_A(a,c) \leq n$ and suppose that $c \equiv_M b$ for some M containing A, then the sequence $a_0,\dots,a_n,b$ witnesses $d_A(a,b) \leq n+1$. □

We write $\mathrm{Autf}(\mathcal{U}/A)$ for the subgroup of $\mathrm{Aut}(\mathcal{U}/A)$ that is generated by the automorphisms that fix point-wise some model M containing A. (The "f" in the symbol stands for *fort*, the French word for *strong*.) It is easy to verify that $\mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$ is a normal subgroup of $\mathrm{Aut}(\mathcal{U}/A)$. The following is a corollary of Proposition 16.22.

16.23 Corollary The following are equivalent

1. $a \overset{\mathrm{L}}{\equiv}_A b$
2. $a = fb$ for some $f \in \mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$.

It may not be immediately obvious that the relation $d_A(x,y) \leq n$ is type-definable.

16.24 Proposition There is a type $p_n(x,y) \subseteq L(A)$ equivalent to $d_A(x,y) \leq n$.

Proof. It suffices to prove the proposition with $n = 1$. Let $\lambda = |L(A)|$ and fix a tuple of distinct variables $w = \langle w_i : i < \lambda\rangle$, then $p_1(x,y) \ = \ \exists w\, p(w,x,y)$ where

$$p(w,x,y) \quad = \quad q(w) \ \cup \ \Big\{\varphi(x,w) \leftrightarrow \varphi(y,w) \ : \ \varphi(x,w) \in L(A)\Big\}$$

and $q(w) \subseteq L(A)$ is a consistent type with the property that all its realizations enumerate a model containing A.

It remains to verify that such a type exist. Let $\langle \psi_i(z,w_{\restriction i}) : i < \lambda\rangle$ be an enumeration of the formulas in $L_{z,w}(A)$, where z is a single variable. Let

$$q(w) \quad = \quad \Big\{\exists z\, \psi_i(z,w_{\restriction i}) \ \rightarrow \ \psi_i(w_i,w_{\restriction i}) \ : \ i < \lambda\Big\}.$$

Any realization of $q(w)$ satisfy the Tarski-Vaught test therefore it enumerates a model containing A. Vice versa it is clear that we can realize $q(w)$ in any model containing A. □

16.25 Exercise Let $p(x) \in S(A)$.

$$Q \quad = \quad \{q(x) \in S(\mathrm{acl}^{\mathrm{eq}} A) \; : \; p(x) \subseteq q(x)\}$$

For $f \in \mathrm{Aut}(\mathcal{U}/A)$ define $fq(x) = \{\varphi(x, fb) : \varphi(x, b) \in q\}$. This is an action of $f \in \mathrm{Aut}(\mathcal{U}/A)$ in Q. Prove that it is transitive.

16.26 Exercise Prove that the equivalence relation $a \overset{\mathrm{L}}{\equiv}_A b$ is the transitive closure of the relation: there is a sequence $\langle c_i : i < \omega \rangle$ indiscernible over A such that $c_0 = a$ and $c_1 = b$. Hint: use Exercise 15.8 to prove symmetry, then reason as in Proposition 16.22.

16.5 Kim-Pillay types

Given a tuple $a \in \mathcal{U}^z$, we write $\mathcal{K}(a/A)$ for the intersection of all type-definable sets containing a that are Lascar invariant over A. Or, more concisely, the intersection of all sets that are type-definable over every model containing A. We call $\mathcal{K}(a/A)$ the Kim-Pillay strong type over A. Clearly $\mathcal{K}(a/A)$ is Lascar invariant over A. It also easy to see that $\mathcal{K}(a/A)$ is type-definable. In fact, by invariance, we can assume that all types in the intersection above are over M, for any fixed model containing A. Hence $\mathcal{K}(a/A)$ is the minimal type-definable set containing a and closed under the relation $\overset{\mathrm{L}}{\equiv}_A$. It follows that if $b \in \mathcal{K}(a/A)$ then $\mathcal{K}(b/A) \subseteq \mathcal{K}(a/A)$.

To summarize, we recall that we have defined a whole hierarchy of strong types obtained from the intersection of different sets with various sots of invariance

$$\mathcal{L}(a/A) \;\subseteq\; \mathcal{K}(a/A) \;\subseteq\; \mathcal{S}(a/A) \;\subseteq\; \mathcal{O}(a/A).$$

Recall that $\mathcal{S}(a/A)$ was defined after Proposition 13.14.

If $\mathcal{K}(a/A) = \mathcal{K}(b/A)$, we say that a and b have the same Kim-Pillay strong type over A. We abbreviate this by $a \overset{\mathrm{KP}}{\equiv}_A b$. In other words, we write $a \overset{\mathrm{KP}}{\equiv}_A b$ when $a \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$ for every type-definable set $\mathcal{D}$ that is Lascar invariant over A.

Warning: the symbol $\mathcal{K}(a/A)$ is not standard. The symbol $a \overset{\mathrm{KP}}{\equiv}_A b$ is not unusual, but some author write KP-stp$(a/A) =$ KP-stp(b/A) or $a\, E_{\mathrm{KP}/A}\, b$.

16.27 Proposition Fix some $a \in \mathcal{U}^z$ and some $A \subseteq \mathcal{U}$. Then there is a type $e(z\,;w) \subseteq L(A)$ such that $\mathcal{K}(b/A) = e(\mathcal{U}\,;b)$ for all $b \in \mathcal{O}(a/A)$ and $e(z\,;w)$ defines an equivalence relation on $\mathcal{O}(a/A)$.

Proof. Notice that $\mathcal{K}(a/A)$ is type-definable over A, a. In fact, if $f \in \mathrm{Aut}(\mathcal{U}/A, a)$ and $\mathcal{D}$ is a set containing a that is type-definable and Lascar invariant over A, then so is $f[\mathcal{D}]$. Therefore $\mathcal{K}(a/A)$ is invariant over A, a. As $\mathcal{K}(a/A)$ is type-definable, invariance implies that it is type-definable over A, a. Let $e(z\,;w) \subseteq L(A)$ be such that $\mathcal{K}(a/A) = e(\mathcal{U}\,;a)$.

We prove that $\mathcal{K}(b/A) = e(\mathcal{U}\,;b)$ for all $b \in \mathcal{O}(a/A)$. Let $f \in \mathrm{Aut}(\mathcal{U}/A)$ be such that $fa = b$. If $\mathcal{D}$ is a type-definable Lascar invariant over A, then so is $f[\mathcal{D}]$. Therefore, f is a bijection between type-definable sets that are Lascar invariant over A and contain a and analogous sets containing b. Then $f[\mathcal{K}(a/A)] = \mathcal{K}(b/A)$ and $\mathcal{K}(b/A) = e(\mathcal{U}\,;b)$ follows.

We prove that $e(b\,;\mathcal{U}) \cap \mathcal{O}(a/A)$ is Lascar invariant over A. Let $f \in \mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$ and $c \in \mathcal{O}(a/A)$. Then $e(b\,;fc)$ is equivalent to $e(f^{-1}b\,;c)$ which in turn is equivalent to

$e(b\,;c)$, by the invariance of $e(\mathcal{U}\,;c)$.

Finally we are ready to prove that $e(z\,;w)$ defines a symmetric relation on $\mathcal{O}(a/A)$. From what proved above, $e(b\,;\mathcal{U}) \cap \mathcal{O}(a/A)$ is a type-definable Lascar invariant set containing b and therefore it contains $\mathcal{K}(b/A)$. We conclude that for all $b, c \in \mathcal{O}(a/A)$

$$e(c\,;b) \quad \leftrightarrow \quad c \in e(\mathcal{U}\,;b) \quad \leftrightarrow \quad c \in \mathcal{K}(b/A) \quad \rightarrow \quad e(b\,;c)$$

Reflexivity is clear; we prove transitivity. As remarked above, $\mathcal{K}(b/A) \subseteq \mathcal{K}(c/A)$, for all $b \in \mathcal{K}(c/A)$ or equivalently $e(b\,;c)$. Hence if $e(b\,;c)$ then

$$e(d\,;b) \quad \rightarrow \quad d \in \mathcal{K}(b/A) \quad \leftrightarrow \quad d \in \mathcal{K}(c/A) \quad \rightarrow \quad e(d\,;c).$$

Which completes the proof. □

16.28 Corollary For every $a, b \in \mathcal{U}^z$ and $A \subseteq \mathcal{U}$ the following are equivalent

1. $a \in \mathcal{K}(b/A)$
2. $b \in \mathcal{K}(a/A)$
3. $\mathcal{K}(a/A) = \mathcal{K}(b/A)$.

The following useful lemma is the key ingredient in the proof of Theorem 16.30.

16.29 Lemma Let $p(z) \subseteq L(A)$ and let $e(z\,;w) \subseteq L(A)$ define a bounded equivalence relation on $p(\mathcal{U})$. Then there is a type $e'(z\,;w) \subseteq L(A)$ which defines a bounded equivalence relation (on $\mathcal{U}$) and refines $e(z\,;w)$ on $p(\mathcal{U})$.

Proof. Let $\mathcal{C} = \langle \mathcal{C}_i : i < \lambda \rangle$ enumerate the partition of $p(\mathcal{U})$ induced by $e(z\,;w)$. Note that each $\mathcal{C}_i$ is type-definable over A, a for any $a \in \mathcal{C}_i$. If $x = \langle x_i : i < \lambda \rangle$, we write $x \in \mathcal{C}$ for the type that is the conjunction of $x_i \in \mathcal{C}_i$ for $i < \lambda$.

We claim that the required type is

$$e'(a\,;b) \quad = \quad \exists x' \in \mathcal{C}\; \exists x'' \in \mathcal{C} \quad a, x' \equiv_A b, x''$$

As the type above is invariant over A, we can assume that $e'(a\,;b)$ is type-definable over A. Moreover, it is clearly a reflexive and symmetric relation, so we only check it is transitive. Suppose $a, x' \equiv_A b, x''$ and $b, y' \equiv_A c, y''$ for some $x', x'', y', y'' \in \mathcal{C}$. Let z be such that $a, x', z \equiv_A b, x'', y'$ then $z \in \mathcal{C}$ and $a, z \equiv_A c, y''$.

The relation $e'(a\,;b)$ clearly refines the equivalence defined by $e(z\,;w)$ when restricted to to $p(\mathcal{U})$. To prove that it is bounded fix some $c \in \mathcal{C}$ and note that $e'(a\,;b)$ is refined by $a \equiv_{A,c} b$, which is bounded. □

Finally, we have the following.

16.30 Theorem Denote by $e_A(z\,;w)$ be the finest bounded equivalence relation on $\mathcal{U}^z$ that is type-definable over A. Then for every $a, b \in \mathcal{U}^z$ the following are equivalent

1. $a \overset{\text{KP}}{\equiv} b$
2. $e_A(a\,;b)$.

Proof. As $a \overset{\text{KP}}{\equiv} b$ is equivalent to $a \in \mathcal{K}(b/A)$, it suffices to prove that $e_A(\mathcal{U}\,;b) = \mathcal{K}(b/A)$ for every $b \in \mathcal{U}^z$.

$\subseteq$. The orbit of $e_A(\mathcal{U};b)$ under $\mathrm{Aut}(\mathcal{U}/A)$ has cardinality $< \kappa$. Hence, by Theorem 16.8, it is Lascar invariant over A. As $\mathcal{K}(b/A)$ is the least of such sets, $\mathcal{K}(b/A) \subseteq e_A(\mathcal{U};b)$.

$\supseteq$. Let $e(z;w)$ be the type-definable equivalence relation given by Proposition 16.27. The orbit of $\mathcal{K}(b/A)$ under $\mathrm{Aut}(\mathcal{U}/A)$ has cardinality $< \kappa$. Hence the equivalence relation $e(z;w)$ that defines on $\mathcal{O}(b/A)$ is bounded. By Lemma 16.29 there is a type-definable bounded equivalence relation $e'(z;w) \subseteq L(A)$ that refines $e(z;w)$ on $\mathcal{O}(b/A)$. As $e'(z;w)$ is refined by $e_A(z;w)$, we obtain $e_A(\mathcal{U};b) \subseteq e(\mathcal{U};b) = \mathcal{K}(b/A)$. □

The logic A-topology, or simply the logic topology when A is empty, is the topology on $\mathcal{U}^z$ whose closed sets are the type-definable Lascar A-invariant sets. It is clearly a compact topology. Its Kolmogorov quotient, that is $\mathcal{U}^z/\overset{\mathrm{KP}}{\equiv}$, is Hausdorff.

16.31 Exercise Prove that the clopen sets in the logic A-topology are exactly the sets that are definable over $\mathrm{acl}^{\mathrm{eq}}A$.

16.6 Notes and references

[1] Artem Chernikov, *Lecture notes on stability theory*, Open Math Notes, Amercan Mathematical Society, 2019.

[2] Pierre Simon, *A guide to NIP theories*, Lecture Notes in Logic, vol. 44, Association for Symbolic Logic, Cambridge University Press, 2015.

Chapter 17

Actions of groups

Chapter under revision.

In this chapter, L is a signature, T is a complete theory without finite models, and $\mathcal{U}$ is a saturated model of inaccessible cardinality κ strictly larger than $|L|$. We use the same notation and make the same implicit assumptions as in Section 9.3.

Assumption 17.1 below is in force throughtout the chapter.

17.1 The dual perspective on invariance

The notions of invariance introduceted in Chapter 14 extend seemlessly to the action of a group G on $\mathcal{U}^z$. It is convenient to work in a more general setting.

In this chapter $\Delta \subseteq L_{x\,z}(\mathcal{U})$. Let $\mathcal{Z} \subseteq \mathcal{U}^z$. We write $\Delta(\mathcal{Z})$ for the set of formulas of the form $\varphi(x\,;b)$ for some $\varphi(x\,;z) \in \Delta$ and some $b \in \mathcal{Z}$. We write $\Delta^{\pm}(\mathcal{Z})$ for the set of formulas in $\Delta(\mathcal{Z})$ or negation thereof. We write $S_\Delta(\mathcal{Z})$ for the set of complete $\Delta^{\pm}(\mathcal{Z})$-types.

We write $\Delta^{\mathsf{B}}(\mathcal{Z})$ for the set of Boolean combinations of formulas in $\Delta(\mathcal{Z})$.

When $\mathcal{A} \subseteq \mathcal{U}$, we may use $\mathcal{A}$ for $\mathcal{A}^z$ in the notation above.

Finally, define $\Delta^{\mathsf{G}}(A)$ to be the set of formulas $\varphi(x) \in L(\mathcal{U})$ that are equivalent to some formula in $\Delta^{\mathsf{B}}(A)$ or, equivalently, that are invariant over A. In the literature these formulas are called *generalized* Δ-formulas over A. Note that when A is a model $\Delta^{\mathsf{G}}(A)$-formulas are equivalent to $\Delta^{\mathsf{B}}(A)$-formulas.

17.1 Assumption Let G be a group that acts on some sets $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$. We require that for every $\varphi(x\,;z) \in \Delta$ the set $\varphi(\mathcal{X};\mathcal{Z})$ is invariant under the action of G. For convenience, we will assume that G is the identity outside $\mathcal{X}$ and $\mathcal{Z}$.

When $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ and $\mathcal{D} \subseteq \mathcal{U}^x$ we write $p(x) \vdash x \in \mathcal{D}$ if the inclusion $\psi(\mathcal{X}) \subseteq \mathcal{D}$ for some $\psi(x)$ that is conjunctions of formulas in $p(x)$.

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that $\mathcal{D}$ is invariant under the action of G, or G-invariant, if $\mathcal{D}$ is fixed setwise by G. That is, $g\,\mathcal{D} = \mathcal{D}$ for every $g \in G$. Yet in other words, if

is1. $\quad a \in \mathcal{D} \;\leftrightarrow\; ga \in \mathcal{D} \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

A formula is G-invariant if the set it defines is G-invariant. We say that $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ is invariant under the action of G, or G-invariant, if for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-formula $\vartheta(x\,;\bar a)$.

it1. $\quad \vartheta(x\,;\bar a) \in p \;\Leftrightarrow\; \vartheta(x\,;g\,\bar a) \in p \qquad$ for every $g \in G$.

It should be clear that invariant under the action of $\mathrm{Aut}(\mathcal{U}/A)$ coincides with invariant over A and Lascar invariant over A coincides with invariant under the action of $\mathrm{Aut}^{\mathsf{f}}(\mathcal{U}/A)$.

Note that $p(x)$ is G-invariant exactly when the sets $\mathcal{D}_{p,\vartheta} \subseteq \mathcal{U}^z$ are. In this section we want to discuss invariance using sets $\mathcal{D} \subseteq \mathcal{U}^x$. We need stronger assumptions on the action of G.

An immediate consequence of the invariance of $\varphi(\mathcal{X}\,;\mathcal{Z})$ is that any G-translate of a $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set is again $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable. In particular, by reasoning as in Remark 9.23 we obtain that for every $\Delta^{\mathrm{B}}(\mathcal{Z})$-formula $\vartheta(x\,;\bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X}\,;\bar{b})] \;=\; \vartheta(\mathcal{X}\,;g\,\bar{b}).$$

Therefore, a type $p(x) \subseteq \Delta^{\mathrm{B}}(\mathcal{Z})$ is G-invariant if

$$p(x) \vdash x \in \mathcal{D} \;\Leftrightarrow\; p(x) \vdash x \in g\,\mathcal{D} \quad \text{for every } \Delta^{\mathrm{B}}(\mathcal{Z})\text{-definable } \mathcal{D} \subseteq \mathcal{X} \text{ and } g \in G.$$

A set $\mathcal{D} \subseteq \mathcal{X}$ is generic under the action of G, or G-generic for short, if finitely many G-translates of $\mathcal{D}$ cover $\mathcal{X}$; we say *n-G-generic* if $\leq n$ translates suffices. Dually, we say that $\mathcal{D}$ is persistent under the action of G, or G-persistent for short, if the intersection of any finitely many G-translates of $\mathcal{D}$ is nonempty; we say *n-G-persistent* when the request is limited to $\leq n$ translates. When $\mathcal{X}$ and/or $\mathcal{Z}$ are not clear from the context, we say that these notions are relative to $\mathcal{X}$ and $\mathcal{Z}$.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$. See Exercise 17.13 for an alternative characterization when $p(x)$ is small.

⚠ The terminology above is non-standard. In [2] the authors write *quasi-non-dividing* for *persistent* under the action of $\mathrm{Aut}(\mathcal{U}/A)$. Their terminology has good motivations, but it would be a mouthful if adapted to our context. In topological dynamics similar notions have been introduced with different terminology: *syndetic* corresponds to *generic* and *thick* corresponds to *persistent*.

17.2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is persistent under the action of $G = \mathrm{Aut}(\mathcal{U}/A)$ relative to any $\mathcal{X} \supseteq A^x$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\mathrm{Aut}(\mathcal{U}/A)$-translate of $\varphi(x)$.

Notation: for $\mathcal{D} \subseteq \mathcal{U}^x$ and $H \subseteq G$ we write $H\,\mathcal{D}$ for $\{h\mathcal{D} : h \in H\}$.

In this notes many proofs require some juggling with negations as epitomized by the following fact.

17.3 Fact The following are equivalent

1. $\mathcal{D}$ is not G-generic
2. $\neg\mathcal{D}$ is G-persistent.

Proof. Immediate by spelling out the definitions

1. there are no finite $H \subseteq G$ such that $\mathcal{X} \;\subseteq\; \cup H\,\mathcal{D}$.
2. $\varnothing \;\neq\; \mathcal{X} \cap \big(\cap\, H\neg\mathcal{D}\big)$ for every finite $H \subseteq G$. □

Define the following type

$$\gamma_G(x) \;=\; \{\vartheta(x) \in \Delta^{\mathrm{B}}(\mathcal{Z}) : \vartheta(x) \text{ is } G\text{-generic}\}$$

17.4 Corollary Let $p(x) \subseteq \Delta^{\mathrm{B}}(\mathcal{Z})$ be such that $\gamma_G(x) \cup p(x)$ is finitely satisfiable in $\mathcal{X}$. Then $p(x)$ is G-persistent.

Proof. Let $\vartheta(x)$ be a conjunction of formulas in $p(x)$. As $\gamma_G(x)$ is finitely satisfiable in $\vartheta(\mathcal{X})$, it cannot be that $\neg\vartheta(x)$ is G-generic. From Fact 17.3, we obtain that $\vartheta(x)$ is G-persistent. □

The converse implication holds for complete types.

17.5 Theorem Let $p(x) \in S_\Delta(\mathcal{Z})$ be finitely satisfiable in $\mathcal{X}$. Then the following are equivalent

1. $p(x)$ is G-invariant
2. $p(x) \vdash \gamma_G(x)$
3. $p(x)$ is G-persistent.

Proof. 1⇒2. Let $H \subseteq G$ be finite such that $\mathcal{X} \subseteq \cup H\,\mathcal{D}$. By completeness and finite satisfiability, $p(x) \vdash x \in \cup H\,\mathcal{D}$. Again by completeness, $p(x) \vdash x \in h\,\mathcal{D}$ for some $h \in H$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

2⇒3. Let $\mathcal{D}$ be defined by a conjunction of formulas in $p(x)$. If $\mathcal{D}$ is not G-persistent then, by Fact 17.3, $\neg\mathcal{D}$ is G-generic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

3⇒1. If $p(x)$ is not G-invariant then, by completeness, $p(x) \vdash \varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ for some $g \in G$. Clearly $\varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ is not 2-G-persistent as it is inconsistent with its g-translate. □

The theorem yields a necessary condition for the existence of G-invariant global $\Delta^{\mathrm{B}}(\mathcal{Z})$-types.

17.6 Corollary If there exists a G-invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in $\mathcal{X}$ then for every $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set $\mathcal{D}$

1. $\mathcal{D}$ and $\neg\mathcal{D}$ are not both G-generic
2. if $\mathcal{D}$ is G-generic then it is G-persistent
3. $\gamma_G(x)$ is finitely satisfiable in $\mathcal{X}$.

Proof. Clearly, 1 and 2 are equivalent by Fact 17.3 and follow from 3. Finally, 3 is an immediate consequences of 2 of Theorem 17.5. □

The following theorem gives a necessary and sufficient condition for the existence of global G-invariant $\Delta^{\mathrm{B}}(\mathcal{Z})$-type. Ideally, we would like to have that every G-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-type extends to a global persitent type. Unfortunately this is not true in general (it requires stronger assumptions, see Section 17.4). A set $\mathcal{D}$ is hereditarily G-persistent if every finite cover of $\mathcal{D}$ by $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets contains a G-persistent set. In [2] a similar property is called *quasi-non-forking*. A type is hereditarily G-persistent if every conjunction of formulas in the type is hereditarily G-persistent.

17.7 Theorem Let $\mathcal{D}$ be a $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set. Then the following are equivalent

1. there exists a G-invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$
2. $\mathcal{D}$ is hereditarily G-persistent.

Proof. 1⇒2. Suppose $\mathcal{C}_1, \ldots, \mathcal{C}_n$ cover $\mathcal{D}$ and pick $p(x)$ as in 1. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i. Then, by Theorem 17.5, $\neg\mathcal{C}_i$ is not G-generic. Therefore, by Fact 17.3, $\mathcal{C}_i$ is G-persistent.

2⇒1. Let $p(x)$ be maximal among the $\Delta^{\mathsf{B}}(\mathcal{Z})$-types that are finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$ and are such that $\vartheta(\mathcal{U}^x)$ is hereditarily G-persistent for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that $p(x)$ is a complete $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\mathcal{C}_1, \ldots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \smallsetminus \vartheta(\mathcal{U}^x)$ and such that no $\mathcal{C}_i$ is G-persistent. As $\mathcal{C}_1, \ldots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is G-invariant. his follows from completeness and Theorem 17.5. □

Let $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ be G-invariant. We say that $p(x)$ is G-prime if for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set $\mathcal{D}$ and every $g \in G$ if $p(x) \vdash x \in (\mathcal{D} \cup g\,\mathcal{D})$ then $p(x) \vdash x \in \mathcal{D}$.

17.8 Proposition Then $\gamma_G(x)$ is G-prime.

Proof. The following are equivalent for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set $\mathcal{D}$

1. $\gamma_G(x) \vdash x \in \mathcal{D}$
2. $p(x) \vdash x \in \mathcal{D}$ for every G-persistent $p(x) \in S_\Delta(\mathcal{Z})$.

Now, assume $\gamma_G(x) \vdash x \in (\mathcal{D} \cup g\,\mathcal{D})$ and pick a G-persistent $p(x) \in S_\Delta(\mathcal{Z})$. By completeness $p(x) \vdash x \in \mathcal{D}$ or $p(x) \vdash x \in g\,\mathcal{D}$. As $p(x)$ is G-invariant, $p(x) \vdash x \in \mathcal{D}$. As $p(x)$ is arbitrary, $\gamma_G(x) \vdash x \in \mathcal{D}$ follows from the above equivalence. □

17.9 Exercise Prove that if $p(x) \in S(\mathcal{U})$ is finitely satisfiable in every $M \supseteq A$ then it is persistent under the action of $\mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$. Is the same true for incomplete types?

17.10 Exercise Let $p(x) \in S_\Delta(\mathcal{U})$ be G-persistent. Prove that the following are equivalent

1. $p(x)$ is G-invariant and finitely satisfiable in $\mathcal{X}$
2. $p(x) \vdash x \in \mathcal{D}$ for every 2-G-generic definable set $\mathcal{D}$
3. $p(x)$ is 2-G-persistent.

17.11 Exercise Prove that the following are equivalent for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set $\mathcal{D} \subseteq \mathcal{U}^x$

1. there is a G-invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$
2. every finite cover of $\mathcal{D}$ by $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets contains a 2-G-persistent set.

17.12 Exercise Let $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ be G-generic. Prove that it is finitely satisfiable in $\mathcal{X}$.

17.13 Exercise Prove that for $p(x) \subseteq \Delta^{\mathsf{B}}(A)$ following are equivalent

1. $p(x)$ is G-persistent
2. $p(\mathcal{U}^x)$ is G-persistent.

17.14 Exercise Give an example of a persistent set that is not hereditarily persistent. Hint: find inspiration in Example 16.17.

17.2 Strong genericity

Unfortunately, G-genericy is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is strongly G-generic if for every finite $H \subseteq G$ the set $\cap H\, \mathcal{D}$ is generic. Dually, we say that $\mathcal{D}$ is weakly G-persistent if for some finite $H \subseteq G$ the set $\cup H\, \mathcal{D}$ is persistent. Again, the same properties may be attributed to formulas and types.

17.15 Lemma The intersection of two strongly G-generic sets is strongly G-generic.

Proof. We may assume that all sets mentioned below are subsets of $\mathcal{X}$. Let $\mathcal{D}$ and $\mathcal{C}$ be strongly G-generic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K\,(\mathcal{C} \cap \mathcal{D})$ is G-generic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K\, \mathcal{C}$ and $\mathcal{D}' = \cap K\, \mathcal{D}$. Note that $\mathcal{C}'$ and $\mathcal{D}'$ are both strongly G-generic. In particular $\mathcal{X} = \cup H\, \mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\begin{aligned}
\cup H\, \mathcal{B} &= \cup H\Big[\mathcal{C}' \cap \mathcal{D}'\Big] \\
\cup H\, \mathcal{B} &\supseteq \cup H\Big[\big(\cap H\, \mathcal{C}'\big) \cap \mathcal{D}'\Big] \\
&= \big(\cap H\, \mathcal{C}'\big) \cap \big(\cup H\, \mathcal{D}'\big) \\
&= \cap H\, \mathcal{C}'
\end{aligned}$$

As $\mathcal{C}'$ is strongly G-generic, $\cap H\, \mathcal{C}'$ is G-generic. Therefore $\cup H\, \mathcal{B}$ is also G-generic. The G-genericity of $\mathcal{B}$ follows. □

Define the following type

$$^{\mathsf{s}}\gamma_G(x) \quad = \quad \{\vartheta(x) \in \Delta^{\mathsf{B}}(\mathcal{Z}) : \vartheta(x) \text{ is strong } G\text{-generic}\}$$

17.16 Corollary The type $^{\mathsf{s}}\gamma_G(x)$ is finitely satisfiable in $\mathcal{X}$, strongly G-generic, and G-invariant.

Proof. The strong G-genericity is an immediate consequence of Lemma 17.15. The finite satisfiability is a consequence of G-genericity, see Exercise 17.12. As for invariance, note that any translate of a strongly G-generic formula is also strongly G-generic. □

17.17 Corollary Let $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ be such that $^{\mathsf{s}}\gamma(x) \cup p(x)$ is finitely satisfiable in $\mathcal{X}$. Then $p(x)$ is weakly G-persistent.

Proof. Similar to Corollary 17.4. Let $\vartheta(x) \in p$. As $^{\mathsf{s}}\gamma(x)$ is finitely satisfiable in $\vartheta(\mathcal{U}^x)$, we cannot have that $\neg\vartheta(x)$ is strongly G-generic. From Fact 17.3, we obtain that $\neg\vartheta(\mathcal{U}^x)$ non strongly G-generic is equivalent to $\vartheta(x)$ weakly G-persistent. □

17.18 Exercise Show that G-generic sets need not be closed under intersection. Hint: find inspiration in Example 16.17.

17.3 The diameter of a Lascar type

As an application we prove an interesting property of the Lascar types. Recall that $\mathcal{L}(a/A)$, the Lascar strong type of $a \in \mathcal{U}^x$, is the union of a chain of type-definable sets of the form $\{x : d_A(a,x) \leq n\}$. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter.

It is convenient to address the problem in more general terms. We work under Assumption 17.1 and also assume that G acts transitively on $\mathcal{X}$ i.e. $G\,a = \mathcal{X}$ for every $a \in \mathcal{X}$. Let $K \subseteq G$ be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugancy invariant i.e. $g\,Kg^{-1} = K$ for every $g \in G$

We define a discrete metric on $\mathcal{X}$. For $a, b \in \mathcal{X}$ let $d(a,b)$ be the minimal n such that $a \in K^n b$. This defines a metric which is G-invariant by 2. The diameter of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a,b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for $\mathcal{X}$ to have finite diameter. The notions introduced in Section 17.2 offer some hint.

17.19 Proposition If $\mathcal{X}$ has a weakly persistent subset of finite diameter, then $\mathcal{X}$ itself has finite diamenter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter n. Let $H \subseteq G$ be finite such that $\cup H\,\mathcal{C}$ is persistent. We claim that also $\cup H\,\mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k, \in H$. Now, let hb and kc, for some $h, k, \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H\,\mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of $\mathcal{C}$,

$$\begin{aligned} d(hb,\, kc) \ &\leq\ d(hb,\, ha) \ +\ d(ha,\, ka) \ +\ d(ka,\, kc) \\ &\leq\ n + m + n. \end{aligned}$$

This proves that $\cup H\,\mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that $\mathcal{C}$ itself is persistent.

By the transitivity of the action, any two elements of $\mathcal{X}$ are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By percistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha,\, ka) \ &\leq\ d(ha,\, c) \ +\ d(c,\, d) \ +\ d(d,\, ka) \\ &\leq\ n + n + n. \end{aligned}$$

Therefore the diameter of $\mathcal{X}$ does not exceed $3n$. □

17.20 Theorem Suppose that $\mathcal{X}$ and the sets $\mathcal{X}_n = K^n a$, for some $a \in \mathcal{X}$ that is $\Delta^{\mathrm{B}}(\mathcal{Z})$-type-definable. Then $\mathcal{X}$ has finite diameter.

Proof. By Proposition 17.19, it suffices to prove that $\mathcal{X}_n$ is weakly persistent. By Corollary 17.17 it suffices to show that for some n the type ${}^{\mathrm{s}}\gamma_G(x)$ is finitely satisfied in $\mathcal{X}_n$. Suppose not. Let $\psi_n(x) \in \gamma_G$ be a formula that is not satisfied in $\mathcal{X}_n$. The type $p(x) = \{\psi_n(x) : n \in \omega\}$ is finitely satisfied in $\mathcal{X}$. Then $p(x)$ has a realization in $\mathcal{X}$. As this realization belongs to some $\mathcal{X}_n$ we contradict the definition of $\psi_n(x)$. □

17.21 Example Let $K \subseteq \mathrm{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A. Then the group G generated by K is $\mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. Let $\Delta = L_{x\,z}$ and $\mathcal{Z} = \mathcal{U}^{z}$. Then $d(a,b)$ coincides with the distance in the Lascar graph. As shown in Proposition 16.24 the sets $K^n \cdot a = \{x : d(x,a) \leq n\}$ are type definable. Then from Theorem 17.20 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

17.4 A tamer landscape

Under suitable assumptions – e.g. see Proposition 18.32 – some of the notions introduced in this chapter coalesce and we are left with a tamer landscape. We prove the following theorem.

17.22 Theorem The following are equivalent

1. G-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are hereditarily G-persistent
2. G-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are closed under intersection
3. G-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are strongly G-generic
4. weakly persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are G-persistent.

Proof. 2⇔3⇔4. Clear.

1⇒2. Let $\mathcal{C}$ and $\mathcal{D}$ be G-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not G-generic. Then $\neg(\mathcal{C} \cap \mathcal{D})$ is G-persistent. By 1 and Theorem 17.7 there is a G-invariant global $\Delta^{\mathrm{B}}(\mathcal{Z})$-type $p(x)$ containing $x \notin \mathcal{C} \cap \mathcal{D}$. By completeness either $p(x) \vdash x \notin \mathcal{C}$ or $p(x) \vdash x \notin \mathcal{D}$. This is a contradiction because by Theorem 17.5 $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

4⇒1. Note that, by 3, the type ${}^{\mathrm{s}}\gamma_G(x)$ coincides with $\gamma_G(x)$, in particular $\gamma_G(x)$ is finitely satisfied in $\mathcal{X}$. Let $\mathcal{D}$ be a G-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set. We show that$\gamma_G(x) = {}^{\mathrm{s}}\gamma_G(x)$ is finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$. Then, by 4 and Corollary 17.17, any global extension of $\gamma_G(x) \cup \{x \in \mathcal{D}\}$ witness 2 of Theorems 17.7. Suppose not, then $\gamma_G(x) \vdash x \notin \mathcal{D}$. Therefore $\neg\mathcal{D}$ is G-generic, contradicting the consistency of ${}^{\mathrm{s}}\gamma_G(x)$. □

17.23 Remark Assume that the equivalent conditions in Theorem 17.22 hold. Then the types $\gamma_G(x)$ and ${}^{\mathrm{s}}\gamma_G(x)$ coincide, and therefore G-invariant global types exist. It is also worth mentioning that every positive Boolean combination of G-generic sets is G-generic.

17.5 Definable groups

In this section we assume that $\mathcal{Z} = G$. Furthermore, we assume that $\mathcal{Z}$ and $\mathcal{X}$ are definable over some set of parameters A. The group operations and the group action are also assumed definable over A. We use the symbol $\cdot$ for both the group multiplication and the group action.

Let $\Psi \subseteq L_x(\mathcal{U})$ be some small set of formulas. In this section Δ contains formulas

$\varphi(x\,;z)$ of the form $\psi(z^{-1}\cdot x)$ for $\psi(x)\in\Psi$. Note that the sets $\varphi(\mathcal{X}\,;\mathcal{Z})$ are $\mathcal{Z}$-invariant. Write 1 be the identity of $\mathcal{Z}$. Clearly, $\varphi(\mathcal{X}\,;g)=g\cdot\varphi(\mathcal{X}\,;1)$.

The following auxiliary structure is useful. Let $\mathcal{U}'=\langle\mathcal{X}\,;\mathcal{Z}\rangle$ be a 2-sorted structure whose signature L' contains only a relation symbol for every formula $\varphi(x\,;z)\in\Delta$. As there is little risk of confusion, these relations symbols are also denoted by $\varphi(x\,;z)$. As $\mathcal{X}$ and $\mathcal{Z}$ are assumed to be definable, $\mathcal{U}'$ is a saturated L'-structure.

To each $h\in\mathcal{Z}$ we associate the L'-automorphism $\langle a\,;g\rangle\mapsto\langle h\,a\,;h\,g\rangle$. Therefore $\mathcal{Z}$ is, up to isomorphism, a subgroup of L'-$\mathrm{Aut}(\mathcal{U}')$. Note that, for any $g\in\mathcal{Z}$ the orbit of $\varphi(\mathcal{X}\,;h)$ under the action of $\mathcal{Z}$ is $\{\varphi(\mathcal{X}\,;g)\,:\,h\in\mathcal{Z}\}$. Therefore the orbit under the action of L'-$\mathrm{Aut}(\mathcal{U}')$ is the same (it cannot be any larger). We conclude that the notions of genericity and persistency under the two actions coincide.

17.24 Fact Let $\varphi(x\,;z)\in\Delta$. Let $p(x)\in L'$-$S_\varphi(\mathcal{Z})$. Then the following are equivalent for any $g\in\mathcal{Z}$

1. $p(x)$ is invariant under the action of L'-$\mathrm{Aut}^{\mathrm{f}}(\mathcal{U})$.
2. $g\,p(x)$ is invariant under the action of L'-$\mathrm{Aut}^{\mathrm{f}}(\mathcal{U})$

Proof. It suffices to prove 1⇒2. Recall that 1 is equivalent to requiring that the orbit of $\mathcal{D}_{p,\varphi}$ under the action of L'-$\mathrm{Aut}^{\mathrm{f}}(\mathcal{U})$ has small cardinality. As $\mathcal{D}_{gp,\varphi}=g\,\mathcal{D}_{p,\varphi}$ the set $\mathcal{D}_{gp,\varphi}$ has the same orbit. □

17.25 Corollary Let $\varphi(x\,;z)\in\Delta$. Assume the existence of a type $p(x)\in L'$-$S_\varphi(\mathcal{Z})$ that is invariant under the action of L'-$\mathrm{Aut}^{\mathrm{f}}(\mathcal{U})$. Then 1⇒2, where

1. $\varphi(x\,;1)$ is $\mathcal{Z}$-generic
2. $\varphi(x\,;g)$ is Lascar hereditarily persistent over A for every $g\in\mathcal{Z}$.

We will prove that under the assumption of stability also the converse implication holds. See Theorem 18.37.

Proof. Let g be given. If $\varphi(x\,;1)$ is $\mathcal{Z}$-generic, then so is $\varphi(x\,;g)$. By completeness, $p(x)\vdash\varphi(x\,;h\,g)$ for some $h\in\mathcal{Z}$. Equivalently, $\varphi(x\,;g)\in h^{-1}p$. By Fact 17.24 also $h^{-1}p(x)$ is invariant under the action of L'-$\mathrm{Aut}^{\mathrm{f}}(\mathcal{U})$. Therefore $\varphi(x\,;g)$ is hereditarily persistent under the action of L'-$\mathrm{Aut}^{\mathrm{f}}(\mathcal{U})$ and, a fortiori, of $\mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$. □

17.6 Notes and references

In Example 17.21 we prove a theorem of Newelski's [3]. The original proof is rather long and complex. A simplified proof (also due, reportedly, to Newelski) appears in Rodrigo Peláez's thesis [4, Section 3.3] and [1, Chapter 9]. The proof here is a streamlined and generalized version of the latter – inspired by [5].

[1] Enrique Casanovas, *Simple theories and hyperimaginaries*, Lecture Notes in Logic, vol. 39, Cambridge University Press, 2011.

[2] Artem Chernikov and Itay Kaplan, *Forking and dividing in* NTP_2 *theories*, J. Symbolic Logic (2012).

[3] Ludomir Newelski, *The diameter of a Lascar strong type*, Fund. Math. (2003).

[4] Rodrigo Peláez, *About the Lascar group*, PhD Thesis, Universitat de Barcelona, Departament de Lógica, História i Filosofia de la Ciéncia, 2008.

[5] Domenico Zambella, *On the diameter of Lascar strong types after Ludomir Newelski*, A tribute to Albert Visser, Coll. Publ., [London], 2016, arXiv:1605.00218.

Chapter 18

Stability

⚠ Chapter under revision.

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality $\kappa > |L|$. The notation and implicit assumptions are as in Section 9.3.

18.1 Externally definable sets

Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{U}^z$. The set $\mathcal{D} \cap A^z$ is called the trace of $\mathcal{D}$ over A. We write $\mathcal{C} =_A \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ have the same trace on A.

Let $p(x) \subseteq L(\mathcal{U})$ be a consistent type. Recall from Section 14.1 that for every formula $\varphi(x\,;z) \in L$ we define

$$\mathcal{D}_{p,\varphi} \quad = \quad \Big\{ a \in \mathcal{U}^z \;:\; \varphi(x\,;a) \in p \Big\}.$$

We say that $\mathcal{D}$ is externally definable if it is of the form $\mathcal{D}_{p,\varphi}$ for a type $p(x)$ in $S(\mathcal{U})$ or $S_\varphi(\mathcal{U})$. We say that $\mathcal{D}$ is externally definable by $p(x)$ and $\varphi(x\,;z)$.

Equivalently, a set $\mathcal{D}$ is externally definable if it is the trace over $\mathcal{U}$ of a set which is definable in some elementary extension of $\mathcal{U}$. More precisely, $\mathcal{D}$ is the trace on $\mathcal{U}$ of a set of the form $\varphi({}^*b\,;{}^*\mathcal{U})$ where ${}^*\mathcal{U}$ is elementary extension of $\mathcal{U}$ and ${}^*b \in {}^*\mathcal{U}^x$. The latter interpretation explains the terminology.

⚠ We prefer to deal with external definability in a different, though equivalent, way. This is not the most common approach.

18.1 Definition We say that $\mathcal{D}$ is approximated by the formula $\varphi(x\,;z)$ if for every finite B there is a tuple $a \in \mathcal{U}^x$ such that $\varphi(a\,;B^z) = \mathcal{D} \cap B^z$. We call $\varphi(x\,;z)$ the sort of $\mathcal{D}$. If in addition $\varphi(a\,;\mathcal{U}^z) \subseteq \mathcal{D}$, we say that $\mathcal{D}$ is approximated from below. Equivalently, we say that $\mathcal{D}$ is approximated from below if for every finite $B^z \subseteq \mathcal{D}$ there is a tuple $a \in \mathcal{U}^x$ such that $B^z \subseteq \varphi(a\,;\mathcal{U}^z) \subseteq \mathcal{D}$. The dual notion of approximation from above is defined as expected (and coincides with $\neg\mathcal{D}$ being approximated by $\neg\varphi(x\,;z)$ from below).

The following proposition is clear by compactness.

18.2 Proposition For every $\mathcal{D}$ the following are equivalent

1. $\mathcal{D}$ is approximated by $\varphi(x\,;z)$
2. $\mathcal{D}$ is externally definable by $\varphi(x\,;z)$.

The rest of this section is only required in Chapter 19.

Approximability from below is an adaptation to our context of the notion of *having* *an honest definition* in [1].

18.3 Example Every definable set is trivially approximable. Sets may be approximable by different formulas. For instance, if $T = T_{\text{dlo}}$, then $\mathcal{D} = \{z \in \mathcal{U} : a \leq z \leq b\}$ is approximable both from below and from above by the formula $x_1 < z < x_2$ though it is not definable by this formula.

Now, let $T = T_{\text{rg}}$. Then every $\mathcal{D} \subseteq \mathcal{U}$ is approximable and, when $\mathcal{D}$ has small infinite cardinality, it is approximable from above but not from below.

In Definition 18.1, the sort $\varphi(x\,;z)$ is fixed (otherwise any set would be approximable) but this requirement of uniformity may be dropped if we allow B to have larger cardinality.

18.4 Proposition For every $\mathcal{D}$ the following are equivalent

1. $\mathcal{D}$ is approximable
2. for every C of cardinality $\leq |L|$ there is $\psi(z) \in L(\mathcal{U})$ such that $\psi(C^z) = \mathcal{D} \cap C^z$.

Proof. To prove 2⇒1 assume 2 and negate 1 for a contradiction. For each formula $\psi(x\,;z) \in L$ choose a finite set B such that $\psi(b\,;\mathcal{U}^z) \neq_B \mathcal{D}$ for every $b \in \mathcal{U}^x$. Let C be the union of all these finite sets. Clearly $|C| \leq |L|$. By 2 there are a formula $\varphi(x\,;z)$ and a tuple c such that $\varphi(c\,;C^z) = \mathcal{D} \cap C^z$, contradicting the definition of C. □

18.5 Remark If $\mathcal{D} \subseteq \mathcal{U}^z$ is approximated by $\varphi(x\,;z)$ then so is any $\mathcal{C}$ such that $\mathcal{C} \equiv \mathcal{D}$, see Section 16.1 for the notation. In fact, if the set $\mathcal{D}$ is approximable by $\varphi(x\,;z)$ then for every n

$$\forall z_1, \ldots, z_n \, \exists x \, \bigwedge_{i=1}^{n} \big[\varphi(x\,;z_i) \;\leftrightarrow\; z_i \in \mathcal{D}\big].$$

So the same holds for any $\mathcal{C} \equiv \mathcal{D}$. A similar remark apply to approximability from below and from above (e.g. for approximability from below, add the conjunct $\forall z \, \big[\varphi(x\,;z) \rightarrow z \in \mathcal{D}\big]$ to the formula above).

From the following easy observation of Chernikov and Simon [1] we obtain an interesting (and mysterious) quantifier elimination result originally due to Shelah, see Corollary 19.6 below.

18.6 Proposition Let $\mathcal{C} \subseteq \mathcal{U}^{y\,z}$ be approximated from below by the formula $\varphi(x\,;y\,z)$. Then $\mathcal{D} = \big\{z : \exists y \, \big(y\,z \in \mathcal{C}\big)\big\}$ is approximated from below by $\exists y \, \varphi(x\,;y\,z)$.

Proof. Let $B \subseteq \mathcal{U}$ be finite. We want $a \in \mathcal{U}^x$ such that

a. $$\exists y \, \big(y\,b \in \mathcal{C}\big) \;\leftrightarrow\; \exists y \, \varphi(a\,;y\,b) \qquad \text{for every } b \in B^z$$

b. $$\forall z \, \Big[\exists y \, \varphi(a\,;y\,z) \;\rightarrow\; \exists y \, \big(y\,z \in \mathcal{C}\big)\Big]$$

Let $D \subseteq \mathcal{U}$ be a finite set such that

c. $$\exists y \in D^y \, \big(y\,b \in \mathcal{C}\big) \;\leftrightarrow\; \exists y \, \big(y\,b \in \mathcal{C}\big) \qquad \text{for every } b \in B^z$$

As $\mathcal{C}$ is approximable from below, there is an a such that

a′. $\quad d\,b \in \mathcal{C} \ \leftrightarrow\ \varphi(a\,;d\,b) \qquad$ for every $d\,b \in (D \cup B)^{y\,z}$

b′. $\quad \forall y\,z\,\Big[\varphi(a\,;y\,z) \ \rightarrow\ y\,z \in \mathcal{C}\Big]$

We obtain b from b′ simply by logic. Implication $\rightarrow$ in a follows from a′ and c. Implication $\leftarrow$ follows from b. □

18.7 Corollary If $p \in S_x(\mathcal{U})$ is honestly definable then the family of sets externally definable by p is closed under quantifiers and Boolean combinations.

Proof. The sets externally definable by $p(x)$ are always closed under Boolean operations. By the proposition above, they are closed under quantifiers. □

18.2 Ladders and definability

Let $\mathcal{R} \subseteq \mathcal{U}^x \times \mathcal{U}^z$ be a binary relation invariant over A. We denote by $\mathcal{R}(x\,;z)$ the predicate associated to $\mathcal{R}$. We say that $\langle a_i\,;b_i : i < \alpha\rangle$ is a ladder of length α for $\mathcal{R}(x\,;z)$ if for every $i < j < \alpha$

$$\mathcal{R}(a_i\,;b_j) \quad\wedge\quad \neg\mathcal{R}(a_j\,;b_i)$$

We say that $\mathcal{R}(x\,;z)$ is stable if there is no ladder of length ω. Otherwise we say it is unstable or that it has the order property. We will mainly deal with relations that are definable or type-definable. We will say that a formula or a type is stable accordingly. The proof of the following easy fact is left to the reader.

18.8 Fact The following are equivalent for every $p(x\,;z) \subseteq L(A)$

1. $p(x\,;z)$ is stable
1. for some finite n, there is no ladder of length n for $p(x\,;z)$
2. there is $q(x\,;z) \subseteq L(A)$ equivalent to $p(x\,;z)$ containing only stable formulas.

We can require that the ladder is an indiscernible sequence.

18.9 Theorem Let $\mathcal{R}(x\,;z)$ be invariant over A. Then the following are equivalent

1. $\mathcal{R}(x\,;z)$ is stable
2. $\mathcal{R}(a_0\,;b_1) \rightarrow \mathcal{R}(a_1\,;b_0)$ for every A-indiscernible sequence $\langle a_i\,;b_i : i < \omega\rangle$.

Proof. (2⇒1) Assume $\mathcal{R}(x\,;z)$ is unstable and that this is witnessed by $\langle a_i\,;b_i : i < \omega\rangle$. Let $\langle a'_i\,;b'_i : i < \omega\rangle$ be a sequence of A-indiscernibles with the same EM-type as $\langle a_i\,;b_i : i < \omega\rangle$. By invariance $\mathcal{R}(a'_0\,;b'_1)$ and $\neg\mathcal{R}(a'_1\,;b'_0)$.

(2⇒1) Immediate by indiscernibility. □

18.10 Lemma Every Boolean combination of stable relations is stable. Moreover, if $\mathcal{R}(x\,;z)$ then $\mathcal{R}^{-1}(x\,;z)$ and $\mathcal{R}(x,x'\,;z,z')$ are also stable formula (in the latter x' and z' are dummy variables).

Proof. It suffices to consider conjunction and negation. Let $\mathcal{R}_i(x\,;z)$, for $i = 1,2$, be stable relations that are invariant over A. By Theorem 18.9, it is immediate that $\mathcal{R}_1(x\,;z) \wedge \mathcal{R}_2(x\,;z)$ is stable. As for negation, let $\langle a_i\,;b_i : i < \omega\rangle$ be a ladder for $\mathcal{R}(x\,;z)$ that is indiscernible over A. By Exercise 15.8, there is a $\langle a'_i\,;b'_i : i < \omega\rangle$ of A-indiscernibles such that $a'_0, b'_0 = a_1 b_1$ and $a'_1, b'_1 = a_0 b_0$. This sequence witnesses the instability of $\neg\mathcal{R}(x\,;z)$.

The second claim is immediate. □

The following theorem claims what is arguably one of the most important properties of stable formulas: any set that is externally definable by a stable formula is definable (by a related formula).

18.11 Theorem Any $\mathcal{D} \subseteq \mathcal{U}^z$ approximated by a stable formula is definable. Precisely, if $\varphi(x\,;z)$ is a stable formula that approximates $\mathcal{D}$ then there are $a_{i,j} \in \mathcal{U}^x$ such that

$$z \in \mathcal{D} \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \varphi(a_{i,j}\,;z)$$

Theorem 18.19 will prove the converse: if every set approxiamted by $\varphi(x\,;z)$ is definable then $\varphi(x\,;z)$ is stable.

Proof. The theorem follows immediately from the two lemmas below. □

18.12 Lemma If $\mathcal{D}$ is approximated from below by a stable formula $\varphi(x\,;z)$ then

$$z \in \mathcal{D} \quad \leftrightarrow \quad \bigvee_{i=0}^{n} \varphi(a_i\,;z)$$

for some $a_0, \ldots, a_n \in \mathcal{U}^x$.

Proof. The elements $a_0, \ldots, a_n$ are defined recursively together with some auxiliary elements $b_0, \ldots, b_{n-1} \in \mathcal{D}$.

Suppose $b_0, \ldots, b_{n-1}$ have been defined (this assumption is empty if $n = 0$). We first define a_n, then b_n. Choose $a_n \in \mathcal{U}^x$ such that $b_0, \ldots, b_{n-1} \in \varphi(a_n\,;\mathcal{U}^z) \subseteq \mathcal{D}$. This is possible because $\mathcal{D}$ is approximated from below. Now, if possible, choose b_n such that

$$b_n \quad \in \quad \mathcal{D} \smallsetminus \bigcup_{i=0}^{n} \varphi(a_i\,;\mathcal{U}^z).$$

Then $\langle a_i\,;b_i : i \leq n\rangle$ is a ladder sequence. By stability, for some n, the tuple b_n does not exist. This yields the required $a_0, \ldots, a_n$. □

18.13 Lemma If $\mathcal{D}$ is approximated by a stable formula $\varphi(x\,;z)$. Then, for some m, the formula

$$\psi(x_0, \ldots, x_m\,;z) \quad = \quad \bigwedge_{j=0}^{m} \varphi(x_j\,;z)$$

approximates $\mathcal{D}$ from below.

Proof. Let m be such that there is no ladder sequence for $\varphi(x\,;z)$ of length greater then m. Let $C^z \subseteq \mathcal{D}$ be finite. We prove that there are some $a_0, \ldots, a_m$ such that $C^z \subseteq \psi(a_0, \ldots, a_m\,; \mathcal{U}^z) \subseteq \mathcal{D}$. As in the proof above, we define by recursion a ladder sequence for $\varphi(x\,;z)$. Suppose that $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1} \notin \mathcal{D}$ have been defined. We first define a_n, then b_n. Choose $a_n \in \mathcal{U}^x$ such that

$$C^z \ \subseteq\ \varphi(a_n\,; \mathcal{U}^z) \ \subseteq\ \mathcal{U}^z \smallsetminus \{b_0, \ldots, b_{n-1}\}.$$

This a_n exists, because $\mathcal{D}$ is approximated by $\varphi(x\,;z)$. (Apply Definition 18.1 with any B such that $C^z \cup \{b_0, \ldots, b_{n-1}\} \subseteq B^z$.) Then, if possible, let b_n such that

$$b_n \quad \in \quad \bigcap_{i=0}^{n} \varphi(a_i, \mathcal{U}^z) \smallsetminus \mathcal{D}$$

This procedure has to stop at some $n \leq m$. Hence the required parameters are $a_1, \ldots, a_n = a_{n+1} = \cdots = a_m$. □

By Lemma 18.10 the formula $\psi(x_0, \ldots, x_m\,; z)$ is stable therefore this concludes the proof of Theorem 18.11.

18.14 Remark Theorem 18.11 is often casted in the following apparently more general form. Let $\mathcal{A} \subseteq \mathcal{U}^x$ and $\mathcal{B} \subseteq \mathcal{U}^z$. We say that $\varphi(x\,;z)$ is stable between $\mathcal{A}$ and $\mathcal{B}$ if for some n no ladder of length n exists with $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$.

Let $\mathcal{D} \subseteq \mathcal{B}$ be approximable by $\varphi(x\,;z) \wedge x \in \mathcal{A}$. Then for any such $\mathcal{D}$ there are $a_{1,1}, \ldots, a_{n,m} \in \mathcal{A}$ such that

$$b \in \mathcal{D} \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \varphi(a_{i,j}\,; b).$$

The proof is essentially the same. In fact, elementarity and saturation as only been used to prove that conjunctions of stable formulas is stable. Exercise 18.18 suggests a finitary proof of this fact.

18.15 Exercise Prove Fact **??**.

18.16 Exercise Prove that if $p(x\,;z)$ admits ladder sequences of arbitrary finite length, then it admits a ladder sequence of infinite length.

18.17 Exercise Let $\varphi(x, y) \in L$, where $|x| = |y| = 1$. Suppose there is an infinite set $A \subseteq \mathcal{U}$ such that $\varphi(a, b) \not\leftrightarrow \varphi(b, a)$ for every two distinct $a, b \in A$. Prove that $\varphi(x\,;y)$ is unstable.

18.18 Exercise Prove that if $p(x\,;z)$ and $q(x\,;z)$ are stable between $\mathcal{A}$ and $\mathcal{B}$, then so is $p(x\,;z) \cup q(x\,;z)$.

18.3 Stability and the number of types

The following proposition highlights the connection between the stability of $\varphi(x\,;z)$ and the cardinality of $S_\varphi(\mathcal{U})$, equivalently the number of sets externally definable by $\varphi(x\,;z)$.

18.19 Theorem The following are equivalent

1. $\varphi(x\,;z)$ is stable
2. every subset of $\mathcal{U}^z$ that is externally definable by $\varphi(x\,;z)$ is definable
3. there are $\leq\kappa$ subsets of $\mathcal{U}^z$ that are externally definable by $\varphi(x\,;z)$
4. there are $<2^\kappa$ subsets of $\mathcal{U}^z$ that are externally definable by $\varphi(x\,;z)$.

Proof. 1⇒2 Clear by Proposition 18.2 and Theorem 18.11.

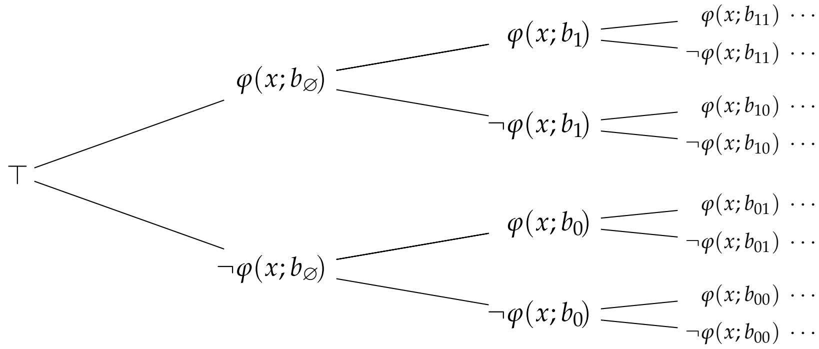
2⇒3⇒4 Obvious.

4⇒1 Suppose that $\varphi(x\,;z)$ is not stable. By compactness there is a ladder sequence $\langle a_i\,;b_i : i\in I\rangle$ where $I,<_I$ a dense linear order of cardinality κ with 2^κ cuts, where by *cut* we mean a subset $c\subseteq I$ that is closed downward. For every such $c\subseteq I$ we pick a global type

$$p_c(x)\quad\supseteq\quad\big\{\varphi(x\,;b_i)\leftrightarrow i\in c\ :\ i\in I\big\}.$$

Clearly the sets $\mathcal{D}_{p_c,\varphi}$ are all distinct. □

Binary trees of formulas have been introduced in Definition 12.19. Here we restrict to trees of a particular form. Namely, $\langle\psi_s : s\in 2^{<\omega}\rangle$ where $\psi_\varnothing=\top$ and for $s\in 2^{<\omega}$ and $i\in 2$ we have $\psi_{s^\frown 0}(x)=\neg\varphi(x\,;b_s)$ and $\psi_{s^\frown 1}(x)=\varphi(x\,;b_s)$. If we define $\varphi^0=\neg\varphi$ and $\varphi^1=\varphi$ the condition of consistency becomes for every $s\in 2^\omega$ the type $\{\varphi^{s_n}(x\,;b_{s\restriction n})\ :\ n<\omega\}$.



When a tree of this form exists, we say that $\varphi(x\,;z)$ has the binary tree property.

18.20 Theorem The following are equivalent

1. $\varphi(x\,;z)$ is unstable
2. $\varphi(x\,;z)$ has the binary tree property.

Proof. 1⇒2. The argument is the same as in the proof of Lemma 12.20. Assume 1. By Theorem 18.19 there are 2^κ sets externally definable by $\varphi(x\,;z)$. Then there is $b_\varnothing\in\mathcal{U}^z$ such that there are 2^κ sets $\mathcal{D}$ externally definable by $\varphi(x\,;z)$ and such that $b_\varnothing\in\mathcal{D}$ and 2^κ sets such that $b_\varnothing\notin\mathcal{D}$.

Assume inductively that $b : 2^n\to\mathcal{U}^z$ is such that for all $s\in 2^n$ and $r\in 2^{n+1}$ there are 2^κ sets $\mathcal{D}$ externally definable by $\varphi(x\,;z)$ and such that $b_{s\restriction i}\in\mathcal{D}\leftrightarrow r(i)=1$. Reasoning as above we can extend b to a map $b' : 2^{n+1}\to\mathcal{U}^z$ with the same property.

2⇒1. From 2, by compactness, there is a binary tree of height κ. Hence there are 2^{κ} sets that are externally definable by $\varphi(x\,;z)$. Therefore, by Theorem 18.19, $\varphi(x\,;z)$ is not stable. $\square$

18.21 Corollary The following are equivalent

1. $\varphi(x\,;z)$ is a stable formula
2. $\big|S_{\varphi}(A)\big| \leq |A|$ for all countable sets A
3. $\big|S_{\varphi}(A)\big| < 2^{|A|}$ for all countable sets A.

Proof. The corollary follows immediately from Lemma 12.20 and Theorem 18.20. $\square$

We use binary trees to prove the following fundamental lemma (attributed by Harnik and Harrington [2] to Martin Ziegler).

18.22 Lemma Let $\varphi(x\,;z), \psi(x\,;z) \in L(A)$. Let $b \in \mathcal{U}^{z}$. Assume that $\varphi(x\,;z)$ is stable and that $\varphi(x\,;b) \vee \psi(x\,;b)$ is finitely satisfied in every $M \supseteq A$. Then $\varphi(x\,;b)$ or $\psi(x\,;b)$ is finitely satisfied in every $M \supseteq A$.

Proof. Negate the theorem. Then there are two models containing A, the first omitting $\varphi(x\,;b)$, the second omitting $\psi(x\,;b)$. It is easy to see that we can expand these models to two substructures $\mathcal{V}_0$ and $\mathcal{V}_1$ that are A-isomorphic to $\mathcal{U}$ and such that $\mathcal{V}_0 = \neg\varphi(\mathcal{V}_0^{x};b)$ and $\mathcal{V}_1 = \neg\psi(\mathcal{V}_1^{x};b)$.

Let $f_i : \mathcal{U} \to \mathcal{V}_i$ be the A-isomorphisms mentioned above. Let $\mathcal{V}_{\varnothing} = \mathcal{U}$ and $b_{\varnothing} = b$. Define inductively for every $s \in 2^{<\omega}$

$$\mathcal{V}_{s\frown i} = f_i[\mathcal{V}_s] \quad \text{and} \quad b_{s\frown i} = f_i(b_s).$$

Then from the inclusion above we obtain

$$\# \qquad \mathcal{V}^{x}_{s\frown 0} \subseteq \neg\varphi(\mathcal{V}^{x}_{s\frown 0};b_s) \quad \text{and} \quad \mathcal{V}^{x}_{s\frown 1} \subseteq \neg\psi(\mathcal{V}^{x}_{s\frown 1};b_s).$$

We prove that the branches of the tree $\langle\varphi(x\,;b_s) \;:\; s \in 2^{<\omega}\rangle$ are (finitely) consistent. By Theorem 18.20, this contradicts the stability of $\varphi(x\,;z)$. Assume provisionally that there is a formula $\vartheta(x) \in L(A)$ such that

$$\#\# \qquad \vartheta(x) \quad \to \quad \varphi(x\,;b) \vee \psi(x\,;b)$$

Let $s \in 2^{n}$ be given. Let $a \in \vartheta(\mathcal{V}^{x}_{s})$ be arbitrary. We claim that a witnesses the concistency of $\{\varphi^{s_i}(x\,;b_{s\restriction i}) \;:\; i < n\}$. As $\mathcal{V}_s \preceq \mathcal{V}_{s\restriction i}$, we have that $a \in \vartheta(\mathcal{V}^{x}_{s\restriction i})$. We prove that $\varphi^{s_i}(a\,;b_{s\restriction i})$. If $s_i = 0$ then from # we obtain $\mathcal{V}^{x}_{s\restriction(i+1)} \subseteq \varphi^{0}(\mathcal{V}^{x}_{s\restriction(i+1)};b_{s\restriction i})$ and $\varphi^{0}(a\,;b_{s\restriction i})$ follows by elementarity. If $s_i = 1$, from the second inclusion in # we obtain $\mathcal{V}^{x}_{s\restriction(i+1)} \subseteq \neg\psi(\mathcal{V}^{x}_{s\restriction(i+1)};b_{s\restriction i})$. Therefore, from $\vartheta(a)$ and ##, we obtain that $\mathcal{V}^{x}_{s\restriction(i+1)} \subseteq \varphi(\mathcal{V}^{x}_{s\restriction(i+1)};b_{s\restriction i})$ and $\varphi^{1}(a\,;b_{s\restriction i})$ follows.

We are left with proving that the provisional assumption is redundant. By Exercise 11.12 there is a formula $\vartheta(\bar{x}) \in L(A)$ such that

$$\vartheta(\bar{x}) \quad \to \quad \varphi'(\bar{x}\,;b) \vee \psi'(\bar{x}\,;b)$$

where $\bar{x} = x_1,\dots,x_n$ and

$$\varphi'(\bar{x}\,;b) \quad = \quad \bigvee_{i=1}^{n} \varphi(x_i\,;b)$$

$$\psi'(\bar{x}\,;b) \quad = \quad \bigvee_{i=1}^{n} \psi(x_i\,;b)$$

Note that $\varphi'(\bar{x}\,;z)$ is a stable formula. Therefore we can apply what proved above to the formulas $\varphi'(\bar{x}\,;b)$ and $\psi'(\bar{x}\,;b)$ and note that these formulas are satisfied in every model $M \supseteq A$ if and only if $\varphi(x\,;b)$, respectively $\psi(x\,;b)$, are. □

18.4 Symmetry and stationarity

We say that T is a stable theory if every formula is stable. Most theorems in this section have a formulation that only requires the stability of a given formula $\varphi(x\,;z) \in L$. This is sometimes called *local* stability. When the global case does not trivially follow from the local one, we use a set $\Delta \subseteq L_{x\,z}$. We say that Δ is stable if all formulas in Δ are stable. When $|x| = |z| = \omega$ and $\Delta = L_{x\,z}$ the stability of Δ is tantamount to the stability of T.

The following corollary of Theorem 18.22 is a fundamental tool. It implies, in particular the existence of Lascar invariant types.

18.23 Corollary Let $\Delta \subseteq L_{x\,z}$ be stable. Let $q(x) \subseteq L(\mathcal{U})$ be finitely satisfiable in every $M \supseteq A$. Then there is a type $p(x) \in S_\Delta(\mathcal{U})$ such that $q(x) \cup p(x)$ is finitely satisfiable in every $M \supseteq A$.

Proof. Let $p(x) \subseteq \Delta^{\pm}(\mathcal{U})$ be maximal such that $q(x) \cup p(x)$ is finitely satisfiable in every $M \supseteq A$. We prove that $p(x)$ is complete. Suppose not than there are a conjunction $\alpha(x)$ of formulas in $q(x)$, a conjunction $\vartheta(x\,;b_1,\dots,b_n)$ of formulas in $p(x)$, and a formula $\varphi(x\,;b)$ for some $\varphi(x\,;z) \in \Delta$ such that

$$\alpha(x) \wedge \vartheta(x\,;b_1,\dots,b_n) \wedge \varphi(x\,;b) \quad \text{and} \quad \alpha(x) \wedge \vartheta(x\,;b_1,\dots,b_n) \wedge \neg\varphi(x\,;b)$$

are not satisfied in every $M \supseteq A$. The disjunction of the two formulas is satisfied in every $M \supseteq A$ by the definition of $p(x)$. Moreover the formulas obtained replacing $b, b_1, \dots, b_n$ by $z, z_1, \dots, z_n$ are stable. This contradicts Theorem 18.22 □

The following theorem shows that in stable theories the heir-coheir relation is symmetric. In fact, when T is stable, it implies that

$$a \mathop{\downfree}_M b \quad \Rightarrow \quad b \mathop{\downfree}_M a.$$

18.24 Theorem (Symmetry) Let $\varphi(x\,;z) \in L(M)$ be stable. Let $a \in \mathcal{U}^x$ and $b \in \mathcal{U}^z$ be such that $a \mathop{\downfree}_M b$. Then

$$\varphi(a\,;b) \quad \Rightarrow \quad \varphi(a\,;M^z) \neq \varnothing.$$

Proof. Assume $\varphi(a\,;b)$. Let $\mathcal{V} \preceq \mathcal{U}$ be isomorphic to $\mathcal{U}$ over M and such that $b \in \mathcal{V}^z$ and $a \mathop{\downfree}_M \mathcal{V}$. Such $\mathcal{V}$ exists by Proposition 14.8. By stability, there is $\psi(z) \in L(\mathcal{V})$ such that $\psi(\mathcal{V}^z) = \varphi(a, \mathcal{V}^z)$. Recall that by Lemma 14.7 (non-splitting) if $b' \equiv_M b''$ are in $\mathcal{V}$ then $b' \equiv_{M,a} b''$. Then $\psi(\mathcal{V}^z)$ is invariant under $\mathrm{Aut}(\mathcal{V}/M)$, so we can assume that $\psi(z) \in L(M)$. Therefore $\psi(z)$ is satisfied in M and so is $\varphi(a, z)$. □

We deduce a version of Harrington's mysterious Lemma cfr. [3, Lemma 8.3.4].

18.25 Corollary Let $\varphi(x\,;z) \in L$ be stable. Let $a \in \mathcal{U}^x$ and $b \in \mathcal{U}^z$. Finally, let $\vartheta(x), \psi(z) \in L(M)$ be such that $\vartheta(M^x) = \varphi(M^x; b)$ and $\psi(M^z) = \varphi(a\,; M^z)$. Such formulas exist by stability. Then $\vartheta(a) \leftrightarrow \psi(b)$.

Proof. We can assume that $a \mathop{\smile\!\!\!\!\!|}_M b$. First, assume $\varphi(a\,;b)$. Then $\vartheta(a)$ otherwise, by $a \mathop{\smile\!\!\!\!\!|}_M b$, we would contradict $\vartheta(M^x) = \varphi(M^x; b)$. We claim that $\psi(b)$. Suppose not. Then $\varphi(a\,;b) \wedge \neg\psi(b)$. As $\varphi(x\,;z) \wedge \neg\psi(z)$ is stable, from Theorem 18.24 we obtain $\varphi(a\,;M^z) \smallsetminus \vartheta(M^z) \neq \varnothing$ which contradicts $\psi(M^z) = \varphi(a\,;M^z)$. Assuming $\neg\varphi(a\,;b)$ we obtain $\neg\vartheta(a)$ and $\neg\psi(b)$ by a symilar argument. Then $\vartheta(a) \leftrightarrow \psi(b)$ as required. □

Now we show that under the assumption of stability Lascar invariance reduces to a tamer form of invariance.

18.26 Proposition Let $\varphi(x\,;z) \in L$ be stable. Let $p(x) \in S_\varphi(\mathcal{U})$. Then the following are equivalent

1. $\mathcal{D}_{p,\varphi} \in \mathrm{acl}^{\mathrm{eq}}A$ – recall that, by stability, $\mathcal{D}_{p,\varphi} \in \mathcal{U}^{\mathrm{eq}}$
2. $p(x)$ is finitely satisfied in every $M \supseteq A$
3. $p(x)$ is invariant over $\mathrm{acl}^{\mathrm{eq}}A$
4. $p(x)$ is Lascar invariant over A.

Proof. 2⇒3⇒4 (and even 1⇒3) are clear – stability is not required.

1⇒2 Let $M \supseteq A$ be given. First, for simplicity, we prove that any $\varphi(x\,;b) \in p$ is satisfied in M. Let $\psi(z) \in L(\mathrm{acl}^{\mathrm{eq}}A)$ be a formula defining $\mathcal{D}_{p,\varphi}$. Pick any $a \models p(x){\restriction}M$ such that $b \mathop{\smile\!\!\!\!\!|}_M a$. As $\varphi(a\,;M^z) = \psi(M^z)$ we have that $\varphi(a\,;b) = \psi(b)$. But $\psi(b)$ holds because $\varphi(x\,;b) \in p$, hence $\varphi(a\,;b)$ and, from Theorem 18.24 we obtain $\varphi(M^x; b) \neq \varnothing$. The same argument extends easily to the conjunction of formulas of the form $\varphi(x\,;b)$ and negation thereof (use that the complement and the Cartesian product of sets in $\mathrm{acl}^{\mathrm{eq}}A$ is again $\mathrm{acl}^{\mathrm{eq}}A$).

3⇒1 By Theorem 18.19 the sets $\mathcal{D}_{p,\varphi}$ are definable (over $\mathcal{U}$). As $p(x)$ is Lascar invariant over A, so are the sets $\mathcal{D}_{p,\varphi}$. By Corollary 16.9, they belong to $\mathrm{acl}^{\mathrm{eq}}A$. □

A type $q(x) \subseteq L(A)$ is stationary if it has a unique global extension that is Lascar invariant over A. The following proposition says that in a stable theory with elimination of imaginaries types over algebraically closed sets are stationary (the actual statement is a local version of this property).

18.27 Theorem (Stationarity) Let $\Delta \subseteq L_{x\,z}$ be stable. Let $q(x)$ be a complete $\Delta^{\mathrm{G}}(\mathrm{acl}^{\mathrm{eq}}A)$-type. Then there is a unique type $p(x) \in S_\Delta(\mathcal{U})$ containing $q(x)$ that is invariant over $\mathrm{acl}^{\mathrm{eq}}A$.

Proof. Existence follows from Corollary 18.23 and Proposition 18.26. Suppose $p_i(x) \in S_\Delta(\mathcal{U})$, for $i = 1, 2$, are two types that extend $q(x)$ and are invariant over $\mathrm{acl}^{\mathrm{eq}}A$. It sufficies to prove that $\mathcal{D}_{p_1,\varphi} = \mathcal{D}_{p_2,\varphi}$ for every $\varphi(x\,;z) \in \Delta$. Let $\psi_i(z) \in L(\mathrm{acl}^{\mathrm{eq}}A)$ define $\mathcal{D}_{p_i,\varphi}$. These formulas exist by stability. We claim that $\psi_1(b) \leftrightarrow \psi_2(b)$, for all $b \in \mathcal{U}^z$. Let $p'(z) \in S_\Delta(\mathcal{U})$ be invariant over $\mathrm{acl}^{\mathrm{eq}}A$ and consistent with $\operatorname{tp}(b/\operatorname{acl}^{\mathrm{eq}}A)$. Such a type exists, again, by Corollary 18.23 and Proposition 18.26. By stability there is a formula $\vartheta(x) \in \Delta^{\mathrm{B}}(\mathcal{U})$ that defines $\mathcal{D}_{t,\varphi^{\mathrm{op}}}$. But $\mathcal{D}_{t,\varphi^{\mathrm{op}}}$ is invariant over $\operatorname{acl}^{\mathrm{eq}}A$, then $\vartheta(x) \in \Delta^{\mathrm{G}}(\operatorname{acl}^{\mathrm{eq}}A)$. Let M be any model containing A. Let $b' \vDash p'(z){\restriction}M$ and $a_i \vDash p_i(x){\restriction}M$. Then $\vartheta(M^x) = \varphi(M^x;b')$ and $\psi_i(M^z) = \varphi(a_i\,;M^z)$. Corollary 18.25 yelds $\psi_i(b') \leftrightarrow \vartheta(a_i)$. As $a_1, a_2 \vDash q(x)$, by completeness they both satisfy $\vartheta(x)$. Hence we obtain $\psi_1(b') \leftrightarrow \psi_2(b')$. As $b \equiv_{\operatorname{acl}^{\mathrm{eq}}A} b'$, the claim follows and with it the theorem. □

18.28 Corollary (T stable) The following are equivalent

1. $a \overset{\mathrm{L}}{\equiv}_A b$, see Definition 16.11
2. $a \overset{\mathrm{Sh}}{\equiv}_A b$, see Definition 13.13

Proof. 1⇒2. This is left as an exercise to the reader – stability is not required.

2⇒1. Assume 2 which, by Proposition 13.14, is equivalent to $a \equiv_{\operatorname{acl}^{\mathrm{eq}}A} b$. Write $q(x)$ for $\operatorname{tp}(a/\operatorname{acl}^{\mathrm{eq}}A) = \operatorname{tp}(b/\operatorname{acl}^{\mathrm{eq}}A)$. Let $p(x) \in S(\mathcal{U}^{\mathrm{eq}})$ be the unique global type that is invariant over $\operatorname{acl}^{\mathrm{eq}}A$ and extends $q(x)$ which we obtain from Theorem 18.27. Let $\bar{c} = \langle c_i : i < \omega \rangle$ be such that $c_i \vDash p(x){\restriction}\operatorname{acl}^{\mathrm{eq}}A,\, a,\, b,\, c_{\restriction i}$. Then $a, \bar{c}$ and $b, \bar{c}$ are A-indiscernible sequences, which proves 1, see Exercise 16.26. □

The following theorem presents a property of definability that seems to go in the opposite direction. It has an application in the proof of Corollary 18.31.

18.29 Theorem Let $\varphi(x\,;z) \in L(\mathcal{U})$ be stable. Let M and $b \in \mathcal{U}^z$ be arbitrary. Then, for some $b_i \equiv_M b$, where $i = 1, \dots, n$, some positive Boolean combination of the sets $\varphi(\mathcal{U}^x; b_i)$ is definable over M.

Proof. Let $\mathcal{V} \preceq \mathcal{U}$ be isomorphic to $\mathcal{U}$ over M. By Theorem 18.11 with $\mathcal{V}$ for $\mathcal{U}$ and the role of the two sorts reversed, there are some $b_i \equiv_M b$ in $\mathcal{V}$, such that $\varphi(\mathcal{V}^x\,; b)$ is a positive Bolean combination of the sets $\varphi(\mathcal{V}^x\,; b_i)$. Let $\vartheta(\mathcal{V}^x\,; b_1, \dots, b_n)$ be such a Boolean combination. Now, by Proposition 14.8, we can choose $\mathcal{V}$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}_M \mathcal{V}$. By non-splitting, i.e. reasoning as in Theorem 18.24, there is a formula $\psi(x) \in L(M)$ such that $\psi(\mathcal{V}^x) = \varphi(\mathcal{V}^x\,; b)$. Finally, by elementarity, we obtain that $\psi(\mathcal{U}^x) = \vartheta(\mathcal{U}^x; b_1, \dots, b_n)$. □

A more general version of the theorem holds which requires a different proof.

18.30 Theorem Let $\varphi(x\,;z) \in L(\mathcal{U})$ be stable. Let A and $b \in \mathcal{U}^z$ be arbitrary. Then, for some $b_i \equiv_{\operatorname{acl}^{\mathrm{eq}}A} b$, where $i = 1, \dots, n$, some positive Boolean combination of the sets $\varphi(\mathcal{U}^x; b_i)$ is definable over A.

Proof. Let $q(z) = \operatorname{tp}(b/\operatorname{acl}^{\mathrm{eq}}A)$. By Corollary 18.23 there is a type $p(z) \in S_\varphi(\mathcal{U})$ consistent with $q(z)$ and invariant over $\operatorname{acl}^{\mathrm{eq}}A$. By Theorem 18.11, with the role of the two sorts reversed, there are some $b_1, \dots, b_n \vDash q(z)$ such that $\mathcal{D}_{p,\varphi^{\mathrm{op}}}$ is equivalent to a positive Bolean combination of the sets $\varphi(\mathcal{U}^x; b_i)$. Let $\vartheta(\mathcal{U}^x; b_1, \dots, b_n)$ be such a Boolean combination. As $\mathcal{D}_{p,\varphi^{\mathrm{op}}} \in \operatorname{acl}^{\mathrm{eq}}A$ by Proposition 18.26, this prove the theorem when $A = \operatorname{acl}^{\mathrm{eq}}A$.

For the general case, reason as follows. Let $f_1, \ldots, f_m \in \mathrm{Aut}(\mathcal{U}/A)$ be such that the sets $f_i\mathcal{D}_{p,\varphi^{\mathrm{op}}}$ span the whole orbit of $\mathcal{D}_{p,\varphi^{\mathrm{op}}}$ over A. Clearly

$$\bigcup_{i=1}^{m} f_i\mathcal{D}_{p,\varphi^{\mathrm{op}}} \quad = \quad \bigvee_{i=1}^{m} \vartheta(\mathcal{U}^x; f_ib_1, \ldots, f_ib_n).$$

The set on the l.h.s. is invariant (hence definable) over A. Therefore the formula on the r.h.s. is the required positive Boolean combination. □

18.5 The action of the Lascar group

We adopt the notation and terminology of Chapter 17 with $G = \mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$ and $\mathcal{Z} = \mathcal{U}^z$. The set $\mathcal{X}$ will specified in the context. We say Lascar generic over A for generic under the action of $\mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$. We define Lascar persistent similarly.

18.31 Corollary Let $\Delta \subseteq L_{xz}$ be stable. Let $\mathcal{X} = q(\mathcal{U}^x)$ for some $q(x) \subseteq L(\mathrm{acl}^{\mathrm{eq}}A)$. Let $\mathcal{D}$ be a $\Delta^{\mathrm{B}}(\mathcal{U})$-definable set. Then the following are equivalent

1. $\mathcal{D}$ is Lascar persistent over A
2. $\{x \in \mathcal{D}\} \cup q(x)$ is finitely satisfied every in $M \supseteq A$.

Proof. As Boolean combinations of stable formulas are stable, we can assume that $\mathcal{D}$ is defined by the formula $\varphi(x\,;b)$ for some $\varphi(x\,;z) \in \Delta$.

1⇒2. Let $M \supseteq A$. Let $\psi(x) \in q$. If $\varphi(x\,;b)$ is Lascar persistent, then it is also persistent under the action of $\mathrm{Aut}(\mathcal{U}/M)$. Then the formula $\vartheta(x\,;b_1, \ldots, b_n)$ in the proof of Theorem 18.29 is consistent with $\psi(x)$. As $\vartheta(x\,;b_1, \ldots, b_n)$ is equivalent to a formula in $L(M)$, by elementarity it satisfiable in $\psi(M^x)$. Hence so is $\varphi(x\,;b_i)$ for some i. As $b_i \equiv_M b$, also $\varphi(x\,;b)$ is satisfiable in $\psi(M^x)$.

2⇒1. From 2 and Corollary 18.23 it follows that $\varphi(x\,;b)$ is contained in a type $p(x) \in S_\Delta(\mathcal{U})$ that is finitely satisfied in every $M \supseteq A$. By Proposition 18.26, $p(x)$ is Lascar invariant. Then, by Theorem 17.5, $p(x)$ and in particular $\varphi(x\,;b)$ is Lascar persistent. □

18.32 Proposition Let $\Delta \subseteq L_{xz}$ be stable. Let $\mathcal{X} = q(\mathcal{U}^x)$ for some $q(x) \subseteq L(\mathrm{acl}^{\mathrm{eq}}A)$. Then the equivalent conditions of Theorem 17.22 are satisfied.

Proof. Let $\mathcal{D}$ be a $\Delta^{\mathrm{B}}(\mathcal{U})$-definable set that is weakly Lascar persistent. We prove that $\mathcal{D}$ is Lascar persistent. Pick a finite $H \subseteq \mathrm{Aut}^{\mathrm{f}}(\mathcal{U}/A)$ such that $\cup H\,\mathcal{D}$ is Lascar persistent. By Corollary 18.31, the type $\{x \in \cup H\,\mathcal{D}\} \cup q(x)$ is finitely satisfiable in every $M \supseteq A$. Then, by Corollary 18.23 there is a Lascar invariant type $p(x) \in S_\Delta(\mathcal{U})$ containing $x \in \cup H\,\mathcal{D}$ and finitely consistent with $q(x)$. By completeness and invariance $x \in \mathcal{D}$ is in $p(x)$. Then $\mathcal{D}$ is persistent by Theorem 17.5. □

18.33 Proposition Let $\Delta \subseteq L_{xz}$ be stable. Let $\mathcal{X} = q(\mathcal{U}^x)$ for some $q(x) \in S(\mathrm{acl}^{\mathrm{eq}}A)$. Then for every $\Delta^{\mathrm{B}}(\mathcal{U})$-definable set $\mathcal{D}$, the following are equivalent

1. $\mathcal{D}$ is Lascar generic over A.
2. $\mathcal{D}$ is Lascar persistent over A.

Proof. 1⇒2. By Theorem 18.27 there is a type $p(x) \in S_\Delta(\mathcal{U})$ consistent with $q(x)$ that is Lascar invariant. Therefore, from 1 and Corollary 17.6 we infer that $\mathcal{D}$ is Lascar persistent.

2⇒1. Assume $\mathcal{D}$ is not Lascar generic. Then $\neg\mathcal{D}$ is Lascar persistent. Assume for a contradiction that also $\mathcal{D}$ is Lascar persistent. Let $\vartheta(x\,;b_1,\ldots,b_n)$ be the $\Delta^{\mathrm{B}}(\mathcal{U})$-formula that defines $\mathcal{D}$. As $\vartheta(x\,;z_1,\ldots,z_n)$ is stable, by Corollary 18.31 both $\vartheta(x\,;b_1,\ldots,b_n)$ and its negation are finitely satisfied in every model $M \supseteq A$. Then there are two distinct types $p_i(x) \in S_\Delta(\mathcal{U})$ finitely consistent with $q(x)$ and finitely satisfied in every $M \supseteq A$. This contradicts stationarity, see Proposition 18.26 and Theorem 18.27. □

18.34 Corollary Let $\Delta \subseteq L_{x\,z}$ be stable. Let $\mathcal{X} = q(\mathcal{U}^x)$ for some $q(x) \in S(\mathrm{acl}^{\mathrm{eq}}A)$. Then for every $\Delta^{\mathrm{B}}(\mathcal{U})$-definable set $\mathcal{D}$, either $\mathcal{D}$ or $\neg\mathcal{D}$ is Lascar generic over A.

Proof. By Theorem 18.27 there is a type $p(x) \in S_\Delta(\mathcal{U})$ finitely consistent with $q(x)$ that is Lascar invariant and therefore persistent. Hence, by Corollary 17.6, $\mathcal{D}$ and $\neg\mathcal{D}$ are not both generic. If $\mathcal{D}$ is not generic then $\neg\mathcal{D}$ is persistent and, by Proposition 18.33, generic. □

The following lemma is applied in the theorem below (it does not require stability).

18.35 Lemma Let $\Delta \subseteq L_{x\,z}$. Let $p_0(x)$ be complete $\Delta^{\mathrm{G}}(A)$-type. Finally, let

$$Q \quad = \quad \big\{q(x) \subseteq \Delta^{\mathrm{G}}(\mathrm{acl}^{\mathrm{eq}}A) \ : \ q(x) \supseteq p_0(x) \text{ and is complete}\big\}.$$

Then $\mathrm{Aut}(\mathcal{U}/A)$ acts on Q transistively, i.e. any two types in Q are conjugated.

Proof. We prove that any two types $q_1, q_2 \in Q$ are realized be some $a_1 \equiv_A a_2$. The claim follows because any automorphism in $\mathrm{Aut}(\mathcal{U}/A)$ that takes a_1 to a_2 takes also q_1 in q_2. It suffices to prove that every type $p(x) \in S(A)$ is consistent with every $q \in Q$. Suppose not, then $p(x) \to \neg\varphi(x)$ for some $\varphi(x) \in q$. Let $\psi(x)$ be the disjunction of all the (finitely many) conjugates of $\varphi(x)$ over A. Then $p(x) \to \neg\psi(x)$. But $\psi(x) \in \Delta^{\mathrm{B}}(\mathrm{acl}^{\mathrm{eq}}A)$ and is invariant over A. Therefore $\psi(x) \in \Delta^{\mathrm{G}}(A)$. Then $\psi(x) \in p$, a contradiction. □

18.36 Theorem (Shelah's Finite Equivalence Relation Theorem) Let $\Delta \subseteq L_{x\,z}$ be stable and finite. Let $p_0(x)$ be complete $\Delta^{\mathrm{G}}(A)$-type. Finally, let

$$P \quad = \quad \big\{p(x) \in S_\Delta(\mathcal{U}) \ : \ p(x) \vdash p_0(x) \text{ and is Lascar invariant over } A\big\}.$$

Then the following hold

1. $\mathrm{Aut}(\mathcal{U}/A)$ acts on P transistively.
2. P is finite
3. there is a finite equivalence relation $\varepsilon(x\,;y) \in \Delta^{\mathrm{G}}(A)$ such that for all $p_i, p_j \in P$
$$p_i = p_j \quad \Leftrightarrow \quad p_i(x) \cup p_j(y) \text{ is consistent with } \varepsilon(x,y).$$

Proof. 1. Let p_i, where $i = 1,2$, be two elements of P. Let $q_i(x)$ be some complete $\Delta^{\mathrm{G}}(\mathrm{acl}^{\mathrm{eq}}A)$-type consistent with $p_i(x)$. By the proposition above $f\,q_1 = q_2$ for some $f \in \mathrm{Aut}(\mathcal{U}/A)$. Then $f\,p_1$ is consistent with q_2. By stationarity $f\,p_1 = p_2$.

2. We may identify a type $p(x) \in S_\Delta(\mathcal{U})$ with the tuple $\langle \mathcal{D}_{p,\varphi} : \varphi(x\,;z) \in \Delta \rangle$. As $\mathcal{D}_{fp,\varphi} = f[\mathcal{D}_{p,\varphi}]$ for any $f \in \mathrm{Aut}(\mathcal{U}/A)$, the orbit of $p(x)$ corresponds to the orbit of $\langle \mathcal{D}_{p,\varphi} : \varphi(x\,;z) \in \Delta \rangle$. As $p \in P$ is Lascar invariant over A, the sets $\mathcal{D}_{p,\varphi}$ are in $\mathrm{acl}^{\mathrm{eq}}A$. Hence have a finite orbit. As Δ is finite the orbit of $\langle \mathcal{D}_{p,\varphi} : \varphi(x\,;z) \in \Delta \rangle$ is finite, and so is the orbit of p. By 1, P is finite.

3. Let $p_1, \ldots, p_n$ be an enumeration without repetitions of P. Let $q_1(x)$ be some complete $\Delta^{\mathrm{G}}(\mathrm{acl}^{\mathrm{eq}}A)$-type consistent with p_1. It is easy to see that the orbit of $q_1(x)$ under the action of $\mathrm{Aut}(\mathcal{U}/A)$ comprise complete $\Delta^{\mathrm{G}}(\mathrm{acl}^{\mathrm{eq}}A)$-types $q_1, \ldots, q_n$ that are complete and consistent with $p_1, \ldots, p_n$. By stationarity the q_i are mutually inconsistent. Pick n mutually inconsistent formulas $\vartheta_i(x) \in q_i$ and define

$$\varepsilon(x,y) \quad = \quad \bigwedge_{i=1}^{n} \big[\vartheta_i(x) \leftrightarrow \vartheta_i(y)\big]$$

This shows that $p_i = p_j$ if and only if $p_i(x) \cup p_j(y)$ is consistent with $\varepsilon(x,y)$. From the definition of $\varepsilon(x,y)$ we know that it belongs to $\Delta^{\mathrm{G}}(\mathrm{acl}^{\mathrm{eq}}A)$, but it is easily seen that $\varepsilon(x,y)$ is invariant over A, hence 3 follows. □

18.6 Stable groups

We assume the hypotheses and the notation of Section 17.5.

18.37 Theorem Let $\varphi(x\,;z) \in \Delta$ be stable. Assume the action of $\mathcal{Z}$ is transitive. Then the following are equivalent

1. $\varphi(x\,;1)$ is $\mathcal{Z}$-generic
2. $\varphi(x\,;g)$ is Lascar (hereditarily) persistent over A for every $g \in \mathcal{Z}$.

Proof. 1⇒2. By stability, there exist a type $p(x) \in L'\text{-}S_\varphi(\mathcal{Z})$ that is invariant under the action of $L'\text{-}\mathrm{Aut}^{\mathrm{f}}(\mathcal{U})$. Then we can apply Corollary 17.25

2⇒1. Let $a \in \mathcal{X}$ and $g \in \mathcal{Z}$ be such that $\varphi(a\,;g)$. Let $q(x) = \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}A)$. Assume 2 then, by Proposition 18.33, $\varphi(x\,;g)$ is Lascar generic relative to $q(\mathcal{U}^x)$, i.e. with $q(\mathcal{U}^x)$ for $\mathcal{X}$. A fortiori $\varphi(x\,;g)$ is G-generic relative to $q(\mathcal{U}^x)$. The same reasoning applies if we replace a and g with $h\,a$ and $h\,g$ for any $h \in \mathcal{Z}$. By the transitivity of the action, $h\,a$ ranges the whole of $\mathcal{X}$. We conclude that there is a set $H \subseteq \mathcal{Z}$ of cardinality $< 2^{|L(A)|}$ such that $\bigcup_{h\in H} \varphi(\mathcal{X}\,;h)$ covers $\mathcal{X}$. By compactness, we can assume H finite, so the $\mathcal{Z}$-genericity follows. □

18.38 Corollary Let $\varphi(x\,;z) \in \Delta$ be stable. Assume the action of $\mathcal{Z}$ is transitive. Then $\varphi(x\,;1)$ or $\neg\varphi(x\,;1)$ is $\mathcal{Z}$-generic.

Proof. Suppose $\varphi(x\,;1)$ is not $\mathcal{Z}$-generic. Then $\varphi(x\,;g)$ is not Lascar persistent for some g. Then $\neg\varphi(x\,;g)$ is Lascar generic. A fortiori it is $\mathcal{Z}$-generic. The $\mathcal{Z}$-genericity of $\neg\varphi(x\,;1)$ follows. □

18.39 Corollary Let $\Delta \subseteq L_{x\,z}$ be stable. Assume the action of $\mathcal{Z}$ is transitive. Let

$$Q \quad = \quad \{q(x) \in S_\Delta(\mathcal{Z}) \ : \ q(x) \text{ is } \mathcal{Z}\text{-generic}\}.$$

Then the following hold

1. $\mathcal{Z}$ acts on Q transitively
2. Q is finite
3. there is a finite equivalence relation $\varepsilon(x\,;y) \in \Delta^{G}(A)$ that if $\mathcal{Z}$-invariant and such that for all $q_i, q_j \in Q$
   $q_i = q_j \quad \Leftrightarrow \quad q_i(x) \cup q_j(y)$ is consistent with $\varepsilon(x,y)$
4. $\varphi(x\,;g)$ is generic if and only if $\varphi(x\,;g) \in q$ for some $q \in Q$.

Proof. Apply Theorem 18.36 in $\mathcal{U}'$ with $x = x$ for $p_0(x)$. Let $\varepsilon(x,y) \in L'$ and P be as defined in the proof of that theorem. As $\varepsilon(x,y)$ is invariant under the action of L'-Aut$(\mathcal{U}')$ it is a fortiori $\mathcal{Z}$-invariant. As $\mathcal{Z}$ acts transitively on the classes of $\varepsilon(x,y)$, it also acts transitively on P.

□

Define $\quad \mathrm{Stab}_G(\mathcal{D}) \quad = \quad \{g \in G \ : \ g \cdot \mathcal{D} = \mathcal{D}\}$

18.40 Proposition The following are equivalent

1. $\mathrm{Stab}_G(\mathcal{D})$ has finite index in G
2. $\mathcal{D}$ has a finite G-orbit.

18.41 Proposition Let $\Delta \subseteq L_{x\,z}$ be stable. $\mathcal{Z}$ acts transitively on the set of

18.7 Stable theories

18.42 Theorem (Pierre Simon) If every formula $\varphi(x\,;y) \in L(\mathcal{U})$, where $|x| = |y| = 1$, is stable then T is stable.

Proof. Suppose $\varphi(x\,;y,z) \in L(\mathcal{U})$ is not stable. We prove that there is a formula $\psi(x\,;y) \in L(\mathcal{U})$ that is not stable. Let $\langle a_i\,;b_i,c_i \ : \ i \in \mathbb{Q}\rangle$ is a sequence of indiscernibles such that

$$i < j \quad \Leftrightarrow \quad \varphi(a_i\,;b_j,c_j) \qquad\qquad \text{for all } i,j \in \mathbb{Q}.$$

Let $\mathbb{Q}^* = \mathbb{Q} \smallsetminus \{0\}$. Assume first that the sequence $\langle a_i \ : \ i \in \mathbb{Q}^*\rangle$ is indiscernible over c_0. Then for every $k \in \mathbb{Q}^*$ the type below is consistent

$$p_k(y) \quad = \quad \{\varphi(a_i\, y, c_0) \leftrightarrow i < k \ : \ i \in \mathbb{Q}^* \smallsetminus \{k\}\}.$$

In fact, by indiscernibility, b_0 witnesses the consistency of all finite subsets of $p_k(y)$. Let $b'_k \vDash p_k(y)$. Then

$$i < k \quad \Leftrightarrow \quad \varphi(a_i\,;b'_k,c_0) \qquad\qquad \text{for all } i,j \in \mathbb{Q}^*, i \neq j$$

From this the instability of $\psi(x\,;y) = \varphi(x\,;y,c_0)$ follows easily.

Now, assume instead that $\langle a_i \ : \ i \in \mathbb{Q}^*\rangle$ is not indiscernible over c_0. Note that the sequences $\langle a_i \ : \ i < 0\rangle$ and $\langle a_i \ : \ i > 0\rangle$ are mutually indiscernible over c_0. Then there is a maximal n such that

$a_{\restriction\{-1,\dots,-n\}} \equiv_{c_0} a_{\restriction\{1,\dots,n\}}$.

Let $A = a \restriction \{\pm 1, \dots, \pm n\}$. By maximality, $a_i \not\equiv_{c_0,A} a_j$ for every $-1 < i < 0 < j < 1$. Let $\psi(x\, y)$ be a formula such that $\psi(a_i\,;c_0)$ and $\neg\psi(a_j\,;c_0)$. We claim that for every $k \in (-1,0) \cup (0,1)$ the type below is consistent

$$q_k(z) \quad = \quad \{\psi(a_i\, y) \leftrightarrow i < k \ \ : \ i \in (-1,0) \cup (0,1)\}.$$

In fact as $\langle a_i \ : \ i \in (-1,0) \cup (0,1)\rangle$ is indiscernible over A, all finite subsets of $q_k(z)$ are realized by c_0. Finally let $c'_k \vDash p_k(z)$. Then $\langle a_i, c'_i \ : \ i \in (-1,0) \cup (0,1)\rangle$ witness the instability of $\psi(x\, z)$. □

18.43 Exercise Prove that if every formula $\varphi(x\,;z) \in L$, where $|x| = 1$, is stable then T is stable.

18.44 Exercise A sequence $\langle a_i : i < \omega\rangle$ is totally indiscernible if $a_1, \dots, a_n \equiv a_{i_1}, \dots, a_{i_n}$ for every distinct $i_1, \dots, i_n$. Prove that the following are equivalent

1. T is stable
2. every indiscernible sequence is totally indiscernible.

18.45 Exercise Prove that the following are equivalent

1. T is unstable
2. there is an infinite set $A \subseteq \mathcal{U}^n$ and a formula $\psi(x\,;y)$, with $|x| = |y| = n$ such that A is linearly ordered by the relation $a < b \leftrightarrow \psi(a\,;b)$.

18.46 Exercise Prove that strongly minimal theories are stable.

18.8 Notes and references

[1] Artem Chernikov and Pierre Simon, *Externally definable sets and dependent pairs*, Israel J. Math. **194** (2013), no. 1, 409–425. ArXiv:1007.4468.

[2] Victor Harnik and Leo Harrington, *Fundamentals of forking*, Ann. Pure Appl. Logic **26** (1984), no. 3, 245–286.

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Chapter 19

Vapnik-Chervonenkis theory

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathfrak{U}$ of inaccessible cardinality $\kappa > |L|$. The notation and implicit assumptions are as in Section 9.3.

19.1 Vapnik-Chervonenkis dimension

We say that the formula $\varphi(x\,;z) \in L$ has Vapnik-Chervonenkis dimension n if this is the largest finite cardinatity of a set $B \subseteq \mathfrak{U}^z$ such that $|S_\varphi(B)| = 2^n$. If such n does not exist, we say that we say that $\varphi(x\,;z)$ has infinite VC-dimension.

Note that the condition $|S_\varphi(B)| = 2^n$ is equivalent to saying that every subset of B is the trace of some definable set of sort $\varphi(x\,;z)$.

For instance, the formula $x_1 < z < x_2$ in $T_{\rm dlo}$ has VC-dimension 2.

Arguing by compactness we obtain the following proposition whose proof is left as an exercise for the reader.

19.1 Proposition The following are equivalent

1. $\varphi(x\,;z) \in L$ has finite VC-dimension
2. there is no infinite set $B \subseteq \mathfrak{U}^z$ such that every subset of B is the trace of some definable set of sort $\varphi(x\,;z)$.

From the proposition above and Proposition 18.20 below it follows that all stable formulas have finite VC-dimension.

We say that the a sequence of sentences $\langle \varphi_i : i < \omega \rangle$ converges if the truth value of φ_i is eventually constant.

19.2 Lemma The following are equivalent

1. $\varphi(x\,;z) \in L$ has finite VC-dimension
2. $\langle \varphi(a\,;b_i) : i < \omega \rangle$ converges for any a and any indiscernible sequence $\langle b_i : i < \omega \rangle$.

Proof. 1⇒2 Negate 2 and let $n < \omega$. It suffices to prove that for every $I \subseteq n$ the formula $\psi_I(x\,;b_0,\dots,b_{n-1})$ that says

$$\varphi(x\,;b_i) \quad\Leftrightarrow\quad i \in I$$

is consistent. If there is a a such that the truth value of $\langle \varphi(a\,;b_i) : i < \omega \rangle$ oscillates at least n times, then we can find $k_0 < \dots < k_{n-1}$ such that

$$\varphi(a\,;b_{k_i}) \quad\Leftrightarrow\quad i \in I.$$

Then the formula $\psi_I(x\,;b_{k_0},\dots,b_{k_{n-1}})$ is consistent. Therefore, by indiscernibility, also the formula $\psi_I(x\,;b_0,\dots,b_{n-1})$ is consistent.

2⇒1 Negate 1 and let $\langle c_i : i < \omega\rangle$ be an infinite sequence that is shattered by $\varphi(x\,;z)$. Let $\langle b_i : i < \omega\rangle$ be an indiscernible sequence that models the EM-type of $\langle c_i : i < \omega\rangle$. Then $\langle b_i : i < \omega\rangle$ satisfies $\exists x\, \psi_{I\restriction n}(x\,;z_0,\ldots,z_{n-1})$ for all n. Let $I \subseteq \omega$ be the set of even integers. By compactness there is a a such that

$$\varphi(a\,;b_i) \quad\Leftrightarrow\quad i \in I.$$

This proves ¬2. □

In the next section we need the following corollary.

19.3 Corollary If $\mathcal{C} \subseteq \mathcal{U}^z$ is a set approximable by a formula with finite VC-dimension, then $\langle b_i \in \mathcal{C} : i < \omega\rangle$ converges for any indiscernible sequence $\langle b_i : i < \omega\rangle$.

19.2 Honest definitions

In this section we present a beautiful theorem by Chernikov and Simon [1] and their alternative proof of a famous quantifier elimination result by Shelah.

We write $\neg^n$ for $\neg \overset{n \text{ times}}{\ldots\ldots} \neg$. We abbreviate $\neg^n(\cdot \in \cdot)$ as $\notin^n$.

Saturated sets have been defined in Definition 16.2.

19.4 Lemma Let $\mathcal{C}$ be saturated set approximable by a formula with finite VC-dimension and let A be a set of parameters. Then every global A-invariant type $p(z)$ contains a formula $\psi(z)$ such that either $\psi(\mathcal{U}^z) \subseteq \mathcal{C}$ or $\psi(\mathcal{U}^z) \subseteq \neg\mathcal{C}$. Moreover, we can require that $\psi(z) \in L(N)$, for any sufficiently saturated model N.

Proof. By Corollary 19.3, there is no infinite sequence $\langle b_i : i < \omega\rangle$

$$b_i \quad\vDash\quad p_{A,b\restriction i}(z) \;\cup\; \{z \notin^i \mathcal{C}\}$$

Let n be the largest integer such that there is a sequence $\langle b_i : i < n\rangle$ that satisfies the condition above. Then

$$p_{A,b\restriction n}(z) \quad\rightarrow\quad z \notin^{n+1} \mathcal{C}$$

and the first claim of the lemma follows by compactness.

As for the second claim note that we can pick $b_i \in N^z$ as soon as $A \subseteq N$ and $\langle N, \mathcal{C}\rangle$ is $|A|^+$-saturated. □

19.5 Corollary Let $\mathcal{C}$ be a set approximable by a formula with finite VC-dimension and let A be a set of parameters. Then there is a definable set $\mathcal{D} \supseteq A^z$ such that $\mathcal{D} \cap \mathcal{C}$ is definable. In particular, $\mathcal{C}$ is approximable from below.

Proof. Let M be a model containing A. Let c enumerate some $|M|^+$-saturated model containing M. For every $b \mathrel{\unicode{x2AEB}\!\!\!\!\!\!\!\;\;\!\!\!}_M c$ the type $\mathrm{tp}(b/c)$ extends to a global coheir over M. By the lemma above, there is a formula $\psi_b(z) \in \mathrm{tp}(b/c)$ such that either $\psi_b(\mathcal{U}^z) \subseteq \mathcal{C}$ or $\psi_b(\mathcal{U}^z) \subseteq \neg\mathcal{C}$, depending on whether $b \in \mathcal{C}$ or $b \notin \mathcal{C}$. Hence

$$z \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!|}_M c \quad\rightarrow\quad \bigvee\Big\{\psi_b(z) \;:\; b \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!|}_M c\Big\}.$$

By compactness,

$$z \mathop{\smile\!\!\!\!\!\!\,|}_M c \quad \rightarrow \quad \bigvee_{i=1}^{n} \psi_{b_i}(z).$$

Again by compactness, there is a formula $\varphi(z)$ such that

$$\varphi(z) \quad \rightarrow \quad \bigvee_{i=1}^{n} \psi_{b_i}(z).$$

Let $\mathcal{D} = \varphi(\mathcal{U}^z)$. Let $\psi(z)$ is the disjunction of those $\psi_{b_i}(z)$ such that $b_i \in \mathcal{C}$. Then $\mathcal{D} \cap \mathcal{C}$ is defined by $\varphi(z) \wedge \psi(z)$.

As $\mathcal{C} =_A \mathcal{D} \cap \mathcal{C}$, we obtain in particular that $\mathcal{C}$ is approximable from below, see Lemma 18.4. □

When all formulas have finite VC-dimension, we say that the theory T has the non-independence property or, for short, that T is nip.

Let $\langle \mathcal{D}_i : i < \lambda \rangle$ be the collection of all subsets of $\mathcal{U}$, of arbitrary finite arity, that are externally definable. The expansion of $\mathcal{U}$ to the language $L(\mathcal{X}_i : i < \lambda)$ is called the Shelah expansion of $\mathcal{U}$ and is denoted by $\mathcal{U}^{\mathrm{Sh}}$.

From Corollary 19.5 and Proposition 18.6 we obtain the following.

19.6 Corollary If T is nip then $\mathcal{U}^{\mathrm{Sh}}$ has L-elimination of quantifiers. (I.e. every formula is Boolean combination of formulas in L and formulas of the form $z \in \mathcal{D}_i$.)

19.7 Exercise (T is nip.) Let $\langle a_i : i < \omega \rangle$ be a sequence of indiscernibels over A. Let

$$p(x) \quad = \quad \{\varphi(x) \in L(\mathcal{U}) \ : \ \varphi(a_i) \text{ holds for almost all } i\}$$

Prove that $p(x)$ is complete.

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