A Crèche Course in Model Theory

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Chapter 1

Preliminaries and notation

This chapter introduces the syntax and semantic of first order logic.

The definitions of terms and formulas we give in Section 1.3 and 1.5 are more formal than required here. Our main objective is to convince the reader that a rigorous definition of language and truth is possible. However, the actual details of such a definition are not relevant for our purposes.

1.1 Tuples

A sequence is a function $a : I \to A$ whose domain is a linear order $I, <_I$. We may use the notation $a = \langle a_i : i \in I \rangle$ for sequences. A tuple is a sequence whose domain is an ordinal, say α, then we write $a = \langle a_i : i < \alpha \rangle$. When α is finite, we may also write $a = a_0, \ldots, a_{\alpha-1}$. The domain of the tuple a, the ordinal α, is denoted by $|a|$ and is called the length of a. In some contexts it may be convenient to confuse $\mathrm{rng}(a)$, the range of a, with a. If $A = \mathrm{rng}(a)$ we say that a is an enumeration of A.

If $J \subseteq I$ is a subset of the domain of the sequence $a = \langle a_i : i \in I \rangle$, we write $a_{\restriction J}$ for the restriction of a to J. When J is well ordered by $<_I$ we identify $a_{\restriction J}$ with a tuple. Sometimes (i.e. not always) we may overline tuples or sequences as mnemonic, as in $\bar{a}, \bar{c}$, etc. Still, restrictions are written as $a_{\restriction J}$, $c_{\restriction J}$ without the bar.

The set of tuples of elements of length α is denoted by A^α. The set of tuples of length $< \alpha$ is denoted by $A^{<\alpha}$. For instance, $A^{<\omega}$ is the set of all finite tuples of elements of A. When α is finite we do not distinguish between A^α and the α-th Cartesian power of A. In particular, we do not distinguish between A^1 and A.

When x is a tuple of variables, see Section 1.3, we will write write A^x for $A^{|x|}$. In fact, this notation is convenient when dealing with many-sorted structures, see Section **??**.

If $a, b \in A^\alpha$ and h is a function defined on A, we write $h(a) = b$ for $h(a_i) = b_i$. We often do not distinguish between the pair $\langle a, b \rangle$ and the tuple of pairs $\langle a_i, b_i \rangle$. The context will resolve the ambiguity.

Note that there is a unique tuple of length 0, the empty set $\varnothing$, which in this context is called empty tuple. Recall that by definition $A^0 = \{\varnothing\}$ for every set A. Therefore, even when A is empty, A^0 contains the empty string.

We often concatenate tuples. If a and b are tuples, we write $a\,b$ for the concatenation or, when emphasis is required, $a^\frown b$.

1.2 Structures

A language L (also called signature) is a triple that consists of

1. a set L_{fun} whose elements are called function symbols;

2. a set L_{rel} whose elements are called relation symbols;

3. a function that assigns to every $f \in L_{\text{fun}}$, respectively $r \in L_{\text{rel}}$, non-negative integers n_f and n_r that we call arity of the function, respectively relation, symbol. We say that f is an n_f-ary function symbol, and similarly for r. A 0-ary function symbol is also called a constant.

Warning: it is customary to use the symbol L to denote both the language and the set of formulas associated to it (to be defined below). We denote by $|L|$ the cardinality of $L_{\text{fun}} \cup L_{\text{rel}} \cup \omega$. Note that, by definition, $|L|$ is always infinite.

A structure M of signature L (for short L-structure) consists of

1. a set called the domain or support of the structure and is denoteed by the same symbol M used for the whole structure;

2. a function that assigns to every $f \in L_{\text{fun}}$ a total map $f^M : M^{n_f} \to M$;

3. a function that assigns to every $r \in L_{\text{rel}}$ a relation $r^M \subseteq M^{n_r}$.

We call f^M and r^M, the interpretation of f, respectively r, in M.

Recall that, by definition, $M^0 = \{\varnothing\}$. Therefore the interpretation of a constant c is a function that maps the unique element of M^0 to an element of M. We identify c^M with $c^M(\varnothing)$.

We may use the word model as a synonym for structure. But beware that, in some contexts, the word is used to denote a particular kind of structure.

If M is an L-structure and $A \subseteq M$ is any subset, we write $L(A)$ for the language obtained by adding to L_{fun} the elements of A as constants. In this context, the elements of A are called parameters. There is a canonical expansion of M to an $L(A)$-structure that is obtained by setting $a^M = a$ for every $a \in A$.

1.1 Example The language of additive groups consists of the following function symbols:

1. a constant (that is, a function symbol of arity 0) $\ 0$

2. a unary function symbol (that is, of arity 1) $\ -$

3. a binary function symbol (that is, of arity 2) $\ +$.

In the language of multiplicative groups the three symbols above are replaced by 1, ${}^{-1}$, and $\cdot$ respectively. Any group is a structure in either of these two signatures with the obvious interpretation. Needless to say, not all structures with these signatures are groups.

The language of (unitary) rings contains all the symbols above except ${}^{-1}$. The language of ordered rings also contains the binary relation symbol $<$.

The following example is less straightforward. The reason for the choice of the language of vector spaces will become clear in Example 1.9 below.

1.2 Example Let F be a field. The language of vector spaces over F, which we denote by L_F, extends that of additive groups by a unary function symbol k for every $k \in F$.

Recall that a vector space over F is an abelian group M together with a function $\mu : F \times M \to M$ satisfying some properties (that we assume to be well-known, see Example 2.6). To view a vector space over F as an L_F-structure, we interpret the group symbols in the obvious way and each $k \in F$ as the function $\mu(k, \text{-})$.

The languages in Examples 1.1 and 1.2, with the exception of that of ordered rings, are functional languages; that is, $L_{\text{rel}} = \varnothing$. In what follows, we consider two important examples of relational languages, that is, languages where $L_{\text{fun}} = \varnothing$.

1.3 Example The language of strict orders only contains a binary relation symbol, usually denoted by $<$. The language of graphs, too, only contains a binary relation symbol $r(\text{-}, \text{-})$.

1.3 Terms

Let V be an infinite set whose elements we call variables. We use the letters x, y, z, etc. to denote variables or tuples of variables. We rarely refer to V explicitly, and we always assume that V is large enough for our needs.

We fix a signature L for the whole section.

1.4 Definition A term is a finite sequence of elements of $L_{\text{fun}} \cup V$ that are obtained inductively as follows

0. every variable, intended as a tuple of length 1, is a term;
i. if $f \in L_{\text{fun}}$ and $t_1, \ldots, t_{n_f}$ are terms, then $f t_1, \ldots, t_{n_f}$ is a term.

We say L-term when we need to specify the language L.

Note that any constant f, intended as a tuple of length 1, is a term (by i, the term f is obtained concatenating $n_f = 0$ terms and prefixing by f). Terms that do not contain variables are called closed terms.

The intended meaning of, for instance, the term $+ + x\, y\, z$ is $(x + y) + z$. The first expression uses prefix notation; the second uses infix notation. When convenient, we informally use infix notation and add parentheses to improve legibility and avoid ambiguity.

The following lemma shows that prefix notation allows to write terms unambiguously without using parentheses.

1.5 Lemma (Unique legibility of terms) Let a be a sequence of terms. Suppose a can be obtained both by concatenating the terms $t_1, \ldots, t_n$ and by concatenating the terms $s_1, \ldots, s_m$. Then $n = m$ and $s_i = t_i$.

Proof. By induction on $|a|$. If $|a| = 0$ than $n = m = 0$ and there is nothing to prove. Suppose the claim holds for tuples of length k and let $a = a_1, \ldots, a_{k+1}$. Then a_1 is the first element of both t_1 and s_1. If a_1 is a variable, say x, then t_1 and s_1 are the term x and $n = m = 1$. Otherwise a_1 is a function symbol, say f. Then $t_1 = f\,\bar{t}$ and $s_1 = f\,\bar{s}$, where $\bar{t}$ and $\bar{s}$ are obtained by concatenating the terms $t'_1, \ldots, t'_p$ and $s'_1, \ldots, s'_p$. Now apply the induction hypothesis to $a_2, \ldots, a_{k+1}$ and to the terms $\bar{t}\,t_2 \ldots t_n$ and $\bar{s}\,s_2 \ldots s_m$. □

If $x = x_1, \ldots, x_n$ is a tuple of distinct variables and $s = s_1, \ldots, s_n$ is a tuple of terms, we write $t[x/s]$ for the sequence obtained by replacing x by s coordinatewise. Proving that $t[x/s]$ is indeed a term is a tedious task that can be safely skipped.

If t is a term and $x_1, \ldots, x_n$ are (tuples of) variables, we write $t(x_1, \ldots, x_n)$ to declare that the variables occurring in t are among those that occur in $x_1, \ldots, x_n$. When a term has been presented as $t(x, y)$, we write $t(s, y)$ for $t[x/s]$.

Finally, we define the interpretation of a term in a structure M. We begin with closed terms. These are interpreted as 0-ary functions, i.e. as elements of the structure.

1.6 Definition Let t be a closed $L(M)$ term. The interpretation of t, denoted by t^M, is defined by induction on the syntax of t as follows

i. if $t = f\,t_1 \ldots t_{n_f}$, where $f \in L_{\text{fun}}$, then $t^M = f^M(t_1^M, \ldots, t_{n_f}^M)$.

Note that in i we have used Lemma 1.5 in an essential way. In fact this ensures that the sequence $t_1, \ldots, t_{n_f}$ uniquely determines the terms $t_1^M, \ldots, t_{n_f}^M$.

The inductive definition above is based on the case $n_f = 0$, that is, the case where f a constant, or a parameter. When $t = c$, a constant, $t_1 \ldots t_{n_f}$ is the empty tuple, and so $t^M = c^M(\varnothing)$, which we abbreviate as c^M. In particular, if $t = a$, a parameter, then $t^M = a^M = a$.

Now we generalize the interpretation to all (not necessarily closed) terms. If $t(x)$ is a term, we define $t^M(x) : M^x \to M$ to be the function that maps a to $t(a)^M$.

1.4 Substructures

In the working practice, a *substructure* is a subset of a structure that is closed under the interpretation of the functions in the language. But there are a few cases when we need the following formal definition.

1.7 Definition Fix a signature L and let M and N be two L-structures. We say that M is a substructure of N, and write $M \subseteq N$, if

1. the domain of M is a subset of the domain of N
2. $f^M = f^N \restriction M^{n_f}$ for every $f \in L_{\text{fun}}$
3. $r^M = r^N \cap M^{n_r}$ for every $f \in L_{\text{rel}}$.

Note that when f is a constant 2 becomes $f^M = f^N$, in particular the substructures of N contains at least all the constants of N.

If a set $A \subseteq N$ is such that

1. $f^N[A^{n_f}] \subseteq A$ for every $f \in L_{\text{fun}}$

then there is a unique substructure $M \subseteq N$ with domain A, namely, the structure with the following interpretation

2. $f^M = f^N \restriction A^{n_f}$ (which is a good definition by assumption 1);
3. $r^M = r^N \cap A^{n_r}$.

It is usual to confuse subsets of N that satisfy 1 with the unique substructure they support.

It is immediate to verify that the intersection of an arbitrary family of substructures of N is a substructure of N. Therefore, for any given $A \subseteq N$ we may define the substructure of N generated A as the intersection of all substructures of N that contain A. We write $\langle A \rangle_N$. The following easy proposition gives more concrete representation of $\langle A \rangle_N$

1.8 Lemma The following hold for every $A \subseteq N$

1. $\langle A \rangle_N = \Big\{ t^N \ :\ t \text{ a closed } L(A)\text{-term} \Big\}$
2. $\langle A \rangle_N = \Big\{ t^N(a) \ :\ t(x) \text{ an } L\text{-term and } a \in A^x \Big\}$
3. $\langle A \rangle_N = \bigcup_{n \in \omega} A_n$, where $A_0 = A$
$$A_{n+1} = A_n \cup \Big\{ f^N(a) \ :\ f \in L_{\text{fun}},\ a \in A_n^{n_f} \Big\}.$$

1.9 Example Let L be the language of groups. Let N be a group, which we consider as an L-structure in the natural way. Then the substructures of N are exactly the subgroups of N and $\langle A \rangle_N$ is the group generated by $A \subseteq N$. A similar claim is true when L_F is the signature of vector spaces over some fixed field F. The choice of the language is more or less fixed if we want that the algebraic and the model theoretic notion of substructure coincide.

1.5 Formulas

Fix a language L and a set of variables V as in Section 1.3. A formula is a finite sequence of symbols in $L_{\text{fun}} \cup L_{\text{rel}} \cup V \cup \{\doteq, \bot, \neg, \vee, \exists\}$. The last set contains the logical symbols that are called respectively

$\doteq$	equality	$\bot$	contradiction	$\neg$	negation
$\vee$	disjunction	$\exists$	existential quantifier.		

Syntactically, $\doteq$ behaves like a binary relation symbol. So, for convenience set $n_{\doteq} = 2$. However $\doteq$ is considered as a logic symbol because its semantic is fixed (it is always interpreted in the diagonal).

The definition below uses the prefix notation which simplifies the proof of the unique legibility lemma. However, in practice we always we use the infix notation: $t \doteq s$, $\varphi \vee \psi$, etc.

1.10 Definition A formula is any finite sequence is obtained with the following inductive procedure

0. if $r \in L_{\rm rel} \cup \{\doteq\}$ and t is a tuple obtained concatenating n_r terms then $r\,t$ is a formula. Formulas of this form are called atomic;

i. if φ e ψ are formulas then the following are formulas: $\bot$, $\neg\,\varphi$, $\vee\,\varphi\,\psi$, and $\exists x\,\varphi$, for any $x \in V$.

We use L to denote both the language and the set of formulas. We write $L_{\rm at}$ for the set of atomic formulas and $L_{\rm qf}$ for the set of quantifier-free formulas i.e. formulas where $\exists$ does not occur.

The proof of the following is similar to the analogous lemma for terms.

1.11 Lemma (Unique legibility of formulas) Let a be a sequence of formulas. Suppose a can be obtained both by the concatenation of the formulas $\varphi_1, \ldots, \varphi_n$ or by the concatenation of the formulas $\psi_1, \ldots, \psi_m$. Then $n = m$ and $\varphi_i = \psi_i$.

A formula is closed if all its variables occur under the scope of a quantifier. Closed formulas are also called sentences. We will do without a formal definition of *occurs under the scope of a quantifier* which is too lengthy. An example suffices: all occurrences of x are under the scope a quantifiers in the formula $\exists x\,\varphi$. These occurrences are called bounded. The formula $x \doteq y \wedge \exists x\,\varphi$ has free (i.e., not bound) occurrences of x and y.

Let x is a tuple of variables and t is a tuple of terms such that $|x| = |t|$. We write $\varphi[x/t]$ for the formula obtained substituting t for all free occurrences of x, coordinatewise.

We write $\varphi(x)$ to declare that the free variables in the formula φ are all among those of the tuple x. In this case we write $\varphi(t)$ for $\varphi[x/t]$.

We often use without explicit mention the following useful syntactic decomposition of formulas with parameters which we state without proof.

1.12 Lemma For every formula $\varphi(x) \in L(A)$ there is a formula $\psi(x\,;z) \in L$ and a tuple of parameters $a \in A^{|z|}$ such that $\varphi(x) = \psi(x\,;a)$.

Just as a term $t(x)$ is a name for a function $t(x)^M : M^x \to M$, a formula $\varphi(x)$ is a name for a subset $\varphi(x)^M \subseteq M^x$ which we call the subset of M defined by $\varphi(x)$. It is also very common to write $\varphi(M)$ for the set defined by $\varphi(x)$. In general sets of the form $\varphi(M)$ for some $\varphi(x) \in M$ are called definable.

1.13 Definition of truth For every formula φ with variables among those of the tuple x we define $\varphi(x)^M$ by induction as follows

o1. $(\doteq t\,s)(x)^M = \Big\{a \in M^x \ :\ t^M(a) = s^M(a)\Big\}$

o2. $(r\,t_1 \ldots t_n)(x)^M = \Big\{a \in M^x \ :\ \langle t_1^M(a), \ldots, t_n^M(a)\rangle \in r^M\Big\}$

i0. $\bot(x)^M = \varnothing$

i1. $(\neg\xi)(x)^M = M^x \smallsetminus \xi(x)^M$

i2. $(\vee\ \xi\ \psi)(x)^M = \xi(x)^M \cup\ \psi(x)^M$

i3. $(\exists y\ \varphi)(x)^M = \bigcup_{a \in M} (\varphi[y/a])(x)^M.$

Condition i2 assumes that ξ and ψ are uniquely determined by $\vee\,\xi\,\psi$. This is a guaranteed by the unique legibility o formulas, Lemma 1.11. Analogously, o1 e o2 assume Lemma 1.5.

The case when x is the empty tuple is far from trivial. Note that $\varphi(\varnothing)^M$ is a subset of $M^0 = \{\varnothing\}$. Then there are two possibilities either $\{\varnothing\}$ or $\varnothing$. We will read them as two truth values: True and False, respectively. If $\varphi^M = \{\varnothing\}$ we say that φ is true in M, if $\varphi^M = \varnothing$, we say that φ is false M and we write $M \vDash \varphi$, respectively $M \nvDash \varphi$. In words we may say that M models φ, respectively M does not model φ. It is immediate to verify that

$$\varphi(M) = \big\{a \in M^x : M \vDash \varphi(a)\big\}.$$

Note that usually, we say *formula* when, strictly speaking, we mean *pair* that consists of a formula and a tuple of variables. Such pairs are interpreted in definable sets (cfr. Definition 1.13). In fact, if the tuple of variables were not given, the arity of the corresponding set is not determined.

In some contexts we also want to distinguish between two sorts of variables that play different roles. Some are placeholder for parameters, some are used to define a set. In the the first chapters this distinction is only a clue for the reader, in the last chapters it is an essential part of the definitions.

1.14 Definition A partitioned formula is a triple $\varphi(x\,;z)$ consisting of a formula and two tuples of variables such that the variables occurring in φ are all among x, z.

We use a semicolon to separate the two tuples of variables. Typically, z is the placeholder for parameters and x runs over the model.

1.6 Yet more notation

Now we abandon the prefix notation in favor of the infix notation. We also use the following logical connectives as abbreviations

$\top$	stands for	$\neg\bot$	tautology
$\varphi \wedge \psi$	stands for	$\neg[\neg\varphi \vee \neg\psi]$	conjunction
$\varphi \rightarrow \psi$	stands for	$\neg\varphi \vee \psi$	implication
$\varphi \leftrightarrow \psi$	stands for	$[\varphi \rightarrow \psi] \ \wedge \ [\psi \rightarrow \varphi]$	bi-implication
$\varphi \not\leftrightarrow \psi$	stands for	$\neg[\varphi \leftrightarrow \psi]$	exclusive disjunction
$\forall x\, \varphi$	stands for	$\neg\exists x \neg\, \varphi$	universal quantifier

We agree that $\rightarrow$ e $\leftrightarrow$ bind less than $\wedge$ e $\vee$. Unary connectives (quantifiers and negation) bind stronger then binarary connectives. For example

$$\exists x\, \varphi \wedge \psi \ \rightarrow \ \neg\xi \vee \vartheta \quad \text{reads as} \quad \Big[[\exists x\, \varphi] \wedge \psi\Big] \ \rightarrow \ \Big[[\neg\xi] \vee \vartheta\Big]$$

We say that $\forall x\, \varphi(x)$ and $\exists x\, \varphi(x)$ are the universal, respectively, existential closure of $\varphi(x)$. We say that $\varphi(x)$ holds in M when its universal closure is true in M. We say that $\varphi(x)$ is consistent in M when its existential closure is true in M.

The semantic of conjunction and disjunction is associative. Then for any finite set of formulas $\{\varphi_i : i \in I\}$ we can write without ambiguities

$$\bigwedge_{i\in I} \varphi_i \qquad\qquad \bigvee_{i\in I} \varphi_i$$

When $x = x_1, \ldots, x_n$ is a tuple of variables we write $\exists x\, \varphi$ or $\exists x_1, \ldots, x_n\, \varphi$ for $\exists x_1 \ldots \exists x_n\, \varphi$. There is a first order sentences that say that $\varphi(M)$ has at least n elements (also, no more than, or exactly n). It is convenient to use the following abbreviations.

$\exists^{\geq n} x\, \varphi(x)$	stands for	$\exists x_1, \ldots, x_n \Big[\bigwedge_{1\leq i\leq n} \varphi(x_i) \ \wedge \ \bigwedge_{1\leq i<j\leq n} x_i \dot{\neq} x_j \Big].$
$\exists^{\leq n} x\, \varphi(x)$	stands for	$\neg\exists^{\geq n+1} x\, \varphi(x)$
$\exists^{= n} x\, \varphi(x)$	stands for	$\exists^{\geq n} x\, \varphi(x) \ \wedge \ \exists^{\leq n} x\, \varphi(x)$

1.15 Exercise Let M be an L-structure and let $\psi(y), \varphi(x, y) \in L$. For each of the following conditions, write a sentence true in M exactly when

a. $\psi(M) \ \in \ \big\{\varphi(a, M) : a \in M\big\}$;

b. $\big\{\varphi(a, M) : a \in M\big\}$ contains at least two sets;

c. $\big\{\varphi(a, M) : a \in M\big\}$ contains only sets that are pairwise disjoint.

1.16 Exercise Let M be a structure in a signature that contains a symbol r for a binary relation. Write a sentence φ such that

a. $M \vDash \varphi$ if and only if there is $A \subseteq M$ such that $r^M \ \subseteq \ A \times \neg A$.

Remark: φ assert an asymmetric version of the property below

b. $M \vDash \psi$ if and only if there is $A \subseteq M$ such that $r^M \ \subseteq \ (A \times \neg A) \ \cup \ (\neg A \times A)$.

Assume M is a graph, what required in b is equivalent to saying that M is a *bipartite graph*, or equivalently that it has *chromatic number* 2 i.e., we can color the vertices with 2 colors so that no two adjacent vertices share the same color.

Chapter 2

Theories and elementarity

2.1 Logical consequences

A theory is a set $T \subseteq L$ of sentences. We write $M \vDash T$ when $M \vDash \varphi$ for every $\varphi \in T$. If $\varphi \in L$ is a sentence we write $T \vdash \varphi$ when

$$M \vDash T \quad \Rightarrow \quad M \vDash \varphi \qquad \text{for every } M.$$

In words, we say that φ is a logical consequence of T or that φ follows from T. If S is a theory $T \vdash S$ has a similar meaning. If $T \vdash S$ and $S \vdash T$ we say that T and S are logically equivalent. We may say that T axiomatizes S (or vice versa).

We say that a theory is consistent if it has a model. With the notation above, T is consistent if and only if $T \nvdash \bot$.

The closure of T under logical consequence is the set $\mathrm{ccl}(T)$ which is defined as follows:

$$\mathrm{ccl}(T) \quad = \quad \Big\{\varphi \in L \ : \ \text{sentence such that } T \vdash \varphi\Big\}$$

If T is a finite set, say $T = \{\varphi_1, \ldots, \varphi_n\}$ we write $\mathrm{ccl}(\varphi_1, \ldots, \varphi_n)$ for $\mathrm{ccl}(T)$. If $T = \mathrm{ccl}(T)$ we say that T is closed under logical consequences.

The theory of M is the set of sentences that hold in M and is denoted by $\mathrm{Th}(M)$. More generally, if $\mathcal{K}$ is a class of structures, $\mathrm{Th}(\mathcal{K})$ is the set of sentences that hold in every model in $\mathcal{K}$. That is

$$\mathrm{Th}(\mathcal{K}) \quad = \quad \bigcap_{M \in \mathcal{K}} \mathrm{Th}(M)$$

The class of all models of T is denoted by $\mathrm{Mod}(T)$. We say that $\mathcal{K}$ is axiomatizable if $\mathrm{Mod}(T) = \mathcal{K}$ for some theory T. If T is finite we say that $\mathcal{K}$ is finitely axiomatizable. To sum up

$$\begin{aligned}
\mathrm{Th}(M) &= \Big\{\varphi \ : \ M \vDash \varphi\Big\} \\
\mathrm{Th}(\mathcal{K}) &= \Big\{\varphi \ : \ M \vDash \varphi \text{ for all } M \in \mathcal{K}\Big\} \\
\mathrm{Mod}(T) &= \Big\{M \ : \ M \vDash T\Big\}
\end{aligned}$$

2.1 Example Let L be the language of multiplicative groups. Let T_g be the set containing the universal closure of following three formulas

1. $(x \cdot y) \cdot z \ = \ x \cdot (y \cdot z);$
2. $x \cdot x^{-1} \ = \ x^{-1} \cdot x \ = \ 1;$
3. $x \cdot 1 \ = \ 1 \cdot x \ = x.$

Then T_g axiomatizes the theory of groups, i.e. $\mathrm{Th}(\mathcal{K})$ for $\mathcal{K}$ the class of all groups. Let φ be the universal closure of the following formula

$$z \cdot x \ = \ z \cdot y \ \rightarrow \ x \ = \ y.$$

As φ formalizes the cancellation property then $T_g \vdash \varphi$, that is, φ is a logical consequence of T_g. Now consider the sentence ψ which is the universal closure of

4. $x \cdot y \ = \ y \cdot x.$

So, commutative groups model ψ and non commutative groups model $\neg\psi$. Hence neither $T_g \vdash \psi$ nor $T_g \vdash \neg\psi$. We say that T_g does not decide ψ.

Note that even when T is a very concrete set, $\mathrm{ccl}(T)$ may be more difficult to grasp. In the example above T_g contains three sentences but $\mathrm{ccl}(T_g)$ is an infinite set containing sentences that code theorems of group theory yet to be proved.

2.2 Remark The following properties say that ccl is a finitary closure operator.

1. $T \subseteq \mathrm{ccl}(T)$ (extensive)
2. $\mathrm{ccl}(T) \ = \ \mathrm{ccl}\big(\mathrm{ccl}(T)\big)$ (idempotent)
3. $T \subseteq S \ \Rightarrow \ \mathrm{ccl}(T) \subseteq \mathrm{ccl}(S)$ (increasing)
4. $\mathrm{ccl}(T) \ = \ \bigcup\big\{\mathrm{ccl}(S) \ : \ S \text{ finite subset of } T\big\}.$ (finitary)

Properties 1-3 are easy to verify while 4 requires the compactness theorem.

In the next example we list a few algebraic theories with straightforward axiomatization.

2.3 Example We write T_{ag} for the theory of abelian groups which contains the universal closure of following

a1. $(x+y)+z \ = \ y+(x+z);$

a2. $x+(-x) \ = \ 0;$

a3. $x+0 \ = \ x;$

a4. $x+y \ = \ y+x.$

2.4 Example The theory T_r of (unitary) rings extends T_ag with

a5. $(x \cdot y) \cdot z \;=\; x \cdot (y \cdot z)$;

a6. $1 \cdot x \;=\; x \cdot 1 \;=\; x$;

a7. $(x + y) \cdot z \;=\; x \cdot z + y \cdot z$;

a8. $z \cdot (x + y) \;=\; z \cdot x + z \cdot y$.

The theory of commutative rings T_cg contains also com of examples 2.1.

2.5 Example The theory of ordered rings T_or extends T_cr with

o1. $x < z \;\rightarrow\; x + y < z + y$;

o2. $0 < x \wedge 0 < z \;\rightarrow\; 0 < x \cdot z$.

2.6 Example The axiomatization of the theory of vector spaces is less straightforward. Fix a field F. The language L_F extends the language of additive groups with a unary function for every element of F. The theory of vector fields over F extends T_ag with the following axioms (for all $h, k, l \in F$)

m1. $h\,(x + y) \;=\; h\,x + h\,y$

m2. $l\,x \;=\; h\,x + k\,x$, where $l \;=\; h +_F k$

m3. $l\,x \;=\; h\,(k\,x)$, where $l \;=\; h \cdot_F k$

m4. $0_F\,x \;=\; 0$

m5. $1_F\,x \;=\; x$

The symbols 0_F and 1_F denote the zero and the unit of F. The symbols $+_F$ and $\cdot_F$ denote the sum and the product in F. These are not part of L_F, they are symbols we use in the metalanguage.

2.7 Example Recall from Example 1.3 that we represent a graph with a symmetric irreflexive relation. Therefore theory of graphs contains the following two axioms

1. $\neg r(x, x)$;
2. $r(x, y) \rightarrow r(y, x)$.

Our last example is a trivial one.

2.8 Example Let L be the empty language The theory of infinite sets is axiomatized by the sentences $\exists^{\geq n} x\; (x = x)$ for all positive integer n.

2.9 Exercise Prove that $\mathrm{ccl}(\varphi \vee \psi) = \mathrm{ccl}(\varphi) \cap \mathrm{ccl}(\psi)$.

2.10 Exercise Prove that $\mathrm{Th}(\mathrm{Mod}(T)) = \mathrm{ccl}(T)$.

2.2 Elementary equivalence

The following is a fundamental notion in model theory.

2.11 Definition We say that M and N are elementarily equivalent if

ee. $N \vDash \varphi \quad \Leftrightarrow \quad M \vDash \varphi, \qquad$ for every sentence $\varphi \in L$.

In this case we write $M \equiv N$. More generally, we write $M \equiv_A N$ and say that M and N are elementarily equivalent over A if the following hold

a. $A \subseteq M \cap N$

ee'. equivalence ee above holds for every sentence $\varphi \in L(A)$.

The case when A is the whole domain of M is particularly important.

2.12 Definition When $M \equiv_M N$ we say that M is an elementary substructure of N and write $M \preceq N$.

The following lemma shows that the use of the term *substructure* in the definition above is appropriate.

2.13 Lemma If M and N are such that $M \equiv_A N$ and A is the domain of a substructure of M then A is also the domain a substructure of N and the two substructures coincide.

Proof. Let f be a function symbol and ler r be a relation symbol. It suffices to prove that $f^M(a) = f^N(a)$ for every $a \in A^{n_f}$ and that $r^M \cap A^{n_r} = r^N \cap A^{n_r}$.

If $b \in A$ is such that $b = f^M a$ then $M \vDash fa = b$. So, from $M \equiv_A N$, we obtain $N \vDash fa = b$, hence $f^N a = b$. This proves $f^M(a) = f^N(a)$.

Now let $a \in A^{n_r}$ and suppose $a \in r^M$. Then $M \vDash ra$ and, by elementarity, $N \vDash ra$, hence $a \in r^N$. By symmetry $r^M \cap A^{n_r} = r^N \cap A^{n_r}$ follows. □

It is not easy to prove that two structures are elementary equivalent. A direct verification is unfeasible even for the most simple structures. It will take a few chapters before we are able to discuss concrete examples.

We generalize the definition of $\mathrm{Th}(M)$ to include parameters

$$\mathrm{Th}(M/A) \quad = \quad \Big\{\varphi \;:\; \text{sentence in } L(A) \text{ such that } \; M \vDash \varphi\Big\}.$$

The following proposition is immediate

2.14 Proposition For every pair of structures M and N and every $A \subseteq M \cap N$ the following are equivalent

a. $M \equiv_A N$;

b. $\mathrm{Th}(M/A) = \mathrm{Th}(N/A)$;

c. $M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(a) \qquad$ for every $\varphi(x) \in L$ and every $a \in A^{|x|}$.

d. $\varphi(M) \cap A^{|x|} \;=\; \varphi(N) \cap A^{|x|}$ for every $\varphi(x) \in L$.

If we restate a and c of the proposition above when $A = M$ we obtain that the following are equivalent

a'. $M \preceq N$;

d'. $\varphi(M) \;=\; \varphi(N) \cap M^{|x|}$ for every $\varphi(x) \in L$.

Note that c' extends to all definable sets what Definition 1.7 requires for a few basic definable sets.

2.15 Example Let G be a group which we consider as a structure in the multiplicative language of groups. We show that if G is simple and $H \preceq G$ then also H is simple. Recall that G is simple if all its normal subgroups are trivial, equivalently, if for every $a \in G \smallsetminus \{1\}$ the set $\{gag^{-1} : g \in G\}$ generates the whole group G.

Assume H is not simple. Then there are $a, b \in H$ such that b is not the product of elements of $\{hah^{-1} : h \in H\}$. Then for every n

$$H \vDash \neg\exists x_1, \ldots, x_n \;\left(b \;=\; x_1 a x_1^{-1} \cdots\; x_n a x_n^{-1}\right)$$

By elementarity the same hold in G. Hence G is not simple.

2.16 Exercise Let $A \subseteq M \cap N$. Prove that $M \equiv_A N$ if and only if $M \equiv_B N$ for every finite $B \subseteq A$.

2.17 Exercise Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if $\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$.

2.18 Exercise Let $M \preceq N$ and let $\varphi(x, z) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M.

2.19 Exercise Consider $\mathbb{Z}^n$ as a structure in the additive language of groups with the natural interpretation. Prove that $\mathbb{Z}^n \not\equiv \mathbb{Z}^m$ for every positive integers $n \neq m$. Hint: in $\mathbb{Z}^n$ there are at most 2^n elements that are not congruent modulo 2.

2.3 A nonstandard example

This section concerns an example that is useful to look at in some detail to get some familiarity with the notion of elementary substructure. The example is culturally interesting, because it formalises rigorously the notions of infinity and infinitesimal, which were used in Newton and Leibniz's time to develop real analysis. These notions were not well defined - in fact, they were inconsistent.

It was only in the mid 19th century that mathematicians of Weiertsrass' generation developed the notion of limit, thus providing rigorous grounds for the development of analysis.

Nonstandard analysis was developed by Abraham Robinson in the 1950s. Robinson found a way to formalise the ideas of infinity and infinitesimals through the concept of elementary extension.

The notation that follows will be used throughout this section.

The language we use contains

1. X, a relation symbol of arity n, for every $n \in \omega$ and every $X \subseteq \mathbb{R}^n$;
2. f, a function symbol of arity n, for every $n \in \omega$ and every $f : \mathbb{R}^n \to \mathbb{R}$.

The standard model of real analysis is $\mathbb{R}$ with the natural interpretation of the symbols in our language, so we use the same symbol for an element of the language and its interpretation in $\mathbb{R}$. This is not an abuse of notation: in this case, elements and interpretations coincide.

Suppose $\mathbb{R}$ has a proper elementary extension ${}^*\mathbb{R}$. The existence of such an extension will be proved later. The interpretations of the symbols f and X in ${}^*\mathbb{R}$ will be denoted by ${}^*f$ and ${}^*X$, respectively. The elements of ${}^*\mathbb{R}$ are called hyperreals, the elements of $\mathbb{R}$ are called standard (hyper)reals, those in ${}^*\mathbb{R} \smallsetminus \mathbb{R}$ are called nonstandard (hyper)reals.

It is easy to verify that ${}^*\mathbb{R}$ is an ordered field. This is because the operations sum and product are in the language, and so is the order relation. The property of being an ordered field can be expressed via a set of sentences that are true in $\mathbb{R}$, and hence also in ${}^*\mathbb{R}$.

A hyperreal c is said to be infinitesimal if $|c| < \varepsilon$ for every positive standard ε. A hyperreal c is infinite if $k < |c|$ for every standard k; the hyperreal c if finite otherwise. Hence if c is infinite, then c^{-1} is infinitesimal. Clearly, all standard reals are finite, and 0 is the only infinitesimal standard real.

The proof of the following lemma is left to the reader.

2.20 Fact Infinitesimals are closed under sum, product and multiplication by standard reals.

We begin with an existence proof.

2.21 Lemma There are infinite nonstandard hyperreals and nonzero infinitesimals. Moreover, for every finite hyperreal c there is a unique standard b such that $b - c$ is infinitesimal.

Proof. If $c \in {}^*\mathbb{R}$ in infinite then c^{-1} is infinitesimal and if c is a nonzero infinitesimal, then c^{-1} is infinite. Then the first claim follows from the second one. Let c be finite. The set $\{a \in \mathbb{R} : c < a\}$ is a nonempty set of (standard) reals that is bounded from below. Let $b \in \mathbb{R}$ the least upper bound. We show that $b - c$ is a infinitesimal. Assume for a contradiction that $\varepsilon < |b - c|$ for some standard positive ε. Then $c < b - \varepsilon$, or $b + \varepsilon < c$, depending on whether $c < b$ or $b < c$. Both cases contradict that b is an infimum. This proves the existence of b.

To prove uniquenes, suppose that $b' \in \mathbb{R}$ and $b' - c$ is also infinitesimal. Then $b - b'$ is also infinitesimal. As 0 is the only standard infinitesimal $b = b'$. □

The existence of infinite non standard hyperreals shows that ${}^*\mathbb{R}$ is not archimedean: standard integers are not cofinal in ${}^*\mathbb{R}$. However, by elementarity, the nonstandard integers ${}^*\mathbb{N}$ are cofinal in ${}^*\mathbb{R}$: from the perspective of an inhabitant of ${}^*\mathbb{R}$, the latter is a normal archimedean field.

Dedekind completeness is another fundamental property of $\mathbb{R}$ that does not hold in ${}^*\mathbb{R}$. An ordered set is Dedekind complete if every subset that is bounded above has a least upper bound. Then ${}^*\mathbb{R}$ is not Dedekind complete: the set of infinitesimals is bounded (both above and below), but, by Lemma 2.20, it does not have a least upper bound. It follows that Dedekind completeness is *not* a first-order property.

However, it sometimes happens that properties that hold of all sets in the standard model hold in ${}^*\mathbb{R}$ for definable sets only.

The following formula $\psi(z, y)$ says that y is an upper bound for the set defined by $\varphi(z, x)$, where z is a tuple of parameters:

We define the following equivalence relation on ${}^*\mathbb{R}$: we write $a \approx b$ if $|a - b|$ is infinitesimal. The fact that this is an equivalence relation follows from Fact 2.20. The equivalence class of c is called monad. By Lemma **??** if c is a finite hyperreal, there is a unique real in the monad of c.

Hyperreals that are not finite are said to be infinite. If c is finite, the unique standard real in the monad of c is called standard part of c and it is denoted by $\mathrm{st}(c)$.

In the following lemma, the expressions on the left can be formalised as first-order sentences, and therefore they hold in $\mathbb{R}$ if and only if they hold in ${}^*\mathbb{R}$.

2.22 Proposition For every $f : \mathbb{R} \to \mathbb{R}$, for every $a, l \in \mathbb{R}$ the following equivalences hold.

a. $\lim\limits_{x\to+\infty} fx = +\infty \quad \Leftrightarrow \quad {}^*f(c)$ is positive and infinite for every infinite $c > 0$

b. $\lim\limits_{x\to+\infty} fx = l \quad \Leftrightarrow \quad {}^*f(c) \approx l$ for every infinite $c > 0$

c. $\lim\limits_{x\to a} fx = +\infty \quad \Leftrightarrow \quad {}^*f(c)$ is positive and infinite for every $c \approx a \neq c$

d. $\lim\limits_{x\to a} fx = l \quad \Leftrightarrow \quad {}^*f(c) \approx l$ for every for every $c \approx a \neq c$.

Proof. We prove part d and we leave the other parts as an exercise.

$\Rightarrow$. The left-hand side of the equivalence d asserts

1 $$\mathbb{R} \vDash \forall \varepsilon > 0\, \exists \delta > 0\, \forall x \Big[0 < |x - a| < \delta \;\rightarrow\; |fx - l| < \varepsilon\Big].$$

For consistency with standard notation in analysis, we use the Greek letters ε and δ as variables. The symbols $\dot{\varepsilon}$ e $\dot{\delta}$ denote parameters either in $\mathbb{R}$ or in ${}^*\mathbb{R}$.

Pick $c \neq a \approx c$ arbitrarily. We now check that ${}^*f(c) \approx l$, that is, $|{}^*fc - l| < \dot{\varepsilon}$ for every positive standard $\dot{\varepsilon}$. Pick $\dot{\varepsilon} \in \mathbb{R}^+$ arbitralily and let $\dot{\delta} \in \mathbb{R}$ be given by 1. By elementarity, we have

$${}^*\mathbb{R} \vDash \forall x \Big[0 < |x - a| < \dot{\delta} \;\rightarrow\; |fx - l| < \dot{\varepsilon}\Big].$$

In particular

$${}^*\mathbb{R} \vDash 0 < |c - a| < \dot{\delta} \;\rightarrow\; |fc - l| < \dot{\varepsilon}.$$

As $0 < |c - a| < \dot{\delta}$ certainly holds (because $\dot{\delta}$ is standard) $|{}^*fc - l| < \dot{\varepsilon}$ follows.

$\Leftarrow$. Assume that 1 is false, that is,

2 $$\mathbb{R} \vDash \exists \varepsilon > 0\, \forall \delta > 0\, \exists x \Big[0 < |x - a| < \delta \;\wedge\; \varepsilon \leq |fx - l|\Big].$$

We want to show that ${}^*f(c) \not\approx l$ for some $c \neq a$. Let $\dot{\varepsilon} \in \mathbb{R}$ witness 2. By elementarity we have

$${}^*\mathbb{R} \vDash \forall \delta > 0\, \exists x \Big[0 < |x - a| < \delta \;\wedge\; \dot{\varepsilon} \leq |fx - l|\Big].$$

Let $\dot{\delta}$ be an arbitrary infinitesimal. Then

$${}^*\mathbb{R} \vDash \exists x \Big[0 < |x - a| < \dot{\delta} \;\wedge\; \dot{\varepsilon} \leq |fx - l|\Big].$$

Any c that witnesses the above formula is such that $c \approx a \neq c$ and, simultaneously, $\varepsilon \leq |{}^*fc - l|$. But $\dot{\varepsilon}$ is standard, hence ${}^*fc \not\approx l$. □

The following corollary is immediate.

2.23 Corollary For every $f : \mathbb{R} \to \mathbb{R}$ the following are equivalent

a. f is continuous;

b. ${}^*f(a) \approx {}^*f(c)$ for every pair of finite hyperreals such that $c \approx a$.

In Corollary 2.23, it is important to restrict c to *finite* hyperreals, otherwise we get a stronger property.

2.24 Proposition For every $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent

a. f is uniformly continuous;

b. ${}^*f(a) \approx {}^*f(b)$ for every pair of hyperreals such that $a \approx b$.

Proof. a⇒b. Assume a, then

1 $$\mathbb{R} \quad \vDash \quad \forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \Big[|x - y| < \delta \quad \to \quad |fx - fy| < \varepsilon \Big].$$

Let $a \approx b$. We want to show that $|{}^*f(a) - {}^*f(b)| < \dot{\varepsilon}$ for every positive standard $\dot{\varepsilon}$. Given a standard positive $\dot{\varepsilon}$, let $\dot{\delta}$ be a standard real obtained from 1. Now elementarity gives

2 $$\mathbb{R} \quad \vDash \quad \forall x, y \Big[|x - y| < \dot{\delta} \quad \to \quad |fx - fy| < \dot{\varepsilon} \Big].$$

Hence, in particular,

$${}^*\mathbb{R} \quad \vDash \quad |a - b| < \dot{\delta} \quad \to \quad |fa - fb| < \dot{\varepsilon}.$$

Since $a \approx b$ andd $\dot{\delta}$ is standard, $|a - b| < \dot{\delta}$ is true. Therefore $|{}^*f(a) - {}^*f(b)| < \dot{\varepsilon}$.

b⇒a. Negate a

3 $$\mathbb{R} \quad \vDash \quad \exists \varepsilon > 0\, \forall \delta > 0\, \exists x, y \Big[|x - y| < \delta \quad \wedge \quad \varepsilon \leq |fx - fy| \Big]$$

We want $a \approx b$ such that $\dot{\varepsilon} \leq |{}^*f(a) - {}^*f(b)|$ for some positive standard $\dot{\varepsilon}$. Let $\dot{\varepsilon}$ be a standard real that witnesses 3. Elementarity gives

$${}^*\mathbb{R} \quad \vDash \quad \forall \delta > 0\, \exists x, y \Big[|x - y| < \delta \quad \wedge \quad \dot{\varepsilon} \leq |fx - fy| \Big].$$

Pick an arbitrary infinitesimal $\dot{\delta} > 0$ and let $a, b \in {}^*\mathbb{R}$ be given by the formula above

$${}^*\mathbb{R} \quad \vDash \quad |a - b| < \delta \quad \wedge \quad \dot{\varepsilon} \leq |fa - fb|.$$

Since $\dot{\delta}$ is infinitesimal, we have $a \approx b$, as required.

The following proposition is an immediate consequence of Proposition 2.22.

2.25 Proposition For every unary function f and every standard a, the following are equivalent.

a. f is differentiable in a;

b. $\operatorname{st}\left(\dfrac{f(a) - f(a+h)}{h}\right)$ exists and is constant for all infinitesimal $h \neq 0$.

2.26 Exercise Every definable (possibly with parameters) subset of ${}^*\mathbb{R}$ that is bounded from above has a least upper bound.

2.27 Exercise Prove that if the function $f : \mathbb{R} \to \mathbb{R}$ is injective, then ${}^*fa$ is nonstantard whenever a is nonstandard.

2.28 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is a finite set;
2. ${}^*X = X$.

2.29 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is an open set in the usual topology on $\mathbb{R}$;
2. $b \approx a \in X \quad \Rightarrow \quad b \in {}^*X$ for every $b \in {}^*\mathbb{R}$.

2.30 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is closed in the usual topology on $\mathbb{R}$;
2. $a \in {}^*X \quad \Rightarrow \quad \operatorname{st} a \in X$ for every finite $a \in {}^*\mathbb{R}$.

2.31 Exercise Prove that the following are equivalent for every subset $X \subseteq \mathbb{R}$

1. X is bounded and closed in the usual topology on $\mathbb{R}$;
2. for every $b \in {}^*X$ there is an $a \in X$ such that $a \approx b$.

2.32 Exercise For what sets $X \subseteq \mathbb{R}$ does the following hold for every $a, b \in {}^*\mathbb{R}$?

$$b \approx a \in {}^*X \;\Rightarrow\; b \in {}^*X.$$

2.33 Exercise Prove that $|\mathbb{R}| \leq |{}^*\mathbb{Q}|$, that is, that the cardinality of ${}^*\mathbb{Q}$ is at least that of the continuum. (Hint: define an injective function $f : \mathbb{R} \to {}^*\mathbb{Q}$ by choosing a nonstandard rational in the monad of every standard real.)

2.34 Exercise Prove that every proper elementary extension of $\mathbb{Q}$ in the language of rings plus a relation symbol for every subset of $\mathbb{Q}$ has infinities and infinitesimals.

2.4 Embeddings and isomorphisms

Here we prove that isomorphic structures are elementarily equivalent and a few related results.

2.35 Definition An embedding of M into N is an injective total map $h : M \hookrightarrow N$ such that

1. $\quad a \in r^M \quad \Leftrightarrow \quad ha \in r^N \qquad$ for every $r \in L_{\text{rel}}$ and $a \in M^{n_r}$;
2. $\quad h\, f^M(a) \quad = \quad f^N(h\,a) \qquad$ for every $f \in L_{\text{fun}}$ and $a \in M^{n_f}$.

Note that when $c \in L_{\text{fun}}$ is a constant 2 reads $h\, c^M = c^N$. Therefore $M \subseteq N$ if and only if $\text{id}_M : M \to N$ is an embedding.

An surjective embedding is an isomorphism or, when domain and codomain coincide, an automorphism.

Condition 1 above and the assumption that h is injective can be summarized in the following

1′. $\quad M \vDash r(a) \quad \Leftrightarrow \quad N \vDash r(ha) \qquad$ for every $r \in L_{\text{rel}} \cup \{\doteq\}$ and every $a \in M^{n_r}$.

Note also that, by straightforward induction on syntax, from 2 we obtain

2′ $\quad h\, t^M(a) \quad = \quad t^N(h\,a) \qquad$ for every term $t(x)$ and every $a \in M^{|x|}$.

Combining 1′ and 2′ we obtain by straightforward induction on the syntax

3. $\quad M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(ha) \qquad$ for every $\varphi(x) \in L_{\text{qf}}$ and every $a \in M^{|x|}$.

Recall that we write L_{qf} for the set of quantifier-free formulas. It is worth noting that when $M \subseteq N$ and $h = \text{id}_M$ then 3 becomes

3′ $\quad M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(a) \qquad$ for every $\varphi(x) \in L_{\text{qf}}$ and for every $a \in M^{|x|}$.

In words this is summarized by saying that the truth of quantifier-free formulas is preserved under sub- and superstructure.

Finally, we prove that first order truth is preserved under isomorphism. We say that a map $h : M \to N$ fixes $A \subseteq M$ (pointwise) if $\text{id}_A \subseteq h$. An isomorphism that fixes A is also called an A-isomorphism.

2.36 Theorem If $h : M \to N$ is an isomorphism then for every $\varphi(x) \in L$

$$M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(ha) \text{ for every } a \in M^{|x|}$$

In particular, if h is an A-isomorphism then $M \equiv_A N$.

Proof. We proceed by induction on the syntax of $\varphi(x)$. When $\varphi(x)$ is atomic # holds by 3 above. Induction for the Boolean connectives is straightforward so we only need to consider the existential quantifier. Assume as induction hypothesis that

$$M \vDash \varphi(a, b) \quad \Leftrightarrow \quad N \vDash \varphi(ha, hb) \text{ for every } a \in M^{|x|} \text{ and } b \in M.$$

We prove that the equivalence in the theorem holds for the formula $\exists y\, \varphi(x, y)$.

$$\begin{aligned}
M \vDash \exists y\, \varphi(a, y) \quad &\Leftrightarrow \quad M \vDash \varphi(a, b) \text{ for some } b \in M \\
&\Leftrightarrow \quad N \vDash \varphi(ha, hb) \text{ for some } b \in M \qquad \text{(by induction hypothesis)} \\
&\Leftrightarrow \quad N \vDash \varphi(ha, c) \text{for some } c \in N \qquad (\Leftarrow \text{ by surjectivity}) \\
&\Leftrightarrow \quad N \vDash \exists y\, \varphi(ha, y).
\end{aligned}$$

□

2.37 Corollary If $h : M \to N$ is an isomorphism then for every $\varphi(x) \in L$.

$$h[\varphi(M)] \quad = \quad \varphi(N)$$

We can now give a few very simple examples of elementarily equivalent structures.

2.38 Example Let L be the language of strict orders. Consider the intervals of $\mathbb{R}$ as structures in the natural way. The intervals $[0,1]$ and $[0,2]$ are isomorphic, hence $[0,1] \equiv [0,2]$ follows from Theorem 2.36. Clearly, $[0,1]$ is a substructure of $[0,2]$. However $[0,1] \not\preceq [0,2]$, in fact the formula $\forall x\,(x \leq 1)$ holds in $[0,1]$ but is false in $[0,2]$. This shows that $M \subseteq N$ and $M \equiv N$ does not imply $M \preceq M$.

Now we prove that $(0,1) \preceq (0,2)$. By Exercise 2.16 it suffices to verify that for every finite $B \subseteq (0,1)$ we have $(0,1) \equiv_B (0,2)$. This follows again from Theorem 2.36 as $(0,1)$ and $(0,2)$ are B-isomorphic for every finite $B \subseteq (0,1)$.

For the sake of completeness we also give the definition of homomorphism.

2.39 Definition A homomorphism is a total map $h : M \to N$ such that

1. $a \in r^M \quad \Rightarrow \quad ha \in r^N$ for every $r \in L_{\text{rel}}$ and $a \in M^{n_r}$;
2. $h\,f^M(a) \quad = \quad f^N(h\,a)$ for every $f \in L_{\text{fun}}$ and $a \in M^{n_f}$.

Note that only one implication is required in 1.

2.40 Exercise Prove that if $h : N \to N$ is an automorphism and $M \preceq N$ then $h[M] \preceq N$.

2.41 Exercise Let L be the empty language. Let $A, D \subseteq M$. Prove that the following are equivalent

1. D is definable over A;
2. either D is finite and $D \subseteq A$, or $\neg D$ is finite and $\neg D \subseteq A$.

Hint: as structures are plain sets, every bijection $f : M \to M$ is an automorphism.

2.42 Exercise Prove that if $\varphi(x)$ is an existential formula and $h : M \hookrightarrow N$ is an embedding then

$$M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha) \qquad\qquad \text{for every } a \in M^{|x|}.$$

Recall that existential formulas as those of the form $\exists y\,\psi(x,y)$ for $\psi(x,y) \in L_{\text{qf}}$. Note that Theorem **??** proves that the property above characterizes existential formulas.

2.43 Exercise Let M be the model with domain $\mathbb{Z}$ in the language of additive groups which is interpreted in the natural way. Prove that there is no existential formula $\varphi(x)$ such that $\varphi(M)$ is the set of odd integers. Hint: use Exercise 2.42.

2.44 Exercise Let $\mathbb{Q}^+$ be the multiplicative group of positive rationals. Let M be the subgroup of the numbers of the form n/m for some odd integers m and n. Prove that $M \preceq \mathbb{Q}^+$. Hint: use the fundamental theorem of arithmetic and reason as in Example 2.38.

2.45 Exercise Prove that, in the language of strict orders, $\mathbb{R} \smallsetminus \{0\} \preceq \mathbb{R}$ and $\mathbb{R} \smallsetminus \{0\} \not\simeq \mathbb{R}$.

2.5 Quotient structure

The content of this section is mainly technical and only required later in the course. Its reading may be postponed.

If E is an equivalence relation on N we write $[c]_E$ for the equivalence class of $c \in N$. We use the same symbol for the equivalence relation on N^n defined as follow: if $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ are n-tuples of elements of N then $a\ E\ b$ means that $a_i\ E\ b_i$ holds for all i. It is easy to see that $b_1, \ldots, b_n \in [a, ..., a_n]_E$ if and only if $b_i \in [a_i]_E$ for all i. Therefore we use the notation $[a]_E$ for both the equivalence class of $a \in N^n$ and the tuple of equivalence classes $[a_1]_E, \ldots, [a_n]_E$.

2.46 Definition We say that the equivalence relation E on a structure N is a congruence if for every $f \in L_{\text{fun}}$

c1. $\quad a\ E\ b \quad \Rightarrow \quad f^N a\ E\ f^N b;$

When E is a congruence on N we write N/E for the a structure that has as domain the set of E-equivalence classes in N and the following interpretation of $f \in L_{\text{fun}}$ and $r \in L_{\text{rel}}$:

c2. $\quad f^{N/E}[a]_E \quad = \quad \big[f^N a\big]_E;$

c3. $\quad [a]_E \ \in\ r^{N/E} \quad \Leftrightarrow \quad [a]_E \cap r^N \neq \varnothing.$

We call N/E the quotient structure.

By c1 the quotient structure is well defined. The reader will recognize it as a familiar notion by the following proposition (which is not required in the following and requires the notion of homomorphism, see Definition 2.39. Recall that the kernel of a total map $h : N \to M$ is the equivalence relation E such that

$$a\ E\ b \quad \Leftrightarrow \quad ha = hb$$

for every $a, b \in N$.

2.47 Proposition Let $h : N \to M$ be a surjective homomorphism and let E be the kernel of h. Then there is an isomorphism k that makes the following diagram commute

$$\begin{tikzcd} N \arrow[r, "h"] \arrow[d, "\pi"'] & M \\ N/E \arrow[ur, "k"] & \end{tikzcd}$$

where $\pi : a \mapsto [a]_E$ is the projection map.

⚠ Quotients clutter the notation with brackets. To avoid the mess, we prefer to reason in N and tweak the satisfaction relation. Warning: this is not standard (though it is what everyone informally does). In a nutshell, we are going to define the satisfaction relation $N/E \overset{*}{\vDash} \varphi(a)$ as an abbreviation of $N/E \vDash \varphi([a]_E)$. Below we spell out this in details for the readers that feel it necessary.

Recall that in model theory, equality is not treated as a all other predicates. In fact, the interpretation of equality is fixed to always be the identity relation. In a few contexts is convinient to allow any congruence to interprete equality. This allows to work in N while thinking of N/E.

We define $N/E \overset{*}{\vDash}$ to be $N \vDash$ but with equality interpreted with E. The proposition below shows that this is the same thing as the regular truth in the quotient structure, $N/E \vDash$.

2.48 Definition For t_2, t_2 closed terms of $L(N)$ define

1* $\quad N/E \overset{*}{\vDash} t_1 = t_2 \quad\Leftrightarrow\quad t_1^N \, E \, t_2^N$

For t a tuple of closed terms of $L(N)$ and $r \in L_{\mathrm{rel}}$ a relation symbol

2* $\quad N/E \overset{*}{\vDash} r\, t \quad\Leftrightarrow\quad t^N \, E \, a \quad$ for some $a \in r^N$

Finally the definition is extended to all sentences $\varphi \in L(N)$ by induction in the usual way

3* $\quad N/E \overset{*}{\vDash} \neg\varphi \quad\Leftrightarrow\quad$ not $\; N/E \overset{*}{\vDash} \varphi$

4* $\quad N/E \overset{*}{\vDash} \varphi \wedge \psi \quad\Leftrightarrow\quad N/E \overset{*}{\vDash} \varphi$ and $N/E \overset{*}{\vDash} \psi$

5* $\quad N/E \overset{*}{\vDash} \exists x\, \varphi(x) \quad\Leftrightarrow\quad N/E \overset{*}{\vDash} \varphi(a) \quad$ for some $a \in N$.

Now, by induction on the syntax of formulas one can prove $\overset{*}{\vDash}$ does what required. In particular, $N/E \overset{*}{\vDash} \varphi(a) \leftrightarrow \varphi(b)$ for every $a \, E \, b$.

2.49 Proposition Let E be a congruence relation of N. Then the following are equivalent for every $\varphi(x) \in L$

1. $\quad N/E \quad \overset{*}{\vDash} \quad \varphi(a)$;
2. $\quad N/E \quad \vDash \quad \varphi([a]_E)$.

2.6 Completeness

A theory T is maximally consistent if it is consistent and there is no consistent theory S such that $T \subset S$. Equivalently, T contains every sentence φ consistent with T, that is, such that $T \cup \{\varphi\}$ is consistent. Clearly a maximally consistent theory is closed under logical consequences.

A theory T is complete if ccl$T$ is maximally consistent. Concrete examples will be given in the next chapters as it is not easy to prove that a theory is complete.

2.50 Proposition The following are equivalent

a. T is maximally consistent;

b. $T = \mathrm{Th}(M)$ for some structure M;

c. T is consistent and $\varphi \in T$ or $\neg\varphi \in T$ for every sentence φ.

Proof. To prove a⇒b, assume that T is consistent. Then there is $M \vDash T$. Therefore $T \subseteq \mathrm{Th}(M)$. As T is maximally consistent $T = \mathrm{Th}(M)$. Implication b⇒c is immediate. As for c⇒a note that if $T \cup \{\varphi\}$ is consistent then $\neg\varphi \notin T$ therefore $\varphi \in T$ follows from c. □

The proof of the proposition below is is left as an exercise for the reader.

2.51 Proposition The following are equivalent

a. T is complete;

b. there is a unique maximally consistent theory S such that $T \subseteq S$;

c. T is consistent and $T \vdash \mathrm{Th}(M)$ for every $M \vDash T$;

d. T is consistent and either $T \vdash \varphi$ or $T \vdash \neg\varphi$ for every sentence φ;

e. T is consistent and $M \equiv N$ for every pair of models of T.

2.52 Exercise Prove that the following are equivalent

a. T is complete;

b. for every sentence φ, o $T \vdash \varphi$ o $T \vdash \neg\varphi$ but not both.

By contrast prove that the following are *not* equivalent

a. T is maximally consistent;

b. for every sentence φ, o $\varphi \in T$ o $\neg\varphi \in T$ but not both.

2.53 Exercise Prove that if T has exactly 2 maximally consistent extension T_1 and T_2 then there is a sentence φ such that $T, \varphi \vdash T_1$ and $T, \neg\varphi \vdash T_2$. State and prove the generalization to finitely many maximally consistent extensions.

2.7 The Tarski-Vaught test

There is no natural notion of *smallest* elementary substructure containing a set of parameters A. The downward Löwenheim-Skolem, which we prove in the next section, is the best result that holds in full generality. Given an arbitrary $A \subseteq N$ we shall construct a model $M \preceq N$ containing A that is small in the sense of cardinality. The construction selects one by one the elements of M that are required to realise the condition $M \preceq N$. Unfortunately, Definition 2.12 supposes full knowledge of the truth in M and it may not be applied during the construction. The following lemma comes to our rescue with a property equivalent to $M \preceq N$ that only mention the truth in N.

2.54 Lemma (Tarski-Vaught test) For every $A \subseteq N$ the following are equivalent

1. A is the domain of a structure $M \preceq N$;
2. for every formula $\varphi(x) \in L(A)$, with $|x| = 1$,
   $N \vDash \exists x\, \varphi(x) \quad \Rightarrow \quad N \vDash \varphi(b)$ for some $b \in A$.

Proof. 1⇒2.

$$\begin{aligned} N \vDash \exists x\, \varphi(x) \quad &\Rightarrow \quad M \vDash \exists x\, \varphi(x) \\ &\Rightarrow \quad M \vDash \varphi(b) \qquad \text{for some } b \in M \\ &\Rightarrow \quad N \vDash \varphi(b) \qquad \text{for some } b \in A. \end{aligned}$$

2⇒1. Firstly, note that A is the domain of a substructure of N, that is, $f^N a \in A$ for every $f \in L_{\text{fun}}$ and every $a \in A^{n_f}$. In fact, this follows from 2 with $fa = x$ for $\varphi(x)$.

Write M for the substructure of N with domain A. By induction on the syntax we prove that for every $\xi(x) \in L$

$$M \vDash \xi(a) \quad \Leftrightarrow \quad N \vDash \xi(a) \qquad\qquad \text{for every } a \in M^{|x|}.$$

If $\xi(x)$ is atomic the claim follows from $M \subseteq N$ and the remarks underneath Definition 2.35. The case of Boolean connectives is straightforward, so only the existential quantifier requires a proof. So, let $\xi(x)$ be the formula $\exists y\, \psi(x,y)$ and assume the induction hypothesis holds for $\psi(x,y)$

$$\begin{aligned} M \vDash \exists y\, \psi(a,y) \quad &\Leftrightarrow \quad M \vDash \psi(a,b) &&\text{for some } b \in M \\ &\Leftrightarrow \quad N \vDash \psi(a,b) &&\text{for some } b \in M \\ &\Leftrightarrow \quad N \vDash \exists y\, \psi(a,y). \end{aligned}$$

The second equivalence holds by induction hypothesis, in the last equivalence we use 2 for the implication $\Leftarrow$. □

2.8 Downward Löwenheim-Skolem

The main theorem of this section was proved by Löwenheim at the beginning of the last century. Skolem gave a simpler proof immediately afterwards. At the time, the result was perceived as paradoxical.

A few years earlier, Zermelo and Fraenkel provided a formalization of set theory in a first order language. The downward Löwenheim-Skolem theorem implies the existence of an infinite countable model M of set theory: this is the so-called Skolem paradox. The existence of M seems paradoxical because, in particular, a sentence that formalises the axiom of power set holds in M. Therefore M contains an element b which, in M, is the set of subsets of the natural numbers. But the set of elements of b is a subset of M, and therefore it is countable.

In fact, this is not a contradiction, because the expression *all subsets of the natural numbers* does not have the same meaning in M as it has in the real world. The notion of cardinality, too, acquires a different meaning. In the language of set theory, there is a first order sentence that formalises the fact that b is uncountable: the sentence says that there is no bijection between b and the natural numbers. Therefore the bijection between the elements of b and the natural numbers (which exists in the real world) does not belong to M. The notion of equinumerosity has a different meaning in M and in the real world, but those who live in M cannot realise this.

2.55 Theorem (Downward Löwenheim-Skolem) Let N be an infinite structure and fix some set $A \subseteq N$. Then there is a structure M of cardinality $\leq |L(A)|$ such that $A \subseteq M \preceq N$.

Proof. Set $\lambda = |L(A)|$. Below we construct a chain $\langle A_i : i < \omega \rangle$ of subsets of N. The chain begins at $A_0 = A$. Finally we set $M = \bigcup_{i<\omega} A_i$. All A_i will have cardinality $\leq \lambda$ so $|M| \leq \lambda$ follows.

Now we construct A_{i+1} given A_i. Assume as induction hypothesis that $|A_i| \leq \lambda$. Then $|L(A_i)| \leq \lambda$. For some fixed variable x let $\langle \varphi_k(x) \ : \ k < \lambda \rangle$ be an enumeration of the formulas in $L(A_i)$ that are consistent in N. For every k pick $a_k \in N$ such that

$N \vDash \varphi_k(a_k)$. Define $A_{i+1} = A_i \cup \{a_k \ : \ k < \lambda\}$. Then $|A_{i+1}| \leq \lambda$ is clear.

We use the Tarski-Vaught test to prove $M \preceq N$. Suppose $\varphi(x) \in L(M)$ is consistent in N. As finitely many parameters occur in formulas, $\varphi(x) \in L(A_i)$ for some i. Then $\varphi(x)$ is among the formulas we enumerated at stage i and $A_{i+1} \subseteq M$ contains a solution of $\varphi(x)$. □

We will need to adapt the construction above to meet more requirements on the model M. To better control the elements that end up in M it is convenient to add one element at the time (above we add λ elements at each stage). We need to enumerate formulas with care if we want to complete the construction by stage λ.

Say, $\varphi(x)$ is the k-th formula in the enumeration above.

Second proof of the downward Löwenheim-Skolem Theorem. Let $\pi : \lambda^2 \to \lambda$ be a bijection such that $j, k \leq \pi(j, k)$ for all $j, k < \lambda$. Suppose we have defined the sets A_j for every $j \leq i$ and let $\langle \varphi_{j,k}(x) \ : \ k < \lambda \rangle$ be an enumeration of the consistent formulas of $L(A_j)$. Let $j, k \leq i$ be such that $\pi(j, k) = i$. Let b be a solution of the formula $\varphi_{j,k}(x)$ and define $A_{i+1} = A_i \cup \{b\}$.

We use Tarski-Vaught test to prove $M \preceq N$. Let $\varphi(x) \in L(M)$ be consistent in N. Then $\varphi(x) \in L(A_j)$ for some j. Then $\varphi(x) = \varphi_{j,k}$ for some k. Hence a witness of $\varphi(x)$ is enumerated in M at stage $\pi(j, k) + 1$. □

2.56 Exercise Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Hint: adapt the construction used to prove the downward Löwenheim-Skolem Theorem.

2.57 Exercise The language contains only $<$, the relation of strict order. Prove that there is a countable ordinal α that is an elementary substructure of ω_1, the first uncountable ordinal.

2.9 Elementary chains

An elementary chain is a chain $\langle M_i \ : \ i < \lambda \rangle$ of structures such that $M_i \preceq M_j$ for every $i < j < \lambda$. The union (or limit) of the chain is the structure with as domain the set $\bigcup_{i<\lambda} M_i$ and as relations and functions the union of the relations and functions of M_i. It is plain that all structures in the chain are substructures of the limit.

2.58 Lemma Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of structures. Let N be the union of the chain. Then $M_i \preceq N$ for every i.

Proof. By induction on the syntax of $\varphi(x) \in L$ we prove

$$M_i \vDash \varphi(a) \ \Leftrightarrow \ N \vDash \varphi(a) \qquad \text{for every } i < \lambda \text{ and every } a \in M_i^{|x|}$$

As remarked in 3' of Section 2.4, the claim holds for quantifier-free formulas. Induction for Boolean connectives is straightforward so we only need to consider the existential quantifier

$$M_i \vDash \exists y \, \varphi(a, y) \ \Rightarrow \ M_i \vDash \varphi(a, b) \qquad \text{for some } b \in M_i.$$

$$\Rightarrow \quad N \vDash \varphi(a, b) \qquad \text{for some } b \in M_i \subseteq N$$

where the second implication follows from the induction hypothesis. Vice versa

$$N \vDash \exists y\, \varphi(a, y) \;\Rightarrow\; N \vDash \varphi(a, b) \qquad \text{for some } b \in N$$

Without loss of generality we can assume that $b \in M_j$ for some $j \geq i$ and obtain

$$\Rightarrow \quad M_j \vDash \varphi(a, b) \qquad \text{for some } b \in M_j$$

Now apply the induction hypothesis to $\varphi(x, y)$ and M_j

$$\Rightarrow \quad M_j \vDash \exists y\, \varphi(a, y)$$

$$\Rightarrow \quad M_i \vDash \exists y\, \varphi(a, y)$$

where the last implication holds because $M_i \preceq M_j$. □

2.59 Exercise Let $\langle M_i : i \in \lambda \rangle$ be an chain of elementary substructures of N. Let M be the union of the chain. Prove that $M \preceq N$ and note that Lemma 2.58 is not required.

2.60 Exercise Give an alternative proof of Exercise 2.56 using the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two countable chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$. The required model is $K = \bigcup_{i \in \omega} K_i$. In fact it is easy to check that $K \cap M = \bigcup_{i \in \omega} M_i$.

Chapter 3

Types and morphisms

There is a lot of notation in this chapter. The reader may skim through and return to it when necessary. In Section 3.1 and 3.2 we introduce distributive lattices and prime filters and prove Stone's representation theorem for distributive lattices. In Section 3.3 we discuss lattices that arise from sets of formulas and their prime filters (prime types). These sections are only required for the discussion of Hilbert's Nullstellensatz, see Section 8.7 below.

3.1 Semilattices and filters

A preorder is a set $\mathbb{P}$ with a reflexive and transitive relation. We usually denote the preorder by $\leq$. If $A, B \subseteq \mathbb{P}$ we write $A \leq B$ if $a \leq b$ for every $a \in A$ and $b \in B$. We write $a \leq B$ and $A \leq b$ for $\{a\} \leq B$ and $A \leq \{b\}$ respectively.

Quotienting a preorder by the equivalence relation

$$a \sim b \quad \Leftrightarrow \quad a \leq b \quad \text{and} \quad b \leq a$$

gives a (partial) order. We often do not distinguish between a preorder and the partial order associated to it. Preorders are very common; here we are mainly interested in preorders induced by the relation of logical consequence

$$\varphi \leq \psi \quad \Leftrightarrow \quad \varphi \vdash \psi$$

or, more generally, induced by the relation of logical consequence modulo a theory T, that is,

$$\varphi \leq \psi \quad \Leftrightarrow \quad T \cup \{\varphi\} \vdash \psi.$$

A partial order $\mathbb{P}$ is a lower semilattice if for each pair $a, b \in \mathbb{P}$ there is a maximal element c such that $c \leq \{a, b\}$. We call c the meet of a and b. The meet is unique and is denoted by $a \wedge b$. Dually, a partial order is an upper semilattice if for each pair of elements a and b there is a minimal element c such that $\{a, b\} \leq c$. This c is called the join of a and b. The join is unique and is denoted by $a \vee b$. A lattice is simultaneously a lower and an upper semilattice.

For instance, the family of subsets of $\mathbb{R}$ that are finite union of left-open, right-closed intervals is an upper-semilattice w.r.t. the relation of inclusion. However, it is not a lattice.

An element c such that $c \leq \mathbb{P}$ is called a lower bound or a bottom. An element such that $\mathbb{P} \leq c$ is called an upper bound or a top. Lower and upper bounds are unique and will be denoted by 0, respectively 1. Other symbols common in the literature are $\perp$, respectively $\top$. A semilattice is bounded if it has both an upper and a lower bound.

For the rest of this section we assume that $\mathbb{P}$ is a bounded lower semilattice.

The meet is associative and commutative

$$\begin{aligned}(a \wedge b) \wedge c &= a \wedge (b \wedge c)\\ a \wedge b &= a \wedge b.\end{aligned}$$

Hence we may unambiguously write $a_1 \wedge \cdots \wedge a_n$. When $C \subseteq \mathbb{P}$ is finite, we write $\wedge C$ for the meet of all the elements of C. We agree that $\wedge\varnothing = 1$.

In an upper semilattice, the dual properties hold for the join. We write $\vee C$ for the join of all elements in C and we agree that $\vee\varnothing = 0$.

A filter of $\mathbb{P}$ is a non-empty set $F \subseteq \mathbb{P}$ that satisfies the following for all $a, b \in \mathbb{P}$

f1. $a \in F$ and $a \leq b \Rightarrow b \in F$

f2. $a, b \in F \Rightarrow a \wedge b \in F.$

We say that F is a proper filter if $F \neq \mathbb{P}$, equivalently if $0 \notin F$. We say that F is principal if $F = \{b \ : \ a \leq b\}$ for some $a \in \mathbb{P}$. More precisely, we say that F is the principal filter generated by a. A proper filter F is maximal if there is no filter H such that $F \subset H \subset \mathbb{P}$.

A set $B \subseteq \mathbb{P}$ has the finite intersection property if $\wedge C \neq \varnothing$ for every finite $C \subseteq B$. For $B \subseteq \mathbb{P}$ we define the filter generated by B to be the intersection of all the filters containing B. It is easy to verify that this is indeed a filter. When B is a finite set, the filter generated by B is the principal filter generated by $\wedge B$. In general we have the following.

3.1 Proposition For every $B \subseteq \mathbb{P}$, the filter generated by B is the set

$$\{a \ : \ \wedge C \leq a \ \text{ for some finite non-empty } \ C \subseteq B\}.$$

In particular B is contained in a proper filter if and only if it has the finite intersection property.

We say that a filter F is maximal relative to c if $c \notin F$ and $c \in H$ for every $H \supset F$. So, a filter F is maximal if it is maximal relative to 0. We say that F is relatively maximal when it is maximal relative to some c.

3.2 Proposition Let $B \subseteq \mathbb{P}$ and let $c \in \mathbb{P}$. If $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$, then B is contained in a maximal filter relative to c.

Proof. Let $\mathcal{F}$ be the set of filters F such that $B \subseteq F$ and $c \notin F$. By Proposition 3.1, $\mathcal{F}$ is non-empty. It is immediate that $\mathcal{F}$ is closed under unions of arbitrary chains. Then, by Zorn's lemma, $\mathcal{F}$ has a maximal element. □

3.3 Exercise Let $B \subseteq \mathbb{P}$. Let $c \in \mathbb{P}$ be such that $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$. Prove that the following are equivalent

1. B is a maximal filter relative to c;
2. $a \notin B \ \Rightarrow \ b \wedge a \leq c$ for some $b \in B$.

3.4 Exercise Let $F \subseteq \mathbb{P}$ be a non-principal filter. Is F always contained in a maximal non-principal filter?

3.2 Distributive lattices and prime filters

Let $\mathbb{P}$ be a lattice. We say that $\mathbb{P}$ is distributive if for every $a, b, c \in \mathbb{P}$

$$
\begin{aligned}
a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\
a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c)
\end{aligned}
$$

Throughout this section we assume that $\mathbb{P}$ is a bounded distributive lattice.

A proper filter F is prime if for every $a, b \in \mathbb{P}$

$$a \vee b \in F \quad \Rightarrow \quad a \in F \quad \text{or} \quad b \in F.$$

3.5 Proposition Every relatively maximal filter of $\mathbb{P}$ is prime.

Proof. Let F be maximal relative to c and assume that $a \notin F$ and $b \notin F$. Then, by Exercise 3.3, there are $d_1, d_2 \in F$ such that $d_1 \wedge a \leq c$ and $d_2 \wedge b \leq c$. Let $d = d_1 \wedge d_2$. Then $d \wedge a \leq c$ and $d \wedge b \leq c$ and therefore $(d \wedge a) \vee (d \wedge b) \leq c$. Hence, by distributivity, $d \wedge (a \vee b) \leq c$. Then $a \vee b \notin F$. □

The Stone space of $\mathbb{P}$ is a topological space that we denote by $S(\mathbb{P})$. The points of $S(\mathbb{P})$ are the prime filters of $\mathbb{P}$. The closed sets of the Stone topology are arbitrary intersections of sets of the form

$$[a]_{\mathbb{P}} = \Big\{ F \ : \ \text{prime filter such that } a \in F \Big\}.$$

for $a \in \mathbb{P}$. In other words, the sets above form a base of closed sets of the Stone topology. Using 1 and 3 in the following proposition the reader can easily check that this is indeed a base for a topology.

3.6 Proposition For every $a, b \in \mathbb{P}$ we have

1. $[0]_{\mathbb{P}} = \varnothing$;
2. $[1]_{\mathbb{P}} = S(\mathbb{P})$;
3. $[a]_{\mathbb{P}} \cup [b]_{\mathbb{P}} = [a \vee b]_{\mathbb{P}}$;
4. $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = [a \wedge b]_{\mathbb{P}}$.

Proof. The verification is immediate. Only 3 requires that the filters in $S(\mathbb{P})$ are prime. □

The closed subsets of $S(\mathbb{P})$ ordered by inclusion form a distributive lattice. The following is a representation theorem for distributive lattices.

3.7 Theorem The map $a \mapsto [a]_{\mathbb{P}}$ is an embedding of $\mathbb{P}$ in the lattice of the closed subsets of $S(\mathbb{P})$. In particular

1. $0 \mapsto \varnothing$;
2. $1 \mapsto S(\mathbb{P})$;
3. $a \vee b \mapsto [a]_{\mathbb{P}} \cup [b]_{\mathbb{P}}$;
4. $a \wedge b \mapsto [a]_{\mathbb{P}} \cap [b]_{\mathbb{P}}$.

Proof. It is immediate that the map above preserves the order and Proposition 3.6 shows that it preserves the lattice operations. We prove that the map is injective.

Let $a \neq b$, say $a \nleq b$. We claim that $[a]_{\mathbb{P}} \nsubseteq [b]_{\mathbb{P}}$. There is a filter F that contains a and is maximal relative to b. By Proposition 3.5 such an F is prime. Then $F \in [a]_{\mathbb{P}} \setminus [b]_{\mathbb{P}}$. □

3.8 Theorem With the Stone topology, $S(\mathbb{P})$ is a compact space.

Proof. Let $\langle [a_i]_{\mathbb{P}} : i \in I \rangle$ be basic closed sets such that for every finite $J \subseteq I$

a. $$\bigcap_{i \in J} [a_i]_{\mathbb{P}} \neq \varnothing.$$

We claim that

b. $$\bigcap_{i \in I} [a_i]_{\mathbb{P}} \neq \varnothing.$$

By 4 of Proposition 3.6 and a we obtain that $\wedge C \nleq 0$ for every finite $C \subseteq \{a_i : i \in I\}$. By Proposition 3.2, there is a maximal (relative to 0) filter containing $\{a_i : i \in I\}$. By Proposition 3.5, such filter is prime and it belongs to the intersection in b. □

Let $a, b \in \mathbb{P}$. If $a \wedge b = 0$ and $a \vee b = 1$, we say that b is the complement of a (and vice versa). The complement of an element need not exist. If the complement exists it is unique, the complement of a is denoted by $\neg a$.

3.9 Lemma Let $U \subseteq S(\mathbb{P})$ be a clopen set. Then $U = [a]_{\mathbb{P}}$ for some $a \in \mathbb{P}$, and $\neg a$ exists.

Proof. As both U and $S(\mathbb{P}) \setminus U$ are closed, for some sets $A, B \subseteq \mathbb{P}$

$$\bigcap_{x \in A} [x]_{\mathbb{P}} = U$$

$$\bigcap_{y \in B} [y]_{\mathbb{P}} = S(\mathbb{P}) \setminus U.$$

By compactness, that is Theorem 3.8, there are some finite $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap_{x \in A_0} [x]_{\mathbb{P}} \cap \bigcap_{y \in B_0} [y]_{\mathbb{P}} = \varnothing$$

Let $a = \wedge A_0$ and $b = \wedge B_0$. From claim 4 of Proposition 3.6 we obtain $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = \varnothing$. Therefore $U \subseteq [a]_{\mathbb{P}} \subseteq S(\mathbb{P}) \setminus [b]_{\mathbb{P}} \subseteq U$. Hence $U = [a]_{\mathbb{P}}$ and $b = \neg a$. □

A Boolean algebra is a bounded distributive lattice where every element has a complement. In a Boolean algebra, the sets $[a]_{\mathbb{P}}$ are clopen and they form also a base of open set of the topology of $S(\mathbb{P})$. A topology that has a base of clopen sets is called zero-dimensional. By the following proposition the Stone topology of a Boolean algebra is Hausdorff.

A proper filter of a Boolean algebra $\mathbb{P}$ is an ultrafilter if either $a \in F$ or $\neg a \in F$ for every $a \in \mathbb{P}$.

3.10 Proposition Let $\mathbb{P}$ be a Boolean algebra. Then the following are equivalent

1. F is maximal;
2. F is prime;
3. F is an ultrafilter.

Proof. For 2⇒3 observe that $a \vee \neg a \in F$. The rest is immediate. □

The most natural example of Boolean algebra is $\mathcal{P}(I)$ where I is any set. Union and intersection are the join, respectively meet of the algebra. We say filter on I for filter of $\mathcal{P}(I)$.

3.11 Exercise Let I be infinite. Let $F = \{a \subseteq I \ : \ |I \smallsetminus a| < \omega\}$. Then F is a filter on I which is called he Fréchet's filter. Prove that every non-principal ultrafilter on I contains Fréchet's filter.

3.12 Exercise Prove that the stone topology on $S(\mathbb{P})$ has a base of open compact sets.

3.13 Exercise Suppose we had defined $S(\mathbb{P})$ as the set of relatively maximal filters. What could possibly go wrong ?

3.3 Types as filters

In this section we work with a fixed set of formulas Δ all with free variables among those of some fixed tuple x. The tuple x is not displayed in the notation. Subsets of Δ are called Δ-types.

We associate to Δ a bounded lattice $\mathbb{P}(\Delta)$. This is the closure under conjunction and disjunction of the formulas in $\Delta \cup \{\bot, \top\}$. The (pre)order relation in $\mathbb{P}(\Delta)$ is given by

$$\psi \leq \varphi \quad \Leftrightarrow \quad T \cup \{\psi\} \ \vdash \ \varphi,$$

for some fixed theory T. In this section, to lighten notation, we absorb T in the symbol $\vdash$. If p is a Δ-type we denote by $\langle p \rangle_{\mathbb{P}}$ the filter in $\mathbb{P}(\Delta)$ generated by p.

A few propositions in this section requires the compactness theorem (for types), Theorem 5.7 which is only proved below.

3.14 Lemma For every Δ-type p

$$\langle p \rangle_{\mathbb{P}} \quad = \quad \Big\{ \ \varphi \ \in \ \mathbb{P}(\Delta) \quad : \quad p \ \vdash \ \varphi \ \Big\}.$$

In particular p is consistent if and only if $\langle p \rangle_{\mathbb{P}}$ is a proper filter.

Proof. Inclusion $\subseteq$ is clear. If $p \vdash \varphi$, by compactness, $\psi \vdash \varphi$ for some formula ψ that is conjunction of formulas in p. Then $\varphi \in \langle p \rangle_{\mathbb{P}}$ follows from $\psi \in \langle p \rangle_{\mathbb{P}}$ and $\psi \leq \varphi$. □

We say that $p \subseteq \Delta$ is a principal Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a principal. The following lemma is an immediate consequence of the Compactness Theorem. Note that in 3 the formula φ is arbitrary, possibly not even in $\mathbb{P}(\Delta)$.

3.15 Lemma For every Δ-type p the following are equivalent

1. p is principal;
2. $\psi \ \vdash \ p \ \vdash \ \psi$ for some formula ψ (here ψ is any formula, it need not be in Δ);
3. $\varphi \ \vdash \ p$ where φ is conjunction of formulas in p.

Proof. Implications 1⇒2 is immediate by Lemma 3.14. To prove 2⇒3 suppose $\psi \vdash p \vdash \psi$. Apply compactness to obtain a formula φ, conjunction of formulas in p, such that $\varphi \vdash \psi$. Implications 3⇒1is trivial. □

3.16 Definition We say that $p \subseteq \Delta$ is a prime Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a prime filter. We say that p is a complete Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a maximal filter.

Though in general neither Δ nor $\mathbb{P}(\Delta)$ are closed under negation, Lemma 3.14 has the following consequence.

3.17 Proposition For every consistent Δ-type p the following are equivalent

1. p is complete;
2. p is consistent and either $p \vdash \varphi$ or $p \vdash \neg\varphi$ for every formula $\varphi \in \Delta$.

Proof. 2⇒1. Assume 2. As p is consistent, $\langle p \rangle_{\mathbb{P}}$ is a proper filter. To prove that it is maximal suppose $\varphi \notin \langle p \rangle_{\mathbb{P}}$. Then $p \nvdash \varphi$ and from 2 it follows that $p \vdash \neg\varphi$. Hence no proper filter contains $p \cup \{\varphi\}$.

1⇒2. As $\langle p \rangle_{\mathbb{P}}$ is proper, p is consistent. Suppose $p \nvdash \varphi$. Then $\varphi \notin \langle p \rangle_{\mathbb{P}}$ and, as $\langle p \rangle_{\mathbb{P}}$ is maximal, $p \cup \{\varphi\}$ generates the improper filter. Then $p \cup \{\varphi\}$ is inconsistent. Hence $p \vdash \neg\varphi$. □

Given a model M and a tuple $c \in M^x$ the Δ-type of c in M is the sets

$$\Delta\text{-tp}_M(c) \quad = \quad \Big\{ \varphi(x) \in \Delta \ : \ M \vDash \varphi(c) \Big\}$$

When the model M is clear from the context we omit the subscript. When x and c are the empty tuple, we write $\mathrm{Th}_\Delta(M)$ for $\Delta\text{-tp}_M(c)$.

3.18 Lemma For every Δ-type p the following are equivalent

1. p is prime;
2. $p \ \cup \ \{\neg\varphi \ : \ \varphi \ \in \ \Delta \ \text{ such that } \ p \nvdash \varphi\}$ is consistent;
3. $\{\varphi \in \Delta : p \vdash \varphi\} \ = \ \Delta\text{-tp}_M(c)$ for some model M and some tuple $c \in M^x$.

Proof. Implications 2⇒3⇒1 are clear, we prove 1⇒2. By compactness if the type in 2 is inconsistent then there are finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Delta$ such that $p \nvdash \varphi_i$ and

$$p \ \vdash \ \bigvee_{i=1}^{n} \varphi_i.$$

hence p is not prime. □

The following corollary is immediate. It simplifies the task of verifying that a given type is prime.

3.19 Corollary For every Δ-type p the following are equivalent

1. p is prime;
2. $p \vdash \bigvee_{i=1}^{n} \varphi_i \quad \Rightarrow \quad p \vdash \varphi_i$ for some $i \leq n$, for every n and every $\varphi_1, \ldots, \varphi_n \in \Delta$.

The set Δ above contains only formulas with variables among those of the tuple x. In the following it is convenient to consider sets Δ that are closed under substitution of variables with any other variable. The set of prime Δ-types is denoted by $S(\Delta)$, and we write $S_x(\Delta)$ when we restrict to types in with variables among those of the tuple x.

The most common Δ used in the sequel is the set of all formulas in $L(A)$. Then the underlying theory is $\mathrm{Th}(M/A)$ for some given model M containing A. In this case we write $\mathrm{tp}_M(c/A)$ for $\Delta\text{-tp}_M(c)$ or, when A is empty, $\mathrm{tp}_M(c)$. The set $S_x(\Delta)$ is denoted by $S_x(A)$. The topology on $S_x(A)$ is generated by the clopen

$$[\varphi(x)] \quad = \quad \Big\{p \in S_x(A) \ : \ \varphi(x) \in p\Big\}.$$

Frequently Δ is the set of all formulas of a given syntactic form. Then we use a more suggestive notation as summarized below.

3.20 Notation The following are some of the most common Δ-types and Δ-theories

1. $\text{at-tp}(c)$, $\quad \mathrm{Th}_{\mathrm{at}}(M)$ $\quad$ when $\Delta = L_{\mathrm{at}}$
2. $\text{at}^{\pm}\text{-tp}(c)$, $\quad \mathrm{Th}_{\mathrm{at}^{\pm}}(M)$ $\quad$ when $\Delta = L_{\mathrm{at}^{\pm}}$
3. $\text{qf-tp}(c)$, $\quad \mathrm{Th}_{\mathrm{qf}}(M)$ $\quad$ when $\Delta = L_{\mathrm{qf}}$.

The types/theories in 2 and 3 are logically equivalent. Most used is the theory $\mathrm{Th}_{\mathrm{at}^{\pm}}(M/M)$ which is called the diagram of M and has a dedicated symbol: $\mathrm{Diag}(M)$. The theory $\mathrm{Th}(M/M)$ is called the elementary diagram of M.

3.21 Remark Let $A \subseteq M \cap N$. The following are equivalent

1. $N \vDash \mathrm{Diag}\langle A \rangle_M$;
2. $\langle A \rangle_M$ is a substructure of N.

3.4 Morphisms

First we set the meaning of the word map.

3.22 Definition A map consists a triple $f : M \to N$ where

1. M is a set (usually a structure) called the domain of the map;
2. N is a set (usually a structure) called the codomain the map;
3. f is a function with domain of definition $\mathrm{dom}\, f \subseteq M$ and image $\mathrm{rng}\, f \subseteq N$.

By cardinality of $f : M \to N$ we understand the cardinality of the function f.

If $\operatorname{dom} f = M$ we say that the map is total; if $\operatorname{rng} f = N$ we say that it is surjective. The composition of two maps and the inverse of a map are defined in the obvious way.

3.23 Definition Let Δ be a set of formulas. The map $h : M \to N$ is a Δ-morphism if it preserves the truth of all formulas in Δ. By this we mean that

p. $\quad M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha) \qquad$ for every $\varphi(x) \in \Delta$ and every $a \in (\operatorname{dom} h)^x$.

When h is total we we say it is a Δ-embedding.

The following is easy remark is useful to simplify notation.

3.24 Remark When Δ is closed under substitution of free variables it is convenient to rewrite p using a fixed tuple a that enumerates $\operatorname{dom} h$ and a fixed tuple a valiables x of the same length. Then condition p can be rephrased as

p′. $\quad M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha) \qquad$ for every $\varphi(x) \in \Delta$.

When $\Delta = L$ we say elementary map and elementary embedding for Δ-morphism, respectively Δ-embedding. When $\Delta = L_{\mathrm{at}}$ we say partial homomorphism and when $\Delta = L_{\mathrm{at}^{\pm}}$ we say either partial embedding or partial isomorphism. The reason for the latter name is explained in Remark 3.25. It is immediate to verify that a partial embedding which is total is an embedding as in Definition 2.35. Similarly, a partial embedding which is total and surjective is an isomorphism. The precise connection between partial and total homo/iso-morphisms is discussed in following remark.

3.25 Remark For every map $h : M \to N$ the following are equivalent

1. $h : M \to N$ is a partial isomorphism;
2. there is an isomorphism $k : \langle \operatorname{dom} h \rangle_M \to \langle \operatorname{rng} h \rangle_N$ that extends h.

Moreover, this extension is unique. The equivalence holds replacing isomorphism by homomorphism (in this case, in 2 we obtain a surjection). The extension k is obtained defining $k\big(t(a)\big) = t(ha)$ for every term $t(x)$.

A similar fact holds for partial homomorphims if we replace isomorphism by epimorphism, i.e. surjective homomorphism.

We use Δ-morphisms to compare, locally, two structures. There are different ways to do this, in the proposition below we list a few synonymous expressions. But first some more notation. When x is a fixed tuple of variables, $a \in M^x$ and $b \in N^x$ we write

$M, a \;\Rrightarrow_\Delta\; N, b \qquad$ if $M \vDash \varphi(a) \Rightarrow N \vDash \varphi(b)$ for every $\varphi(x) \in \Delta$.

$M, a \;\equiv_\Delta\; N, b \qquad$ if $M \vDash \varphi(a) \Leftrightarrow N \vDash \varphi(b)$ for every $\varphi(x) \in \Delta$.

The following equivalences are immediate and will be used without explicit reference.

3.26 Proposition For every given set of formulas Δ and every map $h : M \to N$ the following are equivalent

1. $h : M \to N$ is a Δ-morphism;
2. $M, a \Rrightarrow_\Delta N, ha$ for every $a \in (\operatorname{dom} h)^x$;
3. $\Delta\text{-tp}_M(a) \subseteq \Delta\text{-tp}_N(ha)$ for every $a \in (\operatorname{dom} h)^x$;
4. $N, ha \vDash p(x)$ for every $a \in (\operatorname{dom} h)^x$ and $p(x) = \Delta\text{-tp}_M(a)$.

As in Remark 3.24, one can equivalently require 2, 3, and 4 for some/any fixed tuple a that enumerates $\operatorname{dom} h$.

3.27 Remark Condition p in Definition 3.23 applies to tuples x of any length, in particular to the empty tuple. In this case $\varphi(x)$ is a sentence, $a \in (\operatorname{dom} h)^0 = \{\varnothing\}$ is the empty tuple, and p asserts that $\mathrm{Th}_\Delta(M) \subseteq \mathrm{Th}_\Delta(N)$. When $h = \varnothing$ this is actually all that p says. In fact $\varnothing^x = \varnothing$ unless $|x| = 0$. Still, $\mathrm{Th}_\Delta(M) \subseteq \mathrm{Th}_\Delta(N)$ may be a non trivial requirement.

3.28 Definition We call $\mathrm{Th}_\Delta(M)$ the Δ-theory of M. We say that the theory T is Δ-complete if $\mathrm{Th}_\Delta(M) = \mathrm{Th}_\Delta(N)$ for all $M, N \vDash T$. In other words, for any pair of models of T, the empty map $\varnothing : M \to N$ is a Δ-morphism. When T is $L_{\mathrm{at}^\pm}$-complete we say that T decides the characteristic of its models (by analogy with rings and fields). Note that when T decides the characteristic of its models, we have $\langle\varnothing\rangle_M \simeq \langle\varnothing\rangle_N$ for all $M, N \vDash T$.

We conclude this section with a couple of propositions that break Theorem 2.36 into parts. More interestingly, in Chapter **??** we shall prove a sort of converse of Propositions 3.30 and 3.31.

It is interesting to note that there is a relation between certain properties of Δ-morphisms and the closure of Δ under logical connectives. When $\mathsf{C} \subseteq \{\forall, \exists, \neg, \vee, \wedge\}$ is a set of connectives, we write $\mathsf{C}\Delta$ for the closure of Δ with respect to the connectives in C. We write $\neg\Delta$ for the set containing the negation of the formulas in Δ. Warning: do not confuse $\neg\Delta$ with $\{\neg\}\Delta$. Up to logical equivalence $\{\neg\}\Delta$ coincides with $\Delta \cup \neg\Delta$.

It is clear that Δ-morphisms are $\{\wedge, \vee\}\Delta$-morphisms.

3.29 Proposition For every given set of formulas Δ and every injective map $h : M \to N$ the following are equivalent

a. $h : M \to N$ is a $\neg\Delta$-morphism;

b. $h^{-1} : M \to N$ is a Δ-morphism.

3.30 Proposition For every set of formulas Δ, every total Δ-morphism $h : M \to N$ is a $\{\exists\}\Delta$-morphism.

Proof. Formulas in $\{\exists\}\Delta$ have the form $\exists y\, \varphi(x, y)$ where y is a finite tuples of variables and $\varphi(x, y) \in \Delta$. For every tuple $a \in (\operatorname{dom} h)^x$ we have:

$$
\begin{aligned}
M \vDash \exists y\, \varphi(a, y) \quad &\Rightarrow \quad M \vDash \varphi(a, b) \text{ for } b \in M^{|y|} \\
&\Rightarrow \quad N \vDash \varphi(ha, hb) \\
&\Rightarrow \quad N \vDash \varphi(ha, c) \text{ for } c \in N^{|y|} \\
&\Rightarrow \quad N \vDash \exists y\, \varphi(ha, y).
\end{aligned}
$$

Note that the second implication requires the totality of $h : M \to N$ which guarantees that $\varphi(a, y)$ has a solution in $\operatorname{dom} h$. □

When $h : M \to N$ is injective the following proposition is a corollary of Propositions 3.29 and 3.30. The general proof is the dual version of the proof of Propositions 3.30.

3.31 Proposition For every set of formulas Δ, every surjective Δ-morphism $h : M \to N$ is a $\{\forall\}\Delta$-morphism.

3.32 Exercise Prove that if $h : M \to N$ is an elementary embedding $h[M]$ is an ememen- tary substructure of N and $h : M \to h[M]$ is an isomorphism.

Chapter 4

Ultraproducts

In these notes we only use ultraproducts to prove the compactness theorem. Since a syntactic proof of the compactness theorem is also given, this chapter is, strictly speaking, not required. However, the importance of ultraproducts transcends its application to model theory.

4.1 Direct products

In this and in the next section $\langle M_i : i \in I \rangle$ is a sequence of L-structures. (We are slightly abusing of the word sequence, since I is a just naked set.) The direct product of this sequence is a structure denoted by

$$N \quad = \quad \prod_{i \in I} M_i$$

and defined by conditions 1-3 below. If $M_i = M$ for all $i \in I$, we say that N is a direct power of M and denote it by M^I.

The domain of N is the set containing all functions

1. $$\hat{a} \quad : \quad I \ \rightarrow \ \bigcup_{i \in I} M_i$$ $$\hat{a} \quad : \quad i \ \mapsto \ \hat{a}i \ \in \ M_i$$

We do not distinguish between tuples of elements of N and tuple-valued functions. For instance, the tuple $\hat{a} = \langle \hat{a}_1 \ldots \hat{a}_n \rangle$ is identified with the function $\hat{a} : i \mapsto \hat{a}i = \langle \hat{a}_1 i, \ldots, \hat{a}_n i \rangle$. On a first reading of what follows, it may help to pretend that all functions and relations are unary.

The interpretation of $f \in L_{\text{fun}}$ is defined as follows:

2. $$(f^N \hat{a})i \quad = \quad f^{M_i}(\hat{a}i) \qquad \text{for all } i \in I.$$

The interpretation of $r \in L_{\text{rel}}$ is the product of the relations r^{M_i}, that is, we define

3. $$\hat{a} \in r^N \quad \Leftrightarrow \quad \hat{a}i \in r^{M_i} \qquad \text{for all } i \in I.$$

The following proposition is immediate.

4.1 Proposition If $\doteq, \wedge, \forall, \exists$ are the only logical symbol that occur in $\varphi(x) \in L$, then for every $\hat{a} \in N^{|x|}$

$\sharp$ $$N \vDash \varphi(\hat{a}) \quad \Leftrightarrow \quad M_i \vDash \varphi(\hat{a}i) \qquad \text{for all } i \in I.$$

Proof. By induction on syntax. First note that we can extend 2 to all terms $t(x)$ as follows:

2′. $$(t^N \hat{a})i \quad = \quad t^{M_i}(\hat{a}i) \qquad \text{for all } i \in I.$$

Combining 3 and 2′ gives that for every $r \in L_{\text{rel}} \cup \{=\}$ and every L-term $t(x)$

$$N \vDash rt\hat{a} \quad \Leftrightarrow \quad M_i \vDash rt\hat{a}i \qquad \text{for all } i \in I.$$

This shows that $\sharp$ holds for $\varphi(x)$ atomic. Induction for the connectives $\wedge$, $\forall$ ed $\exists$ is immediate. $\square$

A consequence of Proposition 4.1 is that a direct product of groups, rings or vector spaces is a structure of the same sort. However, a product of fields is not a field.

4.2 Łoś's Theorem

Let I be any set. By filter on I we mean a filter of the Boolean algebra $\mathcal{P}(I)$.

We (temporary) denote by N' the following structure. The domain and the interpretation of the function symbols is as in the structure N defined in 1, respectively 2, of the previous section. As interpretation of $r \in L_{\text{rel}}$ we take

3′. $$\hat{a} \in r^{N'} \quad \Leftrightarrow \quad \big\{ i \in I \ : \ \hat{a}i \in r^{M_i} \big\} \in F\,.$$

Let F be a filter on I. We define the following congruence on N' (see Definition 2.46)

$$\hat{a} \sim_F \hat{c} \quad \Leftrightarrow \quad \big\{ i \in I \ : \ \hat{a}i = \hat{c}i \big\} \in F\,.$$

To check that $\sim_F$ is indeed a congruence, we first need to check that it is an equivalence relation. Reflexivity and symmetry are immediate, and transitivity follows from f2 in Section 3.1. Then we check that $\sim_F$ is compatible with the functions of L, that is, that c1 of Definition 2.46 is satisfied. This follows from f1 in Section 3.1.

For brevity, we write N'/F for $N'/{\sim_F}$ and $[\hat{a}]_F$ for $[\hat{a}]_{\sim_F}$. We write $N'/F \overset{*}{\vDash} \varphi(\hat{a})$ for $N'/F \vDash \varphi([\hat{a}]_F)$. A more formal interpretation of $\overset{*}{\vDash}$ is given in Definition 2.48.

The structure N'/F is called the reduced product of the structures $\langle M_i : i \in I\rangle$ or, when $M_i = M$ for all $i \in I$, the reduced power of M. When F is an ultrafilter we say ultraproduct, respectively ultrapower.

The difference between N and N' is highlighted in Exercise 4.6. For neater notation, below we write N for N'.

The following proposition is almost tautological. Note that it is a special case of Łoš's Theorem below (but it is does not require F to be an *ultra* filter).

4.2 Proposition Let $r \in L_{\text{rel}}$. Let $t(x)$ be tuple of terms of length n_r, the arity of r. Then for every $\hat{a} \in N^{|x|}$ and every filter F the following are equivalent

1. $N/F \overset{*}{\vDash} r\, t(\hat{a})$;
2. $\{i \ : \ M_i \vDash r\, t(\hat{a}i) \} \ \in \ F$.

Proof. 1⇒2. Assume 1. Then, by 2* of Definition 2.48, there is a $\hat{b} \sim t^N(\hat{a})$ such that $N \vDash r\,\hat{b}$. Therefore $\{i : M_i \vDash r\,\hat{b}i\}$ is in F. But $\hat{b} \sim t(\hat{a})$ means that $\big\{i \ : \ \hat{b}i = t(\hat{a}i)\big\}$ is in F, hence 2 follows.

2⇒1. Is clear. $\square$

4.3 Łoś's Theorem Let $\varphi(x) \in L$ and let F be an ultrafilter on I. Then for every $\hat{a} \in N^{|x|}$ the following are equivalent:

1. $N/F \ \overset{*}{\vDash} \ \varphi(\hat{a})$ (see Definition 2.48);
2. $\big\{ i \ : \ M_i \vDash \varphi(\hat{a}i) \big\} \ \in \ F$.

Proof. We proceed by induction on the syntax of $\varphi(x)$. If $\varphi(x)$ is equality, then equivalence holds by definition of $\sim$. If $\varphi(x)$ is of the form $r\,t(x)$ for some tuple of terms $t(x)$ and $r \in L_{\text{rel}}$ then 1⇔2 is Proposition 4.2.

We prove the inductive step for the connectives $\neg$, $\wedge$, and the quantifier $\exists$. We begin with $\neg$. This is the only place in the proof where the assumption that F is an *ultra*filter is required. By the inductive hypothesis,

$$N/F \overset{*}{\nvDash} \neg\varphi(\hat{a}) \quad\Leftrightarrow\quad \{i \,:\, M_i \vDash \varphi(\hat{a}i)\,\} \notin F$$

So, as F is an *ultra*filter

$$\Leftrightarrow\quad \{i \,:\, M_i \vDash \neg\varphi(\hat{a}i)\,\} \in F\,.$$

Now consider $\wedge$. Assume inductively that the equivalence 1⇔2 holds for $\varphi(x)$ and $\psi(x)$. Then

$$N/F \overset{*}{\vDash} \varphi(\hat{a}) \wedge \psi(\hat{a}) \quad\Leftrightarrow\quad \{i \,:\, M_i \vDash \varphi(\hat{a}i)\} \in F \text{ and } \{i \,:\, M_i \vDash \psi(\hat{a}i)\} \in F\,.$$

As filters are closed under intersection, we obtain

$$\Leftrightarrow\quad \{i \,:\, M_i \vDash \varphi(\hat{a}i) \wedge \psi(\hat{a}i)\,\} \in F\,.$$

Finally, consider $\exists y$. Assume inductively that the equivalence 1⇔2 holds for $\varphi(x,y)$. Then

$$\begin{aligned} N/F \overset{*}{\vDash} \exists y\, \varphi(\hat{a},y) \quad&\Leftrightarrow\quad N/F \overset{*}{\vDash} \varphi(\hat{a},\hat{b}) && \text{for some } \hat{b} \in N \\ &\Leftrightarrow\quad \{i \,:\, M_i \vDash \varphi(\hat{a}i,\hat{b}i)\,\} \in F && \text{for some } \hat{b} \in N. \end{aligned}$$

We claim this is equivalent to

$$\Leftrightarrow\quad \{i \,:\, M_i \vDash \exists y\, \varphi(\hat{a}i,y)\,\} \in F.$$

The ⇒ direction is trivial. For ⇐, we choose as $\hat{b}$ a sequence that picks a witness of $M_i \vDash \exists y\, \varphi(\hat{a}i,y)$ if it exists, and some arbitrary element of M_i otherwise. □

Let a^I denote the element of M^I that has constant value a. The following is an immediate consequence of Łoś's theorem.

4.4 Corollary Let $N = M^I$. For every $a \in M$

$$N/F \overset{*}{\vDash} \varphi(a^I) \quad\Leftrightarrow\quad M \vDash \varphi(a)\,.$$

We often identify M with its image under the embedding $h : a \mapsto [a^I]_F$, and say that M^I/F is an elementary extension of M.

The following corollary is an immediate consequence of the compactness theorem that we prove in the next chapter but here we give a direct proof.

4.5 Corollary Every infinite structure has a proper elementary extension.

Proof. Let M be an infinite structure and let F be a *non-principal* ultrafilter on ω. It suffices to show that $h[M]$, the image of the embedding defined above, is a *proper* substructure of M^ω/F. As M is infinite, there is an injective function $\hat{d} \in M^\omega$. Then for every $a \in M$ the set $\{i : \hat{d}\,i = a\}$ is either empty or a singleton and, as F is non principal, it does not belong to F. So, by Łoš Theorem, we have $M^\omega/F \overset{*}{\vDash} \hat{d} \neq a^I$ for every $a \in M$, that is, $[\hat{d}\,]_F \notin h[M]$. □

4.6 Exercise Let N be the structure defined in the previous section. Prove that if $M_i =$

M for all $i \in I$ then $N'/F = N/F$. More generally, prove that if for every $r \in L_{\mathrm{rel}}$, either $r^{M_i} \neq \varnothing$ for all i or $r^{M_i} = \varnothing$ for all i then $N'/F = N/F$.

4.7 Exercise Consider $\mathbb{N}$ as a structure in the language of strict orders. Let F be a non-principal ultrafilter on ω. Prove that in $\mathbb{N}^\omega$ there is a sequence $\langle \hat{a}_i : i \in \omega \rangle$ such that $\mathbb{N}^\omega/F \models^{*} \hat{a}_{i+1} < \hat{a}_i$.

4.8 Exercise Let I be the set of integers $i > 1$. For $i \in I$, let $\mathbb{Z}_i$ denote the additive group of integers modulo i, and let N denote the product $\prod\limits_{i \in I} \mathbb{Z}_i$. Prove that, if F is a non-principal ultrafilter on I,

1. for some F, N/F does not contain any element of finite order;
2. for some F, N/F has some elements of order 2;
3. for all F, N/F contains an element of infinite order;
4. for some F, $N/F \models \forall x\, \exists y\; my = x$ for every integer $m > 0$;
5. for some F, N/F contains an element $\hat{a}$ such that $N/F \models \forall x\; mx \neq \hat{a}$ for every positive integer m.

Chapter 5

Compactness theorem(s)

We present two proofs of the compactness theorem. The first uses ultrapowers the second is syntactic. Finally, we generalize the theorem so that it applies to types.

5.1 Compactness via ultraproducts

A theory is finitely consistent if all its finite subsets are consistent. The following theorem is the *fiat lux* of model theory.

5.1 Compactness Theorem Every finitely consistent theory is consistent.

Proof. Let T be a finitely consistent theory.

We claim that the structure N/F which we define below is a model of T. Let I be the set of consistent sentences $\xi \in L$. For every $\xi \in I$ pick some $M_\xi \vDash \xi$. For any sentence $\varphi \in L$ we define

$$X_\varphi \quad = \quad \Big\{\xi \in I \ : \ \xi \vdash \varphi\Big\}.$$

Clearly φ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi\wedge\psi} \ = \ X_\varphi \cap X_\psi$. Hence, as T is finitely consistent, the set $B = \big\{X_\varphi \ : \ \varphi \in T\big\}$ has the finite intersection property. Therefore B extends to an ultrafilter F on I. Define

$$N \quad = \quad \prod_{\xi\in I} M_\xi.$$

We claim that $N/F \vDash T$. By Łoš Theorem, for every sentence $\varphi \in L$

$$N/F \vDash \varphi \quad \Leftrightarrow \quad \Big\{\xi \ : \ M_\xi \vDash \varphi\Big\} \ \in \ F\,.$$

By the definition of F, for every $\varphi \in T$, the set $X_\varphi \subseteq \{\xi \ : \ M_\xi \vDash \varphi\}$ belongs to F. Therefore $N/F \vDash T$, *et lux fuit*. $\square$

The compactness theorem can be formulated in the following apparently stronger way.

5.2 Corollary If $T \vdash \varphi$ then there is some finite $S \subseteq T$ such that $S \vdash \varphi$.

Proof. Suppose $S \nvdash \varphi$ for every finite $S \subseteq T$. Then for every finite $S \subseteq T$ there is a model $M \vDash S \cup \{\neg\varphi\}$. In other words, $T \cup \{\neg\varphi\}$ is finitely consistent. By compactness $T \cup \{\neg\varphi\}$ is consistent, hence $T \nvDash \varphi$. $\square$

5.3 Exercise Let $\Phi \subseteq L$ be a set of sentences and suppose that $\vdash \psi \leftrightarrow \bigvee \Phi$ for some sentence ψ. Prove that there is a finite $\Phi_0 \subseteq \Phi$ such that $\vdash \psi \leftrightarrow \bigvee \Phi_0$.

5.4 Exercise Let $\mathcal{C}$ be a class of structures. Let $\mathrm{Th}(\mathcal{C}) = \big\{\varphi \mid M \vDash \varphi \text{ for all } M \in \mathcal{C}\big\}$ be the theory of $\mathcal{C}$. Prove the following are equivalent

1. $N \vDash \mathrm{Th}(\mathcal{C})$;
2. N is elementarily equivalent to an ultraproduct of elements of $\mathcal{C}$.

5.2 Compactness for types

Recall that a type is a set of formulas. When we present types we usually declare the variables that may occur in it – we write $p(x)$, $q(x)$, etc. where x is a tuple of variables. When x is the empty tuple, $p(x)$ is just a theory. Often we identify a finite types with the (infinite) conjunction of the formulas contained in it.

We write $M \vDash p(a)$ if $M \vDash \varphi(a)$ for every $\varphi(x) \in p$. We say that a is a solution or a realization of $p(x)$. An equivalent notation is $M, a \vDash p(x)$ or, when M is clear from the context, $a \vDash p(x)$. We say that $p(x)$ is consistent in M if it has a solution in M. In this case we may write $M \vDash \exists x\, p(x)$. We say that $p(x)$ is consistent if it is realized in some model.

We say that a type $p(x)$ is finitely consistent if all its finite subsets are consistent. If its finite subsets are all consistent in the same model M, we say that $p(x)$ is finitely consistent in M.

The following fact is an immediate consequence of the compactness theorem.

5.5 Fact Every finitely consistent type $p(x) \subseteq L$ is consistent.

Proof. Let L' be the expansion of L obtained by adding the fresh symbols c, a tuple of constants of the same length as x. Then $p(c)$ is a finitely consistent theory in the language L'. By the compactness theorem there is an L'-structure $N' \vDash p(c)$. Let N be the reduct of N' to L, that is, the L-structure with the same domain and the same interpretation as N' on the symbols of L. Note that, though the constants c are not in L, the elements of the tuple $c^{N'}$ remain in N. Then $N, c^{N'} \vDash p(x)$. □

The following theorem shows that the notion of finite concistency *in a model*, which is trivial for theories, is instead very interesting for types. Before that, we recall an important particular case of Proposition 3.26.

5.6 Lemma (The elementary diagram method) Let M be an infinite model. Let c be an enumeration of M. Let $q(z) = \mathrm{tp}(c)$. Then, for every model N realizing $q(z)$, there is an elementary embedding $h : M \to N$.

We call $q(z)$ the elementary diagram of M though, stricticly speaking, the elementary diagram of M is theory $\mathrm{Th}(M/M)$.
Proof. Let $N, a \vDash q(z)$. Let $h : M \to N$ map $c \mapsto a$. By Remark 3.24, this is an elementary embeddings. □

5.7 Compactness Theorem for types Let M be an infinite model. If $p(x) \subseteq L(M)$ is finitely consistent in M then it is realised in some elementary extension of M.

Proof. Let a be an enumeration of M. It is convenient to write $p(x)$ as $p(x\,;a)$ where $p(x\,;z) \subseteq L$. Let $q(z) = \mathrm{tp}_M(a)$. Clearly, $p(x\,;z) \cup q(z)$ is finitely consistent. Then, by Fact 5.5, it is realized in some model N' by some $c', a' \in (N')^{x,z}$. Let $h : M \to N'$

map $a \mapsto a'$. By Proposition **??**, h is an elementary map. Note that $hc = c'$. By Exercise 3.32, $h[M]$ is an isomorphic copy of M and is an elementary substructure of N'. By construction $N', hc \vDash p(x\,;ha)$.

We may conclude that there is a structure N (an isomorphic copy of N') that extends elementarily M and realizes $p(x\,;a)$. □

The following corollary is historically important.

5.8 Upward Löwenheim-Skolem Theorem Every infinite structure has arbitrarily large elementary extensions.

Proof. Let $x = \langle x_i : i < \lambda \rangle$ be a tuple of distinct variables, where λ is an arbitrary cardinal. The type $p(x) = \{x_i \neq x_j : i < j < \lambda\}$ is finitely consistent in every infinite structure and every structure that realises p has cardinality $\geq \lambda$. Hence the claim follows from Theorem **??**. □

5.3 Compactness via syntax

Here we prove the compactness theorem using the so-called Henkin method. We divide the proof in two steps. Firstly, we observe that when the language is rich enough to name witnesses of all existential statements of the theory, these witnesses (*Henkin constants*) form a canonical model. Secondly, we show that we can add the required Henkin constants to any finitely consistent theory.

5.9 Definition Fix a language L. Assume for simplicity that formulas use only the connectives $\wedge$, $\neg$ and $\exists$. We say that T is a Henkin theory if for all formulas φ and ψ

0. $\varphi \in T \Rightarrow \neg\varphi \notin T$
1. $\neg\neg\varphi \in T \Rightarrow \varphi \in T$
2. $\varphi \wedge \psi \in T \Rightarrow \varphi \in T$ and $\psi \in T$
3. $\neg(\varphi \wedge \psi) \in T \Rightarrow \neg\varphi \in T$ or $\neg\psi \in T$
4. $\exists x\, \varphi \in T \Rightarrow \varphi[x/a] \in T$ for some closed term a
5. $\neg\exists x\, \varphi \in T \Rightarrow \neg\varphi[x/a] \in T$ for all closed terms a.

Moreover, the following holds for all closed terms a, b, c

a. $a \doteq a \in T$

b. $a \doteq b \in T \Rightarrow b \doteq a \in T$

c. $a \doteq b,\ b \doteq c \in T \Rightarrow a \doteq c \in T$

d. $a \doteq b,\ \varphi[x/a] \in T \Rightarrow \varphi[x/b] \in T$.

Fix a theory T and let M be the structure that has as domain the set of closed terms. Define for every relation symbol r

$$r^M = \{\langle a_1, \ldots, a_n \rangle \in M^n : r(a_1, \ldots, a_n) \in T\},$$

where n is the arity of r. Define for every function symbol f

$$f^M = \{\langle t, a_1, \dots, a_n \rangle \in M^{n+1} : t = fa_1 \dots a_n\}.$$

where n is the arity of r. An easy proof by induction shows that $t^M = t$ for all closed terms t.

Finally, let E be the relation on M that holds when $a \doteq b \in T$.

5.10 Lemma The relation E is a congruence on M (as defined in Section 2.2.5).

Proof. Axioms a-c ensure that E is an equivalence. We claim that that E is a congruence. This is immediate for unary functions: apply e to the formula $fx \doteq fa$. In general the claim is easily proved by induction on the arity of f. □

Condition 0 is the only negative requirement of Definition 5.9 (it requires that T does *not* contain some formula). By condition 0, Henkin theories do not contain any blatant inconsistency. Surprisingly, this is all what is needed for the existence of a model.

5.11 Theorem If T is a Henkin theory then $M/E \vDash T$.

Proof. By induction on the complexity of the formula φ in T we prove that

1. $\varphi \in T \Rightarrow M/E \overset{*}{\vDash} \varphi$ for the notation cf. Definition 2.48
2. $\neg\varphi \in T \Rightarrow M/E \overset{*}{\nvDash} \neg\varphi$

Induction is immediate by 1-5 of Definition 5.9. Hence we only need to verify the claim for atomic formulas. Consider first the formula $\varphi = (t_1 \doteq t_2)$ where the t_i are closed terms. By the definition of M, for every closed term t we have $t^M = t$ so claim 1 is clear. As for 2, suppose $M/E \overset{*}{\vDash} t_1 \doteq t_2$, that is $t_1 E t_2$. Then $t_1 \doteq t_2 \in T$ and $\neg t_1 \doteq t_2 \notin T$ follows from axiom 0.

Now assume $\varphi = rt$ for a relation r and a tuple of closed terms t. The argument is similar: 1 is immediate; to prove 2 suppose that $M/E \overset{*}{\vDash} rt$. Then $tEs \in r^M$ for some tuple of closed terms s. Then $rs \in T$, and by d $rt \in T$. Finally from 0 we obtain $\neg rt \notin T$. □

5.12 Proposition If every finite subset of T has a model then there is a Henkin theory T' containing T. The theory T' may be in an expanded language L'.

Proof. Set $\lambda = |L|$. Let $\langle c_i : i < \lambda \rangle$ be some constants not in L. Let L_i be the language with constants among $c_{\restriction i}$. Fix a variable x and an enumeration $\langle \varphi_i(x) : i < \lambda \rangle$ of the formulas in L_λ. Suppose that the enumeration is such that $\varphi_i(x) \in L_i$.

We now construct a sequence of finitely consistent L_i-theories T_i. If α is 0 or a limit ordinal we define

$$T_\alpha = T \cup \bigcup_{i<\alpha} T_i.$$

As for successor ordinals, let S_i be a maximally finitely consistent set of L_i-formulas containing T_i. (Here we use Zorn's lemma, but see the remark below.) It is immediate that S_i satisfies all requirements in Definition 5.9 but possibly for 4.

Now, if $\exists x\, \varphi_i(x) \in S_i$ set $T_{i+1} = S_i \cup \{\varphi[x/c_i]\}$. As c_i does not occur in S_i, it is evident that T_{i+1} is finitely consistent.

Recall that we assumed $\exists x\, \varphi_i(x) \in L_i$. Then, either $\exists x\, \varphi_i(x) \in T_{i+1}$ or it is not finitely consistent with T_{i+1}. Hence stage i settle requirement 4 in Definition 5.9 as far as $\varphi_i(x)$ is concerned.

At stage λ all possible counterexamples to 4 have been ruled out, then $T' = T_\lambda$ is the required Henkin theory. □

The following theorem is an immediate corollary of the proposition above.

5.13 Compactness Theorem If T is finitely consistent then T is consistent.

To keep the proof of Proposition 5.12 short, we applied Zorn's lemma. This is not strictly necessary. In fact, if we are given a finitely consistent theory T. We can extend T to a theory S that meets Definition 5.9, up to condition 4, by adding systematically all required formulas. The procedure is effective, hence Zorn's lemma is not required.

It is interesting to consider the case when T is finite. Assume also (though this is not really necessary) that the language contains finitely many symbols and no functions other then constants. Then the construction in Proposition 5.12 is an effective procedure that produces in ω steps a model of T. At each step T_n is finite and contains only subformulas of formulas in T or variant on these obtained by substituting constants for variables.

Now suppose instead that we start with an inconsistent T. The procedure above has to come to a halt at same (finite) stage because a model of T does not exist. When the procedure halts, we end up with a finite sequence of finite theories $T_0, \ldots, T_n$ where $T_0 = T$ and T_n contains some blatant inconsistency (i.e. φ and $\neg\varphi$). Many have interpreted $T_0, \ldots, T_n$ as a formal *proof* of the inconsistency of T.

All this has little or no interest to model theory. But it highlights a fascinating phenomenon. When we say that T is inconsistent, we say that no structure models T. This expression uses a (meta linguistic) universal quantifier that ranges over the class of all structures. Yet this is equivalent to an expression that merely asserts the existence of a finite sequence of finite theories.

Chapter 6

Some relational structures

In the first section we prove that theory of dense linear orders without endpoints is ω-categorical. That is, any two such countable orders are isomorphic. This is an easy classical result of Cantor. In this chapter we examine Cantor's construction (a so-called *back-and-forth* construction) in great detail. In the second section we apply the same technique to prove that the theory of the random graph is ω-categorical.

6.1 Dense linear orders

The language of strict orders, which in this section we denote by L, contains only a binary relation symbol $<$. A structure M of signature L is a strict order if it models (the universal closure of) the following formulas

1. $x \not< x$ — irreflexive;
2. $x < z < y \ \rightarrow\ x < y$ — transitive.

Note that the following is an immediate consequence of 1 and 2.

$x < y \ \rightarrow\ y \not< x$ — antisymmetric.

We say that the order is total or linear if

li. $x < y \ \vee\ y < x \ \vee\ x = y$ — linear or total.

An order is dense if

nt. $\exists x, y\ (x < y)$ — non trivial;

d. $x < y \ \rightarrow\ \exists z\ (x < z < y)$ — dense.

We need to require the existence of two comparable elements, then d implies that dense orders are in fact infinite. We say that the ordering has no endpoints if

e. $\exists y\ (x < y)\ \wedge\ \exists y\ (y < x)$ — without endpoints.

We denote by $T_{\rm lo}$ the theory strict linear orders and by $T_{\rm dlo}$ the theory of dense linear orders without endpoints. Clearly, these are consistent theories: $\mathbb{Q}$ with the usual ordering is a model of $T_{\rm dlo}$.

We introduce some notation to improve readability of the proof of the following theorem. Let A and B be subsets of an ordered set. We write $A < B$ if $a < b$ for every $a \in A$ and $b \in B$. We write $a < B$ e $A < b$ for $\{a\} < B$, respectively $A < \{b\}$. Let $M \vDash T_{\rm lo}$. Then $M \vDash T_{\rm dlo}$ if and only if for every finite $A, B \subseteq M$ such that $A < B$ there is a c such that $A < c < B$. In fact axiom d is evident and axioms nt and e are obtained taking replacing A and/or B by the empty set.

Now we prove the first of a series of lemmas that we call extension lemmas. Recall that in the language of strict orders an injective map $k : M \to N$ is a partial isomorphism if

$$M \vDash a < b \Leftrightarrow N \vDash ka < kb \qquad\qquad \text{for every } a, b \in \operatorname{dom} k.$$

When $M, N \vDash T_{\text{lo}}$ the direction $\Rightarrow$ suffices.

6.1 Lemma Fix $M \vDash T_{\text{lo}}$ and $N \vDash T_{\text{dlo}}$. Let $k : M \to N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \to N$ that extends k and is defined in b.

Proof. Given a finite partial isomorphism $k : M \to N$ define

$$A^- = \{a \in \operatorname{dom}(k) : a < b\};$$
$$A^+ = \{a \in \operatorname{dom}(k) : b < a\}.$$

The sets A^- and A^+ are finite and partition $\operatorname{dom} k$, and $A^- < A^+$. As $k : M \to N$ is a partial isomorphism, $k[A^-] < k[A^+]$. Then in N there is an element c such that $k[A^-] < c < k[A^+]$. It is easy to check that setting $h = k \cup \{\langle b, c\rangle\}$ gives the required extension. □

The following is an equivalent version of Lemma 6.1.

6.2 Corollary Let $M \vDash T_{\text{lo}}$ be countable and let $N \vDash T_{\text{dlo}}$. Let $k : M \to N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \hookrightarrow N$ that extends k.

Proof. Let $\langle a_i : i < \omega \rangle$ be an enumeration of M. Define by induction a chain of finite partial isomorphisms $h_i : M \to N$ such that $a_i \in \operatorname{dom} h_{i+1}$. The construction starts with $h_0 = k$. At stage $i + 1$ we chose any finite partial isomorphism $h_{i+1} : M \to N$ that extends h_i and is defined in a_i. This is possible by Lemma 6.1. In the end we set

$$h = \bigcup_{i \in \omega} h_i.$$

It is immediate to verify that $h : M \hookrightarrow N$ is the required embedding. □

We are now ready to prove that any two countable models of T_{dlo} are isomorphic which is a classical result of Cantor's. Actually what we prove is slightly more general than that. In fact Cantor's theorem is obtained from the theorem below by setting $k = \varnothing$, which we are allowed to, because all models have the same empty characteristic (cfr. Remark 3.27).

6.3 Theorem Every finite partial isomorphism $k : M \to N$ between countable models of T_{dlo} extends to an isomorphism $g : M \xrightarrow{\sim} N$.

The following is the archetypal back-and-forth construction. It is important to note that it does not mention linear orders at all. It only uses the extension Lemma 6.1. The same construction can be applied in many other contexts where an extension lemma holds (cfr. Theorem 7.6).

Proof. Let $\langle a_i : i < \omega \rangle$ and $\langle b_i : i < \omega \rangle$ be enumerations of M and N respectively. We define by induction a chain of finite partial isomorphisms $g_i : M \to N$ such that $a_i \in \operatorname{dom} g_{i+1}$ and $b_i \in \operatorname{rng} g_{i+1}$. In the end we set

$$g = \bigcup_{i \in \omega} g_i$$

We begin by letting $g_0 = k$. The inductive step consists of two half-steps that we call the *forth step* and *back step*. In the forth step we define $g_{i+1/2}$ such that $a_i \in \operatorname{dom} g_{i+1/2}$. In the back step to define g_{i+1} such that $b_i \in \operatorname{rng} g_{i+1}$.

By the extension lemma 6.1 there is a finite partial isomorphism $g_{i+1/2} : M \to N$ that extends g_i and is defined in a_i. Now apply the same lemma to extend $\left(g_{i+1/2}\right)^{-1} : N \to M$ to a finite partial isomorphism $(g_{i+1})^{-1} : N \to M$ defined in b_i. □

Let λ be an infinite cardinal. We say that a theory is λ-categorical if any two models of T of cardinality λ are isomorphic. From Theorem 6.3, taking $k = \varnothing$, we obtain the following.

6.4 Corollary The theory T_{dlo} is ω-categorical.

We also obtain that T_{dlo} is a complete theory. This is consequence of the following general fact.

6.5 Proposition If T has no finite models and is λ-categorical for some $\lambda \geq |L|$, then T is complete.

Proof. Let M and N be any two models of T. Applying the upward and/or downward Löwenheim-Skolem theorem, we may assume they both have cardinality λ (here we use that M and N are both infinite and that $\lambda \geq |L|$). Hence $M \simeq N$ and in particular $M \equiv N$. □

6.6 Exercise Prove that the extension Lemma 6.1 characterizes models of T_{dlo} among models of T_{lo}. That is, if N is a model of T_{lo} such that the conclusion of Lemma 6.1 holds, then $M \vDash T_{\mathrm{dlo}}$.

6.7 Exercise Prove that T_{dlo} is not λ-categorical for any uncountable λ.

6.8 Exercise Prove that, in the language of strict orders, $\mathbb{Q} \preceq \mathbb{R}$.

6.9 Exercise Let L be the language of strict orders expanded with countably many constants $\{c_i : i \in \omega\}$. Let T be the theory that extends T_{dlo} by the axioms $c_i < c_{i+1}$ for all i. Prove that T is complete. Find three non isomorphic countable models of this theory. For a suitably chosen model N of T, prove the statement in Lemma 6.1, where M any model of T.

6.10 Exercise Show that in Theorem 6.5 the assumption $\lambda \geq |L|$ is necessary. Hint: let ν be an uncountable cardinal. The language contains only the ordinals $i < \nu$ as constants. The theory T says that there are infinitely many elements and either $i = 0$ for every $i < \nu$, or $i \neq j$ for every $i < j < \nu$. Prove that T is incomplete. Prove that T has countable models and that these are all isomorphic.

6.2 Random graphs

Recall that the language of graphs, which in this section we denote by L, contains only a binary relation r. A graph structure of signature L such that

1. $\neg r(x, x)$ irreflexive;

2. $r(x,y) \to r(y,x)$ symmetric.

An element of a graph M is called a vertex or a node. An edge is an unordered pair of vertices $\{a,b\} \subseteq M$ such that $M \vDash r(a,b)$. In words we may say that a is adjacent to b.

A random graph is a graph that also satisfies the following axioms for every n

nt. $\exists x,y\ \ (x \neq y)$ non trivial;

r_n. $\displaystyle\bigwedge_{i,j=1}^{n} x_i \neq y_j \ \ \to\ \ \exists z \bigwedge_{i=1}^{n} \big[r(x_i,z) \wedge \neg r(z,y_i) \wedge z \neq y_i\big]$ for every $n \in \mathbb{Z}^+$.

The theory of graphs is denoted by $T_{\rm gph}$ and the theory of random graphs is denoted by $T_{\rm rg}$. The scheme of axioms r_n plays the same role as density in the previous section. It says that given two disjoint sets A^+ and A^- of cardinality $\leq n$ there is a vertex z that is adjacent to all vertices in A^+ and to no vertex in A^-. We explicitly required that $z \notin A^-$, by 1 it is clear that $z \notin A^+$.

Strictly speaking, the axioms r_n do not mention the cases when A^+ or A^- are empty. But as it is evident that random graphs are infinite, we can deal with them by adding redundant elements.

The following is the analogous of Lemma 6.1 for random graphs. Recall that in the language of graphs a map $k : M \to N$ is a partial isomorphism if it is injective and

$$M \vDash r(a,b) \Leftrightarrow N \vDash r(ka,kb) \qquad \text{for every } a,b \in \operatorname{dom} k.$$

6.11 Lemma Fix $M \vDash T_{\rm gph}$ and $N \vDash T_{\rm rg}$. Let $k : M \to N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \to N$ that extends k and is defined in b.

Proof. The structure of the proof is the same as in Lemma 6.1, so we use the same notation. Assume $b \notin \operatorname{dom} k$ and define

$$A^+ = \big\{x \in \operatorname{dom} k \ :\ M \vDash r(x,b)\big\} \quad \text{e} \quad A^- = \big\{y \in \operatorname{dom} k \ :\ M \vDash \neg r(y,b)\big\}.$$

These two sets are finite and disjoint, then so are $k[A^+]$ and $k[A^-]$. Then there is a $c \notin \operatorname{rng} k$ such that

$$\bigwedge_{a \in A^+} r(ka,c) \ \ \wedge \ \bigwedge_{a \in A^-} \neg r(ka,c).$$

As $k[A^+] \cup k[A^-] = \operatorname{rng} k$, it is immediate to verify that $h = k \cup \big\{\langle b,c\rangle\big\}$ is the required extension. □

Some readers may doubt that $T_{\rm rg}$ is consistent.

6.12 Proposition There exists a random graph.

Proof. The domain of is the set of natural numbers. Let $r(n,m)$ hold if the n-th prime number divides m or, conversely, the m-th prime number divides n. □

The same proof as that of Corollary 6.2 gives the following.

6.13 Corollary Let $M \vDash T_{\text{gph}}$ be countable and let $N \vDash T_{\text{rg}}$. Let $k : M \to N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \hookrightarrow N$ that extends k.

The proof of Theorem 6.3 gives the following theorem and its corollary.

6.14 Theorem Every finite partial isomorphism $k : M \to N$ between countable models of T_{rg} extends to an isomorphism $g : M \xrightarrow{\sim} N$.

6.15 Corollary The theory T_{rg} is ω-categorical (and therefore complete).

6.16 Exercise Let $a, b, c \in N \vDash T_{\text{rg}}$. Prove that $r(a, N) = r(b, N) \cap r(c, N)$ occurs only in the trivial case $a = b = c$.

6.17 Exercise Let $N \vDash T_{\text{rg}}$ prove that for every $b \in N$ the set $r(b, N)$ is a random graph. Is every random graph $M \subseteq N$ of the form $\varphi(N)$ for some $\varphi(x) \in L(N)$?

6.18 Exercise Let N be free union of two random graphs N_1 and N_2. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$, where $\sqcup$ denotes the disjoint union. Prove that N is not a random graph. Show that N_1 is not definable without parameters. Write a first order formula $\psi(x, y)$ true if x and y belong to the same connected component of N. Axiomatize the class of graphs that are free union of two random graphs.

6.19 Exercise Prove that T_{rg} is not λ-categorical for any uncountable λ. Hint: prove that there is a random graph N of cardinality λ where every vertex is adjacent to $< \lambda$ vertices. Compare it with its complement graph (the graph that has edges between pairs that are non adjacent in N).

6.20 Exercise Prove that T_{rg} is not finitely axiomatizable. Hint: given a random graph N and a set P of cardinality $n + 1$ show that you can add edges to $M \sqcup P$ and make it satisfy axiom r_n but not r_{n+1}.

6.21 Exercise (Peter J. Cameron) Prove that for every infinite countable graph M the following are equivalent

1. M is either random, complete, or empty (i.e. $r^M = M^2$ or $r^M = \varnothing$);
2. if $M_1, M_2 \subseteq M$ are such that $M_1 \sqcup M_2 = M$, then $M_1 \simeq M$ or $M_2 \simeq M$.

Hint: 2⇒1. Assume 2 and show that if $r(a, M) = \varnothing$ for some $a \in M$ then the graph is null; if $\{b\} \cup r(b, M) = M$ for some b, then the graph is complete. Clearly, 2 implies that any finite partition of M contains an element isomorphic to M. Then the claim above generalizes as follows: if there is a finite A such that $\bigcap_{a \in A} r(a, M) = \varnothing$ then M is the empty graph and if there is a finite B such that $B \cup \bigcup_{b \in B} r(b, M) = M$ then M is the complete graph.

Suppose M is not a random graph. Fix some finite, disjoint A and B such that no c satisfies both $r(A, c) = A$ and $r(B, c) = \varnothing$. Let $M_1 = \{c : r(A, c) \neq A\}$ and $M_2 = M \smallsetminus M_1$. Now note that

$$\bigcap_{a \in A} r(a, M_1) = \varnothing \qquad \text{and} \qquad \bigcup_{b \in B} r(b, M_2) = M_2.$$

6.3 Notes and references

We refer the reader to [?Cameron] for a well-written accessible survey on the amazing model theoretic properties of the random graph.

Chapter 7

Rich models

We introduce *Fraïssé limits*, also known as *homogeneous-universal* or *generic* structures, which here we call *rich* models, after Poizat. Rich models generalize the examples in Chapters 6 and the many more to come.

Elimination of quantifiers is briefly discussed at the end of Section 7.1. For the time being we identify quantifier elimination with the property that says that all partial isomorphisms are elementary maps. Proofs are easier with this notion in mind. The equivalence of this property with its syntactic counterpart is only proved in Chapter **??**, when the reader is more familiar with arguments of compactness.

7.1 Models and morphisms

We now define *categories of models and partial morphisms*. These are example of concrete categories as intended in category theory. However, apart from the name, in what follows we dispense with all notions of category theory as they would make the exposition less basic than intended (without providing additional technical tools).

A category (of models and partial morphisms) is a class $\mathcal{M}$ which is disjoint union of two classes: $\mathcal{M}_{\rm ob}$ and $\mathcal{M}_{\rm hom}$. The first is the class of objects and contains structures with a common signature L which we call models. The second is the class of morphisms and contains (partial) maps between models. We require that the identity maps are morphisms and that composition of two morphism is again morphism. This makes $\mathcal{M}$ a well-defined category.

For example, $\mathcal{M}$ could consist of all models of some theory T_0 and of all partial isomorphisms between these. Alternatively, as morphisms we could take elementary maps between models. On a first reading the reader may assume $\mathcal{M}$ is as in one of the two examples above. In the general case we need to make some assumptions on $\mathcal{M}$.

7.1 Definition For ease of reference we list together all properties required below

c1. the (partial) identity map $\mathrm{id}_A : M \to M$ is a morphism, for any $A \subseteq M$;

c2. if $k' : M \to N$ is a morphism for every finite $k' \subseteq k$, then $k : M \to N$ is a morphism.

c3. morphisms are invertible maps and the inverse of a morphism is a morphism;

c4. morphisms preserve the truth of $L_{\rm at}$-formulas;

c5. if M is a model and $N \equiv M$, then also N is a model;

c6. every elementary map between models is a morphism.

The connected component of a model M is the subclass of models N such that there is any morphism with domain M and codomain N (or vice versa, by c3). By axiom c1 the restriction of a morphism is a morphism, therefore M and N are in the same connected component if and only if the empty map $\varnothing : M \to N$ is a morphism. If the whole category $\mathcal{M}$ consists of one connected component we say that $\mathcal{M}$ is connected.

We call c2 the finite character of morphisms. Note that it implies the following

c7. if $k_i : M \to N$ is a chain of morphisms, then $\bigcup_{i<\lambda} k_i : M \to N$ is a morphism.

Notably, the following two definitions require c3. The generalization to non injective morphisms is not straightforward (in fact, there are two generalizations: *projective* and *inductive*). These generalizations are not very common and will not be considered here.

7.2 Definition Assume that $\mathcal{M}$ satisfies c1-c3 of Definition 7.1. We say that a model N is λ-rich if for every model M, every $b \in M$ and every morphism $k : M \to N$ of cardinality $< \lambda$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is a morphism. We say that N is rich if it is λ-rich for $\lambda = |N|$. When $\mathcal{M}_{\text{ob}} = \text{Mod}(T_0)$ for some theory T_0 and $\mathcal{M}_{\text{hom}}$ is clear from the context, we say rich model of T_0.

Rich models are also called *Fraïssé limits* or *homogeneous-universal* for a reason that will soon be clear; they are also called *generic*. Unfortunately these names are either too long or too generic, so we opt for the less common term *rich* that was proposed by Poizat.

The following two notions are closely connected with richness.

7.3 Definition Assume that $\mathcal{M}$ satisfies c1-c3 of Definition 7.1. We say that a model N is λ-universal if for every model M of cardinality $\leq \lambda$ in the same connected component as N there is an embedding $k : M \hookrightarrow N$. We say that a model N is λ-homogeneous if every $k : N \to N$ of cardinality $< \lambda$ extends to a bijective morphism $h : N \xrightarrow{\sim} N$ (an automorphism when c4 holds).

Note that the larger $\mathcal{M}_{\text{hom}}$, the stronger notion of homogeneity. When $\mathcal{M}_{\text{hom}}$ contains all partial isomorphisms between models (the largest class of morphisms considered here), it is common to say λ-ultrahomogeneous for λ-homogeneous.

As above, when $\lambda = |N|$ we say universal, homogeneous and ultrahomogeneous.

In Section 6.6.1 we implicitly used $\mathcal{M}_{\text{ob}} = \text{Mod}(T_{\text{lo}})$ and partial isomorphisms as $\mathcal{M}_{\text{hom}}$. In Section 6.6.2 we used $\mathcal{M}_{\text{ob}} = \text{Mod}(T_{\text{gph}})$ and again partial isomorphisms as $\mathcal{M}_{\text{hom}}$. Corollary 6.2 proves that every model of T_{dlo} is ω-rich. Corollary 6.13 claims the analogous fact for T_{rg}.

In the following we frequently work under the following assumption (even when not all properties are strictly necessary).

7.4 Assumption Assume $|L| \leq \lambda$ and suppose that $\mathcal{M}$ satisfies c1-c6 of Definition 7.1

The assumption on the cardinality of L is necessary to apply the downward Löwenheim-Skolem Theorem when required.

7.5 Proposition (Assume 7.4) The following are equivalent

1. N is a λ-rich model;
2. for every model M of cardinality $\leq \lambda$ and every morphism $k : M \to N$ of cardinality, say $< \lambda$ there is a embedding $h : M \hookrightarrow N$ that extends k.

Proof. Closure under union of chains of morphisms, which is ensured by c7, immediately yields 1⇒2. As for implication 2⇒1 we only need to consider the case $\lambda < |M|$. Let $k : M \to N$ be a morphism of cardinality $< \lambda$ and $b \in M$. By the downward Löwenheim-Skolem theorem there is an $M' \preceq M$ of cardinality λ containing $\operatorname{dom} k \cup \{b\}$. Let $h : M' \hookrightarrow N$ be the embedding obtained from 2. By c4, the map $h : M \to N$ is a composition of morphisms, hence a morphism. □

The following theorem subsumes both Theorem 6.3 and Theorem 6.14.

7.6 Theorem (Assume 7.4) Let M and N be two rich models of the same cardinality λ. Then every morphism $k : M \to N$ of cardinality $< \lambda$ extends to an isomorphism.

Proof. When $\lambda = \omega$, we can take the proof of Theorem 6.3 and replace *partial isomorphism* by *morphism* and the references to Lemma 6.1 by references to Proposition 7.5. As for uncountable λ, we only need to extend the construction through limit stages. By c7 we can simply take the union. □

7.7 Corollary (Assume 7.4) All rich models of cardinality λ in the same connected component are isomorphic.

It is obvious that rich models are universal. By Theorem 7.6, rich models they are homogeneous. These two notions are weaker than richness. For instance, when $\mathcal{M}$ is as in Section 6.6.2, the countable graph with no edge is trivially ultrahomogeneous but it is not universal and a fortiori not rich. On the other hand if we add to a countable random graph an isolated point we obtain a universal graph which is not ultrahomogeneous. However, when taken together, these two properties are equivalent to richness.

7.8 Theorem (Assume 7.4) The following are equivalent:

1. N is rich;
2. N is homogeneous and universal.

Proof. Implication 1⇒2 is clear as noted above, so we prove 2⇒1. We use the characterization of richness given in Proposition 7.5. Let $k : M \to N$ be a morphism such that $|k| < |N|$ and $|M| \leq |N|$. As N is universal, there is a total morphism $f : M \hookrightarrow N$. By c3 the map $k \circ f^{-1} : N \to N$ is a morphism of cardinality $< |N|$. By homogeneity it has an extension to an automorphism $h : N \xrightarrow{\sim} N$. It is immediate that $h \circ f : M \hookrightarrow N$ is the required extension of k. □

7.9 Exercise Let N be the structure obtained by adding to a countable random graph an isolated point. Show that N is homogeneous if morphisms are elementary maps but it is not if morphisms are simply partial isomorphisms.

A consequence of Theorem 7.6 is that morphisms between rich models of the same cardinality are elementary maps. However, the theorem gives no information when the models have different cardinality nor when they are merely λ-rich. This case is dealt with by next theorem, arguably the main result of this section.

To test the theorem below in a simple case we propose the following exercise.

7.10 Exercise Every partial isomorphism $k : M \to N$ between models of T_{dlo} (or models of T_{rg}) is an elementary map. Hint: use downward Löwenheim-Skolem and Theorem 6.3.

7.2 The theory of rich models, and quantifier elimination

Let $\lambda \geq |L|$ be given. The theory of the rich models of $\mathcal{M}$ is the set T_1 of sentences that hold in all λ-rich models. The theorem below proves that T_1 is complete as soon as $\mathcal{M}$ is connected.

7.11 Theorem (Assume 7.4) Every morphism between λ-rich models is elementary. In particular, λ-rich models in the same connected component are elementarily equivalent.

Proof. Let $k : M \to N$ be a morphism between rich models. It suffices to prove that every finite restriction of k is elementary. By c2, we may as well assume that k itself is finite. It suffices to construct $M' \preceq M$ and $N' \preceq N$ together with a morphism $h : M \to N$ that extends k and maps M' bijectively to N'. Then by c6 the map $h : M' \rightrightarrows N'$ is the composition of morphisms, hence it is a morphism. Finally, by c4, it is an isomorphism, in particular an elementary map.

In general, richness is not preserved under elementary equivalence. Therefore M' and N' need to be constructed simultaneously with h. We define a chain of functions $\langle h_i : i{<}\lambda \rangle$ such that $h_i : M \to N$ are morphisms and in the end we set

$$h \;=\; \bigcup_{i<\lambda} h_i, \qquad M' \;=\; \operatorname{dom} h, \qquad N' \;=\; \operatorname{rng} h.$$

We interweave the usual back-and-forth-argument with the Löwenheim-Skolem construction **??** in order to obtain $M' \preceq M$ and $N' \preceq N$.

The chains start with $h_0 = k$. At limit stages we take the union. Now assume we have h_i. Let $\varphi(x) \in L(\operatorname{dom} h_i)$ be some formula consistent in M and pick a solution $b \in M$. By λ-richness there is a $c \in N$ such that $h_i \cup \{\langle b, c\rangle\} : M \to N$ is a morphism. Let $h_{i+1/2} = h_i \cup \{\langle b, c\rangle\}$.

Finally, as in the proof of Theorem 6.3, we extend $h_{i+1/2}$ to obtain h_{i+1} by applying the same procedure with the roles of M and N inverted and $h^{-1}_{i+1/2}$ for h_i.

In the end we obtain $M' \preceq M$ if all formulas $\varphi(x) \in L(M')$ are considered. A similar consideration holds for N'. This is achieved using the same dovetail enumeration as in our second proof the downward Löwenheim-Skolem Theorem **??**. $\square$

Assume for simplicity that $\mathcal{M}$ is connected. Then all λ-rich models belong to $\operatorname{Mod}(T_1)$ for some complete theory T_1. It is interesting to ask if the converse is true: if T_1 is the theory of the λ-rich models of $\mathcal{M}$, do all models in $\operatorname{Mod}(T_1)$ are rich?

7.12 Remark (Assume 7.4) Assume $\mathcal{M}$ is connected. Let T_1 be the theory of the rich models of $\mathcal{M}$. If every model of T_1 is λ-rich then T_1 is λ-categorical. (This is a conseguence of Corollary 7.7)

A very interesting interesting variant of the question asked above is considered in Theorem **??** where it is related to an important phenomenon that we now indroduce.

First, to have a concrete example at hand, we instantiate Theorem 7.11 with the two categories used in Chapter 6.

7.13 Corollary Partial isomorphisms between models of T_{dlo} and between models of T_{rg} are elementary maps.

When partial isomorphism between models of a given theory T coincide with elementary maps, it is always by a fundamental reason. Let us introduce some terminology. Let T be a consistent theory. We say that T has (or admits) elimination of quantifiers if for every $\varphi(x) \in L$ there is a quantifier-free formula $\psi(x) \in L_{\text{qf}}$ such that

$$T \vdash \psi(x) \leftrightarrow \varphi(x).$$

We will discuss general criteria for elimination of quantifiers in Chapter **??**. Here we report without proof the following theorem.

7.14 Theorem The following are equivalent

1. T has elimination of quantifiers;
2. every partial isomorphism between models of T is an elementary map.

⚠ This theorem will be proved only in Chapter **??**, see Exercise **??** or Corollary **??**. For the time being we do not need the syntactic version of elimination of quantifiers, so when saying that T *has quantifier elimination* we mean 2 of the theorem above. For instance we rephrase Corollary 7.13 above by saying that T_{dlo} and T_{rg} have elimination of quantifiers.

In the next chapter we introduce important examples of ω-rich models that do not have an ω-categorical theory. These are algebraic structures (groups, fields etc.) hence more complex than pure relational structures. So, we conclude this section with an example of this phenomenon in an almost trivial context.

7.15 Example Let L contain a unary predicate r_n for every positive integer n. The theory T_0 contains the axioms $\neg\exists x\, \big[r_n(x) \wedge r_m(x)\big]$ for $n \neq m$ and $\exists^{\leq n} x\, r_n(x)$ for every n. Work in the the category of models of T_0 and partial isomorphisms. Let T_1 be the theory that extends T_0 with the axioms $\exists^{=n} x\, r_n(x)$ for every n. Let $q(x)$ be the type $\big\{\neg r_n(x) : n \in \omega\big\}$. There are models of T_1 that do not realize $q(x)$, hence T_1 is not ω-categorical. It is easy to verify that the following are equivalent.

1. N is an ω-rich model;
2. $N \vDash T_1$ and $q(N)$ is infinite.

The reader may use Theorem 7.11 and Compactness Theorem for Types 5.7 to prove that T_1 is complete and has elimination of quantifiers. It is also easy to verify that every uncountable model of T_1 is rich and consequently that T_1 is uncountably categorical.

7.16 Exercise Let T_0 and $\mathcal{M}$ be as in Example 7.15 except that we restrict the language to the relations $r_0, \dots, r_n$ for a fixed n. Do ω-rich models of T_0 exist? If so, let T_1 be the set of sentences that hold in all rich model. Does T_1 has elimination of quantifiers? Is T_1 ω-categorical? Answer the questions above when we add to the language a constant 0. Answer the questions above when we drop the axioms $\neg\exists x\, \big[r_n(x) \wedge r_m(x)\big]$.

7.17 Exercise The language contains only the binary relations $<$ and e. The theory T_0 says that $<$ is a strict linear order and that e is an equivalence relation. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ?

7.18 Exercise The language contains only two binary relations. The theory T_0 says that they are equivalence relations. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ?

7.19 Exercise In the language of graphs let T_0 say that there are no cycles (equivalently, there is at most one path between any two nodes). In combinatorics these graphs are called *forests*, and their connected components are called *trees*. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ?

7.20 Exercise Assuming Theorem 7.14, prove that the following are equivalent

1. T has elimination of quantifiers;
2. every *finite* partial isomorphism between models T is an elementary map.

7.21 Exercise A back-and-forth system between to models M and N is a nonempty set $\mathcal{P}$ of finite functions k such that

0. $k : M \to N$ is a partial embedding;

1a. for every $b \in M$ there is $h \in \mathcal{P}$ such that $k \subseteq h$ and $b \in \operatorname{dom} h$;

1b. for every $c \in N$ there is $h \in \mathcal{P}$ such that $k \subseteq h$ and $c \in \operatorname{rng} h$.

Prove that if L is countable and there is a back-and-forth system between M and N then there are $M' \preceq M$ and $N' \preceq N$ such that $M' \simeq N'$.

7.3 Weaker notions of universality and homogeneity

We want to extend the equivalence in Theorem 7.8 to λ-rich models. For that we need to weaken the notions of λ-homogeneity. This section is more technical and could be skipped at a first reading.

7.22 Definition We say that a structure N is weakly λ-homogeneous if for every $b \in N$ every morphism $k : N \to N$ of cardinality $< \lambda$ extends to one defined in b. The term back-and-forth λ-homogeneous is also used.

The following easy exercise on back-and-forth is required in the sequel.

7.23 Exercise (Assume 7.4) Prove that any weakly λ-homogeneous structure of cardinality λ is homogeneous.

7.24 Lemma (Assume 7.4) Let N be a weakly λ-homogeneous model. Let $A \subseteq N$ have cardinality $\leq \lambda$ and let $k : N \to N$ be a morphism of cardinality $< \lambda$. Then there is a model $M \preceq N$ containing A and an automorphism $h : M \rightleftarrows M$ that extends k.

Proof. Similar to the proof of Theorem 7.11. We shall construct simultaneously a chain $\langle A_i : i < \lambda \rangle$ of subsets of N and a chain functions $\langle h_i : i < \lambda \rangle$, such that $h_i : N \to N$ are morphisms. In the end we will set

$$M \;=\; \bigcup_{i<\lambda} A_i \qquad\qquad \text{and} \qquad\qquad h \;=\; \bigcup_{i<\lambda} h_i$$

The chains start with $A_0 = A \cup \operatorname{dom} k \cup \operatorname{rng} k$ and $h_0 = k$. As usual, at limit stages we take the union. Now we consider successor stages. At stage i we fix some enumerations of A_i and of $L_x(A_i)$, where $|x| = 1$. Let $\langle i_1, i_2 \rangle$ be the i-th pair of ordinals $< \lambda$. If the i_2-th formula in $L_x(A_{i_1})$ is consistent in N, let a be any of its solutions. Also let b be the i_2-th element of A_{i_1}. Let $h_{i+1} : N \to N$ be a minimal morphism that extends h_i and is such that $b \in \operatorname{dom} h_{i+1} \cap \operatorname{rng} h_{i+1}$. Define $A_{i+1} = A_i \cup \{a,\ h_{i+1}b,\ h_{i+1}^{-1}b\}$. $\square$

7.25 Theorem (Assume 7.4) For every model N the following are equivalent

1. N is λ-rich;
2. N is λ-universal and weakly λ-homogeneous.

Proof. Implication 1⇒2 is clear. To prove 2⇒1 we generalize the proof of Theorem 7.8. We assume 2 fix some morphism $k : M \to N$ of cardinality $< \lambda$ and let $b \in M$. By λ-universality and the downward Löwenheim-Skolem theorem, there is a morphism $f : M \to N$ with domain of definition $\operatorname{dom} k \cup \{b\}$. The map $f \circ k^{-1} : N \to N$ has cardinality $< \lambda$ and, by Lemma 7.24, it has an extension to an automorphism $h : N' \rightleftarrows N'$ for some $N' \preceq N$ containing $\operatorname{rng} k \cup \operatorname{rng} f$. Then $h \circ f : M \to N$ extends k and is defined on b. $\square$

7.4 The amalgamation property

In this section we discuss conditions that ensure the existence of rich models.

We say that $\mathcal{M}$ has the amalgamation property if for every pair of morphisms $f_1 : M \to M_1$ and $f_2 : M \to M_2$ there are two of embeddings $g_1 : M_1 \hookrightarrow N$ and $g_2 : M_2 \hookrightarrow N$ such that $g_1 \circ f_1(a) = g_2 \circ f_2(a)$ for every a in the common domain of definition, $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$.

$$\begin{tikzcd}
& M_1 \arrow[dr, hook, "g_1"] & \\
M \arrow[ur, "f_1"] \arrow[dr, "f_1"'] & & N \\
& M_2 \arrow[ur, hook, "g_2"'] &
\end{tikzcd}$$

As we assume that morphisms are invertible, we may express the amalgamation property in a more concise form. Namely, for every morphism $k : M \to N$ there are $N' \supseteq N$ and a morphism $g : M \to N'$ such that the following diagram commutes

$$\begin{tikzcd}
& N \arrow[dr, hook, "\mathrm{id}_N"] & \\
M \arrow[ur, "k"] \arrow[rr, hook, "g"'] & & N'
\end{tikzcd}$$

It is convenient to use the following terminology. We write $M \leq N$ for $M \subseteq N$ and $\mathrm{id}_M : M \hookrightarrow N$ is a morphism. We say that $k : M \to N$ extends to $g : M' \to N'$ if $k \subseteq h$, $M \leq M'$, and $N \leq N'$.

7.26 Proposition Assume c3, then the following are equivalent

1. $\mathcal{M}$ has the amalgamation property;
2. every morphism $k : M \to N$ extends to an embedding $g : M \hookrightarrow N'$.

Proof. 1⇒2 Given $k : M \to N$, the amalgamation property yields the following commutative diagram which can be simplified to the diagram at the right

$$\begin{tikzcd}
& N \arrow[dr, hook, "g_1"] & \\
M \arrow[ur, "k"] \arrow[dr, "\mathrm{id}_M"'] & & N' \\
& M \arrow[ur, hook, "g_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& N \arrow[dr, hook, "g_1"] & \\
M \arrow[ur, "k"] \arrow[rr, hook, "g_2"'] & & N'
\end{tikzcd}$$

Up to isomorphism we can assume $g_2 = \mathrm{id}_M$, i.e. that $N \leq N'$. Hence $g_2 : M \hookrightarrow N'$ is the required extension of $k : M \to N$.

2⇒1 Let $f_1 : M \to M_1$ and $f_2 : M \to M_2$ be given. Let $k = f_2 \circ f_1^{-1} : M_1 \to M_2$ and let $g : M_1 \hookrightarrow N'$ be the extension ensured by 2. Then we obtain

$$\begin{tikzcd}
& M_1 \arrow[dd, "f_2 \circ f_1^{-1}"] \arrow[dr, hook, "g"] & \\
M \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & N' \\
& M_2 \arrow[ur, hook, "\mathrm{id}_{M_2}"'] &
\end{tikzcd}$$

as required. □

We say that $\langle M_i : i < \lambda \rangle$ is a $\leq$-chain if $M_i \leq M_j$ for all $i < j < \lambda$. For the next theorem to hold we need the following property:

7.27 Definition We say that $\mathcal{M}$ is closed under union of $\leq$-chains if

c8. if $\langle M_i : i < \lambda \rangle$ is a $\leq$-chain, then $M_i \leq \bigcup_{j<\lambda} M_j$ for all $i < \lambda$.

The following is a general existence theorem for rich models. This general form requires large cardinalities. We leave to the reader to verify that if the number if finite morphisms is countable (up to isomorphism) then countable rich models exit.

7.28 Theorem Assume 7.4. Assume further c8 and that $\mathcal{M}$ has the amalgamation property. Let λ be such that $|L| < \lambda = \lambda^{<\lambda}$. Then there is a rich model N of cardinality λ.

Proof. We construct N as union of a $\leq$-chain of models $\langle N_i : i < \lambda \rangle$ such that $|N_i| = \lambda$. Let N_0 be any model of cardinality λ. At stage $i+1$, let $f : M \to N_i$ be the least morphism (in a well-ordering that we specify below) such that $|f| \leq |M| < \lambda$ and f has no extension to an embedding $f' : M \hookrightarrow N_i$. Apply the amalgamation property to obtain an total morphism $f' : M \hookrightarrow N'$ that extends $f : M \to N_i$. By the downword Löwenheim-Skolem Theorem we may assume $|N'| = \lambda$. Let $N_{i+1} = N'$. At limit stages take the union.

The well-ordering mentioned needs to be chosen so that in the end we forget nobody. So, first at each stage we well-order the isomorphism-type of the morphisms $f : M \to N_i$ such that $f \leq |M| < \lambda$. Then the required well-ordering is obtained by dovetailing all these well-orderings. The length of this enumeration is at most $\lambda^{<\lambda}$, which is λ by hypothesis.

We check that N is rich. Let $f : M \to N$ be a morphism and $|f| < |M| \leq \lambda$. As $|L| < \lambda$ we can approximate M with an elementary chain of structures of cardinality $< \lambda$. Hence we may as well assume that $|f| \leq |M| < \lambda$. The cofinality of λ is larger than $|f|$, hence $\operatorname{rng} f \subseteq N_i$ for some $i < \lambda$. So $f : M \to N_i$ is a morphism and at some stage j we have ensured the existence of an embedding of $f' : M \hookrightarrow N_{j+1}$ that extends f. □

7.29 Proposition Let $\mathcal{M}$ consist of all structures of some fixed signature and the elementary maps between these. Then $\mathcal{M}$ has the amalgamation property.

Proof. Let $k : M \to N$ be an elementary map. Let a enumerate $\operatorname{dom} k$ and let b enumerate M. Let $p(x\,; z) = \operatorname{tp}_M(b\,; a)$. The type $p(x\,; a)$ is consisent in M, in particular, it is finitely consistent and, by elementarity, $p(x\,; ka)$ is finitely consistent in N. By the compactness theorem, there is $N' \succeq N$ such that $N' \vDash p(c\,; ka)$ for some $c \in N'^{|x|}$. Hence $g = \{\langle b, c \rangle\} : M \to N'$ is the required elementary map that extends $k : M \to N$. □

Chapter 8

Some algebraic structures

The main result in this chapter is the elimination of quantifiers in algebraically closed fields, Corollary 8.22. In commutative algebra this is called Chevalley's Theorem on constructible sets. From it we derive Hilbert's Nullstellensatz. First we state the theorem the model theoretic language, Theorem 8.27, then we tranlate it in the traditional algebraic setting, Theorem 8.31. To obtain the latter theorem we introduce the model theoretic objects that correspond to prime and radical ideals of polynomials.

The first sections of this chapter are not a pre-requisite for Sections 8.4–8.7, at the cost of a few repetitions in the latter.

8.1 Notation Recall that when $A \subseteq M$ we denote by $\langle A\rangle_M$ the substructure of M generated by A. Then $\langle A\rangle_M \subseteq N$ is equivalent to $N \vDash \mathrm{Diag}\,\langle A\rangle_M$. The diagram of a structure has been defined in Notation 3.20.

In this chapter, whenever some $A \subseteq M \vDash T$ are fixed, by *model* we mean superstructures of $\langle A\rangle_M$ that models T. The notions of *logical consequence*, *consistency*, *completeness*, etc. are modified accordingly, and we write $\vdash$ for $T \cup \mathrm{Diag}\langle A\rangle_M \vdash$.

We say that a type $p(x)$ is trivial if $\vdash p(x)$.

8.1 Abelian groups

The language L is that of additive groups. The theory T_{ag} of abelian groups is axiomatized by the universal closure of the usual axioms

a1 $(x+y)+z \;=\; y+(x+z)$;

a2 $x+(-x) \;=\; (-x)+x \;=\; 0$;

a3 $x+0 \;=\; 0+x \;=\; x$;

a4 $x+y \;=\; y+x$.

Let x be a tuple of variables of length α, an ordinal. We write $L_{\mathrm{ter},x}$ for the set of terms $t(x)$ with free variables among x. On this set we define the equivalence relation

$$t(x) \sim s(x) \quad = \quad T_{\mathrm{ag}} \vdash t(x) = s(x).$$

We define the group operations on $L_{\mathrm{ter},x}/\!\sim$ in the obvious way. We denote by $\mathbb{Z}^{\oplus\alpha}$ the set of tuples of integers of length α that are almost always 0. The group operations on $\mathbb{Z}^{\oplus\alpha}$ are defined coordinate-wise. The following immediate proposition implies in particular that $L_{\mathrm{ter},x}/\!\sim$ is isomorphic to $\mathbb{Z}^{\oplus\alpha}$.

8.2 Proposition Let $A \subseteq M \vDash T_{\rm ag}$. Then for every formula $\varphi(x) \in L_{\rm at}(A)$ there are $n \in \mathbb{Z}^{\oplus\alpha}$ and $c \in \langle A\rangle_M$ such that

$$\vdash \quad \varphi(x) \ \leftrightarrow \ \sum_{i<\alpha} n_i x_i \ = \ c.$$

where $n = \langle n_i : i < \alpha\rangle$ and $x = \langle x_i : i < \alpha\rangle$.

Proof. Up to equivalence over $T_{\rm ag}$ the formula $\varphi(x)$ has the form $s(x) = t(a)$ for some parameter-free terms $s(x)$ and $t(z)$. Over Diag $\langle A\rangle_M$, we can replace $t(a)$ with a single $c \in \langle A\rangle_M$ and write $s(x)$ as the linear combination shown above. □

8.3 Definition Let $M \vDash T_{\rm ag}$. For $A \subseteq M$ and $c \in M$, we say that c is independent from A if $\langle A\rangle_M \cap \langle c\rangle_M = \{0\}$. Otherwise we say that c is dependent from A. The rank of M is the least cardinality of a subset $A \subseteq M$ such that all elements in M are dependent from A. We denote it by $\mathrm{rank}(M)$.

Note that when M is a vector space the condition $\langle A\rangle_M \cap \langle c\rangle_M = \{0\}$ is equivalent to saying that c is not a linear combination of vectors in A. Then rank M coincides with the dimension of M. In fact, what we do here for abelian groups could be easily generalized to D-modules, where D is any integral domain, and in particular to vector spaces. In practice, it is more convenient to use the following syntactic characterization of independence.

An element c of an abelian group is a torsion element if $nc = 0$ for some positive integer.

8.4 Proposition Let $A \subseteq M \vDash T_{\rm ag}$. Suppose that $c \in M$ is not a torsion element. Then the following are equivalent

1. c is independent from A;
2. $p(x) = \text{at-tp}_M(c/A)$ is trivial (see Notation 8.1).

Proof. 1⇒2 By Proposition 8.2, formulas in $p(x)$ may be assumed to have the form $nx = a$ for some integer n and some $a \in \langle A\rangle_M$. As this formula is satisfied by c then $a \in \langle A\rangle_M \cap \langle c\rangle_M$. Hence $a = 0$. As c is not a torsion element, $n = 0$ and the equation is trivial.

2⇒1 If $\langle A\rangle_M \cap \langle c\rangle_M \neq \{0\}$ then $nc = a$ for some $a \in \langle A\rangle_M \smallsetminus \{0\}$ and some $n \in \mathbb{Z} \smallsetminus \{0\}$. Then c satisfies the equation $nx = a$. This equation is nontrivial (e.g., in M it is not satisfied by $2c$).

□

8.5 Remark Let $k : M \to N$ be a partial embedding and let a be an enumeration of $\operatorname{dom} k$. We claim that $k \cup \{\langle b, c\rangle\} : M \to N$ is a partial embedding for every $b \in M$ and $c \in N$ that are independent from a, respectively ka. In fact, it suffices to check that $M, b, a \equiv_{\mathrm{at}} N, c, ka$. Suppose $\varphi(x\,;z) \in L_{\mathrm{at}}$ is such that $M \vDash \varphi(b\,;a)$. Then by independence $\varphi(x\,;a)$ is trivial, i.e.

$$T_{\mathrm{ag}} \cup \operatorname{Diag}\langle a\rangle_M \vdash \varphi(x\,;a).$$

As $\langle a\rangle_M$ and $\langle ka\rangle_N$ are isomorphic structures

$$T_{\mathrm{ag}} \cup \operatorname{Diag}\langle ka\rangle_N \vdash \varphi(x\,;ka).$$

Therefore $N \vDash \varphi(c\,;ka)$. This proves $M, b, a \Rrightarrow_{\mathrm{at}} N, c, ka$.

As the same assumptions apply to $k^{-1} : N \to M$, we also have $M, b, a \Lleftarrow_{\mathrm{at}} N, c, ka$.

8.2 Torsion-free abelian groups

The theory of torsion-free abelian groups extends T_{ag} with the following axioms for all positive integers n

tf $\quad nx = 0 \to x = 0.$

We denote this theory by T_{tfag}. It is not difficult to see that in a torsion-free abelian group every equation of the form $nx = a$ has at most one solution.

8.6 Proposition Let $M \vDash T_{\mathrm{tfag}}$ be uncountable. Then $\operatorname{rank} M = |M|$.

Proof. Let $A \subseteq M$ have cardinality $< |M|$. We claim that M contains some element that is independent from A. It suffices to show that the number of elements that are dependent from A is $< |M|$. If $c \in M$ is dependent from A then, by Proposition 8.7, it is a solution of some formula $L_{\mathrm{at}}(A)$. As there is no torsion, such a formula has at most one solution. Therefore the number of elements that are dependent from A is at most $|L_{\mathrm{at}}(A)|$, that is $\max\{|A|, \omega\}$. If M is uncountable the claim follows. $\square$

8.7 Proposition Let $A \subseteq M \vDash T_{\mathrm{tfag}}$. Let $p(x) = \mathrm{at}^{\pm}\text{-tp}(b/A)$, where $b \in M$. Then one of the following holds

1. b is independent from A;
2. $M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\mathrm{at}}(A)$ such that $\vdash \varphi(x) \to p(x)$.

Note the similarity with Example 7.15, where the independent type is $q(x)$ and the isolating formulas are the $r_i(x)$.

It is important to observe that the set A above may be infinite. This is essential to obtain Corollary 8.12, and it is one of the main differences between this example and the examples encountered in Chapter 6.

Proof. If b is dependent from A, then b satisfies a non trivial atomic formula $\varphi(x)$ which we claim is the formula required in 2. It suffices to show that $\varphi(x)$ implies a complete $L_{\mathrm{at}^{\pm}}(A)$-type. Clearly this type must be $p(x)$. Let $\varphi(x)$ have the form $nx = a$ for some $n \in \mathbb{Z} \smallsetminus \{0\}$ and $a \in \langle A\rangle_M \smallsetminus \{0\}$. We show that for every $m \in \mathbb{Z}$ and every $c \in \langle A\rangle_M$ one of the following holds

a. $\vdash\ nx = a \to mx = c$;

b. $\vdash\ nx = a \to mx \neq c$.

Suppose not for a contradiction that neither a nor b holds and fix models N_1, N_2 and some $b_i \in N_i$ such that

a′. $N_1 \vDash nb_1 = a$ and $N_1 \vDash mb_1 \neq c$;

b′. $N_2 \vDash nb_2 = a$ and $N_2 \vDash mb_2 = c$.

From b′ we infer that $N_2 \vDash ma = nc$. As N_1 is torsion-free, from a' we infer that $N_1 \vDash ma \neq nc$. But $ma = nc$ is a formula with parameters in $\langle A\rangle_M$, so it should have the same truth value in all superstructures of $\langle A\rangle_M$, a contradiction. □

8.3 Divisible abelian groups

The theory of divisible abelian groups extends $T_{\rm tfag}$ with the following axioms for all integers $n \neq 0$

div $y \neq 0 \to \exists x\ nx = y$.

We denote this theory by $T_{\rm dag}$.

8.8 Proposition Let $A \subseteq M \vDash T_{\rm tfag}$ and let $\varphi(x) \in L_{\rm at}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\, \varphi(x)$ for every N such that $\langle A\rangle_M \subseteq N \vDash T_{\rm dag}$.

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A\rangle_M \subseteq M' \vDash T_{\rm tfag}$.

The claim in the proposition holds more generally for all $\varphi(x) \in L_{\rm qf}$ and also when x is a tuple of variables. This follows from Lemma 8.10, whose proof uses the proposition.

Proof. We can assume that $\varphi(x)$ has the form $nx = a$ for some $n \in \mathbb{Z}$ and some $a \in \langle A\rangle_M$. If $n = 0$, then $a = 0$ since $\varphi(x)$ is consistent, and the claim is trivial. If $n \neq 0$ then by consistency $a \neq 0$, hence a solution exist in N by axiom div. □

8.9 Exercise Prove a converse of Proposition 8.8. Let $A \subseteq N \vDash T_{\rm ag}$ and let x be a single variable. Prove that if $N \vDash \exists x\, \varphi(x)$ for every consistent $\varphi(x) \in L_{\rm at}(A)$, then $N \vDash T_{\rm dag}$.

We are ready to prove that divisible abelian groups of infinite rank are ω-rich.

8.10 Lemma Let $k : M \to N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\rm tfag}$ and $N \vDash T_{\rm dag}$ is a model of rank $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \{\langle b\,;c\rangle\} : M \to N$ is a partial isomorphism.

Proof. Let a be an enumeration of $\operatorname{dom} k$ and let $p(x\,;z) = {\rm at}^{\pm}\text{-tp}(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is dependent from a, then Proposition 8.7 yields a formula $\varphi(x\,;z) \in L_{\rm at}$ such that

i. $\vdash\ \varphi(x\,;a) \to p(x\,;a)$

ii. $\varphi(x\,;a)$ is consistent.

By isomorphism, i and ii hold with a replaced by ka. Then by Proposition 8.8 the formula $\varphi(x\,;ka)$ has a solution $c \in N$.

The second case, which has no analogue in Lemma 6.1, is when b is independent from a. Then by Remark 8.5 we may choose c to be any element of N independent from ka. Such an element exists because N has rank at least λ. □

Below a few important consequences of this lemma.

8.11 Corollary Work in the category of models of T_{tfag} with partial embeddings as morphisms. Then the following are equivalent

1. N is a λ-rich model
2. $N \vDash T_{\text{dag}}$ and has rank $\geq \lambda$

In particular every uncountable $N \vDash T_{\text{dag}}$ is rich.

8.12 Corollary The theory T_{dag} is uncountably categorical, complete, and has quantifier elimination.

Proof. Categoricity and completeness ar immediate (the category presented in the corollary above is connected). As for quantifier elimination, let $k : M \to N$ is a partial isomorphism between models of T_{dag}. If $M \preceq M'$ and $M \preceq N'$ are elementary superstructures of uncountable cardinality then $k : M' \to N'$ is elementary by Theorem 7.11 and this suffices to conclude that $k : M \to N$ is elementary. □

8.13 Exercise Prove that every model of T_{dag} is ω-ultrahomogeneous (independently of cardinality and rank).

8.4 Commutative rings

In this section L is the language of (unital) rings. It contains two constants 0 and 1 the unary operation $-$ and two binary operations $+$ and $\cdot$. The theory of rings contains the following axioms

a1-a4 as for abelian groups

r1 $\quad (x\cdot y)\cdot z \;=\; y\cdot(x\cdot z),$

r2 $\quad 1\cdot x \;=\; x\cdot 1 \;=\; x,$

r3 $\quad (x+y)\cdot z \;=\; x\cdot z + y\cdot z,$

r4 $\quad z\cdot(x+y) \;=\; z\cdot x + z\cdot y.$

All the rings we consider are commutative

c $\quad x\cdot y \;=\; y\cdot x.$

We denote the theory of commutative rings by T_{cr}.

In what follows the theory $T_{\text{cr}} \cup \text{Diag}\langle A\rangle_M$, for some M clear from the context, is implicit in the sense of Notation 8.1. So it is important to remember that $\text{Diag}\langle A\rangle_M$ is not trivial even when $A = \varnothing$. In fact, $\text{Diag}\langle\varnothing\rangle_M$ determines the characteristic of the models.

Let $A \subseteq M \vDash T_{\rm cr}$ and let x be a tuple of variables. We write $L_{{\rm ter},x}(A)$ for the set of terms $t(x)$ with free variables among x and parameters in A. On this set we define the equivalence relation

$$t(x) \sim s(x) \quad = \quad \vdash\ t(x) = s(x).$$

On $L_{{\rm ter},x}(A)/\!\sim$ we define the ring operations in the obvious way so that $L_{{\rm ter},x}(A)/\!\sim$ is a commutative ring. We denote by $A[x]$ the set of polynomials with variables among x and parameters in $\langle A\rangle_M$. The ring operations on $A[x]$ are defined as usual. The following proposition (which is clear, but tedious to prove) implies in particular that $L_{{\rm ter},x}(A)/\!\sim$ is isomorphic to $A[x]$. For simplicity we state it only for $|x| = 1$.

8.14 Proposition Let $A \subseteq M \vDash T_{\rm cr}$ and let x be a single variable. Then for every formula $\varphi(x) \in L_{\rm at}(A)$ there is a unique $n < \omega$ and a unique tuple $\langle a_i : i \le n\rangle$ of elements of $\langle A\rangle_M$ such that $a_n \neq 0$ and

$$\vdash \quad \varphi(x) \ \leftrightarrow\ \sum_{i\le n} a_i x^i \ =\ 0.$$

The integer n in the proposition above is called the degree of $\varphi(x)$.

8.15 Definition Let $A \subseteq M \vDash T_{\rm cr}$. We say that an element $b \in M$ is transcendental over A if the type $p(x) = \text{at-tp}_M(b/A)$ is trivial (see Notation 3.20 and 8.1). Otherwise we say that b is algebraic over A. The transcendence degree of M is the least cardinality of a subset $A \subseteq M$ such that all the elements of M are algebraic over A.

8.16 Remark Remark 8.5 holds here with 'independent' replaced by 'transcendental' and $T_{\rm ag}$ replaced by $T_{\rm cr}$.

8.5 Integral domains

Let $a \in M \vDash T$. We say that a is a zero divisor if $a\,b = 0$ for some $b \in M \smallsetminus \{0\}$. An integral domain is a commutative ring without zero divisors. The theory of integral domains contains the axioms of commutative rings and the following

nt. $\ 0 \neq 1$

id. $\ x{\cdot}y = 0 \ \rightarrow\ x = 0 \ \vee\ y = 0.$

We denote the theory of integral domains by $T_{\rm id}$.

For a prime p, we define the theory $T^p_{\rm id}$, which contains $T_{\rm id}$ and the axiom

ch$_p$. $\ 1 + \cdots (p \text{ times}) \cdots + 1 = 0.$

The theory $T^0_{\rm id}$ contains the negation of ch$_p$ for all p. Note that all models of $T^p_{\rm id}$ have the same characteristic in the model theoretic sense defined in 3.28. In the remaining section we work in the category of models of $T_{\rm id}$ with partial embeddings as morphisms. This category consists of countably many connected components each containing all models of $T^p_{\rm id}$ for some p.

8.17 Proposition Let $M \vDash T_{\mathrm{id}}$ be uncountable. Then M has transcendence degree $|M|$.

Proof. In an integral domain every polynomial has finitely many solutions and there are $|L(A)|$ polynomials over A. □

8.18 Proposition Let $A \subseteq M \vDash T_{\mathrm{id}}$. For $b \in M$ let $p(x) = \mathrm{at}^{\pm}\text{-tp}(b/A)$. Then one of the following holds

1. b is transcendental over A;
2. $M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\mathrm{at}}(A)$ such that $\vdash \varphi(x) \rightarrow p(x)$.

Note the similarity with Example 7.15, where the transcendental type is $q(x)$ and the isolating formulas are the $r_i(x)$.

As in Proposition 8.7, the set A may be infinite. This is essential to obtain Corollary 8.21.

Proof. Suppose b is not transcendental, i.e. it satisfies a non trivial atomic formula. Let $\varphi(x) \in L_{\mathrm{at}}(A)$ be a non trivial formula with minimal degree such that $\varphi(b)$. We prove that $\varphi(x)$ implies a complete $L_{\mathrm{at}^{\pm}}(A)$-type. Clearly this type must be $p(x)$. We prove that for any $\xi(x) \in L_{\mathrm{at}}(A)$ one of the following holds

1. $\vdash \varphi(x) \rightarrow \xi(x)$
2. $\vdash \varphi(x) \rightarrow \neg\xi(x)$.

Let us write $a(x) = 0$ and $a'(x) = 0$ for the formulas $\varphi(x)$ and $\xi(x)$, respectively. If $\langle A \rangle_M$ is a field, choose a polynomial $d(x)$ of maximal degree such that for some polynomials $t(x)$ and $t'(x)$ the following hold

a. $d(x)\,t(x) = a(x)$

a′. $d(x)\,t'(x) = a'(x)$,

If $\langle A \rangle_M$ is not a field, polynomials $d(x)$, $t(x)$ and $t'(x)$ as above exist with coefficients in the field of fractions of $\langle A \rangle_M$. Then a and a' hold up to a factor in $\langle A \rangle_M$ which we absorb in $a(x)$ and $a'(x)$.

From a we get $d(b) = 0$ or $t(b) = 0$. In the first case, as $a(x)$ has minimal degree, we conclude that $t(x)$ is constant. This implies that any zero of $a(x)$ is also a zero of $a'(x)$, that is, it implies 1.

Now suppose $t(b) = 0$. Then the minimality of the degree of $a(x)$ implies that $d(x) = d$, where d is a nonzero constant. If $\langle A \rangle_M$ is a field, apply Bézout's identity to obtain two polynomials $c(x)$ and $c'(x)$ such that $d = a(x)c(x) + a'(x)c'(x)$. Then $a(x)$ and $a'(x)$ have no common zeros, and 2 follows. If $\langle A \rangle_M$ is not a field, we use Bézout's identity in the field of fractions of $\langle A \rangle_M$ and, for some $d' \in \langle A \rangle_M \smallsetminus \{0\}$, obtain $d'd = a(x)c(x) + a'(x)c'(x)$. Then we reach the same conclusion. □

8.6 Algebraically closed fields

Let $a, b \in M \vDash T_{\mathrm{id}}$. We say that b is the inverse of a if $a{\cdot}b = 1$. A field is a commutative ring where every non-zero element has an inverse. The theory of fields contains T_{id} and the axiom

f. $\exists y\,\big[x \neq 0 \;\rightarrow\; x{\cdot}y = 1\big]$.

Fields are structures in the signature of rings: the language contains no symbol for the multiplicative inverse. So, substructures of fields are merely integral domains.

The theory of algebraically closed field, which we denote by T_{acf}, also contains the following axioms for every positive integer n

ac. $\exists x\;\big(x^n + z_{n-1}x^{n-1} + \cdots + z_1 x + z_0 \;=\; 0\big)$

The theory T^p_{acf} is defined in analogy to T^p_{id} in the previous section.

8.19 Proposition Let $A \subseteq M \vDash T_{\mathrm{id}}$ and let $\varphi(x) \in L_{\mathrm{at}}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\,\varphi(x)$ for every model $N \vDash T_{\mathrm{acf}}$.

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A\rangle_M \subseteq M' \vDash T_{\mathrm{id}}$.

The claim in the proposition holds more generally for all $\varphi(x) \in L_{\mathrm{qf}}$ when x is a tuple of variables. This follows from Lemma 8.20 whose proof uses the proposition.
Proof. Up to equivalence $\varphi(x)$ has the form $a_n x^n + \cdots + a_1 x + a_0 \;=\; 0$ for some $a_i \in \langle A\rangle_N$. Choose n minimal. If $n = 0$ then $a_0 = 0$ by the consistency of $\varphi(x)$ and the claim is trivial. Otherwise $a_n \neq 0$ and the claim follows from f and ac. □

8.20 Lemma Let $k : M \to N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\mathrm{id}}$ and $N \vDash T_{\mathrm{acf}}$ has transcendence degree $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \big\{\langle b, c\rangle\big\} : M \to N$ is a partial isomorphism.

The following is the proof of Lemma 8.10 which we repeat here for convenience.
Proof. Let a be an enumeration of $\operatorname{dom} k$ and let $p(x\,;z) \;=\; \mathrm{at}^{\pm}\text{-}\mathrm{tp}_M(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is algebraic over a, then Proposition 8.18 yields a formula $\varphi(x\,;z) \in L_{\mathrm{at}}$ such that

i. $\varphi(x\,;a) \rightarrow p(x\,;a)$

ii. $\varphi(x\,;a)$ is consistent.

By isomorphism i and ii hold with a replaced by ka. Then by Proposition 8.19 the formula $\varphi(x\,;ka)$ has a solution in $c \in N$.

The second case, which has no analogue in Lemma 6.1, is when b is transcendental over a. Then by Remark 8.16 we may choose c to be any element of N transcendental over ka. This exists because N has transcendence degree $\geq \lambda$. □

Below a few important consequences of this lemma.

8.21 Corollary Work in the category of models of T_{id} with partial embeddings as morphisms. Then the following are equivalent

1. N is a λ-rich model

2. $N \vDash T_{\mathrm{acf}}$ and has transcendence degree $\geq \lambda$

In particular every uncountable $N \vDash T_{\mathrm{acf}}$ is rich.

Proof. Implication 2 ⇒ 1 is an immediate consequence of Lemma 8.20.

In every connected component there is an $M \models T_{\rm acf}$ of cardinality λ and transcendence degree λ (by Proposition 8.17 when $\lambda > \omega$, by compactness for $\lambda = \omega$). As proved above, M is rich and therefore elementarily equivalent to any λ-rich model N in the same connected component. This proves 1 ⇒ 2. □

8.22 Corollary The theory $T_{\rm acf}$ has elimination of quantifiers.

Proof. Let $k : M \to N$ be a partial embedding between models of $T_{\rm acf}$. Let M' and N' be elementary superstructures of M and N respectively of sufficiently large cardinality. As M' and N' are rich, $k : M' \to N'$ is elementary by Theorem 7.11. Hence $k : M \to N$ is also elementary. □

8.23 Corollary The theories $T^p_{\rm acf}$ are complete and uncountably categorical (i.e. λ-categorical for every uncountable λ)

Proof. Two models of $T^p_{\rm acf}$ belong to the same connected component. Then, as every uncountable model of $T^p_{\rm acf}$ is rich, uncountable categoricity and completeness follow. □

8.24 Exercise Prove that every model of $T_{\rm acf}$ is ω-ultrahomogeneous (independently of cardinality and transcendence degree).

8.7 Hilbert's Nullstellensatz

Fix a tuple of variables x and a subset A of an integral domain M. In this section we are interested in formulas in $L_{\rm at}(A)$, hence we redefine the symbol of closure under logical consequence accordingly

$$\operatorname{ccl} p(x) \quad = \quad \Big\{\varphi(x) \in L_{\rm at}(A) \ : \ p(x) \vdash \varphi(x)\Big\}.$$

Recall that we work under the assumptions in Notation 8.1. In particular, in this section we work over the theory $T_{\rm id} \cup \operatorname{Diag}\langle A\rangle_M$, and the symbol $\vdash$ has to be interpreted accordingly.

In general, the notion of closure under logical consequence is elusive. In this respect Propositions 8.25 and 8.27 are useful, as they give a model theoretic characterization of $\operatorname{ccl} p(x)$. Corollary 8.30 gives an algebraic characterization.

8.25 Proposition Let $A \subseteq M \models T_{\rm id}$ and let $p(x) \subseteq L_{\rm at}(A)$. Let N be of sufficiently large cardinality and such that $\langle A\rangle_M \subseteq N \models T_{\rm acf}$. Then

$$\operatorname{ccl} p(x) \quad = \quad \Big\{\varphi(x) \in L_{\rm at}(A) \ : \ N \models \forall x \left[p(x) \to \varphi(x)\right]\Big\}.$$

The cardinality of N is sufficiently large if $|L(A)| < |N|$ and $|x| \leq |N|$; note that here x has possibly infinite length.

Proof. Only the inclusion $\supseteq$ requires a proof. Suppose $\varphi(x) \in L_{\rm at}(A)$ is such that $p(x) \wedge \neg\varphi(x)$ is consistent. Then there is a model M_1 of cardinality $< |L(A)|$ and $\leq |x|$ such that $M_1 \models p(a) \wedge \neg\varphi(a)$ for some $a \in {M_1}^{|x|}$. By Corollary 8.21, if N is large enough, there is a partial isomorphism $h : M_1 \to N$ that extends id_A and is defined on a. Therefore $N \vDash p(ha) \wedge \neg\varphi(ha)$. So $\varphi(x)$ does not belong to the set on the r.h.s. □

In the proposition above we could replace L_{at} by $L_{\mathrm{at}^{\pm}}$. But here we are interesed in a strenghening, the Nullstennensatz, that requires positive formulas.

Hilbert's Nullstellensatz extends the validity of the proposition above to the case $A = M = N$ (assuming x finite). While Proposition 8.25 could be generalized to other theories, the Nullstellensatz rests on an exquisitely algebraic phenomenon. In fact, model theory has no general tools to deal with large sets of parameters. Algebra comes to our aid with the following lemma. Which the reader may wish to compare with Hilbert's Basis Theorem.

8.26 Lemma Let $M \vDash T_{\mathrm{id}}$. Let $p(x) \subseteq L_{\mathrm{at}}(M)$, where x is finite, be closed under logical consequence. Then $\psi(x) \vdash p(x)$, for some $\psi(x)$ conjunction of formulas in $p(x)$.

Proof. By induction on the length of x. If x is the empty tuple, the lemma is trivial. Now, let x be a finite tuple and let y be a single variable. Let $p(x,y) \subseteq L_{\mathrm{at}}(M)$ be closed under logical consequence. Write $p(x)$ for the set of formulas $\varphi(x) \in L_{\mathrm{at}}(M)$ such that $p(x,y) \vdash \varphi(x)$. Assume as induction hypothesis that there is a conjunction of formulas in $p(x)$, say $\psi(x)$, such that $\psi(x) \vdash p(x)$. We prove the lemma with x, y for x.

Let $\psi(x,y) \in L_{\mathrm{at}}(M)$ have minimal degree (in y) among the formulas in $p(x,y)$ such that $\psi(x) \nvdash \psi(x,y)$. Let n be the degree of $\psi(x,y)$, which is non zero, because $\psi(x) \nvdash \psi(x,y)$. So we may write

$$\psi(x,y) \quad = \quad \Big(t_n(x) \cdot y^n + t'(x,y) = 0\Big),$$

where $t'(x,y)$ is a polynomial of degree $< n$. We claim that

1 $\qquad \psi(x) \wedge \psi(x,y) \quad \vdash \quad p(x,y)$

Suppose not for a contradiction. Pick among the formulas in $p(x,y)$ that are a counterexample to 1, one of minimal degree, say $\varphi(x,y)$. By the minimaly of n, the degree of $\varphi(x,y)$ is $n+i$ for some $i \geq 0$. Hence we may write

$$\varphi(x,y) \quad = \quad \Big(s_{n+i}(x) \cdot y^{n+i} + s'(x,y) = 0\Big),$$

for some polynomial $s'(x,y)$ of degree $< n+i$. From $p(x,y) \vdash \varphi(x,y)$ we obtain

$$p(x,y) \quad \vdash \quad \Big(s_{n+i}(x) \cdot t_n(x) \cdot y^{n+i} + t_n(x) \cdot s'(x,y) = 0\Big).$$

Now, using that $p(x,y) \vdash \psi(x,y)$

2. $$p(x,y) \quad \vdash \quad \Big(-s_{n+i}(x) \cdot t'(x,y) \cdot y^i + t_n(x) \cdot s'(x,y) = 0\Big).$$

But the polynomial in 2 has degree $< n+i$ and this contradicts the minimality of the degree of $\varphi(x,y)$. A contradicion that proves the proposition. □

8.27 Hilbert's Nullstellensatz (model theoretic version) Let $N \vDash T_{\mathrm{acf}}$. For every type $p(x) \subseteq L_{\mathrm{at}}(N)$, where x is a finite tuple, we have

$$\operatorname{ccl} p(x) \quad = \quad \Big\{\varphi(x) \in L_{\mathrm{at}}(N) \;:\; N \vDash \forall x\, [p(x) \to \varphi(x)]\Big\}.$$

Proof. By Proposition 8.25, the equality above holds if we replace N by a sufficiently large elementay extension N', i.e.

$$= \Big\{\varphi(x) \in L_{\rm at}(N) \ :\ N' \vDash \forall x\,[p(x) \rightarrow \varphi(x)]\Big\}.$$

Since $p(x)$ and $\operatorname{ccl} p(x)$ are logically equivalent, we can replace one by the other. Then, by Lemma 8.26, we can replace $\operatorname{ccl} p(x)$ by an equivalent formula $\psi(x)$

$$= \Big\{\varphi(x) \in L_{\rm at}(N) \ :\ N' \vDash \forall x\,[\psi(x) \rightarrow \varphi(x)]\Big\}.$$

Now, by elementarity, we replace N back

$$= \Big\{\varphi(x) \in L_{\rm at}(N) \ :\ N \vDash \forall x\,[\psi(x) \rightarrow \varphi(x)]\Big\}.$$

Finally, we replace $p(x)$ back and obtain the desired equality. □

In the rest of this section, we show how to translate the model theoretic version of the Nullstellensatz into a more common algebraic formulation.

Let $A[x]$ be the ring of polynomials with variables in x and coefficients in $\langle A\rangle_M$. We identify $L_{\rm at}(A)$ and $A[x]$ in the obvious way. A type $p(x) \subseteq L_{\rm at}(A)$ is identified with a set $\mathcal{P}_p \subseteq A[x]$. Conversely, for any subset $\mathcal{P} \subseteq A[x]$ we write $p_{\mathcal{P}}(x)$ for the associated type.

We would like to characterize $\operatorname{ccl} p(x)$ in algebraic terms. The following is a preliminary result.

8.28 Proposition Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x) \subseteq L_{\rm at}(A)$. The following are equivalent

1. $p(x)$ is a prime type;
2. $\mathcal{P}_p$ is a prime ideal.

Proof. 1⇒2 Recall that, by our definition, a prime type is closed under logical consequence. Therefore to prove that $\mathcal{P}_p$ is an ideal, it suffices to note that the following entailments hold for every pair of polynomials $t(x)$ and $s(x)$

$$\begin{aligned} t(x) = 0 \ &\vdash\ s(x)t(x) = 0 \\ s(x) = t(x) = 0 \ &\vdash\ s(x) + t(x) = 0 \end{aligned}$$

Finally, to prove that $\mathcal{P}_p$ is prime, suppose that the polynomial $t(x) \cdot s(x)$ belongs to $\mathcal{P}_p$. As we are working over $T_{\rm id}$

$$t(x) \cdot s(x) = 0 \ \vdash\ t(x) = 0 \vee s(x) = 0$$

If $p(x)$ is a prime type, $p(x) \vdash t(x) = 0$ or $p(x) \vdash s(x) = 0$. As $p(x)$ is closed under logical consequence, $t(x) \in \mathcal{P}_p$ or $s(x) \in \mathcal{P}_p$.

2⇒1 First, we prove that $p(x)$ is closed under logical consequence. Equivalently, we assume that $t(x) = 0 \notin p(x)$ and prove that $p(x) \wedge t(x) \neq 0$ is consistent. As $\mathcal{P}_p$ is a prime ideal, $A[x]/\mathcal{P}_p$ is integral domain. We denote by $x + \mathcal{P}_p$ the equivalence class of the polynomial $x \in A[x]$. Then $x + \mathcal{P}_p$ satisfies exactly the formulas in $p(x)$. Hence it witnesses the consistency of $p(x) \wedge t(x) \neq 0$ in $A[x]/\mathcal{P}_p$.

We now prove that $p(x)$ is prime. Let $t_i(x)$ be polynomials such that

$$p(x) \ \vdash \ \bigvee_{i=1}^{n} t_i(x) = 0.$$

Then

$$p(x) \ \vdash \ \prod_{i=1}^{n} t_i(x) \ = \ 0$$

Since $p(x)$ is closed under logical consequence, and $\mathcal{P}_p$ is a prime ideal, $t_i(x) \in \mathcal{P}_p$ for some i. Hence $p(x)$ contains the equation $t_i(x) = 0$. By Corollary 3.19 this suffices to prove that $p(x)$ is a prime type. $\square$

8.29 Proposition Let $A \subseteq M \vDash T_{\text{id}}$ and let $p(x) \subseteq L_{\text{at}}(A)$. Then the following are equivalent for every $\varphi(x) \in L_{\text{at}}(A)$

1. $p(x) \vdash \varphi(x)$;
2. $q(x) \vdash \varphi(x)$ for every prime type $q(x) \subseteq L_{\text{at}}(A)$ containing $p(x)$.

Proof. Only 2⇒1 requires a proof. Assume that $p(x) \nvdash \varphi(x)$. Then there is an N such that $\langle A \rangle_M \subseteq N \vDash T_{\text{id}}$ and $N \vDash p(a) \wedge \neg\varphi(a)$ for some $a \in N^{|x|}$. Let $q(x) = \text{at-tp}_N(a/A)$. Then $q(x)$ is prime and $\varphi(x) \notin q(x)$. $\square$

For any subset $\mathcal{P} \subseteq A[x]$ we write $\sqrt{\mathcal{P}}$ for the intersection of all prime ideals containing $\mathcal{P}$. This is called the radical ideal generated by $\mathcal{P}$. From Propositions 8.28 and 8.29 we obtain the following algebraic characterization of $\operatorname{ccl} p(x)$.

8.30 Corollary Let $A \subseteq M \vDash T_{\text{id}}$. Then $\mathcal{P}_{\operatorname{ccl} p(x)} = \sqrt{\mathcal{P}_p}$ for every $p(x) \subseteq L_{\text{at}}(A)$.
Equivalently, $\operatorname{ccl} p_{\mathcal{P}}(x) = p_{\sqrt{\mathcal{P}}}$ for every $\mathcal{P} \subseteq A[x]$.

We are now ready to translate the Nullstellensatz into a language more familiar to algebraists. Fix $N \vDash T_{\text{acf}}$. Let $\mathcal{P} \subseteq N[x]$. The algebraic variety associated to $\mathcal{P}$, often denoted by $V(\mathcal{P})$, is the set $p_{\mathcal{P}}(N)$. Viceversa, given $A \subseteq N$, let $\mathcal{I}(A)$ be the set of polynomials in $N[x]$ that vanish at all points in A. Clearly, $\mathcal{I}(A)$ is an ideal of $N[x]$.

8.31 Hilbert's Nullstellensatz (standard version) Let $N \vDash T_{\text{acf}}$. Let $\mathcal{P} \subseteq N[x]$ where x is a finite tuple. Then $\mathcal{I}(V(\mathcal{P})) = \sqrt{\mathcal{P}}$.

Proof. Note that $V(\mathcal{P}) = p_{\mathcal{P}}(N)$. Hence $p_{\mathcal{I}}(x)$, where $\mathcal{I} = \mathcal{I}(V(\mathcal{P}))$, is the set of formulas $\varphi(x) \in L_{\text{at}}(N)$ such that $V(\mathcal{P}) \subseteq \varphi(N)$. In other words, $p_{\mathcal{I}}(x)$ is the set containing those formulas such that $N \vDash \forall x[p_{\mathcal{P}}(x) \to \varphi(x)]$. Hence from the model theoretic version of the Nullstellensatz we obtain $p_{\mathcal{I}}(x) = \operatorname{ccl} p_{\mathcal{P}}(x)$, that in turn coincides with $p_{\mathcal{I}}(x) = p_{\sqrt{\mathcal{P}}}(x)$ by the corollary above. $\square$

Chapter 9

Ordered algebraic structures

Work in progress

9.1 Ordered divisible abelian groups

We work in language of ordered additive groups $L = \{0, -, +, <\}$. The theory of ordered abelian groups, denoted by $T_{\rm oag}$, contains $T_{\rm lo} \cup T_{\rm ag}$ and the axiom

oa. $x < y \;\rightarrow\; x + z < y + z.$

It is not difficult to verify that for every positive integer n

$$T_{\rm oag} \;\vdash\; x < y \;\leftrightarrow\; n\,x < n\,y.$$

In particular, every model of $T_{\rm oag}$ is a torsion-free group and its order has no endpoints.

We write $T_{\rm odag}$ for the theory of ordered divisible abelian groups which is obtained by adding to $T_{\rm oag}$ the axiom of divisibility, div of Section 8.8.3. Note that

$$T_{\rm oag} \;\vdash\; x < y \wedge 2z = x + y \;\;\rightarrow\;\; x < z < y.$$

Hence $T_{\rm odag}$ is a dense linear order.

We base our study of $T_{\rm odag}$ on what we know about models of $T_{\rm dag}$ and $T_{\rm dlo}$. For this reason we split $L_{\rm at}$ into equations and inequalities. We use the notation $L_{{\rm at},=}$ and $L_{{\rm at},<}$ respectively.

When $p(x) \subseteq L_{\rm at}(A)$ we write $p_=(x)$ and $p_<(x)$ for the intersection of $p(x)$ with $L_{{\rm at},=}$ and $L_{{\rm at},<}$ respectively. Note that, by the linearity of the order, the types $\text{at-tp}(b/A)$ and $\text{at}^{\pm}\text{-tp}(b/A)$ are equivalent.

If $p_=(x)$ is trivial, as defined in Notation 8.1, then we say that $p(x)$ is equationally trivial. In this case $\vdash p(x) \leftrightarrow p_<(x)$.

For any $A \subseteq M \vDash T_{\rm oag}$ and $c \in M$, we say that c is independent from A if $\text{at-tp}_M(c/A)$ is equationally trivial. Remark 8.5 takes the following form

9.1 Remark Let $k : M \rightarrow N$ be a partial embedding and let a be an enumeration of $\operatorname{dom} k$. We claim that $k \cup \big\{\langle b, c\rangle\big\} : M \rightarrow N$ is a partial embedding for every $b \in M$ and $c \in N$ that are independent from a, respectively ka and such that $\text{at-tp}_{M,<}(b\,;a) = \text{at-tp}_{N,<}(c\,;ka)$.

Dependent elements behave pretty much as those in models of $T_{\rm tfag}$. In fact Proposition 8.7 holds also here. The main difference is that now it is no longer true that all independent elements have the same type.

9.2 Proposition Let $A \subseteq M \vDash T_{\text{oag}}$. Let $p(x) = \text{at-tp}(b/A)$, where $b \in M$. Then one of the following holds

1. b is independent from A;
2. $M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\text{at}}(A)$ such that $\vdash \varphi(x) \to p(x)$.

Proof. If we choose the equation $\varphi(x)$ as in the proof of Proposition 8.7 we obtain $\vdash \varphi(x) \to p_{=}(x)$ with the same proof. We only need to show that $\vdash \varphi(x) \to p_{<}(x)$. We repeat the argument of the proof of Proposition 8.7 with $<$ for $=$. We need to prove that for every $m \in \mathbb{Z}$ and every $c \in \langle A \rangle_M$ one of the following holds

a. $\vdash nx = a \to c < mx$;

b. $\vdash nx = a \to mx < c$;

c. $\vdash nx = a \to mx = c$.

Suppose for a contradiction that neither a nor b holds and fix models N_1, N_2 and some $b_i \in N_i$ such that

a′. $N_1 \vDash nb_1 = a$ and $N_1 \vDash c \nless mb_1$;

b′. $N_2 \vDash nb_2 = a$ and $N_2 \vDash mb_2 \nless c$;

b′. $N_3 \vDash nb_3 = a$ and $N_3 \vDash mb_3 \nless c$.

From a′ we infer that $N_1 \vDash nc \nless ma$. From b′ we infer $N_2 \vDash ma \nless nc$. From c′ we infer $N_3 \vDash ma \neq c$. But this three formulas also hold in $\langle A \rangle_M$. A contradiction. □

As Remark 9.1 suggests, infinite rank alone is not sufficient to guarantee richness. The following property will be required.

9.3 Definition We say that $M \vDash T_{\text{oag}}$ is ω_1-large if for every countable $A \subseteq M$, every finitely consistent $p(x) \subseteq L_{\text{at},<}(A)$ is realized in M.

The term *large* is unimaginative, but it is provisional and will not occur elsewhere. It will be clear in Section **??.??** that largeness captures the notion of ω-saturation for models of T_{oag}. The proof of the following proposition is left to the reader as an exercise.

9.4 Proposition Every $M \vDash T_{\text{odag}}$ has a large elementary extension.

As it happens, large implies infinite rank.

9.5 Proposition Let $M \vDash T_{\text{odag}}$ be ω_1-large. Then for every countable $A \subseteq M$ there is a dense set of elements that are independent of A. In particular M has infinite rank.

Proof. Let $a < b$ be arbitrary elements of M. It suffices to construct a finitely consistent type $p(x) \subseteq L_{\text{at},<}(A, a, b)$ containing $a < x < b$ that has only solutions independent of A. Dependent elements are solution of a non trivial equation with parameters in A. As there are countably many of such equations and each has a unique solution, we can enumerate $\langle a_i : i \in \omega \rangle$ all elements dependent from A. By density, the type $\{a_i \neq x\}$ is finitely consistent with $a < x < b$. We obtain $p(x)$ by replacing $a_i \neq x$ with either $x < a_i$ or $a_i < x$ according to which one preserve finite consistency. □

We are ready to prove that ω_1-large models of $T_{\rm odag}$ are ω_1-rich.

9.6 Lemma Let $N \vDash T_{\rm odag}$ be ω_1-large. Then for every $M \vDash T_{\rm oag}$, every $b \in M$ and every countable partial embedding $k : M \to N$ there is a $c \in N$ such that $k \cup \{\langle b\,;c\rangle\} : M \to N$ is a partial isomorphism.

(That is, N is ω_1-rich in the category of models of $T_{\rm oag}$ with partial embeddings as morphisms.)

Proof. Let a be an enumeration of $\operatorname{dom} k$ and let $p(x\,;z) = \text{at}^{\pm}\text{-tp}(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is dependent from a, then Proposition 9.2 yields a formula $\varphi(x\,;z) \in L_{\rm at,=}$ such that

i. $\vdash\ \varphi(x\,;a) \to p(x\,;a)$;

ii. $\varphi(x\,;a)$ is consistent (the underlying theory is $T_{\rm oag} \cup \langle a\rangle_M$).

By isomorphism, i and ii hold with a replaced by ka. As remarked at the beginning of this section $M \vDash T_{\rm tfag}$ and $N \vDash T_{\rm dag}$. As $\varphi(x\,;z) \in L_{\rm at,=}$, we can apply Proposition 8.8 and infer that the formula $\varphi(x\,;ka)$ has a solution $c \in N$.

The second case is when b is independent from a. Then by Remark 9.1 it suffices to find some $c \in N$ that is independent from ka and realizes $p_<(x\,;ka)$. As $p_<(x\,;ka)$ is finitely consistent, the ω_1-largeness of N and Proposition 9.5 yields the required c. □

Unsurprisingly the lemma above yields no categoricity result. However quantifier elimination follows in the usual way.

9.7 Corollary The theory $T_{\rm odag}$ has elimination of quantifiers.

Proof. Let $k : M \to N$ be a partial embedding between models of $T_{\rm odag}$. If we prove that k is elementary, elimination of quantifiers follows by Theorem 7.14.

The argument is the usual one. We can assume that k is finite and M, N countable. Let M' and N' be the large countable elementary superstructures of M, respectively N given by Proposition 9.4. As M' and N' are rich, $k : M' \to N'$ is elementary by Theorem 7.11. Hence $k : M \to N$ is also elementary. □

9.2 Presburger arithmetic

T.b.c.

9.3 Real closed fields

We work in the language of ordered rings $L = \{0, 1, -, +, \cdot, <\}$. The theory of ordered integral domains, $T_{\rm oid}$ contains $T_{\rm oid} \cup T_{\rm oag}$ and the axioms

1. $0 < 1$

op. $0 < z \wedge x < y \ \to\ z\,x < z\,y$.

It follows immediately that ordered integral domains have characteristic 0.

As in the previous section, we split L_{at} into equations $L_{\text{at},=}$ and inequalities $L_{\text{at},<}$. Note that, by the linearity of the order, the types at-tp(b/A) and at$^{\pm}$-tp(b/A) are equivalent.

The type $p(x)$ is equationally trivial if $p_{=}(x)$ is trivial, as defined in Notation 8.1. In this case $\vdash p(x) \leftrightarrow p_{<}(x)$. If at-tp$(b/A)$ equationally trivial, we say that b is transcendental over A. Otherwise we say that it is algebraic over A.

9.8 Proposition Let $A \subseteq M \vDash T_{\text{oid}}$. For $b \in M$ let $p(x) = \text{at-tp}(b/A)$. Then one of the following holds

1. b is transcendental over A;
2. $M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\text{at}}(A)$ such that $\vdash \varphi(x) \to p(x)$.

Proof. Negate ß1 and let $\varphi(x)$ be as in the proof of Proposition 8.18. Then $\vdash \varphi(x) \to p_{=}(x)$. Let $a, c \in \langle A \rangle_M$ be such that $a < b < c$ is the unique element in $\langle A \rangle_M$ satisfying $a < x < c$ and $\varphi(x)$. We prove that for every polynomial $t(x)$ with parameters in $\langle A \rangle_M$ such that $\vdash \ \varphi(x) \to t(x) \neq 0$ one of the following holds

a. $\vdash \ a < x < c \ \wedge \ \varphi(x) \to t(x) > 0$;

b. $\vdash \ a < x < c \ \wedge \ \varphi(x) \to t(x) < 0$.

Suppose not for a contradiction. Then there are three models $\langle A \rangle_M \subseteq N_i \vDash T_{\text{oid}}$, some $b_i \in N_i$ such that $N_i \vDash a < b_i < c$

a′. $N_1 \vDash \varphi(b_1)$ and $N_1 \vDash t(b_1) > 0$;

b′. $N_2 \vDash \varphi(b_2)$ and $N_2 \vDash t(b_2) < 0$.

□

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Capitolo 5