A Crèche Course in Model Theory

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Preface

This book was written to answer one question "Does a recursion theorist dare to write a book on model theory?"

Gerald E. Sacks
Saturated Model Theory (1972)

These are the notes of a course that I have given for a few years in Amsterdam and many more in Turin. Since then they have grown and other chapters will be added soon. Find the most recent version in

https://github.com/domenicozambella/creche

A warning sign in the margin indicates that the notation is nonstandard. Occasionally, the whole exposition is substantially nonstandard. Below are a few examples.

- Fraïssé limits are presented in a general setting that accommodates a large variety of examples (and I'll add a few more). This setting is used, for example, to discuss saturation.
- Quantifier elimination for ACF and Hilbert's Nullstellensatz are presented in more detail than is usual. (This may annoy some readers, but I hope it will help others.)
- The proof of the Omitting Types Theorem uses a model theoretic construction which highlights the analogy with the Kuratowski-Ulam Theorem.
- Imaginaries and the eq-expansion are introduced from the (equivalent) dual perspective that the canonical name of a definable set is the set itself.
- Ramsey's Theorem is derived from the existence of coheir sequences. This is not the shortest proof, but it is an instructive application of coheirs.
- Along the same lines we prove the theorems of Hindman and of Hales-Jewett. This is an instructive application of the uniqueness of coheir extensions.
- Lascar and Kim-Pillay types are introduced in a slightly unconventional way.
- Newelski's Theorem on the diameter of Lascar types is proved in an elementary self-contained way.
- Stability and NIP are introduced very briefly. We only discuss the properties of externally definable sets, which we identify with *approximable sets*.

Torino, June 2019

Chapter 1

Preliminaries and notation

This chapter introduces the syntax and semantic of first order logic. We assume that the reader has at least some familiarity with first order logic.

The definitions of terms and formulas we give in Section 3 and 5 are more formal than is required in subsequent chapters. Our main objective is to convince the reader that a rigorous definition of language and truth is possible. However, the actual details of such a definition are not relevant for our purposes.

1 Structures

A (first order) language L (also called signature) is a triple that consists of

1. a set L_{fun} whose elements are called function symbols;
2. a set L_{rel} whose elements are called relation symbols;
3. a function that assigns to every $f \in L_{\text{fun}}$, respectively $r \in L_{\text{rel}}$, non-negative integers n_f and n_r that we call arity of the function, respectively relation, symbol. We say that f is an n_f-ary function symbol, and similarly for r. A 0-ary function symbol is also called a constant.

Warning: it is customary to use the symbol L to denote both the language and the set of formulas (to be defined below) associated to it. We denote by $|L|$ the cardinality of $L_{\text{fun}} \cup L_{\text{rel}} \cup \omega$. Note that, by definition, $|L|$ is always infinite.

A (first order) structure M of signature L (for short L-structure) consists of

1. a set that we call the domain or support and denote by the same symbol M used for the structure as a whole;
2. a function that assigns to every $f \in L_{\text{fun}}$ a total map $f^M : M^{n_f} \to M$;
3. a function that assigns to every $r \in L_{\text{rel}}$ a relation $r^M \subseteq M^{n_r}$.

We call f^M, respectively r^M, the interpretation of f, respectively r, in M.

Recall that, by definition, $M^0 = \{\varnothing\}$. Therefore the interpretation of a constant c is a function that maps the unique element of M^0 to an element of M. We identify c^M with $c^M(\varnothing)$.

We may use the word model as a synonym for structure. But beware that, in some contexts, the word is used to denote a particular kind of structure.

If M is an L-structure and $A \subseteq M$ is any subset, we write $L(A)$ for the language obtained by adding to L_{fun} the elements of A as constants. In this context, the elements of A are called parameters. There is a canonical expansion of M to an $L(A)$-structure that is obtained by setting $a^M = a$ for every $a \in A$.

1.1 Example The language of additive groups consists of the following function symbols:

1. a constant (that is, a function symbol of arity 0) 0
2. a unary function symbol (that is, of arity 1) $-$
3. a binary function symbol (that is, of arity 2) $+$.

In the language of multiplicative groups the three symbols above are replaced by 1, $^{-1}$, and $\cdot$ respectively. Any group is a structure in either of these two signatures with the obvious interpretation. Needless to say, not all structures with these signatures are groups.

The language of (unitary) rings contains all the function symbols above except $^{-1}$. The language of ordered rings also contains the binary relation symbol $<$. □

The following example is less straightforward. The reason for the choice of the language of vector spaces will become clear in Example 1.9 below.

1.2 Example Let F be a field. The language of vector spaces over F, which we denote by L_F, extends that of additive groups by a unary function symbol k for every $k \in F$.

Recall that a vector space over F is an abelian group M together with a function $\mu : F \times M \to M$ satisfying some properties (that we assume well-known, see Example 2.4). To view a vector space over F as an L_F-structure, we interpret the group symbols in the obvious way and each $k \in F$ as the function $\mu(k, \text{-})$. □

The languages in Examples 1.1 and 1.2, with the exception of that of ordered rings, are functional languages, that is, $L_{\text{rel}} = \varnothing$. In what follows, we consider two important examples of relational languages, that is, languages where $L_{\text{fun}} = \varnothing$.

1.3 Example The language of strict orders only contains a binary relation symbol, usually denoted By $<$. The language of graphs, too, only contains a binary relation symbol (for which there is no standard notation). □

2 Tuples

A sequence is a function $a : I \to A$ whose domain is a linear order $I, <_I$. We may use the notation $a = \langle a_i : i \in I \rangle$ for sequences. A tuple is a sequence whose domain is an ordinal, say α, then we write $a = \langle a_i : i < \alpha \rangle$. When α is finite, we may also write $a = a_0, \ldots, a_{\alpha-1}$ The domain of the tuple a, the ordinal α, is denoted by $|a|$ and is called the length of a. If a is surjective, it is said to be an enumeration of A.

If $J \subseteq I$ is a subset of the domain of the sequence $a = \langle a_i : i \in I \rangle$, we write $a_{\restriction J}$ for the restriction of a to J. When J is well ordered by $<_I$, e.g. when a is a tuple or when J is finite, we identify $a_{\restriction J}$ with a tuple. This is the tuple $\langle a_{j_k} : k < \beta \rangle$ where $\langle j_k : k < \beta \rangle$ is the unique increasing enumeration on J.

⚠ Sometimes (i.e. not always) we may overline tuples or sequences as mnemonic. When a tuple $\bar{c}$ is introduced, we write c_i for the i-th element of $\bar{c}$. and $c_{\restriction J}$ for the restriction of $\bar{c}$ to $J \subseteq |\bar{c}|$. Note that the bar is dropped for ease of notation.

The set of tuples of elements of length α is denoted by A^α. The set of tuples of length $< \alpha$ is denoted by $A^{<\alpha}$. For instance, $A^{<\omega}$ is the set of all finite tuples of elements of A. When α is finite we do not distinguish between A^α and the α-th Cartesian power of A. In particular, we do not distinguish between A^1 and A.

If $a, b \in A^\alpha$ and h is a function defined on A, we write $h(a) = b$ for $h(a_i) = b_i$. We often do not distinguish between the pair $\langle a, b\rangle$ and the tuple of pairs $\langle a_i, b_i\rangle$. The context will resolve the ambiguity.

Note that there is a unique tuple of length 0, the empty set $\varnothing$, which in this context is called empty tuple. Recall that by definition $A^0 = \{\varnothing\}$ for every set A. Therefore, even when A is empty, A^0 contains the empty string.

We often concatenate tuples. If a and b are tuples, we write $a\,b$ or, equivalently, a, b.

3 Terms

Let V be an infinite set whose elements we call variables. We use the letters x, y, z, etc. to denote variables or tuples of variables. We rarely refer to V explicitly, and we always assume that V is large enough for our needs.

We fix a signature L for the whole section.

1.4 Definition *A term is a finite sequence of elements of $L_{\text{fun}} \cup V$ that are obtained inductively as follows:*

0. *every variable, intended as a tuple of length 1, is a term;*

i. *if $f \in L_{\text{fun}}$ and t is a tuple obtained by concatenating n_f terms, then $f\,t$ is a term. We write $f\,t$ for the tuple obtained by prefixing t by f.*

We say L-term when we need to specify the language L. □

Note that any constant f, intended as a tuple of length 1, is a term (by i, the term f is obtained concatenating $n_f = 0$ terms and prefixing by f). Terms that do not contain variables are called closed terms.

The intended meaning of, for instance, the term $+ + x\,y\,z$ is $(x + y) + z$. The first expression uses prefix notation; the second uses infix notation. When convenient, we informally use infix notation and add parentheses to improve legibility and avoid ambiguity.

The following lemma shows that prefix notation allows to write terms unambiguously without using parentheses.

1.5 Lemma (unique legibility of terms) *Let a be a sequence of terms. Suppose a can be obtained both by concatenating the terms $t_1, \ldots, t_n$ and by concatenating the terms $s_1, \ldots, s_m$. Then $n = m$ and $s_i = t_i$.*

Proof By induction on $|a|$. If $|a| = 0$ than $n = m = 0$ and there is nothing to prove. Suppose the claim holds for tuples of length k and let $a = a_1, \ldots, a_{k+1}$. Then a_1 is the first element of both t_1 and s_1. If a_1 is a variable, say x, then t_1 and s_1 are the term x and $n = m = 1$. Otherwise a_1 is a function symbol, say f. Then $t_1 = f\,\bar{t}$ and $s_1 = f\,\bar{s}$, where $\bar{t}$ and $\bar{s}$ are obtained by concatenating the terms $t'_1, \ldots, t'_p$ and $s'_1, \ldots, s'_p$. Now apply the induction hypothesis to $a_2, \ldots, a_{k+1}$ and to the terms $t'_1, \ldots, t'_p, t_2, \ldots, t_n$ and $s'_1, \ldots, s'_p, s_2, \ldots, s_m$. □

If $x = x_1, \ldots, x_n$ is a tuple of distinct variables and $s = s_1, \ldots, s_n$ is a tuple of terms, we write $t[x/s]$ for the sequence obtained by replacing x by s coordinatewise. Proving that $t[x/s]$ is indeed a term is a tedious task that can be safely skipped.

If t is a term and $x_1, \ldots, x_n$ are (tuples of) variables, we write $t(x_1, \ldots, x_n)$ to declare that the variables occurring in t are among those that occur in $x_1, \ldots, x_n$. When a term has been presented as $t(x, y)$, we write $t(s, y)$ for $t[x/s]$.

Finally, we define the interpretation of a term in a structure M. We begin with closed terms. These are interpreted as 0-ary functions, i.e. as elements of the structure.

1.6 Definition *Let t be a closed $L(M)$ term. The interpretation of t, denoted by t^M, is defined by induction of the syntax of t as follows.*

i. *if $t = f\,\bar{t}$, where $f \in L_{\text{fun}}$ and $\bar{t}$ is a tuple obtained by concatenating the terms $t_1, \ldots, t_{n_f}$, then $t^M = f^M(t_1^M, \ldots, t_{n_f}^M)$.*

Note that in i we have used Lemma 1.5 in an essential way. In fact this ensures that the sequence $\bar{t}$ uniquely determines the terms $t_1, \ldots, t_{n_f}$. □

The inductive definition above is based on the case $n_f = 0$, that is, the case where f a constant, or a parameter. When $t = c$, a constant, $\bar{t}$ is the empty tuple, and so $t^M = c^M(\varnothing)$, which we abbreviate as c^M. In particular, if $t = a$, a parameter, then $t^M = a^M = a$.

Now we generalize the interpretation to all (not necessarily closed) terms. If $t(x)$ is a term, we define $t^M(x) : M^{|x|} \to M$ to be the function that maps a to $t(a)^M$.

4 Substructures

In the working practice, a *substructure* is a subset of a structure that is closed under the interpretation of the functions in the language. But there are a few cases when we need the following formal definition.

1.7 Definition *Fix a signature L and let M and N be two L-structures. We say that M is a substructure of N, and write $M \subseteq N$, if*

1. *the domain of M is a subset of the domain of N*
2. *$f^M = f^N \restriction M^{n_f}$ for every $f \in L_{\text{fun}}$*
3. *$r^M = r^N \cap M^{n_r}$ for every $f \in L_{\text{rel}}$.*

Note that when f is a constant 2 becomes $f^M = f^N$, in particular the substructures of N contains at least all the constants of N.

If a set $A \subseteq N$ is such that

1. $f^N[A^{n_f}] \subseteq A$ for every $f \in L_{\text{fun}}$

then there is a unique substructure $M \subseteq N$ with domain A, namely, the structure with the following interpretation

2. $f^M = f^N \restriction A^{n_f}$ (which is a good definition by the assumption on A);
3. $r^M = r^N \cap A^{n_r}$.

It is usual to confuse subsets of N that satisfy 1 with the unuque substructure they support.

It is immediate to verify that the intersection of an arbitrary family of substructures of N is a substructure of N. Therefore, for any given $A \subseteq N$ we may define the substructure of N generated A as the intersection of all substructures of N that contain A. We write $\langle A\rangle_N$. The following easy proposition gives more concrete representation of $\langle A\rangle_N$

1.8 Lemma *The following hold for every $A \subseteq N$*

1. $\langle A\rangle_N \;=\; \Big\{t^N \;:\; t \text{ a closed } L(A)\text{-term}\Big\}$

2. $\langle A\rangle_N \;=\; \Big\{t^N(a) \;:\; t(x) \text{ an } L\text{-term and } a \in A^{|x|}\Big\}$

3. $\langle A\rangle_N \;=\; \bigcup\limits_{n\in\omega} A_n,$ *where* $A_0 = A$

$$A_{n+1} \;=\; A_n \cup \Big\{f^N(a) \;:\; f \in L_{\text{fun}},\ a \in A_n^{n_f}\Big\}. \qquad \square$$

1.9 Example Let L be the language of groups. Let N be a group, which we consider as an L-structure in the natural way. Then the substructures of N are exactly the subgroups of N and $\langle A\rangle_N$ is the group generated by $A \subseteq N$. A similar claim is true when L_F is the signature of vector spaces over some fixed field F. The choice of the language is more or less fixed if we want that the algebraic and the model theoretic notion of substructure coincide. $\square$

5 Formulas

Fix a language L and a set of variables V as in Section 3. A formula is a finite sequence of symbols in $L_{\text{fun}} \cup L_{\text{rel}} \cup V \cup \{\doteq, \perp, \neg, \vee, \exists\}$. The last set contains the logical symbols that are called respectively

$\doteq$	equality	$\perp$	contradiction	$\neg$	negation
$\vee$	disjunction	$\exists$	existential quantifier.		

Syntactically, $\doteq$ behaves like a binary relation symbol. So, for convenience set $n_{\doteq} = 2$. However $\doteq$ is considered as a logic symbol because its semantic is fixed (it is always interpreted in the diagonal).

The definition below uses the prefix notation which simplifies the proof of the unique legibility lemma. However, in practice we always we use the infix notation: $t \doteq s$, $\varphi \vee \psi$, etc.

1.10 Definition *A formula is any finite sequence is obtained with the following inductive procedure*

0. *if $r \in L_{\text{rel}} \cup \{\doteq\}$ and t is a tuple obtained concatenating n_r terms then $r\,t$ is a formula. Formulas of this form are called atomic;*

i. *if φ e ψ are formulas then the following are formulas: $\perp$, $\neg\,\varphi$, $\vee\,\varphi\,\psi$, and $\exists x\,\varphi$, for any $x \in V$.* $\square$

We use L to denote both the language and the set of formulas. We write L_{at} for the set of atomic formulas and L_{qf} for the set of quantifier-free formulas i.e. formulas where $\exists$ does not occur.

The proof of the following is similar to the analogous lemma for terms.

1.11 Lemma (unique legibility of formulas) *Let a be a sequence of formulas. Suppose a can be obtained both by the concatenation of the formulas $\varphi_1, \ldots, \varphi_n$ or by the concatenation of the formulas $\psi_1, \ldots, \psi_m$. Then $n = m$ and $\varphi_i = \psi_i$.* □

A formula is closed if all its variables occur under the scope of a quantifier. Closed formulas are also called sentences. We will do without a formal definition of *occurs under the scope of a quantifier* which is too lengthy. An example suffices: all occurrences of x are under the scope a quantifiers in the formula $\exists x\, \varphi$. These occurrences are called bonded. The formula $x \doteq y \wedge \exists x\, \varphi$ has free (i.e., not bond) occurrences of x and y.

Let x is a tuple of variables and t is a tuple of terms such that $|x| = |t|$. We write $\varphi[x/t]$ for the formula obtained substituting t for all free occurrences of x, coordinatewise.

We write $\varphi(x)$ to declare that the free variables in the formula φ are all among those of the tuple x. In this case we write $\varphi(t)$ for $\varphi[x/t]$.

We will often use without explicit mention the following useful syntactic decomposition of formulas with parameters.

1.12 Lemma *For every formula $\varphi(x) \in L(A)$ there is a formula $\psi(x\,;z) \in L$ and a tuple of parameters $a \in A^{|z|}$ such that $\varphi(x) = \psi(x\,;a)$.* □

Just as a term $t(x)$ is a name for a function $t(x)^M : M^{|x|} \to M$, a formula $\varphi(x)$ is a name for a subset $\varphi(x)^M \subseteq M^{|x|}$ which we call the subset of M defined by $\varphi(x)$. It is also very common to write $\varphi(M)$ for the set defined by $\varphi(x)$. In general sets of the form $\varphi(M)$ for some $\varphi(x) \in M$ are called definable.

1.13 Definition of truth *For every formula φ with variables among those of the tuple x we define $\varphi(x)^M$ by induction as follows*

$$
\begin{array}{lrcl}
\text{o1.} & (\doteq t\, s)(x)^M & = & \left\{a \in M^{|x|} \ :\ t^M(a) = s^M(a)\right\} \\[2ex]
\text{o2.} & (r\, t_1 \ldots t_n)(x)^M & = & \left\{a \in M^{|x|} \ :\ \langle t_1^M(a), \ldots, t_n^M(a)\rangle \in r^M\right\} \\[2ex]
\text{i0.} & \bot(x)^M & = & \varnothing \\[2ex]
\text{i1.} & (\neg\xi)(x)^M & = & M^{|x|} \smallsetminus \xi(x)^M \\[2ex]
\text{i2.} & (\vee\, \xi\, \psi)(x)^M & = & \xi(x)^M \cup \psi(x)^M \\[2ex]
\text{i3.} & (\exists y\, \varphi)(x)^M & = & \displaystyle\bigcup_{a \in M} (\varphi[y/a])(x)^M
\end{array}
$$

Condition i2 assumes that ξ and ψ are uniquely determined by $\vee\, \xi\, \psi$. This is a guaranteed by the unique legibility o formulas, Lemma 1.11. Analogously, o1 e o2 assume Lemma 1.5.

The case when x is the empty tuple is far from trivial. Note that $\varphi(\varnothing)^M$ is a subset of $M^0 = \{\varnothing\}$. Then there are two possibilities either $\{\varnothing\}$ or $\varnothing$. We wil read them as two truth values: True and False, respectively. If $\varphi^M = \{\varnothing\}$ we say that φ is true in M, if $\varphi^M = \varnothing$, we say that φ is false M. We write $M \vDash \varphi$, respectively $M \nvDash \varphi$. Or we may say that M models φ, respectively M does not model φ. It is immediate to verify at

$$\varphi(M) = \big\{a \in M^{|x|} : M \vDash \varphi(a)\big\}.$$

Note that usually, we say *formula* when, strictly speaking, we mean *pair* that consists of a formula and a tuple of variables. Such pairs are interpreted in definable sets (cfr. Definition 1.13). In fact, if the tuple of variables were not given, the arity of the corresponding set is not determined.

In some contexts we also want to distinguish between two sorts of variables that play different roles. Some are placeholder for parameters, some are used to define a set. In the the first chapters this distinction is only a clue for the reader, in the last chapters it is an essential part of the definitions.

1.14 Definition *A partitioned formula (strictly speaking, we should say a 2-partitioned formula) is a triple $\varphi(x\,;z)$ consisting of a formula and two tuples of variables such that the variables occurring in φ are all among x, z.* □

We use a semicolon to separate the two tuples of variables. Typically, z is the placeholder for parameters and x runs over the elements of the set defined by the formula.

6 Yet more notation

Now we abandon the prefix notation in favor of the infix notation. We also use the following logical connectives as abbreviations

$\top$	stands for	$\neg\bot$	tautology
$\varphi \wedge \psi$	stands for	$\neg[\neg\varphi \vee \neg\psi]$	conjunction
$\varphi \rightarrow \psi$	stands for	$\neg\varphi \vee \psi$	implication
$\varphi \leftrightarrow \psi$	stands for	$[\varphi \rightarrow \psi] \wedge [\psi \rightarrow \varphi]$	bi-implication
$\varphi \nleftrightarrow \psi$	stands for	$\neg[\varphi \leftrightarrow \psi]$	exclusive disjunction
$\forall x\, \varphi$	stands for	$\neg\exists x \neg\, \varphi$	universal quantifier

We agree that $\rightarrow$ e $\leftrightarrow$ bind less than $\wedge$ e $\vee$. Unary connnectives (quantifiers and negation) bind stronger then binarary connectives. For example

$$\exists x\, \varphi \wedge \psi \ \rightarrow\ \neg\xi \vee \vartheta \quad \text{reads as} \quad \Big[[\exists x\, \varphi] \wedge \psi\Big] \ \rightarrow\ \Big[[\neg\xi] \vee \vartheta\Big]$$

We say that $\forall x\, \varphi(x)$ and $\exists x\, \varphi(x)$ are the universal, respectively, existential closure of $\varphi(x)$. We say that $\varphi(x)$ holds in M when its universal closure is true in M. We say that $\varphi(x)$ is consistent in M when its existential closure is true in M.

The semantic of conjunction and disjunction is associative. Then for any finite set of formulas $\{\varphi_i : i \in I\}$ we can write without ambiguities

$$\bigwedge_{i\in I} \varphi_i \qquad\qquad \bigvee_{i\in I} \varphi_i$$

When $x = x_1, \dots, x_n$ is a tuple of variables we write $\exists x\, \varphi$ or $\exists x_1, \dots, x_n\, \varphi$ for $\exists x_1 \dots \exists x_n\, \varphi$. With first order sentences we are able to say that $\varphi(M)$ has at least n elements (also, no more than, or exactly n). It is convenient to use the following abbreviations.

$$\exists^{\geq n} x\; \varphi(x) \quad \text{stands for} \quad \exists x_1, \dots, x_n \Big[\bigwedge_{1\leq i\leq n} \varphi(x_i) \;\wedge \bigwedge_{1\leq i<j\leq n} x_i \dot{\neq} x_j \Big].$$

$$\exists^{\leq n} x\; \varphi(x) \quad \text{stands for} \quad \neg\exists^{\geq n+1} x\; \varphi(x)$$

$$\exists^{= n} x\; \varphi(x) \quad \text{stands for} \quad \exists^{\geq n} x\; \varphi(x) \;\wedge\; \exists^{\leq n} x\; \varphi(x)$$

1.15 Exercise Let M be an L-structure and let $\psi(x), \varphi(x, y) \in L$. For each of the following conditions, write a sentence true in M exactly when

a. $\psi(M) \in \big\{\varphi(a, M) : a \in M\big\}$;

b. $\big\{\varphi(a, M) : a \in M\big\}$ contains at least two sets;

c. $\big\{\varphi(a, M) : a \in M\big\}$ contains only sets that are pairwise disjoint. □

1.16 Exercise Let M be a structure in a signature that contains a symbol r for a binary relation. Write a sentence φ such that

a. $M \vDash \varphi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq A \times \neg A$.

Remark: φ assert an asymmetric version of the property below

b. $M \vDash \psi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq (A \times \neg A) \cup (\neg A \times A)$.

Assume M is a graph, what required in b is equivalent to saying that M is a *bipartite graph*, or equivalently that it has *chromatic number* 2 i.e., we can color the vertices with 2 colors so that no two adjacent vertices share the same color. □

Chapter 2

Theories and elementarity

1 Logical consequences

A theory is a set $T \subseteq L$ of sentences. We write $M \vDash T$ if $M \vDash \varphi$ for every $\varphi \in T$. If $\varphi \in L$ is a sentence we write $T \vdash \varphi$ when

$$M \vDash T \quad \Rightarrow \quad M \vDash \varphi \qquad \text{for every } M.$$

In words, we say that φ is a logical consequence of T or that φ follows from T. If S is a theory $T \vdash S$ has a similar meaning. If $T \vdash S$ and $S \vdash T$ we say that T and S are logically equivalent. We may say that T axiomatizes S (or vice versa).

We say that a theory is consistent if it has a model. With the notation above, T is consistent if and only if $T \nvdash \perp$.

The closure of T under logical consequence is the set $\mathrm{ccl}(T)$ which is defined as follows:

$$\mathrm{ccl}(T) \quad = \quad \Big\{\varphi \in L \ : \ \text{sentence such that } T \vdash \varphi\Big\}$$

If T is a finite set, say $T = \{\varphi_1, \ldots, \varphi_n\}$ we write $\mathrm{ccl}(\varphi_1, \ldots, \varphi_n)$ for $\mathrm{ccl}(T)$. If $T = \mathrm{ccl}(T)$ we say that T is closed under logical consequences.

The theory of M is the set of sentences that hold in M and is denoted by $\mathrm{Th}(M)$. More generally, if $\mathcal{K}$ is a class of structures, $\mathrm{Th}(\mathcal{K})$ is the set of sentences that hold in every model in $\mathcal{K}$. That is

$$\mathrm{Th}(\mathcal{K}) \quad = \quad \bigcap_{M \in \mathcal{K}} \mathrm{Th}(M)$$

The class of all models of T is denoted by $\mathrm{Mod}(T)$. We say that $\mathcal{K}$ is axiomatizable if $\mathrm{Mod}(T) = \mathcal{K}$ for some theory T. If T is finite we say that $\mathcal{K}$ is finitely axiomatizable. To sum up

$$\begin{aligned}
\mathrm{Th}(M) \quad &= \quad \Big\{\varphi \ : \ M \vDash \varphi\Big\} \\
\mathrm{Th}(\mathcal{K}) \quad &= \quad \Big\{\varphi \ : \ M \vDash \varphi \text{ for all } M \in \mathcal{K}\Big\} \\
\mathrm{Mod}(T) \quad &= \quad \Big\{M \ : \ M \vDash T\Big\}
\end{aligned}$$

2.1 Example Let L be the language of multiplicative groups. Let T_{g} be the set containing the universal closure of following three formulas

1. $(x \cdot y) \cdot z \ = \ x \cdot (y \cdot z);$
2. $x \cdot x^{-1} \ = \ x^{-1} \cdot x \ = \ 1;$
3. $x \cdot 1 \ = \ 1 \cdot x \ = x.$

Then T_{g} axiomatizes the theory of groups, i.e. $\mathrm{Th}(\mathcal{K})$ for $\mathcal{K}$ the class of all groups. Let φ be the universal closure of the following formula

$$z \cdot x = z \cdot y \rightarrow x = y.$$

As φ formalizes the cancellation property then $T_g \vdash \varphi$, that is, φ is a logical consequence of T_g. Now consider the sentence ψ which is the universal closure of

4. $x \cdot y = y \cdot x.$

So, commutative groups model ψ and non commutative groups model $\neg\psi$. Hence neither $T_g \vdash \psi$ nor $T_g \vdash \neg\psi$. We say that T_g does not decide ψ. □

Note that even when T is a very concrete set, $\mathrm{ccl}(T)$ may be more difficult to grasp. In the example above T_g contains three sentences but $\mathrm{ccl}(T_g)$ is an infinite set containing sentences that code theorems of group theory yet to be proved.

2.2 Remark The following properties say that ccl is a finitary closure operator.

1. $T \subseteq \mathrm{ccl}(T)$ (extensive)
2. $\mathrm{ccl}(T) = \mathrm{ccl}\big(\mathrm{ccl}(T)\big)$ (idempotent)
3. $T \subseteq S \Rightarrow \mathrm{ccl}(T) \subseteq \mathrm{ccl}(S)$ (increasing)
4. $\mathrm{ccl}(T) = \bigcup\big\{\mathrm{ccl}(S) : S \text{ finite subset of } T\big\}.$ (finitary)

Properties 1-3 are easy to verify while 4 requires the compactness theorem. □

In the next example we list a few algebraic theories with straightforward axiomatization.

2.3 Example We write T_{ag} for the theory of abelian groups which contains the universal closure of following

a1. $(x+y)+z = y+(x+z);$

a2. $x+(-x) = 0;$

a3. $x+0 = x;$

a4. $x+y = y+x.$

The theory T_r of (unitary) rings extends T_{ag} with

a5. $(x \cdot y) \cdot z = x \cdot (y \cdot z);$

a6. $1 \cdot x = x \cdot 1 = x;$

a7. $(x+y) \cdot z = x \cdot z + y \cdot z;$

a8. $z \cdot (x+y) = z \cdot x + z \cdot y.$

The theory of commutative rings T_{cg} contains also com of examples 2.1. The theory of ordered rings T_{or} extends T_{cr} with

o1. $x < z \rightarrow x+y < z+y;$

o2. $0 < x \wedge 0 < z \rightarrow 0 < x \cdot z.$ □

The axiomatization of the theory of vector spaces is less straightforward.

2.4 Example Fix a field F. The language L_F extends the language of additive groups with a unary function for every element of F. The theory of vector fields over F extends T_{ag} with the following axioms (for all $h, k, l \in F$)

m1. $h\,(x+y) \;=\; h\,x + h\,y$

m2. $l\,x \;=\; h\,x + k\,x,$ where $l \;=\; h +_F k$

m3. $l\,x \;=\; h\,(k\,x),$ where $l \;=\; h \cdot_F k$

m4. $0_F\,x \;=\; 0$

m5. $1_F\,x \;=\; x$

The symbols 0_F and 1_F denote the zero and the unit of F. The symbols $+_F$ and $\cdot_F$ denote the sum and the product in F. These are not part of L_F, they are symbols we use in the metalanguage. □

2.5 Example Recall from Example 1.3 that we represent a graph with a symmetric irreflexive relation. Therefore theory of graphs contains the following two axioms

1. $\neg r(x,x)$;
2. $r(x,y) \rightarrow r(y,x)$. □

Our last example is a trivial one.

2.6 Example Let L be the empty language The theory of infinite sets is axiomatized by the sentences $\exists^{\geq n} x\ (x = x)$ for all positive integer n. □

2.7 Exercise Prove that $\mathrm{ccl}(\varphi \vee \psi) = \mathrm{ccl}(\varphi) \cap \mathrm{ccl}(\psi)$. □

2.8 Exercise Prove that $T \cup \{\varphi\} \vdash \psi$ then $T \vdash \varphi \rightarrow \psi$. □

2.9 Exercise Prove that $\mathrm{Th}(\mathrm{Mod}(T)) = \mathrm{ccl}(T)$. □

2 Elementary equivalence

The following is a fundamental notion in model theory.

2.10 Definition *We say that M and N are elementarily equivalent if*

ee. $N \vDash \varphi \;\Leftrightarrow\; M \vDash \varphi,$ *for every sentence $\varphi \in L$.*

In this case we write $M \equiv N$. More generally, we write $M \equiv_A N$ and say that M and N are elementarily equivalent over A if the following hold

a. $A \subseteq M \cap N$

ee'. equivalence ee *above holds for every sentence $\varphi \in L(A)$.* □

The case when A is the whole domain of M is particularly important.

2.11 Definition *When $M \equiv_M N$ we write $M \preceq N$ and say that M is an elementary substructure of N.* □

In the definition above the use of the term *substructure* is appropriate by the following lemma.

2.12 Lemma *If M and N are such that $M \equiv_A N$ and A is the domain of a substructure of M then A is also the domain a substructure of N and the two substructures coincide.*

Proof Let f be a function symbol and ler r be a relation symbol. It suffices to prove that $f^M(a) = f^N(a)$ for every $a \in A^{n_f}$ and that $r^M \cap A^{n_r} = r^N \cap A^{n_r}$.

If $b \in A$ is such that $b = f^M a$ then $M \vDash fa = b$. So, from $M \equiv_A N$, we obtain $N \vDash fa = b$, hence $f^N a = b$. This proves $f^M(a) = f^N(a)$.

Now let $a \in A^{n_r}$ and suppose $a \in r^M$. Then $M \vDash ra$ and, by elementarity, $N \vDash ra$, hence $a \in r^N$. By symmetry $r^M \cap A^{n_r} = r^N \cap A^{n_r}$ follows. □

It is not easy to prove that two structures are elementary equivalent. A direct verification is unfeasible even for the most simple structures. It will take a few chapters before we are able to discuss concrete examples.

We generalize the definition of $\mathrm{Th}(M)$ to include parameters

$$\mathrm{Th}(M/A) \;=\; \Big\{\varphi \;:\; \text{sentence in } L(A) \text{ such that } M \vDash \varphi\Big\}.$$

The following proposition is immediate

2.13 Proposition *For every pair of structures M and N and every $A \subseteq M \cap N$ the following are equivalent*

a. $M \equiv_A N$;

b. $\mathrm{Th}(M/A) = \mathrm{Th}(N/A)$;

c. $M \vDash \varphi(a) \;\Leftrightarrow\; N \vDash \varphi(a)$ *for every $\varphi(x) \in L$ and every $a \in A^{|x|}$.*

d. $\varphi(M) \cap A^{|x|} \;=\; \varphi(N) \cap A^{|x|}$ *for every $\varphi(x) \in L$.* □

If we restate a and c of the proposition above when $A = M$ we obtain that the following are equivalent

a'. $M \preceq N$;

c'. $\varphi(M) \;=\; \varphi(N) \cap M^{|x|}$ for every $\varphi(x) \in L$.

Note that c' extends to all definable sets what Definition 1.7 requires for a few basic definable sets.

2.14 Example Let G be a group which we consider as a structure in the multiplicative language of groups. We show that if G is simple and $H \preceq G$ then also H is simple. Recall that G is simple if all its normal subgroups are trivial, equivalently, if for every $a \in G \smallsetminus \{1\}$ the set $\big\{gag^{-1} : g \in G\big\}$ generates the whole group G.

Assume H is not simple then there $a, b \in H$ be such that b is not product of elements of $\big\{hah^{-1} : h \in H\big\}$, hence for every n

$$H \;\vDash\; \neg\exists x_1, \ldots, x_n \;\left(b \;=\; x_1 a x_1^{-1} \cdots\, x_n a x_n^{-1}\right)$$

By elementarity the same hold in G. Hence G is not simple. □

2.15 Exercise Let $A \subseteq M \cap N$. Prove that $M \equiv_A N$ if and only if $M \equiv_B N$ for every finite $B \subseteq A$. □

2.16 Exercise Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if

$\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$. $\square$

2.17 Exercise Let $M \preceq N$ and let $\varphi(x,z) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M. $\square$

2.18 Exercise Consider $\mathbb{Z}^n$ as a structure in the additive language of groups with the natural interpretation. Prove that $\mathbb{Z}^n \not\equiv \mathbb{Z}^m$ for every positive integers $n \neq m$. Hint: in $\mathbb{Z}^n$ there are at most $2^n - 1$ elements that are not congruent modulo 2. $\square$

3 Embeddings and isomorphisms

Here we prove that isomorphic structures are elementarily equivalent and a few related results.

2.19 Definition *An embedding of M into N is an injective map $h : M \to N$ such that*

1. $a \in r^M \quad\Leftrightarrow\quad ha \in r^N$ *for every $r \in L_{\text{rel}}$ and $a \in M^{n_r}$;*
2. $h\, f^M(a) \quad=\quad f^N(h\, a)$ *for every $f \in L_{\text{fun}}$ and $a \in M^{n_f}$.*

Note that when $c \in L_{\text{fun}}$ is a constant 2 reads $h\, c^M = c^N$. Therefore that $M \subseteq N$ if and only if $\text{id}_M : M \to N$ is an embedding.

An surjective embedding is an isomorphism or, when domain and codomain coincide, an automorphism. $\square$

Condition 1 above and the assumption that h is injective can be summarized in the following

1'. $M \vDash r(a) \quad\Leftrightarrow\quad N \vDash r(ha)$ for every $r \in L_{\text{rel}} \cup \{=\}$ and every $a \in M^{n_r}$.

Note also that, by straightforward induction on syntax, from 2 we obtain

2' $h\, t^M(a) \quad=\quad t^N(h\, a)$ for every term $t(x)$ and every $a \in M^{|x|}$.

Combining these two properties and a straightforward induction on the syntax give

3. $M \vDash \varphi(a) \quad\Leftrightarrow\quad N \vDash \varphi(ha)$ for every $\varphi(x) \in L_{\text{qf}}$ and every $a \in M^{|x|}$.

Recall that we write L_{qf} for the set of quantifier-free formulas. It is worth noting that when $M \subseteq N$ and $h = \text{id}_M$ then 3 becomes

3' $M \vDash \varphi(a) \quad\Leftrightarrow\quad N \vDash \varphi(a)$ for every $\varphi(x) \in L_{\text{qf}}$ and for every $a \in M^{|x|}$.

In words this is summarized by saying that the truth of quantifier-free formulas is preserved under sub- and superstructure.

Finally we prove that first order truth is preserved under isomorphism. We say that a map $h : M \to N$ fixes $A \subseteq M$ (pointwise) if $\text{id}_A \subseteq h$. An isomorphism that fixes A is also called an A-isomorphism.

2.20 Theorem *If $h : M \to N$ is an isomorphism then for every $\varphi(x) \in L$*

\# $\quad M \vDash \varphi(a) \quad\Leftrightarrow\quad N \vDash \varphi(ha)$ *for every $a \in M^{|x|}$*

In particular, if h is an A-isomorphism then $M \equiv_A N$.

Proof We proceed by induction of the syntax of $\varphi(x)$. When $\varphi(x)$ is atomic # holds by 3 above. Induction for the Boolean connectives is straightforward so we only need to consider the existential quantifier. Assume as induction hypothesis that

$$M \vDash \varphi(a, b) \quad \Leftrightarrow \quad N \vDash \varphi(ha, hb) \quad \text{for every tupla } a \in M^{|x|} \text{ and } b \in M.$$

We prove that # holds for the formula $\exists y\, \varphi(x, y)$.

$$\begin{aligned} M \vDash \exists y\, \varphi(a, y) \quad &\Leftrightarrow \quad M \vDash \varphi(a, b) \quad \text{for some } b \in M \\ &\Leftrightarrow \quad N \vDash \varphi(ha, hb) \text{for some } b \in M \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad N \vDash \varphi(ha, c) \quad \text{for some } c \in N \qquad (\Leftarrow \text{ by surjectivity}) \\ &\Leftrightarrow \quad N \vDash \exists y\, \varphi(ha, y). \end{aligned}$$

□

2.21 Corollary *If $h : M \to N$ is an isomorphism then $h\big[\varphi(M)\big] = \varphi(N)$ for every $\varphi(x) \in L$.* □

We can now give a few very simple examples of elementarily equivalent structures.

2.22 Example Let L be the language of strict orders. Consider intervals of $\mathbb{R}$ (or in $\mathbb{Q}$) as structures in the natural way. The intervals $[0, 1]$ and $[0, 2]$ are isomorphic, hence $[0, 1] \equiv [0, 2]$ follows from Theorem 2.20. Clearly, $[0, 1]$ is a substructure of $[0, 2]$. However $[0, 1] \not\preceq [0, 2]$, in fact the formula $\forall x\, (x \leq 1)$ holds in $[0, 1]$ but is false in $[0, 2]$. This shows that $M \subseteq N$ and $M \equiv N$ does not imply $M \preceq M$.

Now we prove that $(0, 1) \preceq (0, 2)$. By Exercise 2.15 above, it suffices to verify that $(0, 1) \equiv_B (0, 2)$ for every finite $B \subseteq (0, 1)$. This follows again by Theorem 2.20 as $(0, 1)$ and $(0, 2)$ are B-isomorphic for every finite $B \subseteq (0, 1)$. □

For the sake of completeness we also give the definition of homomorphism.

2.23 Definition *A homomorphism is a total map $h : M \to N$ such that*

1. $a \in r^M \quad \Rightarrow \quad ha \in r^N \qquad$ *for every $r \in L_{\mathrm{rel}}$ and $a \in M^{n_r}$;*
2. $h\, f^M(a) \quad = \quad f^N(h\, a) \qquad$ *for every $f \in L_{\mathrm{fun}}$ and $a \in M^{n_f}$.*

Note that only one implication is required in 1. □

2.24 Exercise Prove that if $h : N \to N$ is an automorphism and $M \preceq N$ then $h[M] \preceq N$. □

2.25 Exercise Let L be the empty language. Let $A, D \subseteq M$. Prove that the following are equivalent

1. D is definable over A;
2. either D is finite and $D \subseteq A$, or $\neg D$ is finite and $\neg D \subseteq A$.

Hint: as structures are plain sets, every bijection $f : M \to M$ is an automorphism. □

2.26 Exercise Prove that if $\varphi(x)$ is an existential formula and $h : M \to N$ is an embedding then

$$M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha) \qquad \text{for every } a \in M^{|x|}.$$

Recall that existential formulas as those of the form $\exists y\, \psi(x, y)$ for $\psi(x, y) \in L_{\mathrm{qf}}$. Note that Theorem 10.7 proves that the property above characterizes existential formulas. □

2.27 Exercise Let M be the model with domain $\mathbb{Z}$ in the language that contains only the symbol $+$ which is interpreted in the usual way. Prove that there is no existential formula $\varphi(x)$ such that $\varphi(M)$ is the set of odd integers. Hint: use Exercise 2.26. □

2.28 Exercise Let N be the multiplicative group of $\mathbb{Q}$. Let M be the subgroup of those rational numbers that are of the form n/m for some odd integers m and n. Prove that $M \preceq N$. Hint: use the fundamental theorem of arithmetic and reason as in Example 2.22. □

4 Quotient structures

The content of this section is mainly technical and only required later in the course. Its reading may be postponed.

If E is an equivalence relation on N we write $[c]_E$ for the equivalence class of $c \in N$. We use the same symbol for the equivalence relation on N^n defined as follow: if $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ are n-tuples of elements of N then $a\ E\ b$ means that $a_i\ E\ b_i$ holds for all i. It is easy to see that $b_1, \ldots, b_n \in [a, ..., a_n]_E$ if and only if $b_i \in [a_i]_E$ for all i. Therefore we use the notation $[a]_E$ for both the equivalence class of $a \in N^n$ and the tuple of equivalence classes $[a_1]_E, \ldots, [a_n]_E$.

2.29 Definition *We say that the equivalence relation E on a structure N is a congruence if for every $f \in L_{\text{fun}}$*

c1. $$a\ E\ b \quad \Rightarrow \quad f^N a\ E\ f^N b;$$

When E is a congruence on N we write N/E for the a structure that has as domain the set of E-equivalence classes in N and the following interpretation of $f \in L_{\text{fun}}$ and $r \in L_{\text{rel}}$:

c2. $$f^{N/E}[a]_E \quad = \quad \left[f^N a\right]_E;$$

c3. $$[a]_E \ \in\ r^{N/E} \quad \Leftrightarrow \quad [a]_E \cap r^N \neq \varnothing.$$

We call N/E the quotient structure. □

By c1 the quotiont structure is well defined. The reader will recognize it as a familiar notion by the following proposition (which is not required in the following and requires the notion of homomorphism, see Definition 2.23. Recall that the kernel of a total map $h : N \to M$ is the equivalence relation E such that

$$a\ E\ b \quad \Leftrightarrow \quad ha = hb$$

for every $a, b \in N$.

2.30 Proposition *Let $h : N \to M$ be a surjective homomorphism and let E be the kernel of h. Then there is an isomorphism k that makes the following diagram commute*

$$\begin{tikzcd}
N \arrow[r, "h"] \arrow[d, "\pi"'] & M \\
N/E \arrow[ur, "k"'] &
\end{tikzcd}$$

where $\pi : a \mapsto [a]_E$ is the projection map. □

⚠ Quotients clutter the notation with brackets. To avoid the mess, we prefer to reason in N and tweak the satisfaction relation. Warning: this is not standard.

Essentially, we work in N while thinking of N/E. So, we replace equality with E and adapt the interpretation of relation symbols according to c3 above. The proposition below shows that the definition does what is intended to do.

2.31 Definition *For t_2, t_2 closed terms of $L(N)$ define*

1* $\quad N/E \overset{*}{\vDash} t_1 = t_2 \quad \Leftrightarrow \quad t_1^N \, E \, t_2^N$

For t a tuple of closed terms of $L(N)$ and $r \in L_{\mathrm{rel}}$ a relation symbol

2* $\quad N/E \overset{*}{\vDash} rt \quad \Leftrightarrow \quad t^N \, E \, a \quad$ *for some* $a \in r^N$

Finally the definition is extended to all sentences $\varphi \in L(N)$ by induction in the usual way

3* $\quad N/E \overset{*}{\vDash} \neg\varphi \quad \Leftrightarrow \quad$ *not* $N/E \overset{*}{\vDash} \varphi$

4* $\quad N/E \overset{*}{\vDash} \varphi \wedge \psi \quad \Leftrightarrow \quad N/E \overset{*}{\vDash} \varphi$ *and* $N/E \overset{*}{\vDash} \psi$

5* $\quad N/E \overset{*}{\vDash} \exists x\, \varphi(x) \quad \Leftrightarrow \quad N/E \overset{*}{\vDash} \varphi(a) \quad$ *for some* $a \in N$. □

Now, by induction on the syntax of formulas one can prove $\overset{*}{\vDash}$ does what required. In particular, $N/E \overset{*}{\vDash} \varphi(a) \leftrightarrow \varphi(b)$ for every $a \, E \, b$.

2.32 Proposition *Let E be a congruence relation of N. Then the following are equivalent for every $\varphi(x) \in L$*

1. $\quad N/E \overset{*}{\vDash} \varphi(a)$;
2. $\quad N/E \vDash \varphi([a]_E)$. □

5 Completeness

A theory T is maximally consistent if it is consistent and there is no consistent theory S such that $T \subset S$. Equivalently, T contains every sentence φ consistent with T, that is, such that $T \cup \{\varphi\}$ is consistent. Clearly a maximally consistent theory is closed under logical consequences.

A theory T is complete if $\mathrm{ccl}T$ is maximally consistent. Concrete examples will be given in the next chapters as it is not easy to prove that a theory is complete.

2.33 Proposition *The following are equivalent*

a. *T is maximally consistent;*

b. *$T = \mathrm{Th}(M)$ for some structure M;*

c. *T is consistent and $\varphi \in T$ or $\neg\varphi \in T$ for every sentence φ.*

Proof To prove a⇒b, assume that T is consistent. Then there is $M \vDash T$. Therefore $T \subseteq \mathrm{Th}(M)$. As T is maximally consistent $T = \mathrm{Th}(M)$. Implication b⇒c is immediate. As for c⇒a note that if $T \cup \{\varphi\}$ is consistent then $\neg\varphi \notin T$ therefore $\varphi \in T$ follows from c. □

The proof of the proposition below is is left as an exercise for the reader.

2.34 Proposition *The following are equivalent*

a. *T is complete;*

b. *there is a unique maximally consistent theory S such that $T \subseteq S$;*

c. *T is consistent and $T \vdash \mathrm{Th}(M)$ for every $M \vDash T$;*

d. *T is consistent and $T \vdash \varphi$ o $T \vdash \neg\varphi$ for every sentence φ;*

e. *T is consistent and $M \equiv N$ for every pair of models of T.* □

2.35 Exercise Prove that the following are equivalent

a. T is complete;

b. for every sentence φ, o $T \vdash \varphi$ o $T \vdash \neg\varphi$ but not both.

By contrast prove that the following are *not* equivalent

a. T is maximally consistent;

b. for every sentence φ, o $\varphi \in T$ o $\neg\varphi \in T$ but not both.

Hint: consider the theory containing all sentences where the symbol $\neg$ occurs an even number of times. This theory is not consistent as it contains $\perp$. □

2.36 Exercise Prove that if T has exactly 2 maximally consistent extension T_1 and T_2 then there is a sentence φ such that $T, \varphi \vdash T_1$ and $T, \neg\varphi \vdash T_2$. State and prove the generalization to finitely many maximally consistent extensions. □

6 The Tarski-Vaught test

There is no natural notion of *smallest* elementary substructure containing a set of parameters A. The downward Löwenheim-Skolem, which we prove in the next section, is the best result that holds in full generality. Given an arbitrary $A \subseteq N$ we shall construct a model $M \preceq N$ containing A that is small in the sense of cardinality. The construction selects one by one the elements of M that are required to realise the condition $M \preceq N$. Unfortunately, Definition 2.11 supposes full knowledge of the truth in M and it may not be applied during the construction. The following lemma comes to our rescue with a property equivalent to $M \preceq N$ that only mention the truth in N.

2.37 Lemma (Tarski-Vaught test) *For every $A \subseteq N$ the following are equivalent*

1. *A is the domain of a structure $M \preceq N$;*

2. *for every formula $\varphi(x) \in L(A)$, with $|x| = 1$,*
   $N \vDash \exists x\, \varphi(x) \;\Rightarrow\; N \vDash \varphi(b)$ *for some $b \in A$.*

Proof 1⇒2

$$\begin{aligned} N \vDash \exists x\, \varphi(x) \;&\Rightarrow\; M \vDash \exists x\, \varphi(x) \\ &\Rightarrow\; M \vDash \varphi(b) \qquad \text{for some } b \in M \end{aligned}$$

$$\Rightarrow \quad N \vDash \varphi(b) \qquad \text{for some } b \in M.$$

2⇒1 Firstly, note that A is the domain of a substructure of N, that is, $f^N a \in A$ for every $f \in L_{\text{fun}}$ and every $a \in A^{n_f}$. In fact, this follows from 2 with $fa = x$ for $\varphi(x)$.

Write M for the substructure of N with domain A. By induction on the syntax we prove that for every $\xi(x) \in L$

$$M \vDash \xi(a) \;\Leftrightarrow\; N \vDash \xi(a) \qquad \text{for every } a \in M^{|x|}.$$

If $\xi(x)$ is atomic the claim follows from $M \subseteq N$ and the remarks underneath Definition 2.19. The case of Boolean connectives is straightforward, so only the existential quantifier requires a proof. So, let $\xi(x)$ be the formula $\exists y\, \psi(x,y)$ and assume the induction hypothesis holds for $\psi(x,y)$

$$\begin{aligned} M \vDash \exists y\, \psi(a,y) \;&\Leftrightarrow\; M \vDash \psi(a,b) && \text{for some } b \in M \\ &\Leftrightarrow\; N \vDash \psi(a,b) && \text{for some } b \in M \\ &\Leftrightarrow\; N \vDash \exists y\, \psi(a,y). \end{aligned}$$

The second equivalence holds by induction hypothesis, in the last equivalence we use 2 for the implication $\Leftarrow$. □

2.38 Exercise Prove that, in the language of strict orders, $\mathbb{R} \smallsetminus \{0\} \preceq \mathbb{R}$ and $\mathbb{R} \smallsetminus \{0\} \not\preceq \mathbb{R}$. □

7 Downward Löwenheim-Skolem

The main theorem of this section was proved by Löwenheim at the beginning of the last century. Skolem gave a simpler proof immediately afterwards. At the time, the result was perceived as paradoxical.

A few years earlier, Zermelo and Fraenkel provided a formalization of set theory in a first order language. The downward Löwenheim-Skolem theorem implies the existence of an infinite countable model M of set theory: this is the so-called Skolem paradox. The existence of M seems paradoxical because, in particular, a sentence that formalises the axiom of power set holds in M. Therefore M contains an element b which, in M, is the set of subsets of the natural numbers. But the set of elements of b is a subset of M, and therefore it is countable.

In fact, this is not a contradiction, because the expression *all subsets of the natural numbers* does not have the same meaning in M as it has in the real world. The notion of cardinality, too, acquires a different meaning. In the language of set theory, there is a first order sentence that formalises the fact that b is uncountable: the sentence says that there is no bijection between b and the natural numbers. Therefore the bijection between the elements of b and the natural numbers (which exists in the real world) does not belong to M. The notion of equinumerosity has a different meaning in M and in the real world, but those who live in M cannot realise this.

2.39 Downward Löwenheim-Skolem Theorem *Let N be an infinite structure and fix some set $A \subseteq N$. Then there is a structure M of cardinality $\leq |L(A)|$ such that $A \subseteq M \preceq N$.*

Proof Set $\lambda = |L(A)|$. Below we construct a chain $\langle A_i : i < \omega \rangle$ of subsets of N. The chain begins at $A_0 = A$. Finally we set $M = \bigcup_{i<\omega} A_i$. All A_i will have cardinality $\leq \lambda$ so $|M| \leq \lambda$ follows.

Now we construct A_{i+1} given A_i. Assume as induction hypothesis that $|A_i| \leq \lambda$. Then $|L(A_i)| \leq \lambda$. For some fixed variable x let $\langle \varphi_k(x) \ : \ k < \lambda \rangle$ be an enumeration of the formulas in $L(A_i)$ that are consistent in N. For every k pick $a_k \in N$ such that $N \vDash \varphi_k(a_k)$. Define $A_{i+1} = A_i \cup \{a_k \ : \ k < \lambda\}$. Then $|A_{i+1}| \leq \lambda$ is clear.

We use the Tarski-Vaught test to prove $M \preceq N$. Suppose $\varphi(x) \in L(M)$ is consistent in N. As finitely many parameters occur in formulas, $\varphi(x) \in L(A_i)$ for some i. Then $\varphi(x)$ is among the formulas we enumerated at stage i and $A_{i+1} \subseteq M$ contains a solution of $\varphi(x)$. □

We will need to adapt the construction above to meet more requirements on the model M. To better control the elements that enters M it is convenient to add one element at the time (above we add λ elements at each stage). We need to enumerate formulas with care if we want to complete the construction by stage λ.

2.40 Second proof of the downward Löwenheim-Skolem Theorem From set theory we know that there is a bijection $\pi : \lambda^2 \to \lambda$ such that $j, k \leq \pi(j, k)$ for all $j, k < \lambda$. Suppose we have defined the sets A_j for every $j \leq i$ and let $\langle \varphi^j_k(x) \ : \ k < \lambda \rangle$ be an enumeration of the consistent formulas of $L(A_j)$. Let $j, k \leq i$ be such that $\pi(j, k) = i$. Let b be a solution of the formula $\varphi^j_k(x)$ and define $A_{i+1} = A_i \cup \{b\}$.

We use Tarski-Vaught test to prove $M \preceq N$. Let $\varphi(x) \in L(M)$ be consistent in N. Then $\varphi(x) \in L(A_j)$ for some j. Then $\varphi(x) = \varphi^j_k$ for some k. Hence a witness of $\varphi(x)$ is enumerated in M at stage $\pi(j, k) + 1$. □

2.41 Exercise Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Hint: adapt the construction used to prove the downward Löwenheim-Skolem Theorem. □

8 Elementary chains

An elementary chain is a chain $\langle M_i \ : \ i < \lambda \rangle$ of structures such that $M_i \preceq M_j$ for every $i < j < \lambda$. The union (or limit) of the chain is the structure with as domain the set $\bigcup_{i<\lambda} M_i$ and as relations and functions the union of the relations and functions of M_i. It is plain that all structures in the chain are substructures of the limit.

2.42 Lemma *Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of structures. Let N be the union of the chain. Then $M_i \preceq N$ for every i.*

Proof By induction on the syntax of $\varphi(x) \in L$ we prove

$$M_i \vDash \varphi(a) \ \Leftrightarrow \ N \vDash \varphi(a) \qquad \text{for every } i < \lambda \text{ and every } a \in M_i^{|x|}$$

As remarked in 3' of Section 3, the claim holds for quantifier-free formulas. Induction for Boolean connectives is straightforward so we only need to consider the existential quantifier

$$\begin{aligned} M_i \vDash \exists y \, \varphi(a, y) \ &\Rightarrow \ M_i \vDash \varphi(a, b) && \text{for some } b \in M_i. \\ &\Rightarrow \ N \vDash \varphi(a, b) && \text{for some } b \in M_i \subseteq N \end{aligned}$$

where the second implication follows from the induction hypothesis. Vice versa

$$N \vDash \exists y\, \varphi(a, y) \;\; \Rightarrow \;\; N \vDash \varphi(a, b) \qquad \text{for some } b \in N$$

Without loss of generality we can assume that $b \in M_j$ for some $j \geq i$ and obtain

$$\Rightarrow \;\; M_j \vDash \varphi(a, b) \qquad \text{for some } b \in M_j$$

Now apply the induction hypothesis to $\varphi(x, y)$ and M_j

$$\Rightarrow \;\; M_j \vDash \exists y\, \varphi(a, y)$$

$$\Rightarrow \;\; M_i \vDash \exists y\, \varphi(a, y)$$

where the last implication holds because $M_i \preceq M_j$. □

2.43 Exercise Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of substructures of N. Let M be the union of the chain. Prove that $M \preceq N$. □

2.44 Exercise Give an alternative proof of Exercise 2.41 using the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two countable chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$. The required model is $K = \bigcup_{i \in \omega} K_i$. In fact it is easy to check that $K \cap M = \bigcup_{i \in \omega} M_i$. □

Chapter 3

Ultraproducts

In these notes we only use ultraproducts to prove the compactness theorem. Since a syntactic proof of the compactness theorem is also given, this chapter is, strictly speaking, not required. However, the importance of ultraproducts transcends its application to model theory.

1 Filters and ultrafilters

The material in this section will appear again in a more general setting in Chapter 5.

Let I be any set. A filter on I (or a filter of $\mathcal{P}(I)$), is a non empty set $F \subseteq \mathcal{P}(I)$ such that for every $a, b \in \mathcal{P}(I)$:

f1 $\qquad a \in F$ and $a \subseteq b \quad \Rightarrow \quad b \in F$

f2 $\qquad a \in F$ and $b \in F \quad \Rightarrow \quad a \cap b \in F$

A filter F is proper if $F \neq \mathcal{P}(I)$, equivalently if $\varnothing \notin F$. Otherwise it is improper. By f1 above, F is proper if and only if $\varnothing \notin F$. A filter F is principal if $F = \{a \subseteq I : b \subseteq a\}$ for some set $b \subseteq I$. When this happens, we say that $\{b\}$ generates F. When I is finite every filter F is principal and generated by $\{\cap F\}$. Non-principal filters on I exist as soon as I is infinite. In fact, if I is infinite it is easy to check that the following is a filter:

$$F \quad = \quad \Big\{ a \subseteq I \;:\; a \;\text{ is cofinite in }\; I \Big\}.$$

where cofinite (in I) means that $I \smallsetminus a$ is finite. This is called the Fréchet filter on I. The Fréchet filter is the minimal non-principal filter.

A proper filter F is maximal if there is no proper filter H such that $F \subset H$. A proper filter F is an ultrafilter if for every $a \subseteq I$

$$a \notin F \quad \Rightarrow \quad \neg a \in F \qquad\qquad \text{where } \neg a \;=\; I \smallsetminus a\,.$$

Below we prove that the ultrafilters are exactly the maximal filters.

3.1 Exercise Let I be infinite. Prove that every non-principal ultrafilter on I contains Fréchet's filter. Show that this does not hold for plain filters. □

Let $B \subseteq \mathcal{P}(I)$. Then the filter generated by B is the intersection of all the filters that contain B. It is easy to check that the intersection of a family of filters is a filter, so the notion is well defined. The following easy proposition gives a workable characterization of the filter generated by a set.

3.2 Proposition *The filter generated by B is* $\big\{a \subseteq I \;:\; \bigcap C \subseteq a \;$ *for some finite* $\; C \subseteq B\big\}$. □

We say that B has the finite intersection property if $\bigcap C \neq \varnothing$ for every finite $C \subseteq B$. The following proposition is immediate.

3.3 Proposition *The following are equivalent:*

1. *the filter generated by B is non principal;*
2. *B has the finite intersection property.* □

A proper filter F is prime if for every $a, b \subseteq I$.

$$a \cup b \in F \quad \Rightarrow \quad a \in F \quad \text{or} \quad b \in F$$

Prime filters coincide with maximal filters. However, in Chapter 5, we introduce a more general context where primality is distinct from maximality.

3.4 Proposition *For every filter F on I, the following are equivalent:*

1. *F is a maximal filter;*
2. *F is a prime filter;*
3. *F is an ultrafilter.*

Proof 1⇒2. Suppose $a, b \notin F$, where F is maximal. We claim that $a \cup b \notin F$. By maximality, there is a $c \in F$ such that $a \cap c = b \cap c = \varnothing$. Therefore $(a \cup b) \cap c = \varnothing$. Hence $a \cup b \notin F$.

2⇒3. It suffices to note that $I = a \cup \neg a \in F$.

3⇒1. If $a \notin F$, where F is an ultrafilter, then $\neg a \in F$ and no proper filter contains $F \cup \{a\}$. □

3.5 Proposition *Let $B \subseteq \mathcal{P}(I)$ have the finite intersection property. Then B is contained in a maximal filter.*

Proof First we prove that the union of a chain of subsets of $\mathcal{P}(I)$ with the finite intersection property has the finite intersection property. Let $\mathcal{B}$ be such a chain and suppose for a contradiction that $\bigcup \mathcal{B}$ does not have the finite intersection property. Fix a finite $C \subseteq \bigcap \mathcal{B}$ such that $\bigcap C = \varnothing$. As C is finite, we have $C \subseteq B$ for some $B \in \mathcal{B}$. Hence B does not have the finite intersection property, which is a contradiction.

Now apply Zorn's lemma to obtain a $B \subseteq \mathcal{P}(I)$ which is maximal among the sets with the finite intersection property. It is immediate that B is a filter. □

3.6 Exercise Prove that all principal ultrafilters are generated by a singleton. □

2 Direct products

In this and in the next section $\langle M_i : i \in I \rangle$ is a sequence of L-structures. (We are abusing of the word sequence, since I is only a set.) The direct product of this sequence is a structure denoted by

$$\prod_{i \in I} M_i \qquad \text{(below this product is denoted by } N\text{).}$$

and defined by conditions 1-3 below. If $M_i = M$ for all $i \in I$, we say that N is a direct power of M and denote it by M^I.

The domain of N is the set containing all functions

1. $$\hat{a} \ : \ I \to \bigcup_{i\in I} M_i$$
$$\hat{a} \ : \ i \mapsto \hat{a}i \in M_i$$

We do not distinguish between tuples of elements of N and tuple-valued functions. For instance, the tuple $\hat{a} = \langle \hat{a}_1 \ldots \hat{a}_n \rangle$ is identified with the function $\hat{a} : i \mapsto \hat{a}i = \langle \hat{a}_1 i, \ldots, \hat{a}_n i \rangle$. On a first reading of what follows, it may help to pretend that all functions and relations are unary.

The interpretation of $f \in L_{\text{fun}}$ is defined as follows:

2. $$(f^N \hat{a})i \ = \ f^{M_i}(\hat{a}i) \qquad \text{for all } i \in I.$$

The interpretation of $r \in L_{\text{fun}}$ is the product of the relations r^{M_i}, that is, we define

3. $$\hat{a} \in r^N \ \Leftrightarrow \ \hat{a}i \in r^{M_i} \qquad \text{for all } i \in I.$$

The following proposition is immediate.

3.7 Proposition *If $\wedge$ is the only connective and $\forall$, $\exists$ the only quantifiers that occur in $\varphi(x) \in L$, then for every $\hat{a} \in N^{|x|}$*

♯ $$N \vDash \varphi(\hat{a}) \ \Leftrightarrow \ M_i \vDash \varphi(\hat{a}i) \qquad \textit{for all } i \in I.$$

Proof By induction on syntax. First note that we can extend 2 to all terms $t(x)$ as follows:

2′. $$(t^N \hat{a})i \ = \ t^{M_i}(\hat{a}i) \qquad \text{for all } i \in I.$$

Combining 3 and 2′ gives that for every $r \in L_{\text{rel}} \cup \{=\}$ and every L-term $t(x)$

3′. $$N \vDash rt\hat{a} \ \Leftrightarrow \ M_i \vDash rt\hat{a}i \qquad \text{for all } i \in I.$$

This shows that ♯ holds for $\varphi(x)$ atomic. Induction for the connectives $\wedge$, $\forall$ ed $\exists$ is immediate. □

A consequence of Proposition 3.7 is that a direct product of groups, rings or vector spaces is a structure of the same sort. However, a product of fields is not a field.

3 Łoś's Theorem

Assume the notation of the previous sections. In particular, $\langle M_i : i \in I \rangle$ is a sequence of structures and N is the direct product of this sequence.

Let F be a filter on I. We define the following congruence on N (see Definition 2.29):

$$\hat{a} \sim_F \hat{c} \ \Leftrightarrow \ \big\{ i \in I \ : \ \hat{a}i = \hat{c}i \big\} \in F\,.$$

To check that $\sim_F$ is indeed a congruence, first we need to check that it is an equivalence relation. Reflexivity and symmetry are immediate, and transitivity follows from f2 in Section 1. Then we check that $\sim_F$ is compatible with the functions of L, that is, that c1 of Definition 2.29 is satisfied. This follows from f1 in Section 1.

For brevity, we write N/F for $N/{\sim_F}$ and $[\hat{a}]_F$ for $[\hat{a}]_{\sim_F}$. Then c3 of Section 2.4 gives

c3′. $$[\hat{a}]_F \in r^{N/F} \ \Leftrightarrow \ \big\{ i \ : \ \hat{a}i = \hat{b}i \in r^{M_i} \big\} \in F \quad \text{for some } \hat{b} \in N.$$

The structure N/F is called the reduced product of the structures $\langle M_i : i \in I \rangle$ or, when $M_i = M$ for all $i \in I$, the reduced power of M. When F is an ultrafilter we say ultraproduct, respectively ultrapower.

3.8 Łoś's Theorem *Let $\varphi(x) \in L$ and let F be an ultrafilter on I. Then for every $\hat{a} \in N^{|x|}$ the following are equivalent:*

1. $N/F \overset{*}{\vDash} \varphi(\hat{a})$ *(see Definition 2.31);*
2. $\{\, i \,:\, M_i \vDash \varphi(\hat{a}i) \,\} \in F.$

Proof We proceed by induction on the syntax of $\varphi(x)$. Suppose $\varphi(x)$ is of the form $rt(x)$ for some tuple of terms $t(x)$ and $r \in L_{\mathrm{rel}} \cup \{=\}$. Then 1⇔2 is clear.

We prove the inductive step for the connectives $\neg$, $\wedge$, and the quantifier $\exists$. We begin with $\neg$. This is the only place in the proof where the assumption that F is an *ultra*filter is required. By the inductive hypothesis,

$$N/F \overset{*}{\vDash} \neg\varphi(\hat{a}) \quad\Leftrightarrow\quad \{\, i \,:\, M_i \vDash \varphi(\hat{a}i) \,\} \notin F$$

So, as F is an *ultra*filter

$$\Leftrightarrow\quad \{\, i \,:\, M_i \vDash \neg\varphi(\hat{a}i) \,\} \in F\,.$$

Now consider $\wedge$. Assume inductively that the equivalence 1⇔2 holds for $\varphi(x)$ and $\psi(x)$. Then

$$N/F \overset{*}{\vDash} \varphi(\hat{a}) \wedge \psi(\hat{a}) \quad\Leftrightarrow\quad \{i \,:\, M_i \vDash \varphi(\hat{a}i)\} \in F \text{ and } \{i \,:\, M_i \vDash \psi(\hat{a}i)\} \in F\,.$$

As filters are closed under intersection, we obtain

$$\Leftrightarrow\quad \{\, i \,:\, M_i \vDash \varphi(\hat{a}i) \wedge \psi(\hat{a}i) \,\} \in F\,.$$

Finally, consider $\exists y$. Assume inductively that the equivalence 1⇔2 holds for $\varphi(x, y)$. Then

$$\begin{aligned} N/F \overset{*}{\vDash} \exists y\, \varphi(\hat{a}, y) \quad&\Leftrightarrow\quad N/F \overset{*}{\vDash} \varphi(\hat{a}, \hat{b}) && \text{for some } \hat{b} \in N \\ &\Leftrightarrow\quad \{\, i \,:\, M_i \vDash \varphi(\hat{a}i, \hat{b}i) \,\} \in F && \text{for some } \hat{b} \in N. \end{aligned}$$

We claim this is equivalent to

$$\Leftrightarrow\quad \{\, i \,:\, M_i \vDash \exists y\, \varphi(\hat{a}i, y) \,\} \in F.$$

The $\Rightarrow$ direction is trivial. For $\Leftarrow$, we choose as $\hat{b}$ a sequence that picks a witness of $M_i \vDash \exists y\, \varphi(\hat{a}i, y)$ if it exists, and some arbitrary element of M_i otherwise. □

Let a^I denote the element of M^I that has constant value a. The following is an immediate consequence of Łoś's theorem.

3.9 Corollary *For every $a \in M$*

$$M^I/F \overset{*}{\vDash} \varphi(a^I) \quad\Leftrightarrow\quad M \vDash \varphi(a)\,.$$ □

We often identify M with its image under the embedding $h : a \mapsto [a^I]_F$, and say that M^I/F is an elementary extension of M.

The following corollary is an immediate consequence of the compactness theorem that we prove in the next chapter. The construction in the proof uses ultrapowers.

3.10 Corollary *Every infinite structure has a proper elementary extension.*

Proof Let M be an infinite structure and let F be a *non-principal* ultrafilter on ω. It suffices to show that $h[M]$, the image of the embedding defined above, is a *proper* substructure of M^ω/F. As M is infinite, there is an injective function $\hat{d} \in M^\omega$. Then for every $a \in M$ the set $\{i : \hat{d}i = a\}$ is either empty or a singleton and, as F is non principal, it does not belong to F. So, by Łoš Theorem, we have $M^\omega/F \stackrel{*}{\vDash} \hat{d} \neq a^I$ for every $a \in M$, that is, $[\hat{d}]_F \notin h[M]$. □

3.11 Exercise Consider $\mathbb{N}$ as a structure in the language of strict orders. Let F be a non-principal ultrafilter on ω. Prove that in $\mathbb{N}^\omega/F$ there is a sequence $\langle \hat{a}_i : i \in \omega \rangle$ such that $\hat{a}_{i+1} < \hat{a}_i$. □

3.12 Exercise Let I be the set of integers $i > 1$. For $i \in I$, let $\mathbb{Z}_i$ denote the additive group of integers modulo i, and let N denote the product $\prod_{i \in I} \mathbb{Z}_i$. Prove that, if F is a non-principal ultrafilter on I,

1. for some F, N/F does not contain any element of finite order;
2. for some F, N/F has some elements of order 2;
3. for all F, N/F contains an element of infinite order;
4. for some F, $N/F \vDash \forall x \exists y \; my = x$ for every integer $m > 0$;
5. for some F, N/F contains an element $\hat{a}$ such that $N/F \vDash \forall x \; mx \neq \hat{a}$ for every positive integer m. □

Chapter 4

Compactness

We present two proofs of the compactness theorem. The first is syntactic, the second uses ultrapowers.

Somewhat surprisingly, the compactness theorem is not strictly required for the next few chapters (only from Chapter 9). So the reading of this chapter may be postponed.

1 Compactness via syntax

Here we prove the compactness theorem using the so-called Henkin method. We divide the proof in two steps. Firstly, we observe that when the language is rich enough to name witnesses of all existential statements of the theory, these witnesses (*Henkin constants*) form a canonical model. Secondly, we show that we can add the required Henkin constants to any finitely consistent theory.

4.1 Definition *Fix a language L. Assume for simplicity that formulas use only the connectives* $\wedge$, $\neg$ *and* $\exists$. *We say that* T *is a Henkin theory if for all formulas* φ *and* ψ

0. $\varphi \in T \quad \Rightarrow \quad \neg\varphi \notin T$
1. $\neg\neg\varphi \in T \quad \Rightarrow \quad \varphi \in T$
2. $\varphi \wedge \psi \;\in T \quad \Rightarrow \quad \varphi \in T \text{ and } \psi \in T$
3. $\neg(\varphi \wedge \psi) \in T \quad \Rightarrow \quad \neg\varphi \in T \text{ or } \neg\psi \in T$
4. $\exists x\, \varphi \in T \quad \Rightarrow \quad \varphi[x/a] \in T \quad \text{for some closed term } a$
5. $\neg\exists x\, \varphi \in T \quad \Rightarrow \quad \neg\varphi[x/a] \in T \quad \text{for all closed terms } a.$

Moreover, the following holds for all closed terms a, b, c

a. $a \doteq a \in T$
b. $a \doteq b \in T \quad \Rightarrow \quad b \doteq a \in T$
c. $a \doteq b,\ b \doteq c \in T \quad \Rightarrow \quad a \doteq c \in T$
d. $a \doteq b,\ \varphi[x/a] \in T \quad \Rightarrow \quad \varphi[x/b] \in T.$ □

Fix a theory T and let M be the structure that has as domain the set of closed terms. Define for every relation symbol r

$$r^M \quad = \quad \big\{\langle a_1, \ldots, a_n\rangle \in M^n \ : \ r(a_1, \ldots, a_n) \in T\big\},$$

where n is the arity of r. Define for every function symbol f

$$f^M \quad = \quad \big\{\langle t, a_1, \ldots, a_n\rangle \in M^{n+1} \ : \ t = f a_1 \ldots a_n\big\}.$$

where n is the arity of r. An easy proof by induction shows that $t^M = t$ for all closed terms t.

Finally, let E be the relation on M that holds when $a \doteq b \in T$.

4.2 Lemma *The relation E is a congruence on M (as defined in Section 2.4).*

Proof Axioms a-c ensure that E is an equivalence. We claim that that E is a congruence. This is immediate for unary functions: apply e to the formula $fx \doteq fa$. In general the claim is easily proved by induction on the arity of f. □

Condition 0 is the only negative requirement of Definition 4.1 (it requires that T does *not* contain some formula). By condition 0, Henkin theories do not contain any blatant inconsistency. Surprisingly, this is all what is needed for the existence of a model.

4.3 Theorem *If T is Henkin theory then $M/E \vDash T$.*

Proof By induction on the complexity of the formula φ in T we prove that

1. $\varphi \in T \quad \Rightarrow \quad M/E \overset{*}{\vDash} \varphi$ for the notation cf. Definition 2.31
2. $\neg\varphi \in T \quad \Rightarrow \quad M/E \overset{*}{\nvDash} \neg\varphi$

Induction is immediate by 1-5 of Definition 4.1. Hence we only need to verify the claim for atomic formulas. Consider first the formula $\varphi = (t_1 \doteq t_2)$ where the t_i are closed terms. By the definition of M, for every closed term t we have $t^M = t$ so claim 1 is clear. As for 2, suppose $M/E \overset{*}{\vDash} t_1 \doteq t_2$, that is $t_1 E t_2$. Then $t_1 \doteq t_2 \in T$ and $\neg t_1 \doteq t_2 \notin T$ follows from axiom 0.

Now assume $\varphi = rt$ for a relation r and a tuple of closed terms t. The argument is similar: 1 is immediate; to prove 2 suppose that $M/E \overset{*}{\vDash} rt$. Then $tEs \in r^M$ for some tuple of closed terms s. Then $rs \in T$, and by d $rt \in T$. Finally from 0 we obtain $\neg rt \notin T$. □

4.4 Proposition *If every finite subset of T has a model then there is a Henkin theory T' containing T. The theory T' may be in an expanded language L'.*

Proof Set $\lambda = |L|$. Let $\langle c_i : i < \lambda \rangle$ be some constants not in L. Let L_i be the language with constants among $c_{\restriction i}$. Fix a variable x and an enumeration $\langle \varphi_i(x) : i < \lambda \rangle$ of the formulas in L_λ. Suppose that the enumeration is such that $\varphi_i(x) \in L_i$.

We now construct a sequence of finitely consistent L_i-theories T_i. If α is 0 or a limit ordinal we define

$$T_\alpha \quad = \quad T \cup \bigcup_{i<\alpha} T_i.$$

As for successor ordinals, let S_i be a maximally finitely consistent set of L_i-formulas containing T_i. (Here we use Zorn's lemma, but see the remark below.) It is immediate that S_i satisfies all requirements in Definition 4.1 but possibly for 4.

Now, if $\exists x\, \varphi_i(x) \in S_i$ set $T_{i+1} = S_i \cup \{\varphi[x/c_i]\}$. As c_i does not occur in S_i, it is evident that T_{i+1} is finitely consistent.

Recall that we assumed $\exists x\, \varphi_i(x) \in L_i$. Then, either $\exists x\, \varphi_i(x) \in T_{i+1}$ or it is not finitely consistent with T_{i+1}. Hence stage i settle requirement 4 in Definition 4.1 as far as $\varphi_i(x)$ is concerned.

At stage λ all possible counterexamples to 4 have been ruled out, then $T' = T_\lambda$ is the required Henkin theory. □

A theory T is finitely consistent if all its finite subsets are consistent. The following theorem is an immediate corollary of the proposition above.

4.5 Compactness Theorem *If T is finitely consistent then T is consistent.* □

4.6 Remark To keep the proof above short, we applied Zorn's lemma. This is not strictly necessary. In fact, if we are given a finitely consistent theory T. We can extend T to a theory S that meets Definition 4.1, up to condition 4, by adding systematically all required formulas. The procedure is effective, hence Zorn's lemma is not required.

It is interesting to consider the case when T is finite. Assume also (though this is not really necessary) that the language contains finitely many symbols and no functions other then constants. Then the construction in Proposition 4.4 is an effective procedure that produces in ω steps a model of T. At each step T_n is finite and contains only subformulas of formulas in T or variant on these obtained by substituting constants for variables.

Now suppose instead that we start with an inconsistent T. The procedure above has to come to a halt at same (finite) stage because a model of T does not exist. When the procedure halts, we end up with a finite sequence of finite theories $T_0, \ldots, T_n$ where $T_0 = T$ and T_n contains some blatant inconsistency (i.e. φ and $\neg\varphi$). Many have interpreted $T_0, \ldots, T_n$ as a formal *proof* of the inconsistency of T.

All this has little or no interest to model theory. But it highlights a fascinating phenomenon. When we say that T is inconsistent, we say that no structure models T. This expression uses a (meta linguistic) universal quantifier that ranges over the class of all structures. Yet this is equivalent to an expression that merely asserts the existence of a finite sequence of finite theories. □

2 Compactness via ultraproducts

We repeat, a theory is finitely consistent if all its finite subsets are consistent. The following theorem is the *fiat lux* of model theory.

4.7 Compactness Theorem *Every finitely consistent theory is consistent.*

Proof Let T be a finitely consistent theory. We claim that the structure N/F which we define below is a model of T. Let I be the set of consistent sentences I in the language L. For every $\xi \in I$ pick some $M_\xi \vDash \xi$. For any sentence $\varphi \in L$ we write X_φ for the following subset of I

$$X_\varphi \quad = \quad \Big\{\xi \in I \;:\; \xi \vdash \varphi\Big\}$$

Clearly φ is consistent if and only if $X_\varphi \neq \varnothing$. Moreover $X_{\varphi\wedge\psi} \;=\; X_\varphi \cap X_\psi$. hence, as T is finitely consistent, the set $B = \big\{X_\varphi \;:\; \varphi \in T\big\}$ has the finite intersection property. Therefore B extends to an ultrafilter F on I. Define

$$N \quad = \quad \prod_{\xi \in I} M_\xi .$$

We claim that $N/F \vDash T$. By Łoš Theorem, for every sentence $\varphi \in L$

$$N/F \vDash \varphi \quad \Leftrightarrow \quad \Big\{\xi \;:\; M_\xi \vDash \varphi\Big\} \;\in\; F\,.$$

By the definition of F, for every $\varphi \in T$, the set $X_\varphi \subseteq \{\xi \ : \ M_\xi \vDash \varphi\}$ belongs to F. Therefore $N/F \vDash T$, *et lux fuit*. □

The compactness theorem can be formulated in the following apparently stronger way.

4.8 Corollary *If $T \vdash \varphi$ then there is some finite $S \subseteq T$ such that $S \vdash \varphi$.*

Proof Suppose $S \nvdash \varphi$ for every finite $S \subseteq T$. Then for every finite $S \subseteq T$ there is a model $M \vDash S \cup \{\neg\varphi\}$. In other words, $T \cup \{\neg\varphi\}$ is finitely consistent. By compactness $T \cup \{\neg\varphi\}$ hence $T \nvdash \varphi$. □

4.9 Exercise Let $\Phi \subseteq L$ be a set of sentences and suppose that $\vdash \psi \leftrightarrow \bigvee \Phi$ for some sentence ψ. Prove that there is a finite $\Phi_0 \subseteq \Phi$ such that $\vdash \psi \leftrightarrow \bigvee \Phi_0$. □

3 Upward Löwenheim-Skolem

Recall that a type is a set of formulas. When we present types we usually declare the variables that may occur in it – we write $p(x)$, $q(x)$, etc. where x is a tuple of variables. Clearly, when x is the empty tuple, $p(x)$ is just a theory. We identify a finite types with the conjunction of the formulas contained in it.

We write $M \vDash p(a)$ if $M \vDash \varphi(a)$ for every $\varphi(x) \in p$. We say that a is a solution or a realization of $p(x)$. An equivalent notation is $M, a \vDash p(x)$ or, when M is clear from the context, $a \vDash p(x)$. We say that $p(x)$ is consistent in M it has a solution in M. In this case we may write $M \vDash \exists x\, p(x)$. We say that $p(x)$ is consistent if it is consistent in some model.

We say that a type $p(x)$ is finitely consistent if all its finite subsets are consistent. If its finite subsets are all consistent in the same model M, we say that $p(x)$ is finitely consistent in M. The following theorem shows that the latter notion, which is trivial for theories, is very interesting for types.

4.10 Compactness Theorem for types *Every finitely consistent type $p(x) \subseteq L$ is consistent. Moreover, if $p(x) \subseteq L(M)$ is finitely consistent in M then it is realised in some elementary extension of M.*

Proof Let L' be the expansion of L obtained by adding the fresh symbols c, a tuple of constants of the same length as x. Then $p(c)$ is a finitely consistent theory in the language L'. By the compactness theorem there is an L'-structure $N' \vDash p(c)$. Let N be the reduct of N' to L, that is, the L-structure with the same domain and the same interpretation as N' on the symbols of L. Note that, though the constants c are not in L, the elements of the tuple $c^{N'}$ remain in N. Then $N, c^{N'} \vDash p(x)$.

As for the second claim, let a be an enumeration of M. We can assume that $p(x)$ has the form $p'(x\,;a)$ for some $p'(x\,;z) \subseteq L$. Define

$$q(z) \ = \ \{\varphi(z) \ : \ M \vDash \varphi(a)\}$$

Clearly, $p'(x\,;z) \cup q(z)$ is finitely consistent. Then, by the first part of the proof, it is realized in some model N by some $c', a' \in N^{|x,z|}$. Let $h = \{\langle a, a'\rangle\}$. Then for every $\varphi(z) \in L$ we have

$$M \vDash \varphi(a) \quad \Leftrightarrow \quad \varphi(z) \in q \quad \Leftrightarrow \quad N \vDash \varphi(ha).$$

In the last equivalence, $\Leftarrow$ holds because $q(z)$ is complete. (Using a term that will be introduced only in Section 5.4, this shows that $h : M \to N$ is an elementary embedding.)

By the remarks after Definition 2.19, $h : M \to N$ is an embedding. Hence the equivalence above prove that $h[M] \preceq N$. So the theorem follows by identifying M with $h[M]$. □

The following corollary is historically important.

4.11 Upward Löwenheim-Skolem Theorem *Every infinite structure has arbitrarily large elementary extensions.*

Proof Let $x = \langle x_i : i < \lambda \rangle$ be a tuple of distinct variables, where λ is an arbitrary cardinal. The type $p(x) = \{x_i \neq x_j : i < j < \lambda\}$ is finitely consistent in every infinite structure and every structure that realises p has cardinality $\geq \lambda$. Hence the claim follows from Theorem 4.10. □

4 Finite axiomatizability

A theory T is finitely axiomatizable if $\mathrm{ccl}(S) = \mathrm{ccl}(T)$ for some finite S. The following theorem shows that we can restrict the search for S to the subsets of T.

4.12 Proposition *For every theory T the following are equivalent*

1. *T is finitely axiomatizable;*
2. *there a finite $S \subseteq T$ such that $S \vdash T$.*

Proof Only 1⇒2 requires a proof. If T is finitely axiomatizable, there is a sentence φ such that $\mathrm{ccl}(\varphi) = \mathrm{ccl}(T)$. Then $T \vdash \varphi \vdash T$. By Proposition 4.8 there is a finite $S \subseteq T$ such that $S \vdash \varphi$. Then also $S \vdash T$. □

If L is empty, then every structure is a model. The theory of infinite sets is the set of sentences that hold in every infinite structure.

4.13 Example The theory of infinite sets is not finitely axiomatizable. Define

$$T_\infty = \left\{ \exists^{\geq n} x\ (x = x) \right\} : n \in \omega \}$$

Every infinite set is a model of T_∞ and, vice versa, every model of T_∞ is is an infinite set. Then $\mathrm{ccl}(T_\infty)$ is the theory of infinite sets. Suppose for a contradiction that T_∞ is finitely axiomatizable. By Proposition 4.12, $\exists^{\geq n} x (x = x) \vdash T_\infty$ for some n. Any set of cardinality $n + 1$ proves that this is not the case. □

The following is a less trivial example. It proves a claim we made in Exercise 1.16.

4.14 Example Write T_{gph} for the theory of graphs, see Example 2.5. Let $\mathcal{K}$ be the following class of structures

$$\mathcal{K} = \left\{ M \models T_{\mathrm{gph}} : r^M \subseteq (A \times \neg A) \cup (\neg A \times A) \text{ for some } A \subseteq M \right\}$$

We prove that $\mathcal{K}$ is axiomatizable but not finitely axiomatizable, i.e. $\mathcal{K} = \mathrm{Mod}(T)$ for some theory T, but T cannot be choosen finite.

A *path of length n* in M is a sequence $c_0, \ldots, c_n \in M$ such that $M \vDash r(c_i, c_{i+1})$ for every $0 \leq i < n$. A path is *closed* if $c_0 = c_n$. We claim that the following theory axiomatizes $\mathcal{K}$.

$$T \;=\; \left\{ \neg\exists x_0, \ldots x_{2n+1} \left[\bigwedge_{i=0}^{2n} r(x_i, x_{i+1}) \;\wedge\; x_0 = x_{2n+1} \right] \;\;:\;\; n \in \omega \right\}$$

In words, T says that all closed paths have even length. Inclusion $\mathcal{K} \subseteq \mathrm{Mod}(T)$ is clear, we prove $\mathrm{Mod}(T) \subseteq \mathcal{K}$. Let $M \vDash T$ and let $A_o \subseteq M$ contain exactly one point for every connected component of M. Define

$$A \;=\; \left\{ b \;:\; M \vDash \exists x_0, \ldots, x_{2n} \left[\bigwedge_{i=1}^{2n-1} r(x_i, x_{i+1}) \wedge a{=}x_0 \wedge x_{2n}{=}b \right],\; a \in A_o,\;\; n \in \omega \right\}$$

We claim that $r^M \subseteq (A \times \neg A) \cup (\neg A \times A)$, hence $M \in \mathcal{K}$. We need to verify that if $r(b, c)$ then neither $b, c \in A$ nor $b, c \in \neg A$. Suppose for a contradiction that $r(b, c)$ and $b, c \in A$ (the case $b, c \in \neg A$ is similar). As b and c belong to the same connected component, there are two paths $b_0, \ldots, b_{2n}$ and $c_0, \ldots, c_{2m}$ that connect $a = b_0 = c_0 \in A_o$ to $b = b_{2n}$ and $c = c_{2m}$. Then $a, b_1, \ldots, b_{2n}, c_{2m}, \ldots, c_1, a$ is a closed path of odd length. A contradiction.

We now prove that $\mathcal{K}$ is not finitely axiomatizable. By Proposition 4.12 it suffices to note that no finite $S \subseteq T$ axiomatizes $\mathcal{K}$. $\square$

Chapter 5

Types and morphisms

In Section 1 and 2 we introduce distributive lattices and prime filters and prove Stone's representation theorem for distributive lattices. In Section 3 we discuss lattices that arise from sets of formulas and their prime filters (prime types). These sections are mainly required for the discussion of Hilbert's Nullstellensatz in Section 8.7 below.

There is a lot of notation in this chapter. The reader may skim through and return to it when we refer to it.

1 Semilattices and filters

A preorder is a set $\mathbb{P}$ with a transitive and symmetric relation which we usually denote by $\leq$. If $A, B \subseteq \mathbb{P}$ we write $A \leq B$ if $a \leq b$ for every $a \in A$ and $b \in B$. We write $a \leq B$ and $A \leq b$ for $\{a\} \leq B$ and $A \leq \{b\}$ respectively.

Quotienting a preorder by the equivalence relation

$$a \sim b \quad \Leftrightarrow \quad a \leq b \quad \text{and} \quad b \leq a$$

gives a (partial) order. We often do not distinguish between a preorder and the partial order associated to it. Preorders are very common; here we are interested in the one induced by the relation of logical consequence

$$\varphi \leq \psi \quad \Leftrightarrow \quad \varphi \vdash \psi,$$

or, more generally, by the relation of logical consequence over a theory T, that is

$$\varphi \leq \psi \quad \Leftrightarrow \quad T \cup \{\varphi\} \vdash \psi.$$

A partial order $\mathbb{P}$ is a lower semilattice if for each pair $a, b \in \mathbb{P}$ there is a maximal element c such that $c \leq \{a, b\}$. We call c the meet of a and b. The meet is unique and is denoted by $a \wedge b$. Dually, a partial order is an upper semilattice if for each pair of elements a and b there is a minimal element c such that $\{a, b\} \leq c$. This c is called the joint of a and b. The joint is unique and is denoted by $a \vee b$. A lattice is simultaneously a lower and an upper semilattice.

An element c such that $c \leq \mathbb{P}$ is called a lower bound or a bottom. An element such that $\mathbb{P} \leq c$ is called an upper bound or a top. Lower and upper bounds are unique and will be denoted by 0, respectively 1. Other symbols common in the literature are $\perp$, respectively $\top$. A semilattice is bounded if it has both an upper and a lower bound.

For the rest of this section we assume that $\mathbb{P}$ is a bounded lower semilattice.

The meet is associative and commutative

$$\begin{aligned} (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\ a \wedge b &= a \wedge b. \end{aligned}$$

Hence we may unambiguously write $a_1 \wedge \cdots \wedge a_n$. When $C \subseteq \mathbb{P}$ is finite, we write $\wedge C$ for the meet of all the elements of C. We agree that $\wedge\varnothing = 1$.

In an upper semilattice, the dual properties hold for the joint. We write $\vee C$ for the joint of all elements in C and we agree that $\vee\varnothing = 0$.

A filter of $\mathbb{P}$ is a non-empty set $F \subseteq \mathbb{P}$ that satisfies the following for all $a, b \in \mathbb{P}$

f1. $a \in F$ and $a \leq b \;\Rightarrow\; b \in F$

f2. $a, b \in F \;\Rightarrow\; a \wedge b \in F$.

We say that F is a proper filter if $F \neq \mathbb{P}$, equivalently if $0 \notin F$. We say that F is principal if $F = \{b \;:\; a \leq b\}$ for some $a \in \mathbb{P}$. More precisely, we say that F is the principal filter generated by a. A proper filter F is maximal if there is no filter H such that $F \subset H \subset \mathbb{P}$.

We say that a filter F is maximal relative to c if $c \notin F$ and $c \in H$ for every $H \supset F$. So, a filter F is maximal if it is maximal relative to 0. We say that F is relatively maximal when it is maximal relative to some c.

For $B \subseteq \mathbb{P}$ we define the filter generated by B to be the intersection of all the filters containing B. It is easy to verify that this is indeed a filter. When B is a finite set, the filter generated by B is the principal filter generated by $\wedge B$. In general we have the following.

5.1 Proposition *For every $B \subseteq \mathbb{P}$, the filter generated by B is the set*

$$\{a \;:\; \wedge C \leq a \;\; \textit{for some finite non-empty} \;\; C \subseteq B\}. \qquad \square$$

This has the following important consequence.

5.2 Proposition *Let $B \subseteq \mathbb{P}$ and let $c \in \mathbb{P}$. If $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$, then B is contained in a maximal filter relative to c.*

Proof Let $\mathcal{F}$ be the set of filters F such that $B \subseteq F$ and $c \notin F$. By Proposition 5.1, $\mathcal{F}$ is non-empty. It is immediate that $\mathcal{F}$ is closed under unions of arbitrary chains. Then, by Zorn's lemma, $\mathcal{F}$ has a maximal element. $\square$

5.3 Exercise Prove that the following are equivalent

1. F is maximal relative to c;
2. for every $a \notin F$ there is $d \in F$ such that $d \wedge a \leq c$. $\square$

5.4 Exercise (A generalization of the exercise above.) Let $B \subseteq \mathbb{P}$ and let $c \in \mathbb{P}$ be such that $\wedge C \nleq c$ for every finite non-empty $C \subseteq B$. Prove that the following are equivalent

1. B is a maximal filter relative to c;
2. $a \notin B \;\Rightarrow\; b \wedge a \leq c$ for some $b \in B$. $\square$

5.5 Exercise Let $F \subseteq \mathbb{P}$ be a non-principal filter. Is F always contained in a maximal non-principal filter? $\square$

2 Distributive lattices and prime filters

Let $\mathbb{P}$ be a lattice. We say that $\mathbb{P}$ is distributive if for every $a, b, c \in \mathbb{P}$

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \end{aligned}$$

Throughout this section we assume that $\mathbb{P}$ is a bounded distributive lattice.

A proper filter F is prime if for every $a, b \in \mathbb{P}$

$$a \vee b \in F \quad \Rightarrow \quad a \in F \quad \text{or} \quad b \in F.$$

5.6 Proposition *Every relatively maximal filter of $\mathbb{P}$ is prime.*

Proof Let F be maximal relative to c and assume that $a \notin F$ and $b \notin F$. Then, by Exercise 5.3, there are $d_1, d_2 \in F$ such that $d_1 \wedge a \leq c$ and $d_2 \wedge b \leq c$. Let $d = d_1 \wedge d_2$. Then $d \wedge a \leq c$ and $d \wedge b \leq c$ and therefore $(d \wedge a) \vee (d \wedge b) \leq c$. Hence, by distributivity, $d \wedge (a \vee b) \leq c$. Then $a \vee b \notin F$. □

The Stone space of $\mathbb{P}$ is a topological space that we denote by $S(\mathbb{P})$. The points of $S(\mathbb{P})$ are the prime filters of $\mathbb{P}$. The closed sets of the Stone topology are arbitrary intersections of sets of the form

$$[a]_{\mathbb{P}} \quad = \quad \Big\{ F \; : \; \text{prime filter such that } a \in F \Big\}.$$

for $a \in \mathbb{P}$. In other words, the sets above form a base of closed sets of the Stone topology. Using 1 and 3 in the following proposition the reader can easily check that this is indeed a base for a topology.

5.7 Proposition *For every $a, b \in \mathbb{P}$ we have*

1. $[0]_{\mathbb{P}} = \varnothing$;
2. $[1]_{\mathbb{P}} = S(\mathbb{P})$;
3. $[a]_{\mathbb{P}} \cup [b]_{\mathbb{P}} = [a \vee b]_{\mathbb{P}}$;
4. $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = [a \wedge b]_{\mathbb{P}}$.

Proof The verification is immediate. Only 3 requires that the filters in $S(\mathbb{P})$ are prime. □

The closed subsets of $S(\mathbb{P})$ ordered by inclusion form a distributive lattice. The following is a representation theorem for distributive lattices.

5.8 Theorem *The map $a \mapsto [a]_{\mathbb{P}}$ is an embedding of $\mathbb{P}$ in the lattice of the closed subsets of $S(\mathbb{P})$. In particular*

1. $0 \mapsto \varnothing$;
2. $1 \mapsto S(\mathbb{P})$;
3. $a \vee b \mapsto [a]_{\mathbb{P}} \cup [b]_{\mathbb{P}}$;
4. $a \wedge b \mapsto [a]_{\mathbb{P}} \cap [b]_{\mathbb{P}}$.

Proof It is immediate that the map above preserves the order and Proposition 5.7 shows that it preserves the lattice operations. We prove that the map is injective. Let $a \neq b$, say $a \nleq b$. We claim that $[a]_{\mathbb{P}} \nsubseteq [b]_{\mathbb{P}}$. There is a filter F that contains a and is maximal relative to b. By Proposition 5.6 such an F is prime. Then $F \in [a]_{\mathbb{P}} \smallsetminus [b]_{\mathbb{P}}$. □

5.9 Theorem *With the Stone topology, $S(\mathbb{P})$ is a compact space.*

Proof Let $\langle [a_i]_{\mathbb{P}} : i \in I \rangle$ be basic closed sets such that for every finite $J \subseteq I$

a. $$\bigcap_{i \in J} [a_i]_{\mathbb{P}} \neq \varnothing.$$

We claim that

b. $$\bigcap_{i \in I} [a_i]_{\mathbb{P}} \neq \varnothing.$$

By 4 of Proposition 5.7 and a we obtain that $\wedge C \nleq 0$ for every finite $C \subseteq \{a_i : i \in I\}$. By Proposition 5.2, there is a maximal (relative to 0) filter containing $\{a_i : i \in I\}$. By Proposition 5.6, such filter is prime and it belongs to the intersection in b. □

Let $a, b \in \mathbb{P}$. If $a \wedge b = 0$ and $a \vee b = 1$, we say that b is the complement of a (and vice versa). The complement of an element need not exist. If the complement exists it is unique, the complement of a is denoted by $\neg a$.

5.10 Lemma *Let $U \subseteq S(\mathbb{P})$ be a clopen set. Then $U = [a]_{\mathbb{P}}$ for some $a \in \mathbb{P}$, and $\neg a$ exists.*

Proof As both U and $S(\mathbb{P}) \smallsetminus U$ are closed, for some sets $A, B \subseteq \mathbb{P}$

$$\bigcap_{x \in A} [x]_{\mathbb{P}} = U$$

$$\bigcap_{y \in B} [y]_{\mathbb{P}} = S(\mathbb{P}) \smallsetminus U.$$

By compactness, that is Theorem 5.9, there are some finite $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap_{x \in A_0} [x]_{\mathbb{P}} \cap \bigcap_{y \in B_0} [y]_{\mathbb{P}} = \varnothing$$

Let $a = \wedge A_0$ and $b = \wedge B_0$. From claim 4 of Proposition 5.7 we obtain $[a]_{\mathbb{P}} \cap [b]_{\mathbb{P}} = \varnothing$. Therefore $U \subseteq [a]_{\mathbb{P}} \subseteq S(\mathbb{P}) \smallsetminus [b]_{\mathbb{P}} \subseteq U$. Hence $U = [a]_{\mathbb{P}}$ and $b = \neg a$. □

A Boolean algebra is a bounded distributive lattice where every element has a complement. In a Boolean algebra, the sets $[a]_{\mathbb{P}}$ are clopen and they form also a base of open set of the topology of $S(\mathbb{P})$. A topology that has a base of clopen sets is called zero-dimensional. By the following proposition the Stone topology of a Boolean algebra is Hausdorff.

A proper filter of a Boolean algebra is an ultrafilter if either $a \in F$ or $\neg a \in F$ for every a.

5.11 Proposition *Let $\mathbb{P}$ be a Boolean algebra. Then the following are equivalent*

1. *F is maximal;*
2. *F is prime;*
3. *F is an ultrafilter.*

Proof Implication 2 ⇒ 3 is obtained observing that $a \vee \neg a \in F$. The rest is immediate. □

5.12 Exercise Prove that the stone topology on $S(\mathbb{P})$ has a base of open compact sets. □

5.13 Exercise Suppose we had defined $S(\mathbb{P})$ as the set of relatively maximal filters. What could possibly go wrong? □

3 Types as filters

This section we work with a fixed set of formulas Δ all with free variables among those of some fixed tuple x. In this section we do not display x in the notation. Subsets of Δ are called Δ-types.

We associate to Δ a bounded lattice $\mathbb{P}(\Delta)$. This is the closure under conjunction and disjunction of the formulas in $\Delta \cup \{\bot, \top\}$. The (pre)order relation in $\mathbb{P}(\Delta)$ is given by

$$\psi \leq \varphi \quad \Leftrightarrow \quad T \cup \{\psi\} \ \vdash \ \varphi,$$

for some fixed theory T. In this section, to lighten notation, we absorb T in the symbol $\vdash$. If p is a Δ-type we denote by $\langle p \rangle_{\mathbb{P}}$ the filter in $\mathbb{P}(\Delta)$ generated by p.

5.14 Lemma *For every Δ-type p*

$$\langle p \rangle_{\mathbb{P}} \quad = \quad \Big\{ \varphi \ \in \ \mathbb{P}(\Delta) \quad : \quad p \ \vdash \ \varphi \Big\}.$$

In particular p is consistent if and only if $\langle p \rangle_{\mathbb{P}}$ is a proper filter.

Proof Inclusion $\subseteq$ is clear. Inclusion $\supseteq$ is a consequence of the Compactness Theorem. In fact $p \vdash \varphi$ implies that $\psi \vdash \varphi$ for some formula ψ that is conjunction of formulas in p. Then $\varphi \in \langle p \rangle_{\mathbb{P}}$ follows from $\psi \in \langle p \rangle_{\mathbb{P}}$ and $\psi \leq \varphi$. □

We say that $p \subseteq \Delta$ is a principal Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a principal. The following lemma is an immediate consequence of the Compactness Theorem. Note that in 3 the formula φ is arbitrary, possibly not even in $\mathbb{P}(\Delta)$.

5.15 Lemma *For every Δ-type p the following are equivalent*

1. *p is principal;*
2. *$\psi \ \vdash \ p \ \vdash \ \psi$ for some formula ψ (here ψ is any formula, it need not be in Δ);*
3. *$\varphi \ \vdash \ p$ where φ is conjunction of formulas in p.* □

Proof Implications 1⇒2 is immediate by Lemma 5.14. To prove 2⇒3 suppose $\psi \vdash p \vdash \psi$. Apply compactness to obtain a formula φ, conjunction of formulas in p, such that $\varphi \vdash \psi$. Implications 3⇒1is trivial. □

5.16 Definition *We say that $p \subseteq \Delta$ is a prime Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a prime filter. We say that p is a complete Δ-type if $\langle p \rangle_{\mathbb{P}}$ is a maximal filter.* □

Though in general neither Δ nor $\mathbb{P}(\Delta)$ are closed under negation, Lemma 5.14 has the following consequence.

5.17 Proposition *For every consistent Δ-type p the following are equivalent*

1. *p is complete;*

2. *p is consistent and either $p \vdash \varphi$ or $p \vdash \neg\varphi$ for every formula $\varphi \in \Delta$.*

Proof 2⇒1. Assume 2. As p is consistent, $\langle p \rangle_{\mathbb{P}}$ is a proper filter. To prove that it is maximal suppose $\varphi \notin \langle p \rangle_{\mathbb{P}}$. Then $p \nvdash \varphi$ and from 2 it follows that $p \vdash \neg\varphi$. Hence no proper filter contains $p \cup \{\varphi\}$.

1⇒2. As $\langle p \rangle_{\mathbb{P}}$ is proper, p is consistent. Suppose $p \nvdash \varphi$. Then $\varphi \notin \langle p \rangle_{\mathbb{P}}$ and, as $\langle p \rangle_{\mathbb{P}}$ is maximal, $p \cup \{\varphi\}$ generates the improper filter. Then $p \cup \{\varphi\}$ is inconsistent. Hence $p \vdash \neg\varphi$. □

Given a model M and a tuple $c \in M^{|x|}$ the Δ-type of c in M is the sets

$$\Delta\text{-tp}_M(c) \quad = \quad \Big\{\varphi(x) \in \Delta \ : \ M \vDash \varphi(c)\Big\}$$

When the model M is clear from the context we omit the subscript. When x and c are the empty tuple, we write $\mathrm{Th}_\Delta(M)$ for Δ- $\mathrm{tp}_M(c)$.

5.18 Lemma *For every Δ-type p the following are equivalent*

1. *p is prime;*
2. *$p \ \cup \ \big\{\neg\varphi \ : \ \varphi \ \in \ \Delta \ \text{ such that } \ p \nvdash \varphi \ \big\}$ is consistent;*
3. *$\big\{\varphi \in \Delta : p \vdash \varphi\big\} \ = \ \Delta$- $\mathrm{tp}_M(c)$ for some model M and some tuple $c \in M^{|x|}$.*

Proof Implications 2⇒3⇒1 are clear, we prove 1⇒2. By compactness if the type in 2 is inconsistent then there are finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Delta$ such that $p \nvdash \varphi_i$ and

$$p \ \vdash \ \bigvee_{i=1}^{n} \varphi_i.$$

hence p is not prime. □

The following corollary is immediate. When it comes to verifying that a given Δ-type is prime, it simplifies the proof.

5.19 Corollary *For every Δ-type p the following are equivalent*

1. *p is prime;*
2. $p \vdash \bigvee_{i=1}^{n} \varphi_i \quad \Rightarrow \quad p \vdash \varphi_i$ *for some* $i \leq n$, *for every* n *and every* $\varphi_1, \ldots, \varphi_n \in \Delta$. □

The set Δ above contains only formulas with variables among those of the tuple x. In the following it is convenient to consider sets Δ that are closed under substitution of variables with any other variable. The set of prime Δ-types is denoted by $S(\Delta)$, and we write $S_x(\Delta)$ when we restrict to types in with variables among those of the tuple x.

The most common Δ used in the sequel is the set of all formulas in $L(A)$ and the underlying theory is $\mathrm{Th}(M/A)$ for some given model M containing A. In this case we write $\mathrm{tp}_M(c/A)$ for Δ- $\mathrm{tp}_M(c)$ or, when A is empty, $\mathrm{tp}_M(c)$. The set $S_x(\Delta)$ is denoted by $S_x(A)$. The topology on $S_x(A)$ is generated by the clopen

$$[\varphi(x)] \quad = \quad \Big\{p \in S_x(A) \ : \ \varphi(x) \in p\Big\}.$$

Sometimes Δ is the set of all formulas of a given syntactic form. Then we use some more suggestive notation that we summarize below.

5.20 Notation The following are some of the most common Δ-types and Δ-theories

1. $\text{at-tp}(c)$, $\text{Th}_{\text{at}}(M)$ when $\Delta = L_{\text{at}}$
2. $\text{at}^{\pm}\text{-tp}(c)$, $\text{Th}_{\text{at}^{\pm}}(M)$ when $\Delta = L_{\text{at}^{\pm}}$
3. $\text{qf-tp}(c)$, $\text{Th}_{\text{qf}}(M)$ when $\Delta = L_{\text{qf}}$.

Clearly, the types/theories in 2 and 3 are equivalent. Most used is the theory $\text{Th}_{\text{at}^{\pm}}(M/M)$ which is called the diagram of M and has a dedicated symbol: $\text{Diag}(M)$. □

5.21 Remark Let $A \subseteq M \cap N$. The following are equivalent

1. $N \vDash \text{Diag}\langle A \rangle_M$;
2. $\langle A \rangle_M$ is a substructure of N. □

4 Morphisms

First we set the meaning of the word map..

5.22 Definition *A map consists a triple $f : M \to N$ where*

1. *M is a set (usually a structure) called the domain of the map;*
2. *N is a set (usually a structure) called the codomain the map;*
3. *f is a function with domain of definition $\text{dom}\, f \subseteq M$ and image $\text{img}\, f \subseteq N$.*

By cardinality of $f : M \to N$ we understand the cardinality of the function f. □

If $\text{dom}\, f = M$ we say that the map is total; if $\text{img}\, f = N$ we say that it is surjective. The composition of two maps and the inverse of a map are defined in the obvious way.

5.23 Definition *Let Δ be a set of formulas with free variables in the tuple $x = \langle x_i : i < \lambda \rangle$. The map $h : M \to N$ is a Δ-morphism if it preserves the truth of all formulas in Δ. By this we mean that*

p. $M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha)$ *for every $\varphi(x) \in \Delta$ and every $a \in (\text{dom}\, h)^{|x|}$.*

Notation: if a is the tuple $\langle a_i : i < \lambda \rangle$ then ha is the tuple $\langle ha_i : i < \lambda \rangle$. □

When $\Delta = L$ we will say elementary map for Δ-morphism. When $\Delta = L_{\text{at}}$ we say partial homomorphism and when $\Delta = L_{\text{at}^{\pm}}$ we say either partial embedding or partial isomorphism. The reason for the latter name is explained in Remark 5.24.

It is immediate to verify that a partial embedding which is total is an embedding (so, there is not conflict with Definition 2.19 above). Similarly, a partial embedding which is total and surjective is an isomorphism. The precise connection between partial and total homo/iso-morphisms is discussed in following remark.

5.24 Remark For every map $h : M \to N$ the following are equivalent

1. $h : M \to N$ is a partial isomorphism;

2. there is an isomorphism $k : \langle \operatorname{dom} h \rangle_M \to \langle \operatorname{img} h \rangle_N$ that extends h.

Moreover, k unique. The equivalence holds replacing isomorphism by homomorphism (in which case in 2 we obtain an surjection). The extension k is obtained defining $k\big(t(a)\big) = t(ha)$.

A similar fact holds for partial homomorphims if we replace isomorphism by epimorphism, i/e/ surjective (partial) homomorphism. □

We use Δ-morphisms to compare, locally, two structures. There are different ways to do this, in the proposition below we list a few synonymous expressions. But first some more notation. When x is a fixed tuple of variables, $a \in M^{|x|}$ and $b \in N^{|x|}$ we write

$$M, a \Rrightarrow_\Delta N, b \qquad \text{if } M \vDash \varphi(a) \Rightarrow N \vDash \varphi(b) \text{ for every } \varphi(x) \in \Delta.$$
$$M, a \equiv_\Delta N, b \qquad \text{if } M \vDash \varphi(a) \Leftrightarrow N \vDash \varphi(b) \text{ for every } \varphi(x) \in \Delta.$$

The following equivalences are immediate and will be used without explicit reference.

5.25 Proposition *For every given set of formulas Δ and every map $h : M \to N$ the following are equivalent*

1. *$h : M \to N$ is a Δ-morphism;*
2. *$M, a \Rrightarrow_\Delta N, ha$ for every $a \in (\operatorname{dom} h)^{|x|}$;*
3. *$\Delta\text{-}\operatorname{tp}_M(a) \subseteq \Delta\text{-}\operatorname{tp}_N(ha)$ for every $a \in (\operatorname{dom} h)^{|x|}$;*
4. *$N, ha \vDash p(x)$ for every $a \in (\operatorname{dom} h)^{|x|}$ and $p(x) = \Delta\text{-}\operatorname{tp}_M(a)$.* □

5.26 Remark Condition p in Definition 5.23 apply to tuples x of any length, in particular to the empty tuple. In this case $\varphi(x)$ is a sentence, $a \in (\operatorname{dom} h)^0 = \{\varnothing\}$ is the empty tuple, and p asserts that $\operatorname{Th}_\Delta(M) \subseteq \operatorname{Th}_\Delta(N)$. When $h = \varnothing$ this is actually all that p says. In fact $\varnothing^{|x|} = \varnothing$ unless $|x| = 0$. Still, $\operatorname{Th}_\Delta(M) \subseteq \operatorname{Th}_\Delta(N)$ may be a non trivial requirement. □

5.27 Definition *We call $\operatorname{Th}_\Delta(M)$ the Δ-theory of M. We say that the theory T is Δ-complete if $\operatorname{Th}_\Delta(M) = \operatorname{Th}_\Delta(N)$ for all $M, N \vDash T$. In other words, for any pair of models of T, the empty map $\varnothing : M \to N$ is a Δ-morphism. When T is $L_{\mathrm{at}^\pm}$-complete we say that T decides the characteristic of its models (by analogy with rings and fields). Note that when T decides the characteristic of its models, we have $\langle \varnothing \rangle_M \simeq \langle \varnothing \rangle_N$ for all $M, N \vDash T$.* □

We conclude this section with a couple of propositions that break Theorem 2.20 into parts. More interestingly, in Chapter 10 we shall prove a sort of converse of Propositions 5.29 and 5.30.

It is interesting to note that there is a relation between certain properties of Δ-morphisms and the closure of Δ under logical connectives. When $\mathsf{C} \subseteq \{\forall, \exists, \neg, \vee, \wedge\}$ is a set of connectives, we write $\mathsf{C}\Delta$ for the closure of Δ with respect to the connectives in C. We write $\neg\Delta$ for the set containing the negation of the formulas in Δ. Warning: do not confuse $\neg\Delta$ with $\{\neg\}\Delta$. Up to logical equivalence $\{\neg\}\Delta = \Delta \cup \neg\Delta$.

It is clear that Δ-morphisms are $\{\wedge, \vee\}\Delta$-morphisms.

5.28 Proposition *For every given set of formulas Δ and every injective map $h : M \to N$ the following are equivalent*

a. $h : M \to N$ *is a* $\neg\Delta$*-morphism;*

b. $h^{-1} : M \to N$ *is a* Δ*-morphism.* □

5.29 Proposition *For every set of formulas Δ, every Δ-embedding $h : M \to N$ is a $\{\exists\}\Delta$-morphism.*

Proof Formulas in $\{\exists\}\Delta$ have the form $\exists y\, \varphi(x, y)$ where y is a finite tuples of variables and $\varphi(x, y) \in \Delta$. For every tuple $a \in (\operatorname{dom} h)^{|x|}$ we have:

$$\begin{aligned} M \vDash \exists y\, \varphi(a, y) \quad &\Rightarrow \quad M \vDash \varphi(a, b) \text{ for } b \in M^{|y|} \\ &\Rightarrow \quad N \vDash \varphi(ha, hb) \\ &\Rightarrow \quad N \vDash \varphi(ha, c) \text{ for } c \in N^{|y|} \\ &\Rightarrow \quad N \vDash \exists y\, \varphi(ha, y). \end{aligned}$$

Note that the second implication requires the totality of $h : M \to N$ which guarantees that $\varphi(a, y)$ has a solution in $\operatorname{dom} h$. □

When $h : M \to N$ is injective the following is a corollary of Propositions 5.28 and 5.29. The general proof is the 'dual' of that of Propositions 5.29.

5.30 Proposition *For every set of formulas Δ, every surjective Δ-morphism $h : M \to N$ is a $\{\forall\}\Delta$-morphism.* □

Chapter 6

Some relational structures

In the first section we prove that theory of *dense linear orders without endpoints* is *ω-categorical*. That is, any two such countable orders are isomorphic. This is an easy classical result of Cantor. In this chapter we examine Cantor's construction (a so-called *back-and-forth* construction) in great detail. In the second section we apply the same technique to prove that the theory of the *random graph* is ω-categorical.

1 Dense linear orders

The language of strict orders, which in this section we denote by L, contains only a binary relation symbol $<$. A structure M of signature L is a strict order if it models (the universal closure of) the following formulas

1. $x \nless x$ — irreflexive;
2. $x < z < y \;\rightarrow\; x < y$ — transitive.

Note that the following is an immediate consequence of 1 and 2.

$x < y \;\rightarrow\; y \nless x$ — antisymmetric.

We say that the order is total or linear if

li. $x < y \;\vee\; y < x \;\vee\; x = y$ — linear or total.

An order is dense if

nt. $\exists x, y\; (x < y)$ — non trivial;

d. $x < y \;\rightarrow\; \exists z\; (x < z < y)$ — dense.

We need to require the existence of two comparable elements, then d implies that dense orders are in fact infinite. We say that the ordering has no endpoints if

e. $\exists y\; (x < y) \;\wedge\; \exists y\; (y < x)$ — without endpoints.

We denote by $T_{\rm lo}$ the theory strict linear orders and by $T_{\rm dlo}$ the theory of dense linear orders without endpoints. Clearly, these are consistent theories: $\mathbb{Q}$ with the usual ordering is a model of $T_{\rm dlo}$.

We introduce some notation to improve readability of the proof of the following theorem. Let A and B be subsets of an ordered set. We write $A < B$ if $a < b$ for every $a \in A$ and $b \in B$. We write $a < B$ e $A < b$ for $\{a\} < B$, respectively $A < \{b\}$. Let $M \vDash T_{\rm lo}$. Then $M \vDash T_{\rm dlo}$ if and only if for every finite $A, B \subseteq M$ such that $A < B$ there is a c such that $A < c < B$. In fact axiom d is evident and axioms nt and e are obtained taking replacing A and/or B by the empty set.

Now we prove the first of a series of lemmas that we call extension lemmas. Recall that in the language of strict orders an injective map $k : M \rightarrow N$ is a partial isomorphism if

$$M \vDash a < b \Leftrightarrow N \vDash ka < kb \qquad\qquad \text{for every } a, b \in \operatorname{dom} k.$$

(When $M, N \vDash T_{\rm lo}$ the direction $\Rightarrow$ suffices.)

6.1 Lemma *Fix $M \vDash T_{\rm lo}$ and $N \vDash T_{\rm dlo}$. Let $k : M \to N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \to N$ that extends k and is defined in b.*

Proof Given a finite partial isomorphism $k : M \to N$ define

$$\begin{aligned} A^- &= \big\{a \in \operatorname{dom}(k) \ : \ a < b\big\}; \\ A^+ &= \big\{a \in \operatorname{dom}(k) \ : \ b < a\big\}. \end{aligned}$$

The sets A^- and A^+ are finite and partition $\operatorname{dom} k$, and $A^- < A^+$. As $k : M \to N$ is a partial isomorphism, $k[A^-] < k[A^+]$. Then in N there is an element c such that $k[A^-] < c < k[A^+]$. It is easy to check that setting $h = k \cup \big\{\langle b, c\rangle\big\}$ gives the required extension. □

The following is an equivalent version of Lemma 6.1.

6.2 Corollary *Let $M \vDash T_{\rm lo}$ be countable and let $N \vDash T_{\rm dlo}$. Let $k : M \to N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \hookrightarrow N$ that extends k.*

Proof Let $\langle a_i : i < \omega\rangle$ be an enumeration of M. Define by induction a chain of finite partial isomorphisms $h_i : M \to N$ such that $a_i \in \operatorname{dom} h_{i+1}$. The construction starts with $h_0 = k$. At stage $i+1$ we chose any finite partial isomorphism $h_{i+1} : M \to N$ that extends h_i and is defined in a_i. This is possible by Lemma 6.1. In the end we set

$$h \ = \ \bigcup_{i \in \omega} h_i.$$

It is immediate to verify that $h : M \hookrightarrow N$ is the required embedding. □

6.3 Exercise Prove that the extension Lemma 6.1 characterizes models of $T_{\rm dlo}$ among models of $T_{\rm lo}$. That is, if N is a model of $T_{\rm lo}$ such that the conclusion of Lemma 6.1 holds, then $M \vDash T_{\rm dlo}$. □

We are now ready to prove that any two countable models of $T_{\rm dlo}$ are isomorphic which is a classical result of Cantor's. Actually what we prove is slightly more general than that. In fact Cantor's theorem is obtained from the theorem below by setting $k = \varnothing$, which we are allowed to, because all models have the same empty characteristic (cfr. Remark 5.26).

6.4 Theorem *Every finite partial isomorphism $k : M \to N$ between countable models of $T_{\rm dlo}$ extends to an isomorphism $g : M \xrightarrow{\sim} N$.*

The following is the archetypal back-and-forth construction. It is important to note that it does not mention linear orders at all. It only uses the extension Lemma 6.1. The same construction can be applied in many other contexts where an extension lemma holds (cfr. Theorem 7.6).

Proof Let $\langle a_i : i < \omega\rangle$ and $\langle b_i : i < \omega\rangle$ be enumerations of M and N respectively. We define by induction a chain of finite partial isomorphisms $g_i : M \to N$ such that $a_i \in \operatorname{dom} g_{i+1}$ and $b_i \in \operatorname{img} g_{i+1}$. In the end we set

$$g = \bigcup_{i\in\omega} g_i$$

We begin by letting $g_0 = k$. The inductive step consists of two half-steps that we call the *forth step* and *back step*. In the forth step we define $g_{i+1/2}$ such that $a_i \in \operatorname{dom} g_{i+1/2}$. In the back step to define g_{i+1} such that $b_i \in \operatorname{img} g_{i+1}$.

By the extension lemma 6.1 there is a finite partial isomorphism $g_{i+1/2} : M \to N$ that extends g_i and is defined in a_i. Now apply the same lemma to extend $\left(g_{i+1/2}\right)^{-1} : N \to M$ to a finite partial isomorphism $(g_{i+1})^{-1} : N \to M$ defined in b_i. □

Let λ be an infinite cardinal. We say that a theory is λ-categorical if any two models of T of cardinality λ are isomorphic. From Theorem 6.4, taking $k = \varnothing$, we obtain the following.

6.5 Corollary *The theory T_{dlo} is ω-categorical.* □

We also obtain that T_{dlo} is a complete theory. This is consequence of the following general fact.

6.6 Proposition *If T has no finite models and is λ-categorical for some $\lambda \geq |L|$, then T is complete.*

Proof Let M and N be any two models of T. Applying the upward and/or downward Löwenheim-Skolem theorem, we may assume they both have cardinality λ. Note that here we use that N and N are both infinite and that $\lambda \geq |L|$. Hence $M \simeq N$ and in particular $M \equiv N$. □

6.7 Exercise Prove that T_{dlo} is not λ-categorical for any uncountable λ. □

6.8 Exercise Prove that, in the language of strict orders, $\mathbb{Q} \preceq \mathbb{R}$. □

6.9 Exercise Let L be the language of strict orders expanded with countably many constants $\{c_i : i \in \omega\}$. Let T be the theory that extends T_{dlo} by the axioms $c_i < c_{i+1}$ for all i. Prove that T is complete. Find three non isomorphic countable models of this theory. For a suitably chosen model N of T, prove the statement in Lemma 6.1, where M any model of T. □

6.10 Exercise Show that in Theorem 6.6 the assumption $\lambda \geq |L|$ is necessary. (Hint: let ν be an uncountable cardinal. The language contains only the ordinals $i < \nu$ as constants. The theory T says that there are infinitely many elements and either $i = 0$ for every $i < \nu$, or $i \neq j$ for every $i < j < \nu$. Prove that T is ω-categorical but incomplete.) □

2 Random graphs

Recall that the language of graphs, which in this section we denote by L, contains only a binary relation r. A graph structure of signature L such that

1. $\neg r(x, x)$ irreflexive;

2. $r(x,y) \rightarrow r(y,x)$ symmetric.

An element of a graph M is called a vertex or a node. An edge is an unordered pair of vertices $\{a,b\} \subseteq M$ such that $M \vDash r(a,b)$. In words we may say that a is adjacent to b.

A random graph is a graph that also satisfies the following axioms for every n

nt. $\exists x,y \ (x \neq y)$ non trivial;

r_n. $\displaystyle\bigwedge_{i,j=1}^{n} x_i \neq y_j \rightarrow \exists z \bigwedge_{i=1}^{n} \big[r(x_i,z) \wedge \neg r(z,y_i) \wedge z \neq y_i\big]$ for every $n \in \mathbb{Z}^+$.

The theory of graphs is denoted by T_{gph} and the theory of random graphs is denoted by T_{rg}. The scheme of axioms r_n plays the same role as density in the previous section. It says that given two disjoint sets A^+ and A^- of cardinality $\leq n$ there is a vertex z that is adjacent to all vertices in A^+ and to no vertex in A^-. We explicitly required that $z \notin A^-$, by 1 it is clear that $z \notin A^+$.

Strictly speaking, the axioms r_n do not mention the cases when A^+ or A^- are empty. But as it is evident that random graphs are infinite, we can deal with them by adding redundant elements.

The following is the analogous of Lemma 6.1 for random graphs. Recall that in the language of graphs a map $k : M \rightarrow N$ is a partial isomorphism if it is injective and

$$M \vDash r(a,b) \Leftrightarrow N \vDash r(ka,kb) \qquad \text{for every } a,b \in \operatorname{dom} k.$$

6.11 Lemma *Fix $M \vDash T_{\mathrm{gph}}$ and $N \vDash T_{\mathrm{rg}}$. Let $k : M \rightarrow N$ be a finite partial isomorphism and let $b \in M$. Then there is a partial isomorphism $h : M \rightarrow N$ that extends k and is defined in b.*

Proof The structure of the proof is the same as in Lemma 6.1, so we use the same notation. Assume $b \notin \operatorname{dom} k$ and define

$$A^+ = \big\{x \in \operatorname{dom} k \ : \ M \vDash r(x,b)\big\} \quad \text{e} \quad A^- = \big\{y \in \operatorname{dom} k \ : \ M \vDash \neg r(y,b)\big\}.$$

These two sets are finite and disjoint, then so are $k[A^+]$ and $k[A^-]$. Then there is a $c \notin \operatorname{img} k$ such that

$$\bigwedge_{a \in A^+} r(ka,c) \ \wedge \bigwedge_{a \in A^-} \neg r(ka,c).$$

As $k[A^+] \cup k[A^-] = \operatorname{img} k$, it is immediate to verify that $h = k \cup \big\{\langle b,c\rangle\big\}$ is the required extension. □

Some readers may doubt that T_{rg} is consistent.

6.12 Proposition *There exists a random graph.*

Proof The domain of N is the set of natural numbers. Let $N \vDash r(n,m)$ hold if the n-th prime number divides m or, conversely, the m-th prime number divides n. □

The same proof as that of Corollary 6.2 gives the following.

6.13 Corollary *Let $M \vDash T_{\mathrm{gph}}$ be countable and let $N \vDash T_{\mathrm{rg}}$. Let $k : M \rightarrow N$ be a finite partial isomorphism. Then there is a (total) embedding $h : M \hookrightarrow N$ that extends k.* □

The proof of Theorem 6.4 gives the following theorem and its corollary.

6.14 Theorem *Every finite partial isomorphism $k : M \to N$ between models of $T_{\rm rg}$ of the same cardinality extends to an isomorphism $g : M \xrightarrow{\sim} N$.* □

6.15 Corollary *The theory $T_{\rm rg}$ is ω-categorical (and therefore complete).* □

6.16 Exercise Let $a, b, c \in N \vDash T_{\rm rg}$. Prove that $r(a, N) = r(b, N) \cap r(c, N)$ occurs only in the trivial case $a = b = c$. □

6.17 Exercise Let $N \vDash T_{\rm rg}$ prove that for every $b \in N$ the set $r(b, N)$ is a random graph. Is every random graph $M \subseteq N$ of the form $\varphi(N)$ for some $\varphi(x) \in L(N)$? □

6.18 Exercise Let N be free union of two random graphs N_1 and N_2. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$. By $\sqcup$ we denote the disjoint union. Prove that N is not a random graph. Show that N_1 is not definable without parameters (assume $|N_1| = |N_2| = \omega$, otherwise the proof is longer). Write a first order sentence $\psi(x, y)$ true if x and y belong to the same connected component of N. Axiomatize the class $\mathcal{K}$ of graphs that are free union of two random graphs. □

6.19 Exercise Prove that $T_{\rm rg}$ is not λ-categorical for any uncountable λ. Hint: prove that there is a random graph N of cardinality λ where every vertex is adjacent to $< \lambda$ vertices. Compare it with its complement graph (the graph that has edges between pairs that are non adjacent in N). □

6.20 Exercise Prove that $T_{\rm rg}$ is not finitely axiomatizable. Hint: given a random graph N and a set P of cardinality $n + 1$ show that you can add edges to $M \sqcup P$ and make it satisfy axiom r_n but not r_{n+1}. □

6.21 Exercise Let $A \subseteq N \vDash T_{\rm rg}$ and let $\varphi(x) \in L(A)$, where $|x| = 1$. Prove that if $\varphi(N)$ is finite then $\varphi(N) \subseteq A$. □

6.22 Exercise (Peter J. Cameron) Prove that for every infinite countable graph M the following are equivalent

1. M is either random, complete or empty (i.e. $r^M = \varnothing$);
2. if $M_1, M_2 \subseteq M$ are such that $M_1 \sqcup M_2 = M$, then $M_1 \simeq M$ or $M_2 \simeq M$.

Hint: for 2⇒1 first show that if $r(a, M) = \varnothing$ for some $a \in M$ then the graph is null and if $\{b\} \cup r(b, M) = M$ for some b, then the graph is complete. Clearly, 2 implies that any finite partition of M contains an element isomorphic to M. Then the claim above generalizes as follows: if there is a finite A such that $\bigcap_{a \in A} r(a, M) = \varnothing$ then M is the empty graph and if there is a finite B such that $\bigcup_{b \in B} r(b, M) = M$ then M is the complete graph.

Suppose M is not a random graph. Fix some finite, disjoint A and B such that no c satisfies both $r(A, c) = A$ and $r(B, c) = \varnothing$. Let $M_1 = \{c : r(A, c) \neq A\}$ and $M_2 = \neg M_1$. Now note that

$$\bigcap_{a \in A} r(a, M_1) = \varnothing \qquad \text{and} \qquad \bigcup_{b \in B} r(b, M_2) = M_2.$$ □

3 Notes and references

We refer the reader to [2] for a well-written accessible survey on the amazing model theoretic properties of the random graph.

Chapter 7

Rich models

We introduce *Fraïssé limits*, aleas *homogeneous-universal* or *generic* structures, which here we call *rich* models, after Poizat. Rich models generalize the examples in Chapters 6 and the many more to come.

Elimination of quantifiers is briefly discussed at the end of Section 1. For the time being we identify quantifier elimination with the property that says that all partial isomorphisms are elementary maps. Proofs are easier with this notion in mind. The equivalence with its syntactic counterpart is only proved in Chapter 10, when the reader is more familiar with arguments of compactness.

1 Rich models.

We now define *categories of models and partial morphisms*. These are example of concrete categories as intended in category theory. However, apart from the name, in what follows we dispense with all notions of category theory as they would make the exposition less basic than intended (without providing more technical instruments).

⚠ Warning: the terminology introduced in this section is not standard. In the literature many details are left implicit. This is generally safe when the category used is fixed. Here we prefer to be more explicit because, when discussing saturation, quantifier elimination, and model completeness, it helps to compare different categories.

A category (of models and partial morphisms) is a class $\mathcal{M}$ which is disjoint union of two classes: $\mathcal{M}_{\text{ob}}$ and $\mathcal{M}_{\text{hom}}$. The first is the class of objects and contains structures with a common signature L which we call models. The second is the class of morphisms and contains (partial) maps between models. We require that the identity maps are morphisms and that composition of two morphism is again morphism. This makes $\mathcal{M}$ a well-defined category.

For example, $\mathcal{M}$ could consist of all models of some theory T_0 and of all partial isomorphisms between these. Alternatively, as morphisms we could take elementary maps between models. At a first reading the reader may assume $\mathcal{M}$ is as in one of these two examples. In the general case we need to make some assumptions on $\mathcal{M}$.

7.1 Definition *For ease of reference we list together all properties required below*

c1. *the (partial) identity map* $\mathrm{id}_A : M \to M$ *is a morphism, for any* $A \subseteq M$*;*

c2. *if* $k' : M \to N$ *is a morphism for every finite* $k' \subseteq k$*, then* $k : M \to N$ *is a morphism.*

c3. *morphisms are invertible maps and the inverse of a morphism is a morphism;*

c4. *morphisms preserve the truth of* L_{at}*-formulas;*

c5. *if* M *is a model and* $N \equiv M$*, then also* N *is a model;*

c6. *every elementary map between models is a morphism.* □

The connected component of a model M is the subclass of models N such that there is any morphism with domain M and codomain N (or vice versa, by c3). By axiom c1 the restriction of a morphism is a morphism, therefore M and N are in the same connected component if and only if the empty map $\varnothing : M \to N$ is a morphism. If the whole category $\mathcal{M}$ consists of one connected component we say that $\mathcal{M}$ is connected.

We call c2 the finite character of morphisms. Note that it implies the following

c7. if $k_i : M \to N$ is a chain of morphisms, then $\bigcup\limits_{i<\lambda} k_i : M \to N$ is a morphism.

Notably, the following two definitions require c3. The generalization to non injective morphisms is not straightforward (in fact, there are two generalizations: *projective* and *inductive*). These generalizations are not very common and will not be considered here.

7.2 Definition *Assume that $\mathcal{M}$ satisfies c1-c3 of Definition 7.1. We say that a model N is λ-rich if for every model M, every $b \in M$ and every morphism $k : M \to N$ of cardinality $< \lambda$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is a morphism. We say that N is rich if it is λ-rich for $\lambda = |N|$. When $\mathcal{M}_{\rm ob} = \mathrm{Mod}(T_0)$ for some theory T_0 and $\mathcal{M}_{\rm hom}$ is clear from the context, we say rich model of T_0.* □

Rich models are also called *Fraïssé limits* or *homogeneous-universal* for a reason that will soon be clear; they are also called *generic*. Unfortunately these names are either too long or too generic, so we opt for the less common term *rich* that was proposed by Poizat.

The following two notions are closely connected with richness.

7.3 Definition *Assume that $\mathcal{M}$ satisfies c1-c3 of Definition 7.1. We say that a model N is λ-universal if for every model M of cardinality $\leq \lambda$ in the same connected component of N there is an embedding $k : M \hookrightarrow N$. We say that a model N is λ-homogeneous if every $k : N \to N$ of cardinality $< \lambda$ extends to a bijective morphism $h : N \xrightarrow{\sim} N$ (an automorphism when c4 below holds).*

Note that the larger $\mathcal{M}_{\rm hom}$, the stronger notion of homogeneity. When $\mathcal{M}_{\rm hom}$ contains all partial isomorphisms between models (the largest class of morphisms considered here), it is common to say λ-ultrahomogeneous for λ-homogeneous.

As above, when $\lambda = |N|$ we say universal, homogeneous and ultrahomogeneous. □

In Section 6.1 we implicitly used $\mathcal{M}_{\rm ob} = \mathrm{Mod}(T_{\rm lo})$ and partial isomorphisms as $\mathcal{M}_{\rm hom}$. In Section 6.2 we used $\mathcal{M}_{\rm ob} = \mathrm{Mod}(T_{\rm gph})$ and again partial isomorphisms as $\mathcal{M}_{\rm hom}$. Corollary 6.2 proves that every model of $T_{\rm dlo}$ is ω-rich. Corollary 6.13 claims the analogous fact for $T_{\rm rg}$.

In the following we frequently work under the following assumption (even when not all properties are strictly necessary).

7.4 Assumption *Assume $|L| \leq \lambda$ and suppose that $\mathcal{M}$ satisfies c1-c6 of Definition 7.1*

The assumption on the cardinality L is only necessary to apply the downward Löwenheim-Skolem Theorem when required.

7.5 Proposition *(Assume 7.4) The following are equivalent*

1. *N is a λ-rich model;*
2. *for every model M of cardinality $\leq \lambda$ and every morphism $k : M \to N$ of cardinality, say $< \lambda$ there is a embedding $h : M \hookrightarrow N$ that extends k.*

Proof Closure under union of chains of morphisms, which is ensured by c7, immediately yields 1⇒2. As for implication 2⇒1 we only need to consider the case $\lambda < |M|$. Let $k : M \to N$ be a morphism of cardinality $< \lambda$ and $b \in M$. By the downward Löwenheim-Skolem theorem there is an $M' \preceq M$ of cardinality λ containing $\operatorname{dom} k \cup \{b\}$. Let $h : M' \hookrightarrow N$ be the embedding obtained from 2. By c4, the map $h : M \to N$ is a composition of morphisms, hence a morphism. □

The following theorem subsumes both Theorem 6.4 and Theorem 6.14.

7.6 Theorem *(Assume 7.4) Let M and N be two rich models of the same cardinality λ. Then every morphism $k : M \to N$ of cardinality $< \lambda$ extends to an isomorphism.*

Proof When $\lambda = \omega$, we can take the proof of Theorem 6.4 and replace *partial isomorphism* by *morphism* and the references to Lemma 6.1 by references to Proposition 7.5. As for uncountable λ, we only need to extend the construction through limit stages. By c7 we can simply take the union. □

7.7 Corollary *(Assume 7.4) Then all rich models of cardinality λ in the same connected component are isomorphic.* □

It is obvious that λ-rich models are λ-universal and, by Theorem 7.6, they are λ-homogeneous. These two notions are weaker than richness. For instance, when $\mathcal{M}$ is as in Section 6.2, the countable graph with no edge is trivially ultrahomogeneous but it is not universal and a fortiori not rich. On the other hand if we add to a countable random graph an isolated point we obtain a universal graph which is not ultrahomogeneous. However, when taken together, these two properties are equivalent to richness.

7.8 Theorem *(Assume 7.4) The following are equivalent:*

1. *N is rich;*
2. *N is homogeneous and universal.*

Proof Implication 1⇒2 is clear as noted above, so we prove 2⇒1. We use the characterization of richness given in Proposition 7.5. Let $k : M \to N$ be a morphism such that $|k| < |N|$ and $|M| \leq |N|$. As N is universal, there is a total morphism $f : M \hookrightarrow N$. By c3 the map $k \circ f^{-1} : N \to N$ is a morphism of cardinality $< |N|$. By homogeneity it has an extension to an automorphism $h : N \xrightarrow{\sim} N$. It is immediate that $h \circ f : M \hookrightarrow N$ is the required extension of k. □

7.9 Exercise Let N be the structure obtained by adding to a countable random graph an isolated point. Show that N is homogeneous if morphisms are elementary maps but it is not if morphisms are simply partial isomorphisms. □

A consequence of Theorem 7.6 is that morphisms between rich models of the same cardinality are elementary maps. However, the theorem gives no information when the models have different cardinality nor when they are merely λ-rich. This case is dealt with by next theorem, arguably the main result of this section.

To test the theorem below in a simple case we propose the following exercise.

7.10 Exercise Every partial isomorphism $k : M \to N$ between models of $T_{\rm dlo}$ (or models of $T_{\rm rg}$) is an elementary map. Hint: use downward Löwenheim-Skolem and Theorem 6.4. □

2 The theory of rich models and quantifier elimination

Let $\lambda \geq |L|$ be given. The theory of the rich models of $\mathcal{M}$ is the set T_1 of sentences that hold in all λ rich models. The theorem below proves that T_1 does not depends on λ (and we could also restrict to rich models when they exist).

It also follows from the theorem that T_1 is complete as soon as $\mathcal{M}$ is connected.

7.11 Theorem *(Assume 7.4) Every morphism between λ-rich models is elementary. In particular, λ-rich models in the same connected component are elementary equivalent.*

Proof Let $k : M \to N$ be a morphism between rich models. It suffices to prove that every finite restriction of k is elementary. By c2, we may as well assume that k itself is finite. It suffices to construct $M' \preceq M$ and $N' \preceq N$ together with a morphism $h : M \to N$ that extends k and maps M' bijectively to N'. Then by c6 the map $h : M' \xrightarrow{\sim} N'$ is the composition of morphisms, hence it is a morphism. Finally, by c4, it is an isomorphism, in particular an elementary map.

In general, richness is not preserved under elementary equivalence. Therefore M' and N' need to be constructed simultaneously with h. We define a chain of functions $\langle h_i : i{<}\lambda \rangle$ such that $h_i : M \to N$ are morphisms and in the end we set

$$h \;=\; \bigcup_{i<\lambda} h_i, \qquad M' \;=\; \operatorname{dom} h, \qquad N' \;=\; \operatorname{img} h.$$

We interweave the usual back-and-forth-argument with the Löwenheim-Skolem construction 2.40 in order to obtain $M' \preceq M$ and $N' \preceq N$.

The chains start with $h_0 = k$. At limit stages we take the union. Now assume we have h_i. Let $\varphi(x) \in L(\operatorname{dom} h_i)$ some formula consistent in M and pick a solution $b \in M$. By λ-richness there is a $c \in N$ such that $h_i \cup \{\langle b, c\rangle\} : M \to N$ is a morphism. Let $h_{i+1/2} = h_i \cup \{\langle b, c\rangle\}$.

Finally, as in the proof of Theorem 6.4, we extend $h_{i+1/2}$ to obtain h_{i+1}. We use procedure with the roles of M and N inverted and with $h^{-1}_{i+1/2}$ for h_i.

In the end we obtain $M' \preceq M$ if all formulas $\varphi(x) \in L(M')$ are eventually considered. A similar consideration holds for N'. This is achieved using the same dovetail enumeration as in our second proof the downward Löwenheim-Skolem Theorem 2.40. □

Assume for simplicity that $\mathcal{M}$ is connected. Then all λ-rich models belong to $\operatorname{Mod}(T_1)$ for some complete theory T_1. It is interesting to ask if the converse is true: if T_1 is the theory of the λ-rich models of $\mathcal{M}$, do all models in $\operatorname{Mod}(T_1)$ are rich? The answer is affirmative when $\lambda = \omega$ and T_1 is either $T_{\rm dlo}$ or $T_{\rm rg}$, but this is not always the case, see Example 7.15 for an easy to grasp counterexample. But in fact, any non λ-categorical theory counterexamples.

7.12 Remark Assume 7.4) Assume $\mathcal{M}$ is connected. Let T_1 be the theory of the rich models of $\mathcal{M}$. If every model of T_1 is λ-rich then T_1 is λ-categorical. (This is a conseguence of Corollary 10.13) □

A very interesting interesting variant of the question asked above is considered in Theorem 9.9 where it is related to an important phenomenon that we now introduce.

First, to have a concrete example at hand, we instantiate Theorem 7.11 with the two categories used in Chapter 6.

7.13 Corollary *Partial isomorphisms between models of $T_{\rm dlo}$ and between models of $T_{\rm rg}$ are elementary maps.* □

When partial isomorphism between models of a given theory T coincide with elementary maps, it is always by a fundamental reason. Let us introduce some terminology. Let T be a consistent theory. We say that T has (or admits) elimination of quantifiers if for every $\varphi(x) \in L$ there is a quantifier-free formula $\psi(x) \in L_{\rm qf}$ such that

$$T \ \vdash \ \psi(x) \leftrightarrow \varphi(x).$$

We will discuss general criteria for elimination of quantifiers in Chapter 10. Here we report without proof the following theorem.

7.14 Theorem *The following are equivalent*

1. *T has elimination of quantifiers;*
2. *every partial isomorphism between models T is an elementary map.* □

⚠ This theorem will be proved only in Chapter 10, see Exercise 10.4 or Corollary 10.12. For the time being we do not need the syntactic version of elimination of quantifiers, so when saying that T *has quantifier elimination* we intend 2 of the theorem above. For instance we rephrase Corollary 7.13 above by saying that $T_{\rm dlo}$ and $T_{\rm rg}$ have elimination of quantifiers.

In the next chapter we introduce important examples of ω-rich models that do not have an ω-categorical theory. These are algebraic structures (groups, fields etc.) hence more complex than pure relational structures. So, we conclude this section with an example of this phenomenon in an almost trivial context.

7.15 Example Let L contain a unary predicate r_n for every positive integer n. The theory T_0 contains the axioms $\neg\exists x\,\big[r_n(x) \wedge r_m(x)\big]$ for $n \neq m$ and $\exists^{\leq n} x\, r_n(x)$ for every n. Work in the the category of models of T_0 and partial isomorphisms. Let T_1 be the theory that extends T_0 with the axioms $\exists^{=n} x\, r_n(x)$ for every n. Let $q(x)$ be the type $\big\{\neg r_n(x) : n \in \omega\big\}$. There are models of T_1 that do not realize $q(x)$, hence T_1 is not ω-categorical. It is easy to verify that the following are equivalent.

1. N is an ω-rich model;

2. $N \vDash T_1$ and $q(N)$ is infinite.

The reader may use Theorem 7.11 and Compactess Theorem for Types 4.10 to prove that T_1 is complete and has elimination of quantifiers. It is also easy to verify that every uncountable model of T_1 is rich and consequently that T_1 is uncountably categorical. □

7.16 Exercise Let T_0 and $\mathcal{M}$ be as in Example 7.15 except that we restrict the language to the relations $r_0, \ldots, r_n$ for a fixed n. Do ω-rich models of T_0 exist? Is their theory ω-categorical? What if we add to T_0 the axiom $r_0(x) \vee \cdots \vee r_n(x)$? □

7.17 Exercise The language contains only the binary relations $<$ and e. The theory T_0 says that $<$ is a strict linear order and that e is an equivalence relation. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ? □

7.18 Exercise The language contains only two binary relations. The theory T_0 says that they are equivalence relations. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ? □

7.19 Exercise In the language of graphs let T_0 say that there are no cycles (equivalently, there is at most one path between any two nodes). In combinatorics these graphs are called *forests*, and their connected components are called *trees*. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ? □

7.20 Exercise Assuming Theorem 7.14, prove that the following are equivalent

1. T has elimination of quantifiers;
2. every *finite* partial isomorphism between models T is an elementary map. □

3 Weaker notions of universality and homogeneity

We want to extend the equivalence in Theorem 7.8 to λ-rich models. For that we need to weaken the notions of λ-universality and λ-homogeneity. This section is more technical and could be skipped at a first reading.

7.21 Definition *We say that a structure N is weakly λ-homogeneous if for every $b \in N$ every morphism $k : N \to N$ of cardinality $< \lambda$ extends to one defined in b. The term back-and-forth λ-homogeneous is also used.* □

The following easy exercise on back-and-forth is required in the sequel.

7.22 Exercise (Assume 7.4) Prove that any weakly λ-homogeneous structure of cardinality λ is homogeneous. □

7.23 Definition *We say that a structure N is weakly λ-universal if for every model M in the connected component of N and every $A \subseteq M$ of cardinality $< \lambda$ there is a morphism $k : M \to N$ such that $A \subseteq \operatorname{dom} k$.* □

7.24 Lemma *(Assume 7.4) Let N be a weakly λ-homogeneous model. Let $A \subseteq N$ have cardinality $\leq \lambda$ and let $k : N \to N$ be a morphism of cardinality $< \lambda$. Then there is a model $M \preceq N$ containing A and an automorphism $h : M \xrightarrow{\sim} M$ that extends k.*

Proof Similar to the proof of Theorem 7.11. We shall construct simultaneously a chain $\langle A_i : i < \lambda \rangle$ of subsets of N and a chain functions $\langle h_i : i < \lambda \rangle$, such that $h_i : N \to N$ are morphisms. In the end we will set

$$M \;=\; \bigcup_{i<\lambda} A_i \qquad\qquad \text{and} \qquad\qquad h \;=\; \bigcup_{i<\lambda} h_i$$

The chains start with $A_0 = A \cup \operatorname{dom} k \cup \operatorname{img} k$ and $h_0 = k$. As usual, at limit stages we take the union. Now we consider successor stages. At stage i we fix some enumerations of A_i and of $L_x(A_i)$, where $|x| = 1$. Let $\langle i_1, i_2 \rangle$ be the i-th pair of ordinals $< \lambda$. If the i_2-th formula in $L_x(A_{i_1})$ is consistent in N, let a be any of its solutions. Also let b be the i_2-th element of A_{i_1}. Let $h_{i+1} : N \to N$ be a minimal morphism that extends h_i and is such that $b \in \operatorname{dom} h_{i+1} \cap \operatorname{img} h_{i+1}$. Define $A_{i+1} = A_i \cup \{a,\ h_{i+1}b,\ h_{i+1}^{-1}b\}$. □

7.25 Theorem *(Assume 7.4) For every model N the following are equivalent*

1. *N is λ-rich;*
2. *N is weakly λ-universal and weakly λ-homogeneous.*

Proof Implication 1⇒2 is clear. To prove 2⇒1 we generalize the proof of Theorem 7.8. We assume 2 fix some morphism $k : M \to N$ of cardinality $< \lambda$ and let $b \in M$. By weak λ-universality there is a morphism $f : M \to N$ with domain of definition $\operatorname{dom} k \cup \{b\}$. The map $f \circ k^{-1} : N \to N$ has cardinality $< \lambda$ and, by Lemma 7.24, it has an extension to an automorphism $h : N' \xrightarrow{\sim} N'$ for some $N' \preceq N$ containing $\operatorname{img} k \cup \operatorname{img} f$. Then $h \circ f : M \to N$ extends k and is defined on b. □

4 The amalgamation property

In this section we discuss conditions that ensure the existence of rich models.

We say that $\mathcal{M}$ has the amalgamation property if for every pair of morphisms $f_1 : M \to M_1$ and $f_2 : M \to M_2$ there are two of embeddings $g_1 : M_1 \hookrightarrow N$ and $g_2 : M_2 \hookrightarrow N$ such that $g_1 \circ f_1(a) = g_2 \circ f_2(a)$ for every a in the common domain of definition, $\operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$.

$$\begin{tikzcd}
 & M_1 \arrow[dr, hook, "g_1"] & \\
M \arrow[ur, "f_1"] \arrow[dr, "f_1"'] & & N \\
 & M_2 \arrow[ur, hook, "g_2"'] &
\end{tikzcd}$$

As we assume that morphisms are invertible, we may express the amalgamation property in a more concise form. Namely, for every morphism $k : M \to N$ there is morphism $g : M' \to N'$ such that the following diagram commutes

$$\begin{tikzcd} & N \arrow[dr, hook, "\mathrm{id}_N"] & \\ M \arrow[ur, "k"] \arrow[rr, hook, "g"'] & & N' \end{tikzcd}$$

It is convenient to use the following terminology. We write $M \leq N$ for $M \subseteq N$ and $\mathrm{id}_M : M \hookrightarrow N$ is a morphism. We say that $k : M \to N$ extends to $g : M' \to N'$ if $k \subseteq h$, $M \leq M'$, and $N \leq N'$.

7.26 Proposition *Assume* c3*, then the following are equivalent*

1. *$\mathcal{M}$ has the amalgamation property;*
2. *every morphism $k : M \to N$ extends to an embedding $g : M \hookrightarrow N'$.*

Proof 1⇒2 Given $k : M \to N$, the amalgamation property yields the following commutative diagram which can be simplified to the diagram at the right

$$\begin{tikzcd} & N \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "k"] \arrow[dr, "\mathrm{id}_M"'] & & N' \\ & M \arrow[ur, hook, "g_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & N \arrow[dr, hook, "g_1"] & \\ M \arrow[ur, "k"] \arrow[rr, hook, "g_2"'] & & N' \end{tikzcd}$$

Up to isomorphism we can assume $g_2 = \mathrm{id}_M$, i.e. that $N \leq N'$. Hence $g_2 : M \hookrightarrow N'$ is the required extension of $k : M \to N$.

2⇒1 Let $f_1 : M \to M_1$ and $f_2 : M \to M_2$ be given. Let $k = f_2 \circ f_1^{-1} : M_1 \to M_2$ and let $g : M_1 \hookrightarrow N'$ be the extension ensured by 2. Then we obtain

$$\begin{tikzcd} & M_1 \arrow[dd, "f_2 \circ f_1^{-1}"] \arrow[dr, hook, "g"] & \\ M \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & N' \\ & M_2 \arrow[ur, hook, "\mathrm{id}_{M_2}"'] & \end{tikzcd}$$

as required. □

We say that $\langle M_i : i < \lambda \rangle$ is a $\leq$-chain if $M_i \leq M_j$ for all $i < j < \lambda$. For the next theorem to hold we need the following property:

7.27 Definition *We say that $\mathcal{M}$ is closed under union of $\leq$-chains if*

c8. *if $\langle M_i : i < \lambda \rangle$ is a $\leq$-chain, then $M_i \leq \bigcup_{j<\lambda} M_j$ for all $i < \lambda$.* □

The following is a general existence theorem for rich models. This general form requires large cardinalities. We leave to the reader to verify that if the number if finite morphisms is countable (up to isomorphism) then countable rich models exit.

7.28 Theorem *Assume 7.4. Assume further* c8 *and that $\mathcal{M}$ has the amalgamation property. Let λ be such that $|L| < \lambda = \lambda^{<\lambda}$. Then there is a rich model N of cardinality λ.*

Proof We construct N as union of a $\leq$-chain of models $\langle N_i : i < \lambda \rangle$ such that $|N_i| = \lambda$. Let N_0 be any model of cardinality λ. At stage $i+1$, let $f : M \to N_i$ be the least morphism (in a well-ordering that we specify below) such that $|f| \leq |M| < \lambda$

and f has no extension to an embedding $f' : M \hookrightarrow N_i$. Apply the amalgamation property to obtain an total morphism $f' : M \hookrightarrow N'$ that extends $f : M \to N_i$. By the downword Löwenheim-Skolem Theorem we may assume $|N'| = \lambda$. Let $N_{i+1} = N'$. At limit stages take the union.

The well-ordering mentioned needs to be chosen so that in the end we forget nobody. So, first at each stage we well-order the isomorphism-type of the morphisms $f : M \to N_i$ such that $f \leq |M| < \lambda$. Then the required well-ordering is obtained by dovetailing all these well-orderings. The length of this enumeration is at most $\lambda^{<\lambda}$, which is λ by hypothesis.

We check that N is rich. Let $f : M \to N$ be a morphism and $|f| < |M| \leq \lambda$. As $|L| < \lambda$ we can approximate M with an elementary chain of structures of cardinality $< \lambda$. Hence we may as well assume that $|f| \leq |M| < \lambda$. The cofinality of λ is larger than $|f|$, hence $\operatorname{img} f \subseteq N_i$ for some $i < \lambda$. So $f : M \to N_i$ is a morphism and at some stage j we have ensured the existence of an embedding of $f' : M \hookrightarrow N_{j+1}$ that extends f. □

7.29 Proposition *Let $\mathcal{M}$ consist of all structures of some fixed signature and the elementary maps between these. Then $\mathcal{M}$ has the amalgamation property.*

Proof Let $k : M \to N$ be an elementary map. Let a enumerate $\operatorname{dom} k$ and let b enumerate M. Let $p(x\,;z) = \operatorname{tp}_M(b\,;a)$. The type $p(x\,;a)$ is consisent in M, in particular, it is finitely consistent and, by elementarity, $p(x\,;ka)$ is finitely consistent in N. By the compactness theorem, there is $N' \succeq N$ such that $N' \vDash p(c\,;ka)$ for some $c \in N'^{|x|}$. Hence $g = \{\langle b, c\rangle\} : M \to N'$ is the required elementary map that extends $k : M \to N$. □

Chapter 8

Some algebraic structures

The main result in this chapter is Corollary 8.26, the elimination of quantifiers in algebraically closed fields, which in algebra is called Chevalley's Theorem on constructible sets. From it we derive Hilbert's Nullstellensatz 8.26. Finally, we isolate the model theoretic properties of those types that correspond to the algebraic notions of prime and radical ideal of polynomials.

The first sections of this chapter are not a pre-requisite for Sections 4–7, at the cost of a few repetitions in the latter.

8.1 Notation Recall that when $A \subseteq M$ we denote by $\langle A \rangle_M$ the substructure of M generated by A. Then $\langle A \rangle_M \subseteq N$ is equivalent to $N \vDash \mathrm{Diag}\,\langle A \rangle_M$. The diagram of a structure has been defined in Notation 5.20.

In this chapter, whenever some $A \subseteq M \vDash T$ are fixed, by *model* we mean superstructures of $\langle A \rangle_M$ that models T. The notions of *logical consequence*, *consistency*, *completeness*, etc. are modified accordingly, and we write $\vdash$ for $T \cup \mathrm{Diag}\langle A \rangle_M \vdash$.

We say that a type $p(x)$ is trivial if $\vdash p(x)$. □

1 Abelian groups

The language L is that of additive groups. The theory T_{ag} of abelian groups is axiomatized by the universal closure of the following axioms

a1 $(x+y)+z \;=\; y+(x+z)$;

a2 $x+(-x) \;=\; (-x)+x \;=\; 0$;

a3 $x+0 \;=\; 0+x \;=\; x$;

a4 $x+y \;=\; y+x$.

Let x be a tuple of variables of length α, an ordinal. We write $L_{\mathrm{ter},x}$ for the set of terms $t(x)$ with free variables among x. On this set we define the equivalence relation

$$t(x) \sim s(x) \quad = \quad T_{\mathrm{ag}} \vdash t(x) = s(x).$$

We define the group operations on $L_{\mathrm{ter},x}/\!\sim$ in the obvious way. We denote by $\mathbb{Z}^{\oplus\alpha}$ the set of tuples of integers of length α that are almost always 0. The group operations on $\mathbb{Z}^{\oplus\alpha}$ are defined coordinate-wise. The following immediate proposition implies in particular that $L_{\mathrm{ter},x}/\!\sim$ is isomorphic to $\mathbb{Z}^{\oplus\alpha}$.

8.2 Proposition *Let $A \subseteq M \vDash T_{\mathrm{ag}}$. Then for every formula $\varphi(x) \in L_{\mathrm{at}}(A)$ there are $n \in \mathbb{Z}^{\oplus\alpha}$ and $c \in \langle A \rangle_M$ such that*

$$\vdash \quad \varphi(x) \quad \leftrightarrow \quad \sum_{i<\alpha} n_i x_i \;=\; c.$$

where $n = \langle n_i : i < \alpha\rangle$ *and* $x = \langle x_i : i < \alpha\rangle$. □

Proof Up to equivalence over $T_{\rm ag}$ the formula $\varphi(x)$ has the form $s(x) = t(a)$ for some parameter-free terms $s(x)$ and $t(z)$. Over Diag $\langle A\rangle_M$, we can replace $t(a)$ with a single $c \in \langle A\rangle_M$ and write $s(x)$ as the linear combination shown above. □

8.3 Definition *Let* $M \vDash T_{\rm ag}$*. For* $A \subseteq M$ *and* $c \in M$*, we say that c is independent from A if* $\langle A\rangle_M \cap \langle c\rangle_M = \{0\}$*. Otherwise we say that c is dependent from A. The rank of M is the least cardinality of a subset* $A \subseteq M$ *such that all elements in M are dependent from A. We denote it by* $\mathrm{rank}(M)$. □

Note that when M is a vector space the condition $\langle A\rangle_M \cap \langle c\rangle_M = \{0\}$ is equivalent to saying that c is not a linear combination of vectors in A. Then rank M coincides with the dimension of M. In fact, what we do here for abelian groups could be easily generalized to D-modules, where D is any integral domain, and in particular to vector spaces.

The following proposition gives a convenient syntactic characterization of independence. An element c of an abelian group is a torsion element if $nc = 0$ for some positive integer.

8.4 Proposition *Let* $A \subseteq M \vDash T_{\rm ag}$*. Suppose that* $c \in M$ *is not a torsion element. Then the following are equivalent*

1. *c is independent from A;*
2. $p(x) = \text{at-tp}_M(c/A)$ *is trivial (see Notation 5.20 and 8.1).*

Proof 1⇒2 By Proposition 8.2, formulas in $p(x)$ may be assumed to have the form $nx = a$ for some integer n and some $a \in \langle A\rangle_M$. As this formula is satisfied by c then $a \in \langle A\rangle_M \cap \langle c\rangle_M$. Hence $a = 0$. As c is not a torsion element, $n = 0$ and the equation is trivial.

2⇒1 If $\langle A\rangle_M \cap \langle c\rangle_M \neq \{0\}$ then $nc = a$ for some $a \in \langle A\rangle_M \smallsetminus \{0\}$ and some positive integer n. Then c satisfies the non trivial equation $nx = a$. □

8.5 Remark Let $k : M \to N$ be a partial embedding and let a be an enumeration of dom k. We claim that $k \cup \{\langle b, c\rangle\} : M \to N$ is a partial embedding for every $b \in M$ and $c \in N$ that are independent from a, respectively ka. In fact, it suffices to check that $M, b, a \equiv_{\rm at} N, c, ka$. Suppose $\varphi(x\,; z) \in L_{\rm at}$ is such that $M \vDash \varphi(b\,; a)$. Then by independence $\varphi(x\,; a)$ is trivial, i.e.

$$T_{\rm ag} \cup \mathrm{Diag}\langle a\rangle_M \quad\vdash\quad \varphi(x\,; a).$$

As $\langle a\rangle_M$ and $\langle ka\rangle_N$ are isomorphic structures

$$T_{\rm ag} \cup \mathrm{Diag}\langle ka\rangle_N \quad\vdash\quad \varphi(x\,; ka).$$

Therefore $N \vDash \varphi(c\,; ka)$. This proves $M, b, a \Rrightarrow_{\rm at} N, c, ka$. As the same assumptions apply to $k^{-1} : N \to M$ we obtain $M, b, a \Lleftarrow_{\rm at} N, c, ka$.

The proof that $N \vDash \varphi(c\,; ka)$ implies $M \vDash \varphi(b\,; a)$ is similar. □

2 Torsion-free abelian groups

The theory of torsion-free abelian groups extends T_{ag} with the following axioms for all positive integers n

tf $\quad nx = 0 \to x = 0.$

We denote this theory by T_{tfag}. It is not difficult to see that in a torsion-free abelian group every equation of the form $nx = a$ has at most one solution.

8.6 Proposition *Let $M \vDash T_{\mathrm{tfag}}$ be uncountable. Then* $\operatorname{rank} M = |M|$.

Proof Let $A \subseteq M$ have cardinality $< |M|$. We claim that M contains some element that is independent from A. It suffices to show that the number of elements that are dependent from A is $< |M|$. If $c \in M$ is dependent from A then, by Proposition 8.7, it is a solution of some formula $L_{\mathrm{at}}(A)$. As there is no torsion, such a formula has at most one solution. Therefore the number of elements that are dependent from A is at most $|L_{\mathrm{at}}(A)|$, that is $\max\{|A|, \omega\}$. If M is uncountable the claim follows. $\square$

8.7 Proposition *Let $A \subseteq M \vDash T_{\mathrm{tfag}}$. Let $p(x) = \mathrm{at}^{\pm}\text{-}\mathrm{tp}(b/A)$, where $b \in M$. Then one of the following holds*

1. *b is independent from A;*
2. *$M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\mathrm{at}}(A)$ such that $\vdash \varphi(x) \to p(x)$.*

Note the similarity with Example 7.15, where the independent type is $q(x)$ and the isolating formulas are the $r_i(x)$.

It is important to observe that the set A above may be infinite. This is essential to obtain Corollary 8.11, and it is one of the main differences between this example and the examples encountered in Chapter 6.

Proof If b is dependent from A, then b satisfies a non trivial atomic formula $\varphi(x)$ which we claim is the formula required in 2. It suffices to show that $\varphi(x)$ implies a complete $L_{\mathrm{at}^{\pm}}(A)$-type. Clearly this type must be $p(x)$. Let $\varphi(x)$ have the form $nx = a$ for some $n \in \mathbb{Z} \smallsetminus \{0\}$ and $a \in \langle A \rangle_M \smallsetminus \{0\}$. We show that for every $m \in \mathbb{Z}$ and every $c \in \langle A \rangle_M$ one of the following holds

a. $\vdash \; nx = a \to mx = c$;

b. $\vdash \; nx = a \to mx \neq c$.

Suppose not for a contradiction that neither a nor b holds and fix models N_1, N_2 and some $b_i \in N_i$ such that

a′. $N_1 \vDash nb_1 = a$ and $N_1 \vDash mb_1 \neq c$;

b′. $N_2 \vDash nb_2 = a$ and $N_2 \vDash mb_2 = c$.

From b′ we infer that $N_2 \vDash ma = nc$. As N_1 is torsion-free, from a′ we infer that $N_1 \vDash ma \neq nc$. But $ma = nc$ is a formula with parameters in $\langle A \rangle_M$, so it should have the same truth value in all superstructures of $\langle A \rangle_M$, a contradiction. $\square$

3 Divisible abelian groups

The theory of divisible abelian groups extends T_{tfag} with the following axioms for all integers $n \neq 0$

div $\quad y \neq 0 \rightarrow \exists x\; nx = y.$

We denote this theory by T_{dag}.

8.8 Proposition *Let $A \subseteq M \vDash T_{\text{tfag}}$ and let $\varphi(x) \in L_{\text{at}}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\, \varphi(x)$ for every model $N \vDash T_{\text{dag}}$.*

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A \rangle_M \subseteq M' \vDash T_{\text{tfag}}$.

The claim in the proposition holds more generally for all $\varphi(x) \in L_{\text{qf}}$ and also when x is a tuple of variables. This follows from Lemma 8.10, whose proof uses the proposition.

Proof We can assume that $\varphi(x)$ has the form $nx = a$ for some $n \in \mathbb{Z}$ and some $a \in \langle A \rangle_M$. If $n = 0$, then $a = 0$ since $\varphi(x)$ is consistent, and the claim is trivial. If $n \neq 0$ then by consistency $a \neq 0$, hence a solution exist in N by axiom div. □

8.9 Exercise Prove a converse of Proposition 8.8. Let $A \subseteq N \vDash T_{\text{ag}}$ and let x be a single variable. Prove that if $N \vDash \exists x\, \varphi(x)$ for every consistent $\varphi(x) \in L_{\text{at}}(A)$, then $N \vDash T_{\text{dag}}$. □

We are ready to prove that divisible abelian groups of infinite rank are ω-rich.

8.10 Lemma *Let $k : M \rightarrow N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\text{tfag}}$ and $N \vDash T_{\text{dag}}$ is a model of rank $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \big\{\langle b\,; c\rangle\big\} : M \rightarrow N$ is a partial isomorphism.*

Proof Let a be an enumeration of $\operatorname{dom} k$ and let $p(x\,;z) = \text{at}^{\pm}\text{-tp}(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is dependent from a, then Proposition 8.7 yields a formula $\varphi(x\,;z) \in L_{\text{at}}$ such that

i. $\quad \vdash \varphi(x\,;a) \rightarrow p(x\,;a)$

ii. $\quad \varphi(x\,;a)$ is consistent.

By isomorphism, i and ii hold with a replaced by ka. Then by Proposition 8.8 the formula $\varphi(x\,;ka)$ has a solution $c \in N$.

The second case, which has no analogue in Lemma 6.1, is when b is independent from a. Then by Remark 8.5 we may choose c to be any element of N independent from ka. Such an element exists because N has rank at least λ. □

8.11 Corollary *Every uncountable model of T_{dag} is rich in the category of models of T_{tfag} and partial isomorphisms. In particular T_{dag} is uncountably categorical, complete, and has quantifier elimination.*

Proof Any uncountable $N \vDash T_{\text{dag}}$ has rank $|N|$, therefore it is rich by Lemma 8.10; categoricity and completeness follow. As for quantifier elimination, let $k : M \rightarrow N$ is a partial isomorphism between models of T_{dag}. If $M \preceq M'$ and $M \preceq N'$ are elementary superstructures of uncountable cardinality then $k : M' \rightarrow N'$ is elementary by Theorem 7.11 and this suffices to conclude that $k : M \to N$ is elementary. □

8.12 Exercise Prove that every model of $T_{\rm dag}$ is ω-ultraomogeneous (indipendenlty of cardinality and rank). □

4 Commutative rings

In this section L is the language of (unital) rings. It contains two constants 0 and 1 the unary operation $-$ and two binary operations $+$ and $\cdot$. The theory of rings contains the following axioms

a1-a4 as for abelian groups

r1 $(x{\cdot}y){\cdot}z \;=\; y{\cdot}(x{\cdot}z),$

r2 $1{\cdot}x \;=\; x{\cdot}1 \;=\; x,$

r3 $(x+y){\cdot}z \;=\; x{\cdot}z + y{\cdot}z,$

r4 $z{\cdot}(x+y) \;=\; z{\cdot}x + z{\cdot}y.$

All the rings we consider are commutative

c $x{\cdot}y \;=\; y{\cdot}x.$

We denote the theory of commutative rings by $T_{\rm cr}$.

In what follows the theory $T_{\rm cr} \cup {\rm Diag}\langle A\rangle_M$, for some M clear from the context, is implicit in the sense of Notation 8.1. So it is important to remember that ${\rm Diag}\langle A\rangle_M$ is not trivial even when $A = \varnothing$. In fact, ${\rm Diag}\langle\varnothing\rangle_M$ determines the characteristic of the models.

Let $A \subseteq M \models T_{\rm cr}$ and let x be a tuple of variables. We write $L_{{\rm ter},x}(A)$ for the set of terms $t(x)$ with free variables among x and parameters in A. On this set we define the equivalence relation

$$t(x) \sim s(x) \quad = \quad \vdash\; t(x) = s(x).$$

On $L_{{\rm ter},x}(A)/{\sim}$ we define the ring operations in the obvious way so that $L_{{\rm ter},x}(A)/{\sim}$ is a commutative ring. We denote by $A[x]$ the set of polynomials with variables among x and parameters in $\langle A\rangle_M$. The ring operations on $A[x]$ are defined as usual. The following proposition (which is clear, but tedious to prove) implies in particular that $L_{{\rm ter},x}(A)/{\sim}$ is isomorphic to $A[x]$. For simplicity we state it only for $|x| = 1$.

8.13 Proposition *Let $A \subseteq M \models T_{\rm cr}$ and let x be a single variable. Then for every formula $\varphi(x) \in L_{\rm at}(A)$ there is a unique $n < \omega$ and a unique tuple $\langle a_i : i \le n\rangle$ of elements of $\langle A\rangle_M$ such that $a_n \neq 0$ and*

$$\vdash \quad \varphi(x) \;\leftrightarrow\; \sum_{i\le n} a_i x^i \;=\; 0.$$ □

The integer n in the proposition above is called the degree of $\varphi(x)$.

8.14 Definition *Let $A \subseteq M \models T_{\rm cr}$. We say that an element $b \in M$ is transcendent over A if the type $p(x) =$ ${\rm at\text{-}tp}_M(b/A)$ is trivial (see Notation 5.20 and 8.1). Otherwise we say* *that b is algebraic over A. The transcendence degree of M is the least cardinality of a subset $A \subseteq M$ such that all elements of M are algebraic over A.* □

8.15 Remark Remark 8.5 holds here with 'independent' replaced by 'transcendent' and T_{ag} replaced by T_{cr}. □

5 Integral domains

Let $a \in M \vDash T$. We say that a is a zero divisor if $a\,b = 0$ for some $b \in M \smallsetminus \{0\}$. An integral domain is a commutative ring without zero divisors. The theorey of integral domains contains the axioms of commutative rings and the following

nt. $0 \neq 1$

id. $x{\cdot}y = 0 \;\rightarrow\; x = 0 \;\vee\; y = 0.$

We denote the theory of integral domains by T_{id}.

For a prime p, we define the theory T^{p}_{id}, which contains T_{id} and the axiom

ch$_p$. $1 + \ldots (p \text{ times}) \cdots + 1 = 0.$

The theory T^{0}_{id} contains the negation of ch$_p$ for all p. Note that all models of T^{p}_{id} have the same characteristic in the model theoretic sense defined in 5.27. In the remaining section we work in the category of models of T_{id} with partial embeddings as morphisms. This category consists of countably many connected components each containing all models of T^{p}_{id} for some p.

8.16 Proposition *Let $M \vDash T_{\text{id}}$ be uncountable. Then M has transcendence degree $|M|$.*

Proof In an integral domain every polynomial has finitely many solutions and there are $|L(A)|$ polynomials over A. □

8.17 Proposition *Let $A \subseteq M \vDash T_{\text{id}}$. For $b \in M$ let $p(x) = \text{at}^{\pm}\text{-tp}(b/A)$. Then one of the following holds*

1. *b is transcendental over A;*
2. *$M \vDash \varphi(b)$ for some $\varphi(x) \in L_{\text{at}}(A)$ such that $\vdash \;\varphi(x) \rightarrow p(x)$.*

Note the similarity with Example 7.15, where the transcendental type is $q(x)$ and the isolating formulas are the $r_i(x)$.

As in Proposition 8.7, the set A may be infinite. This is essential to obtain Corollary 8.20.

Proof Suppose b is not transcendental, i.e. it satisfies a non trivial atomic formula. Let $\varphi(x) \in L_{\text{at}}(A)$ be a non trivial formula with minimal degree such that $\varphi(b)$. We prove that $\varphi(x)$ implies a complete $L_{\text{at}^{\pm}}(A)$-type. Clearly this type must be $p(x)$. We prove that for any $\xi(x) \in L_{\text{at}}(A)$ one of the following holds

1. $\vdash \;\varphi(x) \rightarrow \;\;\xi(x)$
2. $\vdash \;\varphi(x) \rightarrow \neg\xi(x)$.

Let us write $a(x) = 0$ and $a'(x) = 0$ for the formulas $\varphi(x)$ and $\xi(x)$, respectively. If $\langle A\rangle_M$ is a field, choose a polynomial $d(x)$ of maximal degree such that

a. $d(x)t(x) = a(x)$

a'. $d(x)t'(x) = a'(x)$,

for some polynomials $t(x)$ and $t'(x)$.

If $\langle A\rangle_M$ is not a field, polynomials $d(x)$, $t(x)$ and $t'(x)$ as above exist with coefficients in the field of fractions of $\langle A\rangle_M$. Then a and a' hold up to a factor in $\langle A\rangle_M$ which we absorb in $a(x)$ and $a'(x)$.

From a we get $d(b) = 0$ or $t(b) = 0$. In the first case, as $a(x)$ has minimal degree, we conclude that $t(x)$ is constant. This implies that any zero of $a(x)$ is also a zero of $a'(x)$, that is, it implies 1.

Now suppose $t(b) = 0$. Then the minimality of the degree of $a(x)$ implies that $d(x) = d$, where d is a nonzero constant. If $\langle A\rangle_M$ is a field, apply Bézout's identity to obtain two polynomials $c(x)$ and $c'(x)$ such that $d = a(x)c(x) + a'(x)c'(x)$. Then $a(x)$ and $a'(x)$ have no common zeros, and 2 follows. If $\langle A\rangle_M$ is not a field, we use Bézout's identity in the field of fractions of $\langle A\rangle_M$ and, for some $d' \in \langle A\rangle_M \smallsetminus \{0\}$, obtain $d'd = a(x)c(x) + a'(x)c'(x)$. Then we reach the same conclusion. □

6 Algebraically closed fields

Let $a, b \in M \vDash T_{\rm id}$. We say that b is the inverse of a if $a{\cdot}b = 1$. A field is a commutative ring where every non-zero element has an inverse. The theory of fields contains $T_{\rm id}$ and the axiom

f. $\exists y\, \big[x \neq 0 \;\rightarrow\; x{\cdot}y = 1\big]$.

Fields are structures in the signature of rings: the language contains no symbol for the multiplicative inverse. So, substructures of fields are merely integral domains.

The theory of algebraically closed field, which we denote by $T_{\rm acf}$, also contains the following axioms for every positive integer n

ac. $\exists x\, \big(x^n + z_{n-1}x^{n-1} + \cdots + z_1 x + z_0 \;=\; 0\big)$

The theory $T^p_{\rm acf}$ is defined in analogy to $T^p_{\rm id}$ in the previous section.

8.18 Proposition *Let $A \subseteq M \vDash T_{\rm id}$ and let $\varphi(x) \in L_{\rm at}(A)$, where $|x| = 1$, be consistent. Then $N \vDash \exists x\, \varphi(x)$ for every model $N \vDash T_{\rm acf}$.*

Note that in the proposition above *consistent* means satisfied in some M' such that $\langle A\rangle_M \subseteq M' \vDash T_{\rm id}$.

The claim in the proposition holds more generally for all $\varphi(x) \in L_{\rm qf}$ when x is a tuple of variables. This follows from Lemma 8.19 whose proof uses the proposition.

Proof Up to equivalence $\varphi(x)$ has the form $a_n x^n + \cdots + a_1 x + a_0 \;=\; 0$ for some $a_i \in \langle A\rangle_N$. Choose n minimal. If $n = 0$ then $a_0 = 0$ by the consistency of $\varphi(x)$ and the claim is trivial. Otherwise $a_n \neq 0$ and the claim follows from f and ac. □

8.19 Lemma *Let $k : M \to N$ be a partial isomorphism of cardinality $< \lambda$, where $M \vDash T_{\rm id}$ and $N \vDash T_{\rm acf}$ has transcendence degree $\geq \lambda$. Then for every $b \in M$ there is $c \in N$ such that $k \cup \big\{\langle b, c\rangle\big\} : M \to N$ is a partial isomorphism.*

The following is the proof of Lemma 8.10 which we repeat here for convenience.

Proof Let a be an enumeration of $\operatorname{dom} k$ and let $p(x\,;z) = \text{at}^{\pm}\text{-tp}_M(b\,;a)$. The required c has to realize $p(x\,;ka)$. We consider two cases. If b is algebraic over a, then Proposition 8.17 yields a formula $\varphi(x\,;z) \in L_{\text{at}}$ such that

i. $\varphi(x\,;a) \to p(x\,;a)$

ii. $\varphi(x\,;a)$ is consistent.

By isomorphism i and ii hold with a replaced by ka. Then by Proposition 8.18 the formula $\varphi(x\,;ka)$ has a solution in $c \in N$.

The second case, which has no analogue in Lemma 6.1, is when b is transcendent over a. Then by Remark 8.15 we may choose c to be any element of N transcendent over ka. This exists because N has transcendence degree $\geq \lambda$. □

Below few important consequences of this lemma.

8.20 Corollary *Work in the category of models of T_{id} with partial embeddings as morphisms.Then the following are equivalent.*

1. *N is a λ-rich model*
2. *$N \vDash T_{\text{acf}}$ and has transcendence degree $\geq \lambda$*

Proof Implication 2 ⇒ 1 is an immediate consequence of Lemma 8.19.

In every connected component there is an $M \vDash T_{\text{acf}}$ of cardinality λ and transcendence degree λ (by Proposition 8.16 when $\lambda > \omega$, by compactness for $\lambda = \omega$). As proved above, M is rich and therefore elementarily equivalent to any λ-rich model N in the same connected component. This proves 1 ⇒ 2. □

8.21 Corollary *The theory T_{acf} has elimination of quantifiers.*

Proof Let $k : M \to N$ be a partial embedding between models of T_{acf}. Let M' and N' be elementary superstructures of M and N respectively of sufficiently large cardinality. As M' and N' are rich, $k : M' \to N'$ is elementary by Theorem 7.11. Hence $k : M \to N$ is also elementary. □

8.22 Corollary *The theories T^p_{acf} are complete and uncountably categorical (i.e. λ-categorical for every uncountable λ)*

Proof Two models of T^p_{acf} belong to the same connected component. Then, as every uncountable model of T^p_{acf} is rich, uncountable categoricity and completeness follow. □

8.23 Exercise Prove that every model of T_{acf} is ω-ultrahomogeneous (indipendently of cardinality and transcendence degree). □

7 Hilbert's Nullstellensatz

In this section we fix a tuple of variables x and a subset A of an integral domain. We denote by $\Delta(A)$ the set of formulas of the form $t(x) = 0$ where $t(x)$ is a term with parameters in A. So $\Delta(A)$-types are (possibly infinite) systems of polynomial equations with coefficients in $\langle A\rangle_M$.

For convenience we define the closure of $p(x)$ under logical consequences as follows

$$\operatorname{ccl} p(x) \quad = \quad \Big\{t(x) = 0 \;:\; p(x) \;\vdash\; t(x){=}0\Big\}$$

Remember that we always work under the assumptions made in Notation 8.1. In particular, in this section we work over the theory $T_{\rm id} \cup \operatorname{Diag}\langle A\rangle_M$. In general, closure under logical consequences is an elusive notion. Hence Propositions 8.24 and 8.25 are useful because they give a model theoretical, respectively algebraic, characterisation of $\operatorname{ccl} p(x)$.

8.24 Proposition *Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x)$ be a $\Delta(A)$-type. Fix some N of sufficiently large cardinality such that $\langle A\rangle_M \subseteq N \vDash T_{\rm acf}$. Then*

$$\operatorname{ccl} p(x) \quad = \quad \Big\{t(x) = 0 \;:\; N \vDash \forall x\, \big[p(x) \rightarrow t(x){=}0\big]\Big\}.$$

The cardinality of N is the proposition is sufficiently large when $|A| < |N|$ and $|x| \leq |N|$; note that here x has possibly infinite length. In Corollary 8.26 below we will considerably strengthen this proposition for finite x.

Proof Only the inclusion $\supseteq$ requires a proof, then suppose $t(x){=}0 \notin \operatorname{ccl} p(x)$. As $p(x) \wedge t(x) \neq 0$ is consistent, there is a model M' such that $M' \vDash p(a) \wedge t(a) \neq 0$ for some $a \in M'^{|x|}$. Then there is a partial isomorphism $h : M' \rightarrow N$ that extends id_A and is defined on a, provided N is large enough to accommodate a. This implies that $p(x) \wedge t(x) \neq 0$ has a solution in N. Hence $t(x)$ does not belong to the set on the r.h.s. □

Recall that $A[x]$ denotes the ring of polynomials with variables in x and coefficients in $\langle A\rangle_M$. We identify $\Delta(A)$ nd $A[x]$ in the obvious way. Consequently, a $\Delta(A)$-type $p(x)$ is identified with a set of polynomials $p \subseteq A[x]$. For $p \subseteq A[x]$ we write $\operatorname{rad} p$ for the radical ideal generated by p, that is, the intersection of all prime ideals containing p. When $p = \operatorname{rad} p$, we say that p is a radical ideal. Recall from algebra that if $p \subseteq A[x]$ is an ideal then

$$\operatorname{rad} p \quad = \quad \Big\{t(x) \;:\; t^n(x) \in p \text{ for some positive integer } n\Big\}.$$

An identity that justifies the name.

8.25 Proposition *Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x)$ be a $\Delta(A)$-type. Then $\operatorname{ccl} p(x) \simeq \operatorname{rad} p$.*

The proposition holds with a similar proof for the broader class of rings without nilpotent elements. (Which does not come as a surprise.)

Proof $(\supseteq)$ We claim that $\operatorname{ccl} p(x)$ is an ideal. In fact, for every pair of $L(A)$-terms $t(x)$ and $s(x)$

$$\begin{aligned} t(x) = 0 &\;\vdash\; s(x)t(x) = 0 \\ s(x) = t(x) = 0 &\;\vdash\; s(x) + t(x) = 0 \end{aligned}$$

Moreover, as integral domains do not have nilpotent elements

$$t^n(x) = 0 \;\vdash\; t(x) = 0$$

By # above, $\operatorname{ccl} p(x)$ is a radical ideal which proves $\operatorname{ccl} p(x) \supseteq \operatorname{rad} p$.

($\subseteq$) We fix some $t(x) \notin \operatorname{rad} p$ and prove that $p(x) \wedge t(x) \neq 0$ is consistent. Let q be some prime ideal containing p such that $t(x) \notin q$. As q is prime, the ring $A[x]/q$ is an integral domain. The polynomials that vanish in $A[x]/q$ at $x + q$ are exactly those in q. Hence $A[x]/q$ witnesses the consistency of $q(x) \wedge t(x) \neq 0$. □

When x is a finite tuple of variables, we can extend the validity of Proposition 8.24 to the case $A = M = N$.

8.26 Corollary (Hilbert's Nullstellensatz) *Let $N \vDash T_{\rm acf}$ and let $p(x)$, where $|x| < \omega$, be a $\Delta(N)$-type. Then*

$$\operatorname{rad} p \ \simeq\ \operatorname{ccl} p(x) \ = \ \Big\{t(x) \ : \ N \vDash \forall x\, [p(x) \to t(x) = 0]\Big\}.$$

Proof Let N' be a large elementary extension of N. By Proposition 8.24, the claim holds for N'. By Hilbert's Basis Theorem, the ideal generated by p is finitely generated hence $p(x)$ is equivalent to a formula (cfr. Exercise 8.29). Therefore, by elementarity, the claim holds for N. □

Hilbert's Nullstellensatz comes in two variants. The one in Corollary 8.26 is sometimes referred to as the *strong* Nullstellensatz. The weaker variant is stated in Exercise 8.28.

We conclude this section by showing that the notions of primeness for types and ideals coincide (if we restrict to types closed under logical consequences).

8.27 Proposition *Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x)$ be a $\Delta(A)$-type closed under logical consequences. Then the following are equivalent*

1. *$p(x)$ is a prime $\Delta(A)$-type;*
2. *p is a prime ideal.*

Proof 1⇒2 Assume 1 and suppose that the polynomial $t(x) \cdot s(x)$ belongs to p. Clearly, over $T_{\rm id} \cup \operatorname{Diag}\langle A\rangle_M$ we have $\vdash\ t(x) \cdot s(x) = 0 \to t(x) = 0 \vee s(x) = 0$. As $p(x)$ is a prime $\Delta(A)$-type, $p(x) \vdash t(x) = 0$ or $p(x) \vdash s(x) = 0$. By Proposition 8.25, the type $p(x)$ is closed under logical consequences, therefore $t(x) \in p$ or $s(x) \in p$.

2⇒1 Assume p is a prime ideal and for some $t_i(x) = 0 \in \Delta(A)$

$$p(x) \ \vdash\ \bigvee_{i=1}^{n} t_i(x) = 0.$$

Then

$$p(x) \ \vdash\ \prod_{i=1}^{n} t_i(x) \ = \ 0$$

Since $p(x)$ is closed under logical consequences, and p is a prime ideal, $t_i(x) \in p$ for some i. Hence $p(x)$ contains the equation $t_i(x) = 0$. By Corollary 5.19 this suffices to prove that $p(x)$ is a prime $\Delta(A)$-type. □

8.28 Exercise Let $N \vDash T_{\rm acf}$ and let $p(x)$ be a $\Delta(N)$-type where $|x| < \omega$. Prove that the following are equivalent

1. p is a proper ideal;
2. $p(x)$ has a solution in N. □

8.29 Exercise Let $A \subseteq M \vDash T_{\rm id}$ and let $p(x)$ be a $\Delta(A)$-type. Prove that the following are equivalent

1. $p(x)$ is a principal $\Delta(A)$-type;
2. the ideal generated by p is finitely generated. $\square$

Chapter 9

Saturation and homogeneity

The first two section introduce saturation and homogeneity and Section 3 presents the notation we shall use in the following chapters when working inside a monster model.

1 Saturated structures

Recall that a type $p(x) \subseteq L(M)$ is finitely consistent in M if every conjunction of formulas in $p(x)$ has a solution in M. When $A \subseteq M$ we write $S_x(A)$ for the set of types whose variables are in x which are complete and finitely consistent in M. We never display M in the notation as it will always be clear from the context. When A is empty it usual to write $S_x(T)$ for $S_x(A)$ where $T = \mathrm{Th}(M)$. We write $S(A)$ for the union of $S_x(A)$ as x ranges over all tuples of variables. Similarly for $S(T)$.

The following remark will be used in the sequel without explicit reference.

9.1 Remark Let $k : M \to N$ be an elementary map and let a be an enumeration of $\mathrm{dom}\, k$. Let $p(x\,;z) \subseteq L$. If $p(x\,;a)$ is finitely consistent in M, then $p(x\,;k\,a)$ is finitely consistent in N. (We can drop *finitely* in the antecedent but not in the consequent.) □

9.2 Definition *Let x be a single variable and let λ be an infinite cardinal. We say that a structure N is λ-saturated if it realizes every type $p(x)$ such that*

1. *$p(x) \subseteq L(A)$ for some $A \subseteq N$ of cardinality $< \lambda$;*
2. *$p(x)$ is finitely consistent in N.*

We say that N is saturated if it is λ-saturated and $|N| = \lambda$.

9.3 Exercise Suppose $|L| \leq \omega$ and let M be an infinite structure. Then for every non-principal ultrafilter F on ω the structure M^ω/F is a ω_1-saturated elementary superstructure of M.

Hint: the notation is as in Chapter 3. Let $|x| = 1$ and $|z| = \omega$. It suffices to consider types of the form $p(x\,;\hat{c})$ where $p(x\,;z) = \{\varphi_i(x\,;z) : i < \omega\} \subseteq L$ and $\hat{c} \in (M^\omega)^{|z|}$. Without loss of generality we can also assume that $\varphi_{i+1}(x\,;z) \to \varphi_i(x\,;z)$, and that all formulas $\varphi_i(x\,;\hat{c})$ are consistent in M^ω.

Let $\langle X_i : i < \omega \rangle$ be a strictly decreasing chain of elements of the ultrafilter such that $X_{i+1} \subseteq \{j \ : \ M \vDash \exists x\ \varphi_i(x, \hat{c}j)\}$. Let $\hat{a} \in M^\omega$ be such that $\varphi_i(\hat{a}j, \hat{c}j)$ holds for every $j \in X_i \smallsetminus X_{i+1}$. Then $\hat{a}$ realizes $p(x)$. □

We shall see that some theories have saturated models that are relatively small in size. However, the existence of saturated models of arbitrary theories is problematic. The following theorem states the existence of a saturated model of cardinality λ whenever $|L| < \lambda = \lambda^{<\lambda}$. The existence of cardinals of this kind is independent of ZFC. If the generalized continuum hypothesis (GCH) holds, every successor cardinal is such that $\lambda = \lambda^{<\lambda}$. Without GCH, every inaccessible cardinal has this property. Both GCH and the existence of inaccessible cardinals are not generally accepted axioms.

Nevertheless, saturated models are widely used in model theory, without any worries about their existence. In fact, if consistency is an issue, they can be replaced by models that are both λ-saturated and λ-homogeneous (see next section) for some less problematic large cardinal λ. This is well known, so complications are commonly avoided by simply assuming that saturated models exist.

If T is the theory of the ring $\mathbb{Z}$, or any other sufficiently expressive theory, then one can prove that the cardinality λ of any saturated model of T is such that $\lambda = \lambda^{<\lambda}$. As the existence of cardinals with this property has to be assumed as an extra axiom, one could simply assume the existence of saturated models and skip the proof of the following theorem.

9.4 Theorem *Assume $|L| < \lambda$ where λ is such that $\lambda^{<\lambda} = \lambda$. Then every structure M of cardinality $\leq \lambda$ has a saturated elementary extension of cardinality λ.*

Proof Without loss of generality, we can assume that M has cardinality λ. We construct an elementary chain $\langle M_i : i < \lambda \rangle$ of models of cardinality λ. The chain starts with M and is the union at limit stages. Given M_i we choose as M_{i+1} any model of cardinality λ that realizes all types in $S_x(A)$ for all $A \subseteq M_i$ of cardinality $< \lambda$. The required M_{i+1} exists because there are at most $2^{|L(A)|} \leq \lambda^{<\lambda} = \lambda$ types in $S_x(A)$ and there are $\lambda^{<\lambda} = \lambda$ sets A.

Let N be the union of the chain. We check that N is the required extension. Let $p(x) \in S(A)$ for some $A \subseteq N$ of cardinality $< \lambda$. As $\lambda^{<\lambda} = \lambda$ implies in particular that λ is a regular cardinal, $A \subseteq M_i$ for some $i < \lambda$. Then M_{i+1} realizes $p(x)$, and so does N, by elementarity. □

9.5 Remark The reader who did not read Section 7.1 may replace 2 in the theorem below with the following (and forget about $\mathcal{M}$).

2′ for every $b \in M$, every elementary map $k : M \to N$ of cardinality $< \lambda$ has an extension defined on b;

9.6 Theorem *Assume $|L| \leq \lambda$ and let N be an infinite structure. Let $\mathcal{M}$ be the category (see Section 7.1) that consists of models of a complete theory T and elementary maps between these. Then the following are equivalent*

1 *N is a λ-saturated structure;*

2 *N is a λ-rich model;*

3 *N realizes all types $p(z) \subseteq L(A)$, with $|z| \leq \lambda$ and $|A| < \lambda$, finitely consistent in N.*

Note that it is the completeness of T which makes the category $\mathcal{M}$ connected.

Proof 1⇒2. Let $k : M \to N$ be an elementary map of cardinality $< \lambda$. It suffices to show that for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c \rangle\} : M \to N$ is an elementary map. Let a be an enumeration of $\operatorname{dom} k$ and define $p(x\,; z) = \operatorname{tp}_M(b\,; a)$. As $p(x\,; a)$ is finitely consistent in M then $p(x\,; k\,a)$ is finitely consistent in N. The required c is any element of N such that $N \vDash p(c\,; k\,a)$. Such a c exists by saturation because $|a| < \lambda$.

2⇒3. Let $p(z)$ be as in 3. By the compactness theorem $N \preceq K \vDash p(a)$ for some model K and $a \in K^{|z|}$. By the downward Löwenheim-Skolem theorem there is a model $A, a \subseteq M \preceq K$ of cardinality $\leq \lambda$. (Here we use $|L|, |A|, |z| \leq \lambda$.) By 2, there is an elementary embedding $h : M \hookrightarrow N$ that extends id_A. (Here we use $|A| < \lambda$.) Finally, as $M \vDash p(a)$, elementarity yields $N \vDash p(h\,a)$.

3⇒1. Trivial. □

Two saturated structures of the same cardinality are isomorphic as soon as they are elementarily equivalent (i.e. as soon as $\varnothing : M \to N$ is an elementary map.) In fact, from Theorems 9.6 and 7.6 we obtain the following (reference to Theorems 7.6 may be avoided with an easy back-and-forth construction).

9.7 Corollary *Every elementary map $k : M \to N$ of cardinality $< \lambda$ between saturated models of the same cardinality λ extends to an isomorphism.* □

As it turns out, we already have many examples of saturated structures.

9.8 Corollary *The following models are ω-saturated*

1 *models of T_{dlo};*

2 *models of T_{rg};*

3 *models of T_{dag} with infinite rank;*

4 *models of T_{acf} with infinite degree of transcendence.*

Countable models of T_{dlo} and T_{rg} and uncountable models of T_{dag} or T_{acf} are saturated.

Proof By quantifier elimination embeddings coincide with elementary embeddings. Then saturation is proved applying Theorem 9.6 and the extension lemmas proved in Chapter 6 and 8. □

The following is a useful test for quantifier elimination.

9.9 Theorem *Assume $|L| \leq \lambda$. Consider the category that consists of models of some theory T_0 and partial isomorphism. Suppose λ-rich models exist and denote by T_1 their theory. Then the following are equivalent*

1. *every λ-saturated model of T_1 is λ-rich;*

2. *T_1 has elimination of quantifiers.*

Proof 2⇒1. Let $N \vDash T_1$ be λ-saturated. Fix a partial isomorphism $k : M \to N$ of cardinality $< \lambda$, some $b \in M$ and let $p(x\,;z) = \text{qf-tp}_M(b\,;a)$, where a enumerates $\operatorname{dom} k$. The type $p(x\,;k\,a)$ is realized in any λ-rich model N' that contains $\langle k\,a\rangle_N$. By 2, $N \equiv_{ka} N'$, so $p(x\,;k\,a)$ is finitely consistent in N. By saturation, it is realised by some $c \in N$. Then $k \cup \{\langle b, c\rangle\} : M \to N$ is the required extension.

1⇒2. Let $k : M \to N$ be a finite partial isomorphism between models of T. We claim that it is an elementary map. Let $M' \succeq M$ and $N' \succeq N$ are λ-saturated models of equal cardinality. As these are λ-rich, $k : M' \to N'$ extends to an isomorphism $h : M' \xrightarrow{\sim} N'$ and the claim follows. □

2 Homogeneous structures

Definition 7.3 introduces the notions of universal and homogeneous structures in a general context. When the morphisms of the underlying category are the elementary maps, we refer to these notions as elementary homogeneity and elementary universality. However, one often omits to specify *elementary*. We repeat Definition 7.3 in this specific case.

9.10 Definition *A structure N is (elementarily) λ-universal if every $M \equiv N$ of cardinality $\leq \lambda$ there is an elementary embedding $h : M \hookrightarrow N$. We say universal if it is λ-universal and of cardinality λ.*

We say that N is (elementarily) λ-homogeneous if every elementary map $k : N \to N$ of cardinality $< \lambda$ extends to an automorphism. We say that N is homogeneous if it is λ-homogeneous and of cardinality λ. □

As saturated structures are rich, the following theorem is an instance of Theorem 7.8.

9.11 Theorem *For every structure N of cardinality $\geq |L|$ the following are equivalent*

1. *N is saturated;*
2. *N is elementarily universal and homogeneous.* □

Given $A \subseteq N$ we denote by $\mathrm{Aut}(N/A)$ the group of A-automorphisms of N. That is the group of automorphisms that fix A point-wise. Let a be a tuple of elements of N. The orbit of a over A in N is the set

$$O_N(a/A) \quad = \quad \Big\{ fa \ : \ f \in \mathrm{Aut}(N/A) \Big\}$$

When the model N is clear from the context we omit the subscript.

Orbits in a homogeneous structure are particularly interesting. The following proposition is immediate but its importance cannot be overestimated.

9.12 Proposition *Let N be a λ-homogeneous structure. Let $A \subseteq N$ have cardinality $< \lambda$ and let $a \in N^{<\lambda}$. Then $O_N(a/A) = p(N)$, where $p(x) = \mathrm{tp}_N(a/A)$.* □

Finally, we want to extend the equivalence in Theorem 9.11 to λ-saturated structures. For this we only need to apply Theorem 7.25.

When the morphisms of the underlying category are the elementary maps, we say weakly λ-saturated for weakly λ-universal (cfr. Definition 7.23). This is not the standard definition (that is, 2 of the proposition below) but the reader can easily verify that it is equivalent by reasoning as in the proof of theorem 9.6.

9.13 Proposition *The following are equivalent*

1. *N weakly λ-saturated;*
2. *N realizes every type $p(x) \subseteq L$, where $|x| < \lambda$, that is finitely consistent in N.* □

The following is an instance of Theorem 7.25 that the reader may prove directly as an exercise.

9.14 Corollary *Let $|L| \leq \lambda$. The following are equivalent*

1. *N is λ-saturated;*
2. *N is weakly λ-saturated and weakly λ-homogeneous.* □

9.15 Exercise Let M be an arbitrary structure. Prove that M has an ω-homogeneous elementary extension of the same cardinality. (There is no assumption on the cardinality of the language.) □

9.16 Exercise Let M and N be elementarily homogeneous structures of the same cardinality λ. Suppose that $M \vDash \exists x\, p(x) \;\Leftrightarrow\; N \vDash \exists x\, p(x)$ for every $p(x) \subseteq L$ such that $|x| < \lambda$. Prove that the two structures are isomorphic. □

9.17 Exercise Let L be a language that extends that of strict linear orders with the constants $\{c_i : i \in \omega\}$. Let T be the theory that extends $T_{\rm dlo}$ with the axioms $c_i < c_{i+1}$ for every $i \in \omega$. Prove that T has elimination of quantifiers and is complete (it can be deduced from what is known of $T_{\rm dlo}$). Exhibit a countable saturated model and a countable model that is not homogeneous. □

3 The monster model

In this section we present some notation and terminology frequently adopted when dealing with a complete theory T. We fix a saturated structure $\mathcal{U}$ of cardinality larger than $|L|$. We assume $\mathcal{U}$ to be large enough that among its elementary substructures we can find any model of T we might be interested in. This structure is called the monster model. We denote by κ the cardinality of $\mathcal{U}$. When appropriate, we assume κ to be inaccessible.

Some terms acquire a slightly different meaning when working inside a monster model.

truth	we say that $\varphi(x)$ holds if $\mathcal{U} \vDash \forall x\, \varphi(x)$ and similarly for other expressions such as $p(x) \to \neg q(x)$ or $\exists y\, p(x,y)$ etc., which are not first-order formulas (not even types);
consistency	we say that $\varphi(x)$ is consistent if $\mathcal{U} \vDash \exists x\, \varphi(x)$, similarly for a type $p(x)$ or expressions as those above;
small/large	cardinalities smaller than κ are called small;
models	are elementary substructure of $\mathcal{U}$ of small cardinality, they are denoted by the letters M and N, and derived symbols;
parameters	are always in $\mathcal{U}$; the symbols A, B, C, etc. denote sets of parameters of small cardinality; calligraphic letters as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. are used for sets of arbitrary cardinality;
tuples	have length $< \kappa$ unless otherwise specified;
global types	are complete finitely consistent types over $\mathcal{U}$; the set of global types is denoted by $S(\mathcal{U})$;

formulas	have parameters in $\mathcal{U}$ unless otherwise specified;
definable sets	are sets of the form $\varphi(\mathcal{U})$ for some formula $\varphi(x) \in L(\mathcal{U})$; we may say A-definable if $\varphi(x) \in L(A)$;
type-definable sets	are sets of the form $p(\mathcal{U})$ for some $p(x) \subseteq L(A)$ where, as the symbol suggests, A has small cardinality;
types of tuples	we write $\mathrm{tp}(a/A)$ for $\mathrm{tp}_{\mathcal{U}}(a/A)$ and $a \equiv_A b$ for $\mathcal{U}, a \equiv_A \mathcal{U}, b$;
orbits of tuples	under the action of $\mathrm{Aut}(\mathcal{U}/A)$ are denoted by $\mathcal{O}(a/A)$.

Let x be a tuple of variables. For any fixed $A \subseteq \mathcal{U}$ we introduce a topology on $\mathcal{U}^{|x|}$ that we call the topology induced by A or, for short, A-topology. (This is non standard terminology, not to be confused with the *logic* A-topology in Section 14.4.) The closed sets of the A-topology are those of the form $p(\mathcal{U})$ where $p(x) \subseteq L(A)$ is a type over A.

For $\varphi(x) \in L(A)$ the sets of the form $\varphi(\mathcal{U})$ are clopen in this topology (and vice versa by Proposition 9.18). They form both a base of closed sets and base of open sets, which makes these topologies *zero-dimensional*. By saturation, the topology induced by A is compact. Actually, saturation is equivalent to the compactness of all these topologies as A ranges over the sets of small cardinality.

These topologies are never T_0 as any pair of tuples $a \equiv_A b$ have exactly the same neighborhoods. Such pairs always exist by cardinality reasons. However it is immediate that the topology induced on the quotient $\mathcal{U}^{|x|}/\equiv_A$ is Hausdorff (this is the so-called *Kolmogorov quotient*). Indeed, this quotient corresponds to $S_x(A)$ with the topology introduced in Section 5.3.

The following proposition is an immediate consequence of compactness. When $A = B$ it says that the topology induced by A is *normal*: any two closed sets are separated by open sets. It could be called mutual normality (not a standard name) because the two closed sets belong to different topologies and the separating sets are each found in the corresponding topology.

9.18 Proposition (mutual normality) *Let $p(x) \subseteq L(A)$ and $q(x) \subseteq L(B)$ be such that $p(x) \rightarrow \neg q(x)$. Then there are a conjuction $\varphi(x)$ of formulas in $p(x)$ and a conjuction $\psi(x)$ of formulas in $q(x)$ such that $\varphi(x) \rightarrow \neg\psi(x)$.*

Proof The assumptions say that $p(x) \cup q(x)$ is inconsistent (i.e. not realized in $\mathcal{U}$). Then the formulas $\varphi(x)$ and $\psi(x)$ exist by compactness (i.e. saturation). □

9.19 Remark There are many forms in which the proposition above can be applied. For instance, assuming for brevity that $p(x)$ and $q(x)$ are closed under conjunctions,

a. if $p(x) \leftrightarrow \neg q(x)$ then $p(x) \leftrightarrow \varphi(x)$ for some $\varphi(x) \in p(x)$;

b. if $p(x) \leftrightarrow \psi(x)$ for some $\psi(x) \in L(\mathcal{U})$ then $p(x) \leftrightarrow \varphi(x)$ for some $\varphi(x) \in p$;

c. if $p(x) \rightarrow \psi(x)$ for some $\psi(x) \in L(\mathcal{U})$ then $\varphi(x) \rightarrow \psi(x)$ for some $\varphi(x) \in p$;

d. if $p(x) \rightarrow \bigvee\limits_{\psi \in \Psi} \psi(x)$, where $|\Psi| < \kappa$, then $p(x) \rightarrow \bigvee\limits_{i=1}^{n} \psi_i(x)$ for some $\psi_i \in \Psi$. □

9.20 Remark A definable set has the form $\varphi(\mathcal{U};b)$ for some formula $\varphi(x;z) \in L$ and some $b \in \mathcal{U}^{|z|}$. If $f \in \mathrm{Aut}(\mathcal{U})$ then

$$\begin{aligned} f[\varphi(\mathcal{U};b)] &= \{fa : \varphi(a;b), \quad a \in \mathcal{U}^{|x|}\} \\ &= \{fa : \varphi(fa;fb), a \in \mathcal{U}^{|x|}\} \\ &= \varphi(\mathcal{U};fb). \end{aligned}$$

Hence automorphisms act on definable sets in a very natural way. Their action on type-definable sets is similar. □

We say that a set $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ is invariant over A if $f[\mathcal{D}] = \mathcal{D}$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$ or, equivalently, if $\mathcal{O}(a/A) \subseteq \mathcal{D}$ for every $a \in \mathcal{D}$. By homogeneity this is equivalent to requiring that

$$q(x) \quad \rightarrow \quad x \in \mathcal{D}$$

for every $q(x) = \mathrm{tp}(a/A)$ and $a \in \mathcal{D}$.

Proposition 9.23 below is an important fact about invariant type-definable sets. It may clarify the proof to consider first the particular case of definable sets.

9.21 Proposition *For every $\varphi(x) \in L(\mathcal{U})$ the following are equivalent*

1. *$\varphi(x)$ is equivalent to some formula $\psi(x) \in L(A)$;*
2. *$\varphi(\mathcal{U})$ is invariant over A.*

We give two proofs of this theorem as they are both instructive.

Proof 1⇒2 Obvious.

2⇒1 From 2 and homogeneity we obtain

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{q(x) \in Q} q(x)$$

where Q is the set of the types in $S_x(A)$ such that $q(x) \rightarrow \varphi(x)$. By compactness, we can rewrite this equivalence

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{\vartheta(x) \in \Theta} \vartheta(x)$$

where Θ is the set of the formulas in $L(A)$ such that $\vartheta(x) \rightarrow \varphi(x)$. The latter equivalence says that $\neg\varphi(x)$ is equivalent to a type over A. Again by compactness we obtain

$$\varphi(x) \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \vartheta_i(x)$$

for some formula $\vartheta_i(x) \in L(A)$. □

Second proof of Proposition 9.21 2⇒1 Let $\varphi(\mathcal{U};b)$, where $\varphi(x;z) \in L$, be a set invariant over A. Let $p(z) = \mathrm{tp}(b/A)$. As $f[\varphi(\mathcal{U};b)] = \varphi(\mathcal{U};fb)$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$, homogeneity and invariance yield

$$p(z) \quad \rightarrow \quad \forall x \left[\varphi(x;z) \leftrightarrow \varphi(x;b)\right].$$

By compactness there is a formula $\vartheta(z) \in p$ such that

$$\vartheta(z) \quad \rightarrow \quad \forall x \left[\varphi(x;z) \leftrightarrow \varphi(x;b)\right].$$

Hence $\varphi(\mathfrak{U}\,;b)$ is defined by the formula $\exists z\,[\vartheta(z) \wedge \varphi(x\,;z)]$, which is a formula in $L(A)$ as required. □

9.22 Exercise Let $\varphi(x) \in L$. Prove that the following are equivalent

1. $\varphi(x)$ is equivalent to some $\psi(x) \in L_{\rm qf}$;
2. $\varphi(a) \leftrightarrow \varphi(fa)$ for every partial isomorphism $f : \mathfrak{U} \to \mathfrak{U}$ defined in a.

Use the result to prove Theorem 7.14 for T complete. □

9.23 Proposition *Let $p(x) \subseteq L(B)$. Then the following are equivalent*

1. *$p(x)$ is equivalent to some type $q(x) \subseteq L(A)$;*
2. *$p(\mathfrak{U})$ is invariant over A.*

We give two proofs of this theorem. The second one requires Proposition 9.24 below.

Proof 1⇒2 Obvious.

2⇒1 It suffices to show that for every formula $\psi(x) \in p(x)$ there is a formula $\varphi(x) \in L(A)$ such that $p(x) \to \varphi(x) \to \psi(x)$. Fix $\psi(x) \in p(x)$. By invariance, any $q(x) \in S(A)$ consistent with $p(x)$ implies $p(x)$, hence

$$p(x) \quad\to \bigvee_{q(x)\to\psi(x)} q(x) \quad\to\quad \psi(x)$$

where $q(x)$ above range over all types in $S_x(A)$. By compactness we can rewrite this equivalence as follows

$$p(x) \quad\to \bigvee_{\vartheta(x)\to\psi(x)} \vartheta(x) \quad\to\quad \psi(x)$$

where $\vartheta(x)$ ranges over all formulas in $L(A)$. Applying mutual normality (Proposition 9.18) to the first implication we obtain a finite number of formulas $\vartheta_i(x)$ such that

$$p(x) \quad\to\quad \bigvee_{i=1}^{n} \vartheta_i(x) \quad\to\quad \psi(x).$$

This completes the proof. □

The following easy proposition is very useful. Its proof is left to the reader. Note that it would not hold without saturation. For a counter example consider $\mathbb{R}$ as a structure in the language of strict orders and let $q(x,y) = {\rm tp}(0,1/A)$, where

$$A \quad=\quad \Big\{1+\frac{1}{n+1} \;:\; n\in\omega\Big\}.$$

By quantifier elimination, $0 \equiv_A 1$ but $\mathbb{R}, 1 \nvDash \exists y\, q(x,y)$. However, in any sufficiently saturated elementary extension of $\mathbb{R}$, we have $1 \nvDash \exists y\, q(x,y)$.

9.24 Proposition *Let $p(x\,;z) \subseteq L(A)$. Then $\exists z\, p(x\,;z)$ is equivalent to a type over A, namely to the type $\big\{\exists z\, \varphi(x\,;z) \;:\; \varphi(x\,;z)$ conjunction of formulas in $p(x\,;z)\big\}$. The theorem holds also when x and z have length κ.* □

As an application we give a second proof of the proposition above.

Second proof of Proposition 9.23 2⇒1 Write $p(x)$ as the type $q(x\,;b)$ for some

$q(x\,;z) \subseteq L$ and some $b \in \mathfrak{U}^{|z|}$. Let $s(z) = \text{tp}(b/A)$. By invariance and homogeneity the types $q(x\,;fb)$ for $f \in \text{Aut}(\mathfrak{U}/A)$ are all equivalent. Therefore

$$\begin{aligned} p(x) &\leftrightarrow \bigvee_{f \in \text{Aut}(\mathfrak{U}/A)} q(x\,;fb) \\ p(x) &\leftrightarrow \bigvee_{c \equiv_A b} q(x\,;c) \\ &\leftrightarrow \exists z\,\big[s(z) \wedge q(x\,;z)\big]. \end{aligned}$$

Hence, by Proposition 9.24, $p(x)$ is equivalent to a type over A. □

9.25 Exercise Let $p(x) \subseteq L(A)$, with $|x| < \omega$. Prove that if $p(\mathfrak{U})$ is infinite then it has cardinality κ. Show that this may not be true if x is an infinite tuple. □

9.26 Exercise Let $\varphi(x,y) \in L(\mathfrak{U})$. Prove that if the set $\{\varphi(a,\mathfrak{U}) \;:\; a \in \mathfrak{U}^{|x|}\}$ is infinite then it has cardinality κ. Does the claim remain true with a type $p(x,y) \subseteq L(A)$ for $\varphi(x,y)$? □

9.27 Exercise Let $\varphi(x\,;y) \in L(\mathfrak{U})$. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}\,;a_i) \subset \varphi(\mathfrak{U}\,;a_{i+1})$ for every $i < \omega$;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}\,;a_{i+1}) \subset \varphi(\mathfrak{U}\,;a_i)$ for every $i < \omega$. □

9.28 Exercise Prove that $\mathbb{R}, 1 \nvDash \exists y\, q(x,y)$, as claimed before Proposition 9.24. □

Chapter 10

Preservation theorems

In this chapter we present a few results dating from the 1950s that describe the relationship between syntactic and semantic properties of first-order formulas. These results characterize the classes of formulas that preserved under various sorts of morphisms. Criteria for quantifier-elimination follow from these theorems, see for instance the frequently used back-and-forth method of Corollary 10.13.

1 Lyndon-Robinson Lemma

We refer the reader to Exercise 9.22 for a simpler version of the main result in this section, the Lyndon-Robinson Lemma. In fact, under the additional assumption of completeness, Lemma 10.3 is essentially the same as the claim in Exercise 9.22.

However, we are interested in criteria for quantifier elimination, e.g. Corollary 10.12 below. We often need to prove quantifier elimination in order to prove completeness. Therefore any assumption of completeness would make criteria for quantifier elimination less applicable.

In this section T is a consistent theory without finite models and Δ is a set of formulas closed under renaming of variables. At a first reading the reader is encouraged to assume that Δ is L_{at}.

10.1 Definition *If $C \subseteq \{\forall, \exists, \neg, \vee, \wedge\}$ is a set of connectives, we write $C\Delta$ for the closure of Δ with respect to all connectives in C. We may write $\Delta^{\pm}$ for $\{\neg\}\Delta$.* □

Recall that Δ-morphism is a map $k : M \to N$ that preserves the truth of formulas in Δ. It is immediate that Δ-morphism are automatically $\{\wedge\vee\}\Delta$-morphisms. As Δ is closed under renaming of variables, $\{\exists\}\Delta$-morphism are $\{\exists\wedge\vee\}\Delta$-morphisms and similarly $\{\forall\}\Delta$-morphism are $\{\forall\wedge\vee\}\Delta$-morphisms.

Below we use the following proposition without further reference.

10.2 Proposition *Fix $M \vDash T$ and $b \in M^{|x|}$. Let $q(x) = \Delta\text{-}\operatorname{tp}_M(b)$. Then for every $\varphi(x) \in L$ the following are equivalent*

1. *$N \vDash \varphi(kb)$ for every $k : M \to N \vDash T$ that is a Δ-morphism defined in b;*

1′. *$N \vDash \varphi(c)$ for every $N \vDash T$ such that $N, c \Rrightarrow_\Delta M, b$.*

2. *$T \vdash q(x) \to \varphi(x)$.*

Proof 1⇔1′ In fact, the difference is just in the notation.

2⇒1 Immediate.

1⇒2 Negate 2, then there are $N \vDash T$ and $c \in N^{|x|}$ such that $q(c) \wedge \neg\varphi(c)$. Therefore the map $k : M \to N$, where $k = \{\langle b, c\rangle\}$, contradicts 1. □

The following is sometimes referred to as the Lyndon-Robinson Lemma.

10.3 Lemma *For every $\varphi(x) \in L$ the following are equivalent*

1. *$\varphi(x)$ is equivalent over T to a formula in $\{\wedge\vee\}\Delta$;*

2. *$\varphi(x)$ is preserved by Δ-morphisms between models of T.*

Proof 1⇒2 Immediate.

2⇒1 We claim that 2 implies

$$\# \qquad T \;\vdash\; \varphi(x) \;\leftrightarrow\; \bigvee\{\, p(x) \subseteq \Delta \;:\; T \vdash p(x) \to \varphi(x) \,\}.$$

The implication $\leftarrow$ is clear. To verify the implication $\to$, let $M \vDash T$ and let $b \in M^{|x|}$ be such that $M \vDash \varphi(b)$. From 2 it follows that $\varphi(x)$ satisfies 1 of Proposition 10.2. Therefore $T \vdash q(x) \to \varphi(x)$ for $q(x) = \Delta\text{-}\operatorname{tp}(b)$. Hence $q(x)$ is one of the types that occur in the disjuction in # which therefore is satisfied by b.

From # and compactness we obtain

$$T \;\vdash\; \varphi(x) \;\leftrightarrow\; \bigvee\{\, \psi(x) \in \{\wedge\}\Delta \;:\; T \vdash \psi(x) \to \varphi(x) \,\}.$$

Applying compactness again allows us to replace the infinite disjunction above with a finite one and prove 2. □

The following exercise is immediate but it is arguably the most important result of this chapter.

10.4 Exercise Prove Theorem 7.14, that is, that for every theory T the following are equivalent

1. T has elimination of quantifiers;

2. every partial isomorphism between models T is an elementary map. □

In the rest of these section we ...

10.5 Proposition *Let N be λ-saturated and let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \to N$ is a $\{\exists\}\Delta$-morphism;*

2. *for every $b \in M$ some $\{\exists\}\Delta$-morphism $h : M \to N$ defined in b extends k;*

3. *for every $\bar{b} \in M^{\omega}$ some Δ-morphism $h : M \to N$ defined in $\bar{b}$ extends k.*

Proof 1⇒2 Let a enumerate $\operatorname{dom} k$. Define $p(x\,;z) = \{\exists\}\Delta\text{-}\operatorname{tp}_M(b\,;a)$. By 1, $p(x\,;ka)$ is finitely consistent in N. By saturation there is a $c \in N$ that realizes $p(x\,;ka)$. Therefore, $h : M \to N$ where $h = k \cup \{\langle b, c\rangle\}$, witnesses 2.

2⇒3 Iterate ω-times the extension in 2.

3⇒1 Let a enumerate $\operatorname{dom} k$ and let $|z| = |a|$. Formulas in $\{\exists\}\Delta$ with free variables among z are of the form $\exists\bar{x}\,\varphi(\bar{x}\,;z)$ where $\varphi(\bar{x}\,;z)$ is in Δ and $\bar{x}$ is some fixed tuple of length ω. Assume $M \vDash \exists\bar{x}\,\varphi(\bar{x}\,;a)$ and let $\bar{b}$ be such that $M \vDash \varphi(\bar{b}\,;a)$. By 3, we can extend k to some Δ-morphism $h : M \to N$ defined in $\bar{b}$. Then $N \vDash \varphi(h\bar{b}\,;ha)$ and therefore $N \vDash \exists\bar{x}\,\varphi(\bar{x}\,;ka)$. □

Iterating the lemma above we obtain the following.

10.6 Corollary *Let N be λ-saturated and let $|M| \leq \lambda$. Let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \to N$ is a $\{\exists\}\Delta$-morphism;*
2. *$k : M \to N$ extends to an $\{\exists\}\Delta$-embedding;*
3. *$k : M \to N$ extends to an Δ-embedding.* □

The following theorem is often paraphrased as follows: a formula is existential if and only if (its truth) is preserved under extensions of structures.

10.7 Theorem *For every $\varphi(x) \in L$ the following are equivalent*

1. *$\varphi(x)$ is equivalent over T to a formula in $\{\exists\wedge\vee\}\Delta$;*
2. *$\varphi(x)$ is preserved by Δ-embedding between models of T.*

Proof 1⇒2 Immediate.

2⇒1 Negate 1. By the Lyndon-Robinson Lemma 10.3 there is a $\{\exists\}\Delta$-morphism $k : M \to N$ between models of T that does not preserve $\varphi(x)$. We can assume that N is λ-saturated for some sufficiently large λ. By Corollary 10.6 there is a Δ-embedding $h : M \hookrightarrow N$ that extends k and contradicts 2. □

A dual version of the results above is obtained replacing embeddings by epimorphisms, i.e. surjective (partial) homomorphisms, and $\{\exists\}$ by $\{\forall\}$. If Δ contains the formula $x = y$ and is closed under negation, then $k : M \to N$ is a Δ-morphism if and only is $k^{-1} : N \to M$ is a Δ-morphism. In this case the dual version follows from what proved above. Without these assumptions the results need a similar but independent proof.

10.8 Proposition *Let M be λ-saturated and let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \to N$ is a $\{\forall\}\Delta$-morphism;*
2. *for every $c \in N$ some $\{\forall\}\Delta$-morphism $h : M \to N$ extends k and $c \in \operatorname{img} h$;*
3. *for every $c \in N^\omega$ some Δ-morphism $h : M \to N$ extends k and $c \in (\operatorname{img} h)^\omega$.*

We write $\neg\Delta$ for the set containing the negation of the formulas in Δ. Warning: do not confuse $\neg\Delta$ with $\{\neg\}\Delta$.

Proof Left as an exercise for the reader. Hint: to prove implication 1⇒2 define $p(x,y) = \neg\{\forall\}\Delta\text{-}\operatorname{tp}_N(ka,c)$, where a is a tuple that enumerates $\operatorname{dom} k$. From 1 obtain that $p(a,y)$ is finitely consistent in M. Then proceed as in the proof of Proposition 10.5. □

10.9 Corollary *Let M be λ-saturated and let $|N| \leq \lambda$. Let $k : M \to N$ be a Δ-morphism of cardinality $< \lambda$. Then the following are equivalent*

1. *$k : M \to N$ is a $\{\forall\}\Delta$-morphism;*
2. *$k : M \to N$ extends to an $\{\forall\}\Delta$-epimorphism;*
3. *$k : M \to N$ extends to an Δ-epimorphism.* □

Finally we obtain the following.

10.10 Theorem *The following are equivalent*

1. $\varphi(x)$ *is equivalent to a formula in* $\{\forall\wedge\vee\}\Delta$;
2. *every* Δ*-epimorphism between models of* T *preserves* $\varphi(x)$. □

2 Quantifier elimination by back-and-forth

We say that T admits (or has) positive Δ-elimination of quantifiers if for every formula $\varphi(x)$ in $\{\exists\forall\wedge\vee\}\Delta$ there is a formula $\psi(x)$ in $\{\wedge\vee\}\Delta$ such that

$$T \quad \vdash \quad \varphi(x) \leftrightarrow \psi(x).$$

When Δ is closed under negation the attribute *positive* becomes irrelevant and will be omitted. When Δ is $L_{\mathrm{at}^{\pm}}$ or L_{qf}, we simply say that T admits elimination of quantifiers. This is by far the most common case.

Quantifier elimination is often used to prove that a theory is complete because it reduces it to something much simpler to prove. The following is an immediate consequence of the definition above with x replaced by the empty tuple.

10.11 Remark If T has elimination of quantifiers then the following are equivalent

1. T decides all quantifier free sentences;
2. T is complete.

Hence a theory with quantifier elimination is complete if it decides the characteristic, see Definition 5.26). □

The following is a consequence of Lemma 10.3.

10.12 Corollary *The following are equivalent*

1. T *has* Δ*-elimination of quantifiers;*
2. *every* Δ*-morphism between models of* T *is both a* $\{\exists\}\Delta$ *and a* $\{\forall\}\Delta$*-morphism.*

Proof 1⇒2 Immediate.

2⇒1 We prove by induction of syntax that Δ-morphism preserve the truth of all formulas in $\{\exists\forall\wedge\vee\}\Delta$, this suffices by Lemma 10.3. Induction for the connectives $\vee$ and $\wedge$ is trivial. So assume as induction hypothesis that the truth of $\varphi(x,y)$ is preserved. By Lemma 10.3 $\varphi(x,y)$ is equivalent to a formula in $\{\wedge\vee\}\Delta$, hence by 2 the truth of $\exists y\, \varphi(x,y)$ and $\forall y\, \varphi(x,y)$ is preserved. □

Condition 2 of the corollary above may be difficult to verify directly. The following corollary of Proposition 10.5 and 10.8 gives a back-and-forth condition with is easier to verify.

10.13 Corollary *Let* $|L| \leq \lambda$*. The following are equivalent*

1. T *has* Δ*-elimination of quantifiers;*
2. *for every finite* Δ*-morphism* $k : M \to N$ *between* λ*-saturated models of* T
   a. *for every* $b \in M$ *some* Δ*-morphism* $h : M \to N$ *extends* k *and* $b \in \operatorname{dom} h$*;*

b. *for every $c \in N$ some Δ-morphism $h : M \to N$ extends k and $c \in \operatorname{img} h$.* □

Note that when Δ contains the formula $x = y$ and is closed under negation, then $k : M \to N$ is a Δ-morphism if and only if $k^{-1} : N \to M$ is a Δ-morphism. In this case a and b are equivalent.

10.14 Exercise Let T be a complete theory without finite models in a language that consists only of unary predicates. Prove that T has elimination of quantifiers. □

10.15 Exercise Let T be the theory of discrete linear orders, that is, T extends the theory of linear orders $T_{\rm lo}$ (see Section 6.1) with the following two of axioms with the following two of axioms

dis↑. $\exists z \left[x < z \ \wedge \ \neg\exists y \ x < y < z\right]$;

dis↓. $\exists z \left[z < x \ \wedge \ \neg\exists y \ z < y < x\right]$.

Let Δ be the set of formulas that contains (all alphabetic variants of) the formulas $x <_n y := \exists^{\geq n} z \ (x{<}z{<}y)$ and their negations, for all positive integers n. Prove that the theory of discrete linear orders has Δ-elimination of quantifiers. Prove that the structure $\mathbb{Q} \times \mathbb{Z}$ ordered with the lexicographic order

$$(a_1, a_2) < (b_1, b_2) \quad \Leftrightarrow \quad a_1 < b_1 \ \text{ or } \ (a_1 = b_1 \ \text{ e } \ a_2 < b_2)$$

is a saturated model of T. □

10.16 Exercise Let T be a consistent theory. Suppose that all completions of T are of the form $T \cup S$ for some set S of quantifier-free sentences. Prove that if all completions of T have elimination of quantifiers, so does T. Show that this fails when the completions of T have arbitrary complexity.

Note. Thought the claim follows immediately from Corollary 10.12, a direct proof by compactness is also instructive. Prove that for every formula $\varphi(x)$ there are some quantifier-free sentences σ_i and quantifier-free formulas $\psi_i(x)$ such that

$$\sigma_i \vdash \varphi(x) \leftrightarrow \psi_i(x), \qquad T \vdash \bigvee_{i=1}^{n} \sigma_i, \qquad \text{and} \qquad \sigma_i \vdash \neg\sigma_j \text{ for } i \neq j.$$

For a counter example consider the empty theory in the language with a single unary predicate. □

3 Model-completeness

We say that T is model-complete if every embedding $h : M \hookrightarrow N$ between models of T is an elementary embedding. The terminology, introduced by Abraham Robinson, is inspired by the fact that T is model-complete if and only if $T \cup \operatorname{Diag}(M)$ is a complete theory, in the language $L(M)$, for every $M \vDash T$.

To stress positivity in the next proposition, we generalize the definition as follows. We say that T is Δ-model-complete if every Δ-embedding $h : M \hookrightarrow N$ between models of T is a $\{\forall\exists\}\Delta$-embedding.

Model-completeness is equivalent to a property akin to quantifier elimination.

10.17 Proposition *The following are equivalent*

1. *T is Δ-model-complete;*
2. *T has $\{\exists\}\Delta$-elimination of quantifiers.*

Proof 1⇒2 By 1, every formula $\{\forall\exists\wedge\vee\}\Delta$ is preserved by Δ-embeddings therefore, by Theorem 10.7, it is equivalent to a formula in $\{\exists\wedge\vee\}\Delta$.

2⇒1 Clear, because Δ-embeddings preserve formulas in $\{\exists\wedge\vee\}\Delta$. □

The theory of discrete linear orders defined in Exercise 10.15 is an example of a model-complete theory without elimination of quantifiers.

The difference between quantifier elimination and model-completeness subtle. It boils down to models of T having or not the amalgamation property.

10.18 Proposition *Assume T is model-complete. Let $\mathcal{M}$ be the category that consists of models of T and partial isomorphisms. Then the following are equivalent*

1. *$\mathcal{M}$ has the amalgamation property;*
2. *T has elimination of quantifiers.*

Proof 1⇒2 By Proposition 7.26 every partial morphism $k : M \to N$ extends to an embedding $g : M \hookrightarrow N'$ which, by model-completeness, is an elementary embedding. Model-completeness also implies that $N \preceq N'$. Hence $k : M \to N$ is an elementary map. This proves 2.

2⇒1 If all morphisms are elementary maps, amalgamations follows from Proposition 7.29. □

Note however that the models of a model-complete theory T do have amalgamation when the proper notion of morphism is chosen.

Let $\mathcal{M}'$ be the category that consists of models of T and the maps $k : M \to N$ such that there is a partial isomorphism $h : M' \to N'$ with

1. $k \subseteq h$; $M \preceq M'$; $N \preceq N'$;
2. $\operatorname{dom} h$ contains a substructure of M' that models T (equivalently $\operatorname{img} h$ and N').

Moreover, we add as morphisms the maps that are obtained from those above by composition. It is clear that if T is model-complete then the morphisms of $\mathcal{M}'$ are exactly the elementary maps. In this case $\mathcal{M}'$ has amalgamation. Vice versa if $\mathcal{M}'$ has amalgamation, the theory of rich models is model-complete.

10.19 Exercise Prove that $\mathcal{M}'$ satisfies finite character of morphisms, c2 of Definition 7.1. □

Chapter 11

Geometry and dimension

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathfrak{U}$ of inaccessible cardinality κ larger than $|L|$. The notation and implicit assumptions are as in Section 9.3.

1 Algebraic and definable elements

Let $a \in \mathfrak{U}$ and let $\mathcal{A} \subseteq \mathfrak{U}$ be some set of parameters (of arbitrary cardinality). We say that a is algebraic over $\mathcal{A}$ if $\varphi(a) \wedge \exists^{=k} x\, \varphi(x)$ holds for a formula $\varphi(x) \in L(\mathcal{A})$ and some positive integer k. In particular, when $k = 1$ we say that a is definable over $\mathcal{A}$. We write $\operatorname{acl}(\mathcal{A})$ for the algebraic closure of $\mathcal{A}$, that is, the set of all the elements that are algebraic over $\mathcal{A}$. If $\mathcal{A} = \operatorname{acl}(\mathcal{A})$, we say that $\mathcal{A}$ is algebraically closed. The definable closure of $\mathcal{A}$ is defined similarly and is denoted by $\operatorname{dcl}(\mathcal{A})$.

Let x be a finite tuple of variables. Formulas $\varphi(x) \in L(A)$, or types $p(x) \subseteq L(A)$, with finitely many solutions are called algebraic.

11.1 Proposition *For every $A \subseteq \mathfrak{U}$ and every type $p(x) \subseteq L(A)$, where $|x| < \omega$, the following are equivalent*

1. $\exists^{\leq n} x\, p(x)$;
2. $\exists^{\leq n} x\, \varphi(x)$ *for some $\varphi(x)$ which is a conjunction of formulas in $p(x)$.*

Proof The non trivial implication is 1⇒2. Let $\{a_1, \ldots, a_n\}$ be all the solutions of $p(x)$. Then

$$p(x) \;\leftrightarrow\; \bigvee_{i=1}^{n} a_i = x$$

Then 2 follows by compactness (cfr. Remark 9.19.b). □

11.2 Exercise For every $a \in \mathfrak{U}^n$ and $A \subseteq \mathfrak{U}$, the following are equivalent

1. a is solution of some algebraic formula $\varphi(x) \in L(A)$;
2. $a = a_1, \ldots, a_n$ for some $a_1, \ldots, a_n \in \operatorname{acl}(A)$. □

11.3 Theorem *For every $A \subseteq \mathfrak{U}$ and every $a \in \mathfrak{U}$ the following are equivalent*

1. $a \in \operatorname{dcl}(A)$;
2. $\mathcal{O}(a/A) = \{a\}$.

Proof Implication 1⇒2 is obvious. As for 2⇒1, recall that $\mathcal{O}(a/A)$ is the set of realizations of $\operatorname{tp}(a/A)$, then the theorem follows from Proposition 11.1. □

11.4 Theorem *For every $A \subseteq \mathfrak{U}$ and every $a \in \mathfrak{U}$ the following are equivalent*

1 $a \in \operatorname{acl}(A)$;

2 $\mathcal{O}(a/A)$ *is finite;*

3 a *belongs to every model containing* A.

Proof 1⇔2. This is proved as in Theorem 11.3.

1⇒3. Assume 1. Then there is a formula $\varphi(x) \subseteq L(A)$ such that $\varphi(a) \wedge \exists^{=k} x\, \varphi(x)$ for some k. By elementarity $\exists^{=k} x\, \varphi(x)$ holds in every model M containing A. Again by elementarity, the k solutions of $\varphi(x)$ in M are solutions in $\mathcal{U}$, therefore a is one of these.

3⇒2. Assume $\mathcal{O}(a/A)$ is infinite and fix any model M containing A. By Exercise 9.25, $\mathcal{O}(a/A)$ has cardinality κ, hence $\mathcal{O}(a/A) \nsubseteq M$. Pick any $f \in \operatorname{Aut}(\mathcal{U}/A)$ such that $fa \notin M$. Then $a \notin f^{-1}[M]$, so $f^{-1}[M]$ is a model that contradicts 3. □

11.5 Corollary *For every* $A \subseteq \mathcal{U}$ *and every* $a \in \mathcal{U}$

1 *if* $a \in \operatorname{acl} A$ *then* $a \in \operatorname{acl} B$ *for some finite* $B \subseteq A$; *finite character*

2 $A \subseteq \operatorname{acl} A$; *extensivity*

3 *if* $A \subseteq B$ *then* $\operatorname{acl} A \subseteq \operatorname{acl} B$; *monotonicity*

4 $\operatorname{acl} A = \operatorname{acl}(\operatorname{acl} A)$; *idempotency*

5 $\operatorname{acl} A = \bigcap_{A \subseteq M} M$.

Properties 1-4 say that acl(-) is a closure operator with finite character.

Proof Properties 1-3 are obvious, 4 follows from 5 which in turn follows from Theorem 11.4. □

11.6 Proposition *If* $f \in \operatorname{Aut}(\mathcal{U})$ *then* $f\big[\operatorname{acl}(A)\big] = \operatorname{acl}\big(f[A]\big)$ *for every* $A \subseteq \mathcal{U}$.

Proof We prove $f\big[\operatorname{acl}(A)\big] \subseteq \operatorname{acl}\big(f[A]\big)$. Fix $a \in \operatorname{acl}(A)$ and let $\varphi(x\,;z) \in L$ and $b \in A^{|z|}$ be such that $\varphi(x\,;b)$ is algebraic formula satisfied by a. By elementarity, $\varphi(x\,;fb)$ is algebraic and satisfied by fa. Therefore fa is algebraic over $f[A]$, which proves the inclusion.

The converse inclusion is obtained by substituting f^{-1} for f and $f[A]$ for A. □

11.7 Exercise Let $\varphi(z) \in L(A)$ be a consistent formula. Prove that, if $a \in \operatorname{acl}(A, b)$ for every $b \vDash \varphi(z)$, then $a \in \operatorname{acl}(A)$. Prove the same claim with a type $p(z) \subseteq L(A)$ for $\varphi(z)$. □

11.8 Exercise Let $a \in \mathcal{U} \smallsetminus \operatorname{acl} \varnothing$. Prove that $\mathcal{U}$ is isomorphic to some $\mathcal{V} \preceq \mathcal{U}$ such that $a \notin \mathcal{V}$. Hint: let $\bar{c}$ be an enumeration of $\mathcal{U}$ and let $p(\bar{u}) = \operatorname{tp}(\bar{c})$ prove that $p(\bar{u}) \cup \big\{u_i \neq a : i < |\bar{u}|\big\}$ is realized in $\mathcal{U}$ and that any realization yields the required substructure of $\mathcal{U}$. □

11.9 Exercise Let C be a finite set. Prove that if $C \cap M \neq \varnothing$ for every model M containing A, then $C \cap \operatorname{acl}(A) \neq \varnothing$. Hint: by induction on the cardinality of C. Suppose there is a $c \in C \smallsetminus \operatorname{acl}(A)$, then there is $\mathcal{V} \simeq \mathcal{U}$ such that $A \subseteq \mathcal{V} \preceq \mathcal{U}$ and $c \notin \mathcal{V}$, see Exercise 11.8. Apply the induction hypothesis to $C' = C \cap \mathcal{V}$ with $\mathcal{V}$ for $\mathcal{U}$. □

11.10 Exercise Prove that for every $A \subseteq N$ there is an M such that $\operatorname{acl} A = M \cap N$. Hint: add the requirement $\operatorname{acl}(A_i) \cap N \subseteq \operatorname{acl}(A)$ to the construction used to prove the downward Löwenheim-Skolem theorem. You need to prove that every consistent $\varphi(x) \in L(A_i)$ has a solution a such that $\operatorname{acl}(A_i, a) \cap N \subseteq \operatorname{acl}(A)$. The required a has to realize the type

$$\{\varphi(x)\} \cup \Big\{\neg[\psi(b,x) \wedge \exists^{\leq n} y\, \psi(y,x)] \ : \ b \in N \smallsetminus \operatorname{acl}(A),\ \ \psi(y,x) \in L(A_i),\ \ n < \omega\Big\}$$

whose consistency need to be verified.

11.11 Exercise Prove that for every $A \subseteq N$ there is an automorphism $f \in \operatorname{Aut}(\mathcal{U}/A)$ such that $\operatorname{acl} A = f[N] \cap N$. (This is a stronger version of the claim in Exercise 11.10.) Hint: let $\bar{c}$ be an enumeration of N. Let $p(\bar{x}) = \operatorname{tp}(\bar{c}/A)$. Consider the type

$$p(x) \ \cup \ \Big\{\neg[\psi(b,\bar{x}) \wedge \exists^{\leq n} y\, \psi(y,\bar{x})] \ : \ b \in N \smallsetminus \operatorname{acl}(A),\ \ \psi(y,x) \in L(A),\ \ n < \omega\Big\}$$

Any $\bar{a} \vDash p(\bar{x})$ enumerates a model A-isomorphic to N. □

11.12 Exercise Let $\varphi(x) \in L(\mathcal{U})$ and fix an arbitrary set A. Prove that the following are equivalent

1. there is some model M containing A and such that $M \cap \varphi(\mathcal{U}) = \varnothing$;
2. there is no consistent formula $\psi(z_1, \dots, z_n) \in L(A)$ such that

$$\psi(z_1, \dots, z_n) \to \bigvee_{i=1}^{n} \varphi(z_i).$$

Hint: let $\bar{c}$ be an enumeration of $N^{|x|}$, where N is any model containing A. Let $p(\bar{z}) = \operatorname{tp}(\bar{c}/A)$. Prove that 2 implies the consistency of $p(\bar{z}) \cup \big\{\neg\varphi(z_i) \ : \ i < |\bar{z}|\big\}$ and deduce the existence of the required M. □

2 Strongly minimal theories

Finite and cofinite sets are always (trivially) definable in every structure. We say that M is a minimal structure if all its definable subsets of arity one are finite or cofinite. Unfortunately, this notion is not elementary, i.e. it is not a property of $\operatorname{Th}(M)$. For instance $\mathbb{N}$ with only the order relation in the language is a minimal structure but none of its elementary extensions is. Hence the following definition: we say that M is a strongly minimal structure if it is minimal and all its elementary extensions are minimal.

We say that T, a consistent theory without finite models, is strongly minimal if for every formula $\varphi(x\,;z) \in L$, where x has arity one, there is an $n \in \omega$ tale che

$$T \ \vdash \ \exists^{\leq n} x\ \varphi(x\,;z) \ \vee \ \exists^{\leq n} x\ \neg\varphi(x\,;z).$$

We show that the semantic notion matches the syntactic one.

11.13 Proposition *The following are equivalent*

1. $\operatorname{Th}(M)$ *is a strongly minimal theory;*
2. M *is a strongly minimal structure;*
3. M *has an elementary extension which is minimal and ω-saturated.*

Proof Implications 1⇒2⇒3 are immediate, we prove 3⇒1. Let $\varphi(x\,;z) \in L$ and let N be the elementary extension given by 3. Let $p(z) \subseteq L$ be the following type

$$p(z) \;=\; \Big\{\exists^{>n}x\; \varphi(x\,;z) \;\wedge\; \exists^{>n}x\; \neg\varphi(x\,;z) \;\;:\;\; n\in\omega\Big\}.$$

As N is minimal, $N \nvDash \exists z\, p(z)$. By ω-saturation $p(z)$ is not finitely consistent in M. Hence, for some n

$$M \;\vDash\; \forall z\;\Big[\exists^{\leq n}x\; \varphi(x\,;z) \;\vee\; \exists^{\leq n}x\; \neg\varphi(x\,;z)\Big].$$

which proves that $\mathrm{Th}(M)$ is strongly minimal. □

By quantifier elimination, $T_{\rm acf}$ and $T_{\rm dag}$ are strongly minimal theories.

11.14 Exercise Let T be a complete theory without finite models. Prove that the following are equivalent

1. M is minimal;
2. $a \equiv_M b$ for every $a, b \in \mathfrak{U} \smallsetminus M$. □

3 Independence and dimension

Thoughout this section we assume that T is a complete strongly minimal theory.

When $a \notin \operatorname{acl} B$ we say that a is algebraically independent from B. We say that B is an algebraically independent set if every $a \in B$ is independent from $B \smallsetminus \{a\}$. Below we shall abbreviate $B \cup \{a\}$ by B, a and $B \smallsetminus \{a\}$ by $B \smallsetminus a$.

The following is a pivotal property of independence that holds in strongly minimal structures. It is called symmetry or exchange principle. For every B and every pair of elements $a, b \in \mathfrak{U} \smallsetminus \operatorname{acl} B$

$$b \in \operatorname{acl}(B, a) \;\;\Leftrightarrow\;\; a \in \operatorname{acl}(B, b)$$

Note that when T is the theory of vector spaces (over any fixed field) this principle is the so called Steinitz exchange lemma.

11.15 Theorem *(T strongly minimal.) Independence is symmetric. That is, if $a, b \notin \operatorname{acl} B$ then $b \in \operatorname{acl}(B, a) \Leftrightarrow a \in \operatorname{acl}(B, b)$*

Proof Suppose $b \notin \operatorname{acl}(B, a)$ and $a \in \operatorname{acl}(B, b)$. We prove that $a \in \operatorname{acl} B$. Fix a formula $\varphi(x, y) \in L(B)$ such that $\varphi(x, b)$ witnesses $a \in \operatorname{acl}(B, b)$, i.e. for some n

$$\varphi(a, b) \;\;\wedge\;\; \exists^{\leq n}x\; \varphi(x, b).$$

As $b \notin \operatorname{acl}(B, a)$, the formula

$$\psi(a, y) \;\;=\;\; \varphi(a, y) \;\;\wedge\;\; \exists^{\leq n}x\; \varphi(x, y).$$

is not algebraic. Therefore, by strong minimality, $\psi(a, y)$ has cofinitely many solutions. Hence every model containing B contains a solution of $\psi(a, y)$. As a is algebraic in any of these solutions, a belongs to every model containing B. Therefore, $a \in \operatorname{acl} B$ by Theorem 11.4. □

We say that $B \subseteq C$ is a basis of C if B is an independent set and $C \subseteq \operatorname{acl} B$. The following theorem proves that all bases have the same cardinality, which we call the dimension of C and denote by $\dim C$. First we need the following lemma.

11.16 Lemma *(T strongly minimal.) If B is an independent set and $a \notin \operatorname{acl} B$ then B, a is also an independent set.*

Proof Suppose B, a is not independent and that $a \notin \operatorname{acl} B$. Then $b \in \operatorname{acl}(B \smallsetminus b, a)$ for some $b \in B$. As $a, b \notin \operatorname{acl}(B \smallsetminus b)$, from symmetry we obtain $a \in \operatorname{acl}(B \smallsetminus b, b) = \operatorname{acl} B$. Hence B is not an independent set. □

11.17 Corollary *(T strongly minimal.) For every $B \subseteq C$ the following are equivalent*

1. *B is a basis of C.*
2. *B is a maximally independent subset of C.* □

Finally we prove the main theorem about basis.

11.18 Theorem *(T strongly minimal.) Fix some arbitrary set C, then*

1 *every independent set $B \subseteq C$ can be extended to a basis of C;*

2 *all bases of C have the same cardinality.*

Proof By the finite character of algebraic closure, the independent set form an inductive class. Apply Zorn lemma to obtain a maximally independent subset of C containing B. By Corollary 11.17 this set is a basis of C. This proves 1.

As for 2, assume for a contradiction that $A, B \subseteq C$ are two bases of C and that $|A| < |B|$. First consider the case when B is infinite. For each $a \in A$ fix a finite set $D_a \subseteq B$ such that $a \in \operatorname{acl}(D_a)$. Let

$$D \;=\; \bigcup_{a \in A} D_a.$$

Then $A \subseteq \operatorname{acl} D$ and $|D| < |B|$. By transitivity, $C \subseteq \operatorname{acl} D$ which contradicts the independence of B.

Now we suppose that B is finite. As $|A| < |B|$, there is a $b \in B \smallsetminus A$. As $b \in \operatorname{acl}(A)$ but $b \notin \operatorname{acl}(B \smallsetminus b)$ then $A \nsubseteq \operatorname{acl}(B \smallsetminus b)$. Then there is an $a \in A$ such that $a \notin \operatorname{acl}(B \smallsetminus b)$. By Lemma 11.16 $B \smallsetminus b, a$ is an independent set that, by what proved above, is contained in a base B'. As $|A| < |B'|$, we can iterate the procedure. After $|A| + 1$ iterations we reach a contradiction. □

11.19 Proposition *(T strongly minimal.) Let k be an elementary map. Then $k \cup \{\langle b, c \rangle\}$ is also an elementary map for every $b \notin \operatorname{acl}(\operatorname{dom} k)$ and $c \notin \operatorname{acl}(\operatorname{img} k)$.*

Proof Let a be an enumeration of $\operatorname{dom} k$. We need to show that $\varphi(b\,; a) \leftrightarrow \varphi(c\,; ka)$ holds for every $\varphi(x\,; z) \in L$. As k is elementary, the formulas $\varphi(x\,; a)$ and $\varphi(x\,; ka)$ are either both algebraic or both co-algebraic. As $b \notin \operatorname{acl}(a)$ and $c \notin \operatorname{acl}(ka)$, they are both false or both true respectively. So the proposition follows. □

11.20 Corollary *(T strongly minimal.) Every bijection between independent sets is an elementary map.* □

Finally we show that dimension classifies models of T.

11.21 Theorem *(T strongly minimal.) Models of T with the same dimension are isomorphic.*

Proof Let A e B be bases of M and N respectively. By Corollary 11.20, any bijectiion between A and B is an elementary map. By Proposition 11.6, it extends to the required isomorphism between $\operatorname{acl} A = M$ and $\operatorname{acl} B = N$. □

11.22 Corollary *(T strongly minimal.) Let $|L| < \lambda$. Then T is λ-categorical.*

Proof Let M have cardinality λ. Let $B \subseteq M$ be a base. Then $\lambda = |M| = |\operatorname{acl} B| = |L(B)| = \max\{|L|, |B|\}$. If $|L| < \lambda$, then $\lambda = |B|$. Therefore all models of cardinality λ are isomorphic because they all have the same dimension λ. □

11.23 Proposition *(T strongly minimal.) For every model N of cardinality $\geq |L|$ the following are equivalent*

1. *N is saturated;*
2. $\dim N = |N|$.

Proof 2⇒1. Assume 2 and let $k : M \to N$ be an elementary map of cardinality $< |N|$ and let $b \in M$. We want an extension of k defined in b. If $b \in \operatorname{acl}(\operatorname{dom} k)$ then the required extension exists by Proposition 11.6. Otherwise, we pick any element $c \in N \smallsetminus \operatorname{acl}(\operatorname{img} k)$. Such an element exists as $|k| < \dim N = |N|$. Then $k \cup \{\langle b, c\rangle\}$ is the required extension by Proposition 11.19.

1⇒2. If $B \subseteq N$ is a basis of N the following type is not realized in N

$$p(x) \quad = \quad \Big\{\neg\varphi(x) \ : \ \varphi(x) \in L(B) \text{ is algebraic} \Big\}$$

Therefore, if N is saturated, $|B| = |N|$. □

11.24 Exercise (T strongly minimal.) Prove that every infinite algebraically closed set is a model. □

11.25 Exercise (T strongly minimal, L countable.) Prove that every model is homogeneous. □

11.26 Exercise (T strongly minimal.) Prove that if $\dim N = \dim M + 1$ then there is no model K such that $M \prec K \prec N$. □

Chapter 12

Countable models

In this chapter L is a fix signature, T a complete theory without finite models, and $\mathcal{U}$ is a saturated model of inaccessible cardinality κ larger than $|L|$. We make no blanket assumption on the cardinality of L, but the main theorems require L to be countable. The notation and implicit assumptions are as in Section 9.3.

1 The omitting types theorem

We say that the formula $\varphi(x)$ isolates the type $p(x)$ when $\varphi(x)$ is consistent and $\varphi(x) \rightarrow p(x)$. When Δ is a set of formulas, we say that Δ isolates $p(x)$ if some formula in Δ does. When $\Delta = L_x(A)$, we say that A isolates $p(x)$ or, when A is clear, that $p(x)$ is isolated. We say that a model M omits $p(x)$ if $p(x)$ is not realized in M.

Observe that if $p(x) \subseteq L(M)$ then M realizes $p(x)$ if and only if M isolates $p(x)$. Therefore if A isolates $p(x)$, then every model containing A realizes $p(x)$. Below we prove that the converse holds when L and A are countable. This is a famous classical theorem that is called the *omitting types theorem* because it is proved by constructing a model M that omits a given non-isolated type $p(x)$.

The core of the argument lies in the following lemma.

12.1 Lemma *Assume $L(A)$ is countable. Let $p(x) \subseteq L(A)$ and suppose that A does not isolate $p(x)$. Then, if $\psi(z) \in L(A)$ is consistent, $\psi(z)$ has a solution a such that A, a does not isolate $p(x)$.*

Proof We construct a sequence of formulas $\langle \psi_i(z) : i < \omega \rangle$ such that any realization a of the type $\{\psi_i(z) : i < \omega\}$ is the required solution of $\psi(z)$.

Let $\langle \xi_i(x\,;z) : i < \omega \rangle$ be an enumeration of $L_{x\,;z}(A)$. Take $\psi_0(z) = \psi(z)$. At stage $i+1$:

- ▷ if $\xi_i(x\,;z) \wedge \psi_i(z)$ is inconsistent, let $\psi_{i+1}(z) = \psi_i(z)$;
- ▷ otherwise, pick some $\varphi(x) \in p$ such that $\psi_i(z) \wedge \exists x\,[\xi_i(x\,;z) \wedge \neg\varphi(x)]$ is consistent and let this conjunction be $\psi_{i+1}(z)$.

This guarantees that $\xi_i(x\,;a)$ for any $a \vDash \psi_{i+1}(z)$ does not isolate $p(x)$. The proof is complete if we can show that it is always possible to find the formula $\varphi(x)$ required above.

Suppose for a contradiction that no formula makes $\psi_{i+1}(z)$ consistent, that is,

$$\xi_i(x\,;z) \wedge \psi_i(z) \quad \rightarrow \quad \varphi(x)$$

for every $\varphi(x) \in p$. This immediately implies that

$$\exists z\,\big[\xi_i(x\,;z) \wedge \psi_i(z)\big] \quad \rightarrow \quad p(x),$$

that is, $p(x)$ is isolated by a formula in $L_x(A)$, which contradicts our assumptions. □

12.2 Theorem (Omitting types) *Assume $L(A)$ is countable. Then for every consistent type $p(x) \subseteq L(A)$ the following are equivalent*

1. *all models containing A realize $p(x)$;*
2. *A isolates $p(x)$.*

Proof The implication 2⇒1 is clear. We prove 1⇒2. Assume that A does not isolate $p(x)$. The model M is the union of a chain $\langle A_i : i < \omega\rangle$ of countable subsets of $\mathcal{U}$ where $A_0 = A$. Along the construction we require that A_i does not isolate $p(x)$. At the end, M will not isolate $p(x)$. Since M is a model, this this is equivalent to M omitting $p(x)$.

We proceed as in the proof of the downward Löwenheim-Skolem theorem. Assume that A_i does not isolate $p(x)$. With the notation of Proof 2.40, at stage $i = \pi(j,k)$ apply Lemma 12.1 to find a solution a of $\varphi_k(x)$ such that $A_{i+1} = A_i, a$ does not isolates $p(x)$. □

Gerald Sacks once famously remarked: *Any fool can realize a type but it takes a model theorist to omit one*. However, the diagonalization method in the proof of Lemma 12.1 lean towards descriptive set theory. (We invite the interested reader to compare this lemma with the Kuratowski-Ulam theorem.)

12.3 Example The following example shows that in the omitting types theorem we cannot drop the assumption that $L(A)$ is countable. Let F be the set of all bijections between two uncountable sets, X and Y. Let M be the model whose domain is domain the disjoint union of F, X and Y. The language has a ternary relation symbol for $f(x) = y$ and unary relation symbols for F, X, and Y. Let $\mathcal{U}$ be a saturated elementary extension of M. Then $\mathcal{U}$ is partitioned into three definable sets $\mathcal{U}_F$, $\mathcal{U}_X$ and $\mathcal{U}_Y$. Each element of $\mathcal{U}_F$ defines a bijection between $\mathcal{U}_X$ and $\mathcal{U}_Y$.

Note that for any two elements $a, b \in \mathcal{U}_Y$, there is an automorphism of $\mathcal{U}$ that fixes $\mathcal{U}_X \cup \mathcal{U}_Y \smallsetminus \{a, b\}$ and swaps a and b.

Now, let $Y_1 \subseteq \mathcal{U}_Y$ be countable. Let $c \in \mathcal{U}_Y \smallsetminus Y_1$ and let $p(y) = \mathrm{tp}(c/X, Y_1)$. We claim that $p(y)$ is realized in every model containing X, Y_1. In fact, by the remark above $p(\mathcal{U}) = \mathcal{U}_Y \smallsetminus Y_1$. But every model containing X also contains uncountably many elements of $\mathcal{U}_Y$, hence it contains a conjugate of c which therefore realizes $p(y)$. We also claim that $p(y)$ is not isolated. Suppose for a contradiction there is a consistent formula $\varphi(y)$ such that $\varphi(y) \to p(y)$. Then $\varphi(y)$ has a solution in $\mathcal{U}_Y \smallsetminus Y_1$. By the remark above, this implies that $\varphi(\mathcal{U})$ is a cofinite subset of $\mathcal{U}_Y$ and this contradicts $\varphi(\mathcal{U}) \subseteq p(\mathcal{U})$. □

12.4 Exercise Let $p(x) \subseteq L(B)$ and $p_n(x) \subseteq L(A)$, for $n < \omega$, be consistent types such that

$$p(x) \quad\to\quad \bigvee_{n<\omega} p_n(x)$$

Prove that there is an $n < \omega$ and a formula $\varphi(x) \in L(A)$ consistent with $p(x)$ such that

$$p(x) \wedge \varphi(x) \quad\to\quad p_n(x).$$

□

2 Prime and atomic models

We say that M is prime over A if $A \subseteq M$ and for every N containing A there is an elementary embedding $h : M \to N$ that fixes A. When A is empty we simply say that M is prime.

There is no syntactic analogue of primeness. The closest notion, which works well when everything is countable, is atomicity. For $a \in \mathcal{U}^{|x|}$ we say that$a$ is isolated over A if the type $p(x) = \mathrm{tp}(a/A)$ is isolated. Note that this equivalent to claiming that a is an isolated point in $\mathcal{U}^{|x|}$ w.r.t. the A-topology defined in Section 9.3. We say that M is atomic over A if $A \subseteq M$ and every $a \in M^{<\omega}$ is isolated over A. When A is empty we say that M is atomic.

12.5 Proposition *Let a and b be finite tuples. Then the following are equivalent*

1. *A isolates b, a;*
2. *A, a isolates b and A isolates a.*

Proof Let $p(x,z) = \mathrm{tp}(b,a/A)$. Then $p(x,a) = \mathrm{tp}(b/A,a)$. Note also that $\exists x\, p(x,z) = \mathrm{tp}(a/A)$.

1⇒2 Let $\varphi(x,z) \in p$ be such that $\varphi(x,z) \to p(x,z)$. Then $\varphi(x,a) \to p(x,a)$ and $\exists x\, \varphi(x,z) \to \exists x\, p(x,z)$. Therefore 2 holds by the remark above.

2⇒1 Fix $\varphi(x\,;z), \psi(z) \in L(A)$ such that $\varphi(x\,;a)$ isolates $p(x\,;a)$ and $\psi(z)$ isolates $\exists x\, p(x\,;z)$. Let $\xi(x\,;z) \in p$ be arbitrary. As $\varphi(x\,;a) \to \xi(x\,;a)$, the formula $\forall x\, [\varphi(x\,;z) \to \xi(x\,;z)]$ belongs to $\mathrm{tp}(a/A)$ which, as noted above, coincides with $\exists x\, p(x,z)$. Hence $\psi(z) \to \forall x\, [\varphi(x\,;z) \to \xi(x\,;z)]$. As this holds for all $\xi(x\,;z) \in p$, we conclude that $\psi(z) \wedge \varphi(x\,;z)$ isolates $p(x,z)$. □

The straightforward direction of the proposition above yields the following useful proposition.

12.6 Proposition *If M is atomic over A then M is atomic over A, a for every finite $a \in M^{<\omega}$.*

Proof Let $b \in M^{|x|}$ be a finite tuple. Then A isolates b, a hence A, a isolates b. □

12.7 Proposition *Let $k : M \to N$ be an elementary map and suppose that M is atomic over $\operatorname{dom} k$. Then for every $b \in M$ there is a $c \in N$ such that $k \cup \{\langle b, c\rangle\} : M \to N$ is elementary.*

Proof Let $p(x\,;z) = \mathrm{tp}(b\,;a)$ where a is an enumeration of $\operatorname{dom} k$. Let $\varphi(x\,;z) \in L$ be such that $\varphi(x\,;a) \to p(x\,;a)$. Note that, by elementarity, $\varphi(x\,;ka) \to p(x\,;ka)$. Hence the required c is any solution of $\varphi(x\,;ka)$ in N. □

A limiting assumption in Proposition 12.6 is that a need to be finite. Therefore the following proposition is restricted to countable models.

12.8 Proposition *Any two is countable models atomic over A are isomorphic.*

Proof Easy, using Propositions 12.6 and 12.7 and back-and-forth. □

12.9 Proposition *Assume $L(A)$ is countable. Then for every model M the following are equivalent*

1. *M is countable and atomic over A;*

2. *M is prime over A.*

Proof 1⇒2 By Propositions 12.6 and 12.7.

2⇒1 Some countable model containing A exists, as M embeds in it, M has also to be countable. Now we prove that M is atomic over A. Suppose for a contradiction that there is some $b \in M^{<\omega}$ such that $p(x) = \mathrm{tp}(b/A)$ is not isolated. By the omitting types theorem there is a model N containing A that omits $p(x)$. Then there cannot be any A-elementary embedding of M into N. □

12.10 Proposition *Assume $L(A)$ is countable. Then the following are equivalent*

1. *there are models atomic over A;*

2. *for every $|z| < \omega$, every consistent $\varphi(z) \in L(A)$ has a solution that is isolated over A.*

Note that 2 says that in $\mathcal{U}^{|z|}$ isolated points are dense w.r.t. the topology defined in Section 9.3.

Proof 1⇒2 This holds by elementarity.

2⇒1 We construct by induction a sequence $\langle a_i : i < \omega \rangle$. Reasoning as in (the second proof of) the downward Löverheim-Skolem theorem we can easily ensure that $A \cup \{a_i : i < \omega\}$ is a model. To obtain an atomic model we require that $a_{\restriction i}$ is isolated over A.

Suppose $a_{\restriction i}$ has been defined and assume that some formula $\varphi(z) \in L(A)$ isolates $\mathrm{tp}(a_{\restriction i}/A)$. Let $\psi(x\,;z) \in L(A)$ be such that $\psi(x\,;a_{\restriction i})$ is consistent (we leave to the reader the details of the enumeration of such formulas). Then $\psi(x\,;z) \wedge \varphi(z)$ is also consistent and by assumption it has a solution $b\,;c$ that is isolated over A. As $a_{\restriction i} \equiv_A c$, there is an A-automorphism such that $fc = a_{\restriction i}$. Therefore $fb\,;a_{\restriction i}$ is a solution $\psi(x\,;z)$ that is also isolated over A. Then we can set $a_i = fb$. □

3 Countable categoricity

Here we present some important characterizations of ω-categoricity. The second property below can be stated in different equivalent ways; for convenience, these equivalents are considered in a separate proposition. For the time being we introduce the following generalization (which we will prove is completely unnecessary): we say hat T is ω-categorical over A if any two countable models containing A are isomorphic. We say ω-categorical for ω-categorical over $\varnothing$.

12.11 Theorem (Engeler, Ryll-Nardzewsky, and Svenonius) *Assume $L(A)$ is countable. The following are equivalent:*

1. *T is ω-categorical over A;*

2. *every type $p(x) \subseteq L(A)$ with $|x| < \omega$ is isolated.*

The set A is introduced for convenience. By 3 of Proposition 12.12 below, no theory is ω-categorical over an infinite set, and categoricity over some finite A is equivalent to categoricity over $\varnothing$ (see Exercise 12.13).

Proof 1⇒2 This is an immediate consequence of the omitting types theorem. In fact, if $p(x)$ is a non-isolated A-type, then there are two countable models M and N containing A such that M realizes $p(x) \subseteq L(A)$ while N omits it. Then M and N cannot be isomorphic over A.

2⇒1 Observe that 2 implies that every countable model containing A is atomic over A. But, by Proposition 12.8, countable atomic models are unique up to isomorphism. □

12.12 Proposition *Fix a set A and a finite tuple of variables x. The following are equivalent*

1. *every A-type $p(x)$ is isolated;*
2. *$S_x(A)$ is finite;*
3. *$L_x(A)$ is finite up to equivalence;*
4. *in $\mathcal{U}^{|x|}$ there is a finite number of orbits under* $\mathrm{Aut}(\mathcal{U}/A)$.

Proof To prove the implication 1⇒2 observe that $\mathcal{U}^{|x|}$ is the union of sets of the form $p(\mathcal{U})$ where $p \in S_x(A)$. If these types are isolated then $\mathcal{U}^{|x|}$ is the union of A-definable sets. By compactness this union has to be finite. To prove 2⇒1 let $p \in S_x(A)$. If $S_x(A)$ is finite, $\neg p(\mathcal{U})$ is the union of finitely many type definable sets. A finite union of type definable sets is type definable. So $\neg p(\mathcal{U})$ is type definable. Hence $p(\mathcal{U})$ is isolated. We prove implication 2⇒3 observe that each formula in $L_x(A)$ is equivalent to the disjunction of the types in $S_x(A)$ that contain this formula. If $S_x(A)$ is finite, $L_x(A)$ is finite up to equivalence. Implication 3⇒2 is clear and equivalence 2⇔4 follows from the characterization of orbits as type-definable sets. □

12.13 Exercise Prove that the following are equivalent for every finite set A

1. T is ω-categorical;
2. T is ω-categorical over A. □

12.14 Exercise Prove that the following are equivalent

1. T is ω-categorical;
2. there is countable model that is both saturated and atomic. □

12.15 Exercise Assume L is countable and that T is complete. Suppose that for every finite tuple x there is a model M that realizes only finitely many types in $S_x(T)$. Prove that T is ω-categorical. □

4 Small theories

Let T be, as always in this chapter, a complete theory without finite models. We say that T is small over A if $S_x(A)$ is countable for every x of finite length. When A is empty, we simply say that T is small. The set A is introduced for convenience, in the literature it is used only for $A = \varnothing$. A different term is used in another very interesting case, e.g. a theory small over every countable set A is said to be ω-stable. For this reason, the term 0-stable is sometimes used for small.

12.16 Proposition *No T is small over an uncountable set and if T is small (over $\varnothing$) then it is small over any finite A.*

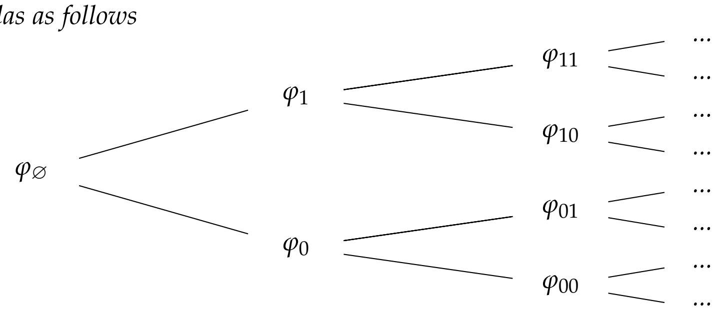
Proof Let a be an enumeration of A. As $S_x(A) = \big\{p(x\,;a) \ : \ p \in S_{x\,;z}(T)\big\}$ the proposition is immediate. $\square$

Below we identify $S_x(A)$ with $\mathcal{U}^{|x|}/{\equiv_A}$

12.17 Definition *Let Δ be a set of formulas (we mainly use $\Delta = L_x(A)$ in this section). A binary tree of formulas in Δ is a sequence $\langle \varphi_s \ : \ s \in 2^{<\lambda}\rangle$ of formulas in $\Delta \cup \{\top\}$ such that*

1. *for each $s \in 2^{\lambda}$ the type $p_s = \big\{\varphi_{s\restriction n} \ : \ n < |s|\big\}$ is consistent;*
2. *$p_s \cup p_r$ is inconsistent for any two distinct $s, r \in 2^{\lambda}$.*

(Condition 2 is usually obtained by taking $\varphi_{s0} \leftrightarrow \neg\varphi_{s1}$ for every s.) We call λ the height of the tree. If the height is not specified, we assume it is ω. We may depict a binary tree of formulas as follows



Where branches are consistent types and distinct branches are inconsistent. $\square$

Let $S(\Delta)$ be denote the set of maximal consistent Δ-types.

12.18 Lemma *Suppose Δ is countable and closed under negation. Then the following are equivalent*

1. *there is a binary tree of formulas in Δ;*
2. *$\big|S(\Delta)\big| = 2^{\omega}$;*
3. *$\big|S(\Delta)\big| > \omega$.*

Proof Since the implications 1⇒2⇒3 are clear, it suffices to prove 3⇒1. We assume that $S(\Delta)$ is uncountable and define a tree of formulas in Δ by induction. Begin with $\varphi_\varnothing = \top$. For $s \in 2^{<\omega}$ define

$$p_s \quad = \quad \big\{\varphi_{s\restriction n} \ : \ n \leq |s|\big\}.$$

Assume inductively that p_s has uncountably many extensions in $S(\Delta)$. This will guarantee the consistency of the branches.

It suffices to show that there is a formula $\psi \in \Delta$ such that both $p_s \cup \{\psi\}$ and $p_s \cup \{\neg\psi\}$ have uncountably many extensions in $S(\Delta)$. Then we define $\varphi_{s0} = \psi$ and $\varphi_{s1} = \neg\psi$.

Consider the following type that extends p_s

$$q \quad = \quad \Big\{\xi \in \Delta \ : \ p_s \cup \{\neg\xi\} \text{ has } \leq \omega \text{ extensions in } S(\Delta)\Big\}.$$

This type is consistent, otherwise $\neg\xi_1 \vee \cdots \vee \neg\xi_n$ would hold for some $\xi_i \in q$. This cannot happen, because p_s has uncountably many extensions in $S(\Delta)$, while by the definition of q each of $p_s \cup \{\neg\xi_i\}$ has countably many extensions.

If the formula ψ required above does not exist, q is complete, hence it belongs to $S(\Delta)$. Every type in $S(\Delta)$ that extends p_s and is distinct from q contains $p_s \cup \{\neg\xi\}$ for some $\xi_i \in q$. By the definition of q, there are countably many such types, so this contradicts the induction hypothesis. □

12.19 Proposition *Suppose $L(A)$ is countable. The following are equivalent*

1. *T is small over A;*
2. *there exists a countable saturated model containing A;*
3. *there is no binary tree of formulas in $L_x(A)$ for any finite x.*

Proof 1⇒2 There is a countable model M containing A that is weakly saturated (see Proposition 9.13). There is a countable homogeneous model N containing M (see Exercise 9.14). Clearly N is also weakly saturated. Then it is saturated by Corollary 9.14 .

2⇒3 Clear.

3⇒1 By Lemma 12.18. □

12.20 Proposition *A small theory has countable atomic models over every countable set A.*

Proof We prove that every formula in $L_x(A)$, where $|x| < \omega$, has a solution isolated over A. Then it suffices to apply Proposition 12.10.

Suppose for a contradiction that $\varphi(x) \in L(A)$ is consistent but has no solution isolated over A. Then there is a formula $\psi(x) \in L(A)$ such that both $\varphi(x) \wedge \psi(x)$ and $\varphi(x) \wedge \neg\psi(x)$ are consistent, otherwise $\varphi(x)$ would imply a complete type and every solution of $\varphi(x)$ would be isolated. Fix such a $\psi(x)$. Clearly neither $\varphi(x) \wedge \psi(x)$ nor $\varphi(x) \wedge \neg\psi(x)$ have a solution isolated over A. This allows to construct a tree of formulas in $L_x(A)$ and prove that T is not small over A. □

12.21 Exercise Suppose Δ is countable and closed under negation. Prove that if there is a binary tree of formulas then there is a binary tree such that $\varphi_{s0} = \neg\varphi_{s1}$ and $\varphi_\varnothing = \top$. □

12.22 Exercise Prove that if T is small over A, then T is small over A, a for every finite tuple a. □

12.23 Exercise Let $|x| = 1$. Prove that if $S_x(A)$ is countable for every finite set A, then T is small. □

12.24 Exercise (Vaught) Prove that no complete theory has exactly 2 countable models (assume L is countable – though it is not really necessary).

Hint: suppose T has exactly two countable models. Then T is small and there are a countable saturated model N and an atomic model $M \subseteq N$. As T is not ω-categorical, $M \not\simeq N$ and there is finite tuple a that is not isolated over $\varnothing$. Let K

be an atomic model over a. Clearly $K \not\simeq M$ and, by Exercises 12.13 and 12.14, also $K \not\simeq N$. □

5 A toy version of a theorem of Zil'ber

As an application we prove that if T is ω-categorical and strongly minimal then it is not finitely axiomatizable.

We say that T has the finite model property if for every sentence $\varphi \in L$ there is a finite substructure $A \subseteq \mathfrak{U}$ such that

fmp $\qquad \mathfrak{U} \vDash \varphi \quad \Leftrightarrow \quad A \vDash \varphi$

The property is interesting because of the following proposition.

12.25 Proposition *If T has the finite model property then it is not finitely axiomatizable.*

Proof Assume fmp and suppose for a contadiction that there is a sentence $\varphi \in L$ such that $T \vdash \varphi \vdash T$. Then $A \vDash T$ for some finite structure A. But $T \vdash \exists^{>k} x\,(x = x)$ for every k. A contradiction. □

We need the following definition. We say that $C \subseteq \mathfrak{U}$ is a homogeneous set if for every pair of tuples $a, c \in C^{<\omega}$ such that $a \equiv c$ and for every $b \in C$ there is a $d \in C$ such that $a, b \equiv c, d$.

12.26 Lemma *Suppose L is countable. If T is ω-categorical and every finite set is contained in a finite homogeneous substructure, then T has the finite model property.*

Proof We prove fmp also for formulas with parameters. We prove that for all n there is a finite structure $A \subseteq \mathfrak{U}$ where fmp holds for all sentences $\varphi \in L(A)$ such that

\# $\qquad$ number of parameters in φ + number of quantifiers in $\varphi \leq n$.

Fix n and pick some finite substructure A that is homogeneous and such that all types $p(z) \subseteq L$ with $|z| \leq n$ have a realization in $A^{|z|}$. Now we prove fmp by induction on the syntax of φ.

The claim for atomic formulas is witnessed by any finite structure that contains the parameters of the formula. Such finite substructure exists in fact it suffices to take the algebraic closure which, in an ω-categorical theory, is finite (by 3 of Lemma 12.12). Induction for Boolean connectives is straightforward. As for induction step for the existential quantifier, consider the formula $\exists x\, \varphi(x\,; c)$, where $c \in A^{<n}$ and $|x| = 1$. Implication $\Leftarrow$ of fmp follows immediately from the induction hypothesis and from the fact that, if $\exists x\, \varphi(x\,; c)$ satisfy #, also $\varphi(d\,; c)$ satisfies it. As for $\Rightarrow$, assume $\mathfrak{U} \vDash \exists x\, \varphi(x\,; c)$. Let $a, b \in A^{<\omega}$ be a solution of $\varphi(x\,; z)$ such that $a \equiv c$. Such a solution exists because all types with $\leq n$ variables are realized in A. By homogeneity there is a $d \in A$ such that $a, b \equiv c, d$ and therefore $A \vDash \varphi(d\,; c)$. □

12.27 Proposition *If T is strongly minimal, then every algebraically closed set is homogeneous.*

Proof Let A be algebraically closed and let $a, c \in A^{<\omega}$ be such that $a \equiv c$. Let b be an element of A. Suppose first that $b \in \operatorname{acl} a$. Let $f \in \operatorname{Aut}(\mathfrak{U})$ be such that $f(a) = c$ and $f[A] = \operatorname{acl} A$. Then $d = fb$ is the required element, in fact $a, b \equiv c, d$. Now, suppose instead that $b \notin \operatorname{acl} a$. Then any $d \notin \operatorname{acl} c$ satisfies $a, b \equiv c, d$. Such a d exists in A, otherwise $A = \operatorname{acl} c \neq \operatorname{acl} a$, which contradicts $a \equiv c$. □

From the propositions above we finally obtain the following.

12.28 Theorem *A theory which is ω-categorical and strongly minimal is not finitely axiomatizable.*

Proof If T is ω-categorical the algebraic closure of a finite set is finite. Therefore from Proposition 12.27 we infer that T satisfies the assumptions of Lemma 12.26. Hence T has the finite model property, so, by Proposition 12.25 it is not finitely axiomatizable. □

12.29 Exercise Assume L is countable and let T be strongly minimal. Prove that the following are equivalent

1. T is ω-categorical;
2. the algebraic closure of a finite set is finite.

Implication 1⇒2 does not require the strong minimality of T. □

6 Notes and references

An uncountable, non-isolated, complete type that cannot be omitted was produced by Gebhard Fuhrken in 1962. Example 12.3 is inspired by a post of Alex Kruckman to StackExhange [5]. I am not aware of other expositions.

A famous theorem of Boris Zil'ber claims that Theorem 12.28 holds for any totally categorical theory. The same theorem has been proved independently by Cherlin, Harrington and Lachlan with a proof that uses the classification of finite simple groups. This theorem signs the birth of a subject known as *geometric stability theory* which studies in depth the geometric properties of model theories which we briefly hinted to in Chapter 11. The interested reader may consult Pillay's monograph [7]. The material in Section 5 comes from [7, Section 2.6]

Chapter 13

Imaginaries

The description of first-order definability is simplified if we allow definable sets to be used as second-order parameters in formulas. This leads to the theory of (elimination of) *imaginaries*. The technical reason that induced Shalah to introduce imaginaries will only be clear later, see Section 17.5, but the theory is of independent interest.

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ strictly larger than $|L|$. The notation and implicit assumptions are as in Section 9.3.

1 Many-sorted structures

A many-sorted language consists of three disjoint sets. Besides the usual $L_{\rm fun}$ and $L_{\rm rel}$, we have a set $L_{\rm srt}$ whose elements are called sorts. The language also includes a (many-sorted) arity function that assigns to function and relation symbols r, f a tuple of sorts of finite positive length which we call arity.

A many-sorted structure M consists of

1. a set M_s, for each $s \in L_{\rm srt}$;
2. a function $f^M : M_{s_1} \times \cdots \times M_{s_n} \to M_{s_0}$, for each $f \in L_{\rm fun}$ of arity $\langle s_0, \ldots, s_n \rangle$;
3. a relation $r^M \subseteq M_{s_0} \times \cdots \times M_{s_n}$, for each $r \in L_{\rm rel}$ of arity $\langle s_0, \ldots, s_n \rangle$.

For every sort s we fix a sufficiently large set of variables V_s. Now we define terms and their respective sorts by induction.

All variables are terms of their respective sort. If $t_1, \ldots, t_n$ are terms of sorts $s_1, \ldots, s_n$ and $f \in L_{\rm fun}$ is of arity $\langle s_0, \ldots, s_n \rangle$ then $f\, t_1, \ldots, t_n$ is a term of sort s_0.

Formulas are defined as follows. If $r \in L_{\rm rel}$ has arity $\langle s_0, \ldots, s_n \rangle$ then $r\, t_0, \ldots, t_n$ is a formula. Also, $t_1 = t_2$ is formula for every pair of terms of equal sort. All other formulas are constructed by induction using the propositional connectives $\neg$ and $\vee$ and the quantifier $\exists x$ (or any other reasonable choice of logical connectives).

Truth of formulas is defined as for one-sorted languages, except that here we require that the witness of the quantifier $\exists x$ belongs to M_s, where s is the sort of the variable x.

Models of second-order logic are arguably the most widely used examples of many-sorted structures. They may be described using a language with a sort n for every $n \in \omega$. The sort 0 is used for the first-order elements; the sort $n > 0$ is used for relations of arity n. For every $n > 0$ the language has a relation symbol $\in_n$ of arity $\langle 0^n, n \rangle$, where $0^n = \overset{n \text{ times}}{0 \ldots\ldots 0}$. There are also arbitrarily many function and relation symbols of sort $\langle 0^n \rangle$ for any $n > 0$.

2 The eq-expansion

⚠ Warning: the structure $\mathcal{U}^{\mathrm{eq}}$ and the theory T^{eq} defined below do not coincide with the standard ones introduced by Shelah. As the difference is merely cosmetic, introducing new notation would be overkill and we prefer to abuse the existing terminology. In Section 6 below we compare our definition with the standard one.

Given a language L, we define a many-sorted language L^{eq} which has a sort for each partitioned formula $\sigma(x\,;z) \in L$ and a sort 0 which we call the home sort. (Partitioned formulas have been introduced in Definition 1.14.)

For legibility, we pretend that all formulas σ depend on the same variables. So we assume that $x\,;z$ are infinite tuples. Hence, with the notation of the previous section $L_{\mathrm{srt}} = \{0\} \cup L_{x\,;z}$.

The home sort is also called the first-order sort and all other sorts are generically called second-order. First of all, L^{eq} contains all relations and functions of the first order language L. The many-sorted arity of a relation r is $\langle 0^{n_r}\rangle$, where n_r is the arity of r in L. Similarly, the many-sorted arity of a function f is $\langle 0^{1+n_f}\rangle$, where n_f is the arity of f in L. Moreover, L^{eq} contains a relation symbol $\in_{\sigma(x\,;z)}$ for each sort $\sigma(x\,;z)$. These relation symbols have arity $\langle 0^{|x_\sigma|}, \sigma(x\,;z)\rangle$, where x_σ are the variables in x that actually occur in σ. As there is no risk of ambiguity, in what follows we omit σ from the subscripts.

In $\mathcal{U}^{\mathrm{eq}}$ the domain of the home sort is $\mathcal{U}$. The domain for the sort $\sigma(x\,;z)$ contains the definable sets $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ as b ranges over $\mathcal{U}^{|z|}$. The symbols in L have the same interpretation as in the one-sorted case, and $\in$ is interpreted as set membership. We write T^{eq} for $\mathrm{Th}(\mathcal{U}^{\mathrm{eq}})$.

As usual L^{eq} also denotes the set of formulas constructed in this language and, if $A \subseteq \mathcal{U}^{\mathrm{eq}}$, we write $L^{\mathrm{eq}}(A)$ for the language and the set of formulas that use elements of A as parameters.

⚠ We write $L(A)$ for the set of formulas in $L^{\mathrm{eq}}(A)$ that contain no second-order variables, neither free nor quantified (when $A \subseteq \mathcal{U}^{\mathrm{eq}}$, it may contain second-order parameters).

We use the symbol $\mathcal{X}$ to denote a generic second-order variable.

It is important to note right away that this expansion of $\mathcal{U}$ is a mild one: the definable subsets of the home sort of $\mathcal{U}^{\mathrm{eq}}$ are the same as those of $\mathcal{U}$. In particular, iterating the expansion would not yield anything new.

13.1 Proposition *Let $\bar{\mathcal{X}} = \mathcal{X}_1, \ldots, \mathcal{X}_n$ be a tuple of second order variables of sort $\sigma_i(x\,;z)$. Then for every formula $\varphi(u\,;\bar{\mathcal{X}}) \in L^{\mathrm{eq}}$ there is a formula $\varphi'(u\,;\bar{z}) \in L$, where $\bar{z}$ is a tuple of n copies of z, such that the following holds in $\mathcal{U}^{\mathrm{eq}}$*

$$\forall \bar{\mathcal{X}}, \bar{z} \left[\bigwedge_{i=1}^{n} \mathcal{X}_i = \{x : \sigma_i(x\,;z_i)\} \rightarrow \forall u \left[\varphi(u\,;\bar{\mathcal{X}}) \leftrightarrow \varphi'(u\,;\bar{z})\right]\right].$$

When $n = 0$ the proposition asserts that L^{eq}_u and L_u have the same expressive power.

Proof (sketch) By induction on syntax. When φ is atomic, we set $\varphi' = \varphi$ unless φ is of the form $t \in \mathcal{X}_i$ for some tuple of terms t or it has the form $\mathcal{X}_i = \mathcal{X}_j$. In the first case φ' is the formula $\sigma_i(t\,;z_i)$. In the second case it is the formula

$\forall x\,[\sigma_i(x\,;z_i) \leftrightarrow \sigma_j(x\,;z_j)]$.

The connectives stay unchanged except for the quantifiers $\exists\mathcal{X}$, where $\mathcal{X}$ is a second-order variable, say of sort $\sigma(x\,;z)$. These quantifiers are replaced by $\exists z$. □

Proposition 13.1 implies in particular that we can always replace $\exists\mathcal{X}$ by $\exists z$ if we substitute $\sigma(t\,;z)$ for $t \in \mathcal{X}$ in the quantified formula.

13.2 Remark Proposition 13.1 should convince the reader that the move from $\mathcal{U}$ to $\mathcal{U}^{\text{eq}}$ is *almost* trivial. For instance, it implies that for every $A \subseteq \mathcal{U}^{\text{eq}}$, there exists a $B \subseteq \mathcal{U}$ such that $L(B)$ is at least as expressive as $L(A)$. By this we mean that every formula in $L(A)$ is equivalent to some formula in $L(B)$. The set $B \subseteq \mathcal{U}$ contains the parameters that define the definable sets in $A \subseteq \mathcal{U}^{\text{eq}}$. The point of $\mathcal{U}^{\text{eq}}$ is that there might not be any $B \subseteq \mathcal{U}$ such that $L(B)$ is *exactly* as expressive as $L^{\text{eq}}(A)$.

For instance, suppose L contains only a binary relation which is interpreted as an equivalence relation with infinitely many infinite classes. Let $\mathcal{A}$ be an equivalence class and let $A = \{\mathcal{A}\}$. Then for $\mathcal{A}$ is definable in $L(B)$ if and only if $B \cap \mathcal{A} \neq \varnothing$. But no element of $\mathcal{A}$ is definable in $L(A)$. □

If $\mathcal{V} \preceq \mathcal{U}$ we write $\mathcal{V}^{\text{eq}}$ for the substructure of $\mathcal{U}^{\text{eq}}$ that has $\mathcal{V}$ as domain of the home sort and the set of definable sets of the form $\sigma(\mathcal{U}\,;b)$ for some $b \in \mathcal{V}^{|z|}$ as domain of the sort $\sigma(x\,;z)$. The following proposition claims that the elementary substructures of $\mathcal{U}^{\text{eq}}$ are exactly those of the form $\mathcal{V}^{\text{eq}}$ for some $\mathcal{V} \preceq \mathcal{U}$.

13.3 Proposition *The following are equivalent for every structure $\mathcal{V}^\dagger$ of signature L^{eq}*

1. $\mathcal{V}^\dagger \preceq \mathcal{U}^{\text{eq}}$;
2. $\mathcal{V}^\dagger = \mathcal{V}^{\text{eq}}$ *for some* $\mathcal{V} \preceq \mathcal{U}$.

Proof Implication 2⇒1 is a direct consequence of Proposition 13.1. We prove 1⇒2. Let $\mathcal{V}$ be the domain of the home sort of $\mathcal{V}^\dagger$. It is clear that $\mathcal{V} \preceq \mathcal{U}$. Let $\mathcal{A} \in \mathcal{V}^{\text{eq}}$ have sort $\sigma(x\,;z)$, say $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ for some $b \in \mathcal{V}^{|z|}$. As $\exists^{=1}\mathcal{X}\,\forall x\,[x \in \mathcal{X} \leftrightarrow \sigma(x\,;b)]$ holds in $\mathcal{U}^{\text{eq}}$, by elementarity it holds in $\mathcal{V}^\dagger$ and therefore $\mathcal{A} \in \mathcal{V}^\dagger$. This proves $\mathcal{V}^{\text{eq}} \subseteq \mathcal{V}^\dagger$. A similar argument proves the converse inclusion. □

13.4 Proposition *Let $A \subseteq \mathcal{U}^{\text{eq}}$. Then every type $p(u\,;\mathcal{X}) \subseteq L^{\text{eq}}(A)$ that is finitely consistent in $\mathcal{U}^{\text{eq}}$ is realized in $\mathcal{U}^{\text{eq}}$. That is, $\mathcal{U}^{\text{eq}}$ is saturated.*

Proof By Remark 13.2, there are some $B \subseteq \mathcal{U}$ and some $q(u\,;\mathcal{X}) \subseteq L^{\text{eq}}(B)$ equivalent to $p(u\,;\mathcal{X})$. This already proves the proposition when $\mathcal{X}$ is the empty tuple. Otherwise, let $q'(u\,;z)$ be obtained by replacing every formula $\varphi(u\,;\mathcal{X})$ in $q(u\,;\mathcal{X})$ with the formula $\varphi'(u\,;z)$ given in Proposition 13.1. Then $q'(u\,;z)$ is finitely consistent in $\mathcal{U}$. Assume for clarity of notation that $\mathcal{X}$ is a single variable of sort $\sigma(x\,;z)$. If $c\,;b \vDash q'(u\,;z)$, then $c\,;\sigma(\mathcal{U}\,;b) \vDash q(u\,;\mathcal{X})$. □

Automorphisms of a many-sorted structure are defined in the obvious way: sorts are preserved and so are functions and relations. Every automorphism $f : \mathcal{U} \to \mathcal{U}$ extends to an automorphism $f : \mathcal{U}^{\text{eq}} \to \mathcal{U}^{\text{eq}}$ as follows. If $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ we define $f\mathcal{A} = \sigma(\mathcal{U}\,;fb) = f[\mathcal{A}]$, which clearly preserves the sort and the relation $\in$. Clearly, this extension is unique.

The homogeneity of $\mathcal{U}^{\text{eq}}$ follows by back-and-forth as in the one-sorted case.

13.5 Proposition *Every elementary map $k : \mathcal{U}^{\mathrm{eq}} \to \mathcal{U}^{\mathrm{eq}}$ of cardinality $< \kappa$ extends to an automorphism of $\mathcal{U}^{\mathrm{eq}}$.* □

3 The definable closure in the eq-expansion

We may safely identify automorphism of $\mathcal{U}$ with automorphisms of $\mathcal{U}^{\mathrm{eq}}$. Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and let a be a tuple of elements of $\mathcal{U}^{\mathrm{eq}}$. We denote by $\mathrm{Aut}(\mathcal{U}/A)$ the set of automorphisms (of $\mathcal{U}^{\mathrm{eq}}$) that fix all elements of A. The symbol $\mathcal{O}(a/A)$ denotes the orbit of a over A. This has been defined in Section 9.2 and now we apply it to $\mathcal{U}^{\mathrm{eq}}$

$$\mathcal{O}(a/A) \quad = \quad \big\{fa \ : \ f \in \mathrm{Aut}(\mathcal{U}/A)\big\}.$$

By homogeneity, $\mathcal{O}(a/A) = p(\mathcal{U}^{\mathrm{eq}})$ where $p(v) = \mathrm{tp}(a/A)$. When $\mathcal{O}(a/A) = \{a\}$ we say that a is invariant over A or A-invariant, for short.

13.6 Definition *Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and $a \in \mathcal{U}^{\mathrm{eq}}$. When $\varphi(a) \wedge \exists^{=1} v\, \varphi(v)$ holds for some formula $\varphi(v) \in L^{\mathrm{eq}}(A)$, we say that a is definable over A. We write $\mathrm{dcl}^{\mathrm{eq}}(A)$ for the set of those $a \in \mathcal{U}^{\mathrm{eq}}$ that are definable over A. We write $\mathrm{dcl}(A)$ for $\mathrm{dcl}^{\mathrm{eq}}(A) \cap \mathcal{U}$. This is the natural generalization of the notion of definability introduced in Section 11.1.* □

The definition above treats first- and second-order elements of $\mathcal{U}^{\mathrm{eq}}$ uniformly. The following propositions proves that when $a \in \mathcal{U}^{\mathrm{eq}}$ is a definable set, the notion of definability coincides with the one usually applied to sets.

13.7 Proposition *Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and let $\mathcal{A} \in \mathcal{U}^{\mathrm{eq}}$ have sort $\sigma(x\,;z)$. Then the following are equivalent*

1. $\mathcal{A} \in \mathrm{dcl}^{\mathrm{eq}}(A)$;
2. $\mathcal{A} = \psi(\mathcal{U})$ *for some* $\psi(x) \in L(A)$.

Proof Implication 2⇒1 is clear because extensionality is implicit in the definition of $\mathcal{U}^{\mathrm{eq}}$. We prove 1⇒2. Let $\varphi(\mathcal{X}) \in L^{\mathrm{eq}}(A)$ be a formula $\mathcal{A}$ is the unique solution of. Then 2 holds with $\exists \mathcal{X}\,\big[x \in \mathcal{X} \wedge \varphi(\mathcal{X})\big]$ for $\psi'(x)$. This $\psi'(x)$ is a formula in $L^{\mathrm{eq}}(A)$. Proposition 13.1 yields the required formula $\psi(x) \in L(A)$. □

The saturation and homogeneity of $\mathcal{U}^{\mathrm{eq}}$ allows us to prove the following proposition with virtually the same proof as for Theorem 11.3

13.8 Theorem *For any $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and $a \in \mathcal{U}^{\mathrm{eq}}$ the following are equivalent*

1. *a is invariant over A;*
2. $a \in \mathrm{dcl}^{\mathrm{eq}}(A)$. □

By Proposition 13.7, Theorem 13.8 when applied to a definable set $\mathcal{A}$ gives an alternative proof of Proposition 9.21.

We conclude this section with a remark about the canonicity of the definitions of sets. The formula $\psi(x)$ in Proposition 13.7 need not be the sort $\sigma(x\,;z)$. For example, consider the theory of a binary equivalence relation $e(x\,;z)$ with two infinite classes, let $\mathcal{A}$ be one of these classes and let $A \neq \varnothing$ be such that $A \cap \mathcal{A} = \varnothing$. Then $\mathcal{A}$ is definable over A though not by some formula of the form $e(x\,;b)$ for some $b \in A$. Things change if we replace A with a model.

13.9 Proposition *Let M be a model and let $\mathcal{A}$ be an element of sort $\sigma(x\,;z)$. Then the following are equivalent*

1. $\mathcal{A} \in \mathrm{dcl}^{\mathrm{eq}}(M)$;
2. $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ *for some* $b \in M^{|z|}$.

In particular $M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$.

Proof Assume 1 and let $\psi(x) \in L(M)$ be such that $\mathcal{A} = \psi(\mathcal{U})$. Such a formula exists by Proposition 13.7. Then $\exists z\, \forall x\, [\psi(x) \leftrightarrow \sigma(x\,;z)]$ holds in $\mathcal{U}$. By elementarity it holds in M, therefore $\exists z$ has a witness in M. This proves 1⇒2, the converse implication is obvious. □

4 The algebraic closure in the eq-expansion

The following is the natural generalization of the notion introduced in Section 11.1.

13.10 Definition *Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and $a \in \mathcal{U}^{\mathrm{eq}}$. We say that a is algebraic over A if a formula of the form $\varphi(a) \wedge \exists^{=k} v\, \varphi(v)$ holds for some $\varphi(v) \in L^{\mathrm{eq}}(A)$ and some positive integer k. We write $\mathrm{acl}^{\mathrm{eq}}(A)$ for the set of those $a \in \mathcal{U}^{\mathrm{eq}}$ that are algebraic over A. We write $\mathrm{acl}(A)$ for $\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathcal{U}$.*

The following proposition is proved with virtually the same proof as Theorem 11.4

13.11 Theorem *For every $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and every $a \in \mathcal{U}^{\mathrm{eq}}$ the following are equivalent*

1. $\mathcal{O}(a/A)$ *is finite;*
2. $a \in \mathrm{acl}^{\mathrm{eq}}(A)$;
3. $a \in M^{\mathrm{eq}}$ *for every model such that* $A \subseteq M^{\mathrm{eq}}$. □

We say finite equivalence relation for an equivalence relation with finitely many classes. A finite equivalence formula or type is a formula, respectively a type, that defines a finite equivalence relation. Theorem 13.12 belows proves that sets algebraic over A are union of classes of a finite equivalence relations definable over A.

13.12 Theorem *Let $A \subseteq \mathcal{U}^{\mathrm{eq}}$ and let $\mathcal{A} \in \mathcal{U}^{\mathrm{eq}}$ be an element of sort $\sigma(x\,;z)$. Then the following are equivalent*

1. $\mathcal{A} \in \mathrm{acl}^{\mathrm{eq}}(A)$
2. *for some finite equivalence formula* $\varepsilon(x\,;y) \in L(A)$ *and some* $c_1, \ldots, c_n \in \mathcal{U}^{|y|}$

$$x \in \mathcal{A} \quad \leftrightarrow \quad \bigvee_{i=1}^{n} \varepsilon(x\,;c_i).$$

Proof 2⇒1 If $\varepsilon(x\,;y)$ has m classes, then $\mathcal{O}(\mathcal{A}/A)$ contains at most $\binom{m}{n}$ sets.

1⇒2 Let $\varphi(\mathcal{X}) \in L^{\mathrm{eq}}(A)$ be an algebraic formula that has $\mathcal{A}$ among its solutions and define

$$\varepsilon(x\,;y) \quad = \quad \forall \mathcal{X} \Big[\varphi(\mathcal{X}) \;\rightarrow\; \big[x \in \mathcal{X} \leftrightarrow y \in \mathcal{X}\big]\Big]$$

If $\varphi(\mathcal{X})$ has n, solutions, then $\varepsilon(x\,;y)$ has at most 2^n equivalence classes. Clearly, $\mathcal{A}$ is union of some these classes. □

13.13 Definition *We write $a \overset{\text{Sh}}{\equiv}_A b$ when $\varepsilon(a\,;b)$ holds for every finite equivalence formula $\varepsilon(x\,;y) \in L(A)$. In words we say that a and b have the same Shelah strong-type over A.* □

By the following proposition, the Shelah strong type of a over A is $\operatorname{tp}(a/\operatorname{acl}^{\text{eq}}A)$.

13.14 Proposition *Let $A \subseteq \mathcal{U}^{\text{eq}}$ and let $a, b \in \mathcal{U}^{|x|}$. Then the following are equivalent*

1. $a \overset{\text{Sh}}{\equiv}_A b$;
2. $a \equiv_{\operatorname{acl}^{\text{eq}}A} b$.

Proof 2⇒1 Assume ¬1 and let $\varepsilon(x\,;y) \in L(A)$ be a finite equivalence formula such that $\neg\varepsilon(a\,;b)$. Let $\mathcal{D} = \varepsilon(\mathcal{U}\,;b)$, then $b \in \mathcal{D}$ and $a \notin \mathcal{D}$. As $\varepsilon(x\,;y)$ is an A-invariant finite equivalence formula, $\mathcal{D} \in \operatorname{acl}^{\text{eq}}(A)$, and ¬2 follows.

1⇒2 Assume ¬2 and let $\varphi(x) \in L\big(\operatorname{acl}^{\text{eq}}(A)\big)$ be such that $\varphi(a) \nleftrightarrow \varphi(b)$. Let $\mathcal{D} = \varphi(\mathcal{U})$, then $\mathcal{D} \in \operatorname{acl}^{\text{eq}}(A)$. Therefore, by Proposition 13.12, the set $\mathcal{D}$ is union of equivalence classes of some finite equivalence formula $\varepsilon(x\,;y) \in L(A)$. Then $\neg\varepsilon(a\,;b)$ and ¬1 follows. □

⚠ We write $\mathcal{S}(a/A)$ for the intersection of all definable sets that contain a and are algebraic over A. By the proposition above $\mathcal{S}(a/A) = \big\{b : b \overset{\text{Sh}}{\equiv}_A a\big\} = \mathcal{O}(a/\operatorname{acl}^{\text{eq}}A)$.

13.15 Exercise Let $p(x) \subseteq L(A)$ and let $\varphi(x\,;y) \in L(A)$ be a formula that defines, when restricted to $p(\mathcal{U})$, an equivalence relation with finitely many classes. Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x\,;y)$ on $p(\mathcal{U})$. □

13.16 Exercise Let $A \subseteq \mathcal{U}$ and let $\mathcal{A}$ be a definable set with finite orbit over A. Without using the eq-expansion, prove that $\mathcal{A}$ is union of classes of a finite equivalence relation definable over A. □

13.17 Exercise Let T be strongly minimal and let $\varphi(x\,;z) \in L(A)$ with $|x| = 1$. For arbitrary $b \in \mathcal{U}^{|z|}$, prove that if the orbit of $\varphi(\mathcal{U}\,;b)$ over A is finite, then $\varphi(\mathcal{U}\,;b)$ is definable over $\operatorname{acl} A$.

Hint: you can use Theorem 13.12. □

5 Elimination of imaginaries

For the time being, we agree that *imaginary* is just another word for definable set. Though this is not formally correct (cfr. Section 6), it is morally true and helps to understand the terminology. The concept of elimination of imaginaries has been introduced by Poizat who also proved Theorem 13.23 below. A theory has elimination of imaginaries if for every $A \subseteq \mathcal{U}^{\text{eq}}$, there is a $B \subseteq \mathcal{U}$ such that $L(B)$ and $L(A)$ have the same expressive power (i.e. they are the same up to to equivalence).

13.18 Definition *We say that T has elimination of imaginaries if for every definable set $\mathcal{A}$ there is a formula $\varphi(x\,;z) \in L$ such that*

ei $$\exists^{=1}z\ \forall x\ \Big[x \in \mathcal{A}\ \leftrightarrow\ \varphi(x\,;z)\Big]$$

We say that the witness of $\exists^{=1}z$ in the formula above is a canonical parameter of $\mathcal{A}$ or a canonical name for $\mathcal{A}$. A set may have different canonical parameters for different formulas $\varphi(x\,;z)$.

We say that T has weak elimination of imaginaries if

wei $$\exists^{=k}z\ \forall x\ \Big[x \in \mathcal{A}\ \leftrightarrow\ \varphi(x\,;z)\Big]$$

for some positive integer k. □

In the formulas above we allow z to be the empty string. In this case we read ei and wei omitting the quantifiers $\exists^{=1}z$, respectively $\exists^{=k}z$. Therefore $\varnothing$-definabile sets have all (at least) the empty string as a canonical parameter.

To show that the notions above are well-defined properties of a theory one needs to check that they are independent of our choice of monster model. We leave this to the reader as an exercise.

We say that two tuples a and b of elements of $\mathcal{U}^{\rm eq}$ are interdefinable if $\mathrm{dcl}^{\rm eq}(a) = \mathrm{dcl}^{\rm eq}(b)$. By Theorem 13.8 this is equivalent to saying that $\mathrm{Aut}(\mathcal{U}/a) = \mathrm{Aut}(\mathcal{U}/b)$, that is, the automorphisms that fix a fix also b, and vice versa.

13.19 Theorem *The following are equivalent*

1. *T has weak elimination of imaginaries;*
2. *every definable set is interdefinable with a finite set;*
3. *every definable set $\mathcal{A}$ is definable over $\mathrm{acl}\{\mathcal{A}\}$;*
4. $\mathrm{dcl}^{\rm eq}\{\mathcal{A}\} = \mathrm{dcl}^{\rm eq}\big(\mathrm{acl}\{\mathcal{A}\}\big)$.

Proof 1⇒2 Assume 1 and let $\mathcal{B}$ be the set of solutions of the formula

$$\forall x\ \Big[x \in \mathcal{A}\ \leftrightarrow\ \sigma(x\,;z)\Big].$$

Hence $\mathcal{B}$ is finite and $\mathcal{B} \in \mathrm{dcl}^{\rm eq}\{\mathcal{A}\}$. We also have $\mathcal{A} \in \mathrm{dcl}^{\rm eq}\{\mathcal{B}\}$ because $\mathcal{A}$ is definibile by the formula $\exists z\,\big[z \in \mathcal{B} \wedge \sigma(x\,;z)\big]$. Therefore $\mathrm{dcl}^{\rm eq}\{\mathcal{A}\} = \mathrm{dcl}^{\rm eq}\{\mathcal{B}\}$.

2⇒3 Assume $\mathrm{dcl}^{\rm eq}\{\mathcal{A}\} = \mathrm{dcl}^{\rm eq}\{\mathcal{B}\}$ for some finite set $\mathcal{B}$. The elements of $\mathcal{B}$, say $b_1,\dots,b_n$, are the (finitely many) solutions of the formula $z \in \mathcal{B}$. Therefore $b_1,\dots,b_n \in \mathrm{acl}\{\mathcal{B}\} = \mathrm{acl}\{\mathcal{A}\}$. Let $\varphi(\mathcal{X}\,;\mathcal{B})$ be a formula that has $\mathcal{A}$ as unique solution. Then $\exists\mathcal{Y}\big[\mathcal{Y} = \{b_1,\dots,b_n\} \wedge \varphi(\mathcal{X}\,;\mathcal{Y})\big]$ is the formula that proves 3.

3⇔4 It suffices to rephrase 3 as $\mathcal{A} \in \mathrm{dcl}^{\rm eq}\big(\mathrm{acl}\{\mathcal{A}\}\big)$.

3⇒1 Assume 3. As wei holds trivially for all $\varnothing$-definable sets, we may assume $\mathcal{A} \neq \varnothing$. Let $\sigma(x\,;z)$ be such that

$$\forall x\ \Big[x \in \mathcal{A}\ \leftrightarrow\ \sigma(x\,;b)\Big].$$

for some tuple b of elements of $\mathrm{acl}\{\mathcal{A}\}$. Fix some algebraic formula $\delta(z\,;\mathcal{A})$ satisfied by b and write $\psi(z\,;\mathcal{X})$ for the formula

$$\forall x\,\Big[x \in \mathcal{X} \;\leftrightarrow\; \sigma(x\,;z)\Big] \;\wedge\; \delta(z\,;\mathcal{X}).$$

The formula $\sharp$ below is clearly satisfied by b therefore, if we can prove that it has finitely many solutions, wei follows from Proposition 13.1

$$\sharp \qquad \forall x\,\Big[x \in \mathcal{A} \;\leftrightarrow\; \sigma(x\,;z) \;\wedge\; \exists\mathcal{X}\,\psi(z\,;\mathcal{X})\Big].$$

We check that any c that satisfies $\sharp$ also satisfies $\delta(z\,;\mathcal{A})$. As $\mathcal{A}$ is non empty, $\psi(c\,;\mathcal{A}')$ holds for some $\mathcal{A}'$. By $\sharp$ and the definition of $\psi(z\,;\mathcal{X})$ we obtain $\mathcal{A}' = \mathcal{A}$ and $\delta(c\,;\mathcal{A})$. □

There are a notions of elimination of imaginaries that are weaker than weak elimination. For instance, we say that T has geometric elimination of imaginaries if for every $A \subseteq \mathcal{U}$

$$\operatorname{acl}^{\rm eq}\{\mathcal{A}\} \;=\; \operatorname{acl}^{\rm eq}\big(\operatorname{acl}\{\mathcal{A}\}\big)$$

This will not be applied in these notes, but see Exercises 13.25 and 13.26.

13.20 Theorem *The following are equivalent*

1. *T has elimination of imaginaries;*
2. *every definable set is interdefinable with a tuple of real elements;*
3. *every definable set $\mathcal{A}$ is definable over $\operatorname{dcl}\{\mathcal{A}\}$;*
4. $\operatorname{dcl}^{\rm eq}\{\mathcal{A}\} = \operatorname{dcl}^{\rm eq}\big(\operatorname{dcl}\{\mathcal{A}\}\big)$.

Proof Implications 1⇒2⇒3⇔4 are immediate. Implication 3⇒1 is identical to the homologous implication in Theorem 13.19, just substitute algebraic with definable. □

We now consider elimination of imaginaries in two concrete structures: algebraically closed fields and real closed fields. The following lemma is required in the proof of Theorem 13.22 below.

13.21 Lemma *The following is a sufficient condition for weak elimination of imaginaries*

$\sharp$ *for every $A \subseteq \mathcal{U}^{\rm eq}$, every consistent $\varphi(z) \in L(A)$ has a solution in* $\operatorname{acl} A$.

Proof Let $\mathcal{A} \in \mathcal{U}^{\rm eq}$ be a definable set of sort $\sigma(x\,;z)$. Then $\forall x\,\big[x \in \mathcal{A} \leftrightarrow \sigma(x\,;z)\big]$ is consistent and, by $\sharp$ it it has a solution in $\operatorname{acl}\{\mathcal{A}\}$. Hence weak elimination follows from Theorem 13.19. □

13.22 Theorem *Let T be a complete, strongly minimal theory. Then, if $\operatorname{acl}\varnothing$ is infinite, T has weak elimination of imaginaries.*

Proof If $\operatorname{acl}\varnothing$ is infinite, $\operatorname{acl} A$ is a model for every A (cfr. Exercise 11.24) so condition $\sharp$ of lemma 13.21 holds by elementarity and the theorem follows. □

13.23 Theorem *The theories $T^p_{\rm acf}$ have elimination of imaginaries.*

Proof By Theorem 13.22 we know that $T^p_{\rm acf}$ has weak elimination of imaginaries. Therefore, by Theorem 13.19 it suffices to prove that every finite set $\mathcal{A}$ is interdefinable with a tuple. Let $\mathcal{A} = \{a_1, \dots, a_n\}$ where each a_i is a tuple $a_{i,1}, \dots, a_{i,m}$ of elements of $\mathcal{U}$. Given $\mathcal{A}$ we define the term

$$t_{\mathcal{A}}(x\,;y) \;=\; \prod_{i=1}^{n}\Big(x - \sum_{k=1}^{m} a_{i,k}\, y_k\Big). \qquad \text{where } y = y_1,\dots,y_m.$$

Note that (the interpretation of) the term $t_{\mathcal{A}}(x\,;y)$ is independent on particular indexing of the set $\mathcal{A}$. So, any automorphism that fixes $\mathcal{A}$, fixes the $t_{\mathcal{A}}(x\,;y)$. Now rewrite $t_{\mathcal{A}}(x\,;y)$ as a sum of monomials and let c be the tuple of coefficients of these monomials. The tuple c uniquely determines $t_{\mathcal{A}}(x\,;y)$ and vice versa. Therefore every automorphism that fixes $\mathcal{A}$ fixes c and vice versa. Hence $\mathcal{A}$ and c are interdefinable. □

13.24 Exercise Let T have elimination of imaginaries and $\varphi(x\,;z) \in L(A)$. For arbitrary $c \in \mathcal{U}^{|z|}$, prove that if the orbit of $\varphi(\mathcal{U}\,;c)$ over A is finite, then $\varphi(\mathcal{U}\,;c)$ is definable over $\operatorname{acl} A$. □

13.25 Exercise Prove that following are equivalent for every $A \subseteq \mathcal{U}$

1. $\operatorname{acl}^{\mathrm{eq}} A = \operatorname{dcl}^{\mathrm{eq}}\big(\operatorname{acl} A\big)$ for every $A \subseteq \mathcal{U}$;
2. $\operatorname{Aut}(\mathcal{U}/\operatorname{acl}^{\mathrm{eq}} A) = \operatorname{Aut}(\mathcal{U}/\operatorname{acl} A)$;
3. $a \equiv_{\operatorname{acl} A} b \;\Leftrightarrow\; a \overset{\mathrm{Sh}}{\equiv}_A b$ for every $A \subseteq \mathcal{U}$ and $a, b \in \mathcal{U}^{<\omega}$. □

13.26 Exercise Prove that the following are equivalent

1. T has weak elimination of imaginaries;
2. T has geometric elimination of imaginaries and 1 of Exercise 13.25 holds. □

6 Immaginaries: the true story

The point of the expansion to $\mathcal{U}^{\mathrm{eq}}$ is to add a canonical parameter for each definable set. In fact, in $\mathcal{U}^{\mathrm{eq}}$ every definable subset of $\mathcal{U}^{|z|}$ is the canonical parameter of itself. This allows us to deal with theories without elimination of imaginaries in the most straightforward way.

The expansion to $\mathcal{U}^{\mathrm{eq}}$ that was originally introduced by Shelah (and still used everywhere else) is slightly different from the one introduced here. For a given set $\mathcal{A} = \sigma(\mathcal{U}\,;b)$ Shelah considers the equivalence relation defined by the formula

$$\varepsilon(z\,;z') \;=\; \forall x\,\Big[\sigma(x\,;z) \;\leftrightarrow\; \sigma(x\,;z')\Big].$$

The equivalence class of b in the relation $\varepsilon(z\,;z')$ is what Shelah uses as canonical parameter of the set $\mathcal{A}$.

Shelah's $\mathcal{U}^{\mathrm{eq}}$ has a sort for each $\varnothing$-definable equivalence relation $\varepsilon(z\,;z')$. The domain of the sort $\varepsilon(z\,;z')$ contains the classes of the equivalence relation defined by $\varepsilon(z\,;z')$. These equivalence classes are called imaginaries. Shelah's L^{eq} contains functions that map tuples in the home sort to their equivalence class.

7 Uniform elimination of imaginaries

Sometimes in the literature elimination of imaginaries is confused with uniform elimination of imaginaries, see e.g. [9, Definition 8.4.2]. Theorem 13.28 below shows that the difference is immaterial. This section is more technical and could be skipped at a first reading.

Let us rephrase the definition of elimination of imaginaries: for every formula $\varphi(x\,;u) \in L$ and every tuple $c \in \mathcal{U}^{|u|}$ there is a formula $\sigma(x\,;z) \in L$ such that

$$\exists^{=1} z\ \forall x\ \Big[\varphi(x\,;c)\ \leftrightarrow\ \sigma(x\,;z)\Big].$$

A priori, the formula $\sigma(x\,;z)$ may depend on c in a very wild manner. We say that T has uniform elimination of imaginaries if for every $\varphi(x\,;u)$ there are a formula $\sigma(x\,;z)$ and a formula $\rho(z)$ such that

$$\text{uei} \qquad \forall u\ \exists^{=1} z\ \Big[\rho(z)\ \wedge\ \forall x\ \big[\varphi(x\,;u)\ \leftrightarrow\ \sigma(x\,;z)\big]\Big].$$

The role of the formula $\rho(z)$ above is mysterious. It is clarified by the following propositions. In fact, uniform elimination of imaginaries is equivalent to a very natural property which in words says: every definable equivalence relation is the kernel of a definable function. Recall that the kernel of the function f is the relation $fa = fb$.

Uniform elimination of imaginaries is convenient when dealing with interpretations of structure inside other structures. Let us consider a simple concrete example. Suppose $\mathcal{U}$ is a group and let $\mathcal{H}$ be a definable normal subgroup of $\mathcal{U}$. The elements of the quotient structure $\mathcal{U}/\mathcal{H}$ are equivalence classes of a definable equivalence relation. If there is uniform elimination of imaginaries we can identify $\mathcal{U}/\mathcal{H}$ with an actual definable subset of $\mathcal{G}$ (the range of the function f above). Moreover, the group operation of $\mathcal{G}$ are definable functions. As working in $\mathcal{U}/\mathcal{H}$ may be notationally cumbersome, $\mathcal{G}$ may offer a convenient alternative.

13.27 Proposition *The following are equivalent*

1. *T has uniform elimination of imaginaries;*
2. *for every $\varphi(x\,;u)$ such that $\forall u\ \exists x\ \varphi(x\,;u)$ there is a formula $\sigma(x\,;z)$ such that*
$$\forall u\ \exists^{=1} z\ \forall x\ \Big[\varphi(x\,;u)\ \leftrightarrow\ \sigma(x\,;z)\Big];$$
3. *for every equivalence formula $\varepsilon(x\,;u) \in L$ there is given by $\sigma(x\,;z) \in L$ such that*
$$\forall u\ \exists^{=1} z\ \forall x\ \Big[\varepsilon(x\,;u)\ \leftrightarrow\ \sigma(x\,;z)\Big].$$

Note that the definable function $fu = z$ mentioned above is defined by the formula

$$\vartheta(u\,;z)\ \ =\ \ \forall x\ \Big[\varepsilon(x\,;u)\ \leftrightarrow\ \sigma(x\,;z)\Big].$$

Proof 1⇒2. If $\forall u\ \exists x\ \varphi(x\,;u)$ we can rewrite uei as

$$\forall u\ \exists^{=1} z\ \forall x\ \Big[\varphi(x\,;u)\ \leftrightarrow\ \rho(z) \wedge \sigma(x\,;z)\Big].$$

2⇒3. Clear.

3⇒1. Apply 3 to the equivalence formula

$$\varepsilon(u\,;v)\ \ =\ \ \forall x\ \Big[\varphi(x\,;u)\ \leftrightarrow\ \varphi(x\,;v)\Big]$$

Let $\vartheta(u\,;z)$ be defined as above. The reader may check that uei holds substituting for $\delta(z)$ the formula $\exists u\ \vartheta(u\,;z)$ and and for $\sigma(x\,;z)$ the formula $\exists u\ \big[\varphi(x\,;u) \wedge \vartheta(u\,;z)\big]$. $\square$

By the following theorem, uniformity comes almost for free.

13.28 Theorem *The following are equivalent*

1. *T has uniform elimination of imaginaries;*
2. $\operatorname{dcl}\varnothing$ *contains at least two elements and T has elimination of imaginaries.*

Proof 1⇒2. Let $\varphi(x,u)$ be the formula $u_1 = u_2$. From 1 we obtain a formula $\sigma(x\,;z) \in L$ such that

$$\forall u_1, u_2\, \exists^{=1} z \left[\rho(z)\ \wedge\ \forall x\ \big[u_1 = u_2\ \leftrightarrow\ \sigma(x\,;z)\big]\right].$$

Therefore the formulas $\exists^{=1} z\, \big[\rho(z) \wedge \forall x\, \sigma(x\,;z)\big]$ and $\exists^{=1} z\, \big[\rho(z) \wedge \forall x \neg\sigma(x\,;z)\big]$ are both true. The witnesses of $\exists^{=1} z$ in these two formulas are two distinct elements of $\operatorname{dcl}\varnothing$.

2⇒1. Assume 2 and fix a formula formula $\varphi(x\,;u)$ such that $\varphi(x,a)$ is consistent for every $a \in \mathfrak{U}^{|u|}$. We prove 2 of Proposition 13.27.

Let $p(u)$ be the type that contains the formulas

$$\neg\, \exists^{=1} z\, \forall x \left[\varphi(x\,;u)\ \leftrightarrow\ \sigma(x;z)\right],$$

where $\sigma(x;z)$ ranges over all formulas in L. By elimination of imaginaries $p(u)$ is not consistent. Therefore, by compactness, there are some formulas $\sigma_i(x,z)$ such that

$$\sharp \qquad \forall u \bigvee_{i=0}^{n} \exists^{=1} z\, \forall x \left[\varphi(x\,;u)\ \leftrightarrow\ \sigma_i(x\,;z)\right].$$

To prove the theorem we need to move the disjunction in front of the $\sigma_i(x\,;z)$.

We can assume that if $\sigma_i(x\,;b) \leftrightarrow \sigma'_i(x\,;b')$ for some $\langle b,i\rangle \neq \langle b',i'\rangle$ then $\sigma_i(x\,;b)$ is inconsistent. Otherwise we can substitute the formula $\sigma_i(x\,;z)$ with

$$\sigma_i(x\,;z)\ \wedge\ \bigwedge_{j\le i} \neg\exists y \neq b\, \forall x \left[\sigma_j(x\,;y) \leftrightarrow \sigma_i(x\,;z)\right].$$

As $\varphi(x,a)$ is consistent for every a, the substitution does not break the validity of $\sharp$.

Fix some distinct $\varnothing$-definable tuples $d_0, \dots, d_n$ of the same length (these are easy to obtain from two $\varnothing$-definable elements). We claim that from $\sharp$ it follows that

$$\forall u\, \exists^{=1} z, y\, \forall x \left[\varphi(x\,;u)\ \leftrightarrow\ \bigvee_{i=1}^{n} \left[\sigma_i(x\,;z)\ \wedge\ y = d_i\right]\right].$$

(The tuple z, y plays the role of z.) We fix some a and check that the formula below has a unique solution

$$\flat \qquad \forall x \left[\varphi(x\,;a)\ \leftrightarrow\ \bigvee_{i=1}^{n} \left[\sigma_i(x\,;z)\ \wedge\ y = d_i\right]\right].$$

Existence follows immediately from $\sharp$. As for uniqueness, note that if b, d_i and $b', d_{i'}$ are two distinct solution of $\flat$ then $\sigma_i(x\,;b) \leftrightarrow \sigma_{i'}(x\,;b')$ for some $\langle b,i\rangle \neq \langle b',i'\rangle$. By what assumed on $\sigma_i(x\,;z)$, we obtain that $\varphi(x\,;a)$ is inconsistent. A contradiction which proves the theorem. □

Chapter 14

Invariant sets

In this chapter, L is a signature, T is a complete theory without finite models, and $\mathcal{U}$ is a saturated model of inaccessible cardinality κ strictly larger than $|L|$. We use the same notation and make the same implicit assumptions as in Section 9.3.

1 Invariant sets and types

Let $\mathcal{D} \subseteq \mathcal{U}^{|z|}$, where z is a tuple of length $< \kappa$. We say that $\mathcal{D}$ is an *A*-invariant set if it is fixed setwise by all A-automorphisms. That is, $f[\mathcal{D}] = \mathcal{D}$ for every automorphism $f \in \mathrm{Aut}(\mathcal{U}/A)$ or, yet in other words,

is1. $\quad a \in \mathcal{D} \;\leftrightarrow\; fa \in \mathcal{D} \qquad$ for every $a \in \mathcal{U}^{|z|}$ and every $f \in \mathrm{Aut}(\mathcal{U}/A)$,

which, by homogeneity, is equivalent to,

is2. $\quad a \in \mathcal{D} \;\leftrightarrow\; b \in \mathcal{D} \qquad$ for all $a, b \in \mathcal{U}^{|z|}$ such that $a \equiv_A b$.

These equivalent conditions yield the following bound on the number of invariant sets.

14.1 Proposition *Let $\lambda = |L_z(A)|$. There are at most 2^{2^λ} sets $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ that are invariant over A.*

Proof By is2, sets that are invariant over A are unions of equivalence classes of the relation $\equiv_A$, that is, unions of sets of the form $p(\mathcal{U})$ where $p(z) \in S(A)$. Then the number of A-invariant sets is at most $2^{|S_z(A)|}$. Clearly $|S_z(A)| \leq 2^\lambda$. □

We say that $\mathcal{D}$ is an invariant set if it is invariant over some (small) set A. Since κ is assumed to be inaccessible, there are exactly κ invariant sets.

In this chapter we work with Δ-types, that is subsets of Δ, where either $\Delta = L(\mathcal{A})$ or

$$\Delta \quad = \quad \Big\{\varphi(x\,;b),\ \neg\varphi(x\,;b)\ :\ b \in \mathcal{A}^{|z|}\Big\}$$

for some given $\varphi(x\,;z) \in L$. In the latter case Δ-types are called φ-types. We denote by $S_\varphi(\mathcal{A})$ the set of maximal φ-types with parameters in $\mathcal{A}$. Typically, $\mathcal{A}$ is either the whole of $\mathcal{U}$ or some small set $A \subseteq \mathcal{U}$. Types in $S_\varphi(\mathcal{U})$ are called global φ-types.

Let $p(x) \subseteq L(\mathcal{U})$ be a consistent type. For every formula $\varphi(x\,;z) \in L$ we define

$$\mathcal{D}_{p,\varphi} \quad = \quad \Big\{a \in \mathcal{U}^{|z|}\ :\ \varphi(x\,;a) \in p\Big\}.$$

We can read the notation in two ways: either the tuple z has infinite length and is the same for all formulas, or it is finite and depends on φ. This is possible because adding or erasing dummy variables to the second tuple of $\varphi(x\,;z)$ does not change $\mathcal{D}_{p,\varphi}$ in any relevant way; in particular, invariance is preserved.

Let $p(x) \subseteq L(\mathcal{U})$ be a consistent type. We say that $p(x)$ is an *A*-invariant type if, for every formula $\varphi(x\,;z) \in L$, it1. $\varphi(x\,;a)\in p \quad\Leftrightarrow\quad \varphi(x\,;fa)\in p \qquad$ for every $a\in\mathcal{U}^{|z|}$ and every $f\in\mathrm{Aut}(\mathcal{U}/A)$.

Hence $p(x)$ is invariant exactly when all the sets $\mathcal{D}_{p,\varphi}$ are. By invariant type we mean a type that is invariant over some (small) set A.

A global φ-type $p(x)$ can be identified with the set $\mathcal{D}_{p,\varphi}$ and a global type $p(x)$ can be identified with the collection of the sets $\mathcal{D}_{p,\varphi}$, where φ ranges over L. The notions of invariance for types and sets coincide.

We say that the type $p(x)\subseteq L(\mathcal{U})$ does not split over A if

it2. $\quad a\equiv_A b \quad\Rightarrow\quad \Big(\varphi(x\,;a)\in p \Leftrightarrow \varphi(x\,;b)\in p\Big) \qquad$ for all $a,b\in\mathcal{U}^{|z|}$

for every formula $\varphi(x\,;z)\in L$. By homogeneity, non splitting is equivalent to invariance. For global types it2 is equivalent to

it2′. $\quad a\equiv_A b \quad\Rightarrow\quad \varphi(x\,;a)\leftrightarrow\varphi(x\,;b)\ \in\ p$

The following is another important equivalent characterization of invariance over A of a global type $p(x)$ that follows easily from it2′

it3. $\quad a\equiv_A b \quad\Rightarrow\quad a\equiv_{A,c} b \qquad$ for all $a,b\in\mathcal{U}^{|z|}$ and for all $c\vDash p_{\restriction A,a,b}$.

Note that it3 applies to global types $p(x)\in S(\mathcal{U})$ but not to φ-types.

2 Invariance from a dual perspective

⚠ The following terminology is non-standard. We say that the set $\mathcal{B}\subseteq\mathcal{U}^{|x|}$, typically a definable set, is quasi-invariant over A if whenever $f_1,\ldots,f_n$ is a finite tuple of automorphisms in $\mathrm{Aut}(\mathcal{U}/A)$, the sets $f_i[\mathcal{B}]$ have non-empty intersection.

We say that the type $p(x)\subseteq L(\mathcal{U})$, typically a global type, is quasi-invariant over A if $\varphi(\mathcal{U})$ is quasi-invariant over A for every conjunction $\varphi(x)$ of formulas in $p(x)$.

For global types, quasi-invariance coincides with invariance. In fact we have the following equivalence.

14.2 Proposition *Let $p(x)\in S_\varphi(\mathcal{U})$ be a global φ-type. Then the following are equivalent*

1. *$p(x)$ is invariant over A;*
2. *$p(x)$ is quasi-invariant over A.*

Proof 1⇒2. Assume $p(x)$ is invariant and let $\psi(x\,;b)\ \in\ p$ where $b\in\mathcal{U}^{|z|}$. Then $\psi(x\,;fb)\ \in\ p$ for every $f\in\mathrm{Aut}(\mathcal{U}/A)$, so 2 follows from the finite consistency of $p(x)$.

2⇒1. Assume $p(x)$ is not invariant. Then there is $b\in\mathcal{U}^{|z|}$ such that $\varphi(x\,;b)\in p$ and $\varphi(x\,;fb)\notin p$ for some $f\in\mathrm{Aut}(\mathcal{U}/A)$. By completeness, $p(x)$ contains the formula $\varphi(x\,;b)\wedge\neg\varphi(x\,;fb)$ which clearly is not consistent with its f-translate. □

14.3 Exercise We say that $\mathcal{B}$ *strongly* quasi-invariant if for every definable set $\mathcal{D}$ at least one of $\mathcal{B}\cap\mathcal{D}$ and $\mathcal{B}\cap\neg\mathcal{D}$ is quasi-invariant. Strongly quasi invariant types are defined similarly to quasi-invariant types. Note incidentally that for global types the two notions coincide. Prove that every strongly quasi-invariant type has an extension to a global invariant type.

Hint: it may help to prove that if $\mathcal{B}$ is strongly quasi-invariant then for every definable $\mathcal{D}$ either $\mathcal{B} \cap \mathcal{D}$ or $\mathcal{B} \cap \neg\mathcal{D}$ is strongly quasi-invariant. Then the maximal strongly quasi-invariant set is the required global extension. □

Unfortunately, a quasi-invariant type does not necessarily extend to a quasi-invariant global type. There are, however, other properties of types that guarantee that an extension to a global type with the property can be found. In the next section we introduce one of these properties, that of being a *coheir*, and in subsequent chapters yet another one, *non-forking*. Being a coheir is stronger that being quasi-invariant, and non-forking is weaker.

3 Heirs and coheirs

The easiest way to obtain types that are invariant over a model M is via types that are finitely satisfiable in M. We say that a type $p(x)$ is finitely satisfiable in a set A if every conjunction of formulas in $p(x)$ has a solution in $A^{|x|}$.

14.4 Proposition *Every type $p(x) \subseteq L(\mathcal{U})$ that is finitely satisfiable in A is quasi-invariant over A.*

Proof Clearly, the same $a \in A^{|x|}$ that satisfies $\varphi(x)$ also satisfies every $\mathrm{Aut}(\mathcal{U}/A)$-translate of $\varphi(x)$. □

Propositions 14.2 and 14.4 yield the following proposition.

14.5 Proposition *Let $p(x) \in S_\varphi(\mathcal{U})$ be a global φ-type that is finitely satisfiable in A. Then $p(x)$ is A-invariant.* □

14.6 Proposition *Every type $q(x) \subseteq L(\mathcal{U})$ that is finitely satisfiable in A has an extension to a global type that is finitely satisfiable in A.*

Proof Let $p(x) \subseteq L(\mathcal{U})$ be maximal among the types containing $q(x)$ and finitely satisfiable in A. We prove that $p(x)$ is complete. Suppose for a contradiction that $p(x)$ contains neither $\psi(x)$ nor $\neg\psi(x)$. Then neither $p(x) \cup \{\psi(x)\}$ nor $p(x) \cup \{\neg\psi(x)\}$ are finitely satisfiable in A. This contradicts the finite satisfiability of $p(x)$. □

In most cases we work with types that are finitely satisfiable over a model. The reason is explained by the next proposition, which is clear by elementarity.

14.7 Proposition *Every consistent type over a model is finitely satisfiable in that model, that is, whenever $p(x) \subseteq L(M)$ is consistent, $p(x)$ is finitely satisfiable in M.* □

14.8 Definition *A type $p(x) \subseteq L(\mathcal{U})$ that is finitely satisfiable in M is said to be a coheir of $p_{\restriction M}(x)$.* □

In many cases it is useful to focus on elements instead of types. We introduce the following notation to express that $\mathrm{tp}(b/M, a)$ is finitely satisfied in M.

14.9 Definition *For every $a \in \mathcal{U}^{|x|}$ and $b \in \mathcal{U}^{|z|}$ we define*

⚠
$$a \mathop{\smile\!\!\!\!\!\!\perp}_M b \quad \Leftrightarrow \quad \varphi(\mathcal{U}, b) \cap M^{|x|} \neq \varnothing \text{ for all } \varphi(x\,; z) \in L(M) \text{ such that } \varphi(a\,; b).$$

We say that $\mathrm{tp}(a/M,b)$ *is a* *coheir* *of* $\mathrm{tp}(a/M)$, *or, equivalently, that* $\mathrm{tp}(b/M,a)$ *is a* *heir* *of* $\mathrm{tp}(b/M)$.

We define the type

$$x \mathrel{\raise0.2ex\hbox{$\downarrow$}\!\!\!\lower0.1ex\hbox{$\smile$}}_M b \quad = \quad \Big\{\varphi(x\,;z) \;:\; \varphi(x\,;b) \in L(M) \text{ and } M^{|x|} \subseteq \varphi(\mathfrak{U}\,;b)\Big\}.$$

We will use the symbol $a \equiv_M x \downarrow_M b$ *for the union of the types* $x \downarrow_M b$ *and* $\mathrm{tp}(a/M)$. □

The tuples a realizing $x \downarrow_M b$ are exactly those such that $a \downarrow_M b$. Note that $\mathrm{tp}(a/M,b)$ is a coheir of $\mathrm{tp}(a/M)$ according to Definition 14.8, so the terminology is consistent.

We think of $a \downarrow_M b$ as saying that a is independent from b over M. In general $b \downarrow_M a$ is not equivalent to $a \downarrow_M b$.

We shall use, sometimes without explicit reference, the following easy lemma.

14.10 Lemma *The following properties hold for all* M, a, b, *and* c

1. $a \downarrow_M b \;\Rightarrow\; f a \downarrow_M f b$ *for every* $f \in \mathrm{Aut}(\mathfrak{U}/M)$ — *invariance*
2. $a \downarrow_M b \;\Leftrightarrow\; a_0 \downarrow_M b_0$ *for all finite* $a_0 \subseteq a$ *and* $b_0 \subseteq b$ — *finite character*
3. $a \downarrow_M c, b$ *and* $c \downarrow_M b \;\Rightarrow\; a, c \downarrow_M b$ — *transitivity*
4. $a \downarrow_M b \;\Rightarrow$ *there exists* $a' \equiv_{M,b} a$ *such that* $a' \downarrow_M b, c$ — *coheir extension* □

Proof Properties 1-3 follow immediately from Definition 14.9. We prove 4. Let $p(x)$ be a global coheir of $\mathrm{tp}(a/A,b)$, which exists by Proposition 14.6. Then any $a' \vDash p_{\restriction A,b,c}(x)$ proves the lemma. □

The type $a \equiv_M x \downarrow_M b$ in Definition 14.9 is the intersection of all global coheirs of $\mathrm{tp}(a/M)$. Its consistency is guaranteed by the fact that M is a model (see Proposition 14.7). However, in general it need not be a complete type over M, b. In fact, completeness in this case is a strong property.

14.11 Definition *We say that* $\downarrow_M$ *is* *stationary* *if* $a \equiv_M x \downarrow_M b$ *is a complete type over* M, b *for all finite tuples* b *and* a.

We say *n-stationary* *if this is restricted to* $|a| = n$. □

An application of stationarity is given in Section 15.3.

Stationarity is often ensured by the following property, which will receive due attention in Section 17.5. Recall that for $\mathcal{D}, \mathcal{C} \subseteq \mathfrak{U}$ we write $\mathcal{D} =_A \mathcal{C}$ for $\mathcal{D} \cap A = \mathcal{C} \cap A$.

14.12 Proposition *Let* x *be a tuple of variables of length* n. *If for every* $\varphi(x) \in L(\mathfrak{U})$ *there is a formula* $\psi(x) \in L(M)$ *such that* $\varphi(\mathfrak{U}) =_M \psi(\mathfrak{U})$ *then* $\downarrow_M$ *is n-stationary.*

Proof Let $b \in \mathfrak{U}^{|z|}$ and $a_1, a_2 \in \mathfrak{U}^{|x|}$ be such that $a_i \downarrow_M b$ and $a_1 \equiv_M a_2$. We claim that $a_1 \equiv_{M,b} a_2$. We need to prove that $\varphi(b; a_1) \leftrightarrow \varphi(b; a_2)$ for every $\varphi(z; x) \in L(M)$. Let $\psi(x) \in L(M)$ be such that $\varphi(b, \mathfrak{U}) =_M \psi(\mathfrak{U})$. From $a_i \downarrow_M b$ we obtain that $\varphi(b, a_i) \leftrightarrow \psi(a_i)$. Finally, the claim follows because $a_1 \equiv_M a_2$. □

14.13 Remark There are many theories where the stationarity of $\downarrow_M$ holds for some particular M. For example, if every subset of M^n is M-definable then $\downarrow_M$ is clearly

n-stationary. This simple observation will help in the proof of Theorem 15.15. For a natural example, let $T = T_{\text{dlo}}$ and let $M \subseteq \mathfrak{U}$ have the order type of $\mathbb{R}$. By quantifier elimination every definable subset of $\mathfrak{U}$ is a union of finitely many intervals. By Dedekind completeness, the trace on M of any interval of $\mathfrak{U}$ coincides with that of an M-definable interval.

4 Morley sequences and indiscernibles

In what follows α is some ordinal $\leq \kappa$, typically ω, and x is a tuple of variables of length $< \kappa$.

Let $p(x) \in S(\mathfrak{U})$ be a global type. We say that $\bar{c} = \langle c_i : i < \alpha \rangle$ is a Morley sequence of $p(x)$ over A if for every $i < \alpha$

Ms. $$c_i \quad \models \quad p_{\restriction A, c_{\restriction i}}(x).$$

We usually require that $p(x)$ is invariant over A. In particular, when $p(x)$ is finitely satisfiable in A we say that $\bar{c}$ is a coheir sequence of $p(x)$ over A.

When we say that $\bar{c}$ is a coheir sequence over A we mean that there is a type $p(x)$ that is finitely satisfiable in A such that $\bar{c}$ is a coheir sequence of $p(x)$.

The following is a convenient characterization of coheir sequences.

14.14 Lemma *The following are equivalent*

1. $\bar{c} = \langle c_n : n < \omega \rangle$ *is a coheir sequence over* M*;*
2. $c_n \mathrel{\smile\mkern-11mu|}_M c_{\restriction n}$ *and* $c_{n+1} \equiv_{M, c_{\restriction n}} c_n$ *for every* $n < \omega$.

Proof 1⇒2. Assume 1 and let $p(x) \in S(\mathfrak{U})$ be a global type that is finitely satisfiable in M and such that $c_i \models p_{\restriction M, c_{\restriction i}}(x)$. The requirement $c_{n+1} \equiv_{M, c_{\restriction n}} c_n$ is clear. Now, suppose $\varphi(c_{n+1})$ for some $\varphi(x) \in L(M, c_{\restriction n+1})$. Then $\varphi(x)$ belongs to $p(x)$, so $\varphi(\mathfrak{U}) \cap M^{|x|} \neq \varnothing$ because $p(x)$ is finitely satisfiable in M. This proves $c_n \mathrel{\smile\mkern-11mu|}_M c_{\restriction n}$.

2⇒1. Let $q(x) = \{\varphi(x) \in L(\bar{c}) \ : \ \varphi(c_n) \text{ holds for cofinitely many } n\}$. From 2 it follows that $q(x)$ is finitely satisfiable in M. Then $\bar{c}$ is a coheir sequence of any global type that extends $q(x)$. □

Let $(I, <_I)$ be a linear order. A function $\bar{a} : I \to \mathfrak{U}^{|x|}$ is said to be an I-sequence, or simply a sequence when I is clear. We will often introduce an I-sequence as $\bar{a} = \langle a_i : i \in I \rangle$.

If $I_0 \subseteq I$ we call $a_{\restriction I_0}$ a subsequence of $\bar{a}$. The subsets $I_0 \subseteq I$ that are well-ordered by $<_I$, in particular the finite ones, are especially relevant. When I_0 has order type α, an ordinal, we identify $a_{\restriction I_0}$ with a tuple of length α.

Recall that $[I]^n$ denotes that the set of n-subsets of I, i.e. the subsets of I of cardinality n. The notation

$$\binom{I}{n} \quad = \quad [I]^n$$

is also common.

14.15 Definition *Let* $(I, <_I)$ *be an infinite linear order and let* $\bar{a}$ *be an* I*-sequence. We say that* $\bar{a}$ *is a sequence of indiscernibles over* A *or, an* A*-indiscernible sequence, if* $a_{\restriction I_0} \equiv_A a_{\restriction I_1}$ *for every* $I_0, I_1 \in [I]^n$ *and* $n < \omega$. □

The indiscernibility condition can be formulated in a number of equivalent ways. For example, we can require that, for every formula $\varphi(x_1, \ldots, x_n) \in L(A)$ and every pair of tuples in I^n such that $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$,

$$\varphi(a_{i_0}, \ldots, a_{i_n}) \quad \leftrightarrow \quad \varphi(a_{j_0}, \ldots, a_{j_n})$$

Alternatively, we can simply say that for all $i_0, \ldots, i_n \in I$ the type $\mathrm{tp}(a_{i_0}, \ldots, a_{i_n}/A)$ only depends on the order type of $i_0, \ldots, i_n$.

14.16 Proposition *Let $p(x) \in S(\mathcal{U})$ be a global A-invariant type and let $\bar{c} = \langle c_i : i < \alpha \rangle$ be a Morley sequence of $p(x)$ over A. Then $\bar{c}$ is a sequence of indiscernibles over A.*

Proof We prove by induction on $n < \omega$ that

$$\sharp \qquad c_{\restriction n} \equiv_A c_{\restriction I_0} \quad \text{for every } I_0 \subseteq \alpha \text{ of cardinality } n.$$

For $n = 0$ the claim is trivial. We assume inductively that $\sharp$ above is true and prove that

$$c_{\restriction n}, c_n \equiv_A c_{\restriction I_0}, c_i \quad \text{for every } I_0 < i < \alpha.$$

As $\bar{c}$ is Morley sequence, $c_n \equiv_{A, c_{\restriction n}} c_i$ whenever $n < i$. Hence we can equivalently prove that

$$c_{\restriction n}, c_i \equiv_A c_{\restriction I_0}, c_i,$$

which is equivalent to

$$c_{\restriction n} \equiv_{A, c_i} c_{\restriction I_0}.$$

The latter holds by induction hypothesis $\sharp$ and the invariance of $p(x)$ as formulated in it3 of Section 1. □

Chapter 15

Ramsey theory

In Section 1 we prove Ramsey's theorem and in Section 2 we present its major application in model theory: Ehrenfeucht-Mostowsky's construction of indiscernibles.

In the remaining sections we prove two important results of Ramsey theory. These results will not be used elsewhere in these notes. Our only purpose is to illustrate a concrete (relatively speaking) application of the notion of coheir.

1 Ramsey's theorem from coheir sequences

In this chapter we are interest in finite partitions. We may represent the partition of a set X into k subsets with a map $f : X \to [k]$. The elements of $[k] = \{1, \ldots, k\}$ are also called colors, and the partition a coloring, or k-coloring, of X. We say that $Y \subseteq X$ is monochromatic if $f_{\restriction Y}$ is constant on Y.

Let M be an arbitrary infinite set. Fix $n, k < \omega$ and fix a coloring f of the set of all n-subsets of M, aleas the complete n-uniform hypergraph with vertex set M,

$$f \quad : \quad \binom{M}{n} \quad \to \quad [k].$$

We say that $H \subseteq M$ is a monochromatic subgraph if the subgraph induced by H is monocromatic. In the literature monochromatic subgraph are also called homogeneous sets.

The following is a very famous theorem which we prove here in unusual way. Its proof will serve as blueprint for other constructions in this chapter.

15.1 Ramsey Theorem *Let M be an infinite set and fix some positive integers n and k. Fix an arbitrary k-coloring of the (edges of the) complete n-uniform hypergraph with vertex set M. Then there is an infinite monochromatic subgraph $H \subseteq M$.*

Proof Let L be a language that contains k relation symbols $r_1, \ldots, r_k$ of arity n. Given a k-coloring f we define a structure with domain M. The interpretation the relation symbols is

$$r_i^M \quad = \quad \Big\{a_1, \ldots, a_n \in M \ : \ f(\{a_1, \ldots, a_n\}) = i\Big\}.$$

We may assume that M is an elementary substructure of some large saturated model $\mathfrak{U}$. Pick any type $p(x) \in S(\mathfrak{U})$ finitely satisfied in M but not realized in M and let $\bar{c} = \langle c_i : i < \omega \rangle$ be a coheir sequence of $p(x)$ over M.

There is a first-order sentence saying that the formulas $r_i(x_1, \ldots, x_n)$ are a coloring of $\binom{M}{n}$. Then by elementarity the same hols in $\mathfrak{U}$. By indiscernibility, all tuples of n distinct elements of $\bar{c}$ have the same color, say 1.

Note parenthetically that no element of $\bar{c}$ is in M. The theorem is proved if we can find in M a sequence $\bar{a} = \langle a_i : i < \omega \rangle$ with color 1. We construct $a_{\restriction i}$ by induction on i as follows.

Assume as induction hypothesis that the subsequences of length n of $a_{\restriction i}, c_{\restriction n}$ have all color 1. Our goal is to find $a_i \in M$ such that the same property holds for $a_{\restriction i}, a_i, c_{\restriction n}$. By the indiscernibility of $\bar{c}$, the property holds for $a_{\restriction i}, c_{\restriction n}, c_n$. And this can be written by a formula $\varphi(a_{\restriction i}, c_{\restriction n}, c_n)$. As $\bar{c}$ is a coheir sequence, by Lemma 14.14 we can find $a_i \in M$ such that $\varphi(a_{\restriction i}, c_{\restriction n}, a_i)$. So, as the order is irrelevant, $a_{\restriction i}, a_i, c_{\restriction n}$ satisfies the induction hypothesis. □

15.2 Exercise Let M be a graph with the property that for every finite $A \subseteq M$ there is a $c \in M$ such that $A \subseteq r(c, \mathfrak{U})$. (This holds in particular when M is a random graph.) Prove that for every finite coloring of the edges of M, there is a subgraph that is complete and monochromatic. □

15.3 Exercise Under the same assumptions of exercise 15.2, but now the coloring may also be infinite. Prove that M has a subgraph that is complete and is either monochromatic or any two edges have different colors. □

2 The Ehrenfeucht-Mostowski theorem

Let $I, <_I$ be an infinite linear order and let $\bar{a}$ be an I-sequence. Fix a tuple of distinct variables $\bar{x} = \langle x_i : i < \omega \rangle$. We write $p(\bar{x}) = \text{EM-tp}(\bar{a}/A)$ and say that $p(\bar{x})$ is the Ehrenfeucht-Mostowski type of $\bar{a}$ over A if

$$p(\bar{x}) \quad = \quad \Big\{ \varphi(x_{\restriction n}) \in L(A) \ : \ \varphi(a_{\restriction I_0}) \text{ holds for every } I_0 \in \binom{I}{n} \Big\}.$$

Note that $\bar{x}$ is always of order-type ω, while $\bar{a}$ is an arbitrary infinite sequence. Clearly, x_i and a_i are tuple of the same sort.

Note also that if $\bar{a}$ is A-indiscernible then $\text{EM-tp}(\bar{a}/A)$ is a complete type, and vice versa. So, if $\bar{a}$ and $\bar{c}$ are two A-indiscernible I-sequences with the same Ehrenfeucht-Mostowski type over A, then $\bar{a} \equiv_A \bar{c}$.

15.4 Ehrenfeucht-Mostowski Theorem *Let $I, <_I$ and $J, <_J$ be two infinite linear orders such that $|J| \leq \kappa$. Then for every sequence $\bar{a} = \langle a_i : i \in I \rangle$ there is an J-sequence of A-indiscernibles $\bar{c}$ such that $\text{EM-tp}(\bar{a}/A) \subseteq \text{EM-tp}(\bar{c}/A)$.*

Proof We prove the theorem for $I = J = \omega$ and leave the general case to the reader. Let $q(\bar{x}) = \text{EM-tp}(\bar{a}/A)$. Let $\bar{c}$ be any realization of the following type yields a J-sequence of A-indiscernibles.

$$q(\bar{x}) \quad \cup \quad \Big\{ \varphi(x_{\restriction I_0}) \leftrightarrow \varphi(x_{\restriction J_0}) \ : \ \varphi(x_{\restriction n}) \in L(A),\ I_0, J_0 \in \binom{\omega}{n},\ n < \omega \Big\}.$$

We will prove that any finite subset of the type above is realized by a finite subsequence of $\bar{a}$. First note that any finite subset of $q(\bar{x})$ is realized by any subsequence of $\bar{a}$ of the proper length by the definition of EM-type. Then we only need to pay attention to the set on the right.

We prove that for k and n arbitrary large and every $\varphi_1, \ldots, \varphi_k \in L_{x_{\restriction n}}(A)$ there is an infinite $H \subseteq \omega$ such that $a_{\restriction H}$ realizes

$$\# \qquad \Big\{ \varphi_i(x_{\restriction I_0}) \leftrightarrow \varphi_i(x_{\restriction J_0}) \ : \ I_0, J_0 \in \binom{\omega}{n} \text{ and } i \in [k] \Big\}.$$

Consider the subsets of $[k]$ as colors and let f be the coloring of $\binom{\omega}{n}$ that maps I_0 to the set $\{i \,:\, \varphi_i(a_{\restriction I_0})\}$. By the Ramsey theorem, there is some infinite monochromatic set $H \subseteq \omega$. Hence

$$\Big\{\varphi_i(a_{\restriction I_0}) \leftrightarrow \varphi_i(a_{\restriction J_0}) \;:\; I_0, J_0 \in \binom{H}{n} \text{ and } i \in [k]\Big\}.$$

As H has order type ω, it is immediate that $a_{\restriction H}$ realizes #, as required to prove the theorem with $I = J = \omega$. □

15.5 Proposition *Let $\bar{a} = \langle a_i : i \in I\rangle$ be a sequence of A-indiscernibles. Then $\bar{a}$ is indiscernible over some model M containing A.*

Proof Fix an model M containing A. By Theorem 15.4 there is an I-sequence of M-indiscernibles $\bar{c}$ such that EM-tp$(\bar{a}/M) \subseteq$ EM-tp$(\bar{c}/M)$. As $\bar{a}$ is an A-indiscernible sequence $\bar{a} \equiv_A \bar{c}$. Therefore $h\bar{c} = \bar{a}$ for some some $h \in \mathrm{Aut}(\mathcal{U}/A)$. Hence $\bar{a}$ is indiscernible over $h[M]$. □

15.6 Exercise Let $\bar{a} = \langle a_i : i \in I\rangle$ be an A-indiscernible sequence and let $J \supseteq I$ with $|J| \leq \kappa$. Then there is an A-indiscernible sequence $\bar{c} = \langle c_i : i \in J\rangle$ such that $c_{\restriction I} = \bar{a}$. □

15.7 Exercise Let $p(x) \in S(\mathcal{U})$ be a global type invariant over A. Let $a, b \vDash p_{\restriction A}(x)$. Prove that there is a sequence $\bar{c} = \langle c_i : i < \omega\rangle$ such that $a, \bar{c}$ and $b, \bar{c}$ are both sequences of A-indiscernibles. □

3 Indempotent orbits in semigroups

In this and the following sections we focus on semigroups definable in a first-order structure. For a lighter notation, we identify our semigroup with $\mathcal{G}$, which here denotes the domain of a sort in a many-sorted monster model. The language contains, among others, the symbol $\cdot$ which is interpreted as a binary associative operation on $\mathcal{G}$.

We fix a set of parameters A, not necessarily all of the same sort. For any two sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$ we define

$$\mathcal{A} \cdot_A \mathcal{B} \quad = \quad \Big\{a{\cdot}b \;:\; a \in \mathcal{A},\ b \in \mathcal{B} \text{ and } a \mathrel{\perp\!\!\!\!\smile}_A b\Big\}$$

In this and the next section we abbreviate $\mathcal{O}(a/A)$, the orbit of a under $\mathrm{Aut}(\mathcal{G}/A)$, with a_A. We write $a \cdot_A \mathcal{B}$ for $\mathcal{O}(a/A) \cdot_A \mathcal{B}$. Similarly for $\mathcal{A} \cdot_A b$ and $a \cdot_A b$.

15.8 Proposition *If $\mathcal{A}$ is type definable over A then so is $\mathcal{A} \cdot_A b$ for any b.*

Proof The set $\mathcal{A} \cdot_A b$ is the union of $\mathcal{A} \cdot_A \{c\}$ as c ranges in b_A. The set $\mathcal{A} \cdot_A \{c\}$ is type definable, say by the the type $\exists y\, p(x, y, c)$ where

$$p(x, y, z) \quad = \quad y \mathrel{\perp\!\!\!\!\smile}_A c \;\;\wedge\;\; y{\cdot}c \equiv_A x \;\;\wedge\;\; y \in \mathcal{A}$$

Note that if f is any A-automorphism, then $\exists y\, p(x, y, fc)$ defines $\mathcal{A} \cdot_A \{fc\}$. Therefore if $q(z) = \mathrm{tp}(b/A)$ then $\exists y, z\, \big[q(z) \cup p(x, y, z)\big]$ defines $\mathcal{A} \cdot_A b$. □

By the invariance of $\mathrel{\perp\!\!\!\!\smile}_A$, for every $f \in \mathrm{Aut}(\mathcal{G}/A)$ we have $f[\mathcal{A} \cdot_A \mathcal{B}] = f[\mathcal{A}] \cdot_A f[\mathcal{B}]$ Therefore, when $\mathcal{A}$ and $\mathcal{B}$ are invariant over A, also $\mathcal{A} \cdot_A \mathcal{B}$ is invariant over A. Below we mainly deal with invariant sets.

15.9 Proposition *For every A-invariant sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.*

$$\mathcal{A} \cdot_A (\mathcal{B} \cdot_A \mathcal{C}) \quad \subseteq \quad (\mathcal{A} \cdot_A \mathcal{B}) \cdot_A \mathcal{C}$$

Proof Let $a \cdot b \cdot c$ be an arbitrary element of the l.h.s. where $a \downarrow_A b \cdot c$ and $b \downarrow_A c$. By extension (Lemma 14.10), there exists a' such that $a \equiv_{A, b \cdot c} a' \downarrow_A b \cdot c, a, b, c$. By transitivity (again Lemma 14.10), $a' \cdot b \downarrow_A c$. Therefore $a' \cdot b \cdot c$ belongs to the r.h.s. Finally, as $a' \equiv_{A, b \cdot c} a$, also $a \cdot b \cdot c$ belongs to the r.h.s. by A-invariance. □

Let $\mathcal{A}$ be a non empty set. When $\mathcal{A} \cdot_A \mathcal{A} \subseteq \mathcal{A}$, we say that it is idempotent (over A).

15.10 Corollary *Assume $\mathcal{B} \subseteq \mathcal{A}$ are both A-invariant. Then if $\mathcal{A}$ is idempotent, also $\mathcal{A} \cdot_A \mathcal{B}$ is idempotent.*

Proof We check that if $\mathcal{A}$ is idempotent so is $\mathcal{A} \cdot_A \mathcal{B}$

$$\begin{aligned}
(\mathcal{A} \cdot_A \mathcal{B}) \cdot_A (\mathcal{A} \cdot_A \mathcal{B}) \quad &\subseteq \quad \mathcal{A} \cdot_A (\mathcal{A} \cdot_A \mathcal{B}) && \text{because } \mathcal{A} \cdot_A \mathcal{B} \subseteq \mathcal{A} \\
&\subseteq \quad (\mathcal{A} \cdot_A \mathcal{A}) \cdot_A \mathcal{B} && \text{by the lemma above} \\
&\subseteq \quad \mathcal{A} \cdot_A \mathcal{B}
\end{aligned}$$

□

We show that, under the assumption of stationarity, the operation $\cdot_A$ is an associative. The quotient map $\mathcal{G} \to \mathcal{G}/{\equiv_A}$ is almost a homeomorphism.

15.11 Proposition *Assume $\downarrow_A$ is 1-stationary, see Definition 14.11. Fix $a \downarrow_A b$ arbitrarily. Then $a' \cdot b' \equiv_A a \cdot b$ for every $a' \equiv_A a$ and $b' \equiv_A b$ such that $a' \downarrow_A b'$. Or, in other words,*

$$(a \cdot b)_A \quad = \quad a \cdot_A b.$$

Proof We prove two inclusions, only the second one requires stationarity.

$\subseteq$ As $a \downarrow_A b$ holds by hypothesis, $a \cdot b \in a \cdot_A b$. The inclusion follows by invariance.

$\supseteq$ By invariance it suffices to show that the l.h.s. contains $a \cdot_A \{b\}$. By extension (Lemma 14.10), there is $a' \in a_A$ such that $a' \downarrow_A b$. We claim that $a' \cdot b \in (a \cdot b)_A$. Both a and a' satisfy $a \equiv_A x \downarrow_A b$. By 1-stationarity, $a \equiv_{A,b} a'$. Hence $a \cdot b \equiv_A a' \cdot b$. □

15.12 Corollary (associativity) *Let M be a model and assume $\downarrow_M$ is 1-stationary. Then for every M-invariant sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.*

$$\mathcal{A} \cdot_M (\mathcal{B} \cdot_M \mathcal{C}) \quad = \quad (\mathcal{A} \cdot_M \mathcal{B}) \cdot_M \mathcal{C}$$

Proof We can assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are M-orbits. Say of a, b, and c respectively. As we are working over a model, we can assume that $a \downarrow_M b \cdot c$ and $b \downarrow_M c$. By Proposition 15.11 the set on the l.h.s. equals $(a \cdot b \cdot c)_M$. By a similar argument the set on the r.h.s. equals $(a' \cdot b' \cdot c')_M$ for some elements a', b', and c'. Proposition 15.9 proves that inclusion $\subseteq$ holds in general. But inclusion between orbits amounts to equality. □

15.13 Lemma *Let M be a model and assume $\downarrow_M$ is 1-stationary. If $\mathcal{A}$ is minimal among the idempotent sets that are type-definable over M, then $\mathcal{A} = b_M$ for some (any) $b \in \mathcal{A}$.*

Proof Fix arbitrarily some $b \in \mathcal{A}$. By Corollary 15.10, the set $\mathcal{A} \cdot_M b$ is contained in $\mathcal{A}$, idempotent and type-definable over M by Proposition 15.8. Therefore by minimality $\mathcal{A} \cdot_M b = \mathcal{A}$. Let $\mathcal{A}' \subseteq \mathcal{A}$ contain those a such that $a \cdot_M b = b_M$. This set is non empty because $b \in \mathcal{A} \cdot_M b$. It is easy to verify that $\mathcal{A}'$ is type-definable over M, b. As it is clearly invariant over M, it is type-definable over M. By associativity it is idempotent. Hence, by minimality, $\mathcal{A}' = \mathcal{A}$. Then $b \in \mathcal{A}'$, which implies $b \cdot_M b = b_M$. That is, b has idempotent orbit. Finally, by minimality, $\mathcal{A} = b_M$. □

15.14 Corollary *Under the same assumptions of the lemma above, every type-definable idempotent set contains an element with an idempotent orbit.* □

4 Hindman theorem

In this section we merge the theory of idempotents presented in Section 3 with the proof of Ramsey's theorem to obtain Hindman's theorem in a straightforward way.

Let $\bar{a}$ be a tuple of elements of $\mathcal{G}$ of length $\leq \omega$. We write $\operatorname{fp}\bar{a}$ for the set of finite products of elements of $\bar{a}$ taken in increasing order. Namely,

$$\operatorname{fp}\bar{a} \quad = \quad \Big\{a_{i_0}\cdot\dots\cdot a_{i_k} \;:\; i_0 < \dots < i_k < |\bar{a}|,\;\; k < |\bar{a}|\Big\}.$$

Let $\lessdot$ be a relation on $\mathcal{G}$. Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{G}$. We say that $\mathcal{A}$ is $\lessdot$-covered by $\mathcal{C}$ if for every $a_1, \dots, a_n \in \mathcal{A}$ there are infinitely many $c \in \mathcal{C}$ such that $a_i \lessdot c$ for all i. When $\mathcal{A} = \mathcal{C}$ we simply sat that $\mathcal{A}$ is $\lessdot$-covered. We say that $\mathcal{A}$ is $\lessdot$-closed if $a \lessdot b \lessdot c$ implies $a \lessdot b{\cdot}c$ for all $a, b, c \in \mathcal{A}$. A $\lessdot$-chain in $\mathcal{G}$ is a tuple $\bar{a} \in \mathcal{G}^{\leq\omega}$ such that $a_i \lessdot a_{i+1}$.

The requirements on $\lessdot$ are hardly restrictive. For example, on a free semigroup we can take the preorder relation given by the length of the words. Or, on any semigroup G, we could take the trivial relation G^2 –the theorem below would remain non trivial.

15.15 Hindman Theorem *Let $\lessdot$ be a relation on a semigroup G. Assume that G is $\lessdot$-closed and $\lessdot$-covered. Then for every finite coloring of G there is an infinite $\lessdot$-chain $\bar{a}$ such that* $\operatorname{fp}\bar{a}$ *is monochromatic.*

Note that this implies that every commutative semigroup G has an infinite monochromatic subset closed under finite sums of distinct elements (order G arbitrarily).

Our proof follows closely the proof of Ramsey's theorem 15.1. The novelty is all in Lemma 15.13.

Proof We interpret G as a structure in a language that extends the natural language of semigroups with a symbol for $\lessdot$ and one for each subset of G. Let $\mathcal{G}$ be a saturated elementary superstucture of G. As observed in Remark 14.13, the language makes $\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\downarrow}_G$ trivially 1-stationary.

We write $\mathcal{G}'$ for the type-definable set $\{x : G \lessdot x\}$, which is non empty because G is $\lessdot$-covered. We claim that $\mathcal{G}'$ is idempotent. In fact, if $a, b \in \mathcal{G}'$ then, as $G \lessdot a, b$ and $a \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\downarrow}_G b$, we must have that $a \lessdot b$. Therefore, from the $\lessdot$-closure of $\lessdot$ we infer $a{\cdot}b \in \mathcal{G}'$.

Let g be an element of $\mathcal{G}'$ with idempotent orbit as given by Corollary 15.14. Let $p(x) \in S(\mathcal{G})$ be a global coheir of $\operatorname{tp}(g/G)$. Let $\bar{g}$ a coheir sequence of $p(x)$, that is,

$$g_i \quad\vDash\quad p_{\restriction G, g_{\restriction i}}(x).$$

We write $\overleftarrow{g}_{\restriction i}$ for the tuple $g_{i-1}, \dots, g_0$. By the idempotency of g_G and Proposition 15.11, $h \equiv_G g$ for all $h \in \operatorname{fp}\overleftarrow{g}_{\restriction i}$ and all i. It follows in particular that $\operatorname{fp}\overleftarrow{g}_{\restriction i}$ is monochromatic, say all its elements have color 1. Now, we use the sequence $\bar{g}$ to define $\bar{a} \in G^\omega$ such that all elements of $\operatorname{fp} \bar{a}$ have color 1.

Assume as induction hypothesis that we have $a_{\restriction i} \in G^i$ such that all elements of $\operatorname{fp}(a_{\restriction i}, g_0)$ have color 1. Our goal is to find a_i such that the same property holds for $\operatorname{fp}(a_{\restriction i+1}, g_0)$.

First we claim that from the induction hypothesis it follows that, for all j, all elements of $\operatorname{fp}(a_{\restriction i}, \bar{g}_{\restriction j})$ have color 1. In fact, the elements of $\operatorname{fp}(a_{\restriction i}, \bar{g}_{\restriction j})$ have the form $b \cdot h$ for some $b \in \operatorname{fp}(a_{\restriction i})$ and $h \in \operatorname{fp}(\bar{g}_{\restriction j})$. As $h \equiv_G g$, we conclude that $b \cdot h \equiv_G b \cdot g_0$, which proves the claim.

Let $\varphi(a_{\restriction i}, g_{i+1}, g_{\restriction i+1})$ say that all elements of $\operatorname{fp}(a_{\restriction i}, \bar{g}_{\restriction i+2})$ have color 1. As $\bar{g}$ is a coheir sequence we can find a_i such that $\varphi(a_{\restriction i}, a_i, g_{\restriction i+1})$. Hence all elements of $\operatorname{fp}(a_{\restriction i+1}, g_{\restriction i+1})$ have color 1. Therefore a_i is as required. □

5 The Hales-Jewett Theorem

The Hales-Jewett Theorem is a purely combinatorial statement that implies the van der Waerden Theorem.

We need a few definitions. We work with the same notation as Section 3. Let $\mathcal{A}$ be an idempotent set that is type-definable over A. We say that an element c is left-minimal (w.r.t. $\mathcal{A}$) if $c \in \mathcal{A} \cdot_A g$ for every $g \in \mathcal{A} \cdot_A c$.

15.16 Proposition *Let $\mathcal{A}$ be idempotent and type-definable over A. Let a be arbitrary. Then $\mathcal{A} \cdot_A a$ contains a left-minimal element c. When $\mathcal{A} \cdot_A a \cap \mathcal{A}$ is non empty, we can also require that c has idempotent orbit.*

Proof If $b \in \mathcal{A} \cdot_A a$ then $\mathcal{A} \cdot_A b \subseteq \mathcal{A} \cdot_A a$. By compacteness we obtain $c' \in \mathcal{A} \cdot_A a$ such that $\mathcal{A} \cdot_A b = \mathcal{A} \cdot_A c'$ for every $b \in \mathcal{A} \cdot_A c'$. Hence every $c \in \mathcal{A} \cdot_A c'$ is left-minimal. Note that if $b \in \mathcal{A}$ then by idempotency $\mathcal{A} \cdot_A b \subseteq \mathcal{A}$. Hence when $\mathcal{A} \cdot_A a$ and $\mathcal{A}$ have non empty intersection, we can also require that $c' \in \mathcal{A}$. Then $\mathcal{A} \cdot_A c'$ is idempotent. Therefore, by Corollary 15.14 there is some $c \in \mathcal{A} \cdot_A c'$ with idempotent orbit. □

15.17 Proposition *Let M be a model and assume $\mathop{\smile\kern-0.9em\mid}_M$ is 1-stationary. Let $\mathcal{A}$ be idempotent and type-definable over M. Let c_M be idempotent and such that $c \cdot_M \mathcal{A}$, $\mathcal{A} \cdot_M c \subseteq \mathcal{A}$. Then*

1. *$c \cdot_M \mathcal{A} \cdot_M c$ contains some g with idempotent orbit*
2. *if moreover c is left-minimal, then $c \equiv_M g$ for every g as in 1.*

Note, parenthetically, that the set in 1 may not be type-definable, therefore Corollary 15.14 does not apply directly and we need an indirect argument.

Proof 1. From $c \cdot_M \mathcal{A} \subseteq \mathcal{A}$ we obtain that $\mathcal{A} \cdot_M c$ is idempotent. As it is also type-definable, by Corollary 15.14 it contains a b with idempotent orbit. Then $b \cdot_M c = b_M$, from which we obtain that $c \cdot_M b$ is idempotent and contained in $c \cdot_M \mathcal{A} \cdot_M c$.

2. From $g \in c \cdot_M \mathcal{A} \cdot_M c$ and the idempotency of c_M we obtain $g_M = c \cdot_M g$. As $g \in \mathcal{A} \cdot_M c$, from the left-minimality of c_M we obtain $c \in \mathcal{A} \cdot_M g$. Hence $c_M = c \cdot_M g$, by the idempotency of g_M. Therefore $c_M = g_M$, which proves 2. □

The following is a technical lemma that is required in the proof of the main theorem.

15.18 Proposition *Let M be a model and assume $\mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.3ex\hbox{$\downarrow$}}_M$ is 1-stationary. Let $\sigma : \mathcal{G} \to \mathcal{G}$ be a semigroup homomorphism definable over M. Then*

1. $\sigma\big[a_M\big] = (\sigma\, a)_M$
2. $\sigma\big[a \cdot_M b\big] = \sigma\, a \cdot_M \sigma\, b.$

Proof 1. As $a \equiv_M a'$ implies $\sigma\, a \equiv_M \sigma\, a'$, inclusion $\subseteq$ is clear. For the converse, note that the type $\exists y\, \big[\sigma\, y = x \wedge y \equiv_M a\big]$ is trivially realized by $\sigma\, a$. By invariance it is equivalent to a type over A. Therefore it is realized by all elements of $(\sigma\, a)_M$. Hence all elements of $(\sigma\, a)_M$ are the image of some element in a_M.

2. We have to prove the following equality

$$\Big\{\sigma(a'{\cdot}b') \;:\; a \equiv_M a' \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.3ex\hbox{$\downarrow$}}_M b' \equiv_M b\Big\} \quad = \quad \Big\{a'{\cdot}b' \;:\; \sigma\, a \equiv_M a' \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.3ex\hbox{$\downarrow$}}_M b' \equiv_M \sigma\, b\Big\}.$$

It suffices to prove one inclusion because by Proposition 15.11 both sides are orbits. We prove $\subseteq$. Note that $a' \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.3ex\hbox{$\downarrow$}}_M b'$ implies $\sigma\, a' \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.3ex\hbox{$\downarrow$}}_M \sigma\, b'$. Hence the set on l.h.s. is contained in the following

$$\Big\{\sigma(a'{\cdot}b') \;:\; \sigma\, a' \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.3ex\hbox{$\downarrow$}}_M \sigma\, b',\; a' \equiv_M a,\; b' \equiv_M b\Big\}.$$

which is in turn contained in the set on the r.h.s. □

Let $\mathcal{G}$ be a semigroup. A nice subsemigroup of $\mathcal{G}$ is a subsemigroup $\mathcal{C}$ with the property that if $a{\cdot}b \in \mathcal{C}$ then both $a, b \in \mathcal{C}$.

15.19 Hales-Jewett Theorem (Koppelberg's version) *Let G be an infinite semigroup and let $C \subset G$ be a nice subsemigroup. Let Σ be a finite set of retractions of G onto C, that is, homomorphisms $\sigma : G \to C$ such that $\sigma_{\restriction C} = \mathrm{id}_C$. Then, for every finite coloring of C, there is an $a \in G \smallsetminus C$ such that $\{\sigma\, a : \sigma \in \Sigma\}$ is monochromatic.*

Proof Let $G \preceq \mathcal{G}$. Here $\mathcal{G}$ is a monster model in a language that expands the natural one with a symbol for all subsets of G. As observed in Remark 14.13, this makes $\mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.3ex\hbox{$\downarrow$}}_G$ trivially 1-stationary. Let $\mathcal{C}$ be the definable set such that $C = G \cap \mathcal{C}$. By elementarity, $\mathcal{C}$ is a nice subsemigroup of $\mathcal{G}$. The language contains also symbols for the retractions $\sigma : \mathcal{G} \to \mathcal{C}$.

By Proposition 15.16, there is a left-minimal $c \in \mathcal{C}$. As $\mathcal{C} \cdot_M c$ is clearly idempotent, we can further require that c has idempotent orbit.

By nicety, $\mathcal{G} \smallsetminus \mathcal{C}$ satisfy the assumptions of Proposition 15.17. Hence, by the first claim of that proposition, there is an idempotent $g \in c \cdot_G (\mathcal{G} \smallsetminus \mathcal{C}) \cdot_G c$. In particular, $g \in \mathcal{G} \smallsetminus \mathcal{C}$. Now apply the second claim of Proposition 15.18, with $\mathcal{C}$ for $\mathcal{A}$, to obtain $\sigma\, g \in c \cdot_G \mathcal{C} \cdot_G c$ for all $\sigma \in \Sigma$. As $\sigma\, g$ is also idempotent, we apply Proposition 15.17 to conclude that $\sigma\, g \equiv_G c$. In particular the set $\{\sigma\, g : \sigma \in \Sigma\}$ is monochromatic.

Though the element g above need not belong to $G \smallsetminus C$, by elementarity $G \smallsetminus C$ contains some a with the same property and this proves the theorem. □

Finally we show how the classical Hales-Jewett theorem follows from its abstract version.

15.20 Hales-Jewett Theorem (Classical version) *Let C be an infinite semigroup. Fix a tuple of variables x and let $F \subseteq C^{|x|}$ be a finite set. Fix also a finite coloring of C. Then there is a non constant term $t(x)$ of the language of semigroups with parameters in C such that*

$\{t(a) : a \in F\}$ *is monochromatic.*

Proof Let G be the set of terms $t(x)$ in the language of semigroups with parameters in C and free variables in x. Then G is a semigroup under the natural operation. For every $a \in C^{|x|}$ the map $\sigma_a : t(x) \mapsto t(a)$ is a retraction. Hence we can apply the theorem above. □

We conclude with a variant of Theorem 15.19 that applies to a broader class of semigroup homomorphisms. For Σ a set of maps $\sigma : G \to C$ and $c \in C$ we define

$$\Sigma^{-1}[c] \quad = \quad \bigcap_{\sigma \in \Sigma} \sigma^{-1}[c]$$

Clearly, when the maps in Σ are retractions, $\Sigma^{-1}[c]$ is non empty for all $c \in C$ because it contains at least c.

15.21 Hales-Jewett Theorem (Yet another variant) *Let Σ be a finite set of homomorphisms $\sigma : G \to C$ between infinite semigroups such that $\Sigma^{-1}[c]$ is non empty for all $c \in C$. Then, for every finite coloring of C, there is a $g \in G$ such that the set $\{\sigma g \ : \ \sigma \in \Sigma\}$ is monochromatic.*

Proof Let $G * C$ be free product of the two semigroups. That is, $G * C$ contains finite sequences of elements of $G \cup C$, below called *words*, that alternate elements in G with elements in C. The product of two words is obtained concatenating them and, when it applies, replacing two contiguous elements of the same group by their product. Note that C is a nice subsemigroups of $G * C$.

Any homomorphism $\sigma : G \to C$ extends canonically to a retraction of $G * C$ to C. In fact, this extension is unique: the elements of G that occur in a word are replaced by their image under σ, finally the elements in the resulting sequence are multiplied. This extension is denoted by the same symbol σ.

Apply Theorem 15.19 to obtain some $w \in G * C$ such that $\{\sigma w : \sigma \in \Sigma\}$ is monochromatic. Suppose $w = c_0 \cdot g_0 \cdot \dots \cdot c_n \cdot g_n$ for some $g_i \in G$ and $c_i \in C$, where one or both of c_0 or g_n could be absent. Pick some $h_i \in \Sigma^{-1}[c_i]$ and let $g = h_0 \cdot g_0 \cdot \dots \cdot h_n \cdot g_n$. Then $\{\sigma g : \sigma \in \Sigma\}$ is monochromatic as required to complete the proof.H □

6 Notes and references

The first application of the algebraic structure of βG (the Stone-Čech compactification of a semigroup G) to Ramsey Theory is the celebrated Galvin-Glazer proof of Hindman's theorem. Here we have used saturated models in place of Stone-Čech compactification. The idea to replace the semigroup βG has been pioneered by Ludomir Newelsi in the study of applications of topological dynamics to model theory.

The original proof of the Hales-Jewett Theorem by Alfred Hales and Robert Jewett is combinatorial. An alternative proof, also combinatorial, has been given by Sheron Shelah. Our proof used ideas taken from [4].

Chapter 16

Lascar invariant sets

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality κ strictly larger than $|L|$. The notation and implicit assumptions are as in Section 9.3.

1 Expansions

This section is only marginally required in the present chapter so it can be postponed with minor consequences.

We will find it convenient to expand the language L with a predicate for a given $\mathcal{D} \subseteq \mathcal{U}^{|z|}$. We denote by $\langle \mathcal{U}, \mathcal{D} \rangle$ the corresponding expansion of $\mathcal{U}$. Generally, we write $L(\mathcal{X})$ for the expanded language but, if the intended interpretation of $\mathcal{X}$ is only going to be $\mathcal{D}$, we may write $L(\mathcal{D})$ and abbreviate $\langle \mathcal{U}\,; \mathcal{D} \rangle \vDash \varphi(\mathcal{X})$ as $\varphi(\mathcal{D})$.

16.1 Remark The definitions above are straightforward when z finite tuple. When z is an infinite tuple the intuition stays the same but a more involved definition is required. In fact, first-order logic does not allow infinitary predicates. We think of $L(\mathcal{X})$ for a two sorted language. The *home sort*, denoted by 0, and the z*-sort*, denoted by z. The expansion $\langle \mathcal{U}, \mathcal{D} \rangle$ has domain $\mathcal{U}$ for the home sort, and $\mathcal{U}^{|z|}$ for the z-sort. Besides the symbols of L, there is a function symbol π_i for every $i < |z|$ which is interpreted as the projection to the i-coordinate. These functions have arity $\langle z, 0 \rangle$ (see Section 13.1 for the notation). There is also a predicate of sort $\langle z \rangle$ interpreted as $\mathcal{D}$. □

What said above is adapted to define the expansion $L(\mathcal{X}_i : i < \lambda)$, where $\mathcal{X}_i$ are predicates of arity $|z_i|$. Again, when the sets $\mathcal{D}_i \subseteq \mathcal{U}^{|z_i|}$ are the only intended interpretation of $\mathcal{X}_i$, we may write $L(\mathcal{D}_i : i < \lambda)$.

If $\mathcal{C}, \mathcal{D} \subseteq \mathcal{U}^{|z|}$ we abbreviate $\langle \mathcal{U}, \mathcal{C} \rangle \equiv_A \langle \mathcal{U}, \mathcal{D} \rangle$ as $\mathcal{C} \equiv_A \mathcal{D}$. We also say that $\mathcal{D}$ is saturated if so is the model $\langle \mathcal{U}\,; \mathcal{D} \rangle$.

16.2 Remark For every $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ there is a saturated $\mathcal{C} \equiv \mathcal{D}$. In fact, it suffices to find a saturated model $\langle \mathcal{U}', \mathcal{D}' \rangle \equiv \langle \mathcal{U}, \mathcal{D} \rangle$ of cardinality κ. By saturation, there is an isomorphism $f : \mathcal{U}' \to \mathcal{U}$. Therefore $f[\mathcal{D}']$ is the required set $\mathcal{C}$. □

16.3 Exercise Prove that if $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ is saturated and invariant over A than it is definable over A. □

16.4 Proposition *If $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ is invariant over A then every A-indiscernible sequence is indiscernible in the language $L(A\,; \mathcal{D})$.*

Proof Let $\bar{c} = \langle c_i : i \in I \rangle$ be an A-indiscernible sequence. For every $I_0, I_1 \in I^{[n]}$ there is an $f \in \mathrm{Aut}(\mathcal{U}/A)$ such that $fc_{\restriction I_0} = c_{\restriction I_1}$, Hence

$$\varphi(c_{\restriction I_0}\,;\mathcal{D}) \quad\leftrightarrow\quad \varphi(fc_{\restriction I_0}\,;f[\mathcal{D}]) \quad\leftrightarrow\quad \varphi(c_{\restriction I_1}\,;\mathcal{D}).$$ □

16.5 Exercise Prove that if $\mathcal{C} \subseteq \mathcal{U}^{|z|}$ is type-definable over B then $\equiv_{A,B}$ implies $\equiv_{A\,;\mathcal{C}}$. □

2 Lascar strong types

Let $\mathcal{D} \subseteq \mathcal{U}^{|z|}$, where z is a tuple of length $< \kappa$. The orbit of $\mathcal{D}$ over A is the set

$$o(\mathcal{D}/A) \quad=\quad \Big\{f[\mathcal{D}] \;:\; f \in \mathrm{Aut}(\mathcal{U}/A)\Big\}$$

So, $\mathcal{D}$ is invariant over A when $o(\mathcal{D}/A) = \big\{\mathcal{D}\big\}$. We say that $\mathcal{D}$ is Lascar invariant over A if it is invariant over every model $M \supseteq A$. Recall that this means that if $a \equiv_M c$ for some model M containing A then $a \in \mathcal{D} \leftrightarrow c \in \mathcal{D}$.

16.6 Proposition *Let $\lambda = |L_z(A)|$. There are at most 2^{2^λ} sets $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ that are Lascar invariant over A.*

Proof Let N be a model containing A of cardinality $\leq \lambda$. Every set that is Lascar invariant over A is invariant over N. As $|L_z(N)| = \lambda$ the bound follows from Proposition 14.1. □

16.7 Theorem *For every $\mathcal{D}$ and every $A \subseteq M$ the following are equivalent*

1. *$\mathcal{D}$ is Lascar invariant over A;*
2. *every set in $o(\mathcal{D}/A)$ is M-invariant;*
3. *$o(\mathcal{D}/A)$ has cardinality $\leq 2^{2^{|L(A)|}}$;*
4. *$o(\mathcal{D}/A)$ has cardinality $< \kappa$;*
5. *$c_0 \in \mathcal{D} \leftrightarrow c_1 \in \mathcal{D}$ for every A-indiscernible sequence $\langle c_i : i < \omega\rangle$.*

Proof 1⇒2. This implication is clear because all sets in $o(\mathcal{D}/A)$ are Lascar invariant over A.

2⇒3. When $|M| \leq |L(A)|$ the implication follows from the bounds discussed in Section 14.1. We temporary add this assumption on M. Once the proof of the proposition is completed, is easily seen to be redundant (by 4 and Proposition 16.6).

3⇒4. This implication holds because κ is a strong limit cardinal.

4⇒5. Assume ¬5. Then we can find an A-indiscernible sequence $\langle c_i : i < \kappa\rangle$ such that $c_0 \in \mathcal{D} \not\leftrightarrow c_1 \in \mathcal{D}$. Define

$$E(u\,;v) \quad\Leftrightarrow\quad u \in \mathcal{C} \leftrightarrow v \in \mathcal{C} \quad \text{for every } \mathcal{C} \in o(\mathcal{D}/A).$$

Then $E(u\,;v)$ is an A-invariant equivalence relation. As $\neg E(c_0\,;c_1)$, by Proposition 16.4, indiscernibility over A implies that $\neg E(c_i\,,c_j)$ for every $i < j < \kappa$. Then $E(u\,;v)$ has κ equivalence classes. As κ is inaccessible, this implies ¬4.

5⇒1. Fix any $a \equiv_N b$ where $A \subseteq N$. It suffices to prove that $a \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$. Let $p(z) \in S(\mathcal{U})$ be a global coheir of $\mathrm{tp}(a/N) = \mathrm{tp}(b/N)$. Let $\bar{c} = \langle c_i : i < \omega\rangle$ be a Morley sequence of $p(z)$ over N, a, b. Then both $a, \bar{c}$ and $b, \bar{c}$ are A-indiscernible sequences. Therefore, from 5 we obtain $a \in \mathcal{D} \leftrightarrow c_0 \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$. □

For definable sets Lascar invariance reduces to definablity over the algebraic closure.

16.8 Corollary *For every definable set $\mathcal{D}$ the following are equivalent*

1. $\mathcal{D}$ *is Lascar invariant over* A;
2. $\mathcal{D}$ *is definable over* $\operatorname{acl}^{\mathrm{eq}}(A)$. □

The following corollary easily follows from Theorem 16.7 and Proposition 16.4. As an exercise the reader may wish to prove it using Proposition 15.5.

16.9 Corollary *The following are equivalent*

1. $\mathcal{D}$ *is Lascar invariant over* A;
2. *every sequence of* A*-indiscernibles is* $L(A\,;\mathcal{D})$*-indiscernible.* □

⚠ Given a tuple $a \in \mathcal{U}^{|z|}$, we write $\mathcal{L}(a/A)$ for the intersection of all sets containing a that are Lascar invariant over A. Clearly $\mathcal{L}(a/A)$ is Lascar invariant over A. The symbol $\mathcal{L}(a/A)$ is not standard.

16.10 Definition *We write* $a \overset{\mathrm{L}}{\equiv}_A b$ *and say that* a *and* b *have the same Lascar strong type over* A *if the equivalence* $a \in \mathcal{D} \leftrightarrow b \in \mathcal{D}$ *holds for every set* $\mathcal{D}$ *that is Lascar invariant over* A *or, in other words, if* $\mathcal{L}(a/A) = \mathcal{L}(b/A)$. *The notation* $\text{L-stp}(a/A) = \text{L-stp}(b/A)$ *and* $a E_{L/A} b$ *are also common in the the literature.* □

16.11 Proposition *The relation* $\overset{\mathrm{L}}{\equiv}_A$ *is the finest equivalence relation with* $< \kappa$ *classes that is invariant over* A. □

Proof Clearly $\overset{\mathrm{L}}{\equiv}_A$ is an equivalence relation invariant over A. Each equivalence class is Lascar invariant over A, hence the number of equivalence classes is bounded by the number of Lascar invariant sets over A. To see that $\overset{\mathrm{L}}{\equiv}_A$ is the finest of such equivalences. Suppose $\mathcal{D}$ is an equivalence class of an A-invariant equivalence relation with $< \kappa$ classes. Then $o(\mathcal{D}/A)$ has also cardinality $< \kappa$. Then $\mathcal{D}$ is Lascar invariant and as such it is union of classes of the relation $\overset{\mathrm{L}}{\equiv}_A$. □

Let $p(x) \in S(\mathcal{U})$ be global type; we say that p is Lascar invariant over A if the sets $\mathcal{D}_{p,\varphi}$ for $\varphi(x\,;z) \in L$ all Lascar invariant over A. (The sets $\mathcal{D}_{p,\varphi}$ are defined in Section 14.1.)

16.12 Proposition *Let* $p(x) \in S(\mathcal{U})$ *be a global type. Then the following are equivalent*

1. $p(x)$ *is Lascar invariant over* A;
2. *every* A*-indiscernible sequence* $\bar{c} = \langle c_i : i < \omega \rangle$ *is also* A, a*-indiscernible for every* $a \vDash p_{\restriction A,\bar{c}}(x)$;
3. *every* A*-indiscernible sequence* $\bar{c} = \langle c_i : i < \omega \rangle$ *is indiscernible over any* $a \vDash p_{\restriction \bar{c}}(x)$.

For convenience all tuples c_i *have length* $|z| = \omega$.

Proof We prove 1⇔3. Equivalence 1⇔2 can be proved similarly because the Lascar invariance of $p(x)$ easily implies the Lascar invariance of the sets $\mathcal{D}_{p,\varphi}$ for all $\varphi(x\,;z) \in L(A)$.

3⇒1 If $p(x)$ is not Lascar invariant over A then $c_0 \in \mathcal{D}_{p,\varphi} \not\leftrightarrow c_1 \in \mathcal{D}_{p,\varphi}$ for some A-indiscernible sequence $\bar{c} = \langle c_i : i < \omega \rangle$ and some $\varphi(x\,;z) \in L$. Then $p(x)$ contains the formula $\varphi(x\,;c_0) \not\leftrightarrow \varphi(x\,;c_1)$. Hence, $\bar{c}$ is not indiscernible over any realization of $p(x)_{\restriction c_0,c_1}$

1⇒3 Assume 1 and fix an A-indiscernible sequence $\bar{c} = \langle c_i : i < \omega \rangle$ and some $a \vDash p_{\restriction \bar{c}}$. We need to prove that for every formula $\varphi(x\,;z') \in L$, where $z' = z_1, \ldots, z_n$,

$$\varphi(a\,;c_0,\ldots,c_{n-1}) \;\leftrightarrow\; \varphi(a\,;c_{i_1},\ldots,c_{i_{n-1}}).$$

holds for every $i_0 < \cdots < i_{n-1} < \omega$. Suppose not and let m be any integer larger than i_{n-1}. Then the following equivalences cannot both be true

$$\varphi(a\,;c_m,\ldots,c_{m+n-1}) \;\leftrightarrow\; \varphi(a\,;c_0,\ldots,c_{n-1});$$

$$\varphi(a\,;c_m,\ldots,c_{m+n-1}) \;\leftrightarrow\; \varphi(a\,;c_{i_0},\ldots,c_{i_{n-1}}).$$

If the first is false, define $c'_k = c_{km}, \ldots, c_{km+n-1}$ for all $k < \omega$. Otherwise, do this only for positive k and set $c'_0 = c_{i_0}, \ldots, c_{i_{n-1}}$. In either cases $\langle c'_k : k < \omega \rangle$ is a sequence of A-indiscernibles and $c'_0 \in \mathcal{D}_{p,\varphi} \not\leftrightarrow c'_1 \in \mathcal{D}_{p,\varphi}$. This contradicts 1. □

16.13 Exercise Prove that the equivalence relation $a \overset{\text{L}}{\equiv}_A b$ is the transitive closure of the relation: there is a sequence $\langle c_i : i < \omega \rangle$ indiscernible over A such that $c_0 = a$ and $c_1 = b$. Hint: apply Theorem 16.7. □

3 The Lascar graph and Newelski's theorem

Here we study Lascar strong types from a different viewpoint. The Lascar graph over A has an arc between all pairs $a, b \in \mathfrak{U}^{|z|}$ such that $a \equiv_M b$ for some model M containing A. We write $d_A(a,b)$ for the distance between a and b in the Lascar graph over A. Let us spell this out: $d_A(a,b) \leq n$ if there is a sequence $a_0, \ldots, a_n$ such that $a = a_0$, $b = a_n$, and $a_i \equiv_{M_i} a_{i+1}$ for some models M_i containing A. We write $d_A(a,b) < \infty$ if a and b are in the same connected component of the Lascar graph over A.

16.14 Proposition *For every* $a \in \mathfrak{U}^{|z|}$

$$\mathcal{L}(a/A) \quad = \quad \{c \,:\, d_A(a,c) < \infty\}.$$

Proof To prove inclusion $\supseteq$ it suffices to show that every Lascar A-invariant set containing a contains the set on the r.h.s. Let $\mathcal{D}$ be Lascar A-invariant, and let $b \in \mathcal{D}$. Then $\mathcal{D}$ contains also every c such that $b \equiv_M c$ for some model M containing A. That is, $\mathcal{D}$ contains every c such that $d_A(b,c) \leq 1$. It follows that $\mathcal{D}$ contains every c such that $d_A(a,c) < \infty$.

To prove inclusion $\subseteq$ we prove the set on the r.h.s. is Lascar A-invariant. Suppose the sequence $a_0, \ldots, a_n$, where $a_0 = a$ and $a_n = c$, witnesses $d_A(a,c) \leq n$ and suppose that $c \equiv_M b$ for some M containing A, then the sequence $a_0, \ldots, a_n, b$ witnesses $d_A(a,b) \leq n+1$. □

We write $\text{Autf}(\mathfrak{U}/A)$ for the subgroup of $\text{Aut}(\mathfrak{U}/A)$ that is generated by the automorphisms that fix point-wise some model M containing A. (The "f" in the symbol stands for *fort*, the French word for *strong*.) It is easy to verify that $\text{Autf}(\mathfrak{U}/A)$ is a normal subgroup of $\text{Aut}(\mathfrak{U}/A)$. The following is a corollary of Proposition 16.14.

16.15 Corollary *The following are equivalent*

1. $a \overset{\text{L}}{\equiv}_A b$

2. $a = fb$ *for some* $f \in \mathrm{Autf}(\mathcal{U}/A)$. □

It may not be immediately obvious that the relation $d_A(z, y) \leq n$ is type-definable.

16.16 Proposition *For every* $n < \omega$ *there is a type* $p_n(z, y) \subseteq L(A)$ *equivalent to* $d_A(z, y) \leq n$.

Proof It suffices to prove the proposition with $n = 1$. Let $\lambda = |L(A)|$ and fix a tuple of distinct variables $w = \langle w_i : i < \lambda \rangle$, then $p_1(z, y) \;=\; \exists w\, p(w, z, y)$ where

$$p(w,z,y) \quad = \quad q(w) \;\cup\; \Big\{\varphi(z,w) \leftrightarrow \varphi(y,w) \;:\; \varphi(z,w) \in L(A)\Big\}$$

and $q(w) \subseteq L(A)$ is a consistent type with the property that all its realizations enumerate a model containing A.

It remains to verify that such a type exist. Let $\langle \psi_i(x, w_{\restriction i}) : i < \lambda \rangle$ be an enumeration of the formulas in $L_{x,w}(A)$, where x is a single variable. Let

$$q(w) \quad = \quad \Big\{\exists x\, \psi_i(x, w_{\restriction i}) \;\rightarrow\; \psi_i(w_i, w_{\restriction i}) \;:\; i < \lambda\Big\}.$$

Any realization of $q(w)$ satisfy the Tarski-Vaught test therefore it enumerates a model containing A. Vice versa it is clear that we can realize $q(w)$ in any model containing A. □

We conclude this section with a theorem of Ludomir Newelski.

The following notions apply generally to any group G acting on some set $\mathcal{X}$ and to any set $\mathcal{D} \subseteq \mathcal{X}$. Below we always have $G = \mathrm{Autf}(\mathcal{U}/A)$ and $\mathcal{X} = \mathcal{U}^{|z|}$. We say that $\mathcal{D}$ is drifting if for every finitely many $f_1, \ldots, f_n \in G$ there is a $g \in G$ such that $g[\mathcal{D}]$ is disjoint from all the $f_i[\mathcal{D}]$. We say that $\mathcal{D}$ is quasi-invariant if for every finitely many $f_1, \ldots, f_n \in G$ the sets $f_i[\mathcal{D}]$ have non-empty intersection. We say that a formula or a type is drifting or quasi-invariant if the set it defines is.

The union of drifting sets need not be drifting. However, by the following lemma it cannot be quasi-invariant.

16.17 Lemma *The union of finitely many drifting sets in not quasi-invariant.*

Proof It is convenient to prove an apparently more general claim. If $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are all drifting and $\mathcal{L}$ is such that for some finite $F \subseteq G$

$$\sharp \qquad\qquad \mathcal{L} \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n],$$

then $\mathcal{L}$ is not quasi-invariant. The claim is vacuously true for $n = 0$. Now, assume n is positive and that the claim holds for $n - 1$. Define $\mathcal{C} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{n-1}$ and rewrite $\sharp$ as follows

$$\mathcal{L} \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{C}] \;\cup\; \bigcup_{h \in F} h[\mathcal{D}_n]$$

Since $\mathcal{D}_n$ is drifting there is a $g \in G$ such that $g[\mathcal{D}_n]$ is disjoint from $h[\mathcal{D}_n]$ for every $h \in F$, which implies that

$$\mathcal{L} \cap g[\mathcal{D}_n] \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{C}].$$

Hence for every h there holds

$$hg^{-1}[\mathcal{L}] \cap h[\mathcal{D}_n] \quad \subseteq \quad \bigcup_{f \in F} hg^{-1}f[\mathcal{C}]$$

So, from $\sharp$ we obtain

$$\mathcal{L} \cap \bigcap_{h \in F} hg^{-1}[\mathcal{L}] \quad \subseteq \quad \bigcup_{f \in F} f[\mathcal{C}] \,\cup\, \bigcup_{h \in F} \bigcup_{f \in F} hg^{-1}f[\mathcal{C}].$$

By the induction hypothesis, the set on the r.h.s. is not quasi-invariant. Hence neither is $\mathcal{L}$, proving the claim and with it the lemma. □

The following is a consequence of Baire's category theorem. We sketch a proof for the convenience of the reader.

16.18 Lemma *Let $p(x) \subseteq L(B)$ and $p_n(x) \subseteq L(A)$, for $n < \omega$, be consistent types such that*

1. $$p(x) \quad \rightarrow \quad \bigvee_{n<\omega} p_n(x)$$

Then there is an $n < \omega$ and a formula $\varphi(x) \in L(A)$ consistent with $p(x)$ such that

2. $$p(x) \wedge \varphi(x) \quad \rightarrow \quad p_n(x)$$

Proof Negate 2 and choose inductively for every $n < \omega$ a formula $\psi_n(x) \in p_n(x)$ such that $p(x) \wedge \neg\psi_0(x) \wedge \cdots \wedge \neg\psi_n(x)$ is consistent. By compactness, this contradicts 1. □

Finally we can prove Newelski's theorem on the diameter of Lascar types.

16.19 Theorem (Newelski) *For every $a \in \mathcal{U}^{|z|}$ the following are equivalent*

1. *$\mathcal{L}(a/A)$ is type-definable;*
2. *$\mathcal{L}(a/A) \;=\; \{c \;:\; d_A(a,c) < n\}$ for some $n < \omega$.*

Proof Implications 2⇒1 holds by Proposition 16.16. We prove 1⇒2. Suppose $\mathcal{L}(a/A)$ is type-definable, say by the type $l(z)$. Let $p(z,y)$ be some consistent type (to be defined below) such that and $p(z,y) \rightarrow l(z) \wedge l(y)$. Then, in particular

$$p(z,y) \quad \rightarrow \quad \bigvee_{n<\omega} d_A(z,y) < n.$$

By Proposition 16.16 and Lemma 16.18, there is some $n < \omega$ and some $\varphi(z,y) \in L(A)$ consistent with $p(z,y)$ such that

$$\sharp_1 \quad p(z,y) \wedge \varphi(z,y) \quad \rightarrow \quad d_A(z,y) < n.$$

Below we define $p(z,y)$ so that for every $\psi(z,y) \in L(A)$

$$\sharp_2 \quad p(z,a) \wedge \psi(z,a) \qquad \text{is non-drifting whenever it is consistent.}$$

Drifting and quasi-invariance are relative to the action of $\mathrm{Autf}(\mathcal{U}/A)$ on $\mathcal{U}^{|z|}$. Then, in particular, $p(z,a) \wedge \varphi(z,a)$ is non-drifting and the theorem follows. In fact, by non-drifting, there are some $a_0, \ldots, a_k \in \mathcal{L}(a/A)$ such that every set $p(\mathcal{U},c) \cap \varphi(\mathcal{U},c)$ for $c \in \mathcal{L}(a/A)$ intersects some $p(\mathcal{U},a_i) \cap \varphi(\mathcal{U},a_i)$. Let m be such that $d_A(a_i,a_j) \leq m$ for every $i,j \leq k$. From $\sharp_1$ we obtain that $d_A(a,c) \leq m + 2n$. As $c \in \mathcal{L}(a/A)$ is arbitrary, the theorem follows.

The required type $p(z,y)$ is union of a chain of types $p_\alpha(z,y)$ defined as follows

$$p_0(z,y) \quad = \quad l(z) \,\cup\, l(y);$$

$$\sharp_3 \qquad \begin{aligned} p_{\alpha+1}(z,y) &= p_\alpha(z,y) \cup \Big\{\neg\psi(z,y) \;:\; p_\alpha(z,a)\wedge\psi(z,a) \text{ is drifting}\Big\};\\ p_\alpha(z,y) &= \bigcup_{n<\alpha} p_n(z,y) \quad \text{for limit } \alpha. \end{aligned}$$

Clearly, the chain stabilizes at some stage $\leq |L(A)|$ yielding a type which satisfies $\sharp_2$. So we only need to prove consistency. We prove that $p_\alpha(z,a)$ is quasi-invariant (so, in particular, consistent). Suppose that $p_n(z,a)$ is quasi-invariant for every $n<\alpha$ but, for a contradiction, $p_\alpha(z,a)$ is not. Then for some $f_1,\ldots,f_k \in \mathrm{Autf}(\mathcal{U}/A)$

$$p_\alpha(z,a) \quad \cup \quad \bigcup_{i=1}^{k} p_\alpha(z,f_i a)$$

is inconsistent. By compactness there is some $n<\alpha$ and some $\psi_i(z,y)$ as in $\sharp_3$ such that

$$p_n(z,a) \quad \rightarrow \quad \neg \bigwedge_{j=1}^{m}\bigwedge_{i=1}^{k} \neg\psi_j(z,f_i a)$$

As $p_n(z,a)$ is quasi-invariant, from Lemma 16.17 we obtain that $p_n(z,f_i a)\wedge\psi_j(z,f_i a)$ is non-drifting for some i,j. Clearly we can replace $f_i a$ with a, then this contradicts the construction of $p_\alpha(z,y)$ and proves the theorem. □

16.20 Exercise Let $\mathcal{L}$ be quasi-invariant and let $\mathcal{D}$ be drifting, prove that $\mathcal{L}\smallsetminus\mathcal{D}$ is quasi-invariant. □

4 Kim-Pillay types

Given a tuple $a\in\mathcal{U}^{|z|}$, we write $\mathcal{K}(a/A)$ for the intersection of all type-definable sets containing a that are Lascar invariant over A. Or, more concisely, the intersection of all sets that are type-definable over a model containing A. We call $\mathcal{K}(a/A)$ the Kim-Pillay strong type over A. Clearly $\mathcal{K}(a/A)$ is Lascar invariant over A. It also easy to see that $\mathcal{K}(a/A)$ is type-definable. In fact, by invariance, we can assume that all types in the intersection above are over M, for any fixed model containing A. Hence $\mathcal{K}(a/A)$ is the minimal type-definable set containing a and closed under the relation $\overset{\mathrm{L}}{\equiv}_A$. It follows that if $b\in\mathcal{K}(a/A)$ then $\mathcal{K}(b/A)\subseteq\mathcal{K}(a/A)$.

To summarize, we recall that we have defined a whole hierarchy of strong types obtained from the intersection of different sets with various sots of invariance

$$\mathcal{L}(a/A) \;\subseteq\; \mathcal{K}(a/A) \;\subseteq\; \mathcal{S}(a/A) \;\subseteq\; \mathcal{O}(a/A).$$

Recall that $\mathcal{S}(a/A)$ was defined after Proposition 13.14.

If $\mathcal{K}(a/A)=\mathcal{K}(b/A)$, we say that a and b have the same Kim-Pillay strong type over A. We abbreviate this by $a\overset{\mathrm{KP}}{\equiv}_A b$. In other words, we write $a\overset{\mathrm{KP}}{\equiv}_A b$ when $a\in\mathcal{D}\leftrightarrow b\in\mathcal{D}$ for every type-definable set $\mathcal{D}$ that is Lascar invariant over A.

⚠ Warning: the symbol $\mathcal{K}(a/A)$ is not standard. The symbol $a\overset{\mathrm{KP}}{\equiv}_A b$ is not unusual, but some author write KP-stp$(a/A)=$ KP-stp(b/A) or $a\;E_{\mathrm{KP}/A}\;b$.

16.21 Proposition *Fix some $a\in\mathcal{U}^{|z|}$ and some $A\subseteq\mathcal{U}$. Then there is a type $e(z;w)\subseteq L(A)$ such that $\mathcal{K}(b/A)=e(\mathcal{U};b)$ for all $b\in\mathcal{O}(a/A)$ and $e(z;w)$ defines an equivalence relation on $\mathcal{O}(a/A)$.*

Proof Notice that $\mathcal{K}(a/A)$ is type-definable over A, a. In fact, if $f \in \operatorname{Aut}(\mathcal{U}/A, a)$ and $\mathcal{D}$ is a set containing a that is type-definable and Lascar invariant over A, then so is $f[\mathcal{D}]$. Therefore $\mathcal{K}(a/A)$ is invariant over A, a. As $\mathcal{K}(a/A)$ is type-definable, invariance implies that it is type-definable over A, a. Let $e(z\,;w) \subseteq L(A)$ be such that $\mathcal{K}(a/A) = e(\mathcal{U}\,;a)$.

We prove that $\mathcal{K}(b/A) = e(\mathcal{U}\,;b)$ for all $b \in \mathcal{O}(a/A)$. Let $f \in \operatorname{Aut}(\mathcal{U}/A)$ be such that $fa = b$. If $\mathcal{D}$ is a type-definable Lascar invariant over A, then so is $f[\mathcal{D}]$. Therefore, f is a bijection between type-definable sets that are Lascar invariant over A and contain a and analogous sets containing b. Then $f[\mathcal{K}(a/A)] = \mathcal{K}(b/A)$ and $\mathcal{K}(b/A) = e(\mathcal{U}\,;b)$ follows.

We prove that $e(b\,;\mathcal{U}) \cap \mathcal{O}(a/A)$ is Lascar invariant over A. Let $f \in \operatorname{Autf}(\mathcal{U}/A)$ and $c \in \mathcal{O}(a/A)$. Then $e(b\,;fc)$ is equivalent to $e(f^{-1}b\,;c)$ which in turn is equivalent to $e(b\,;c)$, by the invariance of $e(\mathcal{U}\,;c)$.

Finally we are ready to prove that $e(z\,;w)$ defines a symmetric relation on $\mathcal{O}(a/A)$. From what proved above, $e(b\,;\mathcal{U}) \cap \mathcal{O}(a/A)$ is a type-definable Lascar invariant set containing b and therefore it contains $\mathcal{K}(b/A)$. We conclude that for all $b, c \in \mathcal{O}(a/A)$

$$e(c\,;b) \quad\leftrightarrow\quad c \in e(\mathcal{U}\,;b) \quad\leftrightarrow\quad c \in \mathcal{K}(b/A) \quad\rightarrow\quad e(b\,;c)$$

Reflexivity is clear; we prove transitivity. As remarked above, $\mathcal{K}(b/A) \subseteq \mathcal{K}(c/A)$, for all $b \in \mathcal{K}(c/A)$ or equivalently $e(b\,;c)$. Hence if $e(b\,;c)$ then

$$e(d\,;b) \quad\rightarrow\quad d \in \mathcal{K}(b/A) \quad\leftrightarrow\quad d \in \mathcal{K}(c/A) \quad\rightarrow\quad e(d\,;c).$$

Which completes the proof. □

16.22 Corollary *For every $a, b \in \mathcal{U}^{|z|}$ and $A \subseteq \mathcal{U}$ the following are equivalent*

1. $a \in \mathcal{K}(b/A)$;
2. $b \in \mathcal{K}(a/A)$;
3. $\mathcal{K}(a/A) = \mathcal{K}(b/A)$. □

The following useful lemma is the key ingredient in the proof of Theorem 16.24.

16.23 Lemma *Let $p(z) \subseteq L(A)$ and let $e(z\,;w) \subseteq L(A)$ define a bounded equivalence relation on $p(\mathcal{U})$. Then there is a type $e'(z\,;w) \subseteq L(A)$ which defines a bounded equivalence relation (on $\mathcal{U}$) and refines $e(z\,;w)$ on $p(\mathcal{U})$.*

Proof Let $\mathcal{C} = \langle \mathcal{C}_i : i < \lambda \rangle$ enumerate the partition of $p(\mathcal{U})$ induced by $e(z\,;w)$. Note that each $\mathcal{C}_i$ is type-definable over A, a for any $a \in \mathcal{C}_i$. If $x = \langle x_i : i < \lambda \rangle$, we write $x \in \mathcal{C}$ for the type that is the conjunction of $x_i \in \mathcal{C}_i$ for $i < \lambda$.

We claim that the required type is

$$e'(a\,;b) \quad\stackrel{\text{def}}{=}\quad \exists x' \in \mathcal{C}\ \exists x'' \in \mathcal{C} \quad a, x' \equiv_A b, x''$$

As the type above is invariant over A, we can assume that $e'(a\,;b)$ is type-definable over A. Moreover, it is clearly a reflexive and symmetric relation, so we only check it is transitive. Suppose $a, x' \equiv_A b, x''$ and $b, y' \equiv_A c, y''$ for some $x', x'', y', y'' \in \mathcal{C}$. Let z be such that $a, x', z \equiv_A b, x'', y'$ then $z \in \mathcal{C}$ and $a, z \equiv_A c, y''$.

The relation $e'(a\,;b)$ clearly refines the equivalence defined by $e(z\,;w)$ when restricted to to $p(\mathcal{U})$. To prove that it is bounded fix some $c \in \mathcal{C}$ and note that $e'(a\,;b)$ is refined by $a \equiv_{A,c} b$, which is bounded. □

Finally, we have the following.

16.24 Theorem *Denote by $e_A(z\,;w)$ be the finest bounded equivalence relation on $\mathcal{U}^{|z|}$ that is type-definable over A. Then for every $a, b \in \mathcal{U}^{|z|}$ the following are equivalent*

1. $a \overset{\mathrm{KP}}{\equiv} b$
2. $e_A(a\,;b)$. □

Proof As $a \overset{\mathrm{KP}}{\equiv} b$ is equivalent to $a \in \mathcal{K}(b/A)$, it suffices to prove that $e_A(\mathcal{U}\,;b) = \mathcal{K}(b/A)$ for every $b \in \mathcal{U}^{|z|}$.

$\subseteq$ The orbit of $e_A(\mathcal{U}\,;b)$ under $\mathrm{Aut}(\mathcal{U}/A)$ has cardinality $< \kappa$. Hence, by Theorem 16.7, it is Lascar invariant over A. As $\mathcal{K}(b/A)$ is the least of such sets, $\mathcal{K}(b/A) \subseteq e_A(\mathcal{U}\,;b)$.

$\supseteq$ Let $e(z\,;w)$ be the type-definable equivalence relation given by Proposition 16.21. The orbit of $\mathcal{K}(b/A)$ under $\mathrm{Aut}(\mathcal{U}/A)$ has cardinality $< \kappa$. Hence the equivalence relation that $e(z\,;w)$ defines on $\mathcal{O}(b/A)$ is bounded. By Lemma 16.23 there is be a type-definable bounded equivalence relation $e'(z\,;w) \subseteq L(A)$ that refines $e(z\,;w)$ on $\mathcal{O}(b/A)$. As $e'(z\,;w)$ is refined by $e_A(z\,;w)$, we obtain $e_A(\mathcal{U}\,;b) \subseteq e(\mathcal{U}\,;b) = \mathcal{K}(b/A)$. □

The logic A-topology, or simply the logic topology when A is empty, is the topology on $\mathcal{U}^{|z|}$ whose closed sets are the type-definable Lascar A-invariant sets. It is clearly a compact topology. Its Kolmogorov quotient, that is $\mathcal{U}^{|z|}/\overset{\mathrm{KP}}{\equiv}$, is Hausdorff. In fact, by Corollary 16.22, the open sets $\neg\mathcal{K}(a/A)$ and $\neg\mathcal{K}(b/A)$ separate $b \overset{\mathrm{KP}}{\not\equiv} a$.

16.25 Exercise Prove that the clopen sets in the logic A-topology are exactly the sets that are definable over $\mathrm{acl}^{\mathrm{eq}} A$. □

5 Notes and references

The original proof of Newelski theorem is rather long and complex. A simplified proof appears in Rodrigo Peláez's thesis [6, Section 3.3]. The proof here is a streamlined version of the latter taken from [10].

Chapter 17

Externally definable sets

In this chapter we fix a signature L, a complete theory T without finite models, and a saturated model $\mathcal{U}$ of inaccessible cardinality $\kappa > |L|$. The notation and implicit assumptions are as in Section 9.3.

1 Approximable sets

Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{U}^{|z|}$. The set $\mathcal{D} \cap A^{|z|}$ is called the trace of $\mathcal{D}$ over A. We write $\mathcal{C} =_A \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ have the same trace on A.

We say that $\mathcal{D}$ is externally definable if it is of the form $\mathcal{D}_{p,\varphi}$ for some global type $p \in S_x(\mathcal{U})$ and some formula $\varphi(x\,;z) \in L$, see Section 14.1 for the notation. We may say that $\mathcal{D}$ is externally definable by $p(x)$ and $\varphi(x\,;z)$. Equivalently, a set $\mathcal{D}$ is externally definable if it is the trace over $\mathcal{U}$ of a set which is definable in some elementary extension of $\mathcal{U}$. Precisely, $\mathcal{D}$ is the trace on $\mathcal{U}$ of a set of the form $\varphi(b'\,;\mathcal{U}')$ where $\mathcal{U}'$ is elementary extension of $\mathcal{U}$ and $b' \in \mathcal{U}'^{|x|}$. The latter interpretation explains the terminology.

⚠ We prefer to deal with external definability in a different, though equivalent, way. This is not the most common approach.

17.1 Definition *We say that $\mathcal{D}$ is approximated by the formula $\varphi(x\,;z)$ if for every finite B there is a $b \in \mathcal{U}^{|x|}$ such that $\varphi(b\,;\mathcal{U}) =_B \mathcal{D}$. If in addition we have that $\varphi(b\,;\mathcal{U}) \subseteq \mathcal{D}$, we say that $\mathcal{D}$ is approximated from below. Similarly, when $\mathcal{D} \subseteq \varphi(b\,;\mathcal{U})$ we say that $\mathcal{D}$ is approximated from above. We may call $\varphi(x\,;z)$ the sort of $\mathcal{D}$.* □

The following proposition is clear by compactness.

17.2 Proposition *For every $\mathcal{D}$ the following are equivalent*

1. *$\mathcal{D}$ is approximated by $\varphi(x\,;z)$;*
2. *$\mathcal{D}$ is externally definable by $\varphi(x\,;z)$.* □

Approximability from below is an adaptation to our context of the notion of *having an honest definition* in [3].

17.3 Definition *We say that the global type $p \in S_x(\mathcal{U})$ is honestly definable if for every $\varphi(x\,;z) \in L$ the set $\mathcal{D}_{p,\varphi}$ is approximated from below (by some formula). We say that p is definable if the sets $\mathcal{D}_{p,\varphi}$ are all definable (over $\mathcal{U}$). Note that the terminology is misleading: honestly definable is weaker than definable.*

17.4 Example Every definable set is trivially approximable. Sets may be approximable by different formulas. For instance, if $T = T_{\mathrm{dlo}}$, then $\mathcal{D} = \{z \in \mathcal{U} : a \leq z \leq b\}$ is approximable both from below and from above by the formula $x_1 < z < x_2$ though it is not definable by this formula.

Now, let $T = T_{\rm rg}$. Then every $\mathcal{D} \subseteq \mathcal{U}$ is approximable and, when $\mathcal{D}$ has small infinite cardinality, it is approximable from above but not from below, see Exercise 6.21. □

In Definition 17.1, the sort $\varphi(x\,;z)$ is fixed (otherwise any set would be approximable) but this requirement of uniformity may be dropped if we allow B to have larger cardinality.

17.5 Proposition *For every $\mathcal{D}$ the following are equivalent*

1. *$\mathcal{D}$ is approximable;*
2. *for every $C \subseteq \mathcal{U}$ of cardinality $\leq |T|$ there is $\psi(z) \in L(\mathcal{U})$ such that $\psi(\mathcal{U}) =_C \mathcal{D}$.*

Similarly, the following are equivalent

3. *$\mathcal{D}$ is approximable from below;*
4. *for every $C \subseteq \mathcal{D}$ of cardinality $\leq |T|$ there is $\psi(z) \in L(\mathcal{U})$ such that $C \subseteq \psi(\mathcal{U}) \subseteq \mathcal{D}$.*

Proof To prove 2⇒1 assume 2 and negate 1 for a contradiction. For each formula $\psi(x\,;z) \in L$ choose a finite set B such that $\psi(b\,;\mathcal{U}) \neq_B \mathcal{D}$ for every $b \in \mathcal{U}^{|x|}$. Let C be the union of all these finite sets. Clearly $|C| \leq |T|$. By 2 there are a formula $\varphi(x\,;z)$ and a tuple c such that $\varphi(c\,;\mathcal{U}) =_C \mathcal{D}$, contradicting the definition of C.

Implication 1⇒2 is obtained by compactness and the equivalence 3⇔4 is proved similarly. □

17.6 Remark If $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ is approximated by $\varphi(x\,;z)$ then so is any $\mathcal{C}$ such that $\mathcal{C} \equiv \mathcal{D}$, see Section 16.1 for the notation. In fact, if the set $\mathcal{D}$ is approximable by $\varphi(x\,;z)$ then for every n

$$\forall z_1,\dots,z_n\ \exists x\ \bigwedge_{i=1}^{n}\big[\varphi(x\,;z_i)\ \leftrightarrow\ z_i \in \mathcal{D}\big].$$

So the same holds for any $\mathcal{C} \equiv \mathcal{D}$. A similar remark apply to approximability from below and from above (e.g. for approximability from below, add the conjunct $\forall z\,\big[\varphi(x\,;z) \to z \in \mathcal{D}\big]$ to the formula above). □

From the following easy observation of Chernikov and Simon [3] we obtain an interesting (and misterious) quantifier elimination result originally due to Shelah, see Corollary 17.21 below.

17.7 Proposition *Let $\mathcal{C} \subseteq \mathcal{U}^{|y\,z|}$ be approximated from below by the formula $\varphi(x\,;y\,z)$. Then $\mathcal{D} = \big\{z : \exists y\ \big(y\,z \in \mathcal{C}\big)\big\}$ is approximated from below by the formula $\exists y\,\varphi(x\,;y\,z)$.*

Proof Let $B \subseteq \mathcal{U}$ be finite. We want $a \in \mathcal{U}^{|x|}$ such that

a. $\qquad \exists y\ \big(y\,b \in \mathcal{C}\big)\ \leftrightarrow\ \exists y\,\varphi(a\,;y\,b) \qquad$ for every $b \in B^{|z|}$

b. $\qquad \forall z\ \Big[\exists y\,\varphi(a\,;y\,z)\ \to\ \exists y\ \big(y\,z \in \mathcal{C}\big)\Big]$

Let $D \subseteq \mathcal{U}$ be a finite set such that

c. $\qquad \exists y \in D^{|y|}\ \big(y\,b \in \mathcal{C}\big)\ \leftrightarrow\ \exists y\ \big(y\,b \in \mathcal{C}\big) \qquad$ for every $b \in B^{|z|}$

As $\mathcal{C}$ is approximable from below, there is an a such that

a'. $\qquad d\,b \in \mathcal{C}\ \leftrightarrow\ \varphi(a\,;d\,b) \qquad$ for every $d\,b \in \big(D \cup B\big)^{|y\,z|}$

b'. $\qquad \forall y\, z\ \Big[\varphi(a\,;y\,z)\ \ \rightarrow\ \ y\,z \in \mathcal{C}\Big]$

We obtain b from b' simply by logic. Implication $\rightarrow$ in a follows from a' and c. Implication $\leftarrow$ follows from b. □

17.8 Corollary *If $p \in S_x(\mathcal{U})$ is honestly definable then the family of sets externally definable by p is closed under quantifiers and Boolean combinations.*

Proof The sets externally definable by $p(x)$ are always closed under Boolean operations. By the proposition above, they are closed under quantifiers. □

2 Ladders and definability

Partitioned formulas have been introduced in Definition 1.14. Here we write $\varphi(x\,;z)^*$ for the partitioned formula where the order of the two tuples is inverted (i.e. it is the opposite of what displayed).

Let $\varphi(x\,;z) \in L(\mathcal{U})$ be a partitioned formula. We say that $\langle a_i\,;b_i\ :\ i < \alpha\rangle$ is a ladder sequence for $\varphi(x\,;z)$ if

$\qquad i \leq j \quad\Leftrightarrow\quad \varphi(a_i\,;b_j) \qquad$ for all $0 \leq i, j \leq \alpha$

We say that $\varphi(x\,;z)$ is stable if for some finite n all ladders have length at most n. Otherwise we say it is unstable or that it has the order property. Note that if a formulas admits ladder sequences of unbounded finite length, then it admits an infinite one, hence it has the order property.

It is clear that $\varphi(x\,;z)$ is stable if and only if $\varphi(x\,;z)^*$ is stable.

The following theorem claims what is arguably one of the most important properties of stable formulas: any set externally definable by a stable formula is definable (by a related formula).

17.9 Theorem *Any $\mathcal{D} \subseteq \mathcal{U}^{|z|}$ approximated by a stable formula is definable. Precisely, if $\varphi(x\,;z)$ is a stable formula that approximates $\mathcal{D}$ then there are $a_{1,1},\ldots,a_{n,m} \in \mathcal{U}^{|x|}$ such that*

$$z \in \mathcal{D} \quad\leftrightarrow\quad \bigvee_{i=1}^{n}\bigwedge_{j=1}^{m}\varphi(a_{i,j}\,;z)$$

Proof The theorem follows immediately from the the three lemmas below. □

The converse of the theorem above holds and is proved separately in Theorem 17.14.

17.10 Remark The conclusion of Theorem 17.9 is often stated in the following apparently more general form: for every $\mathcal{A} \subseteq \mathcal{U}$ there are $a_{1,1},\ldots,a_{n,m} \in \mathcal{A}^{|x|}$ such that

$$z \in \mathcal{D} \quad\leftrightarrow\quad \bigvee_{i=1}^{n}\bigwedge_{j=1}^{m}\varphi(a_{i,j}\,;z)$$

holds for all $z \in \mathcal{A}^{|z|}$.

The proof is exactly the same. In fact, elementarity and saturation are never used. In a sense, no model theory is used, either – just finite combinatorics – unlike in the proof of Theorem 17.14 where compactness is essential. □

17.11 Lemma *If $\mathcal{D}$ is approximated from below by a stable formula $\varphi(x\,;z)$ then*

$$z \in \mathcal{D} \quad \leftrightarrow \quad \bigvee_{i=0}^{n} \varphi(a_i\,;z)$$

for some $a_0, \ldots, a_n \in \mathcal{U}^{|x|}$.

Proof We define by recursion a ladder sequence for $\varphi(x\,;z)$. Suppose that $b_0, \ldots, b_{n-1} \in \mathcal{D}$ have been defined (this assumption is empty if $n = 0$). We first define a_n, then b_n. Choose $a_n \in \mathcal{U}^{|x|}$ such that $b_0, \ldots, b_{n-1} \in \varphi(a_n\,;\mathcal{U}) \subseteq \mathcal{D}$. This is possible because $\mathcal{D}$ is approximated from below. Now, if possible, choose b_n such that

$$b_n \quad \in \quad \mathcal{D} \smallsetminus \bigvee_{i=0}^{n} \varphi(a_i\,;\mathcal{U}).$$

Then $\langle a_i\,;b_i : i \leq n \rangle$ is a ladder sequence. By stability, for some n, the tuple b_n does not exist. This yields the required $a_0, \ldots, a_n$. □

17.12 Lemma *If $\mathcal{D}$ is approximated by a stable formula $\varphi(x\,;z)$. Then, for some m, the formula*

$$\psi(x_0, \ldots, x_m\,;z) \quad = \quad \bigwedge_{j=0}^{m} \varphi(x_j\,;z)$$

approximates $\mathcal{D}$ from below.

Proof Let m be such that there is no ladder sequence for $\varphi(x\,;z)$ of length greater then m. Let $B \subseteq \mathcal{D}$ be finite. We prove that there are some $a_0, \ldots, a_m$ such that $B \subseteq \psi(a_0, \ldots, a_m\,;\mathcal{U}) \subseteq \mathcal{D}$. As in the proof above, we define by recursion a ladder sequence for $\varphi(x\,;z)$. Suppose that $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1} \notin \mathcal{D}$ have been defined. We first define a_n, then b_n. Choose $a_n \in \mathcal{U}^{|x|}$ such that

$$B \ \subseteq \ \varphi(a_n\,;\mathcal{U}) \ \subseteq \ \mathcal{U}^{|z|} \smallsetminus \{b_0, \ldots, b_{n-1}\}.$$

This a_n exists, because $\mathcal{D}$ is approximated by $\varphi(x\,;z)$. Then, if possible, let b_n such that

$$b_n \quad \in \quad \bigwedge_{i=0}^{n} \varphi(a_i, \mathcal{U}) \smallsetminus \mathcal{D}$$

This procedure has to stop at some $n \leq m$. Hence the required parameters are $a_1, \ldots, a_n = a_{n+1} = \cdots = a_m$. □

17.13 Lemma *If $\varphi(x\,;z)$ is a stable formula then for every m the formula*

$$\psi(x_1, \ldots, x_m\,;z) \quad \leftrightarrow \quad \bigwedge_{j=1}^{m} \varphi(x_j\,;z)$$

is stable.

Proof It suffices to prove that if $\varphi_1(x_1\,;z) \wedge \varphi_2(x_2\,;z)$ is unstable then one of the formulas $\varphi_n(x_i\,;z)$ is unstable. For simplicity, we use that instability implies the existence of an infinite ladder (this uses compactness, apparently contradicting Remark 17.10). We leave to the reader to adapt the argument so that compactness is not required.

Let $a_i^1, a_i^2 \in \mathcal{U}^{|x|}$ and $b_i \in \mathcal{U}^{|z|}$ be such that

$$i \leq j \quad \Leftrightarrow \quad \varphi_1(a_i^1\,;b_j) \wedge \varphi_2(a_i^2\,;b_j) \qquad \text{for all } i, j < \omega$$

For $n = 1,2$ let $H_n \subseteq \binom{\omega}{2}$ contain those pairs $j < i$ such that $\neg\varphi_n(a^n_i\,;b_j)$. By the equivalence above $H_1 \cup H_2 = \binom{\omega}{2}$. By the Ramsey Theorem there is an infinite set H such that $\binom{H}{2} \subseteq H_n$ for at least one of $n = 1,2$. Suppose H_1 for definiteness. So, we obtain an infinite sequence a^1_i, b_i such that

$$j < i \quad \Leftrightarrow \quad \neg\varphi_1(a^1_i\,;b_j) \qquad\qquad \text{for all } i,j < \omega$$

hence $\varphi_1(x_1\,;z)$ is unstable. □

This last lemma concludes the proof of Theorem 17.9.

17.14 Theorem *The following are equivalent*

1. $\varphi(x\,;z)$ *is stable;*
2. *every subset of* $\mathcal{U}^{|z|}$ *that is externally definable by* $\varphi(x\,;z)$ *is definable;*
3. *there are* $\leq \kappa$ *subsets of* $\mathcal{U}^{|z|}$ *that are externally definable by* $\varphi(x\,;z)$*;*
4. *there are* $< 2^\kappa$ *subsets of* $\mathcal{U}^{|z|}$ *that are externally definable by* $\varphi(x\,;z)$.

Proof 1⇒2 is clear by Proposition 17.2 and Theorem 17.9.

2⇒3⇒4 are obvious.

4⇒1 is proved by contraposition. Suppose that $\varphi(x\,;z)$ is not stable. By compactness there is a ladder sequence $\langle a_i\,;b_i : i \in I\rangle$ where $I, <_I$ a dense linear order of cardinality κ with 2^κ cuts. By *cut* we mean a subset $c \subseteq I$ that is closed downward. For every such $c \subseteq I$ we pick a global type

$$p_c(x) \quad \supseteq \quad \{\varphi(x\,;b_i) \leftrightarrow i \in c \;:\; i \in I\}$$

Clearly sets $\mathcal{D}_{p_c,\varphi}$ are all distinct. □

17.15 Exercise The following is a version of Harrington's mysterious Lemma (cfr. [9, Lemma 8.3.4]). Let $\varphi(x\,;z) \in L$ be a stable formula and suppose $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ and $\mathcal{A} \subseteq \mathcal{U}^{|x|}$ are approximated by $\varphi(x\,;z)$ and $\varphi(x\,;z)^*$, respectively. Then at least one of the conditions 1 and 2 below occurs

1. a. $\varphi(x\,;z) \;\wedge\; x \in \mathcal{A}$ approximates $\mathcal{B}$ and
   b. $\varphi(x\,;z)^* \wedge\; z \in \mathcal{B}$ approximates $\mathcal{A}$.
2. a. $\varphi(x\,;z) \;\wedge\; x \notin \mathcal{A}$ approximates $\mathcal{B}$ and
   b. $\varphi(x\,;z)^* \wedge\; z \notin \mathcal{B}$ approximates $\mathcal{A}$.

Hint: Note that at least one of 1a or 2a occurs. Hence it suffices to prove that 1a⇒1b and 2a⇒2b. The two implications are essentially equivalent.

3 Vapknik-Chervonenkis dimension

We say that the formula $\varphi(x\,;z) \in L$ has Vapknik-Chervonenkis dimension n if this is the largest finite cardinality of a set $A \subseteq \mathcal{U}^{|x|}$ such that $|S_\varphi(A)| = 2^n$. If such n does not exist, we say that we say that $\varphi(x\,;z)$ has infinite VC-dimension.

Note that the condition $|S_\varphi(A)| = 2^n$ is equivalent to saying that every subset of A is the trace of some definable set of sort $\varphi(x\,;z)$.

For instance, the formula $z_1 < x < z_2$ in $T_{\rm dlo}$ has VC-dimension 1.

Arguing by compactness we obtain the following proposition whose proof is left as an exercise for the reader.

17.16 Proposition *The following are equivalent*

1. $\varphi(x\,;z) \in L$ *has finite VC-dimension;*
2. *there is no infinite set* $A \subseteq \mathcal{U}^{|x|}$ *such that every subset of* A *is the trace of some definable set of sort* $\varphi(x\,;z)$. □

From the proposition above and Proposition 17.30 below it follows that all stable formulas have finite VC-dimension.

We say that the a sequence of sentences $\langle \varphi_i : i < \omega \rangle$ converges if the truth value of φ_i is eventually constant.

17.17 Lemma *The following are equivalent*

1. $\varphi(x\,;z) \in L$ *has finite VC-dimension;*
2. $\langle \varphi(a_i\,;b) : i < \omega \rangle$ *converges for any* b *and any indiscernible sequence* $\langle a_i : i < \omega \rangle$.

Proof 1⇒2 Assume ¬2. It suffices to prove that for every $I \subseteq \omega$ the type that says

$$\varphi(a_i\,;z) \quad \Leftrightarrow \quad i \in I$$

is finitely consistent. Let $\psi_{I\restriction n}(x_0, \ldots, x_{n-1})$ the formula that says that there is a z such that

$$\varphi(x_i\,;z) \quad \Leftrightarrow \quad i \in I \quad \text{for all } i < n.$$

is finitely consistent we need to prove that $\psi_{I\restriction n}(a_0, \ldots, a_{n-1})$ is true. If there is a b such that the truth value of $\langle \varphi(a_i\,;b) : i < \omega \rangle$ oscillates at least n times, then we can find $k_0 < \cdots < k_{n-1}$ such that

$$\varphi(a_{k_i}\,;b) \quad \Leftrightarrow \quad i \in I.$$

This proves that $\psi_{I\restriction n}(a_{k_0}, \ldots, a_{k_{n-1}})$ is true. By indiscernibility, also $\psi_{I\restriction n}(a_0, \ldots, a_{n-1})$ holds true.

2⇒1 Let $I \subseteq \omega$ be the set of even integers. Assume ¬1 and let $\langle c_i : i < \omega \rangle$ be an infinite sequence that is shattered by $\varphi(x\,;z)$. Let $\langle a_i : i < \omega \rangle$ be an indiscernible sequence with the same EM-type as $\langle c_i : i < \omega \rangle$. Then $\langle a_i : i < \omega \rangle$ satisfies $\psi_{I\restriction n}(x_0, \ldots, x_{n-1})$ for all n. By compactness there is a b such that

$$\varphi(a_i\,;z) \quad \Leftrightarrow \quad i \quad \text{is even.}$$

This proves ¬2. □

In the next section we need the following corollary.

17.18 Corollary *If* $\mathcal{C} \subseteq \mathcal{U}^{|x|}$ *is a set approximable by a formula with finite VC-dimension, then* $\langle a_i \in \mathcal{C} : i < \omega \rangle$ *converges for any indiscernible sequence* $\langle a_i : i < \omega \rangle$. □

4 Honest definitions

In this section we present a beautiful theorem of Chernikov and Simon [3] and their alternative proof of a famous quantifier elimination result of Shelah.

We write $\neg^n$ for $\neg \overset{n \text{ times}}{\ldots\ldots} \neg$. We abbreviate $\neg^n(\cdot \in \cdot)$ as $\notin^n$.

17.19 Lemma *Let $\mathcal{C}$ be a set approximable by a formula with finite VC-dimension and let A be any (small) set of parameters. Then every global A-invariant type $p(x)$ contains a formula $\psi(x)$ such that either $\psi(x) \to x \in \mathcal{C}$ or $\psi(x) \to x \notin \mathcal{C}$. Moreover, for every sufficiently saturated model M containing A, we can require that $\psi(x) \in L(M)$*

Proof The property we want to prove is elementary in $L(\mathcal{C})$ therefore, by Remark 16.2 we can assume that $\mathcal{C}$ is saturated. By Corollary 17.18, there is no infinite sequence $\langle a_i : i < \omega \rangle$

$$a_i \quad \vDash \quad p_{A,a\restriction i}(x) \ \cup\ \{x \notin^i \mathcal{C}\}$$

Let n be the largest integer such that $\langle a_i : i < n \rangle$ satisfies the condition above. Then

$$p_{A,a\restriction n}(x) \quad \to \quad x \notin^n \mathcal{C}$$

and the first claim of the lemma follows by compactness.

Now let M be a model that contains A and such that $\langle M, \mathcal{C} \rangle \preceq \langle \mathcal{U}, \mathcal{C} \rangle$. Assume also that $\langle M, \mathcal{C} \rangle$ is saturated, we can further require that all $a_i \in M^{|x|}$. Hence $\psi(x) \in L(M)$. □

17.20 Theorem *Let $\mathcal{C}$ be a set approximable by a formula with finite VC-dimension. Then $\mathcal{C}$ is approximable from above and from below.*

Proof Let $A \subseteq \mathcal{C}$ be any non empty (small) set. As above, by Remark 16.2 we can assume that $\mathcal{C}$ is saturated. Let M be a saturated elementary substructure of $\mathcal{U}$, in the language $L(\mathcal{C})$. Define

$$q(x) \quad = \quad \{\varphi(x) \in L(M) \ : \ A \subseteq \varphi(\mathcal{U})\}.$$

By compactness, it suffices to prove that $q(x) \to x \in \mathcal{C}$. Suppose not and fix some $a \vDash q(x) \wedge x \notin \mathcal{C}$. Let $p(x) = \mathrm{tp}(a/M)$. As $p(x)$ is finitely satisfied in A and M is saturated, from Lemma 17.19 yields a formula $\psi(x) \in p(x)$ such that either $\psi(x) \to x \in \mathcal{C}$ or $\psi(x) \to x \notin \mathcal{C}$. But the first contradicts $a \vDash \psi(x)$ and the second cannot hold because $\psi(x)$ is satisfied in $A \subseteq \mathcal{C}$. □

When all formulas have finite VC-dimension, we say that the theory T has the non-independence property or, for short, that T is nip.

Let $\langle \mathcal{D}_i : i < \lambda \rangle$ be the collection of all subsets of $\mathcal{U}$, of arbitrary finite arity, that are externally definable. The expansion of $\mathcal{U}$ to the language $L(\mathcal{X}_i : i < \lambda)$ is called the Shelah expansion of $\mathcal{U}$ and is denoted by $\mathcal{U}^{\mathrm{Sh}}$.

From Theorem 17.20 and Proposition 17.7 we obtain the following.

17.21 Corollary *If T is nip then $\mathcal{U}^{\mathrm{Sh}}$ has L-elimination of quantifiers. (I.e. every formula is Boolean combination of formulas in L and formulas of the form $z \in \mathcal{D}_i$.)* □

5 Stable theories

We say that T is a stable theory if every formula is stable. By Theorem 17.14 this is equivalent to requiring that all externally definable sets are definable.

If $p(x) \in S(\mathcal{U})$ is a global type, a canonical base of $p(x)$ is a definably closed set

$\mathrm{Cb}(p) \subseteq \mathcal{U}^{\mathrm{eq}}$ such that an automorphism $f \in \mathrm{Aut}(\mathcal{U})$ fixes $p(x)$ if and only if it fixes $\mathrm{Cb}(p)$ pointwise. When they exist, canonical bases are unique, see Exercise 17.29.

Clearly, all definable types (Definition 17.3) have a canonical base, namely

$$\mathrm{Cb}(p) \quad = \quad \mathrm{dcl}^{\mathrm{eq}}\Big(\big\{\mathcal{D}_{p,\varphi} \ : \ \varphi(x\,;z) \in L\big\}\Big).$$

Therefore if T is stable, all global types have a canonical base.

We now turn to Lascar invariance. Quite interestingly when T is stable this reduces to a more manageable kind of invariance.

17.22 Proposition *Let T be stable and let $p(x) \in S(\mathcal{U})$. Then the following are equivalent*

1. *$p(x)$ is Lascar invariant over A;*
2. *$p(x)$ is definable over $\mathrm{acl}^{\mathrm{eq}}A$;*
3. *$\mathcal{D}_{p,\varphi} \in \mathrm{acl}^{\mathrm{eq}}A$ for all $\varphi(x\,;z) \in L$.*

Proof 3⇒2⇒1 are clear (stability is not required).

1⇒3 The sets $\mathcal{D}_{p,\varphi}$ are externally definable therefore, by Theorem 17.14, definable (over $\mathcal{U}$). As $p(x)$ is Lascar invariant over A, so are the sets $\mathcal{D}_{p,\varphi}$. Hence they belong to $\mathrm{acl}^{\mathrm{eq}}A$ by Theorem 13.11. □

A type $q(x) \subseteq L(A)$ is stationary if it has a unique global extension that is Lascar invariant over A. The following proposition says that in a stable theory with elimination of imaginaries types over algebraically closed sets are stationary.

17.23 Proposition *If T is stable then every type $q(x) \in S\big(\mathrm{acl}^{\mathrm{eq}}A\big)$ is stationary.*

Proof Let $p_i(x) \in S(\mathcal{U}^{\mathrm{eq}})$, for $i = 1,2$, be two global types that extend $q(x)$ and are invariant over $\mathrm{acl}^{\mathrm{eq}}A$. To prove that $p_1(x) = p_2(x)$, it suffices to show that for every formula $\varphi(x\,;z) \in L$

$$\# \qquad \mathcal{D}_{p_1,\varphi} \quad = \quad \mathcal{D}_{p_2,\varphi}$$

Note that, by Proposition 17.22 both sets belong to $\mathrm{acl}^{\mathrm{eq}}A$. Clearly, for $i = 1,2$, the formula $\forall z\,\big[\varphi(x\,;z) \leftrightarrow z \in \mathcal{D}_{p_i,\varphi}\big]$ belongs to $p_i(x)$. Then both formulas belong to $q(x)$ and # follows. □

17.24 Corollary *If T is stable then the following are equivalent*

1. *$a \overset{\mathrm{L}}{\equiv}_A b$, see Definition 16.10;*
2. *$a \overset{\mathrm{Sh}}{\equiv}_A b$, see Definition 13.13.*

Proof 1⇒2. This is left as an exercise to the reader (stability is not required).

2⇒1. Assume $a \overset{\mathrm{Sh}}{\equiv}_A b$. By Proposition 13.14 this is equivalent to $a \equiv_{\mathrm{acl}^{\mathrm{eq}}A} b$. Let $q(x) = \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}A) = \mathrm{tp}(b/\mathrm{acl}^{\mathrm{eq}}A)$. Let $p(x) \in S(\mathcal{U}^{\mathrm{eq}})$ be the unique global type that is invariant over $\mathrm{acl}^{\mathrm{eq}}A$ and extends $q(x)$ which we obtain from by Proposition 17.23. Let $\bar{c} = \langle c_i : i < \omega\rangle$ be such that $c_i \vDash p{\restriction}\,\mathrm{acl}^{\mathrm{eq}}(A), a, b, c_{\restriction i}$. Then $a, \bar{c}$ and $b, \bar{c}$ are A-indiscernible sequences, which proves 1, see Exercise 16.13. □

We end this section with a characterization of stability which is not directly related with the properties discussed above.

17.25 Proposition *The following are equivalent*

1. *T is stable;*
2. *every A-indiscernible sequence is totally A-indiscernible.*

Proof 2⇒1. Assume ¬1 and let $\varphi(x\,;z)$ be an unstable formula witnessed by the ladder sequence $\langle a_i\,;b_i : i < \omega\rangle$. Let $\langle a'_i\,;b'_i : i < \omega\rangle$ be indiscernible sequence with the same Ehrenfeucht-Mostowski type as $\langle a_i\,;b_i : i < \omega\rangle$. This is not totally indiscernible because $\varphi(a'_i\,;b'_j)$ if and if $i < j$.

1⇒2. Assume ¬2 and let $\langle a_i : i < \omega\rangle$ be an A-indiscernible sequence, which is not totally A-indiscernible. Then there is a formula $\varphi(x,y) \in L(A)$ and some $i < j$ such that $\varphi(a_i, a_j) \wedge \neg\varphi(a_j, a_i)$. By indiscernibility $\varphi(a_i, a_j)$ holds if and only if $i \leq j$. Hence $\varphi(x\,;y)$ is not stable. □

17.26 Exercise Prove that the following are equivalent

1. T is stable;
2. there is an infinite set $A \subseteq \mathcal{U}^{|x|}$ and a formula $\psi(x\,;y)$ such that A is linearly ordered by the relation $a < b \leftrightarrow \psi(a\,;b)$.

Hint: suppose $\varphi(x\,;z)$ is unstable and let $\langle a_i\,;b_i : i < \omega\rangle$ be an infinite ladder sequence, then $A = \{a_i\,b_i : i < \omega\}$ is linearly ordered by $x < z\,;y\,w) = \varphi(x\,;z)$. □

17.27 Exercise Prove that if every formula $\varphi(x\,;z) \in L$ with $|x| = 1$ is stable then T is stable. Hint: by compactness, if all sets approximable by $\varphi(x\,;y,z)$ are definable, so are the sets approximable by $\varphi(x,y\,;z)$. □

17.28 Exercise Prove that strongly minimal theories are stable. □

17.29 Exercise Let $p(x) \in S(\mathcal{U})$. Prove that there is at most one definably closed set $A \subseteq \mathcal{U}^{\mathrm{eq}}$ such that $\mathrm{Aut}(\mathcal{U}/A)$ is the set of automorphisms that fix $p(x)$. □

6 Stability and the number of types

The following proposition highlights the connection between stability and the cardinality of types.

Binary trees of formulas have been introduced in Definition 12.17. Here we restrict to trees $\langle\psi_s : s \in 2^{<\lambda}\rangle$ of a particular form. Namely, $\psi_\varnothing = \top$ and for every $s \in 2^{<\lambda}$ we have $\psi_{s1} = \varphi(x\,;b_s)$ and $\psi_{s0} = \neg\varphi(x\,;b_s)$. When such a binary tree of height ω exists, we say that $\varphi(x\,;z)$ has the binary tree property.

17.30 Proposition *The following are equivalent*

1. *$\varphi(x\,;z)$ is a stable formula;*
2. *$\big|S_\varphi(A)\big| = |A|$ for all infinite sets A;*
3. *$\big|S_\varphi(A)\big| < 2^{|A|}$ for all infinite sets A;*
4. *$\varphi(x\,;z)$ does not have the binary tree property.*

Proof 2⇔3⇔4. This is proved as in Lemma 12.18.

3⇒1. Assume ¬1. Let $I, <_I$ be a dense linear order of cardinality $|A|$. Let x_i and z_i, for $i \in I$, be a tuples of variables. The type that says that

$$i < j \quad \Leftrightarrow \quad \varphi(x_i\,; z_j) \qquad\qquad \text{for all } i, j \in I$$

is finitely consistent by ¬1. If a_i, b_i is a realization of this type, then for every $<_I$-upward closed set $C \subseteq I$ the type

$$\big\{\varphi(x\,; b_j) \ : \ j \in C\big\} \ \cup \ \big\{\neg\varphi(x\,; b_j) \ : \ j \notin C\big\}$$

is a finitely consistent. Any two such types are mutually inconsistent. As there are $2^{|A|}$ Dedekind's cuts in I, ¬3 follows.

1⇒4 Assume $\varphi(x\,; z)$ has the binary tree property witnessed, say, by $\langle b_s : s \in 2^{<\omega}\rangle$. We show that there is a ladder sequence of length n. For $i = 0, \dots, n$ abbreviate b_{1^i} by b_i. Let a_j be such that

$$a_j \quad \models \quad \neg\varphi(x\,; b_j) \wedge \bigwedge_{i=0}^{j-1} \varphi(x\,; b_i).$$

Then

$$i < j \quad \Leftrightarrow \quad \varphi(a_i\,; b_j),$$

so $\langle a_i\,; b_i : i < n\rangle$ is a ladder sequence of length n. □

17.31 Theorem *The following are equivalent*

1. *T is stable;*
2. *$|S(A)| \leq |A|$ some infinite cardinal λ, and all sets A of cardinality $\leq \lambda$.*
3. *$|S(A)| \leq |A|$ for every set A such that $|L| < \mathrm{cf}(A)$.*

Proof 2⇒1. Suppose a formula $\varphi(x\,; z)$ is unstable. Let $\langle z_s : s \in 2^{<\lambda}\rangle$ be a sequence variables of length $|z|$. Let $p(z_s : s \in 2^{<\lambda})$ be the type that says that $\langle z_s : i < 2^{<\lambda}\rangle$ witnesses a binary tree of height λ. As $\varphi(x\,; z)$ is unstable, p is finitely consistent. As λ is an arbitrary infinite cardinal, this contradicts 2.

3⇒2. Trivial.

1⇒3. Proposition 17.30 implies that $|S(A)| \leq |A|^{|L|}$. Therefore, when $|L| < \mathrm{cf}(A)$, we obtain $|S(A)| \leq |A|$. □

17.32 Exercise Prove that $\varphi(x\,; z)$ is stable if and only if $|S_\varphi(\mathfrak{U})| \leq 2^\kappa$. □

17.33 Exercise Prove that $\varphi(x\,; z)$ is stable if and only if every set approximated by $\varphi(x\,; z)$ is definable. □

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