Fourier transform: why should I care?

Massimo Borsero

Università degli Studi di Torino

Browsing through Mathematics
28 April 2014

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- motivations and ideas behind Fourier analysis

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If you want to know more about the subject or you need references, don't hesitate to come to my office, write me at massimo.borsero@unito.it or skype me at massimo.borsero.

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The Maxwell equations!

$$\begin{aligned} \operatorname{rot} \vec{E} &= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, & \operatorname{div} \vec{B} &= 0, \\ \operatorname{rot} \vec{B} &= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\,\vec{j}, & \operatorname{div} \vec{E} &= 4\pi\rho_\varepsilon. \end{aligned}$$

And so light appeared.

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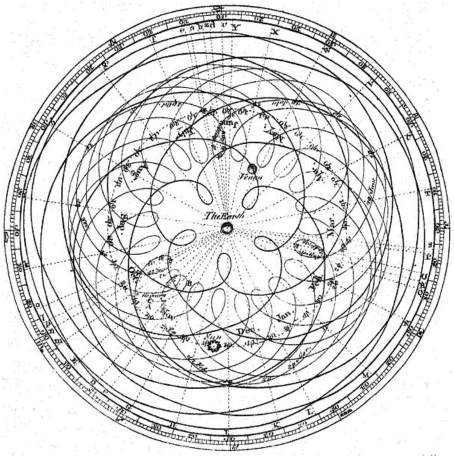
So Ptolemy (A.D. 90 - 168) came up with a new idea: the planets move around in one big circle, but then move around a little circle at the same time.
Well, once they started watching really closely, they realized that even this didn't work, so they put circles on circles on circles...

Eventually, their map of the solar system looked like this:



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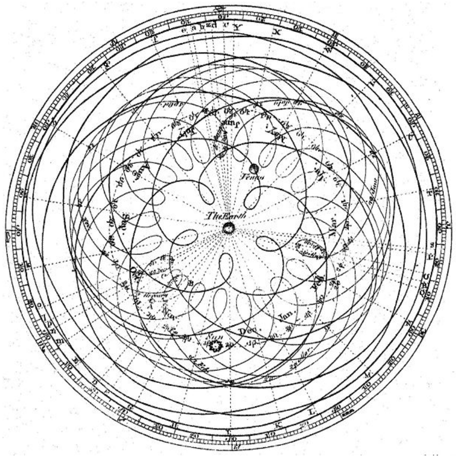

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But it's wrong for an even worse reason:

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Claiming '*planets move around in epicycles*' is mathematically equivalent to saying '*planets move around in two dimensions*'.
Well, that's not saying nothing, but it's not saying much, either!

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The radii R_k are called the **Fourier coefficients** of the signal f.

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'*any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable.*'

Figure: Jean Baptiste Joseph Fourier (21 March 1768 - 16 May 1830)

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Let's see and example.

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If you start by tracing any time-dependent path you want through two-dimensions (your signal), your path can be perfectly-emulated by infinitely many circles of different frequencies, all added up, and the radii of those circles are the Fourier coefficients of your path.

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Be careful, this is not valid in general!

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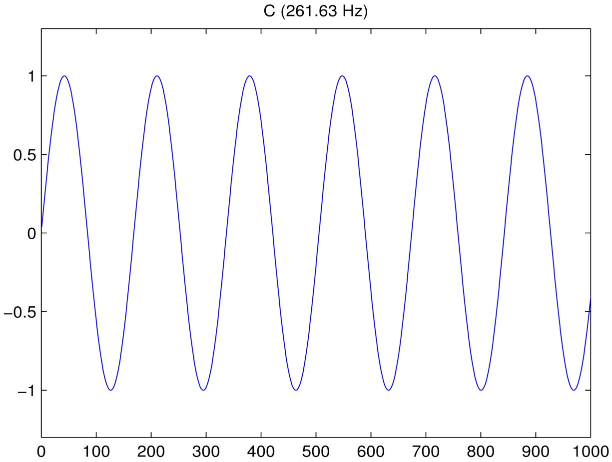
Let's see some examples:

a simple signal...



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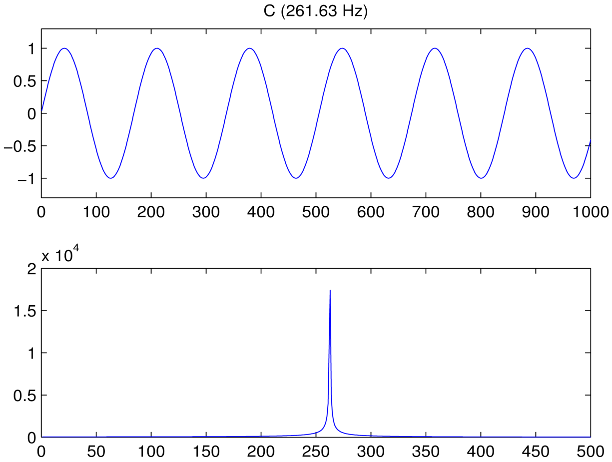

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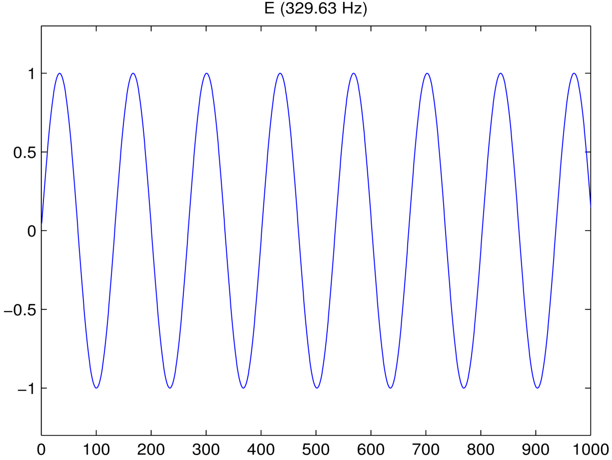

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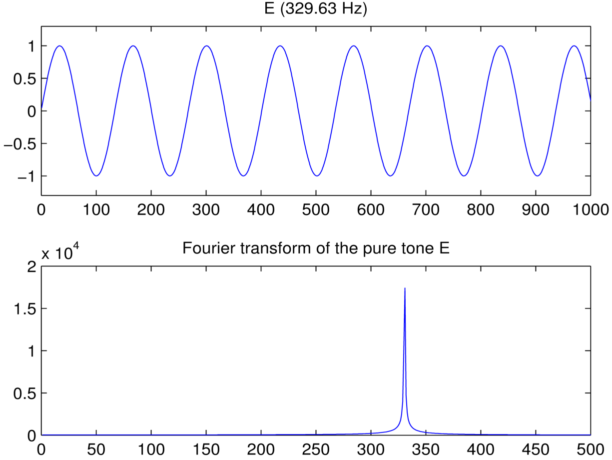

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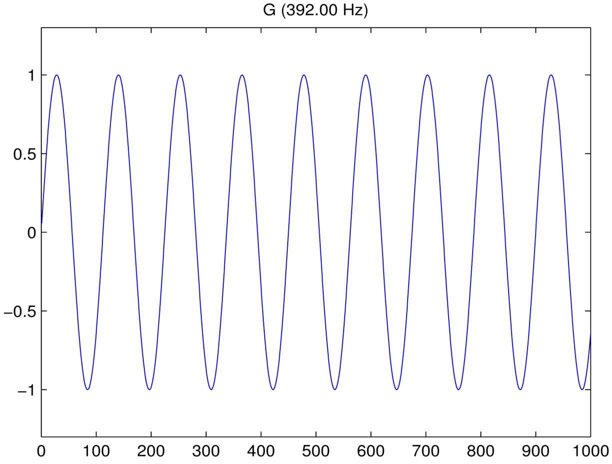

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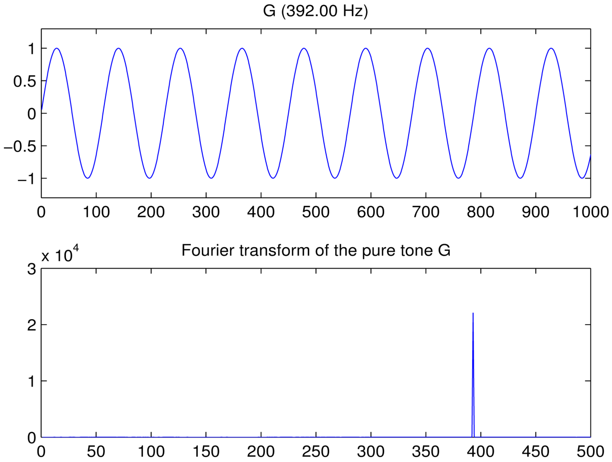

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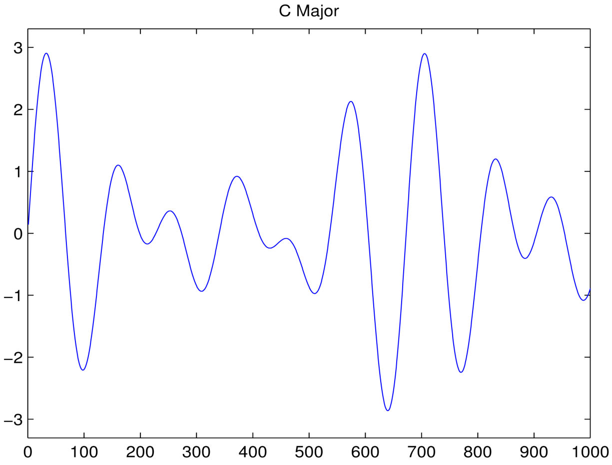

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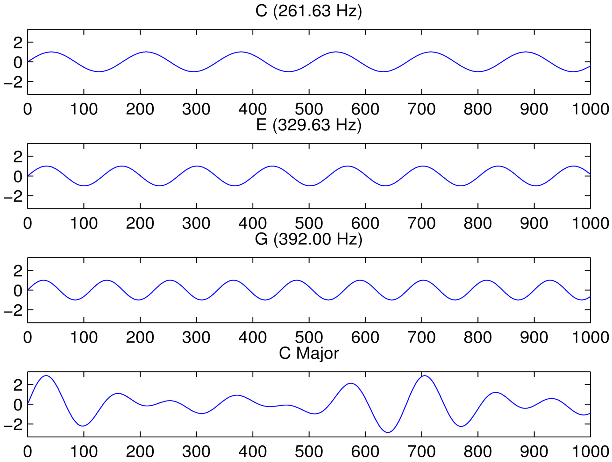

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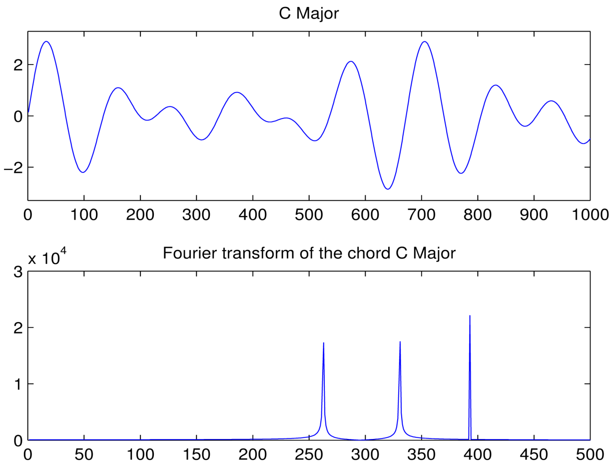

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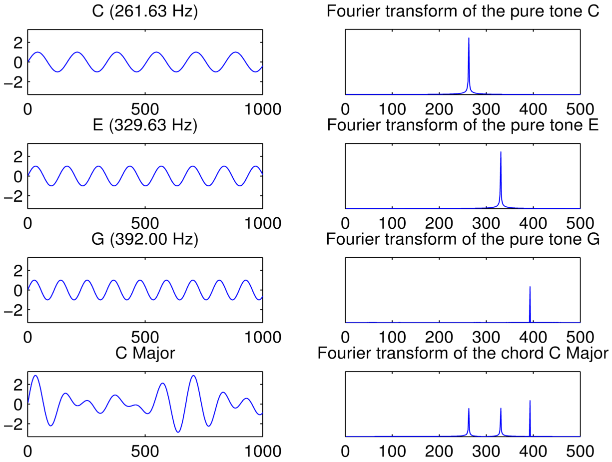

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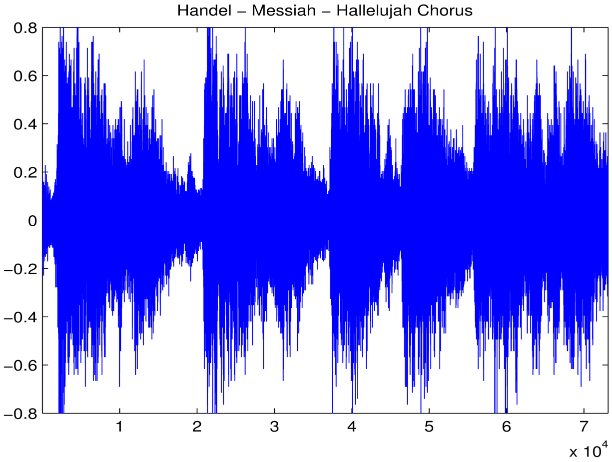

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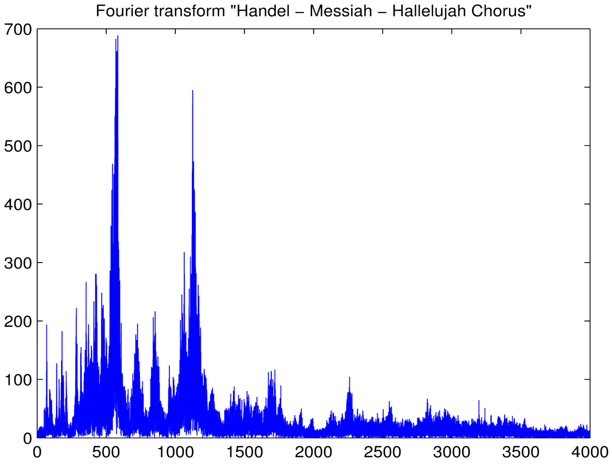

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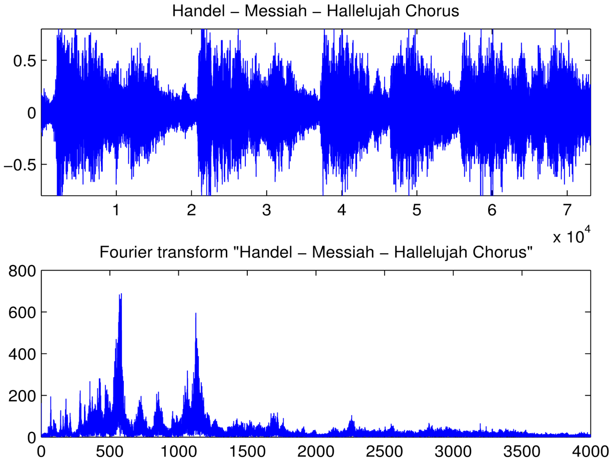

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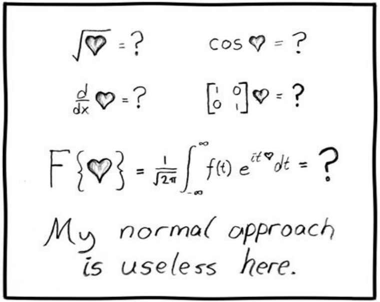

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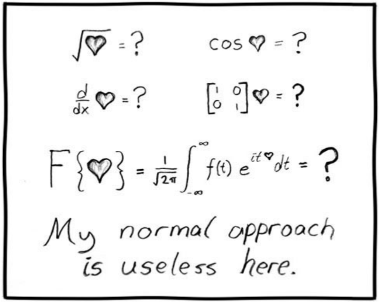

Thank you for your attention! Dank u voor uw aandacht!