1 Pseudofinite yoga

We say that a sentence holds in almost all finite models if it holds in every finite model of sufficienly large cardinality.

Let T_0 be a given theory with arbitrarily large finite models. We define

$$T_{\rm fin} \quad = \quad \{\varphi : M \vDash \varphi \text{ for almost all finite } M \vDash T_0\}.$$

A structure N is a pseudofinite model of T_0 if $N \vDash T_{\rm fin}$. This is equivalent to requiring that every sentence $\varphi \in {\rm Th}(N)$ holds in arbitrarily large finite models of T_0. Note that every model of $T_{\rm fin}$ is infinite.

We now discribe a canonical expansion of a finite model M to a two sorted structure $\langle M, \mathbb{N}\rangle$. The language of $\langle M, \mathbb{N}\rangle$ is denoted by L'.

1. L' expands the language of M and $\mathbb{N}$, where $\mathbb{N}$ is considered as a structure in the language with symbols for all real relations and functions of any finite arity.

2. L' has a function $f_\varphi : M^{|z|} \to \mathbb{N}$ for each formula $\varphi(x\,;z) \in L'$ and every finite tuples of variables $x\,;z$ of the first sort. The interpretation of $f_\varphi(b)$ is the (finite) cardinality of $\varphi(M\,;b)$.

For some application a richer language will be necessary (cfr. Section 2 below). Other times we need to expand M with a few more sorts.

We define

$$T'_{\rm fin} \quad = \quad \{\varphi \in L' : \langle M, \mathbb{N}\rangle \vDash \varphi \text{ for all finite } M \vDash T_0\}.$$

In the sequel $\langle \mathcal{U}, {}^*\mathbb{N}\rangle$ denotes some arbitrary saturated model of $T'_{\rm fin}$.

1 Fact Assume that $\psi \in L'$ holds in every $\langle \mathcal{U}, {}^*\mathbb{N}\rangle$ as above. Then $\langle M, \mathbb{N}\rangle \vDash \psi$ for all sufficiently large finite $M \vDash T_0$. □

Below we write ${}^*|\varphi(x\,;b)|$ for $f_\varphi(b)$. We call this the pseudofinite cardinality of $\varphi(x\,;b)$. By the definition of $T'_{\rm fin}$ it is clear that the pseudofinite cardinality of a definable set $\mathcal{D}$ does not depend on the formula defining it, so we can unambiguously write ${}^*|\mathcal{D}|$.

We will need two preorder relations (a.k.a. quasiorders) on ${}^*\mathbb{N} \smallsetminus \{0,1\}$. We say that the fist linear the second polynomial. For $r, s \in {}^*\mathbb{N} \smallsetminus \{0,1\}$

1. we write $r \le_{\rm l} s$ if $r \le n\,s$ for some positive $n \in \mathbb{N}$.

2. we write $r \le_{\rm p} s$ if $r \le s^n$ for some positive $n \in \mathbb{N}$.

The associated equivalence relation are denoted by $r \underset{\rm l}{\sim} s$ and $r \underset{\rm p}{\sim} s$. Also, we write $r \underset{\rm l}{<} s$ and $r \underset{\rm p}{<} s$ for the associated strict order, i.e. if $n\,r < s$, respectively $r^n < s$ for all $n \in \mathbb{N}$.

2 Proposition Let $\sim$ denote either $\underset{\rm l}{\sim}$ or $\underset{\rm p}{\sim}$. For every non negative $r, s \in {}^*\mathbb{N}$

$$r + s \quad \sim \quad \max\{r,\, s\}$$

In particular, if $\mathcal{D}, \mathcal{C} \subseteq \mathcal{U}^z$ are two definable sets, then

$${}^*|\mathcal{D} \cup \mathcal{C}| \quad \sim \quad \max\left\{{}^*|\mathcal{D}|,\, {}^*|\mathcal{C}|\right\}.$$

Proof In fact

$$\begin{aligned}\max\{r,s\} &\le r+s\\ &\le 2\max\{r,s\}\\ &\le \big(\max\{r,s\}\big)^2\end{aligned}$$

This proves the first equivalence. The second equivalence follows. □

Definable sets $\mathcal{D} \subseteq \mathcal{U}^x$ such that ${}^*|\mathcal{D}| \sim {}^*|\mathcal{U}^x|$ are called large, otherwise we say they are small. The context will clarify if we are referring to the linear or to the polynomial equivalence.

3 Corollary Let $\sim$ denote either $\underset{\mathrm{l}}{\sim}$ or $\underset{\mathrm{p}}{\sim}$. Let $p(x) \subseteq L(\mathcal{U})$ be a type such that $\varphi(\mathcal{U})$ is large for every $\varphi(x)$ that is conjunction of formulas in $p(x)$. Then $p(x)$ has an extension to a complete type with the same property. □

The following fact is immediate. It is stated to illustrate the typical application of the notions introduced above.

4 Fact Let $\sim$ denote either $\underset{\mathrm{l}}{\sim}$ or $\underset{\mathrm{p}}{\sim}$. Let $\varphi(x\,;z) \in L$, where x has finite length. Assume that in every model $\langle \mathcal{U}, {}^*\mathbb{N}\rangle$ there is a $b \in \mathcal{U}^z$ such that $\varphi(\mathcal{U}^x\,;b)$ is large. Then there is an $n \in \mathbb{N}$ such that every sufficiently large finite $M \vDash T_0$ contains a tuple c such that

$$\begin{aligned}\big|M^x\big| &\le n\,\big|\varphi(M^x\,;c)\big| &&\text{if } \sim \text{ is } \underset{\mathrm{l}}{\sim}\\ \big|M^x\big| &\le \big|\psi(M^x\,;c)\big|^n &&\text{if } \sim \text{ is } \underset{\mathrm{p}}{\sim}.\end{aligned}$$ □

The following two proposition only hold for $\underset{\mathrm{p}}{\sim}$.

5 Proposition For every non negative $r,s \in {}^*\mathbb{N}$

$$r\cdot s \ \underset{\mathrm{p}}{\sim}\ \max\{r,s\}$$

Proof In fact

$$\begin{aligned}\max\{r,s\} &\le r\cdot s\\ &\le \big(\max\{r,s\}\big)^2\end{aligned}$$ □

Note that, in particular, ${}^*|\mathcal{U}| \underset{\mathrm{p}}{\sim} {}^*|\mathcal{U}^x|$ for all tuples of variables of finite arity.

6 Corollary Let $\psi(x,z) \in L'$, where x,z are finite tuples of variables of the first sort. Let $\mathcal{A} \subseteq \mathcal{U}$. Assume that

$$\begin{aligned}{}^*|\mathcal{A}| &\underset{\mathrm{p}}{\lesssim} {}^*|\mathcal{U}|\\ {}^*|\psi(a,\mathcal{U}^z)| &\underset{\mathrm{p}}{\lesssim} {}^*|\mathcal{U}| \quad \text{for every } a\in\mathcal{A}.\end{aligned}$$

Then

$${}^*\Big|\bigcup_{a\in\mathcal{A}} \psi(a,\mathcal{U}^z)\Big| \ \underset{\mathrm{p}}{\lesssim}\ {}^*|\mathcal{U}|.$$

Proof Let $r = {}^*|\mathcal{A}|$. Let $s = {}^*|\psi(a,z)|$, where $a \in \mathcal{A}$ is choosen such that s is maximal. Then

$$\begin{aligned}{}^*\Big|\bigcup_{a\in\mathcal{A}} \psi(a,\mathcal{U}^z)\Big| &\le r\cdot s\\ &\underset{\mathrm{p}}{\lesssim} {}^*|\mathcal{U}|\cdot{}^*|\mathcal{U}|\\ &\underset{\mathrm{p}}{\sim} {}^*|\mathcal{U}|.\end{aligned}$$ □

2 Erdős-Hajnal for stable graphs

In this section we assume that the language L' contains a few more symbols (below, for convenience, all finite models considered have as domain a subset of ω).

3. Let X be an n-ary relation symbol of the first sort. Let $\varphi \in L \cup \{X\}$ be a sentence. Then we require that L' has an n-ary predicate A_φ in the first sort. The interpretation of A_φ is a relation of maximal cardinality that makes φ true in M. If more then one relation satisfy the requirement above, we choose the minimal one in the lexicographic order (recall that the domain of M is a subset of ω).

Let $r(\text{-},\text{-})$ be a binary relation symbol. The theory that T_0 says that $r(\text{-},\text{-})$ is an irreflexive and symmetric relation (i.e. a graph) and that it is m-stable, where m is fixed but arbitrary. For the purpose of these notes m-stable means that $R_\Delta(\mathcal{U}) \leq m$, where R_Δ is defined below.

Let $\Delta = \{r(a,x) : a \in \mathcal{U}\}$. Let $\mathcal{S} \subseteq \mathcal{U}$. We denote by $R_\Delta(\mathcal{S})$ the Shelah binary rank of $\mathcal{S}$. That is, the maximal height n of binary tree of $\Delta^{\pm}$-formulas $\langle \varphi_i(x) : s \in {}^{<n}2 \rangle$ such that for every $s \in {}^{n}2$ the type

$$\{\neg^{s_i} \varphi_{s\restriction i}(x) : i \leq n\}$$

has a solution in $\mathcal{S}$.

As T_0 is stable, $R_\Delta(\mathcal{U})$ is finite. Among the large $L(\mathcal{U})$-definable subsets of $\mathcal{U}$ we fix one, say $\mathcal{S} = \sigma(\mathcal{U})$, such that $R_\Delta(\mathcal{S})$ is minimal.

7 Fact For every $\Delta^{\pm}$-definable set $\mathcal{D} \subseteq \mathcal{U}$, exactly one between $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg\mathcal{D}$ is large.

Proof As $\mathcal{S}$ is large, at least one between $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg\mathcal{D}$ is large by Proposition 2. Now, suppose for a contradiction that they both are large. Let $n = R_\Delta(\mathcal{S})$. By the minimality of n, both $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg\mathcal{D}$ have rank n. This contradicts the definition of rank. □

8 Theorem There is a large set $\mathcal{A} \subseteq \mathcal{U}$ that is either a clique or an anticlique.

Proof Let $\mathcal{S} = \sigma(\mathcal{U})$ be as defined above. Let $p(x) \subseteq \Delta^{\pm}$ be maximally consistent among the $\Delta^{\pm}$-types such that $\sigma(x) \wedge \varphi(x)$ is large for every conjunction of formulas in the type. By the fact above, $p(x)$ is $\Delta^{\pm}$-complete, i.e. either $\varphi(x)$ or $\neg\varphi(x)$ is in $p(x)$ for every $\varphi(x) \in \Delta^{\pm}$.

Let $\mathcal{B} = \{a \in \mathcal{S} : r(a,x) \in p\}$. By stability, $\mathcal{B}$ is definable in $L(\mathcal{U})$. At least one between $\mathcal{B}$ and $\mathcal{S} \smallsetminus \mathcal{B}$ is large. Assume the first, the argument when $\mathcal{S} \smallsetminus \mathcal{B}$ is large is similar.

Let $\mathcal{A}$ be a definable subsets of $\mathcal{B}$ that is maximal among those that satisfy the formula $(\forall x, y \in \mathcal{A})\, r(x,y)$. The definition of L' guarantees that such a set exists.

We claim that $\mathcal{A}$ is large, hence it is the clique required by the theorem. So, assume not, and reason for a contradiction.

By the maximality of $\mathcal{A}$, for every $b \in \mathcal{B} \smallsetminus \mathcal{A}$ there is an $a \in \mathcal{A}$ such that $\neg r(a,b)$. We rephrase this as the inclusion

$$\mathcal{B} \smallsetminus \mathcal{A} \ \subseteq \ \bigcup_{a \in \mathcal{A}} \neg r(a, \mathcal{S})$$

By assumption $\mathcal{A}$ is not large, hence $\mathcal{B} \smallsetminus \mathcal{A}$ is large. When $a \in \mathcal{A}$, in particular $r(a, x) \in p$. Therefore, by the fact above, $r(a, \mathcal{S})$ is large and $\neg r(a, \mathcal{S})$ is small. This is a contradiction by Corollary 6. □

Finally the Erdős-Hajnal property is obtained applying Fact 4.

9 **Corollary** There is an n such that in every finite model $M \vDash T_0$ there is a set $A \subseteq M$ of cardinality at least $|M|^{1/n}$ that is either a clique or an anticlique. □