

1 Pseudofinite yoga

As a matter of personal taste I have avoided the use of ultrapowers to deal with pseudofinite structures. Also, I have avoided the notion of pseudofinite dimension, as it is too general for the intended applications.

For convenience, all finite models considered below have as domain a subset of ω . We say that a sentence holds in **almost all finite models** if it holds in every finite model of sufficiently large cardinality.

Let T_0 be a given theory with arbitrarily large finite models. We define

$$T_{\text{fin}} = \{\varphi : M \models \varphi \text{ for almost all finite } M \models T_0\}.$$

A structure N is a **pseudofinite** model of T_0 if $N \models T_{\text{fin}}$. This is equivalent to requiring that every sentence $\varphi \in \text{Th}(N)$ holds in arbitrarily large finite models of T_0 . Note that every model of T_{fin} is infinite. (Other authors include in T_{fin} only sentences true in *all* finite models. Hence they include the finite models among the pseudofinite ones.)

We now describe a canonical expansion of a finite model M to a two sorted structure $\langle M, \mathbb{N} \rangle$. The language of $\langle M, \mathbb{N} \rangle$ is denoted by L' .

1. L' expands the language of M and \mathbb{N} , where \mathbb{N} is considered as a structure in the language with symbols for all real relations and functions of any finite arity.
2. L' has a function $f_\varphi : M^{|z|} \rightarrow \mathbb{N}$ for each formula $\varphi(x; z) \in L'$ and every finite tuples of variables $x; z$ of the first sort. The interpretation of $f_\varphi(b)$ is the (finite) cardinality of $\varphi(M; b)$.
3. (Only required in Section 2.) Let X be an n -ary relation symbol of the first sort. Let $\varphi \in L \cup \{X\}$ be a sentence. Then we require that L' has an n -ary predicate A_φ in the first sort. The interpretation of A_φ is a relation of maximal cardinality that makes φ true in M . If more then one relation satisfy the requirement above, we choose the minimal one in the lexicographic order (recall that the domain of M is a subset of ω).

Finally, we define

$$T'_{\text{fin}} = \{\varphi \in L' : \langle M, \mathbb{N} \rangle \models \varphi \text{ for all finite } M \models T_0\}.$$

In the sequel $\langle \mathcal{U}, * \mathbb{N} \rangle$ denotes some arbitrary saturated model of T'_{fin} .

1 Fact Assume that $\psi \in L'$ holds in every $\langle \mathcal{U}, * \mathbb{N} \rangle$ as above. Then $\langle M, \mathbb{N} \rangle \models \psi$ for all sufficiently large finite $M \models T_0$. □

Below we write $*|\varphi(x; b)|$ for $f_\varphi(b)$. We call this the **pseudofinite cardinality** of $\varphi(x; b)$. By the definition of T'_{fin} it is clear that the pseudofinite cardinality of a definable set \mathcal{D} does not depend on the formula defining it, so we can unambiguously write $*|\mathcal{D}|$.

For $r, s \in * \mathbb{N}$ larger than 1 we write $r \sim s$ if $r \leq m s$ and $s \leq n r$ for some $m, n \in \mathbb{N}$. We write $r \approx s$ if $r \leq s^m$ and $s \leq r^n$ for some $m, n \in \mathbb{N}$. We write $r \prec s$ and $r \prec_p s$ if $n r < s$, respectively $r^n < s$ for all $n \in \mathbb{N}$.

2 Proposition Let \sim denote either \sim or \approx . For every non negative $r, s \in * \mathbb{N}$

$$r + s \sim \max\{r, s\}$$

In particular, if $\mathcal{D}, \mathcal{C} \subseteq \mathcal{U}^z$ are two definable sets, then

$$*|\mathcal{D} \cup \mathcal{C}| \sim \max\{*\mathcal{D}|, *\mathcal{C}|\}.$$

Proof In fact

$$\begin{aligned} \max\{r, s\} &\leq r + s \\ &\leq 2 \max\{r, s\} \\ &\leq (\max\{r, s\})^2 \end{aligned}$$

This proves the first equivalence. The second equivalence follows. \square

Definable sets $\mathcal{D} \subseteq \mathcal{U}^x$ such that $*|\mathcal{D}| \sim *|\mathcal{U}^x|$ are called **large**, otherwise we say they are **small**. The context will clarify if we are referring to the linear or to the polynomial equivalence.

- 3 Corollary** Let \sim denote either \sim or $\sim_{\mathbb{P}}$. Let $p(x) \subseteq L(M)$ be a type such that $\varphi(\mathcal{U})$ is large for every $\varphi(x)$ that is conjunction of formulas in $p(x)$. Then $p(x)$ has an extension to a complete type with the same property. \square

The following fact is immediate. It is stated to illustrate the typical application of the notions introduced above.

- 4 Fact** Let \sim denote either \sim or $\sim_{\mathbb{P}}$. Let $\varphi(x; z) \in L$, where x has finite length. Assume that in every model $\langle \mathcal{U}, *\mathbb{N} \rangle$ there is a $b \in \mathcal{U}^z$ such that $\varphi(\mathcal{U}^x; b)$ is large. Then there is an $n \in \mathbb{N}$ such that every sufficiently large finite $M \models T_0$ contains a tuple c such that

$$\begin{aligned} |M^x| &\leq n |\varphi(M^x; c)| && \text{if } \sim \text{ is } \sim \\ |M^x| &\leq |\psi(M^x; c)|^n && \text{if } \sim \text{ is } \sim_{\mathbb{P}}. \end{aligned}$$

\square

The following two proposition only hold for $\sim_{\mathbb{P}}$.

- 5 Proposition** For every non negative $r, s \in *\mathbb{N}$

$$r \cdot s \sim_{\mathbb{P}} \max\{r, s\}$$

Proof In fact

$$\begin{aligned} \max\{r, s\} &\leq r \cdot s \\ &\leq (\max\{r, s\})^2 \end{aligned}$$

\square

Note that, in particular, $*|\mathcal{U}| \sim_{\mathbb{P}} *|\mathcal{U}^x|$ for all tuples of variables of finite arity.

- 6 Corollary** Let $\psi(x, z) \in L'$, where x, z are finite tuples of variables of the first sort. Let $\mathcal{A} \subseteq \mathcal{U}$. Assume that

$$\begin{aligned} *|\mathcal{A}| &\leq_{\mathbb{P}} *|\mathcal{U}| \\ *|\psi(a, \mathcal{U}^z)| &\leq_{\mathbb{P}} *|\mathcal{U}| \quad \text{for every } a \in \mathcal{A}. \end{aligned}$$

Then

$$*\left| \bigcup_{a \in \mathcal{A}} \psi(a, \mathcal{U}^z) \right| \leq_{\mathbb{P}} *|\mathcal{U}|.$$

Proof Let $r = {}^*\mathcal{A}|$. Let $s = {}^*|\psi(a, z)|$, where $a \in \mathcal{A}$ is chosen such that s is maximal. Then

$$\begin{aligned} {}^*\left|\bigcup_{a \in \mathcal{A}} \psi(a, \mathcal{U}^z)\right| &\leq r \cdot s \\ &\underset{\mathbb{P}}{\leq} {}^*|\mathcal{U}| \cdot {}^*|\mathcal{U}| \\ &\underset{\mathbb{P}}{\approx} {}^*|\mathcal{U}|. \end{aligned}$$

□

2 Erdős-Hajnal for stable graphs

Let $r(-, -)$ be a binary relation symbol. The theory that T_0 says that $r(-, -)$ is an irreflexive and symmetric relation (i.e. a graph) and that it is m -stable, where m is fixed but arbitrary. For the purpose of these notes m -stable means that $R_\Delta(\mathcal{U}) \leq m$, where R_Δ is defined below.

Let $\Delta = \{r(a, x) : a \in \mathcal{U}\}$. Let $\mathcal{S} \subseteq \mathcal{U}$. We denote by $R_\Delta(\mathcal{S})$ the Shelah binary rank of \mathcal{S} . That is, the maximal height n of binary tree of Δ^\pm -formulas $\langle \varphi_i(x) : i \leq n \rangle$ such that for every $s \in {}^n 2$ the type

$$\{\neg^{s_i} \varphi_{s \upharpoonright i}(x) : i \leq n\}$$

has a solution in \mathcal{S} .

As T_0 is stable, $R_\Delta(\mathcal{U})$ is finite. Among the large $L(\mathcal{U})$ -definable subsets of \mathcal{U} we fix one, say $\mathcal{S} = \sigma(\mathcal{U})$, such that $R_\Delta(\mathcal{S})$ is minimal.

7 Fact For every Δ^\pm -definable set $\mathcal{D} \subseteq \mathcal{U}$, exactly one between $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg \mathcal{D}$ is large.

Proof As \mathcal{S} is large, at least one between $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg \mathcal{D}$ is large by Proposition 2. Now, suppose for a contradiction that they both are large. Let $n = R_\Delta(\mathcal{S})$. By the minimality of n , both $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg \mathcal{D}$ have rank n . This contradicts the definition of rank. □

8 Theorem There is a large set $\mathcal{A} \subseteq \mathcal{U}$ that is either a clique or an anticlique.

Proof Let $\mathcal{S} = \sigma(\mathcal{U})$ be as defined above. Let $p(x) \subseteq \Delta^\pm$ be maximally consistent among the Δ^\pm -types such that $\sigma(x) \wedge \varphi(x)$ is large for every conjunction of formulas in the type. By the fact above, $p(x)$ is Δ^\pm -complete, i.e. either $\varphi(x)$ or $\neg \varphi(x)$ is in $p(x)$ for every $\varphi(x) \in \Delta^\pm$.

Let $\mathcal{B} = \{a \in \mathcal{S} : r(a, x) \in p\}$. By stability, \mathcal{B} is definable in $L(\mathcal{U})$. At least one between \mathcal{B} and $\mathcal{S} \setminus \mathcal{B}$ is large. Assume the first, the argument when $\mathcal{S} \setminus \mathcal{B}$ is large is similar.

Let \mathcal{A} be a definable subsets of \mathcal{B} that is maximal among those that satisfy the formula $(\forall x, y \in \mathcal{A}) r(x, y)$. The definition of L' guarantees that such a set exists.

We claim that \mathcal{A} is large, hence it is the clique required by the theorem. So, assume not, and reason for a contradiction.

By the maximality of \mathcal{A} , for every $b \in \mathcal{B} \setminus \mathcal{A}$ there is an $a \in \mathcal{A}$ such that $\neg r(a, b)$. We rephrase this as the inclusion

$$\mathcal{B} \setminus \mathcal{A} \subseteq \bigcup_{a \in \mathcal{A}} \neg r(a, \mathcal{S})$$

By assumption \mathcal{A} is not large, hence $\mathcal{B} \setminus \mathcal{A}$ is large. When $a \in \mathcal{A}$, in particular $r(a, x) \in p$. Therefore, by the fact above, $r(a, \mathcal{S})$ is large and $\neg r(a, \mathcal{S})$ is small. This is a contradiction by Corollary 6. \square

Finally the Erdős-Hajnal property is obtained applying Fact 4.

- 9 Corollary** There is an n such that in every finite model $M \models T_0$ there is a set $A \subseteq M$ of cardinality at least $|M|^{1/n}$ that is either a clique or an anticlique. \square

3 Pseudofinite counting measure

Below, the theory T_0 , the set of formulas Δ , and the rank $m = R_\Delta$ are as defined the previous section.

Let $\mathcal{D} \subseteq \mathcal{U}$ be a definable set. We define

$$\text{Pr}(\mathcal{D}) = \inf \left\{ \frac{m}{n} : m, n \in \mathbb{N} \setminus \{0\} \text{ such that } n^*|\mathcal{D}| \leq m^*|\mathcal{U}| \right\},$$

where the infimum is taken in \mathbb{R} . It is immediate that $\text{Pr}(-)$ is a finite probability measure on the definable subsets of \mathcal{U} . (By the Caratheodory Theorem it can be extended to a probability measure but this is not required here.)

We will write $\mathcal{A} \stackrel{\text{Pr}}{=} \mathcal{D}$ if $\text{Pr}(\mathcal{A} \Delta \mathcal{D}) = 0$.

- 10 Lemma** There are some $\{\wedge\}\Delta^\pm$ -definable sets $\mathcal{B}_1, \dots, \mathcal{B}_n \subseteq \mathcal{U}$, where $n \leq 2^m$, such that for any $\{\wedge\}\Delta^\pm$ -definable set \mathcal{A} ,

$$\mathcal{A} \stackrel{\text{Pr}}{=} \bigcup_{i \in I} \mathcal{B}_i,$$

per qualche $I \subseteq n$. Also, we can require that all \mathcal{B}_i have positive measure and that they form a partition \mathcal{U} .

Proof We prove a slightly more general claim. Let \mathcal{D} be a definable set of positive measure. By induction on $R_\Delta(\mathcal{D}) = m$ we prove that there are $n \leq 2^m$ many $\{\wedge\}\Delta^\pm$ -definable sets $\mathcal{B}_i \subseteq \mathcal{D}$ such that for every $\{\wedge\}\Delta^\pm$ -definable $\mathcal{A} \subseteq \mathcal{D}$ there is an $I \subseteq n$

$$\mathcal{A} \stackrel{\text{Pr}}{=} \bigcup_{i \in I} \mathcal{B}_i.$$

If $R_\Delta(\mathcal{D}) = 0$ then \mathcal{D} is either disjoint of contained in every Δ^\pm -definable set. So the claim holds trivially.

Now, let $R_\Delta(\mathcal{D}) = m + 1$. By the definition of binary rank, there is a Δ -definable set \mathcal{B} such that $R_\Delta(\mathcal{D} \cap \mathcal{B}) = R_\Delta(\mathcal{D} \cap \neg \mathcal{B}) = m$. Apply the induction hypothesis to $\mathcal{D} \cap \mathcal{B}$ if it has positive measure. Do the same for $\mathcal{D} \cap \neg \mathcal{B}$. Note that at least one has positive measure.

It is evident that the sets \mathcal{B}_i obtained in the construction above are disjoint and have positive measure. Also, they cover \mathcal{U} up to a set of measure 0. Hence, replacing \mathcal{B}_1 with some $\mathcal{B}'_1 \stackrel{\text{Pr}}{=} \mathcal{B}_1$ we obtain an actual cover of \mathcal{U} . \square

 Non sono sicuro questa qui sotto sia la corretta traduzione finita.

- 11 Corollary** For every $\varepsilon > 0$ there is an k such that for every finite $M \models T_0$ of cardinality larger than k , there is a partition of M , say $B_0, \dots, B_{n-1} \subseteq M$, where $n \leq 2^m$ such that $|B_i| > \varepsilon |M|$ and for every $\{\wedge\}\Delta^\pm$ -definable set $A \subseteq M$ there is an $I \subseteq n$

such that

$$\left| A \triangle \bigcup_{i \in I} B_i \right| < \varepsilon |M|.$$

4 Stable Szemerédi's regularity

Below, the theory T_0 , the set of formulas Δ , and the rank $m = R_\Delta$ are as defined in Section 2.

12 Lemma There are two partitions of \mathcal{U} , say $\mathcal{B}_1, \dots, \mathcal{B}_n$, where $n \leq 2^m$, and $\mathcal{C}_1, \dots, \mathcal{C}_n$, such that for all i, j either $r(\mathcal{B}_i, \mathcal{C}_j) \equiv_{\overline{\text{Pr}}} \emptyset$ or $r(\mathcal{B}_i, \mathcal{C}_j) \equiv_{\overline{\text{Pr}}} \mathcal{B}_i \times \mathcal{C}_j$.

Proof Let $\mathcal{B}_1, \dots, \mathcal{B}_n$ be the sets given by Lemma 10. For every $J \subseteq n$ let

$$\mathcal{C}_J = \left\{ c \in \mathcal{U} : \bigcup_{i \in J} \mathcal{B}_i \equiv_{\overline{\text{Pr}}} r(\mathcal{U}, c) \right\}$$

By Lemma 10 if $i \in J$ then $r(\mathcal{B}_i, \mathcal{C}_J) \equiv_{\overline{\text{Pr}}} \mathcal{B}_i \times \mathcal{C}_J$, otherwise $r(\mathcal{B}_i, \mathcal{C}_J) \equiv_{\overline{\text{Pr}}} \emptyset$. □

For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{U}$, it is usual call **density** the conditional probability

$$\begin{aligned} d(\mathcal{A}, \mathcal{B}) &= \text{Pr} \left(r(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \times \mathcal{B} \right) \\ &= \frac{\text{Pr} (r(\mathcal{A}, \mathcal{B}))}{\text{Pr}(\mathcal{A} \times \mathcal{B})} \end{aligned}$$

This definition is given with the proviso that \mathcal{A} and \mathcal{B} have non zero probability. The same notation is used for nonempty subsets $A, B \subseteq M$ of a finite model

$$d(A, B) = \frac{|r(A, B)|}{|A| \cdot |B|}$$

13 Corollary There is a partition of \mathcal{U} into sets of positive measure $\mathcal{D}_1, \dots, \mathcal{D}_n$, where $n \leq 2^{2^m+1}$ such that for all i, j either $d(\mathcal{D}_i, \mathcal{D}_j) = 0$ or $d(\mathcal{D}_i, \mathcal{D}_j) = 1$.

Finally, the the stable Szemerédi regularity lemma is obtained applying Fact 4.

 Il corollario qui sotto sembra troppo forte perché n non dipende da ε . Non dev'essere la traduzione al finito giusta.

14 Corollary For every $\varepsilon > 0$ there is an k such that for every finite $M \models T_0$ of cardinality larger than k , there is a partitions of M , say D_1, \dots, D_n , where $n \leq 2^{2^m+1}$ such that for all i, j either $d(D_i, D_j) < \varepsilon$ or $d(D_i, D_j) > 1 - \varepsilon$. □