1 Pseudofinite yoga

As a matter of personal taste I have avoided the use of ultrapowers to deal with pseudofinite structures. Also, I have avoided the notion of pseudofinite dimension, as it is too general for the intended applications.

For convenience, all finite models considered below have as domain a subset of ω. We say that a sentence holds in almost all finite models if it holds in every finite model of sufficienly large cardinality.

Let T_0 be a given theory with arbitrarily large finite models. We define

$$T_{\rm fin} \quad = \quad \{\varphi : M \vDash \varphi \text{ for almost all finite } M \vDash T_0\}.$$

A structure N is a pseudofinite model of T_0 if $N \vDash T_{\rm fin}$. This is equivalent to requiring that every sentence $\varphi \in {\rm Th}(N)$ there holds in arbitrarily large finite models of T_0. Note that every model of $T_{\rm fin}$ is infinite. (Other authors include in $T_{\rm fin}$ only sentences true in *all* finite models. Hence they include the finite models among the pseudofinite ones.)

We now discribe a canonical expansion of a finite model M to a two sorted structure $\langle M, \mathbb{N}\rangle$. The language of $\langle M, \mathbb{N}\rangle$ is denoted by L'.

1. L' expands the language of M and $\mathbb{N}$, where $\mathbb{N}$ is considered as a structure in the language with symbols for all real relations and functions of any finite arity.
2. L' has a function $f_\varphi : M^{|z|} \to \mathbb{N}$ for each formula $\varphi(x\,;z) \in L'$ and every finite tuples of variables $x\,;z$ of the first sort. The interpretation of $f_\varphi(b)$ is the (finite) cardinality of $\varphi(M\,;b)$.
3. (Only reqired in Section 2.) Let X be an n-ary relation symbol of the first sort. Let $\varphi \in L \cup \{X\}$ be a sentence. Then we require that L' has an n-ary predicate A_φ in the first sort. The interpretation of A_φ is a relation of maximal cardinality that makes φ true in M. If more then one relation satisfy the requirement above, we choose the minimal one in the lexicographic order (recall that the domain of M is a subset of ω).

Finally, we define

$$T'_{\rm fin} \quad = \quad \{\varphi \in L' : \langle M, \mathbb{N}\rangle \vDash \varphi \text{ for all finite } M \vDash T_0\}.$$

In the sequel $\langle \mathcal{U}, {}^*\mathbb{N}\rangle$ denotes some arbitrary saturated model of $T'_{\rm fin}$.

1 Fact Assume that $\psi \in L'$ holds in every $\langle \mathcal{U}, {}^*\mathbb{N}\rangle$ as above. Then $\langle M, \mathbb{N}\rangle \vDash \psi$ for all sufficiently large finite $M \vDash T_0$. □

Below we write ${}^*|\varphi(x\,;b)|$ for $f_\varphi(b)$. We call this the pseudofinite cardinality of $\varphi(x\,;b)$. By the definition of $T'_{\rm fin}$ it is clear that the pseudofinite cardinality of a definable set $\mathcal{D}$ does not depend on the formula defining it, so we can unambiguously write ${}^*|\mathcal{D}|$.

For $r, s \in {}^*\mathbb{N}$ larger than 1 we write $r \underset{1}{\sim} s$ if $r \le m\,s$ and $s \le n\,r$ for some $m, n \in \mathbb{N}$. We write $r \underset{\rm p}{\sim} s$ if $r \le s^m$ and $s \le r^n$ for some $m, n \in \mathbb{N}$. We write $r \underset{1}{\lesssim} s$ and $r \underset{\rm p}{\lesssim} s$ if $n\,r < s$, respectively $r^n < s$ for all $n \in \mathbb{N}$.

2 Proposition Let $\sim$ denote either $\underset{1}{\sim}$ or $\underset{\rm p}{\sim}$. For every non negative $r, s \in {}^*\mathbb{N}$

$$r+s \;\sim\; \max\{r,\, s\}$$

In particular, if $\mathcal{D},\mathcal{C}\subseteq\mathcal{U}^z$ are two definable sets, then

$$^*|\mathcal{D}\cup\mathcal{C}| \;\sim\; \max\left\{^*|\mathcal{D}|,\ ^*|\mathcal{C}|\right\}.$$

Proof In fact

$$\begin{aligned}\max\{r,s\} &\leq r+s\\ &\leq 2\max\{r,s\}\\ &\leq \big(\max\{r,s\}\big)^2\end{aligned}$$

This proves the first equivalence. The second equivalence follows. □

Definable sets $\mathcal{D}\subseteq\mathcal{U}^x$ such that $^*|\mathcal{D}| \sim {}^*|\mathcal{U}^x|$ are called large, otherwise we say they are small. The context will clarify if we are referring to the linear or to the polynomial equivalence.

3 Corollary Let $\sim$ denote either $\underset{\mathrm{l}}{\sim}$ or $\underset{\mathrm{p}}{\sim}$. Let $p(x)\subseteq L(M)$ be a type such that $\varphi(\mathcal{U})$ is large for every $\varphi(x)$ that is conjunction of formulas in $p(x)$. Then $p(x)$ has an extension to a complete type with the same property. □

The following fact is immediate. It is stated to illustrate the typical application of the notions introduced above.

4 Fact Let $\sim$ denote either $\underset{\mathrm{l}}{\sim}$ or $\underset{\mathrm{p}}{\sim}$. Let $\varphi(x\,;z)\in L$, where x has finite length. Assume that in every model $\langle\mathcal{U},{}^*\mathbb{N}\rangle$ there is a $b\in\mathcal{U}^z$ such that $\varphi(\mathcal{U}^x\,;b)$ is large. Then there is an $n\in\mathbb{N}$ such that every sufficiently large finite $M\vDash T_0$ contains a tuple c such that

$$\begin{aligned}|M^x| &\leq n\,|\varphi(M^x\,;c)| &&\text{if } \sim \text{ is } \underset{\mathrm{l}}{\sim}\\ |M^x| &\leq |\psi(M^x\,;c)|^n &&\text{if } \sim \text{ is } \underset{\mathrm{p}}{\sim}.\end{aligned}$$ □

The following two proposition only hold for $\underset{\mathrm{p}}{\sim}$.

5 Proposition For every non negative $r,s\in{}^*\mathbb{N}$

$$r\cdot s \;\underset{\mathrm{p}}{\sim}\; \max\{r,s\}$$

Proof In fact

$$\begin{aligned}\max\{r,s\} &\leq r\cdot s\\ &\leq \big(\max\{r,s\}\big)^2\end{aligned}$$ □

Note that, in particular, $^*|\mathcal{U}| \underset{\mathrm{p}}{\sim} {}^*|\mathcal{U}|$ for all tuples of variables of finite arity.

6 Corollary Let $\psi(x,z)\in L'$, where x,z are finite tuples of variables of the first sort. Let $\mathcal{A}\subseteq\mathcal{U}$. Assume that

$$\begin{aligned}^*|\mathcal{A}| &\underset{\mathrm{p}}{\lesssim} {}^*|\mathcal{U}|\\ ^*|\psi(a,\mathcal{U}^z)| &\underset{\mathrm{p}}{\lesssim} {}^*|\mathcal{U}| \quad\text{for every } a\in\mathcal{A}.\end{aligned}$$

Then

$$^*\Big|\bigcup_{a\in\mathcal{A}}\psi(a,\mathcal{U}^z)\Big| \;\underset{\mathrm{p}}{\lesssim}\; {}^*|\mathcal{U}|.$$

Proof Let $r = {}^*|\mathcal{A}|$. Let $s = {}^*|\psi(a,z)|$, where $a \in \mathcal{A}$ is choosen such that s is maximal. Then

$$\begin{aligned} {}^*\Big|\bigcup_{a\in\mathcal{A}}\psi(a,\mathcal{U}^z)\Big| \ &\leq\ r\cdot s \\ &\lessapprox_{\mathrm{P}}\ {}^*|\mathcal{U}|\cdot{}^*|\mathcal{U}| \\ &\approx_{\mathrm{P}}\ {}^*|\mathcal{U}|. \end{aligned}$$

□

2 Erdős-Hajnal for stable graphs

Let $r(\text{-},\text{-})$ be a binary relation symbol. The theory that T_0 says that $r(\text{-},\text{-})$ is an irreflexive and symmetric relation (i.e. a graph) and that it is m-stable, where m is fixed but arbitrary. For the purpose of these notes m-stable means that $R_\Delta(\mathcal{U}) \leq m$, where R_Δ is defined below.

Let $\Delta = \{r(a,x) : a \in \mathcal{U}\}$. Let $\mathcal{S} \subseteq \mathcal{U}$. We denote by $R_\Delta(\mathcal{S})$ the Shelah binary rank of $\mathcal{S}$. That is, the maximal height n of binary tree of $\Delta^{\pm}$-formulas $\langle\varphi_i(x) : s \in {}^{<n}2\rangle$ such that for every $s \in {}^{n}2$ the type

$$\{\neg^{s_i}\varphi_{s\restriction i}(x) : i \leq n\}$$

has a solution in $\mathcal{S}$.

As T_0 is stable, $R_\Delta(\mathcal{U})$ is finite. Among the large $L(\mathcal{U})$-definable subsets of $\mathcal{U}$ we fix one, say $\mathcal{S} = \sigma(\mathcal{U})$, such that $R_\Delta(\mathcal{S})$ is minimal.

7 **Fact** For every $\Delta^{\pm}$-definable set $\mathcal{D} \subseteq \mathcal{U}$, exactly one between $\mathcal{S}\cap\mathcal{D}$ and $\mathcal{S}\cap\neg\mathcal{D}$ is large.

Proof As $\mathcal{S}$ is large, at least one between $\mathcal{S}\cap\mathcal{D}$ and $\mathcal{S}\cap\neg\mathcal{D}$ is large by Proposition 2. Now, suppose for a contradiction that they both are large. Let $n = R_\Delta(\mathcal{S})$. By the minimality of n, both $\mathcal{S}\cap\mathcal{D}$ and $\mathcal{S}\cap\neg\mathcal{D}$ have rank n. This contradicts the definition of rank. □

8 **Theorem** There is a large set $\mathcal{A} \subseteq \mathcal{U}$ that is either a clique or an anticlique.

Proof Let $\mathcal{S} = \sigma(\mathcal{U})$ be as defined above. Let $p(x) \subseteq \Delta^{\pm}$ be maximally consistent among the $\Delta^{\pm}$-types such that $\sigma(x)\wedge\varphi(x)$ is large for every conjunction of formulas in the type. By the fact above, $p(x)$ is $\Delta^{\pm}$-complete, i.e. either $\varphi(x)$ or $\neg\varphi(x)$ is in $p(x)$ for every $\varphi(x) \in \Delta^{\pm}$.

Let $\mathcal{B} = \{a \in \mathcal{S} : r(a,x) \in p\}$. By stability, $\mathcal{B}$ is definable in $L(\mathcal{U})$. At least one between $\mathcal{B}$ and $\mathcal{S}\smallsetminus\mathcal{B}$ is large. Assume the first, the argument when $\mathcal{S}\smallsetminus\mathcal{B}$ is large is similar.

Let $\mathcal{A}$ be a definable subsets of $\mathcal{B}$ that is maximal among those that satisfy the formula $(\forall x,y \in \mathcal{A})\, r(x,y)$. The definition of L' guarantees that such a set exists.

We claim that $\mathcal{A}$ is large, hence it is the clique required by the theorem. So, assume not, and reason for a contradiction.

By the maximality of $\mathcal{A}$, for every $b \in \mathcal{B}\smallsetminus\mathcal{A}$ there is an $a \in \mathcal{A}$ such that $\neg r(a,b)$. We rephrase this as the inclusion

$$\mathcal{B}\smallsetminus\mathcal{A}\ \subseteq\ \bigcup_{a\in\mathcal{A}}\neg r(a,\mathcal{S})$$

By assumption $\mathcal{A}$ is not large, hence $\mathcal{B} \setminus \mathcal{A}$ is large. When $a \in \mathcal{A}$, in particular $r(a, x) \in p$. Therefore, by the fact above, $r(a, \mathcal{S})$ is large and $\neg r(a, \mathcal{S})$ is small. This is a contradiction by Corollary 6. □

Finally the Erdős-Hajnal property is obtained applying Fact 4.

9 Corollary There is an n such that in every finite model $M \vDash T_0$ there is a set $A \subseteq M$ of cardinality at least $|M|^{1/n}$ that is either a clique or an anticlique. □

3 Pseudofinite counting measure

Below, T_0, Δ, $m = R_\Delta$ are as defined the provious section.

Let $\mathcal{D} \subseteq \mathcal{U}$ be a definable set. We define

$$\mu(\mathcal{D}) \quad = \quad \inf\left\{\frac{m}{n} \ : \ m, n \in \mathbb{N} \setminus \{0\} \text{ such that } n\,{}^*|\mathcal{D}| \leq m\,{}^*|\mathcal{U}|\right\},$$

where the infimum is taken in $\mathbb{N}$. It is immediate that $\mu(\text{-})$ is a finite probability measure on the definable subsets of $\mathcal{U}$.

We will write $\mathcal{A} =_\mu \mathcal{D}$ if $\mu(\mathcal{A} \triangle \mathcal{D}) = 0$.

10 Lemma There are some pairwise disjoint $\{\wedge\}\Delta^{\pm}$-definable sets $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1} \subseteq \mathcal{U}$, where $n \leq 2^m$, such that for any $\{\wedge\vee\neg\}\Delta$-definable set $\mathcal{A}$,

$$\mathcal{A} \ =_\mu \ \bigcup_{i \in I} \mathcal{B}_i$$

Per qualche $I \subseteq n$. Finally, we can require that all $\mathcal{B}_i$ have positive measure.

Proof We prove a slightly more general claim. For any definable set $\mathcal{D}$ of positive measure let $\mu_{\mathcal{D}}$ be the contitional measure:

$$\begin{aligned} \mu_{\mathcal{D}}(\mathcal{A}) \quad &= \quad \mu(\mathcal{A}|\mathcal{D}) \\ &= \quad \frac{\mu(\mathcal{A} \cap \mathcal{D})}{\mu(\mathcal{D})} \end{aligned}$$

We prove by induction on $R_\Delta(\mathcal{D}) = m$ that there are $n \leq 2^m$ many $\{\wedge\}\Delta^{\pm}$-definable sets $\mathcal{B}_i$ such that

$$\mathcal{A} \ =_\mu \ \bigcup_{i \in I} \mathcal{B}_i$$

For some $I \subseteq n$. Clearly it suffices to consider $\mathcal{A} \subseteq \mathcal{D}$ that are $\{\wedge\}\Delta^{\pm}$-definable.

If $R_\Delta(\mathcal{D}) = 0$ then $\mathcal{D}$ is either disjoint of contained in every $\Delta^{\pm}$-definable set. So the claim holds trivially.

Now, let $R_\Delta(\mathcal{D}) = m + 1$. By the definition of binary rank, there is a Δ-definable set $\mathcal{B}$ such that $R_\Delta(\mathcal{D} \cap \mathcal{B}) = R_\Delta(\mathcal{D} \cap \neg\mathcal{B}) = m$. Apply the induction hypothesis to $\mathcal{D} \cap \mathcal{B}$ if it has positive measure. Do the same for $\mathcal{D} \cap \neg\mathcal{B}$. Note that at least one has positive measure. □

Non sono sicuro questa sia la corretta traduzione finita.

11 Corollary For every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that for every finite $M \vDash T_0$ of cardinality larger than N, there are $B_0, \ldots, B_{n-1} \subseteq M$, where $n \leq 2^m$ and $\mu(B_i) > \varepsilon$, such that for every $\{\wedge\vee\neg\}\Delta$-definable set $A \subseteq M$ there is an $I \subseteq n$ such that

$$\varepsilon \;>\; \mu\Big(A \,\triangle \bigcup_{i\in I} B_i\Big)$$

4 Stable Szemerédi regularity lemma

12 Lemma Assume that $r(\text{-},\text{-})$ is stable and let $m = R_\Delta(\mathcal{U})$. There are two partitions of $\mathcal{U}$, say $\mathcal{B}_0,\dots,\mathcal{B}_n$, where $n \le 2^m$, and $\mathcal{C}_1,\dots,\mathcal{C}_{2^n} \subseteq \mathcal{U}$ such that for all i,j either $r(\mathcal{B}_i,\mathcal{C}_j) =_\mu \varnothing$ or $r(\mathcal{B}_i,\mathcal{C}_j) =_\mu \mathcal{B}_i \times \mathcal{C}_j$.

Proof Let $\mathcal{B}_0,\dots,\mathcal{B}_{n-1}$ be the sets given by Lemma 10 and define

$$\mathcal{B}_n \;=\; \mathcal{U} \smallsetminus \bigcup_{i=0}^{n-1} \mathcal{B}_i.$$

For every $J \subseteq n$ let

$$\mathcal{C}_J \;=\; \Big\{c \in \mathcal{U} \;:\; \bigcup_{i\in J} \mathcal{B}_i \;=_\mu r(\mathcal{U},c)\Big\}$$

By Lemma 10 if $i \in J$ then $r(\mathcal{B}_i,\mathcal{C}_J) =_\mu \mathcal{B}_i \times \mathcal{C}_J$, otherwise $r(\mathcal{B}_i,\mathcal{C}_J) =_\mu \varnothing$.

It is only left to verify the lemma for $r(\mathcal{B}_n,\mathcal{C}_J)$. Indeed, as $\mathcal{B}_n =_\mu \varnothing$ by Lemma 10, we have that $r(\mathcal{B}_n,\mathcal{C}_J) \;\subseteq\; \mathcal{B}_n \times \mathcal{U} \;=_\mu \varnothing$. □

Finally, the the stable Szemerédi regularity lemma is obtained applying Fact 4.

13 Corollary Assume that $r(\text{-},\text{-})$ is stable and let $m = R_\Delta(\mathcal{U})$. Then for every positive ε and every finite model M, there are two partitions of M, say $B_0,\dots,B_n$, where $n \le 2^m$, and $C_1,\dots,C_{2^n}$ such that

$$r(B_i,C_j) \;\le\; \varepsilon|B_i \times C_j|$$

$$r(B_i,C_j) \;\ge\; \varepsilon|B_i \times C_j|.$$

□