1 Pseudofinite yoga

As a matter of personal taste I have avoided the use of ultrapowers to deal with pseudofinite structures. Also, I have avoided the notion of pseudofinite dimension, as it is too general for the intended applications.

For convenience, all finite models considered below have as domain a subset of ω. We say that a sentence holds in almost all finite models if it holds in every finite model of sufficiently large cardinality.

Let T_0 be a given theory with arbitrarily large finite models. We define

$$T_{\text{fin}} = \{\varphi : M \vDash \varphi \text{ for almost all finite } M \vDash T_0\}.$$

A structure N is a pseudofinite model of T_0 if $N \vDash T_{\text{fin}}$. This is equivalent to requiring that every sentence $\varphi \in \text{Th}(N)$ there holds in arbitrarily large finite models of T_0. Note that every model of T_{fin} is infinite. (Other authors include in T_{fin} only sentences true in *all* finite models. Hence they include the finite models among the pseudofinite ones.)

We now discribe a canonical expansion of a finite model M to a two sorted structure $\langle M, \mathbb{N} \rangle$. The language of $\langle M, \mathbb{N} \rangle$ is denoted by L'.

1. L' expands the language of M and $\mathbb{N}$, where $\mathbb{N}$ is considered as a structure in the language with symbols for all real relations and functions of any finite arity.

2. L' has a function $f_\varphi : M^{|z|} \to \mathbb{N}$ for each formula $\varphi(x;z) \in L'$ and every finite tuples of variables $x;z$ of the first sort. The interpretation of $f_\varphi(b)$ is the (finite) cardinality of $\varphi(M;b)$.

3. (Only reqired in Section 2.) Let X be an n-ary relation symbol of the first sort. Let $\varphi \in L \cup \{X\}$ be a sentence. Then we require that L' has an n-ary predicate A_φ in the first sort. The interpretation of A_φ is a relation of maximal cardinality that makes φ true in M. If more then one relation satisfy the requirement above, we choose the minimal one in the lexicographic order (recall that the domain of M is a subset of ω).

Finally, we define

$$T'_{\text{fin}} = \{\varphi \in L' : \langle M, \mathbb{N} \rangle \vDash \varphi \text{ for all finite } M \vDash T_0\}.$$

In the sequel $\langle \mathcal{U}, {}^*\mathbb{N} \rangle$ denotes some arbitrary saturated model of T'_{fin}.

1 Fact Assume that $\psi \in L'$ holds in every $\langle \mathcal{U}, {}^*\mathbb{N} \rangle$ as above. Then $\langle M, \mathbb{N} \rangle \vDash \psi$ for all sufficiently large finite $M \vDash T_0$. $\quad\square$

Below we write ${}^*|\varphi(x;b)|$ for $f_\varphi(b)$. We call this the pseudofinite cardinality of $\varphi(x;b)$. By the definition of T'_{fin} it is clear that the pseudofinite cardinality of a definable set $\mathcal{D}$ does not depend on the formula defining it, so we can unambiguously write ${}^*|\mathcal{D}|$.

For $r, s \in {}^*\mathbb{N}$ larger than 1 we write $r \mathbin{\widetilde{\curlyvee}} s$ if $r \leq ms$ and $s \leq nr$ for some $m, n \in \mathbb{N}$. We write $r \mathbin{\widetilde{\approx}} s$ if $r \leq s^m$ and $s \leq r^n$ for some $m, n \in \mathbb{N}$. We write $r \mathbin{\widetilde{\lessdot}} s$ and $r \mathbin{\widetilde{\lessapprox}} s$ if $nr < s$, respectively $r^n < s$ for all $n \in \mathbb{N}$.

2 Proposition Let $\sim$ denote either $\widetilde{\curlyvee}$ or $\widetilde{\approx}$. For every non negative $r, s \in {}^*\mathbb{N}$

$$r + s \;\sim\; \max\{r, s\}$$

In particular, if $\mathcal{D}, \mathcal{C} \subseteq \mathcal{U}^z$ are two definable sets, then

$$^*|\mathcal{D} \cup \mathcal{C}| \;\sim\; \max\left\{^*|\mathcal{D}|, {}^*|\mathcal{C}|\right\}.$$

Proof In fact

$$\begin{aligned}
\max\{r, s\} \;&\le\; r + s \\
&\le\; 2\max\{r, s\} \\
&\le\; \big(\max\{r, s\}\big)^2
\end{aligned}$$

This proves the first equivalence. The second equivalence follows. $\qquad\square$

Definable sets $\mathcal{D} \subseteq \mathcal{U}^x$ such that $^*|\mathcal{D}| \sim {}^*|\mathcal{U}^x|$ are called large, otherwise we say they are small. The context will clarify if we are referring to the linear or to the polynomial equivalence.

3 Corollary Let $\sim$ denote either $\underset{\mathrm{L}}{\sim}$ or $\underset{\mathrm{P}}{\sim}$. Let $p(x) \subseteq L(M)$ be a type such that $\varphi(\mathcal{U})$ is large for every $\varphi(x)$ that is conjunction of formulas in $p(x)$. Then $p(x)$ has an extension to a complete type with the same property. $\qquad\square$

The following fact is immediate. It is stated to illustrate the typical application of the notions introduced above.

4 Fact Let $\sim$ denote either $\underset{\mathrm{L}}{\sim}$ or $\underset{\mathrm{P}}{\sim}$. Let $\varphi(x\,; z) \in L$, where x has finite length. Assume that in every model $\langle \mathcal{U}, {}^*\mathbb{N} \rangle$ there is a $b \in \mathcal{U}^z$ such that $\varphi(\mathcal{U}^x\,; b)$ is large. Then there is an $n \in \mathbb{N}$ such that every sufficiently large finite $M \vDash T_0$ contains a tuple c such that

$$\begin{aligned}
\left|M^x\right| \;&\le\; n\left|\varphi(M^x\,; c)\right| && \text{if } \sim \text{ is } \underset{\mathrm{L}}{\sim} \\
\left|M^x\right| \;&\le\; \left|\psi(M^x\,; c)\right|^n && \text{if } \sim \text{ is } \underset{\mathrm{P}}{\sim}.
\end{aligned} \qquad\square$$

The following two proposition only hold for $\underset{\mathrm{P}}{\sim}$.

5 Proposition For every non negative $r, s \in {}^*\mathbb{N}$

$$r \cdot s \;\underset{\mathrm{P}}{\sim}\; \max\{r, s\}$$

Proof In fact

$$\begin{aligned}
\max\{r, s\} \;&\le\; r \cdot s \\
&\le\; \big(\max\{r, s\}\big)^2
\end{aligned} \qquad\square$$

Note that, in particular, $^*|\mathcal{U}| \underset{\mathrm{P}}{\sim} {}^*|\mathcal{U}^x|$ for all tuples of variables of finite arity.

6 Corollary Let $\psi(x, z) \in L'$, where x, z are finite tuples of variables of the first sort. Let $\mathcal{A} \subseteq \mathcal{U}$. Assume that

$$^*|\mathcal{A}| \;\underset{\mathrm{P}}{\lesssim}\; {}^*|\mathcal{U}|$$

$$^*|\psi(a, \mathcal{U}^z)| \;\underset{\mathrm{P}}{\lesssim}\; {}^*|\mathcal{U}| \quad \text{for every } a \in \mathcal{A}.$$

Then

$$^*\left|\bigcup_{a \in \mathcal{A}} \psi(a, \mathcal{U}^z)\right| \;\underset{\mathrm{P}}{\lesssim}\; {}^*|\mathcal{U}|.$$

Proof Let $r = \,^*|\mathcal{A}|$. Let $s = \,^*|\psi(a,z)|$, where $a \in \mathcal{A}$ is choosen such that s is maximal. Then

$$^*\left|\bigcup_{a \in \mathcal{A}} \psi(a, \mathcal{U}^z)\right| \;\leq\; r \cdot s$$

$$\underset{\mathrm{P}}{\lessapprox} \;\; ^*|\mathcal{U}| \cdot \,^*|\mathcal{U}|$$

$$\underset{\mathrm{P}}{\approx} \;\; ^*|\mathcal{U}|. \qquad\qquad \square$$

2 Erdős-Hajnal for stable graphs

Let $r(\text{-},\text{-})$ be a binary relation symbol. The theory that T_0 says that $r(\text{-},\text{-})$ is an irreflexive and symmetric relation (i.e. a graph) and that it is m-stable, where m is fixed but arbitrary. For the purpose of these notes m-stable means that $R_\Delta(\mathcal{U}) \leq m$, where R_Δ is defined below.

Let $\Delta = \{r(a,x) : a \in \mathcal{U}\}$. Let $\mathcal{S} \subseteq \mathcal{U}$. We denote by $R_\Delta(\mathcal{S})$ the Shelah binary rank of $\mathcal{S}$. That is, the maximal height n of binary tree of $\Delta^\pm$-formulas $\langle \varphi_i(x) : s \in {}^{<n}2\rangle$ such that for every $s \in {}^n2$ the type

$$\{\neg^{s_i} \varphi_{s\restriction i}(x) : i \leq n\}$$

has a solution in $\mathcal{S}$.

As T_0 is stable, $R_\Delta(\mathcal{U})$ is finite. Among the large $L(\mathcal{U})$-definable subsets of $\mathcal{U}$ we fix one, say $\mathcal{S} = \sigma(\mathcal{U})$, such that $R_\Delta(\mathcal{S})$ is minimal.

7 Fact For every $\Delta^\pm$-definable set $\mathcal{D} \subseteq \mathcal{U}$, exactly one between $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg\mathcal{D}$ is large.

Proof As $\mathcal{S}$ is large, at least one between $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg\mathcal{D}$ is large by Proposition 2. Now, suppose for a contradiction that they both are large. Let $n = R_\Delta(\mathcal{S})$. By the minimality of n, both $\mathcal{S} \cap \mathcal{D}$ and $\mathcal{S} \cap \neg\mathcal{D}$ have rank n. This contradicts the definition of rank. $\qquad\qquad \square$

8 Theorem There is a large set $\mathcal{A} \subseteq \mathcal{U}$ that is either a clique or an anticlique.

Proof Let $\mathcal{S} = \sigma(\mathcal{U})$ be as defined above. Let $p(x) \subseteq \Delta^\pm$ be maximally consistent among the $\Delta^\pm$-types such that $\sigma(x) \wedge \varphi(x)$ is large for every conjunction of formulas in the type. By the fact above, $p(x)$ is $\Delta^\pm$-complete, i.e. either $\varphi(x)$ or $\neg\varphi(x)$ is in $p(x)$ for every $\varphi(x) \in \Delta^\pm$.

Let $\mathcal{B} = \{a \in \mathcal{S} : r(a,x) \in p\}$. By stability, $\mathcal{B}$ is definable in $L(\mathcal{U})$. At least one between $\mathcal{B}$ and $\mathcal{S} \smallsetminus \mathcal{B}$ is large. Assume the first, the argument when $\mathcal{S} \smallsetminus \mathcal{B}$ is large is similar.

Let $\mathcal{A}$ be a definable subsets of $\mathcal{B}$ that is maximal among those that satisfy the formula $(\forall x, y \in \mathcal{A})\, r(x,y)$. The definition of L' guarantees that such a set exists.

We claim that $\mathcal{A}$ is large, hence it is the clique required by the theorem. So, assume not, and reason for a contradiction.

By the maximality of $\mathcal{A}$, for every $b \in \mathcal{B} \smallsetminus \mathcal{A}$ there is an $a \in \mathcal{A}$ such that $\neg r(a,b)$. We rephrase this as the inclusion

$$\mathcal{B} \smallsetminus \mathcal{A} \;\subseteq\; \bigcup_{a \in \mathcal{A}} \neg r(a, \mathcal{S})$$

By assumption $\mathcal{A}$ is not large, hence $\mathcal{B} \smallsetminus \mathcal{A}$ is large. When $a \in \mathcal{A}$, in particular $r(a, x) \in p$. Therefore, by the fact above, $r(a, \mathcal{S})$ is large and $\neg r(a, \mathcal{S})$ is small. This is a contradiction by Corollary 6. $\qquad\square$

Finally the Erdős-Hajnal property is obtained applying Fact 4.

9 Corollary There is an n such that in every finite model $M \vDash T_0$ there is a set $A \subseteq M$ of cardinality at least $|M|^{1/n}$ that is either a clique or an anticlique. $\qquad\square$

3 Pseudofinite counting measure

Below, the theory T_0, the set of formulas Δ, and the rank $m = R_\Delta$ are as defined the provious section.

Let $\mathcal{D} \subseteq \mathcal{U}$ be a definable set. We define

$$\Pr(\mathcal{D}) \;=\; \inf \left\{ \frac{m}{n} \;:\; m, n \in \mathbb{N} \smallsetminus \{0\} \text{ such that } n^* |\mathcal{D}| \leq m^* |\mathcal{U}| \right\},$$

where the infimum is taken in $\mathbb{R}$. It is immediate that $\Pr(\text{-})$ is a finite probability measure on the definable subsets of $\mathcal{U}$. (By the Caratheodory Theorem it can be extended to a probability measure but this is not required here.)

We will write $\boxed{\mathcal{A} \underset{\Pr}{=\!=} \mathcal{D}}$ if $\Pr(\mathcal{A} \triangle \mathcal{D}) = 0$.

10 Lemma There are some $\{\wedge\}\Delta^\pm$-definable sets $\mathcal{B}_1, \ldots, \mathcal{B}_n \subseteq \mathcal{U}$, where $n \leq 2^m$, such that for any $\{\wedge\}\Delta^\pm$-definable set $\mathcal{A}$,

$$\mathcal{A} \;\underset{\Pr}{=\!=}\; \bigcup_{i \in I} \mathcal{B}_i,$$

per qualche $I \subseteq n$. Also, we can require that all $\mathcal{B}_i$ have positive measure and that they form a partition $\mathcal{U}$.

Proof We prove a slightly more general claim. Let $\mathcal{D}$ be a definable set of positive measure. By induction on $R_\Delta(\mathcal{D}) = m$ we prove that there are $n \leq 2^m$ many $\{\wedge\}\Delta^\pm$-definable sets $\mathcal{B}_i \subseteq \mathcal{D}$ such that for every $\{\wedge\}\Delta^\pm$-definable $\mathcal{A} \subseteq \mathcal{D}$ there is an $I \subseteq n$

$$\mathcal{A} \;\underset{\Pr}{=\!=}\; \bigcup_{i \in I} \mathcal{B}_i.$$

If $R_\Delta(\mathcal{D}) = 0$ then $\mathcal{D}$ is either disjoint of contained in every $\Delta^\pm$-definable set. So the claim holds trivially.

Now, let $R_\Delta(\mathcal{D}) = m + 1$. By the definition of binary rank, there is a Δ-definable set $\mathcal{B}$ such that $R_\Delta(\mathcal{D} \cap \mathcal{B}) = R_\Delta(\mathcal{D} \cap \neg \mathcal{B}) = m$. Apply the induction hypothesis to $\mathcal{D} \cap \mathcal{B}$ if it has positive measure. Do the same for $\mathcal{D} \cap \neg \mathcal{B}$. Note that at least one has positive measure.

It is evident that the sets $\mathcal{B}_i$ obtained in the construction above are disjoint and have positive measure. Also, they cover $\mathcal{U}$ up to a set of measure 0. Hence, replacing $\mathcal{B}_1$ with some $\mathcal{B}'_1 \underset{\Pr}{=\!=} \mathcal{B}_1$ we obtain an actual cover of $\mathcal{U}$. $\qquad\square$

⚠ Non sono sicuro questa qui sotto sia la corretta traduzione finita.

11 Corollary For every $\varepsilon > 0$ there is an k such that for every finite $M \vDash T_0$ of cardinality larger than k, there is a partition of M, say $B_0, \ldots, B_{n-1} \subseteq M$, where $n \leq 2^m$ such that $|B_i| > \varepsilon |M|$ and for every $\{\wedge\}\Delta^\pm$-definable set $A \subseteq M$ there is an $I \subseteq n$

such that

$$\left| A \bigtriangleup \bigcup_{i \in I} B_i \right| \; < \; \varepsilon \, |M|.$$

4 Stable Szemerédi regularity lemma

Below, the theory T_0, the set of formulas Δ, and the rank $m = R_\Delta$ are as defined in Section 2.

12 Lemma There are two partitions of $\mathcal{U}$, say $\mathcal{B}_1, \ldots, \mathcal{B}_n$, where $n \leq 2^m$, and $\mathcal{C}_1, \ldots, \mathcal{C}_{2^n}$ such that for all i, j either $r(\mathcal{B}_i, \mathcal{C}_j) \equiv_{\mathrm{Pr}} \varnothing$ or $r(\mathcal{B}_i, \mathcal{C}_j) \equiv_{\mathrm{Pr}} \mathcal{B}_i \times \mathcal{C}_j$.

Proof Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be the sets given by Lemma 10. For every $J \subseteq n$ let

$$\mathcal{C}_J \;=\; \left\{ c \in \mathcal{U} \;:\; \bigcup_{i \in J} \mathcal{B}_i \equiv_{\mathrm{Pr}} r(\mathcal{U}, c) \right\}$$

By Lemma 10 if $i \in J$ then $r(\mathcal{B}_i, \mathcal{C}_J) \equiv_{\mathrm{Pr}} \mathcal{B}_i \times \mathcal{C}_J$, otherwise $r(\mathcal{B}_i, \mathcal{C}_J) \equiv_{\mathrm{Pr}} \varnothing$. $\square$

For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{U}$, it is usual call density the conditional probability

$$d(\mathcal{A}, \mathcal{B}) \;=\; \mathrm{Pr}\left(r(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \times \mathcal{B} \right)$$

$$=\; \frac{\mathrm{Pr}\left(r(\mathcal{A}, \mathcal{B}) \right)}{\mathrm{Pr}(\mathcal{A} \times \mathcal{B})}$$

This definition is given with the proviso that $\mathcal{A}$ and $\mathcal{B}$ have non zero probability. The same notation is used for nonempty subsets $A, B \subseteq M$ of a finite model

$$d(A, B) \;=\; \frac{|r(A, B)|}{|A| \cdot |B|}$$

13 Corollary There is a partition of $\mathcal{U}$ into sets of positive measure $\mathcal{D}_1, \ldots, \mathcal{D}_n$, where $n \leq 2^{2^m+1}$ such that for all i, j either $d(\mathcal{D}_i, \mathcal{D}_j) = 0$ or $d(\mathcal{D}_i, \mathcal{D}_j) = 1$.

Finally, the the stable Szemerédi regularity lemma is obtained applying Fact 4.

⚠ Il corollario qui sotto sembra troppo forte perché n non dipende da ε. Non dev'essere la traduzione al finito giusta.

14 Corollary For every $\varepsilon > 0$ there is an k such that for every finite $M \vDash T_0$ of cardinality larger than k, there is a partitions of M, say $D_1, \ldots, D_n$, where $n \leq 2^{2^m+1}$ such that for all i, j either $d(D_i, D_j) < \varepsilon$ or $d(D_i, D_j) > 1 - \varepsilon$. $\square$