Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Abstract samples

Let M be a structure of signature L. The multiset expansion of M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, and a sample-sort for the set of functions $s : M \to \mathbb{R}$ that are almot always 0. We write $\overline{M} = \langle M, \mathbb{R}, M^{\mathrm{s}} \rangle$ for the expansion above.

Below x, y, z are always tuples of variables of home-sort, $x^{\mathrm{s}}, y^{\mathrm{s}}, z^{\mathrm{s}}$ are tuples of sample-sort of the same the length of x, y, respectively z.

The language of $\overline{M}$ is denoted by $\bar{L}$. We require that $\bar{L}$ includes

1. L, that applies to the home-sort;
2. the language of ordered rings, that applies to the sort of $\mathbb{R}$;
3. the language of real vector spaces, that applies to the sample-sort.

Moreover, for every $\varphi(x\,; z, z^{\mathrm{s}}) \in \bar{L}$ there is a symbol for a function

$$\bar{\varphi}(x^{\mathrm{s}}; z, z^{\mathrm{s}}) \quad : \quad \big(M^{\mathrm{s}}\big)^{|x|} \times M^{|z|} \times \big(M^{\mathrm{s}}\big)^{|z|} \quad \longrightarrow \quad \mathbb{R}.$$

which is defined as follows

$$\bar{\varphi}(s\,; b, r) \quad = \quad \sum \Big\{ s(a) \ : \ \overline{M} \vDash \varphi(a\,; b, r) \Big\}$$

When $\varphi(x)$ is the formula $x = x$ we write $|x|$ for $\bar{\varphi}(x^{\mathrm{s}})$.

2.1 Definition *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$ and let $\bar{\mathcal{U}} = \langle \mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathrm{s}} \rangle$ be the corresponding expansion. We denote by $\bar{\mathcal{U}}^*$ a saturated elementary extension of $\bar{\mathcal{U}}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\bar{\mathcal{U}}^* = \langle \mathcal{U},\, \mathbb{R}^*, \mathcal{U}^{\mathrm{s}*} \rangle$, where $\mathbb{R}^*$ and $\mathcal{U}^{\mathrm{s}*}$ are suitable saturated extensions of $\mathbb{R}$, respectively $\mathcal{U}^{\mathrm{s}}$. Elements $\mathcal{U}^{\mathrm{s}}$ are called finite samples, elements in $\mathcal{U}^{\mathrm{s}*}$ are called internal samples.*

Finally, a global sample is a type $p(x^{\mathrm{s}}) \in S(\mathcal{U}, \overline{M})$, where M is a model which will be clear from the context. $\square$

2.2 Notation For any global type $p(x^{\mathrm{s}}) \in S(\mathcal{U}, \overline{M})$, formula $\varphi(x^{\mathrm{s}}; z) \in \bar{L}(\overline{M})$, and $b \in \mathcal{U}^{|z|}$ we write $\varphi(p\,; b)$ for $\varphi(x^{\mathrm{s}}; b) \in p$. We also write

$$\varphi(p\,;\mathcal{U}) \;=\; \Big\{b \in \mathcal{U}^{|z|} \;:\; \varphi(p\,;b)\Big\}$$

Sets of this form are called externally definable. This notation intentionally confuses p with any of its realizations in some elementary extension of $\bar{\mathcal{U}}$. □

The following notions are standard

2.3 Definition *Let M be given. A global sample $p(x^{\rm s})$ is*

1. *invariant if for every $\varphi(x^{\rm s};z) \in \bar{L}(\overline{M})$ the set $\varphi(p\,;\mathcal{U})$ is invariant over $\overline{M}$;*
2. *finitely satisfiable if for every $\varphi(x^{\rm s}) \in p$ there is an $r \in M^{\rm s}$ such that $\varphi(p) \leftrightarrow \varphi(r)$;*
3. *definable if for every $\varphi(x^{\rm s};z) \in \bar{L}(\overline{M})$ the set $\varphi(p\,;\mathcal{U})$ is definable over $\overline{M}$.* □

2.4 Definition *Let M be given. A global sample is smooth if it is invariant and realized in $\bar{\mathcal{U}}^*$.* □

We recall the following evident fact.

2.5 Fact *Let M be given. Let $p(x^{\rm s})$ be a finitely satisfiable global sample. Then p is invariant.* □

2.6 Definition *Let M be given. A global sample $p(x^{\rm s})$ is uniformly finitely satisfiable if for every $\varphi(x^{\rm s};z) \in \bar{L}(\overline{M})$ there is a finite $R \subseteq M^{\rm s}$ such that for every $b \in \mathcal{U}^{|z|}$ we have $p \vdash \varphi(x^{\rm s};b) \leftrightarrow \varphi(r\,;b)$ for some $r \in R$.* □

2.7 Lemma *Let M be given. For every smooth global sample $p(x^{\rm s})$ the following hold*

1. *$p(x^{\rm s})$ is definable;*
2. *$p(x^{\rm s})$ is uniformly finitely satisfiable.*

Proof. Let $s \in \bar{\mathcal{U}}^{\rm s*}$ be an internal sample that realizes $p(x^{\rm s})$. Let $\varphi(x^{\rm s};z) \in L$ be given.

1. As $\varphi(p\,;\mathcal{U}) = \varphi(s\,;\mathcal{U})$ is definable in $\bar{\mathcal{U}}^*$, by invariance it is definable in $\overline{M}$.
2. First we prove that $p(x^{\rm s})$ is finitely satisfiable. Let $b \in \mathcal{U}^{|z|}$ be given and pick $b' \equiv_{\overline{M}} b$ be such that $s \downarrow_{\overline{M}} b'$. By invariance, $\varphi(s\,;b') \leftrightarrow \varphi(s\,;b)$. Therefore there is an $r \in M^{\rm s}$ such that $\varphi(r\,;b') \leftrightarrow \varphi(s\,;b)$. This proves finite satisfiability. By finite satisfiability, the following type is inconsistent
$$q(z) \;=\; \Big\{\varphi(s\,;z) \leftrightarrow \varphi(r\,;z) \;:\; r \in M^{\rm s}\Big\}$$
Hence compatness yields the required $R \subseteq M^{\rm s}$. □

2.8 Lemma *Let M be given. Let $p(x^{\rm s})$ be uniformly finitely satisfiable. Then for some finite $A \subseteq M^{\rm s}$ we have $p(x^{\rm s}) \vdash {\rm supp}(x^{\rm s}) = A$.*

Proof. Apply Definition 2.6 to the formula $x^{\rm s}(z) > 0$. Then for every $b \in \mathcal{U}^{|z|}$ there is an $r \in R$.

$$p(x^{\mathrm{s}}) \quad \vdash \quad x^{\mathrm{s}}(b) > 0 \leftrightarrow r(b) > 0.$$

As each $r \in R$ has finite support, and R is finite, the lemma follows. □

3 Recycle bin

Then for every $b \in \mathcal{U}^{|z|}$ there is an $r \in R$

$$p(x^{\mathrm{s}}) \quad \vdash \quad x^{\mathrm{s}}(b) > 0 \leftrightarrow r(b) > 0 \text{ such that}$$

As Write $\mathrm{supp}(R)$ for the union of $\mathrm{supp}(r)$ for $r \in R$. Then $p \vdash \mathrm{supp}(x^{\mathrm{s}}) \subseteq \mathrm{supp}(R)$. In particular $p \vdash \mathrm{supp}(x^{\mathrm{s}}) = A$ for some finite $A \subseteq M$.

$\varphi(p\,;z) \leftrightarrow r(z) > 0$

Let $p(x^{\mathrm{s}})$ be global sample that is uniformly finitely satisfiable w.r.t. all $\varphi(x^{\mathrm{s}}\,;z) \in L$. Then there is an $r \in M^{\mathrm{s}}$ such that

$x^{\mathrm{s}}(z_i) \approx_\varepsilon \mu_i$

3.1 Fact *Let M and $\varphi(x\,;z) \in L$ be given. Let $p(x^{\mathrm{s}})$ be a definable, uniformly finitely satisfiable global sample. Then there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\bar{\vartheta}_i(p) \approx_\varepsilon \bar{\vartheta}_j(p)$ for some i, j.*

Proof. Let $\psi_r(z)$, for $r \in R \subseteq M^{\mathrm{s}}$

Let

Pick finite $F \subseteq \mathbb{R}$ such that for every b we have $\bar{\varphi}(p\,;b) \approx_\varepsilon \mu$ for some $\mu \in F$. By Definition 2.6 there is a finite set $R \subseteq M^{\mathrm{s}}$ such that for every $b \in \mathcal{U}^{|z|}$ and every $\mu \in F$ we have $\varphi(p\,;b) \approx_\varepsilon \mu \;\leftrightarrow\; \varphi(r\,;b) \approx_\varepsilon \mu$ for some $r \in R$. Therefore for every $b \in \mathcal{U}^{|z|}$ we have that $\varphi(p\,;b) \approx_{2\varepsilon} \varphi(r\,;b)$ for some $r \in R$. Let $A \subseteq \mathcal{U}^{|z|}$ be a finite set containing $\mathrm{supp}(r)$ for all $r \in R$. Let $\vartheta_i(x)$ enumerate the formulas $x \in B$ and $x \notin B$, as B ranges over the subsets of A.

We prove that these formulas are as required by the proposition. Fix some $b \in \mathcal{U}^{|z|}$ and let $r \in R$ be such that $\varphi(p\,;b) \approx_{2\varepsilon} \varphi(r\,;b)$. Let

$$\begin{aligned} B_0 &= \{a \in \mathrm{supp}(r) : \varphi(a\,;b)\} \\ B_1 &= \{a \in \mathrm{supp}(r) : \neg\varphi(a\,;b)\} \end{aligned}$$

Clearly, $x \in B_0 \to \varphi(x\,;b) \to x \notin B_1$. □

3.2 Definition *Let $\varphi(x^{\mathrm{s}}\,;z) \in L$ and M be given. We say that $p(x^{\mathrm{s}})$ is weakly approximable if there are some finitely many $r_i \in M^{\mathrm{s}}$, say $0 \leq i < n$, such that for every $b \in \mathcal{U}^{|z|}$ we have $\varphi(p\,;b) \leftrightarrow \varphi(r_i\,;b)$ for some i.* □

Let $\varphi(x\,;z) \in L$ and M be given. Let $\mathcal{B} = \varphi(p;\mathcal{U})$. Let $p(x^{\mathrm{s}})$ be weakly approximable. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1, \ldots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ are consistent.

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

3.3 Fact *Let M be given. Let $p(x^{\rm s})$ be global sample. Suppose for every $\varphi(x) \in L(\mathcal{U})$ and ε there are $\vartheta_1(x), \vartheta_2(x) \in L(M)$ such that $\vartheta_1(x) \to \varphi(x) \to \vartheta_2(x)$ and $\bar\vartheta_2(p) \approx_\varepsilon \bar\vartheta_1(p)$. Then p is finitely satifiable.*

Proof. Let $\varphi_i(x)$, for $0 \le i < n$, be given. We claim that there is an $r \in \overline{M}$ such that $\bar\varphi_i(p) \approx_{3\varepsilon} \bar\varphi_i(r)$ for every i.

Let $\vartheta_{i,1}(x)$ and $\vartheta_{i,2}(x)$ be as in the fact. As $\vartheta_{i,1}(p) \le \varphi_i(p) \le \vartheta_{i,2}(p)$, then $\vartheta_{i,1}(p) \approx_\varepsilon \varphi_i(p)$. By elementarity there is an $r \in \overline{M}$ such that $\vartheta_{i,1}(r) \approx_\varepsilon \vartheta_{i,1}(p)$ and $\vartheta_{i,2}(r) \approx_\varepsilon \vartheta_{i,2}(p)$. Then the claim follows immediately. □

3.4 Lemma *The following are equivalent for every global sample $p(x^{\rm s})$*

1. *p is smooth and invariant;*
2. *for every finite $B \subseteq \mathcal{U}^{|z|}$ and every ε, there is an $r \in \overline{M}$ such that $\bar\varphi(p\,;b) \approx_\varepsilon \bar\varphi(r\,;b)$ for all $b \in B$;*
2. *for every $\varphi_i(x\,;z) \in L$, every $b_i \in \mathcal{U}^{|z|}$, and every ε, there is an $r \in \overline{M}$ such that $\bar\varphi_i(p\,;b_i) \approx_\varepsilon \bar\varphi_i(r\,;b_i)$, for every $0 \le i < n$;*
2. *for every $\psi_0(x), \ldots, \psi_{n-1}(x) \in L(\mathcal{U})$ and every ε, there is an $r \in \overline{M}$ such that $\bar\psi_i(p) \approx_\varepsilon \bar\psi_i(r)$, for every $i < n$;*
4. *there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have that $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\bar\vartheta_j(p) - \bar\vartheta_i(p) < \varepsilon$ for some i, j.*

Proof. Let s be an internal sample that realizes $p(x^{\rm s})$.

1⇒2. Let B and ε be given. Let $\bar b = \langle b_i : i < n\rangle$ be an enumeration of B. Let $\mu_i \in \mathbb{R}$ be such that $\bar\varphi(p\,;b_i) \approx \mu_i$ for every $i < n$. Let $\bar b'$ be such that $s \mathop{\smile\!\!\!\!\!\!\vert}_{\overline{M}} \bar b' \equiv_{\overline{M}} \bar b$. By invariance, $\bar\varphi(s\,;b'_i) \approx \mu_i$. Therefore there is an $r \in \overline{M}$ such that $\bar\varphi(r\,;b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar b' \equiv_{\overline{M}} \bar b$, we obtain $\bar\varphi(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒2. Let $\psi_i(x) = \varphi_i(x\,;b_i)$ for some $\varphi_i(x\,;z) \in L$ and $b_i \in \mathcal{U}^{|z|}$. Let $\bar b = \langle b_i : i < n\rangle$. Let $\mu_i \in \mathbb{R}$ be such that $\bar\varphi_i(p\,;b_i) \approx \mu_i$. Let $\bar b'$ be such that $s \mathop{\smile\!\!\!\!\!\!\vert}_{\overline{M}} \bar b' \equiv_{\overline{M}} \bar b$. By invariance, $\bar\varphi_i(s\,;b'_i) \approx \mu_i$. Therefore there is an $r \in \overline{M}$ such that $\bar\varphi_i(r\,;b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar b' \equiv_{\overline{M}} \bar b$, we obtain $\bar\varphi_i(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒3. By smoothness we it suffices to prove 3 with s for p. The type

$$p(z) \quad = \quad \Big\{\bar\varphi(s\,;z) \not\approx_\varepsilon \bar\varphi(r\,;z) \ :\ r \in \overline{M}\Big\}$$

is inconsistent by 2. Hence compatness yields the required $r_1, \ldots, r_n \in \overline{M}$.

3⇒2. Clear.

2⇒1 Assume 2. We prove that p is invariant. Pick some $b \equiv_{\overline{M}} b'$ and some ε. Let $r, r' \in \overline{M}$ be such that $\bar\varphi(p\,;b) \approx_\varepsilon \bar\varphi(r\,;b)$ and $\bar\varphi(p\,;b') \not\approx_\varepsilon \bar\varphi(r';b')$. As $\bar\varphi(r\,;b) \approx$

$\bar{\varphi}(r';b')$ follows from $b \equiv_{\overline{M}} b'$, we obtain $\bar{\varphi}(p\,;b) \approx_\varepsilon \bar{\varphi}(p\,;b')$.

We prove that p is smooth.

$$p(x^{\mathrm{s}}) \quad = \quad \Big\{\bar{\varphi}(x^{\mathrm{s}};b) \approx_\varepsilon \mu_b \ : \ b \in \mathcal{U}^{|z|}\Big\} \qquad \square$$

3.5 Definition *Let $\varphi(x\,;z) \in L$ and M be given. A global sample $p(x^{\mathrm{s}})$ is*

1. *invariant if $\bar{\varphi}(p\,;b) \approx \bar{\varphi}(p\,;b')$ for every $b \equiv_{\overline{M}} b'$;*
2. *(pointwise) approximable if, for every $b \in \mathcal{U}^{|z|}$ and every ε, there is an $r \in \overline{M}$ such that $\bar{\varphi}(p\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$;*
3. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \bar{\varphi}(p\,;b) < \varepsilon\}$ is definable over $\overline{M}$.* $\square$

The following is a useful characterization of smooth samples.

4 Old stuff/garbage

We write $\mathrm{supp}(p)$ for the support of p, that is, the set of those a in $\mathcal{U}$ such that $p \vdash (a \in x^{\mathrm{s}}) > \varepsilon$ for some standard positive ε. If s is a smooth sample $\mathrm{supp}(s)$ is defined to be $\mathrm{supp}(p)$ for $p(x^{\mathrm{s}}) = \mathrm{tp}(s/\bar{\mathcal{U}})$.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_{x/s}\varphi(x\,;b) \quad = \quad \frac{\big|\big\{a \in s \ : \ \varphi(a\,;b)\big\}\big|}{|s|},$$

When possible we abbreviate $\mathrm{Av}_{x/s}\varphi(x\,;b)$ with $\bar{\varphi}(s\,;b)$.

We write $\mathrm{ft}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

All definitions and facts in this section are relative to some given formula $\varphi(x\,;z) \in L$ and some model M.

4.1 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\bar{\varphi}(s';b) \approx \bar{\varphi}(s\,;b)$.* $\square$

4.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\bar{\varphi}(s\,;b) \approx \bar{\varphi}(s\,;b')$;*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \bar{\varphi}(s\,;b) < \varepsilon\}$ is definable over M;*
3. *finitely satisfiable if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in \overline{M}$ such that $\bar{\varphi}(s\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$;*
4. *generically stable if all of the above hold.* $\square$

Smooth roughly means internal and invariant.

4.3 Fact *Let $\varphi(x;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *there is an invariant internal sample $s' \equiv_M s$.*

Proof. 1⇒2. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathcal{U}^{|z|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \mathrm{Aut}(\bar{\mathcal{U}}^*/M)$. Then $\bar{\varphi}(s';fb) \approx \bar{\varphi}(f^{-1}s';b)$. Moreover, by smothness $\bar{\varphi}(f^{-1}s';b) \approx \bar{\varphi}(s';b)$. The infariace in s' follows.

2⇒1. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ such that $\bar{\varphi}(s';b) \not\approx_\varepsilon \bar{\varphi}(s;b)$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\bar{\varphi}(s'';b) \not\approx_\varepsilon \bar{\varphi}(s;b)$. As $s'' \equiv_M s$ are both in $\bar{\mathcal{U}}^*$, then $s'' = fs$ for some $f \in \mathrm{Aut}(\bar{\mathcal{U}}^*/M)$. Hence we obtan $\bar{\varphi}(s;f^{-1}b) \not\approx_\varepsilon \bar{\varphi}(s;b)$ which contradics the invariance of s. □

4.4 Definition *Let $\varphi(x;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s is (uniformly) approximable on $\mathcal{B}$ if for every ε there is a finite sample $r \in \overline{M}$ such that $\bar{\varphi}(s,b) \approx_\varepsilon \bar{\varphi}(r,b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε, not on b.)* □

4.5 Lemma *Let $\varphi(x;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that $\bar{\varphi}(s;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1, \dots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x;b) : b \in \mathcal{B}_i\}$ are consistent.*

Proof. Let $r \in \overline{M}$ be as in Definition 4.4. As r is finite, we may assume that $\mathrm{supp}(r) = \{a_1, \dots, a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i;b)\}$. As $\bar{\varphi}(r;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

4.6 Corollary *Let $\varphi(x;z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega\rangle$, where $\langle b_i : i < \omega\rangle$ is a sequence of indiscernibles such that $\bar{\varphi}(s;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x;b_i) : i < \omega\}$ is consistent.* □

4.7 Corollary *Let $\varphi(x;z) \in L$ and M be given. Let s be approximable on $\mathcal{C}$. Then for every ε there is a pair of distict $c, c' \in \mathcal{C}$ such that $\mathrm{Av}_{x/s}[\varphi(x;c) \not\leftrightarrow \varphi(x;c')] < \varepsilon$*

Proof. Suppose for a contradiction that $\mathrm{Av}_{x/s}[\varphi(x;c) \not\leftrightarrow \varphi(x;c')] \geq \varepsilon$ for all distict $c, c' \in \mathcal{C}$. Apply Lemma 4.5 to the folmula $\psi(x;z,z') = [\varphi(x;z) \not\leftrightarrow \varphi(x;z')]$ and the set $\mathcal{B} = \{\langle c, c'\rangle \in \mathcal{C}^2 \ : \ c{\neq}c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 4.5 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x;a) \not\leftrightarrow \varphi(x;a') \ : \ a, a'{\in}A,\ a{\neq}a'\}$ is consistent. As $|A| > 2$, this is impossible. □

5 The nip formulas

5.1 Theorem *Let $\varphi(x\,;z) \in L$ and M be given and assume that $\varphi(x\,;z)$ is nip. Then every sample is approximable on M over M.*

Proof. Questo è Vapnik-Chervonenkis. $\square$

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathfrak{U}^{|z|}$.)

5.2 Corollary *Let $\varphi(x\,;z) \in L$ be nip. Every smooth sample s is approximable on $\mathfrak{U}$ over M.*

Proof. By Theorem 5.1, there is an $r \in \overline{M}$ such that $\bar{\varphi}(s\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\bar{\varphi}(s\,;b') \not\approx_\varepsilon \bar{\varphi}(r\,;b')$ for some $b' \in \mathfrak{U}^{|z|}$. Pick a $s' \subseteq \mathfrak{U}^{|x|}$ such that $b' \mathop{\smile\!\!\!\!\!\!\,|}_M s' \equiv_M s$. By the smoothness of s, we obtain $\bar{\varphi}(s'\,;b') \not\approx_\varepsilon \bar{\varphi}(r\,;b')$. As $b' \mathop{\smile\!\!\!\!\!\!\,|}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\bar{\varphi}(s\,;b) \not\approx_\varepsilon \bar{\varphi}(r\,;b)$. A contradiction. $\square$

5.3 Definition ?? *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\ldots z_n) \in L$ such that for every finite set $B \subseteq \mathfrak{U}$ and every $a \in \mathfrak{U}^{|x|}$ there are $b_1,\ldots,b_n \in B$ such that $\psi(a\,;b_1,\ldots,b_n)$ and $\psi(x\,;b_1,\ldots,b_n)$ decides all formulas $\varphi(x\,;b)$ for $b \in B$.* $\square$

5.4 Definition ??* *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x_1,\ldots,x_n\,;z) \in L$ such that for every finite set $A \subseteq \mathfrak{U}^{|x|}$ and every $b \in \mathfrak{U}^{|z|}$ there are $a_1,\ldots,a_n \in A$ such that $\psi(a_1,\ldots,a_n,\ z)$ and $\psi(a_1,\ldots,a_n,\ z)$ decides all formulas $\varphi(a\,;z)$ for $a \in A$.* $\square$

For every finite sample $r \in \overline{M}$ and every $\mu \in \mathbb{R}$ there is a formula $\psi(z) \in L(\operatorname{supp} r)$ such that $\bar{\varphi}(r\,;z) = \mu \leftrightarrow \psi(z)$. The formula $\psi(z)$ depends on r.

When $\varphi(x\,;z)$ is distal

If $\varphi(x\,;z)$ is distal then there is a formula ψ such that for every $r \in M^f$ there is $r_0 \in M$

5.5 Theorem (false) *The following are equivalent*

1. $\varphi(x\,;z) \in L$ *is distal;*
2. *for every sample $s \in \mathfrak{U}^{\mathrm{f}^+_*}$, if s is generically stable then s is smooth.*

There is a formula $\psi(x_1,\ldots,x_n\,;z)$ such that for every $r \in \overline{M}$

there are $a_1,\ldots,a_n$ such that $\varphi(r\,;b) = \mu\ \psi(x_1,\ldots,x_n\,;z)$

We say that $p(x)$ is honestly definable if for every $\varphi(x,y) \in L$ there is a formula $\psi(y,z) \in L$ such that for every finite A, there is a b such that

$$A \subseteq \psi(\mathfrak{U}, b) \subseteq \varphi(p, \mathfrak{U})$$

For $\varphi(x, y) \in L$ and $m \in M$ let

$$\varphi_m(x, y) \quad = \quad \varphi(m, y) \leftrightarrow \varphi(x, y)$$

$$A \subseteq \psi_{m,\varphi}(\mathcal{U}) \subseteq \varphi_m(p, \mathfrak{U})$$

If $p(x)$ is generically stable (definition as in the OP) then for every $\varphi(x, y) \in L$ and $m \in M$ there is a formula $\vartheta_{\varphi,m}(y)$ such that for every a in the monster model

$$\vartheta_{\varphi,m}(a) \quad \Leftrightarrow \quad [\varphi(m, a) \leftrightarrow \varphi(x, a)] \in p$$

Let $q(x)$ be an M-invariant global type such that $q_{\restriction M}(x) = p_{\restriction M}(x)$. Then both q and p contain the formulas

$$\forall y \left[\vartheta_{\varphi,m}(y) \rightarrow \left[\varphi(m, y) \leftrightarrow \varphi(x, y)\right]\right]$$

We write $p(x) \sim_M q(x)$ if

$\varphi(p\,;\mathfrak{U}) \neq \varnothing \Leftrightarrow \varphi(q\,;\mathfrak{U}) \neq \varnothing$

for every $\varphi(x, z) \in L(M)$