Scratch paper

Anonymous
Università di Torino
April 2020

Abstract Poche idee, ben confuse.

1 Introduction

2 Abstract samples

Let M be a semigroup. The finite (fractional) sample expansion of M is a 3-sorted expansion $\bar{M} = \langle M, \mathbb{R}, M^{\mathsf{s}} \rangle$ that has a domain for M, which we call the home-sort, a domain for $\mathbb{R}$, as an ordered field, and a domain M^{s}, which we call sample-sort, which is an $\mathbb{R}$-vector space that we describe below. Variables and elements of sample-sort (or tuple thereof) are denoted with symbols $x^{\mathsf{s}}, a^{\mathsf{s}}$ and variations. The variable x is reserved for the home-sort, for all other variables we the context will clarify the sort.

An element $a^{\mathsf{s}} \in M^{\mathsf{s}}$ is a map $a^{\mathsf{s}} : M \to \mathbb{R}$ which is almost always zero. These functions, which we call samples, are interpreted as (weighted) samples or as (signed) finite measures (up to normalization) that concentrated on the support of a^{s}.

The language of the expansion $\bar{M}$ is denoted by L. It expands the natural language of each sort: on M, the language of semigroups, on $\mathbb{R}$, the language ordered rings, and, on M^{s}, the language of algebras over $\mathbb{R}$. This algebra has the natural sum and the multiplication inherited from M

$$\begin{aligned} (a^{\mathsf{s}} + b^{\mathsf{s}})(z) &= a^{\mathsf{s}}(z) + b^{\mathsf{s}}(z); \\ (a^{\mathsf{s}} \cdot b^{\mathsf{s}})(z) &= \sum_{x \cdot y = z} a^{\mathsf{s}}(x) \cdot b^{\mathsf{s}}(y). \end{aligned}$$

Finally, for every formula $\varphi(x\,;y) \in L$, where y is a tuple a mixed sort, there is a symbol $\overline{\varphi}(x^{\mathsf{s}}\,;y)$ for a function $(M^s)^x \times M^y \to \mathbb{R}$. For $\varphi(x) \in L(\bar{M})$, we interpret $\overline{\varphi}(x^{\mathsf{s}})$ as the function that maps $a^{\mathsf{s}} \in M^{\mathsf{s}}$ to

$$\overline{\varphi}(a^{\mathsf{s}}) = \sum_{x \in \varphi(M)} a^s(x) \Big/ \sum_{x \in M} a^s(x)$$

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

2.1 Definition *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$. Let $\bar{\mathcal{U}} = \langle \mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathsf{s}} \rangle$ be as above and let $\bar{\mathcal{U}}^*$ be a saturated elementary extension of $\mathcal{U}^{\mathsf{s}}$ of cardinality κ. As all saturated* *models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the domain of the home-sort of $\bar{\mathcal{U}}^*$, hence we write $\bar{\mathcal{U}}^* = \langle \mathcal{U}, \mathbb{R}^*, \mathcal{U}^{\mathsf{s}*} \rangle$. Elements of $\mathcal{U}^{\mathsf{s}}$ are called finite samples, elements of $\mathcal{U}^{\mathsf{s}*}$ are called internal samples.*

Finally, an external sample is a maximally consistent set of formulas $p(x^{\mathsf{s}}) \subseteq L(\mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathsf{s}*})$. (Note these are not types over the full of $\bar{\mathcal{U}}$.) □

2.2 Notation Let y and z be tuples of mixed sort. For any type $p(y)$ and formula $\varphi(y, z) \in L(\bar{\mathcal{U}}^*)$ we write

$$\varphi(p\,; \bar{\mathcal{U}}^*) \;=\; \Big\{ a \in (\bar{\mathcal{U}}^*)^z \;:\; p(y) \vdash \varphi(y\,; a) \Big\}$$

This notation intentionally confuses $p(y)$ with any of its realizations in some elementary extension of $\bar{\mathcal{U}}$. □

2.3 Definition *Let M be given. Let y be a tuple of the home-sort. A external sample $p(x^{\mathsf{s}})$ is*

1. *invariant if $\varphi(p\,; \mathcal{U}, \mathcal{U}^{\mathsf{s}*})$ is invariant over $\bar{M}$, for every $\varphi(x\,; y, z^{\mathsf{s}}) \in L$.*
2. *finitely satisfiable if every formula in $p(x^{\mathsf{s}})$ is satisfied by some element of M^{s}.*
3. *definable if $\varphi(p\,; \mathcal{U}, \mathcal{U}^{\mathsf{s}*})$ is definable over $\bar{M}$, for every $\varphi(x\,; y, z^{\mathsf{s}}) \in L$.* □

An external type is generically stable if it is definable and finitely satisfiable in some $\bar{M}$. □

Clearly every finitely satisfiable external sample is invariant.

We will use the symbol $\downfree_{\bar{M}}$ with the usual meaning.

3 Semigroup operation among samples

We work over a model M. We write $a_{\bar{M}}$ for the orbit of a over $\bar{M}$.