Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

The finite (fractional) expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, the set of real numbers, and a sort for the finite fractional multisets of elements of M^n, for any $n < \omega$. A (fractional) multiset is a function $s : M^n \to \mathbb{R}^+ \cup \{0\}$. We say that s is finite if it is almost always 0. We interpret $s(a)$ as the number of occurreces of a in s. For this reason, we use the following suggestive notation:

1. $$a \in s \quad = \quad s(a)$$ this is called the multiplicity of a in s;

2. $$|s| \quad = \quad \sum_{a \in M} a \in s$$ this is called the size of s;

3. $$\big|\{a \in s \ : \ \varphi(a\,;b)\}\big| \quad = \quad \sum_{a \models \varphi(x;b)} a \in s$$

We say sample-sort for the sort of these multisets.

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L and the language of arithmetic. Moreover, L^{f} has a symbols for the functions in 1-3 above. In particular, claim 3 requires a symbol for every $\varphi(x\,;z) \in L$.

Let $\mathcal{U}$ be a monster model of cardinality κ, a cardinal larger than the cardinality of L. We introduce two elementary extension of $\mathcal{U}^{\mathrm{f}}$.

2.1 Definition *We denote by $\mathcal{U}^{\mathrm{f}*}$ a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}*}$. We denote by $\mathcal{U}^{\mathrm{f}+}_{*}$ some elementary extension of $\mathcal{U}^{\mathrm{f}*}$ that realizes all types in $S(\mathcal{U}^{\mathrm{f}*})$. Hence we have $\mathcal{U}^{\mathrm{f}} \prec \mathcal{U}^{\mathrm{f}*} \prec \mathcal{U}^{\mathrm{f}+}_{*}$. Elements of sample-sort in these models are called finite samples, samples, and external samples respectively.* □

The use of $\mathcal{U}^{\mathrm{f}+}_{*}$ is not ideal, it is introduced to replace the use of global types of sample-sort which also have drawbacks.

We write $\mathrm{supp}(s)$ for the support of s, that is, the set of those a in $\mathcal{U}^{\mathrm{f}+}_{*}$ such that $a \in s$ is positive.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\operatorname{Av}_s \varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. We write $\operatorname{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

2.2 Theorem *Let $\varphi(x\,;z) \in L$ be a nip formula. Let M be a model. For every external sample s, and every ε, there is a finite sample $s_{M,\varepsilon} \in M^{\mathrm{f}}$ such that $\operatorname{Av}_s\varphi(x\,;b) \approx_\varepsilon \operatorname{Av}_{s_{M,\varepsilon}} \varphi(x\,;b)$ for all $b \in M^{|z|}$.*

Moreover, the cardinality of $s_{M,\varepsilon}$ soley depends on ε, namely

$$|s_{M,\varepsilon}| \quad\leq\quad c\,\frac{k}{\varepsilon^2}\ln\frac{k}{\varepsilon}$$

where c is an absolute constant and k is the VC-dimension of $\varphi(x\,;z)$.

Proof Questo è Vapnik-Chervonenkis. □

2.3 Lemma *For every nip formula $\varphi(x\,;z) \in L$, every external sample s and every infinite set $B \subseteq \mathcal{U}^{|z|}$ such that $\operatorname{Av}_s\varphi(x\,;b) > \varepsilon$ for every $b \in B$, there is a finite cover of B, say $B_1, \ldots, B_n$, such that all types $p_i(x) = \{\varphi(x\,;b) : b \in B_i\}$ are consistent.*

Proof Let M be any model containing B. Fix some $\varepsilon' < \varepsilon$. Let $s_{M,\varepsilon'}$ be the hyper given by Theorem 2.2. Then $\operatorname{supp}(s_{M,\varepsilon'})$ is finite, say $\operatorname{supp}(s_{M,\varepsilon'}) = \{a_1, \ldots, a_n\}$. Let $B_i = \{b \in B : \varphi(a_i\,;b)\}$. As $\operatorname{Av}_{s_{M.\varepsilon'}}\varphi(x\,;b) > 0$ for all $b \in B$, these B_i are the required cover of B. □

2.4 Corollary *Let $\psi(x\,;z) \in L$ be a nip formula and let s be an external sample. Then there is no infinite set C such that $\operatorname{Av}_s\big[\psi(x\,;c) \not\leftrightarrow \psi(x\,;c')\big] > \varepsilon$ for all $c, c' \in C$, $c \neq c'$.*

Proof Suppose for a contradiction that a set C as above exists. Apply Lemma 2.3 to the folmula $\varphi(x\,;z,z') = [\psi(x\,;z) \not\leftrightarrow \psi(x\,;z')]$ and the set $B = \{\langle c,c'\rangle \in C^2 : c \neq c'\}$. The sets B_i obtained from the lemma induce a finite coloring of the complete graph on C. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq C$. Hence $\{\psi(x\,;a) \not\leftrightarrow \psi(x\,;a') : a, a' \in A, a \neq a'\}$ is consistent. This is impossible for $|A| > 2$. □

The following is another immediate consequence of the lemma (it holds also when $\varphi(x\,;z)$ is not nip, though the proof is lengthier).

2.5 Corollary *For every nip formula $\varphi(x\,;z) \in L$, every external sample s and every sequence of indiscernibles $\langle b_i : i < \omega\rangle$ such that $\operatorname{Av}_s\varphi(x\,;b_i) > \varepsilon$ for every $i < \omega$, the type*

$\{\varphi(x\,;b_i) : i < \omega\}$ *is consistent.* □

2.6 Definition *Let $\varphi(x\,;z) \in L$. We say that the external sample s is smooth for $\varphi(x\,;z)$ over M if, for every $s' \equiv_M s$ and every $b \in \mathfrak{U}^{|z|}$, we have $\mathrm{Av}_{s'}\, \varphi(x\,;b) \approx \mathrm{Av}_s\, \varphi(x\,;b)$.* □

It follows immediately that if s is smooth for $\varphi(x\,;z)$ over M, then there is an internal sample $s' \in \mathfrak{U}^{\mathrm{f}*}$ such that $\mathrm{Av}_{s'}\, \varphi(x\,;b) \approx \mathrm{Av}_s\, \varphi(x\,;b)$ for all $b \in \mathfrak{U}^{|z|}$.

2.7 Definition *Let $\varphi(x\,;z) \in L$. We say that the external sample s is invariant for $\varphi(x\,;z)$ over M if, for every $b' \equiv_M b \in \mathfrak{U}^{|z|}$, we have $\mathrm{Av}_s\, \varphi(x\,;b) \approx \mathrm{Av}_s\, \varphi(x\,;b')$.* □

As authomorphisms of $\mathfrak{U}$ extend to automorphisms of $\mathfrak{U}^{\mathrm{f}+}_{*}$, hence an external sample that is smooth over M is invariant over M. The converse need not be true in general, but we have the following.

2.8 Fact *For every internal sample s following are equivalent*

1. *s is invariant for $\varphi(x\,;z)$ over M;*
2. *s is smooth for $\varphi(x\,;z)$ over M.*

Proof As remarked above, implication 2⇒1 holds for every external sample. We prove 1⇒2. Let $s \in \mathfrak{U}^{\mathrm{f}*}$ be an invariat internal sample. Assume for a contradiction that $\mathrm{Av}_{s'}\, \varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_s\, \varphi(x\,;b)$ for some $b \in \mathfrak{U}^{|x|}$ and some external sample $s' \equiv_M s$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\mathrm{Av}_{s''}\, \varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_s\, \varphi(x\,;b)$. As $s'' \equiv_M s$ are both in $\mathfrak{U}^{\mathrm{f}*}$, this contradicts the invariance of s. □

For smooth external samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathfrak{U}^{|z|}$.)

2.9 Corollary *Let $\varphi(x\,;z) \in L$ be nip. For every external sample s that is smooth for $\varphi(x\,;z)$ over M there is a sample $s_{M,\varepsilon} \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s \varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\, \varphi(x\,;b)$ for all $b \in \mathfrak{U}^{|z|}$.*

Proof By Theorem 2.2, there is an $s_{M,\varepsilon} \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s \varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\, \varphi(x\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\mathrm{Av}_s \varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\, \varphi(x\,;b')$ for some $b' \in \mathfrak{U}^{|z|}$. Pick a $s' \subseteq \mathfrak{U}^{|x|}$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}_M s' \equiv_M s$. By the smoothness of s, we obtain $\mathrm{Av}_{s'} \varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\, \varphi(x\,;b')$. As $b' \mathop{\smile\!\!\!\!\!\!\mid}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\mathrm{Av}_s \varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\, \varphi(x\,;b)$. A contradiction. □

The following non-uniform version of the corollay above holds without assumption of nip.

2.10 Lemma *The following are equivalent for every external sample s*

1. *s is smooth over M;*

2. *for every $b \in \mathcal{U}^{|z|}$ there is an $s_b \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_b}\varphi(x\,;b)$.*

Proof Implication 2⇒1 is clear. We prove 1⇒2, so, fix a $b \in \mathcal{U}^{|z|}$. Let μ be the standard part of $\mathrm{Av}_s\varphi(x\,;b)$ Let s' be an external sample such that $s \equiv_M s' \mathop{\smile\!\!\!\!\mid}_{M^{\mathrm{f}}} b$. By smoothness, $\mathrm{Av}_{s'}\varphi(x\,;b) \approx \mu$. As $s' \mathop{\smile\!\!\!\!\mid}_{M^{\mathrm{f}}} b$ there is an $s_b \in M^{\mathrm{f}}$ such that $\mathrm{Av}_{s_b}\varphi(x\,;b) \approx_\varepsilon \mu$. □

2.11 Definition *A external sample s is finitely satisfiable for $\varphi(x\,;z)$ in M if, for every $b \in \mathcal{U}^{|z|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \not\approx 0$, there is $a \in M^{|z|}$ such that $\varphi(a\,;b)$.* □

2.12 Fact *Let $\varphi(x\,;z) \in L$. Let M be any model. Every external sample s that is smooth for $\varphi(x\,;z)$ over M, is finitely satisfied in M.*

Proof Let $b \in \mathcal{U}^{|z|}$ be such that $\mathrm{Av}_s\varphi(x\,;b) > \varepsilon$. Pick a external sample s' such that $s \equiv_M s' \mathop{\smile\!\!\!\!\mid}_{M^{\mathrm{f}}} b$. By the smoothness of s, we obtain $\mathrm{Av}_{s'}\varphi(x\,;b) > \varepsilon$. Let $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_{s_0}\varphi(x\,;b) > \varepsilon$. Clearly, some element of $\mathrm{supp}(s_0)$ proves the fact. □

2.13 Definition *A external sample s is definable for $\varphi(x\,;z)$ over M if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \mathrm{Av}_s\varphi(x\,;b) < \varepsilon\}$ is definable over M.* □

2.14 Definition *We say that s is generically stable for $\varphi(x\,;z)$ over M if it is both definable and finitely satisfiable for $\varphi(x\,;z)$ over M.* □

2.15 Definition *A formula $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\ldots z_n) \in L$ such that for every finite sample $s \in \mathcal{U}^{\mathrm{f}}$ and every $b \in \mathcal{U}^{|x|}$ there are $a_1,\ldots,a_n \in \mathrm{supp}(s)$ such that $\psi(x\,;a_1,\ldots,a_n) \leftrightarrow p(x)$, where $p(x) = \mathrm{tp}_\varphi(b/\,\mathrm{supp}(s))$.* □

2.16 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *if s is generically stable for $\varphi(x\,;z)$ over M then s is smooth for $\varphi(x\,;z)$ over M.*