

# Scratch paper

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**Abstract** Poche idee, ben confuse.

## 1 Introduction

## 2 Preliminaries

The **finite (fractional) expansion** of a structure  $M$  is a many-sorted expansion that has a sort for  $M$ , which we call the home-sort, a sort for  $\mathbb{R}$ , the set of real numbers, and a sort for the finite fractional multisets of elements of  $M^n$ , for any  $n < \omega$ . A **(fractional) multiset** is a function  $s : M^n \rightarrow \mathbb{R}^+ \cup \{0\}$ . We say that  $s$  is **finite** if it is almost always 0. We interpret  $s(a)$  as the number of occurrences of  $a$  in  $s$ . For this reason, we use the following suggestive notation:

1.  $a \in s = s(a)$  this is called the **multiplicity** of  $a$  in  $s$ ;
2.  $|s| = \sum_{a \in M} a \in s$  this is called the **size** of  $s$ ;
3.  $|\{a \in s : \varphi(a; b)\}| = \sum_{a \models \varphi(x; b)} a \in s$

We call these multisets **samples** of  $M^n$ .

We will write  $M^f$  for the expansion above. The language of  $M^f$  is denoted by  $L^f$ . It expands  $L$  and the language of arithmetic. Moreover,  $L^f$  has a symbols for the functions in 1-3 above. In particular, claim 3 requires a symbol for every  $\varphi(x; z) \in L$ .

Let  $\mathcal{U}$  be a monster model of cardinality  $\kappa$ , an inaccessible cardinal larger than the cardinality of  $L$ . The **hyperfinite expansion** of  $\mathcal{U}$  is a saturated elementary extension of  $\mathcal{U}^f$  of cardinality  $\kappa$ . We will write  $\mathcal{U}^{f*}$  for this expansion. As all saturated models of cardinality  $\kappa$  are isomorphic, we can assume that  $\mathcal{U}$  is the home-sort of  $\mathcal{U}^{f*}$ .

We write  $\text{supp}(s)$  for the **support** of  $s$ , that is, the subset of  $\mathcal{U}^n$  where  $x \in s$  is positive. Note that  $\text{supp}(s)$  is a definable subset of  $\mathcal{U}^{f*}$ , hence its cardinality is either finite or  $\kappa$ . Moreover,  $s \subseteq M^n$  for some small model  $M$ , then  $\text{supp}(s)$  is finite.

If  $s$  is a sample in  $\mathcal{U}^{f*}$ , we write  $s \subseteq \mathcal{U}^n$  as a shorthand of  $\text{supp}(s) \subseteq \mathcal{U}^n$ .

**2.1 Fact** Every model  $M$  is the domain of the home-sort of some  $M^f \preceq \mathcal{U}^{f*}$ .

**Proof** Difficile da scrivere senza hand-waving. □

Let  $s \subseteq \mathcal{U}^{[x]}$  be a sample. For every formula  $\varphi(x; z) \in L$  and every  $b \in \mathcal{U}^{[z]}$  we define

$$\text{Av}_s \varphi(x; b) = \frac{|\{a \in s : \varphi(a; b)\}|}{|s|},$$

Below,  $\varepsilon$  always ranges over the positive standard reals. If  $a$  and  $b$  are hyperreals, we write  $a \approx_\varepsilon b$  for  $|a - b| < \varepsilon$ . We write  $a \approx b$  if  $a \approx_\varepsilon b$  holds for every  $\varepsilon$ . We write  $\text{st}(a)$ , for the standard part of the hyperrational number  $a$ . That is, the unique real number  $\mu$  such that  $\mu \approx a$ .

The standard part of  $\text{Av}_s$  induces a finite probability measure on the algebra of definable subsets of  $\mathcal{U}^{[x]}$ . This measure has a unique extension to a Lebesgue probability measure on a  $\sigma$ -algebra. This is known as the **Loeb measure**.

**2.2 Theorem** *Let  $\varphi(x; z) \in L$  be a nip formula. Let  $M$  be a model. For every sample  $s \subseteq \mathcal{U}^{[x]}$ , and every  $\varepsilon$ , there is a sample  $s_{M, \varepsilon} \subseteq M^{[x]}$  such that  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_{M, \varepsilon}} \varphi(x; b)$  for all  $b \in M^{[z]}$ .*

Moreover, the cardinality of  $s_{M, \varepsilon}$  solely depends on  $\varepsilon$ , namely

$$|s_{M, \varepsilon}| \leq c \frac{k}{\varepsilon^2} \ln \frac{k}{\varepsilon}$$

where  $c$  is an absolute constant and  $k$  is the VC-dimension of  $\varphi(x; z)$ .

**Proof** Questo è Vapnik-Chervonenkis. □

**2.3 Lemma** *For every nip formula  $\varphi(x; z) \in L$ , every sample  $s \subseteq \mathcal{U}^{[x]}$  and every infinite  $B \subseteq \mathcal{U}^{[z]}$  such that  $\text{Av}_s \varphi(x; b) > \varepsilon$  for every  $b \in B$ , there is a finite cover of  $B$ , say  $B_1, \dots, B_n$ , such that all types  $p_i(x) = \{\varphi(x; b) : b \in B_i\}$  are consistent.*

**Proof** Let  $M$  be any model containing  $B$ . Fix some  $\varepsilon' < \varepsilon$ . Let  $s_{M, \varepsilon'}$  be the sample given by Theorem 2.2. Then  $\text{supp}(s_{M, \varepsilon'})$  is finite, say  $\text{supp}(s_{M, \varepsilon'}) = \{a_1, \dots, a_n\}$ . Let  $B_i = \{b \in B : \varphi(a_i; b)\}$ . As  $\text{Av}_{s_{M, \varepsilon'}} \varphi(x; b) > 0$  for all  $b \in B$ , these  $B_i$  are the required cover of  $B$ . □

**2.4 Corollary** *Let  $\psi(x; z) \in L$  be a nip formula and let  $s \subseteq \mathcal{U}^{[x]}$  be a sample. There is no infinite set  $C$  such that  $\text{Av}_s [\psi(x; c) \nleftrightarrow \psi(x; c')] > \varepsilon$  for all  $c, c' \in C$ .*

**Proof** Apply Lemma 2.3 to the formula  $\varphi(x; z, z') = [\psi(x; z) \nleftrightarrow \psi(x; z')]$  and the set  $B = C^2$ . Without loss of generality, we may assume that the sets  $B_i$  given by the lemma form a partition of the complete graph on  $C$ . By the Ramsey theorem there is an infinite  $A \subseteq C$  such that  $\{\psi(x; a) \nleftrightarrow \psi(x; a') : a, a' \in A, a \neq a'\}$  is consistent. Clearly this is a contradiction as soon as  $|A| > 2$ . □

The following is another immediate consequence of the lemma.

**2.5 Corollary** *For every nip formula  $\varphi(x; z) \in L$ , every sample  $s \subseteq \mathcal{U}^{[x]}$  and every sequence of indiscernibles  $\langle b_i : i < \omega \rangle$  such that  $\text{Av}_s \varphi(x; b_i) > \varepsilon$  for every  $i < \omega$ , the type  $\{\varphi(x; b_i) : i < \omega\}$  is consistent.* □

**2.6 Definition** Let  $\varphi(x; z) \in L$ . We say that the sample  $s \subseteq \mathcal{U}^{|x|}$  is *invariant* for  $\varphi(x; z)$  over  $M$  if, for every  $s' \equiv_M s$  and every  $b \in \mathcal{U}^{|z|}$ , we have  $\text{Av}_s \varphi(x; b) \approx \text{Av}_{s'} \varphi(x; b)$ .  $\square$

For invariant samples the Vapnik-Chervonenkis Theorem can be strengthened as follows.

**2.7 Corollary** Let  $\varphi(x; z) \in L$  be a *nip* formula. Let  $M$  be any model. For every sample  $s \subseteq \mathcal{U}^{|x|}$  that is invariant for  $\varphi(x; z)$  over  $M$ , and every  $\varepsilon$ , there is a sample  $s_{M, \varepsilon} \subseteq M^{|x|}$  such that  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_{M, \varepsilon}} \varphi(x; b)$  for all  $b \in \mathcal{U}^{|z|}$ .

**Proof** By the theorem above, there is a sample  $s_{M, \varepsilon} \subseteq M^{|x|}$  such that  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_{M, \varepsilon}} \varphi(x; b)$  for all  $b \in M^{|z|}$ .

Suppose for a contradiction that  $\text{Av}_s \varphi(x; b') \not\approx_\varepsilon \text{Av}_{s_{M, \varepsilon}} \varphi(x; b')$  for some  $b' \in \mathcal{U}^{|z|}$ . Pick a sample  $s' \subseteq \mathcal{U}^{|x|}$  such that  $b' \downarrow_M s' \equiv_M s$ . By invariance,  $\text{Av}_{s'} \varphi(x; b') \not\approx_\varepsilon \text{Av}_{s_{M, \varepsilon}} \varphi(x; b')$ . As  $b' \downarrow_M s'$ , the same formula holds for some  $b \in M^{|z|}$ . Once again by invariance,  $\text{Av}_s \varphi(x; b) \not\approx_\varepsilon \text{Av}_{s_{M, \varepsilon}} \varphi(x; b)$ . A contradiction.  $\square$

The following is stronger form of invariance.

**2.8 Definition** We say that a sample  $s \subseteq \mathcal{U}^{|x|}$  is *smooth* for  $\varphi(x; z)$  over  $M$  if for any other sample  $s'$  such that  $\text{Av}_s \varphi(x; b) \approx \text{Av}_{s'} \varphi(x; b)$  for all  $b \in M^{|z|}$ , the same equivalence holds for all  $b \in \mathcal{U}^{|z|}$ .  $\square$

Clearly, smooth implies invariant.

**2.9 Definition ???** A sample  $s \subseteq \mathcal{U}^{|x|}$  is *finitely satisfiable* for  $\varphi(x; z)$  in  $M$  if, for every  $b \in \mathcal{U}^{|z|}$  such that  $\text{Av}_s \varphi(x; b) \not\approx 0$ , there is  $a \in M^{|z|}$  such that  $\varphi(a; b)$ .

We say that  $s$  is *generically stable* for  $\varphi(x; z)$  over  $M$  if it is both invariant and finitely satisfiable for  $\varphi(x; z)$  over  $M$ .  $\square$

**2.10 Definition ???** A formula  $\varphi(x; z) \in L$  is *distal* if there is a formula  $\psi(x; z_1, \dots, z_n) \in L$  such that for every finite sample  $s$  and every  $b \in \mathcal{U}^{|x|}$  there are  $a_1, \dots, a_n \in s^{|z|}$  such that  $\psi(x; a_1, \dots, a_n) \leftrightarrow p(x)$ , where  $p(x) = \text{tp}_\varphi(b / \dots s)$ .  $\square$

**2.11 Theorem (false)** The following are equivalent

1.  $\varphi(x; z) \in L$  is distal;
2. if  $s$  is generically stable for  $\varphi(x; z)$  over  $M$  then  $s$  is smooth for  $\varphi(x; z)$  over  $M$ .