

# Scratch paper

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**Abstract** Poche idee, ben confuse.

## 1 Introduction

## 2 Abstract samples

Let  $M$  be a structure of signature  $L$ . The **sample-expansion** of  $M$  is a 3-sorted expansion  $\langle M, \omega, \bar{M} \rangle$ , where  $\bar{M} = M^{<\omega}$ . These three sorts are called **home-sort**, **integer-sort**, and **sample-sort**, respectively. The symbol  $x$  always denotes a tuple of variables of the home-sort. We put a bar over the symbols of sample-sort such as  $\bar{x}$  and  $\bar{a}$ . Symbols as  $m, n, i$  are of integer-sort; from the context it should be inferred whether they are variables or parameters.

The **size** of a sample  $\bar{a} \in \bar{M}$  is the length of  $\bar{a}$  as an element of  $M^{<\omega}$ . This is denoted by  $\text{lh } \bar{a}$ . We write  $\bar{a}.i$  for the  $i$ -th element of the sample  $\bar{a}$ . The same symbol  $\bar{a}$  may be used for tuples of samples such as  $\bar{a}_1, \dots, \bar{a}_n$ . Then by  $|\bar{a}|$  we denote the length of this tuple (that is,  $n$ , for the tuple above). We always make the implicit assumption that all  $\bar{a}_i$  in a tuple have the same size. By this assumption,  $\bar{a}.i$  is a well-defined element of  $M^{|\bar{a}|}$ .

The language of this expansion is denoted by  $\bar{L}$ ; it comprises

1. all symbols of  $L$  that apply to the home-sort;
2.  $0, 1, +, \cdot$  that apply to the integer-sort;
3. a ternary relation of sort  $\bar{M}, \omega, M$  that holds if  $\bar{x}.i = y$ ;
4. for every formula  $\varphi(x; y) \in \bar{L}$ , where  $y$  a tuple of mixed sort, there is relation symbol for  $\varphi(\bar{x}.i; y)$ , where  $i$  is a variable of integer-sort.

When  $\varphi(x)$  is the formula  $x = x$ , requirement 4 implies that  $\text{lh } \bar{x}$  is definable. By 2, the cardinality of finite subsets of  $\omega$  is definable. Therefore, from 4, it easily follows that the function  $|\{i < \text{lh } \bar{a} : \varphi(\bar{a}.i)\}|$  is definable, uniformly in the parameters of  $\varphi(x)$ .

As in the integer-sort we can interpret the field of rational numbers, below we will freely use rational numbers when this clarify the notation. For instance we define

$$\text{Fr}_{\bar{a}}[\varphi(x)] = \frac{|\{i < \text{lh } \bar{a} : \varphi(\bar{a}.i)\}|}{\text{lh } \bar{a}}$$

and use that it is a definable function in  $\bar{L}$ . In particular we will use that for every  $s \in \mathbb{Q}$  there is a formula saying  $\text{Fr}_{\bar{a}} [\varphi(x)] \geq s$ , uniformly in the parameter of  $\varphi(x)$ .

**2.1 Definition** Let  $\mathcal{U}$  be a monster model of inaccessible cardinality  $\kappa > |L|$  and let  $\langle \mathcal{U}, \omega, \bar{\mathcal{U}} \rangle$  be the corresponding sample-expansion. We denote by  $\langle \mathcal{U}, \omega^*, \bar{\mathcal{U}}^* \rangle$  a saturated elementary extension of  $\langle \mathcal{U}, \omega, \bar{\mathcal{U}} \rangle$  of cardinality  $\kappa$ . Note that we can assume that the home-sort of this extension is  $\mathcal{U}$ , in fact as all saturated models of cardinality  $\kappa$  are isomorphic. The elements of  $\bar{\mathcal{U}}$  are called *finite samples*, those of  $\bar{\mathcal{U}}^*$  are called *internal samples*.

To any internal sample  $\bar{a}$  we associate a finitely additive measure on definable subsets of  $\mathcal{U}^{|\bar{x}|}$ . For  $\varphi(x) \in \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$  we define

$$\text{Pr}_{\bar{a}} [\varphi(x)] = \sup \left\{ s \in \mathbb{Q} : s < \text{Fr}_{\bar{a}} \varphi(x) \right\}$$

Note that this measure is type-definable, uniformly in the parameters of  $\varphi(x)$ . Namely, let  $r \in \mathbb{R}$  and  $\varphi(x; y) \in \bar{L}$ , where  $y$  is a tuple of mixed sort. Then there is a type  $q(\bar{x}; y) \subseteq \bar{L}$  that defines  $\text{Pr}_{\bar{x}} [\varphi(x; y)] \geq r$ .

Let  $M$  be a given model. By elementarity, for every  $\varepsilon > 0$  and every formula  $\varphi(x) \in L(M, \omega, \bar{M})$  there is an  $\bar{a}' \in \bar{M}$  such that

$$\text{Pr}_{\bar{a}} [\varphi(x)] \approx_{\varepsilon} \text{Fr}_{\bar{a}'} \varphi(x)$$

Note that  $\bar{a}$  need not have the same size of  $\bar{a}'$ . In fact, in general  $\text{lh } \bar{a}$  may be non standard, while  $\text{lh } \bar{a}' \in \omega$ .

**2.2 Definition** An *external sample* is a global type  $p(\bar{x}) \in S(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$  such that, for every formula  $\varphi(x) \in L(\mathcal{U})$ , the set  $\{i \in \omega^* : p(\bar{x}) \vdash \varphi(\bar{x}.i)\}$  is bounded and definable; uniformly in the parameters of  $\varphi(x)$ .  $\square$

For every external sample  $p(\bar{x})$  there is an  $n \in \omega^*$  such that  $p(\bar{x}) \vdash \text{lh } \bar{x} = n$  and  $|\{i \in \omega^* : p(\bar{x}) \vdash \varphi(\bar{x}.i)\}|$  is definable uniformly on the parameters of  $\varphi(x)$ .

In analogy to what done with internal samples, we associate to an external sample  $p(\bar{x})$  a finitely additive probability measure on the definable subsets of  $\mathcal{U}$ . Namely, we define

$$\text{Pr}_{\bar{p}} [\varphi(x)] = \sup \left\{ r \in \mathbb{Q} : r < \frac{|\{i < \text{lh } \bar{x} : p(\bar{x}) \vdash \varphi(\bar{x}.i)\}|}{\text{lh } \bar{x}} \right\}$$

What claimed above for internal samples, apply also to external samples, that is, there is a type  $q(\bar{x}; y) \subseteq \bar{L}$  that defines  $\text{Pr}_{\bar{p}} [\varphi(x; y)] \geq r$ .

**2.3 Notation** Let  $z$  be a tuple of variables of the home-sort. For  $p(\bar{x})$  an external sample,  $\varphi(\bar{x}; z) \in \bar{L}$ , and  $b \in \mathcal{U}^{|z|}$  we write  $\varphi(\bar{p}; b)$  for  $p(\bar{x}) \vdash \varphi(\bar{x}; b)$ . We also write

$$\varphi(\bar{p}; \mathcal{U}) = \left\{ b \in \mathcal{U}^{|z|} : \varphi(\bar{p}; b) \right\}$$

Sets of this form are called *externally definable*. This notation intentionally confuses  $p(\bar{x})$  with any of its realizations (suggestively denoted by  $\bar{p}$ ) in some elementary

extension of  $\bar{\mathcal{U}}$ . □

**2.4 Definition (with warning)** Let  $z$  be a tuple of variables of the home-sort. Let  $M$  be given. An external sample  $p(\bar{x})$  is

1. *invariant* over  $M$  if for every formula  $\varphi(\bar{x}; z) \in \bar{L}(M, \omega, \bar{M})$  the set  $\varphi(\bar{p}; \mathcal{U})$  is invariant over  $M$ ;
2. *finitely satisfiable* in  $M$  if every formula  $\varphi(\bar{x}) \in \bar{L}(\mathcal{U}, \omega, \bar{\mathcal{U}})$  in  $p(x)$  is satisfied by some  $\bar{a} \in \bar{M}$ .

These notions are standard but for the important fact that only parameters in the home-sort are allowed. □

It is clear that satisfiable implies invariant.

If an external sample  $p(\bar{x})$  is weakly satisfiable in  $\bar{M}$  then for every  $\varepsilon > 0$  and every formula  $\varphi(x)$  there is an  $\bar{a} \in \bar{M}$  such that

$$\Pr_{\bar{p}} [\varphi(x)] \approx_{\varepsilon} \text{Fr}_{\bar{a}} \varphi(x)$$

Let  $p(\bar{x})$  external sample finitely satisfiable in  $M$ . A coheir sequence in  $p(\bar{x})$  is sequence  $\langle \bar{a}_i : i < \omega \rangle$  such that

$$\psi(x; b_1, \dots, b_n) \rightarrow [\varphi(a; b) \leftrightarrow \varphi(x; b)]$$

### 3 Distal formulas

The formula  $\varphi(x; z) \in L$  is *distal* if there is a formula  $\psi(x; z_1, \dots, z_n) \in L$  such that for every finite set  $B \subseteq \mathcal{U}^{|z|}$  of cardinality at least 2 and every  $a \in \mathcal{U}^{|x|}$  there are some  $b_1, \dots, b_n \in B$  such that  $\psi(a; b_1, \dots, b_n)$  and

$$\psi(x; b_1, \dots, b_n) \rightarrow [\varphi(a; b) \leftrightarrow \varphi(x; b)] \quad \text{for all } b \in B.$$