# Scratch paper

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**Abstract** Poche idee, ben confuse.

## 1 Introduction

## 2 Samples

The finite (fractional) expansion of a structure $M$ is a many-sorted expansion that has a sort for $M$, which we call the home-sort, a sort for $\mathbb{R}$, the set of real numbers, and, for every $n$, a sort for the functions $s : M^n \to \mathbb{R}$ that are almost always 0.

These functions will be interpreted as (signed and fractional) multisets. Namely $s(a)$ interpreted as the number of times $a$ occurs in $s$. We use the following suggestive notation:

1. $a \in s \;=\; s(a)$ this is called the multiplicity of $a$ in $s$;
2. $|s| \;=\; \sum_{a \in M} a \in s$ this is called the size of $s$;
3. $\big|\{a \in s \,:\, \varphi(a\,;b)\}\big| \;=\; \sum_{a \models \varphi(x;b)} a \in s.$

We call these multisets samples and refer to their sorts collectively as sample-sort.

We will write $M^{\mathrm{f}}$ for the expansion above. The language of $M^{\mathrm{f}}$ is denoted by $L^{\mathrm{f}}$. It expands $L$, the language of rings, and the language of vector spaces over $\mathbb{R}$. Each interpreted in the obvious sort. Moreover, $L^{\mathrm{f}}$ has a symbols for the functions in 1-3 above. In particular, 3 requires a symbol for every $\varphi(x\,;z) \in L$.

Let $\mathcal{U}$ be a monster model of cardinality $\kappa$, a cardinal larger than the cardinality of $L$. We introduce two elementary extension of $\mathcal{U}^{\mathrm{f}}$.

**2.1 Definition** *We denote by $\mathcal{U}^{\mathrm{f}_*}$ a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality $\kappa$. As all saturated models of cardinality $\kappa$ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}_*}$. We denote by $\mathcal{U}^{\mathrm{f}_*^+}$ some elementary extension of $\mathcal{U}^{\mathrm{f}_*}$ that realizes all types in $S(\mathcal{U}^{\mathrm{f}_*})$. Hence we have $\mathcal{U}^{\mathrm{f}} \prec \mathcal{U}^{\mathrm{f}_*} \prec \mathcal{U}^{\mathrm{f}_*^+}$. Elements of sample-sort in these models are called finite samples, internal samples, and (external) samples respectively.* □

The use of $\mathcal{U}^{\mathrm{f}_*^+}$ is not ideal, it is introduced to replace the use of global types of sample-sort which also have drawbacks.

We write $\mathrm{supp}(s)$ for the support of $s$, that is, the set of those $a$ in $\mathcal{U}^{\mathrm{f}+}_{*}$ such that $a \in s$ is positive.

Let $s$ be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_s\, \varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

Below, $\varepsilon$ always ranges over the positive standard reals. If $a$ and $b$ are hyperreals, we write $a \approx_\varepsilon b$ for $|a-b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every $\varepsilon$. We write $\mathrm{st}(a)$, for the standard part of the hyperrational number $a$. That is, the unique real number $\mu$ such that $\mu \approx a$.

The standard part of $\mathrm{Av}_s$ induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a $\sigma$-algebra. This is known as the Loeb measure.

**2.2 Definition** *We say that a sample $s$ is smooth for $\varphi(x\,;z)$ over $M$ if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\mathrm{Av}_{s'}\, \varphi(x\,;b) \approx \mathrm{Av}_s\, \varphi(x\,;b)$.* □

**2.3 Definition** *Let $\varphi(x\,;z) \in L$ and let $M$ be a model. An external $s$ is*

1. *invariant if, for every $b' \equiv_M b \in \mathcal{U}^{|z|}$, we have $\mathrm{Av}_s\, \varphi(x\,;b) \approx \mathrm{Av}_s\, \varphi(x\,;b')$.*
2. *definable if, for every $\varepsilon$, the set $\{b \in \mathcal{U}^{|z|} : \mathrm{Av}_s \varphi(x\,;b) < \varepsilon\}$ is definable over $M$.*
3. *satisfiable if, for all $b \in \mathcal{U}^{|z|}$, if $\mathrm{Av}_s \varphi(x\,;b) \not\approx 0$, then $\varphi(a\,;b)$ for some $a \in M^{|z|}$.*
4. *generically stable if all of the above holds.*

*When $\varphi(x\,;z)$ and $M$ are not clear from the context we add: for $\varphi(x\,;z)$ over/in $M$.*

**2.4 Fact** *Smooth samples are generically stable. (For some given $\varphi(x\,;z)$ and $M$.)*

**Proof** We prove in turn 1-3 of the definition above.

1. If $s$ is smooth for $\varphi(x\,;z)$ over $M$, then there is an internal sample $s' \in \mathcal{U}^{\mathrm{f}_*}$ such that $\mathrm{Av}_{s'}\, \varphi(x\,;b) \approx \mathrm{Av}_s\, \varphi(x\,;b)$ for all $b \in \mathcal{U}^{|z|}$. Then 1 follows.

2. The set $\{b \in \mathcal{U}^{|z|} : \mathrm{Av}_s \varphi(x\,;b) < \varepsilon\}$ is definable in $\mathcal{U}^{\mathrm{f}_*}$ because, by invariance of $s$, we may replace $s$ with any internal sample $s' \equiv_M s$. As this set is invariat over $M$, it is definable over $M^{\mathrm{f}}$ and therefore, over $M$.

3. Let $b \in \mathcal{U}^{|z|}$ be such that $\mathrm{Av}_s \varphi(x\,;b) > \varepsilon$. Pick a sample $s'$ such that $s \equiv_M s' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.3ex\hbox{$|$}}_{M^{\mathrm{f}}}\, b$. By the smoothness of $s$, we obtain $\mathrm{Av}_{s'} \varphi(x\,;b) > \varepsilon$. Let $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_{s_0} \varphi(x\,;b) > \varepsilon$. Clearly, some element of $\mathrm{supp}(s_0)$ proves 3. □

Smooth samples are invariant. The converse need not be true in general, but we have the following.

**2.5 Fact** *For every internal sample $s$, the following are equivalent*

1. *$s$ is invariant for $\varphi(x\,;z)$ over $M$;*

2. *$s$ is smooth for $\varphi(x\,;z)$ over $M$.*

**Proof** By the fact above, implication 2⇒1 holds for every sample. We prove 1⇒2. Let $s \in \mathcal{U}^{\mathrm{f}*}$ be an invariat internal sample. Assume for a contradiction that $\mathrm{Av}_{s'}\,\varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_s\,\varphi(x\,;b)$ for some $b \in \mathcal{U}^{|x|}$ and some sample $s' \equiv_M s$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\mathrm{Av}_{s''}\,\varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_s\,\varphi(x\,;b)$. As $s'' \equiv_M s$ are both in $\mathcal{U}^{\mathrm{f}*}$, this contradicts the invariance of $s$. □

The following is a useful characterization of smoothness.

**2.6 Lemma** *The following are equivalent for every external sample $s$*

1. *$s$ is smooth over $M$;*
2. *for every $b \in \mathcal{U}^{|z|}$ there is a finite sample $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_0}\varphi(x\,;b)$. (The sample $s_0$ depends on $M$, epsilon and $b$.)*
3. *there are some finite samples $s_1,\dots,s_n \in M^{\mathrm{f}}$ such that for every $b \in \mathcal{U}^{|z|}$ we have $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_i}\varphi(x\,;b)$ for some $s_i$. (The samples $s_1,\dots,s_n$ as well as the number $n$ depend on $\varepsilon$ and $M$.)*

**Proof** Implication 3⇒1 is clear. We prove 1⇒2, so, fix a $b \in \mathcal{U}^{|z|}$. Let $\mu$ be the standard part of $\mathrm{Av}_s\varphi(x\,;b)$ Let $s'$ be a sample such that $s \equiv_M s' \mathop{\smile\!\!\!\!\!\perp}_{M^{\mathrm{f}}} b$. By smoothness, $\mathrm{Av}_{s'}\varphi(x\,;b) \approx \mu$. As $s' \mathop{\smile\!\!\!\!\!\perp}_{M^{\mathrm{f}}} b$ there is an $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_{s_0}\varphi(x\,;b) \approx_\varepsilon \mu$.

We prove 2⇒3 by compactness. Suppose not, then the following global type is dinitely consistent, hence realized in $\mathcal{U}^{\mathrm{f}+}_*$

$$p(y) \quad = \quad \Big\{\mathrm{Av}_y\varphi(x,b) \not\approx_\varepsilon \mathrm{Av}_{s_0}\varphi(x\,;b) \;:\; s_0 \in M^{\mathrm{f}},\ b \in \mathcal{U}^{|z|}\Big\}$$

This is a contradiction. □

**2.7 Definition** *Let $\varphi(x\,;z) \in L$. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that $s$ is uniform on $\mathcal{B}$ if for every $\varepsilon$ there is a finite sample $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s\varphi(x,b) \approx_\varepsilon \mathrm{Av}_{s_0}\varphi(x,b)$ for every $b \in \mathcal{B}$. (The sample $s_0$ depends on $\varepsilon$ and $M$.) We add for $\varphi(x\,;z)$ over $M$ when these are not clear from the context.* □

**2.8 Lemma** *Let $s$ be uniform on $\mathcal{B}$ and such that $\mathrm{Av}_s\varphi(x\,;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1,\dots,\mathcal{B}_n$, such that all types $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ are consistent.*

**Proof** Let $s_0 \in M^{\mathrm{f}}$ be as in Definition 2.7. As $s_0$ is finite, we may assume that $\mathrm{supp}(s_0) = \{a_1,\dots,a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i\,;b)\}$. As $\mathrm{Av}_{s_0}\varphi(x\,;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

**2.9 Corollary** *If $s$ be uniform on $\langle b_i : i < \omega\rangle$ for $\varphi(x\,;z)$ over $M$, where $\langle b_i : i < \omega\rangle$ is a sequence of indiscernibles such that $\mathrm{Av}_s\varphi(x\,;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x\,;b_i) : i < \omega\}$ is consistent.* □

**2.10 Corollary** *Let $s$ be uniform on $\mathcal{C}$ for $\varphi(x\,;z)$ over $M$. Then* $\mathrm{Av}_s\big[\psi(x\,;c) \nleftrightarrow \psi(x\,;c')\big] > \varepsilon$ *does not hold for all $c, c' \in \mathcal{C}$, $c \neq c'$.*

**Proof** Suppose for a contradiction that a set $\mathcal{C}$ as above exists. Apply Lemma 2.8 to the folmula $\varphi(x\,;z,z') = [\psi(x\,;z) \nleftrightarrow \psi(x\,;z')]$ and the set $\mathcal{B} = \{\langle c,c'\rangle \in \mathcal{C}^2 : c \neq c'\}$. The sets $\mathcal{B}_i$ obtained from the lemma induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\psi(x\,;a) \nleftrightarrow \psi(x\,;a') : a, a' \in A, a \neq a'\}$ is consistent. As $|A| > 2$ this is impossible. $\square$

# 3 The nip formulas

**3.1 Theorem** *Let $\varphi(x\,;z) \in L$ be a nip formula. Then every sample is uniform on $M$ over $M$.*

**Proof** Questo è Vapnik-Chervonenkis. $\square$

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below $b$ ranges over $\mathcal{U}^{|z|}$.)

**3.2 Corollary** *Let $\varphi(x\,;z) \in L$ be nip. Every smooth sample $s$ is uniform for $\varphi(x\,;z)$ on $\mathcal{U}$ over $M$.*

**Proof** By Theorem 3.1, there is an $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_0}\,\varphi(x\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\mathrm{Av}_s\varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_0}\,\varphi(x\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathop{\smile\!\!\!\!\!\!\!\;|}_M s' \equiv_M s$. By the smoothness of $s$, we obtain $\mathrm{Av}_{s'}\varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_0}\,\varphi(x\,;b')$. As $b' \mathop{\smile\!\!\!\!\!\!\!\;|}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\mathrm{Av}_s\varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_{s_0}\,\varphi(x\,;b)$. A contradiction. $\square$

**3.3 Definition** *A formula $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\ldots z_n) \in L$ such that for every finite set $A \subseteq \mathcal{U}^{\mathrm{f}}$ and every $b \in \mathcal{U}^{|x|}$ there are $a_1,\ldots,a_n \in \mathrm{supp}(s)$ such that $\psi(x\,;a_1,\ldots,a_n) \leftrightarrow p(x)$, where $p(x) = \mathrm{tp}_\varphi(b/A)$.* $\square$

**3.4 Theorem (false)** *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *for every sample $s \in \mathcal{U}^{\mathrm{f}^+_*}$, if $s$ is generically stable for $\varphi(x\,;z)$ over $M$ then $s$ is smooth for $\varphi(x\,;z)$ over $M$.*