Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

The finite (fractional) expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, the set of real numbers, and a sort for the finite fractional multisets of elements of M^n, for any $n < \omega$. A (fractional) multiset is a function $s : M^n \to \mathbb{R}^+ \cup \{0\}$. We say that s is finite if it is almost always 0. We interpret $s(a)$ as the number of occurreces of a in s. For this reason, we use the following suggestive notation:

1.
$$a \in s \;=\; s(a) \qquad \text{this is called the multiplicity of } a \text{ in } s;$$

2.
$$|s| \;=\; \sum_{a \in M} a \in s \qquad \text{this is called the size of } s;$$

3.
$$\big|\{a \in s \;:\; \varphi(a\,;b)\}\big| \;=\; \sum_{a \models \varphi(x;b)} a \in s$$

We call these multisets samples of M^n.

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L and the language of arithmetic. Moreover, L^{f} has a symbols for the functions in 1-3 above. In particular, claim 3 requires a symbol for every $\varphi(x\,;z) \in L$.

Let $\mathcal{U}$ be a monster model of cardinality κ, an inaccessible cardinal larger than the cardinality of L. The hyperfinite expansion of $\mathcal{U}$ is a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. We will write $\mathcal{U}^{\mathrm{f}*}$ for this expansion. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}*}$.

We write $\mathrm{supp}(s)$ for the support of s, that is, the subset of $\mathcal{U}^n$ where $x \in s$ is positive. Note that $\mathrm{supp}(s)$ is a definable subset of $\mathcal{U}^n$, hence its cardinality is either finite of κ. Moreover, $s \subseteq \mathcal{M}^n$ for some small model M, then $\mathrm{supp}(s)$ is finite.

If s is a sample in $\mathcal{U}^{\mathrm{f}*}$, we write $s \subseteq \mathcal{U}^n$ as a shorthand of $\mathrm{supp}(s) \subseteq \mathcal{U}^n$.

2.1 Fact *Every model M is the domain of the home-sort of some $M^{\mathrm{f}} \preceq \mathcal{U}^{\mathrm{f}*}$.*

Proof Difficile da scrivere senza hand-waving. $\qquad\qquad\square$

Let $s \subseteq \mathcal{U}^{|x|}$ be a sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_s\,\varphi(x\,;b) \;=\; \frac{\big|\{a \in s \,:\, \varphi(a\,;b)\}\big|}{|s|},$$

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a-b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. We write $\mathrm{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

2.2 Lemma *For every formula $\varphi(x\,;z) \in L$, every sample $s \subseteq \mathcal{U}^{|x|}$ and every infinite $B \subseteq \mathcal{U}^{|z|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \not\approx 0$ for every $b \in B$, there is a finite cover $B_1,\dots,B_n$ of B such that the types $\{\varphi(x\,;b) : b \in B_i\}$ are consistent.*

Proof Let $M^{\mathrm{f}*} \preceq \mathcal{U}^{\mathrm{f}*}$ be a sufficiently saturated model such that $B \subseteq M^{|z|}$. Define for every ε

$$B_\varepsilon \;=\; \Big\{b \in B \,:\, \mathrm{Av}_s\varphi(x\,;b) > \varepsilon\Big\}.$$

Define also

$$p_\varepsilon(w) \;=\; \Big\{\mathrm{Av}_w\varphi(x\,;b) > \varepsilon : \, b \in B_\varepsilon\Big\}.$$

By saturation there is an $s' \in M^{\mathrm{f}*}$ that realises the union of all these types. As $s' \subseteq M^{|x|}$ then $\mathrm{supp}(s')$ is finite. Clearly, we can assume that $a \in s'$ is standard for all a. So the claim follows. $\qquad\square$

2.3 Theorem *Let $\varphi(x\,;z) \in L$ be a nip formula. Let M be a model. For every sample $s \subseteq \mathcal{U}^{|x|}$, and every ε, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\varphi(x\,;b)$ for all $b \in M^{|z|}$.*

Moreover, the cardinality of $s_{M,\varepsilon}$ soley depends on ε, namely

$$|s_{M,\varepsilon}| \;\leq\; c\,\frac{k}{\varepsilon^2}\,\ln\frac{k}{\varepsilon}$$

where c is an absolute constant.

Proof Questo è Vapnik-Chervonenkis. $\qquad\square$

2.4 Remark ??? When the model M in the theorem above is sufficiently (?) saturated, we can claim that there is a (necessarily finite) sample $s_M \subseteq M^{|x|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx \mathrm{Av}_{s_M}\varphi(x\,;b)$. $\qquad\square$

2.5 Definition *Let $\varphi(x\,;z) \in L$. We say that the sample $s \subseteq \mathcal{U}^{|x|}$ is invariant for $\varphi(x\,;z)$ over A if, for every $s' \equiv_A s$ and every $b \in \mathcal{U}^{|z|}$, we have $\mathrm{Av}_s\,\varphi(x\,;b) \approx \mathrm{Av}_{s'}\,\varphi(x\,;b)$.* $\qquad\square$

For invariant samples the Vapnik-Chervonenkis Theorem can be strengthened as

follows.

2.6 Corollary *Let $\varphi(x;z) \in L$ be a nip formula. Let M be any model. For every sample $s \subseteq \mathcal{U}^{|x|}$ that is invariant for $\varphi(x;z)$ over M, and every ε, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\mathrm{Av}_s\varphi(x;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}} \varphi(x;b)$ for all $b \in \mathcal{U}^{|z|}$.*

Proof By the theorem above, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\mathrm{Av}_s\varphi(x;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}} \varphi(x;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\mathrm{Av}_s\varphi(x;b') \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}} \varphi(x;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a sample $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \underset{M}{\perp} s' \equiv_M s$. By invariance, $\mathrm{Av}_{s'}\varphi(x;b') \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}} \varphi(x;b')$. As $b' \underset{M}{\perp} s'$, the same formula holds for some $b \in M^{|z|}$. Once again by invariance, $\mathrm{Av}_s\varphi(x;b) \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}} \varphi(x;b)$. A contradiction. $\square$

The following is stronger form of invariance.

2.7 Definition ??? *We say that a sample $s \subseteq \mathcal{U}^{|x|}$ is smooth for $\varphi(x;z)$ over M if for any other sample s' such that $\mathrm{Av}_s \varphi(x;b) \approx \mathrm{Av}_{s'} \varphi(x;b)$ for all $b \in M^{|z|}$, the same equivalence holds for all $b \in \mathcal{U}^{|z|}$.* $\square$

2.8 Definition ??? *A sample $s \subseteq \mathcal{U}^{|x|}$ is finitely satisfiable for $\varphi(x;z)$ in M if, for every $b \in \mathcal{U}^{|z|}$ such that $\mathrm{Av}_s\varphi(x;b) \not\approx 0$, there is $a \in M^{|z|}$ such that $\varphi(a;b)$.*

We say that s is generically stable for $\varphi(x;z)$ over M if it is both invariant and finitely satisfiable for $\varphi(x;z)$ over M. $\square$

2.9 Definition ??? *A formula $\varphi(x;z) \in L$ is distal if there is a formula $\psi(x;z_1,\ldots z_n) \in L$ such that for every finite sample s and every $b \in \mathcal{U}^{|x|}$ there are $a_1,\ldots,a_n \in s^{|z|}$ such that $\psi(x;a_1,\ldots,a_n) \leftrightarrow p(x)$, where $p(x) = \mathrm{tp}_\varphi(b/....s)$.* $\square$

2.10 Theorem (false) *The following are equivalent*

1. *$\varphi(x;z) \in L$ is distal;*

2. *if s is generically stable for $\varphi(x;z)$ over M then s is smooth for $\varphi(x;z)$ over M.*