Scratch paper

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Abstract Poche idee, ben confuse.

`branch: Szemeredi`

1 Abstract samples

Let M be a structure. The finite (fractional) sample expansion of M is a 3-sorted expansion $\bar{M} = \langle M, \mathbb{R}, M^{\mathsf{s}} \rangle$ where

1. M is called the home-sort;
2. $\mathbb{R}$ is the set of real numbers in the signature of ordered field;
3. M^{s} is an vector lattice space (a Riesz space) containing all functions $s : M \to \mathbb{R}$ with finite support (i.e. almost always zero); this we call the sample-sort.

The elements M^{s} maybe thought as (signed) discrete measures concentrated on the support.

Variables of sample-sort (or tuple thereof) are denoted with symbols $x^{\mathsf{s}}, y^{\mathsf{s}}, \ldots,$ and variations. Elements of sample-sort are denoted with the symbols s, r, t, and variations. The variable x is reserved for (tuples of) variables of the home-sort. The context will clarify the sort the other variables.

The language of $\bar{M}$ is denoted by L. It expands the language of the three sorts above with a function symbol for each formula $\varphi(x\,;y) \in L$, where y is a tuple a mixed sort. This is interpreted with function $f_{\varphi(x\,;y)} : (M^s)^{|x|} \times M^y \to \mathbb{R}$ as explained below.

For legibility, substitute parameters for y. So, for $\varphi(x) \in L(\bar{M})$ we define

$$\begin{aligned} f_{\varphi(x)}(s) &= \sum_{a \models \varphi(x)} s(a) \\ \text{(or, for short)} &= \sum_{\varphi(x)} s \end{aligned}$$

As usual with vector lattices, we write $s^+ = s \vee 0$ for the nonnegative part of s and $s^- = s \wedge 0$ for the nonpositive part of s. Finally we define $|s| = s^+ + s^-$. The norm of s is

$$\|s\| = \sum_{x=x} |s|.$$

$$= \sum_{a \in \mathcal{U}^{|x|}} |s(a)|.$$

Finally we write

$$\operatorname*{Fr}_{s} \varphi(x) = \frac{1}{\|s\|} \sum_{\varphi(x)} |s|.$$

When ambiguity is of concern we write $\operatorname*{Fr}_{x \in s} \varphi(x)$.

1.1 Definition *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$. Let $\bar{\mathcal{U}} = \langle \mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathsf{s}} \rangle$ be as above and let $\bar{\mathcal{U}}^*$ be a saturated elementary extension of $\bar{\mathcal{U}}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the domain of the home-sort of $\bar{\mathcal{U}}^*$, hence we write $\bar{\mathcal{U}}^* = \langle \mathcal{U}, \mathbb{R}^*, \mathcal{U}^{\mathsf{s}*} \rangle$. Elements of $\mathcal{U}^{\mathsf{s}}$ are called finite samples, elements of $\mathcal{U}^{\mathsf{s}*}$ are called hyperfinite samples.* □

1.2 Remark (pseudofinite structures) Let $\{M_i : i < \omega\}$ be a set of structures and define

$$M = \bigcup_{i<\omega} \{i\} \times M_i$$

We stipulate that $M \vDash r(a)$ if $a = \langle i, a_1 \rangle, \ldots, \langle i, a_n \rangle$ for some i and $M_i \vDash r(a_1, \ldots, a_n)$. Functions symbols are interpreted in a similar way (on mixed tuples function take some fixed but arbitrary value, say the first component of the tuple). In the signature of M we include also a symbol for an equivalence relation that has $\{i\} \times M_i$ as equivalence classes. In $\mathcal{U}$, every equivalence class is a saturated elementary extension of some ultraproducts of $\{M_i : i < \omega\}$. When the M_i have finite unbounded cardinality we call each equivalence class in $\mathcal{U}$ a pseudofinite structure. In this case each class is the support of some 0-1 valued internal sample $s \in \mathcal{U}^{\mathsf{s}*}$. In particular, every class not in M has (properly) hyperfinite cardinality. □

Hyperfinite samples that are 0-1 valued are called hyperfinite sets. We write $a \in s$ for $s(a) = 1$, when s is an hyperfinite set.

Let $s \in \mathcal{U}^{(<\omega)*}$ be a hyperfinite set and fix some formula $\varphi(x\,;y) \in L(\bar{\mathcal{U}}^*)$ where x, y are variables of the home-sort. Define

$$a \sim_s a' = \forall y \in s \big[\varphi(a, y) \leftrightarrow \varphi(a', y)\big]$$

1.3 Notation Let y and z be tuples of mixed sort. For any type $p(y)$ and formula $\varphi(y, z) \in L(\bar{\mathcal{U}}^*)$ we write

$$\varphi(p\,; \bar{\mathcal{U}}^*) = \Big\{ a \in (\bar{\mathcal{U}}^*)^z \ : \ p(y) \vdash \varphi(y\,; a) \Big\}$$

This notation intentionally confuses $p(y)$ with any of its realizations in some elementary extension of $\bar{\mathcal{U}}$. □

1.4 Definition *Let M be given. Let y be a tuple of the home-sort. A external sample $p(x^{\mathsf{s}})$ is*

1. *invariant if $\varphi(p\,; \mathcal{U}, \mathcal{U}^{\mathsf{s}*})$ is invariant over $\bar{M}$, for every $\varphi(x\,; y, z^{\mathsf{s}}) \in L$.*

2. *finitely satisfiable if every formula in $p(x^{\mathsf{s}})$ is satisfied by some element of M^{s}.*

3. *definable if $\varphi(p\,;\mathcal{U},\mathcal{U}^{\mathsf{s}*})$ is definable over $\bar{M}$, for every $\varphi(x\,;y,z^{\mathsf{s}})\in L$.* □

An external type is generically stable if it is definable and finitely satisfiable in some $\bar{M}$. □

Clearly every finitely satisfiable external sample is invariant.

We will use the symbol $\mathop{\smile\!\!\!\!\!\mid}_{\bar{M}}$ with the usual meaning.

2 Type-definable functions

Let $\mathcal{U}$ be a monster model. Let X be a Hausdorff space. We say that the partial map $f:\mathcal{U}^{|x|}\to X$ is bounded if $\operatorname{img} f\subseteq C$ for some compact $C\subseteq X$. We say that f is type-definable if it is bounded and $f^{-1}[C]$ is type-definable for every compact $C\subseteq X$. If $f^{-1}[C]$ is definable for every compact $C\subseteq X$, then we say that f is definable.

We always assume $\mathcal{U}$ has a larger cardinality than the topology of X and that a type-definable function is always type-definable over some small set of parameters, i.e. all the types defining $f^{-1}[C]$ are over the same small set parameters $A\subseteq\mathcal{U}$.

2.1 Fact *Let $f:\mathcal{U}^{|x|}\to X$ be type-definable. Then the following are equivalent*

1. *f is definable;*

2. $\operatorname{img} f$ *is finite.*

Proof By compactness. □

2.2 Fact *Let $p_i(x)\subseteq L(A)$, for $i=1,\dots,n$, be a cover of a type-definable set $\mathcal{D}\subseteq\mathcal{U}^{|x|}$. Then there is a cover of $\mathcal{D}$ by some formulas $p_i(x)\vdash\varphi_i(x)$.*

Proof By compactness. □

2.3 Fact *Let $f:\mathcal{U}^{|x|}\to[0,1]$ be type-definable. Then for every $\varepsilon>0$ there is a definable function $g:\mathcal{U}^{|x|}\to[0,1]$ such that*

$$\big|fa-ga\big| \quad\leq\quad \varepsilon \qquad\qquad \text{for every } a\in\operatorname{dom} f$$

Proof Pick $r_1,\dots,r_n\in[0,1]$ be such that the intervals $[r_i-\varepsilon,r_i+\varepsilon]$ cover $\operatorname{img} f$. Hence, the types defining $f^{-1}[r_i-\varepsilon,r_i+\varepsilon]$ cover $\operatorname{dom} f$. Therefore, by the fact above, there is a definable partition of $\operatorname{dom} f$ that refines $f^{-1}[r_i-\varepsilon,r_i+\varepsilon]$. Define g to be such that $\big\{g^{-1}[r_i]:i\in[n]\big\}$ is the partition above. Then g is as required. □

2.4 Fact *Let $f:\mathcal{U}^{|x|}\to[0,1]$ be type-definable. Then for every $\varepsilon>0$ there are finitely many $a_1,\dots,a_n\in\operatorname{dom} f$ such that*

$$\bigvee_{i=1}^{n}\big|fa-fa_i\big|\leq\varepsilon \qquad\qquad \text{for every } a\in\operatorname{dom} f.$$

Moreover, there are some formulas $\psi_i(x) \in L(\mathfrak{U})$ *that cover* dom f *and*

$$\psi_i(x) \quad \rightarrow \quad \big|fx - fa_i\big| \leq \varepsilon \qquad\qquad \textit{for every } i \in [n].$$

Proof By Fact 2.3 there is a definable function g such that $\big|fa - ga\big| \leq \varepsilon/2$ for every $a \in \operatorname{dom} f$. Pick some $a_1, \dots, a_n \in \mathfrak{U}^{|x|}$ such that $\{ga_1, \dots, ga_n\} = \operatorname{img} g$. Then

$$\begin{aligned}
\min_{i\in[n]} \big|fa - fa_i\big| \quad &= \quad \min_{i\in[n]} \big|fa - ga + ga - fa_i\big| \\
&\leq \quad \big|fa - ga\big| + \min_{i\in[n]} \big|fa_i - ga\big| \\
&\leq \quad \big|fa - ga\big| + \big|fa_i - ga_i\big| \qquad \text{where } i \text{ is such that } ga_i = ga \\
&\leq \quad \varepsilon
\end{aligned}$$

The second claim follows from Fact 2.2 because $|fx - fa_i| \leq \varepsilon$ is equivalent to a type. □

3 Externally definable sets (notation)

4 Szemerédi's regularity lemma

An internal nonnegative sample $s \in \mathcal{U}^{\mathrm{s}*}$ is pseudorandom over M if for every $0 \leq s' \leq s$ such that $\|s'\| \approx \|s\|$ and for every $\varphi(x) \in L(\bar{M})$

$$\operatorname*{Fr}_{s'} \varphi(x) \quad \approx \quad \operatorname*{Fr}_{s} \varphi(x)$$

For every $\varepsilon > 0$ there is an $n = n(\varepsilon)$ such that for every formula $\varphi(x, z)$ and every finite nonnegative sample $a, b \in M^{\mathrm{s}}$ there are $0 \leq s_i \leq a$ and $0 \leq t_i \leq b$, for $i \in [n]$ and a set $\Sigma \subseteq [n]^2$ such that

and for every

$$\big|\varphi(A, B) - \operatorname*{Fr}_{s_i, t_j} \varphi(x, y)\big|$$

$$\big|\varphi(a^{\mathrm{s}}\,; b^{\mathrm{s}})\big|) \quad = \quad \sum_{\varphi(x\,;y)} a^{\mathrm{s}}(x) \cdot b^{\mathrm{s}}(y)$$

$$d_\varphi(r\,; s) \quad = \quad \frac{\big|\varphi(r\,; s)\big|}{\big|r \times s\big|}$$

We say that the pair of samples r, s is ε-regular if for every $r' \subseteq r$ and $s' \subseteq s$ such that $|r'| > \varepsilon|r|$ and $|s'| > \varepsilon|s|$

$$\big|d_\varphi(r\,; s) - d_\varphi(r'; s')\big| \quad < \quad \varepsilon$$

4.1 Szemerédi's Regularity Lemma *For every* $\varepsilon > 0$ *and for every pair of samples* $u, v \in \bar{\mathfrak{U}}^*$ *of hyperfinite cardinality there are some finite partitions of* u *and* v*, say* $r_1, \dots, r_n$ *and* $s_1, \dots, s_m$*, such that*

1. r_i, s_j *is* ε*-regular for every* $i, j \in [n] \times [m] \smallsetminus X$*,*

where $X \subseteq [n] \times [m]$*, called the exceptional set, is such that*

2. $$\sum_{\langle i,j\rangle \in X} |r_i|\cdot|s_j| \quad < \quad \varepsilon\cdot|u|\cdot|v|$$

Proof Let $\mathrm{st}((d(u\,;v))$ □

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