Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Samples

The finite (fractional) expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, the set of real numbers, and, for every n, a sort for the functions $s : M^n \to \mathbb{R}$ that are almost always 0.

These functions will be interpreted as (signed and fractional) multisets. Namely $s(a)$ interpreted as the number of times a occurs in s. We use the following suggestive notation:

1. $a \in s \;=\; s(a)$ this is called the multiplicity of a in s;
2. $|s| \;=\; \sum_{a \in M} a \in s$ this is called the size of s;
3. $\big|\{a \in s \;:\; \varphi(a\,;b)\}\big| \;=\; \sum_{a \models \varphi(x;b)} a \in s.$

We call these multisets samples and refer to their sorts collectively as sample-sort.

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L, the language of rings, and the language of vector spaces over $\mathbb{R}$. Each interpreted in the obvious sort. Moreover, L^{f} has a symbols for the functions in 1-3 above. In particular, 3 requires a symbol for every $\varphi(x\,;z) \in L$.

Let $\mathcal{U}$ be a monster model of cardinality κ, a cardinal larger than the cardinality of L. We introduce two elementary extension of $\mathcal{U}^{\mathrm{f}}$.

2.1 Definition *We denote by $\mathcal{U}^{\mathrm{f}_*}$ a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}_*}$. We denote by $\mathcal{U}^{\mathrm{f}^+_*}$ some elementary extension of $\mathcal{U}^{\mathrm{f}_*}$ that realizes all types in $S(\mathcal{U}^{\mathrm{f}_*})$. Hence we have $\mathcal{U}^{\mathrm{f}} \prec \mathcal{U}^{\mathrm{f}_*} \prec \mathcal{U}^{\mathrm{f}^+_*}$. Elements of sample-sort in these models are called finite samples, internal samples, and (external) samples respectively.* □

The use of $\mathcal{U}^{\mathrm{f}^+_*}$ is not ideal, it is introduced to replace the use of global types of sample-sort which also have drawbacks.

We write $\mathrm{supp}(s)$ for the support of s, that is, the set of those a in $\mathcal{U}^{\mathrm{f}+}_{*}$ such that $a \in s$ is positive.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_s\,\varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a-b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. We write $\mathrm{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

2.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\mathrm{Av}_{s'}\,\varphi(x\,;b) \approx \mathrm{Av}_s\,\varphi(x\,;b)$. When $\varphi(x\,;z)$ and M are not clear from the context we add: for $\varphi(x\,;z)$ over M.* □

2.3 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b' \equiv_M b \in \mathcal{U}^{|z|}$, we have $\mathrm{Av}_s\,\varphi(x\,;b) \approx \mathrm{Av}_s\,\varphi(x\,;b')$.*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \mathrm{Av}_s\varphi(x\,;b) < \varepsilon\}$ is definable over M.*
3. *satisfiable if, for all $b \in \mathcal{U}^{|z|}$, if $\mathrm{Av}_s\varphi(x\,;b) \not\approx 0$, then $\varphi(a\,;b)$ for some $a \in M^{|z|}$.*
4. *generically stable if all of the above holds.*

When $\varphi(x\,;z)$ and M are not clear from the context we add: for $\varphi(x\,;z)$ over/in M.

2.4 Fact *Smooth samples are generically stable.*

Proof We prove in turn 1-3 of the definition above.

1. If s is smooth for $\varphi(x\,;z)$ over M, then there is an internal sample $s' \in \mathcal{U}^{\mathrm{f}_*}$ such that $\mathrm{Av}_{s'}\,\varphi(x\,;b) \approx \mathrm{Av}_s\,\varphi(x\,;b)$ for all $b \in \mathcal{U}^{|z|}$. Then 1 follows.

2. The set $\{b \in \mathcal{U}^{|z|} : \mathrm{Av}_s\varphi(x\,;b) < \varepsilon\}$ is definable in $\mathcal{U}^{\mathrm{f}_*}$ because, by invariance of s, we may replace s with any internal sample $s' \equiv_M s$. As this set is invariat over M, it is definable over M^{f} and therefore, over M.

3. Let $b \in \mathcal{U}^{|z|}$ be such that $\mathrm{Av}_s\varphi(x\,;b) > \varepsilon$. Pick a sample s' such that $s \equiv_M s' \mathop{\perp}_{M^{\mathrm{f}}} b$. By the smoothness of s, we obtain $\mathrm{Av}_{s'}\varphi(x\,;b) > \varepsilon$. Let $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_{s_0}\varphi(x\,;b) > \varepsilon$. Clearly, some element of $\mathrm{supp}(s_0)$ proves 3. □

Smooth samples are invariant. The converse need not be true in general, but we have the following.

2.5 Fact *Let $\varphi(x\,;z) \in L$ and M be given. For every internal sample s, the following are* *equivalent*

1. *s is invariant;*
2. *s is smooth.*

Proof By the fact above, implication 2⇒1 holds for every sample. We prove 1⇒2. Let $s \in \mathcal{U}^{f_*}$ be an invariat internal sample. Assume for a contradiction that $\mathrm{Av}_{s'}\,\varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_s\,\varphi(x\,;b)$ for some $b \in \mathcal{U}^{|x|}$ and some sample $s' \equiv_M s$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\mathrm{Av}_{s''}\,\varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_s\,\varphi(x\,;b)$. As $s'' \equiv_M s$ are both in $\mathcal{U}^{f_*}$, this contradicts the invariance of s. □

The following is a useful characterization of smoothness.

2.6 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *for every $b \in \mathcal{U}^{|z|}$ there is a finite sample $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_0}\varphi(x\,;b)$. (The sample s_0 depends on M, epsilon and b.)*
3. *there are some finite samples $s_1,\dots,s_n \in M^{\mathrm{f}}$ such that for every $b \in \mathcal{U}^{|z|}$ we have $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_i}\varphi(x\,;b)$ for some s_i. (The samples $s_1,\dots,s_n$ as well as the number n depend on ε and M.)*

Proof Implication 3⇒1 is clear. We prove 1⇒2, so, fix a $b \in \mathcal{U}^{|z|}$. Let μ be the standard part of $\mathrm{Av}_s\varphi(x\,;b)$ Let s' be a sample such that $s \equiv_M s' \downfree_{M^{\mathrm{f}}} b$. By smoothness, $\mathrm{Av}_{s'}\varphi(x\,;b) \approx \mu$. As $s' \downfree_{M^{\mathrm{f}}} b$ there is an $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_{s_0}\varphi(x\,;b) \approx_\varepsilon \mu$.

We prove 2⇒3 by compactness. Suppose not, then the following global type is finitely consistent, hence realized in $\mathcal{U}^{f+}_*$

$$p(y) \quad = \quad \Big\{\mathrm{Av}_y\varphi(x,b) \not\approx_\varepsilon \mathrm{Av}_{s_0}\varphi(x\,;b) \ : \ s_0 \in M^{\mathrm{f}},\ b \in \mathcal{U}^{|z|}\Big\}$$

This is a contradiction. □

2.7 Definition *Let $\varphi(x\,;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s is approximable on $\mathcal{B}$ if for every ε there is a finite sample $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s\varphi(x,b) \approx_\varepsilon \mathrm{Av}_{s_0}\varphi(x,b)$ for every $b \in \mathcal{B}$. (The sample s_0 depends on ε and M.) We add for $\varphi(x\,;z)$ over M when these are not clear from the context.* □

2.8 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that $\mathrm{Av}_s\varphi(x\,;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1,\dots,\mathcal{B}_n$, such that all types $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ are consistent.*

Proof Let $s_0 \in M^{\mathrm{f}}$ be as in Definition 2.7. As s_0 is finite, we may assume that $\mathrm{supp}(s_0) = \{a_1,\dots,a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i\,;b)\}$. As $\mathrm{Av}_{s_0}\varphi(x\,;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

2.9 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega \rangle$, where $\langle b_i : i < \omega \rangle$ is a sequence of indiscernibles such that $\mathrm{Av}_s \varphi(x\,;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x\,;b_i) : i < \omega\}$ is consistent.* □

2.10 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{C}$. Then for every ε is a pair of distict $c, c' \in \mathcal{C}$ such that $\mathrm{Av}_s\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] < \varepsilon$*

Proof Suppose for a contradiction that $\mathrm{Av}_s\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] \geq \varepsilon$ for all $c, c' \in \mathcal{C}$. Apply Lemma 2.8 to the folmula $\psi(x\,;z,z') = [\varphi(x\,;z) \not\leftrightarrow \varphi(x\,;z')]$ and the set $\mathcal{B} = \{\langle c, c'\rangle \in \mathcal{C}^2 : c \neq c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 2.8 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x\,;a) \not\leftrightarrow \varphi(x\,;a') : a, a' \in A, a \neq a'\}$ is consistent. As $|A| > 2$, this is impossible. □

3 The nip formulas

3.1 Theorem *Let $\varphi(x\,;z) \in L$ be a nip formula. Then every sample is approximable on M over M.*

Proof Questo è Vapnik-Chervonenkis. □

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|z|}$.)

3.2 Corollary *Let $\varphi(x\,;z) \in L$ be nip. Every smooth sample s is approximable on $\mathcal{U}$ over M.*

Proof By Theorem 3.1, there is an $s_0 \in M^{\mathrm{f}}$ such that $\mathrm{Av}_s \varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_0}\, \varphi(x\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\mathrm{Av}_s \varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_0}\, \varphi(x\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathrel{\raise0.2ex\hbox{$\smile$}\llap{$\mid$}}_M s' \equiv_M s$. By the smoothness of s, we obtain $\mathrm{Av}_{s'} \varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_0}\, \varphi(x\,;b')$. As $b' \mathrel{\raise0.2ex\hbox{$\smile$}\llap{$\mid$}}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\mathrm{Av}_s \varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_{s_0}\, \varphi(x\,;b)$. A contradiction. □

3.3 Definition *A formula $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1, \ldots z_n) \in L$ such that for every finite set $A \subseteq \mathcal{U}^{\mathrm{f}}$ and every $b \in \mathcal{U}^{|x|}$ there are $a_1, \ldots, a_n \in \mathrm{supp}(s)$ such that $\psi(x\,;a_1, \ldots, a_n) \leftrightarrow p(x)$, where $p(x) = \mathrm{tp}_\varphi(b/A)$.* □

3.4 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *for every sample $s \in \mathcal{U}^{\mathrm{f}^+_*}$, if s is generically stable for $\varphi(x\,;z)$ over M then s is smooth for $\varphi(x\,;z)$ over M.*