Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Samples

Let M be a structure of signature L. The finite (fractional) sample expansion of M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, as ordered field, and for each n a sort for F_n, an $\mathbb{R}$-vector space of functions $f : M^n \to \mathbb{R}$. We call the sort of F_n sample-sort. Variables of sample-sort are denoted with sumbolds $x^{\rm s}, y^{\rm s}$ and variations thereof.

We write $M^{\rm f}$ for the expansion above. The language of $M^{\rm f}$ is denoted by $L^{\rm f}$. It expands the natural laguage of each sort: L, the language ordered rings, and the language of $\mathbb{R}$-vector spaces. Moreover, for every $\varphi(x\,;z) \in L$ in $L^{\rm f}$ there is a symbol $\overline{\varphi}(x^{\rm s}\,;z)$ for a function $F_{|x|} \times M^{|z|} \to \mathbb{R}$.

2.1 Definition *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$. We denote by $\mathcal{U}^{\rm s}$ a saturated elementary extension of $\mathcal{U}^{\rm f}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\rm s}$. Elements $r \in \mathcal{U}^{\rm f}$ are called finite samples, elements in $s \in \mathcal{U}^{\rm s}$ are called internal samples. Types $p(x^{\rm s}) \in S(\mathcal{U}^{\rm f})$ are called global samples.* □

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

2.2 Notation In general, for any global type $p(x) \in S(\mathcal{U})$ and formula $\varphi(x\,;z) \in L$ we write

$$\varphi(p\,;\mathcal{U}) \;=\; \Big\{b \in \mathcal{U}^{|z|} \;:\; \varphi(x\,;b) \in p\Big\}$$

This notation intentionally confuses p with any of its realizations in some elementary extension of $\mathcal{U}$. Sets of this form are called externally definable.

We extend this notation to global samples. If $\varphi(x\,;z) \in L$ and $p(x^{\rm s})$ is a global sample, we write $\overline{\varphi}(p\,;b)$ for the unique $\mu \in \mathbb{R}$ such that $p(x^{\rm s}) \vdash \overline{\varphi}(x^{\rm s}\,;b) \approx \mu$. In analogy to the terminology above, we may say that $\overline{\varphi}(p\,;z)$ is an externally definable real valued function. □

2.3 Definition *We say that $p(x^{\rm s}) \in S(\mathfrak{U}^{\rm f})$ is a smooth sample if it is realized in $\mathfrak{U}^{\rm s}$.* □

2.4 Definition *Let $\varphi(x\,;z) \in L$ and M be given. A global sample $p(x^{\rm s})$ is*

1. *invariant if $\bar\varphi(p\,;b) \approx \bar\varphi(p\,;b')$ for every $b \equiv_{M^{\rm f}} b'$;*
2. *(pointwise) approximable if, for every $b \in \mathfrak{U}^{|z|}$ and every ε, there is an $r \in M^{\rm f}$ such that $\bar\varphi(p\,;b) \approx_\varepsilon \bar\varphi(r\,;b)$;*
3. *definable if, for every ε, the set $\{b \in \mathfrak{U}^{|z|} : \bar\varphi(p\,;b) < \varepsilon\}$ is definable over $M^{\rm f}$.* □

2.5 Fact *Let $\varphi(x\,;z) \in L$ and M be given. Let $p(x^{\rm s})$ be a smooth sample. Then if p is invariant, it is approximable and definable.*

Proof Let $s \in \mathfrak{U}^{\rm s}$ be such that $\bar\varphi(s\,;b) \approx \bar\varphi(p\,;b)$ for all $b \in \mathfrak{U}^{|z|}$. Assuming 1 of Definition 3.2, we prove 2 and 3.

1⇒2. Let $\mu \in \mathbb{R}$ be such that $\bar\varphi(p\,;b) \approx \mu \in \mathbb{R}$. Let b' be such that $s \mathop{\smile\!\!\!\!\!\!\;|}_{M^{\rm f}} b' \equiv_{M^{\rm f}} b$. By 1, $\bar\varphi(s\,;b') \approx \mu$. Therefore there is an $r \in M^{\rm f}$ such that $\bar\varphi(r\,;b') \approx_\varepsilon \mu$. As $b' \equiv_{M^{\rm f}} b$, we obtain $\bar\varphi(r\,;b) \approx_\varepsilon \mu$.

1⇒3. The set $\{b : \bar\varphi(p\,;b) < \varepsilon\}$ coincides with $\{b : \bar\varphi(s\,;b) < \varepsilon\}$, hence is definable in $\mathfrak{U}^{\rm s}$. As it is invariat over $M^{\rm f}$, it is definable over $M^{\rm f}$. □

The following is a useful characterization of smooth samples.

2.6 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every global sample $p(x^{\rm s})$*

1. *p is smooth and invariant;*
2. *p is approximable;*
3. *there are some finite samples $r_1,\dots,r_n \in M^{\rm f}$ such that for every $b \in \mathfrak{U}^{|z|}$ we have $\bar\varphi(p\,;b) \approx_\varepsilon \bar\varphi(r_i\,;b)$ for some r_i (the samples $r_1,\dots,r_n$ as well as the number n depend on ε, but not on b);*
4. *there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathfrak{U}^{|z|}$ we have that $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\bar\vartheta_j(p) - \bar\vartheta_i(p) < \varepsilon$ for some i, j.*

Proof Let $s \in \mathfrak{U}^{\rm s}$ be such that $\bar\varphi(s\,;b) \approx \bar\varphi(p\,;b)$ for all $b \in \mathfrak{U}^{|z|}$.

1⇒2. Let $b \in \mathfrak{U}^{|z|}$ be arbitrary. Let $\mu \in \mathbb{R}$ be such that $\bar\varphi(p\,;b) \approx \mu \in \mathbb{R}$. Let b' be such that $s \mathop{\smile\!\!\!\!\!\!\;|}_{M^{\rm f}} b' \equiv_{M^{\rm f}} b$. By 1, $\bar\varphi(s\,;b') \approx \mu$. Therefore there is an $r \in M^{\rm f}$ such that $\bar\varphi(r\,;b') \approx_\varepsilon \mu$. As $b' \equiv_{M^{\rm f}} b$, we obtain $\bar\varphi(r\,;b) \approx_\varepsilon \mu$.

1⇒3. By smoothness we it suffices to prove 3 with s for p. The type

$$p(z) \quad = \quad \Big\{\bar\varphi(s\,;z) \not\approx_\varepsilon \bar\varphi(r\,;z) \ :\ r \in M^{\rm f}\Big\}$$

is inconsistent by 2. Hence compatness yields the required $r_1,\dots,r_n \in M^{\rm f}$.

The proof of equivalence 3 ⇔ 4 is left to the reader. The other implications are evident. $\square$

3 Old stuff/garbage

We write $\mathrm{supp}(p)$ for the support of p, that is, the set of those a in $\mathcal{U}$ such that $p \vdash (a \in x^{\mathrm{s}}) > \varepsilon$ for some standard positive ε. If s is a smooth sample $\mathrm{supp}(s)$ is defined to be $\mathrm{supp}(p)$ for $p(x^{\mathrm{s}}) = \mathrm{tp}(s/\mathcal{U}^{\mathrm{f}})$.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_{x/s}\varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

When possible we abbreviate $\mathrm{Av}_{x/s}\varphi(x\,;b)$ with $\overline{\varphi}(s\,;b)$.

We write $\mathrm{ft}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

All definitions and facts in this section are relative to some given formula $\varphi(x\,;z) \in L$ and some model M.

3.1 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\overline{\varphi}(s';b) \approx \overline{\varphi}(s\,;b)$.* $\square$

3.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\overline{\varphi}(s\,;b) \approx \overline{\varphi}(s\,;b')$;*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(s\,;b) < \varepsilon\}$ is definable over M;*
3. *finitely satisfiable if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$;*
4. *generically stable if all of the above hold.* $\square$

Smooth roughly means internal and invariant.

3.3 Fact *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *there is an invariant internal sample $s' \equiv_M s$.*

Proof 1⇒2. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathcal{U}^{|z|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \mathrm{Aut}(\mathcal{U}^{\mathrm{s}}/M)$. Then $\overline{\varphi}(s';fb) \approx \overline{\varphi}(f^{-1}s'\,;b)$. Moreover, by smothness $\overline{\varphi}(f^{-1}s'\,;b) \approx \overline{\varphi}(s';b)$. The infariace in s' follows.

2⇒1. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ such that $\overline{\varphi}(s';b) \not\approx_\varepsilon \overline{\varphi}(s;b)$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\overline{\varphi}(s'';b) \not\approx_\varepsilon \overline{\varphi}(s;b)$. As $s'' \equiv_M s$ are both in $\mathcal{U}^{\mathrm{s}}$, then $s'' = fs$ for some $f \in \mathrm{Aut}(\mathcal{U}^{\mathrm{s}}/M)$. Hence we obtan $\overline{\varphi}(s;f^{-1}b) \not\approx_\varepsilon \overline{\varphi}(s;b)$ which contradics the invariance of s. □

3.4 Definition *Let $\varphi(x;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s is (uniformly) approximable on $\mathcal{B}$ if for every ε there is a finite sample $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s,b) \approx_\varepsilon \overline{\varphi}(r,b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε, not on b.)* □

3.5 Lemma *Let $\varphi(x;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that $\overline{\varphi}(s;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1, \ldots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x;b) : b \in \mathcal{B}_i\}$ are consistent.*

Proof Let $r \in M^{\mathrm{f}}$ be as in Definition 3.4. As r is finite, we may assume that $\mathrm{supp}(r) = \{a_1, \ldots, a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i;b)\}$. As $\overline{\varphi}(r;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

3.6 Corollary *Let $\varphi(x;z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega\rangle$, where $\langle b_i : i < \omega\rangle$ is a sequence of indiscernibles such that $\overline{\varphi}(s;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x;b_i) : i < \omega\}$ is consistent.* □

3.7 Corollary *Let $\varphi(x;z) \in L$ and M be given. Let s be approximable on $\mathcal{C}$. Then for every ε there is a pair of distict $c, c' \in \mathcal{C}$ such that $\mathrm{Av}_{x/s}\big[\varphi(x;c) \nleftrightarrow \varphi(x;c')\big] < \varepsilon$*

Proof Suppose for a contradiction that $\mathrm{Av}_{x/s}\big[\varphi(x;c) \nleftrightarrow \varphi(x;c')\big] \geq \varepsilon$ for all distict $c, c' \in \mathcal{C}$. Apply Lemma 3.5 to the folmula $\psi(x;z,z') = [\varphi(x;z) \nleftrightarrow \varphi(x;z')]$ and the set $\mathcal{B} = \{\langle c,c'\rangle \in \mathcal{C}^2 \ : \ c{\neq}c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 3.5 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x;a) \nleftrightarrow \varphi(x;a') \ : \ a,a'{\in}A,\ a{\neq}a'\}$ is consistent. As $|A| > 2$, this is impossible. □

4 The nip formulas

4.1 Theorem *Let $\varphi(x;z) \in L$ and M be given and assume that $\varphi(x;z)$ is nip. Then every sample is approximable on M over M.*

Proof Questo è Vapnik-Chervonenkis. □

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|z|}$.)

4.2 Corollary *Let $\varphi(x;z) \in L$ be nip. Every smooth sample s is approximable on $\mathcal{U}$ over M.*

Proof By Theorem 4.1, there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s;b) \approx_\varepsilon \overline{\varphi}(r;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\overline{\varphi}(s\,;b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$ for some $b' \in \mathfrak{U}^{|z|}$. Pick a $s' \subseteq \mathfrak{U}^{|x|}$ such that $b' \mathop{\downfree}_M s' \equiv_M s$. By the smoothness of s, we obtain $\overline{\varphi}(s';b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$. As $b' \mathop{\downfree}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\overline{\varphi}(s\,;b) \not\approx_\varepsilon \overline{\varphi}(r\,;b)$. A contradiction. □

4.3 Definition ?? *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\ldots z_n) \in L$ such that for every finite set $B \subseteq \mathfrak{U}$ and every $a \in \mathfrak{U}^{|x|}$ there are $b_1,\ldots,b_n \in B$ such that $\psi(a\,;b_1,\ldots,b_n)$ and $\psi(x\,;b_1,\ldots,b_n)$ decides all formulas $\varphi(x\,;b)$ for $b \in B$.* □

4.4 Definition ??* *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x_1,\ldots,x_n\,;z) \in L$ such that for every finite set $A \subseteq \mathfrak{U}^{|x|}$ and every $b \in \mathfrak{U}^{|z|}$ there are $a_1,\ldots,a_n \in B$ such that $\psi(a_1,\ldots,a_n,\ z)$ and $\psi(a_1,\ldots,a_n,\ z)$ decides all formulas $\varphi(a\,;z)$ for $a \in A$.* □

For every finite sample $r \in M^{\mathrm{f}}$ and every $\mu \in \mathbb{R}$ there is a formula $\psi(z) \in L(\operatorname{supp} r)$ such that $\overline{\varphi}(r\,;z) = \mu \leftrightarrow \psi(z)$. The formula $\psi(z)$ depends on r.

When $\varphi(x\,;z)$ is distal

If $\varphi(x\,;z)$ is distal then there is a formula ψ such that for every $r \in M^f$ there is $r_0 \in M$

4.5 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *for every sample $s \in \mathfrak{U}^{\mathrm{f}^+_*}$, if s is generically stable then s is smooth.*

There is a formula $\psi(x_1,\ldots,x_n\,;z)$ such that for every $r \in M^{\mathrm{f}}$

there are $a_1,\ldots,a_n$ such that $\varphi(r\,;b) = \mu\ \psi(x_1,\ldots,x_n\,;z)$