Scratch paper

Anonymous
Università di Torino
March 2019

Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

The finite expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for the natural numbers, and a sort for the finite multisets of elements of M^n, for any $n < \omega$. We call these multisets samples of M^n. Recall that a multiset is a function $s : M \to \omega$. We write $a \in s$ for $s(a)$, and this is interpreted as the number of occurreces of $a \in s$. The size of a sample s is defined as follows

$$|s| \;=\; \sum_{a \in M} a \in s$$

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L and the language of arithmetic. Moreover, L^{f} has a function $a \in s$ for the number of occurrences of the tuple a in the sample s, and a funtion $|s|$ for the size of the sample s.

For ease of exposition we may also use rational numbers, though these need not have their own sort as they are interpretable in the naturar numbers.

Let $\mathcal{U}$ be a monster model of cardinality κ, an inaccessible cardinal larger than the cardinality of L. The hyperfinite expansion of $\mathcal{U}$ is a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. We will write $\mathcal{U}^{\mathrm{f}*}$ for this expansion. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}*}$.

We write $\mathcal{S}(s)$ for the support of s, that is, the subset of $\mathcal{U}^n$ where $x \in s$ is positive. Note that its cardinality is either finite of κ. If s is a sample in $\mathcal{U}^{\mathrm{f}*}$, we write $s \subseteq \mathcal{U}^n$ as a shorthand of $\mathcal{S}(s) \subseteq \mathcal{U}^n$.

2.1 Fact *Every model M is the domain of the home-sort of some $M^{\mathrm{f}} \preceq \mathcal{U}^{\mathrm{f}*}$.*

Proof Difficile da scrivere senza hand-waving. □

Let $s \subseteq \mathcal{U}^{|x|}$ be a sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\operatorname{Av}_s \varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

where we use the following suggestive abbreviation

$$\big|\{a \in s \ : \ \varphi(a;b)\}\big| \ = \sum_{a \models \varphi(x;b)} a \in s$$

Below, ε always ranges over the positive standard rational. If a and b are hyperrationals, we write $a \approx_\varepsilon b$ for $|a-b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. If μ is a real number we write $a \approx_\varepsilon \mu$ if there is a standard rational $b \approx_{\varepsilon/2} a$ such that $|b-\mu| < \varepsilon/2$ (this last inequality is evaluated in the standard real line). We write $\mathrm{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

2.2 Theorem *Let $\varphi(x;z) \in L$ be a nip formula. Let M be a model. For every sample $s \subseteq \mathcal{U}^{|x|}$, and every ε, there is a sample $s_\varepsilon \subseteq M^{|x|}$ such that $\mathrm{Av}_s \varphi(x;b) \approx_\varepsilon \mathrm{Av}_{s_\varepsilon} \varphi(x;b)$ for all $b \in M^{|z|}$.*

Moreover, the cardinality of s_ε soley depends on ε, namely

$$|s_\varepsilon| \quad \leq \quad c\,\frac{k}{\varepsilon^2}\ln\frac{k}{\varepsilon}$$

where c is an absolute constant.

Proof Questo è Vapnik-Chervonenkis. □

2.3 Remark ??? When the model M in the theorem above is ω_1-saturated, we can claim that there is a (necessarily finite) sample $s_M \subseteq M^{|x|}$ such that $\mathrm{Av}_s \varphi(x;b) \approx \mathrm{Av}_{s_M} \varphi(x;b)$. □

2.4 Definition ??? *Let $\varphi(x;z) \in L$. We say that the sample $s \subseteq \mathcal{U}^{|x|}$ is invariant for $\varphi(x;z)$ over A if, for every $s' \equiv_A s$ and every $b \in \mathcal{U}^{|z|}$, we have $\mathrm{Av}_s \varphi(x;b) \approx \mathrm{Av}_{s'} \varphi(x;b)$.* □

For invariant samples the Vapnik-Chervonenkis Theorem can be strengthened as follows.

2.5 Corollary *Let $\varphi(x;z) \in L$ be a nip formula. Let M be any model. For every sample $s \subseteq \mathcal{U}^{|x|}$ that is invariant for $\varphi(x;z)$ over M, and every ε, there is a sample $s_\varepsilon \subseteq M^{|x|}$ such that $\mathrm{Av}_s \varphi(x;b) \approx_\varepsilon \mathrm{Av}_{s_\varepsilon} \varphi(x;b)$ for all $b \in M^{|z|}$.*

Proof By the theorem above, there is a sample $s_\varepsilon \subseteq M^{|x|}$ such that $\mathrm{Av}_s \varphi(x;b) \approx_\varepsilon \mathrm{Av}_{s_\varepsilon} \varphi(x;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\mathrm{Av}_s \varphi(x;b') \not\approx_\varepsilon \mathrm{Av}_{s_\varepsilon} \varphi(x;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a sample $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathop{\smile\!\!\!\!\!\mid}_M s' \equiv_M s$. By invariance, $\mathrm{Av}_{s'} \varphi(x;b') \not\approx_\varepsilon \mathrm{Av}_{s_\varepsilon} \varphi(x;b')$. As $b' \mathop{\smile\!\!\!\!\!\mid}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by invariance, $\mathrm{Av}_s \varphi(x;b) \not\approx_\varepsilon \mathrm{Av}_{s_\varepsilon} \varphi(x;b)$. A contradiction. □

The following is stronger form of invariance.

2.6 Definition ??? *We say that a sample* $s \subseteq \mathcal{U}^{|x|}$ *is smooth for* $\varphi(x\,;z)$ *over* A *if for any other sample* s' *such that* $\mathrm{Av}_s\, \varphi(x\,;b) \approx \mathrm{Av}_{s'}\, \varphi(x\,;b)$ *for all* $b \in M^{|z|}$*, the same equivalence holds for all* $b \in \mathcal{U}^{|z|}$. □

2.7 Definition ??? *A sample* $s \subseteq \mathcal{U}^{|x|}$ *is finitely satisfiable for* $\varphi(x\,;z)$ *in* M *if, for every* $b \in \mathcal{U}^{|z|}$ *such that* $\mathrm{Av}_s \varphi(x\,;b) \not\approx_\varepsilon 0$*, there is* $a \in M^{|z|}$ *such that* $\varphi(a\,;b)$.

We say that s *is generically stable for* $\varphi(x\,;z)$ *over* M *if it is both invariant and finitely satisfiable for* $\varphi(x\,;z)$ *over* M. □

2.8 Definition ??? *A formula* $\varphi(x\,;z) \in L$ *is distal if there is a formula* $\psi(x\,;z_1,\ldots z_n) \in L$ *such that for every finite sample* s *and every* $b \in \mathcal{U}^{|x|}$ *there are* $a_1,\ldots,a_n \in s^{|z|}$ *such that* $\psi(x\,;a_1,\ldots,a_n) \leftrightarrow p(x)$*, where* $p(x) = \mathrm{tp}_\varphi(b/....s)$. □

2.9 Theorem *The following are equivalent*

1. $\varphi(x\,;z) \in L$ *is distal;*
2. *if* s *is generically stable for* $\varphi(x\,;z)$ *over* M *then* s *is smooth for* $\varphi(x\,;z)$ *over* M.