Scratch paper

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April 2020

Abstract Poche idee, ben confuse.

1 Type-definable functions

Let $\mathcal{U}$ be a monster model. Let X be a Hausdorff space. We say that the partial map $f : \mathcal{U}^{|x|} \to X$ is bounded if $\operatorname{img} f \subseteq C$ for some compact $C \subseteq X$. We say that f is type-definable if it is bounded and $f^{-1}[C]$ is type-definable for every compact $C \subseteq X$. If $f^{-1}[C]$ is definable for every compact $C \subseteq X$, then we say that f is definable.

As $\mathcal{U}$ has a larger cardinality than the topology of X, a type-definable function is always type-definable over some small set of parameters, i.e. all the types defining $f^{-1}[C]$ are over the same small set parameters $A \subseteq \mathcal{U}$.

1.1 Fact *Let $f : \mathcal{U}^{|x|} \to X$ be type-definable. Then the following are equivalent*

1. *f is definable;*
2. *$\operatorname{img} f$ is finite.*

Proof By compactness. □

1.2 Fact *Let $p_i(x) \subseteq L(A)$, for $i = 1, \dots, n$, be a cover of a type-definable set $\mathcal{D} \subseteq \mathcal{U}^{|x|}$. Then there is a partition of $\mathcal{D}$ by some formulas $p_i(x) \vdash \varphi_i(x)$.*

Proof By compactness. □

1.3 Fact *Let $f : \mathcal{U}^{|x|} \to [0,1]$ be type-definable. Then for every $\varepsilon > 0$ there is a definable function $g : \mathcal{U}^{|x|} \to [0,1]$ such that*

$$\big|fa - ga\big| \quad \leq \quad \varepsilon \qquad\qquad \text{for every } a \in \operatorname{dom} f$$

Proof Pick $r_1, \dots, r_n \in [0,1]$ be such that the intervals $[r_i - \varepsilon, r_i + \varepsilon]$ cover $\operatorname{img} f$. Hence, the types defining $f^{-1}[r_i - \varepsilon, r_i + \varepsilon]$ cover $\operatorname{dom} f$. Therefore, by the fact above, there is a definable partition of $\operatorname{dom} f$ that refines $f^{-1}[r_i - \varepsilon, r_i + \varepsilon]$. Define g to be such that $\big\{g^{-1}[r_i] : i \in [n]\big\}$ is the partition above. Then g is as required. □

1.4 Fact *Let $f : \mathcal{U}^{|x|} \to [0,1]$ be type-definable. Then for every $\varepsilon > 0$ there are finitely many $a_1, \dots, a_n \in \operatorname{dom} f$ such that*

$$\bigvee_{i=1}^{n} |fa - fa_i| \leq \varepsilon \qquad \textit{for every } a \in \operatorname{dom} f$$

Moreover, there are some formulas $\psi_i(x) \in L(\mathcal{U})$ *that cover* $\operatorname{dom} f$ *and every* $i \in [n]$

$$\psi_i(x) \quad \rightarrow \quad |fx - fa_i| \leq \varepsilon$$

Proof By Fact 1.3 there is a definable function g such that $|fa - ga| \leq \varepsilon/2$ for every $a \in \operatorname{dom} f$. Pick some $a_1, \ldots, a_n \in \mathcal{U}^{|x|}$ such that $\{ga_1, \ldots, ga_n\} = \operatorname{img} g$. Then

$$\begin{aligned}
\min_{i \in [n]} |fa - fa_i| &= \min_{i \in [n]} |fa - ga + ga - fa_i| \\
&\leq |fa - ga| + \min_{i \in [n]} |fa_i - ga| \\
&\leq |fa - ga| + |fa_i - ga_i| \qquad \text{for } i \text{ is such that } ga_i = ga \\
&\leq \varepsilon
\end{aligned}$$

The second claim is obtained by compactness. □

2 Externally definable sets (notation)

3 Abstract samples

Let M be a structure. The finite (fractional) sample expansion of M is a 3-sorted expansion $\bar{M} = \langle M, \mathbb{R}, M^{\mathsf{s}} \rangle$ that has a domain for M, which we call the home-sort, a domain for $\mathbb{R}$, as an ordered field, and a domain M^{s}, which we call sample-sort, which is an $\mathbb{R}$-vector space that we describe below. Variables of sample-sort (or tuple thereof) are denoted with symbols $x^{\mathsf{s}}, y^{\mathsf{s}}, \ldots$, and variations. Elements of sample-sort are denoted with the symbols s, r, t, and variations. The variable x is reserved for (tuples of) variables of the home-sort. The context will clarify the sort the other variables.

An element $s \in M^{\mathsf{s}}$ is a map $s : M \to \mathbb{R}$ which is almost always zero. These functions, which we call samples, are interpreted as (weighted) samples or as (signed) finite measures (up to normalization) that are concentrated on the support of s.

The language of the expansion $\bar{M}$ is denoted by L.

For every formula $\varphi(x\,;y) \in L$, where y is a tuple a mixed sort, there is a symbol $\Sigma\varphi(x^{\mathsf{s}}\,;y)$ for a function $(M^{\mathsf{s}})^{|x|} \times M^{y} \to \mathbb{R}$. For $\varphi(x) \in L(\bar{M})$, we interpret $\Sigma\varphi(s)$ as the function that maps $a^{\mathsf{s}} \in M^{\mathsf{s}}$ to

$$\Sigma\varphi(s) \quad = \quad \sum_{a \vDash \varphi(x)} s(x)$$

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

3.1 Definition *Let* $\mathcal{U}$ *be a monster model of cardinality* $\kappa > |L|$. *Let* $\bar{\mathcal{U}} = \langle \mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathsf{s}} \rangle$ *be as above and let* $\bar{\mathcal{U}}^*$ *be a saturated elementary extension of* $\mathcal{U}^{\mathsf{s}}$ *of cardinality* κ. *As all saturated* *models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the domain of the home-sort of $\bar{\mathcal{U}}^*$, hence we write $\bar{\mathcal{U}}^* = \langle \mathcal{U}, \mathbb{R}^*, \mathcal{U}^{\mathsf{s}*}\rangle$. Elements of $\mathcal{U}^{\mathsf{s}}$ are called finite samples, elements of $\mathcal{U}^{\mathsf{s}*}$ are called internal samples.*

Finally, an external sample is a maximally consistent set of formulas $p(x^{\mathsf{s}}) \subseteq L(\mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathsf{s}*})$. (Note these are not types over the full of $\bar{\mathcal{U}}$.) □

3.2 Notation Let y and z be tuples of mixed sort. For any type $p(y)$ and formula $\varphi(y,z) \in L(\bar{\mathcal{U}}^*)$ we write

$$\varphi(p\,;\bar{\mathcal{U}}^*) \quad = \quad \Big\{a \in (\bar{\mathcal{U}}^*)^z \ :\ p(y) \vdash \varphi(y\,;a)\Big\}$$

This notation intentionally confuses $p(y)$ with any of its realizations in some elementary extension of $\bar{\mathcal{U}}$. □

3.3 Definition *Let M be given. Let y be a tuple of the home-sort. A external sample $p(x^{\mathsf{s}})$ is*

1. *invariant if $\varphi(p\,;\mathcal{U}, \mathcal{U}^{\mathsf{s}*})$ is invariant over $\bar{M}$, for every $\varphi(x\,;y,z^{\mathsf{s}}) \in L$.*
2. *finitely satisfiable if every formula in $p(x^{\mathsf{s}})$ is satisfied by some element of M^{s}.*
3. *definable if $\varphi(p\,;\mathcal{U}, \mathcal{U}^{\mathsf{s}*})$ is definable over $\bar{M}$, for every $\varphi(x\,;y,z^{\mathsf{s}}) \in L$.* □

An external type is generically stable if it is definable and finitely satisfiable in some $\bar{M}$. □

Clearly every finitely satisfiable external sample is invariant.

We will use the symbol $\mathop{\smile\!\!\!\!\!\!\;|}_{\bar{M}}$ with the usual meaning.

4 Szemerédi's regularity lemma

Define

$$\big|\varphi(a^{\mathsf{s}}\,;b^{\mathsf{s}})\big|) \quad = \quad \sum_{\varphi(x\,;y)} a^{\mathsf{s}}(x)\cdot b^{\mathsf{s}}(y)$$

$$d_\varphi(r\,;s) \quad = \quad \frac{\big|\varphi(r\,;s)\big|}{\big|r\times s\big|}$$

We say that the pair of samples r, s is ε-regular if for every $r' \subseteq r$ and $s' \subseteq s$ such that $|r'| > \varepsilon|r|$ and $|s'| > \varepsilon|s|$

$$\big|d_\varphi(r\,;s) - d_\varphi(r';s')\big| \quad < \quad \varepsilon$$

4.1 Szemerédi's Regularity Lemma *For every $\varepsilon > 0$ and for every pair of samples $u, v \in {}^*\bar{\mathcal{U}}$ of hyperfinite cardinality there are some finite partitions of u and v, say $r_1, \dots, r_n$ and $s_1, \dots, s_m$, such that*

1. *r_i, s_j is ε-regular for every $i, j \in [n]\times[m] \smallsetminus E$,*

where $E \subseteq [n]\times[m]$, called the exceptional set, is such that

2. $$\sum_{\langle i,j\rangle \in E} |r_i|\cdot|s_j| \quad < \quad \varepsilon\cdot|u|\cdot|v|$$

Proof Let $\mathrm{st}\big((d(u\,;v)\big)$ $\square$

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