Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Samples

Let M be a structure of signature L. The finite (fractional) sample expansion of M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, as ordered field, and for each n a sort for F_n, an $\mathbb{R}$-vector space of functions $f : M^n \to \mathbb{R}$. We call the sort of F_n sample-sort. Variables of sample-sort are denoted with sumbolds $x^{\mathrm{s}}, y^{\mathrm{s}}$ and variations thereof.

We write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands the natural laguage of each sort: L, the language ordered rings, and the language of $\mathbb{R}$-vector spaces. Moreover, for every $\varphi(x\,;z) \in L$ in L^{f} there is a symbol $\overline{\varphi}(x^{\mathrm{s}}\,;z)$ for a function $F_{|x|} \times M^{|z|} \to \mathbb{R}$.

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

2.1 Definition *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$. We denote by $\mathcal{U}^{\mathrm{f}*}$ a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}*}$. Elements $r \in \mathcal{U}^{\mathrm{f}}$ are called finite samples, elements in $s \in \mathcal{U}^{\mathrm{f}*}$ are called internal samples.*

(1) A global sample is a maximally consistent set of formulas of the form $\overline{\varphi}(x^{\mathrm{s}}\,;b) \approx_\varepsilon \mu$, where $b \in \mathcal{U}^{|z|}$, $\varepsilon, \mu \in \mathbb{R}$, and $\varphi(x\,;z) \in L$.

(1) A global sample is a maximally consistent set of formulas of the form $\overline{\varphi}(x^{\mathrm{s}}) \approx_\varepsilon \mu$, where $\varphi(x) \in L(\mathcal{U})$, and $\varepsilon, \mu \in \mathbb{R}$.

(2) A global sample is a global type $p(x^{\mathrm{s}}) \in S(\mathcal{U})$.

(3) A global sample is a global type $p(x^{\mathrm{s}}) \in S(\mathcal{U}, \mathbb{R}))$.

(4) A global sample is a global type $p(x^{\mathrm{s}}) \in S(\mathcal{U}^{\mathrm{f}})$.

(5) A global sample is a global type $p(x^{\mathrm{s}}) \in S(\mathcal{U}^{\mathrm{f}*})$. □

2.2 Notation For any global type $p(x) \in S(\mathcal{U})$ and formula $\varphi(x\,;z) \in L$ we write

$$\varphi(p\,;\mathcal{U}) \;=\; \Big\{ b \in \mathcal{U}^{|z|} \;:\; \varphi(x\,;b) \in p \Big\}$$

This notation intentionally confuses p with any of its realizations in some elementary extension of $\mathcal{U}$. Sets of this form are called externally definable. A global type may be identified with a family of externally definable sets.

We extend this notation to global samples. If $\varphi(x\,;z) \in L$ and $p(x^{\rm s})$ is a global sample, we write $\overline{\varphi}(p\,;b)$ for the unique $\mu \in \mathbb{R}$ such that $p(x^{\rm s}) \vdash \overline{\varphi}(x^{\rm s}\,;b) \approx \mu$. In analogy to the terminology above, we may say that $\overline{\varphi}(p\,;z)$ is an externally definable real valued function. A global sample my be identified with a family of externally definable functions. □

2.3 Definition *A global sample $p(x^{\rm s})$ is smooth if it is realized in $\mathcal{U}^{\rm f*}$.* □

2.4 Lemma *The following are equivalent for every global sample $p(x^{\rm s})$*

1. *p is smooth and invariant;*
2. *for every finite $B \subseteq \mathcal{U}^{|z|}$ and every ε, there is an $r \in M^{\rm f}$ such that $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ for all $b \in B$;*
2. *for every $\varphi_i(x\,;z) \in L$, every $b_i \in \mathcal{U}^{|z|}$, and every ε, there is an $r \in M^{\rm f}$ such that $\overline{\varphi}_i(p\,;b_i) \approx_\varepsilon \overline{\varphi}_i(r\,;b_i)$, for every $0 \leq i < n$;*
2. *for every $\psi_0(x),\dots,\psi_{n-1}(x) \in L(\mathcal{U})$ and every ε, there is an $r \in M^{\rm f}$ such that $\overline{\psi}_i(p) \approx_\varepsilon \overline{\psi}_i(r)$, for every $i < n$;*
4. *there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have that $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\overline{\vartheta}_j(p) - \overline{\vartheta}_i(p) < \varepsilon$ for some i, j.*

Proof Let s be an internal sample that realizes $p(x^{\rm s})$.

1⇒2. Let B and ε be given. Let $\bar{b} = \langle b_i : i < n\rangle$ be an enumeration of B. Let $\mu_i \in \mathbb{R}$ be such that $\overline{\varphi}(p\,;b_i) \approx \mu_i$ for every $i < n$. Let $\bar{b}'$ be such that $s \mathop{\smile\!\!\!\!\!\!\;|}_{M^{\rm f}} \bar{b}' \equiv_{M^{\rm f}} \bar{b}$. By invariance, $\overline{\varphi}(s\,;b'_i) \approx \mu_i$. Therefore there is an $r \in M^{\rm f}$ such that $\overline{\varphi}(r\,;b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{M^{\rm f}} \bar{b}$, we obtain $\overline{\varphi}(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒2. Let $\psi_i(x) = \varphi_i(x\,;b_i)$ for some $\varphi_i(x\,;z) \in L$ and $b_i \in \mathcal{U}^{|z|}$. Let $\bar{b} = \langle b_i : i < n\rangle$. Let $\mu_i \in \mathbb{R}$ be such that $\overline{\varphi}_i(p\,;b_i) \approx \mu_i$. Let $\bar{b}'$ be such that $s \mathop{\smile\!\!\!\!\!\!\;|}_{M^{\rm f}} \bar{b}' \equiv_{M^{\rm f}} \bar{b}$. By invariance, $\overline{\varphi}_i(s\,;b'_i) \approx \mu_i$. Therefore there is an $r \in M^{\rm f}$ such that $\overline{\varphi}_i(r\,;b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{M^{\rm f}} \bar{b}$, we obtain $\overline{\varphi}_i(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒3. By smoothness we it suffices to prove 3 with s for p. The type

$$p(z) \quad = \quad \Big\{ \overline{\varphi}(s\,;z) \not\approx_\varepsilon \overline{\varphi}(r\,;z) \ : \ r \in M^{\rm f} \Big\}$$

is inconsistent by 2. Hence compatness yields the required $r_1,\dots,r_n \in M^{\rm f}$.

3⇒2. Clear.

2⇒1 Assume 2. We prove that p is invariant. Pick some $b \equiv_{M^{\rm f}} b'$ and some ε. Let $r, r' \in M^{\rm f}$ be such that $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ and $\overline{\varphi}(p\,;b') \not\approx_\varepsilon \overline{\varphi}(r'\,;b')$. As $\overline{\varphi}(r\,;b) \approx \overline{\varphi}(r'\,;b')$ follows from $b \equiv_{M^{\rm f}} b'$, we obtain $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(p\,;b')$.

We prove that p is smooth.

$$p(x^{\mathrm{s}}) \quad = \quad \Big\{\overline{\varphi}(x^{\mathrm{s}}\,;b) \approx_{\varepsilon} \mu_b \ : \ b \in \mathfrak{U}^{|z|}\Big\}$$

□

2.5 Definition *Let $\varphi(x\,;z) \in L$ and M be given. A global sample $p(x^{\mathrm{s}})$ is*

1. *invariant if $\bar{\varphi}(p\,;b) \approx \bar{\varphi}(p\,;b')$ for every $b \equiv_{M^{\mathrm{f}}} b'$;*
2. *(pointwise) approximable if, for every $b \in \mathfrak{U}^{|z|}$ and every ε, there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(p\,;b) \approx_{\varepsilon} \overline{\varphi}(r\,;b)$;*
3. *definable if, for every ε, the set $\{b \in \mathfrak{U}^{|z|} : \overline{\varphi}(p\,;b) < \varepsilon\}$ is definable over M^{f}.* □

2.6 Fact *Let $\varphi(x\,;z) \in L$ and M be given. Let $p(x^{\mathrm{s}})$ be a smooth sample. Then if p is invariant, it is approximable and definable.*

Proof Let $s \in \mathfrak{U}^{\mathrm{f}*}$ be such that $\overline{\varphi}(s\,;b) \approx \overline{\varphi}(p\,;b)$ for all $b \in \mathfrak{U}^{|z|}$. Assuming 1 of Definition 3.2, we prove 2 and 3.

1⇒2. Let $\mu \in \mathbb{R}$ be such that $\overline{\varphi}(p\,;b) \approx \mu \in \mathbb{R}$. Let b' be such that $s \mathop{\mathpalette\Ind{}}_{M^{\mathrm{f}}} b' \equiv_{M^{\mathrm{f}}} b$. By 1, $\overline{\varphi}(s\,;b') \approx \mu$. Therefore there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(r\,;b') \approx_{\varepsilon} \mu$. As $b' \equiv_{M^{\mathrm{f}}} b$, we obtain $\overline{\varphi}(r\,;b) \approx_{\varepsilon} \mu$.

1⇒3. The set $\{b : \overline{\varphi}(p\,;b) < \varepsilon\}$ coincides with $\{b : \overline{\varphi}(s\,;b) < \varepsilon\}$, hence is definable in $\mathfrak{U}^{\mathrm{f}*}$. As it is invariat over M^{f}, it is definable over M^{f}. □

The following is a useful characterization of smooth samples.

3 Old stuff/garbage

We write $\mathrm{supp}(p)$ for the support of p, that is, the set of those a in $\mathfrak{U}$ such that $p \vdash (a \in x^{\mathrm{s}}) > \varepsilon$ for some standard positive ε. If s is a smooth sample $\mathrm{supp}(s)$ is defined to be $\mathrm{supp}(p)$ for $p(x^{\mathrm{s}}) = \mathrm{tp}(s/\mathfrak{U}^{\mathrm{f}})$.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathfrak{U}^{|z|}$ we define

$$\mathrm{Av}_{x/s}\varphi(x\,;b) \quad = \quad \frac{\big|\{a \in s \ : \ \varphi(a\,;b)\}\big|}{|s|},$$

When possible we abbreviate $\mathrm{Av}_{x/s}\varphi(x\,;b)$ with $\overline{\varphi}(s\,;b)$.

We write $\mathrm{ft}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathfrak{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

All definitions and facts in this section are relative to some given formula $\varphi(x\,;z) \in L$ and some model M.

3.1 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\overline{\varphi}(s';b) \approx \overline{\varphi}(s\,;b)$.* □

3.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\overline{\varphi}(s\,;b) \approx \overline{\varphi}(s\,;b')$;*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(s\,;b) < \varepsilon\}$ is definable over M;*
3. *finitely satisfiable if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$;*
4. *generically stable if all of the above hold.* □

Smooth roughly means internal and invariant.

3.3 Fact *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *there is an invariant internal sample $s' \equiv_M s$.*

Proof 1⇒2. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathcal{U}^{|z|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \mathrm{Aut}(\mathcal{U}^{\mathrm{f}*}/M)$. Then $\overline{\varphi}(s';fb) \approx \overline{\varphi}(f^{-1}s'\,;b)$. Moreover, by smothness $\overline{\varphi}(f^{-1}s'\,;b) \approx \overline{\varphi}(s';b)$. The infariace in s' follows.

2⇒1. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ such that $\overline{\varphi}(s'\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\overline{\varphi}(s''\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. As $s'' \equiv_M s$ are both in $\mathcal{U}^{\mathrm{f}*}$, then $s'' = fs$ for some $f \in \mathrm{Aut}(\mathcal{U}^{\mathrm{f}*}/M)$. Hence we obtan $\overline{\varphi}(s\,;f^{-1}b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$ which contradics the invariance of s. □

3.4 Definition *Let $\varphi(x\,;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s is (uniformly) approximable on $\mathcal{B}$ if for every ε there is a finite sample $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s,b) \approx_\varepsilon \overline{\varphi}(r,b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε, not on b.)* □

3.5 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that $\overline{\varphi}(s\,;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1, \ldots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ are consistent.*

Proof Let $r \in M^{\mathrm{f}}$ be as in Definition 3.4. As r is finite, we may assume that $\mathrm{supp}(r) = \{a_1, \ldots, a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i\,;b)\}$. As $\overline{\varphi}(r\,;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

3.6 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega\rangle$, where $\langle b_i : i < \omega\rangle$ is a sequence of indiscernibles such that $\overline{\varphi}(s\,;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x\,;b_i) : i < \omega\}$ is consistent.* □

3.7 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{C}$. Then for every ε there is a pair of distict $c, c' \in \mathcal{C}$ such that $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] < \varepsilon$*

Proof Suppose for a contradiction that $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] \geq \varepsilon$ for all distict $c, c' \in \mathcal{C}$. Apply Lemma 3.5 to the folmula $\psi(x\,;z,z') = [\varphi(x\,;z) \not\leftrightarrow \varphi(x\,;z')]$ and the set $\mathcal{B} = \{\langle c,c'\rangle \in \mathcal{C}^2 \ :\ c{\neq}c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 3.5 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x\,;a) \not\leftrightarrow \varphi(x\,;a')\ :\ a,a'{\in}A,\ a{\neq}a'\}$ is consistent. As $|A| > 2$, this is impossible. □

4 The nip formulas

4.1 Theorem *Let $\varphi(x\,;z) \in L$ and M be given and assume that $\varphi(x\,;z)$ is nip. Then every sample is approximable on M over M.*

Proof Questo è Vapnik-Chervonenkis. □

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|z|}$.)

4.2 Corollary *Let $\varphi(x\,;z) \in L$ be nip. Every smooth sample s is approximable on $\mathcal{U}$ over M.*

Proof By Theorem 4.1, there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\overline{\varphi}(s\,;b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathop{\smile\!\!\!\!\!\!\perp}_M s' \equiv_M s$. By the smoothness of s, we obtain $\overline{\varphi}(s';b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$. As $b' \mathop{\smile\!\!\!\!\!\!\perp}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\overline{\varphi}(s\,;b) \not\approx_\varepsilon \overline{\varphi}(r\,;b)$. A contradiction. □

4.3 Definition ?? *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\dots z_n) \in L$ such that for every finite set $B \subseteq \mathcal{U}$ and every $a \in \mathcal{U}^{|x|}$ there are $b_1,\dots,b_n \in B$ such that $\psi(a\,;b_1,\dots,b_n)$ and $\psi(x\,;b_1,\dots,b_n)$ decides all formulas $\varphi(x\,;b)$ for $b \in B$.* □

4.4 Definition ??* *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x_1,\dots,x_n\,;z) \in L$ such that for every finite set $A \subseteq \mathcal{U}^{|x|}$ and every $b \in \mathcal{U}^{|z|}$ there are $a_1,\dots,a_n \in B$ such that $\psi(a_1,\dots,a_n,\ z)$ and $\psi(a_1,\dots,a_n,\ z)$ decides all formulas $\varphi(a\,;z)$ for $a \in A$.* □

For every finite sample $r \in M^{\mathrm{f}}$ and every $\mu \in \mathbb{R}$ there is a formula $\psi(z) \in L(\operatorname{supp} r)$ such that $\overline{\varphi}(r\,;z) = \mu \leftrightarrow \psi(z)$. The formula $\psi(z)$ depends on r.

When $\varphi(x\,;z)$ is distal

If $\varphi(x\,;z)$ is distal then there is a formula ψ such that for every $r \in M^f$ there is $r_0 \in M$

4.5 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *for every sample $s \in \mathcal{U}^{\mathrm{f}^+_*}$, if s is generically stable then s is smooth.*

There is a formula $\psi(x_1,\dots,x_n\,;z)$ such that for every $r \in M^{\mathrm{f}}$

there are $a_1,\dots,a_n$ such that $\varphi(r\,;b) = \mu\ \psi(x_1,\dots,x_n\,;z)$