Scretch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

3 Uniform samples

Fix a tuple of variables x and let $\bar{x} = \langle x_i : i < \omega \rangle$, where $|x_i| = |x|$. For $p(\bar{x}) \in S(\mathcal{U})$ and $\psi(x) \in L(\mathcal{U})$ we define

$$\operatorname{Av}_{p\restriction n} \psi(x) \quad = \quad \frac{1}{n}\Big|\big\{i < n \ : \ p(\bar{x}) \vdash \psi(x_i)\big\}\Big|$$

If $p(\bar{x})$ is the type containing $\bar{x} = \bar{a}$ for some $\bar{a} \in \mathcal{U}^{|x|\cdot\omega}$, we write $\operatorname{Av}_{\bar{a}\restriction n} \psi(x)$.

3.1 Definition *We say that $p(\bar{x}) \in S(\mathcal{U})$ is a sample if the limit below exists for every formula $\psi(x) \in L(\mathcal{U})$.*

$$\operatorname{Av}_p \psi(x) \quad = \quad \lim_{n\to\infty} \operatorname{Av}_{p\restriction n} \psi(x).$$

Let $\varphi(x, z) \in L$ be given. We say that $p(\bar{x})$ is a φ-sample if the formula $\psi(x)$ above is restricted to range over those of the form $\varphi(x, b)$ for $b \in \mathcal{U}^{|z|}$.

We say that $p(\bar{x})$ is a uniform φ-sample if it is a sample and for every $\varepsilon > 0$ there is an k such that $\operatorname{Av}_{p\restriction n} \varphi(x, b)$ is within ε from $\operatorname{Av}_p \varphi(x, b)$ for all $n > k$ and $b \in \mathcal{U}^{|z|}$.

We say that $p(\bar{x})$ is a uniform sample if it is a uniform φ-sample for all $\varphi(x, z) \in L$. □

Define

$$\operatorname{Av}_{p\restriction n} q(x) \quad = \quad \inf\Big\{\operatorname{Av}_{p\restriction n} \psi(x) \ : \ \psi(x) \in q\Big\}$$

Clearly the infimum above is attained.

3.2 Proposition *If $p(\bar{x})$ is a uniform sample and $q(x) \subseteq L(\mathcal{U})$, then the limit below exists and*

$$\lim_{n\to\infty} \operatorname{Av}_{p\restriction n} q(x) \quad = \quad \inf\Big\{\operatorname{Av}_p \varphi(x) \ : \ \varphi(x) \in q\Big\}$$

Proof Fix $\varepsilon > 0$. Let $\varphi(x) \in q$ be such that $\operatorname{Av}_p \varphi(x)$ is within ε from the infimum above. Let k be such that $\operatorname{Av}_{p\restriction n} \varphi(x)$ is within ε from $\operatorname{Av}_p \varphi(x)$ for every $n > k$. Then, for every $n > k$,

$$\operatorname{Av}_{p\restriction n} q(x) \quad \leq \quad \operatorname{Av}_{p\restriction n} \varphi(x)$$

$$\leq \quad \mathrm{Av}_p\, \varphi(x) + \varepsilon$$
$$\leq \quad \inf\Big\{\mathrm{Av}_p\, \varphi(x) \ :\ \varphi(x) \in q\Big\} + 2\varepsilon$$

For the converse inequality, let $\psi_n(x)$ be such that $\mathrm{Av}_p\, \psi_n(x) = \mathrm{Av}_{p\restriction n}\, q(x)$. Then for all n

$$\inf\Big\{\mathrm{Av}_p\, \varphi(x) \ :\ \varphi(x) \in q\Big\} \quad \leq \quad \mathrm{Av}_{p\restriction n}\, \psi(x)$$

$\square$

3.3 Proposition *Let $\bar{x} = \langle x_i : i < \omega\rangle$ and $|x_i| = |x|$ and let $p(\bar{x}) \in S(\mathfrak{U})$. Then the following are equivalent*

1. *$p(\bar{x})$ is a uniform global sample;*
2. *for every $\varepsilon > 0$, there is an n and a formula $\vartheta(\bar{x}) \in p$ such that $\mathrm{Av}_n\big(\bar{a}; \varphi(x,b)\big)$ is within ε from $\mathrm{Av}_p \varphi(x,b)$ for all $b \in \mathfrak{U}^{|z|}$ and all $\bar{a} \vDash \vartheta(\bar{x})$.*

Proof ??? $\square$

Vale anche/solo/nemmeno la versione non uniforme della proposizione?

3.4 Corollary *Let $p(\bar{x})$ bea global sample finitely satisfied in M. Let $q'(x) = q(x) \cup \{\varphi(x)\}$ for $q(x) \in S(M)$ and $\varphi(x) \in L(\mathfrak{U})$. Then $\mathrm{Av}_p\, q'(x)$ is either 0 or 1.*

Proof ??? Under the assumptions of the corollary, for every $\varepsilon > 0$ there is an n and a tuple $\bar{a} \in M^{|x|\cdot\omega}$ such that $\mathrm{Av}_{\bar{a}\restriction n}\, q'(x)$ is within ε from $\mathrm{Av}_p\, q'(x)$. As $\mathrm{Av}_{\bar{a}\restriction n}\, q'(x)$ $\square$

Note that, if $p(\bar{x})$ is definable, say over M, then there is a formula $\psi_{m/n}(z) \in L(M)$ such that

$$\psi_{m/n}(z) \quad \Leftrightarrow \quad \mathrm{Av}_n\big(p(\bar{x}); \varphi(x,z)\big) = \frac{m}{n}.$$

4 References

[1] Pandelis Dodos and Vassilis Kanellopoulos, *Ramsey theory for product spaces*, Mathematical Surveys and Monographs, vol. 212, American Mathematical Society, 2016.

[2] Domenico Zambella, *A crèche course in model theory*, AMS Open Math Notes, 2018. (The link points to the github version).