

Scratch paper

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April 2019

Abstract Poche idee, ben confuse.

1 Introduction

2 Abstract samples

Let M be a structure of signature L . The **finite (fractional) sample expansion** of M is a many-sorted expansion that has a sort for M , which we call the home-sort, a sort for \mathbb{R} , as ordered field, and for each n a sort for F_n , which we say are of **sample-sort**. Variables of sample-sort are denoted with symbols x^s, y^s and variations thereof.

We write M^f for the expansion above. The language of M^f is denoted by L^f . It expands the natural language of each sort: L , the language ordered rings, and the language of \mathbb{R} -vector spaces. Moreover, for every $\varphi(x; z) \in L$ in L^f there is a symbol $\bar{\varphi}(x^s; z)$ for a function $F_{|x|} \times M^{|z|} \rightarrow \mathbb{R}$.

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε .

2.1 Definition Let \mathcal{U} be a monster model of cardinality $\kappa > |L|$. We denote by \mathcal{U}^{f*} a saturated elementary extension of \mathcal{U}^f of cardinality κ . As all saturated models of cardinality κ are isomorphic, we can assume that \mathcal{U} is the home-sort of \mathcal{U}^{f*} . Elements $r \in \mathcal{U}^f$ are called **finite samples**, elements in $s \in \mathcal{U}^{f*}$ are called **internal samples**.

Finally, a **global sample** is a maximally consistent set of formulas of the form $\bar{\varphi}(x^s) \approx_\varepsilon \mu$, where $\varphi(x) \in L(\mathcal{U})$, and $\varepsilon, \mu \in \mathbb{R}$. □

2.2 Notation For any global type $p(x) \in S(\mathcal{U})$ and formula $\varphi(x; z) \in L$ we write

$$\varphi(p; \mathcal{U}) = \left\{ b \in \mathcal{U}^{|z|} : \varphi(x; b) \in p \right\}$$

This notation intentionally confuses p with any of its realizations in some elementary extension of \mathcal{U} . Sets of this form are called **externally definable**. A global type may be identified with a family of externally definable sets.

We extend this notation to global samples. If $\varphi(x; z) \in L$ and $p(x^s)$ is a global sample, we write $\bar{\varphi}(p; b)$ for the unique $\mu \in \mathbb{R}$ such that $p(x^s) \vdash \bar{\varphi}(x^s; b) \approx \mu$. In analogy to the terminology above, we may say that $\bar{\varphi}(p; z)$ is an **externally definable real valued function**. A global sample may be identified with a family of externally definable functions. □

2.3 Definition A global sample $p(x^s)$ is *smooth* if it is realized in \mathcal{U}^{f*} . □

2.4 Definition Let M be given. A global sample $p(x^s)$ is

1. *invariant* if $\bar{\varphi}(p; b) \approx \bar{\varphi}(p; b')$ for every $b \equiv_{M^f} b'$ and every $\varphi(x; z) \in L$.
2. *finitely satisfiable* if for every finite subset of p is realized in M^f .
3. *definable* if, for every ε , the set $\{b \in \mathcal{U}^{[z]} : \bar{\varphi}(p; b) < \varepsilon\}$ is definable over M^f . □

Explicitly, 2 above means that if $\bar{\psi}_i(x^s) \approx_\varepsilon \mu_i$ are some finitely many formulas in p , then there is an $r \in M^f$ such that $\bar{\psi}_i(r) \approx_\varepsilon \mu_i$ for all i . We may also rephrase this by saying that for every finitely many $\psi_i(x) \in L(\mathcal{U})$ there is an $r \in M^f$ such that $\bar{\psi}_i(r) \approx_\varepsilon \bar{\psi}_i(p)$ for all i .

2.5 Fact Let M be given. Then every smooth invariant sample is finitely satisfiable and definable.

Proof Let s be an internal sample that realizes an invariant global sample $p(x^s)$. Let $\varphi_i(x^s; b) \approx_\varepsilon \mu_i$ be some finitely many formulas in p . Let b' be such that $s \downarrow_{M^f} b' \equiv_{M^f} b$. By invariance, $\bar{\varphi}_i(s; b') \approx \mu_i$. Therefore there is an $r \in M^f$ such that $\bar{\varphi}_i(r; b') \approx_\varepsilon \mu_i$. As $b' \equiv_{M^f} b$, we obtain $\bar{\varphi}_i(r; b) \approx_\varepsilon \mu_i$. This proves finite satisfiability. Clearly $\{b : \bar{\varphi}(p; b) < \varepsilon\} = \{b : \bar{\varphi}(s; b) < \varepsilon\}$ is definable in \mathcal{U}^{f*} . By invariance it is definable in M^f . □

2.6 Fact Let M be given. Then every global sample that is finitely satisfiable is invariant.

Proof Let $b \equiv_{M^f} b'$ be given. Let $r \in M^f$ be such that $\bar{\varphi}(p; b) \approx_\varepsilon \bar{\varphi}(r; b)$ and $\bar{\varphi}(p; b') \not\approx_\varepsilon \bar{\varphi}(r; b')$. But $\bar{\varphi}(r; b) \approx \bar{\varphi}(r; b')$ and therefore $\bar{\varphi}(p; b) \approx_\varepsilon \bar{\varphi}(p; b')$. As ε is arbitrary, $\bar{\varphi}(p; b) \approx \bar{\varphi}(p; b')$ follows. □

2.7 Fact Let M be given. Let $p(x^s)$ be a smooth sample. Then for every $\varphi(x; z) \in L$ and every ε there are some finitely many $r_i \in M^f$, say $0 \leq i < n$, such that for every $b \in \mathcal{U}^{[z]}$ we have $\varphi(p; b) \approx_\varepsilon \varphi(r_i; b)$ for some i .

Proof Let s be an internal sample that realizes $p(x^s)$. Let $\varphi(x; z) \in L$ and ε be fixed. The type

$$p(z) = \left\{ \bar{\varphi}(s; z) \not\approx_\varepsilon \bar{\varphi}(r; z) : r \in M^f \right\}$$

is inconsistent by Fact 2.5. Hence compactness yields the required $r_i \in M^f$. □

3 Concrete samples

In this section we assume that for every $\varphi(x), \psi(x) \in L(\mathcal{U})$ if $\varphi(x) \rightarrow \psi(x)$ then $\bar{\varphi}(x^s) \leq \bar{\psi}(x^s)$.

3.1 Fact Let M be given. Let $p(x^s)$ be global sample. Suppose for every $\varphi(x) \in L(\mathcal{U})$ and ε

there are $\vartheta_1(x), \vartheta_2(x) \in L(M)$ such that $\vartheta_1(x) \rightarrow \varphi(x) \rightarrow \vartheta_2(x)$ and $\bar{\vartheta}_2(p) \approx_\varepsilon \bar{\vartheta}_1(p)$. Then p is finitely satisfiable.

Proof By elementarity there is an $r \in M$ such that $\bar{\vartheta}_2(r) \approx_\varepsilon \bar{\vartheta}_1(r) \approx_\varepsilon \vartheta_i(p)$. Then \square

3.2 Lemma *The following are equivalent for every global sample $p(x^s)$*

1. p is smooth and invariant;
2. for every finite $B \subseteq \mathcal{U}^{|z|}$ and every ε , there is an $r \in M^f$ such that $\bar{\varphi}(p; b) \approx_\varepsilon \bar{\varphi}(r; b)$ for all $b \in B$;
2. for every $\varphi_i(x; z) \in L$, every $b_i \in \mathcal{U}^{|z|}$, and every ε , there is an $r \in M^f$ such that $\bar{\varphi}_i(p; b_i) \approx_\varepsilon \bar{\varphi}_i(r; b_i)$, for every $0 \leq i < n$;
2. for every $\psi_0(x), \dots, \psi_{n-1}(x) \in L(\mathcal{U})$ and every ε , there is an $r \in M^f$ such that $\bar{\psi}_i(p) \approx_\varepsilon \bar{\psi}_i(r)$, for every $i < n$;
4. there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have that $\vartheta_i(x) \rightarrow \varphi(x; b) \rightarrow \vartheta_j(x)$ and $\bar{\vartheta}_j(p) - \bar{\vartheta}_i(p) < \varepsilon$ for some i, j .

Proof Let s be an internal sample that realizes $p(x^s)$.

$1 \Rightarrow 2$. Let B and ε be given. Let $\bar{b} = \langle b_i : i < n \rangle$ be an enumeration of B . Let $\mu_i \in \mathbb{R}$ be such that $\bar{\varphi}(p; b_i) \approx \mu_i$ for every $i < n$. Let \bar{b}' be such that $s \downarrow_{M^f} \bar{b}' \equiv_{M^f} \bar{b}$. By invariance, $\bar{\varphi}(s; b'_i) \approx \mu_i$. Therefore there is an $r \in M^f$ such that $\bar{\varphi}(r; b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{M^f} \bar{b}$, we obtain $\bar{\varphi}(r; b_i) \approx_\varepsilon \mu_i$.

$1 \Rightarrow 2$. Let $\psi_i(x) = \varphi_i(x; b_i)$ for some $\varphi_i(x; z) \in L$ and $b_i \in \mathcal{U}^{|z|}$. Let $\bar{b} = \langle b_i : i < n \rangle$. Let $\mu_i \in \mathbb{R}$ be such that $\bar{\varphi}_i(p; b_i) \approx \mu_i$. Let \bar{b}' be such that $s \downarrow_{M^f} \bar{b}' \equiv_{M^f} \bar{b}$. By invariance, $\bar{\varphi}_i(s; b'_i) \approx \mu_i$. Therefore there is an $r \in M^f$ such that $\bar{\varphi}_i(r; b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{M^f} \bar{b}$, we obtain $\bar{\varphi}_i(r; b_i) \approx_\varepsilon \mu_i$.

$1 \Rightarrow 3$. By smoothness we it suffices to prove 3 with s for p . The type

$$p(z) = \left\{ \bar{\varphi}(s; z) \not\approx_\varepsilon \bar{\varphi}(r; z) : r \in M^f \right\}$$

is inconsistent by 2. Hence compactness yields the required $r_1, \dots, r_n \in M^f$.

$3 \Rightarrow 2$. Clear.

$2 \Rightarrow 1$ Assume 2. We prove that p is invariant. Pick some $b \equiv_{M^f} b'$ and some ε . Let $r, r' \in M^f$ be such that $\bar{\varphi}(p; b) \approx_\varepsilon \bar{\varphi}(r; b)$ and $\bar{\varphi}(p; b') \not\approx_\varepsilon \bar{\varphi}(r'; b')$. As $\bar{\varphi}(r; b) \approx \bar{\varphi}(r'; b')$ follows from $b \equiv_{M^f} b'$, we obtain $\bar{\varphi}(p; b) \approx_\varepsilon \bar{\varphi}(p; b')$.

We prove that p is smooth.

$$p(x^s) = \left\{ \bar{\varphi}(x^s; b) \approx_\varepsilon \mu_b : b \in \mathcal{U}^{|z|} \right\}$$

\square

3.3 Definition *Let $\varphi(x; z) \in L$ and M be given. A global sample $p(x^s)$ is*

1. **invariant** if $\bar{\varphi}(p; b) \approx \bar{\varphi}(p; b')$ for every $b \equiv_{M^f} b'$;

2. (pointwise) **approximable** if, for every $b \in \mathcal{U}^{|z|}$ and every ε , there is an $r \in M^f$ such that $\bar{\varphi}(p; b) \approx_\varepsilon \bar{\varphi}(r; b)$;
3. **definable** if, for every ε , the set $\{b \in \mathcal{U}^{|z|} : \bar{\varphi}(p; b) < \varepsilon\}$ is definable over M^f . □

The following is a useful characterization of smooth samples.

4 Old stuff/garbage

We write $\text{supp}(p)$ for the **support** of p , that is, the set of those a in \mathcal{U} such that $p \vdash (a \in x^s) > \varepsilon$ for some standard positive ε . If s is a smooth sample $\text{supp}(s)$ is defined to be $\text{supp}(p)$ for $p(x^s) = \text{tp}(s/\mathcal{U}^f)$.

Let s be an external sample. For every formula $\varphi(x; z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\text{Av}_{x/s} \varphi(x; b) = \frac{|\{a \in s : \varphi(a; b)\}|}{|s|},$$

When possible we abbreviate $\text{Av}_{x/s} \varphi(x; b)$ with $\bar{\varphi}(s; b)$.

We write $\text{ft}(a)$, for the standard part of the hyperrational number a . That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ -algebra. This is known as the **Loeb measure**.

All definitions and facts in this section are relative to some given formula $\varphi(x; z) \in L$ and some model M .

4.1 Definition Let $\varphi(x; z) \in L$ and M be given. We say that a sample s is **smooth** if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\bar{\varphi}(s'; b) \approx \bar{\varphi}(s; b)$. □

4.2 Definition Let $\varphi(x; z) \in L$ and M be given. An external sample s is

1. **invariant** if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\bar{\varphi}(s; b) \approx \bar{\varphi}(s; b')$;
2. **definable** if, for every ε , the set $\{b \in \mathcal{U}^{|z|} : \bar{\varphi}(s; b) < \varepsilon\}$ is definable over M ;
3. **finitely satisfiable** if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in M^f$ such that $\bar{\varphi}(s; b) \approx_\varepsilon \bar{\varphi}(r; b)$;
4. **generically stable** if all of the above hold. □

Smooth roughly means internal and invariant.

4.3 Fact Let $\varphi(x; z) \in L$ and M be given. The following are equivalent for every external sample s

1. s is smooth;
2. there is an invariant internal sample $s' \equiv_M s$.

Proof $1 \Rightarrow 2$. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathcal{U}^{|\mathcal{Z}|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \text{Aut}(\mathcal{U}^{f^*}/M)$. Then $\bar{\varphi}(s'; fb) \approx \bar{\varphi}(f^{-1}s'; b)$. Moreover, by smoothness $\bar{\varphi}(f^{-1}s'; b) \approx \bar{\varphi}(s'; b)$. The invariance in s' follows.

$2 \Rightarrow 1$. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ such that $\bar{\varphi}(s'; b) \not\approx_\varepsilon \bar{\varphi}(s; b)$. Pick a internal sample $s'' \equiv_{M, s} s'$. Then $\bar{\varphi}(s''; b) \not\approx_\varepsilon \bar{\varphi}(s; b)$. As $s'' \equiv_M s$ are both in \mathcal{U}^{f^*} , then $s'' = fs$ for some $f \in \text{Aut}(\mathcal{U}^{f^*}/M)$. Hence we obtain $\bar{\varphi}(s; f^{-1}b) \not\approx_\varepsilon \bar{\varphi}(s; b)$ which contradicts the invariance of s . \square

4.4 Definition Let $\varphi(x; z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|\mathcal{Z}|}$ be arbitrary. We say that s is *(uniformly) approximable* on \mathcal{B} if for every ε there is a finite sample $r \in M^f$ such that $\bar{\varphi}(s, b) \approx_\varepsilon \bar{\varphi}(r, b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε , not on b .) \square

4.5 Lemma Let $\varphi(x; z) \in L$ and M be given. Let s be approximable on \mathcal{B} and such that $\bar{\varphi}(s; b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of \mathcal{B} , say $\mathcal{B}_1, \dots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x; b) : b \in \mathcal{B}_i\}$ are consistent.

Proof Let $r \in M^f$ be as in Definition 4.4. As r is finite, we may assume that $\text{supp}(r) = \{a_1, \dots, a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i; b)\}$. As $\bar{\varphi}(r; b) > 0$ for all $b \in \mathcal{B}$, these \mathcal{B}_i are the required cover of \mathcal{B} . \square

4.6 Corollary Let $\varphi(x; z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega \rangle$, where $\langle b_i : i < \omega \rangle$ is a sequence of indiscernibles such that $\bar{\varphi}(s; b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x; b_i) : i < \omega\}$ is consistent. \square

4.7 Corollary Let $\varphi(x; z) \in L$ and M be given. Let s be approximable on \mathcal{C} . Then for every ε there is a pair of distinct $c, c' \in \mathcal{C}$ such that $\text{Av}_{x/s}[\varphi(x; c) \nleftrightarrow \varphi(x; c')] < \varepsilon$

Proof Suppose for a contradiction that $\text{Av}_{x/s}[\varphi(x; c) \nleftrightarrow \varphi(x; c')] \geq \varepsilon$ for all distinct $c, c' \in \mathcal{C}$. Apply Lemma 4.5 to the formula $\psi(x; z, z') = [\varphi(x; z) \nleftrightarrow \varphi(x; z')]$ and the set $\mathcal{B} = \{\langle c, c' \rangle \in \mathcal{C}^2 : c \neq c'\}$. The sets \mathcal{B}_i obtained from Lemma 4.5 induce a finite coloring of the complete graph on \mathcal{C} . By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x; a) \nleftrightarrow \varphi(x; a') : a, a' \in A, a \neq a'\}$ is consistent. As $|A| > 2$, this is impossible. \square

5 The nip formulas

5.1 Theorem Let $\varphi(x; z) \in L$ and M be given and assume that $\varphi(x; z)$ is nip. Then every sample is approximable on M over M .

Proof Questo è Vapnik-Chervonenkis. \square

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|\mathcal{Z}|}$.)

5.2 Corollary Let $\varphi(x; z) \in L$ be nip. Every smooth sample s is approximable on \mathcal{U} over M .

Proof By Theorem 5.1, there is an $r \in M^f$ such that $\bar{\varphi}(s; b) \approx_\varepsilon \bar{\varphi}(r; b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\bar{\varphi}(s; b') \not\approx_\varepsilon \bar{\varphi}(r; b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \downarrow_M s' \equiv_M s$. By the smoothness of s , we obtain $\bar{\varphi}(s'; b') \not\approx_\varepsilon \bar{\varphi}(r; b')$. As $b' \downarrow_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\bar{\varphi}(s; b) \not\approx_\varepsilon \bar{\varphi}(r; b)$. A contradiction. \square

5.3 Definition ?? We say that $\varphi(x; z) \in L$ is *distal* if there is a formula $\psi(x; z_1, \dots, z_n) \in L$ such that for every finite set $B \subseteq \mathcal{U}$ and every $a \in \mathcal{U}^{|x|}$ there are $b_1, \dots, b_n \in B$ such that $\psi(a; b_1, \dots, b_n)$ and $\psi(x; b_1, \dots, b_n)$ decides all formulas $\varphi(x; b)$ for $b \in B$. \square

5.4 Definition ??* We say that $\varphi(x; z) \in L$ is *distal* if there is a formula $\psi(x_1, \dots, x_n; z) \in L$ such that for every finite set $A \subseteq \mathcal{U}^{|x|}$ and every $b \in \mathcal{U}^{|z|}$ there are $a_1, \dots, a_n \in B$ such that $\psi(a_1, \dots, a_n, z)$ and $\psi(x_1, \dots, x_n, z)$ decides all formulas $\varphi(a; z)$ for $a \in A$. \square

For every finite sample $r \in M^f$ and every $\mu \in \mathbb{R}$ there is a formula $\psi(z) \in L(\text{supp } r)$ such that $\bar{\varphi}(r; z) = \mu \leftrightarrow \psi(z)$. The formula $\psi(z)$ depends on r .

When $\varphi(x; z)$ is distal

If $\varphi(x; z)$ is distal then there is a formula ψ such that for every $r \in M^f$ there is $r_0 \in M$

5.5 Theorem (false) The following are equivalent

1. $\varphi(x; z) \in L$ is distal;
2. for every sample $s \in \mathcal{U}^{f*}$, if s is generically stable then s is smooth.

There is a formula $\psi(x_1, \dots, x_n; z)$ such that for every $r \in M^f$ there are a_1, \dots, a_n such that $\varphi(r; b) = \mu \leftrightarrow \psi(a_1, \dots, a_n; z)$