Scratch paper

Anonymous
Università di Torino
March 2020

Abstract Poche idee, ben confuse.

1 Introduction

2 Abstract samples

Let M be a structure of signature L. The sample-expansion of M is a 3-sorted expansion $\langle M, \omega, \bar{M} \rangle$, where $\bar{M} = M^{<\omega}$. These three sorts are called home-sort, integer-sort, and sample-sort, respectively. The symbol x always denotes a tuple of variables of the home-sort. We put a bar over the symbols of sample-sort such as $\bar{x}$ and $\bar{a}$. Symbols as m, n, i are of integer-sort; from the context it should be inferred whether they are variables or parameters.

The size of a sample $\bar{a} \in \bar{M}$ is the length of $\bar{a}$ as an element of $M^{<\omega}$. This is denoted by $\operatorname{lh} \bar{a}$. The same symbol $\bar{a}$ may be used for tuples of samples $\bar{a}_1, \ldots, \bar{a}_n$ with the implicit assumption that $\operatorname{lh} \bar{a}_i$. The symbol $|\bar{a}|$ is used the length of such tuples, for instance, if $\bar{a} \in \bar{M}^n$ then $|\bar{a}| = n$. We write $\bar{a}.i$ for the i-th element of the sample $\bar{a}$. Note that if $\bar{a} \in \bar{M}^n$ then $\bar{a}.i \in M^n$. The language of this expansion is denoted by $\bar{L}$; it comprises

1. all symbols of L that apply to the home-sort;
2. $0, 1, +, \cdot$ that apply to the integer-sort;
3. a ternary relation of sort $\bar{M}, \omega, M$ that holds if $\bar{x}.i = y$.
4. for every formula $\varphi(x\,; y) \in \bar{L}$, where y a tuple of mixed sort, there is relation symbol for $\varphi(\bar{x}.i\,; y)$, where i is a variable of integer-sort.

When $\varphi(x)$ is the formula $x = x$, requirement 4 implies that $\operatorname{lh} \bar{x}$ is definable. Also, from 4 it easily follows that the function $|\{i < \operatorname{lh} \bar{a} \,:\, \varphi(\bar{a}.i)\}|$ is definable, uniformly in the parameters of $\varphi(x)$.

As in the integer-sort we can interpret the field of rational numbers, below we will freely use rational numbers when this clarify the notation. For instance we define

$$\operatorname*{Fr}_{\bar{a}} \varphi(x) \quad = \quad \frac{|\{i < \operatorname{lh} \bar{a} \,:\, \varphi(\bar{a}.i)\}|}{\operatorname{lh} \bar{a}}$$

and use that it is a definable function in $\bar{L}$. In particular we will use that for every $s \in \mathbb{Q}$ there is a formula saying $\operatorname{Fr}_{\bar{a}} \varphi(x) \geq s$, uniformly in the parameter of $\varphi(x)$.

2.1 Definition *Let $\mathcal{U}$ be a monster model of inaccessible cardinality $\kappa > |L|$ and let $\langle \mathcal{U}, \omega, \bar{\mathcal{U}} \rangle$ be the corresponding sample-expansion. We denote by $\langle \mathcal{U}, \omega^*, \bar{\mathcal{U}}^* \rangle$ a saturated elementary extension of $\langle \mathcal{U}, \omega, \bar{\mathcal{U}} \rangle$ of cardinality κ. Note that we can assume that the home-sort of this extension is $\mathcal{U}$, in fact as all saturated models of cardinality κ are isomorphic. The elements of $\bar{\mathcal{U}}$ are called finite samples, those of $\bar{\mathcal{U}}^*$ are called internal samples.*

To any internal sample $\bar{a}$ we associate a finitely additive measure on definable subsets of $\mathcal{U}^{|x|}$. For $\varphi(x) \in \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ we define

$$\Pr_{\bar{a}} \left[\varphi(x)\right] \quad = \quad \sup \Big\{ s \in \mathbb{Q} \ : \ s < \operatorname*{Fr}_{\bar{a}} \varphi(x) \Big\}$$

Note that this measure is type-definable uniformly in the parameters of $\varphi(x)$. Namely, let $r \in \mathbb{R}$ and $\varphi(x\,;y) \in \bar{L}$, where y is a tuple of mixed sort. Then there is a type $q(\bar{x}\,;y) \subseteq \bar{L}$ that defines $\Pr_{\bar{x}} \left[\varphi(x\,;y)\right] \geq r$.

2.2 Definition *Let x be a tuple of variables of the home-sort. An external sample is a global type $p(\bar{x}) \subseteq \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ such that, for every formula $\varphi(x) \in \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$, the set $\{i \in \omega^* : p(\bar{x}) \vdash \varphi(\bar{x}.i)\}$ is bounded and definable, uniformly in the parameters of $\varphi(x)$.* $\square$

For every external sample $p(\bar{x})$ there is an $n \in \omega^*$ such that $p(\bar{x}) \vdash \operatorname{lh} \bar{x} = n$ and $\big|\{i \in \omega^* : p(\bar{x}) \vdash \varphi(\bar{x}.i)\}\big|$ is definable uniformly on the parameters of $\varphi(x)$.

In analogy to what done with internal samples, we associate to an external sample $p(\bar{x})$ a finitely additive probability measure on the definable subsets of $\mathcal{U}$. Namely, we define

$$\Pr_{\bar{p}} \left[\varphi(\mathcal{U})\right] \quad = \quad \sup \left\{ r \in \mathbb{Q} \ : \ r < \frac{\big|\{i < \operatorname{lh} \bar{x} \ : \ p(\bar{x}) \vdash \varphi(\bar{x}.i)\}\big|}{\operatorname{lh} \bar{x}} \right\}$$

What claimed above for internal samples, apply also to external samples, that is, there is a type $q(\bar{x}\,;y) \subseteq \bar{L}$ that defines $\Pr_{\bar{p}} \left[\varphi(x\,;y)\right] \geq r$.

2.3 Notation Let z be a tuple of variables of the home-sort. For $p(\bar{x})$ an external sample, $\varphi(\bar{x}\,;z) \in \bar{L}$, and $b \in \mathcal{U}^{|z|}$ we write $\varphi(\bar{p}\,;b)$ for $p(\bar{x}) \vdash \varphi(\bar{x};b)$. We also write

$$\varphi(\bar{p}\,;\mathcal{U}) \quad = \quad \Big\{ b \in \mathcal{U}^{|z|} \ : \ \varphi(\bar{p}\,;b) \Big\}$$

Sets of this form are called externally definable. This notation intentionally confuses $p(\bar{x})$ with any of its realizations (suggestively denoted by $\bar{p}$) in some elementary extension of $\bar{\mathcal{U}}$. $\square$

The additional *weakly* in the definition below refers to the fact that we only to parameters in the home-sort. Beside that the definition is standard.

2.4 Definition *Let z be a tuple of variables of the home-sort. Let M be given. An external sample $p(\bar{x})$ is*

1. *weakly invariant over M if $\varphi(\bar{p}\,;\mathcal{U})$ is invariant over M for every $\varphi(\bar{x}\,;z) \in \bar{L}$;*
2. *weakly finitely satisfiable in $\bar{M}$ if for every formula $\varphi(\bar{x}) \in \bar{L}(M)$ if $\varphi(\bar{x}) \in p$, then*

$\varphi(\bar{a})$ for some $\bar{a} \in \bar{M}$.

$\square$

Note that in 2 above the requirement $\bar{a} \in \bar{M}$ entails in particular $\operatorname{lh} \bar{a} \in \omega$.

If an external sample $p(\bar{x})$ is weakly finitely satisfiable in $\bar{M}$ then for every $\varepsilon > 0$ and every formula $\varphi(x)$ there is an $\bar{a} \in M$ such that

$$\Pr_{\bar{p}}\left[\varphi(x)\right] \quad \approx_{\varepsilon} \quad \operatorname*{Fr}_{\bar{a}} \varphi(x)$$