

# Scratch paper

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**Abstract** Poche idee, ben confuse.

## 1 Introduction

## 2 Preliminaries

The **finite (fractional) expansion** of a structure  $M$  is a many-sorted expansion that has a sort for  $M$ , which we call the home-sort, a sort for  $\mathbb{R}$ , the set of real numbers, and, for every  $n$ , a sort for the functions  $s : M^n \rightarrow \mathbb{R}$  that are almost always 0.

We will be mainly interested in function that take non negative values. These will be interpreted as **(fractional) multisets**. Namely  $s(a)$  interpreted as the number of times  $a$  occurs in  $s$ . We use the following suggestive notation:

1.  $a \in s = s(a)$  this is called the **multiplicity** of  $a$  in  $s$ ;
2.  $|s| = \sum_{a \in M} a \in s$  this is called the **size** of  $s$ ;
3.  $|\{a \in s : \varphi(a; b)\}| = \sum_{a \models \varphi(x; b)} a \in s$ .

We call these multisets **samples** and refer to their sorts collectively as **sample-sort**.

We will write  $M^f$  for the expansion above. The language of  $M^f$  is denoted by  $L^f$ . It expands  $L$ , the language of rings, and the language of vector spaces over  $\mathbb{R}$ . Each interpreted in the obvious sort. Moreover,  $L^f$  has a symbols for the functions in 1-3 above. In particular, 3 requires a symbol for every  $\varphi(x; z) \in L$ .

Let  $\mathcal{U}$  be a monster model of cardinality  $\kappa$ , a cardinal larger than the cardinality of  $L$ . We introduce two elementary extension of  $\mathcal{U}^f$ .

**2.1 Definition** We denote by  $\mathcal{U}^{f*}$  a saturated elementary extension of  $\mathcal{U}^f$  of cardinality  $\kappa$ . As all saturated models of cardinality  $\kappa$  are isomorphic, we can assume that  $\mathcal{U}$  is the home-sort of  $\mathcal{U}^{f*}$ . We denote by  $\mathcal{U}^{f*+}$  some elementary extension of  $\mathcal{U}^{f*}$  that realizes all types in  $S(\mathcal{U}^{f*})$ . Hence we have  $\mathcal{U}^f \prec \mathcal{U}^{f*} \prec \mathcal{U}^{f*+}$ . Elements of sample-sort in these models are called **finite samples**, **internal samples**, and **external samples** respectively.  $\square$

The use of  $\mathcal{U}^{f*+}$  is not ideal, it is introduced to replace the use of global types of sample-sort which also have drawbacks.

We write  $\text{supp}(s)$  for the **support** of  $s$ , that is, the set of those  $a$  in  $\mathcal{U}^{f*}$  such that  $a \in s$  is positive.

Let  $s$  be an external sample. For every formula  $\varphi(x; z) \in L$  and every  $b \in \mathcal{U}^{|z|}$  we define

$$\text{Av}_s \varphi(x; b) = \frac{|\{a \in s : \varphi(a; b)\}|}{|s|},$$

Below,  $\varepsilon$  always ranges over the positive standard reals. If  $a$  and  $b$  are hyperreals, we write  $a \approx_\varepsilon b$  for  $|a - b| < \varepsilon$ . We write  $a \approx b$  if  $a \approx_\varepsilon b$  holds for every  $\varepsilon$ . We write  $\text{st}(a)$ , for the standard part of the hyperrational number  $a$ . That is, the unique real number  $\mu$  such that  $\mu \approx a$ .

The standard part of  $\text{Av}_s$  induces a finite probability measure on the algebra of definable subsets of  $\mathcal{U}^{|x|}$ . This measure has an unique extension to a Lebesgue probability measure on a  $\sigma$ -algebra. This is known as the **Loeb measure**.

**2.2 Definition** We say that the external sample  $s$  is **smooth** for  $\varphi(x; z)$  over  $M$  if, for every  $s' \equiv_M s$  and every  $b \in \mathcal{U}^{|z|}$ , we have  $\text{Av}_{s'} \varphi(x; b) \approx \text{Av}_s \varphi(x; b)$ .  $\square$

**2.3 Definition** Let  $\varphi(x; z) \in L$  and let  $M$  be a model. An external sample  $s$  is

1. **invariant** if, for every  $b' \equiv_M b \in \mathcal{U}^{|z|}$ , we have  $\text{Av}_s \varphi(x; b) \approx \text{Av}_s \varphi(x; b')$ .
2. **definable** if, for every  $\varepsilon$ , the set  $\{b \in \mathcal{U}^{|z|} : \text{Av}_s \varphi(x; b) < \varepsilon\}$  is definable over  $M$ .
3. **satisfiable** if, for all  $b \in \mathcal{U}^{|z|}$ , if  $\text{Av}_s \varphi(x; b) \not\approx 0$ , then  $\varphi(a; b)$  for some  $a \in M^{|x|}$ .
4. **generically stable** if all of the above holds.

When  $\varphi(x; z)$  and  $M$  are not clear from the context we add: **for  $\varphi(x; z)$  over/in  $M$ .**

**2.4 Fact** Smooth samples are generically stable. (For some given  $\varphi(x; z)$  and  $M$ .)

**Proof** We prove in turn 1-3 of the definition above.

1. If  $s$  is smooth for  $\varphi(x; z)$  over  $M$ , then there is an internal sample  $s' \in \mathcal{U}^{f*}$  such that  $\text{Av}_{s'} \varphi(x; b) \approx \text{Av}_s \varphi(x; b)$  for all  $b \in \mathcal{U}^{|z|}$ . Then 1 follows.
2. The set  $\{b \in \mathcal{U}^{|z|} : \text{Av}_s \varphi(x; b) < \varepsilon\}$  is definable in  $\mathcal{U}^{f*}$  because, by invariance of  $s$ , we may replace  $s$  with any internal sample  $s' \equiv_M s$ . As this set is invariant over  $M$ , it is definable over  $M^f$  and therefore, over  $M$ .
3. Let  $b \in \mathcal{U}^{|z|}$  be such that  $\text{Av}_s \varphi(x; b) > \varepsilon$ . Pick a external sample  $s'$  such that  $s \equiv_M s' \perp_{M^f} b$ . By the smoothness of  $s$ , we obtain  $\text{Av}_{s'} \varphi(x; b) > \varepsilon$ . Let  $s_0 \in M^f$  such that  $\text{Av}_{s_0} \varphi(x; b) > \varepsilon$ . Clearly, some element of  $\text{supp}(s_0)$  proves 3.  $\square$

Smooth samples are invariant. The converse need not be true in general, but we have the following.

**2.5 Fact** For every internal sample  $s$ , the following are equivalent

1.  $s$  is invariant for  $\varphi(x; z)$  over  $M$ ;

2.  $s$  is smooth for  $\varphi(x; z)$  over  $M$ .

**Proof** By the fact above, implication  $2 \Rightarrow 1$  holds for every external sample. We prove  $1 \Rightarrow 2$ . Let  $s \in \mathcal{U}^{\text{f}*}$  be an invariant internal sample. Assume for a contradiction that  $\text{Av}_{s'} \varphi(x; b) \not\approx_\varepsilon \text{Av}_s \varphi(x; b)$  for some  $b \in \mathcal{U}^{|x|}$  and some external sample  $s' \equiv_M s$ . Pick an internal sample  $s'' \equiv_{M, s} s'$ . Then  $\text{Av}_{s''} \varphi(x; b) \not\approx_\varepsilon \text{Av}_s \varphi(x; b)$ . As  $s'' \equiv_M s$  are both in  $\mathcal{U}^{\text{f}*}$ , this contradicts the invariance of  $s$ .  $\square$

The following is a useful characterization of smoothness.

**2.6 Lemma** *The following are equivalent for every external sample  $s$*

1.  $s$  is smooth over  $M$ ;
2. for every  $b \in \mathcal{U}^{|z|}$  there is a finite sample  $s_0 \in M^{\text{f}}$  such that  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_0} \varphi(x; b)$ .  
(The sample  $s_0$  depends on  $M$ , epsilon and  $b$ .)
3. there are some finite samples  $s_1, \dots, s_n \in M^{\text{f}}$  such that for every  $b \in \mathcal{U}^{|z|}$  we have  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_i} \varphi(x; b)$  for some  $s_i$ . (The samples  $s_1, \dots, s_n$  as well as the number  $n$  depend on  $\varepsilon$  and  $M$ .)

**Proof** Implication  $3 \Rightarrow 1$  is clear. We prove  $1 \Rightarrow 2$ , so, fix a  $b \in \mathcal{U}^{|z|}$ . Let  $\mu$  be the standard part of  $\text{Av}_s \varphi(x; b)$ . Let  $s'$  be an external sample such that  $s \equiv_M s' \perp_{M^{\text{f}}} b$ . By smoothness,  $\text{Av}_{s'} \varphi(x; b) \approx \mu$ . As  $s' \perp_{M^{\text{f}}} b$  there is an  $s_0 \in M^{\text{f}}$  such that  $\text{Av}_{s_0} \varphi(x; b) \approx_\varepsilon \mu$ .

We prove  $2 \Rightarrow 3$  by compactness. Suppose not, then the following global type is finitely consistent, hence realized in  $\mathcal{U}^{\text{f}*}$

$$p(y) = \left\{ \text{Av}_y \varphi(x, b) \not\approx_\varepsilon \text{Av}_{s_0} \varphi(x; b) : s_0 \in M^{\text{f}}, b \in \mathcal{U}^{|z|} \right\}$$

This is a contradiction.  $\square$

**2.7 Definition** Let  $\mathcal{B} \subseteq \mathcal{U}^{|z|}$  be arbitrary. We say that  $s$  is **uniform** on  $\mathcal{B}$  for  $\varphi(x; z)$  over  $M$  if for every  $\varepsilon$  there is a finite sample  $s_0 \in M^{\text{f}}$  such that  $\text{Av}_s \varphi(x, b) \approx_\varepsilon \text{Av}_{s_0} \varphi(x, b)$  for every  $b \in \mathcal{B}$ . Reference to  $\varphi(x; z)$  and  $M$  will be omitted when clear from the context.  $\square$

**2.8 Lemma** Let  $s$  be uniform on  $\mathcal{B}$  for  $\varphi(x; z)$  over  $M$  and such that  $\text{Av}_s \varphi(x; b) > \varepsilon$  for every  $b \in \mathcal{B}$ . Then there is a finite cover of  $\mathcal{B}$ , say  $\mathcal{B}_1, \dots, \mathcal{B}_n$ , such that all types  $p_i(x) = \{\varphi(x; b) : b \in \mathcal{B}_i\}$  are consistent.

**Proof** Let  $s_0 \in M^{\text{f}}$  be as in Definition 2.7. As  $s_0$  is finite, we may assume that  $\text{supp}(s_0) = \{a_1, \dots, a_n\}$ . Let  $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i; b)\}$ . As  $\text{Av}_{s_0} \varphi(x; b) > 0$  for all  $b \in \mathcal{B}$ , these  $\mathcal{B}_i$  are the required cover of  $\mathcal{B}$ .  $\square$

**2.9 Corollary** If  $s$  be uniform on  $\langle b_i : i < \omega \rangle$  for  $\varphi(x; z)$  over  $M$ , where  $\langle b_i : i < \omega \rangle$  is a sequence of indiscernibles such that  $\text{Av}_s \varphi(x; b_i) > \varepsilon$  for every  $i < \omega$ , then the type  $\{\varphi(x; b_i) : i < \omega\}$  is consistent.  $\square$

**2.10 Corollary** Let  $s$  be uniform on  $\mathcal{C}$  for  $\varphi(x; z)$  over  $M$ . Then  $\text{Av}_s[\psi(x; c) \nleftrightarrow \psi(x; c')] > \varepsilon$  does not hold for all  $c, c' \in \mathcal{C}, c \neq c'$ .

**Proof** Suppose for a contradiction that a set  $\mathcal{C}$  as above exists. Apply Lemma 2.8 to the formula  $\varphi(x; z, z') = [\psi(x; z) \nleftrightarrow \psi(x; z')]$  and the set  $\mathcal{B} = \{\langle c, c' \rangle \in \mathcal{C}^2 : c \neq c'\}$ . The sets  $\mathcal{B}_i$  obtained from the lemma induce a finite coloring of the complete graph on  $\mathcal{C}$ . By the Ramsey theorem there is an infinite monochromatic set  $A \subseteq \mathcal{C}$ . Hence  $\{\psi(x; a) \nleftrightarrow \psi(x; a') : a, a' \in A, a \neq a'\}$  is consistent. As  $|A| > 2$  this is impossible.  $\square$

**2.11 Theorem** Let  $\varphi(x; z) \in L$  be a nip formula. For every external sample  $s$  there is a finite sample  $s_0 \in M^f$  such that  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_0} \varphi(x; b)$  for all  $b \in M^{|z|}$ . (The sample  $s_0$  depends on  $\varepsilon$  and  $M$ .)

Moreover, the cardinality of  $s_0$  solely depends on  $\varepsilon$ , namely

$$|s_0| \leq c \frac{k}{\varepsilon^2} \ln \frac{k}{\varepsilon}$$

where  $c$  is an absolute constant and  $k$  is the VC-dimension of  $\varphi(x; z)$ .

**Proof** Questo è Vapnik-Chervonenkis.  $\square$

For smooth external samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below  $b$  ranges over  $\mathcal{U}^{|z|}$ .)

**2.12 Corollary** Let  $\varphi(x; z) \in L$  be nip. For every smooth external sample  $s$  there is a finite sample  $s_0 \in M^f$  such that  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_0} \varphi(x; b)$  for all  $b \in \mathcal{U}^{|z|}$ .

**Proof** By Theorem 2.11, there is an  $s_0 \in M^f$  such that  $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_0} \varphi(x; b)$  for all  $b \in M^{|z|}$ .

Suppose for a contradiction that  $\text{Av}_s \varphi(x; b') \not\approx_\varepsilon \text{Av}_{s_0} \varphi(x; b')$  for some  $b' \in \mathcal{U}^{|z|}$ . Pick a  $s' \subseteq \mathcal{U}^{|x|}$  such that  $b' \downarrow_M s' \equiv_M s$ . By the smoothness of  $s$ , we obtain  $\text{Av}_{s'} \varphi(x; b') \not\approx_\varepsilon \text{Av}_{s_0} \varphi(x; b')$ . As  $b' \downarrow_M s'$ , the same formula holds for some  $b \in M^{|z|}$ . Once again by smoothness,  $\text{Av}_s \varphi(x; b) \not\approx_\varepsilon \text{Av}_{s_0} \varphi(x; b)$ . A contradiction.  $\square$

The following is another immediate consequence of the lemma (it holds also when  $\varphi(x; z)$  is not nip, though the proof is lengthier).

**2.13 Definition** A formula  $\varphi(x; z) \in L$  is *distal* if there is a formula  $\psi(x; z_1, \dots, z_n) \in L$  such that for every finite set  $A \subseteq \mathcal{U}^f$  and every  $b \in \mathcal{U}^{|x|}$  there are  $a_1, \dots, a_n \in \text{supp}(s)$  such that  $\psi(x; a_1, \dots, a_n) \leftrightarrow p(x)$ , where  $p(x) = \text{tp}_\varphi(b/A)$ .  $\square$

**2.14 Theorem (false)** The following are equivalent

1.  $\varphi(x; z) \in L$  is distal;
2. for every external sample  $s \in \mathcal{U}^{f+}$ , if  $s$  is generically stable for  $\varphi(x; z)$  over  $M$  then  $s$  is smooth for  $\varphi(x; z)$  over  $M$ .