Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

The finite (fractional) expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for the natural numbers, and a sort for the finite fractional multisets of elements of M^n, for any $n < \omega$. A (fractional) multiset is a function $s : M^n \to \mathbb{R}^+ \cup \{0\}$. We say that s is finite if it is almost always 0. We interpret $s(a)$ as the number of occurreces of a in s. For this reason, we use the following suggestive notation:

1. $a \in s \;=\; s(a)$ this is called the multiplicity of a in s;
2. $|s| \;=\; \sum_{a \in M} a \in s$ this is called the size of s;
3. $\big|\{a \in s \;:\; \varphi(a\,;b)\}\big| \;=\; \sum_{a \models \varphi(x;b)} a \in s$

We call these multisets samples of M^n.

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L and the language of arithmetic. Moreover, L^{f} has a symbols for the functions in 1-3 above. In particular, 3 requires a symbol for every $\varphi(x\,;z) \in L$.

Let $\mathcal{U}$ be a monster model of cardinality κ, an inaccessible cardinal larger than the cardinality of L. The hyperfinite expansion of $\mathcal{U}$ is a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. We will write $\mathcal{U}^{\mathrm{f}*}$ for this expansion. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}*}$.

We write $\mathrm{supp}(s)$ for the support of s, that is, the subset of $\mathcal{U}^n$ where $x \in s$ is positive. Note that its cardinality is either finite of κ. If s is a sample in $\mathcal{U}^{\mathrm{f}*}$, we write $s \subseteq \mathcal{U}^n$ as a shorthand of $\mathrm{supp}(s) \subseteq \mathcal{U}^n$.

2.1 Fact *Every model M is the domain of the home-sort of some $M^{\mathrm{f}} \preceq \mathcal{U}^{\mathrm{f}*}$.*

Proof Difficile da scrivere senza hand-waving. □

Let $s \subseteq \mathcal{U}^{|x|}$ be a sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\text{Av}_s\,\varphi(x\,;b) \quad = \quad \frac{\big|\{a\in s\ :\ \varphi(a\,;b)\}\big|}{|s|},$$

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a-b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. If μ is a real number we write $a \approx_\varepsilon \mu$ if there is a standard rational $b \approx_{\varepsilon/2} a$ such that $|b-\mu| < \varepsilon/2$ (this last inequality is evaluated in the standard real line). We write $\text{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

2.2 Lemma *For every formula $\varphi(x\,;z)\in L$, every sample $s\subseteq\mathcal{U}^{|x|}$ and every infinite $B\subseteq\mathcal{U}^{|z|}$ such that $\text{Av}_s\varphi(x\,;b)\not\approx 0$ for every $b\in B$, there is a finite cover $B_1,\ldots B_n$ of B such that the types $\{\varphi(x\,;b) : b\in B_i\}$ are consistent.*

Proof Let $M^{\text{f}*}\preceq\mathcal{U}^{\text{f}*}$ be a sufficiently saturated model containing B. Define for every ε

$$B_\varepsilon \quad = \quad \Big\{b\in B\ :\ \text{Av}_s\varphi(x\,;b)>\varepsilon\Big\}.$$

Define also

$$p_\varepsilon(w) \quad = \quad \Big\{\text{Av}_w\varphi(x\,;b)>\varepsilon :\ b\in B_\varepsilon\Big\}.$$

By saturation there is an $s'\in M^{\text{f}*}$ that realises all these types. As $s'\subseteq M^{|x|}$ then $\text{supp}(s')$ is finite. Clearly, we can assume that $a\in s'$ is standard for all a. So the claim follows. □

2.3 Theorem *Let $\varphi(x\,;z)\in L$ be a nip formula. Let M be a model. For every sample $s\subseteq\mathcal{U}^{|x|}$, and every ε, there is a sample $s_{M,\varepsilon}\subseteq M^{|x|}$ such that $\text{Av}_s\varphi(x\,;b)\approx_\varepsilon \text{Av}_{s_{M,\varepsilon}}\,\varphi(x\,;b)$ for all $b\in M^{|z|}$.*

Moreover, the cardinality of $s_{M,\varepsilon}$ soley depends on ε, namely

$$|s_{M,\varepsilon}| \quad \leq \quad c\,\frac{k}{\varepsilon^2}\ln\frac{k}{\varepsilon}$$

where c is an absolute constant.

Proof Questo è Vapnik-Chervonenkis. □

2.4 Remark ??? When the model M in the theorem above is sufficiently (?) saturated, we can claim that there is a (necessarily finite) sample $s_M\subseteq M^{|x|}$ such that $\text{Av}_s\varphi(x\,;b)\approx\text{Av}_{s_M}\,\varphi(x\,;b)$. □

2.5 Definition *Let $\varphi(x\,;z)\in L$. We say that the sample $s\subseteq\mathcal{U}^{|x|}$ is invariant for $\varphi(x\,;z)$ over A if, for every $s'\equiv_A s$ and every $b\in\mathcal{U}^{|z|}$, we have $\text{Av}_s\,\varphi(x\,;b)\approx\text{Av}_{s'}\,\varphi(x\,;b)$.* □

For invariant samples the Vapnik-Chervonenkis Theorem can be strengthened as follows.

2.6 Corollary *Let $\varphi(x\,;z) \in L$ be a nip formula. Let M be any model. For every sample $s \subseteq \mathcal{U}^{|x|}$ that is invariant for $\varphi(x\,;z)$ over M, and every ε, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\,\varphi(x\,;b)$ for all $b \in \mathcal{U}^{|z|}$.*

Proof By the theorem above, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\,\varphi(x\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\mathrm{Av}_s\varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\,\varphi(x\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a sample $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathop{\smile\!\!\!\!\!\!\!\;\,\perp}\!\!_M\, s' \equiv_M s$. By invariance, $\mathrm{Av}_{s'}\varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\,\varphi(x\,;b')$. As $b' \mathop{\smile\!\!\!\!\!\!\!\;\,\perp}\!\!_M\, s'$, the same formula holds for some $b \in M^{|z|}$. Once again by invariance, $\mathrm{Av}_s\varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\,\varphi(x\,;b)$. A contradiction. □

The following is stronger form of invariance.

2.7 Definition ??? *We say that a sample $s \subseteq \mathcal{U}^{|x|}$ is smooth for $\varphi(x\,;z)$ over M if for any other sample s' such that $\mathrm{Av}_s\,\varphi(x\,;b) \approx \mathrm{Av}_{s'}\,\varphi(x\,;b)$ for all $b \in M^{|z|}$, the same equivalence holds for all $b \in \mathcal{U}^{|z|}$.* □

2.8 Definition ??? *A sample $s \subseteq \mathcal{U}^{|x|}$ is finitely satisfiable for $\varphi(x\,;z)$ in M if, for every $b \in \mathcal{U}^{|z|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \not\approx 0$, there is $a \in M^{|z|}$ such that $\varphi(a\,;b)$.*

We say that s is generically stable for $\varphi(x\,;z)$ over M if it is both invariant and finitely satisfiable for $\varphi(x\,;z)$ over M. □

2.9 Definition ??? *A formula $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\ldots z_n) \in L$ such that for every finite sample s and every $b \in \mathcal{U}^{|x|}$ there are $a_1,\ldots,a_n \in s^{|z|}$ such that $\psi(x\,;a_1,\ldots,a_n) \leftrightarrow p(x)$, where $p(x) = \mathrm{tp}_\varphi(b/....s)$.* □

2.10 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *if s is generically stable for $\varphi(x\,;z)$ over M then s is smooth for $\varphi(x\,;z)$ over M.*