Scretch paper

A.A. & V.V.
Università di Torino
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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

3 Uniform samples

Fix a tuple of variables x and let $\bar{x} = \langle x_i : i < \omega \rangle$, where $|x_i| = |x|$. For $p(\bar{x}) \in S(\mathcal{U})$ and $\psi(x) \in L(\mathcal{U})$ we define ($n > 1$)

$$\operatorname{Av}_{p\restriction n} \psi(x) \quad = \quad \frac{1}{n}\Big|\big\{i < n \ :\ p(\bar{x}) \vdash \psi(x_i)\big\}\Big|$$

If $p(\bar{x})$ is the type containing $\bar{x} = \bar{a}$ for some $\bar{a} \in \mathcal{U}^{|x|\cdot\omega}$, we write $\operatorname{Av}_{\bar{a}\restriction n} \psi(x)$.

3.1 Definition *We say that $p(\bar{x}) \in S(\mathcal{U})$ is a sample if the limit below exists for every formula $\psi(x) \in L(\mathcal{U})$.*

$$\operatorname{Av}_p \psi(x) \quad = \quad \lim_{n\to\infty} \operatorname{Av}_{p\restriction n} \psi(x).$$

Let $\varphi(x, z) \in L$ be given. We say that $p(\bar{x})$ is a φ-sample if the formula $\psi(x)$ above is restricted to range over those of the form $\varphi(x, b)$ for $b \in \mathcal{U}^{|z|}$.

We say that $p(\bar{x})$ is a uniform φ-sample if it is a sample and for every $\varepsilon > 0$ there is a k such that $\operatorname{Av}_{p\restriction n} \varphi(x, b)$ is within ε from $\operatorname{Av}_p \varphi(x, b)$ for all $n > k$ and $b \in \mathcal{U}^{|z|}$.

We say that $p(\bar{x})$ is a uniform sample if it is a uniform φ-sample for all $\varphi(x, z) \in L$. □

Define

$$\operatorname{Av}_{p\restriction n} q(x) \quad = \quad \inf\Big\{\operatorname{Av}_{p\restriction n} \psi(x) \ :\ \psi(x) \in q\Big\}$$

Clearly the infimum above is attained.

3.2 Proposition *If $p(\bar{x})$ is a sample and $q(x) \subseteq L(\mathcal{U})$, then*

$$\lim_{n\to\infty} \operatorname{Av}_{p\restriction n} q(x) \quad = \quad \inf\Big\{\operatorname{Av}_p \varphi(x) \ :\ \varphi(x) \in q\Big\}$$

Proof Fix $\varepsilon > 0$. Let $\varphi(x) \in q$ be such that $\operatorname{Av}_p \varphi(x)$ is within ε from the infimum above. Let k be such that $\operatorname{Av}_{p\restriction n} \varphi(x)$ is within ε from $\operatorname{Av}_p \varphi(x)$ for every $n > k$. Then, for every $n > k$,

$$\operatorname{Av}_{p\restriction n} q(x) \quad \leq \quad \operatorname{Av}_{p\restriction n} \varphi(x)$$

$$
\begin{aligned}
&\leq \quad \mathrm{Av}_p\, \varphi(x) + \varepsilon \\
&\leq \quad \inf \Big\{ \mathrm{Av}_p\, \varphi(x) \ :\ \varphi(x) \in q \Big\} + 2\varepsilon
\end{aligned}
$$

For the converse inequality we pick a sequence of integers n_i a formula $\psi_i(x) \in q$ as follows. Start with $n_0 = 1$ and $\psi_0 = \top$. Then, inductively let n_{i+1} be such that $\mathrm{Av}_{p\restriction n_{i+1}}\, q(x) \leq \mathrm{Av}_p\, \psi_i(x) + \varepsilon$. Let $\mathrm{Av}_{p\restriction n_{i+1}}\, \psi_i(x) = \mathrm{Av}_{p\restriction n_{i+1}}\, q(x)$.

such that $\mathrm{Av}_{p\restriction n_i}\, \psi_i(x) = \mathrm{Av}_{p\restriction n_i}\, q(x)$. We can also require that $\mathrm{Av}_{p\restriction n_{i+1}}\, \psi_{i+1}(x) \leq \mathrm{Av}_p\, \psi_i(x) + \varepsilon$. In fact, first chose n_{i+1} such that $\mathrm{Av}_{p\restriction n_{i+1}}\, q(x) \leq \mathrm{Av}_p\, \psi_i(x) + \varepsilon$ then

Then for all n

$$
\begin{aligned}
\inf \Big\{ \mathrm{Av}_p\, \varphi(x) \ :\ \varphi(x) \in q \Big\} \quad &\leq \quad \inf \Big\{ \mathrm{Av}_p\, \psi_n(x) \ :\ n < \omega \Big\} \\
&\leq \quad \mathrm{Av}_{p\restriction n}\, q(x) - \varepsilon
\end{aligned}
$$

□

3.3 Corollary *Let $p(\bar{x})$ be a sample finitely satisfied in M. Let $q'(x) = q(x) \cup \{\varphi(x)\}$ for $q(x) \in S(M)$ and $\varphi(x) \in L(\mathfrak{U})$. Then $\mathrm{Av}_p\, q'(x)$ is either 0 or 1.*

Proof Under the assumptions of the corollary, for every $\varepsilon > 0$ there is an n and a tuple $\bar{a} \in M^{|x|\cdot\omega}$ such that $\mathrm{Av}_{\bar{a}\restriction n}\, q'(x)$ is within ε from $\mathrm{Av}_p\, q'(x)$. As $\mathrm{Av}_{\bar{a}\restriction n}\, q'(x)$ □

3.4 Proposition *Let $\bar{x} = \langle x_i : i < \omega \rangle$ and $|x_i| = |x|$ and let $p(\bar{x}) \in S(\mathfrak{U})$. Then the following are equivalent*

1. *$p(\bar{x})$ is a uniform global sample;*
2. *for every $\varepsilon > 0$, there is an n and a formula $\vartheta(\bar{x}) \in p$ such that $\mathrm{Av}_n\big(\bar{a}; \varphi(x, b)\big)$ is within ε from $\mathrm{Av}_p \varphi(x, b)$ for all $b \in \mathfrak{U}^{|z|}$ and all $\bar{a} \vDash \vartheta(\bar{x})$.*

Proof ??? □

Vale anche/solo/nemmeno la versione non uniforme della proposizione?

Note that, if $p(\bar{x})$ is definable, say over M, then there is a formula $\psi_{m/n}(z) \in L(M)$ such that

$$
\psi_{m/n}(z) \quad \Leftrightarrow \quad \mathrm{Av}_n\big(p(\bar{x}); \varphi(x, z)\big) = \frac{m}{n}.
$$

4 References

[1] Pandelis Dodos and Vassilis Kanellopoulos, *Ramsey theory for product spaces*, Mathematical Surveys and Monographs, vol. 212, American Mathematical Society, 2016.

[2] Domenico Zambella, *A crèche course in model theory*, AMS Open Math Notes, 2018. (The link points to the github version).