Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

The finite (fractional) expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, the set of real numbers, and a sort for the finite fractional multisets of elements of M^n, for any $n < \omega$. A (fractional) multiset is a function $s : M^n \to \mathbb{R}^+ \cup \{0\}$. We say that s is finite if it is almost always 0. We interpret $s(a)$ as the number of occurreces of a in s. For this reason, we use the following suggestive notation:

1. $a \in s \quad = \quad s(a)$ this is called the multiplicity of a in s;

2. $|s| \quad = \quad \sum_{a \in M} a \in s$ this is called the size of s;

3. $\big|\{a \in s \ : \ \varphi(a\,;b)\}\big| \quad = \quad \sum_{a \vDash \varphi(x;b)} a \in s$

We call these multisets samples of M^n.

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L and the language of arithmetic. Moreover, L^{f} has a symbols for the functions in 1-3 above. In particular, claim 3 requires a symbol for every $\varphi(x\,;z) \in L$.

Let $\mathcal{U}$ be a monster model of cardinality κ, an inaccessible cardinal larger than the cardinality of L. The hyperfinite expansion of $\mathcal{U}$ is a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. We will write $\mathcal{U}^{\mathrm{f}*}$ for this expansion. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}*}$.

We write $\mathrm{supp}(s)$ for the support of s, that is, the subset of $\mathcal{U}^n$ where $x \in s$ is positive. Note that $\mathrm{supp}(s)$ is a definable subset of $\mathcal{U}^{\mathrm{f}*}$, hence its cardinality is either finite or κ. Moreover, $s \subseteq M^n$ for some small model M, then $\mathrm{supp}(s)$ is finite.

If s is a sample in $\mathcal{U}^{\mathrm{f}*}$, we write $s \subseteq \mathcal{U}^n$ as a shorthand of $\mathrm{supp}(s) \subseteq \mathcal{U}^n$.

2.1 Fact *Every model M is the domain of the home-sort of some $M^{\mathrm{f}} \preceq \mathcal{U}^{\mathrm{f}*}$.*

Proof Difficile da scrivere senza hand-waving. □

Let $s \subseteq \mathfrak{U}^{|x|}$ be a sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathfrak{U}^{|z|}$ we define

$$\mathrm{Av}_s\,\varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. We write $\mathrm{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathfrak{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

2.2 Theorem *Let $\varphi(x\,;z) \in L$ be a nip formula. Let M be a model. For every sample $s \subseteq \mathfrak{U}^{|x|}$, and every ε, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\mathrm{Av}_s\varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_{M,\varepsilon}}\,\varphi(x\,;b)$ for all $b \in M^{|z|}$.*

Moreover, the cardinality of $s_{M,\varepsilon}$ soley depends on ε, namely

$$|s_{M,\varepsilon}| \quad\leq\quad c\,\frac{k}{\varepsilon^2}\ln\frac{k}{\varepsilon}$$

where c is an absolute constant and k is the VC-dimension of $\varphi(x\,;z)$.

Proof Questo è Vapnik-Chervonenkis. $\square$

2.3 Lemma *For every nip formula $\varphi(x\,;z) \in L$, every sample $s \subseteq \mathfrak{U}^{|x|}$ and every infinite $B \subseteq \mathfrak{U}^{|z|}$ such that $\mathrm{Av}_s\varphi(x\,;b) > \varepsilon$ for every $b \in B$, there is a finite cover of B, say $B_1, \ldots, B_n$, such that all types $p_i(x) = \{\varphi(x\,;b) : b \in B_i\}$ are consistent.*

Proof Let M be any model containing B. Fix some $\varepsilon' < \varepsilon$. Let $s_{M,\varepsilon'}$ be the sample given by Theorem 2.2. Then $\mathrm{supp}(s_{M,\varepsilon'})$ is finite, say $\mathrm{supp}(s_{M,\varepsilon'}) = \{a_1, \ldots, a_n\}$. Let $B_i = \{b \in B : \varphi(a_i\,;b)\}$. As $\mathrm{Av}_{s_{M.\varepsilon'}}\varphi(x\,;b) > 0$ for all $b \in B$, these B_i are the required cover of B. $\square$

2.4 Corollary *Let $\psi(x\,;z) \in L$ be a nip formula and let $s \subseteq \mathfrak{U}^{|x|}$ be a sample. Then there is no infinite set C such that $\mathrm{Av}_s\big[\psi(x\,;c) \not\leftrightarrow \psi(x\,;c')\big] > \varepsilon$ for all $c, c' \in C$, $c \neq c'$.*

Proof Suppose for a contradiction that a set C as above exists. Apply Lemma 2.3 to the folmula $\varphi(x\,;z,z') = [\psi(x\,;z) \not\leftrightarrow \psi(x\,;z')]$ and the set $B = \{\langle c, c'\rangle \in C^2 : c \neq c'\}$. We may assume that the sets B_i given by the lemma form a partition of the complete graph on C. Hence, by the Ramsey theorem there is an infinite set $A \subseteq C$ such that $\{\psi(x\,;a) \not\leftrightarrow \psi(x\,;a') : a, a' \in A, a \neq a'\}$ is consistent. This is impossible if $|A| > 2$. $\square$

The following is another immediate consequence of the lemma (it holds also when $\varphi(x\,;z)$ is not nip, though the proof is lengthier).

2.5 Corollary *For every nip formula $\varphi(x\,;z) \in L$, every sample $s \subseteq \mathfrak{U}^{|x|}$ and every sequence of indiscernibles $\langle b_i : i < \omega\rangle$ such that $\mathrm{Av}_s\varphi(x\,;b_i) > \varepsilon$ for every $i < \omega$, the type*

$\{\varphi(x\,;b_i) : i < \omega\}$ is consistent. □

2.6 Definition *Let $\varphi(x\,;z) \in L$. We say that the sample $s \subseteq \mathcal{U}^{|x|}$ is invariant for $\varphi(x\,;z)$ over M if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\operatorname{Av}_s \varphi(x\,;b) \approx \operatorname{Av}_{s'} \varphi(x\,;b)$.* □

For invariant samples the Vapnik-Chervonenkis Theorem can be strengthened as follows.

2.7 Corollary *Let $\varphi(x\,;z) \in L$ be a nip formula. Let M be any model. For every sample $s \subseteq \mathcal{U}^{|x|}$ that is invariant for $\varphi(x\,;z)$ over M, and every ε, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\operatorname{Av}_s\varphi(x\,;b) \approx_\varepsilon \operatorname{Av}_{s_{M,\varepsilon}} \varphi(x\,;b)$ for all $b \in \mathcal{U}^{|z|}$.*

Proof By the theorem above, there is a sample $s_{M,\varepsilon} \subseteq M^{|x|}$ such that $\operatorname{Av}_s\varphi(x\,;b) \approx_\varepsilon \operatorname{Av}_{s_{M,\varepsilon}} \varphi(x\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\operatorname{Av}_s\varphi(x\,;b') \not\approx_\varepsilon \operatorname{Av}_{s_{M,\varepsilon}} \varphi(x\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a sample $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \downarrow_M s' \equiv_M s$. By invariance, $\operatorname{Av}_{s'}\varphi(x\,;b') \not\approx_\varepsilon \operatorname{Av}_{s_{M,\varepsilon}} \varphi(x\,;b')$. As $b' \downarrow_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by invariance, $\operatorname{Av}_s\varphi(x\,;b) \not\approx_\varepsilon \operatorname{Av}_{s_{M,\varepsilon}} \varphi(x\,;b)$. A contradiction. □

The following is stronger form of invariance.

2.8 Definition *We say that a sample $s \subseteq \mathcal{U}^{|x|}$ is smooth for $\varphi(x\,;z)$ over M if for any other sample s' such that $\operatorname{Av}_s \varphi(x\,;b) \approx \operatorname{Av}_{s'} \varphi(x\,;b)$ for all $b \in M^{|z|}$, the same equivalence holds for all $b \in \mathcal{U}^{|z|}$.* □

Clearly, smooth impies invariant.

2.9 Definition ??? *A sample $s \subseteq \mathcal{U}^{|x|}$ is finitely satisfiable for $\varphi(x\,;z)$ in M if, for every $b \in \mathcal{U}^{|z|}$ such that $\operatorname{Av}_s\varphi(x\,;b) \not\approx 0$, there is $a \in M^{|z|}$ such that $\varphi(a\,;b)$.*

We say that s is generically stable for $\varphi(x\,;z)$ over M if it is both invariant and finitely satisfiable for $\varphi(x\,;z)$ over M. □

2.10 Definition ??? *A formula $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\dots z_n) \in L$ such that for every finite sample s and every $b \in \mathcal{U}^{|x|}$ there are $a_1,\dots,a_n \in s^{|z|}$ such that $\psi(x\,;a_1,\dots,a_n) \leftrightarrow p(x)$, where $p(x) = \operatorname{tp}_\varphi(b/....s)$.* □

2.11 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *if s is generically stable for $\varphi(x\,;z)$ over M then s is smooth for $\varphi(x\,;z)$ over M.*