Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Samples

The finite (fractional) expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, the set of real numbers, and, for every n, a sort for the functions $s : M^n \to \mathbb{R}$ that are almost always 0.

These functions are interpreted as (signed and fractional) multisets. Namely $s(a)$ interpreted as the number of times a occurs in s. We use the following suggestive notation:

1. $a \in s \;=\; |s(a)|$ this is called the multiplicity of a in s;
2. $|s| \;=\; \sum_{a \in M} a \in s$ this is called the size of s;
3. $\big|\{a \in s \;:\; \varphi(a\,;b)\}\big| \;=\; \sum_{a \models \varphi(x;b)} a \in s.$

We call these multisets samples and refer to their sorts collectively as sample-sort.

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L, the language of rings, and the language of vector spaces over $\mathbb{R}$. Each interpreted in the obvious sort. Moreover, L^{f} has a symbols for the functions in 1-3 above. In particular, 3 requires a symbol for every $\varphi(x\,;z) \in L$.

Let $\mathcal{U}$ be a monster model of cardinality κ, a cardinal larger than the cardinality of L. We introduce two elementary extension of $\mathcal{U}^{\mathrm{f}}$.

2.1 Definition *We denote by $\mathcal{U}^{\mathrm{f}_*}$ a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}_*}$. We denote by $\mathcal{U}^{\mathrm{f}^+_*}$ some elementary extension of $\mathcal{U}^{\mathrm{f}_*}$ that realizes all types in $S(\mathcal{U}^{\mathrm{f}_*})$. Hence we have $\mathcal{U}^{\mathrm{f}} \prec \mathcal{U}^{\mathrm{f}_*} \prec \mathcal{U}^{\mathrm{f}^+_*}$. Elements of sample-sort in these models are called finite samples, internal samples, and (external) samples respectively.* □

The use of $\mathcal{U}^{\mathrm{f}^+_*}$ is not ideal, it is introduced to replace the use of global types of sample-sort which also have drawbacks.

We write $\mathrm{supp}(s)$ for the support of s, that is, the set of those a in $\mathcal{U}^{\mathrm{f}+}_{*}$ such that $a \in s$ is positive.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_{x/s}\varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

When possible we abbreviate $\mathrm{Av}_{x/s}\varphi(x\,;b)$ with $\overline{\varphi}(s\,;b)$.

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a-b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. We write $\mathrm{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

2.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\overline{\varphi}(s'\,;b) \approx \overline{\varphi}(s\,;b)$. When $\varphi(x\,;z)$ and M are not clear from the context we will say smooth for $\varphi(x\,;z)$ over M.* □

2.3 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\overline{\varphi}(s\,;b) \approx \overline{\varphi}(s\,;b')$.*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(s\,;b) < \varepsilon\}$ is definable over M.*
3. *satisfiable if, for all $b \in \mathcal{U}^{|z|}$, if $\overline{\varphi}(s\,;b) > \varepsilon$, then $\varphi(r\,;b) > \varepsilon$ for some $r \in M^{\mathrm{f}}$.*
4. *generically stable if all of the above hold.* □

Smooth roughly means internal and invariant.

2.4 Fact *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *there is an invariant internal sample $s' \equiv_M s$.*

Proof 1⇒2. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathcal{U}^{|z|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \mathrm{Aut}(\mathcal{U}^{\mathrm{f}_*}/M)$. Then $\overline{\varphi}(s'\,;fb) \approx \overline{\varphi}(f^{-1}s'\,;b)$. Moreover, by smothness $\overline{\varphi}(f^{-1}s'\,;b) \approx \overline{\varphi}(s'\,;b)$. The infariace in s' follows.

2⇒1. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ busuch that $\overline{\varphi}(s'\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\overline{\varphi}(s''\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. As $s'' \equiv_M s$ are both in $\mathcal{U}^{\mathrm{f}_*}$, then $s'' = fs$ for some $f \in \mathrm{Aut}(\mathcal{U}^{\mathrm{f}_*}/M)$. Hence we obtan $\overline{\varphi}(s\,;f^{-1}b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$ which contradics the invariance of s. □

2.5 Fact *For any given $\varphi(x\,;z) \in L$ and M, smooth samples are generically stable.*

Proof Assume s is smooth and prove in turn 1-3 of Definition 2.3.

1. Invariance follows immediately from Fact 2.4.

2. Note that the set $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(s\,;b) < \varepsilon\}$ is definable in $\mathcal{U}^{\mathrm{f}*}$. In fact, by the smoothness of s, we may replace s with any internal sample $s' \equiv_M s$. By what proved above, this set is invariat over M. Therefore it is definable over M^{f}, or equivalently, over M.

3. Let $b \in \mathcal{U}^{|z|}$ be such that $\overline{\varphi}(s\,;b) > \varepsilon$. Pick a sample s' such that $s \equiv_M s' \mathop{\smile\!\!\!\!\!\!\,|}_{M^{\mathrm{f}}} b$. By the smoothness of s, we obtain $\overline{\varphi}(s';b) > \varepsilon$. Let $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(r\,;b) > \varepsilon$. □

The following is a useful characterization of smoothness.

2.6 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *for every $b \in \mathcal{U}^{|z|}$ there is a finite sample $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ (the sample r depends on ε and b);*
3. *there are some finite samples $r_1, \ldots, r_n \in M^{\mathrm{f}}$ such that for every $b \in \mathcal{U}^{|z|}$ we have $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r_i\,;b)$ for some r_i (the samples $r_1, \ldots, r_n$ as well as the number n depend on ε, but not on b);*
4. *there are finitely many formulas $\psi_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have that $\psi_i(x) \to \varphi(x\,;b) \to \psi_j(x)$ and $\mathrm{Av}_{x/s}[\psi_j(x) \not\to \psi_i(x)] < \varepsilon$ for some i, j.*

Proof 1⇒2. Let $b \in \mathcal{U}^{|z|}$ be arbitrary. Let μ be the standard part of $\overline{\varphi}(s\,;b)$. Let s' be a sample such that $s \equiv_M s' \mathop{\smile\!\!\!\!\!\!\,|}_{M^{\mathrm{f}}} b$. By smoothness, $\overline{\varphi}(s';b) \approx \mu$. As $s' \mathop{\smile\!\!\!\!\!\!\,|}_{M^{\mathrm{f}}} b$ there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(r\,;b) \approx_\varepsilon \mu$.

1⇒3. Let $s' \equiv_M s$ be an internal sample. By smoothness we it suffices to prove 3 with s' for s. The type

$$p(z) \quad = \quad \Big\{\overline{\varphi}(s';z) \not\approx_\varepsilon \overline{\varphi}(r\,;z) \;:\; r \in M^{\mathrm{f}}\Big\}$$

is inconsistent by 2 (clearly s' is smooth). Hence compatness yields the required $r_1, \ldots, r_n \in M^{\mathrm{f}}$.

The proof of equivalence 3 ⇔ 4 is left to the reader. The other implications are evident. □

2.7 Definition *Let $\varphi(x\,;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s is approximable on $\mathcal{B}$ if for every ε there is a finite sample $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s, b) \approx_\varepsilon \overline{\varphi}(r, b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε.) We add for $\varphi(x\,;z)$ over M when these are not clear from the context.* □

2.8 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that*

$\overline{\varphi}(s\,;b) > \varepsilon$ *for every* $b \in \mathcal{B}$*. Then there is a finite cover of* $\mathcal{B}$*, say* $\mathcal{B}_1, \dots, \mathcal{B}_n$*, such that all types* $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ *are consistent.*

Proof Let $r \in M^{\mathrm{f}}$ be as in Definition 2.7. As r is finite, we may assume that $\mathrm{supp}(r) = \{a_1, \dots, a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i\,;b)\}$. As $\overline{\varphi}(r\,;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. $\square$

2.9 Corollary *Let* $\varphi(x\,;z) \in L$ *and* M *be given. If* s *be approximable on* $\langle b_i : i < \omega\rangle$*, where* $\langle b_i : i < \omega\rangle$ *is a sequence of indiscernibles such that* $\overline{\varphi}(s\,;b_i) > \varepsilon$ *for every* $i < \omega$*, then the type* $\{\varphi(x\,;b_i) : i < \omega\}$ *is consistent.* $\square$

2.10 Corollary *Let* $\varphi(x\,;z) \in L$ *and* M *be given. Let* s *be approximable on* $\mathcal{C}$*. Then for every* ε *there is a pair of distict* $c, c' \in \mathcal{C}$ *such that* $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] < \varepsilon$

Proof Suppose for a contradiction that $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] \geq \varepsilon$ for all $c, c' \in \mathcal{C}$. Apply Lemma 2.8 to the folmula $\psi(x\,;z,z') = [\varphi(x\,;z) \not\leftrightarrow \varphi(x\,;z')]$ and the set $\mathcal{B} = \{\langle c, c'\rangle \in \mathcal{C}^2 : c \neq c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 2.8 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x\,;a) \not\leftrightarrow \varphi(x\,;a') : a, a' \in A, a \neq a'\}$ is consistent. As $|A| > 2$, this is impossible. $\square$

3 The nip formulas

3.1 Theorem *Let* $\varphi(x\,;z) \in L$ *and* M *be given and assume that* $\varphi(x\,;z)$ *is nip. Then every sample is approximable on* M *over* M*.*

Proof Questo è Vapnik-Chervonenkis. $\square$

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|z|}$.)

3.2 Corollary *Let* $\varphi(x\,;z) \in L$ *be nip. Every smooth sample* s *is approximable on* $\mathcal{U}$ *over* M*.*

Proof By Theorem 3.1, there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\overline{\varphi}(s\,;b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathop{\raise0.2ex\hbox{$\bigcup$}\kern-0.75em\lower0.2ex\hbox{$\smile$}}_M s' \equiv_M s$. By the smoothness of s, we obtain $\overline{\varphi}(s'\,;b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$. As $b' \mathop{\raise0.2ex\hbox{$\bigcup$}\kern-0.75em\lower0.2ex\hbox{$\smile$}}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\overline{\varphi}(s\,;b) \not\approx_\varepsilon \overline{\varphi}(r\,;b)$. A contradiction. $\square$

3.3 Definition *A formula* $\varphi(x\,;z) \in L$ *is* distal *if there is a formula* $\psi(x\,;z_1, \dots z_n) \in L$ *such that for every finite set* $A \subseteq \mathcal{U}^{\mathrm{f}}$ *and every* $b \in \mathcal{U}^{|x|}$ *there are* $a_1, \dots, a_n \in \mathrm{supp}(s)$ *such that* $\psi(x\,;a_1, \dots, a_n) \leftrightarrow p(x)$*, where* $p(x) = \mathrm{tp}_\varphi(b/A)$*.* $\square$

3.4 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *for every sample $s \in \mathcal{U}^{\mathrm{f}^+_*}$, if s is generically stable for $\varphi(x\,;z)$ over M then s is smooth for $\varphi(x\,;z)$ over M.*