

Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

The **finite (fractional) expansion** of a structure M is a many-sorted expansion that has a sort for M , which we call the home-sort, a sort for \mathbb{R} , the set of real numbers, and a sort for the finite fractional multisets of elements of M^n , for any $n < \omega$. A **(fractional) multiset** is a function $s : M^n \rightarrow \mathbb{R}^+ \cup \{0\}$. We say that s is **finite** if it is almost always 0. We interpret $s(a)$ as the number of occurrences of a in s . For this reason, we use the following suggestive notation:

1. $a \in s = s(a)$ this is called the **multiplicity** of a in s ;
2. $|s| = \sum_{a \in M} a \in s$ this is called the **size** of s ;
3. $|\{a \in s : \varphi(a; b)\}| = \sum_{a \models \varphi(x; b)} a \in s$

We call these multisets **samples** of M^n .

We will write M^f for the expansion above. The language of M^f is denoted by L^f . It expands L and the language of arithmetic. Moreover, L^f has a symbols for the functions in 1-3 above. In particular, claim 3 requires a symbol for every $\varphi(x; z) \in L$.

Let \mathcal{U} be a monster model of cardinality κ , an inaccessible cardinal larger than the cardinality of L . The **hyperfinite expansion** of \mathcal{U} is a saturated elementary extension of \mathcal{U}^f of cardinality κ . We will write \mathcal{U}^{f*} for this expansion. As all saturated models of cardinality κ are isomorphic, we can assume that \mathcal{U} is the home-sort of \mathcal{U}^{f*} .

We write $\text{supp}(s)$ for the **support** of s , that is, the subset of \mathcal{U}^n where $x \in s$ is positive. Note that $\text{supp}(s)$ is a definable subset of \mathcal{U}^{f*} , hence its cardinality is either finite or κ . Moreover, $s \subseteq M^n$ for some small model M , then $\text{supp}(s)$ is finite.

If s is a sample in \mathcal{U}^{f*} , we write $s \subseteq \mathcal{U}^n$ as a shorthand of $\text{supp}(s) \subseteq \mathcal{U}^n$.

2.1 Fact Every model M is the domain of the home-sort of some $M^f \preceq \mathcal{U}^{f*}$.

Proof Difficile da scrivere senza hand-waving. □

Let $s \subseteq \mathcal{U}^{|x|}$ be a sample. For every formula $\varphi(x; z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\text{Av}_s \varphi(x; b) = \frac{|\{a \in s : \varphi(a; b)\}|}{|s|},$$

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε . We write $\text{st}(a)$, for the standard part of the hyperrational number a . That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has a unique extension to a Lebesgue probability measure on a σ -algebra. This is known as the **Loeb measure**.

2.2 Theorem *Let $\varphi(x; z) \in L$ be a nip formula. Let M be a model. For every sample $s \subseteq \mathcal{U}^{|x|}$, and every ε , there is a sample $s_{M, \varepsilon} \subseteq M^{|x|}$ such that $\text{Av}_s \varphi(x; b) \approx_\varepsilon \text{Av}_{s_{M, \varepsilon}} \varphi(x; b)$ for all $b \in M^{|z|}$.*

Moreover, the cardinality of $s_{M, \varepsilon}$ solely depends on ε , namely

$$|s_{M, \varepsilon}| \leq c \frac{k}{\varepsilon^2} \ln \frac{k}{\varepsilon}$$

where c is an absolute constant and k is the VC-dimension of $\varphi(x; z)$.

Proof Questo è Vapnik-Chervonenkis. □

2.3 Lemma *For every nip formula $\varphi(x; z) \in L$, every sample $s \subseteq \mathcal{U}^{|x|}$ and every infinite $B \subseteq \mathcal{U}^{|z|}$ such that $\text{Av}_s \varphi(x; b) > \varepsilon$ for every $b \in B$, there is a finite cover of B , say B_1, \dots, B_n , such that all types $p_i(x) = \{\varphi(x; b) : b \in B_i\}$ are consistent.*

Proof Let M be any model containing B . Fix some $\varepsilon' < \varepsilon$. Let $s_{M, \varepsilon'}$ be the sample given by Theorem 2.2. Then $\text{supp}(s_{M, \varepsilon'})$ is finite, say $\text{supp}(s_{M, \varepsilon'}) = \{a_1, \dots, a_n\}$. Let $B_i = \{b \in B : \varphi(a_i; b)\}$. As $\text{Av}_{s_{M, \varepsilon'}} \varphi(x; b) > 0$ for all $b \in B$, these B_i are the required cover of B . □

2.4 Corollary *Let $\psi(x; z) \in L$ be a nip formula and let $s \subseteq \mathcal{U}^{|x|}$ be a sample. Then there is no infinite set C such that $\text{Av}_s [\psi(x; c) \nleftrightarrow \psi(x; c')] > \varepsilon$ for all $c, c' \in C, c \neq c'$.*

Proof Apply Lemma 2.3 to the formula $\varphi(x; z, z') = [\psi(x; z) \nleftrightarrow \psi(x; z')]$ and the set $B = \{\langle c, c' \rangle \in C^2 : c \neq c'\}$. Without loss of generality, we may assume that the sets B_i given by the lemma form a partition of the complete graph on C . By the Ramsey theorem there is an infinite $A \subseteq C$ such that $\{\psi(x; a) \nleftrightarrow \psi(x; a') : a, a' \in A, a \neq a'\}$ is consistent. Clearly this is a contradiction as soon as $|A| > 2$. □

The following is another immediate consequence of the lemma (it holds also when $\varphi(x; z)$ is not nip, though the proof is lengthier).

2.5 Corollary *For every nip formula $\varphi(x; z) \in L$, every sample $s \subseteq \mathcal{U}^{|x|}$ and every sequence of indiscernibles $\langle b_i : i < \omega \rangle$ such that $\text{Av}_s \varphi(x; b_i) > \varepsilon$ for every $i < \omega$, the type*

$\{\varphi(x; b_i) : i < \omega\}$ is consistent. \square

2.6 Definition Let $\varphi(x; z) \in L$. We say that the sample $s \subseteq \mathcal{U}^{|x|}$ is *invariant* for $\varphi(x; z)$ over M if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\text{Av}_s \varphi(x; b) \approx \text{Av}_{s'} \varphi(x; b)$. \square

For invariant samples the Vapnik-Chervonenkis Theorem can be strengthened as follows.

2.7 Corollary Let $\varphi(x; z) \in L$ be a nip formula. Let M be any model. For every sample $s \subseteq \mathcal{U}^{|x|}$ that is invariant for $\varphi(x; z)$ over M , and every ϵ , there is a sample $s_{M, \epsilon} \subseteq M^{|x|}$ such that $\text{Av}_s \varphi(x; b) \approx_\epsilon \text{Av}_{s_{M, \epsilon}} \varphi(x; b)$ for all $b \in \mathcal{U}^{|z|}$.

Proof By the theorem above, there is a sample $s_{M, \epsilon} \subseteq M^{|x|}$ such that $\text{Av}_s \varphi(x; b) \approx_\epsilon \text{Av}_{s_{M, \epsilon}} \varphi(x; b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\text{Av}_s \varphi(x; b') \not\approx_\epsilon \text{Av}_{s_{M, \epsilon}} \varphi(x; b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a sample $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \downarrow_M s' \equiv_M s$. By invariance, $\text{Av}_{s'} \varphi(x; b') \not\approx_\epsilon \text{Av}_{s_{M, \epsilon}} \varphi(x; b')$. As $b' \downarrow_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by invariance, $\text{Av}_s \varphi(x; b) \not\approx_\epsilon \text{Av}_{s_{M, \epsilon}} \varphi(x; b)$. A contradiction. \square

The following is stronger form of invariance.

2.8 Definition We say that a sample $s \subseteq \mathcal{U}^{|x|}$ is *smooth* for $\varphi(x; z)$ over M if for any other sample s' such that $\text{Av}_s \varphi(x; b) \approx \text{Av}_{s'} \varphi(x; b)$ for all $b \in M^{|z|}$, the same equivalence holds for all $b \in \mathcal{U}^{|z|}$. \square

Clearly, smooth implies invariant.

2.9 Definition ??? A sample $s \subseteq \mathcal{U}^{|x|}$ is *finitely satisfiable* for $\varphi(x; z)$ in M if, for every $b \in \mathcal{U}^{|z|}$ such that $\text{Av}_s \varphi(x; b) \not\approx 0$, there is $a \in M^{|z|}$ such that $\varphi(a; b)$.

We say that s is *generically stable* for $\varphi(x; z)$ over M if it is both invariant and finitely satisfiable for $\varphi(x; z)$ over M . \square

2.10 Definition ??? A formula $\varphi(x; z) \in L$ is *distal* if there is a formula $\psi(x; z_1, \dots, z_n) \in L$ such that for every finite sample s and every $b \in \mathcal{U}^{|x|}$ there are $a_1, \dots, a_n \in s^{|z|}$ such that $\psi(x; a_1, \dots, a_n) \leftrightarrow p(x)$, where $p(x) = \text{tp}_\varphi(b / \dots s)$. \square

2.11 Theorem (false) The following are equivalent

1. $\varphi(x; z) \in L$ is distal;
2. if s is generically stable for $\varphi(x; z)$ over M then s is smooth for $\varphi(x; z)$ over M .