Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Samples

Let M be a structure of signature L. The finite (fractional) sample expansion of M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, the set of real numbers, and for each n a sort for F_n, an $\mathbb{R}$-vector space of functions $f : M^n \to \mathbb{R}$. We refer collectively to these vector spaces as sample-sort. Variabels (or tuples of variables) of sample-sort are denoted with sumbolds $x^{\mathrm{s}}, y^{\mathrm{s}}$ and variations thereof.

We will write M^{f} for the expansion above. The language of M^{f} is denoted by L^{f}. It expands L, the language of $\mathbb{R}$ as an ordered rings, and the language of vector spaces over $\mathbb{R}$. Each interpreted in the obvious sort. Moreover, for every $\varphi(x\,;z) \in L$ in L^{f} there is a symbol $\overline{\varphi}(x^{\mathrm{s}}\,;z)$ for a function $F_{|x|} \times M^{|z|} \to \mathbb{R}$.

2.1 Definition *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$. We denote by $\mathcal{U}^{\mathrm{s}}$ a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{s}}$. Elements $r \in \mathcal{U}^{\mathrm{f}}$ are called finite samples, elements in $s \in \mathcal{U}^{\mathrm{s}}$ are called smooth samples.* □

Below, ε always ranges over the positive standard reals. If a and b are hyperreals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

2.2 Notation In general, for any global type $p(x) \in S(\mathcal{U})$ and formula $\varphi(x\,;z) \in L$ we write

$$\varphi(p\,;\mathcal{U}) \;=\; \Big\{b \in \mathcal{U}^{|z|} \;:\; \varphi(x,\, b) \in p\Big\}$$

This notation intentionally confuses p with any of its realizations in some elementary extension of $\mathcal{U}$. Sets of this form are called externally definable.

We extend this notation to global samples. If $\varphi(x\,;z) \in L$ and $p(x^{\mathrm{s}})$ is a global sample, we write $\overline{\varphi}(p\,;b)$ for the unique $\mu \in \mathbb{R}$ such that $p(x^{\mathrm{s}}) \vdash \overline{\varphi}(x^{\mathrm{s}}\,;b) \approx \mu$. □

2.3 Definition *Let $\varphi(x\,;z) \in L$. We say that the global type $p(x^{\mathrm{s}})$ is smooth if there is a smooth sample s such that $\overline{\varphi}(s\,;b) \approx \overline{\varphi}(p\,;b)$ for all $b \in \mathcal{U}^{|z|}$.* □

2.4 Definition *Let $\varphi(x\,;z) \in L$ and M be given. A global sample $p(x^{\rm s})$ is*

1. *invariant if $\bar{\varphi}(p\,;z)$ is invariant under $Aut(\mathcal{U}/M)$, that is, $\bar{\varphi}(p\,;b) \approx \bar{\varphi}(p\,;fb)$ for every $b \in \mathcal{U}^{|z|}$ and $f \in Aut(\mathcal{U}/M)$;*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(p\,;b) < \varepsilon\}$ is definable over M;*
3. *finitely satisfiable if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in M^{\rm f}$ such that $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$;*
4. *generically stable if all of the above hold.* □

3 Old stuff

We write $\operatorname{supp}(p)$ for the support of p, that is, the set of those a in $\mathcal{U}$ such that $p \vdash (a \in x^{\rm s}) > \varepsilon$ for some standard positive ε. If s is a smooth sample $\operatorname{supp}(s)$ is defined to be $\operatorname{supp}(p)$ for $p(x^{\rm s}) = \operatorname{tp}(s/\mathcal{U}^{\rm f})$.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\operatorname{Av}_{x/s}\varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

When possible we abbreviate $\operatorname{Av}_{x/s}\varphi(x\,;b)$ with $\overline{\varphi}(s\,;b)$.

4 Old stuff

We write $\operatorname{ft}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

All definitions and facts in this section are relative to some given formula $\varphi(x\,;z) \in L$ and some model M.

4.1 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\overline{\varphi}(s';b) \approx \overline{\varphi}(s\,;b)$.* □

4.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\overline{\varphi}(s\,;b) \approx \overline{\varphi}(s\,;b')$;*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(s\,;b) < \varepsilon\}$ is definable over M;*
3. *finitely satisfiable if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in M^{\rm f}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$;*
4. *generically stable if all of the above hold.* □

Smooth roughly means internal and invariant.

4.3 Fact *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *there is an invariant internal sample $s' \equiv_M s$.*

Proof 1⇒2. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathfrak{U}^{|z|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \mathrm{Aut}(\mathfrak{U}^{\mathrm{s}}/M)$. Then $\overline{\varphi}(s'; fb) \approx \overline{\varphi}(f^{-1}s'\,;b)$. Moreover, by smothness $\overline{\varphi}(f^{-1}s'\,;b) \approx \overline{\varphi}(s';b)$. The infariace in s' follows.

2⇒1. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ such that $\overline{\varphi}(s'\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\overline{\varphi}(s''\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. As $s'' \equiv_M s$ are both in $\mathfrak{U}^{\mathrm{s}}$, then $s'' = fs$ for some $f \in \mathrm{Aut}(\mathfrak{U}^{\mathrm{s}}/M)$. Hence we obtan $\overline{\varphi}(s\,;f^{-1}b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$ which contradics the invariance of s. □

4.4 Fact *For any given $\varphi(x\,;z) \in L$ and M, smooth samples are generically stable.*

Proof Assume s is smooth and prove in turn 1-3 of Definition 4.2.

1. Invariance follows immediately from Fact 4.3.

2. Note that the set $\{b \in \mathfrak{U}^{|z|} : \overline{\varphi}(s\,;b) < \varepsilon\}$ is definable in $\mathfrak{U}^{\mathrm{s}}$. In fact, by the smoothness of s, we may replace s with any internal sample $s' \equiv_M s$. By what proved above, this set is invariat over M. Therefore it is definable over M^{f}, or equivalently, over M.

3. Let $b \in \mathfrak{U}^{|z|}$ be such that $\overline{\varphi}(s\,;b) > \varepsilon$. Pick a sample s' such that $s \equiv_M s' \downarrow_{M^{\mathrm{f}}} b$. By the smoothness of s, we obtain $\overline{\varphi}(s';b) > \varepsilon$. Let $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(r\,;b) > \varepsilon$. □

The following is a useful characterization of smoothness.

4.5 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *for every $b \in \mathfrak{U}^{|z|}$ there is a finite sample $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ (the sample r depends on ε and b);*
3. *there are some finite samples $r_1, \ldots, r_n \in M^{\mathrm{f}}$ such that for every $b \in \mathfrak{U}^{|z|}$ we have $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r_i\,;b)$ for some r_i (the samples $r_1, \ldots, r_n$ as well as the number n depend on ε, but not on b);*
4. *there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathfrak{U}^{|z|}$ we have that $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\overline{\vartheta}_j(s) - \overline{\vartheta}_i(s) < \varepsilon$ for some i, j.*

Proof 1⇒2. Let $b \in \mathfrak{U}^{|z|}$ be arbitrary. Let μ be the standard part of $\overline{\varphi}(s\,;b)$. Let s' be a sample such that $s \equiv_M s' \downarrow_{M^{\mathrm{f}}} b$. By smoothness, $\overline{\varphi}(s';b) \approx \mu$. As $s' \downarrow_{M^{\mathrm{f}}} b$ there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(r\,;b) \approx_\varepsilon \mu$.

1⇒3. Let $s' \equiv_M s$ be an internal sample. By smoothness we it suffices to prove 3 with s' for s. The type

$$p(z) \quad = \quad \Big\{ \overline{\varphi}(s';z) \not\approx_\varepsilon \overline{\varphi}(r;z) \ : \ r \in M^{\mathrm{f}} \Big\}$$

is inconsistent by 2 (clearly s' is smooth). Hence compatness yields the required $r_1, \ldots, r_n \in M^{\mathrm{f}}$.

The proof of equivalence 3 ⇔ 4 is left to the reader. The other implications are evident. □

4.6 Definition *Let $\varphi(x;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s is (uniformly) approximable on $\mathcal{B}$ if for every ε there is a finite sample $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s,b) \approx_\varepsilon \overline{\varphi}(r,b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε, not on b.)* □

4.7 Lemma *Let $\varphi(x;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that $\overline{\varphi}(s;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1, \ldots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x;b) : b \in \mathcal{B}_i\}$ are consistent.*

Proof Let $r \in M^{\mathrm{f}}$ be as in Definition 4.6. As r is finite, we may assume that $\mathrm{supp}(r) = \{a_1, \ldots, a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i;b)\}$. As $\overline{\varphi}(r;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

4.8 Corollary *Let $\varphi(x;z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega \rangle$, where $\langle b_i : i < \omega \rangle$ is a sequence of indiscernibles such that $\overline{\varphi}(s;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x;b_i) : i < \omega\}$ is consistent.* □

4.9 Corollary *Let $\varphi(x;z) \in L$ and M be given. Let s be approximable on $\mathcal{C}$. Then for every ε there is a pair of distict $c, c' \in \mathcal{C}$ such that $\mathrm{Av}_{x/s}\big[\varphi(x;c) \not\leftrightarrow \varphi(x;c')\big] < \varepsilon$*

Proof Suppose for a contradiction that $\mathrm{Av}_{x/s}\big[\varphi(x;c) \not\leftrightarrow \varphi(x;c')\big] \geq \varepsilon$ for all distict $c, c' \in \mathcal{C}$. Apply Lemma 4.7 to the folmula $\psi(x;z,z') = [\varphi(x;z) \not\leftrightarrow \varphi(x;z')]$ and the set $\mathcal{B} = \{\langle c, c' \rangle \in \mathcal{C}^2 \ : \ c{\neq}c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 4.7 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x;a) \not\leftrightarrow \varphi(x;a') \ : \ a, a'{\in}A,\ a{\neq}a'\}$ is consistent. As $|A| > 2$, this is impossible. □

5 The nip formulas

5.1 Theorem *Let $\varphi(x;z) \in L$ and M be given and assume that $\varphi(x;z)$ is nip. Then every sample is approximable on M over M.*

Proof Questo è Vapnik-Chervonenkis. □

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|z|}$.)

5.2 Corollary *Let $\varphi(x\,;z) \in L$ be nip. Every smooth sample s is approximable on $\mathcal{U}$ over M.*

Proof By Theorem 5.1, there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\overline{\varphi}(s\,;b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathop{\downarrow}_M s' \equiv_M s$. By the smoothness of s, we obtain $\overline{\varphi}(s';b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$. As $b' \mathop{\downarrow}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\overline{\varphi}(s\,;b) \not\approx_\varepsilon \overline{\varphi}(r\,;b)$. A contradiction. □

5.3 Definition ?? *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1, \ldots z_n) \in L$ such that for every finite set $B \subseteq \mathcal{U}$ and every $a \in \mathcal{U}^{|x|}$ there are $b_1, \ldots, b_n \in B$ such that $\psi(a\,;b_1, \ldots, b_n)$ and $\psi(x\,;b_1, \ldots, b_n)$ decides all formulas $\varphi(x\,;b)$ for $b \in B$.* □

5.4 Definition ??* *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x_1, \ldots, x_n\,;z) \in L$ such that for every finite set $A \subseteq \mathcal{U}^{|x|}$ and every $b \in \mathcal{U}^{|z|}$ there are $a_1, \ldots, a_n \in B$ such that $\psi(a_1, \ldots, a_n,\ z)$ and $\psi(a_1, \ldots, a_n,\ z)$ decides all formulas $\varphi(a\,;z)$ for $a \in A$.* □

For every finite sample $r \in M^{\mathrm{f}}$ and every $\mu \in \mathbb{R}$ there is a formula $\psi(z) \in L(\operatorname{supp} r)$ such that $\overline{\varphi}(r\,;z) = \mu \leftrightarrow \psi(z)$. The formula $\psi(z)$ depends on r.

When $\varphi(x\,;z)$ is distal

If $\varphi(x\,;z)$ is distal then there is a formula ψ such that for every $r \in M^f$ there is $r_0 \in M$

5.5 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,;z) \in L$ is distal;*
2. *for every sample $s \in \mathcal{U}^{\mathrm{f}^+}_*$, if s is generically stable then s is smooth.*

There is a formula $\psi(x_1, \ldots, x_n\,;z)$ such that for every $r \in M^{\mathrm{f}}$

there are $a_1, \ldots, a_n$ such that $\varphi(r\,;b) = \mu\ \psi(x_1, \ldots, x_n\,;z)$