Scretch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Preliminaries

The finite expansion of a structure M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for the natural numbers, and a sort for the finite multisets of elements of M^n, for any $n < \omega$. We call these multisets samples of M^n. We will write M^{f} for this expansion. The language of M^{f} is denoted by L^{f}. It expands L and the language of arithmetic. Moreover, L^{f} has a function for the number of occurrences of a tuple a in a sample s, which we denote by $a \in s$ and a funtion for the cardinality

$$|s| \;=\; \sum_{a \in M} a \in s$$

For ease of exposition we may also use rational numbers, though these need not have their own sort as they are interpretable in the naturar numbers.

Let $\mathcal{U}$ be a monster model of cardinality κ, an inaccessible cardinal larger than the cardinality of L. The hyperfinite expansion of $\mathcal{U}$ is a saturated elementary extension of $\mathcal{U}^{\mathrm{f}}$ of cardinality κ. We will write $\mathcal{U}^{\mathrm{f}*}$ for this expansion. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\mathrm{f}*}$.

We write $\mathcal{R}(s)$ for the range of s, that is, the subset of $\mathcal{U}^n$ where $x \in s$ is positive. Note that its cardinality is either finite of κ. If s is a sample in $\mathcal{U}^{\mathrm{f}*}$, we write $s \subseteq \mathcal{U}^n$ to specify that $\mathcal{R}(s) \subseteq \mathcal{U}^n$.

2.1 Fact *Every model M is the domain of the home-sort of some $M^{\mathrm{f}} \preceq \mathcal{U}^{\mathrm{f}*}$.*

Proof Difficile da scrivere senza hand-waving. □

Let $s \subseteq \mathcal{U}^{|x|}$ be a sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_s\, \varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

where we use the following suggestive abbreviation

$$\big|\{a \in s \;:\; \varphi(a\,;b)\}\big| \;=\; \sum_{a \vDash \varphi(x;b)} a \in s$$

Below, ε always ranges over the positive standard rational. If a and b are hyperrationals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε. If μ is a real number we write $a \approx_\varepsilon \mu$ if there is a standard rational $b \approx_{\varepsilon/2} a$ such that $|b - \mu| < \varepsilon/2$ (this last inequality is evaluated in the standard real line). We write $\mathrm{st}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

We say that the sample $s \subseteq \mathcal{U}^{|x|}$ is invariant over A if for every $\varphi(x) \in L(\mathcal{U})$ and every $s' \equiv_A s$ we have $\mathrm{Av}_s\, \varphi(x) \approx \mathrm{Av}_{s'}\, \varphi(x)$.

The following is a weaker form of invariance.

2.2 Definition *We say that a sample $s \subseteq \mathcal{U}^{|x|}$ is smooth over A if for any other sample s', if $\mathrm{Av}_s\, \varphi(x) \approx \mathrm{Av}_{s'}\, \varphi(x)$ holds for all $\varphi(x) \in L(A)$ then it holds for all $\varphi(x) \in L(\mathcal{U})$.*

2.3 Theorem *Let $\varphi(x\,;z) \in L$ be a nip formula. For every sample $s \subseteq \mathcal{U}^{|x|}$, and every ε, there is a finite sample s_ε such that $\mathrm{Av}_s \varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_\varepsilon}\, \varphi(x\,;b)$ for all $b \in \mathcal{U}^{|z|}$. Moreover, the cardinality of s_ε soley depends on ε, namely*

$$|s_\varepsilon| \quad \leq \quad c\,\frac{k}{\varepsilon^2} \ln \frac{k}{\varepsilon}$$

where c is an absolute constant.

Proof Questo è Vapnik-Chrevonenkis. □

2.4 Corollary *Let $\varphi(x\,;z) \in L$ be a nip formula. Let M be any model. For every sample s of $\mathcal{U}^{|x|}$ that is invariant over M, and every ε, there is a sample s_ε of $M^{|x|}$ such that $\mathrm{Av}_s \varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_\varepsilon}\, \varphi(x\,;b)$ for all $b \in M^{|z|}$.*

Proof By elementarity there is a sample s_ε of $M^{|x|}$ such that $\mathrm{Av}_s \varphi(x\,;b) \approx_\varepsilon \mathrm{Av}_{s_\varepsilon}\, \varphi(x\,;b)$ for all $b \in M^{|z|}$. We claim that this holds for all $b \in \mathcal{U}^{|z|}$. Suppose not, so let $\mathrm{Av}_s \varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_{s_\varepsilon}\, \varphi(x\,;b)$. Let $s' \equiv_M s$ be such that $b \mathop{\smile\!\!\!\!\!\mid\;} s'$. As $\mathrm{Av}_{s'} \varphi(x\,;b) \not\approx_\varepsilon \mathrm{Av}_{s_\varepsilon}\, \varphi(x\,;b)$ holds, then $\mathrm{Av}_{s'} \varphi(x\,;b') \not\approx_\varepsilon \mathrm{Av}_{s_\varepsilon}\, \varphi(x\,;b')$ holds for some $b' \in M^{|z|}$. □

2.5 Fact *If $\mathrm{Av}_A\, \varphi(x\,;b) \approx \mathrm{Av}_B\, \varphi(x\,;b)$ for all $b \in M^{|z|}$ then the same holds for all $b \in \mathcal{U}^{|z|}$.*

A hypersample is invariant over M if for every

2.6 Definition *An hypersample s is generically stable over M if .* □

2.7 Definition *A formula $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1, \ldots z_n) \in L$ such that for every hyperfinite sample s and every $b \in \mathcal{U}^{|x|}$ there are $a_1, \ldots, a_n \in s^{|z|}$ such that $\psi(x\,;a_1, \ldots, a_n) \leftrightarrow p(x)$, where $p(x) = \mathrm{tp}_\varphi(b/....s)$.* □

2.8 Proposition *Let s be an hyperfinite sample. Suppose that for every ther*

There is a formula $\psi(x\,; z_1, \ldots z_n) \in L$ *such that for every hyperfinite sample* S *and every* $b \in \mathcal{U}^{|x|}$ *there are* $a_1, \ldots, a_n \in S^{|z|}$ *such that* $\psi(x\,; a_1, \ldots, a_n) \leftrightarrow p(x)$*, where* $p(x) = \mathrm{tp}_{\varphi}(b/{\ldots}s)$*.* □