# Scratch paper

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April 2020

**Abstract** Poche idee, ben confuse.

## 1 Introduction

## 2 Abstract samples

Let $M$ be a semigroup. The finite (fractional) sample expansion of $M$ is a 3-sorted expansion $\bar{M} = \langle M, \mathbb{R}, M^{\mathsf{s}} \rangle$ that has a domain for $M$, which we call the home-sort, a domain for $\mathbb{R}$, as an ordered field, and a domain $M^{\mathsf{s}}$, which we call sample-sort, which is an $\mathbb{R}$-vector space that we describe below. Variables and elements of sample-sort (or tuple thereof) are denoted with symbols $x^{\mathsf{s}}, a^{\mathsf{s}}$ and variations. The variable $x$ is reserved for the home-sort, for all other variables we the context will clarify the sort.

An element $a^{\mathsf{s}} \in M^{\mathsf{s}}$ is a map $a^{\mathsf{s}} : M \to \mathbb{R}$ which is almost always zero. These functions, which we call samples, are interpreted as (weighted) samples or as (signed) finite measures (up to normalization) that concentrated on the support of $a^{\mathsf{s}}$.

The language of the expansion $\bar{M}$ is denoted by $L$. It expands the natural language of each sort: on $M$, the language of semigroups, on $\mathbb{R}$, the language ordered rings, and, on $M^{\mathsf{s}}$, the language of algebras over $\mathbb{R}$. This algebra has the natural sum and the multiplication inherited from $M$

$$\begin{aligned} (a^{\mathsf{s}} + b^{\mathsf{s}})(z) &= a^{\mathsf{s}}(z) + b^{\mathsf{s}}(z); \\ (a^{\mathsf{s}} \cdot b^{\mathsf{s}})(z) &= \sum_{x \cdot y = z} a^{\mathsf{s}}(x) \cdot b^{\mathsf{s}}(y). \end{aligned}$$

Finally, for every formula $\varphi(x\,; y) \in L$, where $y$ is a tuple a mixed sort, there is a symbol $\overline{\varphi}(x^{\mathsf{s}}\,; y)$ for a function $(M^s)^x \times M^y \to \mathbb{R}$. For $\varphi(x) \in L(\bar{M})$, we interpret $\overline{\varphi}(x^{\mathsf{s}})$ as the function that maps $a^{\mathsf{s}} \in M^{\mathsf{s}}$ to

$$\overline{\varphi}(a^{\mathsf{s}}) = \sum_{x \in \varphi(M)} a^s(x) \Big/ \sum_{x \in M} a^s(x)$$

Below, $\varepsilon$ always ranges over the positive standard reals. If $a$ and $b$ are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every $\varepsilon$.

**2.1 Definition** *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$. Let $\bar{\mathcal{U}} = \langle \mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathsf{s}} \rangle$ be as above and let $\bar{\mathcal{U}}^*$ be a saturated elementary extension of $\mathcal{U}^{\mathsf{s}}$ of cardinality $\kappa$. As all saturated* *models of cardinality $\kappa$ are isomorphic, we can assume that $\mathcal{U}$ is the domain of the home-sort of $\bar{\mathcal{U}}^*$, hence we write $\bar{\mathcal{U}}^* = \langle \mathcal{U}, \mathbb{R}^*, \mathcal{U}^{\mathsf{s}*} \rangle$. Elements of $\mathcal{U}^{\mathsf{s}}$ are called finite samples, elements of $\mathcal{U}^{\mathsf{s}*}$ are called internal samples.*

*Finally, an external sample is a maximally consistent set of formulas $p(x^{\mathsf{s}}) \subseteq L(\mathcal{U}, \mathbb{R}, \mathcal{U}^{\mathsf{s}*})$. (Note these are not types over the full of $\bar{\mathcal{U}}$.)* □

**2.2 Notation** Let $y$ and $z$ be tuples of mixed sort. For any type $p(y)$ and formula $\varphi(y, z) \in L(\bar{\mathcal{U}}^*)$ we write

$$\varphi(p\,;\bar{\mathcal{U}}^*) \quad = \quad \Big\{ a \in (\bar{\mathcal{U}}^*)^z \ : \ p(y) \vdash \varphi(y\,;a) \Big\}$$

This notation intentionally confuses $p(y)$ with any of its realizations in some elementary extension of $\bar{\mathcal{U}}$. □

**2.3 Definition** *Let $M$ be given. Let $y$ be a tuple of the home-sort. A external sample $p(x^{\mathsf{s}})$ is*

1. *invariant if $\varphi(p\,;\mathcal{U}, \mathcal{U}^{\mathsf{s}*})$ is invariant over $\bar{M}$, for every $\varphi(x\,;y, z^{\mathsf{s}}) \in L$.*
2. *finitely satisfiable if every formula in $p(x^{\mathsf{s}})$ is satisfied by some element of $M^{\mathsf{s}}$.*
3. *definable if $\varphi(p\,;\mathcal{U}, \mathcal{U}^{\mathsf{s}*})$ is definable over $\bar{M}$, for every $\varphi(x\,;y, z^{\mathsf{s}}) \in L$.* □

*An external type is generically stable if it is definable and finitely satisfiable in some $\bar{M}$.* □

Clearly every finitely satisfiable external sample is invariant.

We will use the symbol $\mathop{\smile\!\!\!\!\!\!\mid}_{\bar{M}}$ with the usual meaning.

# 3 Semigroup operation among samples

We work over a model $M$. We write $a_{\bar{M}}$ for the orbit of $a$ over $\bar{M}$.

# 4 Szemerédi's regularity lemma

Define

$$\big|\varphi(a^{\mathsf{s}}\,;b^{\mathsf{s}})\big|) \quad = \quad \sum_{\varphi(x\,;y)} a^{\mathsf{s}}(x) \cdot b^{\mathsf{s}}(y)$$

$$d_\varphi(r\,;s) \quad = \quad \frac{\big|\varphi(r\,;s)\big|}{\big|r \times s\big|}$$

We say that the pair of samples $r, s$ is $\varepsilon$-regular if for every $r' \subseteq r$ and $s' \subseteq s$ such that $|r'| > \varepsilon|r|$ and $|s'| > \varepsilon|s|$

$$\big|d_\varphi(r\,;s) - d_\varphi(r'; s')\big| \quad < \quad \varepsilon$$

**4.1 Szemerédi's Regularity Lemma** *For every $\varepsilon > 0$ and for every pair of samples $u, v \in {}^*\bar{\mathcal{U}}$*

*of hyperfinite cardinality there are some finite partitions of $u$ and $v$, say $r_1, \ldots, r_n$ and $s_1, \ldots, s_m$, such that*

1. *$r_i, s_j$ is $\varepsilon$-regular for every $i, j \in [n] \times [m] \smallsetminus E$,*

*where $E \subseteq [n] \times [m]$, called the exceptional set, is such that*

2. $$\sum_{\langle i,j\rangle \in E} |r_i| \cdot |s_j| \quad < \quad \varepsilon \cdot |u| \cdot |v|$$

**Proof** Let $\mathrm{st}((d(u\,;v))$ □