Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

`master`

2 Abstract samples

Let M be a structure of signature L. The sample-expansion of M is a 3-sorted expansion $\langle M, \omega, \bar{M}\rangle$, where $\bar{M} = M^{<\omega}$. These three sorts are called home-sort, integer-sort, and sample-sort, respectively. The symbol x always denotes a tuple of variables of the home-sort. We put a bar over the symbols of sample-sort such as $\bar{x}$ and $\bar{a}$. Symbols as m, n, i are of integer-sort; from the context it should be inferred whether they are variables or parameters.

The size of a sample $\bar{a} \in \bar{M}$ is the length of $\bar{a}$ as an element of $M^{<\omega}$. This is denoted by $\mathrm{lh}\,\bar{a}$. We write $\bar{a}.i$ for the i-th element of the sample $\bar{a}$. The same symbol $\bar{a}$ may be used for tuples of samples such as $\bar{a}_1, \ldots, \bar{a}_n$. Then by $|\bar{a}|$ we denote the length of this tuple (that is, n, for the tuple above). We always make the the implicit assumption that all $\bar{a}_i$ in a tuple have the same size. By this assumption, $\bar{a}.i$ is a well-defined element of $M^{|\bar{a}|}$.

The language of this expansion is denoted by $\bar{L}$; it comprises

1. all symbols of L that apply to the home-sort;
2. $0, 1, +, \cdot$ that apply to the integer-sort;
3. a ternary relation of sort $\bar{M}, \omega, M$ that holds if $\bar{x}.i = y$;
4. for every formula $\varphi(x\,;y) \in \bar{L}$, where y a tuple of mixed sort, there is relation symbol for $\varphi(\bar{x}.i\,;y)$, where i is a variable of integer-sort.

When $\varphi(x)$ is the formula $x = x$, requirement 4 implies that $\mathrm{lh}\,\bar{x}$ is definable. By 2, the cardinality of finite subsets of ω is definable. Therefore, from 4, it easily follows that the function $|\{i < \mathrm{lh}\,\bar{a} \;:\; \varphi(\bar{a}.i)\}|$ is definable, uniformly in the parameters of $\varphi(x)$.

As in the integer-sort we can interpret the field of rational numbers, below we will freely use rational numbers when this clarify the notation. For instance we define

$$\Pr_{\bar a}\left[\varphi(x)\right] \quad = \quad \frac{|\{i < \mathrm{lh}\,\bar a \ :\ \varphi(\bar a.i)\}|}{\mathrm{lh}\,\bar a}$$

and use that it is a definable function in $\bar L$. In particular we will use that for every $s \in \mathbb{Q}$ there is a formula saying $\mathrm{Fr}_{\bar a}\left[\varphi(x)\right] \geq s$, uniformly in the parameter of $\varphi(x)$.

2.1 Definition *Let $\mathcal{U}$ be a monster model of inaccessible cardinality $\kappa > |L|$ and let $\langle \mathcal{U}, \omega, \bar{\mathcal{U}}\rangle$ be the corresponding sample-expansion. We denote by $\langle \mathcal{U}, \omega^*, \bar{\mathcal{U}}^*\rangle$ a saturated elementary extension of $\langle \mathcal{U}, \omega, \bar{\mathcal{U}}\rangle$ of cardinality κ. Note that we can assume that the home-sort of this extension is $\mathcal{U}$, in fact as all saturated models of cardinality κ are isomorphic. The elements of $\bar{\mathcal{U}}$ are called finite samples, those of $\bar{\mathcal{U}}^*$ are called internal samples.*

To any internal sample $\bar a$ we associate a finitely additive measure on definable subsets of $\mathcal{U}^{|x|}$. For $\varphi(x) \in \bar L(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ we define

$$\Pr_{\bar a}\left[\varphi(x)\right] \quad = \quad \sup\Big\{s \in \mathbb{Q} \ :\ s < \operatorname*{Fr}_{\bar a}\varphi(x)\Big\}$$

Note that this measure is type-definable, uniformly in the parameters of $\varphi(x)$. Namely, let $r \in \mathbb{R}$ and $\varphi(x\,;y) \in \bar L$, where y is a tuple of mixed sort. Then there is a type $q(\bar x\,;y) \subseteq \bar L$ that defines $\Pr_{\bar x}\left[\varphi(x\,;y)\right] \geq r$.

Let M be a given model. By elementarity, for every $\varepsilon > 0$ and every formula $\varphi(x) \in L(M, \omega, \bar M)$ there is an $\bar a' \in \bar M$ such that

$$\Pr_{\bar a}\left[\varphi(x)\right] \quad \approx_\varepsilon \quad \operatorname*{Fr}_{\bar a'}\varphi(x)$$

Note that $\bar a$ need not have the same size of $\bar a'$. In fact, in general $\mathrm{lh}\,\bar a$ may be non standard, while $\mathrm{lh}\,\bar a' \in \omega$.

2.2 Definition *An external sample is a global type $p(\bar x) \in S(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ such that, for every formula $\varphi(x) \in L(\mathcal{U})$, the set $\{i \in \omega^* \ :\ p(\bar x) \vdash \varphi(\bar x.i)\}$ is bounded and definable; uniformly in the parameters of $\varphi(x)$.* □

For every external sample $p(\bar x)$ there is an $n \in \omega^*$ such that $p(\bar x) \vdash \mathrm{lh}\,\bar x = n$ and $\big|\{i \in \omega^* : p(\bar x) \vdash \varphi(\bar x.i)\}\big|$ is definable uniformly on the parameters of $\varphi(x)$.

In analogy to what done with internal samples, we associate to an external sample $p(\bar x)$ a finitely additive probability measure on the definable subsets of $\mathcal{U}$. Namely, we define

$$\Pr_{\bar p}\left[\varphi(x)\right] \quad = \quad \sup\left\{r \in \mathbb{Q} \ :\ r < \frac{\big|\{i < \mathrm{lh}\,\bar x \ :\ p(\bar x) \vdash \varphi(\bar x.i)\}\big|}{\mathrm{lh}\,\bar x}\right\}$$

What claimed above for internal samples, apply also to external samples, that is, there is a type $q(\bar x\,;y) \subseteq \bar L$ that defines $\Pr_{\bar p}\left[\varphi(x\,;y)\right] \geq r$.

2.3 Notation Let z be a tuple of variables of the home-sort. For $p(\bar x)$ an external sample, $\varphi(\bar x\,;z) \in \bar L$, and $b \in \mathcal{U}^{|z|}$ we write $\varphi(\bar p\,;b)$ for $p(\bar x) \vdash \varphi(\bar x\,;b)$. We also write

$$\varphi(\bar p\,;\mathcal{U}) \quad = \quad \Big\{b \in \mathcal{U}^{|z|} \ :\ \varphi(\bar p\,;b)\Big\}$$

Sets of this form are called externally definable. This notation intentionally confuses $p(\bar{x})$ with any of its realizations (suggestively denoted by $\bar{p}$) in some elementary extension of $\bar{\mathcal{U}}$. □

2.4 Definition (with warning) *Let z be a tuple of variables of the home-sort. Let M be given. An external sample $p(\bar{x})$ is*

1. *invariant over M if for every formula $\varphi(\bar{x}\,;z) \in \bar{L}(M,\omega,\bar{M})$ the set $\varphi(\bar{p}\,;\mathcal{U})$ is invariant over M;*
2. *finitely satisfiable in M if every formula $\varphi(\bar{x}) \in \bar{L}(\mathcal{U},\omega,\bar{\mathcal{U}})$ in $p(x)$ is satisfied by some $\bar{a} \in \bar{M}$.*

These notions are standard but for the important fact that only parameters in the home-sort are allowed. □

It is clear that satisfiable implies invariant.

If an external sample $p(\bar{x})$ is weakly satisfiable in $\bar{M}$ then for every $\varepsilon > 0$ and every formula $\varphi(x)$ there is an $\bar{a} \in \bar{M}$ such that

$$\Pr_{\bar{p}}\big[\varphi(x)\big] \quad \approx_\varepsilon \quad \operatorname*{Fr}_{\bar{a}} \varphi(x)$$

Let $p(\bar{x})$ external sample finitely satisfiable in M. A coheir sequence in $p(\bar{x})$ is sequence $\langle \bar{a}_i : i < \omega \rangle$ such that

$$\psi(x\,;b_1,\dots,b_n) \quad\rightarrow\quad \big[\varphi(a\,;b) \;\leftrightarrow\; \varphi(x\,;b)\big]$$

3 Distal formulas

The formula $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\dots,z_n) \in L$ such that for every finite set $B \subseteq \mathcal{U}^{|z|}$ of cardinality at least 2 and every $a \in \mathcal{U}^{|x|}$ there are some $b_1,\dots,b_n \in B$ such that $\psi(a\,;b_1,\dots,b_n)$ and

$$\psi(x\,;b_1,\dots,b_n) \quad\rightarrow\quad \big[\varphi(a\,;b) \;\leftrightarrow\; \varphi(x\,;b)\big] \qquad \text{for all } b \in B.$$