Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Abstract samples

Let M be a structure of signature L. The sample-expansion of M is a 3-sorted expansion $\langle M, \omega, \bar{M}\rangle$, where $\bar{M} = M^{<\omega}$. These three sorts are called home-sort, integer-sort, and sample-sort, respectively. We put a bar over the symbols that denotes elements of sample-sort. The size of a sample $\bar{a} \in \bar{M}$ is the length of $\bar{a}$ as an element of $M^{<\omega}$. This is denoted by $\operatorname{lh}\bar{a}$. The same symbol $\bar{a}$ may be used for tuples of samples of the same size. The usual notation $|\bar{a}|$ is used the length of such tuples, for instance, if $\bar{a} \in \bar{M}^n$ then $|\bar{a}| = n$. We write $\bar{a}.i$ for the i-th element of the sample $\bar{a}$. Then $\bar{a}_i \in M^{|\bar{a}|}$. The language of this expansion is denoted by $\bar{L}$; it comprises

1. all symbols of L, these apply to the home-sort;
2. $0, 1, +, \cdot$ which apply to the integer-sort;
3. a ternary relation of sort $\bar{M}, \omega, M$ that holds if $\bar{x}.i = y$.
4. for every $\varphi(x) \in \bar{L}$, where here for clarity $|x| = 1$, there is relation symbol of sort $\bar{M}, \omega$ that holds for those $\bar{a}$ and $i < \operatorname{lh}\bar{a}$ such that $\varphi(\bar{a}.i)$;

When $\varphi(x)$ is the formula $x = x$, then 4 above implies that $\operatorname{lh}\bar{x}$ is definable. Also, from 4 it easily follows that the function $|\{i < \operatorname{lh}\bar{a} \,:\, \varphi(\bar{a}.i)\}|$ is definable.

As in the integer-sort we can interpret the field of rational numbers, below we will freely use rational numbers when this clarify the notation. For instance we define

$$\operatorname*{Fr}_{\bar{a}} \varphi(x) \quad = \quad \frac{|\{i < \operatorname{lh}\bar{a} \,:\, \varphi(\bar{a}.i)\}|}{\operatorname{lh}\bar{a}}$$

and use that it is a definable function in $\bar{L}$.

2.1 Definition *Let $\mathcal{U}$ be a monster model of inaccessible cardinality $\kappa > |L|$ and let $\langle \mathcal{U}, \omega, \bar{\mathcal{U}}\rangle$ be the corresponding sample-expansion. We denote by $\langle \mathcal{U}, \omega^*, \bar{\mathcal{U}}^*\rangle$ a saturated elementary extension of $\langle \mathcal{U}, \omega, \bar{\mathcal{U}}\rangle$ of cardinality κ. Note that we can assume that the home-sort of this extension is $\mathcal{U}$, in fact as all saturated models of cardinality κ are isomorphic. The elements of $\bar{\mathcal{U}}$ are called finite samples, those of $\bar{\mathcal{U}}^*$ are called internal samples.*

To any internal sample $\bar{a}$ we associate a finitely additive measure on the definable subsets of $\mathcal{U}$. Define

$$\Pr_{\bar{a}}\big[\varphi(\mathcal{U})\big] \quad = \quad \sup\Big\{r \in \mathbb{Q} \ : \ r < \operatorname*{Fr}_{\bar{a}} \varphi(x)\Big\}$$

2.2 Definition *Let x be a tuple of variables of the home-sort. An external sample is a global type $p(\bar{x}) \subseteq \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ such that for every formula $\varphi(x) \in \bar{L}$ the set $\{i \in \omega^* : p(\bar{x}) \vdash \varphi(\bar{x}.i)\}$ is definable and bounded.* □

For every external sample $p(\bar{x})$ there is an $n \in \omega^*$ such that $p(\bar{x}) \vdash \operatorname{lh} \bar{x} = n$ and the (nonstandard) cardinality of the set $\{i \in \omega^* : p(\bar{x}) \vdash \varphi(\bar{x}.i)\}$ is well-defined.

Therefore, to any external sample $p(\bar{x})$ we associate a finitely additive probability measure on the definable subsets of $\mathcal{U}$. Define

$$\Pr_{\bar{p}}\big[\varphi(\mathcal{U})\big] \quad = \quad \sup\left\{r \in \mathbb{Q} \ : \ r < \frac{\big|\big\{i < \operatorname{lh} \bar{x} \ : \ p(\bar{x}) \vdash \varphi(\bar{x}.i)\big\}\big|}{\operatorname{lh} \bar{x}}\right\}$$

2.3 Notation Let z be a tuple of variables of the home-sort. For any external sample $p(\bar{x})$, and every formula $\varphi(\bar{x}\,;z) \in \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$, and $b \in \mathcal{U}^{|z|}$ we write $\varphi(\bar{p}\,;b)$ for $p(\bar{x}) \vdash \varphi(\bar{x};b)$. We also write

$$\varphi(\bar{p}\,;\mathcal{U}) \quad = \quad \Big\{b \in \mathcal{U}^{|z|} \ : \ \varphi(\bar{p}\,;b)\Big\}$$

Sets of this form are called externally definable. This notation intentionally confuses $p(\bar{x})$ with any of its realizations (suggestively denoted by $\bar{p}$) in some elementary extension of $\bar{\mathcal{U}}$. □

The following notions are standard

2.4 Definition *Let z be a tuple of variables of the home-sort. Let M be given. An external sample $p(\bar{x})$ is*

1. *invariant if for every $\varphi(\bar{x};z) \in \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ the set $\varphi(\bar{p}\,;\mathcal{U})$ is invariant over M;*
2. *finitely satisfiable if for every formula $\varphi(\bar{x},\ z) \in \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ if $\varphi(\bar{p}\,;\mathcal{U}) \neq \varnothing$, then $\varphi(\bar{a}\,;\mathcal{U}) \neq \varnothing$ for some $a \in \bar{M}$;*
3. *definable over M if for every $\varphi(\bar{x};z) \in \bar{L}(\mathcal{U}, \omega^*, \bar{\mathcal{U}}^*)$ the set $\varphi(p\,;\mathcal{U})$ is definable over $\bar{M}$.* □

2.5 Fact *If an external sample $p(\bar{x})$ is finitely satisfiable in M, then for every $\varepsilon > 0$ there is an $\bar{a} \in \bar{M}$ such that*

$$\Pr_{\bar{p}}\big[\varphi(\mathcal{U})\big] \quad \approx_\varepsilon \quad \operatorname*{Fr}_{\bar{a}} \varphi(x)$$

As usual, finite satisfiability implies invariance.

2.6 Fact *Let M be given. Let $p(\bar{x})$ be a finitely satisfiable external sample. Then p is invariant.* □

2.7 Definition *Let M be given. A external sample $p(\bar{x})$ is uniformly satisfiable if for every*

$\varphi(\bar{x};z) \in \bar{L}(\bar{M})$ *there is an* $r \subseteq M^{\mathrm{s}}$ *such that* $\varphi(p\,;b) \leftrightarrow \varphi(r\,;b)$ *for every* $b \in \mathcal{U}^{|z|}$. $\square$

Note that uniform satisfiability implies definability.

2.8 Definition *Let* M *be given. An external sample is* *smooth* *if it is invariant and realized in* $\bar{\mathcal{U}}^*$ *(by some internal sample).* $\square$

2.9 Lemma *Let* M *be given. Then every smooth external sample* $p(\bar{x})$ *is uniformly satisfiable.*

Proof. Let $s \in \bar{\mathcal{U}}^{\mathrm{s}*}$ be an internal sample that realizes $p(\bar{x})$. Let $\varphi(\bar{x};z) \in L$ be given. Firstly, note that $\varphi(p\,;\mathcal{U}) = \varphi(s\,;\mathcal{U})$ is definable in $\bar{\mathcal{U}}^*$ hence, by invariance, it is definable in $\bar{M}$.

Secondly, we prove that $p(\bar{x})$ is finitely satisfiable. Let $b \in \mathcal{U}^{|z|}$ be given and pick $b' \equiv_{\bar{M}} b$ be such that $s \mathop{\smile\!\!\!\!\!\!\downarrow}_{\bar{M}} b'$. By invariance, $\varphi(s\,;b') \leftrightarrow \varphi(s\,;b)$. Therefore there is an $r \in M^{\mathrm{s}}$ such that $\varphi(r\,;b') \leftrightarrow \varphi(s\,;b)$. This proves finite satisfiability. By finite satisfiability, the following type is inconsistent

$$q(z) \quad = \quad \Big\{\varphi(s\,;z) \not\leftrightarrow \varphi(r\,;z) \;:\; r \in M^{\mathrm{s}}\Big\}$$

Hence compatness yields some finitely many $r_i \subseteq M^{\mathrm{s}}$ such that

$$\forall z \bigvee_{i=1}^{n} \big[\varphi(s\,;z) \leftrightarrow \varphi(r_i\,;z)\big]$$

By the definability of $p(\bar{x})$, which we proved in 1, there are some formulas $\vartheta_i(z) \in L(\bar{M})$ equivalent to $\varphi(s\,;z) \leftrightarrow \varphi(r_i\,;z)$. Hence we have

$$\forall z \bigwedge_{i=1}^{n} \Big[\vartheta_i(z) \leftrightarrow \big[\varphi(s\,;z) \leftrightarrow \varphi(r_i\,;z)\big]\Big]$$

By elementarity, there is an $r \in M^{\mathrm{s}}$ such that the formula above holds with s replaced by r. Therefore $\forall z\,\big[\varphi(s\,;z) \leftrightarrow \varphi(r\,;z)\big]$. Hence r is as required by Definition 2.7. $\square$

Let $p(\bar{x})$ be a external sample. The support of $p(\bar{x})$ is the set of those $a \in \mathcal{U}^{|x|}$ such that $p(\bar{x}) \vdash \bar{x}(a) \neq 0$.

2.10 Lemma *Let* M *be given. Let* $p(\bar{x})$ *be a uniformly satisfiable external sample. Then the support of* $p(\bar{x})$ *is a finite subset of* M.

Proof. Apply Definition 2.7 to the formula $\bar{x}(z) \neq 0$. Then there is an $r \in M^{\mathrm{s}}$ such that for every $b \in \mathcal{U}^{|z|}$.

$$p(\bar{x}) \quad \vdash \quad \bar{x}(b) \neq 0 \rightarrow r(b) \neq 0.$$

As r has finite support, the lemma follows. $\square$

2.11 Fact *Let* M *be given. Let* $p(\bar{x})$ *be a global type with a finite support enumerated by the tuple* a. *Then there is type* $q(y)$, *where* y *is a tuple of variables of real-sort of length* $|a|$, *with parameters in* $\mathcal{U}, \bar{M}$ *and no variable of sample-sort such that*

$$p(\bar{x}) \quad \leftrightarrow \quad \exists y \left[q(y) \;\cup\; \{\bar{x}(a) = y\}\right]$$

Proof. □

2.12 Lemma *Let M be given. Let $p(\bar{x})$ be a global type, then the following are equivalent*

1. *$p(\bar{x})$ is smooth;*
2. *$p(\bar{x})$ is uniformly satisfiable.*

Proof. Implication 1⇒2 has been proved in Lemma **??**. Here we prove 2⇒1. By Lemma 2.10, $p(\bar{x})$ has a finite support $A \subseteq M$. Now apply Fact 2.11. As the type $q(y)$ is relized in $\bar{\mathcal{U}}^*$, so is $p(\bar{x})$. Invariance is clear by finite satisfiability. □

3 The Erdős-Hajnal property

Let $\varphi(x\,;z) \in L$. We say that $\varphi(x\,;z)$ has the Erdős-Hajnal property if there is a constant $\delta \in (0,1)$, which depends soley on $\varphi(x\,;z)$, such that for every finite $A \subseteq \mathcal{U}^{|x|}$ and $B \subseteq \mathcal{U}^{|z|}$ there are $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $|A_0| > \delta|A|$, $|B_0| > \delta|B|$ and $A_0 \times B_0$ is either disjoint from or contained in $\varphi(\mathcal{U}\,;\mathcal{U})$.

3.1 Definition *We say that $\varphi(x\,;z) \in L$ is a distal formula if there are some formulas $\eta_i(z\,;\bar{z}) \in L$, where $i = 0,1$ and $|\bar{z}| = n{\cdot}|z|$ for some n, such that for every $a \in \mathcal{U}^{|x|}$ and every finite set $B \subseteq \mathcal{U}^{|z|}$ there is a tuple $\bar{b} \in B^n$ such that $\varphi(a\,;B) = \eta_i(B\,;\bar{b})$ and $\eta_0(z\,;\bar{b}) \rightarrow \varphi(a\,;z) \rightarrow \eta_1(z,\bar{b})$.*

3.2 Theorem *Assume $\eta(x\,;z) \in L$ has* vc*-dimension $k < \infty$. Then for every $\varepsilon > 0$ and every finite $B \subseteq \mathcal{U}^{|z|}$ there is a $B_0 \subseteq B$ of cardinality $< (k/\varepsilon^2)\ln(k/\varepsilon)$.*

3.3 Theorem (Chernikov-Starchenko) *Assume T is nip. Then every distal formula has the Erdős-Hajnal property.*

Proof. Let $\psi(x\,;\bar{z})$ be the formula given by distality. Write $\vartheta(z\,;\bar{z})$ for the formula

$$\forall x\; \left[\psi(x\,;\bar{z}) \rightarrow \varphi(x\,;z)\right] \;\vee\; \forall x\; \left[\psi(x\,;\bar{z}) \rightarrow \neg\varphi(x\,;z)\right].$$

Let $s \in \mathcal{U}^s$ be the indicator function of A. Let $\bar{b} \in \mathcal{U}^{|\bar{z}|}$ be such that $\psi(s\,;\bar{b}) > \varepsilon$. □

3.4 Proposition *Let $\varphi(x\,;z) \in L$ be given. Below, we write $\bar{z}$ for the tuple of variables $z_1, \ldots, z_n$, where $|z_i| = |z|$. The following are equivalent*

1. *there is a formula $\psi(x\,;\bar{z}) \in L$ such that for every finite set $B \subseteq \mathcal{U}^{|z|}$ and every $a \in \mathcal{U}^{|x|}$ there is a $\bar{b} \in B^n$ such that $\psi(a\,;\bar{b})$ and $\psi(x\,;\bar{b})$ decides all formulas $\varphi(x\,;b)$ for $b \in B$;*
2. *there are two formulas $\eta_i(z\,;\bar{z})$, where $i = 0,1$, such that for every $a \in \mathcal{U}^{|x|}$ there is a $\bar{b} \in B^n$ such that $\varphi(a\,;B) = \eta_i(B\,;\bar{b})$ and $\eta_0(z\,;\bar{b}) \rightarrow \varphi(a\,;z) \rightarrow \eta_1(z,\bar{b})$.*

4 Recycle bin

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

4.1 Lemma *Let M be given. Let $p(x^{\mathrm{s}})$ be smooth. Then for every ε there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\mathrm{Fr}\,\vartheta_i(p) \approx_\varepsilon \mathrm{Fr}\,\vartheta_j(p)$ for some i, j.*

Proof. □

Then for every $b \in \mathcal{U}^{|z|}$ there is an $r \in R$

$$p(x^{\mathrm{s}}) \quad \vdash \quad x^{\mathrm{s}}(b) > 0 \leftrightarrow r(b) > 0 \text{ such that}$$

As Write $\mathrm{supp}(R)$ for the union of $\mathrm{supp}(r)$ for $r \in R$. Then $p \vdash \mathrm{supp}(x^{\mathrm{s}}) \subseteq \mathrm{supp}(R)$. In particular $p \vdash \mathrm{supp}(x^{\mathrm{s}}) = A$ for some finite $A \subseteq M$.

$\varphi(p\,;z) \leftrightarrow r(z) > 0$

Let $p(x^{\mathrm{s}})$ be external sample that is uniformly finitely satisfiable w.r.t. all $\varphi(x^{\mathrm{s}}\,;z) \in L$. Then there is an $r \in M^{\mathrm{s}}$ such that

$x^{\mathrm{s}}(z_i) \approx_\varepsilon \mu_i$

4.2 Fact *Let M and $\varphi(x\,;z) \in L$ be given. Let $p(x^{\mathrm{s}})$ be a definable, uniformly finitely satisfiable external sample. Then there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\bar{\vartheta}_i(p) \approx_\varepsilon \bar{\vartheta}_j(p)$ for some i, j.*

Proof. Let $\psi_r(z)$, for $r \in R \subseteq M^{\mathrm{s}}$

Let

Pick finite $F \subseteq \mathbb{R}$ such that for every b we have $\bar{\varphi}(p\,;b) \approx_\varepsilon \mu$ for some $\mu \in F$. By Definition **??** there is a finite set $R \subseteq M^s$ such that for every $b \in \mathcal{U}^{|z|}$ and every $\mu \in F$ we have $\varphi(p\,;b) \approx_\varepsilon \mu \;\leftrightarrow\; \varphi(r\,;b) \approx_\varepsilon \mu$ for some $r \in R$. Therefore for every $b \in \mathcal{U}^{|z|}$ we have that $\varphi(p\,;b) \approx_{2\varepsilon} \varphi(r\,;b)$ for some $r \in R$. Let $A \subseteq \mathcal{U}^{|z|}$ be a finite set containing $\mathrm{supp}(r)$ for all $r \in R$. Let $\vartheta_i(x)$ enumerate the formulas $x \in B$ and $x \notin B$, as B ranges over the subsets of A.

We prove that these formulas are as required by the proposition. Fix some $b \in \mathcal{U}^{|z|}$ and let $r \in R$ be such that $\varphi(p\,;b) \approx_{2\varepsilon} \varphi(r\,;b)$. Let

$$B_0 \quad = \quad \{a \in \mathrm{supp}(r) : \varphi(a\,;b)\}$$

$$B_1 \quad = \quad \{a \in \mathrm{supp}(r) : \neg\varphi(a\,;b)\}$$

Clearly, $x \in B_0 \to \varphi(x\,;b) \to x \notin B_1$. □

4.3 Definition *Let $\varphi(x^{\mathrm{s}}\,;z) \in L$ and M be given. We say that $p(x^{\mathrm{s}})$ is weakly approximable if there are some finitely many $r_i \in M^{\mathrm{s}}$, say $0 \leq i < n$, such that for every $b \in \mathcal{U}^{|z|}$ we have $\varphi(p\,;b) \leftrightarrow \varphi(r_i\,;b)$ for some i.* □

Let $\varphi(x\,;z) \in L$ and M be given. Let $\mathcal{B} = \varphi(p;\mathcal{U})$. Let $p(x^{\rm s})$ be weakly approximable. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1, \dots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ are consistent.

4.4 Fact *Let M be given. Let $p(x^{\rm s})$ be external sample. Suppose for every $\varphi(x) \in L(\mathcal{U})$ and ε there are $\vartheta_1(x), \vartheta_2(x) \in L(M)$ such that $\vartheta_1(x) \to \varphi(x) \to \vartheta_2(x)$ and $\bar{\vartheta}_2(p) \approx_\varepsilon \bar{\vartheta}_1(p)$. Then p is finitely satifiable.*

Proof. Let $\varphi_i(x)$, for $0 \leq i < n$, be given. We claim that there is an $r \in \bar{M}$ such that $\bar{\varphi}_i(p) \approx_{3\varepsilon} \bar{\varphi}_i(r)$ for every i.

Let $\vartheta_{i,1}(x)$ and $\vartheta_{i,2}(x)$ be as in the fact. As $\vartheta_{i,1}(p) \leq \varphi_i(p) \leq \vartheta_{i,2}(p)$, then $\vartheta_{i,1}(p) \approx_\varepsilon \varphi_i(p)$. By elementarity there is an $r \in \bar{M}$ such that $\vartheta_{i,1}(r) \approx_\varepsilon \vartheta_{i,1}(p)$ and $\vartheta_{i,2}(r) \approx_\varepsilon \vartheta_{i,2}(p)$. Then the claim follows immediately. □

4.5 Lemma *The following are equivalent for every external sample $p(x^{\rm s})$*

1. *p is smooth and invariant;*
2. *for every finite $B \subseteq \mathcal{U}^{|z|}$ and every ε, there is an $r \in \bar{M}$ such that $\bar{\varphi}(p\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$ for all $b \in B$;*
2. *for every $\varphi_i(x\,;z) \in L$, every $b_i \in \mathcal{U}^{|z|}$, and every ε, there is an $r \in \bar{M}$ such that $\bar{\varphi}_i(p\,;b_i) \approx_\varepsilon \bar{\varphi}_i(r\,;b_i)$, for every $0 \leq i < n$;*
2. *for every $\psi_0(x), \dots, \psi_{n-1}(x) \in L(\mathcal{U})$ and every ε, there is an $r \in \bar{M}$ such that $\bar{\psi}_i(p) \approx_\varepsilon \bar{\psi}_i(r)$, for every $i < n$;*
4. *there are finitely many formulas $\vartheta_i(x) \in L(M)$ such that for every $b \in \mathcal{U}^{|z|}$ we have that $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ and $\bar{\vartheta}_j(p) - \bar{\vartheta}_i(p) < \varepsilon$ for some i, j.*

Proof. Let s be an internal sample that realizes $p(x^{\rm s})$.

1⇒2. Let B and ε be given. Let $\bar{b} = \langle b_i : i < n \rangle$ be an enumeration of B. Let $\mu_i \in \mathbb{R}$ be such that $\bar{\varphi}(p\,;b_i) \approx \mu_i$ for every $i < n$. Let $\bar{b}'$ be such that $s \mathop{\perp}_{\bar{M}} \bar{b}' \equiv_{\bar{M}} \bar{b}$. By invariance, $\bar{\varphi}(s\,;b'_i) \approx \mu_i$. Therefore there is an $r \in \bar{M}$ such that $\bar{\varphi}(r\,;b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{\bar{M}} \bar{b}$, we obtain $\bar{\varphi}(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒2. Let $\psi_i(x) = \varphi_i(x\,;b_i)$ for some $\varphi_i(x\,;z) \in L$ and $b_i \in \mathcal{U}^{|z|}$. Let $\bar{b} = \langle b_i : i < n \rangle$. Let $\mu_i \in \mathbb{R}$ be such that $\bar{\varphi}_i(p\,;b_i) \approx \mu_i$. Let $\bar{b}'$ be such that $s \mathop{\perp}_{\bar{M}} \bar{b}' \equiv_{\bar{M}} \bar{b}$. By invariance, $\bar{\varphi}_i(s\,;b'_i) \approx \mu_i$. Therefore there is an $r \in \bar{M}$ such that $\bar{\varphi}_i(r\,;b'_i) \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{\bar{M}} \bar{b}$, we obtain $\bar{\varphi}_i(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒3. By smoothness we it suffices to prove 3 with s for p. The type

$$p(z) \quad = \quad \Big\{ \bar{\varphi}(s\,;z) \not\approx_\varepsilon \bar{\varphi}(r\,;z) \ : \ r \in \bar{M} \Big\}$$

is inconsistent by 2. Hence compatness yields the required $r_1, \dots, r_n \in \bar{M}$.

3⇒2. Clear.

2⇒1 Assume 2. We prove that p is invariant. Pick some $b \equiv_{\bar{M}} b'$ and some ε. Let

$r, r' \in \bar{M}$ be such that $\bar{\varphi}(p\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$ and $\bar{\varphi}(p\,;b') \not\approx_\varepsilon \bar{\varphi}(r';b')$. As $\bar{\varphi}(r\,;b) \approx \bar{\varphi}(r';b')$ follows from $b \equiv_{\bar{M}} b'$, we obtain $\bar{\varphi}(p\,;b) \approx_\varepsilon \bar{\varphi}(p\,;b')$.

We prove that p is smooth.

$$p(x^{\mathrm{s}}) \quad = \quad \Big\{\bar{\varphi}(x^{\mathrm{s}};b) \approx_\varepsilon \mu_b \ : \ b \in \mathcal{U}^{|z|}\Big\}$$ □

4.6 Definition *Let $\varphi(x\,;z) \in L$ and M be given. A external sample $p(x^{\mathrm{s}})$ is*

1. *invariant if $\bar{\varphi}(p\,;b) \approx \bar{\varphi}(p\,;b')$ for every $b \equiv_{\bar{M}} b'$;*
2. *(pointwise) approximable if, for every $b \in \mathcal{U}^{|z|}$ and every ε, there is an $r \in \bar{M}$ such that $\bar{\varphi}(p\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$;*
3. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \bar{\varphi}(p\,;b) < \varepsilon\}$ is definable over $\bar{M}$.* □

The following is a useful characterization of smooth samples.

5 Old stuff/garbage

We write $\mathrm{supp}(p)$ for the support of p, that is, the set of those a in $\mathcal{U}$ such that $p \vdash (a \in x^{\mathrm{s}}) > \varepsilon$ for some standard positive ε. If s is a smooth sample $\mathrm{supp}(s)$ is defined to be $\mathrm{supp}(p)$ for $p(x^{\mathrm{s}}) = \mathrm{tp}(s/\bar{\mathcal{U}})$.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_{x/s}\varphi(x\,;b) \quad = \quad \frac{\big|\{a \in s \ : \ \varphi(a\,;b)\}\big|}{|s|},$$

When possible we abbreviate $\mathrm{Av}_{x/s}\varphi(x\,;b)$ with $\bar{\varphi}(s\,;b)$.

We write $\mathrm{ft}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

All definitions and facts in this section are relative to some given formula $\varphi(x\,;z) \in L$ and some model M.

5.1 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\bar{\varphi}(s';b) \approx \bar{\varphi}(s\,;b)$.* □

5.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\bar{\varphi}(s\,;b) \approx \bar{\varphi}(s\,;b')$;*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \bar{\varphi}(s\,;b) < \varepsilon\}$ is definable over M;*
3. *finitely satisfiable if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in \bar{M}$ such that $\bar{\varphi}(s\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$;*

4. *generically stable if all of the above hold.* □

Smooth roughly means internal and invariant.

5.3 Fact *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *there is an invariant internal sample $s' \equiv_M s$.*

Proof. 1⇒2. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathcal{U}^{|z|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \mathrm{Aut}(\bar{\mathcal{U}}^*/M)$. Then $\bar{\varphi}(s'; fb) \approx \bar{\varphi}(f^{-1}s'\,;b)$. Moreover, by smothness $\bar{\varphi}(f^{-1}s'\,;b) \approx \bar{\varphi}(s';b)$. The infariace in s' follows.

2⇒1. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ such that $\bar{\varphi}(s'\,;b) \not\approx_\varepsilon \bar{\varphi}(s\,;b)$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\bar{\varphi}(s''\,;b) \not\approx_\varepsilon \bar{\varphi}(s\,;b)$. As $s'' \equiv_M s$ are both in $\bar{\mathcal{U}}^*$, then $s'' = fs$ for some $f \in \mathrm{Aut}(\bar{\mathcal{U}}^*/M)$. Hence we obtan $\bar{\varphi}(s\,;f^{-1}b) \not\approx_\varepsilon \bar{\varphi}(s\,;b)$ which contradics the invariance of s. □

5.4 Definition *Let $\varphi(x\,;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s is (uniformly) approximable on $\mathcal{B}$ if for every ε there is a finite sample $r \in \bar{M}$ such that $\bar{\varphi}(s, b) \approx_\varepsilon \bar{\varphi}(r, b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε, not on b.)* □

5.5 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that $\bar{\varphi}(s\,;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1, \ldots, \mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ are consistent.*

Proof. Let $r \in \bar{M}$ be as in Definition 5.4. As r is finite, we may assume that $\mathrm{supp}(r) = \{a_1, \ldots, a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i\,;b)\}$. As $\bar{\varphi}(r\,;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

5.6 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega\rangle$, where $\langle b_i : i < \omega\rangle$ is a sequence of indiscernibles such that $\bar{\varphi}(s\,;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x\,;b_i) : i < \omega\}$ is consistent.* □

5.7 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{C}$. Then for every ε there is a pair of distict $c, c' \in \mathcal{C}$ such that $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] < \varepsilon$*

Proof. Suppose for a contradiction that $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] \geq \varepsilon$ for all distict $c, c' \in \mathcal{C}$. Apply Lemma 5.5 to the folmula $\psi(x\,;z,z') = [\varphi(x\,;z) \not\leftrightarrow \varphi(x\,;z')]$ and the set $\mathcal{B} = \{\langle c, c'\rangle \in \mathcal{C}^2 \;:\; c{\neq}c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 5.5 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x\,;a) \not\leftrightarrow \varphi(x\,;a') \;:\; a, a'{\in}A,\ a{\neq}a'\}$ is consistent. As $|A| > 2$, this is impossible. □

6 The nip formulas

6.1 Theorem *Let $\varphi(x\,;z) \in L$ and M be given and assume that $\varphi(x\,;z)$ is nip. Then every sample is approximable on M over M.*

Proof. Questo è Vapnik-Chervonenkis. □

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|z|}$.)

6.2 Corollary *Let $\varphi(x\,;z) \in L$ be nip. Every smooth sample s is approximable on $\mathcal{U}$ over M.*

Proof. By Theorem 6.1, there is an $r \in \bar{M}$ such that $\bar{\varphi}(s\,;b) \approx_\varepsilon \bar{\varphi}(r\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\bar{\varphi}(s\,;b') \not\approx_\varepsilon \bar{\varphi}(r\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \perp_M s' \equiv_M s$. By the smoothness of s, we obtain $\bar{\varphi}(s';b') \not\approx_\varepsilon \bar{\varphi}(r\,;b')$. As $b' \perp_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\bar{\varphi}(s\,;b) \not\approx_\varepsilon \bar{\varphi}(r\,;b)$. A contradiction. □

6.3 Definition ?? *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\dots z_n) \in L$ such that for every finite set $B \subseteq \mathcal{U}$ and every $a \in \mathcal{U}^{|x|}$ there are $b_1,\dots,b_n \in B$ such that $\psi(a\,;b_1,\dots,b_n)$ and $\psi(x\,;b_1,\dots,b_n)$ decides all formulas $\varphi(x\,;b)$ for $b \in B$.* □

6.4 Definition ??* *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x_1,\dots,x_n\,;z) \in L$ such that for every finite set $A \subseteq \mathcal{U}^{|x|}$ and every $b \in \mathcal{U}^{|z|}$ there are $a_1,\dots,a_n \in A$ such that $\psi(a_1,\dots,a_n,\ z)$ and $\psi(a_1,\dots,a_n,\ z)$ decides all formulas $\varphi(a\,;z)$ for $a \in A$.* □

For every finite sample $r \in \bar{M}$ and every $\mu \in \mathbb{R}$ there is a formula $\psi(z) \in L(\operatorname{supp} r)$ such that $\bar{\varphi}(r\,;z) = \mu \leftrightarrow \psi(z)$. The formula $\psi(z)$ depends on r.

When $\varphi(x\,;z)$ is distal

If $\varphi(x\,;z)$ is distal then there is a formula ψ such that for every $r \in M^f$ there is $r_0 \in M$

6.5 Theorem (false) *The following are equivalent*

1. $\varphi(x\,;z) \in L$ *is distal;*
2. *for every sample $s \in \mathcal{U}^{f_*^+}$, if s is generically stable then s is smooth.*

There is a formula $\psi(x_1,\dots,x_n\,;z)$ such that for every $r \in \bar{M}$

there are $a_1,\dots,a_n$ such that $\varphi(r\,;b) = \mu\ \psi(x_1,\dots,x_n\,;z)$

We say that $p(x)$ is honestly definable if for every $\varphi(x,y) \in L$ there is a formula $\psi(y,z) \in L$ such that for every finite A, there is a b such that

$$A \subseteq \psi(\mathfrak{U}, b) \subseteq \varphi(p, \mathfrak{U})$$

For $\varphi(x, y) \in L$ and $m \in M$ let

$$\varphi_m(x, y) \quad = \quad \varphi(m, y) \leftrightarrow \varphi(x, y)$$

$$A \subseteq \psi_{m,\varphi}(\mathcal{U}) \subseteq \varphi_m(p, \mathfrak{U})$$

If $p(x)$ is generically stable (definition as in the OP) then for every $\varphi(x, y) \in L$ and $m \in M$ there is a formula $\vartheta_{\varphi,m}(y)$ such that for every a in the monster model

$$\vartheta_{\varphi,m}(a) \quad \Leftrightarrow \quad [\varphi(m, a) \leftrightarrow \varphi(x, a)] \in p$$

Let $q(x)$ be an M-invariant global type such that $q_{\restriction M}(x) = p_{\restriction M}(x)$. Then both q and p contain the formulas

$$\forall y \left[\vartheta_{\varphi,m}(y) \to \left[\varphi(m, y) \leftrightarrow \varphi(x, y)\right]\right]$$

We write $p(x) \sim_M q(x)$ if

$\varphi(p\,; \mathfrak{U}) \neq \varnothing \Leftrightarrow \varphi(q\,; \mathfrak{U}) \neq \varnothing$

for every $\varphi(x, z) \in L(M)$