

Scratch paper

Anonymous
Università di Torino
May 2020

Abstract Poche idee, ben confuse.

(Branch: Szemeredi)

1 Type-definable functions

Let \mathcal{U} be a monster model. Let X be a Hausdorff space. We say that the partial map $f : \mathcal{U}^{[x]} \rightarrow X$ is **bounded** if $\text{img } f \subseteq C$ for some compact $C \subseteq X$. We say that f is **type-definable** if it is bounded and $f^{-1}[C]$ is type-definable for every compact $C \subseteq X$. If $f^{-1}[C]$ is definable for every compact $C \subseteq X$, then we say that f is **definable**.

We always assume \mathcal{U} has a larger cardinality than the topology of X and that a type-definable function is always type-definable over some small set of parameters, i.e. all the types defining $f^{-1}[C]$ are over the same small set parameters $A \subseteq \mathcal{U}$.

1.1 Fact Let $f : \mathcal{U}^{[x]} \rightarrow X$ be type-definable. Then the following are equivalent

1. f is definable;
2. $\text{img } f$ is finite.

Proof By compactness. □

1.2 Fact Let $p_i(x) \subseteq L(A)$, for $i = 1, \dots, n$, be a cover of a type-definable set $\mathcal{D} \subseteq \mathcal{U}^{[x]}$. Then there is a cover of \mathcal{D} by some formulas $p_i(x) \vdash \varphi_i(x)$.

Proof By compactness. □

1.3 Fact Let $f : \mathcal{U}^{[x]} \rightarrow [0, 1]$ be type-definable. Then for every $\varepsilon > 0$ there is a definable function $g : \mathcal{U}^{[x]} \rightarrow [0, 1]$ such that

$$|fa - ga| \leq \varepsilon \quad \text{for every } a \in \text{dom } f$$

Proof Pick $r_1, \dots, r_n \in [0, 1]$ be such that the intervals $[r_i - \varepsilon, r_i + \varepsilon]$ cover $\text{img } f$. Hence, the types defining $f^{-1}[r_i - \varepsilon, r_i + \varepsilon]$ cover $\text{dom } f$. Therefore, by the fact above, there is a definable partition of $\text{dom } f$ that refines $f^{-1}[r_i - \varepsilon, r_i + \varepsilon]$. Define g to be such that $\{g^{-1}[r_i] : i \in [n]\}$ is the partition above. Then g is as required. □

1.4 Fact Let $f : \mathcal{U}^{[x]} \rightarrow [0, 1]$ be type-definable. Then for every $\varepsilon > 0$ there are finitely many $a_1, \dots, a_n \in \text{dom } f$ such that

$$\bigvee_{i=1}^n |fa - fa_i| \leq \varepsilon \quad \text{for every } a \in \text{dom } f.$$

Moreover, there are some formulas $\psi_i(x) \in L(\mathcal{U})$ that cover $\text{dom } f$ and

$$\psi_i(x) \rightarrow |fx - fa_i| \leq \varepsilon \quad \text{for every } i \in [n].$$

Proof By Fact 1.3 there is a definable function g such that $|fa - ga| \leq \varepsilon/2$ for every $a \in \text{dom } f$. Pick some $a_1, \dots, a_n \in \mathcal{U}^{[x]}$ such that $\{ga_1, \dots, ga_n\} = \text{img } g$. Then

$$\begin{aligned} \min_{i \in [n]} |fa - fa_i| &= \min_{i \in [n]} |fa - ga + ga - fa_i| \\ &\leq |fa - ga| + \min_{i \in [n]} |ga_i - ga| \\ &\leq |fa - ga| + |fa_i - ga_i| \quad \text{where } i \text{ is such that } ga_i = ga \\ &\leq \varepsilon \end{aligned}$$

The second claim follows from Fact 1.2 because $|fx - fa_i| \leq \varepsilon$ is equivalent to a type. \square

2 Externally definable sets (notation)

3 Abstract samples

Let M be a structure. The **finite (fractional) sample expansion** of M is a 3-sorted expansion $\bar{M} = \langle M, \mathbb{R}, M^s \rangle$ that has a domain for M , which we call the home-sort, a domain for \mathbb{R} , as an ordered field, and a domain M^s , which we call **sample-sort**, which is an \mathbb{R} -vector space that we describe below. Variables of sample-sort (or tuple thereof) are denoted with symbols x^s, y^s, \dots , and variations. Elements of sample-sort are denoted with the symbols s, r, t , and variations. The variable x is reserved for (tuples of) variables of the home-sort. The context will clarify the sort the other variables.

An element $s \in M^s$ is a map $s : M \rightarrow \mathbb{R}$ which is almost always zero. These functions, which we call **samples**, are interpreted as (weighted) samples or as (signed) finite measures (up to normalization) that are concentrated on the support of s .

The language of the expansion \bar{M} is denoted by L .

For every formula $\varphi(x; y) \in L$, where y is a tuple a mixed sort, there is a symbol $\Sigma\varphi(x^s; y)$ for a function $(M^s)^{|x|} \times M^y \rightarrow \mathbb{R}$. For $\varphi(x) \in L(\bar{M})$, we interpret $\Sigma\varphi(s)$ as the function that maps $a^s \in M^s$ to

$$\Sigma\varphi(s) = \sum_{a \models \varphi(x)} s(x)$$

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals,

we write $a \approx_\varepsilon b$ for $|a - b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε .

3.1 Definition Let \mathcal{U} be a monster model of cardinality $\kappa > |L|$. Let $\bar{\mathcal{U}} = \langle \mathcal{U}, \mathbb{R}, \mathcal{U}^s \rangle$ be as above and let $\bar{\mathcal{U}}^*$ be a saturated elementary extension of $\bar{\mathcal{U}}$ of cardinality κ . As all saturated models of cardinality κ are isomorphic, we can assume that \mathcal{U} is the domain of the home-sort of $\bar{\mathcal{U}}^*$, hence we write $\bar{\mathcal{U}}^* = \langle \mathcal{U}, \mathbb{R}^*, \mathcal{U}^{s*} \rangle$. Elements of \mathcal{U}^s are called *finite samples*, elements of \mathcal{U}^{s*} are called *internal samples*.

Finally, an *external sample* is a maximally consistent set of formulas $p(x^s) \subseteq L(\mathcal{U}, \mathbb{R}, \mathcal{U}^{s*})$. (Note these are not types over the full of $\bar{\mathcal{U}}$.) \square

3.2 Notation Let y and z be tuples of mixed sort. For any type $p(y)$ and formula $\varphi(y, z) \in L(\bar{\mathcal{U}}^*)$ we write

$$\varphi(p; \bar{\mathcal{U}}^*) = \left\{ a \in (\bar{\mathcal{U}}^*)^z : p(y) \vdash \varphi(y; a) \right\}$$

This notation intentionally confuses $p(y)$ with any of its realizations in some elementary extension of $\bar{\mathcal{U}}$. \square

3.3 Definition Let M be given. Let y be a tuple of the home-sort. A external sample $p(x^s)$ is

1. *invariant* if $\varphi(p; \mathcal{U}, \mathcal{U}^{s*})$ is invariant over \bar{M} , for every $\varphi(x; y, z^s) \in L$.
2. *finitely satisfiable* if every formula in $p(x^s)$ is satisfied by some element of M^s .
3. *definable* if $\varphi(p; \mathcal{U}, \mathcal{U}^{s*})$ is definable over \bar{M} , for every $\varphi(x; y, z^s) \in L$. \square

An external type is *generically stable* if it is definable and finitely satisfiable in some \bar{M} . \square

Clearly every finitely satisfiable external sample is invariant.

We will use the symbol $\perp_{\bar{M}}$ with the usual meaning.

4 Szemerédi's regularity lemma

Define

$$\begin{aligned} |\varphi(a^s; b^s)| &= \sum_{\varphi(x; y)} a^s(x) \cdot b^s(y) \\ d_\varphi(r; s) &= \frac{|\varphi(r; s)|}{|r \times s|} \end{aligned}$$

We say that the pair of samples r, s is ε -regular if for every $r' \subseteq r$ and $s' \subseteq s$ such that $|r'| > \varepsilon|r|$ and $|s'| > \varepsilon|s|$

$$|d_\varphi(r; s) - d_\varphi(r'; s')| < \varepsilon$$

4.1 Szemerédi's Regularity Lemma For every $\varepsilon > 0$ and for every pair of samples $u, v \in \bar{\mathcal{U}}^*$ of hyperfinite cardinality there are some finite partitions of u and v , say r_1, \dots, r_n and

s_1, \dots, s_m , such that

1. r_i, s_j is ε -regular for every $i, j \in [n] \times [m] \setminus X$,

where $X \subseteq [n] \times [m]$, called the exceptional set, is such that

2.
$$\sum_{(i,j) \in X} |r_i| \cdot |s_j| < \varepsilon \cdot |u| \cdot |v|$$

Proof Let $\text{st}((d(u; v))$

□

5 Szemerédi's regularity lemma 2