Scratch paper

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Abstract Poche idee, ben confuse.

1 Introduction

2 Abstract samples

Let M be a structure of signature L. The finite (fractional) sample expansion of M is a many-sorted expansion that has a sort for M, which we call the home-sort, a sort for $\mathbb{R}$, as ordered field, and a sort, called the sample-sort, for the set of functions $f : M \to \mathbb{R}$ that are almot always 0. We write $M^{\rm f} = \langle M, \mathbb{R}, F\rangle$ for the expansion above.

Variables of sample-sort are denoted with symbols $x^{\rm s}, y^{\rm s}$ and variations thereof. The language of $M^{\rm f}$ is denoted by $L^{\rm f}$. It expands the laguage of each sort, that is, L, the language of orders, the language of real vector spaces. Moreover, for every $\varphi(x\,; z, z^{\rm s}) \in L^{\rm f}$ there is a symbol for a function

$$\overline{\varphi}(x^{\rm s}\,; z, z^{\rm s}) \quad : \quad F^{|x|} \times M^{|z|} \times F^{|z^{\rm s}|} \quad \longrightarrow \quad \mathbb{R}.$$

Below, ε always ranges over the positive standard reals. If a and b are (hyper)reals, we write $a \approx_\varepsilon b$ for $|a-b| < \varepsilon$. We write $a \approx b$ if $a \approx_\varepsilon b$ holds for every ε.

2.1 Definition *Let $\mathcal{U}$ be a monster model of cardinality $\kappa > |L|$ and let $\mathcal{U}^{\rm f} = \langle \mathcal{U}, \mathbb{R}, \mathcal{F}\rangle$ be the corresponding expansion. We denote by $\mathcal{U}^{\rm f*}$ a saturated elementary extension of $\mathcal{U}^{\rm f}$ of cardinality κ. As all saturated models of cardinality κ are isomorphic, we can assume that $\mathcal{U}$ is the home-sort of $\mathcal{U}^{\rm f*}$. Elements $r \in \mathcal{U}^{\rm f}$ are called finite samples, elements in $s \in \mathcal{U}^{\rm f*}$ are called internal samples.*

Finally, a global sample is a type $p(x^{\rm s}) \in S(\mathcal{U}^{\rm f})$. □

Note that $\mathcal{U}^{\rm f}$ is not saturated but it is homogeneous.

2.2 Definition *A global sample $p(x^{\rm s})$ is smooth if it is realized in $\mathcal{U}^{\rm f*}$.* □

The following notions are standard

2.3 Definition *Let M be given. A global sample $p(x^{\rm s})$ is*

1. *invariant if for every formula $\varphi(x^{\rm s}\,; z, z^{\rm s})$ the set $\varphi(p\,; \mathcal{U}, \mathcal{F})$ is invariant over $M^{\rm f}$;*

2. *finitely satisfiable if for every finite subset of $p(x^{\rm s})$ is realized in $M^{\rm f}$.*

3. *definable if, for every $\psi(x^{\rm s}\,;z,z^{\rm s})\in L^{\rm f}$, the set $\psi(p\,;\mathcal{U},\mathcal{F})$ is definable over $M^{\rm f}$.* $\square$

2.4 Notation For any global type $p(x^{\rm s})\in S(\mathcal{U})$ and formula $\psi(x^{\rm s}\,;z)\in L^{\rm f}$ we write

$$\psi(p\,;\mathcal{U}) \;=\; \Big\{b\in\mathcal{U}^{|z|}\;:\;\psi(x^{\rm s}\,;b)\in p\Big\}$$

(the same notation applies with x for $x^{\rm s}$). This notation intentionally confuses p with any of its realizations in some elementary extension of $\mathcal{U}^{\rm f}$. Sets of this form are called externally definable. The type $p(x^{\rm s})$ may be identified with a collection of externally definable sets.

We extend this notation to functions. If $\varphi(x\,;z)\in L$ and $p(x^{\rm s})$ is a global sample, we write $\overline{\varphi}(p\,;b)$ for the unique $\mu\in\mathbb{R}$ such that $p(x^{\rm s})\vdash\bar{\varphi}(x^{\rm s}\,;b)\approx\mu$. In analogy to the terminology above, we may say that $\overline{\varphi}(p\,;z)$ is an externally definable real valued function. A global sample may be identified with a family of externally definable functions. $\square$

Explicitely, 2 above means that if $\bar{\psi}_i(x^{\rm s})\approx_\varepsilon\mu_i$ are some finitely many formulas in p, then there is an $r\in M^{\rm f}$ such that $\bar{\psi}_i(r)\approx_\varepsilon\mu_i$ for all i. We may also rephrase this by saying that for every finitely many $\psi_i(x)\in L(\mathcal{U})$ there is an $r\in M^{\rm f}$ such that $\bar{\psi}_i(r)\approx_\varepsilon\bar{\psi}_i(p)$ for all i.

2.5 Fact *Let M be given. Let $p(x^{\rm s})$ be a global sample. Then, if p is finitely satisfiable , it is is invariant.*

Proof Let $b\equiv_{M^{\rm f}}b'$ be given. Let $r\in M^{\rm f}$ be such that $\overline{\varphi}(p\,;b)\approx_\varepsilon\overline{\varphi}(r\,;b)$ and $\overline{\varphi}(p\,;b')\not\approx_\varepsilon\overline{\varphi}(r;b')$. But $\overline{\varphi}(r\,;b)\approx\overline{\varphi}(r;b')$ and therefore $\overline{\varphi}(p\,;b)\approx_\varepsilon\overline{\varphi}(p\,;b')$. As ε is arbitrary, $\overline{\varphi}(p\,;b)\approx\overline{\varphi}(p\,;b')$ follows. $\square$

2.6 Fact *Let M be given. Let $p(x^{\rm s})$ be a smooth sample. Then, if p is invariant, it is also finitely satisfiable and definable.*

Proof Let s be an internal sample that realizes an invariant global sample $p(x^{\rm s})$. Let $\overline{\varphi}_i(x^{\rm s}\,;b)\approx_\varepsilon\mu_i$ be some finitely many formulas in p. Let b' be such that $s\mathop{\smile\!\!\!\!\!\!\,|}\,_{M^{\rm f}}b'\equiv_{M^{\rm f}}b$. By invariance, $\overline{\varphi}_i(s\,;b')\approx\mu_i$. Therefore there is an $r\in M^{\rm f}$ such that $\overline{\varphi}_i(r\,;b')\approx_\varepsilon\mu_i$. As $b'\equiv_{M^{\rm f}}b$, we obtain $\overline{\varphi}_i(r\,;b)\approx_\varepsilon\mu_i$. This proves finite satisfiability.

Now, let $\psi(x^{\rm s}\,;z)\in L^{\rm f}$. As $\psi(p\,;\mathcal{U})=\psi(s\,;\mathcal{U})$ is definable in $\mathcal{U}^{\rm f*}$. By invariance it is definable in $M^{\rm f}$. $\square$

2.7 Fact *Let M be given. Let $p(x^{\rm s})$ be a smooth sample. Then for every $\varphi(x\,;z)\in L$ and every ε there are some finitely many $r_i\in M^{\rm f}$, say $0\le i<n$, such that for every $b\in\mathcal{U}^{|z|}$ we have $\overline{\varphi}(p\,;b)\approx_\varepsilon\overline{\varphi}(r_i\,;b)$ for some i.*

Proof Let s be an internal sample that realizes $p(x^{\rm s})$. Let $\varphi(x\,;z)\in L$ and ε be fixed. The type

$$p(z) \quad = \quad \Big\{\overline{\varphi}(s\,;z) \not\approx_\varepsilon \overline{\varphi}(r\,;z) \ : \ r \in M^{\rm f}\Big\}$$

is inconsistent by Fact 2.6. Hence compatness yields the required $r_i \in M^{\rm f}$. □

2.8 Fact *Let M be given. For every global sample $p(x^{\rm s})$ the following are equivalent*

1. *p is smooth;*
2. *p is definable and for every $\varphi(x\,;z) \in L$ and every ε there are some finitely many $r_i \in M^{\rm f}$, say $0 \leq i < n$, such that for every $b \in \mathcal{U}^{|z|}$ we have $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(r_i\,;b)$ for some i.*
3. *p is definable and for every $\varphi(x) \in L(\mathcal{U})$ and every ε there is an $r \in M^{\rm f}$ such that $\overline{\varphi}(p) \approx_\varepsilon \overline{\varphi}(r)$.*

Proof 1⇒2 definability has been proved above. Let s be an internal sample that realizes $p(x^{\rm s})$. Let $\varphi(x\,;z) \in L$ and ε be fixed. The type

$$p(z) \quad = \quad \Big\{\overline{\varphi}(s\,;z) \not\approx_\varepsilon \overline{\varphi}(r\,;z) \ : \ r \in M^{\rm f}\Big\}$$

is inconsistent by Fact 2.6. Hence compatness yields the required $r_i \in M^{\rm f}$.

2⇒1 For every $r \in M^{\rm f}$ and $\varphi(x\,;z) \in L$ let $\vartheta_{r,\varphi}(z) \in$ the formula defining the set $\{b : \overline{\varphi}(p\,;b) \not\approx_\varepsilon \overline{\varphi}(r\,;b)\}$. By assumption, the following type is finitely consistent

$$p(x^{\rm s}) \quad = \quad \Big\{\forall z\Big[\vartheta_{r,\varphi}(z) \rightarrow \overline{\varphi}(x^{\rm s}\,;z) \not\approx_\varepsilon \overline{\varphi}(r\,;z)\Big] \ : \ r \in M^{\rm f}\Big\}$$ □

3 Less abstract samples

In this section we assume that for every $\varphi(x), \psi(x) \in L(\mathcal{U})$ if $\varphi(x) \rightarrow \psi(x)$ then $\overline{\varphi}(x^{\rm s}) \leq \overline{\psi}(x^{\rm s})$.

3.1 Fact *Let M be given. Let $p(x^{\rm s})$ be global sample. Suppose for every $\varphi(x) \in L(\mathcal{U})$ and ε there are $\vartheta_1(x), \vartheta_2(x) \in L(M)$ such that $\vartheta_1(x) \rightarrow \varphi(x) \rightarrow \vartheta_2(x)$ and $\overline{\vartheta}_2(p) \approx_\varepsilon \overline{\vartheta}_1(p)$. Then p is finitely satifiable.*

Proof Let $\varphi_i(x)$, for $0 \leq i < n$, be given. We claim that there is an $r \in M^{\rm f}$ such that $\overline{\varphi}_i(p) \approx_{3\varepsilon} \overline{\varphi}_i(r)$ for every i.

Let $\vartheta_{i,1}(x)$ and $\vartheta_{i,2}(x)$ be as in the fact. As $\vartheta_{i,1}(p) \leq \varphi_i(p) \leq \vartheta_{i,2}(p)$, then $\vartheta_{i,1}(p) \approx_\varepsilon \varphi_i(p)$. By elementarity there is an $r \in M^{\rm f}$ such that $\vartheta_{i,1}(r) \approx_\varepsilon \vartheta_{i,1}(p)$ and $\vartheta_{i,2}(r) \approx_\varepsilon \vartheta_{i,2}(p)$. Then the claim follows immediately. □

3.2 Lemma *The following are equivalent for every global sample $p(x^{\rm s})$*

1. *p is smooth and invariant;*
2. *for every finite $B \subseteq \mathcal{U}^{|z|}$ and every ε, there is an $r \in M^{\rm f}$ such that $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ for all $b \in B$;*
2. *for every $\varphi_i(x\,;z) \in L$, every $b_i \in \mathcal{U}^{|z|}$, and every ε, there is an $r \in M^{\rm f}$ such that*

$\overline{\varphi}_i(p\,;b_i) \approx_\varepsilon \overline{\varphi}_i(r\,;b_i)$, *for every* $0 \leq i < n$;

2. *for every* $\psi_0(x), \dots, \psi_{n-1}(x) \in L(\mathcal{U})$ *and every* ε, *there is an* $r \in M^{\mathrm{f}}$ *such that* $\overline{\psi}_i(p) \approx_\varepsilon \overline{\psi}_i(r)$, *for every* $i < n$;

4. *there are finitely many formulas* $\vartheta_i(x) \in L(M)$ *such that for every* $b \in \mathcal{U}^{|z|}$ *we have that* $\vartheta_i(x) \to \varphi(x\,;b) \to \vartheta_j(x)$ *and* $\overline{\vartheta}_j(p) - \overline{\vartheta}_i(p) < \varepsilon$ *for some* i, j.

Proof Let s be an internal sample that realizes $p(x^{\mathrm{s}})$.

1⇒2. Let B and ε be given. Let $\bar{b} = \langle b_i : i < n \rangle$ be an enumeration of B. Let $\mu_i \in \mathbb{R}$ be such that $\overline{\varphi}(p\,;b_i) \approx \mu_i$ for every $i < n$. Let $\bar{b}'$ be such that $s \mathop{\smile\!\!\!\!\!\!|}_{M^{\mathrm{f}}} \bar{b}' \equiv_{M^{\mathrm{f}}} \bar{b}$. By invariance, $\overline{\varphi}(s\,;b_i') \approx \mu_i$. Therefore there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(r\,;b_i') \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{M^{\mathrm{f}}} \bar{b}$, we obtain $\overline{\varphi}(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒2. Let $\psi_i(x) = \varphi_i(x\,;b_i)$ for some $\varphi_i(x\,;z) \in L$ and $b_i \in \mathcal{U}^{|z|}$. Let $\bar{b} = \langle b_i : i < n \rangle$. Let $\mu_i \in \mathbb{R}$ be such that $\overline{\varphi}_i(p\,;b_i) \approx \mu_i$. Let $\bar{b}'$ be such that $s \mathop{\smile\!\!\!\!\!\!|}_{M^{\mathrm{f}}} \bar{b}' \equiv_{M^{\mathrm{f}}} \bar{b}$. By invariance, $\overline{\varphi}_i(s\,;b_i') \approx \mu_i$. Therefore there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}_i(r\,;b_i') \approx_\varepsilon \mu_i$ for every $i < n$. As $\bar{b}' \equiv_{M^{\mathrm{f}}} \bar{b}$, we obtain $\overline{\varphi}_i(r\,;b_i) \approx_\varepsilon \mu_i$.

1⇒3. By smoothness we it suffices to prove 3 with s for p. The type

$$p(z) \quad = \quad \Big\{ \overline{\varphi}(s\,;z) \not\approx_\varepsilon \overline{\varphi}(r\,;z) \ : \ r \in M^{\mathrm{f}} \Big\}$$

is inconsistent by 2. Hence compatness yields the required $r_1, \dots, r_n \in M^{\mathrm{f}}$.

3⇒2. Clear.

2⇒1 Assume 2. We prove that p is invariant. Pick some $b \equiv_{M^{\mathrm{f}}} b'$ and some ε. Let $r, r' \in M^{\mathrm{f}}$ be such that $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ and $\overline{\varphi}(p\,;b') \not\approx_\varepsilon \overline{\varphi}(r';b')$. As $\overline{\varphi}(r\,;b) \approx \overline{\varphi}(r';b')$ follows from $b \equiv_{M^{\mathrm{f}}} b'$, we obtain $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(p\,;b')$.

We prove that p is smooth.

$$p(x^{\mathrm{s}}) \quad = \quad \Big\{ \overline{\varphi}(x^{\mathrm{s}}\,;b) \approx_\varepsilon \mu_b \ : \ b \in \mathcal{U}^{|z|} \Big\}$$

□

3.3 Definition *Let* $\varphi(x\,;z) \in L$ *and* M *be given. A global sample* $p(x^{\mathrm{s}})$ *is*

1. *invariant if* $\bar{\varphi}(p\,;b) \approx \bar{\varphi}(p\,;b')$ *for every* $b \equiv_{M^{\mathrm{f}}} b'$;
2. *(pointwise) approximable if, for every* $b \in \mathcal{U}^{|z|}$ *and every* ε, *there is an* $r \in M^{\mathrm{f}}$ *such that* $\overline{\varphi}(p\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$;
3. *definable if, for every* ε, *the set* $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(p\,;b) < \varepsilon\}$ *is definable over* M^{f}. □

The following is a useful characterization of smooth samples.

4 Old stuff/garbage

We write $\mathrm{supp}(p)$ for the support of p, that is, the set of those a in $\mathcal{U}$ such that $p \vdash (a \in x^{\mathrm{s}}) > \varepsilon$ for some standard positive ε. If s is a smooth sample $\mathrm{supp}(s)$ is defined to be $\mathrm{supp}(p)$ for $p(x^s) = \mathrm{tp}(s/\mathcal{U}^f)$.

Let s be an external sample. For every formula $\varphi(x\,;z) \in L$ and every $b \in \mathcal{U}^{|z|}$ we define

$$\mathrm{Av}_{x/s}\varphi(x\,;b) \;=\; \frac{\big|\{a \in s \;:\; \varphi(a\,;b)\}\big|}{|s|},$$

When possible we abbreviate $\mathrm{Av}_{x/s}\varphi(x\,;b)$ with $\overline{\varphi}(s\,;b)$.

We write $\mathrm{ft}(a)$, for the standard part of the hyperrational number a. That is, the unique real number μ such that $\mu \approx a$.

The standard part of Av_s induces a finite probability measure on the algebra of definable subsets of $\mathcal{U}^{|x|}$. This measure has an unique extension to a Lebesgue probability measure on a σ-algebra. This is known as the Loeb measure.

All definitions and facts in this section are relative to some given formula $\varphi(x\,;z) \in L$ and some model M.

4.1 Definition *Let $\varphi(x\,;z) \in L$ and M be given. We say that a sample s is smooth if, for every $s' \equiv_M s$ and every $b \in \mathcal{U}^{|z|}$, we have $\overline{\varphi}(s';b) \approx \overline{\varphi}(s\,;b)$.* □

4.2 Definition *Let $\varphi(x\,;z) \in L$ and M be given. An external sample s is*

1. *invariant if, for every $b, b' \in \mathcal{U}^{|z|}$ such that $b \equiv_M b'$, we have $\overline{\varphi}(s\,;b) \approx \overline{\varphi}(s\,;b')$;*
2. *definable if, for every ε, the set $\{b \in \mathcal{U}^{|z|} : \overline{\varphi}(s\,;b) < \varepsilon\}$ is definable over M;*
3. *finitely satisfiable if, for all $b \in \mathcal{U}^{|z|}$ there is an $r \in M^f$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$;*
4. *generically stable if all of the above hold.* □

Smooth roughly means internal and invariant.

4.3 Fact *Let $\varphi(x\,;z) \in L$ and M be given. The following are equivalent for every external sample s*

1. *s is smooth;*
2. *there is an invariant internal sample $s' \equiv_M s$.*

Proof 1⇒2. Let s be smooth and let $s' \equiv_M s$ be any internal sample. If $b, b' \in \mathcal{U}^{|z|}$ and $b \equiv_M b'$, then $b' = fb$ for some $f \in \mathrm{Aut}(\mathcal{U}^{f*}/M)$. Then $\overline{\varphi}(s';fb) \approx \overline{\varphi}(f^{-1}s'\,;b)$. Moreover, by smothness $\overline{\varphi}(f^{-1}s'\,;b) \approx \overline{\varphi}(s';b)$. The infariace in s' follows.

2⇒1. It suffices to prove that if s is internal and invariant then it is smooth. Suppose for a contradiction that there is an $s' \equiv_M s$ such that $\overline{\varphi}(s'\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. Pick a internal sample $s'' \equiv_{M,s} s'$. Then $\overline{\varphi}(s''\,;b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$. As $s'' \equiv_M s$ are both in $\mathcal{U}^{f*}$, then $s'' = fs$ for some $f \in \mathrm{Aut}(\mathcal{U}^{f*}/M)$. Hence we obtan $\overline{\varphi}(s\,;f^{-1}b) \not\approx_\varepsilon \overline{\varphi}(s\,;b)$ which contradics the invariance of s. □

4.4 Definition *Let $\varphi(x\,;z) \in L$ and M be given. Let $\mathcal{B} \subseteq \mathcal{U}^{|z|}$ be arbitrary. We say that s*

is (uniformly) approximable on $\mathcal{B}$ if for every ε there is a finite sample $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s,b) \approx_\varepsilon \overline{\varphi}(r,b)$ for every $b \in \mathcal{B}$. (The sample r depends on ε, not on b.) □

4.5 Lemma *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{B}$ and such that $\overline{\varphi}(s\,;b) > \varepsilon$ for every $b \in \mathcal{B}$. Then there is a finite cover of $\mathcal{B}$, say $\mathcal{B}_1,\dots,\mathcal{B}_n$, such that all the types $p_i(x) = \{\varphi(x\,;b) : b \in \mathcal{B}_i\}$ are consistent.*

Proof Let $r \in M^{\mathrm{f}}$ be as in Definition 4.4. As r is finite, we may assume that $\mathrm{supp}(r) = \{a_1,\dots,a_n\}$. Let $\mathcal{B}_i = \{b \in \mathcal{B} : \varphi(a_i\,;b)\}$. As $\overline{\varphi}(r\,;b) > 0$ for all $b \in \mathcal{B}$, these $\mathcal{B}_i$ are the required cover of $\mathcal{B}$. □

4.6 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. If s be approximable on $\langle b_i : i < \omega\rangle$, where $\langle b_i : i < \omega\rangle$ is a sequence of indiscernibles such that $\overline{\varphi}(s\,;b_i) > \varepsilon$ for every $i < \omega$, then the type $\{\varphi(x\,;b_i) : i < \omega\}$ is consistent.* □

4.7 Corollary *Let $\varphi(x\,;z) \in L$ and M be given. Let s be approximable on $\mathcal{C}$. Then for every ε there is a pair of distict $c, c' \in \mathcal{C}$ such that $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] < \varepsilon$*

Proof Suppose for a contradiction that $\mathrm{Av}_{x/s}\big[\varphi(x\,;c) \not\leftrightarrow \varphi(x\,;c')\big] \geq \varepsilon$ for all distict $c, c' \in \mathcal{C}$. Apply Lemma 4.5 to the folmula $\psi(x\,;z,z') = [\varphi(x\,;z) \not\leftrightarrow \varphi(x\,;z')]$ and the set $\mathcal{B} = \{\langle c,c'\rangle \in \mathcal{C}^2 \ :\ c{\neq}c'\}$. The sets $\mathcal{B}_i$ obtained from Lemma 4.5 induce a finite coloring of the complete graph on $\mathcal{C}$. By the Ramsey theorem there is an infinite monochromatic set $A \subseteq \mathcal{C}$. Hence $\{\varphi(x\,;a) \not\leftrightarrow \varphi(x\,;a')\ :\ a,a'{\in}A,\ a{\neq}a'\}$ is consistent. As $|A| > 2$, this is impossible. □

5 The nip formulas

5.1 Theorem *Let $\varphi(x\,;z) \in L$ and M be given and assume that $\varphi(x\,;z)$ is nip. Then every sample is approximable on M over M.*

Proof Questo è Vapnik-Chervonenkis. □

For smooth samples the Vapnik-Chervonenkis Theorem can be strengthened as follows. (The difference is that below b ranges over $\mathcal{U}^{|z|}$.)

5.2 Corollary *Let $\varphi(x\,;z) \in L$ be nip. Every smooth sample s is approximable on $\mathcal{U}$ over M.*

Proof By Theorem 5.1, there is an $r \in M^{\mathrm{f}}$ such that $\overline{\varphi}(s\,;b) \approx_\varepsilon \overline{\varphi}(r\,;b)$ for all $b \in M^{|z|}$.

Suppose for a contradiction that $\overline{\varphi}(s\,;b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$ for some $b' \in \mathcal{U}^{|z|}$. Pick a $s' \subseteq \mathcal{U}^{|x|}$ such that $b' \mathop{\smile\!\!\!\!\!\!\,|}_M s' \equiv_M s$. By the smoothness of s, we obtain $\overline{\varphi}(s';b') \not\approx_\varepsilon \overline{\varphi}(r\,;b')$. As $b' \mathop{\smile\!\!\!\!\!\!\,|}_M s'$, the same formula holds for some $b \in M^{|z|}$. Once again by smoothness, $\overline{\varphi}(s\,;b) \not\approx_\varepsilon \overline{\varphi}(r\,;b)$. A contradiction. □

5.3 Definition ?? *We say that $\varphi(x\,;z) \in L$ is distal if there is a formula $\psi(x\,;z_1,\dots z_n) \in L$*

such that for every finite set $B \subseteq \mathcal{U}$ and every $a \in \mathcal{U}^{|x|}$ there are $b_1, \ldots, b_n \in B$ such that $\psi(a\,; b_1, \ldots, b_n)$ and $\psi(x\,; b_1, \ldots, b_n)$ decides all formulas $\varphi(x\,; b)$ for $b \in B$. □

5.4 Definition ??* *We say that $\varphi(x\,; z) \in L$ is distal if there is a formula $\psi(x_1, \ldots, x_n\,; z) \in L$ such that for every finite set $A \subseteq \mathcal{U}^{|x|}$ and every $b \in \mathcal{U}^{|z|}$ there are $a_1, \ldots, a_n \in B$ such that $\psi(a_1, \ldots, a_n,\ z)$ and $\psi(a_1, \ldots, a_n,\ z)$ decides all formulas $\varphi(a\,; z)$ for $a \in A$.* □

For every finite sample $r \in M^{\mathrm{f}}$ and every $\mu \in \mathbb{R}$ there is a formula $\psi(z) \in L(\operatorname{supp} r)$ such that $\bar{\varphi}(r\,; z) = \mu \leftrightarrow \psi(z)$. The formula $\psi(z)$ depends on r.

When $\varphi(x\,; z)$ is distal

If $\varphi(x\,; z)$ is distal then there is a formula ψ such that for every $r \in M^f$ there is $r_0 \in M$

5.5 Theorem (false) *The following are equivalent*

1. *$\varphi(x\,; z) \in L$ is distal;*
2. *for every sample $s \in \mathcal{U}^{\mathrm{f}+}_{*}$, if s is generically stable then s is smooth.*

There is a formula $\psi(x_1, \ldots, x_n\,; z)$ such that for every $r \in M^{\mathrm{f}}$

there are $a_1, \ldots, a_n$ such that $\varphi(r\,; b) = \mu\ \psi(x_1, \ldots, x_n\,; z)$