1 Loeb samples (bootstrapping)

In this section we introduce hypefinite samples and prove in Lemma 2 that all Keisler measures are generated by some Loeb sample.

Below $\mathfrak{U}$ is a saturated model of a complete theory T in the language L. We write κ for the cardinality of $\mathfrak{U}$ and assume that κ an inaccessible cardinal larger than $|L|$.

For every $n \in \omega$ define

$$S_n \quad = \quad \Big\{ s : \mathfrak{U}^n \to \mathbb{R} \quad : \quad s\, a = 0 \text{ for all but finitely many } a \Big\}$$

The elements of S_n are called standard samples. These will interpreted as signed measures concentrated on a finite set. We denote by $\bar{\mathfrak{U}}$ the multi-sorted structure $\langle \mathfrak{U}, \mathbb{R}, (S_n)_{n\in\omega} \rangle$. Clearly, $|\bar{\mathfrak{U}}| = \kappa$. We call the first sort the home sort; the second one the real sort and the remaining ones the sample sorts.

The language of $\bar{\mathfrak{U}}$ is denoted by $\bar{L}$. It contains L and a symbol for every function $\mathbb{R}^n \to \mathbb{R}$. Moreover, for every formula in $\varphi(x,z) \in L$ the language $\bar{L}$ contains a function symbol of sort

$$S_{|x|} \times \mathfrak{U}^{|z|} \quad \to \quad \mathbb{R}$$

that we interpret as the function that maps

1. $$(s, b) \quad \mapsto \quad \sum_{\varphi(x,b)} s\, x.$$

As the functions in $S_{|x|}$ are null almost everywhere, the sum in 1 is well-defined. We will use two informal but suggestive symbols for this function: $\sum_{\varphi(x,b)} s\, x$ or $\mu_s \varphi(x,b)$. When $\varphi(x,b)$ is the formula $x = b$, we write $s\, b$.

Let ${}^*\bar{\mathfrak{U}} = \langle {}^*\mathfrak{U}, {}^*\mathbb{R}, ({}^*S_n)_{n\in\omega} \rangle$ be some fixed elementary extension of $\bar{\mathfrak{U}}$ that is saturated and has cardinality $> \kappa$.

The elements of $\bigcup_{n\in\omega} {}^*S_n$ are called (Loeb) samples. The support of $s \in {}^*S_{|x|}$ is the definable (hyperfinite) set $\{a \in {}^*\mathfrak{U}^{|x|} : s\, a \neq 0\}$ which we denote by $\operatorname{supp} s$.

If $M \preceq \mathfrak{U}$ we write $S_n{\restriction}M$ for the set of functions $s \in S_n$ such that $\operatorname{supp} s \subseteq M^n$. We define $\bar{M}$ to be the structure $\langle M, \mathbb{R}, (S_n{\restriction}M)_{n\in\omega} \rangle$.

1 Fact Let $M \preceq \mathfrak{U}$ be ω-saturated. Then $\bar{M} \preceq \bar{\mathfrak{U}}$. In general, when M is not saturated, for all sentences $\varphi \in \bar{L}(\bar{M})$ with no quantifiers of sample sort

$$\bar{M} \vDash \varphi \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi.$$

Proof We prove the second claim first. We can assume that the function $\mu_s \varphi(x,z)$ only occurs in atomic formulas of the form $\mu_s \varphi(x,z) = y$.

Fix $s \in S_{|x|}{\restriction}M$. Let $a_1, \dots, a_n$ be an enumeration of $\operatorname{supp} s$ and define $\alpha_i = s\, a_i$. The formula $\mu_s \varphi(x,z) = y$ is easily seen to be equivalent, both in $\bar{M}$ and in $\bar{\mathfrak{U}}$, to the conjunction of the formulas

$$\bigwedge_{i=1}^{n} \neg^{\varepsilon_i} \varphi(a_i, z) \quad \to \quad y = \sum_{i=1}^{n} [1 - \varepsilon_i] \cdot \alpha_i$$

as ε ranges over ${}^n 2$. Hence, every sentence $\varphi \in \bar{L}(\bar{M})$ is equivalent to some sentence in $\psi \in L(M, \mathbb{R})$. As ψ does not contain parameters nor quantifiers of sample sort, its truth in $\bar{M}$ and $\bar{\mathfrak{U}}$ depends only on the structures $M, \mathbb{R}$, respectvely $\mathfrak{U}, \mathbb{R}$. Then the equivalence above is a consequence of $M \preceq \mathfrak{U}$.

Now assume that M is ω-saturated. We need to prove that for every tuples a, t in $\bar{M}$ of home, respectively sample sort we have

$$\bar{M} \vDash \varphi(a,t) \quad \Leftrightarrow \quad \bar{\mathcal{U}} \vDash \varphi(a,t) \qquad\qquad \text{for all } \varphi(x,w) \in \bar{L}.$$

(There is no need to mention parameters in $\mathbb{R}$ because they occur as constant in $\bar{L}$.)

Reason by induction on the syntax. The only interesting case concern the existential quantifier of samle sort, say $\exists u$ where u has the sort of $S_{|x|}$. If $\bar{\mathcal{U}} \vDash \exists u\, \varphi(u,a,t)$ then $\bar{\mathcal{U}} \vDash \varphi(s,a,t)$ for some finite sample $s \in S_{|x|}$. Let $b_1,\dots,b_n \in \mathcal{U}^{|x|}$ enumerate the support of s. By ω-saturation, there are $c_1,\dots,c_n \in M^{|x|}$ such that $b_1,\dots,b_n \equiv_{a,\operatorname{supp} t} c_1,\dots,c_n$. By homogeneity, there is an $f \in \operatorname{Aut}(\mathcal{U}/a, \operatorname{supp} t)$ such that $fb_i = c_i$. Extend f to an automorphism of $\bar{\mathcal{U}}$ by requiring that f is the identity on $\mathbb{R}$ and $f(s\,b) = (f\,s)(f\,b)$. Then $\bar{\mathcal{U}} \vDash \varphi(fs,a,t)$, so $\bar{M} \vDash \varphi(fs,a,t)$ follows by induction hypothesis. □

2 **Lemma** Let μ be finitely additive signed measures on $L_x(\mathcal{U})$. Then there is $s \in {}^*S_{|x|}$ such that

$$\# \qquad \mu_s\varphi(x) \quad = \quad \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let u be a variable of sample sort. We claim that the type $p(u)$ defined below is finitely consistent

$$p(u) \quad = \quad \Big\{ \sum_{\varphi(x)} u\,x = \mu\varphi(x) \quad : \quad \varphi(x) \in L(\mathcal{U}) \Big\}$$

Let $\{\varphi_1(x),\dots,\varphi_n(x)\} \subseteq L(\mathcal{U})$. It suffices to show that there is $s \in S_{|x|}$ such that

$$1. \qquad \sum_{\varphi_i(x)} s\,x \quad = \quad \mu\varphi_i(x) \qquad\qquad \text{for } i = 1,\dots,n.$$

Without loss of generality we can assume that $\{\varphi_1(x),\dots,\varphi_n(x)\}$ is a Boolean algebra with atoms $\varphi_1(x),\dots,\varphi_k(x)$ for some $k \le n$. Pick some $a_1,\dots,a_k \in \mathcal{U}^{|x|}$ such that $a_i \vDash \varphi_i(x)$. Pick $s \in S_{|x|}$ with support $\{a_1,\dots,a_k\}$ and such that

$$s\,a_i \quad = \quad \mu\varphi_i(x) \qquad\qquad \text{for } i = 1,\dots,k.$$

Clearly 1 above is satisfied by the finite additivity of the measure. □

We say that μ_s is bounded if there in an $\alpha \in \mathbb{R}$ such that $|\mu_s| < \alpha$.

3 **Corollary** Let $s \in {}^*S_{|x|}$ be such that μ_s is bounded. Then there is a $t \in {}^*S_{|x|}$ such that $\mu_t = \operatorname{st}(\mu_s)$, where st denotes the standard part. □

We write $\mu_s{\restriction}M = \mu_t{\restriction}M$ if $\mu_s\varphi(x) = \mu_t\varphi(x)$ for every $\varphi(x) \in L(M)$. The expression $\mu_s{\restriction}M \approx \mu_t{\restriction}M$ has a similar meaning.

4 **Conjecture** For every bounded $s, t \in {}^*S_{|x|}$ the following are equivalent

1. $s \equiv_M t$;
2. $\mu_s{\restriction}M \approx \mu_t{\restriction}M$.

2 Smooth samples

The notion of smooth measure has been introduced by Keisler in his seminal article. It perfectly translates to samples. With samples the intuition behind smoothness is clarified. Smooth samples are those that are realized in a smaller model.

We say that a non-negative sample $s \in {}^*S_{|x|}$ is smooth over A, if for every $\varphi(x,z) \in L$, every $b \in \mathfrak{U}^{|z|}$ and every $\varepsilon \in \mathbb{R}^+$ there are two formulas $\psi_1(x), \psi_2(x) \in L(M)$ such that $\psi_1(x) \to \varphi(x,b) \to \psi_2(x)$ and

$$\mu_s\psi_1(x) \quad \approx_\varepsilon \quad \mu_s\psi_2(x)$$

The following notion will be proven redundant but it is important to point it out.

If for a given $\varphi(x,z)$ and ε, finitely many pairs of formulas $\psi_1(x), \psi_2(x)$ suffices for all $b \in \mathfrak{U}^{|z|}$, we say that s is uniformly smooth. More precisely, s is uniformly smooth if for every $\varphi(x,z) \in L$ and every $\varepsilon \in \mathbb{R}^+$ there is an n and some formulas $\psi_{1,i}(x), \psi_{2,i}(x) \in L(M)$, for $i = 1, \dots, n$, such that

$$\forall z \bigvee_{i=1}^{n} \left[\psi_{1,i}(x) \to \varphi(x,z) \to \psi_{2,i}(x)\right]$$

and

$$\mu_s\psi_{1,i}(x) \quad \approx_\varepsilon \quad \mu_s\psi_{2,i}(x) \qquad \text{for } i = 1, \dots, n.$$

5 Fact The following are equivalent for every non-negative sample $s \in {}^*S_{|x|}$

1. s is smooth over M;
2. s is uniformly smooth over M;
3. if $t \in {}^*S_{|x|}$ is non-negative and $\mu_s{\restriction}M \approx \mu_t{\restriction}M$, then $\mu_s \approx \mu_t$.

Proof A compactness argument easily proves 1⇔2.

1⇒3 Fix arbitrarily $\varepsilon \in \mathbb{R}$ and $\varphi(x) \in L(\mathfrak{U})$. We claim that $\mu_s\varphi(x) \approx \mu_t\varphi(x)$. Assume that s is smooth and let $\psi_1(x), \psi_2(x) \in L(M)$ be as above but with $\varepsilon/3$ for ε. Then $\mu_t\big(\psi_2(x) \smallsetminus \psi_1(x)\big) < \varepsilon/3$. Hence $\mu_t\varphi(x) \approx_{\varepsilon/3} \mu_t\psi_2(x) = \mu_s\psi_2(x) \approx_{\varepsilon/3} \mu_s\varphi(x)$ and the claim follows.

3⇒1 Negate 1. Compactness and the fact that $\bar{\mathfrak{U}} \preceq {}^*\bar{\mathfrak{U}}$ ensure the existence of s_1 and s_2 such that $s_1 \equiv_M s_2 \equiv_M s$ and

$$\begin{aligned}
&\big(\forall a \in \operatorname{supp} s_1 \smallsetminus \operatorname{supp} s\big) \quad \varphi(a) \quad \wedge \quad \big(\forall a \in \operatorname{supp} s \smallsetminus \operatorname{supp} s_1\big)\, \neg\varphi(a);\\
&\big(\forall a \in \operatorname{supp} s_2 \smallsetminus \operatorname{supp} s\big)\, \neg\varphi(a) \quad \wedge \quad \big(\forall a \in \operatorname{supp} s \smallsetminus \operatorname{supp} s_2\big) \quad \varphi(a).
\end{aligned}$$

Moreover we require that the cardinalities of $\varphi(\operatorname{supp} s_1)$ and $\neg\varphi(\operatorname{supp} s_2)$ are maximal given the properties above.

$$\begin{aligned}
\text{Let} \quad r_1 &= \inf\big\{r \in \mathbb{R} \;:\; \mu_s\psi(x) \leq r \text{ and } \psi(x) \to \varphi(x)\big\};\\
r_2 &= \sup\big\{r \in \mathbb{R} \;:\; \mu_s\psi(x) \geq r \text{ and } \varphi(x) \to \psi(x)\big\}.
\end{aligned}$$

Then μ_{s_1} and μ_{s_2} coincide with μ_s on $L(M)$ but $\mu_{s_2}\varphi(x) - \mu_{s_1}\varphi(x) = r_2 - r_1 \geq \varepsilon$. This contradicts 3. □

6 Fact Let $s \in {}^*S_{|x|}$ be a non-negative bounded sample. Then the following are equivalent

1. s is smooth over M;
2. every saturated elementary extension of $\bar{M}$ contains a sample t such that $\mu_t \approx \mu_s$.

Proof 1⇒2 As s is bounded, there is a type $p(u) \subseteq \bar{L}(M)$ that says $\mu_u{\restriction}M \approx \mu_s{\restriction}M$. Let $t \vDash p(u)$. By 3 of the fact above $\mu_t \approx \mu_s$.

2⇒1

□

3 Pseudofinite samples

In this section we define pseudofiniteness, a very strong form of finite satisfiability. Note that there are a few distinct notions of finite satisfiabilty that apply to this context, however pseudofiniteness is one of the most natural. (It might be stronger than the notion of finite satisfiability that applies to Keisler measures.)

We say that a Loeb sample $s \in {}^*S_{|x|}$ is weakly pseudofinite over M if

1. $$\varphi(s) \quad\Rightarrow\quad \varphi(t) \text{ for some } t \in \bar{M} \qquad\qquad \text{for every } \varphi(u) \in \bar{L}(M).$$

By Fact 1, every sample is weakly pseudofinite over any ω-saturated model.

If 1 holds for every $\varphi(u) \in \bar{L}(\mathfrak{U})$ then we say that s is pseudofinite. By the standard argument of existence of global coheirs, for every weakly pseudofinite $s \in {}^*S_{|x|}$ there is a pseudofinite $s' \in {}^*S_{|x|}$ such that $s' \equiv_M s$.

To illustrate the notion of pseudofiniteness we prove the following simple fact.

7 **Fact** Let $s \in {}^*S_{|x|}$ be pseudofinite and let $M \preceq \mathfrak{U}$. Then every formula $\varphi(x) \in L(\mathfrak{U})$ such that $\varphi(M) \subseteq \operatorname{supp} s$ is algebraic. In particular, if $\operatorname{supp} s$ is definable by a formula in $L(\mathfrak{U})$ then it is finite.

Proof If $\varphi(M) \subseteq \operatorname{supp} s$ the formula $\varphi(u)$ below is satisfied by s

$$\forall x \Big[\varphi(x) \quad\rightarrow\quad x \in \operatorname{supp} u\Big]$$

By pseudofiniteness this formula is satisfied in $\bar{M}$ hence $\varphi(M)$ is finite. □

8 **Question** For what theories does the following hold? For every pseudofinite $s \in {}^*S_{|x|}$ there is an $s' \equiv_M s$ such that for every $\varphi(u) \in \bar{L}(\mathfrak{U})$

2. $$\varphi(s') \quad\Rightarrow\quad \text{there is a finite } A \subseteq M \text{ such that } \varphi(s' \cdot \mathbb{1}_A).$$

A similar question may be asked for s weakly pseudofinite and $\varphi(u) \in \bar{L}(M)$. □

4 Definable samples

The following notion is apparently unrelated to the homonimous notion for Keisler measures, still it is quite natural.

We say that the sample $s \in {}^*S_{|x|}$ is definable over M if for every $\varphi(u,x) \in \bar{L}(\mathfrak{U})$ there is a formula $\psi(u,x) \in \bar{L}(M)$ such that $\psi(t,\mathfrak{U}) = \varphi(s,\mathfrak{U})$ for some $t \in \bar{M}$.

9 **Fact ???** If $s \in {}^*S_{|x|}$ is smooth over M then it is definable over M.

5 Generically stable samples

A sample $s \in {}^*S_{|x|}$ is called generically stable over M if it is both peudofinite and definable over M.

10 Fact ??? Let $s \in {}^*S_{|x|}$ be bounded and generically stable. Then for every formula $\varphi(x,z) \in L(M)$ and every $\varepsilon \in \mathbb{R}^+$ there is a $t \in \bar{M}$ such that $\mu_t\varphi(x,b) \approx_\varepsilon \mu_s\varphi(x,b)$ for every $b \in \mathcal{U}^{|z|}$.

Proof For every $\alpha \in \mathbb{R}$ let $\psi_\alpha(x) \in L(M)$ be such that $\psi_\alpha(\mathcal{U}) = \{b : \mu_s\varphi(x,b) \approx_{\varepsilon/2} \alpha\}$ there a $t \in \bar{M}$ such that $\mu_t\varphi(x,b) \approx_{\varepsilon/2} \alpha$. □

6 Invariant samples I

We introduce the notions of *invariant* and *finitely satisfiable* samples. There are two sensible variants of these notions. Here we consider the most stringent variant, the less stringent one is discussed in the following section.

We write $\mathrm{Aut}({}^*\mathcal{U}/A, \{\mathcal{U}\})$ for the set of automorphisms of ${}^*\mathcal{U}$ that fix A pointwise and $\mathcal{U}$ setwise. Note that every automorphism $f \in \mathrm{Aut}({}^*\mathcal{U})$ has a canonical extension to an automorphism in $\mathrm{Aut}({}^*\bar{\mathcal{U}})$, which we denote by the same symbol f. Namely, this is the extension that is the identity on ${}^*\mathbb{R}$ and that maps $s \in {}^*S_n$ to the unique $fs \in {}^*S_n$ such that $(fs)(fa) = f(s\,a)$.

Given $s \in {}^*S_{|x|}$, we say that μ_s is invariant over A if for every $f \in \mathrm{Aut}({}^*\mathcal{U}/A, \{\mathcal{U}\})$

$$\mu_s \quad = \quad \mu_{f\,s}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad = \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z) \in L \text{ and } b \in \mathcal{U}^{|z|}.$$

11 Definition We say that s is finitely satisfiable in M if for every $\varphi(x) \in L(\mathcal{U})$

$$\varphi(\mathrm{supp}\, s) \neq \varnothing \quad \Rightarrow \quad \varphi(M) \neq \varnothing.$$ □

The following lemma shows that the finite satisfiability of a sample corresponds (in a sense) to the finite satisfiability of the associated Keisler measure.

12 Lemma Let μ be as in Lemma 2 with $\Delta = L(\mathcal{U})$. Assume that

$$\mu\varphi(x) \neq 0 \quad \Rightarrow \quad \varphi(M) \neq \varnothing \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Then there is $s \in {}^*S_{|x|}$ that is finitely satisfied in M and

$$\mu_s\varphi(x) \quad = \quad \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let $p(u)$ be as in the proof of Lemma 2. Define

$$q(u) \quad = \quad \Big\{\forall x\, [\varphi(x) \rightarrow ux = 0] \quad : \quad \varphi(x) \in \Delta,\ \varphi(M) = \varnothing\Big\}$$

We need to show that $p(u) \cup q(u)$ is finitely consistent. Apply the same reasoning as in the proof of Lemma 2. □

13 Fact Every sample $s \in {}^*S_{|x|}$ that is finitely satisfiable in M is invariant over M.

Proof If s is not M-invariant then for some $f \in \mathrm{Aut}({}^*\mathcal{U}/M, \{\mathcal{U}\})$, some $\varphi(x,z) \in L$ and $b \in \mathcal{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \neq \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \neq \quad \mu_s\Big(\varphi(x,b) \not\leftrightarrow \varphi(x,fb)\Big)$$

Then there is $a \in \operatorname{supp} s$ such that $\varphi(a,b) \not\leftrightarrow \varphi(a,fb)$. Hence, from the finite satisfiability of s, we obtain that $\varphi(M,b) \neq \varphi(M,fb)$. This contradicts the M-invarance of μ_s. □

7 Invariant samples II

The exposition is parallel to that of the previous section with no significant differences.

We write $\operatorname{Aut}({}^*\bar{\mathfrak{U}}/A,\{\mathfrak{U}\})$ for the set of automorphisms of ${}^*\bar{\mathfrak{U}}$ that fix A pointwise and $\mathfrak{U}$ setwise. Given $s \in {}^*S_{|x|}$, we say that μ_s is weakly invariant over A if for every $f \in \operatorname{Aut}({}^*\bar{\mathfrak{U}}/A,\{\mathfrak{U}\})$

$$\mu_s \quad \approx \quad \mu_{fs}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad \approx \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z) \in L \text{ and } b \in \mathfrak{U}^{|z|}.$$

14 Definition We say that s is weakly finitely satisfiable in M if for every $\varphi(x) \in L(\mathfrak{U})$

$$\mu_s\varphi(x) \not\approx 0 \quad \Rightarrow \quad \varphi(M) \neq \varnothing.$$ □

15 Fact Every sample $s \in {}^*S_{|x|}$ that is weakly finitely satisfiable in M is weakly invariant over M.

Proof If s is not weakly M-invariant then for some $f \in \operatorname{Aut}({}^*\bar{\mathfrak{U}}/A,\{\mathfrak{U}\})$, some $\varphi(x,z) \in L$ and $b \in \mathfrak{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \not\approx \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \not\approx \quad \mu_s\Big(\varphi(x,b) \not\leftrightarrow \varphi(x,fb)\Big)$$

Then, by the finite satisfiability of s, we obtain that $\varphi(M,b) \neq \varphi(M,fb)$. This contradicts the M-invariance of μ_s. □