1 Hyperfinite samples

In this section we introduce hypefinite samples and prove in Lemma **??** that all global Keisler measures are generated by some hyperfinite sample.

Below $\mathfrak{U}$ is a saturated model of a complete theory T in the language L. We write κ for the cardinality of $\mathfrak{U}$ and assume that κ an inaccessible cardinal larger than $|L|$.

For every $n \in \omega$ define

$$S_n \quad = \quad \Big\{ s : \mathfrak{U}^n \to \mathbb{R} \quad : \quad s\,a = 0 \text{ for all but finitely many } a \Big\}$$

The elements of S_n are called standard samples. These will interpreted as signed measures concentrated on a finite set. We denote by $\bar{\mathfrak{U}}$ the multi-sorted structure $\langle \mathfrak{U}, \mathbb{R}, (S_n)_{n\in\omega} \rangle$. Clearly, $|\bar{\mathfrak{U}}| = \kappa$. We call the first sort the home sort; the second the real sort; the others are collectively named sample sorts.

The language of $\bar{\mathfrak{U}}$ is denoted by $\bar{L}$. It contains L and a symbol for every function $\mathbb{R}^n \to \mathbb{R}$. Moreover, for every formula in $\varphi(x,z) \in L$ the language $\bar{L}$ contains a function symbol of sort

$$S_{|x|} \times \mathfrak{U}^{|z|} \quad \to \quad \mathbb{R}$$

that we interpret as the function that maps

$$(s, b) \quad \mapsto \quad \sum_{\varphi(x,b)} s\,x. \tag{1}$$

As the functions in $S_{|x|}$ are null almost everywhere, the sum in 1 is well-defined.

We will use two informal but suggestive symbols for this function: $\sum_{\varphi(x,b)} s\,x$ or $\mu_s \varphi(x,b)$. When $\varphi(x,b)$ is the formula $x = b$, we write $s\,b$.

Let ${}^*\bar{\mathfrak{U}} = \langle {}^*\mathfrak{U}, {}^*\mathbb{R}, ({}^*S_n)_{n\in\omega} \rangle$ be some fixed elementary extension of $\bar{\mathfrak{U}}$ that is saturated and has cardinality $> \kappa$. The elements of $\bigcup_{n\in\omega} {}^*S_n$ are called (hyperfinite) samples.

1 Lemma Let μ be finitely additive signed measures on $L_x(\mathfrak{U})$. Then there is $s \in {}^*S_{|x|}$ such that

$$\mu_s \varphi(x) \quad = \quad \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x) \in L(\mathfrak{U}). \tag{\#}$$

Proof Let u be a variable of sample sort. We claim that the type $p(u)$ defined below is finitely consistent

$$p(u) \quad = \quad \Big\{ \sum_{\varphi(x)} u\,x = \mu\varphi(x) \quad : \quad \varphi(x) \in L(\mathfrak{U}) \Big\}$$

Let $\{\varphi_1(x), \ldots, \varphi_n(x)\} \subseteq L(\mathfrak{U})$. It suffices to show that there is $s \in S_{|x|}$ such that

$$\sum_{\varphi_i(x)} s\,x \quad = \quad \mu\varphi_i(x) \qquad\qquad \text{for } i = 1, \ldots, n. \tag{1}$$

Without loss of generality we can assume that $\{\varphi_1(x), \ldots, \varphi_n(x)\}$ is a Boolean algebra with atoms $\varphi_1(x), \ldots, \varphi_k(x)$ for some $k \leq n$. Pick some $a_1, \ldots, a_k \in \mathfrak{U}^{|x|}$ such that $a_i \vDash \varphi_i(x)$. Pick $s \in S_{|x|}$ with support $\{a_1, \ldots, a_k\}$ and such that

$$s\,a_i \quad = \quad \mu\varphi_i(x) \qquad\qquad \text{for } i = 1, \ldots, k.$$

Clearly 1 above is satisfied by the finite additivity of the measure. □

We say that μ_s is bounded if there in an $\alpha \in \mathbb{R}$ such that $|\mu_s| < \alpha$.

2 Corollary Let $s \in {}^*S_{|x|}$ be such that μ_s is bounded. Then there is a $t \in {}^*S_{|x|}$ such that

$\mu_t = \mathrm{st}(\mu_s)$, where st denotes the standard part. □

2 Bootstrapping

The support of $s \in {}^*S_{|x|}$ is the definable (hyperfinite) set $\{a \in {}^*\mathfrak{U}^{|x|} : s\,a \neq 0\}$ which we denote by $\mathrm{supp}\, s$. If $M \preceq \mathfrak{U}$ we write $S_n{\restriction}M$ for the set of functions $s \in S_n$ such that $\mathrm{supp}\, s \subseteq M^n$. We define $\bar{M}$ to be the structure $\langle M, \mathbb{R}, (S_n{\restriction}M)_{n\in\omega}\rangle$.

3 Fact Let $M \preceq \mathfrak{U}$ be ω-saturated. Then $\bar{M} \preceq \bar{\mathfrak{U}}$. In general, when M is not saturated, for all sentences $\varphi \in \bar{L}(\bar{M})$ with no quantifiers of sample sort

$$\bar{M} \vDash \varphi \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi.$$

Proof We prove the second claim first. We can assume that the function $\mu_s\varphi(x,z)$ only occurs in atomic formulas of the form $\mu_s\varphi(x,z) = y$.

Fix $s \in S_{|x|}{\restriction}M$. Let $a_1, \dots, a_n$ be an enumeration of $\mathrm{supp}\, s$ and define $\alpha_i = s\,a_i$. The formula $\mu_s\varphi(x,z) = y$ is easily seen to be equivalent, both in $\bar{M}$ and in $\bar{\mathfrak{U}}$, to the conjunction of the formulas

$$\bigwedge_{i=1}^{n} \neg^{\varepsilon_i}\varphi(a_i,z) \quad \to \quad y = \sum_{i=1}^{n}[1-\varepsilon_i]\cdot\alpha_i$$

as ε ranges over ${}^n2$. Hence, every sentence $\varphi \in \bar{L}(\bar{M})$ is equivalent to some sentence in $\psi \in L(M,\mathbb{R})$. As ψ does not contain parameters nor quantifiers of sample sort, its truth in $\bar{M}$ and $\bar{\mathfrak{U}}$ depends only on the structures $M, \mathbb{R}$, respectvely $\mathfrak{U}, \mathbb{R}$. Then the equivalence above is a consequence of $M \preceq \mathfrak{U}$.

Now assume that M is ω-saturated. We need to prove that for every tuples a, t in $\bar{M}$ of home, respectively sample sort we have

$$\bar{M} \vDash \varphi(a,t) \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi(a,t) \qquad\qquad \text{for all } \varphi(x,w) \in \bar{L}.$$

(There is no need to mention parameters in $\mathbb{R}$ because they occur as constant in $\bar{L}$.)

Reason by induction on the syntax. The only interesting case concern the existential quantifier of samle sort, say $\exists u$ where u has the sort of $S_{|x|}$. If $\bar{\mathfrak{U}} \vDash \exists u\, \varphi(u,a,t)$ then $\bar{\mathfrak{U}} \vDash \varphi(s,a,t)$ for some finite sample $s \in S_{|x|}$. Let $b_1, \dots, b_n \in \mathfrak{U}^{|x|}$ enumerate the support of s. By ω-saturation, there are $c_1, \dots, c_n \in M^{|x|}$ such that $b_1, \dots, b_n \equiv_{a,\mathrm{supp}\, t} c_1, \dots, c_n$. By homogeneity, there is an $f \in \mathrm{Aut}(\mathfrak{U}/a, \mathrm{supp}\, t)$ such that $fb_i = c_i$. Extend f to an automorphism of $\bar{\mathfrak{U}}$ by requiring that f is the identity on $\mathbb{R}$ and $f(s\,b) = (f\,s)(f\,b)$. Then $\bar{\mathfrak{U}} \vDash \varphi(fs,a,t)$, so $\bar{M} \vDash \varphi(fs,a,t)$ follows by induction hypothesis. □

4 Example This example shows that in the fact above would fail without the assumption of saturation. Let $\mathfrak{U} = {}^*\mathbb{N}$ and $M = \mathbb{N}$. Then $\bar{\mathfrak{U}} = \langle {}^*\mathbb{N}, \mathbb{R}, (S_n)_{n\in\omega}\rangle$ and $\bar{M} = \langle \mathbb{N}, \mathbb{R}, (S_n{\restriction}\mathbb{N})_{n\in\omega}\rangle$. Let x, y and u be variables of sort $\mathfrak{U}$ and S_1, respectively. Then $\bar{\mathfrak{U}} \nvDash \forall x\ \exists u\ \forall y < x\ (u\,y > 0)$ but this sentence holds in $\bar{M}$. □

We write $\mu_s{\restriction}M = \mu_t{\restriction}M$ if $\mu_s\varphi(x) = \mu_t\varphi(x)$ for every $\varphi(x) \in L(M)$. The expression $\mu_s{\restriction}M \approx \mu_t{\restriction}M$ has a similar meaning.

5 Conjecture (of fundamental importance, I guess) For every bounded $s, t \in {}^*S_{|x|}$ the following are equivalent

1. $s \equiv_M t$;

2. $\mu_s{\restriction}M \approx \mu_t{\restriction}M$.

3 Smooth samples

The notion of smooth measure has been introduced by Keisler in his seminal article. It perfectly translates to samples.

We say that a non-negative sample $s \in {}^*S_{|x|}$ is smooth over M, if for every $\varphi(x,z) \in L$, every $b \in \mathfrak{U}^{|z|}$ and every $\varepsilon \in \mathbb{R}^+$ there is a formula $\psi(x) \in L(M)$ such that

$$\begin{aligned} \psi(x) &\rightarrow \varphi(x,b) \\ \mu_s\psi(x) &\approx_\varepsilon \mu_s\varphi(x,b). \end{aligned}$$

The following notion will be proven redundant in Fact **??** but it is introduced for emphasis. If for a given $\varphi(x,z)$ and ε, finitely many formulas $\psi(x)$ suffices for all $b \in \mathfrak{U}^{|z|}$, we say that s is uniformly smooth. Precisely, s is uniformly smooth if for every $\varphi(x,z) \in L$ and every $\varepsilon \in \mathbb{R}^+$ there are some finitely many formulas $\psi_1(x),\dots,\psi_n(x) \in L(M)$ such that for every $b \in \mathfrak{U}^{|z|}$ there is an $i \in \{1,\dots,n\}$ such that

$$\begin{aligned} \psi_i(x) &\rightarrow \varphi(x,b) \\ \mu_s\psi_i(x) &\approx_\varepsilon \mu_s\varphi(x,b). \end{aligned}$$

The following is an important consequence of uniformity.

6 Fact Let $s \in {}^*S_{|x|}$ be a non-negative bounded sample that is uniformly smooth over M. Then for every $\varepsilon \in \mathbb{R}^+$ there is a $t \in \bar{M}$ such that $\mu_t \approx_\varepsilon \mu_s$.

Proof For any given $\varepsilon \in \mathbb{R}^+$ choose $t \in \bar{M}$ such that μ_t coincides with $\mathrm{st}(\mu_s)$ on the Boolean algebra generated by $\psi_1(x),\dots,\psi_n(x)$. □

7 Fact Let $s \in {}^*S_{|x|}$ be a non-negative bounded sample. Then the following are equivalent

1. s is smooth over M;
2. s is uniformly smooth over M;
3. if $t \in {}^*S_{|x|}$ is non-negative and $\mu_s{\restriction}M \approx \mu_t{\restriction}M$, then $\mu_s \approx \mu_t$.

Proof A compactness argument easily proves 1⇔2.

1⇒3 Fix $\varepsilon \in \mathbb{R}$ and $\varphi(x) \in L(\mathfrak{U})$. We claim that $\mu_s\varphi(x) \approx_\varepsilon \mu_t\varphi(x)$. Assume that s is smooth and let $\psi(x) \in L(M)$ be as in the definition of smooth but with $\varepsilon/2$ for ε. Then $\mu_t\psi(x) \approx \mu_s\psi(x) \approx_{\varepsilon/2} \mu_s\varphi(x)$. Then $\mu_s\varphi(x) - \mu_t\varphi(x) \leq \mu_s\varphi(x) - \mu_t\psi(x) \leq \varepsilon$. The same argument applied to $\neg\varphi(x)$ yields $\mu_t\varphi(x) - \mu_s\varphi(x) \leq \varepsilon$. This proves the claim and 3 follows.

3⇒1 Suppose $\varphi(x,b)$ witness the failure of 1. That is, $\mu_s\psi(x) \not\approx_\varepsilon \varphi(x,b)$ for some $\varepsilon \in \mathbb{R}^+$ and all $\psi(x) \in L(M)$. Compactness and the fact that $\bar{\mathfrak{U}} \preceq {}^*\bar{\mathfrak{U}}$ ensure the existence of t of maximal (hypefinite) cardinality such that $t \equiv_M s$ and

$$(\forall x \in \operatorname{supp} t \smallsetminus \operatorname{supp} s)\neg\varphi(x,b) \quad \wedge \quad (\forall x \in \operatorname{supp} s \smallsetminus \operatorname{supp} t)\, \varphi(x,b);$$

We claim that

$$\mu_t\varphi(x,b) \quad\approx\quad \sup\big\{\mu_t\psi(x)\ :\ \psi(x)\to\varphi(x)\big\}.$$

This contraddicts 3. □

4 Analytic samples

A stronger notion of smoothness is natural in our context. Apparently, it has no parallel for Keisler measures. Tentatively, we say that a sample $s\in{}^*S_{|x|}$ is analytic (just a hyperbole of *smooth*) if $t\equiv_M s$ implies $t\equiv_{\mathfrak{U}} s$ for every $t\in{}^*S_{|x|}$.

I do not know if this notion has non trivial examples. On the other hand, if Conjecture **??** is true, it may be equivalent to smoothness.

5 Pseudofinite samples

In this section we define pseudofiniteness, a very strong form of finite satisfiability. There are a few distinct notions of finite satisfiabilty that apply to this context. Pseudofiniteness it is the notion that best fit with that of analytic samples. (A weaker notion of finite satisfiability that corresponds to that of Keisler measures will be introduced later.)

We say that a sample $s\in{}^*S_{|x|}$ is elementary over M if

1. $\varphi(s)\quad\Rightarrow\quad\varphi(t)$ for some $t\in\bar{M}$ $\qquad\qquad$ for every $\varphi(u)\in\bar{L}(M)$.

By Fact **??**, every sample is elementary over any ω-saturated model. If 1 holds for every $\varphi(u)\in\bar{L}(\mathfrak{U})$ then we say that s is pseudofinite. By the standard argument of existence of global coheirs, for every elementary $s\in{}^*S_{|x|}$ there is a pseudofinite $s'\in{}^*S_{|x|}$ such that $s'\equiv_M s$.

To illustrate the notion of pseudofiniteness we prove the following simple fact.

8 Fact Let $s\in{}^*S_{|x|}$ be pseudofinite and let $M\preceq\mathfrak{U}$. Then every formula $\varphi(x)\in L(\mathfrak{U})$ such that $\varphi(M)\subseteq\operatorname{supp}s$ is algebraic. In particular, if $\operatorname{supp}s$ is definable by a formula in $L(\mathfrak{U})$ then it is finite.

Proof If $\varphi(M)\subseteq\operatorname{supp}s$ then

$$s\quad\vDash\quad\forall x\Big[\varphi(x)\to x\in\operatorname{supp}u\Big]$$

By pseudofiniteness this formula is satisfied also by some $t\in\bar{M}$, hence $\varphi(M)$ is finite. □

9 Fact If s is elementary and analytic then it is pseudofinite.

Proof Let $p(u)=\operatorname{tp}(s/M)$. As s is elementary, $p(u)$ is finitely satisfied in M. Then it hase an extension to a complete type over $\mathfrak{U}$ that is finitely satisfied in M. By analyticity, $s\vDash p(u)$. □

10 Question For what theories does the following hold? For every pseudofinite $s\in{}^*S_{|x|}$ there is an $s'\equiv_M s$ such that for every $\varphi(u)\in\bar{L}(\mathfrak{U})$

2. $\varphi(s')\quad\Rightarrow\quad$ there is a finite $A\subseteq M$ such that $\varphi(s'\cdot\mathbb{1}_A)$.

A similar question may be asked for s elementary and $\varphi(u)\in\bar{L}(M)$. □

6 Definable samples

Again, the following notion does not literally translare the homonymous notion for Keisler measures; it is a stronger notion introduced in analogy to those in the two paragraphs above.

We say that the sample $s \in {}^*S_{|x|}$ is definable over M if for every $\varphi(u,x) \in \bar{L}(\mathfrak{U})$ there is a type $p(x) \subseteq \bar{L}(M)$ such that $p(\mathfrak{U}) = \varphi(s,\mathfrak{U})$.

11 Fact If $s \in {}^*S_{|x|}$ is analytic over M then it is definable over M.

Proof Let $q(u) = \mathrm{tp}(s/M)$ and define $p(x) = \exists u\, \big[q(u) \wedge \varphi(u,x)\big]$. By analyticity $p(\mathfrak{U}) = \varphi(s,\mathfrak{U})$. □

Qui sotto solo scemenze. Magari anche sopra.

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7 Generically stable samples

A sample $s \in {}^*S_{|x|}$ is called generically stable over M if it is both pseudofinite and definable over M. Again, this notion is stronger than the homonymous notion for Keisler measures.

12 Fact (T is NIP?) Let $s \in {}^*S_{|x|}$ be non negative and generically stable. Let $\varphi_i(u,x) \in \bar{L}(M)$, for $i = 1,\dots,n$, be such that $\varphi(s,\mathfrak{U})$ covers $\mathfrak{U}^{|x|}$. Then there is a $t \in \bar{M}$ such that $\varphi_i(t,\mathfrak{U}) = \varphi_i(s,\mathfrak{U})$ for all i.

Proof Let $p_i(x) \subseteq \bar{L}(M)$ be such that $p_i(\mathfrak{U}) = \varphi_i(s,\mathfrak{U})$. by compactness there are some formulas $\psi_i(x) \in p_i(x)$ such that $\psi_i(\mathfrak{U}) = \varphi_i(s,\mathfrak{U})$. □

13 Fact (T is NIP?) Let $s \in {}^*S_{|x|}$ be non negative, bounded, and generically stable. Then for every formula $\varphi(x,z) \in L(M)$ and every $\varepsilon \in \mathbb{R}^+$ there is a $t \in \bar{M}$ such that $\mu_t\varphi(x,b) \approx_\varepsilon \mu_s\varphi(x,b)$ for every $b \in \mathfrak{U}^{|z|}$.

Proof Fix n and define $\varepsilon = 1/(n+1)$ and $\alpha_i = i/(n+1)$ for $i = 1,\dots,n$. Let $p_i(x) \subseteq \bar{L}(M)$ be such that $p_i(\mathfrak{U}) = \{b : \mu_s\varphi(x,b) \approx_\varepsilon \alpha_i\}$. As the sets $p_i(\mathfrak{U})$ cover $\mathfrak{U}$, by compactness there are some formulas $\psi_i(x) \in \bar{L}(M)$ such that $\psi_i(\mathfrak{U}) = \{b : \mu_s\varphi(x,b) \approx_\varepsilon \alpha_i\}$. By pseudofiniteness there is a $t \in \bar{M}$ such that $\mu_t\varphi(x,b) \approx_\varepsilon \mu_s\varphi(x,b)$ for all b. □

8 Invariant samples I

We introduce the notions of *invariant* and *finitely satisfiable* samples. There are two sensible variants of these notions. Here we consider the most stringent variant, the less stringent one is discussed in the following section.

We write $\mathrm{Aut}({}^*\mathfrak{U}/A,\{\mathfrak{U}\})$ for the set of automorphisms of ${}^*\mathfrak{U}$ that fix A pointwise and $\mathfrak{U}$ setwise. Note that every automorphism $f \in \mathrm{Aut}({}^*\mathfrak{U})$ has a canonical extension to an automorphism in $\mathrm{Aut}({}^*\bar{\mathfrak{U}})$, which we denote by the same symbol f. Namely, this is the extension that is the identity on ${}^*\mathbb{R}$ and that maps $s \in {}^*S_n$ to the unique $fs \in {}^*S_n$ such that $(fs)(fa) = f(s\,a)$.

Given $s \in {}^*S_{|x|}$, we say that μ_s is invariant over A if for every $f \in \operatorname{Aut}({}^*\mathfrak{U}/A, \{\mathfrak{U}\})$

$$\mu_s \quad = \quad \mu_{f\,s}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad = \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z) \in L \text{ and } b \in \mathfrak{U}^{|z|}.$$

14 Definition We say that s is finitely satisfiable in M if for every $\varphi(x) \in L(\mathfrak{U})$

$$\varphi(\operatorname{supp} s) \neq \varnothing \quad \Rightarrow \quad \varphi(M) \neq \varnothing. \qquad \square$$

The following lemma shows that the finite satisfiability of a sample corresponds (in a sense) to the finite satisfiability of the associated Keisler measure.

15 Lemma Let μ be as in Lemma **??** with $\Delta = L(\mathfrak{U})$. Assume that

$$\mu\varphi(x) \neq 0 \quad \Rightarrow \quad \varphi(M) \neq \varnothing \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Then there is $s \in {}^*S_{|x|}$ that is finitely satisfied in M and

$$\mu_s\varphi(x) \quad = \quad \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let $p(u)$ be as in the proof of Lemma **??**. Define

$$q(u) \quad = \quad \Big\{\forall x\,[\varphi(x) \rightarrow ux = 0] \ \ : \ \ \varphi(x) \in \Delta,\ \varphi(M) = \varnothing\Big\}$$

We need to show that $p(u) \cup q(u)$ is finitely consistent. Apply the same reasoning as in the proof of Lemma **??**. $\square$

16 Fact Every sample $s \in {}^*S_{|x|}$ that is finitely satisfiable in M is invariant over M.

Proof If s is not M-invariant then for some $f \in \operatorname{Aut}({}^*\mathfrak{U}/M, \{\mathfrak{U}\})$, some $\varphi(x,z) \in L$ and $b \in \mathfrak{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \neq \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \neq \quad \mu_s\Big(\varphi(x,b) \not\leftrightarrow \varphi(x,fb)\Big)$$

Then there is $a \in \operatorname{supp} s$ such that $\varphi(a,b) \not\leftrightarrow \varphi(a,fb)$. Hence, from the finite satisfiability of s, we obtain that $\varphi(M,b) \neq \varphi(M,fb)$. This contradicts the M-invarance of μ_s. $\square$

9 Invariant samples II

The exposition is parallel to that of the previous section with no significant differences.

We write $\operatorname{Aut}({}^*\bar{\mathfrak{U}}/A, \{\mathfrak{U}\})$ for the set of automorphisms of ${}^*\bar{\mathfrak{U}}$ that fix A pointwise and $\mathfrak{U}$ setwise. Given $s \in {}^*S_{|x|}$, we say that μ_s is weakly invariant over A if for every $f \in \operatorname{Aut}({}^*\bar{\mathfrak{U}}/A, \{\mathfrak{U}\})$

$$\mu_s \quad \approx \quad \mu_{f\,s}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad \approx \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z) \in L \text{ and } b \in \mathfrak{U}^{|z|}.$$

17 Definition We say that s is weakly finitely satisfiable in M if for every $\varphi(x) \in L(\mathcal{U})$

$$\mu_s\varphi(x) \not\approx 0 \quad \Rightarrow \quad \varphi(M) \neq \varnothing. \qquad \square$$

18 Fact Every sample $s \in {}^*S_{|x|}$ that is weakly finitely satisfiable in M is weakly invariant over M.

Proof If s is not weakly M-invariant then for some $f \in \mathrm{Aut}({}^*\bar{\mathcal{U}}/A, \{\mathcal{U}\})$, some $\varphi(x,z) \in L$ and $b \in \mathcal{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \not\approx \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \not\approx \quad \mu_s\Big(\varphi(x,b) \not\leftrightarrow \varphi(x,fb)\Big)$$

Then, by the finite satisfiability of s, we obtain that $\varphi(M,b) \neq \varphi(M,fb)$. This contradicts the M-invariance of μ_s. $\square$