1 Hyperfinite samples

In this section we introduce hypefinite samples and prove that all Keisler measures are generated by some hypefinite sample (Lemma 3).

Below $\mathfrak{U}$ is a saturated model of a complete theory T in the language L. We write κ for the cardinality of $\mathfrak{U}$ and assume that κ is larger thean $|L|$.

For every $n \in \omega$ define

$$S_n \quad = \quad \Big\{ s : \mathfrak{U}^n \to \mathbb{R} \ \ : \ \ s\,a = 0 \text{ for all but finitely many } a \Big\}$$

The elements of S_n are called finite samples (we will mainly use $\mathbb{N}$-valued samples and interpret these as finite multisets). Let $\bar{\mathfrak{U}}$ be the multi-sorted structure $\langle \mathfrak{U}, \mathbb{R}, (S_n)_{n\in\omega}\rangle$. We call the first sort the home sort; the second is called the real sort. The remaining sorts are called sample sorts.

The language of $\bar{\mathfrak{U}}$ is denoted by $\bar{L}$. It contains L and a symbol for every subset of $\mathbb{R}^n$ and every function $\mathbb{R}^n \to \mathbb{R}$.

Moreover, for every formula in $\varphi(x,z) \in L$, where x is a variable of the home sort, the language $\bar{L}$ contains a function symbol of sort

$$S_{|x|} \times \mathfrak{U}^{|z|} \quad \to \quad \mathbb{R}$$

that we interpret as the function that maps

1. $$(s, b) \quad \mapsto \quad \sum_{\varphi(x,b)} s\,x.$$

As the functions in $S_{|x|}$ are null almost everywhere, the sum in 1 is well-defined. We will use two informal but suggestive symbols for this function: $\sum_{\varphi(x,b)} s\,x$ or $\mu_s\varphi(x,b)$. When $\varphi(x,b)$ is the formula $x = b$, we write $s\,b$.

Fix an elementary extension of $\bar{\mathfrak{U}}$ that is saturated and of larger cardinality. This elementary extension is denoted by ${}^*\bar{\mathfrak{U}} = \langle {}^*\mathfrak{U}, {}^*\mathbb{R}, ({}^*S_n)_{n\in\omega}\rangle$.

The elements of $\bigcup_{n\in\omega} {}^*S_n$ are called (hyperfinite) samples. The support of $s \in {}^*S_{|x|}$ is the definable hyperfinite set $\{a \in {}^*\mathfrak{U}^{|x|} : s\,a \neq 0\}$which we denote by $\operatorname{supp} s$.

If $M \preceq \mathfrak{U}$ we write $S_n{\restriction}M$ for the set of functions $s \in S_n$ such that $\operatorname{supp} s \subseteq M^n$. We define $\bar{M}$ to be the structure $\langle M, \mathbb{R}, (S_n{\restriction}M)_{n\in\omega}\rangle$.

1 Fact Let $M \preceq \mathfrak{U}$ be ω-saturated. Then $\bar{M} \preceq \bar{\mathfrak{U}}$. In general, when M is not saturated, we have that for all sentences $\varphi \in \bar{L}(\bar{M})$ with no quantifiers of sample sort

$$\bar{M} \vDash \varphi \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi.$$

Proof We prove the second claim first. We can assume that the function $\mu_s\varphi(x,y)$ only occurs in atomic formulas of the form $\mu_s\varphi(x,y) = w$.

Fix $s \in S_n{\restriction}M$. Let $a_1, \ldots, a_n$ be an enumeration of $\operatorname{supp} s$ and define $r_i = s\,a_i$. The formula $\mu_s\varphi(x,y) = w$ is easily seen to be equivalent, both in $\bar{M}$ and in $\bar{\mathfrak{U}}$, to the conjunction of the formulas

$$\bigwedge_{i=1}^{n} \neg^{\varepsilon(i)}\varphi(a_i, y) \quad \to \quad \sum_{i=1}^{n} \varepsilon(i)\cdot r_i = w$$

as ε ranges over ${}^n2$. Hence, every sentence $\varphi \in \bar{L}(\bar{M})$ is equivalent to some sentence in $\psi \in L(M,\mathbb{R})$. As ψ does not contain parameters nor quantifiers of sample sort, its truth in $\bar{M}$ and $\bar{\mathfrak{U}}$ depends only on the structures $M, \mathbb{R}$, respectvely $\mathfrak{U}, \mathbb{R}$. Then the fact is a consequence of $M \preceq \mathfrak{U}$.

Now assume that M is ω-saturated. We need to prove that for every tuples a, r, t in $\bar{M}$ of home, real, respectively sample sort we have

$$\bar{M} \vDash \varphi(a,r,t) \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi(a,r,t) \qquad \text{for all } \varphi(x,y,w) \in \bar{L}.$$

Reason by induction of the syntax. The only interesting case concern the existential quantifier of samle sort, say $\exists u$ where u has the sort of $S_{|x|}$. If $\bar{\mathfrak{U}} \vDash \exists u\, \varphi(u,a,r,t)$ then $\bar{\mathfrak{U}} \vDash \varphi(s,a,r,t)$ for some finite sample $s \in S_n$. Let $b_1,\dots,b_n \in \mathfrak{U}^{|x|}$ enumerate the support of s. Let $c_1,\dots,c_n \in M^{|x|}$ be such that $b_i \equiv_{a,\operatorname{supp} t} c_i$. By homogeneity, there is an $f \in \operatorname{Aut}(\mathfrak{U}/a, \operatorname{supp} t)$ such that $fb_i = c_i$. Extend f to an automorphism of $\bar{\mathfrak{U}}$ by requiring that f is the identity on $\mathbb{R}$ and $s\,b = (f\,s)(f\,b)$. Then $\bar{\mathfrak{U}} \vDash \varphi(f\,s,a,r,t)$, so $\bar{M} \vDash \varphi(f\,s,a,r,t)$ follows by induction hypothesis. □

2 Notation Throughout the following Δ is a collection of $L_x(\mathfrak{U})$ formulas or, depending on the context, the collection of sets defined by these. □

3 Lemma *Let μ be finitely additive signed measures on Δ, a Boolean algebra. Then there is $s \in {}^*S_{|x|}$ such that*

$$\mu_s\varphi(x) \quad = \quad \mu\varphi(x) \qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let u be a variable of same sort as ${}^*S_{|x|}$. We claim that the type $p(u)$ defined below is finitely consistent

$$p(u) \quad = \quad \Big\{ \sum_{\varphi(x)} ux = \mu\varphi(x) \quad : \quad \varphi(x) \in \Delta \Big\}$$

Let $\{\varphi_1(x),\dots,\varphi_n(x)\} \subseteq \Delta$. It suffices to show that there is $s \in {}^*S_{|x|}$ such that

1. $$\sum_{\varphi_i(x)} s\,x \quad = \quad \mu\varphi_i(x) \qquad \text{for } i = 1,\dots,n.$$

Without loss of generality we can assume that $\{\varphi_1(x),\dots,\varphi_n(x)\}$ is a Boolean algebra with atoms $\varphi_1(x),\dots,\varphi_k(x)$ for some $k \leq n$. Pick some $a_1,\dots,a_k \in \mathfrak{U}^{|x|}$ such that $a_i \vDash \varphi_i(x)$. Pick $s \in {}^*S_{|x|}$ with support $\{a_1,\dots,a_k\}$ and such that

$$s\,a_i \quad = \quad \mu\varphi_i(x) \qquad \text{for } i = 1,\dots,k.$$

Clearly 1 above is satisfied by the finite additivity of the measure. □

2 Pseudofinite sample

We say that a hyperfinite sample $s \in {}^*S_{|x|}$ is pseudofinite over M

1. $$\varphi(s) \quad \Rightarrow \quad \varphi(\bar{M}) \neq \varnothing \qquad \text{for every } \varphi(u) \in \bar{L}(\bar{M})$$

In other words, if $\operatorname{tp}(s/\bar{\mathfrak{U}})$ is finitely satisfied in $\bar{M}$. We prefer not to mention finite satisfiability in this context to avoid clash with the terminology in the next section. Note also that the condition above is equivalent to

2. $$\varphi(\bar{M}) = S_{|x|}{\restriction}M \quad \Rightarrow \quad \varphi(s) \qquad \text{for every } \varphi(u) \in \bar{L}(\bar{M})$$

The following example should justify the terminology.

4 Example We prove the following claim. Let L be the language of graphs. Let T be the theory of the random graph. Fix some $M \preceq \mathfrak{U}$ and let $s \in {}^*S_1$ be a pseudofinite sample over M. Then $\operatorname{supp} s$ is a pseudofinite graph.

Firstly, recall the definition of pseudofinite graph. Let $T_{\rm fg}$ the set of sentences in L that hold in every finite graph. A *pseudofinite graph* is any structure that models $T_{\rm fg}$.

If $\varphi \in L$ is a sentence, we denote by $\bar{\varphi}(u)$ the formula obtained by replacing in φ the quantifiers $\exists x$ and $\forall x$ with their bounded form: $\exists x \in \operatorname{supp} u$, respectively $\forall x \in \operatorname{supp} u$. Then for all $t \in {}^*S_1$, we have that $\bar{\varphi}(t)$ if and only if $\operatorname{supp} t \vDash \varphi$.

To prove the claim, suppose that $\neg\bar{\varphi}(s)$. Then, by 1 above, $\neg\bar{\varphi}(t)$ holds for some $t \in S_1{\restriction}M$. As $\operatorname{supp} t$ is a finite graph, $\varphi \notin T_{\rm fg}$. □

We present a stronger version of pseudofiniteness. This is obtained by strengthening 1 as follows. Require that for every $\varphi(u) \in \bar{L}(\bar{M})$

3. $\varphi(s) \quad \Rightarrow \quad \varphi(s \cdot \mathbb{1}_A)$ for some finite $A \subseteq M$

where $\mathbb{1}_M$ is the indicator of A.

3 Invariant samples

Here we introduce the notions of invariant and finitely satisfiable samples. There are two sensible variants. We consider the most stringent variant, which is also the easier to deal with.

We write $\operatorname{Aut}({}^*\mathfrak{U}/A, \{\mathfrak{U}\})$ for the set of automorphisms of ${}^*\mathfrak{U}$ that fix A pointwise and $\mathfrak{U}$ setwise. Every automorphism $f \in \operatorname{Aut}({}^*\mathfrak{U})$ has a canonical extension to an automorphism in $\operatorname{Aut}(\bar{\mathfrak{U}})$, which we denote by the same symbol f. Namely, this extension is the identity on ${}^*\mathbb{R}$ and maps $s \in {}^*S_n$ to the unique $fs \in {}^*S_n$ such that $(fs)\,(fa) = s\,a$.

Given $s \in {}^*S_{|x|}$, we say that μ_s is A-invariant if for every $f \in \operatorname{Aut}(\mathfrak{U}/A, \{\mathfrak{U}\})$

$$\mu_s \quad = \quad \mu_{fs}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad = \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z) \in L \text{ and } b \in \mathfrak{U}^{|z|}.$$

5 Definition We say that f is finitely satisfiable in M if for every $\varphi(x) \in L(\mathfrak{U})$

$$\varphi(\operatorname{supp} f) \neq \varnothing \quad \Rightarrow \quad \varphi(M) \neq \varnothing. \qquad □$$

The following lemma shows that the finite satisfiability of a sample corresponds to the finite satisfiability of the associated Keisler measure.

6 Lemma Let μ be as in Lemma 3 with $\Delta = L(\mathfrak{U})$. Assume that

$$\mu\varphi(x) \neq 0 \quad \Rightarrow \quad \varphi(M) \neq \varnothing \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Then there is $s \in {}^*S_{|x|}$ that is finitely satisfied in M and

$$\mu_s\varphi(x) \quad = \quad \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let $p(u)$ be as in the proof of Lemma 3. Define

$$q(u) \quad = \quad \Big\{\forall x\,[\varphi(x) \rightarrow ux = 0] \ \ : \ \ \varphi(x) \in \Delta,\ \varphi(M) = \varnothing\Big\}$$

We need to show that $p(u) \cup q(u)$ is finitely consistent. Apply the same reasoning as in the proof of Lemma 3 and note that Definition 9 is automatically saisfied. □

7 Fact Every sample $s \in {}^*S_{|x|}$ that is finitely satisfiable in M is invariant over M.

Proof If s is not M-invariant then for some $f \in \operatorname{Aut}({}^*\bar{\mathcal{U}}/M, \{\mathcal{U}\})$, some $\varphi(x,z) \in L$ and $b \in \mathcal{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \neq \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \neq \quad \mu_s\Big(\varphi(x,b) \not\leftrightarrow \varphi(x,fb)\Big)$$

Then there is $a \in \operatorname{supp} s$ such that $\varphi(a,b) \not\leftrightarrow \varphi(a,fb)$. Hence, from the finite satisfiability of s, we obtain that $\varphi(M,b) \neq \varphi(M,fb)$. This contradicts the M-invarance of μ_s. □

4 Smooth samples

We say that $s \in {}^*S_{|x|}$ is smooth over Δ, or over A when $\Delta = L(A)$, if for every $\varphi(x) \in L(\mathcal{U})$ and every $\varepsilon \in \mathbb{R}^+$ there are $\psi_i(x) \in \Delta$ such that $\psi_1(x) \to \varphi(x) \to \psi_2(x)$ and

$$\mu_s\psi_1(x) \quad \approx_\varepsilon \quad \mu_s\psi_2(x)$$

An easy argument of compactness yields this uniform version fo the condition above.

8 **Fact** The following are equivalent

1. $s \in {}^*S_{|x|}$ is smooth;
2. for every $t \in {}^*S_{|x|}$, if $\mu_s\varphi(x) \approx \mu_t\varphi(x)$ for all $\varphi(x) \in L(M)$ then the same holds for all $\varphi(x) \in L(\mathcal{U})$;
3. for every $\varphi(x,z) \in L$ and every $\varepsilon \in \mathbb{R}^+$ there is a

Proof Only 1⇒2 requires a proof. Negate 2. □

5 Altro

9 **Definition** We say that s is weakly independent of Δ over M if

$$\varphi(a) \quad \Rightarrow \quad \varphi(M) \neq \varnothing$$

for every $\varphi(x) \in \Delta$ and $a \in \operatorname{supp} s$. □

We write

6 The Radon-Nikodyn theorem

10 **Definition** Let μ, ν be finitely additive measures on Δ. We write $\nu \ll_\Delta \mu$ if for every $\varepsilon \in \mathbb{R}^+$ there is a $\delta \in \mathbb{R}^+$ such that

1. $$|\mu|\varphi(x) \leq \delta \quad \to \quad |\nu|\varphi(x) \leq \varepsilon$$

holds for every $\varphi(x) \in \Delta$. We write $\nu \ll \mu$ if 1 holds for all $\varphi(x) \in \bar{L}(\bar{\mathcal{U}})$. □

11 **Lemma** Let μ, ν be finitely additive signed measures on Δ, a Boolean algebra of small cardinality. Assume also that μ, ν are bounded, $\nu \ll_\Delta \mu$ and $\mu \geq 0$. Then there are $f \in \mathcal{F}_{|x|}$ and, for every $\varepsilon \in \mathbb{R}^+$ a simple function $g_\varepsilon \in \mathcal{F}_{|x|}$ such that

$$\begin{aligned}\mu_f\varphi(x) &\approx \mu\varphi(x)\\ \mu_{g_\varepsilon\cdot f}\varphi(x) &\approx_\varepsilon \nu\varphi(x) \qquad\text{for every } \varphi(x)\in\Delta.\end{aligned}$$

Proof As we can work separatey with ν^+ and ν^-, we can assume $\nu \geq 0$. Let δ_ε be a function that witnesses $\nu \ll_\Delta \mu$. Without loss of generality, we assume that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.

For every ε fix a maximal partition $\vartheta_{\varepsilon,1}(x),\dots,\vartheta_{\varepsilon,n_\varepsilon}$ such that $\mu\,\vartheta_{\varepsilon,i}(x) > \delta_\varepsilon$. Define, for $a \vDash \vartheta_{\varepsilon,i}(x)$

$$g_\varepsilon(a) \;=\; \nu\,\vartheta_{\varepsilon,i}(x)/\mu\,\vartheta_{\varepsilon,i}(x)$$

Let u be variable of sort $\mathcal{F}_{|x|}$. We claim that the type $p(u)$ defined below is finitely consistent.

$$\begin{aligned}p(u) \;=\; &\Big\{\sum_{\varphi(x)} ux \approx_{\delta_\varepsilon} \mu\varphi(x) \quad:\; \varphi(x)\in\Delta\Big\}\\ \cup\; &\Big\{\sum_{\varphi(x)} g_\varepsilon x\cdot ux \approx_\varepsilon \nu\varphi(x) \quad:\; \varphi(x)\in\Delta,\ \varepsilon\in\mathbb{R}^+\Big\}\end{aligned}$$

Let $\{\varphi_1(x),\dots,\varphi_n(x)\} \subseteq \Delta$. We need to show that there are $f,g \in \mathcal{F}_{|x|}$ such that

1. $$\sum_{\varphi_i(x)} fx \;=\; \mu\varphi_i(x) \qquad\text{for } i=1,\dots,n;$$
2. $$\sum_{\varphi_i(x)} gx \;=\; \nu\varphi_i(x) \qquad\text{for } i=1,\dots,n;$$

and

3. $$\forall a \subseteq \operatorname{supp} u \Big[\sum_{x\in a} fx \leq \delta(\varepsilon) \;\to\; \sum_{x\in a} gx \leq \varepsilon\Big] \qquad\text{for every } \varepsilon\in\mathbb{R}^+.$$

Without loss of generality we can assume that $\{\varphi_1(x),\dots,\varphi_n(x)\}$ is a Boolean algebra with atoms $\varphi_1(x),\dots,\varphi_k(x)$. Pick some $a_1,\dots,a_k \in \mathcal{U}^{|x|}$ such that $a_i \vDash \varphi(x)$. Pick f,g with support $\{a_1,\dots,a_k\}$ and such that

$$\begin{aligned}f(a_i) &= \mu\varphi_i(x) \qquad\text{for } i=1,\dots,k;\\ g(a_i) &= \nu\varphi_i(x) \qquad\text{for } i=1,\dots,k.\end{aligned}$$

Clearly 1 and 2 above are satisfied by the finite additivity of the measure. As for 3, it suffices to verify that

$$\sum_{j\in J} fa_i \leq \delta(\varepsilon) \;\to\; \sum_{j\in J} ga_j \leq \varepsilon \qquad\text{for every } J\subseteq\{1,\dots,k\}.$$

By the definition of f,g this is equivalent to

$$\mu\varphi_i(x) \leq \delta(\varepsilon) \;\to\; \nu\varphi_i(x) \leq \varepsilon \qquad\text{for some } i=1,\dots,n$$

which is holds because $\nu \ll_\Delta \mu$. □

The norm of $f \in {}^*S_{|x|}$ is defined as

$$\|f\| \;=\; \sum_{x=x} |fx|. \qquad □$$

The (internal) cardinality of the supprt is denoted by $|\operatorname{supp} f|$. In general, $|\operatorname{supp} f|$ is a hyperfinite integer. If $a,b \in {}^*S_{|x|}$ are $\{0,1\}$-valued we confused them with their support. E.g., we write $a \subseteq b$ for $\operatorname{supp} a \subseteq \operatorname{supp} b$.

12 Fact Assume Δ is a Boolean algebra of small cardinality. Let $f,g \in \mathcal{F}_{|x|}$ be such that $f \geq 0$ and $\|g\|_\infty$ is finite. Then for every $\varepsilon \in \mathbb{R}^+$ there are $h \in \mathcal{F}_{|x|}$ and $\lambda \in \mathbb{R}^+$ such that $|h| \leq \lambda f$ and $\|g - h\| \leq \varepsilon$.

Proof Given ε, let δ be as given by the assumption $g \ll l$. Let $n \in \mathbb{N}$ be such that $|g| \leq n$. □

13 Definition For $g \in \mathcal{F}_{|x|}$ we say that g is Δ-simple if it is (finite) linear combination of indicator functions of sets in Δ. □

14 Theorem Assume Δ is a Boolean algebra of small cardinality. Let $f, g \in \mathcal{F}_{|x|}$ and assume $f \geq 0$. If $g \ll_\Delta f$ than for every $\varepsilon \in \mathbb{R}^+$ there is a Δ-simple h such that $\|g - h \cdot f\| \leq \varepsilon$.