1 Hyperfinite samples

In this section we introduce hypefinite samples and prove that all Keisler measures are generated by some hypefinite sample (Lemma 3).

Below $\mathfrak{U}$ is a saturated model of a complete theory T in the language L. We write κ for the cardinality of $\mathfrak{U}$ and assume that κ is larger thean $|L|$.

For every $n \in \omega$ define

$$S_n \quad = \quad \Big\{ s : \mathfrak{U}^n \to \mathbb{R} \quad : \quad s\,a = 0 \text{ for all but finitely many } a \Big\}$$

The elements of S_n are called standard samples (we will mainly use $\mathbb{N}$-valued samples and interpret these as finite multisets). Let $\bar{\mathfrak{U}}$ be the multi-sorted structure $\langle \mathfrak{U}, \mathbb{R}, (S_n)_{n\in\omega} \rangle$. We call the first sort the home sort; the second is called the real sort. The remaining sorts are called sample sorts.

The language of $\bar{\mathfrak{U}}$ is denoted by $\bar{L}$. It contains L and a symbol for every subset of $\mathbb{R}^n$ and every function $\mathbb{R}^n \to \mathbb{R}$.

Moreover, for every formula in $\varphi(x, z) \in L$, where x is a variable of the home sort, the language $\bar{L}$ contains a function symbol of sort

$$S_{|x|} \times \mathfrak{U}^{|z|} \quad \to \quad \mathbb{R}$$

that we interpret as the function that maps

$$1. \qquad (s, b) \quad \mapsto \quad \sum_{\varphi(x,b)} s\,x.$$

As the functions in $S_{|x|}$ are null almost everywhere, the sum in 1 is well-defined. We will use two informal but suggestive symbols for this function: $\sum_{\varphi(x,b)} s\,x$ or $\mu_s \varphi(x, b)$. When $\varphi(x, b)$ is the formula $x = b$, we write $s\,b$.

Fix an elementary extension of $\bar{\mathfrak{U}}$ that is saturated and of larger cardinality. This elementary extension is denoted by ${}^*\bar{\mathfrak{U}} = \langle {}^*\mathfrak{U}, {}^*\mathbb{R}, ({}^*S_n)_{n\in\omega} \rangle$.

The elements of $\bigcup_{n\in\omega} {}^*S_n$ are called (hyperfinite) samples. The support of $s \in {}^*S_{|x|}$ is the definable hyperfinite set $\{a \in {}^*\mathfrak{U}^{|x|} : s\,a \neq 0\}$ which we denote by $\operatorname{supp} s$.

If $M \preceq \mathfrak{U}$ we write $S_n{\restriction}M$ for the set of functions $s \in S_n$ such that $\operatorname{supp} s \subseteq M^n$. We define $\bar{M}$ to be the structure $\langle M, \mathbb{R}, (S_n{\restriction}M)_{n\in\omega} \rangle$.

1 Fact Let $M \preceq \mathfrak{U}$ be ω-saturated. Then $\bar{M} \preceq \bar{\mathfrak{U}}$. In general, when M is not saturated, we have that for all sentences $\varphi \in \bar{L}(\bar{M})$ with no quantifiers of sample sort

$$\bar{M} \vDash \varphi \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi.$$

Proof We prove the second claim first. We can assume that the function $\mu_s \varphi(x, y)$ only occurs in atomic formulas of the form $\mu_s \varphi(x, y) = w$.

Fix $s \in S_{|x|}{\restriction}M$. Let $a_1, \ldots, a_n$ be an enumeration of $\operatorname{supp} s$ and define $r_i = s\,a_i$. The formula $\mu_s \varphi(x, y) = w$ is easily seen to be equivalent, both in $\bar{M}$ and in $\bar{\mathfrak{U}}$, to the conjunction of the formulas

$$\bigwedge_{i=1}^{n} \neg^{\varepsilon(i)} \varphi(a_i, y) \quad \to \quad \sum_{i=1}^{n} \varepsilon(i) \cdot r_i = w$$

as ε ranges over ${}^n2$. Hence, every sentence $\varphi \in \bar{L}(\bar{M})$ is equivalent to some sentence in $\psi \in L(M, \mathbb{R})$. As ψ does not contain parameters nor quantifiers of sample sort, its truth in $\bar{M}$ and $\bar{\mathfrak{U}}$ depends only on the structures $M, \mathbb{R}$, respectvely $\mathfrak{U}, \mathbb{R}$. Then the fact is a consequence of $M \preceq \mathfrak{U}$.

Now assume that M is ω-saturated. We need to prove that for every tuples a, r, t in $\bar{M}$ of home, real, respectively sample sort we have

$$\bar{M} \vDash \varphi(a,r,t) \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi(a,r,t) \qquad \text{for all } \varphi(x,y,w) \in \bar{L}.$$

Reason by induction of the syntax. The only interesting case concern the existential quantifier of samle sort, say $\exists u$ where u has the sort of $S_{|x|}$. If $\bar{\mathfrak{U}} \vDash \exists u\, \varphi(u,a,r,t)$ then $\bar{\mathfrak{U}} \vDash \varphi(s,a,r,t)$ for some finite sample $s \in S_n$. Let $b_1, \dots, b_n \in \mathfrak{U}^{|x|}$ enumerate the support of s. Let $c_1, \dots, c_n \in M^{|x|}$ be such that $b_i \equiv_{a, \operatorname{supp} t} c_i$. By homogeneity, there is an $f \in \operatorname{Aut}(\mathfrak{U}/a, \operatorname{supp} t)$ such that $fb_i = c_i$. Extend f to an automorphism of $\bar{\mathfrak{U}}$ by requiring that f is the identity on $\mathbb{R}$ and $f(s\,b) = (f\,s)(f\,b)$. Then $\bar{\mathfrak{U}} \vDash \varphi(fs,a,r,t)$, so $\bar{M} \vDash \varphi(fs,a,r,t)$ follows by induction hypothesis. □

2 Notation Throughout the following Δ is a collection of $L_x(\mathfrak{U})$ formulas or, depending on the context, the collection of sets defined by these. □

3 Lemma Let μ be finitely additive signed measures on Δ, a Boolean algebra. Then there is $s \in {}^*S_{|x|}$ such that

$$\mu_s \varphi(x) \quad = \quad \mu\varphi(x) \qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let u be a variable of sample sort. We claim that the type $p(u)$ defined below is finitely consistent

$$p(u) \quad = \quad \Big\{ \sum_{\varphi(x)} u\,x = \mu\varphi(x) \quad : \quad \varphi(x) \in \Delta \Big\}$$

Let $\{\varphi_1(x), \dots, \varphi_n(x)\} \subseteq \Delta$. It suffices to show that there is $s \in S_{|x|}$ such that

1. $$\sum_{\varphi_i(x)} s\,x \quad = \quad \mu\varphi_i(x) \qquad \text{for } i = 1, \dots, n.$$

Without loss of generality we can assume that $\{\varphi_1(x), \dots, \varphi_n(x)\}$ is a Boolean algebra with atoms $\varphi_1(x), \dots, \varphi_k(x)$ for some $k \leq n$. Pick some $a_1, \dots, a_k \in \mathfrak{U}^{|x|}$ such that $a_i \vDash \varphi_i(x)$. Pick $s \in S_{|x|}$ with support $\{a_1, \dots, a_k\}$ and such that

$$s\,a_i \quad = \quad \mu\varphi_i(x) \qquad \text{for } i = 1, \dots, k.$$

Clearly 1 above is satisfied by the finite additivity of the measure. □

We say that μ_s is bounded if there an $r \in \mathbb{R}$ such that $|\mu_s| < r$.

4 Corollary Let $s \in {}^*S_{|x|}$ be such that μ_s is bounded. Then there is a $t \in {}^*S_{|x|}$ such that $\mu_t = \operatorname{st}(\mu_s)$, where st denotes the standard part. □

2 Pseudofinite samples

We say that a hyperfinite sample $s \in {}^*S_{|x|}$ is pseudofinite over M

1. $$\varphi(s) \quad \Rightarrow \quad \varphi(\bar{M}) \neq \varnothing \qquad \text{for every } \varphi(u) \in \bar{L}(\bar{M})$$

or, equivalently,

2. $$\varphi(\bar{M}) = S_{|x|}{\restriction} M \quad \Rightarrow \quad \varphi(s) \qquad \text{for every } \varphi(u) \in \bar{L}(\bar{M})$$

In other words, s is pseudofinite if $\bar{\operatorname{tp}}(s/\bar{M})$ is finitely satisfied in $\bar{M}$ (we will not further mention finite satisfiability in this context to avoid clash with the terminology below). By Fact 1, every sample is pseudofinite over any ω-saturated model.

If 1 holds for every $\varphi(u) \in \bar{L}(\bar{\mathcal{U}})$ then we say that s is strongly pseudofinite. By the standard argument of existence of coheirs, for every $s \in {}^*S_{|x|}$ pseudofinite over M, there is a strongly pseudofinite $t \in {}^*S_{|x|}$ such that $s \equiv_{\bar{M}} t$.

The following example should justify the terminology.

5 Example We prove the following claim. Let L be the language of graphs. Let T be the theory of the random graph. Fix some $M \preceq \mathcal{U}$ and let $s \in {}^*S_1$ be a pseudofinite sample over M. Then $\operatorname{supp} s$ is a pseudofinite graph.

Firstly, recall the definition of pseudofinite graph. Let $T_{\rm fg}$ the set of sentences in L that hold in every finite graph. A *pseudofinite graph* is any structure that models $T_{\rm fg}$.

If $\varphi \in L$ is a sentence, we denote by $\bar{\varphi}(u)$ the formula obtained by replacing in φ the quantifiers $\exists x$ and $\forall x$ with their bounded form: $\exists x \in \operatorname{supp} u$, respectively $\forall x \in \operatorname{supp} u$. Then for all $t \in {}^*S_1$, we have that $\bar{\varphi}(t)$ if and only if $\operatorname{supp} t \vDash \varphi$.

To prove the claim, suppose that $\neg\bar{\varphi}(s)$. Then, by 1 above, $\neg\bar{\varphi}(t)$ holds for some $t \in S_1 {\restriction} M$. As $\operatorname{supp} t$ is a finite graph, $\varphi \notin T_{\rm fg}$. □

We use the notion above to prove the following.

6 Fact Let $s \in {}^*S_{|x|}$. Then every formula $\varphi(x) \in L(\mathcal{U})$ such that $\varphi(\mathcal{U}) \subseteq \operatorname{supp} s$ is algebraic. In particular, if $\operatorname{supp} s$ is definable by a formula in $L(\mathcal{U})$ then it is finite.

Proof Let M be a saturated model containing all parameters of $\varphi(x)$. As noted above, s is pseudofinite over M. Therefore the formula below is satisfied in $\bar{M}$

$$\forall x \Big[\varphi(x) \quad \rightarrow \quad x \in \operatorname{supp} u\Big]$$

The fact follows immediately. □

7 Question Let $s \in {}^*S_{|x|}$ be such that for every $\varphi(u) \in \bar{L}(\bar{M})$

3. $$\varphi(s) \quad \Rightarrow \quad \varphi(s \cdot \mathbb{1}_A) \text{ for some finite } A \subseteq M.$$

Does it follows that $\operatorname{supp} s$ is finite? □

3 Invariant samples I

We introduce the notions of invariant and finitely satisfiable samples. There are two sensible variants. Here we consider the most stringent variant, the less stringent one is discussed in the following section.

We write $\operatorname{Aut}({}^*\mathcal{U}/A, \{\mathcal{U}\})$ for the set of automorphisms of ${}^*\mathcal{U}$ that fix A pointwise and $\mathcal{U}$ setwise. Every automorphism $f \in \operatorname{Aut}({}^*\mathcal{U})$ has a canonical extension to an automorphism in $\operatorname{Aut}({}^*\bar{\mathcal{U}})$, which we denote by the same symbol f. Namely, this extension is the identity on ${}^*\mathbb{R}$ and maps $s \in {}^*S_n$ to the unique $fs \in {}^*S_n$ such that $(fs)\,(fa) = f(s\,a)$.

Given $s \in {}^*S_{|x|}$, we say that μ_s is invariant over A if for every $f \in \operatorname{Aut}({}^*\mathcal{U}/A, \{\mathcal{U}\})$

$$\mu_s \quad = \quad \mu_{f\,s}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad = \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z)\in L \text{ and } b\in\mathfrak{U}^{|z|}.$$

8 **Definition** We say that s is finitely satisfiable in M if for every $\varphi(x)\in L(\mathfrak{U})$

$$\varphi(\operatorname{supp} s)\neq\varnothing \quad\Rightarrow\quad \varphi(M)\neq\varnothing.$$ □

The following lemma shows that the finite satisfiability of a sample corresponds (in a sense) to the finite satisfiability of the associated Keisler measure.

9 **Lemma** Let μ be as in Lemma 3 with $\Delta = L(\mathfrak{U})$. Assume that

$$\mu\varphi(x)\neq 0 \quad\Rightarrow\quad \varphi(M)\neq\varnothing \qquad\qquad \text{for every } \varphi(x)\in\Delta.$$

Then there is $s\in {}^*S_{|x|}$ that is finitely satisfied in M and

$$\mu_s\varphi(x) \quad = \quad \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x)\in\Delta.$$

Proof Let $p(u)$ be as in the proof of Lemma 3. Define

$$q(u) \quad = \quad \Big\{\forall x\,[\varphi(x)\rightarrow ux=0] \ \ : \ \ \varphi(x)\in\Delta,\ \varphi(M)=\varnothing\Big\}$$

We need to show that $p(u)\cup q(u)$ is finitely consistent. Apply the same reasoning as in the proof of Lemma 3. □

10 **Fact** Every sample $s\in {}^*S_{|x|}$ that is finitely satisfiable in M is invariant over M.

Proof If s is not M-invariant then for some $f\in\operatorname{Aut}({}^*\mathfrak{U}/M,\{\mathfrak{U}\})$, some $\varphi(x,z)\in L$ and $b\in\mathfrak{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \neq \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \neq \quad \mu_s\Big(\varphi(x,b)\not\leftrightarrow\varphi(x,fb)\Big)$$

Then there is $a\in\operatorname{supp} s$ such that $\varphi(a,b)\not\leftrightarrow\varphi(a,fb)$. Hence, from the finite satisfiability of s, we obtain that $\varphi(M,b)\neq\varphi(M,fb)$. This contradicts the M-invarance of μ_s. □

4 Invariant samples II

The exposition is parallel to that of the previous section with no significant differences.

We write $\operatorname{Aut}({}^*\bar{\mathfrak{U}}/A,\{\mathfrak{U}\})$ for the set of automorphisms of ${}^*\bar{\mathfrak{U}}$ that fix A pointwise and $\mathfrak{U}$ setwise. Given $s\in {}^*S_{|x|}$, we say that μ_s is weakly invariant over A if for every $f\in\operatorname{Aut}({}^*\bar{\mathfrak{U}}/A,\{\mathfrak{U}\})$

$$\mu_s \quad \approx \quad \mu_{fs}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad \approx \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z)\in L \text{ and } b\in\mathfrak{U}^{|z|}.$$

11 **Definition** We say that s is weakly finitely satisfiable in M if for every $\varphi(x)\in L(\mathfrak{U})$

$$\mu_s\varphi(x)\not\approx 0 \quad\Rightarrow\quad \varphi(M)\neq\varnothing.$$ □

12 **Fact** Every sample $s\in {}^*S_{|x|}$ that is weakly finitely satisfiable in M is weakly invariant over M.

Proof If s is not weakly M-invariant then for some $f \in \operatorname{Aut}({}^*\bar{\mathfrak{U}}/A, \{\mathfrak{U}\})$, some $\varphi(x,z) \in L$ and $b \in \mathfrak{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \not\approx \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \not\approx \quad \mu_s\Big(\varphi(x,b) \not\leftrightarrow \varphi(x,fb)\Big)$$

Then, by the finite satisfiability of s, we obtain that $\varphi(M,b) \neq \varphi(M,fb)$. This contradicts the M-invariance of μ_s. □

5 Smooth samples

We say that $s \in {}^*S_{|x|}$ is smooth over A, if for every $\varphi(x,z) \in L$, every $b \in \mathfrak{U}^{|z|}$ and every $\varepsilon \in \mathbb{R}^+$ there are $\psi_i(x) \in L(M)$ such that $\psi_1(x) \rightarrow \varphi(x,b) \rightarrow \psi_2(x)$ and

$$\mu_s\psi_1(x) \quad \approx_\varepsilon \quad \mu_s\psi_2(x)$$

If for a given $\varphi(x,z) \in L$ and $\varepsilon \in \mathbb{R}^+$, finitely many pairs of formulas $\psi_i(x)$ suffices for all $b \in \mathfrak{U}^{|z|}$, we say that s is uniformly smooth.

13 Fact The following are equivalent for every $s \in {}^*S_{|x|}$

1. s is smooth over A;
2. s is uniformly smooth over A;
3. for every $t \in {}^*S_{|x|}$, if $\mu_s\varphi(x) \approx \mu_t\varphi(x)$ for all $\varphi(x) \in L(A)$, then $\mu_s \approx \mu_t$.

Proof Compactness easily proves 1⇔2.

1⇒3 Negate 3. Then for $\mu_s \not\approx_\varepsilon \mu_t$ for some $\varepsilon \in \mathbb{R}^+$ and some $\varphi(x) \in L(\mathfrak{U})$. Suppose s is smooth and let $\psi_i(x)$ as above but with $\varepsilon/3$ for ε. Then also $\mu_t\big(\psi_2(x) \smallsetminus \psi_1(x)\big) < \varepsilon/3$. Therefore $\mu_t\varphi(x) \approx_{\varepsilon/3} \mu_t\psi_2(x) = \mu_s\psi_2(x) \approx_{\varepsilon/3} \mu_s\varphi(x)$. A contradiction.

3⇒1 Negate 1. Compactness and the fact that $\bar{\mathfrak{U}} \preceq {}^*\bar{\mathfrak{U}}$ ensure the existence of s_1 and s_2 such that $s_1 \equiv_M s_2 \equiv_M s$ and

$$\begin{aligned}
&(\forall a \in \operatorname{supp} s_1 \smallsetminus \operatorname{supp} s) \quad \varphi(a) \quad \wedge \quad (\forall a \in \operatorname{supp} s \smallsetminus \operatorname{supp} s_1)\, \neg\varphi(a);\\
&(\forall a \in \operatorname{supp} s_2 \smallsetminus \operatorname{supp} s)\, \neg\varphi(a) \quad \wedge \quad (\forall a \in \operatorname{supp} s \smallsetminus \operatorname{supp} s_2) \quad \varphi(a).
\end{aligned}$$

Moreover we require that the cardinalities of $\varphi(\operatorname{supp} s_1)$ and $\neg\varphi(\operatorname{supp} s_2)$ are maximal given the properties above.

Let

$$\begin{aligned}
r_1 \quad &= \quad \inf\big\{r \in \mathbb{R} \;:\; \mu_s\psi(x) \leq r \text{ and } \psi(x) \rightarrow \varphi(x)\big\};\\
r_2 \quad &= \quad \sup\big\{r \in \mathbb{R} \;:\; \mu_s\psi(x) \geq r \text{ and } \varphi(x) \rightarrow \psi(x)\big\}.
\end{aligned}$$

Then μ_{s_1} and μ_{s_2} coincide with μ_s on $L(M)$ but $\mu_{s_2}\varphi(x) - \mu_{s_1}\varphi(x) = r_2 - r_1 \geq \varepsilon$. This contraddicts 3. □