1 Hyperfinite samples

In this section we introduce hypefinite samples and prove that all Keisler measures are generated by some hypefinite sample (Lemma 4).

Below $\mathfrak{U}$ is a saturated model of a complete theory T in the language L. We write κ for the cardinality of $\mathfrak{U}$ and assume that κ an inaccessible cardinal larger than $|L|$.

For every $n \in \omega$ define

$$S_n \quad = \quad \Big\{ s : \mathfrak{U}^n \to \mathbb{R} \quad : \quad s\,a = 0 \text{ for all but finitely many } a \Big\}$$

The elements of S_n are called standard samples (we will mainly use $\mathbb{N}$-valued samples and interpret these as finite multisets i.e. sets where elements can occur with different multiplicity). Let $\bar{\mathfrak{U}}$ be the multi-sorted structure $\langle \mathfrak{U}, \mathbb{R}, (S_n)_{n\in\omega}\rangle$. Clearly, $|\bar{\mathfrak{U}}| = \kappa$. We call the first sort the home sort; the second is called the real sort. The remaining sorts are called sample sorts.

The language of $\bar{\mathfrak{U}}$ is denoted by $\bar{L}$. It contains L and a symbol for every subset of $\mathbb{R}^n$ and for every function $\mathbb{R}^n \to \mathbb{R}$. Moreover, for every formula in $\varphi(x, z) \in L$ the language $\bar{L}$ contains a function symbol of sort

$$S_{|x|} \times \mathfrak{U}^{|z|} \quad \to \quad \mathbb{R}$$

that we interpret as the function that maps

$$\text{1.} \qquad (s, b) \quad \mapsto \quad \sum_{\varphi(x,b)} s\,x.$$

As the functions in $S_{|x|}$ are null almost everywhere, the sum in 1 is well-defined. We will use two informal but suggestive symbols for this function: $\sum_{\varphi(x,b)} s\,x$ or $\mu_s\varphi(x, b)$. When $\varphi(x, b)$ is the formula $x = b$, we write $s\,b$.

There are two extensions of $\bar{\mathfrak{U}}$ that are relevant in the following. The first one is denoted by ${}^*\bar{\mathfrak{U}} = \langle {}^*\mathfrak{U}, {}^*\mathbb{R}, ({}^*S_n)_{n\in\omega}\rangle$. This is an elementary extension of $\bar{\mathfrak{U}}$ that is saturated and has cardinality $> \kappa$. The second extension is an intermediate saturated model ${}^\circ\mathfrak{U}$ such that $\bar{\mathfrak{U}} \preceq {}^\circ\bar{\mathfrak{U}} \preceq {}^*\bar{\mathfrak{U}}$ and that $|{}^\circ\mathfrak{U}| = \kappa$. As κ is inaccessible such model exist and we can assume that the home sort ${}^\circ\bar{\mathfrak{U}}$ is $\mathfrak{U}$. The other sorts are donoted by ${}^\circ\mathbb{R}$ and $({}^\circ S_n)_{n\in\omega}$.

1 Remark The model ${}^\circ\bar{\mathfrak{U}}$ plays the role of the moster model of $\bar{T} = \mathrm{Th}(\bar{\mathfrak{U}})$, while ${}^*\bar{\mathfrak{U}}$ is a model where all global types (i.e. types over ${}^\circ\bar{\mathfrak{U}}$) are realized. In fact, for notational reasons we will only mention global types through their realizations in ${}^*\bar{\mathfrak{U}}$. Intuitively the samples in ${}^*\bar{\mathfrak{U}}$ corresponds to the global Keisler measures while the samples in ${}^\circ\bar{\mathfrak{U}}$ corresponds to the smoohth measures. □

The elements of $\bigcup_{n\in\omega} {}^*S_n$ are called (hyperfinite) samples. The support of $s \in {}^*S_{|x|}$ is the definable hyperfinite set $\{a \in {}^*\mathfrak{U}^{|x|} : s\,a \neq 0\}$ which we denote by $\operatorname{supp} s$.

If $M \preceq \mathfrak{U}$ we write $S_n{\restriction}M$ for the set of functions $s \in S_n$ such that $\operatorname{supp} s \subseteq M^n$. We define $\bar{M}$ to be the structure $\langle M, \mathbb{R}, (S_n{\restriction}M)_{n\in\omega}\rangle$.

2 Fact Let $M \preceq \mathfrak{U}$ be ω-saturated. Then $\bar{M} \preceq \bar{\mathfrak{U}}$. In general, when M is not saturated, we have that for all sentences $\varphi \in \bar{L}(\bar{M})$ with no quantifiers of sample sort

$$\bar{M} \vDash \varphi \quad \Leftrightarrow \quad \bar{\mathfrak{U}} \vDash \varphi.$$

Proof We prove the second claim first. We can assume that the function $\mu_s\varphi(x, y)$

only occurs in atomic formulas of the form $\mu_s\varphi(x,y) = w$.

Fix $s \in S_{|x|}{\restriction}M$. Let $a_1,\dots,a_n$ be an enumeration of $\operatorname{supp} s$ and define $r_i = s\,a_i$. The formula $\mu_s\varphi(x,y) = w$ is easily seen to be equivalent, both in $\bar{M}$ and in $\bar{\mathcal{U}}$, to the conjunction of the formulas

$$\bigwedge_{i=1}^{n} \neg^{\varepsilon(i)}\varphi(a_i,y) \quad\to\quad \sum_{i=1}^{n}\varepsilon(i)\cdot r_i = w$$

as ε ranges over ${}^n2$. Hence, every sentence $\varphi \in \bar{L}(\bar{M})$ is equivalent to some sentence in $\psi \in L(M,\mathbb{R})$. As ψ does not contain parameters nor quantifiers of sample sort, its truth in $\bar{M}$ and $\bar{\mathcal{U}}$ depends only on the structures $M,\mathbb{R}$, respectvely $\mathcal{U},\mathbb{R}$. Then # is a consequence of $M \preceq \mathcal{U}$.

Now assume that M is ω-saturated. We need to prove that for every tuples a,r,t in $\bar{M}$ of home, real, respectively sample sort we have

$$\bar{M} \vDash \varphi(a,r,t) \quad\Leftrightarrow\quad \bar{\mathcal{U}} \vDash \varphi(a,r,t) \qquad\qquad \text{for all } \varphi(x,y,w) \in \bar{L}.$$

Reason by induction of the syntax. The only interesting case concern the existential quantifier of samle sort, say $\exists u$ where u has the sort of $S_{|x|}$. If $\bar{\mathcal{U}} \vDash \exists u\, \varphi(u,a,r,t)$ then $\bar{\mathcal{U}} \vDash \varphi(s,a,r,t)$ for some finite sample $s \in S_n$. Let $b_1,\dots,b_n \in \mathcal{U}^{|x|}$ enumerate the support of s. Let $c_1,\dots,c_n \in M^{|x|}$ be such that $b_i \equiv_{a,\operatorname{supp} t} c_i$. By homogeneity, there is an $f \in \operatorname{Aut}(\mathcal{U}/a, \operatorname{supp} t)$ such that $f b_i = c_i$. Extend f to an automorphism of $\bar{\mathcal{U}}$ by requiring that f is the identity on $\mathbb{R}$ and $f(s\,b) = (f\,s)(f\,b)$. Then $\bar{\mathcal{U}} \vDash \varphi(fs,a,r,t)$, so $\bar{M} \vDash \varphi(fs,a,r,t)$ follows by induction hypothesis. □

3 Notation Throughout the following Δ is a collection of $L_x(\mathcal{U})$ formulas or, depending on the context, the collection of sets defined by these formulas. □

4 Lemma Let μ be finitely additive signed measures on Δ, a Boolean algebra. Then there is $s \in {}^*S_{|x|}$ such that

$$\# \qquad \mu_s\varphi(x) \;=\; \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let u be a variable of sample sort. We claim that the type $p(u)$ defined below is finitely consistent

$$p(u) \;=\; \Big\{ \sum_{\varphi(x)} u\,x = \mu\varphi(x) \quad:\quad \varphi(x) \in \Delta \Big\}$$

Let $\{\varphi_1(x),\dots,\varphi_n(x)\} \subseteq \Delta$. It suffices to show that there is $s \in S_{|x|}$ such that

$$1. \qquad \sum_{\varphi_i(x)} s\,x \;=\; \mu\varphi_i(x) \qquad\qquad \text{for } i = 1,\dots,n.$$

Without loss of generality we can assume that $\{\varphi_1(x),\dots,\varphi_n(x)\}$ is a Boolean algebra with atoms $\varphi_1(x),\dots,\varphi_k(x)$ for some $k \le n$. Pick some $a_1,\dots,a_k \in \mathcal{U}^{|x|}$ such that $a_i \vDash \varphi_i(x)$. Pick $s \in S_{|x|}$ with support $\{a_1,\dots,a_k\}$ and such that

$$s\,a_i \;=\; \mu\varphi_i(x) \qquad\qquad \text{for } i = 1,\dots,k.$$

Clearly 1 above is satisfied by the finite additivity of the measure. □

We say that μ_s is bounded if there an $r \in \mathbb{R}$ such that $|\mu_s| < r$.

5 Corollary Let $s \in {}^*S_{|x|}$ be such that μ_s is bounded. Then there is a $t \in {}^*S_{|x|}$ such that $\mu_t = \operatorname{st}(\mu_s)$, where st denotes the standard part. □

2 Pseudofinite samples

We say that a hyperfinite sample $s \in {}^*S_{|x|}$ is pseudofinite over M

1. $$\varphi(s) \quad \Rightarrow \quad \varphi(\bar{M}) \neq \varnothing \qquad\qquad \text{for every } \varphi(u) \in \bar{L}(\bar{M})$$

or, equivalently,

2. $$\varphi(\bar{M}) = S_{|x|}{\restriction}M \quad \Rightarrow \quad \varphi(s) \qquad\qquad \text{for every } \varphi(u) \in \bar{L}(\bar{M})$$

In other words, s is pseudofinite if $\bar{\text{tp}}(s/\bar{M})$ is finitely satisfied in $\bar{M}$ (we will not further mention finite satisfiability in this context to avoid clash with the terminology in the following sections). By Fact 2, every sample is pseudofinite over any ω-saturated model.

The following example should justify the terminology.

6 Example We prove the following claim. Let L be the language of graphs. Let T be the theory of some infinite graph. Fix some $M \preceq \mathfrak{U}$ and let $s \in {}^*S_1$ be a pseudofinite sample over M. Then $\operatorname{supp} s$ is a pseudofinite graph.

Recall the definition of pseudofinite graph. Let T_{fg} the set of sentences in L that hold in every finite graph. A *pseudofinite graph* is any structure that models T_{fg}.

If $\varphi \in L$ is a sentence, we denote by $\bar{\varphi}(u)$ the formula obtained by replacing in φ the quantifiers $\exists x$ and $\forall x$ with their bounded form: $\exists x \in \operatorname{supp} u$, respectively $\forall x \in \operatorname{supp} u$. Then for all $s \in {}^*S_1$, we have that $\bar{\varphi}(s)$ if and only if $\operatorname{supp} s \vDash \varphi$.

To prove the claim, suppose that $\operatorname{supp} s \nvDash \varphi$. From the paragraph above we obtain that $\neg\bar{\varphi}(s)$. Hence, by pseudofiniteness, $\neg\bar{\varphi}(t)$ holds for some $t \in S_1{\restriction}M$. As $\operatorname{supp} t$ is a finite graph, $\varphi \notin T_{\text{fg}}$. □

If 1 holds for every $\varphi(u) \in \bar{L}(\bar{\mathfrak{U}})$ then we say that s is strongly pseudofinite. By the standard argument of existence of coheirs, for every $s \in {}^*S_{|x|}$ there is a strongly pseudofinite $t \in {}^*S_{|x|}$ such that $s \equiv_{\bar{M}} t$.

We use the notion of pseudofiniteness to prove the following.

7 Fact Let $s \in {}^*S_{|x|}$ and let $M \preceq \mathfrak{U}$. Then every formula $\varphi(x) \in L(\bar{M})$ such that $\varphi(M) \subseteq \operatorname{supp} s$ is algebraic. In particular, if $\operatorname{supp} s$ is definable by a formula in $L(\mathfrak{U})$ then it is finite.

Proof Let s' be such that $s \equiv_{\bar{M}} s'$ and s' is pseudofinite over M. By elementarity $\varphi(M) \subseteq \operatorname{supp} s'$. Therefore the formula $\varphi(u)$ below is satisfied by s'

$$\forall x \Big[\varphi(x) \quad \rightarrow \quad x \in \operatorname{supp} u\Big]$$

By pseudofiniteness this formula is satisfied in $\bar{M}$ hence $\varphi(M)$ is finite. □

8 Question Let $s \in {}^*S_{|x|}$ be such that for every $\varphi(u) \in \bar{L}(\bar{M})$

3. $$\varphi(s) \quad \Rightarrow \quad \varphi(s \cdot \mathbb{1}_A) \text{ for some finite } A \subseteq M.$$

Does it follows that $\operatorname{supp} s$ is finite? □

3 Invariant samples I

We introduce the notions of *invariant* and *finitely satisfiable* samples. There are two sensible variants of these notions. Here we consider the most stringent variant, the less stringent one is discussed in the following section.

We write $\mathrm{Aut}({}^*\mathfrak{U}/A, \{\mathfrak{U}\})$ for the set of automorphisms of ${}^*\mathfrak{U}$ that fix A pointwise and $\mathfrak{U}$ setwise. Note that every automorphism $f \in \mathrm{Aut}({}^*\mathfrak{U})$ has a canonical extension to an automorphism in $\mathrm{Aut}({}^*\bar{\mathfrak{U}})$, which we denote by the same symbol f. Namely, this is the extension that is the identity on ${}^*\mathbb{R}$ and that maps $s \in {}^*S_n$ to the unique $fs \in {}^*S_n$ such that $(fs)\,(fa) = f(s\,a)$.

Given $s \in {}^*S_{|x|}$, we say that μ_s is invariant over A if for every $f \in \mathrm{Aut}({}^*\mathfrak{U}/A, \{\mathfrak{U}\})$

$$\mu_s \quad = \quad \mu_{f\,s}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x, b) \quad = \quad \mu_s\varphi(x, fb) \qquad\qquad \text{for every } \varphi(x, z) \in L \text{ and } b \in \mathfrak{U}^{|z|}.$$

9 Definition We say that s is finitely satisfiable in M if for every $\varphi(x) \in L(\mathfrak{U})$

$$\varphi(\mathrm{supp}\, s) \neq \varnothing \quad \Rightarrow \quad \varphi(M) \neq \varnothing. \qquad \square$$

The following lemma shows that the finite satisfiability of a sample corresponds (in a sense) to the finite satisfiability of the associated Keisler measure.

10 Lemma Let μ be as in Lemma 4 with $\Delta = L(\mathfrak{U})$. Assume that

$$\mu\varphi(x) \neq 0 \quad \Rightarrow \quad \varphi(M) \neq \varnothing \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Then there is $s \in {}^*S_{|x|}$ that is finitely satisfied in M and

$$\mu_s\varphi(x) \quad = \quad \mu\varphi(x) \qquad\qquad \text{for every } \varphi(x) \in \Delta.$$

Proof Let $p(u)$ be as in the proof of Lemma 4. Define

$$q(u) \quad = \quad \Big\{\forall x\,[\varphi(x) \to ux = 0] \;\;:\;\; \varphi(x) \in \Delta,\ \varphi(M) = \varnothing\Big\}$$

We need to show that $p(u) \cup q(u)$ is finitely consistent. Apply the same reasoning as in the proof of Lemma 4. $\square$

11 Fact Every sample $s \in {}^*S_{|x|}$ that is finitely satisfiable in M is invariant over M.

Proof If s is not M-invariant then for some $f \in \mathrm{Aut}({}^*\mathfrak{U}/M, \{\mathfrak{U}\})$, some $\varphi(x, z) \in L$ and $b \in \mathfrak{U}^{|z|}$

$$\mu_s\varphi(x, b) \quad \neq \quad \mu_s\varphi(x, fb)$$

In particular

$$0 \quad \neq \quad \mu_s\Big(\varphi(x, b) \not\leftrightarrow \varphi(x, fb)\Big)$$

Then there is $a \in \mathrm{supp}\, s$ such that $\varphi(a, b) \not\leftrightarrow \varphi(a, fb)$. Hence, from the finite satisfiability of s, we obtain that $\varphi(M, b) \neq \varphi(M, fb)$. This contradicts the M-invarance of μ_s. $\square$

4 Invariant samples II

The exposition is parallel to that of the previous section with no significant differences.

We write $\mathrm{Aut}({}^*\bar{\mathfrak{U}}/A, \{\mathfrak{U}\})$ for the set of automorphisms of ${}^*\bar{\mathfrak{U}}$ that fix A pointwise and $\mathfrak{U}$ setwise. Given $s \in {}^*S_{|x|}$, we say that μ_s is weakly invariant over A if for every $f \in \mathrm{Aut}({}^*\bar{\mathfrak{U}}/A, \{\mathfrak{U}\})$

$$\mu_s \quad \approx \quad \mu_{f\,s}$$

Note that this is equivalent to requiring that

$$\mu_s\varphi(x,b) \quad \approx \quad \mu_s\varphi(x,fb) \qquad\qquad \text{for every } \varphi(x,z) \in L \text{ and } b \in \mathfrak{U}^{|z|}.$$

12 Definition We say that s is weakly finitely satisfiable in M if for every $\varphi(x) \in L(\mathfrak{U})$

$$\mu_s\varphi(x) \not\approx 0 \quad \Rightarrow \quad \varphi(M) \neq \varnothing. \qquad \square$$

13 Fact Every sample $s \in {}^*S_{|x|}$ that is weakly finitely satisfiable in M is weakly invariant over M.

Proof If s is not weakly M-invariant then for some $f \in \mathrm{Aut}({}^*\bar{\mathfrak{U}}/A, \{\mathfrak{U}\})$, some $\varphi(x,z) \in L$ and $b \in \mathfrak{U}^{|z|}$

$$\mu_s\varphi(x,b) \quad \not\approx \quad \mu_s\varphi(x,fb)$$

In particular

$$0 \quad \not\approx \quad \mu_s\Big(\varphi(x,b) \not\leftrightarrow \varphi(x,fb)\Big)$$

Then, by the finite satisfiability of s, we obtain that $\varphi(M,b) \neq \varphi(M,fb)$. This contradicts the M-invariance of μ_s. $\square$

5 Smooth samples

We say that a non-negative sample $s \in {}^*S_{|x|}$ is smooth over A, if for every $\varphi(x,z) \in L$, every $b \in \mathfrak{U}^{|z|}$ and every $\varepsilon \in \mathbb{R}^+$ there are two formulas $\psi_1(x), \psi_2(x) \in L(M)$ such that $\psi_1(x) \to \varphi(x,b) \to \psi_2(x)$ and

$$\mu_s\psi_1(x) \quad \approx_\varepsilon \quad \mu_s\psi_2(x)$$

If for a given $\varphi(x,z) \in L$ and $\varepsilon \in \mathbb{R}^+$, finitely many pairs of formulas $\psi_1(x), \psi_2(x)$ suffices for all $b \in \mathfrak{U}^{|z|}$, we say that s is uniformly smooth.

14 Fact The following are equivalent for every non-negative sample $s \in {}^*S_{|x|}$

1. s is smooth over A;
2. s is uniformly smooth over A;
3. if $t \in {}^*S_{|x|}$ is non-negative and $\mu_s\psi(x) \approx \mu_t\psi(x)$ for all $\psi(x) \in L(A)$, then $\mu_s \approx \mu_t$.

Proof A compactness argument easily proves 1⇔2.

1⇒3 Negate 3. Then for $\mu_s \not\approx_\varepsilon \mu_t$ for some $\varepsilon \in \mathbb{R}^+$ and some $\varphi(x) \in L(\mathfrak{U})$. Suppose s is smooth and let $\psi_1(x), \psi_2(x)$ as above but with $\varepsilon/3$ for ε. Then also $\mu_t\big(\psi_2(x) \smallsetminus \psi_1(x)\big) < \varepsilon/3$. Therefore $\mu_t\varphi(x) \approx_{\varepsilon/3} \mu_t\psi_2(x) = \mu_s\psi_2(x) \approx_{\varepsilon/3} \mu_s\varphi(x)$. A contradiction.

3⇒1 Negate 1. Compactness and the fact that $\bar{\mathfrak{U}} \preceq {}^*\bar{\mathfrak{U}}$ ensure the existence of s_1 and s_2 such that $s_1 \equiv_M s_2 \equiv_M s$ and

$$\begin{aligned}
&\big(\forall a \in \operatorname{supp} s_1 \smallsetminus \operatorname{supp} s\big) \quad \varphi(a) \quad \wedge \quad \big(\forall a \in \operatorname{supp} s \smallsetminus \operatorname{supp} s_1\big) \,\neg\varphi(a);\\
&\big(\forall a \in \operatorname{supp} s_2 \smallsetminus \operatorname{supp} s\big) \,\neg\varphi(a) \quad \wedge \quad \big(\forall a \in \operatorname{supp} s \smallsetminus \operatorname{supp} s_2\big) \quad \varphi(a).
\end{aligned}$$

Moreover we require that the cardinalities of $\varphi(\operatorname{supp} s_1)$ and $\neg\varphi(\operatorname{supp} s_2)$ are maximal given the properties above.

$$\begin{aligned}
\text{Let} \qquad r_1 \quad &= \quad \inf\big\{r \in \mathbb{R} \;:\; \mu_s\psi(x) \leq r \text{ and } \psi(x) \to \varphi(x)\big\};\\
r_2 \quad &= \quad \sup\big\{r \in \mathbb{R} \;:\; \mu_s\psi(x) \geq r \text{ and } \varphi(x) \to \psi(x)\big\}.
\end{aligned}$$

Then μ_{s_1} and μ_{s_2} coincide with μ_s on $L(M)$ but $\mu_{s_2}\varphi(x) - \mu_{s_1}\varphi(x) = r_2 - r_1 \geq \varepsilon$. This contraddicts 3. □

15 Fact ??? The following are equivalent

1. s is smooth;
2. every saturated elementary extension of $\bar{\mathfrak{U}}$ contains a sample t such that $\mu_t \approx \mu_s$.