

Topics around Vapnik-Chevronenkis theory

1	The Sauer-Shelah Lemma	2
1	Two equivalent frameworks	2
2	The Sauer-Shelah Lemma	4
3	Pajor variant and the method of shifting	5
4	Notes and references	7
2	Samples and approximations of measures	8
1	Samples and subsamples	8
2	Discrepancy	8
3	Random colorings	9
4	Important inequalities	10
5	A uniform law of large numbers	11
3	Packings and transversals	14
1	Linear programming / convex optimization	14
2	Transversals and packings	14
4	A piercing problem	16
1	Appendix: Farkas' lemma	17

Chapter 1

The Sauer-Shelah Lemma

1 Two equivalent frameworks

Let \mathcal{U} and \mathcal{V} be two sets and let $\varphi \subseteq \mathcal{U} \times \mathcal{V}$. In other words φ is a **bipartite graph**. We may call φ an (abstract) **incidence relation** and write $\varphi(x, y)$ for $\langle x, y \rangle \in \varphi$. Sets of the form

$$\varphi(\mathcal{U}, b) = \{a \in \mathcal{U} : \varphi(a, b)\}$$

are called **definable sets** or, when other relations are involved, **φ -definable sets**. The collection of all definable sets is denoted by $\varphi(\mathcal{U}, b)_{b \in \mathcal{V}}$.

Often we need to consider the **trace** of a definable set on some arbitrary $A \subseteq \mathcal{U}$

$$\varphi(A, b) = \{a \in A : \varphi(a, b)\}.$$

We call this a **definable subset of A** and denote by $\varphi(A, b)_{b \in \mathcal{V}}$ the collection of definable subsets on A .

Given $A \subseteq \mathcal{U}$ define the equivalence relation $\equiv_{\varphi, A}$ defined by

$$\begin{aligned} b \equiv_{\varphi, A} b' &\Leftrightarrow \varphi(a, b) \leftrightarrow \varphi(a, b') \quad \text{for all } a \in A; \\ &\Leftrightarrow \varphi(A, b) = \varphi(A, b'). \end{aligned}$$

The $\equiv_{\varphi, A}$ -equivalence class are called **types over A** . We denote by $S_\varphi(A)$ the set types over A . As there is a one-to-one correspondence between definable subsets and types, $|S_\varphi(A)| = |\varphi(A, b)_{b \in \mathcal{V}}|$.

If all subsets of A are definable, that is $\mathcal{P}A = \varphi(A, b)_{b \in \mathcal{V}}$, we say that A is **shattered** by φ . The following is called the **shatter function**

$$\begin{aligned} \pi_\varphi(n) &= \max \left\{ |\varphi(A, b)_{b \in \mathcal{V}}| : A \in \binom{\mathcal{U}}{n} \right\}. \\ &= \max \left\{ |S_\varphi(A)| : A \in \binom{\mathcal{U}}{n} \right\}. \end{aligned}$$

So, $\pi_\varphi(n)$ gives the maximal number of definable subsets that a set of cardinality n may have. Trivially, $\pi_\varphi(n) \leq 2^n$ for all n . Moreover, if $\pi_\varphi(k) = 2^k$ for some k , then $\pi_\varphi(n) = 2^n$ for every $n \leq k$.

The dual incidence relation φ^* is the relation on $\mathcal{V} \times \mathcal{U}$ which is sometimes denoted by φ^{-1} . Then dual scattering function is defined as follows (with the obvious meaning of the notation)

$$\begin{aligned} \pi_\varphi^*(n) &= \max \left\{ |\varphi(a, B)_{a \in \mathcal{U}}| : B \in \binom{\mathcal{V}}{n} \right\}. \\ &= \max \left\{ |S_{\varphi^*}(B)| : B \in \binom{\mathcal{V}}{n} \right\}. \end{aligned}$$

1.1 Definition The **Vapnik-Chervonenkis dimension** of φ , abbreviated by **VC-dimension**,

is the maximal cardinality of a finite set $A \subseteq \mathcal{U}$ that is shattered by φ . Equivalently, it is the maximal k such that $\pi_\varphi(k) = 2^k$. If such a maximum does not exist, we say that φ has infinite VC-dimension.

We will say **dual VC-dimension** for the VC-dimension of φ^* .

The **VC-density** of φ is the infimum over all real number r such that $\pi_\varphi(n) \in O(n^r)$. It is infinite if no such r exist. The **dual VC-density** is defined accordingly. \square

Then if the VC-density is finite so is the VC-dimension. The converse is also true. In fact, in the next section we show that the VC-dimension bounds the VC-density.

1.2 Proposition If φ has VC-dimension $< k$ then its dual VC-dimension is $< 2^{k+1}$.

Proof Assume that the VC-dimension of φ^* is $\geq 2^k$. We prove that the VC-dimension of φ is $\geq k$. Let $B = \{b_I : I \subseteq [k]\}$ be a set of cardinality 2^k shattered by φ^* . That is, for every $\mathcal{J} \subseteq \mathcal{P}([k])$ there is $a_{\mathcal{J}}$ such that

$$\varphi(a_{\mathcal{J}}, b_I) \Leftrightarrow I \in \mathcal{J}$$

Let $a_i = a_{\{I \subseteq [k] : i \in I\}}$. Then from the equivalence above we obtain

$$\varphi(a_i, b_I) \Leftrightarrow i \in I$$

That is, φ shatters $\{a_i : i < k\}$. \square

An alternative formalism uses **set systems** in place of incidence relations. A set system is a collection Φ of subsets of some set \mathcal{U} . We denote a set system by (\mathcal{U}, Φ) . Given a set system, we immediately obtain an incidence relation. Fix any \mathcal{V} containing Φ and define $\varphi(x, y)$ as $x \in y \in \Phi$.

Vice versa, to an incidence relation φ we associate $\Phi = \{\varphi(\mathcal{U}, b) : b \in \mathcal{V}\}$, the set system of the definable subsets of \mathcal{U} . The **VC-dimension of Φ** is defined to be that of the associated incidence relation. The shatter function $\pi_\Phi(n)$ is defined in a similar way.

An incidence relation is **extensional** if $\varphi(\mathcal{U}, b) = \varphi(\mathcal{U}, b')$ implies $b, b' \in \mathcal{V}$. The correspondence between extensional incidence relations and set systems is one-to-one. In general, the correspondence is many-to-one. However, as we are mainly interested the set system associated, we may switch from one formalism to the other according to which one is more convenient.

- 1.3 Example**
- a. Set systems of cardinality 1 are those with an incidence relation of the form $A \times \mathcal{V}$ for some $A \subseteq \mathcal{U}$. They shatter only the empty set, therefore they have VC-dimension 0. Their shatter function is identically 1.
 - b. Let Φ be a non trivial partition of \mathcal{U} . Then only singletons are shattered, so the VC-dimension is 1. The shatter function is $\pi_\Phi(n) = \min \{n, |\Phi|\}$.
 - c. If Φ is a non trivial chain of subsets of \mathcal{U} the situation is identical to that described in b.
 - d. Let $\mathcal{U} = \mathbb{R}$ and let Φ be the collection of open intervals. Any set of 2 points is shattered but no set with 3 points can. So the VC-dimension is 2.
 - e. Let $\mathcal{U} = \mathbb{R}^2$ and let Φ be the collection of half planes. Any set of 3 non collinear points is shattered but no set with 4 points can (by Radon's Theorem). So the VC-dimension is 3.

- f. Let $\Phi = \mathcal{U}^{[\leq k]}$ be the collection of all subsets of \mathcal{U} of cardinality $\leq k$. Then Φ has VC-dimension k and

$$\pi_{\Phi}(n) = \sum_{i=0}^k \binom{n}{i}.$$

- g. Let $\mathcal{U} = \mathbb{R}^2$ and let Φ be the collection of polygons. Then Φ has VC-dimension ∞ . □

2 The Sauer-Shelah Lemma

According to Gil Kalai in [3], Sauer-Shelah's Lemma can be described as an *eigen-theorem* because it is important in many different areas of mathematic (model theory, learning theory, probability theory, ergodic theory, Banach spaces, to name a few). No wonder it has been discovered and rediscovered many times.

It has been proved independently by Shelah [6], Sauer [5], and Vapnik-Chervonenkis [7] around 1970 (Shelah gives credit to Micha Perles). Saharon Shelah was working in model theory while Norbert Sauer, Vladimir Vapnik and Alexey Chervonenkis were in statistical learning theory.

We shall present three proofs of this lemma, one in this section and two in the next section. I am aware of a forth proof which uses linear algebra, see e.g. [2].

1.4 Proposition (Sauer-Shelah's Lemma) *If φ has VC-dimension k then for every $n \geq k$*

$$\pi_{\varphi}(n) \leq \sum_{i=0}^k \binom{n}{i}.$$

The set system presented in f of Example 1.3 shows that the bound is optimal.

Proof If $k = 0$, both sides of the inequality are 1. Now, assume the lemma is true for $k - 1$. We prove by induction on n that for every A of cardinality n

$$|\varphi(A, b)_{b \in \mathcal{V}}| \leq \sum_{i=0}^k \binom{n}{i}.$$

If $n = k$ the r.h.s. of the inequality above is 2^n and the claim is trivial. So, assume the claim is true for $n - 1$ and let A have cardinality n . Fix some $a \in A$ and let $A' = A \setminus \{a\}$. We can assume that $\varphi(A, b) \triangle \varphi(A, b') = \{a\}$ for some b, b' , otherwise $|\varphi(A, b)_{b \in \mathcal{V}}| = |\varphi(A', b)_{b \in \mathcal{V}}|$ and the claim follows immediately from the induction hypothesis.

Define a new incidence relation

$$\psi(x, y) = x \in A' \wedge \varphi(x, y) \wedge \exists y' [\varphi(A, y) \triangle \varphi(A, y') = \{a\}].$$

Note that if A'' is shattered by ψ then $A'' \cup \{a\}$ it is shattered by φ . Then the VC-dimension of ψ is at most $k - 1$. We also have that

$$|\varphi(A, b)_{b \in \mathcal{V}}| = |\varphi(A', b)_{b \in \mathcal{V}}| + |\psi(A', b)_{b \in \mathcal{V}}|.$$

Hence by induction hypothesis

$$|\varphi(A, b)_{b \in \mathcal{V}}| \leq \sum_{i=0}^k \binom{n-1}{i} + \sum_{i=0}^{k-1} \binom{n-1}{i}$$

$$\begin{aligned}
&= \binom{n-1}{0} + \sum_{i=1}^k \left[\binom{n-1}{i} + \binom{n-1}{i-1} \right] \\
&= \binom{n-1}{0} + \sum_{i=1}^k \binom{n}{i} \\
&\leq \sum_{i=0}^k \binom{n}{i}
\end{aligned}$$

which completes the proof of the proposition. \square

Next corollary states an important dichotomy. It says that the shatter function grows exponentially unless the VC-dimension is finite. In this case the growth is only polynomial. Therefore the VC-dimension is an upper bound to the VC-density.

1.5 Corollary *For every incidence relation φ one of the following obtains*

1. *the VC-dimension is infinite and $\pi_\varphi(n) = 2^n$ for every positive integer n ;*
2. *the VC-dimension is k and $\pi_\varphi(n) \in O(n^k)$.*

Proof If VC-dimension is infinite claim 1 is obvious. So assume φ has VC-dimension is k . Then

$$\pi_\varphi(n) \leq \sum_{i=0}^k \binom{n}{i} \leq \sum_{i=0}^k \frac{n^i}{i!} \leq e n^k \quad \square$$

3 Pajor variant and the method of shifting

An alternative proof of the Sauer-Shelah's Lemma derives it as corollary of a lemma by Alain Pajor [4].

1.6 Proposition (Pajor's Lemma) *Let $A \subseteq \mathcal{U}$ be finite. Then φ shatters at least $|S_\varphi(A)|$ subsets of A .*

We show how Sauer-Shelah's Lemma follows from Pajor's Lemma. Fix a set $A \subseteq \mathcal{U}$ of cardinality n such that $\pi_\varphi(n) = |S_\varphi(A)|$. By Pajor's Lemma there are $|S_\varphi(A)|$ subsets of A shattered by φ . These subsets cannot have cardinality larger than the VC-dimension of φ , then

$$\pi_\varphi(n) = |\varphi(A, b)_{b \in \mathcal{V}}| \leq \left| \bigcup_{i=0}^k \binom{A}{i} \right| = \sum_{i=0}^k \binom{n}{i}.$$

Proof If A is empty then $|S_\varphi(A)| = 1$ and \emptyset is the only subset of A that φ shatters. Fix $a \in A$ and assume the lemma holds for $A' = A \setminus \{a\}$. Let ψ be the relation defined in the proof of Proposition 1.4. Recall that

$$|\varphi(A, b)_{b \in \mathcal{V}}| = |\varphi(A', b)_{b \in \mathcal{V}}| + |\psi(A', b)_{b \in \mathcal{V}}|.$$

and that if $A'' \subseteq A'$ is shattered by ψ then $A'' \cup \{a\}$ is shattered by φ . By induction hypothesis φ shatters least $|\varphi(A', b)_{b \in \mathcal{V}}|$ subsets of A' and at least $|\psi(A', b)_{b \in \mathcal{V}}|$ containing a . The lemma follows.

We present a different wording of essentially the same proof. Consider the set system $\Phi = \varphi(A, b)_{b \in \mathcal{V}}$ and reason by induction of $|\Phi|$. The proposition holds

trivially if $|\Phi| = 1$. Now, suppose it holds for set systems of cardinality $< |\Phi|$. Fix $a \in A$ and let $\Phi_0 = \{B \in \Phi : a \notin B\}$ and $\Phi_1 = \Phi \setminus \Phi_0$. As $|\Phi| > 1$, we can choose a such that $\Phi_i \neq \Phi$. By the inductive hypothesis, both Φ_i shatter at least $|\Phi_i|$ subsets of A . If no set is shattered by both Φ_i , the claim follows immediately. Otherwise, note that for each set shattered by both Φ_i there are two sets shattered by Φ , one containing a and one not containing a . The claim follows. \square

We give different proof of Pajor's Lemma by a method which is interesting in itself because of its many applications. The method has been introduced by Erdős, Ko and Rado to prove their eponymous theorem. They named it *compression*, but is also known as *shifting*.

When $A' \subseteq A$ and $A \setminus A' = \{a\}$, we write $A' \subseteq_a A$. We write

$$B_{\varphi,a} = \{b : \varphi(a,b) \text{ and } \neg \exists b' \varphi(\mathcal{U}, b') \subseteq_a \varphi(\mathcal{U}, b)\}$$

We say that φ is **compressed** if $B_{\varphi,a} = \emptyset$ for every $a \in \mathcal{U}$.

1.7 Proposition *If φ is compressed then any co-finite subset of a definable set is definable. In particular φ shatters every finite subset of a definable set.*

Proof If for contradiction $\psi(\mathcal{U}, b) \setminus \{a\}$ is not definable, then $b \in B_{\varphi,a}$. \square

1.8 Proposition *Let $\psi = \varphi \setminus \{a\} \times B_{\varphi,a}$. Then*

$$\varphi(\mathcal{U}, b) = \varphi(\mathcal{U}, b') \Leftrightarrow \varphi(\mathcal{U}, b) = \varphi(\mathcal{U}, b') \quad \text{for all } b, b' \in \mathcal{V}.$$

Proof \Rightarrow Assume $\varphi(\mathcal{U}, b) \neq \varphi(\mathcal{U}, b')$. We may also assume that $\varphi(\mathcal{U} \setminus \{a\}, b) = \varphi(\mathcal{U} \setminus \{a\}, b')$, otherwise $\psi(\mathcal{U}, b) \neq \psi(\mathcal{U}, b')$ is immediate. Then $\varphi(a, b) \nleftrightarrow \varphi(a, b')$, say $\varphi(a, b)$ and $\neg \varphi(a, b')$. Then $\varphi(\mathcal{U}, b') \subseteq_a \varphi(\mathcal{U}, b)$ and $b \notin B_{\varphi,a}$. So $\psi(a, b)$ and $\neg \psi(a, b')$.

\Leftarrow Assume $\psi(\mathcal{U}, b) \neq \psi(\mathcal{U}, b')$. Again, we may also assume that $\psi(\mathcal{U} \setminus \{a\}, b) = \psi(\mathcal{U} \setminus \{a\}, b')$. Then $\psi(a, b) \nleftrightarrow \psi(a, b')$, say $\psi(a, b)$ and $\neg \psi(a, b')$. As $\varphi(a, b)$ is clear, we only have to prove that $\neg \varphi(a, b')$. Suppose for a contradiction that $\varphi(a, b')$. Then $b' \in B_{\varphi,a}$. This is a contradiction as $\psi(\mathcal{U}, b) \subseteq_a \psi(\mathcal{U}, b')$. \square

1.9 Proposition *Let $\psi = \varphi \setminus \{a\} \times B_{\varphi,a}$. Then every set shattered by ψ is shattered by φ .*

Proof Assume A is shattered by ψ . We prove that for every b here is a b' such that $\psi(A, b) = \varphi(A, b')$. We may assume that $a \in A$. If $\psi(a, b)$ then we may chose $b = b'$. So, assume $\neg \psi(a, b)$. As ψ shatters A then there is c such that $\psi(A, c) = \psi(A, b) \cup \{a\}$. Then $\exists b' \varphi(\mathcal{U}, b') \subseteq_a \varphi(\mathcal{U}, c)$. Therefore $\psi(A, b) = \varphi(A, b')$. \square

1.10 Second proof of Pajor's Lemma Let φ be a finite extensional incidence relation associated to Φ . Let $\varphi_0 = \varphi$ and $\varphi_{i+1} = \varphi_i \setminus \{a\} \times B_{\varphi_i,a}$ for some a . As φ is finite, we can assume that at some stage n we obtain a compressed relation $\psi = \varphi_n$. Let Ψ be the set of ψ -definable sets. By Proposition 1.7, Ψ shatters at least $|\Psi|$ sets. By Proposition 1.9, every set shattered by Ψ is shattered by Φ . By Proposition 1.8, $|\Psi| = |\Phi|$. \square

Define $E_{\varphi,a} \subseteq \mathcal{V}^2$ as the set of e pairs $\langle b', b \rangle$ such that $\varphi(\mathcal{U}, b') \subseteq_a \varphi(\mathcal{U}, b)$. Note incidentally that $E_{\varphi,a}$ is the graph of a partial injections of \mathcal{V} into itself.

1.11 Proposition Let $\psi = \varphi \setminus \{a\} \times B_{\varphi,a}$. Then $E_\varphi \subseteq E_\psi$

Proof Both the range and the domain of $E_{\varphi,a}$ are disjoint from $B_{\varphi,a}$. □

We call $E_\varphi = \bigcup_{a \in U} E_{\varphi,a}$ the unit distance diagram of φ .

1.12 Proposition Let φ be a finite extensional incidence relation with VC-dimension k . Then $|E_\varphi| \leq k |\mathcal{V}|$.

Proof Let ψ be as defined in Proof 1.10. By Propositions 1.7 and 1.9, every $b \in \mathcal{V}$ has at most k ancestors. Therefore

$$\frac{|E_\varphi|}{|\mathcal{V}|} \leq \frac{|E_\psi|}{|\mathcal{V}|} \leq k \quad \square$$

4 Notes and references

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Chapter 2

Samples and approximations of measures

1 Samples and subsamples

A sample is a just sequence of elements of \mathcal{U} where we disregard the order and only consider the number of times they appear. Formally, a **sample** is a **multi-subset of \mathcal{U}** , that is is a function $A : \mathcal{U} \rightarrow \mathbb{N}$.

The definitions in this and the next sections easily generalize to **fractional multi-sets** i.e., function $A : \mathcal{U} \rightarrow \mathbb{R}$ with non negative values. These will have some applications in the next chapters.

We interpret $A(x)$ as the **multiplicity** of the element $x \in \mathcal{U}$. The **support** of A is the set $\text{supp}(A) = \{x : A(x) \neq 0\}$. The **size of A** is defined as

$$|A| = \sum_{x \in \mathcal{U}} A(x).$$

If we identify sets with $\{0, 1\}$ -valued samples, the size generalizes cardinality.

We say that C is a **subsample** of A , and write $C \subseteq A$, if $C(x) \leq A(x)$ for all $x \in \mathcal{U}$. We also define the **intersection** and the **difference** of two samples. The element $x \in \mathcal{U}$ has multiplicity $A(x) \wedge C(x)$ in $A \cap C$ and multiplicity $A(x) \dot{-} C(x)$ in $A \setminus C$. Note that $|A| = |A \cap C| + |A \setminus C|$.

If $\mathcal{B} \subseteq \mathcal{U}$ we define the **frequency** of \mathcal{B} over A

$$\text{Fr}(\mathcal{B}/A) = \frac{|\mathcal{B} \cap A|}{|A|}$$

Note that $\text{Fr}(\cdot/A)$ is a probability measure on \mathcal{U} .

An **ε -approximation** of a sample A is a subsample $C \subseteq A$ such that for every definable set \mathcal{B}

$$\left| \text{Fr}(\mathcal{B}/A) - \text{Fr}(\mathcal{B}/C) \right| \leq \varepsilon$$

As an alternative to multi-sets, we could use the following approach. Replace \mathcal{U} with $\mathcal{U}' = \mathcal{U} \times \omega$, which we think of as the union of infinitely many disjoint copies of \mathcal{U} . Replace Φ with $\Phi' = \{\mathcal{B} \times \omega : \mathcal{B} \in \Phi\}$. Then \mathcal{U}' contains infinitely many copies of each element of \mathcal{U} which are indistinguishable by sets in Φ' . Then to every finite subset of $A \subseteq \mathcal{U}'$ we can associate a multi-subset of \mathcal{U} . (The correspondence is not one-to-one, but this is not relevant in the following.)

2 Discrepancy

Given ε we are interested in the least n such that some ε -approximations of size n exist. The idea is to start with an approximation of large size and reduce size at the cost of slightly enlarging ε . We now introduce a powerful technique to achieve this.

In general \mathcal{U} may not be among the definable sets. As it is often convenient to

include it, we write Φ' for $\Phi \cup \{\mathcal{U}\}$.

Let $C \subseteq A$ and $\mathcal{B} \in \Phi'$. We call the quantity

$$\Delta_{A,C,\mathcal{B}} = |C \cap \mathcal{B}| - |(A \setminus C) \cap \mathcal{B}|$$

the **discrepancy** of C in \mathcal{B} . The **discrepancy** of C is

$$\Delta_{A,C} = \max_{\mathcal{B} \in \Phi'} |\Delta_{A,C,\mathcal{B}}|.$$

Finally, the **discrepancy** of A is

$$\Delta_A = \min_{C \subseteq A} \Delta_{A,C}.$$

We will use the same notation but with a lowercase δ for the **relative discrepancy**. This is obtained dividing the discrepancy by the size of A .

The next lemma is intuitive, if an ε -approximation has small discrepancy then we can halve its size at a small cost.

2.1 Lemma *Let A be a sample of size n . Let $C \subseteq A$ have relative discrepancy $\delta_{A,C}$. Then either C or $A \setminus C$ is an ε -approximation of size $\leq n/2$ for $\varepsilon = 2\delta_{A,C}$.*

Proof Define $n^+ = |C|$ and $n^- = |A \setminus C|$. We may assume that $n^+ \leq n/2$, otherwise swap C and $A \setminus C$. Then $\delta_{A,C,\mathcal{U}} = (n^+ - n^-)/n < 0$. Now, let $\mathcal{B} \in \Phi'$ be arbitrary

$$\begin{aligned} 1. \quad \frac{|A \cap \mathcal{B}|}{n} &= \frac{|C \cap \mathcal{B}| + |(A \setminus C) \cap \mathcal{B}|}{n} \\ (*) &= \frac{2|C \cap \mathcal{B}|}{n} - \delta_{A,C,\mathcal{B}} \\ &\leq \frac{|C \cap \mathcal{B}|}{n^+} + \delta_{A,C} \end{aligned}$$

We also have

$$\begin{aligned} 2. \quad (*) &= \frac{|C \cap \mathcal{B}|}{n^+} (1 + \delta_{A,C,\mathcal{U}}) - \delta_{A,C,\mathcal{B}} \\ &\geq \frac{|C \cap \mathcal{B}|}{n^+} - 2\delta_{A,C} \end{aligned}$$

Combining 1 and 2 we obtain

$$\begin{aligned} |\text{Fr}(\mathcal{B}/A) - \text{Fr}(\mathcal{B}/C)| &\leq \left| \frac{|\mathcal{B} \cap A|}{n} - \frac{|\mathcal{B} \cap C|}{n^+} \right| \\ &\leq 2\delta_{A,C} \end{aligned}$$

as claimed by the lemma. □

3 Random colorings

We say that a tuple $a = \langle a_1, \dots, a_n \rangle \in \mathcal{U}^n$ is an **enumeration** of the sample A if for all $x \in \mathcal{U}$

$$A(x) = |\{i : a_i = x\}|.$$

Clearly, all enumerations of A have length $n = |A|$. We write **rng**(a) for the sample enumerated by a .

Fix an enumeration $a = \langle a_1, \dots, a_n \rangle$ of A . A tuple $c = \langle c_1, \dots, c_n \rangle \in \{-1, +1\}^n$

is called a **coloring**. To each coloring c we associate a subsample of A , which we denote by $\text{smpl}(a, c)$. It assigns to $x \in \mathcal{U}$ multiplicity $|\{i : a_i = x, c_i = +1\}|$.

Note that the quantity

$$\Delta_{a,c,\mathcal{B}} = \sum_{a_i \in \mathcal{B}} c_i.$$

coincide with the discrepancy of $\text{smpl}(a, c)$ over \mathcal{B} .

There is probability measure on the subsamples of A which is computationally convenient. We assume that $C = \text{smpl}(a, c)$ where $c = \langle c_1, \dots, c_n \rangle \in \{\pm 1\}^n$ is obtained tossing n times a fair coin.

That is, we assume an uniform distribution on the colorings and this induces a multivariate binomial distribution on the subsamples. Note that this distribution is independent of the enumeration of A .

4 Important inequalities

2.2 Proposition (Chebychev's inequality) *Let X be a random variable with finite mean and finite variance.*

$$1 - \frac{\text{Var}(X)}{\varepsilon^2} < \Pr(|E(X) - X| \leq \varepsilon)$$

Proof $1 - \frac{\text{Var}(X)}{\varepsilon^2} < \Pr(|X - E(X)| \leq \varepsilon)$

□

2.3 Proposition (Weak Law of Large Numbers) *Let \Pr be a probability measure on \mathcal{U} . Then, for every event $\mathcal{B} \subseteq \mathcal{U}$ and every $n, \varepsilon > 0$*

$$1 - \frac{1}{4n\varepsilon^2} < \Pr^n(c \in \mathcal{U}^n : |\Pr(\mathcal{B}) - \text{Fr}(\mathcal{B}/c)| \leq \varepsilon)$$

Proof Note that $X = X(c) = |\text{rng}(c) \cap \mathcal{B}|$ is a binomial random variable with success probability $p = \mu\mathcal{B}$. Hence $E(X) = p$ and $\text{Var}(X) = p(1-p)/n$. Apply Chebyshev's inequality. □

2.4 Lemma (Chernoff's bound, special case) *For $i = 1, \dots, n$ let X_i be independent identically distributed random variables such that $\Pr(X_i = \pm 1) = 1/2$. Then for every $\varepsilon > 0$*

$$\Pr(M \geq \varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{2n}\right) \quad \text{where } M = \sum_{i=1}^n X_i$$

Proof Let $t > 0$ be arbitrary. Then

$$\begin{aligned} \# \quad \Pr(M \geq \varepsilon) &= \Pr(e^{tM} \geq e^{t\varepsilon}) \\ &\leq e^{-t\varepsilon} E(e^{tM}) \end{aligned}$$

In fact, the equality follows because the exponential is an increasing function and the inequality is Markov's inequality, which says that $\Pr(X \geq a) \leq a^{-1}E(X)$ for every a and is immediate to verify. Now observe that

$$E(e^{tX_i}) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i=0}^{\infty} \frac{t^i}{i!} + \frac{1}{2} \sum_{i=0}^{\infty} \frac{(-t)^i}{i!} \\
&= \sum_{i=0}^{\infty} \frac{t^{2i}}{(2i)!} \\
&\leq \sum_{i=0}^{\infty} \frac{(t^2/2)^i}{i!} \\
&= e^{t^2/2}
\end{aligned}$$

From this, by independence we have

$$E(e^{tM}) = \prod_{i=1}^n e^{tX_i} = e^{nt^2/2}$$

Substituting in \sharp gives $\Pr(M \geq \varepsilon) \leq e^{nt^2/2 - t\varepsilon}$. Finally Chernoff's inequality is obtained substituting ε/n for t . \square

5 A uniform law of large numbers

We want to estimate the minimal size of an ε -approximation of A . We want a bound that depends solely on ε , not on the size of A . It is also important to note that the definition of approximation includes a requirement of uniformity: the same subsample works for all definable sets.

If we allow $C \subseteq A$ to depend on \mathcal{B} , the existence of a subsample that satisfies the inequality

$$\left| \text{Fr}(\mathcal{B}/A) - \text{Fr}(\mathcal{B}/C) \right| \leq \varepsilon$$

follows easily from Chebychev's inequality and its size only depends on ε . This is essentially the weak law of large numbers for the probability measure $\text{Fr}(\cdot/A)$.

2.5 Proposition (Weak law of large numbers) *For every sample A , every $\mathcal{B} \subseteq \mathcal{U}$ and every $\varepsilon > 0$ there is a C of size $n = 1/4\varepsilon^2$ such that $|\text{Fr}(\mathcal{B}/A) - \text{Fr}(\mathcal{B}/C)| \leq \varepsilon$.*

Note that to avoid technicalities we do not insist in requiring $C \subseteq A$.

Proof Interpret $\text{Fr}(\cdot/A)$ as a probability measure on $\text{supp } A$. Fix \mathcal{B} and set $p = \text{Fr}(\mathcal{B}/A)$. Consider the Bernoulli random variables X_i that on input $c \in (\text{supp } A)^n$ output

$$X_i(c) = \begin{cases} 1 & \text{if } c_i \in \mathcal{B} \\ 0 & \text{if } c_i \notin \mathcal{B} \end{cases}$$

Define also $\bar{X} = \frac{1}{n} \sum_i X_i$. Then $E(\bar{X}) = p$ and $\text{Var}(\bar{X}) = \frac{p(1-p)}{n} \leq \frac{1}{4n}$.

If we set $C = \text{rng}(c)$, we obtain

$$|E(\bar{X}) - \bar{X}(c)| = |\text{Fr}(\mathcal{B}/A) - \text{Fr}(\mathcal{B}/C)|$$

So it suffices that $|E(\bar{X}) - \bar{X}| \leq \varepsilon$ has positive probability for some large enough n . By Chebychev's inequality

$$1 - \frac{1}{4n\varepsilon^2} < \Pr(|E(\bar{X}) - \bar{X}| \leq \varepsilon)$$

Hence the probability above is positive for $\frac{1}{4\epsilon^2} \leq n$. \square

The main result of this chapter is the following theorem of Vapnik and Chervonenkis which we proof in the next section.

2.6 Theorem (Uniform law of large numbers) *Assume (\mathcal{U}, Φ) has VC-density d . Then every sample A has an ϵ -approximation of size*

$$n = c \frac{d}{\epsilon^2} \ln \frac{1}{\epsilon},$$

where c is an absolute constant.

The lemma above tells that ϵ -approximations with small discrepancy are useful, but as yet we have no clue as to finding one. We are going to prove that when the number of definable subsets of A is relatively small, then the discrepancy of A is not too large. We use a probabilistic argument to prove this bound (when you don't have a clue how to do something, you might as well do it randomly).

2.7 Lemma *Let A be a sample of size $\leq n$. Assume the support of A has $\leq m$ definable subsets. Then $\Delta_A \leq \sqrt{2n \ln(m+2)}$.*

Proof Let $\epsilon = \sqrt{2n \ln(m+2)}$. Then we need to prove that $\Delta_A \leq \epsilon$ which is equivalent to

$$\exists C \subseteq A \forall \mathcal{B} \in \Phi' \Delta_{A,C,\mathcal{B}} \leq \epsilon$$

Let $\Pr(\cdot)$ denote the probability measure on the subsamples of A discussed in Section 3 (though, clearly, any other probability measure would do for the implication)

\uparrow

$$\Pr(\forall \mathcal{B} \in \Phi' \Delta_{A,C,\mathcal{B}} \leq \epsilon) > 0.$$

As Φ' has at most $m+1$ elements

\uparrow

$$\forall \mathcal{B} \in \Phi' \Pr(\Delta_{A,C,\mathcal{B}} \geq \epsilon) \leq \frac{1}{m+2}.$$

Fix an enumeration a of A . Suppose $c = \langle c_1, \dots, c_n \rangle \in \{\pm 1\}^n$ is obtained tossing n times a fair coin. Consider the Bernoulli random variables X_i that on input c output c_i . For a given \mathcal{B} define $M_{\mathcal{B}} = \sum_{a_i \in \mathcal{B}} X_i$. Then $\Delta_{A,C,\mathcal{B}} = \Delta_{a,c,\mathcal{B}} = M_{\mathcal{B}}(c)$. Therefore

$$\Pr(\Delta_{A,C,\mathcal{B}} \geq \epsilon) \leq \frac{1}{m+2}.$$

\Downarrow

$$\Pr(M_{\mathcal{B}} \geq \epsilon) \leq \frac{1}{m+2}.$$

By the Chernoff's bound

\uparrow

$$\exp\left(-\frac{\epsilon^2}{2|\mathcal{B}|}\right) \leq \frac{1}{m+2}.$$

\Downarrow

$$2|\mathcal{B}| \ln(m+2) \leq \epsilon^2$$

As $|\mathcal{B}| \leq n$, the latter is clearly true. \square

2.8 Proof of Proposition 2.6 Set $A_0 = A$ and $\varepsilon_0 = 0$. We construct a decreasing chain A_i of ε_i -approximations. We denote by n_i and δ_i the cardinality, respectively the discrepancy, of A_i . By lemma 2.1, we can require that $\varepsilon_{i+1} = \varepsilon_i + 2\delta_i$ and $n_{i+1} \leq n_i/2$. Then

$$\varepsilon_h = 2 \sum_{i=1}^h \delta_i$$

Let h be the largest such that $\varepsilon_h \leq \varepsilon$. The theorem claims that $2^{-h}n_0 \leq c(d/\varepsilon^2) \log(1/\varepsilon)$, independently of n_0 .

$$2 \sum_{i=1}^h \delta_i \leq 2 \sum_{i=1}^h \sqrt{(2/n_i) \ln(n_i^d + 2)}$$

and, as $n_i = 2^{-i}n_0$, the sum above is proportional to its largest term,

$$\leq c \sqrt{(d/n_h) \ln(n_h)}$$

Hence a sufficient condition for $\varepsilon_h < \varepsilon$ is

$$c \sqrt{(d/n_h) \ln(n_h)} \leq \varepsilon$$

which is equivalent to

$$\frac{c^2 d}{\varepsilon^2} \leq \frac{n_h}{\ln n_h}$$

this in turn is ensured if

$$n_h \leq 2 \frac{c^2 d}{\varepsilon^2} \ln \frac{c^2 d}{\varepsilon^2}$$

Chapter 3

Packings and transversals

1 Linear programming / convex optimization

The following is a classical result. The proof may be found in any introductory text of linear programming. There are many (sometimes non equivalent) ways to state it. The following somewhat unusual phrasing is taken from Terence Tao's blog. It stresses the analogies between Farkas' lemma and Hilbert's Nullstellensatz.

3.1 Proposition (Farkas' Lemma) For $i \in [m]$ let $P_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be affine linear functions. Then the following are equivalent

1. $\bigwedge_{i=1}^m P_i(x) \geq 0$ has a solution $x \in \mathbb{R}^n$;
2. there are no $0 \leq y \in \mathbb{R}^m$ such that $\sum_{i=1}^m y_i P_i = -1$. □

3.2 Proposition (Farkas' Lemma, second formulation) Let A be a $m \times n$ real matrix. Let $b \in \mathbb{R}^m$ be a column vector. Then the following are equivalent

1. $Ax \geq b$ has a solution $x \in \mathbb{R}^n$;
2. all $0 \leq y \in \mathbb{R}^m$ such that $A^T y = 0$ also satisfy $b^T y \geq 0$. □

3.3 Proposition (Duality for LP) Let A be a $m \times n$ real matrix. Let $c \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ be column vectors. Then the following maximum and minimum exist and coincide.

1. $\min \{ c^T x : Ax \geq b, 0 \leq x \}$;
2. $\max \{ b^T y : A^T y \leq c, 0 \leq y \}$. □

2 Transversals and packings

A subset $A \subseteq \mathcal{U}$ is a **transversal** if $\varphi(A, b) \neq \emptyset$ for every $b \in \mathcal{V}$. Equivalently, if $\varphi(a, \mathcal{V})_{a \in A}$ covers \mathcal{V} . The **transversal number** of φ , denoted by $\tau(\varphi)$, is the smallest cardinality of a transversal of φ .

A subset $B \subseteq \mathcal{V}$ is a **packing** if $\varphi(\mathcal{U}, b) \cap \varphi(\mathcal{U}, b') = \emptyset$ for every $b, b' \in B$. Equivalently if $|\varphi(a, B)| \leq 1$ for every $a \in \mathcal{U}$. The **packing number** of φ , denoted by $\nu(\varphi)$, is the largest cardinality of a packing $B \subseteq \mathcal{V}$.

As any transversal is at least as large as any packing, we always have $\nu(\varphi) \leq \tau(\varphi)$. Very little can be said in the reverse direction in general.

A fractional multi-set A over \mathcal{U} is a **fractional transversal** if $|\varphi(A, b)| \geq 1$ for every $b \in \mathcal{V}$. The **fractional transversal number** of φ , denoted by $\tau^*(\varphi)$, is the infimum of the size of the fractional transversals of φ .

A fractional multi-set B over \mathcal{V} is a **fractional packing** if $|\varphi(a, B)| \leq 1$ for every $a \in \mathcal{U}$. The **fractional packing number** of φ , denoted by $\nu^*(\varphi)$, is the supremum of the size of the fractional packings of φ .

3.4 Example Let $\mathcal{U} = \mathbb{R}^2$ and \mathcal{V} is a set of n lines in general position. Let φ be the incidence relation. Then $\nu(\varphi) = 1$, as any two lines intersect. And $\tau(\varphi) = \lceil n/2 \rceil$, as each point belongs to at most two lines.

Let B assign $1/2$ to every line. Then $|\varphi(a, B)| \leq 1$ holds because each point is contained in at most two lines. Then $\nu^*(\varphi) \geq n/2$. It is easy to see that $\tau^*(\varphi) = n/2$. If n is even, it is clear. If n is odd, take 3 any lines, and assign $1/2$ to the three intersection points. Proceed as in the even case with the other lines. \square

3.5 Exercise Let $\mathcal{U} = \mathbb{R}$ and \mathcal{V} is a set of finitely many closed intervals. Let φ be the incidence relation. Then $\nu(\varphi) = \tau(\varphi)$. Hint: by induction on $\nu(\varphi)$.

3.6 Theorem For every finite incidence relation $\nu^*(\varphi) = \tau^*(\varphi)$.

Proof Let $\mathcal{U} = \{a_1, \dots, a_m\}$ and $\mathcal{V} = \{b_1, \dots, b_n\}$. Let F be the $\{0, 1\}$ -valued incidence matrix $\varphi(a_i, b_j)$. A multi-set over \mathcal{U} is a naturally associated to a vector $0 \leq x \in \mathbb{R}^m$. A multi-set over \mathcal{V} is associated to a vector $0 \leq y \in \mathbb{R}^n$. Then

$$\begin{aligned}\tau^*(\varphi) &= \min \{ 1_n^T x : Fx \geq 1_m, 0 \leq x \}. \\ \nu^*(\varphi) &= \max \{ 1_n^T y : F^T y \leq 1_m, 0 \leq y \};\end{aligned}$$

\square

A subset $A \subseteq \mathcal{U}$ is a **transversal** if $A \cap \mathcal{B} \neq \emptyset$ for every $\mathcal{B} \in \Phi$. The **transversal number** of Φ , denoted by $\tau(\Phi)$, is the smallest cardinality of a transversal of Φ .

A collection $\Psi \subseteq \Phi$ is a **packing** if it contain pairwise disjoint sets. The **packing number** of Φ , denoted by $\nu(\Phi)$, is the smallest cardinality of a packing $\Psi \subseteq \Phi$.

As any transversal is at least as large as any packing, we always have $\nu(\Phi) \leq \tau(\Phi)$. Very little can be said in the reverse direction in general.

A fractional multi-set A of elements of \mathcal{U} is a **fractional transversal** if $|A \cap \mathcal{B}| \geq 1$ for every $\mathcal{B} \in \Phi$. The **fractional transversal number** of Φ , denoted by $\tau^*(\Phi)$, is the infimum of the size of the fractional transversals of Φ .

A fractional multi-set Ψ of elements of Φ is a **fractional packing** if $|\Psi \cap \Phi_x| \leq 1$ for every $x \in \mathcal{U}$, where $\Phi_x = \{\mathcal{B} : x \in \mathcal{B}\}$. The **fractional packing number** of Φ , denoted by $\nu^*(\Phi)$, is the supremum of the size of the fractional packings of Φ .

Then we have $\tau^* \leq \tau \leq \nu \leq \nu^*$.

Chapter 4

A piercing problem

After Pierre Simon, after Jiri Matousek, after Noga Alon and Daniel Kleitman.

Let $q \leq p < \omega$ we say that Φ has the (p, q) -property if out of every p sets in Φ some q have non empty intersection. It has the dual (p, q) -property if for every p point in \mathcal{U} some q are belong to the same set in Φ .

The following is a particular case of a theorem of Matousek (based on a proof of Noga Alon and Daniel Kleitman). The proof by Pierre Simon is simpler.

4.1 Theorem *Let \mathcal{U} be finite and let Φ have VC-dimension $k < \omega$. There are some integers q and h that depends only on k such that, if every $B \subseteq \mathcal{U}$ of cardinality q is a subset of some $\varphi \in \Phi$, then \mathcal{U} is covered by some $X \subseteq \Phi$ of cardinality h .*

This theorem is often stated in the dual form which explains why it is sometimes called a piercing (or Helly-type) theorem: there are some integers q and h that depends only on k such that, if every q sets in Φ have non-empty intersection, then there is a $B \subseteq \mathcal{U}$ of cardinality h that intersects every $\varphi \in \Phi$. See Exercise 4.2.

Recall that the dual system is $\mathcal{U}^* = \Phi$ and $\Phi^* = \{\Phi_i : i < m\}$, where $\Phi_i = \{\varphi \in \Phi : a_i \in \varphi\}$. By Proposition ?? the dual system has VC-dimension 2^k . Choose some $\varepsilon = 1/2$. From Theorem 2.6 we obtain an h such that every probability measure μ^* on Φ has an ε -approximation of cardinality h . As ε is fixed, this h only depends on k . Then there is a multi-set $X : \Phi \rightarrow \omega$ of cardinality h such that Moreover $|Y| \leq h$ as required. Suppose we can find some $x_j \in \mathbb{R}$ such that for every $i < m$ In fact, We can also rescale solutions so, setting $\mu^*(\varphi_j) \stackrel{\text{def}}{=} x_j$, we obtain a well-defined probability measure It suffices to show that clause 2 of Farkas' Lemma cannot obtain, that is, for every $\lambda_i \in \mathbb{R}_+$ there are some $x_j \in \mathbb{R}_+$ such that Note that for every j we have By assumption, there is a j such that $\{a : B(a) \neq \emptyset\} \subseteq \varphi_j$ hence $|B \cap \varphi_j| = q$. Therefore $\mu(\varphi_j) > 1 - \varepsilon$. Let \check{x}_j be 1 for $j = \check{j}$ and 0 otherwise. We claim that the tuple $\langle x_j : j < n \rangle$ is a solution of $\mathbb{b}\mathbb{b}$. In fact

Proof Let $\langle a_i : i < m \rangle$ and $\langle \varphi_j : j < n \rangle$ enumerate \mathcal{U} and Φ without repetitions. Recall that the dual system is $\mathcal{U}^* = \Phi$ and $\Phi^* = \{\Phi_i : i < m\}$, where $\Phi_i = \{\varphi \in \Phi : a_i \in \varphi\}$. By Proposition ?? the dual system has VC-dimension 2^k . Choose some $\varepsilon = 1/2$. From Corollary ?? we obtain an h such that every probability measure μ^* on Φ admits an ε -net of cardinality h . As ε is fixed, this h only depends on k . Restating this explicitly there is a set $X \subseteq \mathcal{U}^*$ of cardinality h such that

$$\# \quad |\mu^*(\Phi_i)| > \varepsilon \Rightarrow X \cap \Phi_i \neq \emptyset$$

So, if the probability measure on Φ is such that $\mu^*(\Phi_i) > \varepsilon$ for all $i < m$, then $X \subseteq \Phi$ cover \mathcal{U} as required by the theorem. So we only need to find such a probability measure.

For $i < m$ and $j < n$ let $P_{i,j} = 1$ if $a_i \in \varphi_j$ and 0 otherwise. Suppose we can find some $x_j \in \mathbb{R}$ such that for every $i < m$

$$\flat \quad \sum_{j < n} (P_{i,j} - \varepsilon) x_j > 0$$

Then we can assume that all x_j are positive. In fact,

$$\sum_{j < n} P_{i,j} \geq 1$$

(because every a_i belongs to some φ_j) so we can translate solutions of any positive quantity. We can also rescale solutions so, setting $\mu^*(\varphi_j) \stackrel{\text{def}}{=} x_j$, we obtain a well-defined probability measure

$$\mu^*(\Phi_i) = \sum_{j < n} P_{i,j} x_j > \varepsilon$$

We only have to prove that equation \flat has a solution. It suffices to show that clause 2 of Farkas' Lemma cannot obtain, that is, for every $\lambda_i \in \mathbb{R}_+$ there are some $x_j \in \mathbb{R}_+$ such that

$$\flat\flat \quad \sum_{i < m} \lambda_i \sum_{j < n} (P_{i,j} - \varepsilon) x_j > 0.$$

As we can assume the λ_i add to 1, setting $\mu(a_i) \stackrel{\text{def}}{=} \lambda_i$, we obtain a probability measure on \mathcal{U} . Note that for every j we have

$$\mu(\varphi_j) = \sum_{i < m} \lambda_i P_{i,j}.$$

Apply Corollary ?? once again to the set-system $\langle \mathcal{U}, \neg\Phi \rangle$. There is a q such that for every probability measure μ on \mathcal{U} there is a set $B \subseteq \mathcal{U}$ of cardinality q such that

$$\sharp\sharp \quad \mu(\neg\varphi_j) > \varepsilon \Rightarrow B \not\subseteq \varphi_j$$

As ε is fixed, q only depends on k . By assumption, there is a \check{j} such that $B \subseteq \varphi_{\check{j}}$ and therefore $\mu(\varphi_{\check{j}}) \geq 1 - \varepsilon$. Let \check{x}_j be 1 for $j = \check{j}$ and 0 otherwise. We claim that the tuple $\langle x_j : j < n \rangle$ is a solution of $\flat\flat$. In fact

$$\sum_{i < m} \lambda_i \sum_{j < n} (P_{i,j} - \varepsilon) \check{x}_j = \sum_{i < m} \lambda_i (P_{i,\check{j}} - \varepsilon) = \mu(\varphi_{\check{j}}) - \varepsilon \geq 1 - \varepsilon > 0.$$

This concludes the proof. \square

It will help induction Up to rescaling we can assume that $P_{i,j}(x, y)$ have the form

4.2 Exercise Let k, q and h be some integers as in Theorem 4.1 and let Φ be a set-system with dual VC-dimension k . Suppose that any q sets in Φ have non-empty intersection and prove that there is some set $B \subseteq \mathcal{U}$ of cardinality h that intersects every $\varphi \in \Phi$. \square

1 Appendix: Farkas' lemma

Below, an affine version of Farkas' Lemma tailored to our purposes.

4.3 Proposition Fix some $v_1, \dots, v_n, u \in \mathbb{Q}^k$ and let $r_1, \dots, r_n, s \in \mathbb{Q}$. Let

$$X(r_1, \dots, r_n) = \{x \in \mathbb{Q}^k : r_i \leq v_i \cdot x, \text{ for every } i = 1, \dots, n\}.$$

Then the following are equivalent

1. $s \leq u \cdot x$ for all $x \in X(r_1, \dots, r_n)$;
2. there exist $q_1, \dots, q_n \in \mathbb{Q}^+$ such that $\sum_{i=1}^n q_i v_i = u$.

Proof Implication $2 \Rightarrow 1$ is immediate. To prove $1 \Rightarrow 2$ assume 1. We claim that

3. $0 \leq u \cdot x$ for all $x \in X(0, \dots, 0)$.

Suppose not and let $x \in X(0, \dots, 0)$ be such that $u \cdot x < 0$. Let $y \in X(r_1, \dots, r_n)$. Then $y + ax \in X(r_1, \dots, r_n)$ for every $a \in \mathbb{Q}^+$. From 1 we obtain $s \leq u \cdot (y + ax)$ which, for a is sufficiently large, is a contradiction.

From 3 it follows that if $x \cdot v_i = 0$ for $i = 1, \dots, n$ then $0 = u \cdot x$. We can assume without loss of generality that v_1, \dots, v_n are linearly independent. Then by linear algebra $\sum q_i v_i = u$ for some $q_i \in \mathbb{Q}$. Now fix i and verify that q_i is non-negative. Let x_i such that $x_i \cdot v_j = 1$, if $i = j$, and 0 otherwise. Then $0 \leq x_i \cdot u = q_i$. \square