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# Group actions on models

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Abstract. A set is *strongly generic* if the intersection of any finitely many of its translation is generic. To demostrate the convenience of this notion I use it for a short proof of (a generalization of) Newelski's theorem on the diamter of the Lascar graph, see Theorem 12.

Theorem 14 shows that the condition *strongly generic* = *generic* is robust. It might be of some interest (it is reminiscent of *forking* = *dividing*). Is it worth investigating?

Section **??** is incomplete. I would like to recover in a natural way the classical theory of stable groups – but something does not add up.

The connections with topological dynamics are commented at the end of the notes.

## 1. The two perspectives on the invariance of types

This section I review well-known matter and set the terminology.

Below $\Delta \subseteq L_{x\,z}(\mathcal{U})$. Let $\mathcal{Z} \subseteq \mathcal{U}^z$. We write $\Delta(\mathcal{Z})$ for the set of formulas of the form $\varphi(x\,;b)$ for some $\varphi(x\,;z) \in \Delta$ and some $b \in \mathcal{Z}$. We write $\Delta^{\pm}(\mathcal{Z})$ for the set of formulas in $\Delta(\mathcal{Z})$ or negation thereof. We write $S_\Delta(\mathcal{Z})$ for the set of complete $\Delta^{\pm}(\mathcal{Z})$-types.

We write $\Delta^{\mathsf{B}}(\mathcal{Z})$ for the set of Boolean combinations of formulas in $\Delta(\mathcal{Z})$.

When $\mathcal{A} \subseteq \mathcal{U}$, we may use $\mathcal{A}$ for $\mathcal{A}^z$ in the notation above.

Finally, define $\Delta^{\mathsf{G}}(A)$ to be the set of formulas $\varphi(x) \in L(\mathcal{U})$ that are equivalent to some formula in $\Delta^{\mathsf{B}}(A)$ or, equivalently, that are invariant over $A$. In the literature these formulas are called *generalized* $\Delta$-formulas over $A$. Note that when $A$ is a model $\Delta^{\mathsf{G}}(A)$-formulas are equivalent to $\Delta^{\mathsf{B}}(A)$-formulas.

**1 Assumption** Let $G$ be a group that acts on some sets $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$. We require that for every $\varphi(x\,;z) \in \Delta$ the set $\varphi(\mathcal{X};\mathcal{Z})$ is invariant under the action of $G$.

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that $\mathcal{D}$ is invariant under the action of $G$, or $G$-invariant, if $\mathcal{D} \cap \mathcal{Z}$ is fixed setwise by $G$. Yet in other words, if

is1. $\quad a \in \mathcal{D} \;\leftrightarrow\; ga \in \mathcal{D} \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

A formula is invariant if the set it defines is invariant. We say that $p(x) \subseteq L(\mathcal{U})$ is invariant under the action of $G$, or $G$-invariant, if for every formula $\varphi(x\,;z) \in L$

it1. $\quad \varphi(x\,;a) \in p \;\Leftrightarrow\; \varphi(x\,;ga) \in p \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

It should be evident that invariant under the action of $\mathrm{Aut}(\mathcal{U}/A)$ coincides with invariant over $A$ and that Lascar invariant over $A$ coincides with invariant under the action of $\mathrm{Autf}(\mathcal{U}/A)$.

We have just defined invariance using the subsets of $\mathcal{Z}$ (externally) defined by $p$. Now we discuss invariance using the subsets of $\mathcal{X}$ that are in $p$.

An immediate consequence of Assumption 1 is that any $G$-translate of a $\Delta$-definable set is again $\Delta$-definable. In particular for every $\Delta$-formula $\vartheta(x\,;\bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X}\,;\bar{b})] \quad = \quad \vartheta(\mathcal{X}\,;g\bar{b}).$$

Therefore $p(x) \subseteq L_\Delta(\mathcal{Z})$ is invariant if

$$p(x) \vdash x \in \mathcal{D} \;\Leftrightarrow\; p(x) \vdash x \in g\mathcal{D} \qquad \text{for every } \Delta\text{-definable } \mathcal{D} \subseteq \mathcal{U}^x \text{ and } g \in G,$$

where by $p(x) \vdash x \in \mathcal{D}$ we understand $\vartheta(\mathcal{X}) \subseteq \mathcal{D}$ for some $\vartheta(x)$ that is conjunction of formulas in $p(x)$.

A set $\mathcal{D} \subseteq \mathcal{X}$ is generic under the action of $G$, or $G$-generic for short, if finitely many $G$-translates of $\mathcal{D}$ cover $\mathcal{X}$; we say *$n$-$G$-generic* if $\leq n$ translates suffices. Dually, we say that $\mathcal{D}$ is persistent under the action of $G$, or $G$-persistent for short, if the intersection of any finitely many $G$-translates of $\mathcal{D}$ is nonempty; we say *$n$-$G$-persistent* when the request is limited to $\leq n$ translates. When $\mathcal{X}$ is not clear from the context, we say that these notions are relative to $\mathcal{X}$.

The terminology above is non-standard. In [1] the authors write *quasi-non-dividing* for *persistent* under the action of $\mathrm{Aut}(\mathcal{U}/A)$. Their terminology has good motivations, but it would be a mouthful if adapted to our context. In topological dynamics similar notions have been introduced with different terminology: *syndetic* corresponds to *generic* and *thick* corresponds to *persistent*.

**2 Example** If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in $A$ then $p(x)$ is persistent under the action of $\mathrm{Aut}(\mathcal{U}/A)$ relative to any $\mathcal{X} \supseteq A^x$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\mathrm{Aut}(\mathcal{U}/A)$-translate of $\varphi(x)$.

Notation: for $\mathcal{D} \subseteq \mathcal{U}^x$ and $H \subseteq G$ we write $H\,\mathcal{D}$ for $\{h\mathcal{D} : h \in H\}$.

In this notes many proofs require some juggling with negations as epitomized by the following fact.

**3 Fact** (Assume 1) The following are equivalent

1. $\mathcal{D}$ is not $G$-generic
2. $\neg\mathcal{D}$ is $G$-persistent.

**Proof.** Immediate by spelling out the definitions

1. there are no finite $H \subseteq G$ such that $\mathcal{X} \subseteq \cup H\,\mathcal{D}$.
2. $\varnothing \neq \mathcal{X} \cap \big(\cap\, H \neg\mathcal{D}\big)$ for every finite $H \subseteq G$. □

**4 Theorem** (Assume 1) Let $p(x) \in S_\Delta(\mathcal{Z})$ be finitely satisfiable in $\mathcal{X}$. Then the following are equivalent

1. $p(x)$ is $G$-invariant
2. $p(x) \vdash x \in \mathcal{D}$ for every $G$-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set $\mathcal{D}$
3. $p(x)$ is $G$-persistent.

**Proof.** 1⇒2. Let $H \subseteq G$ be finite such that $\mathcal{X} \subseteq \cup H\,\mathcal{D}$. By completeness and finite satisfiability, $p(x) \vdash x \in \cup H\,\mathcal{D}$. Again by completeness, $p(x) \vdash x \in h\mathcal{D}$ for some $h \in H$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

2⇒3. Let $\mathcal{D}$ be defined by a conjunction of formulas in $p(x)$. If $\mathcal{D}$ is not $G$-persistent then, by Fact 3, $\neg\mathcal{D}$ is $G$-generic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

3⇒1. If $p(x)$ is not $G$-invariant then, by completeness, $p(x) \vdash \varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ for some $g \in G$. Clearly $\varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ is not 2-$G$-persistent as it is inconsistent with its $g$-translate. □

The theorem yields a necessary condition for the existence of $G$-invariant global $\Delta^{\mathsf{B}}(\mathcal{Z})$-types.

**5 Corollary** (Assume 1) If there exists a $G$-invariant global type finitely satisfiable in $\mathcal{X}$ then for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set $\mathcal{D}$

1. $\mathcal{D}$ and $\neg\mathcal{D}$ are not both $G$-generic
2. if $\mathcal{D}$ is $G$-generic then it is $G$-persistent
3. the type $\gamma_G(x) = \{\vartheta(x) \in \Delta^{\mathsf{B}}(\mathcal{Z}) : \vartheta(x) \text{ is } G\text{-generic}\}$ is finitely satisfiable in $\mathcal{X}$.

**Proof.** Clearly, 1 and 2 are equivalent, moreover 1 and 3 are immediate consequences of 2 of Theorem 4. □

The following theorem gives a necessary and sufficient condition for the existence of global $G$-invariant $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Ideally, we would like to have that every $G$-persistent $\Delta^{\mathsf{B}}(\mathcal{Z})$-type extends to a global persitent type. Unfortunately this is not true – we need a stronger property. A set $\mathcal{D}$ is hereditarely $G$-persistent if every finite cover of $\mathcal{D}$ by $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets contains a $G$-persistent set. In [1] a similar property is called *quasi-non-forking*. A type is hereditarely $G$-persistent if every conjunction of formulas in the type is hereditarely $G$-persistent.

**6 Theorem** (Assume 1) Let $q(x) \subseteq L(\mathcal{U})$. Then the following are equivalent

1. $q(x)$ is consistent with a $G$-invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in $\mathcal{X}$
2. $q(x)$ is hereditarely $G$-persistent.

**Proof.** 1⇒2. Let $\vartheta(x)$ be a conjunction of formulas in $q(x)$. Suppose $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\vartheta(\mathcal{U}^x)$ and pick $p(x)$ as in 1. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some $i$. Then, by Theorem 4, $\neg\mathcal{C}_i$ is not $G$-generic. Therefore, by Fact 3, $\mathcal{C}_i$ is $G$-persistent.

2⇒1. Let $p(x)$ be maximal among the $\Delta^{\mathsf{B}}(\mathcal{Z})$-types that are consistent with $q(x)$ and are such that $\vartheta(\mathcal{U}^x)$ is hereditarely $G$-persistent for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that $p(x)$ is a complete $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\mathcal{C}_1, \dots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \smallsetminus \vartheta(\mathcal{U}^x)$ and such that no $\mathcal{C}_i$ is $G$-persistent. As $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is finitely satisfiable in $\mathcal{X}$ and $G$-invariant. Finite satisfiability follows from persistency. From completeness and Theorem 4 we obtain invariance. □

We concude with a fact that reminds of Lemma 2.10 in [2]. It is not used below.

**7 Fact** (Assume 1) Let $\mathcal{D}$ and $\mathcal{C}$ be $\Delta$-definable sets. The relation on $G$ defined by

$$R(h\,;k) \quad\Leftrightarrow\quad h\,\mathcal{D}\cap k\,\mathcal{C} \text{ is persistent}$$

is stable.

**Proof.** Let $\langle h_i\,;k_i : i<3\rangle$ be a sequence of elements of $G^2$. Assume $h_0\mathcal{D}\cap k_1\mathcal{C}$ is persistent. Note that if a set $\mathcal{B}$ is persistent then $\mathcal{B}\cap g\mathcal{B}$ is also persistent for any $g\in G$. Therefore $h_0\mathcal{D}\cap k_1\mathcal{C}\cap h_2\mathcal{D}\cap h_2h_0^{-1}k_1\mathcal{C}$ is persistent. A fortiori $h_2\mathcal{D}\cap k_1\mathcal{C}$ is peristent. Therefore $R(h_i\,;k_j)\Leftrightarrow i<j$ fails for some $i,j$. □

## 2. Strong genericity

Unfortunatelly, $G$-genericity is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set $\mathcal{D}\subseteq\mathcal{U}^x$ is strongly $G$-generic if for every finite $H\subseteq G$ the set $\cap H\,\mathcal{D}$ is generic. Dually, we say that $\mathcal{D}$ is weakly $G$-persistent if for some finite $H\subseteq G$ the set $\cup H\,\mathcal{D}$ is persistent. Again, the same properties may be attributed to formulas and types.

**8 Lemma** (Assume 1) The intersection of two strongly $G$-generic sets is strongly $G$-generic.

**Proof.** We may assume that all sets mentioned below are subsets of $\mathcal{X}$. Let $\mathcal{D}$ and $\mathcal{C}$ be strongly $G$-generic and let $K\subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B}=\cap K\,(\mathcal{C}\cap\mathcal{D})$ is $G$-generic. Clearly $\mathcal{B}=\mathcal{C}'\cap\mathcal{D}'$, where $\mathcal{C}'=\cap K\,\mathcal{C}$ and $\mathcal{D}'=\cap K\,\mathcal{D}$. Note that $\mathcal{C}'$ and $\mathcal{D}'$ are both strongly $G$-generic. In particular $\mathcal{X}=\cup H\,\mathcal{D}'$ for some finite $H\subseteq G$. Now, from

$$\begin{aligned}
\cup H\,\mathcal{B} &= \cup H\Big[\mathcal{C}'\ \cap\ \mathcal{D}'\Big]\\
\cup H\,\mathcal{B} &\supseteq \cup H\Big[\big(\cap\ H\,\mathcal{C}'\big)\ \cap\ \mathcal{D}'\Big]\\
&= \big(\cap\ H\,\mathcal{C}'\big)\ \cap\ \big(\cup\ H\,\mathcal{D}'\big)\\
&= \cap H\,\mathcal{C}'
\end{aligned}$$

As $\mathcal{C}'$ is strongly $G$-generic, $\cap H\,\mathcal{C}'$ is $G$-generic. Therefore $\cup H\,\mathcal{B}$ is also $G$-generic. The $G$-genericity of $\mathcal{B}$ follows. □

**9 Corollary** (Assume 1) Let ${}^{\mathrm{s}}\gamma_G(x)=\{\vartheta(x)\in L_\varphi(\mathcal{U}) : \vartheta(x) \text{ strongly } G\text{-generic}\}$. Then ${}^{\mathrm{s}}\gamma_G(x)$ is finitely satisfiable in $\mathcal{X}$, strongly $G$-generic, and $G$-invariant.

**Proof.** Strong $G$-genericity is an immediate consequence of Lemma 8. Finite satisfiability is a consequence of $G$-genericity. As for invariance, note that any translate of a strongly $G$-generic formula is also strongly $G$-generic. □

**10 Corollary** (Assume 1) Let ${}^{\mathrm{s}}\gamma(x)$ be as in Corollary 9. Let $p(x)\subseteq L(\mathcal{U})$ be such that ${}^{\mathrm{s}}\gamma(x)\cup p(x)$ is finitely satisfied in $\mathcal{X}$. Then $p(x)$ is weakly $G$-persistent.

**Proof.** Let $\vartheta(x) \in p$. As ${}^{\mathrm{s}}\gamma(x)$ is finitely satisfiable in $\vartheta(\mathcal{U}^x)$, we cannot have that $\neg\vartheta(x)$ is strongly $G$-generic. From Fact 3, we obtain that $\neg\vartheta(\mathcal{U}^x)$ non strongly $G$-generic is equivalent to $\vartheta(x)$ weakly $G$-persistent. $\square$

## 3. The diameter of a Lascar type

As an application we prove an interesting property of the Lascar types. Recall that $\mathcal{L}(a/A)$, the Lascar strong type of $a \in \mathcal{U}^x$, is the union of a chain of type-definable sets of the form $\{x : d_A(a,x) \leq n\}$. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of $a$ in the Lascar graph has finite diameter.

It is convenient to address the problem in more general terms. We work under Assumption 1 with $\Delta = L_{xz}$ and $G \trianglelefteq \mathrm{Aut}(\mathcal{U})$. Let $K \subseteq G$ be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugancy invariant i.e. $gKg^{-1} = K$ for every $g \in G$

Assume $G$ acts transitively on $\mathcal{X}$ i.e. $G\,a = \mathcal{X}$ for every $a \in \mathcal{X}$. We define a discrete metric on $\mathcal{X}$. For $a, b \in \mathcal{X}$ let $d(a,b)$ be the minimal $n$ such that $a \in K^n b$. This defines a metric which is $G$-invariant by 2. The diameter of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a,b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for $\mathcal{X}$ to have finite diameter. The notions introduced in Section 2 offer some hint.

**11 Proposition** If $\mathcal{X}$ has a weakly persistent subset of finite diameter, then $\mathcal{X}$ itself has finite diamenter.

**Proof.** Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter $n$. Let $H \subseteq G$ be finite such that $\cup H\,\mathcal{C}$ is persistent. We claim that also $\cup H\,\mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let $m$ be larger than $d(ha, ka)$ for all $h, k, \in H$. Now, let $hb$ and $kc$, for some $h, k, \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H\,\mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of $\mathcal{C}$,

$$\begin{aligned} d(hb,\, kc) &\leq d(hb,\, ha) \;+\; d(ha,\, ka) \;+\; d(ka,\, kc) \\ &\leq n + m + n. \end{aligned}$$

This proves that $\cup H\,\mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that $\mathcal{C}$ itself is persistent.

By the transitivity of the action, any two elements of $\mathcal{X}$ are of the form $ha$, $ka$ for some $h, k \in G$ and some $a \in \mathcal{C}$. By percistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha,\, ka) &\leq d(ha,\, c) \;+\; d(c,\, d) \;+\; d(d,\, ka) \\ &\leq n + n + n. \end{aligned}$$

Therefore the diameter of $\mathcal{X}$ does not exceed $3n$. $\square$

**12 Theorem** Suppose that $\mathcal{X}$ and the sets $\mathcal{X}_n = K^n a$, for some $a \in \mathcal{X}$, are type-definable. Then $\mathcal{X}$ has finite diameter.

**Proof.** By Proposition 11, it suffices to prove that $\mathcal{X}_n$ is weakly persistent. Let ${}^{\mathrm{s}}\gamma_G(x)$ be as in Corollary 9, with $L_{x,z}$ for $\Delta$. It suffices to prove that for some $n$ the type ${}^{\mathrm{s}}\gamma_G(x)$ is finitely satisfied in $\mathcal{X}_n$. Suppose not. Let $\psi_n(x) \in q$ be a formula that is not satisfied in $\mathcal{X}_n$. The type $p(x) = \{\psi_n(x) : n \in \omega\}$ is finitely satisfied in $\mathcal{X}$. Then $p(x)$ has a realization in $\mathcal{X}$. As this realization belongs to some $\mathcal{X}_n$ we contradict the definition of $\psi_n(x)$. □

**13 Example** Let $K \subseteq \mathrm{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing $A$. Then the group $G$ generated by $K$ is $\mathrm{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. Then $d(a,b)$ concides with the dinstance in the Lascar graph. As the sets $K^n \cdot a = \{x : d(x,a) \leq n\}$ are type definable from Theorem 12 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

## 4. A tamer landscape

Under suitable assumptions some of the notions introduced in this chapter coalesce and we are left with a tamer landscape. We prove the following theorem.

**14 Theorem** (Assume 1) The following are equivalent

1. $G$-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are hereditarely $G$-persistent
2. $G$-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are closed under intersection
3. $G$-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are strongly $G$-generic
4. weakly persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are $G$-persistent.

**Proof.** 2⇔3⇔4. Clear.

1⇒2. Let $\mathcal{C}$ and $\mathcal{D}$ be $G$-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not $G$-generic. Then $\neg(\mathcal{C} \cap \mathcal{D})$ is $G$-persistent. By 1 and Theorem 6 there is a $G$-invariant global $\Delta^{\mathrm{B}}(\mathcal{Z})$-type $p(x)$ containing $x \notin \mathcal{C} \cap \mathcal{D}$. By completeness either $p(x) \vdash x \notin \mathcal{C}$ or $p(x) \vdash x \notin \mathcal{D}$. This is a contradiction because by Theorem 4 $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

4⇒1. Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) : \vartheta(x)\ G\text{-generic}\}$. By 2 this is the same type defined in Corollary 10. Therefore, any completion of $q(x)$ is, by 4, $G$-persistent. Let $\mathcal{D}$ be a $G$-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set. By Theorems 4 and 6 it suffices to show that $\mathcal{D}$ is consistent with $q(x)$. Suppose not, then $q(x) \vdash x \notin \mathcal{D}$. Therefore, by 3, $\neg\mathcal{D}$ is $G$-generic. This is a contradiction by Fact 3. □

**15 Example** It is not difficult to verify that the equivalent conditions in the theorem hold when $\Delta$ is a set of stable formulas, $G = \mathrm{Aut}(\mathcal{U}/A)$, and $\mathcal{X}$ is the solution set of a complete type over $\mathrm{acl}^{\mathrm{eq}} A$.

**16 Assumption** For $G$, $\mathcal{X}$, $\mathcal{Z}$ and $\Delta$ as in Assumption 1 we also require that the equivalent conditions in Theorem 14 hold.

**17 Remark** (Assume 16) Note that the types $\gamma_G(x)$ and ${}^{\mathrm{s}}\gamma_G(x)$ defined in corollary 5 and 10 coincide. Then for every $p(x) \in S_\Delta(\mathcal{Z})$ the following are equivalent

1. $p(x)$ is $G$-persistent (equivalently, $G$-invariant)
2. $p(x)$ extends $\gamma_G(x)$.

Note also that under Assumption 16 $G$-invariant global types exist because $\gamma_G(x)$ is finitely consistent in $\mathcal{X}$.

It is also worth mentioning that when $\mathcal{D}$ is $G$-generic then every positive Bolean combination of $G$-translates of $\mathcal{D}$ is $G$-generic.

## 5. Notes and references

Connnections with topological dynamics are mentioned everywhere but I ignored them until the very last. I just realized that *persistent* = *thick* and that *weakly persistent* = *piecewise syndetic.* Of course, *generic* = *syndetic.* The notion of *hereditarely persistent* may also have an analogon in topological dynamics, but could not find it yet.

[1] Artem Chernikov and Itay Kaplan, *Forking and dividing in* $\mathrm{NTP}_2$ *theories*, J. Symbolic Logic **77** (2012), 1–20.
[2] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.