

Group actions on models

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ABSTRACT. A set is *strongly generic* if the intersection of any finitely many of its translation is generic. To demonstrate the convenience of this notion I use it for a short proof of (a generalization of) Newelski's theorem on the diameter of the Lascar graph, see Theorem 12.

Theorem 14 shows that the condition *strongly generic* = *generic* is robust. It might be of some interest (it is reminiscent of *forking* = *dividing*). Is it worth investigating?

Section 5 is incomplete. I would like to recover in a natural way the classical theory of stable groups – but something does not add up.

The connections with topological dynamics are commented at the end of the notes.

1. The two perspectives on the invariance of types

This section I review well-known matter and set the terminology.

Below $\Delta \subseteq L_{xz}(\mathcal{U})$, $\mathcal{X} \subseteq \mathcal{U}^x$, and $\mathcal{Z} \subseteq \mathcal{U}^z$ are some arbitrary nonempty sets (at some point we will require that \mathcal{X} and \mathcal{Z} are type-definable). We write $L_\Delta(\mathcal{Z})$ for the set of formulas $\vartheta(x)$ that are Boolean combination of formulas $\varphi(x; b)$ for some $\varphi(x; z) \in \Delta$ and some $b \in \mathcal{Z}$. Such formulas are called Δ -formulas. A Δ -definable set is a set of the form $\vartheta(\mathcal{U}^x)$ for some Δ -formula $\vartheta(x)$. Subsets of $L_\Delta(\mathcal{Z})$ are called Δ -types. We write $S_\Delta(\mathcal{Z})$ for the set of complete Δ -types with parameters in \mathcal{Z} . Beware that there may be other parameters hidden in Δ . When convenient, we may assume that complete types only contain formulas of the form $\pm\varphi(x; b)$ for $\varphi(x; z) \in \Delta$.

1 Assumption Let G be a group that acts on \mathcal{X} and on \mathcal{Z} from the left. We require that for every $\varphi(x; z) \in \Delta$ the set $\varphi(\mathcal{X}; \mathcal{Z})$ is invariant under the action of G .

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that \mathcal{D} is **invariant** under the action of G , or **G -invariant**, if $\mathcal{D} \cap \mathcal{Z}$ is fixed setwise by G . Yet in other words, if

$$\text{is1.} \quad a \in \mathcal{D} \leftrightarrow ga \in \mathcal{D} \quad \text{for every } a \in \mathcal{Z} \text{ and every } g \in G.$$

A formula is invariant if the set it defines is invariant. We say that $p(x) \subseteq L(\mathcal{U})$ is **invariant** under the action of G , or **G -invariant**, if for every formula $\varphi(x; z) \in L$

$$\text{it1.} \quad \varphi(x; a) \in p \Leftrightarrow \varphi(x; ga) \in p \quad \text{for every } a \in \mathcal{Z} \text{ and every } g \in G.$$

It should be evident that invariant under the action of $\text{Aut}(\mathcal{U}/A)$ coincides with invariant over A and Lascar invariant over A coincides with invariant under the action of $\text{Autf}(\mathcal{U}/A)$.

We have just defined invariance using the subsets of \mathcal{Z} (externally) defined by p . Now we discuss invariance using the subsets of \mathcal{X} that are in p .

An immediate consequence of Assumption 1 is that any G -translate of a Δ -definable set is again Δ -definable. In particular for every Δ -formula $\vartheta(x; \bar{b})$ and every $g \in G$

$$g[\vartheta(x; \bar{b})] = \vartheta(x; g\bar{b}).$$

Therefore $p(x) \subseteq L_\Delta(\mathbb{Z})$ is invariant if

$$p(x) \vdash x \in \mathcal{D} \Leftrightarrow p(x) \vdash x \in g\mathcal{D} \quad \text{for every } \Delta\text{-definable } \mathcal{D} \subseteq \mathcal{U}^x \text{ and } g \in G,$$

where by $p(x) \vdash x \in \mathcal{D}$ we understand $\vartheta(x) \subseteq \mathcal{D}$ for some $\vartheta(x)$ that is conjunction of formulas in $p(x)$.

A set $\mathcal{D} \subseteq \mathcal{X}$ is **generic** under the action of G , or **G -generic** for short, if finitely many G -translates of \mathcal{D} cover \mathcal{X} ; we say **n - G -generic** if $\leq n$ translates suffices. Dually, we say that \mathcal{D} is **persistent** under the action of G , or **G -persistent** for short, if the intersection of any finitely many G -translates of \mathcal{D} is nonempty; we say **n - G -persistent** when the request is limited to $\leq n$ translates. We will drop reference to G when it is clear from the context.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

The terminology is mine. In [1] the authors write *quasi-non-dividing* for *persistent* when $G = \text{Aut}(\mathcal{U}/A)$.

2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is persistent (in any $\mathcal{X} \supseteq A^x$) under the action of $\text{Aut}(\mathcal{U}/A)$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\text{Aut}(\mathcal{U}/A)$ -translate of $\varphi(x)$.

In this chapter many proofs require some juggling with negations.

3 Fact (Assume 1) The following are equivalent

1. \mathcal{D} is not n -generic
2. $\neg \mathcal{D}$ is n -persistent.

Proof. Immediate by spelling out the definitions

1. there are no $g_1, \dots, g_n \in G$ such that $\mathcal{X} \subseteq \bigcup_{i=1}^n g_i \mathcal{D}$
2. $\emptyset \neq \mathcal{X} \cap \bigcap_{i=1}^n \neg g_i \mathcal{D}$ for every $g_1, \dots, g_n \in G$. □

4 Theorem (Assume 1) Let $p(x) \in S_\Delta(\mathbb{Z})$ be finitely satisfiable in \mathcal{X} . Then the following are equivalent

1. $p(x)$ is invariant
2. $p(x) \vdash x \in \mathcal{D}$ for every generic Δ -definable set \mathcal{D}
3. $p(x)$ is persistent.

Proof. 1 \Rightarrow 2. Let $g_1 \mathcal{D}, \dots, g_n \mathcal{D}$ be translations of \mathcal{D} that cover \mathcal{X} . Negate 2. By completeness, $p(x) \vdash x \notin \mathcal{D}$. Hence, from invariance we obtain

$$p(x) \vdash x \notin \bigcup_{i=1}^n g_i \mathcal{D}.$$

This contradicts the finite satisfiability of $p(x)$ in \mathcal{X} .

2 \Rightarrow 3. Let \mathcal{D} be defined by a conjunction of formulas in $p(x)$. If \mathcal{D} is not persistent then, by Fact 3, $\neg \mathcal{D}$ is generic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

3 \Rightarrow 1. If $p(x)$ is not invariant then, by completeness, $p(x) \vdash \varphi(x; b) \wedge \neg \varphi(x; gb)$ for some $g \in G$. Clearly $\varphi(x; b) \wedge \neg \varphi(x; gb)$ is not persistent as it is inconsistent with its g -translate. \square

5 Remark In the theorem above, 2 and 3 can be replaced by

- 2'. $p(x) \vdash x \in \mathcal{D}$ for every 2-generic Δ -definable set \mathcal{D}
- 3'. $p(x)$ is 2-persistent.

The theorem yields an immediate necessary condition for the existence of an invariant global Δ -type.

6 Corollary (Assume 1) If there exists an invariant global Δ -type then for every Δ -definable set \mathcal{D}

- 1. \mathcal{D} and $\neg \mathcal{D}$ cannot be both generic
- 2. if \mathcal{D} is generic then it is persistent.

Proof. By Fact 3, 1 and 2 are equivalent; 1 is an immediate consequence of 2 of Theorem 4. \square

The following theorem gives a necessary and sufficient condition for the existence of global invariant Δ -type. Ideally, we would like to prove that every persistent Δ -type extends to a global persistent type. Unfortunately this is not true – we need a stronger property. A Δ -definable set \mathcal{D} is **hereditarily persistent** if every finite cover of \mathcal{D} by Δ -definable sets contains a persistent set. A type is hereditarily persistent if every conjunction of formulas in the type is hereditarily persistent.

The terminology is provisional. In [1] a related property is called *quasi-non-forking*.

7 Theorem (Assume 1) Let $q(x) \subseteq L(\mathcal{U})$. Then the following are equivalent

- 1. $q(x)$ extends to an invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in \mathcal{X}
- 2. $q(x)$ is hereditarily persistent.

Proof. 1 \Rightarrow 2. Let $\vartheta(x)$ be a conjunction of formulas in $q(x)$. Suppose $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\vartheta(\mathcal{U}^x)$ and pick $p(x)$ as in 1. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i . Then, by Theorem 4, $\neg \mathcal{C}_i$ is not generic. Therefore, by Fact 3, \mathcal{C}_i is persistent.

2 \Rightarrow 1. Let $p(x)$ be maximal among the Δ -types that contain $q(x)$ and are such that $\vartheta(\mathcal{U}^x)$ is hereditarily persistent for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that p is a complete Δ -type. Suppose for a contradiction that $\vartheta(x), \neg \vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$ and some $\mathcal{C}_1, \dots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \setminus \vartheta(\mathcal{U}^x)$ and such that no \mathcal{C}_i is persistent. As $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction.

It is only left to show that $p(x)$ is finitely satisfiable in \mathcal{X} and invariant. Finite satisfiability follows from persistency. From completeness and Theorem 4 we obtain invariance. \square

2. Strong genericity

Unfortunately, genericity is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is **strongly generic** if the intersection of \mathcal{D} with any of its translations \mathcal{D} is generic. Dually, we say that \mathcal{D} is **weakly persistent** if the union of \mathcal{D} with one of its translations is persistent. Again, the same properties may be attributed to formulas and types.

Notation: for $\mathcal{B} \subseteq \mathcal{X}$ and $H \subseteq G$ we write $H\mathcal{B}$ for $\{h\mathcal{B} : h \in H\}$.

8 Lemma (Assume 1) The intersection of strongly generic sets is strongly generic.

Proof. We may assume that all sets mentioned below are subsets of \mathcal{X} . Let \mathcal{D} and \mathcal{C} be strongly generic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K(\mathcal{C} \cap \mathcal{D})$ is generic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K\mathcal{C}$ and $\mathcal{D}' = \cap K\mathcal{D}$. Note that \mathcal{C}' and \mathcal{D}' are both strongly generic. In particular $\mathcal{X} = \cup H\mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\begin{aligned} \cup H\mathcal{B} &= \cup H[\mathcal{C}' \cap \mathcal{D}'] \\ \cup H\mathcal{B} &\supseteq \cup H[(\cap H\mathcal{C}') \cap \mathcal{D}'] \\ &= (\cap H\mathcal{C}') \cap (\cup H\mathcal{D}') \\ &= \cap H\mathcal{C}' \end{aligned}$$

As \mathcal{C}' is strongly generic, $\cap H\mathcal{C}'$ is generic. Therefore $\cup H\mathcal{B}$ is also generic. The genericity of \mathcal{B} follows. \square

9 Corollary (Assume 1) Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) : \vartheta(x) \text{ strongly generic}\}$. Then $q(x)$ is finitely satisfiable in \mathcal{X} , strongly generic, and invariant.

Proof. Strong genericity is an immediate consequence of Lemma 8. Finite satisfiability follows easily from genericity. As for invariance, note that any translate of a strongly generic formula is also strongly generic. \square

10 Corollary (Assume 1) Let $q(x)$ be as in Corollary 9. Let $p(x) \subseteq L(\mathcal{U})$ be such that $p(x) \cup q(x)$ is finitely satisfied in \mathcal{X} . Then $p(x)$ is weakly persistent.

Proof. Let $\vartheta(x) \in p$. As $q(x)$ is finitely satisfiable in $\vartheta(\mathcal{U}^x)$, we cannot have that $\neg\vartheta(x)$ is strongly generic. From Fact 3, we obtain that $\neg\vartheta(\mathcal{U}^x)$ non strongly generic is equivalent to $\vartheta(x)$ weakly persistent. \square

3. The diameter of a Lascar type

Recall that $\mathcal{L}(a/A)$, the Lascar strong type of $a \in \mathcal{U}^x$, is the union of a chain of type-definable sets of the form $\{x : d_A(a, x) \leq n\}$. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter. It is convenient to address the problem in more general terms.

Assume $G \trianglelefteq \text{Aut}(\mathcal{U})$. Let $K \subseteq G$ be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugacy invariant i.e. $gKg^{-1} = K$ for every $g \in G$

Assume G acts transitively on \mathcal{X} i.e., $Ga = \mathcal{X}$ for every $a \in \mathcal{X}$. We define a discrete metric on \mathcal{X} . For $a, b \in \mathcal{X}$ let $d(a, b)$ be the minimal n such that $a \in K^n b$. This defines a metric which is G -invariant by 2. The **diameter** of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a, b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for \mathcal{X} to have finite diameter. The notions introduced in Section 2 offer some hint.

11 Proposition If \mathcal{X} has a weakly persistent subset of finite diameter, then \mathcal{X} itself has finite diameter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter n . Let $H \subseteq G$ be finite such that $\bigcup H\mathcal{C}$ is persistent. We claim that also $\bigcup H\mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k \in H$. Now, let hb and kc , for some $h, k \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\bigcup H\mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of \mathcal{C} ,

$$\begin{aligned} d(hb, kc) &\leq d(hb, ha) + d(ha, ka) + d(ka, kc) \\ &\leq n + m + n. \end{aligned}$$

This proves that $\bigcup H\mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that \mathcal{C} itself is persistent.

By the transitivity of the action, any two elements of \mathcal{X} are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By persistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha, ka) &\leq d(ha, c) + d(c, d) + d(d, ka) \\ &\leq n + n + n. \end{aligned}$$

Therefore the diameter of \mathcal{X} does not exceed $3n$. □

12 Theorem Suppose that \mathcal{X} and the sets $\mathcal{X}_n = K^n a$, for some $a \in \mathcal{X}$, are type-definable. Then \mathcal{X} has finite diameter.

Proof. By Proposition 11, it suffices to prove that \mathcal{X}_n is weakly persistent. Define

$$q(x) = \{\vartheta(x) \in L(\mathcal{U}) : \vartheta(x) \text{ strongly generic}\}.$$

By Corollary 10, with L for L_φ , it suffices to prove that for some n the type $q(x)$ is finitely satisfied in \mathcal{X}_n . Suppose not. Let $\psi_n(x) \in q$ be a formula that is not satisfied in \mathcal{X}_n . The type $p(x) = \{\psi_n(x) : n \in \omega\}$ is finitely satisfied in \mathcal{X} . Then $p(x)$ has a

realization in \mathcal{X} . As this realization belongs to some \mathcal{X}_n we contradict the definition of $\psi_n(x)$. \square

13 Example Let $K \subseteq \text{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A . Then the group G generated by K is $\text{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. Then $d(a, b)$ coincides with the distance in the Lascar graph. It is not difficult to see that the sets $K^n \cdot a$ are type definable. Then from Theorem 12 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

4. A simplified landscape

Under suitable assumptions – e.g. the stability of $\varphi(x; z)$ – some of the notions introduced above coalesce and we are left with cleaner theory. We prove the following theorem.

14 Theorem (Assume 1) The following are equivalent

1. persistent Δ -definable sets are hereditarily persistent
2. generic Δ -definable sets are strongly generic
3. generic Δ -definable sets are closed under intersection
4. weakly persistent Δ -definable sets are persistent.

Proof. $1 \Rightarrow 2$. It suffices to prove that generic sets are closed under intersection. Let \mathcal{C} and \mathcal{D} be generic Δ -definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not generic. By 1 and Theorem 7 there is an invariant global Δ -type $p(x)$ containing $x \in \neg\mathcal{C} \cup \neg\mathcal{D}$. By completeness either $p(x) \vdash x \in \neg\mathcal{C}$ or $p(x) \vdash x \in \neg\mathcal{D}$. This is a contradiction because, by Theorem 4, $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

$2 \Leftrightarrow 3 \Leftrightarrow 4$. Clear.

$4 \Rightarrow 1$. Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) : \vartheta(x) \text{ generic}\}$. By 2 this is the same type defined in Corollary 10. Therefore, any completion of $q(x)$ is, by 4, persistent. Let \mathcal{D} be a persistent Δ -definable set. By Theorems 4 and 7 it suffices to show that \mathcal{D} is consistent with $q(x)$. Suppose not, then $q(x) \vdash x \in \neg\mathcal{D}$. Therefore, by 3, $\neg\mathcal{D}$ is generic. This is a contradiction by Fact 3. \square

15 Remark Assume the equivalent conditions in Theorem 14. Then there is a unique maximal generic type $q(x) \subseteq L_\Delta(\mathbb{Z})$. Moreover the following are equivalent for every $p(x) \in S_\Delta(\mathbb{Z})$

1. $p(x)$ is persistent (equivalently, invariant)
2. $p(x)$ extends $q(x)$.

In particular, invariant global types exist.

5. Definable groups

In this section we work as always under Assumption 1 but with add some important requirement on G and Δ . We set $G = \mathbb{Z}$ and require that \mathbb{Z} and \mathcal{X} are type-definable over A . We assume that the group operations and the group action

are definable over A . We use the symbol \cdot for both the group multiplication and the group action. Clearly, \mathbb{Z} also acts on itself by left multiplication.

In this section we deal with the actions of two groups: \mathbb{Z} and $\text{Aut}(\mathcal{U}/A)$. Generic and persistent only refer to the action of \mathbb{Z} . We will always be explicit when we mention invariance.

Let $\psi(x; y) \in L(A)$. We write $\varphi(x; z; y)$ for the formula $\psi(z^{-1} \cdot x; y)$. In this section Δ contains the formulas $\varphi(x; z; a)$ where a ranges over some given $\mathcal{Y} \subseteq \mathcal{U}^{\mathcal{Y}}$ that is invariant over A . Note that $\varphi(\mathcal{X}; \mathbb{Z}; a)$ is \mathbb{Z} -invariant for every a . Therefore the above G and Δ are compatible with Assumption 1.

Let 1 be the identity of \mathbb{Z} which, for simplicity, we think as a constant of L . Clearly, $\varphi(\mathcal{X}; g; a) = g \cdot \varphi(\mathcal{X}; 1; a)$.

16 Fact There are some formulas $\gamma(y), \pi(y) \in L(A)$ such that, for every $a \in \mathcal{Y}$ and every $g \in \mathbb{Z}$

$$\gamma(a) \Leftrightarrow \varphi(x; g; a) \text{ is generic}$$

$$\pi(a) \Leftrightarrow \varphi(x; g; a) \text{ is persistent.}$$

Similar claims holds for $\neg\varphi(x; 1; a)$.

Proof. As $\varphi(x; g; a)$ is generic/persistent if and only if $\varphi(x; 1; a)$, we can replace g by 1 . By an easy argument of compactness, there is an n such that if $\varphi(x; 1; a)$ is generic then it is also n -generic. Then

$$\gamma(y) = \exists z_1, \dots, z_n \forall x \bigvee_{i=1}^n \varphi(x; z_i; y)$$

The other claims follow easily. \square

An immediate consequence is that $\text{Aut}(\mathcal{U}/A)$ maps generic/persistent sets of the form $\pm\varphi(\mathcal{X}; g; a)$ to generic/persistent sets of the same form.

17 Fact Let $p(x) \in S_{\Delta}(\mathbb{Z})$ be persistent – equivalently, \mathbb{Z} -invariant. Then $p(x)$ is invariant over A .

Proof. We may assume that $p(x)$ only contains formulas of the form $\pm\varphi(x; b)$ for $\varphi(x; z) \in \Delta$. As $p(x)$ is \mathbb{Z} -invariant, $\varphi(x; g; a) \in p \Leftrightarrow \varphi(x; 1; a) \in p$. Then, as $\varphi(x; 1; a) \leftrightarrow \varphi(x; 1; fa)$ for every $f \in \text{Aut}(\mathcal{U}/A)$, invariance over A follows. \square

18 Assumption Assume that the three equivalent conditions in Theorem 14 hold for $G = \mathbb{Z}, \mathcal{X}$, and Δ introduced in this section.

Under this assumption the types $p(x) \in S_{\Delta}(\mathbb{Z})$ that extend the maximal generic type $q(x)$, see Remark 15, are invariant under the action of both \mathbb{Z} and $\text{Aut}(\mathcal{U}/A)$.

6. Notes and references

Connections with topological dynamics are mentioned everywhere but I ignored them until the very last. I just realized that *persistent* = *thick* and that *weakly persistent*

= *piecewise syndetic*. Of course, *generic* = *syndetic*. The notion of *hereditarily persistent* may also have an analogon in topological dynamics, but could not find it yet.

- [1] Artem Chernikov and Itay Kaplan, *Forking and dividing in NTP_2 theories*, J. Symbolic Logic **77** (2012), 1–20.