Group actions on models

ABSTRACT. A set is *strongly generic* if the intersection of any finitely many of its translations is generic.

Theorem 18 shows that the condition *strongly generic = generic* is roboust. It might be of some interest (it is reminiscent of *forking = dividing*). Does it have interesting examples?

Proposition 25 reminds of Hrushovski's stabilizer theorem [2]. But how, exactly? And, can it be strengthened to obtain a type definable group?

The connections with topological dynamics have not been explored :(

1. The dual perspective on invariance

The notions in this section are well-known but sometimes the terminology differs.

In this chapter $\Delta \subseteq L_{xz}(\mathcal{U})$. Let $\mathcal{Z} \subseteq \mathcal{U}^z$. We write $\Delta(\mathcal{Z})$ for the set of formulas of the form $\varphi(x;b)$ for some $\varphi(x;z) \in \Delta$ and some $b \in \mathcal{Z}$. We write $\Delta^{\pm}(\mathcal{Z})$ for the set of formulas in $\Delta(\mathcal{Z})$ or negation thereof. Furthermore, we write $S_\Delta(\mathcal{Z})$ for the set of complete $\Delta^{\pm}(\mathcal{Z})$-types.

We write $\Delta^{\mathsf{B}}(\mathcal{Z})$ for the set of Boolean combinations of formulas in $\Delta(\mathcal{Z})$.

When $\mathcal{Z} = \mathcal{A}^z$ for some $\mathcal{A} \subseteq \mathcal{U}$, we may write $\mathcal{A}$ for $\mathcal{A}^z$ in the notation above.

> **1 Assumption** Let G be a group that acts on some sets $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$. We require that for every $\varphi(x;z) \in \Delta$ the set $\varphi(\mathcal{X};\mathcal{Z})$ is invariant under the action of G. For convenience, we will assume that G is the identity outside $\mathcal{X}$ and $\mathcal{Z}$.
>
> When $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ and $\mathcal{D} \subseteq \mathcal{U}^x$ we write $p(x) \vdash x \in \mathcal{D}$ if the inclusion $\psi(\mathcal{X}) \subseteq \mathcal{D}$ for some $\psi(x)$ that is conjunctions of formulas in $p(x)$.

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that $\mathcal{D}$ is invariant under the action of G, or *G-invariant*, if $\mathcal{D}$ is fixed setwise by G. That is, $g\mathcal{D} = \mathcal{D}$ for every $g \in G$. Yet in other words, if

is1. $a \in \mathcal{D} \ \leftrightarrow \ ga \in \mathcal{D}$ for every $a \in \mathcal{Z}$ and every $g \in G$.

A formula is G-invariant if the set it defines is G-invariant. We say that $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ is invariant under the action of G, or *G-invariant*, if for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-formula $\vartheta(x;\bar{a})$.

it1. $\vartheta(x;\bar{a}) \in p \ \Leftrightarrow \ \vartheta(x;g\,\bar{a}) \in p$ for every $g \in G$.

It should be clear that invariant under the action of $\mathrm{Aut}(\mathcal{U}/A)$ coincides with invariant over A and Lascar invariant over A coincides with invariant under the action of $\mathrm{Autf}(\mathcal{U}/A)$.

Note that $p(x)$ is G-invariant exactly when the sets $\mathcal{D}_{p,\vartheta} \subseteq \mathcal{U}^z$ are. In this section we want to discuss invariance using sets $\mathcal{D} \subseteq \mathcal{U}^x$. We need stronger assumptions on the action of G.

An immediate consequence of the invariance of $\varphi(\mathcal{X}; \mathcal{Z})$ is that any G-translate of a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set is again $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable. In particular, by reasoning as in Remark **??** we obtain that for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-formula $\vartheta(x; \bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X}; \bar{b})] \;=\; \vartheta(\mathcal{X}; g\,\bar{b}).$$

Therefore, a type $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ is G-invariant if

$$p(x) \vdash x \in \mathcal{D} \;\Leftrightarrow\; p(x) \vdash x \in g\cdot\mathcal{D} \text{ for every } \Delta^{\mathsf{B}}(\mathcal{Z})\text{-definable } \mathcal{D} \subseteq \mathcal{X} \text{ and } g \in G.$$

A set $\mathcal{D} \subseteq \mathcal{X}$ is generic under the action of G, or G-generic for short, if finitely many G-translates of $\mathcal{D}$ cover $\mathcal{X}$; we say n-G-generic if $\leq n$ translates suffices. Dually, we say that $\mathcal{D}$ is persistent under the action of G, or G-persistent for short, if the intersection of any finitely many G-translates of $\mathcal{D}$ is nonempty; we say n-G-persistent when the request is limited to $\leq n$ translates. When $\mathcal{X}$ and/or $\mathcal{Z}$ are not clear from the context, we say that these notions are relative to $\mathcal{X}$ and $\mathcal{Z}$.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

⚠ The terminology above is non-standard. In [1] the authors write *quasi-non-dividing* for *persistent* under the action of $\mathrm{Aut}(\mathcal{U}/A)$. Their terminology has good motivations, but it would be a mouthful if adapted to our context. In topological dynamics similar notions have been introduced with different terminology: *syndetic* corresponds to *generic* and *thick* corresponds to *persistent*.

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2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is persistent under the action of $G = \mathrm{Aut}(\mathcal{U}/A)$ relative to any $\mathcal{X} \supseteq A^x$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\mathrm{Aut}(\mathcal{U}/A)$-translate of $\varphi(x)$.

Notation: for $\mathcal{D} \subseteq \mathcal{U}^x$ and $H \subseteq G$ we write $H\,\mathcal{D}$ for $\{h\mathcal{D} : h \in H\}$.

In this chapter many proofs require some juggling with negations as epitomized by the following fact.

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3 Fact The following are equivalent

1. $\mathcal{D}$ is not G-generic
2. $\neg\mathcal{D}$ is G-persistent.

Proof. Immediate by spelling out the definitions

1. there are no finite $H \subseteq G$ such that $\mathcal{X} \subseteq \cup H\,\mathcal{D}$.
2. $\varnothing \neq \mathcal{X} \cap \left(\cap H\neg\mathcal{D} \right)$ for every finite $H \subseteq G$.   □

Define the following type

$$\gamma_G(x) \;=\; \{\vartheta(x) \in \Delta^{\mathsf{B}}(\mathcal{Z}) : \vartheta(x) \text{ is } G\text{-generic}\}$$

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4 Corollary Let $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ be such that $\gamma_G(x) \cup p(x)$ is finitely satisfiable in $\mathcal{X}$. Then $p(x)$ is G-persistent.

Proof. Let $\vartheta(x)$ be a conjunction of formulas in $p(x)$. As $\gamma_G(x)$ is finitely satisfiable in $\vartheta(\mathfrak{X})$, it cannot be that $\neg\vartheta(x)$ is G-generic. From Fact 3, we obtain that $\vartheta(x)$ is G-persistent. $\square$

The converse implication holds for complete types.

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5 Theorem Let $p(x) \in S_\Delta(\mathfrak{Z})$. Then the following are equivalent

1. $p(x)$ is G-invariant and finitely satisfiable in $\mathfrak{X}$
2. $p(x) \vdash \gamma_G(x)$
3. $p(x)$ is G-persistent.

Proof. 1$\Rightarrow$2. Let $H \subseteq G$ be finite such that $\mathfrak{X} \subseteq \cup H\,\mathfrak{D}$. By completeness and finite satisfiability, $p(x) \vdash x \in \cup H\,\mathfrak{D}$. Again by completeness, $p(x) \vdash x \in h\,\mathfrak{D}$ for some $h \in H$. Finally, by invariance, $p(x) \vdash x \in \mathfrak{D}$.

2$\Rightarrow$3. Let $\mathfrak{D}$ be defined by a conjunction of formulas in $p(x)$. If $\mathfrak{D}$ is not G-persistent then, by Fact 3, $\neg\mathfrak{D}$ is G-generic. By 2, $p(x) \vdash x \notin \mathfrak{D}$, a contradiction.

3$\Rightarrow$1. First note that G-persistent types are finitely satisfiable in $\mathfrak{X}$. Now, suppose $p(x)$ is not G-invariant. Then, by completeness, $p(x) \vdash \varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ for some $g \in G$. Clearly $\varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ is not 2-G-persistent as it is inconsistent with its g-translate. $\square$

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6 Corollary The following are equivalent for every $\Delta^B(\mathfrak{Z})$-definable set $\mathfrak{D}$

1. $\gamma_G(x) \vdash x \in \mathfrak{D}$
2. $p(x) \vdash x \in \mathfrak{D}$ for every G-persistent $p(x) \in S_\Delta(\mathfrak{Z})$.

Proof. 1$\Rightarrow$2. This is an immediate consequence of Theorem 5.

2$\Rightarrow$1. Suppose $\gamma_G(x) \nvdash x \in \mathfrak{D}$. Then there is a type $p(x) \in S_\Delta(\mathfrak{Z})$ consistent with $\gamma_G(x) \cup \{x \notin \mathfrak{D}\}$. By Corollary 4 $p(x)$ is G-persistent. Then $\neg 2$. $\square$

The theorem yields a necessary condition for the existence of G-invariant global $\Delta^B(\mathfrak{Z})$-types.

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7 Corollary If there exists a G-invariant type $p(x) \in S_\Delta(\mathfrak{Z})$ finitely satisfiable in $\mathfrak{X}$ then for every $\Delta^B(\mathfrak{Z})$-definable set $\mathfrak{D}$

1. $\mathfrak{D}$ and $\neg\mathfrak{D}$ are not both G-generic
2. if $\mathfrak{D}$ is G-generic then it is G-persistent
3. $\gamma_G(x)$ is finitely satisfiable in $\mathfrak{X}$.

Proof. Clearly, 1 and 2 are equivalent by Fact 3 and follow from 3. Finally, 3 is an immediate consequence of 2 of Theorem 5. $\square$

The following theorem gives a necessary and sufficient condition for the existence of global G-invariant $\Delta^B(\mathfrak{Z})$-type. Ideally, we would like to have that every G-persistent $\Delta^B(\mathfrak{Z})$-type extends to a global persistent type. Unfortunately this is not true in general (it requires stronger assumptions, see Section 4). A set $\mathfrak{D}$ is G-wide if every finite cover of $\mathfrak{D}$ by $\Delta^B(\mathfrak{Z})$-definable sets contains a G-persistent set. In [1] a similar property is called *quasi-non-forking.* Our use of the term *wide* is consistent with [2],

though we apply it to a narrow context. A type is *G*-wide if every conjunction of formulas in the type is *G*-wide.

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8 Theorem Let $\mathcal{D}$ be a $\Delta^B(\mathcal{Z})$-definable set. Then the following are equivalent

1. $\gamma_G(x)$ is finitely satisfied in $\mathcal{D} \cap \mathcal{X}$
2. there exists a *G*-persistent type $p(x) \in S_\Delta(\mathcal{Z})$ containing $x \in \mathcal{D}$
3. $\mathcal{D}$ is *G*-wide.

Proof. 1⇒2. By Corollary 4, it suffices to pick any $p(x) \in S_\Delta(\mathcal{Z})$ containing $\gamma_G(x)$ and finitely satisfied in $\mathcal{D} \cap \mathcal{X}$.

2⇒1. By Theorem 5.

2⇒3. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be $\Delta^B(\mathcal{Z})$-definable sets that cover $\mathcal{D}$. Pick $p(x)$ as in 2. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i. Then, by Theorem 5, $\neg\mathcal{C}_i$ is not *G*-generic. Therefore, by Fact 3, $\mathcal{C}_i$ is *G*-persistent.

3⇒2. Let $p(x)$ be maximal among the $\Delta^B(\mathcal{Z})$-types that are finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$ and are such that $\vartheta(\mathcal{U}^x)$ is *G*-wide for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that $p(x)$ is a complete $\Delta^B(\mathcal{Z})$-type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\Delta^B(\mathcal{Z})$-definable sets $\mathcal{C}_1, \ldots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \smallsetminus \vartheta(\mathcal{U}^x)$ and such that no $\mathcal{C}_i$ is *G*-persistent. As $\mathcal{C}_1, \ldots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is *G*-invariant. This follows from completeness and Theorem 5. □

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9 Corollary Let $\mathcal{D}$ be a *G*-wide $\Delta^B(\mathcal{Z})$-definable set. Then $\mathcal{D} \cap g\cdot\mathcal{D}$ is *G*-wide for every $g \in G$.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be a *G*-persistent type such that $p(x) \vdash x \in \mathcal{D}$. By *G*-invariance $p(x) \vdash x \in g\cdot\mathcal{D}$. □

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10 Fact Let $\mathcal{D}$ be a $\Delta^B(\mathcal{Z})$-definable set. The following are equivalent

1. $\mathcal{D}$ is *G*-wide;
2. every finite cover of $\mathcal{D}$ by $\Delta^\pm(\mathcal{Z})$-definable sets contains a *G*-persistent set.

Let $q(x) \subseteq \Delta^B(\mathcal{Z})$ be *G*-invariant. We say that $q(x)$ is G-prime if for every $\Delta^B(\mathcal{Z})$-definable set $\mathcal{D}$ and every $g \in G$ if $q(x) \vdash x \in (\mathcal{D} \cup g\cdot\mathcal{D})$ then $q(x) \vdash x \in \mathcal{D}$.

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11 Proposition The type $\gamma_G(x)$ is *G*-prime.

Proof. Suppose $\gamma_G(x) \vdash x \in (\mathcal{D} \cup g\cdot\mathcal{D})$. Assume that $\gamma_G(x)$ is consistent, otherwise the claim is trivial. Let $p(x) \in S_\Delta(\mathcal{Z})$ be *G*-persistent. We claim that $p(x) \vdash x \in \mathcal{D}$. The proposition follows from the claim by Corollary 6. By completeness $p(x) \vdash x \in \mathcal{D}$ or $p(x) \vdash x \in g\cdot\mathcal{D}$. If the latter occurs, $p(x) \vdash x \in \mathcal{D}$ follows from invariance. □

2. Strong genericity

Unfortunately, *G*-genericy is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is strongly G-generic if for every finite $H \subseteq G$ the set $\cap H \mathcal{D}$ is generic (recall that $H \mathcal{D}$ stands for $\{h \cdot \mathcal{D} : h \in H\}$). Dually, we say that $\mathcal{D}$ is weakly G-persistent if for some finite $H \subseteq G$ the set $\cup H \mathcal{D}$ is persistent. Again, the same properties may be attributed to formulas and types.

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12 Lemma The intersection of two strongly G-generic sets is strongly G-generic.

Proof. We may assume that all sets mentioned below are subsets of $\mathcal{X}$. Let $\mathcal{D}$ and $\mathcal{C}$ be strongly G-generic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K (\mathcal{C} \cap \mathcal{D})$ is G-generic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K \mathcal{C}$ and $\mathcal{D}' = \cap K \mathcal{D}$. Note that $\mathcal{C}'$ and $\mathcal{D}'$ are both strongly G-generic. In particular $\mathcal{X} = \cup H \mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\cup H \mathcal{B} \;=\; \cup H \Big[\mathcal{C}' \cap \mathcal{D}' \Big]$$

$$\cup H \mathcal{B} \;\supseteq\; \cup H \Big[(\cap H \mathcal{C}') \cap \mathcal{D}' \Big]$$

$$=\; (\cap H \mathcal{C}') \cap (\cup H \mathcal{D}')$$

$$=\; \cap H \mathcal{C}'$$

As $\mathcal{C}'$ is strongly G-generic, $\cap H \mathcal{C}'$ is G-generic. Therefore $\cup H \mathcal{B}$ is also G-generic. The G-genericity of $\mathcal{B}$ follows. $\square$

Define the following type

$$^s\gamma_G(x) \;=\; \{\vartheta(x) \in \Delta^{\mathsf{B}}(\mathcal{Z}) : \vartheta(x) \text{ is strong } G\text{-generic}\}$$

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13 Corollary The type $^s\gamma_G(x)$ is finitely satisfiable in $\mathcal{X}$, strongly G-generic, and G-invariant.

Proof. The strong G-genericity is an immediate consequence of Lemma 12. The finite satisfiability is a consequence of G-genericity, see Exercise **??**. As for invariance, note that any translate of a strongly G-generic formula is also strongly G-generic. $\square$

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14 Corollary Let $p(x) \subseteq L(\mathcal{U})$ be such that $^s\gamma(x) \cup p(x)$ is finitely satisfiable in $\mathcal{X}$. Then $p(x)$ is weakly G-persistent.

Proof. Similar to Corollary 4. Let $\vartheta(x)$ be a conjunction of formulas in $p(x)$. As $^s\gamma(\mathcal{X})$ is finitely satisfiable in $\vartheta(\mathcal{X})$, it cannot be that $\neg\vartheta(x)$ is strongly G-generic. From Fact 3, we obtain that $\neg\vartheta(x)$ not being strongly G-generic is equivalent to $\vartheta(x)$ being weakly G-persistent. $\square$

3. The diameter of a Lascar type

As an application we prove an interesting property of the Lascar types. Let $\mathcal{L}(a/A)$, the set of tuples with the same Lascar strong type as a over A. This set is the union of a chain of type-definable sets of the form $\{x : d_A(a, x) \leq n\}$, where d_A is the distance in the Lascar graph. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter.

It is convenient to address the problem in more general terms. We work under Assumption 1 and also assume that G acts transitively on $\mathcal{X}$ i.e. $G\,a = \mathcal{X}$ for every $a \in \mathcal{X}$. Let $K \subseteq G$ be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugacy invariant i.e. $g\,Kg^{-1} = K$ for every $g \in G$

We define a discrete metric on $\mathcal{X}$. For $a, b \in \mathcal{X}$ let $d(a, b)$ be the minimal n such that $a \in K^n b$. This defines a metric which is G-invariant by 2. The diameter of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a, b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for $\mathcal{X}$ to have finite diameter. The notions introduced in Section 2 offer some hint.

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15 Proposition If $\mathcal{X}$ has a weakly persistent subset of finite diameter, then $\mathcal{X}$ itself has finite diamenter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter n. Let $H \subseteq G$ be finite such that $\cup H\,\mathcal{C}$ is persistent. We claim that also $\cup H\,\mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k, \in H$. Now, let hb and kc, for some $h, k, \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H\,\mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of $\mathcal{C}$,

$$d(hb, kc) \ \leq \ d(hb, ha) \ + \ d(ha, ka) \ + \ d(ka, kc)$$

$$\leq \ n + m + n.$$

This proves that $\cup H\,\mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that $\mathcal{C}$ itself is persistent.

By the transitivity of the action, any two elements of $\mathcal{X}$ are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By persistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$d(ha, ka) \ \leq \ d(ha, c) \ + \ d(c, d) \ + \ d(d, ka)$$

$$\leq \ n + n + n.$$

Therefore the diameter of $\mathcal{X}$ does not exceed $3n$. $\qquad\square$

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16 Theorem Suppose that $\mathcal{X}$ and the sets $\mathcal{X}_n = K^n a$, for some $a \in \mathcal{X}$, are type-definable. Then $\mathcal{X}$ has finite diameter.

Proof. By Proposition 15, it suffices to prove that $\mathcal{X}_n$ is weakly persistent. By Corollary 14 it suffices to show that for some n the type ${}^s\gamma_G(x)$ is finitely satisfied in $\mathcal{X}_n$. Suppose not. Let $\psi_n(x) \in {}^s\gamma_G$ be a formula that is not satisfied in $\mathcal{X}_n$. Then the type $p(x) = \{\psi_n(x) : n \in \omega\}$ is finitely satisfied in $\mathcal{X}$. From the type-definablity of $\mathcal{X}$ it follows that $p(x)$ has a realization in $\mathcal{X}$. As this realization belongs to some $\mathcal{X}_n$ we contradict the definition of $\psi_n(x)$. $\qquad\square$

17 Example Let $K \subseteq \operatorname{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A. Then the group G generated by K is $\operatorname{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. Let $\Delta = L_{xz}$ and $\mathcal{Z} = \mathcal{U}^z$. Then $d(a,b)$ coincides with the distance in the Lascar graph. The sets $K^n \cdot a = \{x : d(x,a) \leq n\}$ are type definable. Then from Theorem 16 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

4. A tamer landscape

Under suitable assumptions – e.g. stability – some notion introduced in this chapter coalesce, and we are left with a tamer landscape. We prove the following theorem.

18 Theorem The following are equivalent
1. G-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are G-wide
2. G-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are closed under intersection
3. G-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are strongly G-generic
4. weakly persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets are G-persistent.

Proof. 2⇔3⇔4. Clear.

1⇒2. Let $\mathcal{C}$ and $\mathcal{D}$ be G-generic $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not G-generic. Then $\neg(\mathcal{C} \cap \mathcal{D})$ is G-persistent. By 1 and Theorem 8 there is a G-invariant global $\Delta^{\mathrm{B}}(\mathcal{Z})$-type $p(x)$ containing $x \notin \mathcal{C} \cap \mathcal{D}$. By completeness either $p(x) \vdash x \notin \mathcal{C}$ or $p(x) \vdash x \notin \mathcal{D}$. This is a contradiction because by Theorem 5 $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

4⇒1. Note that, by 3, the type ${}^s\gamma_G(x)$ coincides with $\gamma_G(x)$, in particular $\gamma_G(x)$ is finitely satisfied in $\mathcal{X}$. Let $\mathcal{D}$ be a G-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set. We show that $\gamma_G(x) = {}^s\gamma_G(x)$ is finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$. Then, by 4 and Corollary 14, any global extension of $\gamma_G(x) \cup \{x \in \mathcal{D}\}$ witness 2 of Theorems 8. Suppose not, then $\gamma_G(x) \vdash x \notin \mathcal{D}$. Therefore $\neg\mathcal{D}$ is G-generic, contradicting the consistency of ${}^s\gamma_G(x)$. $\square$

19 Remark Assume that the equivalent conditions in Theorem 18 hold. Then the types $\gamma_G(x)$ and ${}^s\gamma_G(x)$ coincide, and therefore G-invariant global types exist. It is also worth mentioning that every positive Boolean combination of G-generic sets is G-generic.

5. The action of normal subgroups

Let $H \unlhd G$. The following is an immediate consequence of normality.

20 Remark For every $\mathcal{D} \subseteq \mathcal{U}^x$ and every $g \in G$

$$\mathcal{D} \text{ is } H\text{-foo} \quad \Leftrightarrow \quad g \cdot \mathcal{D} \text{ is } H\text{-foo},$$

where *foo* can be replaced by *generic, invariant, persistent, wide.* In particular, the type $\gamma_H(x)$ is G-invariant.

Recall that if $\gamma_H(x)$ is finitely satisfiable in $\mathfrak{X}$, then H-generic sets are H-wide, cf. Theorem 8. As it happens, we can slightly strengthen this fact.

21 Proposition Assume that $\gamma_H(x)$ is consistent. Let $\mathcal{D}$ be a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set. Then if $\mathcal{D}$ is G-generic it is also H-wide.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be consistent with $\gamma_H(x)$. As $\mathcal{D}$ is G-generic, by completeness $p(x) \vdash x \in g{\cdot}\mathcal{D}$ for some $g \in G$. Equivalently, $g^{-1}p \vdash x \in \mathcal{D}$. As $p(x)$ is H-persistent, by Remark 20 $g^{-1}p(x)$ is also H-persistent. Then the proposition follows from Theorem 8. $\hspace{2em}\square$

6. Definable groups

In this section we assume that $\mathcal{Z}$ and $\mathfrak{X}$ are type-definable over some set of parameters A. Moreover we assume that $\mathcal{Z}$ is a group that act on $\mathfrak{X}$. The group operations and the group action are assumed definable over A. We use the symbol $\cdot$ for both the group multiplication and the group action.

Let $\Psi \subseteq L_x(\mathcal{U})$ be some small set of formulas. In this section Δ contains formulas $\varphi(x\,;z)$ of the form $\psi(z^{-1}{\cdot}\,x)$ for $\psi(x) \in \Psi$. The sets $\varphi(\mathfrak{X}\,;\mathcal{Z})$ are $\mathcal{Z}$-invariant. We write 1 for the identity of $\mathcal{Z}$. If $\varphi(\mathfrak{X}\,;1) \in \Delta^{\mathsf{B}}(1)$ then $\varphi(\mathfrak{X}\,;g) = g \cdot \varphi(\mathfrak{X}\,;1)$.

The following auxiliary structure is useful. Let $\mathcal{U}^\Delta = \langle \mathfrak{X}\,;\mathcal{Z} \rangle$ be a 2-sorted structure whose signature L^Δ contains only relation symbols for every formula $\varphi(x\,;z) \in \Delta$. As there is little risk of confusion, these relations symbols are also denoted by $\varphi(x\,;z)$. As $\mathfrak{X}$ and $\mathcal{Z}$ are assumed to be type-definable, $\mathcal{U}^\Delta$ is a saturated L^Δ-structure.

In this section $G = \mathrm{Aut}(\mathcal{U}^\Delta)$.

22 Remark It is worth noticing that automorphisms of $\mathcal{U}^\Delta$ do not preserve the group operations nor the group action. However, if $\mathcal{D} = \varphi(\mathfrak{X}\,;1)$ and $g \in \mathcal{Z}$, then $f(g){\cdot}\mathcal{D} = f[g{\cdot}\mathcal{D}]$ for any $f \in \mathrm{Aut}(\mathcal{U}^\Delta)$.

To each $h \in \mathcal{Z}$ we associate the L^Δ-automorphism $\langle a\,;g \rangle \mapsto \langle h{\cdot}a\,;h{\cdot}g \rangle$. Therefore $\mathcal{Z}$ is, up to isomorphism, a subgroup of G. In fact, it is a normal subgroup. Note that, for any $g \in \mathcal{Z}$, the orbit of $\varphi(\mathfrak{X}\,;g)$ under the action of $\mathcal{Z}$ is $\{\varphi(\mathfrak{X}\,;h) : h \in \mathcal{Z}\}$. Therefore it coincides with the orbit under the action of G (it cannot get any larger). We conclude that for formulas in $\Delta^\pm(\mathcal{Z})$ the notions of generic and persistent under the two actions coincide.

We also consider the action some other normal subgroup $H \trianglelefteq G$. In the applications H will be either $\mathrm{Autf}(\mathcal{U}^\Delta)$ or $\mathrm{Aut}(\mathcal{U}^\Delta/M)$.

23 Proposition Let $\mathcal{D}$ be a Δ^{B}-definable set. Assume that $\gamma_H(x)$ is consistent. Then 1$\Rightarrow$2 holds, where

1. $\mathcal{D}$ is $\mathcal{Z}$-generic
2. $g{\cdot}\mathcal{D}$ is H-wide for every $g \in \mathcal{Z}$.

Under the assumption of stability and with $H = \mathrm{Autf}(\mathcal{U}^\Delta)$ a stronger claim obtains – the consistency of $\gamma_H(x)$ is guaranteed, and also the converse implication holds

Proof. Let g be given. If $\mathcal{D}$ is $\mathcal{Z}$-generic, then so is $g \cdot \mathcal{D}$. Then $g \cdot \mathcal{D}$ is, a fortiori, G-generic. Therefore 2 follows from Proposition 21. $\qquad\square$

We write $(g)_H$ for the H-orbit of g, that is, the set $\{f(g) \;:\; f \in H\}$.

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24 Proposition Let $\vartheta(x; z_1, \ldots, z_n)$ be a Boolean combination of formulas $\varphi_i(x; z_i)$ for some $\varphi_i(x; z) \in \Delta$. Then for every $h_i \in (g_i)_H$ the following are equivalent

1. $\vartheta(x; g_1, \ldots, g_n)$ is H-wide
2. $\vartheta(x; h_1, \ldots, h_n)$ is H-wide.

Proof. Let $f_i \in H$ be such that $h_i \in f_i(g_i)$. Without loss of generality we can assume that $\vartheta(x; g_1, \ldots, g_n)$ is the conjunction of the formulas $\varphi_i(x; g_i)$. Let $\mathcal{C}_i = \varphi_i(\mathcal{X}; 1)$. Then 1 says that $\mathcal{C} = g_1 \cdot \mathcal{C}_1 \cap \cdots \cap g_n \cdot \mathcal{C}_n$ is H-wide. Let $f_i \in H$ be such that $h_i = f_i(g_i)$. Then, by Corollary 9 also the intersection of the sets $f_i[\mathcal{C}]$ is H-wide. A fortiori the intersection of the sets $f_i[g_i \cdot \mathcal{C}_i]$ is H-wide. Then $1 \Rightarrow 2$ follows from Remark 22. By symmetry, this proves the equivalence. $\qquad\square$

If $\mathcal{A} \subseteq \mathcal{Z}$, write $\langle \mathcal{A} \rangle$ for the subgroup generated by $\mathcal{A}$.

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25 Proposition Let $\vartheta(x; z_1, \ldots, z_n)$ be a Boolean combination of formulas $\varphi_i(x; z_i)$ for some $\varphi_i(x; z) \in \Delta$. Let $g \in \mathcal{Z}$ be arbitrary. Assume that $\vartheta(x; 1, \ldots, 1)$ is H-wide. Then $\vartheta(x; h_1, \ldots, h_n)$ is H-wide for every $h_i \in \big\langle (g)_H^{-1} \cdot (g)_H \ \cup \ (g)_H \cdot (g)_H^{-1} \big\rangle$.

Proof. We proceed by induction on the number of factors $a^{-1} \cdot b$ or $a \cdot b^{-1}$, for some $a, b \in (g)_H$, that occur in $h_1, \ldots, h_n$. Without loss of generality we can assume that $\vartheta(x; z_1, \ldots, z_n)$ is the conjunction of the formulas $\varphi_i(x; z_i)$ for some $\varphi_i(x; z) \in \Delta$. Let $\mathcal{C}_i = \varphi_i(\mathcal{X}; 1)$. Assume inductively that $h_1 \cdot \mathcal{C}_1 \cap \cdots \cap h_n \cdot \mathcal{C}_n$ is H-wide. Pick two arbitrary $a, b \in (g)_H$. Then

$$a \cdot \mathcal{C}_1 \ \cap \ a \cdot h_1^{-1} \cdot h_2 \cdot \mathcal{C}_2 \ \cap \ \ldots \ldots \ \cap \ a \cdot h_1^{-1} \cdot h_n \cdot \mathcal{C}_n \ \text{is } H\text{-wide.}$$

By Proposition 24, in this intersection we can replace $a \cdot \mathcal{C}_1$ by $b \cdot \mathcal{C}_1$. Then finally

$$h_1 \cdot a^{-1} \cdot b \cdot \mathcal{C}_1 \ \cap \ h_2 \cdot \mathcal{C}_2 \ \cap \ \ldots \ldots \ \cap \ h_n \cdot \mathcal{C}_n \ \text{is } H\text{-wide.}$$

A similar argument applies to $a \cdot b^{-1}$. $\qquad\square$

7. Notes and references

[1] Artem Chernikov and Itay Kaplan, *Forking and dividing in* NTP_2 *theories*, J. Symbolic Logic **77** (2012), 1–20.

[2] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.