Group actions on models

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ABSTRACT. In a nutshell, a set is *strongly generic* if the intersection of any finitely many of its translation is generic. I do not know if the notion has been studied in the context presented here. To demostrate its interest we use it for a short proof of (a generalization of) Newelski's theorem on the diamter of the Lascar graph, see Theorem 12.

Theorem 14 shows that the phenomenon *strongly generic* = *generic* has some interest (it is reminiscent of *forking* = *dividing*).

In the last section I try to apply these notions to definable groups hoping to recover part of the stable group theory but I fail. I think I am missing something important (or commit an error somewhere).

1. The two perspectives on the invariance of types

This section review well-known matter and set some terminology.

Below $\Delta \subseteq L_{x\,z}(\mathcal{U})$, $\mathcal{X} \subseteq \mathcal{U}^x$, and $\mathcal{Z} \subseteq \mathcal{U}^z$ are some arbitrary nonempty sets (at some point we will require that $\mathcal{X}$ and $\mathcal{Z}$ are type-definable). We write $L_\Delta(\mathcal{Z})$ for the set of formulas $\vartheta(x)$ that are Boolean combination of formulas $\varphi(x\ b)$ for some $\varphi(x\ z) \in \Delta$ and some $b \in \mathcal{Z}$. Such formulas a called Δ-formulas. A relatively Δ-definable set is a set of the form $\vartheta(\mathcal{X})$ for some Δ-formula $\vartheta(x)$. Subsets of $L_\Delta(\mathcal{Z})$ are called Δ-types. We write $S_\Delta(\mathcal{Z})$ for the set of complete Δ-types with parameters in $\mathcal{Z}$ (there may be other parameters hidden ion Δ).

1 Assumption Let G be a group that acts on $\mathcal{X}$ and on $\mathcal{Z}$ from the left. We require that for every $\varphi(x\,;z) \in \Delta$ the set $\varphi(\mathcal{X};\mathcal{Z})$ is invariant under the action of G.

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that $\mathcal{D}$ is invariant under the action of G, or G-invariant, if $\mathcal{D} \cap \mathcal{Z}$ is fixed setwise by G. Yet in other words, if

is1. $\quad a \in \mathcal{D} \;\leftrightarrow\; ga \in \mathcal{D} \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

A formula is invariant if the set it defines is invariant. We say that $p(x) \subseteq L(\mathcal{U})$ is invariant under the action of G, or G-invariant, if for every formula $\varphi(x\,;z) \in L$

it1. $\quad \varphi(x\,;a) \in p \;\Leftrightarrow\; \varphi(x\,;ga) \in p \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

It should be evident that invariant under the action of $\mathrm{Aut}(\mathcal{U}/A)$ coincides with invariant over A and Lascar invariant over A coincides with invariant under the action of $\mathrm{Autf}(\mathcal{U}/A)$.

We have just defined invariance using the subsets of $\mathcal{Z}$ (externally) defined by p. Now we discuss invariance using the subsets of $\mathcal{X}$ that are in p.

An immediate consequence of Assumption 1 is that any G-translate of a Δ-definable set is again Δ-definable. In particular for every Δ-formula $\vartheta(x\,;\bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X};\bar{b})] \;=\; \vartheta(\mathcal{X};g\bar{b}).$$

Therefore $p(x) \subseteq L_\Delta(\mathcal{Z})$ is invariant if

$$p(x) \vdash x \in \mathcal{D} \;\Leftrightarrow\; p(x) \vdash x \in g\mathcal{D} \qquad \text{for every } \Delta\text{-definable } \mathcal{D} \subseteq \mathcal{U}^x \text{ and } g \in G,$$

where by $p(x) \vdash x \in \mathcal{D}$ we understand $\vartheta(\mathcal{X}) \subseteq \mathcal{D}$ for some $\vartheta(x)$ that is conjunction of formulas in $p(x)$.

A set $\mathcal{D} \subseteq \mathcal{X}$ is generic under the action of G, or G-generic for short, if finitely many G-translates of $\mathcal{D}$ cover $\mathcal{X}$; we say n-G-generic if $\leq n$ translates suffices. Dually, we say that $\mathcal{D}$ is persistent under the action of G, or G-persistent for short, if the intersection of any finitely many G-translates of $\mathcal{D}$ is nonempty; we say n-G-persistent when the request is limited to $\leq n$ translates. We may drop reference to G when it is clear from the context.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

⚠ The terminology above non-standard. In [1] the authors write *quasi-non-dividing* for *persistent* when $G = \mathrm{Aut}(\mathcal{U}/A)$. Their terminology has good motivations, but it would be a mouthful if adapted to our context.

2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is persistent (in any $\mathcal{X} \supseteq A^x$) under the action of $\mathrm{Aut}(\mathcal{U}/A)$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\mathrm{Aut}(\mathcal{U}/A)$-translate of $\varphi(x)$.

In this chapter many proofs require some juggling with negations.

3 Fact (Assume 1) The following are equivalent

1. $\mathcal{D}$ is not n-generic
2. $\neg\mathcal{D}$ is n-persistent.

Proof. Immediate by spelling out the definitions

1. there are no $g_1,\dots,g_n \in G$ such that $\mathcal{X} \;\subseteq\; \bigcup_{i=1}^{n} g_i\mathcal{D}$
2. $\varnothing \;\neq\; \mathcal{X} \cap \bigcap_{i=1}^{n} \neg g_i\mathcal{D}$ for every $g_1,\dots,g_n \in G$. □

4 Theorem (Assume 1) Let $p(x) \;\in\; S_\Delta(\mathcal{Z})$ be finitely satisfiable in $\mathcal{X}$. Then the following are equivalent

1. $p(x)$ is invariant
2. $p(x) \vdash x \in \mathcal{D}$ for every generic relatively Δ-definable set $\mathcal{D}$
3. $p(x)$ is persistent.

Proof. 1⇒2. Let $g_1\mathcal{D},\dots,g_n\mathcal{D}$ be translations of $\mathcal{D}$ that cover $\mathcal{X}$. Negate 2. By completeness, $p(x) \vdash x \notin \mathcal{D}$. Hence, from invariance we obtain

$$p(x) \;\vdash\; x \;\notin\; \bigcup_{i=1}^{n} g_i\mathcal{D}.$$

This contradicts the finite satisfiability of $p(x)$ in $\mathcal{X}$.

2⇒3. Let $\mathcal{D}$ be defined by a conjunction of formulas in $p(x)$. If $\mathcal{D}$ is not persistent then, by Fact 3, $\neg\mathcal{D}$ is generic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

3⇒1. If $p(x)$ is not invariant then, by completeness, $p(x) \vdash \varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ for some $g \in G$. Clearly $\varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ is not persistent as it is inconsistent with its g-translate. □

5 Remark In the theorem above, 2 and 3 can be replaced by

2′. $p(x) \vdash x \in \mathcal{D}$ for every 2-generic relatively Δ-definable set $\mathcal{D}$

3′. $p(x)$ is 2-persistent.

The theorem yields an immediate necessary condition for the existence of an invariant global Δ-type.

6 Corollary (Assume 1) If there exists an invariant global Δ-type then for every Δ-definable set $\mathcal{D}$

1. $\mathcal{D}$ and $\neg\mathcal{D}$ cannot be both generic
2. if $\mathcal{D}$ is generic than it is persistent.

Proof. By Fact 3, 1 and 2 are equivalent; 1 is an immediate consequenc of 2 of Theorem 4. □

The following theorem gives a necessary and sufficient condition for the existence of global invariant Δ-type. Ideally, we would like to prove that every persistent Δ-type extends to a global persitent type. Unfortunately this is not true – we need a stronger property. A Δ-definable set $\mathcal{D}$ is hereditarely persistent if every finite cover of $\mathcal{D}$ by Δ-definable sets contains a persistent set. In [1] a related property is called *quasi-non-forking*, but we persist in the unorthodoxy. A type is hereditarely persistent if every conjunction of formulas in the type is hereditarely persistent.

7 Theorem (Assume 1) Let $q(x) \subseteq L(\mathcal{U})$. Then the following are equivalent

1. $q(x)$ extends to an invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in $\mathcal{X}$
2. $q(x)$ is hereditarely persistent.

Proof. 1⇒2. Let $\vartheta(x)$ be a conjunction of formulas in $q(x)$. Suppose $\mathcal{C}_1,\dots,\mathcal{C}_n$ cover $\vartheta(\mathcal{U}^x)$ and pick $p(x)$ as in 1. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i. Then, by Theorem 4, $\neg\mathcal{C}_i$ is not generic. Therefore, by Fact 3, $\mathcal{C}_i$ is persistent.

2⇒1. Let $p(x)$ be maximal among the Δ-types that contain $q(x)$ and are such that $\vartheta(\mathcal{U}^x)$ is hereditarely persistent for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that p is a complete Δ-type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$ and some $\mathcal{C}_1,\dots,\mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \smallsetminus \vartheta(\mathcal{U}^x)$ and such that no $\mathcal{C}_i$ is persistent. As $\mathcal{C}_1,\dots,\mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is finitely satisfiable in $\mathcal{X}$ and invariant. Finite satisfiability follows from persistency. From completeness and Theorem 4 we obtain invariance. □

2. Strong genericity

Unfortunatelly, genericity is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity. This is only required in Section 3.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is strongly generic if the intersection of $\mathcal{D}$ with any of its tranlations $\mathcal{D}$ is generic. Dually, we say that $\mathcal{D}$ is weakly persistent if the union of $\mathcal{D}$ with one of its translations is peristent. Again, the same properties may be attributed to formulas and types.

Notation: for $\mathcal{B} \subseteq \mathcal{X}$ and $H \subseteq G$ we write $H\,\mathcal{B}$ for $\{h\mathcal{B} : h \in H\}$.

8 Lemma (Assume 1) The intersection of strongly generic sets is strongly generic.

Proof. We may assume that all sets mentioned below are subsets of $\mathcal{X}$. Let $\mathcal{D}$ and $\mathcal{C}$ be strongly generic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K\,(\mathcal{C} \cap \mathcal{D})$ is generic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K\,\mathcal{C}$ and $\mathcal{D}' = \cap K\,\mathcal{D}$. Note that $\mathcal{C}'$ and $\mathcal{D}'$ are both strongly generic. In particular $\mathcal{X} = \cup H\,\mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\begin{aligned}
\cup H\,\mathcal{B} &= \cup H\Big[\mathcal{C}' \cap \mathcal{D}'\Big]\\
\cup H\,\mathcal{B} &\supseteq \cup H\Big[\big(\cap H\,\mathcal{C}'\big) \cap \mathcal{D}'\Big]\\
&= \big(\cap H\,\mathcal{C}'\big) \cap \big(\cup H\,\mathcal{D}'\big)\\
&= \cap H\,\mathcal{C}'
\end{aligned}$$

As $\mathcal{C}'$ is strongly generic, $\cap H\,\mathcal{C}'$ is generic. Therefore $\cup H\,\mathcal{B}$ is also generic. The genericity of $\mathcal{B}$ follows. □

9 Corollary (Assume 1) Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) : \vartheta(x) \text{ strongly generic}\}$. Then $q(x)$ is finitely satisfiable in $\mathcal{X}$, strongly generic, and invariant.

Proof. Strong genericity is an immediate consequence of Lemma 8. Finite satisfiability follows easily from genericity. As for invariance, note that any translate of a strongly generic formula is also strongly generic. □

10 Corollary (Assume 1) Let $q(x)$ be as in Corollary 9. Let $p(x) \subseteq L(\mathcal{U})$ be such that $p(x) \cup q(x)$ is finitely satisfied in $\mathcal{X}$. Then $p(x)$ is weakly persistent.

Proof. Let $\vartheta(x) \in p$. As $q(x)$ is finitely satisfiable in $\vartheta(\mathcal{U}^x)$, we cannot have that $\neg\vartheta(x)$ is strongly generic. From Fact 3, we obtain that $\neg\vartheta(\mathcal{U}^x)$ non strongly generic is equivalent to $\vartheta(x)$ weakly persistent. □

3. The diameter of a Lascar type

Recall that $\mathcal{L}(a/A)$, the Lascar strong type of $a \in \mathcal{U}^x$, is the union of a chain of type-definable sets of the form $\{x \ :\ d_A(a,x) \leq n\}$. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter. It is convenient to address the problem in more general terms.

Let G be a normal subgroup of $\text{Aut}(\mathcal{U})$. Assume G is the union of a countable chain of sets $\langle G_n : n \in \omega \rangle$ with the following properties

1. every G_n is symmetric i.e. it contains the unit and is closed under inverse
2. every G_n is conjugancy invariant i.e. $g\, G_n g^{-1} = G_n$ for every $g \in G$
3. $G_n G_m \subseteq G_{n+m}$ for every $n, m \in \omega$.

Assume G acts transitively on $\mathcal{X}$ i.e., $G\, a = \mathcal{X}$ for every $a \in \mathcal{X}$. We define a discrete metric on $\mathcal{X}$. For $a, b \in \mathcal{X}$ let $d(a, b)$ be the minimal n such that $a \in G_n b$. This defines a metric by 1 and 3. By 2, this metric in G-invariant. The diameter of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a, b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for $\mathcal{X}$ to have finite diameter. The notions introduced in Section 2 offer some hints.

11 Proposition If $\mathcal{X}$ has a weakly persistent subset of finite diameter, then $\mathcal{X}$ itself has finite diamenter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter n. Let $H \subseteq G$ be finite such that $\cup H\, \mathcal{C}$ is persistent. We claim that also $\cup H\, \mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k, \in H$. Now, let hb and kc, for some $h, k, \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H\, \mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of $\mathcal{C}$,

$$\begin{aligned} d(hb,\, kc) \;&\leq\; d(hb,\, ha) \;+\; d(ha,\, ka) \;+\; d(ka,\, kc) \\ &\leq\; n + m + n. \end{aligned}$$

This proves that $\cup H\, \mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that $\mathcal{C}$ itself is persistent.

By the transitivity of the action, any two elements of $\mathcal{X}$ are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By percistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha,\, ka) \;&\leq\; d(ha,\, c) \;+\; d(c,\, d) \;+\; d(d,\, ka) \\ &\leq\; n + n + n. \end{aligned}$$

Therefore the diameter of $\mathcal{X}$ does not exceed $3n$. □

12 Theorem Suppose that $\mathcal{X}$ and the sets $\mathcal{X}_n = G_n a$, for some $a \in \mathcal{X}$, are type-definable. Then $\mathcal{X}$ has finite diameter.

Proof. By Proposition 11, it suffices to prove that $\mathcal{X}_n$ is weakly persistent. Define

$$q(x) \;=\; \{\vartheta(x) \in L(\mathcal{U}) \;:\; \vartheta(x) \text{ strongly generic}\}.$$

By Corollary 10, with L for L_φ, it suffices to prove that for some n the type $q(x)$ is finitely satisfied in $\mathcal{X}_n$. Suppose not. Let $\psi_n(x) \in q$ be a formula that is not satisfied in $\mathcal{X}_n$. The type $\{\psi_n(x) : n \in \omega\}$ is finitely satisfied in $\mathcal{X}$. Then it has a realization in $\mathcal{X}$. As this realization belongs to some $\mathcal{X}_n$ this contradicts the definition of $\psi_n(x)$. □

13 Example Let $G_1 \subseteq \mathrm{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A. Thenthe group G generated by G_1 is $\mathrm{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. If we let G_1 be the set of automorphisms that fix a model $M \supseteq A$ and $G_n = G_1^n$, then $d(a,b)$ concides with the dinstance in the Lascar graph. It is not difficult to see that the sets $G_n \cdot a$ are type definable. Then from Theorem 12 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

4. A simplified landscape

Under suitable assumptions – e.g. the sability of $\varphi(x\,;z)$ – some of the notions introduced above coalesce and we are left with cleaner theory. We prove the following theorem.

14 Theorem (Assume 1) The following are equivalent

1. persistent Δ-definable sets are hereditarely persistent
2. generic Δ-definable sets are strongly generic
3. generic Δ-definable sets are closed under intersection
4. weakly persistent Δ-definable sets are persistent.

Proof. 1⇒2. It suffices to prove that generic sets are closed under intersection. Let $\mathcal{C}$ and $\mathcal{D}$ be generic Δ-definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not generic. By 1 and Theorem 7 there is an invariant global Δ-type $p(x)$ containing $x \in \neg\mathcal{C} \cup \neg\mathcal{D}$. By completeness either $p(x) \vdash x \in \neg\mathcal{C}$ or $p(x) \vdash x \in \neg\mathcal{D}$. This is a contradiction because, by Theorem 4, $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

2⇔3⇔4. Clear.

4⇒1. Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) \,:\, \vartheta(x) \text{ generic}\}$. By 2 this is the same type defined in Corollary 10. Therefore, any completion of $q(x)$ is, by 4, persistent. Let $\mathcal{D}$ be a persistent Δ-definable set. By Theorems 4 and 7 it suffices to show that $\mathcal{D}$ is consistent with $q(x)$. Suppose not, then $q(x) \vdash x \in \neg\mathcal{D}$. Therefore, by 3, $\neg\mathcal{D}$ is generic. This is a contradiction by Fact 3. □

5. Definable groups

In this section we set $G = \mathcal{Z}$ and require that $\mathcal{Z}$ and $\mathcal{X}$ are type-definable over A. The group operations and the group action are required to be definable over A. Clearly, $\mathcal{Z}$ also acts on itself by left multiplication. We use the symbol $\cdot$ for both the group multiplication and the group action.

In this section we deal with the actions of two groups: $\mathcal{Z}$ and $\mathrm{Aut}(\mathcal{U}/A)$. Generic and persistent only refer to the action of $\mathcal{Z}$, we will always be explicit about invariance.

Let $\psi(x\,;y) \in L(A)$. We write $\varphi(x\,;z\,;y)$ for the formula $\psi(z^{-1} \cdot x\,;y)$. In this section Δ contains the formulas $\varphi(x\,;z\,;a)$ where a ranges over the realizations of a given $q(y) \in S(A)$. Note that $\varphi(\mathcal{X}\,;\mathcal{Z}\,;a)$ is invariant under the action of $\mathcal{Z}$ and $\varphi(\mathcal{X}\,;g\,;a) = g \cdot \varphi(\mathcal{X}\,;1\,;a)$ for every a. Therefore the above G and Δ satisfy Assumption 1.

Let 1 be the identity of $\mathcal{Z}$ which, for simplicity, we assume is a constant of L.

15 Fact There are some formulas $\gamma(y), \pi(y) \in L(A)$ such that, for every $a \in \mathcal{U}^y$

$$\begin{aligned} \gamma(a) &\Leftrightarrow \varphi(x\,;1\,;a) \text{ is generic} \\ \pi(a) &\Leftrightarrow \varphi(x\,;1\,;a) \text{ is persistent.} \end{aligned}$$

Similar claims holds for $\neg\varphi(x\,;1\,;a)$.

Proof. By an easy argument of compactness, there is an n such that if $\varphi(x\,;1\,;a)$ is generic then it is also n-generic. Then

$$\gamma(y) \quad = \quad \exists z_1, \ldots, z_n \, \forall x \bigvee_{i=1}^{n} \varphi(x\,;z_i\,;y)$$

The other claims follow easily. $\square$

As $\varphi(x\,;g\,;a)$ is generic/persistent if and only if $\varphi(x\,;1\,;a)$ is such, $\mathrm{Aut}(\mathcal{U}/A)$ maps generic/persistent sets of the form $\varphi(\mathcal{X}\,;g\,;a)$ to generic/persistent sets of the same form.

16 Fact If $p(x) \in S_\Delta(\mathcal{Z})$ is persistent (equivalently, $\mathcal{Z}$-invariant) then it is invariant over A – i.e. under the action of $\mathrm{Aut}(\mathcal{U}/A)$.

Proof. By Theorem 4, $p(x)$ is $\mathcal{Z}$-invariant. Therefore $\varphi(x\,;g\,;a) \in p$ if and only if $\varphi(x\,;1\,;a) \in p$. As $\varphi(x\,;1\,;a) \leftrightarrow \varphi(x\,;1\,;fa)$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$, invariance over A follows. $\square$

17 Assumption We assume that the three equivalent conditions in Theorem 14 hold for $G = \mathcal{Z}$, $\mathcal{X}$, and Δ introduced in this section.

Under this assumption there is a unique maximal generic type $q(x) \subseteq L_\Delta(\mathcal{Z})$. The types $p(x) \in S_\Delta(\mathcal{Z})$ that extend $q(x)$ are persistent, hence invariant under the action of both $\mathcal{Z}$ and $\mathrm{Aut}(\mathcal{U}/A)$.

6. Notes and references

[1] Artem Chernikov and Itay Kaplan, *Forking and dividing in* NTP_2 *theories*, J. Symbolic Logic **77** (2012), 1–20.