

Group actions on models

G. A. M. Polymath

ABSTRACT. A set is *strongly generic* if the intersection of any finitely many of its translation is generic. To demonstrate the convenience of this notion I use it for a short proof of (a generalization of) Newelski's theorem on the diameter of the Lascar graph, see Theorem 12.

Theorem 14 shows that the condition *strongly generic* = *generic* is robust. It might be of some interest (it is reminiscent of *forking* = *dividing*). Is it worth investigating?

Section ?? is incomplete. I would like to recover in a natural way the classical theory of stable groups – but something does not add up.

The connections with topological dynamics are commented at the end of the notes.

1. The two perspectives on the invariance of types

This section I review well-known matter and set the terminology.

Below $\Delta \subseteq L_{xz}(\mathcal{U})$. Let $\mathcal{Z} \subseteq \mathcal{U}^z$. We write $\Delta(\mathcal{Z})$ for the set of formulas of the form $\varphi(x; b)$ for some $\varphi(x; z) \in \Delta$ and some $b \in \mathcal{Z}$. We write $\Delta^\pm(\mathcal{Z})$ for the set of formulas in $\Delta(\mathcal{Z})$ or negation thereof. We write $S_\Delta(\mathcal{Z})$ for the set of complete $\Delta^\pm(\mathcal{Z})$ -types.

We write $\Delta^B(\mathcal{Z})$ for the set of Boolean combinations of formulas in $\Delta(\mathcal{Z})$.

When $\mathcal{A} \subseteq \mathcal{U}$, we may use \mathcal{A} for \mathcal{A}^z in the notation above.

Finally, define $\Delta^G(A)$ to be the set of formulas $\varphi(x) \in L(\mathcal{U})$ that are equivalent to some formula in $\Delta^B(A)$ or, equivalently, that are invariant over A . In the literature these formulas are called *generalized* Δ -formulas over A . Note that when A is a model $\Delta^G(A)$ -formulas are equivalent to $\Delta^B(A)$ -formulas.

1 Assumption Let G be a group that acts on some sets $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$. We require that for every $\varphi(x; z) \in \Delta$ the set $\varphi(\mathcal{X}; \mathcal{Z})$ is invariant under the action of G .

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that \mathcal{D} is *invariant* under the action of G , or *G-invariant*, if $\mathcal{D} \cap \mathcal{Z}$ is fixed setwise by G . Yet in other words, if

$$\text{is1.} \quad a \in \mathcal{D} \leftrightarrow ga \in \mathcal{D} \quad \text{for every } a \in \mathcal{Z} \text{ and every } g \in G.$$

A formula is invariant if the set it defines is invariant. We say that $p(x) \subseteq L(\mathcal{U})$ is *invariant* under the action of G , or *G-invariant*, if for every formula $\varphi(x; z) \in L$

$$\text{it1.} \quad \varphi(x; a) \in p \leftrightarrow \varphi(x; ga) \in p \quad \text{for every } a \in \mathcal{Z} \text{ and every } g \in G.$$

It should be evident that invariant under the action of $\text{Aut}(\mathcal{U}/A)$ coincides with invariant over A and that Lascar invariant over A coincides with invariant under the action of $\text{Autf}(\mathcal{U}/A)$.

We have just defined invariance using the subsets of \mathbb{Z} (externally) defined by p . Now we discuss invariance using the subsets of \mathcal{X} that are in p .

An immediate consequence of Assumption 1 is that any G -translate of a Δ -definable set is again Δ -definable. In particular for every Δ -formula $\vartheta(x; \bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X}; \bar{b})] = \vartheta(\mathcal{X}; g\bar{b}).$$

Therefore $p(x) \subseteq L_\Delta(\mathbb{Z})$ is invariant if

$$p(x) \vdash x \in \mathcal{D} \Leftrightarrow p(x) \vdash x \in g\mathcal{D} \quad \text{for every } \Delta\text{-definable } \mathcal{D} \subseteq \mathcal{U}^x \text{ and } g \in G,$$

where by $p(x) \vdash x \in \mathcal{D}$ we understand $\vartheta(x) \subseteq \mathcal{D}$ for some $\vartheta(x)$ that is conjunction of formulas in $p(x)$.

A set $\mathcal{D} \subseteq \mathcal{X}$ is **generic** under the action of G , or **G -generic** for short, if finitely many G -translates of \mathcal{D} cover \mathcal{X} ; we say **n - G -generic** if $\leq n$ translates suffices. Dually, we say that \mathcal{D} is **persistent** under the action of G , or **G -persistent** for short, if the intersection of any finitely many G -translates of \mathcal{D} is nonempty; we say **n - G -persistent** when the request is limited to $\leq n$ translates. When \mathcal{X} is not clear from the context, we say that these notions are **relative** to \mathcal{X} .

The terminology above is non-standard. In [1] the authors write *quasi-non-dividing* for *persistent* under the action of $\text{Aut}(\mathcal{U}/A)$. Their terminology has good motivations, but it would be a mouthful if adapted to our context. In topological dynamics similar notions have been introduced with different terminology: *syndetic* corresponds to *generic* and *thick* corresponds to *persistent*.

2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is persistent under the action of $\text{Aut}(\mathcal{U}/A)$ relative to any $\mathcal{X} \supseteq A^x$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\text{Aut}(\mathcal{U}/A)$ -translate of $\varphi(x)$.

Notation: for $\mathcal{D} \subseteq \mathcal{U}^x$ and $H \subseteq G$ we write $H\mathcal{D}$ for $\{h\mathcal{D} : h \in H\}$.

In this notes many proofs require some juggling with negations as epitomized by the following fact.

3 Fact (Assume 1) The following are equivalent

1. \mathcal{D} is not G -generic
2. $\neg\mathcal{D}$ is G -persistent.

Proof. Immediate by spelling out the definitions

1. there are no finite $H \subseteq G$ such that $\mathcal{X} \subseteq \bigcup H\mathcal{D}$.
2. $\emptyset \neq \mathcal{X} \cap (\bigcap H\neg\mathcal{D})$ for every finite $H \subseteq G$. □

4 Theorem (Assume 1) Let $p(x) \in S_\Delta(\mathbb{Z})$ be finitely satisfiable in \mathcal{X} . Then the following are equivalent

1. $p(x)$ is G -invariant
2. $p(x) \vdash x \in \mathcal{D}$ for every G -generic $\Delta^B(\mathbb{Z})$ -definable set \mathcal{D}
3. $p(x)$ is G -persistent.

Proof. $1 \Rightarrow 2$. Let $H \subseteq G$ be finite such that $\mathcal{X} \subseteq \bigcup H \mathcal{D}$. By completeness and finite satisfiability, $p(x) \vdash x \in \bigcup H \mathcal{D}$. Again by completeness, $p(x) \vdash x \in h\mathcal{D}$ for some $h \in H$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

$2 \Rightarrow 3$. Let \mathcal{D} be defined by a conjunction of formulas in $p(x)$. If \mathcal{D} is not G -persistent then, by Fact 3, $\neg \mathcal{D}$ is G -generic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

$3 \Rightarrow 1$. If $p(x)$ is not G -invariant then, by completeness, $p(x) \vdash \varphi(x; b) \wedge \neg \varphi(x; gb)$ for some $g \in G$. Clearly $\varphi(x; b) \wedge \neg \varphi(x; gb)$ is not 2- G -persistent as it is inconsistent with its g -translate. \square

The theorem yields a necessary condition for the existence of G -invariant global $\Delta^B(\mathbb{Z})$ -types.

5 Corollary (Assume 1) If there exists a G -invariant global type finitely satisfiable in \mathcal{X} then for every $\Delta^B(\mathbb{Z})$ -definable set \mathcal{D}

1. \mathcal{D} and $\neg \mathcal{D}$ are not both G -generic
2. if \mathcal{D} is G -generic then it is G -persistent
3. the type $\gamma_G(x) = \{\vartheta(x) \in \Delta^B(\mathbb{Z}) : \vartheta(x) \text{ is } G\text{-generic}\}$ is finitely satisfiable in \mathcal{X} .

Proof. Clearly, 1 and 2 are equivalent, moreover 1 and 3 are immediate consequences of 2 of Theorem 4. \square

The following theorem gives a necessary and sufficient condition for the existence of global G -invariant $\Delta^B(\mathbb{Z})$ -type. Ideally, we would like to have that every G -persistent $\Delta^B(\mathbb{Z})$ -type extends to a global persistent type. Unfortunately this is not true – we need a stronger property. A set \mathcal{D} is **hereditarily** G -persistent if every finite cover of \mathcal{D} by $\Delta^B(\mathbb{Z})$ -definable sets contains a G -persistent set. In [1] a similar property is called *quasi-non-forking*. A type is hereditarily G -persistent if every conjunction of formulas in the type is hereditarily G -persistent.

6 Theorem (Assume 1) Let $q(x) \subseteq L(\mathcal{U})$. Then the following are equivalent

1. $q(x)$ is consistent with a G -invariant type $p(x) \in S_\Delta(\mathbb{Z})$ finitely satisfiable in \mathcal{X}
2. $q(x)$ is hereditarily G -persistent.

Proof. $1 \Rightarrow 2$. Let $\vartheta(x)$ be a conjunction of formulas in $q(x)$. Suppose $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\vartheta(\mathcal{U}^x)$ and pick $p(x)$ as in 1. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i . Then, by Theorem 4, $\neg \mathcal{C}_i$ is not G -generic. Therefore, by Fact 3, \mathcal{C}_i is G -persistent.

$2 \Rightarrow 1$. Let $p(x)$ be maximal among the $\Delta^B(\mathbb{Z})$ -types that are consistent with $q(x)$ and are such that $\vartheta(\mathcal{U}^x)$ is hereditarily G -persistent for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that $p(x)$ is a complete $\Delta^B(\mathbb{Z})$ -type. Suppose for a contradiction that $\vartheta(x), \neg \vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\mathcal{C}_1, \dots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \setminus \vartheta(\mathcal{U}^x)$ and such that no \mathcal{C}_i is G -persistent. As $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is finitely satisfiable in \mathcal{X} and G -invariant. Finite satisfiability follows from persistency. From completeness and Theorem 4 we obtain invariance. \square

We conclude with a fact that reminds of Lemma 2.10 in [2]. It is not used below.

7 Fact (Assume 1) Let \mathcal{D} and \mathcal{C} be Δ -definable sets. The relation on G defined by

$$R(h; k) \Leftrightarrow h\mathcal{D} \cap k\mathcal{C} \text{ is persistent}$$

is stable.

Proof. Let $\langle h_i; k_i : i < 3 \rangle$ be a sequence of elements of G^2 . Assume $h_0\mathcal{D} \cap k_1\mathcal{C}$ is persistent. Note that if a set \mathcal{B} is persistent then $\mathcal{B} \cap g\mathcal{B}$ is also persistent for any $g \in G$. Therefore $h_0\mathcal{D} \cap k_1\mathcal{C} \cap h_2\mathcal{D} \cap h_2h_0^{-1}k_1\mathcal{C}$ is persistent. A fortiori $h_2\mathcal{D} \cap k_1\mathcal{C}$ is persistent. Therefore $R(h_i; k_j) \Leftrightarrow i < j$ fails for some i, j . \square

2. Strong genericity

Unfortunately, G -genericity is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is **strongly G -generic** if for every finite $H \subseteq G$ the set $\cap H \mathcal{D}$ is generic. Dually, we say that \mathcal{D} is **weakly G -persistent** if for some finite $H \subseteq G$ the set $\cup H \mathcal{D}$ is persistent. Again, the same properties may be attributed to formulas and types.

8 Lemma (Assume 1) The intersection of two strongly G -generic sets is strongly G -generic.

Proof. We may assume that all sets mentioned below are subsets of \mathcal{X} . Let \mathcal{D} and \mathcal{C} be strongly G -generic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K (\mathcal{C} \cap \mathcal{D})$ is G -generic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K \mathcal{C}$ and $\mathcal{D}' = \cap K \mathcal{D}$. Note that \mathcal{C}' and \mathcal{D}' are both strongly G -generic. In particular $\mathcal{X} = \cup H \mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\begin{aligned} \cup H \mathcal{B} &= \cup H [\mathcal{C}' \cap \mathcal{D}'] \\ \cup H \mathcal{B} &\supseteq \cup H [(\cap H \mathcal{C}') \cap \mathcal{D}'] \\ &= (\cap H \mathcal{C}') \cap (\cup H \mathcal{D}') \\ &= \cap H \mathcal{C}' \end{aligned}$$

As \mathcal{C}' is strongly G -generic, $\cap H \mathcal{C}'$ is G -generic. Therefore $\cup H \mathcal{B}$ is also G -generic. The G -genericity of \mathcal{B} follows. \square

9 Corollary (Assume 1) Let ${}^s\gamma_G(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) : \vartheta(x) \text{ strongly } G\text{-generic}\}$. Then ${}^s\gamma_G(x)$ is finitely satisfiable in \mathcal{X} , strongly G -generic, and G -invariant.

Proof. Strong G -genericity is an immediate consequence of Lemma 8. Finite satisfiability is a consequence of G -genericity. As for invariance, note that any translate of a strongly G -generic formula is also strongly G -generic. \square

10 Corollary (Assume 1) Let ${}^s\gamma(x)$ be as in Corollary 9. Let $p(x) \subseteq L(\mathcal{U})$ be such that ${}^s\gamma(x) \cup p(x)$ is finitely satisfied in \mathcal{X} . Then $p(x)$ is weakly G -persistent.

Proof. Let $\vartheta(x) \in p$. As $\text{sg}_\gamma(x)$ is finitely satisfiable in $\vartheta(\mathcal{U}^x)$, we cannot have that $\neg\vartheta(x)$ is strongly G -generic. From Fact 3, we obtain that $\neg\vartheta(\mathcal{U}^x)$ non strongly G -generic is equivalent to $\vartheta(x)$ weakly G -persistent. \square

3. The diameter of a Lascar type

As an application we prove an interesting property of the Lascar types. Recall that $\mathcal{L}(a/A)$, the Lascar strong type of $a \in \mathcal{U}^x$, is the union of a chain of type-definable sets of the form $\{x : d_A(a, x) \leq n\}$. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter.

It is convenient to address the problem in more general terms. We work under Assumption 1 with $\Delta = L_{xz}$ and $G \trianglelefteq \text{Aut}(\mathcal{U})$. Let $K \subseteq G$ be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugacy invariant i.e. $gKg^{-1} = K$ for every $g \in G$

Assume G acts transitively on \mathcal{X} i.e. $Ga = \mathcal{X}$ for every $a \in \mathcal{X}$. We define a discrete metric on \mathcal{X} . For $a, b \in \mathcal{X}$ let $d(a, b)$ be the minimal n such that $a \in K^n b$. This defines a metric which is G -invariant by 2. The **diameter** of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a, b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for \mathcal{X} to have finite diameter. The notions introduced in Section 2 offer some hint.

11 Proposition If \mathcal{X} has a weakly persistent subset of finite diameter, then \mathcal{X} itself has finite diameter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter n . Let $H \subseteq G$ be finite such that $\cup H\mathcal{C}$ is persistent. We claim that also $\cup H\mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k \in H$. Now, let hb and kc , for some $h, k \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H\mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of \mathcal{C} ,

$$\begin{aligned} d(hb, kc) &\leq d(hb, ha) + d(ha, ka) + d(ka, kc) \\ &\leq n + m + n. \end{aligned}$$

This proves that $\cup H\mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that \mathcal{C} itself is persistent.

By the transitivity of the action, any two elements of \mathcal{X} are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By persistence, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha, ka) &\leq d(ha, c) + d(c, d) + d(d, ka) \\ &\leq n + n + n. \end{aligned}$$

Therefore the diameter of \mathcal{X} does not exceed $3n$. \square

12 Theorem Suppose that \mathcal{X} and the sets $\mathcal{X}_n = K^n a$, for some $a \in \mathcal{X}$, are type-definable. Then \mathcal{X} has finite diameter.

Proof. By Proposition 11, it suffices to prove that \mathcal{X}_n is weakly persistent. Let ${}^s\gamma_G(x)$ be as in Corollary 9, with $L_{x,z}$ for Δ . It suffices to prove that for some n the type ${}^s\gamma_G(x)$ is finitely satisfied in \mathcal{X}_n . Suppose not. Let $\psi_n(x) \in q$ be a formula that is not satisfied in \mathcal{X}_n . The type $p(x) = \{\psi_n(x) : n \in \omega\}$ is finitely satisfied in \mathcal{X} . Then $p(x)$ has a realization in \mathcal{X} . As this realization belongs to some \mathcal{X}_n we contradict the definition of $\psi_n(x)$. \square

13 Example Let $K \subseteq \text{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A . Then the group G generated by K is $\text{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. Then $d(a, b)$ coincides with the distance in the Lascar graph. As the sets $K^n \cdot a = \{x : d(x, a) \leq n\}$ are type definable from Theorem 12 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

4. A tamer landscape

Under suitable assumptions some of the notions introduced in this chapter coalesce and we are left with a tamer landscape. We prove the following theorem.

14 Theorem (Assume 1) The following are equivalent

1. G -persistent $\Delta^B(\mathcal{Z})$ -definable sets are hereditarily G -persistent
2. G -generic $\Delta^B(\mathcal{Z})$ -definable sets are closed under intersection
3. G -generic $\Delta^B(\mathcal{Z})$ -definable sets are strongly G -generic
4. weakly persistent $\Delta^B(\mathcal{Z})$ -definable sets are G -persistent.

Proof. $2 \Leftrightarrow 3 \Leftrightarrow 4$. Clear.

$1 \Rightarrow 2$. Let \mathcal{C} and \mathcal{D} be G -generic $\Delta^B(\mathcal{Z})$ -definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not G -generic. Then $\neg(\mathcal{C} \cap \mathcal{D})$ is G -persistent. By 1 and Theorem 6 there is a G -invariant global $\Delta^B(\mathcal{Z})$ -type $p(x)$ containing $x \notin \mathcal{C} \cap \mathcal{D}$. By completeness either $p(x) \vdash x \notin \mathcal{C}$ or $p(x) \vdash x \notin \mathcal{D}$. This is a contradiction because by Theorem 4 $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

$4 \Rightarrow 1$. Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) : \vartheta(x) \text{ } G\text{-generic}\}$. By 2 this is the same type defined in Corollary 10. Therefore, any completion of $q(x)$ is, by 4, G -persistent. Let \mathcal{D} be a G -persistent $\Delta^B(\mathcal{Z})$ -definable set. By Theorems 4 and 6 it suffices to show that \mathcal{D} is consistent with $q(x)$. Suppose not, then $q(x) \vdash x \notin \mathcal{D}$. Therefore, by 3, $\neg\mathcal{D}$ is G -generic. This is a contradiction by Fact 3. \square

15 Example It is not difficult to verify that the equivalent conditions in the theorem hold when Δ is a set of stable formulas, $G = \text{Aut}(\mathcal{U}/A)$, and \mathcal{X} is the solution set of a complete type over $\text{acl}^{\text{eq}}A$.

16 Assumption For $G, \mathcal{X}, \mathcal{Z}$ and Δ as in Assumption 1 we also require that the equivalent conditions in Theorem 14 hold.

17 Remark (Assume 16) Note that the types $\gamma_G(x)$ and ${}^s\gamma_G(x)$ defined in corollary 5 and 10 coincide. Then for every $p(x) \in S_\Delta(\mathbb{Z})$ the following are equivalent

1. $p(x)$ is G -persistent (equivalently, G -invariant)
2. $p(x)$ extends $\gamma_G(x)$.

Note also that under Assumption 16 G -invariant global types exist because $\gamma_G(x)$ is finitely consistent in \mathcal{X} .

It is also worth mentioning that when \mathcal{D} is G -generic then every positive Boolean combination of G -translates of \mathcal{D} is G -generic.

««««< HEAD In this section we deal with the actions of two groups: $G = \mathbb{Z}$ and $\text{Aut}(\mathcal{U}/A)$. Generic and persistent only refer to the action of G . We will be explicit about invariance. =====>>>> refs/remotes/origin/revision01

5. Notes and references

Connections with topological dynamics are mentioned everywhere but I ignored them until the very last. I just realized that *persistent* = *thick* and that *weakly persistent* = *piecewise syndetic*. Of course, *generic* = *syndetic*. The notion of *hereditarily persistent* may also have an analogon in topological dynamics, but could not find it yet.

- [1] Artem Chernikov and Itay Kaplan, *Forking and dividing in NTP_2 theories*, J. Symbolic Logic **77** (2012), 1–20.
- [2] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.