Group actions on models

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Abstract. In a nutshell, a set is *strongly syndetic* if the intersection of any finitely many of its translation is syndetic (a.k.a. generic). I do not know if the notion has been studied in the context presented here. To demostrate its interest I use it for a short proof of (a generalization of) Newelski's theorem on the diamter of the Lascar graph, see Theorem 12.

I only recently realized the connnections with topological dynamics (though, I see, it is written on the wall). I only adopted some terminologi and did not investigate further.

Theorem 14 shows that the phenomenon *strongly syndetic* = *syndetic* has some interest (it is reminiscent of *forking* = *dividing*).

In the last section I try to apply these notions to definable groups hoping to recover part of the stable group theory but I fail. I think I am missing something important (or commit errors somewhere).

1. The two perspectives on the invariance of types

This section I review well-known matter and set the terminology.

Below $\Delta \subseteq L_{x\,z}(\mathcal{U})$, $\mathcal{X} \subseteq \mathcal{U}^x$, and $\mathcal{Z} \subseteq \mathcal{U}^z$ are some arbitrary nonempty sets (at some point we will require that $\mathcal{X}$ and $\mathcal{Z}$ are type-definable). We write $L_\Delta(\mathcal{Z})$ for the set of formulas $\vartheta(x)$ that are Boolean combination of formulas $\varphi(x\ b)$ for some $\varphi(x\ z) \in \Delta$ and some $b \in \mathcal{Z}$. Such formulas a called Δ-formulas. A relatively Δ-definable set is a set of the form $\vartheta(\mathcal{X})$ for some Δ-formula $\vartheta(x)$. Subsets of $L_\Delta(\mathcal{Z})$ are called Δ-types. We write $S_\Delta(\mathcal{Z})$ for the set of complete Δ-types with parameters in $\mathcal{Z}$. Beware that there may be other parameters hidden ion Δ).

1 Assumption Let G be a group that acts on $\mathcal{X}$ and on $\mathcal{Z}$ from the left. We require that for every $\varphi(x\,;z) \in \Delta$ the set $\varphi(\mathcal{X};\mathcal{Z})$ is invariant under the action of G.

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that $\mathcal{D}$ is invariant under the action of G, or G-invariant, if $\mathcal{D} \cap \mathcal{Z}$ is fixed setwise by G. Yet in other words, if

is1. $\quad a \in \mathcal{D} \ \leftrightarrow\ ga \in \mathcal{D} \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

A formula is invariant if the set it defines is invariant. We say that $p(x) \subseteq L(\mathcal{U})$ is invariant under the action of G, or G-invariant, if for every formula $\varphi(x\,;z) \in L$

it1. $\quad \varphi(x\,;a) \in p \ \Leftrightarrow\ \varphi(x\,;ga) \in p \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

It should be evident that invariant under the action of $\mathrm{Aut}(\mathcal{U}/A)$ coincides with invariant over A and Lascar invariant over A coincides with invariant under the action of $\mathrm{Autf}(\mathcal{U}/A)$.

We have just defined invariance using the subsets of $\mathcal{Z}$ (externally) defined by p. Now we discuss invariance using the subsets of $\mathcal{X}$ that are in p.

An immediate consequence of Assumption 1 is that any G-translate of a Δ-definable set is again Δ-definable. In particular for every Δ-formula $\vartheta(x\,;\bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X};\bar{b})] \quad = \quad \vartheta(\mathcal{X};g\bar{b}).$$

Therefore $p(x) \subseteq L_\Delta(\mathcal{Z})$ is invariant if

$$p(x) \vdash x \in \mathcal{D} \;\Leftrightarrow\; p(x) \vdash x \in g\mathcal{D} \qquad \text{for every } \Delta\text{-definable } \mathcal{D} \subseteq \mathcal{U}^x \text{ and } g \in G,$$

where by $p(x) \vdash x \in \mathcal{D}$ we understand $\vartheta(\mathcal{X}) \subseteq \mathcal{D}$ for some $\vartheta(x)$ that is conjunction of formulas in $p(x)$.

A set $\mathcal{D} \subseteq \mathcal{X}$ is syndetic under the action of G, or G-syndetic for short, if finitely many G-translates of $\mathcal{D}$ cover $\mathcal{X}$; we say n-G-syndetic if $\leq n$ translates suffices. Dually, we say that $\mathcal{D}$ is thick under the action of G, or G-thick for short, if the intersection of any finitely many G-translates of $\mathcal{D}$ is nonempty; we say n-G-thick when the request is limited to $\leq n$ translates. We may drop reference to G when it is clear from the context.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

The terminology is taken from topological dynamics (I hope correctly). Model theorists say generic for syndetic. In [1] the authors write *quasi-non-dividing* for *thick* when $G = \mathrm{Aut}(\mathcal{U}/A)$.

2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is thick (in any $\mathcal{X} \supseteq A^x$) under the action of $\mathrm{Aut}(\mathcal{U}/A)$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\mathrm{Aut}(\mathcal{U}/A)$-translate of $\varphi(x)$.

In this chapter many proofs require some juggling with negations.

3 Fact (Assume 1) The following are equivalent

1. $\mathcal{D}$ is not n-syndetic
2. $\neg\mathcal{D}$ is n-thick.

Proof. Immediate by spelling out the definitions

1. there are no $g_1, \ldots, g_n \in G$ such that $\mathcal{X} \subseteq \bigcup_{i=1}^{n} g_i\mathcal{D}$
2. $\varnothing \neq \mathcal{X} \cap \bigcap_{i=1}^{n} \neg g_i\mathcal{D}$ for every $g_1, \ldots, g_n \in G$. □

4 Theorem (Assume 1) Let $p(x) \in S_\Delta(\mathcal{Z})$ be finitely satisfiable in $\mathcal{X}$. Then the following are equivalent

1. $p(x)$ is invariant
2. $p(x) \vdash x \in \mathcal{D}$ for every syndetic relatively Δ-definable set $\mathcal{D}$
3. $p(x)$ is thick.

Proof. 1⇒2. Let $g_1\mathcal{D}, \ldots, g_n\mathcal{D}$ be translations of $\mathcal{D}$ that cover $\mathcal{X}$. Negate 2. By completeness, $p(x) \vdash x \notin \mathcal{D}$. Hence, from invariance we obtain

$$p(x) \quad \vdash \quad x \ \notin \ \bigcup_{i=1}^{n} g_i\mathcal{D}.$$

This contradicts the finite satisfiability of $p(x)$ in $\mathcal{X}$.

2⇒3. Let $\mathcal{D}$ be defined by a conjunction of formulas in $p(x)$. If $\mathcal{D}$ is not thick then, by Fact 3, $\neg\mathcal{D}$ is syndetic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

3⇒1. If $p(x)$ is not invariant then, by completeness, $p(x) \vdash \varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ for some $g \in G$. Clearly $\varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ is not thick as it is inconsistent with its g-translate. □

5 Remark In the theorem above, 2 and 3 can be replaced by

2′. $p(x) \vdash x \in \mathcal{D}$ for every 2-syndetic relatively Δ-definable set $\mathcal{D}$

3′. $p(x)$ is 2-thick.

The theorem yields an immediate necessary condition for the existence of an invariant global Δ-type.

6 Corollary (Assume 1) If there exists an invariant global Δ-type then for every Δ-definable set $\mathcal{D}$

1. $\mathcal{D}$ and $\neg\mathcal{D}$ cannot be both syndetic
2. if $\mathcal{D}$ is syndetic than it is thick.

Proof. By Fact 3, 1 and 2 are equivalent; 1 is an immediate consequenc of 2 of Theorem 4. □

The following theorem gives a necessary and sufficient condition for the existence of global invariant Δ-type. Ideally, we would like to prove that every thick Δ-type extends to a global persitent type. Unfortunately this is not true – we need a stronger property. A Δ-definable set $\mathcal{D}$ is hereditarely thick if every finite cover of $\mathcal{D}$ by Δ-definable sets contains a thick set. A type is hereditarely thick if every conjunction of formulas in the type is hereditarely thick.

⚠ I do not know the term used in topological dynamics for hereditarely thick – the terminology is provisional. In [1] a related property is called *quasi-non-forking*.

7 Theorem (Assume 1) Let $q(x) \subseteq L(\mathcal{U})$. Then the following are equivalent

1. $q(x)$ extends to an invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in $\mathcal{X}$
2. $q(x)$ is hereditarely thick.

Proof. 1⇒2. Let $\vartheta(x)$ be a conjunction of formulas in $q(x)$. Suppose $\mathcal{C}_1, \ldots, \mathcal{C}_n$ cover $\vartheta(\mathcal{U}^x)$ and pick $p(x)$ as in 1. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i. Then, by Theorem 4, $\neg\mathcal{C}_i$ is not syndetic. Therefore, by Fact 3, $\mathcal{C}_i$ is thick.

2⇒1. Let $p(x)$ be maximal among the Δ-types that contain $q(x)$ and are such that $\vartheta(\mathcal{U}^x)$ is hereditarely thick for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that p is a complete Δ-type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$ and some $\mathcal{C}_1, \ldots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \smallsetminus \vartheta(\mathcal{U}^x)$ and such that no $\mathcal{C}_i$ is thick. As $\mathcal{C}_1, \ldots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is finitely satisfiable in $\mathcal{X}$ and invariant. Finite satisfiability follows from thickness. From completeness and Theorem 4 we obtain invariance. $\square$

2. Strong syndeticty

Unfortunatelly, genericy is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity. This is only required in Section 3.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is strongly syndetic if the intersection of $\mathcal{D}$ with any of its tranlations $\mathcal{D}$ is syndetic. Dually, we say that $\mathcal{D}$ is weakly thick if the union of $\mathcal{D}$ with one of its translations is peristent. Again, the same properties may be attributed to formulas and types.

Notation: for $\mathcal{B} \subseteq \mathcal{X}$ and $H \subseteq G$ we write $H\,\mathcal{B}$ for $\{h\mathcal{B} : h \in H\}$.

8 Lemma (Assume 1) The intersection of strongly syndetic sets is strongly syndetic.

Proof. We may assume that all sets mentioned below are subsets of $\mathcal{X}$. Let $\mathcal{D}$ and $\mathcal{C}$ be strongly syndetic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K\,(\mathcal{C} \cap \mathcal{D})$ is syndetic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K\,\mathcal{C}$ and $\mathcal{D}' = \cap K\,\mathcal{D}$. Note that $\mathcal{C}'$ and $\mathcal{D}'$ are both strongly syndetic. In particular $\mathcal{X} = \cup H\,\mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\begin{aligned}
\cup H\,\mathcal{B} &= \cup H\Big[\mathcal{C}' \,\cap\, \mathcal{D}'\Big] \\
\cup H\,\mathcal{B} &\supseteq \cup H\Big[(\cap\, H\,\mathcal{C}') \,\cap\, \mathcal{D}'\Big] \\
&= (\cap\, H\,\mathcal{C}') \,\cap\, (\cup\, H\,\mathcal{D}') \\
&= \cap H\,\mathcal{C}'
\end{aligned}$$

As $\mathcal{C}'$ is strongly syndetic, $\cap H\,\mathcal{C}'$ is syndetic. Therefore $\cup H\,\mathcal{B}$ is also syndetic. The syndeticty of $\mathcal{B}$ follows. $\square$

9 Corollary (Assume 1) Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) \;:\; \vartheta(x) \text{ strongly syndetic}\}$. Then $q(x)$ is finitely satisfiable in $\mathcal{X}$, strongly syndetic, and invariant.

Proof. Strong syndeticty is an immediate consequence of Lemma 8. Finite satisfiability follows easily from syndeticty. As for invariance, note that any translate of a strongly syndetic formula is also strongly syndetic. $\square$

10 Corollary (Assume 1) Let $q(x)$ be as in Corollary 9. Let $p(x) \subseteq L(\mathcal{U})$ be such that $p(x) \cup q(x)$ is finitely satisfied in $\mathcal{X}$. Then $p(x)$ is weakly thick.

Proof. Let $\vartheta(x) \in p$. As $q(x)$ is finitely satisfiable in $\vartheta(\mathcal{U}^x)$, we cannot have that $\neg\vartheta(x)$ is strongly syndetic. From Fact 3, we obtain that $\neg\vartheta(\mathcal{U}^x)$ non strongly syndetic is equivalent to $\vartheta(x)$ weakly thick. $\square$

3. The diameter of a Lascar type

Recall that $\mathcal{L}(a/A)$, the Lascar strong type of $a \in \mathcal{U}^x$, is the union of a chain of type-definable sets of the form $\{x \ : \ d_A(a, x) \leq n\}$. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter. It is convenient to address the problem in more general terms.

Let G be a normal subgroup of $\mathrm{Aut}(\mathcal{U})$. Assume G is the union of a countable chain of sets $\langle G_n : n \in \omega \rangle$ with the following properties

1. every G_n is symmetric i.e. it contains the unit and is closed under inverse
2. every G_n is conjugancy invariant i.e. $g\, G_n g^{-1} = G_n$ for every $g \in G$
3. $G_n G_m \subseteq G_{n+m}$ for every $n, m \in \omega$.

Assume G acts transitively on $\mathcal{X}$ i.e., $G\, a = \mathcal{X}$ for every $a \in \mathcal{X}$. We define a discrete metric on $\mathcal{X}$. For $a, b \in \mathcal{X}$ let $d(a, b)$ be the minimal n such that $a \in G_n b$. This defines a metric by 1 and 3. By 2, this metric in G-invariant. The diameter of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a, b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for $\mathcal{X}$ to have finite diameter. The notions introduced in Section 2 offer some hints.

11 Proposition If $\mathcal{X}$ has a weakly thick subset of finite diameter, then $\mathcal{X}$ itself has finite diamenter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly thick set of diameter n. Let $H \subseteq G$ be finite such that $\cup H\, \mathcal{C}$ is thick. We claim that also $\cup H\, \mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k, \in H$. Now, let hb and kc, for some $h, k, \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H\, \mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of $\mathcal{C}$,

$$\begin{aligned} d(hb,\, kc) \ &\leq\ d(hb,\, ha)\ +\ d(ha,\, ka)\ +\ d(ka,\, kc) \\ &\leq\ n + m + n. \end{aligned}$$

This proves that $\cup H\, \mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that $\mathcal{C}$ itself is thick.

By the transitivity of the action, any two elements of $\mathcal{X}$ are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By percistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha,\, ka) \ &\leq\ d(ha,\, c)\ +\ d(c,\, d)\ +\ d(d,\, ka) \\ &\leq\ n + n + n. \end{aligned}$$

Therefore the diameter of $\mathcal{X}$ does not exceed $3n$. □

12 Theorem Suppose that $\mathcal{X}$ and the sets $\mathcal{X}_n = G_n a$, for some $a \in \mathcal{X}$, are type-definable. Then $\mathcal{X}$ has finite diameter.

Proof. By Proposition 11, it suffices to prove that $\mathcal{X}_n$ is weakly thick. Define

$$q(x) \ = \ \{\vartheta(x) \in L(\mathcal{U}) \ : \ \vartheta(x) \text{ strongly syndetic}\}.$$

By Corollary 10, with L for L_φ, it suffices to prove that for some n the type $q(x)$ is finitely satisfied in $\mathcal{X}_n$. Suppose not. Let $\psi_n(x) \in q$ be a formula that is not satisfied in $\mathcal{X}_n$. The type $\{\psi_n(x) : n \in \omega\}$ is finitely satisfied in $\mathcal{X}$. Then it has a realization in $\mathcal{X}$. As this realization belongs to some $\mathcal{X}_n$ this contradicts the definition of $\psi_n(x)$. $\square$

13 Example Let $G_1 \subseteq \mathrm{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A. Thenthe group G generated by G_1 is $\mathrm{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. If we let G_1 be the set of automorphisms that fix a model $M \supseteq A$ and $G_n = G_1^n$, then $d(a, b)$ concides with the dinstance in the Lascar graph. It is not difficult to see that the sets $G_n \cdot a$ are type definable. Then from Theorem 12 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

4. A simplified landscape

Under suitable assumptions – e.g. the sability of $\varphi(x\,;z)$ – some of the notions introduced above coalesce and we are left with cleaner theory. We prove the following theorem.

14 Theorem (Assume 1) The following are equivalent

1. thick Δ-definable sets are hereditarely thick
2. syndetic Δ-definable sets are strongly syndetic
3. syndetic Δ-definable sets are closed under intersection
4. weakly persisent Δ-definable sets are thick.

Proof. 1⇒2. It suffices to prove that syndetic sets are closed under intersection. Let $\mathcal{C}$ and $\mathcal{D}$ be syndetic Δ-definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not syndetic. By 1 and Theorem 7 there is an invariant global Δ-type $p(x)$ containing $x \in \neg\mathcal{C} \cup \neg\mathcal{D}$. By completeness either $p(x) \vdash x \in \neg\mathcal{C}$ or $p(x) \vdash x \in \neg\mathcal{D}$. This is a contradiction because, by Theorem 4, $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

2⇔3⇔4. Clear.

4⇒1. Let $q(x) = \{\vartheta(x) \in L_\varphi(\mathcal{U}) : \vartheta(x) \text{ syndetic}\}$. By 2 this is the same type defined in Corollary 10. Therefore, any completion of $q(x)$ is, by 4, thick. Let $\mathcal{D}$ be a thick Δ-definable set. By Theorems 4 and 7 it suffices to show that $\mathcal{D}$ is consistent with $q(x)$. Suppose not, then $q(x) \vdash x \in \neg\mathcal{D}$. Therefore, by 3, $\neg\mathcal{D}$ is syndetic. This is a contradiction by Fact 3. $\square$

5. Definable groups

In this section we set $G = \mathcal{Z}$ and require that $\mathcal{Z}$ and $\mathcal{X}$ are type-definable over A. We assume that the group operations and the group action are definable over A. We use the symbol $\cdot$ for both the group multiplication and the group action. Clearly, $\mathcal{Z}$ also acts on itself by left multiplication.

In this section we deal with the actions of two groups: $\mathcal{Z}$ and $\mathrm{Aut}(\mathcal{U}/A)$. Syndetic and thick only refer to the action of $\mathcal{Z}$. We will always be explicit about invariance.

Let $\psi(x\,;y) \in L(A)$. We write $\varphi(x\,;z\,;y)$ for the formula $\psi(z^{-1} \cdot x\,;y)$. In this section Δ contains the formulas $\varphi(x\,;z\,;a)$ where a ranges over the realizations of a given $q(y) \in S(A)$. Note that $\varphi(\mathcal{X};\mathcal{Z};a)$ is invariant under the action of $\mathcal{Z}$ and $\varphi(\mathcal{X};g;a) = g\cdot\varphi(\mathcal{X};1;a)$ for every a. Therefore the above G and Δ satisfy Assumption 1.

Let 1 be the identity of $\mathcal{Z}$ which, for simplicity, we assume is a constant of L.

15 Fact There are some formulas $\gamma(y), \pi(y) \in L(A)$ such that, for every $a \in \mathcal{U}^y$

$$\gamma(a) \;\Leftrightarrow\; \varphi(x;1;a) \text{ is syndetic}$$

$$\pi(a) \;\Leftrightarrow\; \varphi(x;1;a) \text{ is thick.}$$

Similar claims holds for $\neg\varphi(x;1;a)$.

Proof. By an easy argument of compactness, there is an n such that if $\varphi(x;1;a)$ is syndetic then it is also n-syndetic. Then

$$\gamma(y) \;=\; \exists z_1,\ldots,z_n\, \forall x \bigvee_{i=1}^{n} \varphi(x;z_i;y)$$

The other claims follow easily. □

As $\varphi(x;g;a)$ is syndetic/thick if and only if $\varphi(x;1;a)$ is such, $\mathrm{Aut}(\mathcal{U}/A)$ maps syndetic/thick sets of the form $\varphi(\mathcal{X};g;a)$ to syndetic/thick sets of the same form.

16 Fact If $p(x) \in S_\Delta(\mathcal{Z})$ is thick (equivalently, $\mathcal{Z}$-invariant) then it is invariant over A – i.e. under the action of $\mathrm{Aut}(\mathcal{U}/A)$.

Proof. By Theorem 4, $p(x)$ is $\mathcal{Z}$-invariant. Therefore $\varphi(x;g;a) \in p$ if and only if $\varphi(x;1;a) \in p$. As $\varphi(x;1;a) \leftrightarrow \varphi(x;1;fa)$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$, invariance over A follows. □

17 Assumption We assume that the three equivalent conditions in Theorem 14 hold for $G = \mathcal{Z}$, $\mathcal{X}$, and Δ introduced in this section.

Under this assumption there is a unique maximal syndetic type $q(x) \subseteq L_\Delta(\mathcal{Z})$. The types $p(x) \in S_\Delta(\mathcal{Z})$ that extend $q(x)$ are thick, hence invariant under the action of both $\mathcal{Z}$ and $\mathrm{Aut}(\mathcal{U}/A)$.

6. Notes and references

[1] Artem Chernikov and Itay Kaplan, *Forking and dividing in* NTP_2 *theories*, J. Symbolic Logic **77** (2012), 1–20.