

## Group actions on models

ABSTRACT. A set is *strongly generic* if the intersection of any finitely many of its translations is generic.

Theorem 18 shows that the condition *strongly generic* = *generic* is robust. It might be of some interest (it is reminiscent of *forking* = *dividing*). Does it have interesting examples?

Proposition 25 looks interesting. Can it be strengthened to obtain a type-definable group?

The connections with topological dynamics have not been explored :(

### 1. The dual perspective on invariance

The notions in this section are well-known but sometimes the terminology differs.

In this chapter  $\Delta \subseteq L_{\mathcal{X}\mathcal{Z}}(\mathcal{U})$ . Let  $\mathcal{Z} \subseteq \mathcal{U}^{\mathcal{Z}}$ . We write  $\Delta(\mathcal{Z})$  for the set of formulas of the form  $\varphi(\mathcal{X}; b)$  for some  $\varphi(\mathcal{X}; z) \in \Delta$  and some  $b \in \mathcal{Z}$ . We write  $\Delta^{\pm}(\mathcal{Z})$  for the set of formulas in  $\Delta(\mathcal{Z})$  or negation thereof. Furthermore, we write  $S_{\Delta}(\mathcal{Z})$  for the set of complete  $\Delta^{\pm}(\mathcal{Z})$ -types.

We write  $\Delta^{\mathbb{B}}(\mathcal{Z})$  for the set of Boolean combinations of formulas in  $\Delta(\mathcal{Z})$ .

**1 Assumption** Let  $G$  be a group that acts on some sets  $\mathcal{X} \subseteq \mathcal{U}^{\mathcal{X}}$  and  $\mathcal{Z} \subseteq \mathcal{U}^{\mathcal{Z}}$ . We require that for every  $\varphi(\mathcal{X}; z) \in \Delta$  the set  $\varphi(\mathcal{X}; \mathcal{Z})$  is invariant under the action of  $G$ . For convenience, we will assume that  $G$  is the identity outside  $\mathcal{X}$  and  $\mathcal{Z}$ .

When  $p(\mathcal{X}) \subseteq \Delta^{\mathbb{B}}(\mathcal{Z})$  and  $\mathcal{D} \subseteq \mathcal{U}^{\mathcal{X}}$  we write  $p(\mathcal{X}) \vdash \mathcal{X} \in \mathcal{D}$  if the inclusion  $\psi(\mathcal{X}) \subseteq \mathcal{D}$  for some  $\psi(\mathcal{X})$  that is conjunctions of formulas in  $p(\mathcal{X})$ .

Let  $\mathcal{D} \subseteq \mathcal{U}^{\mathcal{Z}}$ . We say that  $\mathcal{D}$  is **invariant** under the action of  $G$ , or **G-invariant**, if  $\mathcal{D}$  is fixed setwise by  $G$ . That is,  $g\mathcal{D} = \mathcal{D}$  for every  $g \in G$ . Yet in other words, if

$$\text{is1.} \quad a \in \mathcal{D} \leftrightarrow ga \in \mathcal{D} \quad \text{for every } a \in \mathcal{Z} \text{ and every } g \in G.$$

A formula is  $G$ -invariant if the set it defines is  $G$ -invariant. We say that  $p(\mathcal{X}) \subseteq \Delta^{\mathbb{B}}(\mathcal{Z})$  is **invariant** under the action of  $G$ , or **G-invariant**, if for every  $\Delta^{\mathbb{B}}(\mathcal{Z})$ -formula  $\vartheta(\mathcal{X}; \bar{a})$ .

$$\text{it1.} \quad \vartheta(\mathcal{X}; \bar{a}) \in p \Leftrightarrow \vartheta(\mathcal{X}; g\bar{a}) \in p \quad \text{for every } g \in G.$$

It should be clear that invariant under the action of  $\text{Aut}(\mathcal{U}/A)$  coincides with invariant over  $A$  and Lascar invariant over  $A$  coincides with invariant under the action of  $\text{Autf}(\mathcal{U}/A)$ .

Note that  $p(\mathcal{X})$  is  $G$ -invariant exactly when the sets  $\mathcal{D}_{p, \vartheta} = \{b : \vartheta(\mathcal{X}; b) \in p\} \subseteq \mathcal{U}^{\mathcal{Z}}$  are. Now we would like to discuss invariance using sets  $\mathcal{D} \subseteq \mathcal{U}^{\mathcal{X}}$ .

An immediate consequence of the invariance of  $\varphi(\mathcal{X}; \mathcal{Z})$  is that any  $G$ -translate of a  $\Delta^{\mathbb{B}}(\mathcal{Z})$ -definable set is again  $\Delta^{\mathbb{B}}(\mathcal{Z})$ -definable. In particular for every  $\Delta^{\mathbb{B}}(\mathcal{Z})$ -formula  $\vartheta(\mathcal{X}; \bar{b})$  and every  $g \in G$

$$g[\vartheta(\mathcal{X}; \bar{b})] = \vartheta(\mathcal{X}; g\bar{b}).$$

Therefore, a type  $p(x) \subseteq \Delta^B(\mathcal{Z})$  is  $G$ -invariant if

$$p(x) \vdash x \in \mathcal{D} \Leftrightarrow p(x) \vdash x \in g \cdot \mathcal{D} \text{ for every } \Delta^B(\mathcal{Z})\text{-definable } \mathcal{D} \subseteq \mathcal{X} \text{ and } g \in G.$$

A set  $\mathcal{D} \subseteq \mathcal{X}$  is **generic** under the action of  $G$ , or  **$G$ -generic** for short, if finitely many  $G$ -translates of  $\mathcal{D}$  cover  $\mathcal{X}$ ; we say  **$n$ - $G$ -generic** if  $\leq n$  translates suffices. Dually, we say that  $\mathcal{D}$  is **persistent** under the action of  $G$ , or  **$G$ -persistent** for short, if the intersection of any finitely many  $G$ -translates of  $\mathcal{D}$  is nonempty; we say  **$n$ - $G$ -persistent** when the request is limited to  $\leq n$  translates. When  $\mathcal{X}$  and/or  $\mathcal{Z}$  are not clear from the context, we say that these notions are **relative** to  $\mathcal{X}$  and  $\mathcal{Z}$ .

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type  $p(x)$ , we understand that they hold for every conjunction of formulas in  $p(x)$ .

⚠ The terminology above is non-standard. In [1] the authors write *quasi-non-dividing* for *persistent* under the action of  $\text{Aut}(\mathcal{U}/A)$ . Their terminology has good motivations, but it would be a mouthful if adapted to our context. In topological dynamics similar notions have been introduced with different terminology: *syndetic* corresponds to *generic* and *thick* corresponds to *persistent*.

**2 Example** If  $p(x) \subseteq L(\mathcal{U})$  is finitely satisfiable in  $A$  then  $p(x)$  is persistent under the action of  $G = \text{Aut}(\mathcal{U}/A)$  relative to any  $\mathcal{X} \supseteq A^x$ . In fact, the same  $a \in A^x$  that satisfies  $\varphi(x)$  also satisfies every  $\text{Aut}(\mathcal{U}/A)$ -translate of  $\varphi(x)$ .

Notation: for  $\mathcal{D} \subseteq \mathcal{U}^x$  and  $H \subseteq G$  we write  **$H\mathcal{D}$**  for  $\{h\mathcal{D} : h \in H\}$ .

In this chapter many proofs require some juggling with negations as epitomized by the following fact.

**3 Fact** The following are equivalent

1.  $\mathcal{D}$  is not  $G$ -generic
2.  $\neg \mathcal{D}$  is  $G$ -persistent.

**Proof.** Immediate by spelling out the definitions

1. there are no finite  $H \subseteq G$  such that  $\mathcal{X} \subseteq \bigcup H\mathcal{D}$ .
2.  $\emptyset \neq \mathcal{X} \cap (\bigcap H\neg\mathcal{D})$  for every finite  $H \subseteq G$ . □

Define the following type

$$\gamma_G(x) = \{\vartheta(x) \in \Delta^B(\mathcal{Z}) : \vartheta(x) \text{ is } G\text{-generic}\}$$

**4 Corollary** Let  $p(x) \subseteq \Delta^B(\mathcal{Z})$  be such that  $\gamma_G(x) \cup p(x)$  is finitely satisfiable in  $\mathcal{X}$ . Then  $p(x)$  is  $G$ -persistent.

**Proof.** Let  $\vartheta(x)$  be a conjunction of formulas in  $p(x)$ . As  $\gamma_G(x)$  is finitely satisfiable in  $\vartheta(\mathcal{X})$ , it cannot be that  $\neg\vartheta(x)$  is  $G$ -generic. From Fact 3, we obtain that  $\vartheta(x)$  is  $G$ -persistent. □

The converse implication holds for complete types.

**5 Theorem** Let  $p(x) \in S_\Delta(\mathbb{Z})$ . Then the following are equivalent

1.  $p(x)$  is  $G$ -invariant and finitely satisfiable in  $\mathcal{X}$
2.  $p(x) \vdash \gamma_G(x)$
3.  $p(x)$  is  $G$ -persistent.

**Proof.**  $1 \Rightarrow 2$ . Let  $H \subseteq G$  be finite such that  $\mathcal{X} \subseteq \bigcup H \mathcal{D}$ . By completeness and finite satisfiability,  $p(x) \vdash x \in \bigcup H \mathcal{D}$ . Again by completeness,  $p(x) \vdash x \in h \mathcal{D}$  for some  $h \in H$ . Finally, by invariance,  $p(x) \vdash x \in \mathcal{D}$ .

$2 \Rightarrow 3$ . Let  $\mathcal{D}$  be defined by a conjunction of formulas in  $p(x)$ . If  $\mathcal{D}$  is not  $G$ -persistent then, by Fact 3,  $\neg \mathcal{D}$  is  $G$ -generic. By 2,  $p(x) \vdash x \notin \mathcal{D}$ , a contradiction.

$3 \Rightarrow 1$ . First note that  $G$ -persistent types are finitely satisfiable in  $\mathcal{X}$ . Now, suppose  $p(x)$  is not  $G$ -invariant. Then, by completeness,  $p(x) \vdash \varphi(x; b) \wedge \neg \varphi(x; gb)$  for some  $g \in G$ . Clearly  $\varphi(x; b) \wedge \neg \varphi(x; gb)$  is not 2- $G$ -persistent as it is inconsistent with its  $g$ -translate.  $\square$

**6 Corollary** The following are equivalent for every  $\Delta^B(\mathbb{Z})$ -definable set  $\mathcal{D}$

1.  $\gamma_G(x) \vdash x \in \mathcal{D}$
2.  $p(x) \vdash x \in \mathcal{D}$  for every  $G$ -persistent  $p(x) \in S_\Delta(\mathbb{Z})$ .

**Proof.**  $1 \Rightarrow 2$ . This is an immediate consequence of Theorem 5.

$2 \Rightarrow 1$ . Suppose  $\gamma_G(x) \not\vdash x \in \mathcal{D}$ . Then there is a type  $p(x) \in S_\Delta(\mathbb{Z})$  consistent with  $\gamma_G(x) \cup \{x \notin \mathcal{D}\}$ . By Corollary 4  $p(x)$  is  $G$ -persistent. Then  $\neg 2$ .  $\square$

The theorem yields a necessary condition for the existence of  $G$ -invariant global  $\Delta^B(\mathbb{Z})$ -types.

**7 Corollary** If there exists a  $G$ -invariant type  $p(x) \in S_\Delta(\mathbb{Z})$  finitely satisfiable in  $\mathcal{X}$  then for every  $\Delta^B(\mathbb{Z})$ -definable set  $\mathcal{D}$

1.  $\mathcal{D}$  and  $\neg \mathcal{D}$  are not both  $G$ -generic
2. if  $\mathcal{D}$  is  $G$ -generic then it is  $G$ -persistent
3.  $\gamma_G(x)$  is finitely satisfiable in  $\mathcal{X}$ .

**Proof.** Clearly, 1 and 2 are equivalent by Fact 3 and follow from 3. Finally, 3 is an immediate consequence of 2 of Theorem 5.  $\square$

The following theorem gives a necessary and sufficient condition for the existence of global  $G$ -invariant  $\Delta^B(\mathbb{Z})$ -type. Ideally, we would like to have that every  $G$ -persistent  $\Delta^B(\mathbb{Z})$ -type extends to a global persistent type. Unfortunately this is not true in general (it requires stronger assumptions, see Section 4). A set  $\mathcal{D}$  is  **$G$ -wide** if every finite cover of  $\mathcal{D}$  by  $\Delta^B(\mathbb{Z})$ -definable sets contains a  $G$ -persistent set. In [1] a similar property is called *quasi-non-forking*. Our use of the term *wide* is consistent with [2], though we apply it to a narrow context. A type is  $G$ -wide if every conjunction of formulas in the type is  $G$ -wide.

**8 Theorem** Let  $\mathcal{D}$  be a  $\Delta^B(\mathcal{Z})$ -definable set. Then the following are equivalent

1.  $\gamma_G(x)$  is finitely satisfied in  $\mathcal{D} \cap \mathcal{X}$
2. there exists a  $G$ -persistent type  $p(x) \in S_\Delta(\mathcal{Z})$  containing  $x \in \mathcal{D}$
3.  $\mathcal{D}$  is  $G$ -wide.

**Proof.**  $1 \Rightarrow 2$ . By Corollary 4, it suffices to pick any  $p(x) \in S_\Delta(\mathcal{Z})$  containing  $\gamma_G(x)$  and finitely satisfied in  $\mathcal{D} \cap \mathcal{X}$ .

$2 \Rightarrow 1$ . By Theorem 5.

$2 \Rightarrow 3$ . Let  $\mathcal{C}_1, \dots, \mathcal{C}_n$  be  $\Delta^B(\mathcal{Z})$ -definable sets that cover  $\mathcal{D}$ . Pick  $p(x)$  as in 2. By completeness,  $p(x) \vdash x \in \mathcal{C}_i$  for some  $i$ . Then, by Theorem 5,  $\neg \mathcal{C}_i$  is not  $G$ -generic. Therefore, by Fact 3,  $\mathcal{C}_i$  is  $G$ -persistent.

$3 \Rightarrow 2$ . Let  $p(x)$  be maximal among the  $\Delta^B(\mathcal{Z})$ -types that are finitely satisfiable in  $\mathcal{X} \cap \mathcal{D}$  and are such that  $\vartheta(u^x)$  is  $G$ -wide for every  $\vartheta(x)$  that is conjunction of formulas in  $p(x)$ . We claim that  $p(x)$  is a complete  $\Delta^B(\mathcal{Z})$ -type. Suppose for a contradiction that  $\vartheta(x), \neg \vartheta(x) \notin p$ . By maximality there is some formula  $\psi(x)$ , a conjunction of formulas in  $p(x)$ , and some  $\Delta^B(\mathcal{Z})$ -definable sets  $\mathcal{C}_1, \dots, \mathcal{C}_n$  that cover both  $\psi(u^x) \cap \vartheta(u^x)$  and  $\psi(u^x) \setminus \vartheta(u^x)$  and such that no  $\mathcal{C}_i$  is  $G$ -persistent. As  $\mathcal{C}_1, \dots, \mathcal{C}_n$  cover  $\psi(u^x)$  this is a contradiction. It is only left to show that  $p(x)$  is  $G$ -invariant. This follows from completeness and Theorem 5.  $\square$

**9 Corollary** Let  $\mathcal{D}$  be a  $G$ -wide  $\Delta^B(\mathcal{Z})$ -definable set. Then  $\mathcal{D} \cap g \cdot \mathcal{D}$  is  $G$ -wide for every  $g \in G$ .

**Proof.** Let  $p(x) \in S_\Delta(\mathcal{Z})$  be a  $G$ -persistent type such that  $p(x) \vdash x \in \mathcal{D}$ . By  $G$ -invariance  $p(x) \vdash x \in g \cdot \mathcal{D}$ .  $\square$

**10 Fact** Let  $\mathcal{D}$  be a  $\Delta^B(\mathcal{Z})$ -definable set. The following are equivalent

1.  $\mathcal{D}$  is  $G$ -wide;
2. every finite cover of  $\mathcal{D}$  by  $\Delta^\pm(\mathcal{Z})$ -definable sets contains a  $G$ -persistent set.

Let  $q(x) \subseteq \Delta^B(\mathcal{Z})$  be  $G$ -invariant. We say that  $q(x)$  is  **$G$ -prime** if for every  $\Delta^B(\mathcal{Z})$ -definable set  $\mathcal{D}$  and every  $g \in G$  if  $q(x) \vdash x \in (\mathcal{D} \cup g \cdot \mathcal{D})$  then  $q(x) \vdash x \in \mathcal{D}$ .

**11 Proposition** The type  $\gamma_G(x)$  is  $G$ -prime.

**Proof.** Suppose  $\gamma_G(x) \vdash x \in (\mathcal{D} \cup g \cdot \mathcal{D})$ . Assume that  $\gamma_G(x)$  is consistent, otherwise the claim is trivial. Let  $p(x) \in S_\Delta(\mathcal{Z})$  be  $G$ -persistent. We claim that  $p(x) \vdash x \in \mathcal{D}$ . The proposition follows from the claim by Corollary 6. By completeness  $p(x) \vdash x \in \mathcal{D}$  or  $p(x) \vdash x \in g \cdot \mathcal{D}$ . If the latter occurs,  $p(x) \vdash x \in \mathcal{D}$  follows from invariance.  $\square$

## 2. Strong genericity

Unfortunately,  $G$ -genericity is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set  $\mathcal{D} \subseteq \mathcal{U}^x$  is **strongly G-generic** if for every finite  $H \subseteq G$  the set  $\cap H \mathcal{D}$  is generic (recall that  $H \mathcal{D}$  stands for  $\{h \cdot \mathcal{D} : h \in H\}$ ). Dually, we say that  $\mathcal{D}$  is **weakly G-persistent** if for some finite  $H \subseteq G$  the set  $\cup H \mathcal{D}$  is persistent. Again, the same properties may be attributed to formulas and types.

**12 Lemma** The intersection of two strongly G-generic sets is strongly G-generic.

**Proof.** We may assume that all sets mentioned below are subsets of  $\mathcal{X}$ . Let  $\mathcal{D}$  and  $\mathcal{C}$  be strongly G-generic and let  $K \subseteq G$  be an arbitrary finite set. It suffices to prove that  $\mathcal{B} = \cap K (\mathcal{C} \cap \mathcal{D})$  is G-generic. Clearly  $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$ , where  $\mathcal{C}' = \cap K \mathcal{C}$  and  $\mathcal{D}' = \cap K \mathcal{D}$ . Note that  $\mathcal{C}'$  and  $\mathcal{D}'$  are both strongly G-generic. In particular  $\mathcal{X} = \cup H \mathcal{D}'$  for some finite  $H \subseteq G$ . Now, from

$$\begin{aligned} \cup H \mathcal{B} &= \cup H [\mathcal{C}' \cap \mathcal{D}'] \\ \cup H \mathcal{B} &\supseteq \cup H [(\cap H \mathcal{C}') \cap \mathcal{D}'] \\ &= (\cap H \mathcal{C}') \cap (\cup H \mathcal{D}') \\ &= \cap H \mathcal{C}' \end{aligned}$$

As  $\mathcal{C}'$  is strongly G-generic,  $\cap H \mathcal{C}'$  is G-generic. Therefore  $\cup H \mathcal{B}$  is also G-generic. The G-genericity of  $\mathcal{B}$  follows.  $\square$

Define the following type

$${}^s\gamma_G(x) = \{\vartheta(x) \in \Delta^B(\mathcal{Z}) : \vartheta(x) \text{ is strong G-generic}\}$$

**13 Corollary** The type  ${}^s\gamma_G(x)$  is finitely satisfiable in  $\mathcal{X}$ , strongly G-generic, and G-invariant.

**Proof.** The strong G-genericity is an immediate consequence of Lemma 12. The finite satisfiability is a consequence of G-genericity. As for invariance, note that any translate of a strongly G-generic formula is also strongly G-generic.  $\square$

**14 Corollary** Let  $p(x) \subseteq L(\mathcal{U})$  be such that  ${}^s\gamma(x) \cup p(x)$  is finitely satisfiable in  $\mathcal{X}$ . Then  $p(x)$  is weakly G-persistent.

**Proof.** Similar to Corollary 4. Let  $\vartheta(x)$  be a conjunction of formulas in  $p(x)$ . As  ${}^s\gamma(\mathcal{X})$  is finitely satisfiable in  $\vartheta(\mathcal{X})$ , it cannot be that  $\neg\vartheta(x)$  is strongly G-generic. From Fact 3, we obtain that  $\neg\vartheta(x)$  not being strongly G-generic is equivalent to  $\vartheta(x)$  being weakly G-persistent.  $\square$

### 3. The diameter of a Lascar type

As an application we prove an interesting theorem of Newelski's on Lascar types. Let  $\mathcal{L}(a/A)$ , the set of tuples with the same Lascar strong type as  $a$  over  $A$ . This set is the union of a chain of type-definable sets of the form  $\{x : d_A(a, x) \leq n\}$ , where  $d_A$  is the distance in the Lascar graph. In this section we prove that  $\mathcal{L}(a/A)$  is type-definable (if and) only this chain is finite. In other words, only if the connected component of  $a$  in the Lascar graph has finite diameter.

It is convenient to address the problem in more general terms. We work under Assumption 1 and also assume that  $G$  acts transitively on  $\mathcal{X}$  i.e.  $Ga = \mathcal{X}$  for every  $a \in \mathcal{X}$ . Let  $K \subseteq G$  be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugacy invariant i.e.  $gKg^{-1} = K$  for every  $g \in G$

We define a discrete metric on  $\mathcal{X}$ . For  $a, b \in \mathcal{X}$  let  $d(a, b)$  be the minimal  $n$  such that  $a \in K^n b$ . This defines a metric which is  $G$ -invariant by 2. The **diameter** of a set  $\mathcal{C} \subseteq \mathcal{X}$  is the supremum of  $d(a, b)$  for  $a, b \in \mathcal{C}$ .

We are interested in sufficient conditions for  $\mathcal{X}$  to have finite diameter. The notions introduced in Section 2 offer some hint.

**15 Proposition** If  $\mathcal{X}$  has a weakly persistent subset of finite diameter, then  $\mathcal{X}$  itself has finite diameter.

**Proof.** Let  $\mathcal{C} \subseteq \mathcal{X}$  be a weakly persistent set of diameter  $n$ . Let  $H \subseteq G$  be finite such that  $\cup H \mathcal{C}$  is persistent. We claim that also  $\cup H \mathcal{C}$  has finite diameter. Let  $a \in \mathcal{C}$  be arbitrary. Let  $m$  be larger than  $d(ha, ka)$  for all  $h, k \in H$ . Now, let  $hb$  and  $kc$ , for some  $h, k \in H$  and  $b, c \in \mathcal{C}$ , be two arbitrary elements of  $\cup H \mathcal{C}$ . As  $h\mathcal{C}$  and  $k\mathcal{C}$  have the same diameter of  $\mathcal{C}$ ,

$$\begin{aligned} d(hb, kc) &\leq d(hb, ha) + d(ha, ka) + d(ka, kc) \\ &\leq n + m + n. \end{aligned}$$

This proves that  $\cup H \mathcal{C}$  has finite diameter. Therefore, without loss of generality, we may assume that  $\mathcal{C}$  itself is persistent.

By the transitivity of the action, any two elements of  $\mathcal{X}$  are of the form  $ha, ka$  for some  $h, k \in G$  and some  $a \in \mathcal{C}$ . By persistency, there are  $c \in \mathcal{C} \cap h\mathcal{C}$  and  $d \in \mathcal{C} \cap k\mathcal{C}$ . Then

$$\begin{aligned} d(ha, ka) &\leq d(ha, c) + d(c, d) + d(d, ka) \\ &\leq n + n + n. \end{aligned}$$

Therefore the diameter of  $\mathcal{X}$  does not exceed  $3n$ . □

**16 Theorem** Suppose that  $\mathcal{X}$  and the sets  $\mathcal{X}_n = K^n a$ , for some  $a \in \mathcal{X}$ , are type-definable. Then  $\mathcal{X}$  has finite diameter.

**Proof.** By Proposition 15, it suffices to prove that  $\mathcal{X}_n$  is weakly persistent. By Corollary 14 it suffices to show that for some  $n$  the type  ${}^s\gamma_G(x)$  is finitely satisfied in  $\mathcal{X}_n$ . Suppose not. Let  $\psi_n(x) \in {}^s\gamma_G$  be a formula that is not satisfied in  $\mathcal{X}_n$ . Then the type  $p(x) = \{\psi_n(x) : n \in \omega\}$  is finitely satisfied in  $\mathcal{X}$ . From the type-definability of  $\mathcal{X}$  it follows that  $p(x)$  has a realization in  $\mathcal{X}$ . As this realization belongs to some  $\mathcal{X}_n$  we contradict the definition of  $\psi_n(x)$ . □

**17 Example** Let  $K \subseteq \text{Aut}(\mathcal{U}/A)$  be the set of automorphisms that fix a model containing  $A$ . Then the group  $G$  generated by  $K$  is  $\text{Autf}(\mathcal{U}/A)$  and  $G \cdot a = \mathcal{X}$  is  $\mathcal{L}(a/A)$ . Let  $\Delta = L_{xz}$  and  $\mathcal{Z} = \mathcal{U}^{\mathcal{Z}}$ . Then  $d(a, b)$  coincides with the distance in the Lascar graph. The sets  $K^n \cdot a = \{x : d(x, a) \leq n\}$  are type definable. Then from Theorem 16 it follows that  $\mathcal{L}(a/A)$  is type definable (if and) only if it has a finite diameter.

#### 4. A tamer landscape

Under suitable assumptions some notion introduced in this chapter coalesce, and we are left with a tamer landscape. In general, we have the following.

**18 Theorem** Assume

1.  $G$ -persistent  $\Delta^{\mathcal{B}}(\mathcal{Z})$ -definable sets are  $G$ -wide.
- Then the following hold
2.  $G$ -generic  $\Delta^{\mathcal{B}}(\mathcal{Z})$ -definable sets are closed under intersection
  3.  $G$ -generic  $\Delta^{\mathcal{B}}(\mathcal{Z})$ -definable sets are strongly  $G$ -generic
  4. weakly persistent  $\Delta^{\mathcal{B}}(\mathcal{Z})$ -definable sets are  $G$ -persistent.

**Proof.** Clearly  $2 \Leftrightarrow 3 \Leftrightarrow 4$ .

$1 \Rightarrow 2$ . Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $G$ -generic  $\Delta^{\mathcal{B}}(\mathcal{Z})$ -definable sets. Suppose for a contradiction that  $\mathcal{C} \cap \mathcal{D}$  is not  $G$ -generic. Then  $\neg(\mathcal{C} \cap \mathcal{D})$  is  $G$ -persistent. By 1 and Theorem 8 there is a  $G$ -invariant global  $\Delta^{\mathcal{B}}(\mathcal{Z})$ -type  $p(x)$  containing  $x \notin \mathcal{C} \cap \mathcal{D}$ . By completeness either  $p(x) \vdash x \notin \mathcal{C}$  or  $p(x) \vdash x \notin \mathcal{D}$ . This is a contradiction because by Theorem 5  $p(x) \vdash x \in \mathcal{C}$  and  $p(x) \vdash x \in \mathcal{D}$ .  $\square$

**19 Question** Is 1 in Theorem 18 equivalent to 2-4?

**20 Remark** Assume 2-4 in Theorem 18 hold. Then the types  $\gamma_G(x)$  and  ${}^s\gamma_G(x)$  coincide, and therefore  $G$ -invariant global types exist. It is also worth mentioning that every positive Boolean combination of  $G$ -generic sets is  $G$ -generic.

#### 5. The action of normal subgroups

Let  $H \trianglelefteq G$ . The following is an immediate consequence of normality.

**21 Remark** For every  $\mathcal{D} \subseteq \mathcal{U}^x$  and every  $g \in G$

$$\mathcal{D} \text{ is } H\text{-foo} \Leftrightarrow g \cdot \mathcal{D} \text{ is } H\text{-foo},$$

where *foo* can be replaced by *generic*, *invariant*, *persistent*, *wide*. In particular, the type  $\gamma_H(x)$  is  $G$ -invariant.

Recall that if  $\gamma_H(x)$  is finitely satisfiable in  $\mathcal{X}$ , then  $H$ -generic sets are  $H$ -wide, cf. Theorem 8. As it happens, we can slightly strengthen this fact.

**22 Proposition** Assume that  $\gamma_H(x)$  is consistent. Let  $\mathcal{D}$  be a  $\Delta^B(\mathbb{Z})$ -definable set. Then if  $\mathcal{D}$  is  $G$ -generic it is also  $H$ -wide.

**Proof.** Let  $p(x) \in S_\Delta(\mathbb{Z})$  be consistent with  $\gamma_H(x)$ . As  $\mathcal{D}$  is  $G$ -generic, by completeness  $p(x) \vdash x \in g \cdot \mathcal{D}$  for some  $g \in G$ . Equivalently,  $g^{-1} \cdot p \vdash x \in \mathcal{D}$ . As  $p(x)$  is  $H$ -persistent, by Remark 21  $g^{-1} \cdot p(x)$  is also  $H$ -persistent. Then the proposition follows from Theorem 8.  $\square$

## 6. Definable groups

In this section we assume that  $\mathbb{Z}$  and  $\mathbb{X}$  are type-definable over some set of parameters  $A$ . Moreover we assume that  $\mathbb{Z}$  is a group that act on  $\mathbb{X}$ . The group operations and the group action are assumed definable over  $A$ . We use the symbol  $\cdot$  for both the group multiplication and the group action.

Let  $\Psi \subseteq L_x(\mathcal{U})$  be some small set of formulas. In this section  $\Delta$  contains formulas  $\varphi(x; z)$  of the form  $\psi(z^{-1} \cdot x)$  for  $\psi(x) \in \Psi$ . The sets  $\varphi(\mathbb{X}; \mathbb{Z})$  are  $\mathbb{Z}$ -invariant. We write  $1$  for the identity of  $\mathbb{Z}$ . If  $\varphi(\mathbb{X}; 1) \in \Delta^B(1)$  then  $\varphi(\mathbb{X}; g) = g \cdot \varphi(\mathbb{X}; 1)$ .

The following auxiliary structure is useful. Let  $\mathcal{U}^\Delta = \langle \mathbb{X}; \mathbb{Z} \rangle$  be a 2-sorted structure whose signature  $L^\Delta$  contains relation symbols for every formula  $\varphi(x; z) \in \Delta$ . As there is little risk of confusion, these relations symbols are also denoted by  $\varphi(x; z)$ .

To each  $h \in \mathbb{Z}$  we associate the  $L^\Delta$ -automorphism  $\langle a; g \rangle \mapsto \langle h \cdot a; h \cdot g \rangle$ . Therefore  $\mathbb{Z}$  is, up to isomorphism, a subgroup of  $G$ . In fact, it is a normal subgroup. For any  $g \in \mathbb{Z}$ , the orbit of  $\varphi(\mathbb{X}; g)$  under the action of  $\mathbb{Z}$  is  $\{\varphi(\mathbb{X}; h) : h \in \mathbb{Z}\}$ . Therefore it coincides with the orbit under the action of  $G$ .

Now, we consider the action some other normal subgroup  $H \trianglelefteq G$ .

**23 Proposition** Let  $\mathcal{D}$  be a  $\Delta^B$ -definable set. Assume that  $\gamma_H(x)$  is consistent. Then  $1 \Rightarrow 2$  holds, where

1.  $\mathcal{D}$  is  $\mathbb{Z}$ -generic
2.  $g \cdot \mathcal{D}$  is  $H$ -wide for every  $g \in \mathbb{Z}$ .

Under the assumption of stability and with  $H = \text{Autf}(\mathcal{U}^\Delta)$  a stronger claim obtains – the consistency of  $\gamma_H(x)$  is guaranteed, and also the converse implication holds

**Proof.** Let  $g$  be given. If  $\mathcal{D}$  is  $\mathbb{Z}$ -generic, then so is  $g \cdot \mathcal{D}$ . Then  $g \cdot \mathcal{D}$  is, a fortiori,  $G$ -generic. Therefore 2 follows from Proposition 22.  $\square$

We write  $(g)_H$  for the  $H$ -orbit of  $g$ , that is, the set  $\{f(g) : f \in H\}$ .

**24 Proposition** Let  $\vartheta(x; z_1, \dots, z_n)$  be a Boolean combination of formulas  $\varphi_i(x; z_i)$  for some  $\varphi_i(x; z) \in \Delta$ . Then for every  $h_i \in (g_i)_H$  the following are equivalent

1.  $\vartheta(x; g_1, \dots, g_n)$  is  $H$ -wide
2.  $\vartheta(x; h_1, \dots, h_n)$  is  $H$ -wide.



**Proof.** Without loss of generality we can assume that only conjunctions occur in  $\vartheta(x; g_1, \dots, g_n)$ . Let  $\mathcal{C}_i = \varphi_i(\mathcal{X}; 1)$ . Then 1 says that  $\mathcal{C} = g_1 \cdot \mathcal{C}_1 \cap \dots \cap g_n \cdot \mathcal{C}_n$  is  $H$ -wide. Let  $f_i \in H$  be such that  $h_i = f_i(g_i)$ . Then, by Corollary 9 also the intersection of the sets  $f_i[\mathcal{C}]$  is  $H$ -wide. A fortiori the intersection of the sets  $f_i[g_i \cdot \mathcal{C}_i]$  is  $H$ -wide. As  $f_i[g_i \cdot \mathcal{C}_i] = h_i \cdot \mathcal{C}_i$ , implication  $1 \Rightarrow 2$  follows and, by symmetry, this proves the equivalence.  $\square$

If  $\mathcal{A} \subseteq \mathbb{Z}$ , we write  $\langle \mathcal{A} \rangle$  for the subgroup generated by  $\mathcal{A}$ .

**25 Proposition** Let  $\vartheta(x; z_1, \dots, z_n)$  be a Boolean combination of formulas  $\varphi_i(x; z_i)$  for some  $\varphi_i(x; z) \in \Delta$ . Let  $g \in \mathbb{Z}$  be arbitrary. Assume that  $\vartheta(x; 1, \dots, 1)$  is  $H$ -wide. Then  $\vartheta(x; h_1, \dots, h_n)$  is  $H$ -wide for every

$$h_1, \dots, h_n \in \left\langle \bigcup_{g \in \mathbb{Z}} (g)_H^{-1} \cdot (g)_H \right\rangle.$$

**Proof.** We proceed by induction on the number of factors of the form  $a^{-1} \cdot b$ , for some  $a, b \in (g)_H$ , that occur in  $h_1, \dots, h_n$ . Without loss of generality we can assume that only conjunctions occur in  $\vartheta(x; 1, \dots, 1)$ . Let  $\mathcal{C}_i = \varphi_i(\mathcal{X}; 1)$ . Assume inductively that  $h_1 \cdot \mathcal{C}_1 \cap \dots \cap h_n \cdot \mathcal{C}_n$  is  $H$ -wide. Pick two arbitrary  $a, b \in (g)_H$ . By Remark 21

$$a \cdot \mathcal{C}_1 \cap a \cdot h_1^{-1} \cdot h_2 \cdot \mathcal{C}_2 \cap \dots \cap a \cdot h_1^{-1} \cdot h_n \cdot \mathcal{C}_n \text{ is } H\text{-wide.}$$

By Proposition 24, we can replace  $a \cdot \mathcal{C}_1$  by  $b \cdot \mathcal{C}_1$ . Then, again by Remark 21

$$h_1 \cdot a^{-1} \cdot b \cdot \mathcal{C}_1 \cap h_2 \cdot \mathcal{C}_2 \cap \dots \cap h_n \cdot \mathcal{C}_n \text{ is } H\text{-wide.} \quad \square$$

**26 Question** Let  $H = \text{Autf}(\mathcal{U}^\Delta)$ . Under the assumption of stability the group in Proposition 25 has finite index in  $\mathbb{Z}$  and is contained in the connected component of  $\mathbb{Z}$ . Does it coincide with it? I.e. is it definable?

**27 Question** Let  $H = \text{Autf}(\mathcal{U}^\Delta)$ . Describe the group in Proposition 25.

**28 Question** Is the group in Proposition 25  $H$ -invariant?

## 7. Notes and references

- [1] Artem Chernikov and Itay Kaplan, *Forking and dividing in NTP<sub>2</sub> theories*, J. Symbolic Logic **77** (2012), 1–20.
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