Group actions on models

ABSTRACT. A set is *strongly generic* if the intersection of any finitely many of its translations is generic.

Theorem 18 shows that the condition *strongly generic* = *generic* is robust. It might be of some interest (it is reminiscent of *forking* = *dividing*). Does it have interesting examples?

Proposition 25 reminds of Hrushovski's stabilizer theorem [2]. But how, exactly? And, can it be strengthened to obtain a type definable group?

The connections with topological dynamics have not been explored :(

1. The dual perspective on invariance

The notions in this section are well-known but sometimes the terminology differs.

In this chapter $\Delta \subseteq L_{x\,z}(\mathcal{U})$. Let $\mathcal{Z} \subseteq \mathcal{U}^z$. We write $\Delta(\mathcal{Z})$ for the set of formulas of the form $\varphi(x\,;b)$ for some $\varphi(x\,;z) \in \Delta$ and some $b \in \mathcal{Z}$. We write $\Delta^{\pm}(\mathcal{Z})$ for the set of formulas in $\Delta(\mathcal{Z})$ or negation thereof. Furthermore, we write $S_\Delta(\mathcal{Z})$ for the set of complete $\Delta^{\pm}(\mathcal{Z})$-types.

We write $\Delta^{\mathsf{B}}(\mathcal{Z})$ for the set of Boolean combinations of formulas in $\Delta(\mathcal{Z})$.

1 Assumption Let G be a group that acts on some sets $\mathcal{X} \subseteq \mathcal{U}^x$ and $\mathcal{Z} \subseteq \mathcal{U}^z$. We require that for every $\varphi(x\,;z) \in \Delta$ the set $\varphi(\mathcal{X};\mathcal{Z})$ is invariant under the action of G. For convenience, we will assume that G is the identity outside $\mathcal{X}$ and $\mathcal{Z}$.

When $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ and $\mathcal{D} \subseteq \mathcal{U}^x$ we write $p(x) \vdash x \in \mathcal{D}$ if the inclusion $\psi(\mathcal{X}) \subseteq \mathcal{D}$ for some $\psi(x)$ that is conjunctions of formulas in $p(x)$.

Let $\mathcal{D} \subseteq \mathcal{U}^z$. We say that $\mathcal{D}$ is invariant under the action of G, or G-invariant, if $\mathcal{D}$ is fixed setwise by G. That is, $g\,\mathcal{D} = \mathcal{D}$ for every $g \in G$. Yet in other words, if

is1. $\quad a \in \mathcal{D} \;\leftrightarrow\; ga \in \mathcal{D} \qquad$ for every $a \in \mathcal{Z}$ and every $g \in G$.

A formula is G-invariant if the set it defines is G-invariant. We say that $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ is invariant under the action of G, or G-invariant, if for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-formula $\vartheta(x\,;\bar{a})$.

it1. $\quad \vartheta(x\,;\bar{a}) \in p \;\Leftrightarrow\; \vartheta(x\,;g\,\bar{a}) \in p \qquad$ for every $g \in G$.

It should be clear that invariant under the action of $\mathrm{Aut}(\mathcal{U}/A)$ coincides with invariant over A and Lascar invariant over A coincides with invariant under the action of $\mathrm{Autf}(\mathcal{U}/A)$.

Note that $p(x)$ is G-invariant exactly when the sets $\mathcal{D}_{p,\vartheta} = \{b : \vartheta(x\,;b) \in p\} \subseteq \mathcal{U}^z$ are. Now we would like to discuss invariance using sets $\mathcal{D} \subseteq \mathcal{U}^x$. For that, we need stronger assumptions on the action of G.

An immediate consequence of the invariance of $\varphi(\mathcal{X}\,;\mathcal{Z})$ is that any G-translate of a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set is again $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable. In particular, by reasoning as in Remark **??** we obtain that for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-formula $\vartheta(x\,;\bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X};\bar{b})] \quad = \quad \vartheta(\mathcal{X};g\,\bar{b}).$$

Therefore, a type $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ is G-invariant if

$$p(x) \vdash x \in \mathcal{D} \;\;\Leftrightarrow\;\; p(x) \vdash x \in g{\cdot}\mathcal{D} \;\text{ for every } \Delta^{\mathsf{B}}(\mathcal{Z})\text{-definable } \mathcal{D} \subseteq \mathcal{X} \text{ and } g \in G.$$

A set $\mathcal{D} \subseteq \mathcal{X}$ is generic under the action of G, or G-generic for short, if finitely many G-translates of $\mathcal{D}$ cover $\mathcal{X}$; we say *n-G-generic* if $\leq n$ translates suffices. Dually, we say that $\mathcal{D}$ is persistent under the action of G, or G-persistent for short, if the intersection of any finitely many G-translates of $\mathcal{D}$ is nonempty; we say *n-G-persistent* when the request is limited to $\leq n$ translates. When $\mathcal{X}$ and/or $\mathcal{Z}$ are not clear from the context, we say that these notions are relative to $\mathcal{X}$ and $\mathcal{Z}$.

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

⚠ The terminology above is non-standard. In [1] the authors write *quasi-non-dividing* for *persistent* under the action of $\mathrm{Aut}(\mathcal{U}/A)$. Their terminology has good motivations, but it would be a mouthful if adapted to our context. In topological dynamics similar notions have been introduced with different terminology: *syndetic* corresponds to *generic* and *thick* corresponds to *persistent*.

2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is persistent under the action of $G = \mathrm{Aut}(\mathcal{U}/A)$ relative to any $\mathcal{X} \supseteq A^x$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\mathrm{Aut}(\mathcal{U}/A)$-translate of $\varphi(x)$.

Notation: for $\mathcal{D} \subseteq \mathcal{U}^x$ and $H \subseteq G$ we write $H\,\mathcal{D}$ for $\{h\mathcal{D} : h \in H\}$.

In this chapter many proofs require some juggling with negations as epitomized by the following fact.

3 Fact The following are equivalent

1. $\mathcal{D}$ is not G-generic
2. $\neg\mathcal{D}$ is G-persistent.

Proof. Immediate by spelling out the definitions

1. there are no finite $H \subseteq G$ such that $\mathcal{X} \subseteq \bigcup H\,\mathcal{D}$.
2. $\varnothing \neq \mathcal{X} \cap \big(\bigcap H\neg\mathcal{D}\big)$ for every finite $H \subseteq G$. □

Define the following type

$$\gamma_G(x) \quad = \quad \{\vartheta(x) \in \Delta^{\mathsf{B}}(\mathcal{Z}) : \vartheta(x) \text{ is } G\text{-generic}\}$$

4 Corollary Let $p(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ be such that $\gamma_G(x) \cup p(x)$ is finitely satisfiable in $\mathcal{X}$. Then $p(x)$ is G-persistent.

Proof. Let $\vartheta(x)$ be a conjunction of formulas in $p(x)$. As $\gamma_G(x)$ is finitely satisfiable in $\vartheta(\mathcal{X})$, it cannot be that $\neg\vartheta(x)$ is G-generic. From Fact 3, we obtain that $\vartheta(x)$ is G-persistent. □

The converse implication holds for complete types.

5 Theorem Let $p(x) \in S_\Delta(\mathcal{Z})$. Then the following are equivalent

1. $p(x)$ is G-invariant and finitely satisfiable in $\mathcal{X}$
2. $p(x) \vdash \gamma_G(x)$
3. $p(x)$ is G-persistent.

Proof. 1⇒2. Let $H \subseteq G$ be finite such that $\mathcal{X} \subseteq \cup H\,\mathcal{D}$. By completeness and finite satisfiability, $p(x) \vdash x \in \cup H\,\mathcal{D}$. Again by completeness, $p(x) \vdash x \in h\,\mathcal{D}$ for some $h \in H$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

2⇒3. Let $\mathcal{D}$ be defined by a conjunction of formulas in $p(x)$. If $\mathcal{D}$ is not G-persistent then, by Fact 3, $\neg\mathcal{D}$ is G-generic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

3⇒1. First note that G-persistent types are finitely satisfiable in $\mathcal{X}$. Now, suppose $p(x)$ is not G-invariant. Then, by completeness, $p(x) \vdash \varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ for some $g \in G$. Clearly $\varphi(x\,;b) \wedge \neg\varphi(x\,;gb)$ is not 2-G-persistent as it is inconsistent with its g-translate. □

6 Corollary The following are equivalent for every $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set $\mathcal{D}$

1. $\gamma_G(x) \vdash x \in \mathcal{D}$
2. $p(x) \vdash x \in \mathcal{D}$ for every G-persistent $p(x) \in S_\Delta(\mathcal{Z})$.

Proof. 1⇒2. This is an immediate consequence of Theorem 5.

2⇒1. Suppose $\gamma_G(x) \nvdash x \in \mathcal{D}$. Then there is a type $p(x) \in S_\Delta(\mathcal{Z})$ consistent with $\gamma_G(x) \cup \{x \notin \mathcal{D}\}$. By Corollary 4 $p(x)$ is G-persistent. Then ¬2. □

The theorem yields a necessary condition for the existence of G-invariant global $\Delta^{\mathrm{B}}(\mathcal{Z})$-types.

7 Corollary If there exists a G-invariant type $p(x) \in S_\Delta(\mathcal{Z})$ finitely satisfiable in $\mathcal{X}$ then for every $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable set $\mathcal{D}$

1. $\mathcal{D}$ and $\neg\mathcal{D}$ are not both G-generic
2. if $\mathcal{D}$ is G-generic then it is G-persistent
3. $\gamma_G(x)$ is finitely satisfiable in $\mathcal{X}$.

Proof. Clearly, 1 and 2 are equivalent by Fact 3 and follow from 3. Finally, 3 is an immediate consequence of 2 of Theorem 5. □

The following theorem gives a necessary and sufficient condition for the existence of global G-invariant $\Delta^{\mathrm{B}}(\mathcal{Z})$-type. Ideally, we would like to have that every G-persistent $\Delta^{\mathrm{B}}(\mathcal{Z})$-type extends to a global persistent type. Unfortunately this is not true in general (it requires stronger assumptions, see Section 4). A set $\mathcal{D}$ is G-wide if every finite cover of $\mathcal{D}$ by $\Delta^{\mathrm{B}}(\mathcal{Z})$-definable sets contains a G-persistent set. In [1] a similar property is called *quasi-non-forking*. Our use of the term *wide* is consistent with [2], though we apply it to a narrow context. A type is G-wide if every conjunction of formulas in the type is G-wide.

8 Theorem Let $\mathcal{D}$ be a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set. Then the following are equivalent

1. $\gamma_G(x)$ is finitely satisfied in $\mathcal{D} \cap \mathcal{X}$
2. there exists a G-persistent type $p(x) \in S_\Delta(\mathcal{Z})$ containing $x \in \mathcal{D}$
3. $\mathcal{D}$ is G-wide.

Proof. 1⇒2. By Corollary 4, it suffices to pick any $p(x) \in S_\Delta(\mathcal{Z})$ containing $\gamma_G(x)$ and finitely satisfied in $\mathcal{D} \cap \mathcal{X}$.

2⇒1. By Theorem 5.

2⇒3. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets that cover $\mathcal{D}$. Pick $p(x)$ as in 2. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i. Then, by Theorem 5, $\neg\mathcal{C}_i$ is not G-generic. Therefore, by Fact 3, $\mathcal{C}_i$ is G-persistent.

3⇒2. Let $p(x)$ be maximal among the $\Delta^{\mathsf{B}}(\mathcal{Z})$-types that are finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$ and are such that $\vartheta(\mathcal{U}^x)$ is G-wide for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that $p(x)$ is a complete $\Delta^{\mathsf{B}}(\mathcal{Z})$-type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets $\mathcal{C}_1, \ldots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \smallsetminus \vartheta(\mathcal{U}^x)$ and such that no $\mathcal{C}_i$ is G-persistent. As $\mathcal{C}_1, \ldots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is G-invariant. This follows from completeness and Theorem 5. □

9 Corollary Let $\mathcal{D}$ be a G-wide $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set. Then $\mathcal{D} \cap g\cdot\mathcal{D}$ is G-wide for every $g \in G$.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be a G-persistent type such that $p(x) \vdash x \in \mathcal{D}$. By G-invariance $p(x) \vdash x \in g\cdot\mathcal{D}$. □

10 Fact Let $\mathcal{D}$ be a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set. The following are equivalent

1. $\mathcal{D}$ is G-wide;
2. every finite cover of $\mathcal{D}$ by $\Delta^{\pm}(\mathcal{Z})$-definable sets contains a G-persistent set.

Let $q(x) \subseteq \Delta^{\mathsf{B}}(\mathcal{Z})$ be G-invariant. We say that $q(x)$ is G-prime if for every $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set $\mathcal{D}$ and every $g \in G$ if $q(x) \vdash x \in (\mathcal{D} \cup g\cdot\mathcal{D})$ then $q(x) \vdash x \in \mathcal{D}$.

11 Proposition The type $\gamma_G(x)$ is G-prime.

Proof. Suppose $\gamma_G(x) \vdash x \in (\mathcal{D} \cup g\cdot\mathcal{D})$. Assume that $\gamma_G(x)$ is consistent, otherwise the claim is trivial. Let $p(x) \in S_\Delta(\mathcal{Z})$ be G-persistent. We claim that $p(x) \vdash x \in \mathcal{D}$. The proposition follows from the claim by Corollary 6. By completeness $p(x) \vdash x \in \mathcal{D}$ or $p(x) \vdash x \in g\cdot\mathcal{D}$. If the latter occurs, $p(x) \vdash x \in \mathcal{D}$ follows from invariance. □

2. Strong genericity

Unfortunately, G-genericy is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is strongly G-generic if for every finite $H \subseteq G$ the set $\bigcap H\mathcal{D}$ is generic (recall that $H\mathcal{D}$ stands for $\{h\cdot\mathcal{D} : h \in H\}$). Dually, we say that $\mathcal{D}$ is weakly G-persistent if for some finite $H \subseteq G$ the set $\cup H\,\mathcal{D}$ is persistent. Again, the same properties may be attributed to formulas and types.

12 Lemma The intersection of two strongly G-generic sets is strongly G-generic.

Proof. We may assume that all sets mentioned below are subsets of $\mathcal{X}$. Let $\mathcal{D}$ and $\mathcal{C}$ be strongly G-generic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K\,(\mathcal{C} \cap \mathcal{D})$ is G-generic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K\,\mathcal{C}$ and $\mathcal{D}' = \cap K\,\mathcal{D}$. Note that $\mathcal{C}'$ and $\mathcal{D}'$ are both strongly G-generic. In particular $\mathcal{X} = \cup H\,\mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\begin{aligned}
\cup H\,\mathcal{B} &= \cup H\Big[\mathcal{C}' \cap \mathcal{D}'\Big] \\
\cup H\,\mathcal{B} &\supseteq \cup H\Big[(\cap\, H\,\mathcal{C}') \cap \mathcal{D}'\Big] \\
&= (\cap\, H\,\mathcal{C}') \cap (\cup\, H\,\mathcal{D}') \\
&= \cap H\,\mathcal{C}'
\end{aligned}$$

As $\mathcal{C}'$ is strongly G-generic, $\cap H\,\mathcal{C}'$ is G-generic. Therefore $\cup H\,\mathcal{B}$ is also G-generic. The G-genericity of $\mathcal{B}$ follows. □

Define the following type

$$^{\mathrm{s}}\gamma_G(x) \;=\; \{\vartheta(x) \in \Delta^{\mathrm{B}}(\mathcal{Z}) : \vartheta(x) \text{ is strong } G\text{-generic}\}$$

13 Corollary The type $^{\mathrm{s}}\gamma_G(x)$ is finitely satisfiable in $\mathcal{X}$, strongly G-generic, and G-invariant.

Proof. The strong G-genericity is an immediate consequence of Lemma 12. The finite satisfiability is a consequence of G-genericity. As for invariance, note that any translate of a strongly G-generic formula is also strongly G-generic. □

14 Corollary Let $p(x) \subseteq L(\mathcal{U})$ be such that $^{\mathrm{s}}\gamma(x) \cup p(x)$ is finitely satisfiable in $\mathcal{X}$. Then $p(x)$ is weakly G-persistent.

Proof. Similar to Corollary 4. Let $\vartheta(x)$ be a conjunction of formulas in $p(x)$. As $^{\mathrm{s}}\gamma(\mathcal{X})$ is finitely satisfiable in $\vartheta(\mathcal{X})$, it cannot be that $\neg\vartheta(x)$ is strongly G-generic. From Fact 3, we obtain that $\neg\vartheta(x)$ not being strongly G-generic is equivalent to $\vartheta(x)$ being weakly G-persistent. □

3. The diameter of a Lascar type

As an application we prove an interesting property of the Lascar types. Let $\mathcal{L}(a/A)$, the set of tuples with the same Lascar strong type as a over A. This set is the union of a chain of type-definable sets of the form $\{x \;:\; d_A(a,x) \leq n\}$, where d_A is the distance in the Lascar graph. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter.

It is convenient to address the problem in more general terms. We work under Assumption 1 and also assume that G acts transitively on $\mathcal{X}$ i.e. $G\,a = \mathcal{X}$ for every $a \in \mathcal{X}$. Let $K \subseteq G$ be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugacy invariant i.e. $gKg^{-1} = K$ for every $g \in G$

We define a discrete metric on $\mathcal{X}$. For $a, b \in \mathcal{X}$ let $d(a,b)$ be the minimal n such that $a \in K^n b$. This defines a metric which is G-invariant by 2. The diameter of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a,b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for $\mathcal{X}$ to have finite diameter. The notions introduced in Section 2 offer some hint.

15 Proposition If $\mathcal{X}$ has a weakly persistent subset of finite diameter, then $\mathcal{X}$ itself has finite diameter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter n. Let $H \subseteq G$ be finite such that $\cup H\, \mathcal{C}$ is persistent. We claim that also $\cup H\, \mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k, \in H$. Now, let hb and kc, for some $h, k, \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H\, \mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of $\mathcal{C}$,

$$\begin{aligned} d(hb, kc) &\leq d(hb, ha) + d(ha, ka) + d(ka, kc) \\ &\leq n + m + n. \end{aligned}$$

This proves that $\cup H\, \mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that $\mathcal{C}$ itself is persistent.

By the transitivity of the action, any two elements of $\mathcal{X}$ are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By persistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha, ka) &\leq d(ha, c) + d(c, d) + d(d, ka) \\ &\leq n + n + n. \end{aligned}$$

Therefore the diameter of $\mathcal{X}$ does not exceed $3n$. □

16 Theorem Suppose that $\mathcal{X}$ and the sets $\mathcal{X}_n = K^n a$, for some $a \in \mathcal{X}$, are type-definable. Then $\mathcal{X}$ has finite diameter.

Proof. By Proposition 15, it suffices to prove that $\mathcal{X}_n$ is weakly persistent. By Corollary 14 it suffices to show that for some n the type ${}^{\mathrm{s}}\gamma_G(x)$ is finitely satisfied in $\mathcal{X}_n$. Suppose not. Let $\psi_n(x) \in {}^{\mathrm{s}}\gamma_G$ be a formula that is not satisfied in $\mathcal{X}_n$. Then the type $p(x) = \{\psi_n(x) : n \in \omega\}$ is finitely satisfied in $\mathcal{X}$. From the type-definablity of $\mathcal{X}$ it follows that $p(x)$ has a realization in $\mathcal{X}$. As this realization belongs to some $\mathcal{X}_n$ we contradict the definition of $\psi_n(x)$. □

17 Example Let $K \subseteq \mathrm{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A. Then the group G generated by K is $\mathrm{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. Let $\Delta = L_{xz}$ and $\mathcal{Z} = \mathcal{U}^z$. Then $d(a,b)$ coincides with the distance in the Lascar graph. The sets $K^n \cdot a = \{x : d(x,a) \leq n\}$ are type definable. Then from Theorem 16 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

4. A tamer landscape

Under suitable assumptions – e.g. stability – some notion introduced in this chapter coalesce, and we are left with a tamer landscape. We prove the following theorem.

18 Theorem The following are equivalent

1. G-persistent $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets are G-wide
2. G-generic $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets are closed under intersection
3. G-generic $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets are strongly G-generic
4. weakly persistent $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets are G-persistent.

Proof. 2⇔3⇔4. Clear.

1⇒2. Let $\mathcal{C}$ and $\mathcal{D}$ be G-generic $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not G-generic. Then $\neg(\mathcal{C} \cap \mathcal{D})$ is G-persistent. By 1 and Theorem 8 there is a G-invariant global $\Delta^{\mathsf{B}}(\mathcal{Z})$-type $p(x)$ containing $x \notin \mathcal{C} \cap \mathcal{D}$. By completeness either $p(x) \vdash x \notin \mathcal{C}$ or $p(x) \vdash x \notin \mathcal{D}$. This is a contradiction because by Theorem 5 $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$.

4⇒1. Note that, by 3, the type ${}^{\mathsf{s}}\gamma_G(x)$ coincides with $\gamma_G(x)$, in particular $\gamma_G(x)$ is finitely satisfied in $\mathcal{X}$. Let $\mathcal{D}$ be a G-persistent $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set. We show that $\gamma_G(x) = {}^{\mathsf{s}}\gamma_G(x)$ is finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$. Then, by 4 and Corollary 14, any global extension of $\gamma_G(x) \cup \{x \in \mathcal{D}\}$ witness 2 of Theorems 8. Suppose not, then $\gamma_G(x) \vdash x \notin \mathcal{D}$. Therefore $\neg\mathcal{D}$ is G-generic, contradicting the consistency of ${}^{\mathsf{s}}\gamma_G(x)$. □

19 Remark Assume that the equivalent conditions in Theorem 18 hold. Then the types $\gamma_G(x)$ and ${}^{\mathsf{s}}\gamma_G(x)$ coincide, and therefore G-invariant global types exist. It is also worth mentioning that every positive Boolean combination of G-generic sets is G-generic.

5. The action of normal subgroups

Let $H \trianglelefteq G$. The following is an immediate consequence of normality.

20 Remark For every $\mathcal{D} \subseteq \mathcal{U}^x$ and every $g \in G$

$$\mathcal{D} \text{ is } H\text{-foo} \quad \Leftrightarrow \quad g{\cdot}\mathcal{D} \text{ is } H\text{-foo},$$

where *foo* can be replaced by *generic, invariant, persistent, wide.* In particular, the type $\gamma_H(x)$ is G-invariant.

Recall that if $\gamma_H(x)$ is finitely satisfiable in $\mathcal{X}$, then H-generic sets are H-wide, cf. Theorem 8. As it happens, we can slightly strengthen this fact.

21 Proposition Assume that $\gamma_H(x)$ is consistent. Let $\mathcal{D}$ be a $\Delta^{\mathsf{B}}(\mathcal{Z})$-definable set. Then if $\mathcal{D}$ is G-generic it is also H-wide.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be consistent with $\gamma_H(x)$. As $\mathcal{D}$ is G-generic, by completeness $p(x) \vdash x \in g{\cdot}\mathcal{D}$ for some $g \in G$. Equivalently, $g^{-1}{\cdot}\, p \vdash x \in \mathcal{D}$. As $p(x)$ is

H-persistent, by Remark 20 $g^{-1}\cdot p(x)$ is also H-persistent. Then the proposition follows from Theorem 8. □

6. Definable groups

In this section we assume that $\mathcal{Z}$ and $\mathcal{X}$ are type-definable over some set of parameters A. Moreover we assume that $\mathcal{Z}$ is a group that act on $\mathcal{X}$. The group operations and the group action are assumed definable over A. We use the symbol $\cdot$ for both the group multiplication and the group action.

Let $\Psi \subseteq L_x(\mathcal{U})$ be some small set of formulas. In this section Δ contains formulas $\varphi(x\,;z)$ of the form $\psi(z^{-1}\cdot x)$ for $\psi(x) \in \Psi$. The sets $\varphi(\mathcal{X}\,;\mathcal{Z})$ are $\mathcal{Z}$-invariant. We write 1 for the identity of $\mathcal{Z}$. If $\varphi(\mathcal{X}\,;1) \in \Delta^{\mathrm{B}}(1)$ then $\varphi(\mathcal{X}\,;g) = g\cdot\varphi(\mathcal{X}\,;1)$.

The following auxiliary structure is useful. Let $\mathcal{U}^\Delta = \langle\mathcal{X}\,;\mathcal{Z}\rangle$ be a 2-sorted structure whose signature L^Δ contains only relation symbols for every formula $\varphi(x\,;z) \in \Delta$. As there is little risk of confusion, these relations symbols are also denoted by $\varphi(x\,;z)$. As $\mathcal{X}$ and $\mathcal{Z}$ are assumed to be type-definable, $\mathcal{U}^\Delta$ is a saturated L^Δ-structure.

In this section $G = \mathrm{Aut}(\mathcal{U}^\Delta)$.

22 Remark It is worth noticing that automorphisms of $\mathcal{U}^\Delta$ do not preserve the group operations nor the group action. However, if $\mathcal{D} = \varphi(\mathcal{X}\,;1)$ and $g \in \mathcal{Z}$, then $f(g)\cdot\mathcal{D} = f[g\cdot\mathcal{D}]$ for any $f \in \mathrm{Aut}(\mathcal{U}^\Delta)$.

To each $h \in \mathcal{Z}$ we associate the L^Δ-automorphism $\langle a\,;g\rangle \mapsto \langle h\cdot a\,;h\cdot g\rangle$. Therefore $\mathcal{Z}$ is, up to isomorphism, a subgroup of G. In fact, it is a normal subgroup. Note that, for any $g \in \mathcal{Z}$, the orbit of $\varphi(\mathcal{X}\,;g)$ under the action of $\mathcal{Z}$ is $\{\varphi(\mathcal{X}\,;h) : h \in \mathcal{Z}\}$. Therefore it coincides with the orbit under the action of G (it cannot get any larger). We conclude that for formulas in $\Delta^{\pm}(\mathcal{Z})$ the notions of generic and persistent under the two actions coincide.

We also consider the action some other normal subgroup $H \trianglelefteq G$. In the applications H will be either $\mathrm{Autf}(\mathcal{U}^\Delta)$ or $\mathrm{Aut}(\mathcal{U}^\Delta/M)$.

23 Proposition Let $\mathcal{D}$ be a Δ^{B}-definable set. Assume that $\gamma_H(x)$ is consistent. Then 1⇒2 holds, where

1. $\mathcal{D}$ is $\mathcal{Z}$-generic
2. $g\cdot\mathcal{D}$ is H-wide for every $g \in \mathcal{Z}$.

Under the assumption of stability and with $H = \mathrm{Autf}(\mathcal{U}^\Delta)$ a stronger claim obtains – the consistency of $\gamma_H(x)$ is guaranteed, and also the converse implication holds

Proof. Let g be given. If $\mathcal{D}$ is $\mathcal{Z}$-generic, then so is $g\cdot\mathcal{D}$. Then $g\cdot\mathcal{D}$ is, a fortiori, G-generic. Therefore 2 follows from Proposition 21. □

We write $(g)_H$ for the H-orbit of g, that is, the set $\{f(g) \;:\; f \in H\}$.

24 Proposition Let $\vartheta(x\,;z_1,\dots,z_n)$ be a Boolean combination of formulas $\varphi_i(x\,;z_i)$ for some $\varphi_i(x\,;z)\in\Delta$. Then for every $h_i\in(g_i)_H$ the following are equivalent

1. $\vartheta(x\,;g_1,\dots,g_n)$ is H-wide
2. $\vartheta(x\,;h_1,\dots,h_n)$ is H-wide.

Proof. Let $f_i\in H$ be such that $h_i\in f_i(g_i)$. Without loss of generality we can assume that $\vartheta(x\,;g_1,\dots,g_n)$ is the conjunction of the formulas $\varphi_i(x\,;g_i)$. Let $\mathcal{C}_i=\varphi_i(\mathcal{X}\,;1)$. Then 1 says that $\mathcal{C}=g_1\cdot\mathcal{C}_1\cap\dots\cap g_n\cdot\mathcal{C}_n$ is H-wide. Let $f_i\in H$ be such that $h_i=f_i(g_i)$. Then, by Corollary 9 also the intersection of the sets $f_i[\mathcal{C}]$ is H-wide. A fortiori the intersection of the sets $f_i[g_i\cdot\mathcal{C}_i]$ is H-wide. Then 1⇒2 follows from Remark 22. By symmetry, this proves the equivalence. □

If $\mathcal{A}\subseteq\mathcal{Z}$, write $\langle\mathcal{A}\rangle$ for the subgroup generated by $\mathcal{A}$.

25 Proposition Let $\vartheta(x\,;z_1,\dots,z_n)$ be a Boolean combination of formulas $\varphi_i(x\,;z_i)$ for some $\varphi_i(x\,;z)\in\Delta$. Let $g\in\mathcal{Z}$ be arbitrary. Assume that $\vartheta(x\,;1,\dots,1)$ is H-wide. Then $\vartheta(x\,;h_1,\dots,h_n)$ is H-wide for every $h_i\in\big\langle\,(g)_H^{-1}\cdot(g)_H\;\cup\;(g)_H\cdot(g)_H^{-1}\big\rangle$.

Proof. We proceed by induction on the number of factors $a^{-1}\cdot b$ or $a\cdot b^{-1}$, for some $a,b\in(g)_H$, that occur in $h_1,\dots,h_n$. Without loss of generality we can assume that $\vartheta(x\,;z_1,\dots,z_n)$ is the conjunction of the formulas $\varphi_i(x\,;z_i)$ for some $\varphi_i(x\,;z)\in\Delta$. Let $\mathcal{C}_i=\varphi_i(\mathcal{X}\,;1)$. Assume inductively that $h_1\cdot\mathcal{C}_1\cap\dots\cap h_n\cdot\mathcal{C}_n$ is H-wide. Pick two arbitrary $a,b\in(g)_H$. Then

$$a\cdot\mathcal{C}_1\ \cap\ a\cdot h_1^{-1}\cdot h_2\cdot\mathcal{C}_2\ \cap\ \dots\dots\ \cap\ a\cdot h_1^{-1}\cdot h_n\cdot\mathcal{C}_n \text{ is } H\text{-wide.}$$

By Proposition 24, in this intersection we can replace $a\cdot\mathcal{C}_1$ by $b\cdot\mathcal{C}_1$. Then finally

$$h_1\cdot a^{-1}\cdot b\cdot\mathcal{C}_1\ \cap\ h_2\cdot\mathcal{C}_2\ \cap\ \dots\dots\ \cap\ h_n\cdot\mathcal{C}_n \text{ is } H\text{-wide.}$$

A similar argument applies to $a\cdot b^{-1}$. □

7. Notes and references

[1] Artem Chernikov and Itay Kaplan, *Forking and dividing in* NTP_2 *theories*, J. Symbolic Logic **77** (2012), 1–20.

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