

Group actions on models

ABSTRACT. A set is *strongly generic* if the intersection of any finitely many of its translations is generic.

Theorem 17 shows that the condition *strongly generic* = *generic* is robust. It might be of some interest (it is reminiscent of *forking* = *dividing*). Does it have interesting examples?

Proposition 24 looks interesting. Can it be strengthened to obtain a type-definable group?

The connections with topological dynamics have not been explored :(

1. The dual perspective on invariance

The notions in this section are well-known but sometimes the terminology differs.

In this chapter $\Delta \subseteq L_{\mathcal{X}\mathcal{Z}}(\mathcal{U})$. Let $\mathcal{Z} \subseteq \mathcal{U}^{\mathcal{Z}}$. We write $\Delta(\mathcal{Z})$ for the set of formulas of the form $\varphi(\mathcal{X}; b)$ for some $\varphi(\mathcal{X}; z) \in \Delta$ and some $b \in \mathcal{Z}$. We write $\Delta^{\pm}(\mathcal{Z})$ for the set of formulas in $\Delta(\mathcal{Z})$ or negation thereof. Furthermore, we write $S_{\Delta}(\mathcal{Z})$ for the set of complete $\Delta^{\pm}(\mathcal{Z})$ -types.

We write $\Delta^{\mathbb{B}}(\mathcal{Z})$ for the set of Boolean combinations of formulas in $\Delta(\mathcal{Z})$.

1 Assumption Let G be a group that acts on some sets $\mathcal{X} \subseteq \mathcal{U}^{\mathcal{X}}$ and $\mathcal{Z} \subseteq \mathcal{U}^{\mathcal{Z}}$. We require that for every $\varphi(\mathcal{X}; z) \in \Delta$ the set $\varphi(\mathcal{X}; \mathcal{Z})$ is invariant under the action of G . For convenience, we will assume that G is the identity outside \mathcal{X} and \mathcal{Z} .

When $p(\mathcal{X}) \subseteq \Delta^{\mathbb{B}}(\mathcal{Z})$ and $\mathcal{D} \subseteq \mathcal{U}^{\mathcal{X}}$ we write $p(\mathcal{X}) \vdash \mathcal{X} \in \mathcal{D}$ if the inclusion $\psi(\mathcal{X}) \subseteq \mathcal{D}$ for some $\psi(\mathcal{X})$ that is conjunctions of formulas in $p(\mathcal{X})$.

Let $\mathcal{D} \subseteq \mathcal{U}^{\mathcal{Z}}$. We say that \mathcal{D} is **invariant** under the action of G , or **G-invariant**, if \mathcal{D} is fixed setwise by G . That is, $g\mathcal{D} = \mathcal{D}$ for every $g \in G$. Yet in other words, if

$$\text{is1.} \quad a \in \mathcal{D} \leftrightarrow ga \in \mathcal{D} \quad \text{for every } a \in \mathcal{Z} \text{ and every } g \in G.$$

A formula is G -invariant if the set it defines is G -invariant. We say that $p(\mathcal{X}) \subseteq \Delta^{\mathbb{B}}(\mathcal{Z})$ is **invariant** under the action of G , or **G-invariant**, if for every $\Delta^{\mathbb{B}}(\mathcal{Z})$ -formula $\vartheta(\mathcal{X}; \bar{a})$.

$$\text{it1.} \quad \vartheta(\mathcal{X}; \bar{a}) \in p \Leftrightarrow \vartheta(\mathcal{X}; g\bar{a}) \in p \quad \text{for every } g \in G.$$

It should be clear that invariant under the action of $\text{Aut}(\mathcal{U}/A)$ coincides with invariant over A and Lascar invariant over A coincides with invariant under the action of $\text{Autf}(\mathcal{U}/A)$.

Note that $p(\mathcal{X})$ is G -invariant exactly when the sets $\mathcal{D}_{p, \vartheta} = \{b : \vartheta(\mathcal{X}; b) \in p\} \subseteq \mathcal{U}^{\mathcal{Z}}$ are. Now we would like to discuss invariance using sets $\mathcal{D} \subseteq \mathcal{U}^{\mathcal{X}}$.

An immediate consequence of the invariance of $\varphi(\mathcal{X}; \mathcal{Z})$ is that any G -translate of a $\Delta^{\mathbb{B}}(\mathcal{Z})$ -definable set is again $\Delta^{\mathbb{B}}(\mathcal{Z})$ -definable. In particular for every $\Delta^{\mathbb{B}}(\mathcal{Z})$ -formula $\vartheta(\mathcal{X}; \bar{b})$ and every $g \in G$

$$g[\vartheta(\mathcal{X}; \bar{b})] = \vartheta(\mathcal{X}; g\bar{b}).$$

Therefore, a type $p(x) \subseteq \Delta^B(\mathcal{Z})$ is G -invariant if

$$p(x) \vdash x \in \mathcal{D} \Leftrightarrow p(x) \vdash x \in g \cdot \mathcal{D} \text{ for every } \Delta^B(\mathcal{Z})\text{-definable } \mathcal{D} \subseteq \mathcal{X} \text{ and } g \in G.$$

A set $\mathcal{D} \subseteq \mathcal{X}$ is **generic** under the action of G , or **G -generic** for short, if finitely many G -translates of \mathcal{D} cover \mathcal{X} ; we say **n - G -generic** if $\leq n$ translates suffices. Dually, we say that \mathcal{D} is **persistent** under the action of G , or **G -persistent** for short, if the intersection of any finitely many G -translates of \mathcal{D} is nonempty; we say **n - G -persistent** when the request is limited to $\leq n$ translates. When \mathcal{X} and/or \mathcal{Z} are not clear from the context, we say that these notions are **relative** to \mathcal{X} and \mathcal{Z} .

The same properties may be attributed to formulas (as these are identified with the set they define). When these properties are attributed to a type $p(x)$, we understand that they hold for every conjunction of formulas in $p(x)$.

⚠ The terminology above is non-standard. In [1] the authors write *quasi-non-dividing* for *persistent* under the action of $\text{Aut}(\mathcal{U}/A)$. Their terminology has good motivations, but it would be a mouthful if adapted to our context. In topological dynamics similar notions have been introduced with different terminology: *syndetic* corresponds to *generic* and *thick* corresponds to *persistent*.

2 Example If $p(x) \subseteq L(\mathcal{U})$ is finitely satisfiable in A then $p(x)$ is persistent under the action of $G = \text{Aut}(\mathcal{U}/A)$ relative to any $\mathcal{X} \supseteq A^x$. In fact, the same $a \in A^x$ that satisfies $\varphi(x)$ also satisfies every $\text{Aut}(\mathcal{U}/A)$ -translate of $\varphi(x)$.

Notation: for $\mathcal{D} \subseteq \mathcal{U}^x$ and $H \subseteq G$ we write **$H\mathcal{D}$** for $\{h\mathcal{D} : h \in H\}$.

In this chapter many proofs require some juggling with negations as epitomized by the following fact.

3 Fact The following are equivalent

1. \mathcal{D} is not G -generic
2. $\neg \mathcal{D}$ is G -persistent.

Proof. Immediate by spelling out the definitions

1. there are no finite $H \subseteq G$ such that $\mathcal{X} \subseteq \bigcup H\mathcal{D}$.
2. $\emptyset \neq \mathcal{X} \cap (\bigcap H\neg \mathcal{D})$ for every finite $H \subseteq G$. □

Define the following type

$$\gamma_G(x) = \{\vartheta(x) \in \Delta^B(\mathcal{Z}) : \vartheta(x) \text{ is } G\text{-generic}\}$$

4 Corollary Let $p(x) \subseteq \Delta^B(\mathcal{Z})$ be such that $\gamma_G(x) \cup p(x)$ is finitely satisfiable in \mathcal{X} . Then $p(x)$ is G -persistent.

Proof. Let $\vartheta(x)$ be a conjunction of formulas in $p(x)$. As $\gamma_G(x)$ is finitely satisfiable in $\vartheta(\mathcal{X})$, it cannot be that $\neg \vartheta(x)$ is G -generic. From Fact 3, we obtain that $\vartheta(x)$ is G -persistent. □

The converse implication holds for complete types.

5 Theorem Let $p(x) \in S_\Delta(\mathbb{Z})$. Then the following are equivalent

1. $p(x)$ is G -invariant and finitely satisfiable in \mathcal{X}
2. $p(x) \vdash \gamma_G(x)$
3. $p(x)$ is G -persistent.

Proof. $1 \Rightarrow 2$. Let $H \subseteq G$ be finite such that $\mathcal{X} \subseteq \bigcup H \mathcal{D}$. By completeness and finite satisfiability, $p(x) \vdash x \in \bigcup H \mathcal{D}$. Again by completeness, $p(x) \vdash x \in h \mathcal{D}$ for some $h \in H$. Finally, by invariance, $p(x) \vdash x \in \mathcal{D}$.

$2 \Rightarrow 3$. Let \mathcal{D} be defined by a conjunction of formulas in $p(x)$. If \mathcal{D} is not G -persistent then, by Fact 3, $\neg \mathcal{D}$ is G -generic. By 2, $p(x) \vdash x \notin \mathcal{D}$, a contradiction.

$3 \Rightarrow 1$. First note that G -persistent types are finitely satisfiable in \mathcal{X} . Now, suppose $p(x)$ is not G -invariant. Then, by completeness, $p(x) \vdash \varphi(x; b) \wedge \neg \varphi(x; gb)$ for some $g \in G$. Clearly $\varphi(x; b) \wedge \neg \varphi(x; gb)$ is not 2- G -persistent as it is inconsistent with its g -translate. \square

6 Corollary The following are equivalent for every $\Delta^B(\mathbb{Z})$ -definable set \mathcal{D}

1. $\gamma_G(x) \vdash x \in \mathcal{D}$
2. $p(x) \vdash x \in \mathcal{D}$ for every G -persistent $p(x) \in S_\Delta(\mathbb{Z})$.

Proof. $1 \Rightarrow 2$. This is an immediate consequence of Theorem 5.

$2 \Rightarrow 1$. Suppose $\gamma_G(x) \not\vdash x \in \mathcal{D}$. Then there is a type $p(x) \in S_\Delta(\mathbb{Z})$ consistent with $\gamma_G(x) \cup \{x \notin \mathcal{D}\}$. By Corollary 4 $p(x)$ is G -persistent. Then $\neg 2$. \square

The theorem yields a necessary condition for the existence of G -invariant global $\Delta^B(\mathbb{Z})$ -types.

7 Corollary If there exists a G -invariant type $p(x) \in S_\Delta(\mathbb{Z})$ finitely satisfiable in \mathcal{X} then for every $\Delta^B(\mathbb{Z})$ -definable set \mathcal{D}

1. \mathcal{D} and $\neg \mathcal{D}$ are not both G -generic
2. if \mathcal{D} is G -generic then it is G -persistent
3. $\gamma_G(x)$ is finitely satisfiable in \mathcal{X} .

Proof. Clearly, 1 and 2 are equivalent by Fact 3 and follow from 3. Finally, 3 is an immediate consequence of 2 of Theorem 5. \square

The following theorem gives a necessary and sufficient condition for the existence of global G -invariant $\Delta^B(\mathbb{Z})$ -type. Ideally, we would like to have that every G -persistent $\Delta^B(\mathbb{Z})$ -type extends to a global persistent type. Unfortunately this is not true in general (it requires stronger assumptions, see Section 4). A set \mathcal{D} is **G -wide** if every finite cover of \mathcal{D} by $\Delta^B(\mathbb{Z})$ -definable sets contains a G -persistent set. In [1] a similar property is called *quasi-non-forking*. Our use of the term *wide* is consistent with [2], though we apply it to a narrow context. A type is G -wide if every conjunction of formulas in the type is G -wide.

8 Theorem Let \mathcal{D} be a $\Delta^B(\mathcal{Z})$ -definable set. Then the following are equivalent

1. $\gamma_G(x)$ is finitely satisfied in $\mathcal{D} \cap \mathcal{X}$
2. there exists a G -persistent type $p(x) \in S_\Delta(\mathcal{Z})$ containing $x \in \mathcal{D}$
3. \mathcal{D} is G -wide.

Proof. $1 \Rightarrow 2$. By Corollary 4, it suffices to pick any $p(x) \in S_\Delta(\mathcal{Z})$ containing $\gamma_G(x)$ and finitely satisfied in $\mathcal{D} \cap \mathcal{X}$.

$2 \Rightarrow 1$. By Theorem 5.

$2 \Rightarrow 3$. Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be $\Delta^B(\mathcal{Z})$ -definable sets that cover \mathcal{D} . Pick $p(x)$ as in 2. By completeness, $p(x) \vdash x \in \mathcal{C}_i$ for some i . Then, by Theorem 5, $\neg \mathcal{C}_i$ is not G -generic. Therefore, by Fact 3, \mathcal{C}_i is G -persistent.

$3 \Rightarrow 2$. Let $p(x)$ be maximal among the $\Delta^B(\mathcal{Z})$ -types that are finitely satisfiable in $\mathcal{X} \cap \mathcal{D}$ and are such that $\vartheta(\mathcal{U}^x)$ is G -wide for every $\vartheta(x)$ that is conjunction of formulas in $p(x)$. We claim that $p(x)$ is a complete $\Delta^B(\mathcal{Z})$ -type. Suppose for a contradiction that $\vartheta(x), \neg \vartheta(x) \notin p$. By maximality there is some formula $\psi(x)$, a conjunction of formulas in $p(x)$, and some $\Delta^B(\mathcal{Z})$ -definable sets $\mathcal{C}_1, \dots, \mathcal{C}_n$ that cover both $\psi(\mathcal{U}^x) \cap \vartheta(\mathcal{U}^x)$ and $\psi(\mathcal{U}^x) \setminus \vartheta(\mathcal{U}^x)$ and such that no \mathcal{C}_i is G -persistent. As $\mathcal{C}_1, \dots, \mathcal{C}_n$ cover $\psi(\mathcal{U}^x)$ this is a contradiction. It is only left to show that $p(x)$ is G -invariant. This follows from completeness and Theorem 5. \square

9 Corollary Let \mathcal{D} be a G -wide $\Delta^B(\mathcal{Z})$ -definable set. Then $\mathcal{D} \cap g \cdot \mathcal{D}$ is G -wide for every $g \in G$.

Proof. Let $p(x) \in S_\Delta(\mathcal{Z})$ be a G -persistent type such that $p(x) \vdash x \in \mathcal{D}$. By G -invariance $p(x) \vdash x \in g \cdot \mathcal{D}$. \square

Let $q(x) \subseteq \Delta^B(\mathcal{Z})$ be G -invariant. We say that $q(x)$ is **G -prime** if for every $\Delta^B(\mathcal{Z})$ -definable set \mathcal{D} and every $g \in G$ if $q(x) \vdash x \in (\mathcal{D} \cup g \cdot \mathcal{D})$ then $q(x) \vdash x \in \mathcal{D}$.

10 Proposition The type $\gamma_G(x)$ is G -prime.

Proof. Suppose $\gamma_G(x) \vdash x \in (\mathcal{D} \cup g \cdot \mathcal{D})$. Assume that $\gamma_G(x)$ is consistent, otherwise the claim is trivial. Let $p(x) \in S_\Delta(\mathcal{Z})$ be G -persistent. We claim that $p(x) \vdash x \in \mathcal{D}$. The proposition follows from the claim by Corollary 6. By completeness $p(x) \vdash x \in \mathcal{D}$ or $p(x) \vdash x \in g \cdot \mathcal{D}$. If the latter occurs, $p(x) \vdash x \in \mathcal{D}$ follows from invariance. \square

2. Strong genericity

Unfortunately, G -genericity is not preserved under intersection. To obtain closure under intersection, we need to push the concept to a higher level of complexity.

A set $\mathcal{D} \subseteq \mathcal{U}^x$ is **strongly G -generic** if for every finite $H \subseteq G$ the set $\bigcap H \mathcal{D}$ is generic (recall that $H \mathcal{D}$ stands for $\{h \cdot \mathcal{D} : h \in H\}$). Dually, we say that \mathcal{D} is **weakly G -persistent** if for some finite $H \subseteq G$ the set $\bigcup H \mathcal{D}$ is persistent. Again, the same properties may be attributed to formulas and types.

11 Lemma The intersection of two strongly G -generic sets is strongly G -generic.

Proof. We may assume that all sets mentioned below are subsets of \mathcal{X} . Let \mathcal{D} and \mathcal{C} be strongly G -generic and let $K \subseteq G$ be an arbitrary finite set. It suffices to prove that $\mathcal{B} = \cap K(\mathcal{C} \cap \mathcal{D})$ is G -generic. Clearly $\mathcal{B} = \mathcal{C}' \cap \mathcal{D}'$, where $\mathcal{C}' = \cap K\mathcal{C}$ and $\mathcal{D}' = \cap K\mathcal{D}$. Note that \mathcal{C}' and \mathcal{D}' are both strongly G -generic. In particular $\mathcal{X} = \cup H\mathcal{D}'$ for some finite $H \subseteq G$. Now, from

$$\begin{aligned} \cup H\mathcal{B} &= \cup H[\mathcal{C}' \cap \mathcal{D}'] \\ \cup H\mathcal{B} &\supseteq \cup H[(\cap H\mathcal{C}') \cap \mathcal{D}'] \\ &= (\cap H\mathcal{C}') \cap (\cup H\mathcal{D}') \\ &= \cap H\mathcal{C}' \end{aligned}$$

As \mathcal{C}' is strongly G -generic, $\cap H\mathcal{C}'$ is G -generic. Therefore $\cup H\mathcal{B}$ is also G -generic. The G -genericity of \mathcal{B} follows. \square

Define the following type

$${}^s\gamma_G(x) = \{\vartheta(x) \in \Delta^{\mathbb{P}}(\mathcal{Z}) : \vartheta(x) \text{ is strong } G\text{-generic}\}$$

12 Corollary The type ${}^s\gamma_G(x)$ is finitely satisfiable in \mathcal{X} , strongly G -generic, and G -invariant.

Proof. The strong G -genericity is an immediate consequence of Lemma 11. The finite satisfiability is a consequence of G -genericity. As for invariance, note that any translate of a strongly G -generic formula is also strongly G -generic. \square

13 Corollary Let $p(x) \subseteq L(\mathcal{U})$ be such that ${}^s\gamma(x) \cup p(x)$ is finitely satisfiable in \mathcal{X} . Then $p(x)$ is weakly G -persistent.

Proof. Similar to Corollary 4. Let $\vartheta(x)$ be a conjunction of formulas in $p(x)$. As ${}^s\gamma(\mathcal{X})$ is finitely satisfiable in $\vartheta(\mathcal{X})$, it cannot be that $\neg\vartheta(x)$ is strongly G -generic. From Fact 3, we obtain that $\neg\vartheta(x)$ not being strongly G -generic is equivalent to $\vartheta(x)$ being weakly G -persistent. \square

3. The diameter of a Lascar type

As an application we prove an interesting theorem of Newelski's on Lascar types. Let $\mathcal{L}(a/A)$, the set of tuples with the same Lascar strong type as a over A . This set is the union of a chain of type-definable sets of the form $\{x : d_A(a, x) \leq n\}$, where d_A is the distance in the Lascar graph. In this section we prove that $\mathcal{L}(a/A)$ is type-definable (if and) only this chain is finite. In other words, only if the connected component of a in the Lascar graph has finite diameter.

It is convenient to address the problem in more general terms. We work under Assumption 1 and also assume that G acts transitively on \mathcal{X} i.e. $Ga = \mathcal{X}$ for every $a \in \mathcal{X}$. Let $K \subseteq G$ be a set of generators that is

1. symmetric i.e. it contains the unit and is closed under inverse
2. conjugacy invariant i.e. $gKg^{-1} = K$ for every $g \in G$

We define a discrete metric on \mathcal{X} . For $a, b \in \mathcal{X}$ let $d(a, b)$ be the minimal n such that $a \in K^n b$. This defines a metric which is G -invariant by 2. The **diameter** of a set $\mathcal{C} \subseteq \mathcal{X}$ is the supremum of $d(a, b)$ for $a, b \in \mathcal{C}$.

We are interested in sufficient conditions for \mathcal{X} to have finite diameter. The notions introduced in Section 2 offer some hint.

14 Proposition If \mathcal{X} has a weakly persistent subset of finite diameter, then \mathcal{X} itself has finite diameter.

Proof. Let $\mathcal{C} \subseteq \mathcal{X}$ be a weakly persistent set of diameter n . Let $H \subseteq G$ be finite such that $\cup H \mathcal{C}$ is persistent. We claim that also $\cup H \mathcal{C}$ has finite diameter. Let $a \in \mathcal{C}$ be arbitrary. Let m be larger than $d(ha, ka)$ for all $h, k \in H$. Now, let hb and kc , for some $h, k \in H$ and $b, c \in \mathcal{C}$, be two arbitrary elements of $\cup H \mathcal{C}$. As $h\mathcal{C}$ and $k\mathcal{C}$ have the same diameter of \mathcal{C} ,

$$\begin{aligned} d(hb, kc) &\leq d(hb, ha) + d(ha, ka) + d(ka, kc) \\ &\leq n + m + n. \end{aligned}$$

This proves that $\cup H \mathcal{C}$ has finite diameter. Therefore, without loss of generality, we may assume that \mathcal{C} itself is persistent.

By the transitivity of the action, any two elements of \mathcal{X} are of the form ha, ka for some $h, k \in G$ and some $a \in \mathcal{C}$. By persistency, there are $c \in \mathcal{C} \cap h\mathcal{C}$ and $d \in \mathcal{C} \cap k\mathcal{C}$. Then

$$\begin{aligned} d(ha, ka) &\leq d(ha, c) + d(c, d) + d(d, ka) \\ &\leq n + n + n. \end{aligned}$$

Therefore the diameter of \mathcal{X} does not exceed $3n$. □

15 Theorem Suppose that \mathcal{X} and the sets $\mathcal{X}_n = K^n a$, for some $a \in \mathcal{X}$, are type-definable. Then \mathcal{X} has finite diameter.

Proof. By Proposition 14, it suffices to prove that \mathcal{X}_n is weakly persistent. By Corollary 13 it suffices to show that for some n the type $s_{\gamma_G}(x)$ is finitely satisfied in \mathcal{X}_n . Suppose not. Let $\psi_n(x) \in s_{\gamma_G}$ be a formula that is not satisfied in \mathcal{X}_n . Then the type $p(x) = \{\psi_n(x) : n \in \omega\}$ is finitely satisfied in \mathcal{X} . From the type-definability of \mathcal{X} it follows that $p(x)$ has a realization in \mathcal{X} . As this realization belongs to some \mathcal{X}_n we contradict the definition of $\psi_n(x)$. □

16 Example Let $K \subseteq \text{Aut}(\mathcal{U}/A)$ be the set of automorphisms that fix a model containing A . Then the group G generated by K is $\text{Autf}(\mathcal{U}/A)$ and $G \cdot a = \mathcal{X}$ is $\mathcal{L}(a/A)$. Let $\Delta = L_{xz}$ and $\mathcal{Z} = \mathcal{U}^{\mathcal{Z}}$. Then $d(a, b)$ coincides with the distance in the Lascar graph. The sets $K^n \cdot a = \{x : d(x, a) \leq n\}$ are type definable. Then from Theorem 15 it follows that $\mathcal{L}(a/A)$ is type definable (if and) only if it has a finite diameter.

4. A tamer landscape

Under suitable assumptions some notion introduced in this chapter coalesce, and we are left with a tamer landscape. In general, we have the following.

17 Theorem Assume

1. G -persistent $\Delta^B(\mathbb{Z})$ -definable sets are G -wide.
- Then the following hold
2. G -generic $\Delta^B(\mathbb{Z})$ -definable sets are closed under intersection
3. G -generic $\Delta^B(\mathbb{Z})$ -definable sets are strongly G -generic
4. weakly persistent $\Delta^B(\mathbb{Z})$ -definable sets are G -persistent.

Proof. Clearly $2 \Leftrightarrow 3 \Leftrightarrow 4$.

$1 \Rightarrow 2$. Let \mathcal{C} and \mathcal{D} be G -generic $\Delta^B(\mathbb{Z})$ -definable sets. Suppose for a contradiction that $\mathcal{C} \cap \mathcal{D}$ is not G -generic. Then $\neg(\mathcal{C} \cap \mathcal{D})$ is G -persistent. By 1 and Theorem 8 there is a G -invariant global $\Delta^B(\mathbb{Z})$ -type $p(x)$ containing $x \notin \mathcal{C} \cap \mathcal{D}$. By completeness either $p(x) \vdash x \notin \mathcal{C}$ or $p(x) \vdash x \notin \mathcal{D}$. This is a contradiction because by Theorem 5 $p(x) \vdash x \in \mathcal{C}$ and $p(x) \vdash x \in \mathcal{D}$. \square

18 Question Is 1 in Theorem 17 equivalent to 2-4?

19 Remark Assume 2-4 in Theorem 17 hold. Then the types $\gamma_G(x)$ and $^s\gamma_G(x)$ coincide, and therefore G -invariant global types exist. It is also worth mentioning that every positive Boolean combination of G -generic sets is G -generic.

5. The action of normal subgroups

Let $H \leq G$. The following is an immediate consequence of normality.

20 Remark For every $\mathcal{D} \subseteq \mathcal{U}^x$ and every $g \in G$

$$\mathcal{D} \text{ is } H\text{-foo} \Leftrightarrow g \cdot \mathcal{D} \text{ is } H\text{-foo},$$

where *foo* can be replaced by *generic*, *invariant*, *persistent*, *wide*. In particular, the type $\gamma_H(x)$ is G -invariant.

Recall that if $\gamma_H(x)$ is finitely satisfiable in \mathcal{X} , then H -generic sets are H -wide, cf. Theorem 8. As it happens, we can slightly strengthen this fact.

21 Proposition Assume that $\gamma_H(x)$ is consistent. Let \mathcal{D} be a $\Delta^B(\mathbb{Z})$ -definable set. Then if \mathcal{D} is G -generic it is also H -wide.

Proof. Let $p(x) \in S_\Delta(\mathbb{Z})$ be consistent with $\gamma_H(x)$. As \mathcal{D} is G -generic, by completeness $p(x) \vdash x \in g \cdot \mathcal{D}$ for some $g \in G$. Equivalently, $g^{-1} \cdot p \vdash x \in \mathcal{D}$. As $p(x)$ is H -persistent, by Remark 20 $g^{-1} \cdot p(x)$ is also H -persistent. Then the proposition follows from Theorem 8. \square

6. Definable groups

In this section we assume that \mathbb{Z} and \mathcal{X} are type-definable over some set of parameters A . Moreover we assume that \mathbb{Z} is a group that act on \mathcal{X} . The group operations and the group action are assumed definable over A . We use the symbol \cdot for both the group multiplication and the group action.

Let $\Psi \subseteq L_{\mathcal{X}}(\mathcal{U})$ be some small set of formulas. In this section Δ contains formulas $\varphi(\mathbf{x}; \mathbf{z})$ of the form $\psi(\mathbf{z}^{-1} \cdot \mathbf{x})$ for $\psi(\mathbf{x}) \in \Psi$. The sets $\varphi(\mathcal{X}; \mathcal{Z})$ are \mathcal{Z} -invariant. We write $\mathbf{1}$ for the identity of \mathcal{Z} . If $\varphi(\mathcal{X}; \mathbf{1}) \in \Delta^{\mathbb{B}}(\mathbf{1})$ then $\varphi(\mathcal{X}; \mathbf{g}) = \mathbf{g} \cdot \varphi(\mathcal{X}; \mathbf{1})$.

The following auxiliary structure is useful. Let $\mathcal{U}^{\Delta} = \langle \mathcal{X}; \mathcal{Z} \rangle$ be a 2-sorted structure whose signature L^{Δ} contains relation symbols for every formula $\varphi(\mathbf{x}; \mathbf{z}) \in \Delta$. As there is little risk of confusion, these relations symbols are also denoted by $\varphi(\mathbf{x}; \mathbf{z})$.

To each $h \in \mathcal{Z}$ we associate the L^{Δ} -automorphism $\langle \mathbf{a}; \mathbf{g} \rangle \mapsto \langle h \cdot \mathbf{a}; h \cdot \mathbf{g} \rangle$. Therefore \mathcal{Z} is, up to isomorphism, a subgroup of G . In fact, it is a normal subgroup. For any $\mathbf{g} \in \mathcal{Z}$, the orbit of $\varphi(\mathcal{X}; \mathbf{g})$ under the action of \mathcal{Z} is $\{\varphi(\mathcal{X}; h) : h \in \mathcal{Z}\}$. Therefore it coincides with the orbit under the action of G .

Now, we consider the action some other normal subgroup $H \trianglelefteq G$.

22 Proposition Let \mathcal{D} be a $\Delta^{\mathbb{B}}$ -definable set. Assume that $\gamma_H(\mathbf{x})$ is consistent. Then $1 \Rightarrow 2$ holds, where

1. \mathcal{D} is \mathcal{Z} -generic
2. $\mathbf{g} \cdot \mathcal{D}$ is H -wide for every $\mathbf{g} \in \mathcal{Z}$.

Under the assumption of stability and with $H = \text{Autf}(\mathcal{U}^{\Delta})$ a stronger claim obtains – the consistency of $\gamma_H(\mathbf{x})$ is guaranteed, and also the converse implication holds

Proof. Let \mathbf{g} be given. If \mathcal{D} is \mathcal{Z} -generic, then so is $\mathbf{g} \cdot \mathcal{D}$. Then $\mathbf{g} \cdot \mathcal{D}$ is, a fortiori, G -generic. Therefore 2 follows from Proposition 21. \square

We write $(\mathbf{g})_H$ for the H -orbit of \mathbf{g} , that is, the set $\{f(\mathbf{g}) : f \in H\}$.

23 Proposition Let $\vartheta(\mathbf{x}; \mathbf{z}_1, \dots, \mathbf{z}_n)$ be a Boolean combination of formulas $\varphi_i(\mathbf{x}; \mathbf{z}_i)$ for some $\varphi_i(\mathbf{x}; \mathbf{z}) \in \Delta$. Then for every $h_i \in (g_i)_H$ the following are equivalent

1. $\vartheta(\mathbf{x}; \mathbf{g}_1, \dots, \mathbf{g}_n)$ is H -wide
2. $\vartheta(\mathbf{x}; h_1, \dots, h_n)$ is H -wide.

Proof. Without loss of generality we can assume that only conjunctions occur in $\vartheta(\mathbf{x}; \mathbf{g}_1, \dots, \mathbf{g}_n)$. Let $\mathcal{C}_i = \varphi_i(\mathcal{X}; \mathbf{1})$. Then 1 says that $\mathcal{C} = \mathbf{g}_1 \cdot \mathcal{C}_1 \cap \dots \cap \mathbf{g}_n \cdot \mathcal{C}_n$ is H -wide. Let $f_i \in H$ be such that $h_i = f_i(\mathbf{g}_i)$. Then, by Corollary 9 also the intersection of the sets $f_i[\mathcal{C}]$ is H -wide. A fortiori the intersection of the sets $f_i[\mathbf{g}_i \cdot \mathcal{C}_i]$ is H -wide. As $f_i[\mathbf{g}_i \cdot \mathcal{C}_i] = h_i \cdot \mathcal{C}_i$, implication $1 \Rightarrow 2$ follows and, by symmetry, this proves the equivalence. \square

If $\mathcal{A} \subseteq \mathcal{Z}$, we write $\langle \mathcal{A} \rangle$ for the subgroup generated by \mathcal{A} .

24 Proposition Let $\vartheta(\mathbf{x}; \mathbf{z}_1, \dots, \mathbf{z}_n)$ be a Boolean combination of formulas $\varphi_i(\mathbf{x}; \mathbf{z}_i)$ for some $\varphi_i(\mathbf{x}; \mathbf{z}) \in \Delta$. Let $\mathbf{g} \in \mathcal{Z}$ be arbitrary. Assume that $\vartheta(\mathbf{x}; \mathbf{1}, \dots, \mathbf{1})$ is H -wide. Then $\vartheta(\mathbf{x}; h_1, \dots, h_n)$ is H -wide for every

$$h_1, \dots, h_n \in \left\langle \bigcup_{\mathbf{g} \in \mathcal{Z}} (\mathbf{g})_H^{-1} \cdot (\mathbf{g})_H \right\rangle.$$

Proof. We proceed by induction on the number of factors of the form $a^{-1} \cdot b$, for some $a, b \in (g)_H$, that occur in h_1, \dots, h_n . Without loss of generality we can assume that only conjunctions occur in $\vartheta(x; 1, \dots, 1)$. Let $\mathcal{C}_i = \varphi_i(\mathcal{X}; 1)$. Assume inductively that $h_1 \cdot \mathcal{C}_1 \cap \dots \cap h_n \cdot \mathcal{C}_n$ is H -wide. Pick two arbitrary $a, b \in (g)_H$. By Remark 20

$$a \cdot \mathcal{C}_1 \cap a \cdot h_1^{-1} \cdot h_2 \cdot \mathcal{C}_2 \cap \dots \cap a \cdot h_1^{-1} \cdot h_n \cdot \mathcal{C}_n \text{ is } H\text{-wide.}$$

By Proposition 23, we can replace $a \cdot \mathcal{C}_1$ by $b \cdot \mathcal{C}_1$. Then, again by Remark 20

$$h_1 \cdot a^{-1} \cdot b \cdot \mathcal{C}_1 \cap h_2 \cdot \mathcal{C}_2 \cap \dots \cap h_n \cdot \mathcal{C}_n \text{ is } H\text{-wide.} \quad \square$$

25 Question Let $H = \text{Autf}(\mathcal{U}^\Delta)$. Under the assumption of stability the group in Proposition 24 has finite index in \mathbb{Z} and is contained in the connected component of \mathbb{Z} . Does it coincide with it? I.e. is it definable?

26 Question Let $H = \text{Autf}(\mathcal{U}^\Delta)$. Describe the group in Proposition 24.

27 Question Is the group in Proposition 24 H -invariant?

7. Notes and references

- [1] Artem Chernikov and Itay Kaplan, *Forking and dividing in NTP₂ theories*, J. Symbolic Logic **77** (2012), 1–20.
- [2] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.