µ-Random thoughts

R.T. Polymath

Let $\mathcal{U}$ be a monster model of signature L and let $T = \mathrm{Th}(\mathcal{U})$. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|x|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters from $\mathcal{U}$). Global types are types over $\mathcal{U}$ that are complete. Unless otherwise stated, any other type is partial.

We denote by ${}^*\mathcal{U}$ an elementary extension of $\mathcal{U}$ where all global type are realized. If $\mathcal{D} = \varphi(\mathcal{U})$, we write ${}^*\mathcal{D}$ for $\varphi({}^*\mathcal{U})$. If $p(x) \subseteq L(\mathcal{U})$ we write ${}^*p$ for $p({}^*\mathcal{U})$.

Below by $\mu(x) \subseteq L(\mathcal{U})$ we always denote a consistent type closed under conjunctions and logical consequences modulo T that is, if $\varphi(x) \in \mu$ and $\varphi(x) \to \psi(x)$ then $\psi(x) \in \mu$. We denote by 1_μ the filter $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal that is, the ideal $\big\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\big\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and ${}^*\mu \subseteq {}^*\mathcal{D}$ are synonymous.

The elements of ${}^*\mu$ are called μ-random. Let $p(x) \subseteq L(\mathcal{U})$. We say that $p(x) \subseteq L(\mathcal{U})$ is μ-wide if $\mu(x) \cup p(x)$ is finitely consistent. A global μ-wide type is also called a μ-random type. In other words, μ-random types are the type over $\mathcal{U}$ of μ-random elements.

The following is tautological but worth noticing.

Remark 1. A type is μ-wide if and only if it is realized by a random element. Every μ-wide type extends to a μ-random type.

Let $\kappa = \kappa^{<\kappa}$ be the cardinality of $\mathcal{U}$. We say that μ is κ-prime if for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. By the regularity of κ this equivalent to requiring that $\mathcal{D}_i \in 1_\mu$ for every $i < \kappa$.

Sometimes we use the dual version of this property which sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 2. Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|x|}$. Let 1_μ be the set of subsets of measure 1. Then μ is κ-prime.

Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino.

Proof. Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa \rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As κ is regular and uncountable, we may assume that for some n all sets $\mathcal{D}_i$ have measure $\geq 1/n$. Then, up to a set measure 0, the sets $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are pairwise disjoint and contained in $\neg\mathcal{D}_0$. This is a contradiction because $\neg\mathcal{D}_0$ has measure < 1. □

We say that μ is S1 over A if for every A-indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu \quad \Rightarrow \quad \mathcal{D}_0 \in 1_\mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

Fact 3. For any μ that is Lascar-invariant over A the following are equivalent

1. μ is κ-prime;
2. μ is S1 over A.

Proof. (1⇒2) Let $\langle \mathcal{D}_i : i < \omega \rangle$ be a sequence of A-indiscernibles such that $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$. Then $\langle \mathcal{D}_i : i < \omega \rangle$ is M-indiscernible for some model M containing A. As μ is invarianty over M. By compactness, we can strech this sequence one to length κ. From indiscernibility and the M-invariance of μ we obtain that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Then $\mathcal{D}_i \in 1_\mu$ for some $i < \kappa$ and, again by indiscernibility and invariance, $\mathcal{D}_0 \in 1_\mu$.

(2⇒1) The following fact is well-known.

Fact. For every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ there is an A-indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that for every $n < \omega$ there is some $I \subseteq \kappa$ of cardinality n such that $\mathcal{C}_{\restriction n} \equiv_A \mathcal{D}_{\restriction I}$.

Suppose $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Pick a model M containing A. Then μ is invariant over M. Apply the fact with $n = 2$ and M for A to obtain A-indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that $\mathcal{C}_0, \mathcal{C}_1 \equiv_M \mathcal{D}_i, \mathcal{D}_j$ for some $i < j < \kappa$. From the M-invariance of μ we obtain $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. Then $\mathcal{C}_0 \in 1_\mu$. Again by M-invariance, $\mathcal{D}_i \in 1_\mu$. □

Example 4. Let $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} \ : \ A^{|x|} \subseteq \mathcal{X}\}$. Then μ is S1 over A. Clearly, a type $p(x) \subseteq L(\mathcal{U})$ is μ-wide if and only if it is finitely satisfied in A.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A-indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{|x|}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{|x|}$. Hence $\mathcal{D}_0 \in 1_\mu$. □

For every small set A we define

$$\mu^A(x) \quad = \quad \{\varphi(x) \ : \ \varphi(x) \in p \ \text{ for every } p \in S_x(\mathcal{U}) \text{ that is Lascar-invariant over } A\}$$

Clearly, a type is μ^A-wide if and only if it has an extension to a global type Lascar-invariant over A.

Lemma 5. μ^A is S1 over A.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_{\mu^A}$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A-indiscernibles. Then $\mathcal{D}_0, \mathcal{D}_1$ are conjugated over some model M containing A. If p is any global type Lascar-invariant over A, then p is invariant over M. Therefore p contains $x \in \mathcal{D}_0$ if and only if it contains $x \in \mathcal{D}_1$. Therefore $\mathcal{D}_0 \in 1_{\mu^A}$. □

We say that $\mathcal{D}$ is generic over A if for every finitely many translates of $\mathcal{D}$ under $\mathrm{Aut}(\mathcal{U}/A)$ cover $\mathcal{U}^{|x|}$. Note that $\mathcal{D}$ is generic over A if and only if $\mathrm{orb}(\neg\mathcal{D}/A)$ does not have fip (i.e. the finite intersection propery).

The definition of Lascar-generic over A is similar, we only replace $\mathrm{Aut}(\mathcal{U}/A)$ by $\mathrm{Autf}(\mathcal{U}/A)$. Again, this is equivalent to requiring that $\mathrm{orbf}(\neg\mathcal{D}/M)$ does not have fip.

Fact 6. 1_{μ^A} is the filter generated by sets that are Lascar-generic over A.

Proof. (⊇) It suffices to prove if $\mathcal{D}$ is Lascar-generic over A then $\mathcal{D} \in 1_{\mu^A}$. That is, every μ^A-random type p contains $x \in \mathcal{D}$. Let p be μ^A-random. By Lascar-genericity, there are some strong A-automorphisms $f_1, \ldots, f_n$ be such that $\mathcal{U}^{|x|} = \bigcup_i f_i\mathcal{D}$. By completeness, p contains $x \in f_i\mathcal{D}$ for some i. By Lascar-invariance, p contains also $x \in \mathcal{D}$.

(⊆) Assume that there are no Lascar-generic sets $\mathcal{B}_i$ such that

$$\bigcap_{i=1}^{n} \mathcal{B}_i \quad \subseteq \quad \mathcal{D}$$

Hence, by taking complements, for any $\mathcal{C}_i$ such that

$$\neg\mathcal{D} \quad \subseteq \quad \bigcup_{i=1}^{n} \mathcal{C}_i,$$

at least one $\mathrm{orbf}(\mathcal{C}_i/A)$ has the fip. By Fact 8 below, we obtain that $x \notin \mathcal{D}$ belongs to some global type that Lascar-invariant over A. Therefore $\mathcal{D} \notin 1_{\mu^A}$. □

Fact 7. For every $\mathcal{D}$ the following are equivalent

1. there is a global type containing $x \in \mathcal{D}$ that is invariant over A;
2. every finite cover $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of $\mathcal{D}$ contains some $\mathcal{C}_i$ such that $\mathrm{orb}(\mathcal{C}_i/A)$ has fip.

Proof. (1⇒2) Let p be a global A-invariant type containing $\varphi(x)$. Then p contains $\mathcal{C}_i$ for some i. By invariance, p contains $x \in \mathcal{C}$ for every $\mathcal{C}$ conjugated to $\mathcal{C}_i$ hence (2) follows.

(2⇒1) Let p be maximally finitely consistent type containing $\varphi(x)$ such that all formulas p have the property in (2). We claim that p is complete type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some $\psi(x) \in p$ and some $\mathcal{C}_1, \ldots, \mathcal{C}_n$ covering both $\psi(\mathcal{U}) \cap \mathcal{D}$ and $\psi(\mathcal{U}) \smallsetminus \mathcal{D}$ such that no $\mathrm{orb}(\mathcal{C}_i/A)$ has fip. This is a contradiction because $\mathcal{C}_1, \ldots, \mathcal{C}_n$ also cover $\psi(\mathcal{U})$. Finally, (1) follows if we can prove that p is invariant over A. Suppose not then, by completeness, p contains $x \in \mathcal{B} \smallsetminus f\mathcal{B}$ for some $\mathcal{B}$ and some $f \in \mathrm{Aut}(\mathcal{U}/A)$. But $\mathcal{B} \smallsetminus f\mathcal{B}$ is disjoint of $f\mathcal{B} \smallsetminus f^2\mathcal{B}$, hence it cannot satisfy (2). A contradiction. $\square$

By essentially the same proof we obtain the following.

Fact 8. For every $\mathcal{D}$ the following are equivalent

1. there is a global type containing $x \in \mathcal{D}$ that is Lascar-invariant over A;
2. every finite cover $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of $\mathcal{D}$ contains some $\mathcal{C}_i$ such that $\mathrm{orbf}(\mathcal{C}_i/A)$ has fip.

In Example 4 the elements of $A^{|x|}$ are, trivially, μ-random. Clearly, there are many other μ-random types: all global types that are finitely satisfied in A are μ-random. If we allow elements outside $\mathcal{U}$, the filter 1_{μ^A} has an expression similar to 1_μ in Example 4. In fact, as the number of invariant types is bounded, it is immediate that there is a small set of μ^A-random elements R such that $1_{\mu^A} = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} : R \subseteq {}^*\mathcal{X}\}$.

Definition 9. Let 1_{ν^A} be the filter generated by the sets $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ such that there is an A-indiscerible sequence $\langle \mathcal{D}_i : i < \omega \rangle$ such that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_0, \ldots, \mathcal{D}_n$ cover $\mathcal{U}^{|x|}$ for some n. We call 1_{ν^A} the nonforking filter over A. We call 0_{ν^A} the forking ideal over A.

Fact 10. Let μ be S1 over A. Then $1_{\nu^A} \subseteq 1_\mu$.

An immediate consequence that μ-random elements are ν^A-random.

Proof. Let $\mathcal{D} \in 1_{\nu^A}$. Let $\langle \mathcal{D}_i : i < \omega \rangle$ be as in Definition 9. Then $\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_n \in 1_\mu$ for some n. Assume n is minimal, we prove that $n = 0$. Otherwise, let $\mathcal{C}_i = \mathcal{D}_i \cup \cdots \cup \mathcal{D}_{i+n-1}$. Then $\langle \mathcal{C}_i : i < \omega \rangle$ is an A-indiscernible sequence and $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. From S1 we obtain $\mathcal{C}_0 \in 1_\mu$ which contradicts the minimality of n. $\square$

A type $p(x\,;y)$ that is Lascar-invariant over A is stable if $p(a_0\,;b_1) \to p(a_1\,;b_0)$ for every A-indiscernible sequence $\langle a_i; b_i : i < \omega \rangle$.

Fact 11. Every stable type $p(x\,;y)$ is equivalent to a type $q(x\,;y) \subseteq p$ containing only stable formulas.

Proof. Let $i(\bar{x}\,;\bar{y})$ be the type that says that $\langle x_i; y_i : i < \omega \rangle$ is a sequence of indiscernibles over A. The required type $q(x\,;y)$ contains the formulas $\psi(x\,;y) \in p$ such that $i(\bar{x}\,;\bar{y}) \wedge \psi(x_0; y_1) \to \psi(x_1; y_0)$. $\square$

Theorem 12. Let $p(x;z), q(x;z) \subseteq L(\mathfrak{U})$ be Lascar-invariant over A. Let μ be S1 and Lascar-invariant over A. Then the relation

$$R(a,b) \quad \Leftrightarrow \quad p(x;a) \cup q(x;b) \text{ is wide.}$$

is stable.

Proof. Let $\langle a_i; b_i : i < \omega \rangle$ be a sequence of A-indiscernibles such that $p(x;a_0) \cup q(x;b_1)$ is μ-wide. By S1, also $\big[p(x;a_0) \cup q(x;b_1)\big] \cup \big[p(x;a_2) \cup q(x;b_3)\big]$ is μ-wide. A fortiori $p(x;a_2) \cup q(x;b_1)$ is μ-wide and, by indiscernibility and the Lascar-invariance of μ, also $p(x;a_1) \cup q(x;b_0)$ is wide. □

Fact 13. Let $q(x\ y) \subseteq L(M)$ be stable. Assume $a \equiv_M a'$ and $b \equiv_M b'$ are such that $a \mathop{\perp}_M b$ and $a' \mathop{\perp}_M b'$. Then $q(a\ b) \leftrightarrow q(a'; b')$.

Proof. By invariance we may assume that $b' = b$. Let $\varphi(x;y) \in q$. By Fact 11 we can assume that $\varphi(x;y)$ is stable. Then $\varphi(a;b) \leftrightarrow \varphi(a';b)$ by stationarity. □

Definition 14. We say that μ is type-definable over A if for every formula $\varphi(x;z) \in L$ the set $\{b \in \mathfrak{U}^{|z|} : \varphi(x;z) \in \mu\}$ is type-definable over A.

For instance, μ in Example 4 is type definable.

Note that if μ is type-definable over A and $\varphi(x;b) \notin \mu$ the there is a formula $\vartheta(z) \in L(A)$ such that $\vartheta(b)$ and $\varphi(x;b') \notin \mu$ for every $b' \models \vartheta(z)$.

1. Independence (tentative ramble)

Let $\mu \subseteq L_v(\mathfrak{U})$, where $|v| = 1$, be A-invariant. When x is the tuple $\langle x_i : i < |x| \rangle$, we say that $p(x) \subseteq L(\mathfrak{U})$ is μ-wide if it is finitely consistent with $\bigcup_i \mu(x_i)$.

Let $a \in \mathfrak{U}^{|x|}$. We write $a \mathop{\perp}_\mu b$ if the type $\mathrm{tp}(a/A, b)$ is μ-wide. Clearly, this is equivalent to saying that, $\varphi(x;b)$ is μ-wide for every $\varphi(x;z) \in L(A)$ such that $\varphi(a;b)$.

We write $b \equiv_\mu b'$ if $\mu(x)$ implies $\varphi(x;b) \leftrightarrow \varphi(x;b')$ for every $\varphi(x;z) \in L(A)$.

Lemma 15. The following properties hold for all a, b, c

1. $a \mathop{\perp}_\mu b \ \Rightarrow\ f a \mathop{\perp}_\mu f b$ for every $f \in \mathrm{Aut}(\mathfrak{U}/A)$ — *invariance*
2. $a \mathop{\perp}_\mu b \ \Leftarrow\ a_0 \mathop{\perp}_\mu b_0$ for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$ — *finite character*
3. $a \mathop{\perp}_\mu c, b$ and $c \mathop{\perp}_\mu b \ \Rightarrow\ a, c \mathop{\perp}_\mu b$ — *transitivity*
4. $a \mathop{\perp}_\mu b \ \Rightarrow$ there exists $a' \equiv_{A,b} a$ such that $a' \mathop{\perp}_\mu b, c$ — *extension*
5. $a \mathop{\perp}_\mu b_1, b_2$ and $b_1 \equiv_\mu b_2 \ \Rightarrow\ b_1 \equiv_{\mu, a} b_2$ — *non-splitting*

Proof. Properties 1,2,3 are immediate. We prove 4. As $\mathrm{tp}(a/A,b)$ is μ-wide, it extends to a μ-random type $p(x)$. Then we can take any $a' \models p_{\restriction A,b,c}(x)$. $\square$

Definition 16. We say that $\mathop{\smile\!\!\!\!\!\!\mid}_\mu$ is stationary if $a \equiv_M x \mathop{\smile\!\!\!\!\!\!\mid}_\mu b$ is a complete type over M, b for all finite tuples b and a.

We say n-stationary if this is restricted to $|a| = n$.

Stationarity is often ensured by the following property.

Proposition 17. If for every $\varphi(x) \in L(\mathfrak{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(\mathfrak{U}) =_\mu \psi(\mathfrak{U})$ then $\mathop{\smile\!\!\!\!\!\!\mid}_\mu$ is stationary.

Proof. Let $b \in \mathfrak{U}^{|z|}$ and $a_1, a_2 \in \mathfrak{U}^{|x|}$ be such that $a_i \mathop{\smile\!\!\!\!\!\!\mid}_\mu b$ and $a_1 \equiv_M a_2$. We claim that $a_1 \equiv_{M,b} a_2$. We need to prove that $\varphi(b;a_1) \leftrightarrow \varphi(b;a_2)$ for every $\varphi(z;x) \in L(M)$. Let $\psi(x) \in L(M)$ be such that $\varphi(b;\mathfrak{U}) =_M \psi(\mathfrak{U})$. From $a_i \mathop{\smile\!\!\!\!\!\!\mid}_\mu b$ we obtain that $\varphi(b;a_i) \leftrightarrow \psi(a_i)$. Finally, the claim follows because $a_1 \equiv_\mu a_2$. $\square$

References

[MOS] Samaria Montenegro, Alf Onshuus, and Pierre Simon, *Stabilizers,* NTP$_2$ *groups with f-generics, and PRC fields*, J. Inst. Math. Jussieu **19** (2020), no. 3, 821–853.

[Hr] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.

[MP] Amador Martin-Pizarro and Daniel Palacín, *Stabilizers, Measures and IP-sets* (2020), arXiv:1912.07252