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μ-Random thoughts

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Let $\mathcal{U}$ be a monster model. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|x|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters from $\mathcal{U}$).

We denote by ${}^*\mathcal{U}$ an elementary extension of $\mathcal{U}$ where all global type (over $\mathcal{U}$) are realized. If $\mathcal{D} = \varphi(\mathcal{U})$, we write ${}^*\mathcal{D}$ for $\varphi({}^*\mathcal{U})$. If $p(x) \subseteq L(\mathcal{U})$ we write ${}^*p$ for $p({}^*\mathcal{U})$.

Below by $\mu(x) \subseteq L(\mathcal{U})$ we always denote a consistent type closed under conjunctions and logical consequences that is, if $\varphi(x) \in \mu$ and $\varphi(x) \to \psi(x)$ then $\psi(x) \in \mu$. We denote by 1_μ the filter generated by $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal that is, the ideal generated by $\big\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\big\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and ${}^*\mu \subseteq {}^*\mathcal{D}$ are synonymous.

The elements of ${}^*\mu$ are called μ-random. Let $p(x) \subseteq L(\mathcal{U})$. We say that $p(x) \subseteq L(\mathcal{U})$ is μ-wide if $\mu(x) \cup p(x)$ is finitely consistent. A type is μ-random if it is complete and μ-wide, i.e. it is the type over $\mathcal{U}$ of a μ-random element.

The following is tautological but worthwhile to notice.

Remark 1. Every μ-wide type $p(x) \subseteq L(\mathcal{U})$ extends to a μ-random type.

Let $\kappa = \kappa^{<\kappa}$ be the cardinality of $\mathcal{U}$. We say that μ is κ-prime if for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. By the regularity of κ this equivalent to requiring that $\mathcal{D}_i \in 1_\mu$ for every $i < \kappa$.

Sometimes we use the dual version of this property which sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 2. Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|x|}$. Let 1_μ be the set of subsets of measure 1. Then μ is κ-prime.

Proof. Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa\rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As κ is regular and uncountable, we may assume that for some n all sets $\mathcal{D}_i$ have measure $\geq 1/n$. Then, up to a set measure 0, the sets $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are pairwise disjoint and contained in $\neg\mathcal{D}_0$. This is a contradiction because $\neg\mathcal{D}_0$ has measure < 1. □

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We say that μ is S1 over A if for every A-indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu \quad \Rightarrow \quad \mathcal{D}_0 \in 1_\mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

Fact 3. For any μ that is Lascar-invariant over A the following are equivalent

1. μ is κ-prime;
2. μ is S1 over A.

Proof. (1⇒2) Let $\langle \mathcal{D}_i : i < \omega \rangle$ be a sequence of A-indiscernibles such that $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$. By compactness, we can strech this sequence one to length κ. From indiscernibility we obtain that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Then $\mathcal{D}_i \in 1_\mu$ for some $i < \kappa$ and, again by indiscernibility, $\mathcal{D}_0 \in 1_\mu$.

(2⇒1) The following fact is well-known.

Fact. For every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ there is an A-indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that for every $n < \omega$ there is some $I \subseteq \kappa$ of cardinality n such that $\mathcal{C}_{\restriction n} \equiv_A \mathcal{D}_{\restriction I}$.

Suppose $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Pick a model M containing A. Then μ is invariant over M. Apply the fact with $n = 2$ and M for A to obtain A-indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that $\mathcal{C}_0, \mathcal{C}_1 \equiv_M \mathcal{D}_i, \mathcal{D}_j$ for some $i < j < \kappa$. From the M-invariance of μ we obtain $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. Then $\mathcal{C}_0 \in 1_\mu$. Again by M-invariance, $\mathcal{D}_i \in 1_\mu$. □

Example 4. Let $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} \ : \ A^{|x|} \subseteq \mathcal{X}\}$. Then μ is S1 over A. Clearly, a type $p(x) \subseteq L(\mathcal{U})$ is μ-wide if and only if it is finitely satisfied in A.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A-indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{|x|}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{|x|}$. Hence $\mathcal{D}_0 \in 1_\mu$. □

For every small set A we define

$$\mu^A(x) \quad = \quad \{\varphi(x) \ : \ \varphi(x) \in p \ \text{ for every } p \in S_x(\mathcal{U}) \text{ that is Lascar-invariant over } A\}$$

Clearly, a type is μ^A-wide if and only if it has an extension to a global type Lascar-invariant over A.

Lemma 5. μ^A is S1 over A.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_{\mu^A}$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A-indiscernibles. Then $\mathcal{D}_0, \mathcal{D}_1$ are conjugated over some model M containing A. If p is any global type Lascar-invariant over A, then p is invariant over M. Therefore $p \in \mathcal{D}_0$ if and only if $p \in \mathcal{D}_1$. Therefore $\mathcal{D}_0 \in 1_{\mu^A}$. □

We say that $\mathcal{D}$ is Lascar-generic over A if finitely many translates of $\mathcal{D}$ under $\mathrm{Autf}(\mathcal{U}/A)$ cover $\mathcal{U}^{|x|}$.

Fact 6. 1_{μ^A} is the filter generated by the A-generic sets.

Proof. It is easy to see that if $\mathcal{D}$ is A-generic then $\mathcal{D} \in 1_\mu$. Vice versa, assume that there are no A-generic sets $\mathcal{B}_i$ such that

$$\bigcap_{i=1}^{n} \mathcal{B}_i \quad \subseteq \quad \mathcal{D}$$

Note that a set $\mathcal{B}$ is not A-generic if and only if the orbit over A of $\neg\mathcal{B}$ has the finite intersection property (fip). Hence, by taking complements, for any $\mathcal{C}_i$ such that

$$\neg\mathcal{D} \quad \subseteq \quad \bigcup_{i=1}^{n} \mathcal{C}_i,$$

there is at least one i such that the A-orbit of $\mathcal{C}_i$ has the fip. By the fact below we obtain that there is an A-invariant type $p \notin \mathcal{D}$. Therefore $\mathcal{D} \notin 1_\mu$. □

Fact 7. For every $\varphi(x) \in L(\mathcal{U})$ the following are equivalent

1. there is an global A-invariant type containing $\varphi(x)$;
2. for every finitely many $\mathcal{C}_1, \ldots, \mathcal{C}_n$ covering $\varphi(\mathcal{U})$ there is some $\mathcal{C}_i$ whose orbit over A has the fip.

Proof. (1⇒2) Let p be a global A-invariant type containing $\varphi(x)$. Then p contains $\mathcal{C}_i$ for some i. By invariance, p contains $x \in \mathcal{C}$ for every $\mathcal{C}$ conjugated to $\mathcal{C}_i$ hence (2) follows.

(2⇒1) Let p be maximal type containing $\varphi(x)$ such that all formulas p have the property in (2). We claim that p is complete type. Suppose $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some $\psi(x) \in p$ and some $\mathcal{C}_1, \ldots, \mathcal{C}_n$ covering both $\psi(\mathcal{U}) \cap \vartheta(\mathcal{U})$ and $\psi(\mathcal{U}) \smallsetminus \vartheta(\mathcal{U})$ such that no $\mathcal{C}_i$ has the fip. This is a contradiction because $\mathcal{C}_1, \ldots, \mathcal{C}_n$ also cover $\psi(\mathcal{U})$. □

In Example 4 the elements of $A^{|x|}$ are, trivially, μ-random. Clearly, there are many other μ-random types: all global types that are finitely satisfied in A are μ-random. If we allow elements outside $\mathcal{U}$, the filter 1_{μ^A} has an expression similar to 1μ in Example 4. In fact, as the number of invariant types is bounded, it is immediate that there is a small set $R \subseteq {}^*\mu$ of μ^A-random elements such that $1_{\mu^A} = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} \,:\, R \subseteq \mathcal{X}\}$.

Definition 8. Let 1_A be the filter generated by the sets $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ such that there is an A-indiscerible sequence $\langle \mathcal{D}_i : i < \omega \rangle$ such that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_0, \ldots, \mathcal{D}_n$ cover $\mathcal{U}^{|x|}$ for some n. We call 1_A the nonforking filter over A.

Fact 9. Let μ be S1 and invariant over A. Then 1_μ contains 1_A.

Proof. Let $\mathcal{D} \in 1_A$. Let $\langle \mathcal{D}_i : i < \omega \rangle$ be as in Definition 8. Then $\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_n \in 1_\mu$ for some n. Assume n is minimal, we prove that $n = 0$. Otherwise, let $\mathcal{C}_i = \mathcal{D}_i \cup \cdots \cup \mathcal{D}_{i+n-1}$. Then $\langle \mathcal{C}_i : i < \omega \rangle$ is an A-indiscernible sequence and $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. From S1 we obtain $\mathcal{C}_0 \in 1_\mu$ which contradicts the minimality of n. □

Theorem 10. Let $p(x;z), q(x;z) \subseteq L(A)$. Let μ be S1 and A-invariant. Then

$$R(a,b) \quad \Leftrightarrow \quad p(x;a) \cup q(x;b) \text{ is wide}$$

is a stable relation.

Proof. Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence of A-indiscernibles such that $p(x;a_0) \cup q(x;b_1)$ is μ-wide. By S1, also $\big[p(x;a_0) \cup q(x;b_1)\big] \cup \big[p(x;a_2) \cup q(x;b_3)\big]$ is μ-wide. A fortiori $p(x;a_2) \cup q(x;b_1)$ is μ-wide and, by indiscernibility, so is $p(x;a_1) \cup q(x;b_0)$. □

Definition 11. We say that μ is type-definable over A if for every formula $\varphi(x;z) \in L(\mathcal{U})$ the set $\{b \in \mathcal{U}^{|z|} : \phi(x\,z) \in \mu\}$ is type-definable over A.

For instance, in general μ in Example 4 is type definable.

Note that if μ is type-definable over A and $\varphi(x;b) \notin \mu$ the there is a formula $\vartheta(z) \in L(A)$ such that $\vartheta(b)$ and $\varphi(x;b') \notin \mu$ for every $b' \models \vartheta(z)$.

1. Independence (tentative ramble)

Let $\mu \subseteq L_v(\mathcal{U})$, where $|v| = 1$, be A-invariant. When x is the tuple $\langle x_i : i < |x| \rangle$, we say that $p(x) \subseteq L(\mathcal{U})$ is μ-wide if it is finitely consistent with $\bigcup_i \mu(x_i)$.

Let $a \in \mathcal{U}^{|x|}$. We write $a \perp_\mu b$ if the type $\mathrm{tp}(a/A,b)$ is μ-wide. Clearly, this is equivalent to saying that, $\varphi(x;b)$ is μ-wide for every $\varphi(x;z) \in L(A)$ such that $\varphi(a;b)$.

We write $b \equiv_\mu b'$ if $\mu(x)$ implies $\varphi(x;b) \leftrightarrow \varphi(x;b')$ for every $\varphi(x;z) \in L(A)$.

Lemma 12. The following properties hold for all a, b, c

1. $a \perp_\mu b \;\Rightarrow\; f a \perp_\mu f b$ for every $f \in \mathrm{Aut}(\mathcal{U}/A)$ *invariance*
2. $a \perp_\mu b \;\Leftarrow\; a_0 \perp_\mu b_0$ for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$ *finite character*
3. $a \perp_\mu c, b$ and $c \perp_\mu b \;\Rightarrow\; a, c \perp_\mu b$ *transitivity*
4. $a \perp_\mu b \;\Rightarrow\;$ there exists $a' \equiv_{A,b} a$ such that $a' \perp_\mu b, c$ *extension*
5. $a \perp_\mu b_1, b_2$ and $b_1 \equiv_\mu b_2 \;\Rightarrow\; b_1 \equiv_{\mu,a} b_2$ *non-splitting*

Proof. Properties 1,2,3 are immediate. We prove 4. As $\mathrm{tp}(a/A,b)$ is μ-wide, it extends to a μ-random type $p(x)$. Then we can take any $a' \models p_{\restriction A,b,c}(x)$. □

Definition 13. We say that $\mathop{\smile\!\!\!\!\!\!\!\perp}_\mu$ is stationary if $a \equiv_M x \mathop{\smile\!\!\!\!\!\!\!\perp}_\mu b$ is a complete type over M, b for all finite tuples b and a.

We say n-stationary if this is restricted to $|a| = n$.

Stationarity is often ensured by the following property.

Proposition 14. If for every $\varphi(x) \in L(\mathfrak{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(\mathfrak{U}) =_\mu \psi(\mathfrak{U})$ then $\mathop{\smile\!\!\!\!\!\!\!\perp}_\mu$ is stationary.

Proof. Let $b \in \mathfrak{U}^{|z|}$ and $a_1, a_2 \in \mathfrak{U}^{|x|}$ be such that $a_i \mathop{\smile\!\!\!\!\!\!\!\perp}_\mu b$ and $a_1 \equiv_M a_2$. We claim that $a_1 \equiv_{M,b} a_2$. We need to prove that $\varphi(b; a_1) \leftrightarrow \varphi(b; a_2)$ for every $\varphi(z; x) \in L(M)$. Let $\psi(x) \in L(M)$ be such that $\varphi(b; \mathfrak{U}) =_M \psi(\mathfrak{U})$. From $a_i \mathop{\smile\!\!\!\!\!\!\!\perp}_\mu b$ we obtain that $\varphi(b; a_i) \leftrightarrow \psi(a_i)$. Finally, the claim follows because $a_1 \equiv_\mu a_2$. □

References

[MOS] Samaria Montenegro, Alf Onshuus, and Pierre Simon, *Stabilizers,* NTP_2 *groups with f-generics, and PRC fields*, J. Inst. Math. Jussieu **19** (2020), no. 3, 821–853.

[Hr] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.

[MP] Amador Martin-Pizarro and Daniel Palacín, *Stabilizers, Measures and IP-sets* (2020), arXiv:1912.07252