

μ -Random thoughts

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Let \mathcal{U} be a monster model of signature L and let $T = \text{Th}(\mathcal{U})$. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|\mathcal{U}|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters from \mathcal{U}). Global types are types over \mathcal{U} that are complete. Unless otherwise stated, any other type is partial.

We denote by $^*\mathcal{U}$ an elementary extension of \mathcal{U} where all global type are realized. If $\mathcal{D} = \varphi(\mathcal{U})$, we write $^*\mathcal{D}$ for $\varphi(^*\mathcal{U})$. If $p(x) \subseteq L(\mathcal{U})$ we write *p for $p(^*\mathcal{U})$.

Below by $\mu(x) \subseteq L(\mathcal{U})$ we always denote a consistent type closed under conjunctions and logical consequences modulo T that is, if $\varphi(x) \in \mu$ and $\varphi(x) \rightarrow \psi(x)$ then $\psi(x) \in \mu$. We denote by 1_μ the filter $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal that is, the ideal $\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and $^*\mu \subseteq ^*\mathcal{D}$ are synonymous.

The elements of $^*\mu$ are called μ -random. We say that $\varphi(x) \in L(\mathcal{U})$ is μ -wide if it is finitely consistent with $\mu(x)$. In other words, if $\varphi(\mathcal{U}) \notin 0_\mu$. A type $p(x) \subseteq L(\mathcal{U})$ is μ -wide if all the formulas in p are μ -wide. A global μ -wide type is also called a μ -random type. In other words, μ -random types are the type over \mathcal{U} of μ -random elements.

The following is tautological but worth noticing.

Remark 1. A type is μ -wide if and only if it is realized by a random element. Every μ -wide type extends to a μ -random type.

Let $\kappa = \kappa^{<\kappa}$ be the cardinality of \mathcal{U} . We say that μ is κ -prime if for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. By the regularity of κ this equivalent to requiring that $\mathcal{D}_i \in 1_\mu$ for every $i < \kappa$.

Sometimes we use the dual version of this property which sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 2. Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|\mathcal{U}|}$. Let 1_μ be the set of subsets of measure 1. Then μ is κ -prime.

Proof. Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa \rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As κ is regular and uncountable, we may assume that for some n all sets \mathcal{D}_i have measure $\geq 1/n$. Then, up to a set measure 0, the sets $\mathcal{D}_1, \dots, \mathcal{D}_n$ are pairwise disjoint and contained in $\neg \mathcal{D}_0$. This is a contradiction because $\neg \mathcal{D}_0$ has measure < 1 . \square

We say that μ is **S1** over A if for every A -indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu \Rightarrow \mathcal{D}_0 \in 1_\mu.$$

Which may also be formulated as follows

$$\mathcal{D}_0 \text{ is } \mu\text{-wide} \Rightarrow \mathcal{D}_0 \cap \mathcal{D}_1 \text{ is } \mu\text{-wide}.$$

The terminology originated in some obscure corner of Hrushovski's mind.

Fact 3. For any μ that is Lascar-invariant over A the following are equivalent

1. μ is κ -prime;
2. μ is S1 over A .

Proof. (1 \Rightarrow 2) Let $\langle \mathcal{D}_i : i < \omega \rangle$ be a sequence of A -indiscernibles such that $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$. Then $\langle \mathcal{D}_i : i < \omega \rangle$ is M -indiscernible for some model M containing A . As μ is invariantly over M . By compactness, we can stretch this sequence one to length κ . From indiscernibility and the M -invariance of μ we obtain that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Then $\mathcal{D}_i \in 1_\mu$ for some $i < \kappa$ and, again by indiscernibility and invariance, $\mathcal{D}_0 \in 1_\mu$.

(2 \Rightarrow 1) The following fact is well-known.

Fact. For every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ there is an A -indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that for every $n < \omega$ there is some $I \subseteq \kappa$ of cardinality n such that $\mathcal{C}_{\upharpoonright n} \equiv_A \mathcal{D}_{\upharpoonright I}$.

Suppose $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Pick a model M containing A . Then μ is invariant over M . Apply the fact with $n = 2$ and M for A to obtain A -indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that $\mathcal{C}_0, \mathcal{C}_1 \equiv_M \mathcal{D}_i, \mathcal{D}_j$ for some $i < j < \kappa$. From the M -invariance of μ we obtain $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. Then $\mathcal{C}_0 \in 1_\mu$. Again by M -invariance, $\mathcal{D}_i \in 1_\mu$. \square

Example 4. Let $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{|\mathcal{X}|} : A^{|\mathcal{X}|} \subseteq \mathcal{X}\}$. Then μ is S1 over A . Clearly, a type $p(x) \subseteq L(\mathcal{U})$ is μ -wide if and only if it is finitely satisfied in A .

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A -indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{|\mathcal{X}|}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{|\mathcal{X}|}$. Hence $\mathcal{D}_0 \in 1_\mu$. \square

For every small set A we define

$$\lambda^A(x) = \{\varphi(x) : \varphi(x) \in p \text{ for every } p \in S_x(\mathcal{U}) \text{ that is Lascar-invariant over } A\}$$

Clearly, a type is λ^A -wide if and only if it has an extension to a global type that is Lascar-invariant over A .

Lemma 5. λ^A is S1 over A .

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_{\lambda^A}$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A -indiscernibles. Then $\mathcal{D}_0, \mathcal{D}_1$ are conjugated over some model M containing A . If p is any global type Lascar-invariant over A , then p is invariant over M . Therefore p contains $x \in \mathcal{D}_0$ if and only if it contains $x \in \mathcal{D}_1$. Therefore $\mathcal{D}_0 \in 1_{\lambda^A}$. \square

We say that \mathcal{D} is **generic** over A if for every finitely many translates of \mathcal{D} under $\text{Aut}(\mathcal{U}/A)$ cover $\mathcal{U}^{|\mathcal{D}|}$. Note that \mathcal{D} is generic over A if and only if $\text{orb}(\neg\mathcal{D}/A)$ does not have **fip** (i.e. the finite intersection property).

The definition of **Lascar-generic** over A is similar, we only replace $\text{Aut}(\mathcal{U}/A)$ by $\text{Autf}(\mathcal{U}/A)$. Again, this is equivalent to requiring that $\text{orbf}(\neg\mathcal{D}/M)$ does not have fip.

Fact 6. 1_{λ^A} is the filter generated by sets that are Lascar-generic over A .

Proof. (\supseteq) It suffices to prove if \mathcal{D} is Lascar-generic over A then $\mathcal{D} \in 1_{\lambda^A}$. That is, every λ^A -random type p contains $x \in \mathcal{D}$. Let p be λ^A -random. By Lascar-genericity, there are some strong A -automorphisms f_1, \dots, f_n be such that $\mathcal{U}^{|\mathcal{D}|} = \bigcup_i f_i \mathcal{D}$. By completeness, p contains $x \in f_i \mathcal{D}$ for some i . By Lascar-invariance, p contains also $x \in \mathcal{D}$.

(\subseteq) Assume that there are no Lascar-generic sets \mathcal{B}_i such that

$$\bigcap_{i=1}^n \mathcal{B}_i \subseteq \mathcal{D}$$

Hence, by taking complements, for any \mathcal{C}_i such that

$$\neg\mathcal{D} \subseteq \bigcup_{i=1}^n \mathcal{C}_i,$$

at least one $\text{orbf}(\mathcal{C}_i/A)$ has the fip. By Fact 8 below, we obtain that $x \notin \mathcal{D}$ belongs to some global type that is Lascar-invariant over A . Therefore $\mathcal{D} \notin 1_{\lambda^A}$. \square

Fact 7. For every \mathcal{D} the following are equivalent

1. there is a global type containing $x \in \mathcal{D}$ that is invariant over A ;
2. every finite cover $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ of \mathcal{D} contains some \mathcal{C}_i such that $\text{orb}(\mathcal{C}_i/A)$ has fip.

Proof. ($1 \Rightarrow 2$) Let p be a global A -invariant type containing $\varphi(x)$. Then p contains \mathcal{C}_i for some i . By invariance, p contains $x \in \mathcal{C}$ for every \mathcal{C} conjugated to \mathcal{C}_i hence (2) follows.

($2 \Rightarrow 1$) Let p be maximally finitely consistent type containing $\varphi(x)$ such that all formulas p have the property in (2). We claim that p is complete type. Suppose for a contradiction that $\vartheta(x), \neg\vartheta(x) \notin p$. By maximality there is some $\psi(x) \in p$ and some $\mathcal{C}_1, \dots, \mathcal{C}_n$ covering

both $\psi(\mathcal{U}) \cap \mathcal{D}$ and $\psi(\mathcal{U}) \setminus \mathcal{D}$ such that no $\text{orb}(\mathcal{C}_i/A)$ has fip. This is a contradiction because $\mathcal{C}_1, \dots, \mathcal{C}_n$ also cover $\psi(\mathcal{U})$. Finally, (1) follows if we can prove that p is invariant over A . Suppose not then, by completeness, p contains $x \in \mathcal{B} \setminus f\mathcal{B}$ for some \mathcal{B} and some $f \in \text{Aut}(\mathcal{U}/A)$. But $\mathcal{B} \setminus f\mathcal{B}$ is disjoint of $f\mathcal{B} \setminus f^2\mathcal{B}$, hence it cannot satisfy (2). A contradiction. \square

By essentially the same proof we obtain the following.

Fact 8. For every \mathcal{D} the following are equivalent

1. there is a global type containing $x \in \mathcal{D}$ that is Lascar-invariant over A ;
2. every finite cover $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ of \mathcal{D} contains some \mathcal{C}_i such that $\text{orbf}(\mathcal{C}_i/A)$ has fip.

In Example 4 the elements of $A^{|x|}$ are, trivially, μ -random. Clearly, there are many other μ -random types: all global types that are finitely satisfied in A are μ -random. If we allow elements outside \mathcal{U} , the filter 1_{λ^A} has an expression similar to 1_μ in Example 4. In fact, as the number of invariant types is bounded, it is immediate that there is a small set of λ^A -random elements R such that $1_{\lambda^A} = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} : R \subseteq {}^*\mathcal{X}\}$.

Definition 9. Let 1_{v^A} be the filter generated by the sets $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ such that there is an A -indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$ such that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_0, \dots, \mathcal{D}_n$ cover $\mathcal{U}^{|x|}$ for some n . We call 1_{v^A} the **nonforking filter** over A . We call 0_{v^A} the **forking ideal** over A .

Fact 10. Let μ be S1 over A . Then $1_{v^A} \subseteq 1_\mu$.

An immediate consequence that μ -random elements are v^A -random.

Proof. Let $\mathcal{D} \in 1_{v^A}$. Let $\langle \mathcal{D}_i : i < \omega \rangle$ be as in Definition 9. Then $\mathcal{D}_0 \cup \dots \cup \mathcal{D}_n \in 1_\mu$ for some n . Assume n is minimal, we prove that $n = 0$. Otherwise, let $\mathcal{C}_i = \mathcal{D}_i \cup \dots \cup \mathcal{D}_{i+n-1}$. Then $\langle \mathcal{C}_i : i < \omega \rangle$ is an A -indiscernible sequence and $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. From S1 we obtain $\mathcal{C}_0 \in 1_\mu$ which contradicts the minimality of n . \square

A type $p(x; y)$ that is Lascar-invariant over A is **stable** if $p(a_0; b_1) \rightarrow p(a_1; b_0)$ for every A -indiscernible sequence $\langle a_i; b_i : i < \omega \rangle$.

Fact 11. Every stable type $p(x; y)$ is equivalent to a type $q(x; y) \subseteq p$ containing only stable formulas.

Proof. Let $i(\bar{x}; \bar{y})$ be the type that says that $\langle x_i; y_i : i < \omega \rangle$ is a sequence of indiscernibles over A . The required type $q(x; y)$ contains the formulas $\psi(x; y) \in p$ such that $i(\bar{x}; \bar{y}) \wedge \psi(x_0; y_1) \rightarrow \psi(x_1; y_0)$. \square

Theorem 12. Let $p(x; z), q(x; z) \subseteq L(\mathcal{U})$ be Lascar-invariant over A . Let μ be S1 and Lascar-invariant over A . Then the relation

$$R(a, b) \Leftrightarrow p(x; a) \cup q(x; b) \text{ is wide.}$$

is stable.

Proof. Pick $\varphi(x; z) \in p$ and $\psi(x; z) \in q$. Let $\langle a_i; b_i : i < \omega \rangle$ be a sequence of A -indiscernibles such that $\varphi(x; a_0) \wedge \psi(x; b_1)$ is μ -wide. By S1, also $[\varphi(x; a_0) \wedge \psi(x; b_1)] \wedge [\varphi(x; a_2) \wedge \psi(x; b_3)]$ is μ -wide. A fortiori $\varphi(x; a_2) \wedge \psi(x; b_1)$ is μ -wide and, by indiscernibility and the Lascar-invariance of μ , also $\varphi(x; a_1) \wedge \psi(x; b_0)$ is wide. \square

Fact 13. Let $q(x; y) \subseteq L(M)$ be stable. Assume $a \equiv_M a'$ and $b \equiv_M b'$ are such that $a \perp_M b$ and $a' \perp_M b'$. Then $q(a; b) \leftrightarrow q(a'; b')$.

Proof. By invariance we may assume that $b' = b$. Let $\varphi(x; y) \in q$. By Fact 11 we can assume that $\varphi(x; y)$ is stable. Then $\varphi(a; b) \leftrightarrow \varphi(a'; b)$ by stationarity. \square

Definition 14. We say that μ is type-definable over A if for every formula $\varphi(x; z) \in L$ the set $\{b \in \mathcal{U}^{|z|} : \varphi(x; z) \in \mu\}$ is type-definable over A .

For instance, μ in Example 4 is type definable.

Note that if μ is type-definable over A and $\varphi(x; b) \notin \mu$ then there is a formula $\vartheta(z) \in L(A)$ such that $\vartheta(b)$ and $\varphi(x; b') \notin \mu$ for every $b' \models \vartheta(z)$.

1. Stable groups

In this section \mathcal{U} is a group; the group operations, denoted by $1, \cdot, ^{-1}$, are among the symbols of L .

Let $\mathcal{D} \subseteq \mathcal{U}$ be a definable set.

Theorem 15. (T stable) The following are equivalent

1. \mathcal{D} is Lascar-generic over \emptyset ;
2. for some $g_1, \dots, g_n \in \mathcal{U}$, the sets $g_i \cdot \mathcal{D}$ cover \mathcal{U} .

Proof (?). Let L' be a language that has only one binary relation symbol $r(x, y)$. In \mathcal{U} we interpret $r(x, y)$ by $y^{-1} \cdot x \in \mathcal{D}$. Note that \mathcal{D} is definable by the formula $r(x, 1)$. By stability exactly one of the formulas $r(x, 1)$ and $\neg r(x, 1)$ is almost finitely satisfied in \emptyset . Assume the first, then $\neg r(\mathcal{U}, 1)$ has no extension to a global Lascar-invariant L' -type. \square

Then L' -orbf($\neg r(\mathcal{U}, 1)$) does not have fip. Equivalently, there are $f_i \in L'$ -Autf(\mathcal{U}), where $i = 1, \dots, n$ such that the sets $r(\mathcal{U}, f_i 1)$ cover \mathcal{U} . Let $g_i = f_i \mathcal{U}$, then the sets $g_i \cdot \mathcal{D}$ covers \mathcal{U} .

2. Subgroups with bounded orbit (tentative ramble)

Let $G \trianglelefteq \text{Aut}(\mathcal{U}/A)$. We write $\text{orb}(a/G)$ and $\text{orb}(a/A)$ for the orbit of a under G , respectively $\text{Aut}(\mathcal{U}/A)$. We define $\text{orb}(\mathcal{D}/G)$ and $\text{orb}(\mathcal{D}/A)$ similarly.

We say that G is bounded if the action of G on \mathcal{U} has $< \kappa$ orbits. If G is bounded, then the number of G -invariant sets \mathcal{D} is also $< \kappa$.

3. Independence

Let $\mu \subseteq L_\nu(\mathcal{U})$, where $|\nu| = 1$, be A -invariant. When x is the tuple $\langle x_i : i < |x| \rangle$, we say that $p(x) \subseteq L(\mathcal{U})$ is μ -wide if it is finitely consistent with $\bigcup_i \mu(x_i)$.

Let $a \in \mathcal{U}^{|x|}$. We write $a \perp_\mu b$ if the type $\text{tp}(a/A, b)$ is μ -wide. Clearly, this is equivalent to saying that, $\varphi(x; b)$ is μ -wide for every $\varphi(x; z) \in L(A)$ such that $\varphi(a; b)$.

We write $b \equiv_\mu b'$ if $\mu(x)$ implies $\varphi(x; b) \leftrightarrow \varphi(x; b')$ for every $\varphi(x; z) \in L(A)$.

Lemma 16. The following properties hold for all a, b, c

- | | |
|--|-------------------------|
| 1. $a \perp_\mu b \Rightarrow f a \perp_\mu f b$ for every $f \in \text{Aut}(\mathcal{U}/A)$ | <i>invariance</i> |
| 2. $a \perp_\mu b \Leftarrow a_0 \perp_\mu b_0$ for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$ | <i>finite character</i> |
| 3. $a \perp_\mu c, b$ and $c \perp_\mu b \Rightarrow a, c \perp_\mu b$ | <i>transitivity</i> |
| 4. $a \perp_\mu b \Rightarrow$ there exists $a' \equiv_{A, b} a$ such that $a' \perp_\mu b, c$ | <i>extension</i> |
| 5. $a \perp_\mu b_1, b_2$ and $b_1 \equiv_\mu b_2 \Rightarrow b_1 \equiv_{\mu, a} b_2$ | <i>non-splitting</i> |

Proof. Properties 1,2,3 are immediate. We prove 4. As $\text{tp}(a/A, b)$ is μ -wide, it extends to a μ -random type $p(x)$. Then we can take any $a' \models p \upharpoonright_{A, b, c}(x)$. \square

Definition 17. We say that \perp_μ is **stationary** if $a \equiv_M x \perp_\mu b$ is a complete type over M, b for all finite tuples b and a .

We say **n -stationary** if this is restricted to $|a| = n$.

Stationarity is often ensured by the following property.

Proposition 18. If for every $\varphi(x) \in L(\mathcal{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(\mathcal{U}) =_\mu \psi(\mathcal{U})$ then \perp_μ is stationary.

Proof. Let $b \in \mathcal{U}^{|z|}$ and $a_1, a_2 \in \mathcal{U}^{|x|}$ be such that $a_i \perp_\mu b$ and $a_1 \equiv_M a_2$. We claim that $a_1 \equiv_{M, b} a_2$. We need to prove that $\varphi(b; a_1) \leftrightarrow \varphi(b; a_2)$ for every $\varphi(z; x) \in L(M)$. Let $\psi(x) \in L(M)$ be such that $\varphi(b; \mathcal{U}) =_M \psi(\mathcal{U})$. From $a_i \perp_\mu b$ we obtain that $\varphi(b; a_i) \leftrightarrow \psi(a_i)$. Finally, the claim follows because $a_1 \equiv_\mu a_2$. \square

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