

## $\mu$ -Random thoughts

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Let  $\mathcal{U}$  be a monster model. We confuse formulas  $\varphi(x) \in L(\mathcal{U})$  with the subset of  $\mathcal{U}^{|\mathcal{U}|}$  that they define. If  $\mathcal{D} = \varphi(\mathcal{U})$ , we may write  $x \in \mathcal{D}$  for  $\varphi(x)$ . Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters from  $\mathcal{U}$ ). Global type are types over  $\mathcal{U}$  that are complete. Unless otherwise stated all other types are partial.

We denote by  ${}^*\mathcal{U}$  an elementary extension of  $\mathcal{U}$  where all global type are realized. If  $\mathcal{D} = \varphi(\mathcal{U})$ , we write  ${}^*\mathcal{D}$  for  $\varphi({}^*\mathcal{U})$ . If  $p(x) \in L(\mathcal{U})$  we write  ${}^*p$  for  $p({}^*\mathcal{U})$ .

Below by  $\mu(x) \in L(\mathcal{U})$  we always denote a consistent type closed under conjunctions and logical consequences that is, if  $\varphi(x) \in \mu$  and  $\varphi(x) \rightarrow \psi(x)$  then  $\psi(x) \in \mu$ . We denote by  $1_\mu$  the filter  $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$ . We write  $0_\mu$  for the corresponding ideal that is, the ideal  $\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$ . Note that the expressions  $\mathcal{D} \in 1_\mu$  and  ${}^*\mu \subseteq {}^*\mathcal{D}$  are synonymous.

The elements of  ${}^*\mu$  are called  $\mu$ -random. Let  $p(x) \in L(\mathcal{U})$ . We say that  $p(x) \in L(\mathcal{U})$  is  $\mu$ -wide if  $\mu(x) \cup p(x)$  is finitely consistent. A global  $\mu$ -wide type is also called a  $\mu$ -random type. In other words,  $\mu$ -random types are the type over  $\mathcal{U}$  of  $\mu$ -random elements.

The following is tautological but worth noticing.

**Remark 1.** A type is  $\mu$ -wide if and only if it is realized by a random element. Every  $\mu$ -wide type extends to a  $\mu$ -random type.

Let  $\kappa = \kappa^{<\kappa}$  be the cardinality of  $\mathcal{U}$ . We say that  $\mu$  is  $\kappa$ -prime if for every sequence  $\langle \mathcal{D}_i : i < \kappa \rangle$  of definable sets such that  $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$  for every  $i < j < \kappa$ , there is an  $i < \kappa$  such that  $\mathcal{D}_i \in 1_\mu$ . By the regularity of  $\kappa$  this equivalent to requiring that  $\mathcal{D}_i \in 1_\mu$  for every  $i < \kappa$ .

Sometimes we use the dual version of this property which sounds: for every sequence  $\langle \mathcal{D}_i : i < \kappa \rangle$  of definable sets such that  $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$  for every  $i < j < \kappa$ , there is an  $i < \kappa$  such that  $\mathcal{D}_i \in 0_\mu$ .

**Example 2.** Assume there is a finitely additive probability measure on the definable subsets of  $\mathcal{U}^{|\mathcal{U}|}$ . Let  $1_\mu$  be the set of subsets of measure 1. Then  $\mu$  is  $\kappa$ -prime.

*Proof.* Assume for a contradiction that the sets  $\langle \mathcal{D}_i : i < \kappa \rangle$  have positive measure but that  $\mathcal{D}_i \cap \mathcal{D}_j$  has measure 0 for every  $i < j < \kappa$ . As  $\kappa$  is regular and uncountable, we may assume that for some  $n$  all sets  $\mathcal{D}_i$  have measure  $\geq 1/n$ . Then, up to a set measure 0,

the sets  $\mathcal{D}_1, \dots, \mathcal{D}_n$  are pairwise disjoint and contained in  $\neg\mathcal{D}_0$ . This is a contradiction because  $\neg\mathcal{D}_0$  has measure  $< 1$ .  $\square$

We say that  $\mu$  is **S1** over  $A$  if for every  $A$ -indiscernible sequence  $\langle \mathcal{D}_i : i < \omega \rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu \Rightarrow \mathcal{D}_0 \in 1_\mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

**Fact 3.** For any  $\mu$  that is Lascar-invariant over  $A$  the following are equivalent

1.  $\mu$  is  $\kappa$ -prime;
2.  $\mu$  is S1 over  $A$ .

*Proof.* (1 $\Rightarrow$ 2) Let  $\langle \mathcal{D}_i : i < \omega \rangle$  be a sequence of  $A$ -indiscernibles such that  $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$ . Then  $\langle \mathcal{D}_i : i < \omega \rangle$  is  $M$ -indiscernible for some model  $M$  containing  $A$ . As  $\mu$  is invariantly over  $M$ . By compactness, we can stretch this sequence one to length  $\kappa$ . From indiscernibility and the  $M$ -invariance of  $\mu$  we obtain that  $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$  for every  $i < j < \kappa$ . Then  $\mathcal{D}_i \in 1_\mu$  for some  $i < \kappa$  and, again by indiscernibility and invariance,  $\mathcal{D}_0 \in 1_\mu$ .

(2 $\Rightarrow$ 1) The following fact is well-known.

Fact. For every sequence  $\langle \mathcal{D}_i : i < \kappa \rangle$  there is an  $A$ -indiscernible sequence  $\langle \mathcal{C}_i : i < \omega \rangle$  such that for every  $n < \omega$  there is some  $I \subseteq \kappa$  of cardinality  $n$  such that  $\mathcal{C}_{\upharpoonright n} \equiv_A \mathcal{D}_{\upharpoonright I}$ .

Suppose  $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$  for every  $i < j < \kappa$ . Pick a model  $M$  containing  $A$ . Then  $\mu$  is invariant over  $M$ . Apply the fact with  $n = 2$  and  $M$  for  $A$  to obtain  $A$ -indiscernible sequence  $\langle \mathcal{C}_i : i < \omega \rangle$  such that  $\mathcal{C}_0, \mathcal{C}_1 \equiv_M \mathcal{D}_i, \mathcal{D}_j$  for some  $i < j < \kappa$ . From the  $M$ -invariance of  $\mu$  we obtain  $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$ . Then  $\mathcal{C}_0 \in 1_\mu$ . Again by  $M$ -invariance,  $\mathcal{D}_i \in 1_\mu$ .  $\square$

**Example 4.** Let  $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{[x]} : A^{[x]} \subseteq \mathcal{X}\}$ . Then  $\mu$  is S1 over  $A$ . Clearly, a type  $p(x) \in L(\mathcal{U})$  is  $\mu$ -wide if and only if it is finitely satisfied in  $A$ .

*Proof.* Assume  $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$ , where  $\mathcal{D}_0, \mathcal{D}_1$  start a sequence of  $A$ -indiscernibles. Then  $a \in \mathcal{D}_0 \cup \mathcal{D}_1$  for every  $a \in A^{[x]}$ . By indiscernibility,  $a \in \mathcal{D}_0$  holds for every  $a \in A^{[x]}$ . Hence  $\mathcal{D}_0 \in 1_\mu$ .  $\square$

For every small set  $A$  we define

$$\mu^A(x) = \{\varphi(x) : \varphi(x) \in p \text{ for every } p \in S_x(\mathcal{U}) \text{ that is Lascar-invariant over } A\}$$

Clearly, a type is  $\mu^A$ -wide if and only if it has an extension to a global type Lascar-invariant over  $A$ .

**Lemma 5.**  $\mu^A$  is S1 over  $A$ .

*Proof.* Assume  $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_{\mu^A}$ , where  $\mathcal{D}_0, \mathcal{D}_1$  start a sequence of  $A$ -indiscernibles. Then  $\mathcal{D}_0, \mathcal{D}_1$  are conjugated over some model  $M$  containing  $A$ . If  $p$  is any global type Lascar-invariant over  $A$ , then  $p$  is invariant over  $M$ . Therefore  $p$  contains  $x \in \mathcal{D}_0$  if and only if it contains  $x \in \mathcal{D}_1$ . Therefore  $\mathcal{D}_0 \in 1_{\mu^A}$ .  $\square$

We say that  $\mathcal{D}$  is **generic** over  $A$  if for every finitely many translates of  $\mathcal{D}$  under  $\text{Aut}(\mathcal{U}/A)$  cover  $\mathcal{U}^{|\mathcal{X}|}$ . Note that  $\mathcal{D}$  is generic over  $A$  if and only if  $\text{orb}(\neg\mathcal{D}/A)$  does not have **fip** (i.e. the finite intersection property).

The definition of **Lascar-generic** over  $A$  is similar, we only replace  $\text{Aut}(\mathcal{U}/A)$  by  $\text{Autf}(\mathcal{U}/A)$ . Again, this is equivalent to requiring that  $\text{orbf}(\neg\mathcal{D}/M)$  does not have fip.

**Fact 6.**  $1_{\mu^A}$  is the filter generated by sets that are Lascar-generic over  $A$ .

*Proof.*  $(\supseteq)$  It suffices to prove if  $\mathcal{D}$  is Lascar-generic over  $A$  then  $\mathcal{D} \in 1_{\mu^A}$ . That is, every  $\mu^A$ -random type  $p$  contains  $x \in \mathcal{D}$ . Let  $p$  be  $\mu^A$ -random. By Lascar-genericity, there are some strong  $A$ -automorphisms  $f_1, \dots, f_n$  be such that  $\mathcal{U}^{|\mathcal{X}|} = \bigcup_i f_i \mathcal{D}$ . By completeness,  $p$  contains  $x \in f_i \mathcal{D}$  for some  $i$ . By Lascar-invariance,  $p$  contains also  $x \in \mathcal{D}$ .

$(\subseteq)$  Assume that there are no Lascar-generic sets  $\mathcal{B}_i$  such that

$$\bigcap_{i=1}^n \mathcal{B}_i \subseteq \mathcal{D}$$

Hence, by taking complements, for any  $\mathcal{C}_i$  such that

$$\neg\mathcal{D} \subseteq \bigcup_{i=1}^n \mathcal{C}_i,$$

at least one  $\text{orbf}(\mathcal{C}_i/A)$  has the fip. By Fact 8 below, we obtain that  $x \notin \mathcal{D}$  belongs to some global type that Lascar-invariant over  $A$ . Therefore  $\mathcal{D} \notin 1_{\mu^A}$ .  $\square$

**Fact 7.** For every  $\mathcal{D}$  the following are equivalent

1. there is a global type containing  $x \in \mathcal{D}$  that is invariant over  $A$ ;
2. for every  $\mathcal{C}_1, \dots, \mathcal{C}_n$  covering  $\mathcal{D}$  there is some  $i$  such that  $\text{orb}(\mathcal{C}_i/A)$  has fip.

*Proof.*  $(1 \Rightarrow 2)$  Let  $p$  be a global  $A$ -invariant type containing  $\varphi(x)$ . Then  $p$  contains  $\mathcal{C}_i$  for some  $i$ . By invariance,  $p$  contains  $x \in \mathcal{C}$  for every  $\mathcal{C}$  conjugated to  $\mathcal{C}_i$  hence (2) follows.

$(2 \Rightarrow 1)$  Let  $p$  be maximally finitely consistent type containing  $\varphi(x)$  such that all formulas  $p$  have the property in (2). We claim that  $p$  is complete type. Suppose for a contradiction that  $\vartheta(x), \neg\vartheta(x) \notin p$ . By maximality there is some  $\psi(x) \in p$  and some  $\mathcal{C}_1, \dots, \mathcal{C}_n$  covering both  $\psi(\mathcal{U}) \cap \mathcal{D}$  and  $\psi(\mathcal{U}) \setminus \mathcal{D}$  such that no  $\text{orb}(\mathcal{C}_i/A)$  has fip. This is a contradiction because  $\mathcal{C}_1, \dots, \mathcal{C}_n$  also cover  $\psi(\mathcal{U})$ . Finally, (1) follows if we can prove that  $p$  is invariant over  $A$ . Suppose not then, by completeness,  $p$  contains  $x \in \mathcal{B} \setminus f\mathcal{B}$  for some  $\mathcal{B}$  and some  $f \in \text{Aut}(\mathcal{U}/A)$ . But  $\mathcal{B} \setminus f\mathcal{B}$  is disjoint of  $f\mathcal{B} \setminus f^2\mathcal{B}$ , hence it cannot satisfy (2). A contradiction.  $\square$

By essentially the same proof we obtain the following.

**Fact 8.** For every  $\mathcal{D}$  the following are equivalent

1. there is a global type containing  $x \in \mathcal{D}$  that is Lascar-invariant over  $A$ ;
2. for every  $\mathcal{C}_1, \dots, \mathcal{C}_n$  covering  $\mathcal{D}$  there is some  $i$  such that  $\text{orbf}(\mathcal{C}_i / A)$  has fip.

In Example 4 the elements of  $A^{|x|}$  are, trivially,  $\mu$ -random. Clearly, there are many other  $\mu$ -random types: all global types that are finitely satisfied in  $A$  are  $\mu$ -random. If we allow elements outside  $\mathcal{U}$ , the filter  $1_{\mu^A}$  has an expression similar to  $1_\mu$  in Example 4. In fact, as the number of invariant types is bounded, it is immediate that there is a small set of  $\mu^A$ -random elements  $R$  such that  $1_{\mu^A} = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} : R \subseteq {}^*\mathcal{X}\}$ .

**Definition 9.** Let  $1_{v^A}$  be the filter generated by the sets  $\mathcal{D} \subseteq \mathcal{U}^{|x|}$  such that there is an  $A$ -indiscernible sequence  $\langle \mathcal{D}_i : i < \omega \rangle$  such that  $\mathcal{D} = \mathcal{D}_0$  and  $\mathcal{D}_0, \dots, \mathcal{D}_n$  cover  $\mathcal{U}^{|x|}$  for some  $n$ . We call  $1_{v^A}$  the **nonforking filter** over  $A$ . We call  $0_{v^A}$  the **forking ideal** over  $A$ .

**Fact 10.** Let  $\mu$  be S1 over  $A$ . Then  $1_\mu$  contains  $1_{v^A}$ .

*Proof.* Let  $\mathcal{D} \in 1_{v^A}$ . Let  $\langle \mathcal{D}_i : i < \omega \rangle$  be as in Definition 9. Then  $\mathcal{D}_0 \cup \dots \cup \mathcal{D}_n \in 1_\mu$  for some  $n$ . Assume  $n$  is minimal, we prove that  $n = 0$ . Otherwise, let  $\mathcal{C}_i = \mathcal{D}_i \cup \dots \cup \mathcal{D}_{i+n-1}$ . Then  $\langle \mathcal{C}_i : i < \omega \rangle$  is an  $A$ -indiscernible sequence and  $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$ . From S1 we obtain  $\mathcal{C}_0 \in 1_\mu$  which contradicts the minimality of  $n$ .  $\square$

**Theorem 11.** Let  $p(x; z), q(x; z) \subseteq L(A)$ . Let  $\mu$  be S1 and Lascar-invariant over  $A$ . Then the relation below is stable

$$R(a, b) \Leftrightarrow p(x; a) \cup q(x; b) \text{ is wide.}$$

*Proof.* Let  $\langle a_i, b_i : i < \omega \rangle$  be a sequence of  $A$ -indiscernibles such that  $p(x; a_0) \cup q(x; b_1)$  is  $\mu$ -wide. By S1, also  $[p(x; a_0) \cup q(x; b_1)] \cup [p(x; a_2) \cup q(x; b_3)]$  is  $\mu$ -wide. A fortiori  $p(x; a_2) \cup q(x; b_1)$  is  $\mu$ -wide and, by indiscernibility and the Lascar-invariance of  $\mu$ , also  $p(x; a_1) \cup q(x; b_0)$  is wide.  $\square$

**Definition 12.** We say that  $\mu$  is type-definable over  $A$  if for every formula  $\varphi(x; z) \in L$  the set  $\{b \in \mathcal{U}^{|z|} : \varphi(x; z) \in \mu\}$  is type-definable over  $A$ .

For instance,  $\mu$  in Example 4 is type definable.

Note that if  $\mu$  is type-definable over  $A$  and  $\varphi(x; b) \notin \mu$  then there is a formula  $\vartheta(z) \in L(A)$  such that  $\vartheta(b)$  and  $\varphi(x; b') \notin \mu$  for every  $b' \models \vartheta(z)$ .

**Fact 13.** Let  $q(x, y) \subseteq L(M)$  be stable. Assume  $a \equiv_M a'$  and  $b \equiv_M b'$  are such that  $a \perp_M b$  and  $a' \perp b'$ . Then  $q(a, b) \leftrightarrow q(a', b')$ .

*Proof.* Suppose not. By invariance we may assume that  $b' = b$ . Let  $\varphi(x; y) \in q$  be such that  $q(a, b)$  and  $\neg\varphi(a'; b)$ . Let  $p(x)$  be a global coheir of  $\text{tp}(a/M, b)$ . Let  $p'(x)$  be a global coheir of  $\text{tp}(a'/M, b)$ .

Let  $\langle a_i, b_i : i < \omega \rangle$  be defined as follows. Set  $b_0 = b$ . Then let  $b_{i+1} \equiv_{M, b \upharpoonright i} b_i$  be such that  $b_{i+1} \perp_M b \upharpoonright (i+1)$ ,  $a \upharpoonright i$  and  $a_i \models p(x) \upharpoonright M, b \upharpoonright i$ .  $\square$

### 1. Independence (tentative ramble)

Let  $\mu \subseteq L_v(\mathcal{U})$ , where  $|v| = 1$ , be  $A$ -invariant. When  $x$  is the tuple  $\langle x_i : i < |x| \rangle$ , we say that  $p(x) \subseteq L(\mathcal{U})$  is  $\mu$ -wide if it is finitely consistent with  $\bigcup_i \mu(x_i)$ .

Let  $a \in \mathcal{U}^{|x|}$ . We write  $a \perp_\mu b$  if the type  $\text{tp}(a/A, b)$  is  $\mu$ -wide. Clearly, this is equivalent to saying that,  $\varphi(x; b)$  is  $\mu$ -wide for every  $\varphi(x; z) \in L(A)$  such that  $\varphi(a; b)$ .

We write  $b \equiv_\mu b'$  if  $\mu(x)$  implies  $\varphi(x; b) \leftrightarrow \varphi(x; b')$  for every  $\varphi(x; z) \in L(A)$ .

**Lemma 14.** The following properties hold for all  $a, b, c$

1.  $a \perp_\mu b \Rightarrow f a \perp_\mu f b$  for every  $f \in \text{Aut}(\mathcal{U}/A)$  *invariance*
2.  $a \perp_\mu b \Leftarrow a_0 \perp_\mu b_0$  for all finite  $a_0 \subseteq a$  and  $b_0 \subseteq b$  *finite character*
3.  $a \perp_\mu c, b$  and  $c \perp_\mu b \Rightarrow a, c \perp_\mu b$  *transitivity*
4.  $a \perp_\mu b \Rightarrow$  there exists  $a' \equiv_{A, b} a$  such that  $a' \perp_\mu b, c$  *extension*
5.  $a \perp_\mu b_1, b_2$  and  $b_1 \equiv_\mu b_2 \Rightarrow b_1 \equiv_{\mu, a} b_2$  *non-splitting*

*Proof.* Properties 1,2,3 are immediate. We prove 4. As  $\text{tp}(a/A, b)$  is  $\mu$ -wide, it extends to a  $\mu$ -random type  $p(x)$ . Then we can take any  $a' \models p \upharpoonright_{A, b, c}(x)$ .  $\square$

**Definition 15.** We say that  $\perp_\mu$  is **stationary** if  $a \equiv_M x \perp_\mu b$  is a complete type over  $M, b$  for all finite tuples  $b$  and  $a$ .

We say  **$n$ -stationary** if this is restricted to  $|a| = n$ .

Stationarity is often ensured by the following property.

**Proposition 16.** If for every  $\varphi(x) \in L(\mathcal{U})$  there is a formula  $\psi(x) \in L(M)$  such that  $\varphi(\mathcal{U}) =_\mu \psi(\mathcal{U})$  then  $\perp_\mu$  is stationary.

*Proof.* Let  $b \in \mathcal{U}^{|z|}$  and  $a_1, a_2 \in \mathcal{U}^{|x|}$  be such that  $a_i \perp_\mu b$  and  $a_1 \equiv_M a_2$ . We claim that  $a_1 \equiv_{M, b} a_2$ . We need to prove that  $\varphi(b; a_1) \leftrightarrow \varphi(b; a_2)$  for every  $\varphi(z; x) \in L(M)$ . Let

$\psi(x) \in L(M)$  be such that  $\varphi(b; \mathcal{U}) =_M \psi(\mathcal{U})$ . From  $a_i \perp_\mu b$  we obtain that  $\varphi(b; a_i) \leftrightarrow \psi(a_i)$ . Finally, the claim follows because  $a_1 \equiv_\mu a_2$ .  $\square$

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