Branch: main 2024-01-21 17:01:07+01:00

Subgroups

Domenico Zambella

Let $\mathcal{U}$ be a monster model. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|x|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless stated otherwise, calligraphic capital letters denote definable sets (with parameters). If $p \in S_x(\mathcal{U})$ we write $p \in \mathcal{D}$ for $p(x) \to x \in \mathcal{D}$.

Let κ be the cardinality of $\mathcal{U}$. For $\mu(x) \subseteq L(\mathcal{U})$ we denote by 1_μ the filter generated by $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal, the ideal generated by $\big\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\big\}$.

We say that μ is κ-prime if for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. The dual version of this property sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 1. Assume there is a finitely additive probability measure on the definable subsets of and let μ be the set of formulas of measure 1. Then μ is κ-prime.

Proof. ??? Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa\rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. We can assume that for some $\varepsilon > 0$ all sets have measure $< 1-\epsilon$. Up to a set measure 0, the sets $\mathcal{D}_i$ are pairwise disjoint and $\mathcal{D}_i$ contains $\neg\mathcal{D}_j$ for every $j \neq i$. This is clearly a contradiction. □

We say that μ is S1 if for every A-indiscernible sequence $\langle \mathcal{D}_i : i < \omega\rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in \mu \quad \Rightarrow \quad \mathcal{D}_0 \in \mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

Fact 2. For any A-invariant filter μ the following are equivalent

1. μ is κ-prime;
2. μ is S1.

Proof. ... □

Example 3. Let $1_\mu = \big\{\mathcal{X} \subseteq \mathcal{U}^{|x|} \ : \ A^{|x|} \subseteq \mathcal{X}\big\}$. Then μ is S1.

Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in \mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A-indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{|x|}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{|x|}$. Hence $\mathcal{D}_0 \in \mu$. □

Example 4. Let $\mu = \big\{\mathcal{D} \subseteq \mathcal{U}^{|x|} \ : \ p \in \mathcal{D} \ \text{ for every } A\text{-invariant } p \in S_x(\mathcal{U})\big\}$. Then μ is κ-prime.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in \mu$, where $\mathcal{D}_0, \mathcal{D}_1$ are conjugate over A. By invariance $p \in \mathcal{D}_0$ if and only if $p \in \mathcal{D}_1$. Therefore $\mathcal{D}_0 \in \mu$. □

Example 5. We say that $\mathcal{D}$ is A-generic if finitely many A-translates of $\mathcal{D}$ cover $\mathcal{U}^{|x|}$. Then the filter generated by the A-generic definable sets is the filter μ in Example 4.

Proof. It is easy to see that if $\mathcal{D}$ is A-generic then $\mathcal{D} \in \mu$. Vice versa, assume that there are no A-generic sets $\mathcal{B}_i$ such that

$$\bigcap_{i=1}^{n} \mathcal{B}_i \quad \subseteq \quad \mathcal{D}$$

By taking complements, for any $\mathcal{C}_i$ such that

$$\neg\mathcal{D} \quad \subseteq \quad \bigcup_{i=1}^{n} \mathcal{C}_i$$

there is at least one i such that the A-orbit of $\mathcal{C}_i$ has the finite intersection property. By a standard argument we obtain that there is an A-invariant type $p \in \not{\mathcal{D}}$. Therefore $\mathcal{D} \notin \mu$ □

Proof. Let $\langle \mathcal{D}_i : i < \kappa \rangle$ be a sequence of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in \mu$ for every $i < j < \kappa$...... □

Definition 6. Let $p(x) \subseteq L(\mathcal{U})$. If $\mu(x) \cup p(x)$ is finitely consistent, then we say that $p(x)$ is wide.

Example 7. If μ is as in Example 3 then the following are equivalent

1. $p(x)$ is wide;
2. $p(x)$ is finitely satisfied in B.

Proof. (1⇒2) If $\varphi(x)$ is not finitely satisfiable in B, then $\neg\varphi(x)$ is in μ and $p(x)$ is not consistent with $\mu(x)$. (2⇒1) If $p(x) \rightarrow \neg\varphi(x)$ for some $\varphi(x) \in \mu$, ten $p(x)$ is not finitely satisfied in B. □

Theorem 8. Let $p(x;z), q(x;z) \subseteq L(A)$. Let μ be a k-prime and A-invariant. Then

$$R(a,b) \quad \Leftrightarrow \quad p(x;a) \cup q(x;b) \text{ is wide}$$

is a stable relation.

Proof. Let $\langle a_i, b_i : i < \omega\rangle$ be a sequence of A-indiscernibles such that $p(x;a_0) \cup q(x;b_1)$ is wide. By κ-primality, also $\big[p(x;a_0) \cup q(x;b_1)\big] \cup \big[p(x;a_2) \cup q(x;b_3)\big]$ is wide. A fortiori $p(x;a_2) \cup q(x;b_1)$ is wide and, by indiscernibility, so is $p(x;a_1) \cup q(x;b_0)$. □

Definition 9. The nonforking filter ν is the filter generated by the sets $\mathcal{D}$ such that some $\mathcal{D} = \mathcal{D}_1, \ldots, \mathcal{D}_n$ that starts a sequence of indiscernibles cover $\mathcal{U}$.