

μ -Random thoughts

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Let \mathcal{U} be a monster model. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|\mathcal{U}|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters). If $p \subseteq L_x(\mathcal{U})$ we write $p \subseteq \mathcal{D}$ for $p(x) \rightarrow x \in \mathcal{D}$. If $p \in S_x(\mathcal{U})$ we may also write $p \in \mathcal{D}$. In other words, when convenient we identify \mathcal{D} with a subset of $S_x(\mathcal{U})$ and incomplete types with the set of their completions.

When $\mu(x) \subseteq L(\mathcal{U})$ we denote by 1_μ the filter generated by $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal, the ideal generated by $\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and $\mu \subseteq \mathcal{D}$ are synonymous.

Let $p \subseteq L_x(\mathcal{U})$. We say that p is μ -wide if $\mu(x) \cup p(x)$ is finitely consistent. If $p \in S_x(\mathcal{U})$ we may also say that p is μ -random, in other words, p is μ -random if $p \in \mathcal{D}$ for every $\mathcal{D} \in 1_\mu$ or, equivalently, if $p \notin \mathcal{D}$ for every $\mathcal{D} \in 0_\mu$. That is, when convenient we identify μ with the sets of μ -random (global) types. The following is immediate.

Remark 1. Every μ -wide type $p \subseteq L_x(\mathcal{U})$ extends to a μ -random type.

Let $\kappa = \kappa^{<\kappa}$ be the cardinality of \mathcal{U} . We say that μ is κ -prime if for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. By the regularity of κ this equivalent to requiring that $\mathcal{D}_i \in 1_\mu$ for every $i < \kappa$.

Sometimes we use the dual version of this property which sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 2. Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|\mathcal{U}|}$. Let 1_μ be the set of subsets of measure 1. Then μ is κ -prime. Clearly, a global type $p \in S_x(\mathcal{U})$ is μ -random if and only if it is contained in all definable sets of measure 1.

Proof. Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa \rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As κ is regular and uncountable, we may assume that for some n all sets \mathcal{D}_i have measure $\geq 1/n$. Then, up to a set measure 0, the sets $\mathcal{D}_1, \dots, \mathcal{D}_n$ are pairwise disjoint and contained in $\neg\mathcal{D}_0$. This is a contradiction because $\neg\mathcal{D}_0$ has measure < 1 . \square

We say that μ is **S1** over A if for every A -indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu \Rightarrow \mathcal{D}_0 \in 1_\mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

Fact 3. For any A -invariant μ the following are equivalent

1. μ is κ -prime;
2. μ is S1 over A .

Proof. (1 \Rightarrow 2) Let $\langle \mathcal{D}_i : i < \omega \rangle$ be a sequence of A -indiscernibles such that $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$. By compactness, we can stretch this sequence to length κ . By indiscernibility $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Then $\mathcal{D}_i \in 1_\mu$ for some $i < \kappa$ and, by indiscernibility, $\mathcal{D}_0 \in 1_\mu$.

(2 \Rightarrow 1) The following fact is well-known.

Fact. For every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ be there is an A -indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that for every $n < \omega$ there is some $I \subseteq \kappa$ of cardinality n such that $\mathcal{C}_{\upharpoonright n} \equiv_A \mathcal{D}_{\upharpoonright I}$.

Suppose $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Apply the fact with $n = 2$ to obtain A -indiscernible sequence $\langle \mathcal{C}_i : i < \omega \rangle$ such that $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. Then $\mathcal{C}_0 \in 1_\mu$ implies that $\mathcal{D}_i \in 1_\mu$ for some $i < \kappa$. \square

Example 4. Let $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{|\mathcal{X}|} : A^{|\mathcal{X}|} \subseteq \mathcal{X}\}$. Then μ is S1 over A . Clearly, a type $p(x) \subseteq L(\mathcal{U})$ is μ -wide if and only if it is finitely satisfied in A .

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A -indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{|\mathcal{X}|}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{|\mathcal{X}|}$. Hence $\mathcal{D}_0 \in 1_\mu$. \square

Exercise 5. In Example 4 the elements of $A^{|\mathcal{X}|}$ are, trivially, μ -random. Then μ has the following property: there is a small set $R \subseteq S_x(\mathcal{U})$ of μ -random types such that

$$\mathcal{D} \in 1_\mu \Leftrightarrow R \subseteq \mathcal{D}$$

Characterize the types μ such that the above property holds.

Example 6. Let $\mu = \{\varphi(x) \in L(\mathcal{U}) : \varphi(x) \in p \text{ for every } A\text{-invariant } p \in S_x(\mathcal{U})\}$. Then μ is S1 over A . Clearly, a type $p \subseteq L_x(\mathcal{U})$ is μ -wide if and only if it has an extension to an A -invariant global type.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ are conjugate over A . Then $p \in \mathcal{D}_0$ if and only if $p \in \mathcal{D}_1$ for every A -invariant global type p . Therefore $\mathcal{D}_0 \in 1_\mu$. \square

Example 7. We say that \mathcal{D} is A -generic if finitely many A -translates of \mathcal{D} cover $\mathcal{U}^{[x]}$. Let μ be as in Example 6. Then 1_μ is the filter generated by the A -generic sets.

Proof. It is easy to see that if \mathcal{D} is A -generic then $\mathcal{D} \in 1_\mu$. Vice versa, assume that there are no A -generic sets \mathcal{B}_i such that

$$\bigcap_{i=1}^n \mathcal{B}_i \subseteq \mathcal{D}$$

Note that a set \mathcal{B} is not A -generic if and only if the orbit over A of $\neg \mathcal{B}$ has the finite intersection property (fip). Hence, by taking complements, for any \mathcal{C}_i such that

$$\neg \mathcal{D} \subseteq \bigcup_{i=1}^n \mathcal{C}_i,$$

there is at least one i such that the A -orbit of \mathcal{C}_i has the fip. By the fact below we obtain that there is an A -invariant type $p \notin \mathcal{D}$. Therefore $\mathcal{D} \notin 1_\mu$. \square

Fact 8. Assume that for every finitely many $\mathcal{C}_1, \dots, \mathcal{C}_n$ covering \mathcal{D} there is some \mathcal{C}_i whose orbit over A has the fip. Then there is an A -invariant global type $p \in S_x(\mathcal{U})$.

Proof. Let $p \subseteq L_x(\mathcal{U})$ be maximal type containing $x \in \mathcal{D}$ such that all formulas in p have the same property as \mathcal{D} in the fact. We claim that p is a global type. Suppose $\varphi(x), \neg \varphi(x) \notin p$. Then by maximality there is some $\psi(x) \in p$ and some $\mathcal{C}_1, \dots, \mathcal{C}_n$ covering both $\psi(\mathcal{U}) \cap \varphi(\mathcal{U})$ and $\psi(\mathcal{U}) \setminus \varphi(\mathcal{U})$ such that no \mathcal{C}_i has the fip. This is a contradiction because $\mathcal{C}_1, \dots, \mathcal{C}_n$ also cover $\psi(\mathcal{U})$. \square

Definition 9. Let 1_A be the filter generated by the sets $\mathcal{D} \subseteq \mathcal{U}^{[x]}$ such that there is an A -indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$ such that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_0, \dots, \mathcal{D}_n$ cover $\mathcal{U}^{[x]}$ for some n . We call 1_A the **nonforking filter** over A .

Fact 10. Let μ be S1 and invariant over A . Then 1_μ contains 1_A .

Proof. Let $\mathcal{D} \in 1_A$. Let $\langle \mathcal{D}_i : i < \omega \rangle$ be as in Definition 9. Then $\mathcal{D}_0 \cup \dots \cup \mathcal{D}_n \in 1_\mu$ for some n . Assume n is minimal, we prove that $n = 0$. Otherwise, let $\mathcal{C}_i = \mathcal{D}_i \cup \dots \cup \mathcal{D}_{i+n-1}$. Then $\langle \mathcal{C}_i : i < \omega \rangle$ is an A -indiscernible sequence and $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. From S1 we obtain $\mathcal{C}_0 \in 1_\mu$ which contradicts the minimality of n . \square

Theorem 11. Let $p(x; z), q(x; z) \subseteq L(A)$. Let μ be S1 and A -invariant. Then

$$R(a, b) \Leftrightarrow p(x; a) \cup q(x; b) \text{ is wide}$$

is a stable relation.

Proof. Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence of A -indiscernibles such that $p(x; a_0) \cup q(x; b_1)$ is μ -wide. By S1, also $[p(x; a_0) \cup q(x; b_1)] \cup [p(x; a_2) \cup q(x; b_3)]$ is μ -wide. A fortiori $p(x; a_2) \cup q(x; b_1)$ is μ -wide and, by indiscernibility, so is $p(x; a_1) \cup q(x; b_0)$. \square

We say that μ is **definable** if for every $\varphi(x; z) \in L$ the set $\{b \in \mathcal{U}^{|z|} : \mu(x) \rightarrow \varphi(x; b)\}$ is definable. The definition of μ **type-definable** is similar. For instance, in general μ in Example 4 is type definable, but it is definable if the ambient theory is stable.

1. Independence (tentative ramble)

Let $\mu \subseteq L_\nu(\mathcal{U})$, where $|\nu| = 1$, be A -invariant. When x is the tuple $\langle x_i : i < |x| \rangle$, we say that $p(x) \subseteq L(\mathcal{U})$ is μ -wide if it is finitely consistent with $\bigcup_i \mu(x_i)$.

Let $a \in \mathcal{U}^{|x|}$. We write $a \perp_\mu b$ if the type $\text{tp}(a/A, b)$ is μ -wide. Clearly, this is equivalent to saying that, $\varphi(x; b)$ is μ -wide for every $\varphi(x; z) \in L(A)$ such that $\varphi(a; b)$.

We write $b \equiv_\mu b'$ if $\mu(x)$ implies $\varphi(x; b) \leftrightarrow \varphi(x; b')$ for every $\varphi(x; z) \in L(A)$.

Lemma 12. The following properties hold for all a, b, c

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|--|-------------------------|
| 1. $a \perp_\mu b \Rightarrow f a \perp_\mu f b$ for every $f \in \text{Aut}(\mathcal{U}/A)$ | <i>invariance</i> |
| 2. $a \perp_\mu b \Leftarrow a_0 \perp_\mu b_0$ for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$ | <i>finite character</i> |
| 3. $a \perp_\mu c, b$ and $c \perp_\mu b \Rightarrow a, c \perp_\mu b$ | <i>transitivity</i> |
| 4. $a \perp_\mu b \Rightarrow$ there exists $a' \equiv_{A, b} a$ such that $a' \perp_\mu b, c$ | <i>extension</i> |
| 5. $a \perp_\mu b_1, b_2$ and $b_1 \equiv_\mu b_2 \Rightarrow b_1 \equiv_{\mu, a} b_2$ | <i>non-splitting</i> |

Proof. Properties 1,2,3 are immediate. We prove 4. As $\text{tp}(a/A, b)$ is μ -wide, it extends to a μ -random type $p(x)$. Then we can take any $a' \models p \upharpoonright_{A, b, c}(x)$. \square

Definition 13. We say that \perp_μ is **stationary** if $a \equiv_M x \perp_\mu b$ is a complete type over M, b for all finite tuples b and a .

We say **n -stationary** if this is restricted to $|a| = n$.

Stationarity is often ensured by the following property.

Proposition 14. If for every $\varphi(x) \in L(\mathcal{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(\mathcal{U}) =_\mu \psi(\mathcal{U})$ then \perp_μ is stationary.

Proof. Let $b \in \mathcal{U}^{|z|}$ and $a_1, a_2 \in \mathcal{U}^{|x|}$ be such that $a_i \perp_\mu b$ and $a_1 \equiv_M a_2$. We claim that $a_1 \equiv_{M, b} a_2$. We need to prove that $\varphi(b; a_1) \leftrightarrow \varphi(b; a_2)$ for every $\varphi(z; x) \in L(M)$. Let $\psi(x) \in L(M)$ be such that $\varphi(b; \mathcal{U}) =_M \psi(\mathcal{U})$. From $a_i \perp_\mu b$ we obtain that $\varphi(b; a_i) \leftrightarrow \psi(a_i)$. Finally, the claim follows because $a_1 \equiv_\mu a_2$. \square

References

- [MOS] Samaria Montenegro, Alf Onshuus, and Pierre Simon, *Stabilizers, NTP₂ groups with f -generics, and PRC fields*, J. Inst. Math. Jussieu **19** (2020), no. 3, 821–853.
- [Hr] Ehud Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. **25** (2012), no. 1, 189–243.
- [MP] Amador Martin-Pizarro and Daniel Palacín, *Stabilizers, Measures and IP-sets* (2020), arXiv:1912.07252