

μ -Random thoughts

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Let \mathcal{U} be a monster model. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|\mathcal{U}|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless stated otherwise, calligraphic capital letters denote definable sets (with parameters). If $p \in L_x(\mathcal{U})$ we write $p \subseteq \mathcal{D}$ for $p(x) \rightarrow x \in \mathcal{D}$. If $p \in S_x(\mathcal{U})$ we may also write $p \in \mathcal{D}$. In other words, when convenient, we identify p and \mathcal{D} with subsets (and/or elements) of $S_x(\mathcal{U})$.

When $\mu(x) \subseteq L(\mathcal{U})$ we denote by 1_μ the filter generated by $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal, the ideal generated by $\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and $\mu \subseteq \mathcal{D}$ are synonymous.

Let $p \in L_x(\mathcal{U})$. We say that p is μ -wide if $\mu(x) \cup p(x)$ is finitely consistent. If $p \in S_x(\mathcal{U})$ we may also say that p is μ -random, in other words, p is μ -random if $p \in \mathcal{D}$ for every $\mathcal{D} \in 1_\mu$ or, equivalently, if $p \notin \mathcal{D}$ for every $\mathcal{D} \in 0_\mu$. That is, when convenient we identify μ with the sets of μ -random (global) types.

Let κ be the cardinality of \mathcal{U} . We say that μ is κ -prime if for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. The dual version of this property sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 1. Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|\mathcal{U}|}$. Let 1_μ be the set of subsets of measure 1. Then μ is κ -prime. Clearly, $p \in S_x(\mathcal{U})$ is μ -random, if it is contained in all definable sets of measure 1.

Proof. Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa \rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As κ is uncountable, we can assume that all sets have measure $\geq 1/n$ for some n . Then, up to a set measure 0, the sets \mathcal{D}_i , for $0 < i \leq n$ are pairwise disjoint and contained in $\neg\mathcal{D}_0$. This is a contradiction because $\neg\mathcal{D}_0$ has measure < 1 . \square

We say that μ is **S1** if for every A -indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in \mu \Rightarrow \mathcal{D}_0 \in \mu,$$

a terminology that originated in some obscure corner of Hrushovski's mind.

Fact 2. For any A -invariant filter μ the following are equivalent

1. μ is κ -prime;
2. μ is S1.

Proof. ... □

Example 3. Let $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{[x]} : A^{[x]} \subseteq \mathcal{X}\}$. Then μ is S1. Clearly, a type $p(x) \subseteq L(\mathcal{U})$ is μ -wide if and only if it is finitely satisfied in A .

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A -indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{[x]}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{[x]}$. Hence $\mathcal{D}_0 \in 1_\mu$. □

Example 4. Let $\mu = \{\varphi(x) \in L(\mathcal{U}) : \varphi(x) \in p \text{ for every } A\text{-invariant } p \in S_x(\mathcal{U})\}$. Then μ is S1. Clearly, a type $p \subseteq L_x(\mathcal{U})$ is μ -wide if and only if it has an extension to an A -invariant global type.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ are conjugate over A . By invariance $p \in \mathcal{D}_0$ if and only if $p \in \mathcal{D}_1$. Therefore $\mathcal{D}_0 \in 1_\mu$. □

Example 5. We say that \mathcal{D} is A -generic if finitely many A -translates of \mathcal{D} cover $\mathcal{U}^{[x]}$. Let μ be as in Example 4. Then 1_μ is the filter generated by the A -generic sets.

Proof. It is easy to see that if \mathcal{D} is A -generic then $\mathcal{D} \in 1_\mu$. Vice versa, assume that there are no A -generic sets \mathcal{B}_i such that

$$\bigcap_{i=1}^n \mathcal{B}_i \subseteq \mathcal{D}$$

Note that a set \mathcal{B} is A -generic if and only if the orbit over A of $\neg\mathcal{B}$ has the finite intersection property (fip). Hence, by taking complements, for any \mathcal{C}_i such that

$$\neg\mathcal{D} \subseteq \bigcup_{i=1}^n \mathcal{C}_i$$

there is at least one i such that the A -orbit of \mathcal{C}_i has the fip. By a standard argument we obtain that there is an A -invariant type $p \in \mathcal{D}$. Therefore $\mathcal{D} \notin 1_\mu$. □

Theorem 6. Let $p(x; z), q(x; z) \subseteq L(A)$. Let μ be a k -prime and A -invariant. Then

$$R(a, b) \Leftrightarrow p(x; a) \cup q(x; b) \text{ is wide}$$

is a stable relation.

Proof. Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence of A -indiscernibles such that $p(x; a_0) \cup q(x; b_1)$ is wide. By κ -primality, also $[p(x; a_0) \cup q(x; b_1)] \cup [p(x; a_2) \cup q(x; b_3)]$ is wide. A fortiori $p(x; a_2) \cup q(x; b_1)$ is wide and, by indiscernibility, so is $p(x; a_1) \cup q(x; b_0)$. \square

Definition 7. The nonforking filter ν is the filter generated by the sets \mathcal{D} such that some $\mathcal{D} = \mathcal{D}_1, \dots, \mathcal{D}_n$ that starts a sequence of indiscernibles cover \mathcal{U} .