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# μ-Random thoughts

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Let $\mathcal{U}$ be a monster model. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|x|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters). If $p \subseteq L_x(\mathcal{U})$ we write $p \subseteq \mathcal{D}$ for $p(x) \rightarrow x \in \mathcal{D}$. If $p \in S_x(\mathcal{U})$ we may also write $p \in \mathcal{D}$. In other words, when convenient, we identify $\mathcal{D}$ with a subset of $S_x(\mathcal{U})$ and incomplete types with the set of their completions.

When $\mu(x) \subseteq L(\mathcal{U})$ we denote by $1_\mu$ the filter generated by $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write $0_\mu$ for the corresponding ideal, the ideal generated by $\big\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\big\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and $\mu \subseteq \mathcal{D}$ are synonymous.

Let $p \subseteq L_x(\mathcal{U})$. We say that $p$ is $\mu$-wide if $\mu(x) \cup p(x)$ is finitely consistent. If $p \in S_x(\mathcal{U})$ we may also say that $p$ is $\mu$-random, in other words, $p$ is $\mu$-random if $p \in \mathcal{D}$ for every $\mathcal{D} \in 1_\mu$ or, equivalently, if $p \notin \mathcal{D}$ for every $\mathcal{D} \in 0_\mu$. That is, when convenient we identify $\mu$ with the sets of $\mu$-random (global) types.

Let $\kappa = \kappa^{<\kappa}$ be the cardinality of $\mathcal{U}$. We say that $\mu$ is $\kappa$-prime if for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. The dual version of this property sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

**Example 1.** Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|x|}$. Let $1_\mu$ be the set of subsets of measure 1. Then $\mu$ is $\kappa$-prime. Clearly, $p \in S_x(\mathcal{U})$ is $\mu$-random, if and only if it is contained in all definable sets of measure 1.

*Proof.* Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa \rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As $\kappa$ is regular and uncountable, we can assume that all sets $\mathcal{D}_i$ have measure $\geq 1/n$ for some $n$. Then, up to a set measure 0, the sets $\mathcal{D}_i$, for $0 < i \leq n$ are pairwise disjoint and contained in $\neg\mathcal{D}_0$. This is a contradiction because $\neg\mathcal{D}_0$ has measure $< 1$. □

We say that $\mu$ is S1 if for every $A$-indiscernible sequence $\langle \mathcal{D}_i : i < \omega \rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu \quad \Rightarrow \quad \mathcal{D}_0 \in 1_\mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

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**Fact 2.** For any $A$-invariant filter $\mu$ the following are equivalent

1. $\mu$ is $\kappa$-prime;
2. $\mu$ is S1.

*Proof.* (2⇒1) By compactness we can find an indiscernible sequence of length $\kappa$ such that $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$. By indiscernibility $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Then $\mathcal{D}_i \in 1_\mu$ for some $i < \kappa$ and, by indiscernibility, $\mathcal{D}_0 \in 1_\mu$. (1⇒2) □

**Example 3.** Let $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} \ : \ A^{|x|} \subseteq \mathcal{X}\}$. Then $\mu$ is S1. Clearly, a type $p(x) \subseteq L(\mathcal{U})$ is $\mu$-wide if and only if it is finitely satisfied in $A$.

*Proof.* Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of $A$-indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{|x|}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{|x|}$. Hence $\mathcal{D}_0 \in 1_\mu$. □

**Remark 4.** Note that $\mu$ in the Example 3 has the following property. There is a small set $R$ of $\mu$-random types such that

$$\mathcal{D} \in 1_\mu \quad \Leftrightarrow \quad R \subseteq \mathcal{D}$$

In fact, in Example 3 the elements of $A^{|x|}$ are, trivially, $\mu$-random.

**Example 5.** Let $\mu = \{\varphi(x) \in L(\mathcal{U}) \ : \ \varphi(x) \in p \ \text{ for every } A\text{-invariant } p \in S_x(\mathcal{U})\}$. Then $\mu$ is S1. Clearly, a type $p \subseteq L_x(\mathcal{U})$ is $\mu$-wide if and only if it has an extension to an $A$-invariant global type.

*Proof.* Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ are conjugate over $A$. By invariance $p \in \mathcal{D}_0$ if and only if $p \in \mathcal{D}_1$. Therefore $\mathcal{D}_0 \in 1_\mu$. □

**Example 6.** We say that $\mathcal{D}$ is $A$-generic if finitely many $A$-translates of $\mathcal{D}$ cover $\mathcal{U}^{|x|}$. Let $\mu$ be as in Example 5. Then $1_\mu$ is the filter generated by the $A$-generic sets.

*Proof.* It is easy to see that if $\mathcal{D}$ is $A$-generic then $\mathcal{D} \in 1_\mu$. Vice versa, assume that there are no $A$-generic sets $\mathcal{B}_i$ such that

$$\bigcap_{i=1}^{n} \mathcal{B}_i \quad \subseteq \quad \mathcal{D}$$

Note that a set $\mathcal{B}$ is $A$-generic if and only if the orbit over $A$ of $\neg\mathcal{B}$ has the finite intersection property (fip). Hence, by taking complements, for any $\mathcal{C}_i$ such that

$$\neg\mathcal{D} \quad \subseteq \quad \bigcup_{i=1}^{n} \mathcal{C}_i$$

there is at least one $i$ such that the $A$-orbit of $\mathcal{C}_i$ has the fip. By a standard (easy) argument we obtain that there is an $A$-invariant type $p \notin \mathcal{D}$. Therefore $\mathcal{D} \notin 1_\mu$. □

**Definition 7.** Let $1_A$ be the filter generated by the sets $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ such that there is an $A$-indiscerible sequence $\langle \mathcal{D}_i : i < \omega \rangle$ such that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_0, \dots, \mathcal{D}_n$ cover $\mathcal{U}^{|x|}$ for some $n$. We call $1_A$ the nonforking filter over $A$.

**Fact 8.** Let $\mu$ be an S1 and invariant over $A$. Then $1_\mu$ contains $1_A$.

*Proof.* Let $\mathcal{D} \in 1_A$. If $\langle \mathcal{D}_i : i < \omega \rangle$ is as above, $\mathcal{D}_0 \cup \dots \cup \mathcal{D}_n \in 1_\mu$ for some $n$. Assume $n$ is minimal, we prove that $n = 0$. Otherwise, let $\mathcal{C}_i = \mathcal{D}_i \cup \dots \cup \mathcal{D}_{i+n-1}$. Then $\langle \mathcal{C}_i : i < \omega \rangle$ is an $A$-indiscernible sequence and $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. From S1 we obtain $\mathcal{C}_0 \in 1_\mu$ which contradicts the minimality of $n$. □

**Theorem 9.** Let $p(x;z), q(x;z) \subseteq L(A)$. Let $\mu$ be S1 and $A$-invariant. Then

$$R(a,b) \quad \Leftrightarrow \quad p(x;a) \cup q(x;b) \text{ is wide}$$

is a stable relation.

*Proof.* Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence of $A$-indiscernibles such that $p(x;a_0) \cup q(x;b_1)$ is $\mu$-wide. By S1, also $\big[p(x;a_0) \cup q(x;b_1)\big] \cup \big[p(x;a_2) \cup q(x;b_3)\big]$ is $\mu$-wide. A fortiori $p(x;a_2) \cup q(x;b_1)$ is $\mu$-wide and, by indiscernibility, so is $p(x;a_1) \cup q(x;b_0)$. □