μ-Random thoughts

M.R.T. Polymath

Let $\mathcal{U}$ be a monster model. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|x|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters). If $p \subseteq L_x(\mathcal{U})$ we write $p \subseteq \mathcal{D}$ for $p(x) \rightarrow x \in \mathcal{D}$. If $p \in S_x(\mathcal{U})$ we may also write $p \in \mathcal{D}$. In other words, when convenient we identify $\mathcal{D}$ with a subset of $S_x(\mathcal{U})$ and incomplete types with the set of their completions.

When $\mu(x) \subseteq L(\mathcal{U})$ we denote by 1_μ the filter generated by $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal, the ideal generated by $\big\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\big\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and $\mu \subseteq \mathcal{D}$ are synonymous.

Let $p \subseteq L_x(\mathcal{U})$. We say that p is μ-wide if $\mu(x) \cup p(x)$ is finitely consistent. If $p \in S_x(\mathcal{U})$ we may also say that p is μ-random, in other words, p is μ-random if $p \in \mathcal{D}$ for every $\mathcal{D} \in 1_\mu$ or, equivalently, if $p \notin \mathcal{D}$ for every $\mathcal{D} \in 0_\mu$. That is, when convenient we identify μ with the sets of μ-random (global) types.

Let $\kappa = \kappa^{<\kappa}$ be the cardinality of $\mathcal{U}$. We say that μ is κ-prime if for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. Sometimes we use the dual version of this property which sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa\rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 1. Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|x|}$. Let 1_μ be the set of subsets of measure 1. Then μ is κ-prime. Clearly, a global type $p \in S_x(\mathcal{U})$ is μ-random, if and only if it is contained in all definable sets of measure 1.

Proof. Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa\rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As κ is regular and uncountable, we can assume that for some n all sets $\mathcal{D}_i$ have measure $\geq 1/n$. Then, up to a set measure 0, the sets $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are pairwise disjoint and contained in $\neg\mathcal{D}_0$. This is a contradiction because $\neg\mathcal{D}_0$ has measure < 1. □

We say that μ is S1 over A if for every A-indiscernible sequence $\langle \mathcal{D}_i : i < \omega\rangle$

$$\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu \quad \Rightarrow \quad \mathcal{D}_0 \in 1_\mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino.

Fact 2. For any A-invariant filter μ the following are equivalent

1. μ is κ-prime;
2. μ is S1 over A.

Proof. (1⇒2) Let $\langle \mathcal{D}_i : i < \omega\rangle$ be a sequence of A-indiscernibles. By compactness we can find an indiscernible sequence of length κ such that $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$. By indiscernibility $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$. Then $\mathcal{D}_i \in 1_\mu$ for some $i < \kappa$ and, by indiscernibility, $\mathcal{D}_0 \in 1_\mu$.

(2⇒1) ..... □

Example 3. Let $1_\mu = \{\mathcal{X} \subseteq \mathcal{U}^{|x|} \ : \ A^{|x|} \subseteq \mathcal{X}\}$. Then μ is S1 over A. Clearly, a type $p(x) \subseteq L(\mathcal{U})$ is μ-wide if and only if it is finitely satisfied in A.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ start a sequence of A-indiscernibles. Then $a \in \mathcal{D}_0 \cup \mathcal{D}_1$ for every $a \in A^{|x|}$. By indiscernibility, $a \in \mathcal{D}_0$ holds for every $a \in A^{|x|}$. Hence $\mathcal{D}_0 \in 1_\mu$. □

Exercise 4. In Example 3 the elements of $A^{|x|}$ are, trivially, μ-random. Then μ has the following property: there is a small set R of μ-random types such that

$$\mathcal{D} \in 1_\mu \quad \Leftrightarrow \quad R \subseteq \mathcal{D}$$

Does this hold for every A-invariant μ?

Example 5. Let $\mu = \{\varphi(x) \in L(\mathcal{U}) \ : \ \varphi(x) \in p \ \text{ for every } A\text{-invariant } p \in S_x(\mathcal{U})\}$. Then μ is S1 over A. Clearly, a type $p \subseteq L_x(\mathcal{U})$ is μ-wide if and only if it has an extension to an A-invariant global type.

Proof. Assume $\mathcal{D}_0 \cup \mathcal{D}_1 \in 1_\mu$, where $\mathcal{D}_0, \mathcal{D}_1$ are conjugate over A. By invariance $p \in \mathcal{D}_0$ if and only if $p \in \mathcal{D}_1$. Therefore $\mathcal{D}_0 \in 1_\mu$. □

Example 6. We say that $\mathcal{D}$ is A-generic if finitely many A-translates of $\mathcal{D}$ cover $\mathcal{U}^{|x|}$. Let μ be as in Example 5. Then 1_μ is the filter generated by the A-generic sets.

Proof. It is easy to see that if $\mathcal{D}$ is A-generic then $\mathcal{D} \in 1_\mu$. Vice versa, assume that there are no A-generic sets $\mathcal{B}_i$ such that

$$\bigcap_{i=1}^{n} \mathcal{B}_i \quad \subseteq \quad \mathcal{D}$$

Note that a set $\mathcal{B}$ is A-generic if and only if the orbit over A of $\neg\mathcal{B}$ has the finite intersection property (fip). Hence, by taking complements, for any $\mathcal{C}_i$ such that

$$\neg\mathcal{D} \quad \subseteq \quad \bigcup_{i=1}^{n} \mathcal{C}_i$$

there is at least one i such that the A-orbit of $\mathcal{C}_i$ has the fip. By a standard (easy) argument we obtain that there is an A-invariant type $p \notin \mathcal{D}$. Therefore $\mathcal{D} \notin 1_\mu$. □

Definition 7. Let 1_A be the filter generated by the sets $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ such that there is an A-indiscerible sequence $\langle \mathcal{D}_i : i < \omega \rangle$ such that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_0, \ldots, \mathcal{D}_n$ cover $\mathcal{U}^{|x|}$ for some n. We call 1_A the nonforking filter over A.

Fact 8. Let μ be S1 and invariant over A. Then 1_μ contains 1_A.

Proof. Let $\mathcal{D} \in 1_A$. If $\langle \mathcal{D}_i : i < \omega \rangle$ is as in Definition 7, $\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_n \in 1_\mu$ for some n. Assume n is minimal, we prove that $n = 0$. Otherwise, let $\mathcal{C}_i = \mathcal{D}_i \cup \cdots \cup \mathcal{D}_{i+n-1}$. Then $\langle \mathcal{C}_i : i < \omega \rangle$ is an A-indiscernible sequence and $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. From S1 we obtain $\mathcal{C}_0 \in 1_\mu$ which contradicts the minimality of n. □

Theorem 9. Let $p(x;z), q(x;z) \subseteq L(A)$. Let μ be S1 and A-invariant. Then

$$R(a,b) \quad \Leftrightarrow \quad p(x;a) \cup q(x;b) \text{ is wide}$$

is a stable relation.

Proof. Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence of A-indiscernibles such that $p(x;a_0) \cup q(x;b_1)$ is μ-wide. By S1, also $\big[p(x;a_0) \cup q(x;b_1)\big] \cup \big[p(x;a_2) \cup q(x;b_3)\big]$ is μ-wide. A fortiori $p(x;a_2) \cup q(x;b_1)$ is μ-wide and, by indiscernibility, so is $p(x;a_1) \cup q(x;b_0)$. □