μ-Random thoughts

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Let $\mathcal{U}$ be a monster model. We confuse formulas $\varphi(x) \in L(\mathcal{U})$ with the subset of $\mathcal{U}^{|x|}$ that they define. If $\mathcal{D} = \varphi(\mathcal{U})$, we may write $x \in \mathcal{D}$ for $\varphi(x)$. Unless otherwise stated, calligraphic capital letters denote definable sets (with parameters). If $p \subseteq L_x(\mathcal{U})$ we write $p \subseteq \mathcal{D}$ for $p(x) \to x \in \mathcal{D}$. If $p \in S_x(\mathcal{U})$ we may also write $p \in \mathcal{D}$. In other words, when convenient we identify $\mathcal{D}$ with a subset of $S_x(\mathcal{U})$ and incomplete types with the set of their completions.

When $\mu(x) \subseteq L(\mathcal{U})$ we denote by 1_μ the filter generated by $\{\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. We write 0_μ for the corresponding ideal, the ideal generated by $\{\neg\varphi(\mathcal{U}) : \varphi(x) \in \mu\}$. Note that the expressions $\mathcal{D} \in 1_\mu$ and $\mu \subseteq \mathcal{D}$ are synonymous.

Let $p \subseteq L_x(\mathcal{U})$. We say that p is μ-wide if $\mu(x) \cup p(x)$ is finitely consistent. If $p \in S_x(\mathcal{U})$ we may also say that p is μ-random, in other words, p is μ-random if $p \in \mathcal{D}$ for every $\mathcal{D} \in 1_\mu$ or, equivalently, if $p \notin \mathcal{D}$ for every $\mathcal{D} \in 0_\mu$. That is, when convenient we identify μ with the sets of μ-random (global) types. The following is immediate.

Remark 1. Every μ-wide type $p \subseteq L_x(\mathcal{U})$ extends to a μ-random type.

Let $\kappa = \kappa^{<\kappa}$ be the cardinality of $\mathcal{U}$. We say that μ is κ-prime if for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cup \mathcal{D}_j \in 1_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 1_\mu$. By the regularity of κ this equivalent to requiring that $\mathcal{D}_i \in 1_\mu$ for every $i < \kappa$.

Sometimes we use the dual version of this property which sounds: for every sequence $\langle \mathcal{D}_i : i < \kappa \rangle$ of definable sets such that $\mathcal{D}_i \cap \mathcal{D}_j \in 0_\mu$ for every $i < j < \kappa$, there is an $i < \kappa$ such that $\mathcal{D}_i \in 0_\mu$.

Example 2. Assume there is a finitely additive probability measure on the definable subsets of $\mathcal{U}^{|x|}$. Let 1_μ be the set of subsets of measure 1. Then μ is κ-prime. Clearly, a global type $p \in S_x(\mathcal{U})$ is μ-random, if and only if it is contained in all definable sets of measure 1.

Proof. Assume for a contradiction that the sets $\langle \mathcal{D}_i : i < \kappa \rangle$ have positive measure but that $\mathcal{D}_i \cap \mathcal{D}_j$ has measure 0 for every $i < j < \kappa$. As κ is regular and uncountable, we can assume that for some n all sets $\mathcal{D}_i$ have measure $\geq 1/n$. Then, up to a set measure 0, the sets $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are pairwise disjoint and contained in $\neg\mathcal{D}_0$. This is a contradiction because $\neg\mathcal{D}_0$ has measure < 1. □

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We say that μ is S1 over A if for every A-indiscernible sequence $\langle\mathcal{D}_i : i<\omega\rangle$

$$\mathcal{D}_0\cup\mathcal{D}_1\in 1_\mu \quad\Rightarrow\quad \mathcal{D}_0\in 1_\mu.$$

The terminology originated in some obscure corner of Hrushovski's mind.

Fact 3. For any A-invariant μ the following are equivalent

1. μ is κ-prime;
2. μ is S1 over A.

Proof. (1⇒2) Let $\langle\mathcal{D}_i : i<\omega\rangle$ be a sequence of A-indiscernibles. By compactness, we can find an indiscernible sequence of length κ such that $\mathcal{D}_0\cup\mathcal{D}_1\in 1_\mu$. By indiscernibility $\mathcal{D}_i\cup\mathcal{D}_j\in 1_\mu$ for every $i<j<\kappa$. Then $\mathcal{D}_i\in 1_\mu$ for some $i<\kappa$ and, by indiscernibility, $\mathcal{D}_0\in 1_\mu$.

(2⇒1) The following fact is well-known.

Fact. For every sequence $\langle\mathcal{D}_i : i<\kappa\rangle$ be there is an A-indiscernible sequence $\langle\mathcal{C}_i : i<\omega\rangle$ such that for every $n<\omega$ there is some $I\subseteq\kappa$ of cardinality n such that $\mathcal{C}_{\restriction n}\equiv_A\mathcal{D}_{\restriction I}$.

Suppose $\mathcal{D}_i\cup\mathcal{D}_j\in 1_\mu$ for every $i<j<\kappa$. Apply the fact with $n=2$ to obtain A-indiscernible sequence $\langle\mathcal{C}_i : i<\omega\rangle$ such that $\mathcal{C}_0\cup\mathcal{C}_1\in 1_\mu$. Then $\mathcal{C}_0\in 1_\mu$ implies that $\mathcal{D}_i\in 1_\mu$ for some $i<\kappa$. □

Example 4. Let $1_\mu=\big\{\mathcal{X}\subseteq\mathcal{U}^{|x|} \;:\; A^{|x|}\subseteq\mathcal{X}\big\}$. Then μ is S1 over A. Clearly, a type $p(x)\subseteq L(\mathcal{U})$ is μ-wide if and only if it is finitely satisfied in A.

Proof. Assume $\mathcal{D}_0\cup\mathcal{D}_1\in 1_\mu$, where $\mathcal{D}_0,\mathcal{D}_1$ start a sequence of A-indiscernibles. Then $a\in\mathcal{D}_0\cup\mathcal{D}_1$ for every $a\in A^{|x|}$. By indiscernibility, $a\in\mathcal{D}_0$ holds for every $a\in A^{|x|}$. Hence $\mathcal{D}_0\in 1_\mu$. □

Exercise 5. In Example 4 the elements of $A^{|x|}$ are, trivially, μ-random. Then μ has the following property: there is a small set $R\subseteq S_x(\mathcal{U})$ of μ-random types such that

$$\mathcal{D}\in 1_\mu \quad\Leftrightarrow\quad R\subseteq\mathcal{D}$$

Does this hold for every A-invariant μ ?

Example 6. Let $\mu=\big\{\varphi(x)\in L(\mathcal{U}) \;:\; \varphi(x)\in p \text{ for every } A\text{-invariant } p\in S_x(\mathcal{U})\big\}$. Then μ is S1 over A. Clearly, a type $p\subseteq L_x(\mathcal{U})$ is μ-wide if and only if it has an extension to an A-invariant global type.

Proof. Assume $\mathcal{D}_0\cup\mathcal{D}_1\in 1_\mu$, where $\mathcal{D}_0,\mathcal{D}_1$ are conjugate over A. Then $p\in\mathcal{D}_0$ if and only if $p\in\mathcal{D}_1$ for every A-invariant global type p. Therefore $\mathcal{D}_0\in 1_\mu$. □

Example 7. We say that $\mathcal{D}$ is A-generic if finitely many A-translates of $\mathcal{D}$ cover $\mathcal{U}^{|x|}$. Let μ be as in Example 6. Then 1_μ is the filter generated by the A-generic sets.

Proof. It is easy to see that if $\mathcal{D}$ is A-generic then $\mathcal{D} \in 1_\mu$. Vice versa, assume that there are no A-generic sets $\mathcal{B}_i$ such that

$$\bigcap_{i=1}^{n} \mathcal{B}_i \quad \subseteq \quad \mathcal{D}$$

Note that a set $\mathcal{B}$ is not A-generic if and only if the orbit over A of $\neg\mathcal{B}$ has the finite intersection property (fip). Hence, by taking complements, for any $\mathcal{C}_i$ such that

$$\neg\mathcal{D} \quad \subseteq \quad \bigcup_{i=1}^{n} \mathcal{C}_i,$$

there is at least one i such that the A-orbit of $\mathcal{C}_i$ has the fip. By the fact below we obtain that there is an A-invariant type $p \notin \mathcal{D}$. Therefore $\mathcal{D} \notin 1_\mu$. □

Fact 8. Assume that for every finitely many $\mathcal{C}_1, \dots, \mathcal{C}_n$ covering $\mathcal{D}$ there is some $\mathcal{C}_i$ whose orbit over A has the fip. Then there is an A-invariant global type $p \in S_x(\mathcal{U})$.

Proof. Let $p \subseteq L_x(\mathcal{U})$ be maximal type containing $x \in \mathcal{D}$ such that all formulas in p have the same property as $\mathcal{D}$ in the fact. We claim that p is a global type. Suppose $\varphi(x), \neg\varphi(x) \notin p$. Then by maximality there is some $\psi(x) \in p$ and some $\mathcal{C}_1, \dots, \mathcal{C}_n$ covering both $\psi(\mathcal{U}) \cap \varphi(\mathcal{U})$ and $\psi(\mathcal{U}) \smallsetminus \varphi(\mathcal{U})$ such that no $\mathcal{C}_i$ has the fip. This is a contradiction because $\mathcal{C}_1, \dots, \mathcal{C}_n$ also cover $\psi(\mathcal{U})$. □

Definition 9. Let 1_A be the filter generated by the sets $\mathcal{D} \subseteq \mathcal{U}^{|x|}$ such that there is an A-indisceribile sequence $\langle \mathcal{D}_i : i < \omega \rangle$ such that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_0, \dots, \mathcal{D}_n$ cover $\mathcal{U}^{|x|}$ for some n. We call 1_A the nonforking filter over A.

Fact 10. Let μ be S1 and invariant over A. Then 1_μ contains 1_A.

Proof. Let $\mathcal{D} \in 1_A$. Let $\langle \mathcal{D}_i : i < \omega \rangle$ be as in Definition 9. Then $\mathcal{D}_0 \cup \dots \cup \mathcal{D}_n \in 1_\mu$ for some n. Assume n is minimal, we prove that $n = 0$. Otherwise, let $\mathcal{C}_i = \mathcal{D}_i \cup \dots \cup \mathcal{D}_{i+n-1}$. Then $\langle \mathcal{C}_i : i < \omega \rangle$ is an A-indiscernible sequence and $\mathcal{C}_0 \cup \mathcal{C}_1 \in 1_\mu$. From S1 we obtain $\mathcal{C}_0 \in 1_\mu$ which contradicts the minimality of n. □

Theorem 11. Let $p(x;z), q(x;z) \subseteq L(A)$. Let μ be S1 and A-invariant. Then

$$R(a,b) \quad \Leftrightarrow \quad p(x;a) \cup q(x;b) \text{ is wide}$$

is a stable relation.

Proof. Let $\langle a_i, b_i : i < \omega \rangle$ be a sequence of A-indiscernibles such that $p(x;a_0) \cup q(x;b_1)$ is μ-wide. By S1, also $\big[p(x;a_0) \cup q(x;b_1)\big] \cup \big[p(x;a_2) \cup q(x;b_3)\big]$ is μ-wide. A fortiori $p(x;a_2) \cup q(x;b_1)$ is μ-wide and, by indiscernibility, so is $p(x;a_1) \cup q(x;b_0)$. □

We say that μ is definable if for every $\varphi(x;z) \in L$ the set $\{b \in \mathfrak{U}^{|z|} : \mu(x) \to \varphi(x;b)\}$ is definable (type-definable). The definition of μ type-definable is similar. For instance, in general μ in Example 4 is type definable, but it is definable if the ambient theory is stable.

1. Independence (tentative ramble)

Let $\mu \subseteq L_v(\mathfrak{U})$, where $|v| = 1$, be A-invariant. When x is the tuple $\langle x_i : i < |x| \rangle$, we say that $p(x) \subseteq L(\mathfrak{U})$ is μ-wide if it is finitely consistent with $\bigcup_i \mu(x_i)$.

Let $a \in \mathfrak{U}^{|x|}$. We write $a \mathop{\perp\!\!\!\perp}_\mu b$ if the type $\mathrm{tp}(a/A,b)$ is μ-wide. Clearly, this is equivalent to saying that, $\varphi(x;b)$ is μ-wide for every $\varphi(x;z) \in L(A)$ such that $\varphi(a;b)$.

We write $b \equiv_\mu b'$ if $\mu(x)$ implies $\varphi(x;b) \leftrightarrow \varphi(x;b')$ for every $\varphi(x;z) \in L(A)$.

Lemma 12. The following properties hold for all a, b, c

1. $a \mathop{\perp\!\!\!\perp}_\mu b \;\Rightarrow\; f a \mathop{\perp\!\!\!\perp}_\mu f b$ for every $f \in \mathrm{Aut}(\mathfrak{U}/A)$ — *invariance*
2. $a \mathop{\perp\!\!\!\perp}_\mu b \;\Leftarrow\; a_0 \mathop{\perp\!\!\!\perp}_\mu b_0$ for all finite $a_0 \subseteq a$ and $b_0 \subseteq b$ — *finite character*
3. $a \mathop{\perp\!\!\!\perp}_\mu c, b$ and $c \mathop{\perp\!\!\!\perp}_\mu b \;\Rightarrow\; a, c \mathop{\perp\!\!\!\perp}_\mu b$ — *transitivity*
4. $a \mathop{\perp\!\!\!\perp}_\mu b \;\Rightarrow\;$ there exists $a' \equiv_{A,b} a$ such that $a' \mathop{\perp\!\!\!\perp}_\mu b, c$ — *extension*
5. $a \mathop{\perp\!\!\!\perp}_\mu b_1, b_2$ and $b_1 \equiv_\mu b_2 \;\Rightarrow\; b_1 \equiv_{\mu,a} b_2$ — *non-splitting*

Proof. Properties 1,2,3 are immediate. We prove 4. As $\mathrm{tp}(a/A,b)$ is μ-wide, it extends to a μ-random type $p(x)$. Then we can take any $a' \models p_{\restriction A,b,c}(x)$. □

Definition 13. We say that $\mathop{\perp\!\!\!\perp}_\mu$ is stationary if $a \equiv_M x \mathop{\perp\!\!\!\perp}_\mu b$ is a complete type over M, b for all finite tuples b and a.

We say n-stationary if this is restricted to $|a| = n$.

Stationarity is often ensured by the following property.

Proposition 14. If for every $\varphi(x) \in L(\mathfrak{U})$ there is a formula $\psi(x) \in L(M)$ such that $\varphi(\mathfrak{U}) =_\mu \psi(\mathfrak{U})$ then $\mathop{\perp\!\!\!\perp}_\mu$ is stationary.

Proof. Let $b \in \mathfrak{U}^{|z|}$ and $a_1, a_2 \in \mathfrak{U}^{|x|}$ be such that $a_i \mathop{\perp\!\!\!\perp}_\mu b$ and $a_1 \equiv_M a_2$. We claim that $a_1 \equiv_{M,b} a_2$. We need to prove that $\varphi(b;a_1) \leftrightarrow \varphi(b;a_2)$ for every $\varphi(z;x) \in L(M)$. Let $\psi(x) \in L(M)$ be such that $\varphi(b;\mathfrak{U}) =_M \psi(\mathfrak{U})$. From $a_i \mathop{\perp\!\!\!\perp}_\mu b$ we obtain that $\varphi(b;a_i) \leftrightarrow \psi(a_i)$. Finally, the claim follows because $a_1 \equiv_\mu a_2$. □

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