

**Exercise 1.** Assume  $L$  is countable and let  $M \leq N$  have arbitrary (large) cardinality. Let  $A \subseteq N$  be countable. Adapt the construction used for the downward Löwenheim-Skolem Theorem to prove that there is a countable model  $K$  such that  $A \subseteq K \leq N$  and  $K \cap M \leq N$  (in particular,  $K \cap M$  is a model).

**Exercise 2.** Let  $\langle M_i : i \in \lambda \rangle$  be an elementary chain of substructures of  $N$ . Let  $M$  be the union of the chain. Prove that  $M \leq N$ .

**Exercise 3.** Give an alternative proof of Exercise 1 using the elementary chain lemma and the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two chains of countable models such that  $K_i \cap M \subseteq M_i \leq N$  and  $A \cup M_i \subseteq K_{i+1} \leq N$ .

**Exercise 4.** Let  $N$  consist of 3 copies of  $\mathbb{Z}$  that share the same non negative part. The language has only a function symbol for the successor function (which will be non invertible in 0). Let  $M$  be the substructure of  $N$  obtained erasing one copy of  $\mathbb{Z}$ .

1. Prove that  $M \neq N$ .
2. Suppose instead that  $N$  is build from infinitely many copies of  $\mathbb{Z}$ , prove that  $M \equiv N$ .
3. If  $N$  is like in 2 is  $M \leq N$ ?

**Exercise 5.** Prove that  $T_{\text{dlo}}$  is not  $\lambda$ -categorical for any uncountable  $\lambda$ .

**Exercise 6.** Show that there is an  $\omega$ -categorical theory that is not complete (the language need to be uncountable). Hint. Let  $\nu$  be an uncountable cardinal. The language contains only the ordinals  $i < \nu$  as constants. The theory  $T$  says that there are infinitely many elements and either  $i = 0$  for every  $i < \nu$ , or  $i \neq j$  for every  $i < j < \nu$ . Prove that  $T$  is  $\omega$ -categorical but incomplete.

**Exercise 7.** Let  $N$  be free union of two random graphs  $N_1$  and  $N_2$ . That is,  $N = N_1 \sqcup N_2$  and  $r^N = r^{N_1} \sqcup r^{N_2}$ . By  $\sqcup$  we denote the disjoint union. Prove that  $N$  is not a random graph. Show that  $N_1$  is not definable without parameters (assume  $|N_1| = |N_2| = \omega$ , otherwise the proof is involved). Write a first order sentence  $\psi(x, y)$  true if  $x$  and  $y$  belong to the same connected component of  $N$ . Axiomatize the class  $\mathcal{K}$  of graphs that are free union of two random graphs.

**Exercise 8.** The language contains only the binary relations  $<$  and  $e$ . The theory  $T_0$  says that  $<$  is a strict linear order and that  $e$  is an equivalence relation. Let  $\mathcal{M}$  consists of models of  $T_0$  and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it  $\lambda$ -categorical for some  $\lambda$ ?

**Exercise 9.** Prove that for every infinite graph  $M$  the following are equivalent

1.  $M$  is either random, empty, or complete;
2. if  $M_1, M_2 \subseteq M$  are such that  $M_1 \sqcup M_2 = M$ , then  $M_1 \simeq M$  or  $M_2 \simeq M$ .

With  $\sqcup$  we denote the disjoint union.