Esercizio 1. Assume L is countable and both $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be a countable set. Adapt the construction used to prove the downward Löwenheim-Skolem Theorem to prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Esercizio 2. Consider $\mathbb{R}$ in the language of strict orders. Prove that $\mathbb{R} \smallsetminus \{0\} \preceq \mathbb{R}$. (You may use the downward Löwenheim-Skolem Theorem.) Are these two structures isomorphic?

Esercizio 3. Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if $\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$.

Esercizio 1. Assume L is countable and both $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be a countable set. Apply the downward Löwenheim-Skolem Theorem to prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Hint: contruct an elementary chain and apply the elementary chain lemma.

Esercizio 2. Let $M \preceq N$ and let $\varphi(x, z) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M.

Esercizio 3. Consider $\mathbb{Z}^n$ as a structure in the additive language of groups with the natural interpretation. Prove that $\mathbb{Z}^n \not\equiv \mathbb{Z}^m$ for every positive integers $n \neq m$.

Esercizio 1. Prove that if T has exactly 2 maximally consistent extension T_1 and T_2 then there is a sentence φ such that $T, \varphi \vdash T_1$ and $T, \neg\varphi \vdash T_2$. State and prove the generalization to finitely many maximally consistent extensions (this seems easy to prove, but difficult to write down well).

Esercizio 2. Prove that the following are equivalent for every consistent theory T

1. T is complete;
2. if $M, N \models T$ then $M \equiv N$.

Esercizio 3. Let L contain the symbols $0, 1, \cdot$ with the usual interpretation. Prove that $\mathbb{Q} \not\preceq \mathbb{R}$. N.B. this contrasts with the (non obvious) fact that $\mathbb{Q} \preceq \mathbb{R}$ in the language $0, 1, +$.