Exercise 1. Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Adapt the construction used for the downward Löwenheim-Skolem Theorem to prove that there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Exercise 2. Give an alternative proof of Exercise 1 using the elementary chain lemma and the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$.

Exercise 3. Let N be free union of two countable random graphs N_1 and N_2. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$. By $\sqcup$ we denote the disjoint union. Prove that N is not a random graph. Write a first-order sentence $\psi(x, y) \in L$ true if x and y belong both to N_1 or both to N_2.

Exercise 4. Let T_{lo} be the theory of linear orders in the language $L = \{<\}$. Prove that for every $b \in M \models T_{\text{lo}}$, every $N \models T_{\text{dlo}}$, every finite partial isomorphism $k : M \to N$ has an extension to a partial isomorphism defined in b.

Exercise 5. Let T_{grph} be the theory of graphs that is, the theory that says that $r(x, y)$ is a irreflexive, symmetric relation. Let T_{rg} be the theory of random graphs. Prove the claim in Exercise 4 with T_{grph} and T_{rg} for T_{lo}, respectively T_{dlo}.

Exercise 6. Prove the converse of Exercises 4 and 5. E.g., $N \models T_{\text{grph}}$ is a random graph whenever the following holds: for every $b \in M \models T_{\text{grph}}$, every finite partial isomorphism $k : M \to N$ has an extension to a partial isomorphism defined in $b \in M$.

Exercise 7. The language contains only the binary relations $<$ and e. The theory T_0 says that $<$ is a strict linear order and that e is an equivalence relation. Axiomatize a theory T_1 such that what claimed in Exercise 4 holds for T_0 and T_1.

Proof the claim for yourself, hand in only the axiomatization.

Exercise 8. Prove that the theory T_1 in Exercise 5 is ω-categorical. Try to write a proof that works simultaneously for T_1, T_{rg}, and T_{dlo}.