Exercise 1. Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Adapt the construction used for the downward Löwenheim-Skolem Theorem to prove that there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Exercise 2. Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of substructures of N. Let M be the union of the chain. Prove that $M \preceq N$.

Exercise 3. Give an alternative proof of Exercise 1 using the elementary chain lemma and the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$.

Exercise 4. Prove that T_{dlo} is not λ-categorical for any uncountable λ.

Exercise 5. Show that there is an ω-categorical theory that is not complete (the language need to be uncountable). Hint. Let ν be an uncountable cardinal. The language contains only the ordinals $i < \nu$ as constants. The theory T says that there are infinitely many elements and either $i = 0$ for every $i < \nu$, or $i \neq j$ for every $i < j < \nu$. Prove that T is ω-categorical but incomplete.

Exercise 6. Let N be free union of two random graphs N_1 and N_2. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$. By $\sqcup$ we denote the disjoint union. Prove that N is not a random graph. Show that N_1 is not definable without parameters (assume $|N_1| = |N_2| = \omega$, otherwise the proof is involved). Write a first order sentence $\psi(x, y)$ true if x and y belong to the same connected component of N. Axiomatize the class $\mathcal{K}$ of graphs that are free union of two random graphs.

Exercise 7. The language contains only the binary relations $<$ and e. The theory T_0 says that $<$ is a strict linear order and that e is an equivalence relation. Let $\mathcal{M}$ consists of models of T_0 and partial isomorphisms. Do rich models exist? Can we axiomatize their theory? If so, does it have elimination of quantifiers? Is it λ-categorical for some λ?

Exercise 8. Prove that for every infinite graph M the following are equivalent

1. M is either random, empty, or complete;
2. if $M_1, M_2 \subseteq M$ are such that $M_1 \sqcup M_2 = M$, then $M_1 \simeq M$ or $M_2 \simeq M$.

With $\sqcup$ we denote the disjoint union.