Exercise 1. Let L be a language that extends that of strict linear orders with the constants $\{c_i : i \in \omega\}$. Let T be the theory that extends $T_{\rm dlo}$ with the axioms $c_i < c_{i+1}$ for every $i \in \omega$. From what is known of $T_{\rm dlo}$ we deduce that T has elimination of quantifiers and is complete. Exhibit a countable models that are

1. saturated;
2. homogeneous but not saturated;
3. not homogeneous.

Exercise 2. Let L be the language of strict orders Prove that the structure $\mathbb{Q} \times \mathbb{Z}$ ordered with the lexicographic order

$$(a_1, a_2) < (b_1, b_2) \quad \Leftrightarrow \quad a_1 < b_1 \text{ or } (a_1 = b_1 \text{ e } a_2 < b_2)$$

is a saturated model.

Exercise 3. Let M and N be elementarily homogeneous structures of the same cardinality λ. Suppose that $M \models \exists x\, p(x) \Leftrightarrow N \models \exists x\, p(x)$ for every $p(x) \subseteq L$ such that $|x| < \lambda$. Prove that the two structures are isomorphic.

Exercise 4. Let $\varphi(z) \in L(A)$ be a consistent formula. Prove that, if $a \in \operatorname{acl}(A, b)$ for every $b \models \varphi(z)$, then $a \in \operatorname{acl}(A)$. Prove the same claim with a type $p(z) \subseteq L(A)$ for $\varphi(z)$.

Exercise 5. Let $\varphi(x; z) \in L$. Prove that if the set $\{\varphi(a; \mathfrak{U}) \;:\; a \in \mathfrak{U}^{|x|}\}$ is infinite then it has cardinality κ. Does the claim remains true with a type $p(x; z) \subseteq L$ for $\varphi(x; z)$? Hint: consider $\operatorname{Th}(\mathbb{N}, <)$.

Exercise 6. Let C be a finite set. Prove that if $C \cap M \neq \varnothing$ for every model M containing A, then $C \cap \operatorname{acl}(A) \neq \varnothing$.

Exercise 7. Prove that for every $A \subseteq N$ there is an M such that $\operatorname{acl} A = M \cap N$.

Exercise 8. Let $\mathfrak{U}$ be some large saturated model and let $a \in \mathfrak{U} \smallsetminus \operatorname{acl}\varnothing$. Prove that $\mathfrak{U}$ is isomorphic to some $\mathcal{V} \preceq \mathfrak{U}$ such that $a \notin \mathcal{V}$. Hint: let c be an enumeration of $\mathfrak{U}$ and let $p(u) = \operatorname{tp}(c)$ prove that $p(u) \cup \{u_i \neq a : i < \kappa\}$ is realized in $\mathfrak{U}$ and that any realization yields the required substructure of $\mathfrak{U}$.

Exercise 9. Let T be a consistent theory. Suppose that all completions of T are of the form $T \cup S$ for some set S of quantifier-free sentences. Prove that, if all completion of T have elimination of quantifiers, so does T. Show that this fails when the completions of T have arbitrary complexity.