Exercise 1. Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Adapt the construction used for the downward Löwenheim-Skolem Theorem to prove that there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Exercise 2. Give an alternative proof of Exercise 1 using the elementary chain lemma and the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$.

Exercise 3. Let N be a countable random graph. Prove that if we erase from N finitely many vertices and all their direct neighbours we obtain a graph isomorphic to N.

Exercise 4. Let N be free union of two countable random graphs N_1 and N_2. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$. By $\sqcup$ we denote the disjoint union. Prove that N is not a random graph. Write a first-order sentence $\psi(x, y) \in L$ true if x and y belong both to N_1 or both to N_2.

Exercise 5. Let T_{grph} be the theory of graphs that is, the theory that says that $r(x, y)$ is a irreflexive, symmetric relation. Assume $N \models T_{\text{grph}}$ is such that for every $b \in M \models T_{\text{grph}}$, every finite partial isomorphism $k : M \to N$ has an extension to a partial isomorphism defined in b. Prove that N is a random graph.

Exercise 6. The language contains only two binary relations: $<$ and e. Let T_0 be the theory that says that $<$ is a strict linear order and that e is an equivalence relation. Axiomatize a theory $T_1 \supseteq T_0$ such that every $N \models T_1$ has the following property: for every $b \in M \models T_0$, every finite partial isomorphism $k : M \to N$ has an extension to a partial isomorphism defined in b. Solution:

Axioms of linear orders without endpoints

Axioms of equivalence relations

There are infinitely many equivalence classes

$$\forall x, y, v \left[x < y \ \rightarrow \ \exists z \left[x < z < y \ \wedge \ e(v, z) \right] \right]$$

Exercise 7. Prove that the theory T_1 in Exercise 6 is ω-categorical.

Exercise 8. Let N be a saturated model and let $\varphi(x; y) \in L(N)$. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(N; a_i) \subset \varphi(N; a_{i+1})$ for every $i < \omega$;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(N; a_{i+1}) \subset \varphi(N; a_i)$ for every $i < \omega$.

Exercise 9. Let N be a saturated model and let $\varphi(x) \in L(N)$. Prove that the following are equivalent

1. $\varphi(N)$ is infinite;
2. $\varphi(N)$ has the cardinality of N.

Exercise 10. Let N be a saturated model and let $p(x) \subseteq L(A)$, for some $A \subseteq N$ of cardinality $< |N|$, be a type closed under conjunction (for simplicity). Prove that the following are equivalent

1. $p(x)$ has finitely many solution;
2. $p(x)$ contains a formula with finitely many solutions.

Exercise 11. Let N be a saturated model of T_{lo}. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i < \omega \rangle$ such that $a_i < a_{i+1}$ for every $i < \omega$;
2. there is a sequence $\langle a_i : i < \omega \rangle$ such that $a_i > a_{i+1}$ for every $i < \omega$.

Solution:

2⇒1

Let $p(\bar{x}) = \{x_i < x_{i+1} : i < \omega\}$. Any solution of $p(\bar{x})$ is as required by 1. It suffices to show that $p(\bar{x})$ is finitely consistent. Any finite subset of $p(\bar{x})$ is contained in the type $p_n(x_0, \ldots, x_{n-1}) = \{x_i < x_{i+1} : i < n\}$. Let a_i be as in 2. Then $a_{n-1}, \ldots, a_0 \models p_n(x_0, \ldots, x_{n-1})$.