Esercizio 1. Let M be an L-structure and let $\psi(x), \varphi(x, y) \in L$. For each of the following conditions, write a sentence true in M exactly when

a. $\psi(M) \in \{\varphi(a, M) : a \in M\}$;

b. $\{\varphi(a, M) : a \in M\}$ contains at least two sets;

c. $\{\varphi(a, M) : a \in M\}$ contains only sets that are pairwise disjoint.

Esercizio 2. Let M be a structure in a signature that contains a symbol r for a binary relation. Write a sentence φ such that

a. $M \models \varphi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq A \times \neg A$.

Esercizio 3. Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if $\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$.

Esercizio 4. Let $M \preceq N$ and let $\varphi(x, z) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M.

Esercizio 5. Let N be the multiplicative group $\mathbb{Q} \smallsetminus \{0\}$. Let M be the subgroup of those rational numbers that are of the form n/m for some odd integers m and n. Prove that $M \preceq N$.

Esercizio 6. Let ${}^*\mathbb{Q}$ be an elementary extension of $\mathbb{Q}$ in a language that contains symbols for all relation and function of $\mathbb{Q}$. Prove that ${}^*\mathbb{Q}$ contains infinite and infinitesimal language.