Exercise 1. Let $p(x) \subseteq L(B)$ and $p_n(x) \subseteq L(A)$, for $n < \omega$, be consistent types such that

$$p(x) \rightarrow \bigvee_{n<\omega} p_n(x)$$

Prove that there is an $n < \omega$ and a formula $\varphi(x) \in L(A)$ consistent with $p(x)$ such that

$$p(x) \wedge \varphi(x) \rightarrow p_n(x).$$

Exercise 2. Let M be a second countable topological space (i.e. the topology has a countable base). We say that $A \subseteq M$ is meager if it is the countable union of nowhere dense sets. Use Lemma 12.1 to prove the Kuratowski-Ulam Theorem, i.e. that for $A \subseteq M^2$ the following are equivalent (w.r.t. the product topology)

1. A is meager in M^2;
2. $\Big\{x \in M : A \cap \{x\} \times M \text{ is not meager}\Big\}$ is meager in M.

Hint: Use the base of the topology as predicates of a first-order language.

Exercise 3. Let $p(x) \subseteq L(A)$ and let $\varphi(x;y) \in L(A)$ be a formula that defines, when restricted to $p(\mathfrak{U})$, an equivalence relation with finitely many classes. Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x;y)$ on $p(\mathfrak{U})$.

Exercise 4. Let $p(x) \subseteq L(A)$ and let $\varphi(x;y) \in L(A)$ be a formula that defines, when restricted to $p(\mathfrak{U})$, an equivalence relation with finitely many classes. Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x;y)$ on $p(\mathfrak{U})$.

Exercise 5. Let T be a consistent theory. Suppose that all completions of T are of the form $T \cup S$ for some set S of quantifier-free sentences. (As, for example, T_{acf}.) Prove that, if all completion of T have elimination of quantifiers, so does T.

Hint: prove that for every formula $\varphi(x)$ there are some quantifier-free sentences σ_i and quantifier-free formulas $\psi_i(x)$ such that

$$\sigma_i \vdash \varphi(x) \leftrightarrow \psi_i(x), \qquad T \vdash \bigvee_{i=1}^{n} \sigma_i, \qquad \text{and} \qquad \sigma_i \vdash \neg\sigma_j \text{ for } i \neq j.$$

For a counter example consider the empty theory in the language with a single unary predicate.

Exercise 6. Show that the claim in the exercise above fails when the theories S have arbitrary complexity.

Hint: consider the empty theory in the language with a single unary predicate.