Exercise 1. Let $p(x) \subseteq L(A)$ and let $\varphi(x;y) \in L(A)$ be a formula that defines, when restricted to $p(\mathfrak{U})$, an equivalence relation with finitely many classes (a *finite equivalence relation*). Prove that there is a finite equivalence relation definable over A that coincides with $\varphi(x;y)$ on $p(\mathfrak{U})$.

Exercise 2. Let T be a consistent theory. Suppose that all completions of T are of the form $T \cup S$ for some set S of quantifier-free sentences (for example, $T = T_{\mathrm{acf}}$). Prove that, if all completion of T have elimination of quantifiers, so does T.

Hint: prove that for every formula $\varphi(x)$ there are some quantifier-free sentences σ_i and quantifier-free formulas $\psi_i(x)$ such that

$$\sigma_i \vdash \varphi(x) \leftrightarrow \psi_i(x), \qquad T \vdash \bigvee_{i=1}^{n} \sigma_i, \qquad \text{and} \qquad \sigma_i \vdash \neg\sigma_j \text{ for } i \neq j.$$

For a counter example consider the empty theory in the language with a single unary predicate.

Exercise 3. Show that the claim in the exercise above fails when the theories S have arbitrary complexity.

Hint: let T be the empty theory in the language with a single unary predicate.

Exercise 4. Let $\varphi(z) \in L(A)$ be a consistent formula. Prove that, if $a \in \mathrm{acl}(A, b)$ for every $b \models \varphi(z)$, then $a \in \mathrm{acl}(A)$. Prove the same claim with a type $p(z) \subseteq L(A)$ for $\varphi(z)$.

Exercise 5. Let c be en enumeration of $\mathfrak{U}$. Let $\mathcal{V}$ be the set enumerated by some $a \equiv c$. Prove that $\mathcal{V} \preceq \mathfrak{U}$.

Exercise 6. Let $a \in \mathfrak{U} \smallsetminus \mathrm{acl} A$. Prove that $\mathfrak{U}$ is A-isomorphic to some $\mathcal{V} \preceq \mathfrak{U}$ such that $a \notin \mathcal{V}$.

Exercise 7. Let C be a finite set. Prove that if $C \cap M \neq \varnothing$ for every model M containing A, then $C \cap \mathrm{acl}(A) \neq \varnothing$.

Hint: by induction on the cardinality of C. Suppose there is a $c \in C \smallsetminus \mathrm{acl}(A)$, then there is $\mathcal{V} \simeq \mathfrak{U}$ such that $A \subseteq \mathcal{V} \preceq \mathfrak{U}$ and $c \notin \mathcal{V}$. Apply the induction hypothesis to $C' = C \cap \mathcal{V}$ with $\mathcal{V}$ for $\mathfrak{U}$.

Exercise 8. Prove that for every $A \subseteq N$ there is an M such that $\mathrm{acl} A = M \cap N$.

Hint: let $\bar{c}$ be an enumeration of N. Let $p(\bar{x}) = \mathrm{tp}(\bar{c}/A)$. Consider the type

$$p(x) \cup \left\{ \neg\left[\psi(b,\bar{x}) \wedge \exists^{\leq n} y\, \psi(y,\bar{x})\right] \;:\; b \in N \smallsetminus \mathrm{acl}(A),\;\; \psi(y,x) \in L(A),\;\; n < \omega \right\}$$

Exercise 9. Let $\varphi(x) \in L(\mathfrak{U})$ and fix an arbitrary set A. Prove that the following are equivalent

1. there is some model M containing A and such that $M \cap \varphi(\mathfrak{U}) = \varnothing$;
2. there is no consistent formula $\psi(z_1, \dots, z_n) \in L(A)$ such that
$$\psi(z_1, \dots, z_n) \to \bigvee_{i=1}^{n} \varphi(z_i).$$

Exercise 10. Prove that a in strongly minimal theory every infinite algebraically closed set is a model.

Exercise 11. Prove that a countable strongly minimal theory is either ω-categorical or has infinitely many countable models.