Esercizio 1. Let T be a consistent theory. Suppose that all completions of T are of the form $T \cup S$ for some set S of quantifier-free sentences. Prove (in the most direct possible way) that, if all completion of T have elimination of quantifiers, so does T. Show that this fails when the completions of T have arbitrary complexity.

Esercizio 2. Let L be the language of strict orders. Let T be the theory of *discrete linear orders*. Namely, T extends T_{dlo} with the following two of axioms

dis↑. $\exists z \left[x < z \wedge \neg\exists y\, x < y < z \right]$;

dis↓. $\exists z \left[z < x \wedge \neg\exists y\, z < y < x \right]$.

Let Δ be the set of formulas that contains (all alphabetic variants of) the formulas

$$x <_n y \ := \ \exists^{\geq n} z\, (x<z<y)$$

and their negations (read $<_0$ as $<$).

1. Prove that the structure $\mathbb{Q} \times \mathbb{Z}$ ordered with the lexicographic order
$$(a_1, a_2) < (b_1, b_2) \quad \Leftrightarrow \quad a_1 < b_1 \text{ or } (a_1 = b_1 \text{ e } a_2 < b_2)$$
is a saturated model of T.
2. Prove that the theory of discrete linear orders has Δ-elimination of quantifiers.
3. Prove that T is not model-complete.

Esercizio 3. Let $\varphi(z) \in L(A)$ be a consistent formula. Prove that, if $a \in \operatorname{acl}(A, b)$ for every $b \models \varphi(z)$, then $a \in \operatorname{acl}(A)$. Prove the same claim with a type $p(z) \subseteq L(A)$ for $\varphi(z)$.

Esercizio 1. Dimostrare che se N è (elementarmente) saturo allora è (elementarmente) omogeneo.

La dimostrazione usa la tecnica dell'andirivieni (back-and-forth). Nelle note è esposta nel caso generale. Si provi a fare una dimostrazione nel caso specifico.

Esercizio 2. Let $\varphi(x;z) \in L$. Prove that if the set $\{\varphi(a;\mathcal{U}) \,:\, a \in \mathcal{U}^{|x|}\}$ is infinite then it has cardinality κ. Does the claim remains true with a type $p(x;z) \subseteq L$ for $\varphi(x;z)$?

Suggerimento per la seconda domanda: potrebbe esserci un controesempio in $\mathcal{U} \equiv \mathbb{N}$ nel linguaggio degli ordini.

Esercizio 3. Si dia una dimostrazione sintattica di $\operatorname{acl}(\operatorname{acl} A) \subseteq \operatorname{acl} A$