Exercise 1. Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Adapt the construction used for the downward Löwenheim-Skolem Theorem to prove that there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model).

Exercise 2. Give an alternative proof of Exercise 1 using the elementary chain lemma and the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$.

Exercise 3. Let N be a countable random graph. Prove that if we erase from N finitely many vertices and all their direct neighbours we obtain a graph isomorphic to N.

Exercise 4. Let N be free union of two countable random graphs N_1 and N_2. That is, $N = N_1 \sqcup N_2$ and $r^N = r^{N_1} \sqcup r^{N_2}$. By $\sqcup$ we denote the disjoint union. Prove that N is not a random graph. Write a first-order sentence $\psi(x, y) \in L$ true if x and y belong both to N_1 or both to N_2.

Exercise 5. Let T_{lo} be the theory of linear orders in the language $L = \{<\}$. Prove that for every $b \in M \models T_{\mathrm{lo}}$, every $N \models T_{\mathrm{dlo}}$, every finite partial isomorphism $k : M \to N$ has an extension to a partial isomorphism defined in b.

Exercise 6. Let T_{grph} be the theory of graphs that is, the theory that says that $r(x, y)$ is a irreflexive, symmetric relation. Assume $N \models T_{\mathrm{grph}}$ is such that for every $b \in M \models T_{\mathrm{grph}}$, every finite partial isomorphism $k : M \to N$ has an extension to a partial isomorphism defined in b. Prove that N is a random graph.

Exercise 7. The language contains only two binary relations: $<$ and e. Let T_0 be the theory that says that $<$ is a strict linear order and that e is an equivalence relation. Axiomatize a theory $T_1 \supseteq T_0$ such that every $N \models T_1$ has the following property: for every $b \in M \models T_0$, every finite partial isomorphism $k : M \to N$ has an extension to a partial isomorphism defined in b.

Proof the claim for yourself, hand in only the axiomatization.

Exercise 8. Prove that the theory T_1 in Exercise 7 is ω-categorical.

Exercise 9. Let $\varphi(x; y) \in L(\mathfrak{U})$. Prove that the following are equivalent

1. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}; a_i) \subset \varphi(\mathfrak{U}; a_{i+1})$ for every $i < \omega$;
2. there is a sequence $\langle a_i : i \in \omega \rangle$ such that $\varphi(\mathfrak{U}; a_{i+1}) \subset \varphi(\mathfrak{U}; a_i)$ for every $i < \omega$.