FREQUENCY-DOMAIN ANALYSIS OF DIGITAL FILTERS

- Frequency response of LTI systems
- Frequency response of LTI systems, example
- Frequency response of LTI systems, discrete case
- FIR filters implemented as IIR filters, integer multiplier filters
- Laboratory work
- (Frequency response of LTI systems, example)
- (FIR filters implemented as IIR filters, integer multiplier filters)
- (Examples)

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Frequency response of LTI systems

- Transfer function, *H(z)*, is complex function of a complex variable over entire *z* plane, *h*[*n*] (impuse response) → *Z* transform → *H(z)*
- Frequency Response, *H(ω)*, is Transfer Function, *H(z)*, evaluated on the unit circle

$$H(\omega) = H(z)\big|_{z=e^{j\omega} \text{ or } |z|=1} = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}, \quad \omega = 2\pi f$$

- *H(ω)* - Frequency response of the system (in the frequency domain)
- |*H(ω)*| - Amplitude (magnitude) response of the system
- *Θ(ω)* - Phase response of the system

$$H(\omega) = H_R(\omega) + j H_I(\omega) \quad \text{(DTFT)}$$

$$H(\omega) = |H(\omega)|\,.\,e^{j\theta(\omega)} \quad \text{(Polar notation)}$$

$$|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}$$

$$\theta(\omega) = \sphericalangle H(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right)$$

Frequency response of LTI systems, example

- Example $y[n] = 0.8\,y[n-1] + x[n]$

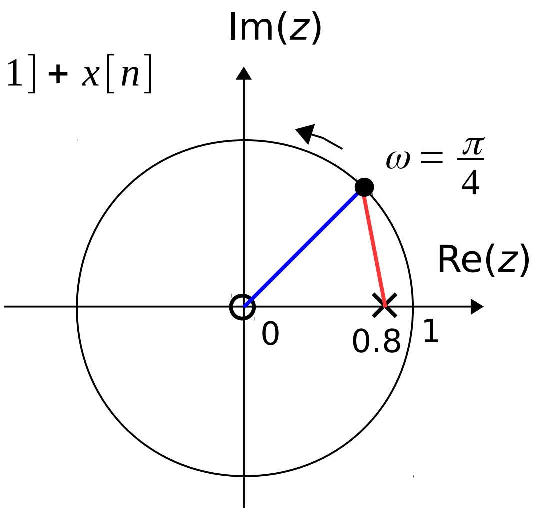
$$H(z) = \frac{1}{1 - 0.8\,z^{-1}} = \frac{z}{z - 0.8}$$

$$z_1 = 0, \qquad p_1 = 0.8$$

$$H(\omega) = \frac{e^{j\omega}}{e^{j\omega} - 0.8}$$

$$|H(\omega)| = \frac{V_1(\omega)}{U_1(\omega)} = \frac{|e^{j\omega}|}{|e^{j\omega} - 0.8|}$$

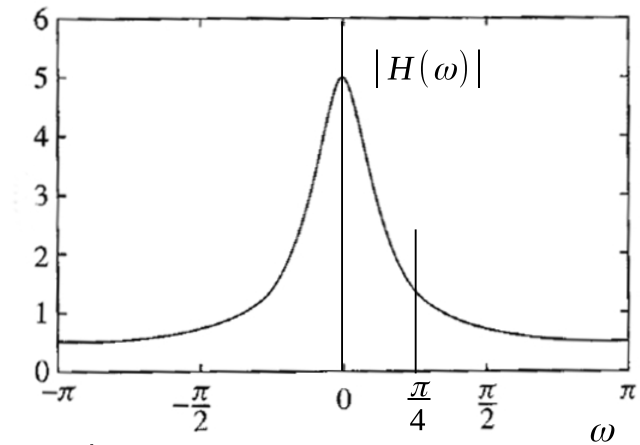
$$|H(\omega)| = \frac{1}{\sqrt{1.64 - 1.6\cos\omega}}$$



$$|H(0)| = \frac{1}{0.2} = 5$$

$$\left|H\left(\frac{\pi}{4}\right)\right| = \frac{1}{0.71} = 1.4$$

Amplitude response



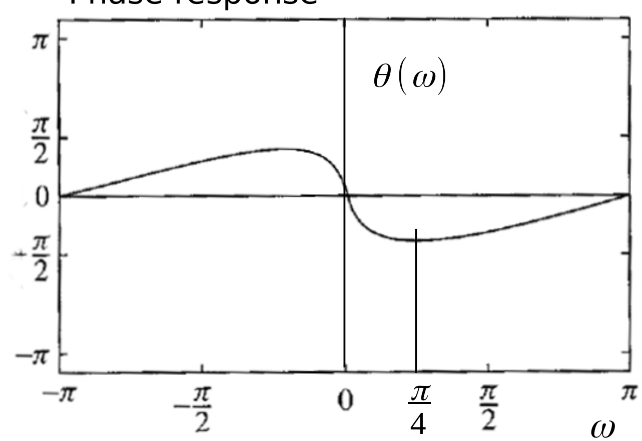
$$\theta(\omega) = \Theta_1(\omega) - \Phi_1(\omega)$$

$$\theta(\omega) = \omega - \arctan\frac{\sin\omega}{\cos\omega - 0.8}$$

$$\theta(0) = 0 - 0 = 0$$

$$\theta\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1.7 = -0.91$$

Phase response



[Proakis, Manolakis]

Frequency response of LTI systems, discrete case

- The Z-transform is equivalent to DTFT transform on the unit circle in the *Z* plane

$$z = e^{j\omega} = e^{j2\pi f} = e^{j2\pi F/F_S}$$

- Discrete case, DFT (in terms of sampling frequency):

$$z_k = e^{j\omega_k} = e^{j2\pi f_k} = e^{j2\pi F_k/F_S}$$

$$\omega_k = 2\pi f_k = 2\pi F_k/F_S$$

$$2\pi F_k/F_S = 2\pi k/N, \qquad k = 0,1,2,\ldots,N-1$$

$$F_k = (k/N).F_S \qquad\qquad f_k = k/N$$

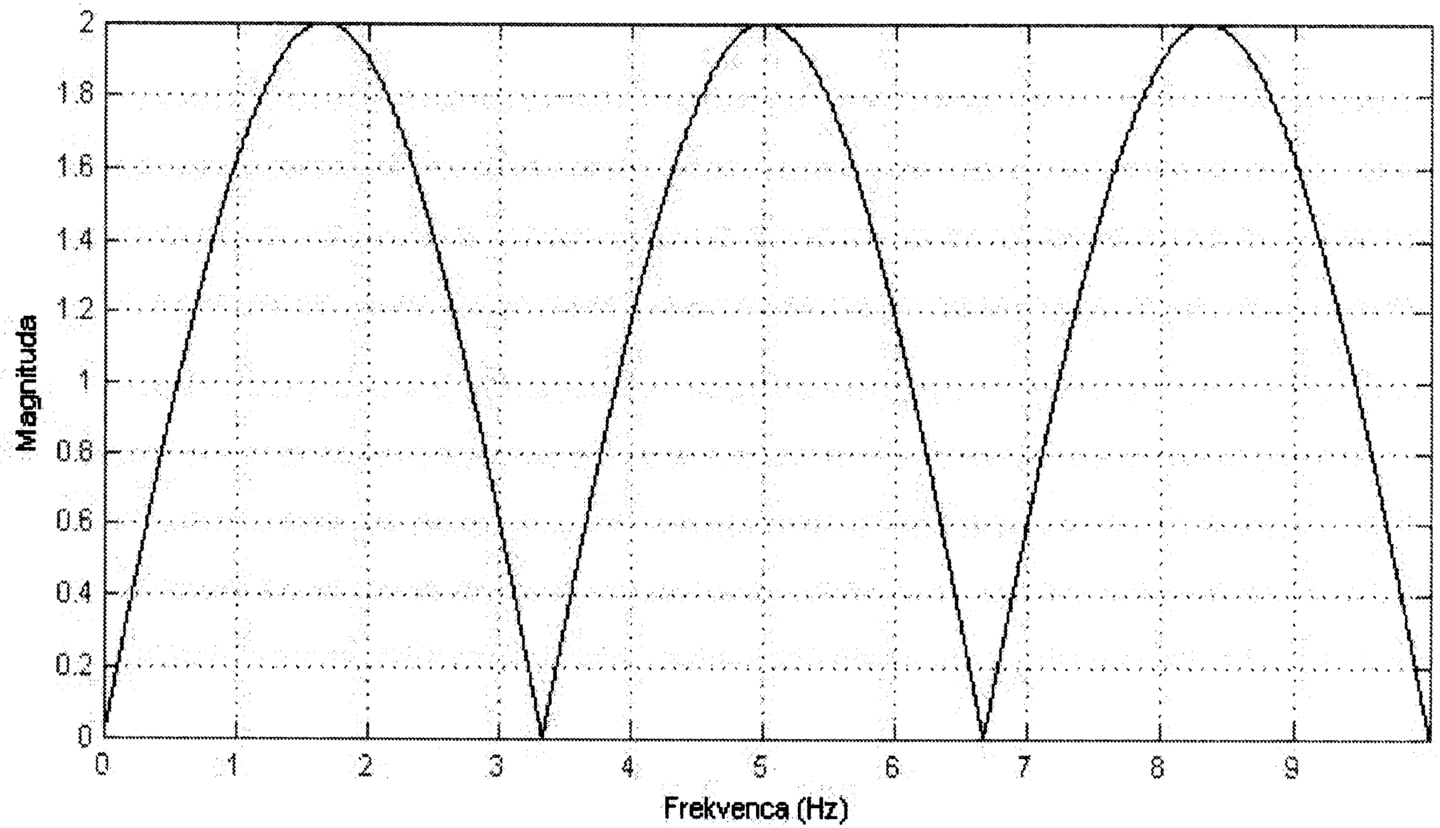
FIR filters implemented as IIR filters, integer multiplier filters

- Amplitude response characteristic of a differentiator ($Fs = 20$ smp/sec)

$$H(z) = (1 - z^{-6})$$

$$\omega_k = 2\pi (k/N)\, Fs, \qquad F_k = (k/N)\, Fs$$

$$|H(\omega)|$$



FIR filters implemented as IIR filters, integer multiplier filters

- Example, recall moving average

$M = 8$

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l]$$

$$h[n]=\begin{cases}\frac{1}{M}, & 0 \le n \le M-1\\ 0, & \text{otherwise}\end{cases}$$

- The transfer function is

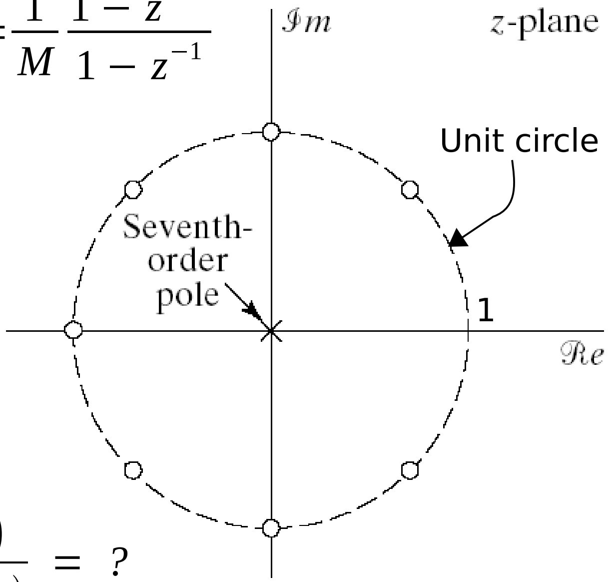
$$H(z)=\sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1}{M}\left(1+z^{-1}+\ldots+z^{-M+1}\right) = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}}$$

- The zeros, $z_{(k+1)}$, can be written as

$$z_{(k+1)} = a e^{j2\pi k/M}, \quad k = 0,1,\ldots,M-1$$

- For $k = 0$ we have a zero at $z_1 = 1$
- The zero cancels the pole at $p_1 = 1$

$$H(z) = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}} = \frac{1}{M}\frac{z^M}{z^M}\frac{\left(1-z^{-M}\right)}{\left(1-z^{-1}\right)} = \frac{1}{M}\frac{\left(z^M-1\right)}{z^{M-1}(z-1)} = ?$$



[Oppenheim, Schafer]

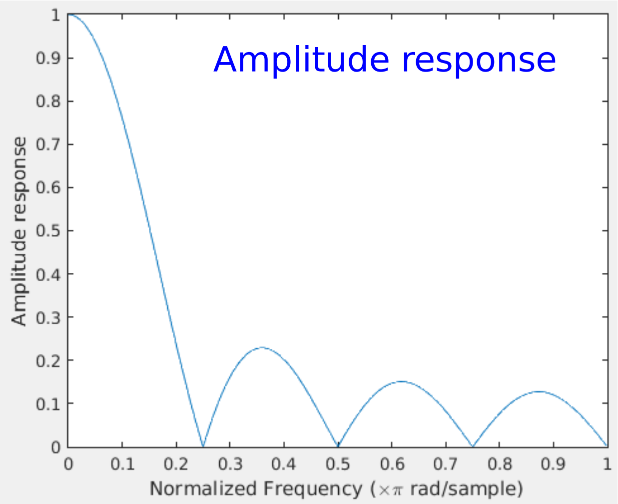
FIR filters implemented as IIR filters, integer multiplier filters

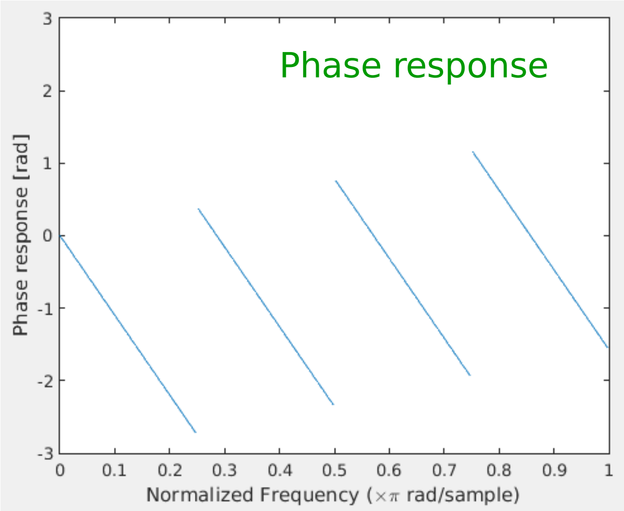
- Example, recall moving average

$$H(\omega) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$

$M = 8$

$$H(\omega) = |H(\omega)| \,.\, e^{j\theta(\omega)}$$





$$|H(\omega)| = \left|\frac{1}{M}\right| \left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

$$\theta(\omega) = -\frac{(M-1)}{2}\omega + \pi r$$

FIR filters implemented as IIR filters, integer multiplier filters

- Example, recall moving average

$M = 8$, $M = 1000$?

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} z^{-n} = \frac{1}{M}(1 + z^{-1} + \ldots + z^{-M+1}) = \frac{1}{M} \frac{1 - z^{-M}}{1 - z^{-1}}$$

The output, $y[n]$

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$

$$y[n] = y[n-1] + \frac{1}{M}(x[n] - x[n-M])$$

$$y'[n] = y'[n-1] + (x[n] - x[n-M])$$

$$y[n] \leftarrow y'[n] \cdot \frac{1}{M}$$

FIR filters implemented as IIR filters, integer multiplier filters

- $M \rightarrow m$

$$H(z) = \frac{(1 - z^{-M})}{(1 - z^{-1})} \quad (M \rightarrow m) \rightarrow \quad H(z) = \frac{(1 - z^{-m})}{(1 - z^{-1})}$$

- Introducing another parameter, M

$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M}$$

What are the amplitude and phase response characteristics of the following moving average filter? What are the delays of output signal? How do m and M influence the characteristics and delays?

$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M} = e^{-j\omega T\left(\frac{m}{2} - \frac{1}{2}\right)M} \cdot \left(\frac{\sin(m/2\,\omega T)}{\sin(1/2\,\omega T)}\right)^M$$

Laboratory work

- Study the frequency, amplitude and phase response characteristics for the following filters. What are the delays of output signals? How do l, m, n, N, M, a_k, b_k and c_k influence the characteristics and delays?

$$H(z) = \frac{\left(1 - z^{-m}\right)^M}{\left(1 - z^{-1}\right)^M}$$

$$H(z) = \frac{\left(1 + a_k z^{-m}\right)^M}{\left(1 + b_k z^{-n}\right)^M} \qquad a_k, b_k \in \mathbb{Z}$$

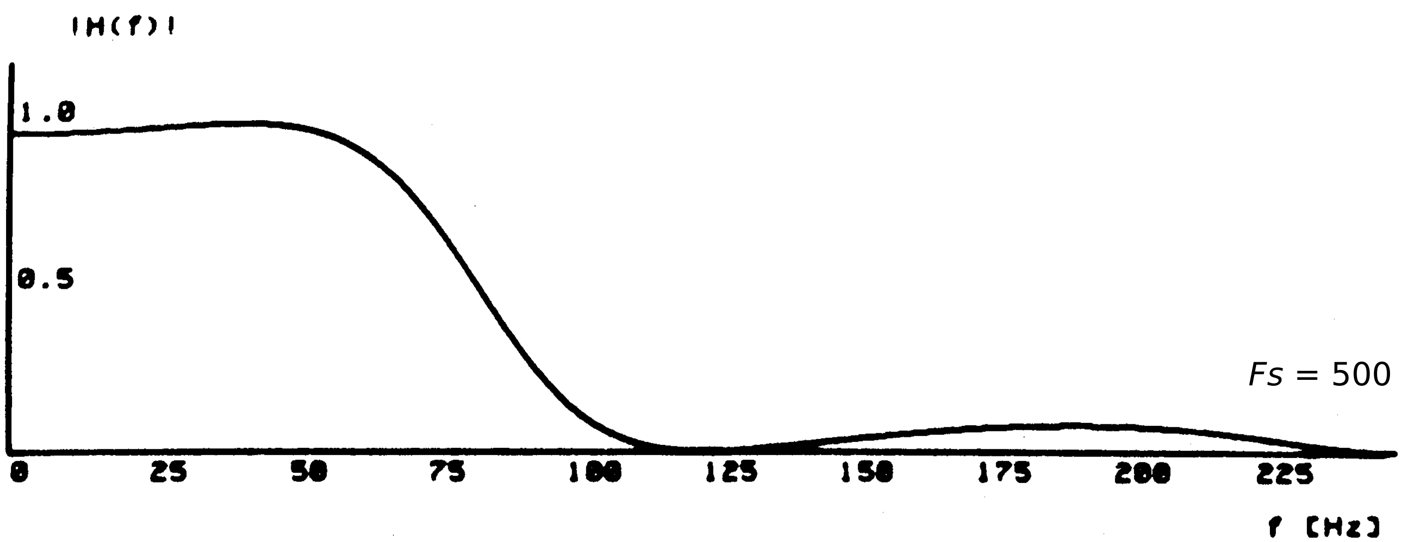
$$H(z) = \left(1 + c_k z^{-l}\right)^N \qquad c_k \in \mathbb{R}$$

$$H(z) = \frac{\left(1 - z^{-m}\right)^M}{\left(1 - z^{-1}\right)^M} \cdot \frac{1}{\left(1 + c_k z^{-l}\right)^N}$$

Laboratory work

- Low-pass filter ($m = 4$, $l = 3$, $M = 2$, $ck = 0.25$, $N = 2$)

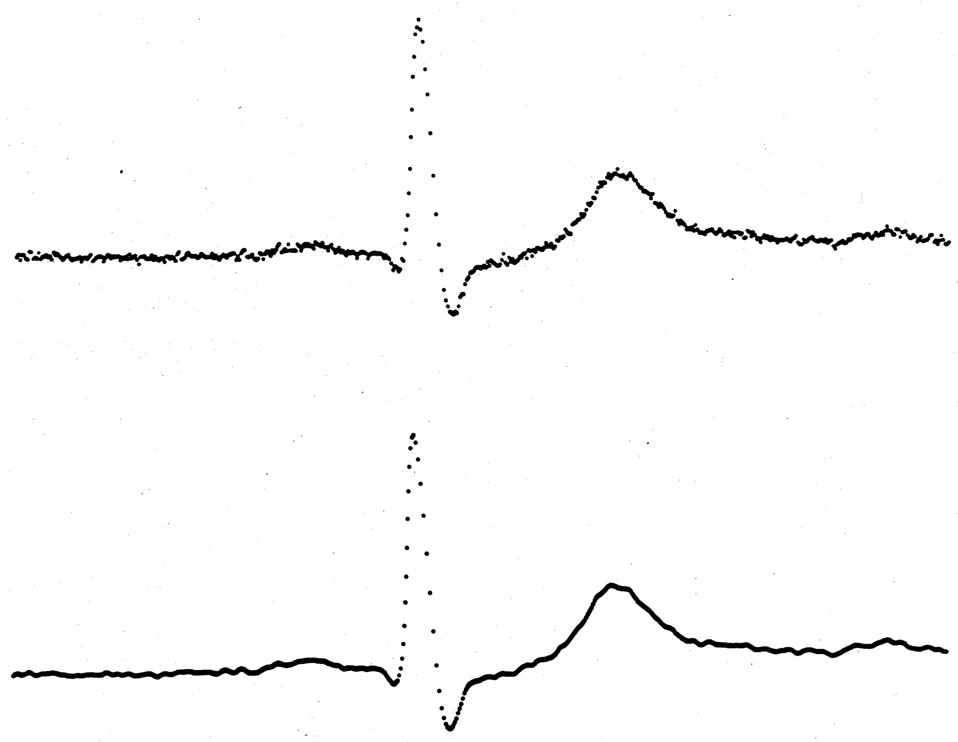
$$H(z) = H_L(z) = \frac{\left(1 - z^{-m}\right)^M}{\left(1 - z^{-1}\right)^M} \cdot \frac{1}{\left(1 + c_k z^{-l}\right)^N}$$



Fs = 500 *smp/sec*

Laboratory work

- Original signal and signal after low-pass filtering



(Frequency response of LTI systems, example)

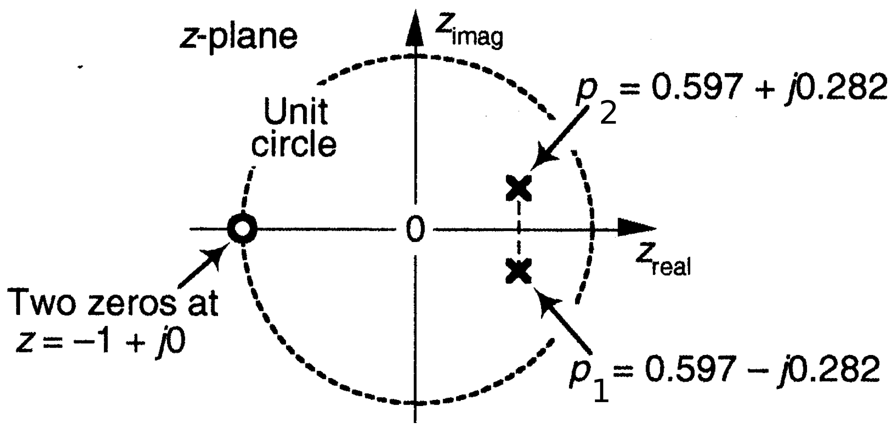
- Transfer function, *H(z),* is complex function of a complex variable over entire z plane
- Frequency Response is Transfer Function, *H(z)*, evaluated on the unit circle

$$H(\omega) = H(z)\big|_{z=e^{j\omega} \text{ or } |z|=1}$$

- Example

$$H(z) = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = \frac{(z+1)(z+1)}{(z-0.597+j0.282)(z-0.597-j0.282)}$$

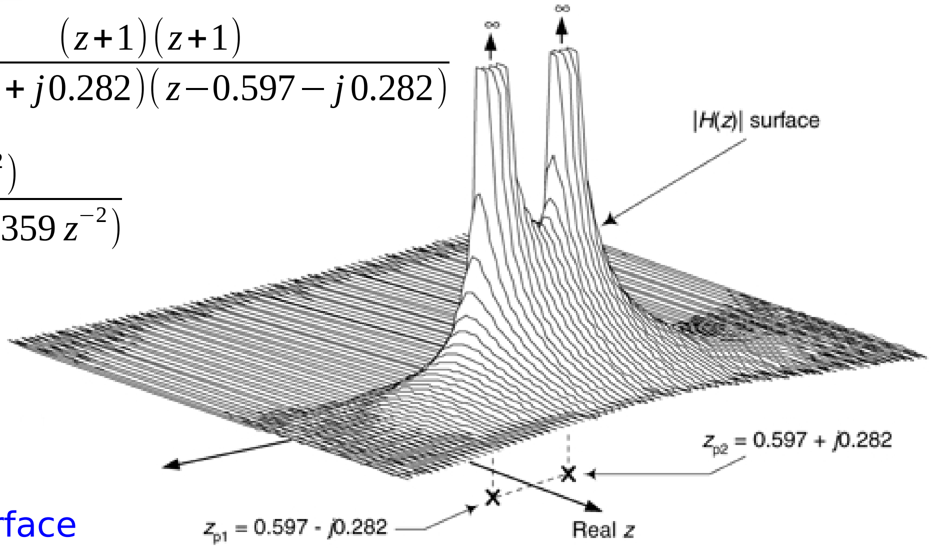
$$H(z) = \frac{(1+2z^{-1}+z^{-2})}{(1-1.194z^{-1}+0.4359z^{-2})}$$



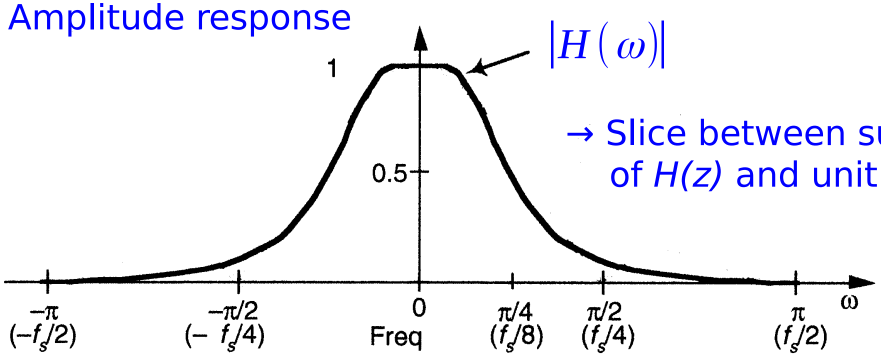
(Frequency response of LTI systems, example)

$$H(z) = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = \frac{(z+1)(z+1)}{(z-0.597+j0.282)(z-0.597-j0.282)}$$

$$H(z) = \frac{(1+2z^{-1}+z^{-2})}{(1-1.194z^{-1}+0.4359z^{-2})}$$

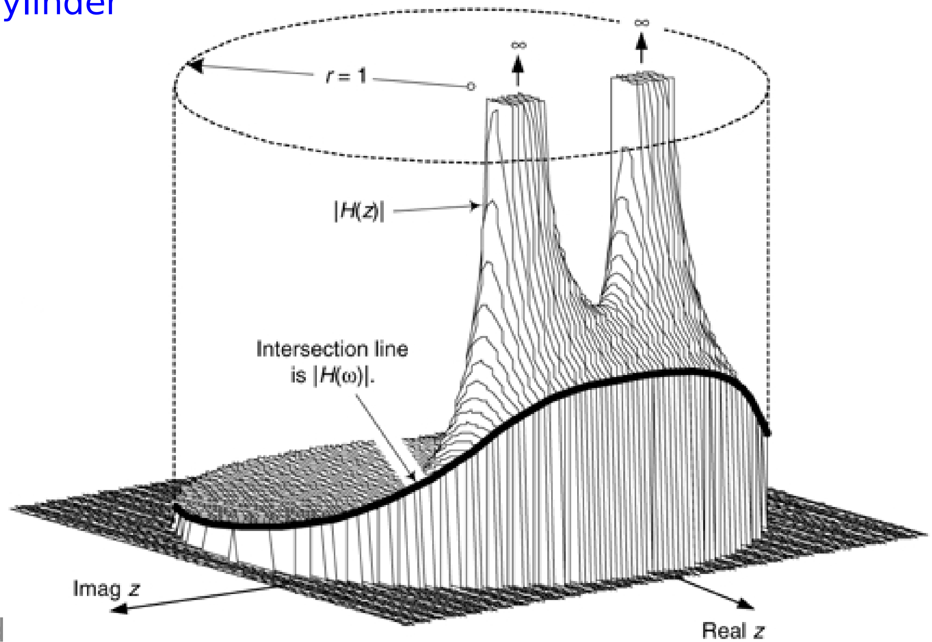


Amplitude response



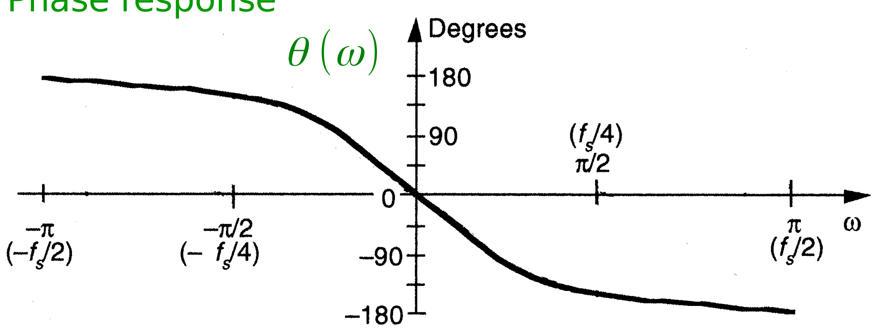
$|H(\omega)|$

→ Slice between surface of *H(z)* and unit cylinder



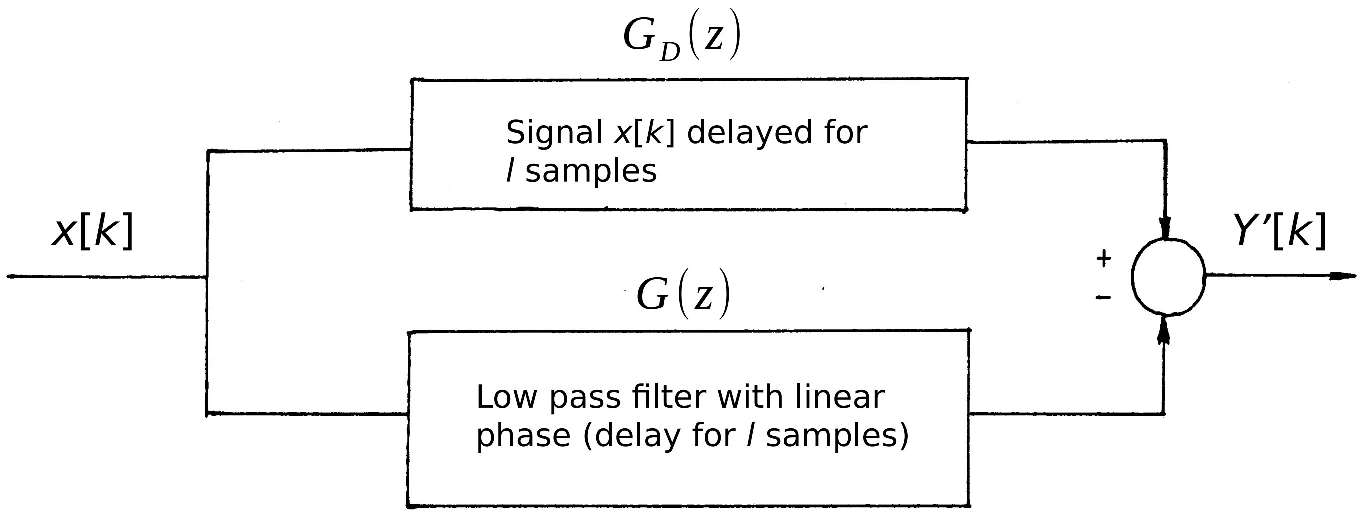
Phase response

$\theta(\omega)$



(FIR filters implemented as IIR filters, integer multiplier filters)

- How to obtain band-stop characteristic using band-pass filter (or, vice versa)?



$$G(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M}$$

$$H(z) = G_D(z) - G(z) = \left(\frac{m}{n}\right)^M . z^{-j\omega\left(\frac{m}{2}.M - \frac{n}{2}.M\right)} - G(z)$$

(FIR filters implemented as IIR filters, integer multiplier filters)

- High-pass filtering using low-pass filter, *HLP(z)*

$$H_{LP}(z) = \frac{\left(1 - z^{-344}\right)^2}{\left(1 - z^{-1}\right)^2} \qquad G(z) = \frac{\left(1 + a_k z^{-m}\right)^M}{\left(1 + b_k z^{-n}\right)^M}$$

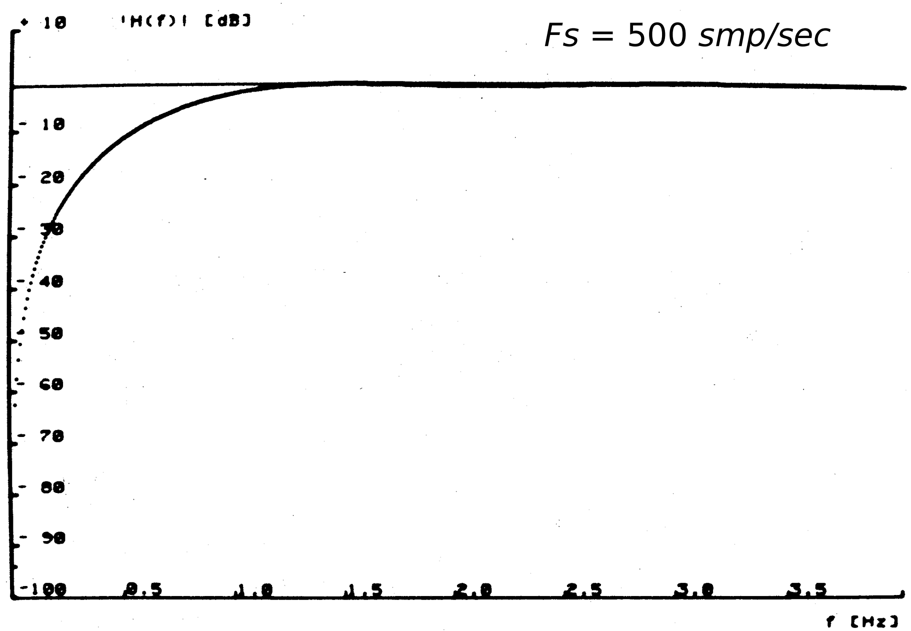
ak = -1, bk = -1, m = 344, n = 1 in M = 2

$$y(k) = 2 \cdot y(k-1) - y(k-2) + x(k) - 2 \cdot x(k-344) + x(k-688)$$

$$y'(k) = k_v \cdot x(k-343) - y(k)$$

$$y'(k) \leftarrow y'(k)/k_v$$

$$k_v = 344^2$$



(FIR filters implemented as IIR filters, integer multiplier filters)

- High-pass and 50, 100, 150, 200, 250 Hz notch filtering using combined band-pass filter, i.e., low-pass and 50, 100, 150, 200, 250 Hz band-pass filter, *HL,50(z)*

$$G(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M}$$

ak = -1, bk = -1, m = 330, n = 10 in M = 2

$$H_{L,50}(z) = \frac{(1 - z^{-330})^2}{(1 - z^{-10})^2}$$

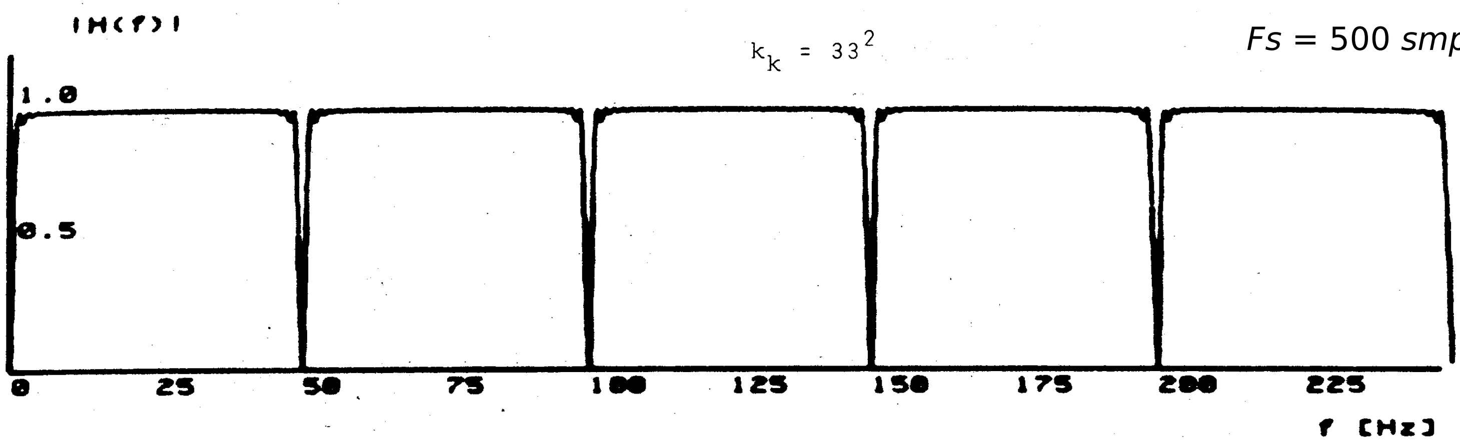
y(k) = 2·y(k - 10) - y(k - 20) + x(k) - 2·x(k - 330) + + x(k - 660)

y'(k) = k_k x(k - 320) - y(k)

y'(k) ← y'(k)/k_k

$k_k = 33^2$

Fs = 500 *smp/sec*



(FIR filters implemented as IIR filters, integer multiplier filters)

- 50 Hz notch filtering using 50, 150, 250 Hz band-pass filter, *$H_{BP}(z)$*

$$H_{BP}(z) = \frac{\left(1 + z^{-185}\right)^2}{\left(1 + z^{-5}\right)^2} \qquad G(z) = \frac{\left(1 + a_k z^{-m}\right)^M}{\left(1 + b_k z^{-n}\right)^M}$$

ak = 1, bk = 1, m = 185, n = 5 in M = 2

y(k) = -2y(k - 5) - y(k- 10) + x(k) + 2x(k - 185

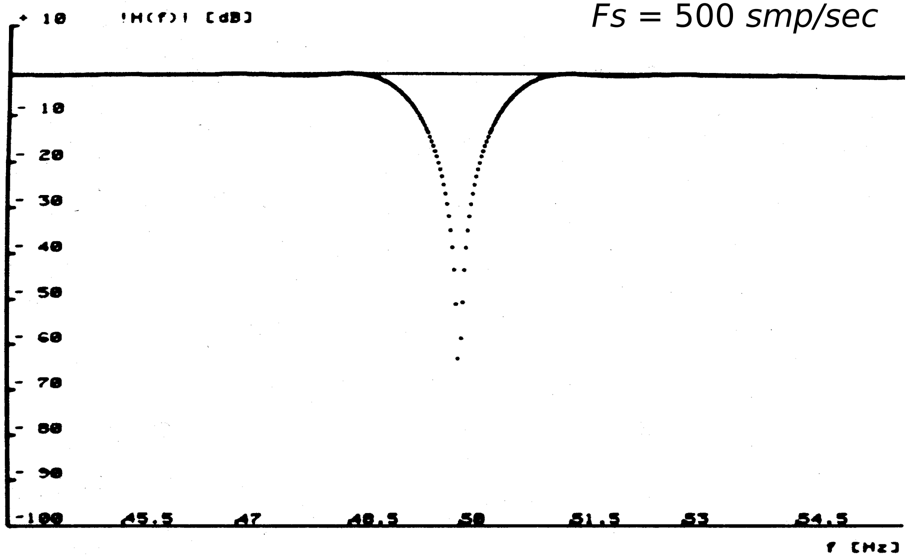
+ x(k - 370)

$y'(k) = k_{50} \cdot x(k - 180) - y(k)$

$y'(k) \leftarrow y'(k)/k_{50}$

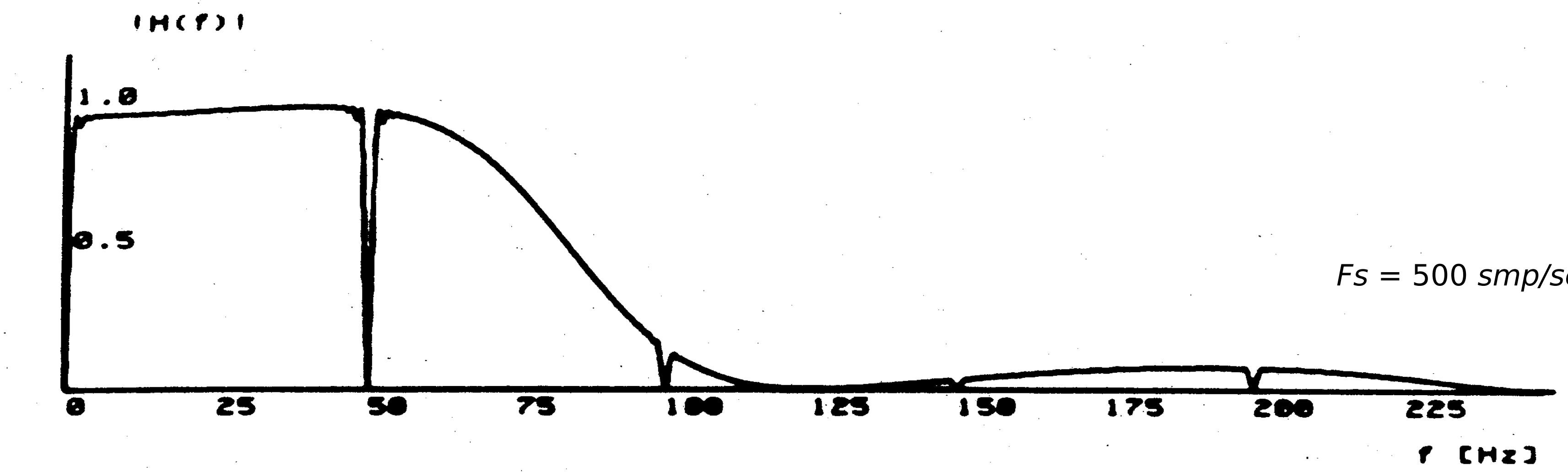
$k_{50} = 37^2$

Fs = 500 smp/sec



(Examples)

- Amplitude response of low-pass filter, *HL(z)*, and, high-pass and 50, 100, 150, 200, 250 Hz notch filter, *HL,50(z)*



Fs = 500 *smp/sec*

(Examples)

- Original signal and signal after using $H_L(z)$ and $H_{L,50}(z)$