# SPECTRAL ANALYSIS

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# Fourier-based power spectrum analysis

- The estimated power spectrum (periodogram) of a *stationary* signal *x[n]* (*x[n]* is assumed to be zero outside the interval *[0, N - 1]*), is obtained by computing the squared magnitude of the N-point DTFT of *x[n]*

$$\hat{S}_x(\omega) = \frac{1}{N}|X(\omega)|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j\omega n}\right|^2$$

- Since N-point DFT completely specifies the DTFT of a finite duration sequence of length N samples, the estimated power spectrum (periodogram) of a *stationary* signal *x[n]* can simply be computed following

$$P[k] = \frac{1}{N}|X(k)|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{j\left(\frac{2\pi k}{N}\right)n}\right|^2$$

- Parseval's theorem expresses the energy in the finite duration sequence *x[n]* in terms of the frequency components *X[k]*

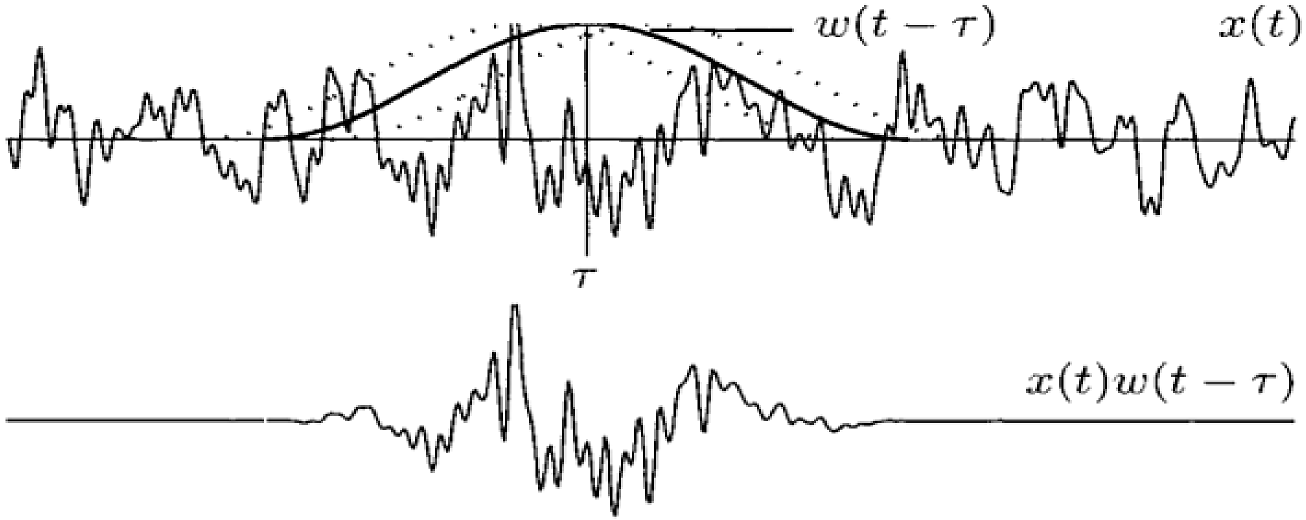
$$\sum_{n=0}^{N-1} x[n]^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

# Joint time-frequency analysis

- A major limitation of Fourier-based spectral analysis is its inability to provide information on when in time different frequencies of a signal occur
- The Fourier transform only reflects which frequencies exist during the total observation interval, because the Fourier transform integrates frequency components over the total observation interval
- While such spectral analysis is adequate for stationary signals whose frequencies, on average, are equally spread in time, it is inadequate for non-stationary signals with time dependent spectral content
- There is strong motivation for the development of methods that analyze signals with regard to both time and frequency so that the frequencies present at each instant in time can be displayed
- Joint time-frequency information has been found extremely valuable for many types of biomedical signals exhibiting non-stationary characteristics

# The short-time Fourier transform

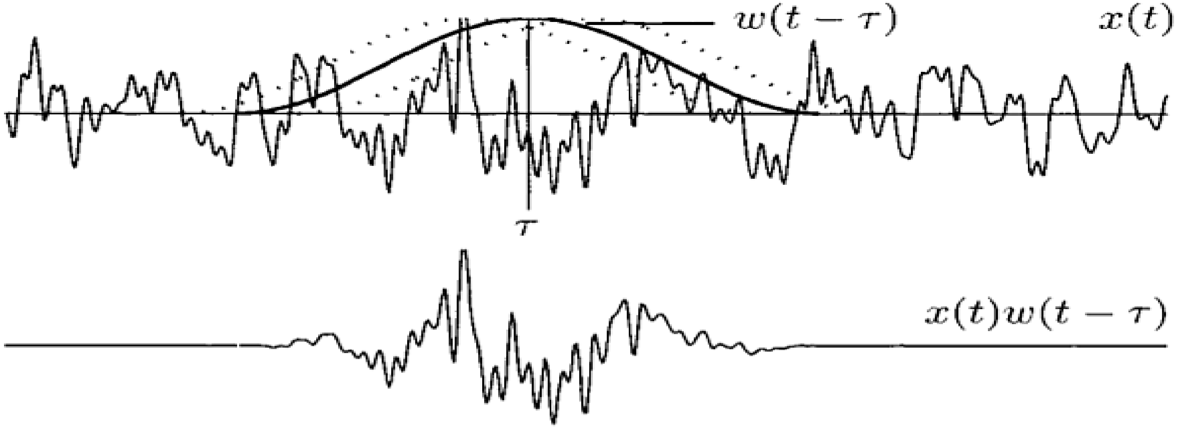
- The short-time Fourier transform is a classical non-parametric method to obtain time-frequency representation by linear filtering operation




- In the short-time Fourier transform, a sliding time window *w (t - τ)* is used for excerpting successive parts of the signal *x(t)*

(Sornmo, Laguna)

# The short-time Fourier transform




- In this approach, the definition of the Fourier transform is modified so that a sliding time window *w(t)* is included that defines each time segment to be analyzed, thus resulting in a two-dimensional function *X (t, Ω)* defined by:

$$X(t,\Omega) = \int_{-\infty}^{\infty} x(\tau) w(\tau - t)\, e^{-j\Omega\tau}\, d\tau$$

where *Ω* denotes analog frequency

(Sornmo, Laguna)

# Spectrogram

- Analogous to the computation of the periodogram, which was obtained as the squared magnitude of the Fourier transform of the signal:

$$\hat{S}_x(\omega) = \frac{1}{N}\left|X(\omega)\right|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j\omega n}\right|^2$$

the spectrogram, $S_x(t, \Omega)$, of $x(t)$ is obtained by computing the squared magnitude of the short-time Fourier transform:

$$S_x(t,\Omega) = \left|X(t,\Omega)\right|^2$$

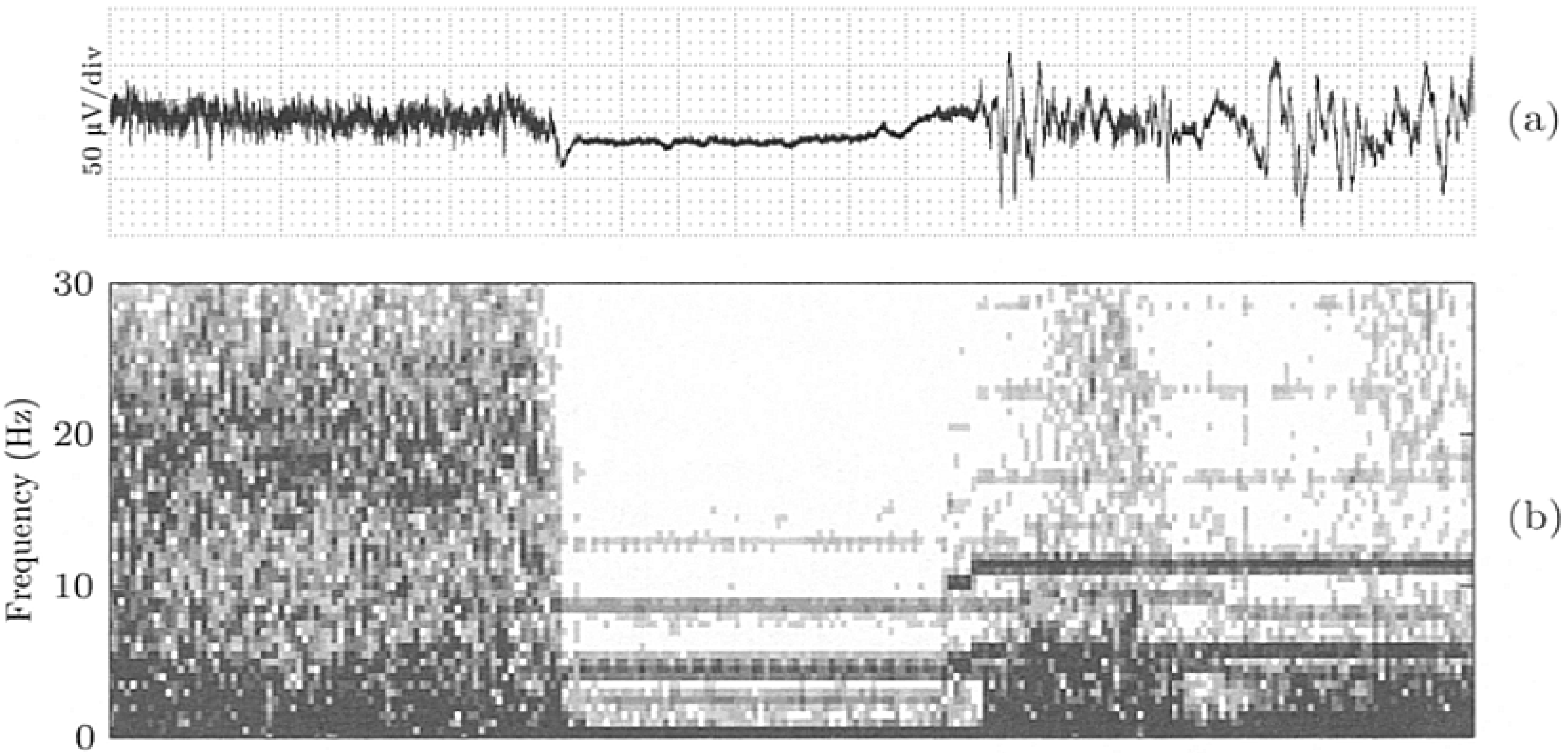
where

$$X(t,\Omega) = \int_{-\infty}^{\infty} x(\tau)w(\tau - t)\, e^{-j\Omega\tau}\, d\tau$$

- The spectrogram, $S_x(t, \Omega)$, is a real-valued, nonnegative distribution which provides a signal representation in the time-frequency domain

# Example

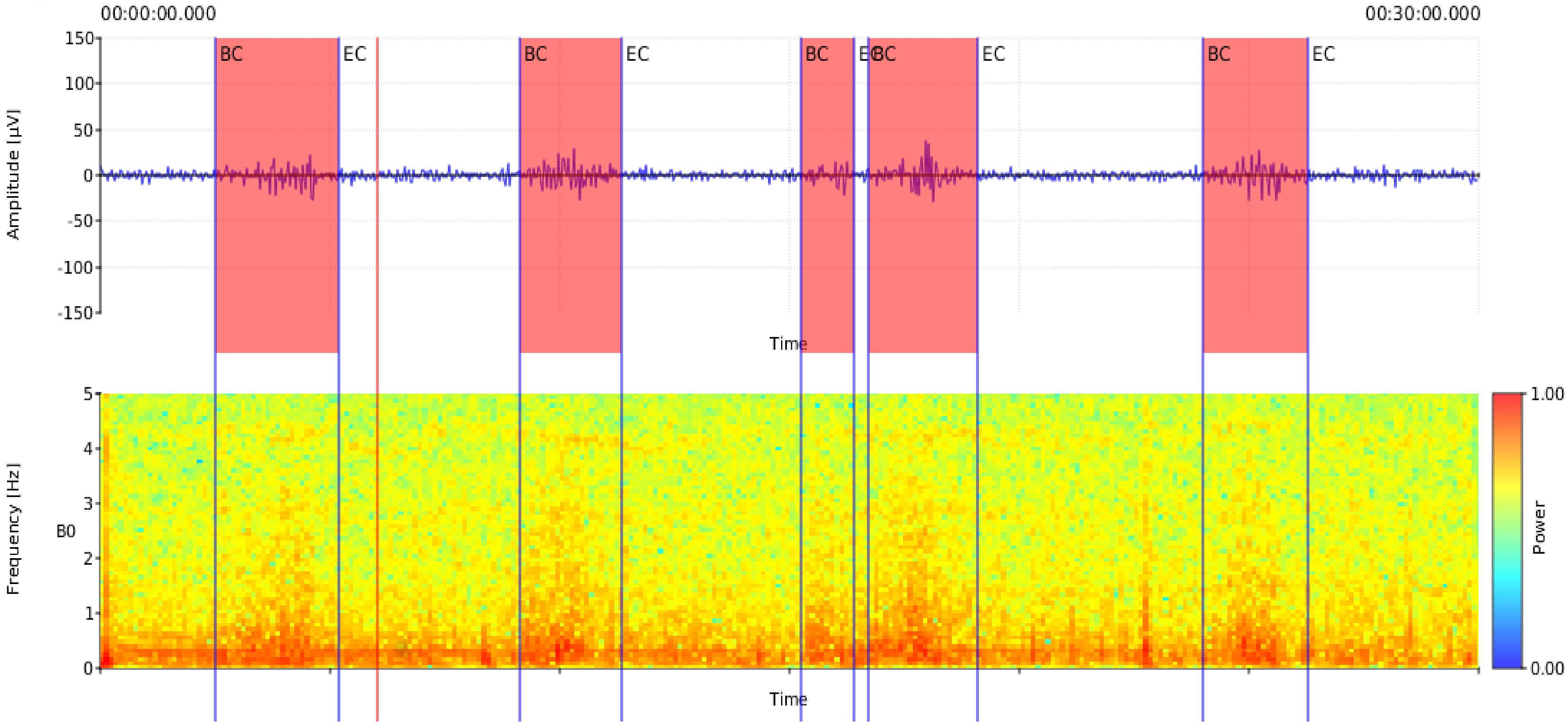
- (a) The EEG recorded during heart surgery of an infant (time scale from 0 to 300 sec)
  (b) The spectrogram displays a drastic reduction in high-frequency content after 100 s, partially reverting at about 200 s




(Sornmo, Laguna)

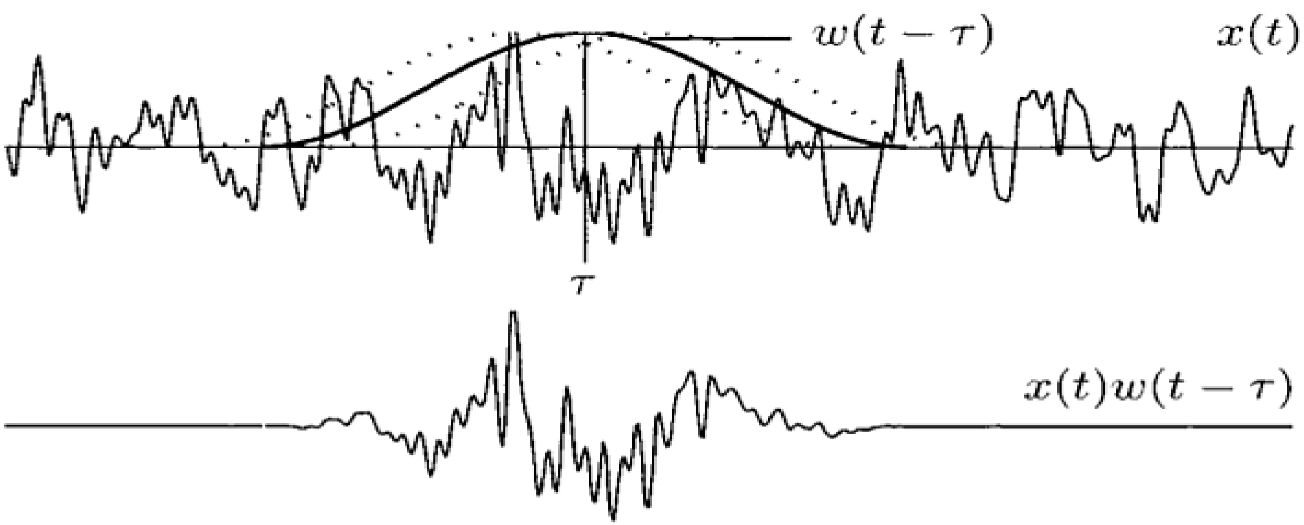
# Example

- (Upper) 20-minute excerpt of the EMG of uterus recorded from the abdomen showing contractions (the pregnancy ended in pre-term delivery)
- (Lower) The spectrogram, the power spectra, from 0 Hz to 5.0 Hz (the width of sliding window is 256 samples, 12.8 sec, Fs = 20 smp/sec)



# Window functions

- The length of the sliding time window *w(t)* determines the resolution in time and frequency such that a short window yields good time resolution but poor frequency resolution, and the opposite when a long window is used
- Rectangular window *w(t)* causes less reliable measurements on spectral power
- Better estimates of power spectra are obtained by using weighted window functions like Blackman, Hamming, and Hanning windows




(Sornmo, Laguna)

# Window functions

- Coefficients of window functions

Rectangular window $$w[n] = 1, \quad n = 0,1,2,\ldots,L-1$$

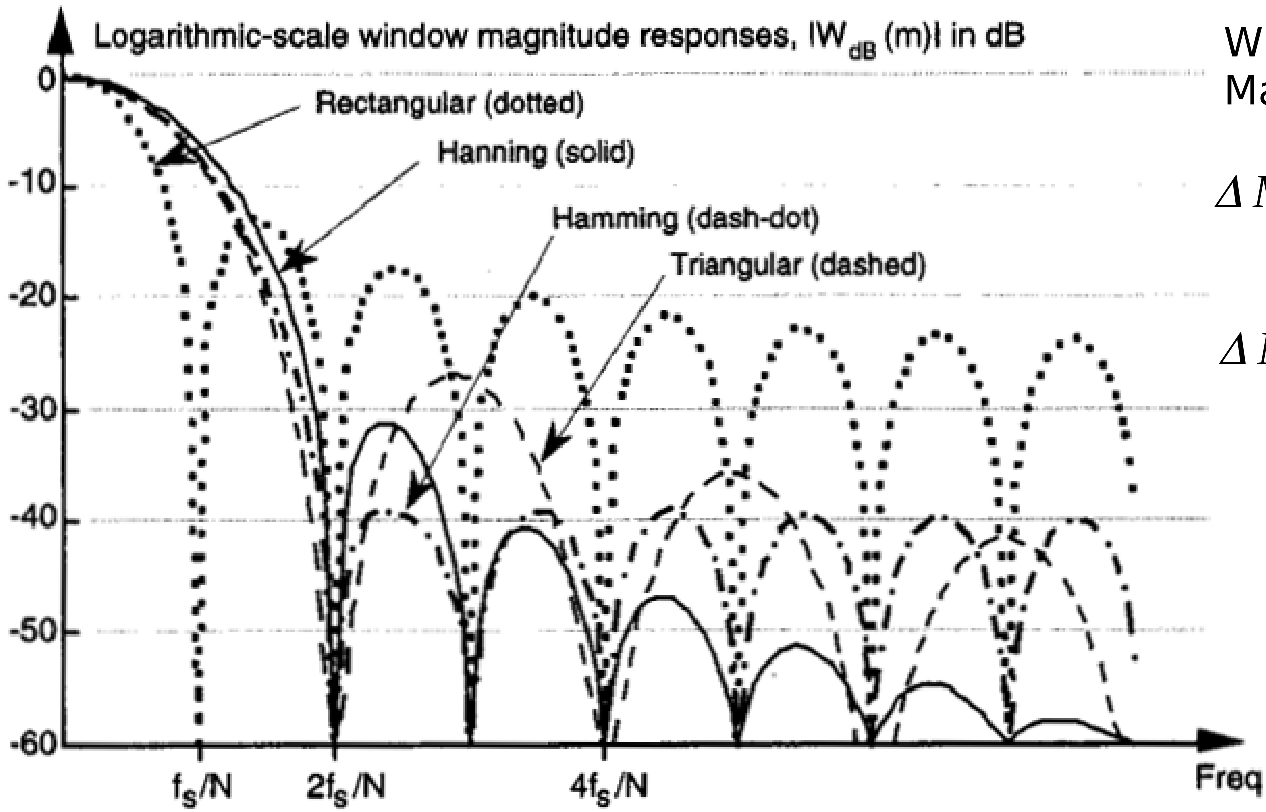
Triangular window $$w[n] = \begin{cases} \dfrac{n}{L/2}, & n = 0,1,2,\ldots,L/2 \\ 2-\dfrac{n}{L/2}, & n = L/2+1, L/2+2,\ldots,L-1 \end{cases}$$

Hanning (Hann) window $$w[n] = 0.5 - 0.5\cos\left(\frac{2\pi n}{L}\right), \quad n = 0,1,2,\ldots,L-1$$

Hamming window $$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right), \quad n = 0,1,2,\ldots,L-1$$

# Window functions

- Logarithmic-scale window amplitude spectra, $|W_{dB}[k]|$ in dB




(Lyons)

Width of the Main Lobe (*ML*)

$$\Delta ML_{Rect} = 2.\frac{F_S}{N}$$

$$\Delta ML_{Other} = 4.\frac{F_S}{N}$$

# Example of spectrogram

- *Example of spectrogram using Hamming windows (next slide)*

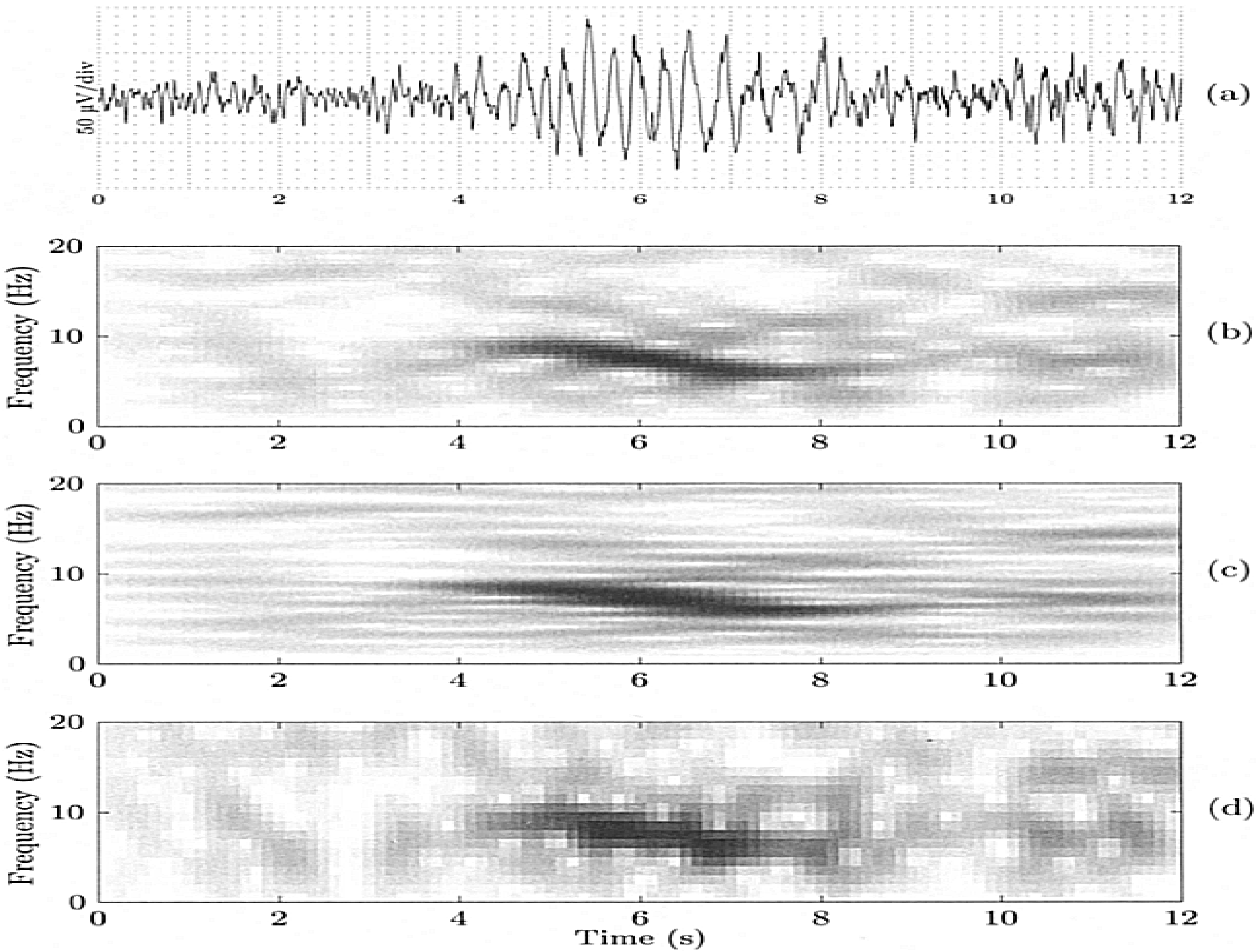
(a) The EEG at the onset of an epileptic seizure. The corresponding spectrogram is computed using a Hamming window with a length of
(b) 1 s, (c) 2 s, and (d) 0.5 s.

(c) The spectrogram is obtained with the longest time window (2 s) and therefore exhibits the poorest time resolution of the three lengths; property is reflected by a ridge which extends longer in time than does the ridge in figure (d)

(c) The spectrogram shows the best frequency resolution is due to the longer time window (2 s), while the frequency resolution in figure (d) is worse

- There is always a trade-off with respect to resolution in time and frequency

(Sornmo, Laguna)


50 μV/div
(a)
0
2
4
6
8
10
12
Frequency (Hz)
20
10
0
(b)
(c)
(d)
Time (s)

# (Coherence function)

- The coherence function (magnitude-squared coherence), $C_{xy}(\omega)$, allows us to find common frequencies and to evaluate the similarity of signals
- Coherence (Latin – *cohaerentia*) means natural or logical connection or consistency
- The coherence function estimates the extent to which $y(t)$ may be predicted from $x(t)$

$$C_{xy}(\omega) := \frac{P_{xy}(\omega)}{\sqrt{P_{xx}(\omega) P_{yy}(\omega)}}$$

$$C_{xy}^{2}(\omega) = \frac{|\ P_{xy}(\omega)\ |}{P_{xx}(\omega) P_{yy}(\omega)}$$

$$P_{xx}(\omega) := |\hat{x}(\omega)|^2 = \hat{x}(\omega)\overline{\hat{x}(\omega)}$$

$$P_{xy}(\omega) := \hat{x}(\omega)\overline{\hat{y}(\omega)}$$

$$\hat{x}(\omega) := \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

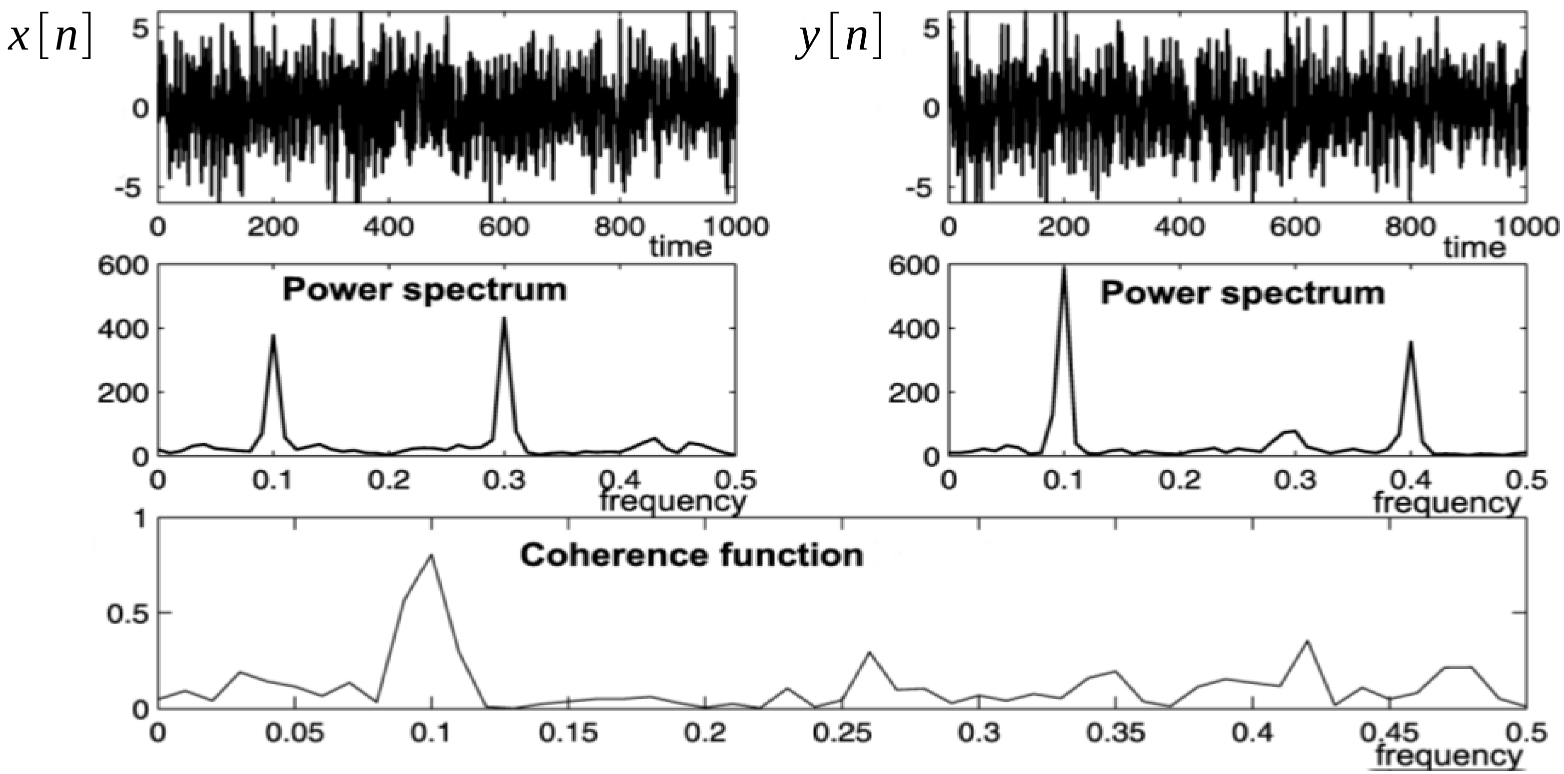
*where* $P_{xx}(\omega)$ and $P_{yy}(\omega)$ are power spectra of signals $x(t)$ and $y(t)$, $P_{xy}(\omega)$ is cross-power spectrum for these signals, and $\hat{x}(\omega)$ and $\hat{y}(\omega)$ are the Fourier transforms of $x(t)$ *and* $y(t)$

- The value of coherence will always satisfy $0 \leq C_{xy}(\omega) \leq 1$
- If $y[n] = h[n] * x[n]$, then $C_{xy}(\omega) = 1$

# (Example of coherence function)

$$x[n] = k_1[n] + \cos(2\pi \,.\, 0.1n) + \cos(2\pi \,.\, 0.3n)$$

$$y[n] = k_2[n] + \cos(2\pi(0.1n + \psi)) + \cos(2\pi(0.4n + \phi))$$




(Golinska AK, Coherence function in biomedical signal processing)