THE Z TRANSFORM

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Introduction

- The Z-transform is a generalization of the DTFT
- The Z-transform can be used to characterize the response of linear, time-invariant filters to complex exponential signals

Definition

- Let consider the response of a filter with unit-sample response *h[n]* to the complex exponential *z(^n)* , where *z* is an arbitrary complex number:

$$y[n] = h[n] * z^n = \sum_{m=-\infty}^{\infty} h[m] z^{n-m}$$

$$y[n] = z^n \sum_{m=-\infty}^{\infty} h[m] z^{-m} = x[n] H(z)$$

$$H(z) \triangleq \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- For LTI systems described by a unit-sample response *h[n]*, *H(z)* is also referred to as the *system function*. The system function is a generalization of the frequency response, since the DTFT is a special case of the Z-transform, that is:

$$H(f) = H(z)\big|_{z=e^{j2\pi f}}$$

Z-transform of filters defined by a difference equation

- Digital filters described by linear, constant-coefficient difference equations (LCCDEs):

$$y[n] = \sum_{k=1}^{K} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

- The Z-transform of these digital filters follows directly from the definition of *H(z)* as the ratio of the output to the input for complex exponential inputs
- Specifically, if *x[n] = z(^n)* , then *y[n] = H(z) . z(^n)*. Substituting produces:

$$y[n] = H(z) z^n = \sum_{k=1}^{K} a_k H(z) z^{n-k} + \sum_{m=0}^{M} b_m z^{n-m}$$

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1 - \sum_{k=1}^{K} a_k z^{-k}}$$

Z-transform of filters defined by a difference equation

- The Z-transform of a digital filter defined by a finite difference equation is a rational function in *z*
- Since any reasonably well-behaved function of *z* can be approximated by a rational function, any LTI filter can be approximated by a finite difference equation that can be implemented

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1 - \sum_{k=1}^{K} a_k z^{-k}} \qquad y[n] = \sum_{k=1}^{K} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

- The *K* complex roots of the denominator of *H(z)* are called the *poles* of the filter
- The *M* complex roots of the numerator are called *zeros* of the filter
- For FIR filters, the denominator is unity, so that the system function has only zeros.
- For purely recursive filters, the numerator is the constant *b0*, so that the system function has only poles (IIR, AR; otherwise IIR, ARMA)
- This explains the terms *all-zero* and *all-pole* to designate FIR filters and purely-recursive filters

Properties

- The most important property of Z-transforms is the *convolution theorem*

$$x1[n] \ast x2[n] \longleftrightarrow X1(z) \,.\, X2(z)$$

* Because it replaces the complicated convolution operation by a simpler multiplication, this theorem is:

(1) useful for computing the responses of digital filters to signals given by analytic expressions

(2) for finding the unit-sample response of cascade combinations of filters

Properties

Property	Time Domain	z-Domain
Notation	$x(n)$	$X(z)$
	$x_1(n)$	$X_1(z)$
	$x_2(n)$	$X_2(z)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$
Time shifting	$x(n-k)$	$z^{-k}X(z)$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$
Time reversal	$x(-n)$	$X(z^{-1})$
Conjugation	$x^*(n)$	$X^*(z^*)$
Real part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$
Imaginary part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$

(Proakis, Manolakis)

Some common Z-transform pairs

	Signal, $x(n)$	z-Transform, $X(z)$
1	$\delta(n)$	1
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$

(Proakis, Manolakis)

Examples of Z transform

- The relation between a signal *h[n]* and its Z-transform *H(z)*: *h[n]* ←→ *H(z)*

1. Gain:
$$G\delta[n] \longleftrightarrow G$$
In particular, the Z-transform of the unit sample $\delta[n]$ is the constant 1.

2. Delay by n_0 samples:
$$\delta[n-n_0] \longleftrightarrow z^{-n_0}$$
This explains why the notation z^{-1} is often used to designate a unit delay.

3. Rectangular ("boxcar") filter of length N:
$$R_N[n] \stackrel{\triangle}{=} u[n] - u[n-N] \longleftrightarrow \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}}$$
This filter has $N-1$ zeroes equally spaced on the unit circle, except for $z=1$ where the zero is cancelled by a pole.

4. First-order recursive lowpass filter $y[n] = ay[n-1] + x[n]$:
$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \quad \text{for} \quad |z| > |a|$$

The system function of LTI systems

- The output (response) $y[n]$ of an LTI system to an input sequence $x[n]$ can be obtained by computing the convolution of $x[n]$ with the unit-sample response of the system $h[n]$ (Note that $h[n]$ is the response of the system to unit-sample signal $\delta[n]$)
  The convolution property allows us to express this relationship in the Z-domain as:

$$Y(z) = H(z) \, . \, X(z)$$

  where $Y(z)$ is the Z-transform of the output sequence $y[n]$,

  $X(z)$ is the Z-transform of the input sequence $x[n]$, and

  $H(z)$ is the Z-transform of the unit-sample response $h[n]$.

  We than determine output (response) $y[n]$ by evaluating the inverse Z-transform of $Y(z)$.
- Alternatively, if we know $x[n]$ and we observe the output $y[n]$ of the system, we can determine the unit-sample response by first solving for $H(z)$ from the relation:

$$H(z) = Y(z) \; / \, X(z)$$

  and then evaluating the inverse Z-transform of $H(z)$ to get $h[n]$,
  where $H(z)$ represents the Z-domain characterization of the system and the unit-sample response $h[n]$ is the corresponding time-domain characterization of the system.
- The transform $H(z)$ is called the *system function* or *transfer characteristic* of the system

Finding response of LTI systems (example)

- The system characterized by diff. eq.: → $y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$
- The input signal: → $x(n) = \delta(n) - \frac{1}{3}\delta(n-1).$
- The system function: → $H(z) = \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$

  Poles at z1 = 1/2 and z2 = 1/3

$$= \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

- The Z transform of the input signal: → $X(z) = 1 - \frac{1}{3}z^{-1}$

$$Y(z) = H(z)X(z)$$

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

- The response (output) of the system: → $y(n) = (\frac{1}{2})^n u(n)$

Finding unit-sample response of LTI systems (example)

- The system characterized by difference equation: → $y(n) = 2.5y(n-1) - y(n-2) + x(n) - 5x(n-1) + 6x(n-2)$

- The system function: → $$H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}} = \frac{1 - 5z^{-1} + 6z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}$$
  Poles at $z1 = 2$ and $z2 = 1/2$

The zeros occur at $z = 2$ and $z = 3$

The zero at $z = 2$ cancels the pole at $z = 2$

Finding unit-sample response of LTI systems (example)

- Consequently, *H(Z)* reduces to: →

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{z-3}{z-\frac{1}{2}}$$

$$= 1 - \frac{2.5z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- The unit-sampe (impulse) response: →

$$h(n) = \delta(n) - 2.5(\frac{1}{2})^{n-1}u(n-1)$$

- The system is characterized by: →

$$y(n) = \frac{1}{2}y(n-1) + x(n) - 3x(n-1)$$