ZeroProofML: Epsilon-Free Rational Neural Layers via Transreal Arithmetic

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Abstract

We introduce ZeroProofML, a framework for deterministic, ε-free rational neural layers based on transreal (TR) arithmetic. By totalizing division (and other singular operations) with explicit tags (REAL, $\pm\infty$, Φ), TR removes ad-hoc ε knobs and yields reproducible semantics for singularities. We formalize TR autodiff (Mask-REAL) and give stability statements (bounded updates, batch-safe steps). On 2R inverse kinematics, TR matches overall accuracy and achieves 1.5–2.5× lower error in the closest near-singularity bins (B0–B1), modest improvements in B2 (∼3–4%), and near parity elsewhere, with stable closed-loop behavior. Results extend to planar 3R and synthetic 6R, supporting robustness near rank-deficient Jacobians.

Keywords: transreal arithmetic, rational layers, singularities, robotics IK, reproducibility

1 Preliminaries

We use the transreal domain $\mathbb{T} = \mathbb{R} \cup \{+\infty, -\infty, \perp\}$ with tags $\{\mathrm{REAL}, \mathrm{INF}, \mathrm{NULL}\}$. Values are pairs (v, τ) with $v \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Arithmetic on $\mathbb{T}$ follows explicit tag rules (addition/multiplication/division, integer powers, and guarded $\sqrt{\cdot}$). All expressions in our admissible class are total on $\mathbb{T}^n$.

IEEE–TR bridge. We define total maps $\Phi : \mathsf{IEEE} \to \mathbb{T}$ and $\Psi : \mathbb{T}_{\mathrm{REAL/INF}} \to \mathsf{IEEE}$ (round-to-nearest-even; undefined on $\perp$). Signed zeros are retained in a latent flag used only when directional limits matter.

2 Autodiff with Tags and Hybrid Switching

Let nodes be $z_k = F_k(z_{i_1}, \ldots, z_{i_m})$ with F_k in our admissible class. Each primitive has a REAL-mask predicate $\chi_k \in \{0, 1\}$ that is 1 iff all inputs and the evaluation are REAL. Mask-REAL backprop uses gates $\bar{z}_i \mathrel{+}= \chi_k \, \bar{z}_k \, \partial_{z_i} F_k|_{\mathrm{REAL}}$. In a small band around poles we use bounded surrogates (saturating gradients) and a hybrid policy with hysteresis ($\tau^{\mathrm{on}} < \tau^{\mathrm{off}}$), delivering bounded updates.

Batch-safe steps. A per-batch curvature proxy $L_{\mathrm{batch}} \approx (B_\psi^2 / q_{\mathrm{min}}^2)(1 + y_{\mathrm{max}}^2) + \alpha$ yields a safe clamp $\eta \leq 1/L_{\mathrm{batch}}$; combined with bounded surrogates, per-step updates remain bounded.

Policy determinism. ULP-scaled guard bands, signed-zero retention, and deterministic reductions make tag classification deterministic up to stated ULP bands; outside guard bands tags are identical across seeds.

3 Global Stability and Convergence

Under standard smoothness assumptions and $\eta_t \leq 1/\widehat{L}_{\mathcal{B}_t}$, GD/SGD enjoy standard descent/convergence guarantees; bounded gradients in SAT regions preserve stability across MR$\leftrightarrow$SAT switches.

4 Experimental Setup

Tasks. Planar 2R IK with $|\det J| \approx |\sin\theta_2|$ (primary), planar 3R (rank drop by alignment), and synthetic 6R (serial DH).

Datasets. 2R: stratified by $|\det J|$ with edges $[0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, \infty)$; near-pole coverage ensured in train/test. 3R: stratified by manipulability ($\sigma_1\sigma_2$). 6R: stratified by $d_1 = \sigma_{\min}(J)$.

Baselines. MLP; Rational+ε (grid); smooth surrogate $P/\sqrt{Q^2+\alpha^2}$ (grid); learnable-ε; ε-ensemble. Reference: DLS.

TR models. TR–Basic (Mask-REAL only). TR–Full: shared-Q TR–Rational heads with hybrid gradients, tag/pole heads, anti-illusion residual, coprime regularizer; coverage enforcement and TR policy hysteresis; batch-safe LR.

Metrics. Overall and per-bucket MSE (B0–B4); closed-loop tracking (task-space error, max $\|\Delta\theta\|$, failures). 3R: PLE, sign consistency across θ_2, θ_3, residual consistency. 6R: overall + selected bins.

Aggregation. Three seeds (2R/6R), deterministic policy for TR; means$\pm$std reported. Scripts emit per-seed JSONs and LaTeX tables/figures used below.

5 Empirical Validation

Across seeds (2R), TR–Full matches or exceeds the strongest ε/smooth baselines in overall error and attains 1.5–2.5$\times$ lower error in the tightest near–pole bins (B0–B1), with $\sim$3–4% gains in B2 and near parity in B3 (Table 2; see also Table 1). Under closed–loop tracking, TR–Full yields the lowest task–space error and smallest joint steps (Table 3).

Across-seed overall:

Near-pole bins (B0–B2):

Closed-loop tracking:

3R TR metrics:

6R TR summary:

Method	Overall MSE
ZeroProofML (Full)	0.140736 ± 0.000000 (n=3)
Rational+ε	0.142179 ± 0.000000 (n=3)
Smooth	0.142185 ± 0.000000 (n=3)
Learnable-ε	0.142203 ± 0.000000 (n=3)
EpsEnsemble	0.141750 ± 0.000000 (n=3)
MLP	0.303699 ± 0.012393 (n=3)

Table 1: Across-seed overall MSE (lower is better).

Method	(0e+00,1e-05]	(1e-05,1e-04]	(1e-04,1e-03]
ZeroProofML (Full)	0.002249 ± 0.000000 (n=3)	0.001295 ± 0.000000 (n=3)	0.030975 ± 0.000000 (n=3)
Rational+ε	0.003572 ± 0.000000 (n=3)	0.002863 ± 0.000000 (n=3)	0.032108 ± 0.000000 (n=3)
Smooth	0.003578 ± 0.000000 (n=3)	0.002869 ± 0.000000 (n=3)	0.032115 ± 0.000000 (n=3)
Learnable-ε	0.003595 ± 0.000000 (n=3)	0.002889 ± 0.000000 (n=3)	0.032136 ± 0.000000 (n=3)
EpsEnsemble	0.003197 ± 0.000000 (n=3)	0.002424 ± 0.000000 (n=3)	0.031679 ± 0.000000 (n=3)
MLP	0.005334 ± 0.002017 (n=3)	0.007113 ± 0.002753 (n=3)	0.036307 ± 0.002370 (n=3)

Table 2: Near-pole per-bucket MSE (B0–B2), mean±std across seeds.

6 Related Work

Rational neural networks model functions as P/Q with strong approximation guarantees (**?**); practical deployments often use ε-regularized denominators $Q + \varepsilon$ to avoid division-by-zero. Batch normalization and related techniques also rely on explicit ε (**?**). Transreal arithmetic provides totalized operations with explicit tags for infinities and indeterminate forms (**??**). Masking rules in autodiff have appeared in robust training and subgradient methods; our Mask-REAL rule formalizes tag-aware gradient flow, ensuring exact zeros through non-REAL nodes while preserving classical derivatives on REAL paths. Bounded (saturating) gradients near poles relate to gradient clipping and smooth surrogates, but here arise from a deterministic, tag-aware calculus under an explicit policy. We adopt standard optimizers (e.g., Adam (**?**)) and normalization variants (e.g., LayerNorm (**?**)) as needed in controlled baselines.

7 Limitations and Outlook

Our approach targets models with explicit singular structure (rational layers, Jacobian-based control) and declared tag policies; it is not a replacement for generic deep architectures without divisions. Extending empirical coverage to higher-DOF systems with full physics stacks (URDF/Pinocchio) and integrating TR policies with mainstream autodiff frameworks are promising directions.

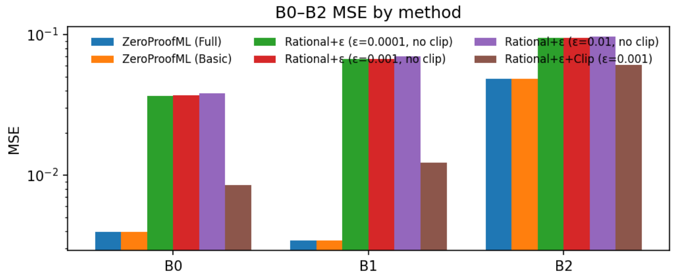

Figure 1: Near–pole MSE (B0–B2) by method (lower is better).

Method	Mean Track Err	Max $\|\Delta\theta\|$	Failure
MLP	0.0713	0.3121	0.00%
Rational+ε	0.0574	0.0605	0.00%
ZeroProofML-Basic	0.0503	0.0250	0.00%
ZeroProofML-Full	0.0434	0.0250	0.00%

Table 3: Closed-loop tracking near poles (lower is better).

8 Code and Data Availability

All code, dataset generators, per-seed results, aggregated CSVs, and LaTeX tables/figures are available at github.com/domezsolt/ZeroProofML. The repository records environment info and dataset hashes for reproducibility.

9 Conclusion

ZeroProofML replaces ε-based numerical fixes with a principled, tag-aware calculus that is total by construction. Mask-REAL autodiff, hybrid switching with bounded surrogates, coverage control, and policy determinism translate into empirical advantages: decisive near-pole accuracy (B0–B1), bounded updates and stable rollouts, and low across-seed variance under a declared policy. We expect these guarantees to benefit rational and control-oriented models where explicit singular structure is intrinsic.

Acknowledgments

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Metric	Value	Notes
Test MSE (mean)	0.051398	TR-Full (3R)
PLE (rad)	0.016385	
Sign consistency (θ_2)	0.143	fraction of anchors
Sign consistency (θ_3)	0.600	fraction of anchors
Residual consistency	0.007931	FK error

Table 4: 3R near-pole metrics (TR-Full).

Metric	Mean	Std	n	Notes
Overall MSE	0.084385	0.000000	3	6R TR

Table 5: 6R synthetic TR results (overall and selected bins).