$x^2 f'' + f = (x^2+1)e^x$ ①, $f(0)=1$, $f(1)=e^x$ exact solution: $f(x)=e^x$

we estimate $f(x)$ with the Legendra polynomials. $f(x)=\sum_{n=0}^{N} a_n P_n(x)$

Recursion: $(2n+1)P_n(x) = \frac{d}{dx}[P_{n+1}(x) - P_{n-1}(x)] \Rightarrow P_n'(x) = D_{nk} P_k(x)$

and: $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_n(x) \Rightarrow xP_n(x) = X_{nk} P_k(x)$

LHS: $f'' = \sum_{n=0}^{N} a_n P_n''(x)$, $x^2 f''(x) = \sum_{n=0}^{N} a_n x^2 P_n''(x) \Rightarrow x^2 f''(x) + f(x) = \sum_{n=0}^{N} a_n x^2 P_n''(x) + \sum_{n=0}^{N} a_n P_n(x)$

RHS: also expand with Legendra Polynomial: $(x^2+1)e^x = \sum_{n=0}^{N} b_n P_n(x)$, $b_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)(x^2+1)e^x dx$

Comparing both side of coefficients of $P_n(x)$ $\Rightarrow (X^T X^T D^T D^T + I^T)a = b$ ②

if we can solve equation ②, and find values of a_n, we find the estimation of $f(x) = \sum_{n=0}^{N} a_n P_n(x)$

$n=3$, the D and X matrix:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 5 & 0 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & \frac{3}{7} & 0 \end{pmatrix}$$

based on ②: $\underset{\Downarrow}{(D^2X^2+I)^T} a = b \Rightarrow$ the coefficients on both side of $P_n(x)$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1.6146435049 44718 \\ 1.7778994262518077 \\ 1.3306064973487786 \\ 0.5481437588941276 \end{pmatrix} \quad (1)$$

boundary condition:

$$\left.\begin{aligned} f(0) &= a_0P_0(0) + a_1P_1(0) + a_2P_2(0) + a_3P_3(0) = a_0 - \tfrac{1}{2}a_2 = 1 \\ f(1) &= a_0 + a_1 + a_2 + a_3 = e \end{aligned}\right\}$$ add this to (1), we get solutions with boundary condition.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 9 \\ 1 & 0 & -\frac{1}{2} & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1.614643504944718 \\ 1.7778994262518077 \\ 1.000 \\ e^1 \end{pmatrix} \quad (2)$$

Solve this (2), we find the coefficients a_n of $P_n(x)$ for $f(x) = \sum_{n=0}^{3} a_n P_n(x)$:

$a_0 = 1.2048811683149 06$

$a_1 = 1.0193556856721 42$

$a_2 = 0.409762336629812$

$a_3 = 0.0842826378421 85$

Solution: $f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + a_3P_3(x) = 0.210707x^3 + 0.61464x^2 + 0.89293x + 1.000000$

L2 error: $L_2 = \sqrt{\int_{-1}^{1}(f(x)-e^x)^2 dx} = \sqrt{7.65\times10^{-3}} = 8.75\times10^{-2}$

$n=5$, the $\mathcal{D}$ and $\mathcal{X}$ matrix:

$$\mathcal{D}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 7 & 0 & 0 \\ 1 & 0 & 5 & 0 & 9 & 0 \end{pmatrix} \qquad \mathcal{X}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{7} & 0 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & \frac{4}{9} & 0 & \frac{5}{9} \\ 0 & 0 & 0 & 0 & \frac{5}{11} & 0 \end{pmatrix}$$

By solving $(\mathcal{D}^2\mathcal{X}^2+I)^T a = b$, we get the coefficients of $P_n(x)$, which are $a_0, a_1, a_2, a_3, a_4, a_5$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 9 & 0 & 36 \\ 0 & 0 & 3 & 0 & 25 & 0 \\ 0 & 0 & 0 & 7 & 0 & 49 \\ 0 & 0 & 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}=\begin{pmatrix} 1.614643504944718 \\ 1.777899426251807 7 \\ 1.330606491348 7786 \\ 0.548143758894 1276 \\ 0.137712471330 81488 \\ 0.019654844320 02995 \end{pmatrix}$$

The boundary condition is:

$$f(0)=a_0-\frac{1}{2}a_2+\frac{3}{8}a_4=1$$

$$f(1)=a_0+a_1+a_2+a_3+a_4+a_5=e$$

New equations with boundary condition:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 9 & 0 & 36 \\ 0 & 0 & 3 & 0 & 25 & 0 \\ 0 & 0 & 0 & 7 & 0 & 49 \\ 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}=\begin{pmatrix} 1.614643504944718 \\ 1.7778994262518077 \\ 1.3306064913487786 \\ 0.5481437588941276 \\ 1 \\ e \end{pmatrix}$$

Coefficients of $P_n(x)$ when we expand $f(x)=\sum_{n=0}^{5} a_n P_n(x)$:

$$\begin{cases} a_0=1.174571919929633 \\ a_1=1.105197440811793 \\ a_2=0.356937646586907 \\ a_3=0.069975883419958 \\ a_4=0.010391742303522 \\ a_5=0.001187195407233 \end{cases}$$

$$f(x)=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+a_3P_3(x)+a_4P_4(x)+a_5P_5(x)$$

Estimation of $f(x)$: $f(x)=0.009349x^5+0.045464x^4+0.164602x^3+0.496437x^2+1.002430x+1.000000$

L2 Error: $L_2=\sqrt{\int_0^1 (f(x)-e^x)^2 dx}=\sqrt{2.824\times10^{-6}}=1.68\times10^{-3}$