For ODE $x^2 f'' + x = (x^2+1)e^x$, we have the extra conditions as $f(0) = 1$ and $f(1) = e$
It's easy to get that the exact solution is $f(x) = e^x$

First we solve for the spectral solution for expansion up to x^3
Based on recursion relationship, we have D matrix for derivation recursion relationship and X matrix for non-constant coefficients.
Thus the matrix realization for the ODE is

$$(DDXX + I)^T a = b$$

Matrix D truncated at x^3 is:
$D =$

0	0	0	0
1	0	0	0
0	3	0	0
1	0	5	0

Matrix X truncated at x^3 is:
$X =$

0	1	0	0
1/3	0	2/3	0
0	2/5	0	3/5
0	0	3/7	0

Thus, $A = (DDXX + I)^T = X^T X^T D^T D^T + I =$

1	0	1	0
0	1	0	9
0	0	3	0
0	0	0	7

The elements of column b are found from

$$b_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)(x^2+1)e^x dx$$

b_n for n=3 is:

1.614643504944718
1.777899426251808
1.330606497348779
0.548143758894128

Now we put in the auxiliary conditions $f(0) = a_0 - \frac{1}{2}a_2 = 1, f(1) = a_0 + a_1 + a_2 + a_3 = e$ to replace the last two rows of A with the equivalent restrictions on the coefficients:
$A' =$

1	0	1	0
0	1	0	9
1	0	-1/2	0
1	1	1	1

$b_n' =$

1.614643504944718
1.777899426251808
1.000000000000000
2.718281828459046

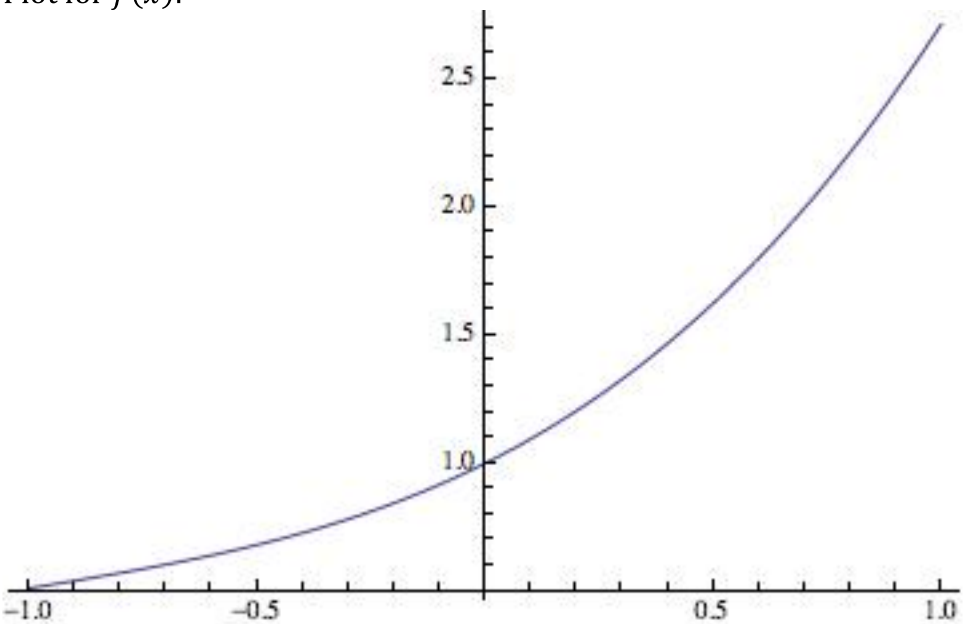
$a_n =$

1.204881168314906
1.019355685672142
0.409762336629812
0.084282637842185

So we can expand

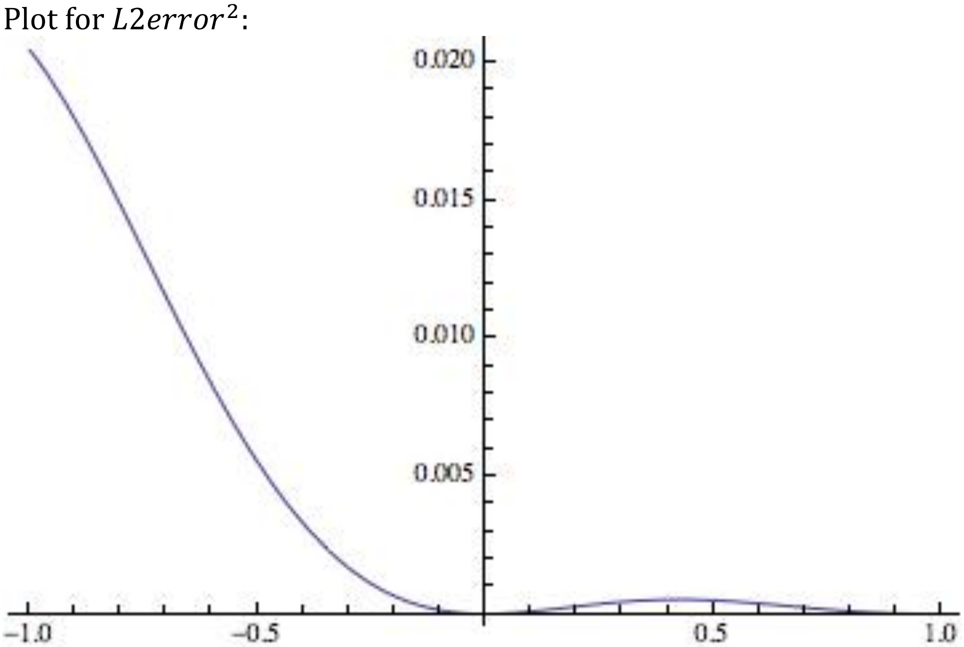
$$f(x) = \sum_{n=0}^{3} a_n P_n(x)$$
$$= 1.204881 + 1.019356\,x + 0.204881(-1 + 3x^2) + 0.042141(-3x + 5x^3)$$
$$= 1 + 0.892932\,x + 0.614644\,x^2 + 0.210707\,x^3$$

Plot for $f(x)$:



$$L2error = \sqrt{\int_{-1}^{1} (f(x) - e^x)^2 dx} = 0.08749$$

Plot for $L2error^2$:



Secondly, we solve for the spectral solution for expansion up to x^5

Matrix D truncated at x^5 is:

$D =$

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 7 & 0 & 0 \\ 1 & 0 & 5 & 0 & 9 & 0 \end{matrix}$$

Matrix X truncated at x^5 is:

$X =$

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 2/5 & 0 & 3/5 & 0 & 0 \\ 0 & 0 & 3/7 & 0 & 4/7 & 0 \\ 0 & 0 & 0 & 4/9 & 0 & 5/9 \\ 0 & 0 & 0 & 0 & 5/11 & 0 \end{matrix}$$

Thus, $A = (DDXX + I)^T = X^T X^T D^T D^T + I =$

$$\begin{matrix} 1 & 0 & 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 9 & 0 & 36 \\ 0 & 0 & 3 & 0 & 25 & 0 \\ 0 & 0 & 0 & 7 & 0 & 49 \\ 0 & 0 & 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21 \end{matrix}$$

The elements of column b are found from

$$b_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)(x^2+1)e^x dx$$

b_n for n=5 is:

1.614643504944718
1.777899426251808
1.330606497348779
0.548143758894128
0.137712471330815
0.019654844320030

Now we put in the auxiliary conditions $f(0) = a_0 - \frac{1}{2}a_2 + \frac{3}{8}a_4 = 1, f(1) = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = e$ to replace the last two rows of A with the equivalent restrictions on the coefficients:

$A' =$

$$\begin{matrix} 1 & 0 & 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 9 & 0 & 36 \\ 0 & 0 & 3 & 0 & 25 & 0 \\ 0 & 0 & 0 & 7 & 0 & 49 \\ 1 & 0 & -0.5 & 0 & 0.375 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$b_n' =$

1.614643504944718
1.777899426251808

0.548143758894128
1.000000000000000
2.718281828459046

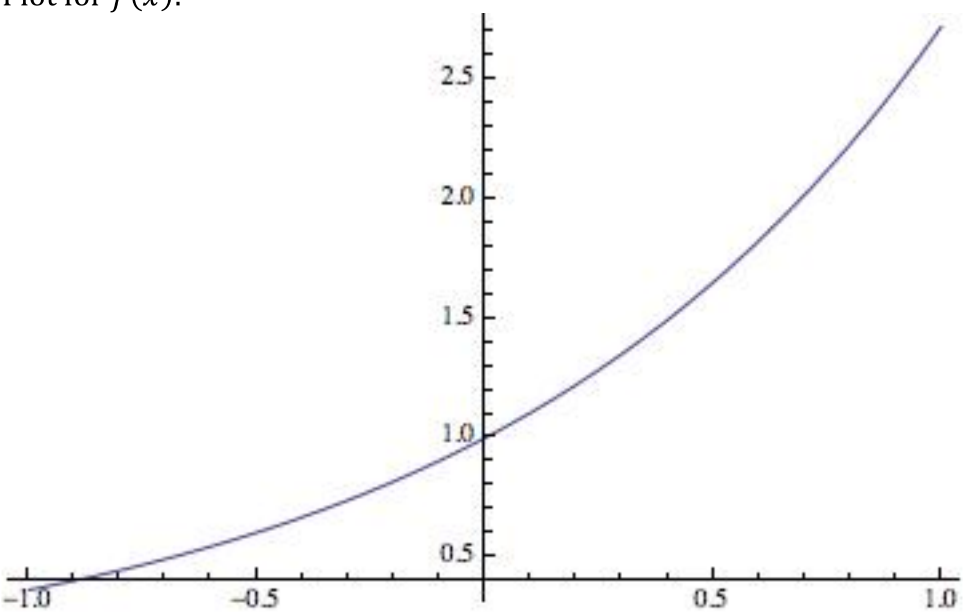
Solve $A'a = b'$ for a, we get:
$a_n =$
1.174571919929633
1.105197440811793
0.356937646586907
0.069995883419958
0.010391742303522
0.001187195407233

So we can expand

$$f(x) = \sum_{n=0}^{3} a_n P_n(x)$$

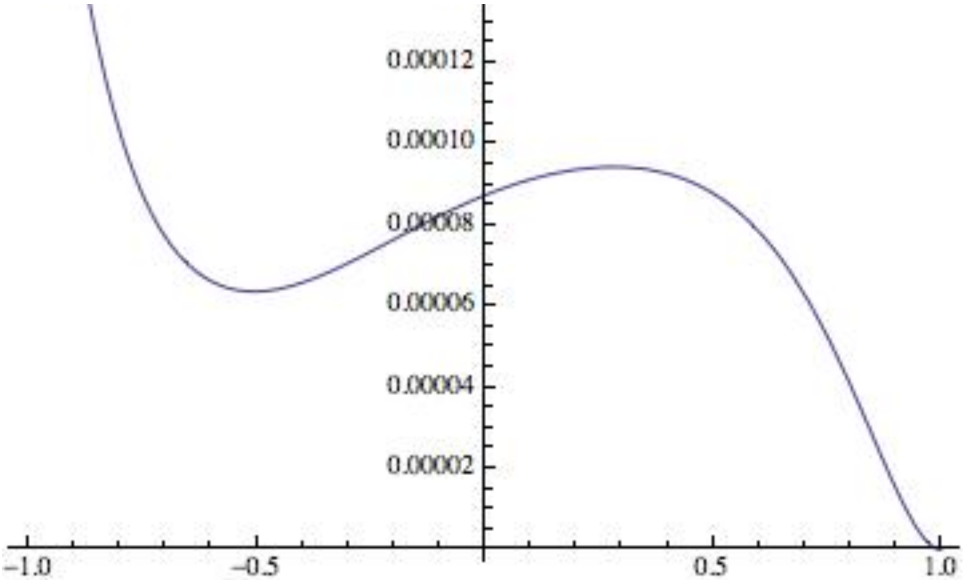
$$= 1.174572 + 1.105197\,x + 0.178469\,(-1 + 3x^2) + 0.034998\,(-3x + 5x^3) + 0.001299\,(3 - 30x^2 + 35x^4) + 0.000148\,(15x - 70x^3 + 63x^5)$$
$$= 1 + 1.002430\,x + 0.496437\,x^2 + 0.164601\,x^3 + 0.045464\,x^4 + 0.009349\,x^5$$

Plot for $f(x)$:



$$L2error = \sqrt{\int_{-1}^{1} (f(x) - e^x)^2 dx} = 0.00168$$

Plot for $L2error^2$:


0.00012
0.00010
0.00008
0.00006
0.00004
0.00002
-1.0
-0.5
0.5
1.0