Cal 3 2019/01

Q1.

a)

Rewrite $f(t)$ as unit step function we have:

$$f(t) = \sin t\, u(t) - \sin t\, u(t-\pi) = \sin t\, u(t) + \sin(t-\pi)\, u(t-\pi)$$

$$\rightarrow F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + \frac{1}{s^2+1} e^{-\pi s}$$

b)

$$\mathcal{L}^{-1}\left\{\frac{(1-e^{-s})^2}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{2}{s^2} e^{-s} + \frac{1}{s^2} e^{-2s}\right\}$$

$$= tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$$

Q2.

Given that:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = P_0\delta(t-t_0) \ \ (*), \quad x(0)=0, \quad x'(0)=0$$

Let $X(s) = \mathcal{L}\{x(t)\}$, it holds that:

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0) = sX(s)$$

$$\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0) = s^2X(s)$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$ms^2X(s) + csX(s) + kX(s) = P_0e^{-t_0s}$$

$$\leftrightarrow X(s)(ms^2+cs+k) = P_0e^{-t_0s}$$

$$\leftrightarrow X(s) = \frac{P_0e^{-t_0s}}{ms^2+cs+k}$$

$$\leftrightarrow X(s) = \frac{P_0}{m}\frac{1}{\left(s+\frac{c}{2m}\right)^2 + \frac{4km-c^2}{4m^2}}e^{-t_0s}$$

$$\rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{P_0}{m}\frac{1}{\frac{\sqrt{4km-c^2}}{2m}}e^{-\frac{c}{2m}(t-t_0)}\sin\left[\frac{\sqrt{4km-c^2}}{2m}(t-t_0)\right]u(t-t_0)$$

Thus, the solution of the given differential equation is:

$$x(t) = \frac{2P_0}{\sqrt{4km-c^2}}e^{-\frac{c}{2m}(t-t_0)}\sin\left[\frac{\sqrt{4km-c^2}}{2m}(t-t_0)\right]u(t-t_0)$$

Hint: Use the below formula and shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\} = e^{-at}\sin(\omega t)$$

Q3.

a)

Given that:

$$y_{n+2} - 5y_{n+1} + 6y_n = 2^n \ \ (*), \quad y_0 = 1, \quad y_1 = 0$$

Let $Y(z) = \mathcal{Z}\{y_n\}$, it holds that:

$$\mathcal{Z}\{y_{n+1}\} = zY(z) - zy_0 = zY(z) - z$$

$$\mathcal{Z}\{y_{n+2}\} = z^2Y(z) - z^2y_0 - zy_1 = z^2Y(z) - z^2$$

Taking $\mathcal{Z}$-transform both side of $(*)$, we obtain:

$$[z^2Y(z) - z^2] - 5[zY(z) - z] + 6[Y(z)] = \frac{z}{z-2}$$

$$\leftrightarrow Y(z)(z^2-5z+6)=\frac{z}{z-2}+z^2+5z$$
$$\rightarrow \frac{Y(z)}{z}=\frac{1}{(z-2)(z^2-5z+6)}+\frac{z+5}{z^2-5z+6}$$
$$\leftrightarrow \frac{Y(z)}{z}=-\frac{1}{(z-2)^2}-\frac{8}{z-2}+\frac{9}{z-3}$$
$$\rightarrow Y(z)=-\frac{z}{(z-2)^2}-\frac{8z}{z-2}+\frac{9z}{z-3}$$

$\rightarrow y_n=\mathcal{Z}^{-1}\{Y(z)\}=-n.2^{n-1}-8.2^n+9.3^n$

Thus, the solution of the given system difference equations is:

$$y_n=-n.2^{n-1}-8.2^n+9.3^n$$

b)

Using definition of $\mathcal{Z}$-transform, we have:

$$\mathcal{Z}\left\{\frac{1}{n!}\right\}=\sum_{n=0}^{+\infty}\frac{z^{-n}}{n!}=e^{-z}$$

Since, we know that:

$$e^x=\sum_{n=0}^{+\infty}\frac{x^n}{n!},\qquad x\in\mathbb{R}$$

Thus,

$$\mathcal{Z}\left\{\frac{1}{n!}\right\}=e^{-z}$$

Q4.

Given that: $f(x)=\begin{cases}0, & -\pi<x\le 0\\ \pi-x, & 0<x\le\pi\end{cases},\qquad T=2\pi\rightarrow\omega=\frac{2\pi}{T}=1$

- $a_0=\dfrac{2}{T}\displaystyle\int_{x_0}^{x_0+T}f(x)dx=\frac{2}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_0^{\pi}(\pi-x)dx\right]=\frac{\pi}{2}$
- $a_n=\dfrac{2}{T}\displaystyle\int_{x_0}^{x_0+T}f(x)\cos(n\omega x)\,dx=\frac{2}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_0^{\pi}(\pi-x)\cos(nx)\,dx\right]$

$$=\frac{1}{\pi}\int_0^{\pi}(\pi-x)\cos(nx)\,dx$$
$$=\frac{1}{\pi}\left[\frac{\pi-x}{n}\sin(nx)-\frac{1}{n^2}\cos(nx)\right]\Bigg|_0^{\pi}$$
$$=\frac{1-(-1)^n}{\pi n^2}$$

- $b_n=\dfrac{2}{T}\displaystyle\int_{x_0}^{x_0+T}f(x)\sin(n\omega x)\,dx=\frac{2}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_0^{\pi}(\pi-x)\sin(nx)\,dx\right]$

$$=\frac{1}{\pi}\int_0^{\pi}(\pi-x)\sin(nx)\,dx$$
$$=\frac{1}{\pi}\left[-\frac{\pi-x}{n}\cos(nx)-\frac{1}{n^2}\sin(nx)\right]\Bigg|_0^{\pi}$$
$$=\frac{1}{n}$$

The Fourier series is given by:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\omega x) + \sum_{n=1}^{+\infty} b_n \sin(n\omega x)$$
$$= \frac{\pi}{4} + \sum_{n=1}^{+\infty} \frac{1-(-1)^n}{\pi n^2}\cos(nx) + \sum_{n=1}^{+\infty} \frac{1}{n}\sin(nx)$$

Since we have: $f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ \pi - x, & 0 < x \le \pi \end{cases} \rightarrow f(\pi) = 0$

Therefore,

$$f(\pi) = \frac{\pi}{4} + \sum_{n=1}^{+\infty} \frac{1-(-1)^n}{\pi n^2}(-1)^n = 0$$
$$\rightarrow -\sum_{n=1}^{+\infty} \frac{(-1)^n - 1}{\pi n^2} = \frac{\pi}{4}$$
$$\leftrightarrow \sum_{n=1}^{+\infty} \frac{1-(-1)^n}{2n^2} = \frac{\pi^2}{8}$$
$$\rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^3} + \cdots = \frac{\pi^2}{4}$$
$$\rightarrow \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$