Cal 3 2018/04

Q1.

a)

Rewrite $f(t)$ as unit step function we have:

$$f(t) = (t^2 - 2t + 2)u(t-1) = [(t-1)^2 + 1]u(t-1)$$
$$\to F(s) = \mathcal{L}\{f(t)\} = \left(\frac{2}{s^3} + \frac{1}{s^2}\right)e^{-s}$$

b)

Given that: $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$, $T = 2\pi \to \omega = \dfrac{2\pi}{T} = 1$

- $a_0 = \dfrac{2}{T}\displaystyle\int_{t_0}^{t_0+T} f(t)dt = \dfrac{2}{2\pi}\left[\int_{-\pi}^{0} 1dx + \int_0^{\pi} 0dx\right] = 1$
- $$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega t)\,dt = \frac{2}{2\pi}\left[\int_{-\pi}^{0} 1\cos(nt)\,dt + \int_0^{\pi} 0dt\right]$$
$$= \frac{1}{\pi}\int_{-\pi}^{0}\cos(nt)\,dt$$
$$= \frac{1}{\pi}\left[\frac{1}{n}\sin(nt)\right]\bigg|_{-\pi}^{0}$$
$$= 0$$
- $$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega t)\,dt = \frac{2}{2\pi}\left[\int_{-\pi}^{0} 1\sin(nt)\,dt + \int_0^{\pi} 0dt\right]$$
$$= \frac{1}{\pi}\int_{-\pi}^{0}\sin(nt)\,dt$$
$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos(nt)\right]\bigg|_{-\pi}^{0}$$
$$= \frac{(-1)^n - 1}{n}$$

The Fourier series is given by:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n\cos(n\omega t) + \sum_{n=1}^{+\infty} b_n\sin(n\omega t)$$
$$= 1 + \sum_{n=1}^{+\infty}\frac{(-1)^n - 1}{n}\sin(nt)$$

Q2.

a)

$$\mathcal{Z}\{x_k\} = \mathcal{Z}\left\{1 - \left(-\frac{1}{3}\right)^{-k}\right\} = \mathcal{Z}\{1 - (-3)^k\} = \frac{z}{z-1} - \frac{z}{z+3}$$

b)

Given that:

$$y_{k+2} - 3y_{k+1} + 2y_k = 1\ \ (*),\qquad y_0 = 0,\quad y_1 = 0$$

Let $Y(z) = \mathcal{Z}\{y_k\}$, it holds that:

$$\mathcal{Z}\{y_{k+1}\} = zY(z) - zy_0 = zY(z)$$
$$\mathcal{Z}\{y_{k+2}\} = z^2Y(z) - z^2y_0 - zy_1 = z^2Y(z)$$

Taking $\mathcal{Z}$-transform both side of $(*)$, we obtain:

$$z^2Y(z) - 3zY(z) + 2Y(z) = \frac{z}{z-1}$$
$$\leftrightarrow Y(z)(z^2 - 3z + 2) = \frac{z}{z-1}$$
$$\to \frac{Y(z)}{z} = \frac{1}{(z-1)(z^2-3z+2)}$$
$$\leftrightarrow \frac{Y(z)}{z} = -\frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{z-2}$$
$$\to Y(z) = -\frac{z}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-2}$$

$\to y_k = \mathcal{Z}^{-1}\{Y(z)\} = -k - 1 + 2^k$

Thus, the solution of the given system difference equations is:

$$y_k = 2^k - k - 1$$

Q3.

Given that:

$$y'' - 3y' - 4y = u(t-1) + u(t-2) \ \ (*), \qquad y(0) = 0, \quad y'(0) = 1$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$
$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s) - 1] - 3[sY(s)] - 4[Y(s)] = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$
$$\leftrightarrow Y(s)(s^2 - 3s - 4) = 1 + \frac{e^{-s} + e^{-2s}}{s}$$
$$\leftrightarrow Y(s) = \frac{1}{s^2-3s-4} + \frac{e^{-s}+e^{-2s}}{s(s^2-3s-4)}$$
$$\leftrightarrow Y(s) = \frac{1}{5}\left(\frac{1}{s-4} - \frac{1}{s+1}\right) + \frac{1}{20}\left(\frac{4}{s+1} + \frac{1}{s-4} - \frac{5}{s}\right)(e^{-s} + e^{-2s})$$

$\to y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \frac{1}{5}(e^{4t} - e^{-t})u(t) + \frac{1}{20}\left(4e^{-(t-1)} + e^{4(t-1)} - 5\right)u(t-1) + \frac{1}{20}\left(4e^{-(t-2)} + e^{4(t-2)} - 5\right)u(t-2)$$

Thus, the solution of the given differential equation is:

$$y(t) = \frac{1}{5}(e^{4t} - e^{-t})u(t) + \frac{1}{20}\left(4e^{-(t-1)} + e^{4(t-1)} - 5\right)u(t-1)$$
$$+ \frac{1}{20}\left(4e^{-(t-2)} + e^{4(t-2)} - 5\right)u(t-2)$$

Q4.

a)

Let $f(t) = t * t^2$

We have:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t * t^2\} = \mathcal{L}\{t\}.\mathcal{L}\{t^2\} = \frac{1}{s^2}\frac{2!}{s^3} = \frac{2}{s^5}$$
$$\to f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\} = \frac{t^4}{12}$$

Thus,

$$t * t^2 = \frac{t^4}{12}$$

b)

Given that:

$$y'' + 4y = \delta(t - 4\pi) \ (*), \qquad y(0) = \frac{1}{2}, \quad y'(0) = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - \frac{1}{2}s$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$s^2Y(s) - \frac{1}{2}s + 4Y(s) = e^{-4\pi s}$$

$$\leftrightarrow Y(s)(s^2 + 4) = \frac{1}{2}s + e^{-4\pi s}$$

$$\leftrightarrow Y(s) = \frac{1}{2}\frac{s}{s^2 + 2^2} + \frac{1}{2}\frac{2}{s^2 + 2^2}e^{-4\pi s}$$

$$\rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\cos(2t)\, u(t) + \frac{1}{2}\sin\big(2(t - 4\pi)\big)u(t - 4\pi)$$

$$= \frac{1}{2}\cos(2t)\, u(t) + \frac{1}{2}\sin(2t)u(t - 4\pi)$$

Thus, the solution of the given differential equation is:

$$y(t) = \frac{1}{2}\cos(2t)\, u(t) + \frac{1}{2}\sin(2t)u(t - 4\pi)$$

Q5.

Given that: $f(x) = \sin x$, $\quad 0 < x < \pi$, $\quad L = \pi$

The half range cosine series is given by:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where:

- $a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{\pi}\int_0^\pi \sin x\, dx = \frac{4}{\pi}$
- $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\int_0^\pi \sin x\cos(nx)\, dx$

$$= \frac{2}{\pi}\left(-\frac{1}{2}\right)\left[\frac{\cos\big(x(1-n)\big)}{1-n} + \frac{\cos\big(x(1+n)\big)}{1+n}\right]\Bigg|_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n - 1}{1-n} + \frac{(-1)^n - 1}{1+n}\right]$$

$$= \frac{2(1 - (-1)^n)}{\pi(n^2 - 1)}$$

Thus,

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{+\infty}\frac{2(1 - (-1)^n)}{\pi(n^2 - 1)}\cos(nx)$$