Principles of EE2 – Spring 2019
Midterm exam – SOLUTION

Prob. 1 (15 marks) The switch in Fig. 1 has been closed for a long time, and it opens at $t = 0$. Find $v(t)$ for $t \geq 0$.

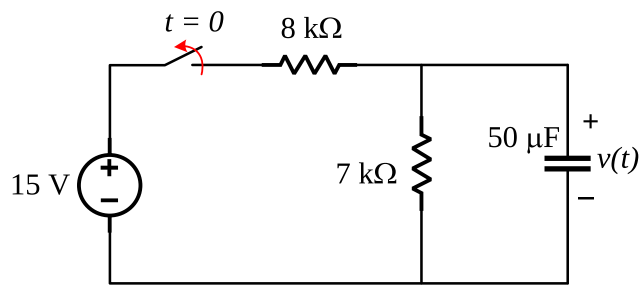

Fig. 1

<u>Sol.</u>

At the time $t = 0$, voltage of the capacitor is: $v_0 = v(0) = \dfrac{7}{7+8} \times 15 = 7 \quad (V)$

At the time $t \geq 0$, voltage of the capacitor is $v(t) = v_0 e^{-t/\tau}$,

$$\tau = RC = 7 \times 10^3 \times 50 \times 10^{-6} = 0.35 \quad (s)$$

Thus, $v(t)$ for $t \geq 0$:

$$v(t) = 7e^{-t/0.35} \qquad \text{(V)}$$

Prob. 2 (30 marks): In the parallel RLC circuit shown in Fig. 2, given R = 2/3 Ω, L = 1 H, C = 0.5 F.

a) Write down the second-order differential equation for this circuit?

b) Find the characteristic equation of the circuit? Solve it to obtain the characteristic roots?

c) Find the natural response of $v(t)$ for $t > 0$ for $v(0) = 10$ V, and $i(0) = 2$ A.

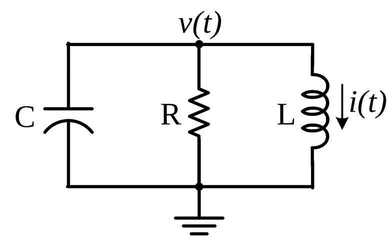

Fig. 2

<u>Sol.:</u>

a) The second-order differential equation for this circuit $\dfrac{d^2v}{dt^2} + \dfrac{1}{RC}\dfrac{dv}{dt} + \dfrac{v}{LC} = 0$

b) The characteristic equation of the circuit: $s^2 + \frac{s}{RC} + \frac{1}{LC} = 0 \quad or \quad s^2 + 3s + 2 = 0$

`Therefore, the roots of the characteristic equation are: $s_1 = -1$ and $s_2 = -2$.

c) The natural response is $v(t) = A_1 e^{-t} + A_2 e^{-2t}$

To find A_1 and A_2 we have:

+ The initial capacitor voltage is $v(0) = 10$,

so we have $v(0) = A_1 + A_2$ or $10 = A_1 + A_2$ (*)

+ $\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2$, and $\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = -\frac{v(0)}{RC} - \frac{i(0)}{C}$

Then $s_1 A_1 + s_2 A_2 = -\frac{v(0)}{RC} - \frac{i(0)}{C}$ or $-A_1 - 2A_2 = -\frac{10}{1/3} - \frac{2}{1/2} \Rightarrow -A_1 - 2A_2 = -34$ (**)

Solving (*) & (**) we obtain $A_2 = 24$ and $A_1 = -14$.

Therefore, the natural response is $v(t) = \left(-14e^{-t} + 24e^{-2t}\right) \quad (V)$

Prob. 3 (15 marks): Find the inverse Laplace transform $f(t)$ if $F(s)$ is

$$F(s) = \frac{2(5s^2+2)}{s(s+1)(s+2)^2}$$

<u>Sol.:</u>

Let $F(s) = \frac{2(5s^2+2)}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$

Thus,
$$A = \left[\frac{10s^2+4}{(s+1)(s+2)^2}\right]_{s=0} = \frac{4}{1\times 2^2} = 1$$

$$B = \left[\frac{10s^2+4}{s(s+2)^2}\right]_{s=-4} = \frac{14}{(-1)\times(1)^2} = -14$$

$$C = \left[\frac{10s^2+4}{s(s+1)}\right]_{s=-2} = \frac{44}{(-2)\times(-1)} = 22$$

$$D = \frac{d}{ds}\left[\frac{10s^2+4}{s^2+s}\right]_{s=-2} = \left[\frac{(s^2+s)(20s)-(10s^2+4)(2s+1)}{(s^2+s)^2}\right]_{s=-2} = 13$$

So that

$$F(s) = \frac{1}{s} - \frac{14}{s+1} + \frac{13}{s+2} + \frac{22}{(s+2)^2}$$

Taking the inverse transform of each term, we get

$$f(t) = \left(1 - 14e^{-t} + 13e^{-2t} + 22te^{-2t}\right)u(t)$$

Prob. 4 (20 marks): Find the initial and final values of the function whose Laplace transform is

$$F(s) = \frac{s^2 + 10s + 6}{s(s+1)^2(s+2)}$$

Sol.:

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s^2 + 10s + 6}{(s+1)^2(s+2)} = \lim_{s\to\infty} \frac{1/s + 10/s^2 + 6/s^3}{(1+1/s)(1+2/s)} = \frac{0}{1} = 0$$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s^2 + 10s + 6}{(s+1)^2(s+2)} = \frac{6}{1\times 2} = 3$$

Prob. 5 (30 marks): For the s-domain circuit in Fig. 3, find:

a) The transfer function $H(s) = V_0/V_i$.

b) The impulse response.

c) The response when $v_i(t) = u(t)$ V.

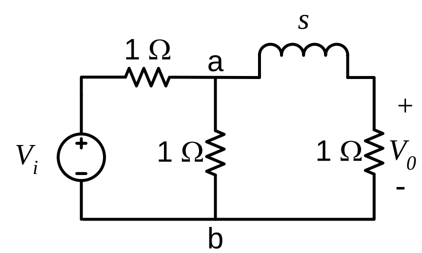

Fig. 3

Sol.:

a) Using voltage division, $V_0 = \frac{1}{s+1} V_{ab}$ (1)

But

$$V_{ab} = \frac{1 \| (s+1)}{1 + 1 \| (s+1)} V_i = \frac{(s+1)/(s+2)}{1 + (s+1)/(s+2)} V_i = \frac{s+1}{2s+3} V_i \qquad (2)$$

Substituting Eq. (2) into Eq. (1) results in $V_0 = \frac{V_i}{2s+3}$

Thus, the transfer function is: $H(s) = \frac{V_0}{V_i} = \frac{1}{2s+3}$

b) We may write H(s) as $H(s) = \frac{1}{2} \frac{1}{s + \frac{3}{2}}$

Its inverse Laplace transform is the required impulse response: $h(t) = \frac{1}{2} e^{-3t/2} u(t)$

c) When $v_i(t) = u(t)$, $V_i(s) = 1/s$, and $V_0(s) = H(s)V_i(s) = \frac{1}{2s\left(s + \frac{3}{2}\right)} = \frac{A}{s} + \frac{B}{s + \frac{3}{2}}$

Where

$$A = sV_0(s)\Big|_{s=0} = \left.\frac{1}{2\left(s+\frac{3}{2}\right)}\right|_{s=0} = \frac{1}{3}$$

$$B = \left(s+\frac{3}{2}\right)V_0(s)\Big|_{s=-3/2} = \left.\frac{1}{2s}\right|_{s=-3/2} = -\frac{1}{3}$$

Hence, for $v_i(t) = u(t)$,

$$V_0(s) = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+\frac{3}{2}}\right)$$

And its inverse Laplace transform is $v_0(t) = \frac{1}{3}\left(1 - e^{-3t/2}\right)u(t)$ (V)