CALCULUS 2 (MA003IU) – FINAL EXAMINATION

Semester 2, 2022-23 • Duration: 120 minutes • Date: August 03, 2023

SUBJECT: CALCULUS 2	
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INSTRUCTIONS:

- Use of calculator is allowed. Each student is allowed two double-sided sheets of notes (size A4 or similar). All other documents and electronic devices are forbidden.
- Write the steps you use to arrive at the answers to each question. No marks will be given for the answer alone.
- There are a total of 10 (ten) questions. Each one carries 10 points.

1. Show that the function $u(x,y) = \ln\left(x^2+y^2\right)$ satisfies the Laplace equation $u_{xx}(x,y)+u_{yy}(x,y)=0$ for $(x,y) \neq (0,0)$.
2. Let $f(x,y) = \dfrac{e^{-2x}}{1+y^2}$. Find the gradient vector $\nabla f(x,y)$ and the directional derivative $D_{\mathbf{u}}f(0,0)$, where $\mathbf{u}$ is a unit vector of $\mathbf{u} = \dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$.
3. Find an equation of the tangent plane to the surface given by $\sin(xyz) = x+2y+3z$ at the point $(2,-1,0)$.
4. Find the critical points of the function $f(x,y) = 3xy - x^2y - xy^2$, and determine whether each critical point corresponds to a local maximum, local minimum or a saddle point.
5. Find the absolute maximum and minimum values of function $f(x,y) = 5x+2y$ within the domain $D = \{(x,y) : x^2+y^2 \leq 25\}$.
6. Evaluate $\displaystyle\iint\limits_D \frac{dA}{4+x^2+y^2}$, where D is the disk $x^2+y^2 \leqslant 4$.
7. Evaluate $\iiint_E xy\,dV$ where E lies under the plane $z = 1+x+y$ and above the region in the xy-plane bounded by curves $y = \sqrt{x}$, $y = 0$, and $x = 2$.
8. Evaluate the line integral $\int_C yz\cos x\,ds$ where C is parameterized by $x = t$, $y = 3\cos t$, $z = 3\sin t$ with $0 \leq t \leq \pi$.
9. Use Green's theorem to evaluate the line integral $\displaystyle\oint\limits_C \left(xy+e^{-2x}\right)dx + \left(x^2+x+ye^y\right)dy$, where C is the boundary of the triangle D with vertices $(-1,0)$, $(1,0)$, and $(0,1)$, oriented counterclockwise.
10. Let $\mathbf{F} = (2xy+5)\,\mathbf{i} + \left(x^2-4z\right)\mathbf{j} - 4y\mathbf{k}$. Find a function V (if any) such that $\nabla V = \mathbf{F}$.

—END OF THE QUESTION PAPERS. GOOD LUCK!—

Answer keys

1. For $(x,y) \neq (0,0)$, we have

$$u_x(x,y) = \frac{2x}{x^2+y^2}, u_{xx}(x,y) = \frac{2y^2-2x^2}{(x^2+y^2)^2},$$

and

$$u_y(x,y) = \frac{2y}{x^2+y^2}, u_{yy}(x,y) = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Therefore, for $(x,y) \neq (0,0)$, one has $u_{xx}(x,y)+u_{yy}(x,y)=0$.

2. We have

$$\nabla f(x,y) = \langle \frac{-2e^{-x}}{1+y^2}, -\frac{2ye^{-2x}}{(1+y^2)^2} \rangle$$

Thus,

$$D_{\mathbf{u}} f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = \langle -2,0 \rangle \cdot \langle 3/5, -4/5 \rangle = -\frac{6}{5}$$

3. Let $F(x,y,z) = x+2y+3z-\sin(xyz)$. Note that $F_x(2,-1,0)=1$, $F_y(2,-1,0)=2$, $F_z(2,-1,0)=5$. This implies an equation for the tangent plane: $x+2y+5z=0$.

4. $f_x = y(3-2x-y)$ and $f_y = x(3-x-2y)$. Critical points: $(0,0)$, $(0,3)$, $(3,0)$, $(1,1)$. By second derivative test, $(0,0)$, $(0,3)$, and $(3,0)$ are saddle points; $(1,1)$ is a local maximum.

5. First, note that $f_x = 5$ and $f_y = 2$, so there is no local Min/Max in the interior of D. Next, use the Lagrange's method to find the Min/Max of f on the boundary. The absolute minimum is $f(\frac{-25}{\sqrt{29}}, \frac{-10}{\sqrt{29}}) = -5\sqrt{29} = -26.9258$. The absolute maximum is $f(\frac{25}{\sqrt{29}}, \frac{10}{\sqrt{29}}) = 5\sqrt{29} = 26.9258$.

6. Using polar coordinates: $x = r\cos\theta, y = r\sin\theta, 0 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant 2\pi$. We obtain

$$I = \iint_D \frac{dA}{4+x^2+y^2} = \int_0^{2\pi} d\theta \int_0^2 \frac{rdr}{4+r^2} = (2\pi)\int_4^8 \frac{du}{2u} = \pi \ln 2.$$

7.

$$I = \int_0^2 dx \int_0^{\sqrt{x}} dy \int_0^{1+x+y} (xy)\,dz = \int_0^2 dx \int_0^{\sqrt{x}} (xy+x^2y+xy^2)dy = \frac{4}{3}+2+\frac{16\sqrt{2}}{21} = 4.4108$$

8. $\int_C yz\cos x\,ds = \int_0^{\pi} 3\cos(t)3\sin(t)\cos(t)\sqrt{1+9\sin^2(t)+9\cos^2(t)}dt = 6\sqrt{10}$

9. Let D be the triangle with vertices $(-1,0)$, $(1,0)$, and $(0,1)$. The equation in Green's theorem can be written as

$$\oint_{\partial D} Pdx+Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

$$I = \int_0^1 dy \int_{x=y-1}^{x=1-y} (2x+1-x)dx = \int_0^1 (2-2y)dy = 1.$$

10. A potential function satisfying $\nabla V = \mathbf{F}$ is

$$V = x^2y+5x-4yz+C.$$

—END OF THE ANSWER KEY—