P1.1. **(a)** Total distance traveled by the bug

$$= 1 + s + s^2 + s^3 + \ldots$$

$$= \frac{1}{1-s} = 1.5$$

$$1.5 - 1.5s = 1$$

$$s = \frac{1}{3}$$

(b) Distance traveled northward

$$= 1 - s^2 + s^4 - \ldots = \frac{1}{1+s^2}$$

Distance traveled eastward

$$= s - s^3 + s^5 - \ldots = \frac{s}{1+s^2}$$

Straight line distance from the initial position of the bug

$$= \frac{\sqrt{1+s^2}}{1+s^2} = 0.8$$

$$1 + s^2 = \frac{1}{0.64}$$

$$s^2 = \frac{0.36}{0.64}$$

$$s = 0.75$$

(c) $\text{Tan } 30^\circ = \dfrac{s/(1+s^2)}{1/(1+s^2)}$

$$s = 1/\sqrt{3}$$

P1.2.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 5 & 5 \\ 1 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 & 0 \\ -2 & 5 & 5 \\ 1 & 5 & 0 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 5 & 3 & -4 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ -2 & 5 & 5 \\ 1 & 5 & 0 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 15 & 30 & 0 \\ 0 & 0 & 15 \\ -15 & 15 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Thus

$\mathbf{A} = \mathbf{a}_1 + 2\mathbf{a}_2$

$\mathbf{B} = \mathbf{a}_3$

$\mathbf{C} = -\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$

P1.3. **(a)** From $\mathbf{C} \cdot \mathbf{C} = (\mathbf{B} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{A})$, we obtain

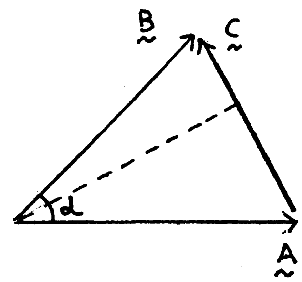
$$C^2 = B^2 - \mathbf{B} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} + A^2$$

$$= A^2 + B^2 - 2\mathbf{A} \cdot \mathbf{B}$$

$$= A^2 + B^2 - 2AB \cos \alpha$$

(b) The required distance

$$= \frac{1}{C} \,(2 \times \text{area of the triangle})$$



$$= \frac{1}{|\mathbf{A} - \mathbf{B}|}\left(2 \times \frac{1}{2} \times A \times B \sin \alpha\right)$$

$$= \frac{|\mathbf{A}|\,|\mathbf{B}| \sin \alpha}{|\mathbf{A} - \mathbf{B}|}$$

$$= \frac{|\mathbf{A} \times \mathbf{B}|}{|\mathbf{A} - \mathbf{B}|}$$

P1.4. (a) $\mathbf{A} \bullet \mathbf{B} = 2m - m + 2 = 0$

$$m + 2 = 0$$

$$m = -2$$

(b) $\mathbf{B} \times \mathbf{C} = \mathbf{0}$

$$\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ m & 1 & -2 \\ 1 & m & 2 \end{vmatrix} = (2 + 2m)\mathbf{a}_1 + (-2 - 2m)\mathbf{a}_2 + (m^2 - 1)\mathbf{a}_3$$

$$2 + 2m = 0,\ m = -1$$

$$m^2 - 1 = 0,\ m = \pm 1$$

$$\therefore\ m = -1$$

(c) $\mathbf{A} \bullet \mathbf{B} \times \mathbf{C} = 0$

$$\begin{vmatrix} 2 & -m & -1 \\ m & 1 & -2 \\ 1 & m & 2 \end{vmatrix} = 2(2 + 2m) - m(-2 - 2m) - 1(m^2 - 1) = 0$$

$$m^2 + 6m + 5 = 0$$

$$(m + 1)(m + 5) = 0$$

$$m = -1, -5$$

(d) $\mathbf{D} \bullet \mathbf{A} = 2m^2 - m^2 - 1 = 0$

$$(m^2 - 1) = 0,\ m = \pm 1$$

$$\mathbf{D} \bullet \mathbf{B} = m^3 + m - 2 = 0$$

$$(m - 1)(m^2 + m + 2) = 0$$

P1.4. (continued)

$$m = 1,\ \frac{-1 \pm \sqrt{1-8}}{2}$$

$$\therefore\ m = 1$$

or,

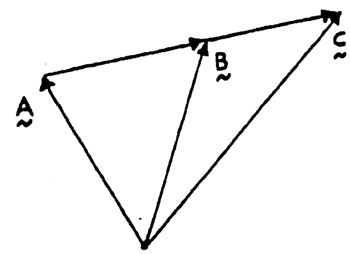
$$\mathbf{D} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$$

$$\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ m^2 & m & 1 \\ 2m+1 & -m+4 & 2+m^2 \end{vmatrix} = \mathbf{0}$$

Also gives $m = 1$.

P1.5. $(\mathbf{B} - \mathbf{A})$ and $(\mathbf{C} - \mathbf{A})$

are along a straight line.



$\therefore\ (\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) = \mathbf{0}$

$$\mathbf{B} \times \mathbf{C} - \mathbf{B} \times \mathbf{A} - \mathbf{A} \times \mathbf{C} + \mathbf{A} \times \mathbf{A} = \mathbf{0}$$

$$\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} = \mathbf{0}$$

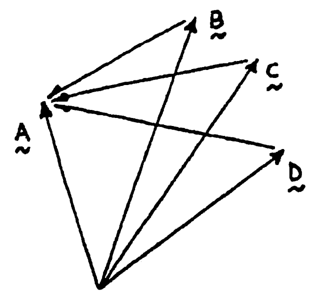
$\mathbf{A} \times \mathbf{B}$ is a vector having magnitude twice the area of the triangle formed by **A** and **B** and directed into the paper.

$\mathbf{B} \times \mathbf{C}$ is a vector having magnitude twice the area of the triangle formed by **B** and **C** and directed into the paper.

$\mathbf{C} \times \mathbf{A}$ is a vector having magnitude twice the area of the triangle formed by **C** and **A** and directed out of the paper.

$\therefore$ For the tips of **A**, **B** and **C** to lie along a straight line, $(\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A})$ must be equal to the null vector.

P1.6. For the tips of the vectors **A**, **B**, **C**, and **D** to lie in a plane, the vectors (**A** – **B**), (**A** – **C**), and (**A** – **D**) must lie in a plane.



∴ The volume of the parallelepiped formed by (**A** – **B**), (**A** – **C**), and (**A** – **D**) must be zero. Thus

$$(\mathbf{A}-\mathbf{B}) \cdot (\mathbf{A}-\mathbf{C}) \times (\mathbf{A}-\mathbf{D}) = 0$$

For $\mathbf{A} = \mathbf{a}_1$, $\mathbf{B} = 2\mathbf{a}_2$, $\mathbf{C} = 2\mathbf{a}_3$, and $\mathrm{D} = \mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3$,

$$\mathbf{A}-\mathbf{B} = \mathbf{a}_1 - 2\mathbf{a}_2$$

$$\mathbf{A}-\mathbf{C} = \mathbf{a}_1 - 2\mathbf{a}_3$$

$$\mathbf{A}-\mathbf{D} = -2\mathbf{a}_2 + 2\mathbf{a}_3$$

$$(\mathbf{A}-\mathbf{B}) \cdot (\mathbf{A}-\mathbf{C}) \times (\mathbf{A}-\mathbf{D}) = \begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 2 \end{vmatrix}$$

$$= -4 + 4 + 0 = 0$$

∴ The answer is "yes."

P1.7.

(a) $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ A_1 & A_2 & A_3 \\ (B_2C_3 - B_3C_2) & (B_3C_1 - B_1C_3) & (B_1C_2 - B_2C_1) \end{vmatrix}$$

$$= (A_2B_1C_2 - A_2B_2C_1 - A_3B_3C_1 + A_3B_1C_3)\mathbf{a}_1$$

$$+ (A_3B_2C_3 - A_3B_3C_2 - A_1B_1C_2 + A_1B_2C_1)\mathbf{a}_2$$

$$+ (A_1B_3C_1 - A_1B_1C_3 - A_2B_2C_3 + A_2B_3C_2)\mathbf{a}_3$$

$$= (A_1C_1 + A_2C_2 + A_3C_3)B_1\mathbf{a}_1 + (A_1C_1 + A_2C_2 + A_3C_3)B_2\mathbf{a}_2$$

$$+ (A_1C_1 + A_2C_2 + A_3C_3)B_3\mathbf{a}_3 - (A_1B_1 + A_2B_2 + A_3B_3)C_1\mathbf{a}_1$$

$$- (A_1B_1 + A_2B_2 + A_3B_3)C_2\mathbf{a}_2 - (A_1B_1 + A_2B_2 + A_3B_3)C_3\mathbf{a}_3$$

$$= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

(b) (i) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B})$

$$= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} + (\mathbf{B} \cdot \mathbf{A})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} + (\mathbf{C} \cdot \mathbf{B})\mathbf{A} - (\mathbf{C} \cdot \mathbf{A})\mathbf{B}$$

$$= \mathbf{0}$$

(ii) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})$

$$= (\mathbf{A} \times \mathbf{B}) \cdot [(\mathbf{B} \times \mathbf{C} \cdot \mathbf{A})\mathbf{C} - (\mathbf{B} \times \mathbf{C} \cdot \mathbf{C})\mathbf{A}]$$

$$= (\mathbf{B} \times \mathbf{C} \cdot \mathbf{A})(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})$$

$$= (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})^2$$

P1.8. **(a)** $BC = \sqrt{(0-0)^2 + (0-15)^2 + (-20-0)^2}$

$$= \sqrt{15^2 + 20^2} = \sqrt{625} = 25$$

(b) Vector from A to $C = -12\mathbf{a}_x - 20\mathbf{a}_z$

Vector from B to $C = -15\mathbf{a}_y - 20\mathbf{a}_z$

Required component

$$= \frac{(-12\mathbf{a}_x - 20\mathbf{a}_z) \bullet (-15\mathbf{a}_y - 20\mathbf{a}_z)}{|-15\mathbf{a}_y - 20\mathbf{a}_z|}$$

$$= \frac{400}{25} = 16$$

(c) Required perpendicular distance

$$= \frac{2 \times \text{area of triangle formed by } A,\ B, \text{ and } C}{BC}$$

$$= \frac{|(-12\mathbf{a}_x - 20\mathbf{a}_z) \times (-15\mathbf{a}_y - 20\mathbf{a}_z)|}{25}$$

$$= \frac{|180\mathbf{a}_z - 240\mathbf{a}_y - 300\mathbf{a}_x|}{25}$$

$$= \frac{60}{25}\sqrt{9+16+25}$$

$$= 12\sqrt{2}$$

P1.9. Equating the distance from (x_2, y_2, z_2) to (x_0, y_0, z_0) to the distance from (x_1, y_1, z_1) to (x_0, y_0, z_0), we have

$$(x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2$$

or,

$$x_2^2 - 2x_2x_0 + y_2^2 - 2y_2y_0 + z_2^2 - 2z_2z_0$$

$$= x_1^2 - 2x_1x_0 + y_1^2 - 2y_1y_0 + z_1^2 - 2z_1z_0$$

Rearranging, we obtain

$$2[(x_2 - x_1)x_0 + (y_2 - y_1)y_0 + (z_2 - z_1)z_0]$$

$$= (x_2^2 + y_2^2 + z_2^2) - (x_1^2 + y_1^2 + z_1^2)$$

Similarly, equating the distances from (x_3, y_3, z_3) to (x_0, y_0, z_0) and from (x_4, y_4, z_4) to (x_0, y_0, z_0) to the distance from (x_1, y_1, z_1) to (x_0, y_0, z_0), simplifying, and rearranging, we obtain two more equations. Together, the three equations can be expressed as

$$2\begin{bmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

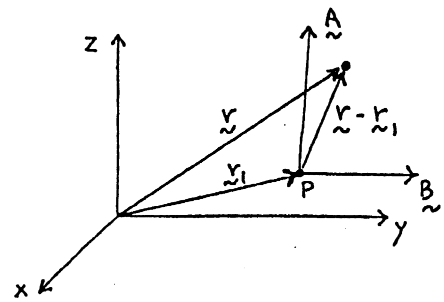
$$= \begin{bmatrix} (x_2^2 + y_2^2 + z_2^2) - (x_1^2 + y_1^2 + z_1^2) \\ (x_3^2 + y_3^2 + z_3^2) - (x_1^2 + y_1^2 + z_1^2) \\ (x_4^2 + y_4^2 + z_4^2) - (x_1^2 + y_1^2 + z_1^2) \end{bmatrix}$$

For the four points (1, 1, 4), (3, 3, 2), (2, 3, 3), and (3, 2, 3), this equation gives

$$2\begin{bmatrix} 2 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

Solving, we obtain the center point of the sphere to be (1, 1, 1). The radius is $\sqrt{0 + 0 + 3^2}$, or, 3.

P1.10. (a) For an arbitrary point (x, y, z) on the plane, the vector from P to that point is $(\mathbf{r} - \mathbf{r}_1)$. Thus the vectors $\mathbf{A}$, $\mathbf{B}$, and $(\mathbf{r} - \mathbf{r}_1)$ must lie on the plane. Hence, the condition is $\mathbf{A} \times \mathbf{B} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$.



(b) Let P be (1, 1, 2). Then

$\mathbf{r} - \mathbf{r}_1 = (x-1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-2)\mathbf{a}_z$

$\mathbf{A}$ = Vector from P to (2, 2, 0) = $\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$

$\mathbf{B}$ = Vector from P to (3, 0, 1) = $2\mathbf{a}_x - \mathbf{a}_y - \mathbf{a}_z$

The equation is given by

$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & -1 \\ (x-1) & (y-1) & (z-2) \end{vmatrix} = 0$$

$-z + 2 + y - 1 - x + 1 - 2z + 4 - 2(2y - 2 + x - 1) = 0$

$x + y + z = 4$

P1.11. For $x + y = 2$, $y = z^2$,

$$dx + dy = 0,\ dy = 2z\,dz$$

$$\therefore\ dx = -dy = -2z\,dz$$

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

$$= -2z\,dz\,\mathbf{a}_x + 2z\,dz\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

$$= (-2z\mathbf{a}_x + 2z\mathbf{a}_y + \mathbf{a}_z)\,dz$$

(a) At the point (2, 0, 0), $z = 0$

$$d\mathbf{l} = dz\,\mathbf{a}_z$$

(b) At the point (1, 1, 1), $z = 1$

$$d\mathbf{l} = (-2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)\,dz$$

(c) At the point (–2, 4, 2), $z = 2$

$$d\mathbf{l} = (-\ 4\mathbf{a}_x + 4\mathbf{a}_y + \mathbf{a}_z)\,dz$$

P1.12. For $x = y^2 = z^3$,

$$dx = 2y\,dy = 3z^2\,dz$$

At (1, 1, 1), $dx = 2\,dy = 3\,dz$

$$d\mathbf{l} = dz\,(3\mathbf{a}_x + 1.5\mathbf{a}_y + \mathbf{a}_z)$$

For $x = y = z$, $dx = dy = dz$, $d\mathbf{l} = dz\,(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$

The required unit vector is

$$\pm \frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \times (3\mathbf{a}_x + 1.5\mathbf{a}_y + \mathbf{a}_z)}{|(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \times (3\mathbf{a}_x + 1.5\mathbf{a}_y + \mathbf{a}_z)|}$$

$$= \pm \frac{-0.5\mathbf{a}_x + 2\mathbf{a}_y - 1.5\mathbf{a}_z}{|-0.5\mathbf{a}_x + 2\mathbf{a}_y - 1.5\mathbf{a}_z|}$$

$$= \pm \frac{\mathbf{a}_x - 4\mathbf{a}_y + 3\mathbf{a}_z}{\sqrt{26}}$$

P1.13. Consider two curves on the surface

$$z = 1,\ x^2 + y^2 = 4 - 2 = 2 \qquad \text{(curve 1)}$$
$$y = 1,\ x^2 + 2z^2 = 4 - 1 = 3 \qquad \text{(curve 2)}$$

For curve 1,

$dz = 0,\ 2x\,dx + 2y\,dy = 0$

$$dy = -\frac{x}{y}\,dx = -dx$$

$$d\mathbf{l}_1 = dx\,\mathbf{a}_x - dx\,\mathbf{a}_y = dx\,(\mathbf{a}_x - \mathbf{a}_y)$$

For curve 2,

$dy = 0,\ 2x\,dx + 4z\,dz = 0$

$$dz = -\frac{x}{2z}dx = -\frac{1}{2}dx$$

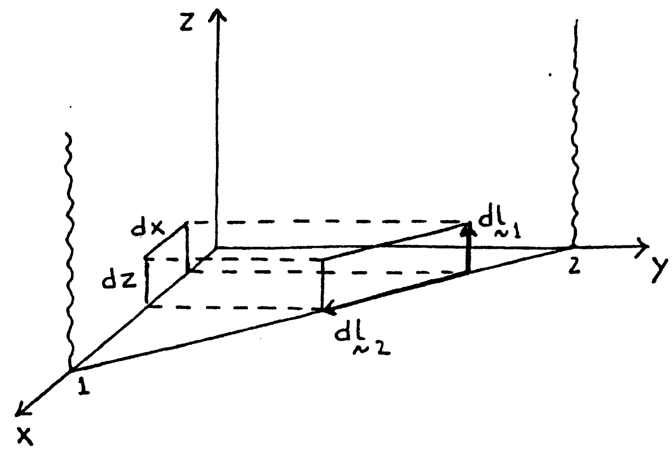
$$d\mathbf{l}_2 = dx\,\mathbf{a}_x - \frac{1}{2}dx\,\mathbf{a}_z = dx\left(\mathbf{a}_x - \frac{1}{2}\,\mathbf{a}_z\right)$$

$$\therefore\ \mathbf{a}_n = \frac{d\mathbf{l}_1 \times d\mathbf{l}_2}{|d\mathbf{l}_1 \times d\mathbf{l}_2|}$$

$$= \frac{(\mathbf{a}_x - \mathbf{a}_y) \times \left(\mathbf{a}_x - \frac{1}{2}\,\mathbf{a}_z\right)}{\left|(\mathbf{a}_x - \mathbf{a}_y) \times \left(\mathbf{a}_x - \frac{1}{2}\,\mathbf{a}_z\right)\right|}$$

$$= \frac{\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z}{\sqrt{6}}$$

P1.14. $d\mathbf{l}_1 = dz\,\mathbf{a}_z$



For $2x + y = 2$,

$2\,dx + dy = 0$

$dy = -2\,dx$

$$\begin{aligned}
d\mathbf{l}_2 &= dx\,\mathbf{a}_x + dy\,\mathbf{a}_y \\
&= dx\,\mathbf{a}_x - 2\,dx\,\mathbf{a}_y \\
&= (\mathbf{a}_x - 2\mathbf{a}_y)\,dx
\end{aligned}$$

$$\begin{aligned}
d\mathbf{S} &= \pm\, d\mathbf{l}_1 \times d\mathbf{l}_2 \\
&= \pm\, dz\,\mathbf{a}_z \times (\mathbf{a}_x - 2\mathbf{a}_y)\,dx \\
&= \pm\,(2\mathbf{a}_x + \mathbf{a}_y)\,dx\,dz
\end{aligned}$$

P1.15. $A(2, \pi/3, 1) \rightarrow (1, \sqrt{3}, 1)$
$B(2\sqrt{3}, \pi/6, -2) \rightarrow (3, \sqrt{3}, -2)$
$C(2, 5\pi/6, 0) \rightarrow (-\sqrt{3}, 1, 0)$

(a) Volume of parallelepiped
= Absolute value of

$$\begin{vmatrix} 1 & \sqrt{3} & 1 \\ 3 & \sqrt{3} & -2 \\ -\sqrt{3} & 1 & 0 \end{vmatrix}$$

$= \left| 2 + \sqrt{3}(2\sqrt{3}) + 6 \right|$

$= 14$

(b) $D(\sqrt{3}, \pi/2, 2.5) \rightarrow (0, \sqrt{3}, 2.5)$

Vector from A to $B = 2\mathbf{a}_x - 3\mathbf{a}_z$
Vector from A to $C = (-\sqrt{3} - 1)\mathbf{a}_x + (1 - \sqrt{3})\mathbf{a}_y - \mathbf{a}_z$
Vector from A to $D = -\mathbf{a}_x + 1.5\mathbf{a}_z$

$$\begin{vmatrix} 2 & 0 & -3 \\ (-\sqrt{3}-1) & (1-\sqrt{3}) & -1 \\ -1 & 0 & 1.5 \end{vmatrix}$$

$= 3(1 - \sqrt{3}) - 3(1 - \sqrt{3})$

$= 0$

$\therefore$ D lies in the plane containing A, B, and C.

P1.16.

spherical		Cartesian
$A(1, \pi/2, 0)$	$\longrightarrow$	$A(1, 0, 0)$
$B(\sqrt{8}, \pi/4, \pi/3)$	$\longrightarrow$	$B(1, \sqrt{3}, 2)$
$C(1, 0, 0)$	$\longrightarrow$	$C(0, 0, 1)$
$D(\sqrt{12}, \pi/6, \pi/2)$	$\longrightarrow$	$D(0, \sqrt{3}, 3)$

Vector from A to $C = -\mathbf{a}_x + \mathbf{a}_z$, say, $\mathbf{R}_1$

Vector from B to $D = -\mathbf{a}_x + \mathbf{a}_z$, say, $\mathbf{R}_2$

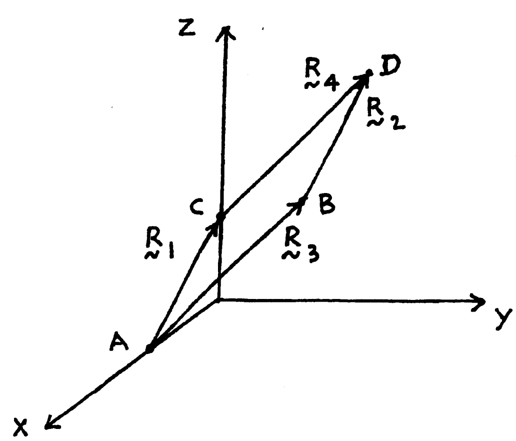
Vector from A to $B = \sqrt{3}\,\mathbf{a}_y + 2\mathbf{a}_z$, say, $\mathbf{R}_3$

Vector from C to $D = \sqrt{3}\,\mathbf{a}_y + 2\mathbf{a}_z$, say, $\mathbf{R}_4$

$\mathbf{R}_1 = \mathbf{R}_2$ and $\mathbf{R}_3 = \mathbf{R}_4$

$\therefore$ *ABDC* is a parallelogram.

$$
\begin{aligned}
\text{Area } &= |\mathbf{R}_3 \times \mathbf{R}_1| \\
&= \left|(\sqrt{3}\mathbf{a}_y + 2\mathbf{a}_z) \times (-\mathbf{a}_x + \mathbf{a}_z)\right| \\
&= \left|\sqrt{3}\mathbf{a}_x - 2\mathbf{a}_y + \sqrt{3}\mathbf{a}_z\right| \\
&= \sqrt{3+4+3} \\
&= \sqrt{10}
\end{aligned}
$$



P1.17. $\mathbf{A} = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$

$$= \frac{\sqrt{3}}{2}\,\mathbf{a}_x + \frac{1}{2}\,\mathbf{a}_y$$

$\mathbf{B} = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

$$= -\frac{\sqrt{3}}{2}\,\mathbf{a}_x + \frac{1}{2}\,\mathbf{a}_y$$

$\mathbf{C} = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

$$= -\frac{1}{2}\,\mathbf{a}_x - \frac{\sqrt{3}}{2}\,\mathbf{a}_y$$

(a) $$\mathbf{A}\bullet\mathbf{B} = \left(\frac{\sqrt{3}}{2}\,\mathbf{a}_x + \frac{1}{2}\,\mathbf{a}_y\right)\bullet\left(-\frac{\sqrt{3}}{2}\,\mathbf{a}_x + \frac{1}{2}\,\mathbf{a}_y\right)$$

$$= -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

(b) $$\mathbf{B}\bullet\mathbf{C} = \left(-\frac{\sqrt{3}}{2}\,\mathbf{a}_x + \frac{1}{2}\,\mathbf{a}_y\right)\bullet\left(-\frac{1}{2}\,\mathbf{a}_x - \frac{\sqrt{3}}{2}\,\mathbf{a}_y\right)$$

$$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

(c) $$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{vmatrix}$$

$$= \mathbf{a}_z$$

P1.18. $\mathbf{A} = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$

$$= \sin\frac{\pi}{6}\cos\frac{\pi}{2}\,\mathbf{a}_x + \sin\frac{\pi}{6}\sin\frac{\pi}{2}\,\mathbf{a}_y + \cos\frac{\pi}{6}\,\mathbf{a}_z$$

$$= \frac{1}{2}\mathbf{a}_y + \frac{\sqrt{3}}{2}\mathbf{a}_z$$

$\mathbf{B} = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$

$$= \cos\frac{\pi}{3}\cos 0\,\mathbf{a}_x + \cos\frac{\pi}{3}\sin 0\,\mathbf{a}_y - \sin\frac{\pi}{3}\,\mathbf{a}_z$$

$$= \frac{1}{2}\mathbf{a}_x - \frac{\sqrt{3}}{2}\mathbf{a}_z$$

$\mathbf{C} = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

$$= -\sin\frac{3\pi}{2}\,\mathbf{a}_x + \cos\frac{3\pi}{2}\,\mathbf{a}_y$$

$$= \mathbf{a}_x$$

(a) $\mathbf{A}\bullet\mathbf{B} = \left(\frac{1}{2}\mathbf{a}_y + \frac{\sqrt{3}}{2}\mathbf{a}_z\right)\bullet\left(\frac{1}{2}\mathbf{a}_x - \frac{\sqrt{3}}{2}\mathbf{a}_z\right) = -\frac{3}{4}$

(b) $\mathbf{A}\bullet\mathbf{C} = \left(\frac{1}{2}\mathbf{a}_y + \frac{\sqrt{3}}{2}\mathbf{a}_z\right)\bullet\mathbf{a}_x = 0$

(c) $\mathbf{B}\bullet\mathbf{C} = \left(\frac{1}{2}\mathbf{a}_x - \frac{\sqrt{3}}{2}\mathbf{a}_z\right)\bullet\mathbf{a}_x = \frac{1}{2}$

(d) $\mathbf{A}\times\mathbf{B}\bullet\mathbf{C} = \mathbf{C}\bullet\mathbf{A}\times\mathbf{B}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{vmatrix}$$

$$= -\frac{\sqrt{3}}{4}$$

P1.19. The spherical coordinates of $(1, 1, \sqrt{2})$ are

$$r_s = \sqrt{1+1+2} = 2$$

$$\theta = \tan^{-1} \frac{\sqrt{1+1}}{\sqrt{2}} = \pi/4$$

$$\phi = \tan^{-1} \frac{1}{1} = \pi/4$$

Then

$$\mathbf{a}_x = \sin \frac{\pi}{4} \cos \frac{\pi}{4} \mathbf{a}_{rs} + \cos \frac{\pi}{4} \cos \frac{\pi}{4} \mathbf{a}_\theta - \sin \frac{\pi}{4} \mathbf{a}_\phi$$

$$= \frac{1}{2} \mathbf{a}_{rs} + \frac{1}{2} \mathbf{a}_\theta - \frac{1}{\sqrt{2}} \mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin \frac{\pi}{4} \sin \frac{\pi}{4} \mathbf{a}_{rs} + \cos \frac{\pi}{4} \sin \frac{\pi}{4} \mathbf{a}_\theta + \cos \frac{\pi}{4} \mathbf{a}_\phi$$

$$= \frac{1}{2} \mathbf{a}_{rs} + \frac{1}{2} \mathbf{a}_\theta + \frac{1}{\sqrt{2}} \mathbf{a}_\phi$$

$$-\sqrt{2}\,\mathbf{a}_z = -\sqrt{2} \left(\cos \frac{\pi}{4} \mathbf{a}_{rs} - \sin \frac{\pi}{4} \mathbf{a}_\theta \right)$$

$$= -\mathbf{a}_{rs} + \mathbf{a}_\theta$$

$$\therefore \left[\mathbf{a}_x + \mathbf{a}_y - \sqrt{2}\mathbf{a}_z\right]_{(1,\,1,\,\sqrt{2})} = 2\mathbf{a}_\theta$$

P1.20. $\left[\mathbf{a}_{rc} - \sqrt{3}\mathbf{a}_\phi + 3\mathbf{a}_z\right]_{(3,\, \pi/3,\, 5)}$

$$= (\cos\phi + \sqrt{3}\sin\phi)\mathbf{a}_x + (\sin\phi - \sqrt{3}\cos\phi)\mathbf{a}_y + 3\mathbf{a}_z$$

$$= \left(\cos\frac{\pi}{3} + \sqrt{3}\sin\frac{\pi}{3}\right)\mathbf{a}_x + \left(\sin\frac{\pi}{3} - \sqrt{3}\cos\frac{\pi}{3}\right)\mathbf{a}_y + 3\mathbf{a}_z$$

$$= 2\mathbf{a}_x + 3\mathbf{a}_z$$

$\left[3\mathbf{a}_{rs} - \sqrt{3}\mathbf{a}_\theta - \mathbf{a}_\phi\right]_{(1,\, \pi/3,\, \pi/6)}$

$$= \left(3\sin\frac{\pi}{3}\cos\frac{\pi}{6} - \sqrt{3}\cos\frac{\pi}{3}\cos\frac{\pi}{6} + \sin\frac{\pi}{6}\right)\mathbf{a}_x$$

$$+ \left(3\sin\frac{\pi}{3}\sin\frac{\pi}{6} - \sqrt{3}\cos\frac{\pi}{3}\sin\frac{\pi}{6} - \cos\frac{\pi}{6}\right)\mathbf{a}_y$$

$$+ \left(3\cos\frac{\pi}{3} + \sqrt{3}\sin\frac{\pi}{3}\right)\mathbf{a}_z$$

$$= \left(\frac{9}{4} - \frac{3}{4} + \frac{1}{2}\right)\mathbf{a}_x + \left(\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\right)\mathbf{a}_y$$

$$+ \left(\frac{3}{2} + \frac{3}{2}\right)\mathbf{a}_z$$

$$= 2\mathbf{a}_x + 3\mathbf{a}_z$$

$\therefore$ The two vectors are equal.

P1.21. For $r^2 \sin 2\phi = 1, z = 0$

$$2r\,dr \sin 2\phi + 2r^2 \cos 2\phi\, d\phi = 0,\ dz = 0$$

$$d\phi = -\frac{1}{r} \tan 2\phi\, dr,\ dz = 0$$

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{i}_\phi + dz\,\mathbf{a}_z$$

$$= dr\,\mathbf{a}_r - \tan 2\phi\, dr\,\mathbf{a}_\phi$$

Unit vector tangential to the curve is

$$\mathbf{a}_t = \pm\frac{d\mathbf{l}}{dl} = \pm\frac{\mathbf{a}_r - \tan 2\phi\,\mathbf{a}_\phi}{\sqrt{1+\tan^2 2\phi}}$$

$$= \pm(\cos 2\phi\,\mathbf{a}_r - \sin 2\phi\,\mathbf{a}_\phi)$$

(a) At $(1, \pi/4, 0)$

$$\mathbf{a}_t = \pm\left(\cos\frac{\pi}{2}\,\mathbf{a}_r - \sin\frac{\pi}{2}\,\mathbf{a}_\phi\right)$$

$$= \pm\,\mathbf{a}_\phi$$

(b) At $(\sqrt{2}, \pi/12, 0)$

$$\mathbf{a}_t = \pm\left(\cos\frac{\pi}{6}\,\mathbf{a}_r - \sin\frac{\pi}{6}\,\mathbf{a}_\phi\right)$$

$$= \pm\left(\frac{\sqrt{3}}{2}\,\mathbf{a}_r - \frac{1}{2}\,\mathbf{a}_\phi\right)$$

P1.22. For $r = 1$, $\phi = 2\theta$,

$dr = 0$, $d\phi = 2\,d\theta$

$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

$$= d\theta\,(\mathbf{a}_\theta + 2\sin\theta\,\mathbf{a}_\phi)$$

$\therefore$ The required unit vector is

$$\mathbf{a}_t = \pm\frac{d\mathbf{l}}{dl} = \pm\frac{\mathbf{a}_\theta + 2\sin\theta\,\mathbf{a}_\phi}{\sqrt{1+4\sin^2\theta}}$$

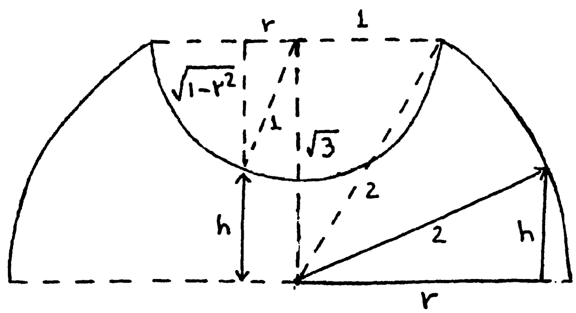
(a) At $(1, \pi/4, \pi/2)$,

$$\mathbf{a}_t = \pm\frac{\mathbf{a}_\theta + \sqrt{2}\,\mathbf{a}_\phi}{\sqrt{3}}$$

(b) At $(1, \pi/2, \pi)$,

$$\mathbf{a}_t = \frac{\mathbf{a}_\theta + 2\mathbf{a}_\phi}{\sqrt{5}}$$

P1.23.



For $r \le 1$, $h = \sqrt{3} - \sqrt{1-r^2}$

For $r \ge 1$, $h = \sqrt{4-r^2}$

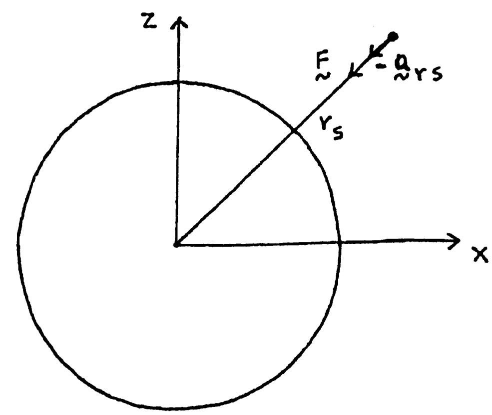
(a)

$$h = \begin{cases} \sqrt{3} - \sqrt{1-x^2-y^2} & \text{for } x^2+y^2 \le 1 \\ \sqrt{4-x^2-y^2} & \text{for } x^2+y^2 \ge 1 \end{cases}$$

(b)

$$h = \begin{cases} \sqrt{3} - \sqrt{1-r^2} & \text{for } r \le 1 \\ \sqrt{4-r^2} & \text{for } r \ge 1 \end{cases}$$

P1.24.



(a) $$\mathbf{F} = -\frac{mMG}{r_s^2}\mathbf{a}_{rs} = -\frac{mMG}{r_s^2}\frac{\mathbf{r}_s}{r_s}$$

$$= -\frac{mMG}{r_s^3}\mathbf{r}_s = -mMG\frac{x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{(x^2 + y^2 + z^2)^{3/2}}$$

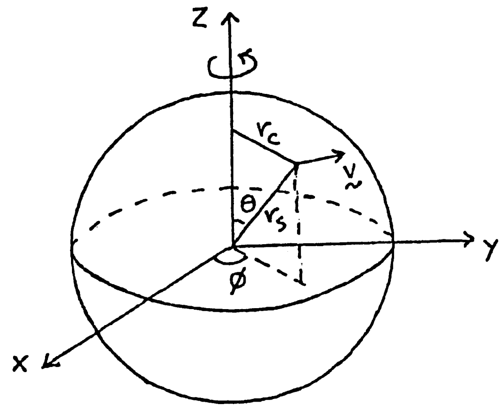
(b) $$\mathbf{F} = -\frac{mMG\,\mathbf{r}_s}{r_s^3}$$

$$= -mMG\frac{r_c\mathbf{a}_{rc} + z\mathbf{a}_z}{(r_c^2 + z^2)^{3/2}}$$

(c) $$\mathbf{F} = -\frac{mMG}{r_s^2}\mathbf{a}_{rs}$$

Constant magnitude surfaces are spherical surfaces concentric with the earth. Direction lines are radial lines directed toward the center of the earth.

P1.25.



$\mathbf{v} = \omega r_c \mathbf{a}_\phi$

where $\omega = \dfrac{2\pi}{24 \times 60 \times 60} = 7.2722 \times 10^{-5}$ rad/s

is the angular velocity of the spin motion.

(a) Cartesian:

$$r_c \mathbf{a}_\phi = \sqrt{x^2 + y^2}\ (-\sin\phi\, \mathbf{a}_x + \cos\phi\, \mathbf{a}_y)$$

$$= \sqrt{x^2 + y^2}\left(-\frac{y}{\sqrt{x^2 + y^2}}\, \mathbf{a}_x + \frac{x}{\sqrt{x^2 + y^2}}\, \mathbf{a}_y\right)$$

$$= -y\mathbf{a}_x + x\mathbf{a}_y$$

$$\mathbf{v} = \omega(-y\mathbf{a}_x + x\mathbf{a}_y)$$

(b) cylindrical:

$$\mathbf{v} = \omega r_c \mathbf{a}_\phi$$

(c) spherical:

$$r_c \mathbf{a}_\phi = r_s \sin\theta\, \mathbf{a}_\phi$$

$$\mathbf{v} = \omega r_s \sin\theta\, \mathbf{a}_\phi$$

Constant magnitude surfaces are cylinders having the z-axis as their axes. Direction lines are circles in the z = constant planes and centered on the z-axis.

P1.26. **(a)** For $(2y\mathbf{a}_x - x\mathbf{a}_y)$,

$$\frac{dx}{2y} = \frac{dy}{-x} = \frac{dz}{0}$$

$$x\,dx + 2y\,dy = 0,\ dz = 0$$

$$d\left(\frac{x^2}{2} + y^2\right) = 0,\ dz = 0$$

$$\frac{x^2}{2} + y^2 = C_1,\ z = C_2$$

For the point (1, 2, 3),

$$C_1 = \frac{1}{2} + 4 = \frac{9}{2} \text{ and } C_2 = 3,$$

and the equation is given by

$$x^2 + 2y^2 = 9,\ z = 3$$

(b) For $(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)$,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\ln x = \ln y + \ln C_1 = \ln z + \ln C_2$$

$$\ln x = \ln C_1 y = \ln C_2 z$$

$$x = C_1 y = C_2 z$$

For the point (1, 2, 3),

$$1 = 2C_1 = 3C_2$$

$$C_1 = \frac{1}{2},\ C_2 = \frac{1}{3}$$

and the direction line is given by

$$x = \frac{1}{2}y = \frac{1}{3}z$$

$$6x = 3y = 2z$$

P1.27. For ($\sin\phi\,\mathbf{a}_r + \cos\phi\,\mathbf{a}_\phi$),

$$\frac{dr}{\sin\phi} = \frac{r\,d\phi}{\cos\phi} = \frac{dz}{0}$$

$$\frac{dr}{r} = \tan\phi\,d\phi,\ dz = 0$$

$$\ln r = -\ln\cos\phi + \ln C_1,\ z = C_2$$

$$\ln r = \ln\frac{C_1}{\cos\phi},\ z = C_2$$

$$r\cos\phi = C_1,\ z = C_2$$

For the point ($2, \pi/3, 1$),

$$C_1 = 2\cos\frac{\pi}{3} = 1,\ C_2 = 1$$

and the direction line is given by

$$r\cos\phi = 1,\ z = 1$$

P1.28. For $(2\cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta)$,

$$\frac{dr}{2\cos\theta} = \frac{r\,d\theta}{-\sin\theta} = \frac{r\sin\theta\,d\phi}{0}$$

$$\frac{dr}{r} = -2\cot\theta\,d\theta,\ d\phi = 0$$

$$\ln r = -2\ln\sin\theta + \ln C_1,\ \phi = C_2$$

$$\ln r + \ln\sin^2\theta = \ln C_1,\ \phi = C_2$$

$$r\sin^2\theta = C_1,\ \phi = C_2$$

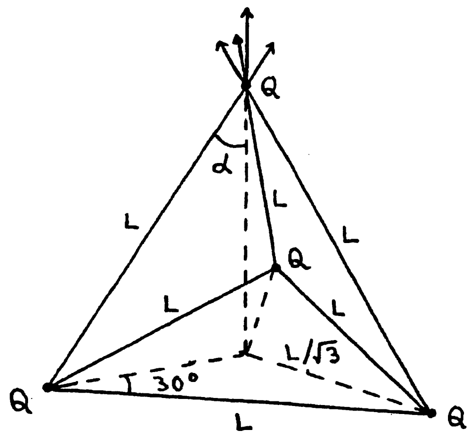
For the point $(2, \pi/4, \pi/6)$,

$$C_1 = 2\sin^2\frac{\pi}{4} = 1,\ C_2 = \frac{\pi}{6}$$

and the direction line is given by

$$r\sin^2\theta = 1,\ \phi = \frac{\pi}{6}$$

P1.29. From symmetry considerations, it is sufficient to consider one of the four charges, as shown in the figure. Thus the force on each point charge



$$= 3 \times \frac{Q^2}{4\pi\varepsilon_0 L^2} \cos\alpha$$

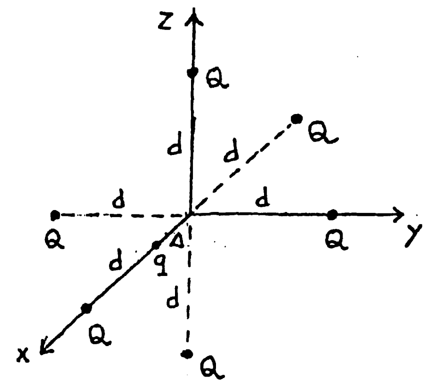
$$= 3 \times \frac{Q^2}{4\pi\varepsilon_0 L^2}\sqrt{1 - \frac{1}{3}}$$

$$= \frac{3Q^2}{4\pi\varepsilon_0 L^2}\sqrt{\frac{2}{3}}$$

$$= \frac{0.1949Q^2}{\varepsilon_0 L^2}$$

Direction is away from the center of the tetrahedron.

P1.30. $\mathbf{F} = \dfrac{Qq}{4\pi\varepsilon_0}\left[-\dfrac{1}{(d-\Delta)^2}\,\mathbf{a}_x + \dfrac{1}{(d+\Delta)^2}\,\mathbf{a}_x + \dfrac{4\Delta}{(d^2+\Delta^2)^{3/2}}\,\mathbf{a}_x\right]$



$$= \frac{4Qq}{4\pi\varepsilon_0}\left[-\frac{d\Delta}{(d^2-\Delta^2)^2} + \frac{\Delta}{(d^2+\Delta^2)^{3/2}}\right]\mathbf{a}_x$$

$$= \frac{Qq\Delta}{\pi\varepsilon_0 d^3}\left[\left(1+\frac{\Delta^2}{d^2}\right)^{-3/2} - \left(1-\frac{\Delta^2}{d^2}\right)^{-2}\right]\mathbf{a}_x$$

$$= \frac{Qq\Delta}{\pi\varepsilon_0 d^3}\left[\left(1-\frac{3}{2}\frac{\Delta^2}{d^2}+\ldots\right)-\left(1+\frac{2\Delta^2}{d^2}+\ldots\right)\right]\mathbf{a}_x$$

$$\approx -\frac{7Qq\Delta^3}{2\pi\varepsilon_0 d^5}\,\mathbf{a}_x$$

$\therefore\ \mathbf{F} \approx \dfrac{7Qq\Delta^3}{2\pi\varepsilon_0 d^5}$ toward the origin

P1.31. (a) For a solution to exist, a necessary (but not sufficient) condition is that there must be a point of intersection between the straight lines along the field vectors. Thus the two vectors $\mathbf{E}_1 = (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$ V/m at (2, 2, 3) and $\mathbf{E}_2 = (\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)$ at (–1, 0, 3) must lie in a plane, or, the determinant

$$\begin{vmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 0 \end{vmatrix}$$

must be zero. Since it is equal to (–8 + 12 – 4) = 0, the two vectors do lie in a plane and hence there is a point of intersection. The equations of the two straight lines are

$$\frac{2-x}{2} = \frac{2-y}{2} = \frac{3-z}{1} \quad \text{or,} \quad x = y, x - 2z = -4$$

and

$$\frac{-1-x}{1} = \frac{0-y}{2} = \frac{3-z}{2} \quad \text{or,} \quad 2x - y = -2, 2x - z = -5$$

and hence the point of intersection is (–2, –2, 1).

Assuming a point charge Q at (–2, –2, 1), its value required to produce $\mathbf{E}_1$ is given by

$$\frac{Q(4\mathbf{a}_x + 4\mathbf{a}_y + 2\mathbf{a}_z)}{4\pi\varepsilon_0(16+16+4)^{3/2}} = 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$$

or, $Q = 432\pi\varepsilon_0$. Value of Q required to produce $\mathbf{E}_2$ is given by

$$\frac{Q(\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)}{4\pi\varepsilon_0(1+4+4)^{3/2}} = \mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z$$

or, $Q = 108\pi\varepsilon_0$. Since the two values of Q are not the same, there is no solution to the problem.

(b) Following in the same manner as in **(a)**, we first check to see if $\mathbf{E}_1$ and $\mathbf{E}_2$ lie in a plane. Since

P1.31. (continued)

$$\begin{vmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = -6+4+2=0$$

the two vectors do lie in a plane and hence there is a point of intersection. The equations of the two straight lines are

$$\frac{1-x}{2}=\frac{1-y}{2}=\frac{1-z}{1} \qquad \text{or,} \quad x=y,\ y-2z=-1$$

and

$$\frac{1-x}{2}=\frac{2-y}{1}=\frac{0-z}{2} \qquad \text{or,} \quad x-2y=-3,\ 2y-z=4$$

and the point of intersection is (3, 3, 2). Value of Q required to produce $\mathbf{E}_1$ is given by

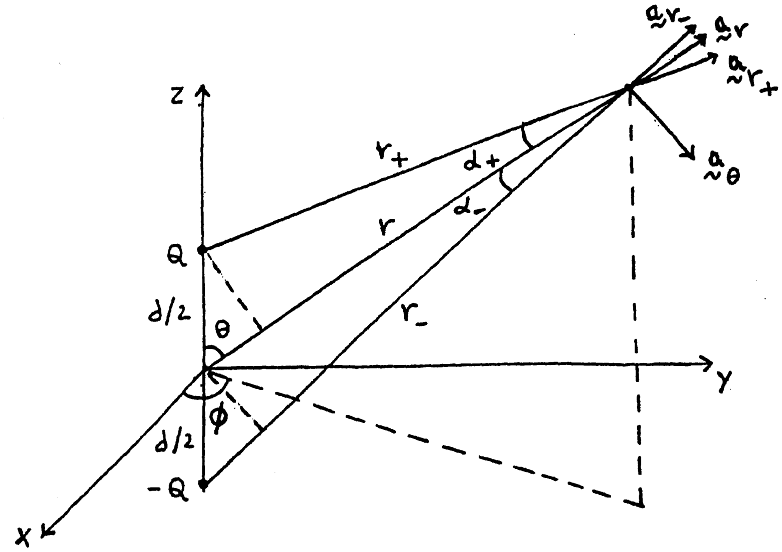
$$\frac{Q(-2\mathbf{a}_x-2\mathbf{a}_y-\mathbf{a}_z)}{4\pi\varepsilon_0(4+4+1)^{3/2}}=2\mathbf{a}_x+2\mathbf{a}_y+\mathbf{a}_z$$

or, $Q=-108\pi\varepsilon_0$. Value of Q required to produce $\mathbf{E}_2$ is given by

$$\frac{Q(-2\mathbf{a}_x-\mathbf{a}_y-2\mathbf{a}_z)}{4\pi\varepsilon_0(4+1+4)^{3/2}}=2\mathbf{a}_x+\mathbf{a}_y+2\mathbf{a}_z$$

or, $Q=-108\pi\varepsilon_0$. Thus the solution is $-108\pi\varepsilon_0$ C at the point (3, 3, 2).

P1.32.



$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r_+^2}\,\mathbf{a}_{r_+} - \frac{Q}{4\pi\varepsilon_0 r_-^2}\,\mathbf{a}_{r_-}$$

$$E_r = \mathbf{E}\cdot\mathbf{a}_r = \frac{Q}{4\pi\varepsilon_0 r_+^2}\cos\alpha_+ - \frac{Q}{4\pi\varepsilon_0 r_-^2}\cos\alpha_-$$

$$= \frac{Q}{4\pi\varepsilon_0 r_+^3}\left(r - \frac{d}{2}\cos\theta\right) - \frac{Q}{4\pi\varepsilon_0 r_-^2 r}\left(r_- - \frac{d}{2}\cos\theta\right)$$

For $r >> d$,

$$r_+ \approx r - \frac{d}{2}\cos\theta$$

$$r_- \approx r + \frac{d}{2}\cos\theta$$

$$E_r \approx \frac{Q}{4\pi\varepsilon_0\left(r - \frac{d}{2}\cos\theta\right)^2} - \frac{Q}{4\pi\varepsilon_0\left(r + \frac{d}{2}\cos\theta\right)^2}$$

$$= \frac{Q(2rd\cos\theta)}{4\pi\varepsilon_0\left(r^2 - \frac{d^2}{4}\cos^2\theta\right)^2}$$

$$\approx \frac{Q}{2\pi\varepsilon_0 r^3}\,d\cos\theta$$

P1.32. (continued)

$$E_\theta = \mathbf{E} \bullet \mathbf{a}_\theta$$

$$= \frac{Q}{4\pi\varepsilon_0 r_+^2} \sin\alpha_+ + \frac{Q}{4\pi\varepsilon_0 r_-^2} \sin\alpha_-$$

$$= \frac{Q\left(\frac{d}{2}\sin\theta\right)}{4\pi\varepsilon_0 r_+^3} + \frac{Q\left(\frac{d}{2}\sin\theta\right)}{4\pi\varepsilon_0 r_-^2 r}$$

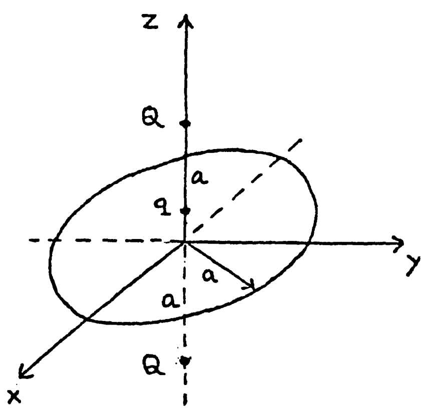
$$\approx \frac{Q\left(\frac{d}{2}\sin\theta\right)}{4\pi\varepsilon_0\left(r - \frac{d}{2}\cos\theta\right)^3} + \frac{Q\left(\frac{d}{2}\sin\theta\right)}{4\pi\varepsilon_0 r\left(r + \frac{d}{2}\cos\theta\right)^2}$$

$$\approx \frac{Q}{4\pi\varepsilon_0 r^3} d\sin\theta$$

Thus

$$\mathbf{E} \approx \frac{Qd}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

P1.33. Electric force on the point charge q



$$= \left[\frac{Qz}{4\pi\varepsilon_0(a^2+z^2)^{3/2}} - \frac{Q}{4\pi\varepsilon_0(a-z)^2} + \frac{Q}{4\pi\varepsilon_0(a+z)^2}\right]\mathbf{a}_z$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{z}{(a^2+z^2)^{3/2}} - \frac{4az}{(a^2-z^2)^2}\right]\mathbf{a}_z$$

$$\approx -\frac{3Qz}{4\pi\varepsilon_0 a^3}\mathbf{a}_z$$

The equation of motion of the charge q is

$$m\frac{d^2z}{dt^2} = -\frac{3Qz}{4\pi\varepsilon_0 a^3}$$

$$\frac{d^2z}{dt^2} + \frac{3Qz}{4\pi\varepsilon_0 m a^3} = 0$$

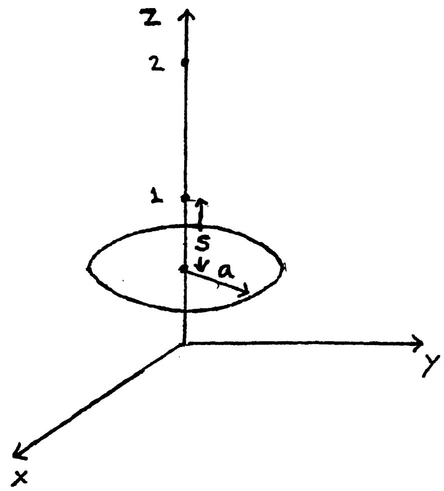
The solution subject to initial condition $z = z_0$ and $\frac{dz}{dt} = 0$ at $t = 0$ is

$$z = z_0 \sin\sqrt{\frac{3Q}{4\pi\varepsilon_0 m a^3}}\, t$$

Thus the frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{3Q}{4\pi\varepsilon_0 m a^3}}$$

P1.34.



From geometrical considerations, the circular ring charge must lie parallel to the xy-plane and with its center on the z-axis. Then, with reference to the notation shown in the figure, and from the expression for the electric field due to a ring charge along its axis given in Ex. 1.7, we require that

$$\frac{10^{-6}s}{4\pi\varepsilon_0(a^2+s^2)^{3/2}} = 10^3 \qquad (1)$$

$$\frac{10^{-6}(s+1)}{4\pi\varepsilon_0[a^2+(s+1)^2]^{3/2}} = 10^3 \qquad (2)$$

Dividing (2) by (1), we then have

$$\left[\frac{a^2+(s+1)^2}{a^2+s^2}\right]^{3/2}\left(\frac{s}{s+1}\right) = 1$$

$$\frac{a^2+s^2+2s+1}{a^2+s^2} = \left(\frac{s+1}{s}\right)^{2/3} \qquad (3)$$

But from (1),

$$a^2+s^2 = \left(\frac{10^{-6}s}{4\pi\varepsilon_0\times 10^3}\right)^{2/3} = (9s)^{2/3} \qquad (4)$$

Substituting (4) in (3), we have

$$\frac{(9s)^{2/3}+2s+1}{(9s)^{2/3}} = \left(\frac{s+1}{s}\right)^{2/3}$$

$$9^{2/3}[(s+1)^{2/3} - s^{2/3}] = 2s+1$$

Solving for s, we obtain

$$s = 0.82707$$

Then from (4),

$$a = \sqrt{(9s)^{2/3} - s^2}$$

$$= 1.76869$$

Thus the circular ring charge is centered at (0, 0, 0.82707), with a radius of 1.76869 and parallel to the xy-plane.

For any Q, (3) remains the same, but (4) becomes

$$a^2 + s^2 = \left(\frac{Qs}{4\pi\varepsilon_0 \times 10^3}\right)^{2/3} = (9 \times 10^6 Qs)^{2/3} \qquad (5)$$

Substituting (5) in (3), we have

$$\frac{(9 \times 10^6 Qs)^{2/3} + 2s + 1}{(9 \times 10^6 Qs)^{2/3}} = \left(\frac{s+1}{s}\right)^{2/3}$$

$$(9 \times 10^6 Q)^{2/3}\,[(s+1)^{2/3} - s^{2/3}] = 2s + 1$$

$$(9 \times 10^6 Q)^{2/3} = \frac{2s+1}{(s+1)^{2/3} - s^{2/3}}$$

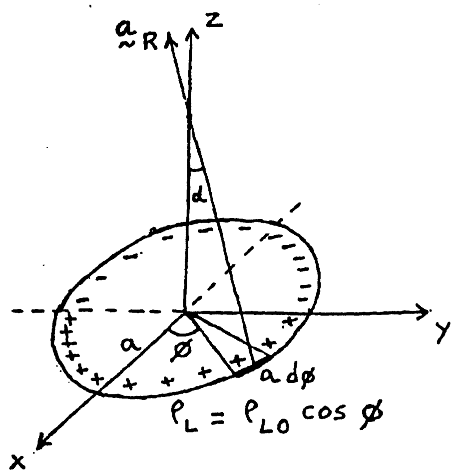
Since s has to be greater than zero for both fields to be in the $+z$-direction, and the right side is a monotonically increasing function of s, the value of Q has to be greater than that corresponding to $s = 0$, that is,

$$(9 \times 10^6 Q)^{2/3} > \left[\frac{2s+1}{(s+1)^{2/3} - s^{2/3}}\right]_{s=0}$$

$$9 \times 10^6 Q > 1$$

$$Q > \frac{1}{9} \times 10^{-6}\,\text{C} \quad \text{or} \quad \frac{1}{9}\,\mu\text{C}$$

P1.35.



Because of the dependence of ρ_L on ϕ in the manner cos ϕ, there is only a negative x-component of **E** at a point on the z-axis.

$$[dE_x]_{(0,\,0,\,z)} = \frac{(\rho_{L0}\cos\phi)(a\,d\phi)}{4\pi\varepsilon_0(a^2+z^2)}\,\mathbf{a}_R \cdot \mathbf{a}_x$$

$$= -\frac{\rho_{L0}\,a\cos\phi\,d\phi}{4\pi\varepsilon_0(a^2+z^2)}\sin\alpha\cos\phi$$

$$= -\frac{\rho_{L0}\,a^2}{4\pi\varepsilon_0(a^2+z^2)^{3/2}}\cos^2\phi\,d\phi$$

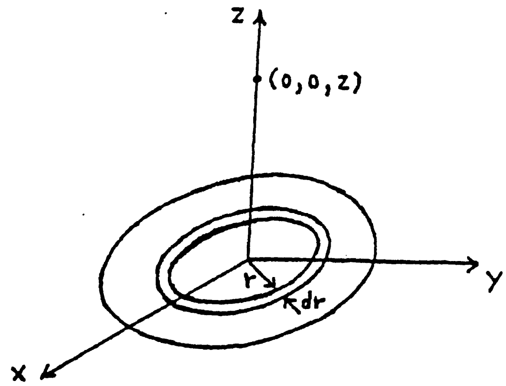
$$[E_x]_{(0,\,0,\,z)} = 4\int_{\phi=0}^{\pi/2}[dE_x]_{(0,0,z)}$$

$$= -\frac{\rho_{L0}\,a^2}{\pi\varepsilon_0(a^2+z^2)^{3/2}}\int_{\phi=0}^{\pi/2}\cos^2\phi\,d\phi$$

$$= -\frac{\rho_{L0}\,a^2}{\pi\varepsilon_0(a^2+z^2)^{3/2}}\left[\frac{\phi}{2}+\frac{\sin 2\phi}{4}\right]_0^{\pi/2}$$

$$[\mathbf{E}]_{(0,\,0,\,z)} = -\frac{\rho_{L0}\,a^2}{4\varepsilon_0(a^2+z^2)^{3/2}}\,\mathbf{a}_x$$

P1.36.



Using the procedure employed in Ex. 1.8, we have

$$[d\mathbf{E}]_{(0,0,z)} = \frac{[(4\pi\varepsilon_0/r)\,2\pi r\,dr]z}{4\pi\varepsilon_0(r^2+z^2)^{3/2}}\,\mathbf{a}_z$$

$$= \frac{2\,\pi z\,dr}{(r^2+z^2)^{3/2}}\,\mathbf{a}_z$$

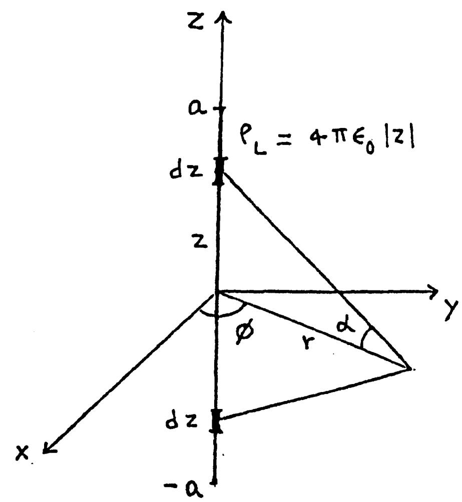
Thus,

$$[\mathbf{E}]_{(0,0,z)} = \int_{r=0}^{a} \frac{2\pi z\,dr}{(r^2+z^2)^{3/2}}\,\mathbf{a}_z$$

$$= 2\pi z\left[\frac{r}{z^2\sqrt{r^2+z^2}}\right]_{r=0}^{a}\mathbf{a}_z$$

$$= \frac{2\pi a}{z\sqrt{a^2+z^2}}\,\mathbf{a}_z$$

P1.37. Because of |z| dependence of ρ_L, and symmetry about the origin, there is only an r-component of $\mathbf{E}$ at $(r, \phi, 0)$. Considering two differential length elements symmetrically situated about the origin, we have



$$[dE_r]_{(r,\,\phi,\,0)} = \frac{2\rho_L\,dz}{4\pi\varepsilon_0(r^2+z^2)}\cos\alpha$$

$$= \frac{4\pi\varepsilon_0 z\,dz}{2\pi\varepsilon_0(r^2+z^2)}\frac{r}{\sqrt{r^2+z^2}}$$

$$= \frac{2rz\,dz}{(r^2+z^2)^{3/2}}$$

$$[E_r]_{(r,\,\phi,\,0)} = \int_{z=0}^{z=a}\frac{2rz\,dz}{(r^2+z^2)^{3/2}}$$

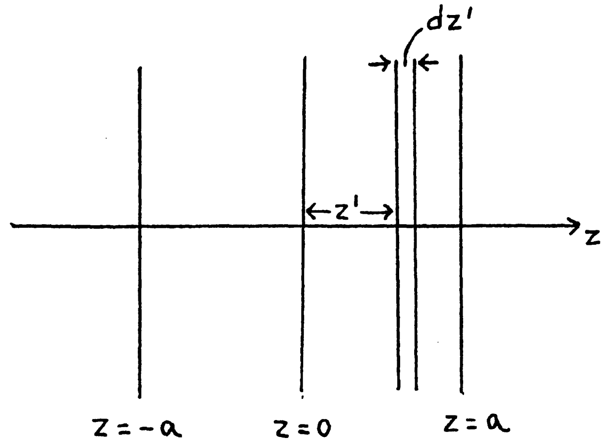
$$= 2r\left[\frac{-1}{\sqrt{r^2+z^2}}\right]_{z=0}^{z=a}$$

$$= 2r\left[\frac{1}{r}-\frac{1}{\sqrt{r^2+a^2}}\right]$$

$$= 2\left(1-\frac{r}{\sqrt{r^2+a^2}}\right)$$

$$[\mathbf{E}]_{(r,\,\phi,\,0)} = 2\left(1-\frac{r}{\sqrt{r^2+a^2}}\right)\mathbf{a}_r$$

P1.38.



From the result of Ex. 1.8, the field due to the thin slab of charge of thickness dz' shown in the figure is given by

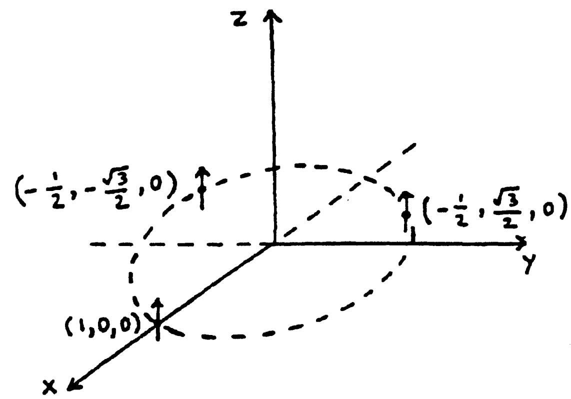
$$d\mathbf{E} = \pm \frac{\rho_0 \, dz'}{2\varepsilon_0} \mathbf{a}_z \qquad \text{for } z \gtrless z'$$

Then using superposition, we have for the entire slab charge

$$\mathbf{E} = \begin{cases} \displaystyle\int_{-a}^{a} -\frac{\rho_0 \, dz'}{2\varepsilon_0} \mathbf{a}_z & \text{for } z < -a \\ \displaystyle\int_{z}^{a} -\frac{\rho_0 \, dz'}{2\varepsilon_0} \mathbf{a}_z + \int_{-a}^{z} \frac{\rho_0 \, dz'}{2\varepsilon_0} \mathbf{a}_z & \text{for } -a < z < a \\ \displaystyle\int_{-a}^{a} \frac{\rho_0 \, dz'}{2\varepsilon_0} \mathbf{a}_z & \text{for } z > a \end{cases}$$

$$= \begin{cases} -\dfrac{\rho_0 a}{\varepsilon_0} \mathbf{a}_z & \text{for } z < -a \\ \dfrac{\rho_0 z}{\varepsilon_0} \mathbf{a}_z & \text{for } -a < z < a \\ \dfrac{\rho_0 a}{\varepsilon_0} \mathbf{a}_z & \text{for } z > a \end{cases}$$

P1.39.



From symmetry considerations, it is sufficient to find the force on one element. Hence, we shall consider the element at (1, 0, 0).

$$\mathbf{F} = \frac{\mu_0}{4\pi} I\,dz\,\mathbf{a}_z$$

$$\mathbf{x}\left[\frac{I\,dz\,\mathbf{a}_z\,\mathbf{x}\left(\frac{3}{2}\mathbf{a}_x - \frac{\sqrt{3}}{2}\mathbf{a}_y\right)}{\left(\frac{9}{4}+\frac{3}{4}\right)^{3/2}} + \frac{I\,dz\,\mathbf{a}_z\,\mathbf{x}\left(\frac{3}{2}\mathbf{a}_x + \frac{\sqrt{3}}{2}\mathbf{a}_y\right)}{\left(\frac{9}{4}+\frac{3}{4}\right)^{3/2}}\right]$$

$$= -\frac{\mu_0}{4\sqrt{3}\pi}(I\,dz)^2\,\mathbf{a}_x = -0.046\mu_0(I\,dz)^2\,\mathbf{a}_x$$

$\therefore$ The force on each element is $0.046\mu_0(I\,dz)^2$ directed toward the origin.

P1.40. From Biot-Savart Law,

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, d\mathbf{l} \times \mathbf{a}_R}{R^2} = \frac{\mu_0}{4\pi} \frac{I\, d\mathbf{l} \times \mathbf{R}}{R^3}$$

where **R** is the vector drawn from the current element to the point at which **B** is to be computed.

(a) For the point (2, –1, 3),

$$\mathbf{R} = (2-1)\mathbf{a}_x + (-1+2)\mathbf{a}_y + (3-2)\mathbf{a}_z$$

$$= \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, dx\, (\mathbf{a}_x + \mathbf{a}_y) \times (\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{(\sqrt{3})^3}$$

$$= \frac{\mu_0 I\, dx}{12\sqrt{3}\pi} (\mathbf{a}_x - \mathbf{a}_y)$$

(b) For the point (2, –3, 4),

$$\mathbf{R} = (2-1)\mathbf{a}_x + (-3+2)\mathbf{a}_y + (4-2)\mathbf{a}_z$$

$$= \mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, dx\, (\mathbf{a}_x + \mathbf{a}_y) \times (\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z)}{(\sqrt{6})^3}$$

$$= \frac{\mu_0 I\, dx}{12\sqrt{6}\pi} (\mathbf{a}_x - \mathbf{a}_y - \mathbf{a}_z)$$

(c) For the point (3, 0, 2),

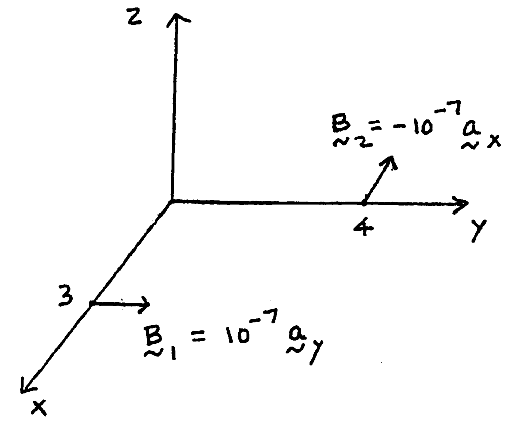
$$\mathbf{R} = (3-1)\mathbf{a}_x + (0+2)\mathbf{a}_y + (2-2)\mathbf{a}_z$$

$$= 2\mathbf{a}_x + 2\mathbf{a}_y$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, dx\, (\mathbf{a}_x + \mathbf{a}_y) \times (2\mathbf{a}_x + 2\mathbf{a}_y)}{(\sqrt{8})^3}$$

$$= \mathbf{0}$$

P1.41. (a) Since the magnetic field due to an infinitely long filamentary wire is circular surrounding the wire, the wire must be in a plane perpendicular to the field at the point under consideration. Thus for $\mathbf{B}_1 = 10^{-7}\mathbf{a}_y$ Wb/m^2 at (3, 0, 0), the wire must lie in the plane $y = 0$, and for $\mathbf{B}_2 = -10^{-7}\mathbf{a}_x$ Wb/m^2 at (0, 4, 0), the wire must lie in the $x = 0$ plane.



Therefore the wire must lie at the intersection of these two planes, which is the z-axis. Let the current be I in the $+z$-direction. Then

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi(3)}\mathbf{a}_y = 10^{-7}\mathbf{a}_y \quad \text{or} \quad I = 1.5 \text{ A}$$

$$\mathbf{B}_2 = \frac{\mu_0 I}{2\pi(4)}(-\mathbf{a}_x) = -10^{-7}\mathbf{a}_x \quad \text{or,} \quad I = 2 \text{ A}$$

Since the two values are not the same, there is no solution to the problem.

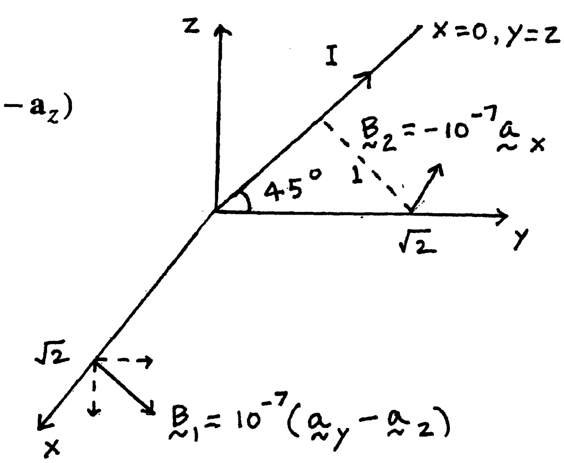
(b) By inspection, plane normal to $\mathbf{B}_1$ is $y = z$, and plane normal to $\mathbf{B}_2$ is $x = 0$. Therefore the wire must lie along the line $x = 0$, $y = z$. Let the current be I in the direction of increasing z. Then

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi\sqrt{2}}\left(\frac{\mathbf{a}_y - \mathbf{a}_z}{\sqrt{2}}\right) = \frac{10^{-7}}{2}(\mathbf{a}_y - \mathbf{a}_z)$$

or, $I = 0.5$ A

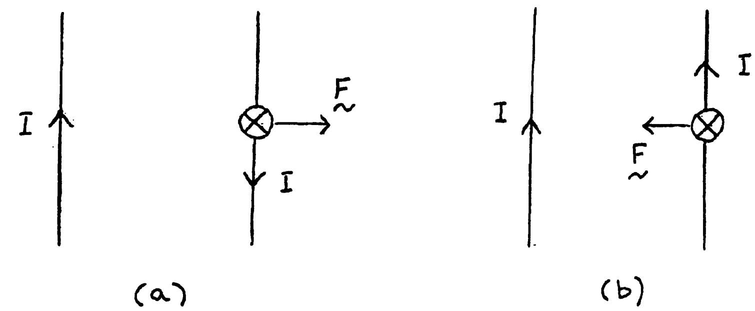
$$\mathbf{B}_2 = \frac{\mu_0 I}{2\pi(1)}(-\mathbf{a}_x) = -10^{-7}\mathbf{a}_x$$

or, $I = 0.5$ A



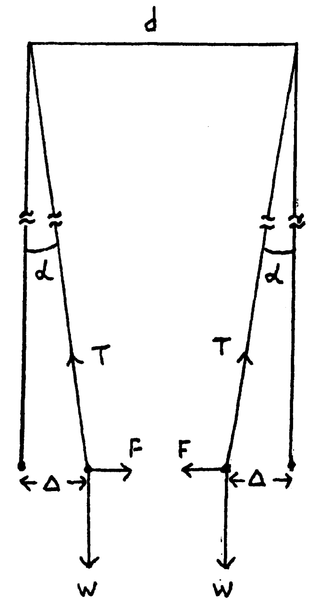
Since the two values of I are the same, the solution is $I = 0.5$ A along the wire having the orientation $x = 0$, $y = z$.

P1.42. **(a)**



As shown in the figure, for currents flowing in opposite senses, the force is one of repulsion, and for currents flowing in the same sense, it is one of attraction. Hence, the currents must be in the same sense.

(b) Since $d \ll l$, the wires can be considered to be infinitely long for the purpose of computing the magnetic field. Let the swing from the vertical position be Δ, as shown in the figure. Then, from the application of $d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}$ to an entire rod, the magnetic force on it due to the magnetic field of the other rod is



$$F = Il \times \frac{\mu_0 I}{2\pi(d-2\Delta)} = \frac{\mu_0 I^2 l}{2\pi(d-2\Delta)}$$

For equilibrium,

$$\tan\alpha \approx \frac{\Delta}{L} = \frac{F}{w} = \frac{\mu_0 I^2 l}{2\pi(d-2\Delta)w}$$

or,

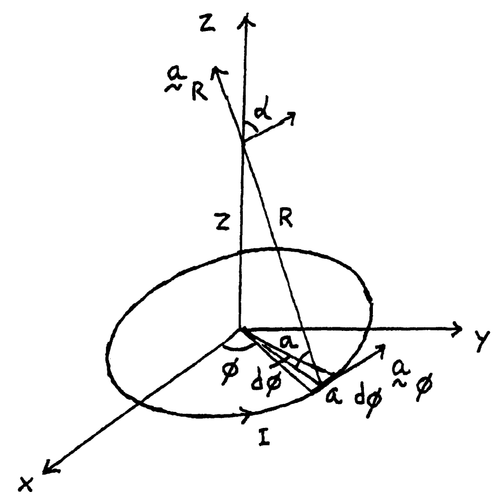
$$\frac{\mu_0 I^2 lL}{2\pi w} = (d-2\Delta)\Delta \qquad (1)$$

For $I = 0$, $\Delta = 0$. As I increases, Δ increases, $(d-2\Delta)$ decreases and hence there is a maximum value for $(d-2\Delta)\Delta$, beyond which the equation cannot be satisfied. This maximum value occurs for $\Delta = \frac{d}{4}$. For any further increase in the current, the force F becomes greater than $T \sin\alpha$, or the left side of (1) becomes greater than the right side, and hence the rods swing and touch each other. Thus the critical value of the current is given by

$$\frac{\mu_0 I^2 l L}{2\pi w} = \left(d - \frac{d}{2}\right)\frac{d}{4} = \frac{d^2}{8}$$

$$I = \sqrt{\frac{\pi d^2 w}{4\mu_0 l L}}$$

P1.43. From the symmetry of the current distribution about the z-axis, we note that only the z-component of $\mathbf{B}$ exists at (0, 0, z).



Considering an infinitesimal segment $a\, d\phi\, \mathbf{a}_\phi$ of the loop, we obtain the z-component due to it to be

$$dB_z = \left(\frac{\mu_0 I a\, d\phi\, \mathbf{a}_\phi \times \mathbf{a}_R}{4\pi R^2}\right) \cdot \mathbf{a}_z$$

$$= \frac{\mu_0 I a\, d\phi}{4\pi R^2} \cos\alpha$$

$$= \frac{\mu_0 I a^2\, d\phi}{4\pi (a^2 + z^2)^{3/2}}$$

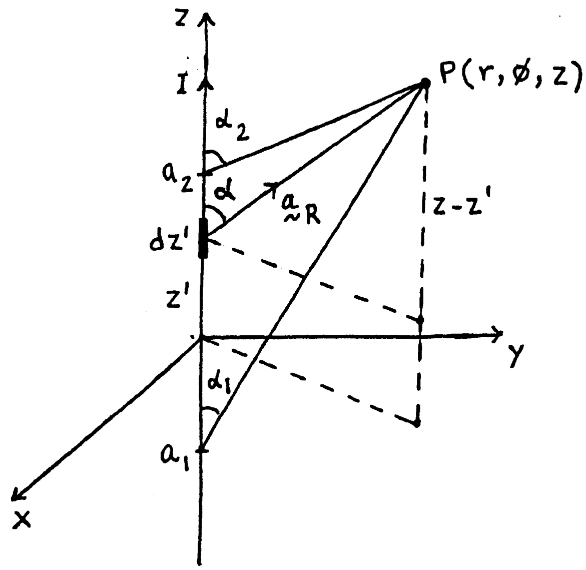
Thus, $\mathbf{B}$ at (0, 0, z) due to the entire current loop

$$= \left(\int_{\phi=0}^{2\pi} dB_z\right) \mathbf{a}_z$$

$$= \int_{\phi=0}^{2\pi} \frac{\mu_0 I a^2\, d\phi}{4\pi (a^2 + z^2)^{3/2}}\, \mathbf{a}_z$$

$$= \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \mathbf{a}_z$$

P1.44.



$$\mathbf{B} = \int_{z'=a_1}^{a_2} \frac{\mu_0 I \, dz' \, \mathbf{a}_z \times \mathbf{a}_R}{4\pi[r^2 + (z-z')^2]}$$

$$= \int_{z'=a_1}^{a_2} \frac{\mu_0 I \, dz' \sin\alpha \, \mathbf{a}_\phi}{4\pi r^2\{1+[(z-z')/r]^2\}}$$

Substituting $z - z' = r \cot\alpha$, $dz' = r \operatorname{cosec}^2 \alpha \, d\alpha$, we obtain

$$\mathbf{B} = \frac{\mu_0 I}{4\pi r} \int_{\alpha=\alpha_1}^{\alpha_2} \frac{\operatorname{cosec}^2 \alpha \sin\alpha \, d\alpha}{\operatorname{cosec}^2 \alpha} \mathbf{a}_\phi$$

$$= \frac{\mu_0 I}{4\pi r} [-\cos\alpha]_{\alpha=\alpha_1}^{\alpha_2} \mathbf{a}_\phi$$

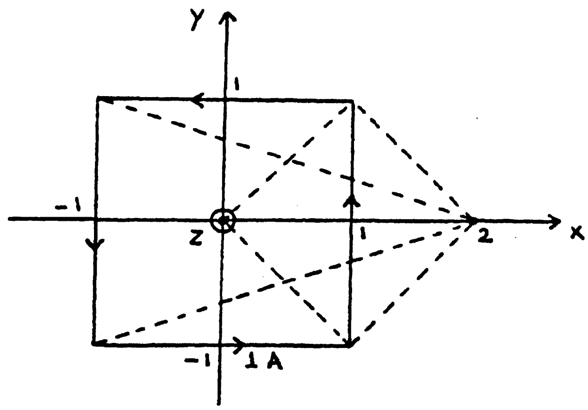
$$= \frac{\mu_0 I}{4\pi r} (\cos\alpha_1 - \cos\alpha_2) \, \mathbf{a}_\phi$$

In the limit $a_1 \to -\infty$ and $a_2 \to \infty$, $\alpha_1 \to 0$ and $\alpha_2 \to \pi$,

$$\mathbf{B} \to \frac{\mu_0 I}{4\pi r} [1 - (-1)] \, \mathbf{a}_\phi = \frac{\mu_0 I}{2\pi r} \mathbf{a}_\phi$$

which is the same as that for the infinitely long, straight wire.

P1.45.



(a) From symmetry considerations, it is sufficient to find **B** due to one side and multiply by 4. Thus,

$$[\mathbf{B}]_{(0,\,0,\,0)} = 4\,\frac{\mu_0}{4\pi}\,(\cos 45^\circ - \cos 135^\circ)\,\mathbf{a}_z$$

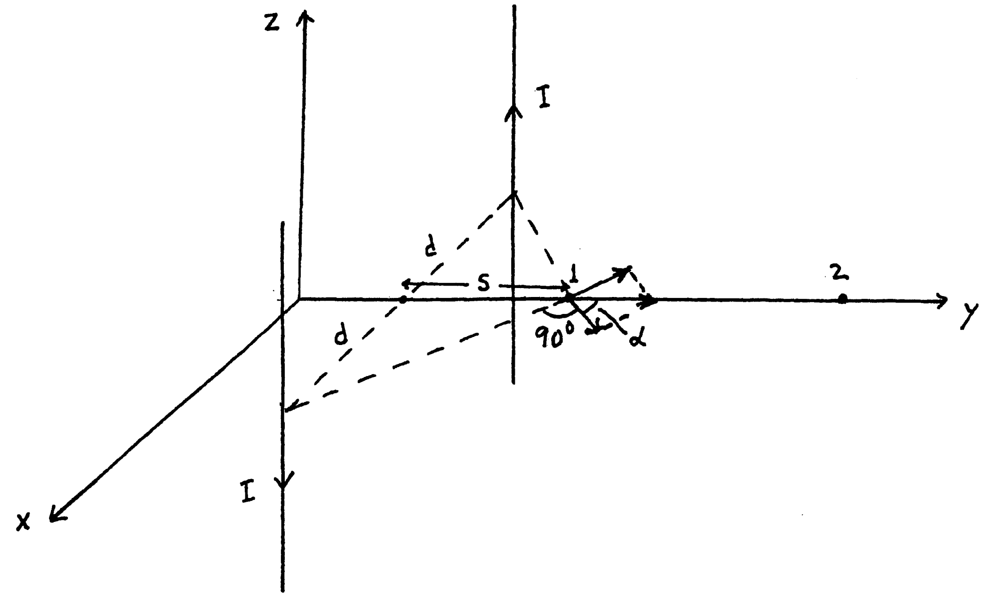
$$= \frac{\sqrt{2}\mu_0}{\pi}\,\mathbf{a}_z = 0.45\mu_0\mathbf{a}_z$$

(b) $$[\mathbf{B}]_{(2,\,0,\,0)} = -\frac{\mu_0}{4\pi}(\sqrt{2})\,\mathbf{a}_z + \frac{\mu_0}{12\pi}\left(\frac{2}{\sqrt{10}}\right)\mathbf{a}_z + \frac{2\mu_0}{4\pi}\left(\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)\mathbf{a}_z$$

$$= \frac{\mu_0}{\pi}(-0.3536 + 0.0527 + 0.1208)\mathbf{a}_z$$

$$= -\frac{0.18\mu_0}{\pi}\,\mathbf{a}_z = -0.057\mu_0\mathbf{a}_z$$

P1.46.



From geometrical considerations of magnetic field due to an infinitely long, straight wire, the two wires must be symmetrically situated about the y-axis with the currents flowing as shown in the figure. With reference to the notation shown in the figure, the magnetic flux density at (0, 1, 0) is given by

$$\mathbf{B}_1 = \frac{2\mu_0 I}{2\pi\sqrt{d^2+s^2}} \cos\alpha\, \mathbf{a}_y$$

$$= \frac{\mu_0 I d}{\pi(d^2+s^2)} \mathbf{a}_y = 10^{-7}\mathbf{a}_y$$

or,

$$\frac{4Id}{(d^2+s^2)} = 1 \qquad (1)$$

Likewise, the magnetic flux density at (0, 2, 0) is given by

$$\mathbf{B}_2 = \frac{\mu_0 I d}{\pi[d^2+(s+1)^2]} \mathbf{a}_y = 0.5 \times 10^{-7}\mathbf{a}_y$$

or,

$$\frac{4Id}{d^2+(s+1)^2} = 0.5 \qquad (2)$$

Dividing (1) by (2), we have

P1.46. (continued)

$$\frac{d^2+(s+1)^2}{d^2+s^2}=2$$

$$d^2+s^2-2s-1=0 \tag{3}$$

$$d=\sqrt{-s^2+2s+1} \tag{4}$$

From (1), we have for $I = 1$,

$$d^2+s^2=4d \tag{5}$$

Using (3) and (4), we then can write

$$2s+1=4\sqrt{-s^2+2s+1} \tag{6}$$

$$4s^2+4s+1=-16s^2+32s+16 \tag{7}$$

$$20s^2-28s-15=0$$

$$s=\frac{28\pm\sqrt{28^2+1200}}{40}$$

$$=\frac{28\pm 44.54}{40}$$

$$= 1.8136 \text{ or } -0.41355$$

The corresponding values of d are

$$d = 1.1567,\ 0.0432$$

Thus for $I = 1$ A, the wires must pass through the pair of points ($\pm$ 1.1567, –0.8136, 0) or through the pair of points ($\pm$ 0.0432, 1.41355, 0).

For any I, we have in the place of (5) and (6),

$$d^2+s^2=4Id \tag{8}$$

$$2s+1=4I\sqrt{-s^2+2s+1} \tag{9}$$

P1.46. (continued)

$$4s^2 + 4s + 1 = -16I^2s^2 + 32I^2s + 16I^2$$

$$(4 + 16I^2)s^2 + (4 - 32I^2)s + (1 - 16I^2) = 0$$

For a solution to exist,

$$(4 - 32I^2)^2 - 4(4 + 16I^2)(1 - 16I^2) \text{ must be} \geq 0$$

$$(1 - 8I^2)^2 - (1 + 4I^2)(1 - 16I^2) \geq 0$$

$$1 - 16I^2 + 64I^4 - (1 - 12I^2 - 64I^4) \geq 0$$

$$128I^4 - 4I^2 \geq 0$$

$$I^2 \geq \frac{1}{32}$$

$$I \geq 0.1768$$

$\therefore$ There is a minimum value of I equal to 0.1768 A, below which there is no solution to the design problem.

P1.47. Let the current densities on the sheets be $\mathbf{J}_{S1}$, $\mathbf{J}_{S2}$, and $\mathbf{J}_{S3}$ on the $x = 0$, $y = 0$, and $z = 0$ planes, respectively. Then,

$$\frac{\mu_0}{2}(\mathbf{J}_{S1} \times \mathbf{a}_x + \mathbf{J}_{S2} \times \mathbf{a}_y + \mathbf{J}_{S3} \times \mathbf{a}_z) = 3B_0\mathbf{a}_x \qquad (1)$$

$$\frac{\mu_0}{2}(\mathbf{J}_{S1} \times \mathbf{a}_x - \mathbf{J}_{S2} \times \mathbf{a}_y + \mathbf{J}_{S3} \times \mathbf{a}_z) = B_0(-\mathbf{a}_x + 2\mathbf{a}_z) \qquad (2)$$

$$\frac{\mu_0}{2}(\mathbf{J}_{S1} \times \mathbf{a}_x + \mathbf{J}_{S2} \times \mathbf{a}_y - \mathbf{J}_{S3} \times \mathbf{a}_z) = B_0(\mathbf{a}_x + 2\mathbf{a}_y) \qquad (3)$$

$$(1) - (2) \rightarrow \mathbf{J}_{S2} \times \mathbf{a}_y = \frac{B_0}{\mu_0}(4\mathbf{a}_x - 2\mathbf{a}_z) \qquad (4)$$

$$(1) - (3) \rightarrow \mathbf{J}_{S3} \times \mathbf{a}_z = \frac{B_0}{\mu_0}(2\mathbf{a}_x - 2\mathbf{a}_y) \qquad (5)$$

$$2 \times (1) - (4) - (5) \rightarrow \mathbf{J}_{S1} \times \mathbf{a}_x = \frac{B_0}{\mu_0}(2\mathbf{a}_y + 2\mathbf{a}_z)$$

(a) At $(-6, -2, -3)$,

$$\mathbf{B} = \frac{\mu_0}{2}(-\mathbf{J}_{S1} \times \mathbf{a}_x - \mathbf{J}_{S2} \times \mathbf{a}_y - \mathbf{J}_{S3} \times \mathbf{a}_z)$$

$$= -\mathbf{B}_{(1, 2, 3)} = -3B_0\mathbf{a}_x$$

(b) At $(-4, -5, 7)$,

$$\mathbf{B} = \frac{\mu_0}{2}(-\mathbf{J}_{S1} \times \mathbf{a}_x - \mathbf{J}_{S2} \times \mathbf{a}_y + \mathbf{J}_{S3} \times \mathbf{a}_z)$$

$$= -\mathbf{B}_{(8, 9, -4)} = -B_0(\mathbf{a}_x + 2\mathbf{a}_y)$$

(c) At $(6, -3, -5)$,

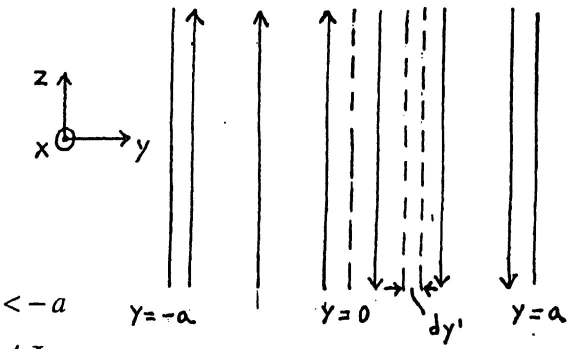
$$\mathbf{B} = \frac{\mu_0}{2}(\mathbf{J}_{S1} \times \mathbf{a}_x - \mathbf{J}_{S2} \times \mathbf{a}_y - \mathbf{J}_{S3} \times \mathbf{a}_z)$$

$$= \frac{\mu_0}{2}\frac{B_0}{\mu_0}[(2\mathbf{a}_y + 2\mathbf{a}_z) - (4\mathbf{a}_x - 2\mathbf{a}_z) - (2\mathbf{a}_x - 2\mathbf{a}_y)]$$

$$= B_0(-3\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)$$

P1.48.

$$\mathbf{J} = \begin{cases} J_0\mathbf{a}_z & \text{for } -a < y < 0 \\ -J_0\mathbf{a}_z & \text{for } 0 < y < a \\ \mathbf{0} & \text{otherwise} \end{cases}$$



$$\mathbf{B} = \begin{cases} \displaystyle\int_{y'=-a}^{a} \frac{\mu_0\, dy'\, \mathbf{J}}{2} \times (-\mathbf{a}_y) & \text{for } y < -a \\ \displaystyle\int_{y'=-a}^{y} \frac{\mu_0\, dy'\, \mathbf{J}}{2} \times \mathbf{a}_y + \int_{y'=y}^{a} \frac{\mu_0\, dy'\, \mathbf{J}}{2} \times (-\mathbf{a}_y) & \text{for } -a < y < a \\ \displaystyle\int_{y'=-a}^{a} \frac{\mu_0\, dy'\, \mathbf{J}}{2} \times \mathbf{a}_y & \text{for } y > a \end{cases}$$

$$= \begin{cases} \displaystyle\int_{y'=-a}^{0} \frac{\mu_0 J_0 \mathbf{a}_z \times (-\mathbf{a}_y)}{2} dy' + \int_{y'=0}^{a} \frac{-\mu_0 J_0 \mathbf{a}_z \times (-\mathbf{a}_y)}{2} dy' & \text{for } y < -a \\ \displaystyle\int_{y'=-a}^{y} \frac{\mu_0 J_0 \mathbf{a}_z \times \mathbf{a}_y}{2} dy' + \int_{y'=y}^{0} \frac{\mu_0 J_0 \mathbf{a}_z \times (-\mathbf{a}_y)}{2} dy' & \\ \qquad \displaystyle + \int_{y'=0}^{a} \frac{-\mu_0 J_0 \mathbf{a}_z \times (-\mathbf{a}_y)}{2} dy' & \text{for } -a < y < 0 \\ \displaystyle\int_{y'=-a}^{0} \frac{\mu_0 J_0 \mathbf{a}_z \times \mathbf{a}_y}{2} dy' + \int_{y'=0}^{y} \frac{-\mu_0 J_0 \mathbf{a}_z \times \mathbf{a}_y}{2} dy' & \\ \qquad \displaystyle + \int_{y'=y}^{a} \frac{-\mu_0 J_0 \mathbf{a}_z \times (-\mathbf{a}_y)}{2} dy' & \text{for } 0 < y < a \\ \displaystyle\int_{y'=-a}^{0} \frac{\mu_0 J_0 \mathbf{a}_z \times \mathbf{a}_y}{2} dy' + \int_{y'=0}^{a} \frac{-\mu_0 J_0 \mathbf{a}_z \times \mathbf{a}_y}{2} dy' & \text{for } y > a \end{cases}$$

$$= \begin{cases} \dfrac{\mu_0 J_0 \mathbf{a}_x}{2}(a - a) = \mathbf{0} & \text{for } y < -a \\ \dfrac{\mu_0 J_0 \mathbf{a}_x}{2}(-y - a - y - a) = -\mu_0 J_0 (y + a)\mathbf{a}_x & \text{for } -a < y < 0 \\ \dfrac{\mu_0 J_0 \mathbf{a}_x}{2}(-a + y - a + y) = \mu_0 J_0 (y - a)\mathbf{a}_x & \text{for } 0 < y < a \\ \dfrac{\mu_0 J_0 \mathbf{a}_x}{2}(-a + a) = \mathbf{0} & \text{for } y > a \end{cases}$$

$$\mathbf{B} = \begin{cases} \mu_0 J_0 (|y| - a)\mathbf{a}_x & \text{for } |y| \le a \\ \mathbf{0} & \text{otherwise} \end{cases}$$

P1.49. $\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_2v_2^2$

$$\frac{m_1}{m_2} = \frac{v_2^2}{v_1^2}$$

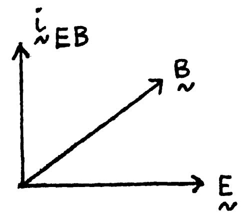
$$\frac{R_1}{R_2} = \frac{m_1v_1/qB}{m_2v_2/qB}$$

$$= \frac{m_1}{m_2}\frac{v_1}{v_2}$$

$$= \frac{m_1}{m_2}\sqrt{\frac{m_2}{m_1}}$$

$$= \sqrt{\frac{m_1}{m_2}}$$

P1.50. $q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$



$$= q\mathbf{E} + q\left(\frac{\mathbf{E} \times \mathbf{B}}{B^2}\right) \times \mathbf{B}$$

$$= q\mathbf{E} + \frac{q}{B^2}(EB\mathbf{a}_{EB} \times \mathbf{B})$$

where $\mathbf{a}_{EB}$ is unit vector perpendicular to both $\mathbf{E}$ and $\mathbf{B}$ and in the right-hand sense.

Noting that $\mathbf{a}_{EB} \times \mathbf{B} = -B\,\frac{\mathbf{E}}{E}$ and proceeding further, we have

$q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

$$= q\mathbf{E} + \frac{q}{B^2}(EB)\left(-B\,\frac{\mathbf{E}}{E}\right)$$

$$= q\mathbf{E} - q\mathbf{E}$$

$$= \mathbf{0}$$

Hence, the test charge moves with constant velocity equal to the initial value.

For $\mathbf{E} = E_0(2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$ and $\mathbf{B} = B_0(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$,

$$\mathbf{v} = \frac{E_0(2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z) \times B_0(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)}{|B_0(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)|^2}$$

$$= \frac{E_0 B_0}{9B_0^2}\,(6\mathbf{a}_x - 3\mathbf{a}_y - 6\mathbf{a}_z)$$

$$= \frac{E_0}{3B_0}\,(2\mathbf{a}_x - \mathbf{a}_y - 2\mathbf{a}_z)$$

P1.51. $q\mathbf{E} + q\mathbf{v}_1 \times \mathbf{B} = \mathbf{F}_1 \qquad (1)$

$q\mathbf{E} + q\mathbf{v}_2 \times \mathbf{B} = \mathbf{F}_2 \qquad (2)$

$q\mathbf{E} + q\mathbf{v}_3 \times \mathbf{B} = \mathbf{F}_3 \qquad (3)$

$(2) - (1) \rightarrow q(\mathbf{v}_2 - \mathbf{v}_1) \times \mathbf{B} = \mathbf{F}_2 - \mathbf{F}_1 \qquad (4)$

$(3) - (1) \rightarrow q(\mathbf{v}_3 - \mathbf{v}_1) \times \mathbf{B} = \mathbf{F}_3 - \mathbf{F}_1 \qquad (5)$

From (4) and (5), $\mathbf{B}$ is perpendicular to both $(\mathbf{F}_2 - \mathbf{F}_1)$ and $(\mathbf{F}_3 - \mathbf{F}_1)$. Hence we can write

$$\begin{aligned} \mathbf{B} &= k(\mathbf{F}_2 - \mathbf{F}_1) \times (\mathbf{F}_3 - \mathbf{F}_1) \\ &= k(\mathbf{F}_1 \times \mathbf{F}_2 + \mathbf{F}_2 \times \mathbf{F}_3 + \mathbf{F}_3 \times \mathbf{F}_1) \\ &= k\mathbf{A} \end{aligned}$$

provided $(\mathbf{F}_2 - \mathbf{F}_1)$ and $(\mathbf{F}_3 - \mathbf{F}_1)$ are not collinear, that is, $\mathbf{A} \neq \mathbf{0}$. Substituting in (4), we get

$$q(\mathbf{v}_2 - \mathbf{v}_1) \times k\mathbf{A} = (\mathbf{F}_2 - \mathbf{F}_1)$$

$$k = \frac{1}{q} \frac{\mathbf{F}_2 - \mathbf{F}_1}{(\mathbf{v}_2 - \mathbf{v}_1) \times \mathbf{A}}$$

Thus

$$\mathbf{B} = \frac{1}{q}\left[\frac{\mathbf{F}_2 - \mathbf{F}_1}{(\mathbf{v}_2 - \mathbf{v}_1) \times \mathbf{A}}\right]\mathbf{A}$$

P1.52. From Lorentz force equation,

$$q\mathbf{E} + qv_0\mathbf{a}_x \times \mathbf{B} = qE_0(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z) \qquad (1)$$

$$q\mathbf{E} + qv_0\mathbf{a}_y \times \mathbf{B} = qE_0(\mathbf{a}_x - \mathbf{a}_y - \mathbf{a}_z) \qquad (2)$$

$$q\mathbf{E} + qv_0\mathbf{a}_z \times \mathbf{B} = \mathbf{0} \qquad (3)$$

(2) – (1) gives

$$v_0(\mathbf{a}_y - \mathbf{a}_x) \times \mathbf{B} = -2E_0\mathbf{a}_z \qquad (4)$$

(2) – (3) gives

$$v_0(\mathbf{a}_y - \mathbf{a}_z) \times \mathbf{B} = E_0(\mathbf{a}_x - \mathbf{a}_y - \mathbf{a}_z) \qquad (5)$$

From (4) and (5),

$$\mathbf{B} = C\mathbf{a}_z \times (\mathbf{a}_x - \mathbf{a}_y - \mathbf{a}_z) = C(\mathbf{a}_x + \mathbf{a}_y) \qquad (6)$$

where C is a constant. Substituting (6) in (4), we obtain

$$v_0C(\mathbf{a}_y - \mathbf{a}_x) \times (\mathbf{a}_x + \mathbf{a}_y) = -2E_0\mathbf{a}_z$$

or $\quad C = E_0/v_0$

$$\therefore \mathbf{B} = \frac{E_0}{v_0}(\mathbf{a}_x + \mathbf{a}_y)$$

Then from (3),

$$\mathbf{E} = -v_0\mathbf{a}_z \times \mathbf{B}$$

$$= E_0(\mathbf{a}_x - \mathbf{a}_y)$$

P1.53. From Lorentz force equation,

$$\mathbf{F}_1 = q\mathbf{E} + qv_0\mathbf{a}_x \times \mathbf{B} = \mathbf{0} \tag{1}$$

$$\mathbf{F}_2 = q\mathbf{E} + qv_0\mathbf{a}_y \times \mathbf{B} = \mathbf{0} \tag{2}$$

$$\mathbf{F}_3 = q\mathbf{E} + qv_0(\mathbf{a}_x + 2\mathbf{a}_y) \times \mathbf{B} = qE_0\mathbf{a}_z \tag{3}$$

(a) $(1) + 2 \times (2) - (3) \rightarrow 2q\mathbf{E} = -qE_0\mathbf{a}_z$

$$\therefore \mathbf{F}_4 = q\mathbf{E} = -\frac{qE_0}{2}\,\mathbf{a}_z$$

(b)
$$\begin{aligned}
\mathbf{F}_5 &= q\mathbf{E} + qv_0(\mathbf{a}_x + \mathbf{a}_y) \times \mathbf{B} \\
&= (q\mathbf{E} + qv_0\mathbf{a}_x \times \mathbf{B}) + (q\mathbf{E} + qv_0\mathbf{a}_y \times \mathbf{B}) - q\mathbf{E} \\
&= \mathbf{F}_1 + \mathbf{F}_2 - \mathbf{F}_4 \\
&= \frac{qE_0}{2}\,\mathbf{a}_z
\end{aligned}$$

(c)
$$\begin{aligned}
\mathbf{F}_6 &= q\mathbf{E} + q\,\frac{v_0}{4}\,(3\mathbf{a}_x + \mathbf{a}_y) \\
&= \frac{3}{4}\,(q\mathbf{E} + qv_0\mathbf{a}_x \times \mathbf{B}) + \frac{1}{4}\,(q\mathbf{E} + qv_0\mathbf{a}_y \times \mathbf{B}) \\
&= \frac{3}{4}\,\mathbf{F}_1 + \frac{1}{4}\,\mathbf{F}_2 \\
&= \mathbf{0}
\end{aligned}$$

P1.54. Constant velocity equal to initial velocity means no acceleration and hence **F** is zero. Thus

$$q\mathbf{E} + q\mathbf{v}_1 \times \mathbf{B} = \mathbf{0} \qquad (1)$$

$$q\mathbf{E} + q\mathbf{v}_2 \times \mathbf{B} = \mathbf{0} \qquad (2)$$

Multiplying Eq. (1) by m and Eq. (2) by n and adding them and then dividing by $(m + n)$, provided $(m + n) \neq 0$, we have

$$\frac{(m+n)q\mathbf{E} + q(m\mathbf{v}_1 \times \mathbf{B} + n\mathbf{v}_2 \times \mathbf{B})}{m+n} = \mathbf{0}$$

or $$q\mathbf{E} + q\,\frac{m\mathbf{v}_1 + n\mathbf{v}_2}{m+n} \times \mathbf{B} = \mathbf{0}$$

Thus when released with initial velocity $\dfrac{m\mathbf{v}_1 + n\mathbf{v}_2}{m+n}$, the test charge moves with constant velocity equal to that value.

R1.1. If $\mathbf{A} \times \mathbf{F} = \mathbf{C}$ and $\mathbf{B} \times \mathbf{F} = \mathbf{D}$, then $\mathbf{F}$ is perpendicular to both $\mathbf{C}$ and $\mathbf{D}$. Therefore, let $\mathbf{F} = m\mathbf{C} \times \mathbf{D}$. Substituting then in one of the equations, we have

$$m\mathbf{A} \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}$$
$$m[(\mathbf{A} \bullet \mathbf{D})\mathbf{C} - (\mathbf{A} \bullet \mathbf{C})\mathbf{D}] = \mathbf{C}$$

Since $\mathbf{A} \bullet \mathbf{C} = 0$, $m = \dfrac{1}{\mathbf{A} \bullet \mathbf{D}}$

$$\therefore\ \mathbf{F} = \frac{\mathbf{C} \times \mathbf{D}}{\mathbf{A} \bullet \mathbf{D}}$$

Similarly, using the second equation,

$$m\mathbf{B} \times (\mathbf{C} \times \mathbf{D}) = \mathbf{D}$$
$$m[(\mathbf{B} \bullet \mathbf{D})\mathbf{C} - (\mathbf{B} \bullet \mathbf{C})\mathbf{D}] = \mathbf{D}$$

Since $\mathbf{B} \bullet \mathbf{D} = 0$, $m = -\dfrac{1}{\mathbf{B} \bullet \mathbf{C}}$

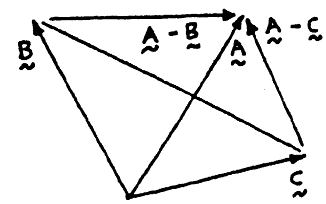
$$\therefore\ \mathbf{F} = -\frac{\mathbf{C} \times \mathbf{D}}{\mathbf{B} \bullet \mathbf{C}}$$

For $\mathbf{A} = (\mathbf{a}_x + \mathbf{a}_y)$, $\mathbf{B} = (\mathbf{a}_x + 2\mathbf{a}_y - 2\mathbf{a}_z)$, $\mathbf{C} = (\mathbf{a}_x - \mathbf{a}_y)$, and $\mathbf{D} = (6\mathbf{a}_x - 5\mathbf{a}_y - 2\mathbf{a}_z)$,

$$\mathbf{C} \times \mathbf{D} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 1 & -1 & 0 \\ 6 & -5 & -2 \end{vmatrix} = 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$$

$$\mathbf{A} \bullet \mathbf{D} = 1(6) + 1(-5) = 1$$
$$\mathbf{B} \bullet \mathbf{C} = 1(1) + 2(-1) = -1$$
$$\mathbf{F} = 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$$

R1.2. (a) From the construction shown, it can be seen that $(\mathbf{A}-\mathbf{B})$ and $(\mathbf{A}-\mathbf{C})$ are two vectors in the plane determined by the tips of **A**, **B**, and **C**.



$\therefore$ Unit vector normal to the plane is

$$\mathbf{a}_n = \frac{(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})}{|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})|}$$

But
$$\begin{aligned}(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C}) &= \mathbf{A}\times\mathbf{A}-\mathbf{B}\times\mathbf{A}-\mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{C}\\ &= \mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\end{aligned}$$

Shortest distance from the common point to the plane

= Perpendicular distance from the common point to the plane

$$= |\mathbf{A}\bullet\mathbf{a}_n| = \left|\mathbf{A}\bullet\frac{\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}}{|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}|}\right|$$

$$= \left|\frac{\mathbf{A}\bullet\mathbf{A}\times\mathbf{B}+\mathbf{A}\bullet\mathbf{B}\times\mathbf{C}+\mathbf{A}\bullet\mathbf{C}\times\mathbf{A}}{|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}|}\right|$$

$$= \frac{|\mathbf{A}\bullet\mathbf{B}\times\mathbf{C}|}{|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}|}$$

(b) $(\sqrt{3}, 3, 2) \rightarrow (\sqrt{12}, \pi/3, 2)$ in cylindrical coordinates
$\rightarrow (4, \pi/3, \pi/3)$ in spherical coordinates

$$\mathbf{B} = \frac{1}{2}(\mathbf{a}_r - \sqrt{3}\mathbf{a}_\phi)$$

$$= \frac{1}{2}\left[\left(\cos\frac{\pi}{3}\,\mathbf{a}_x + \sin\frac{\pi}{3}\,\mathbf{a}_y\right) - \sqrt{3}\left(-\sin\frac{\pi}{3}\,\mathbf{a}_x + \cos\frac{\pi}{3}\,\mathbf{a}_y\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}+\frac{3}{2}\right)\mathbf{a}_x + \left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)\mathbf{a}_y\right]$$

$$= \mathbf{a}_x$$

$$\mathbf{C} = \frac{1}{4}(3\mathbf{a}_r - \sqrt{3}\mathbf{a}_\theta + 2\mathbf{a}_\phi)$$

R1.2. (continued)

$$= \frac{1}{4}\left[3\left(\sin\frac{\pi}{3}\cos\frac{\pi}{3}\,\mathbf{a}_x + \sin\frac{\pi}{3}\sin\frac{\pi}{3}\,\mathbf{a}_y + \cos\frac{\pi}{3}\,\mathbf{a}_z\right)\right.$$

$$-\sqrt{3}\left(\cos\frac{\pi}{3}\cos\frac{\pi}{3}\,\mathbf{a}_x + \cos\frac{\pi}{3}\sin\frac{\pi}{3}\,\mathbf{a}_y - \sin\frac{\pi}{3}\,\mathbf{a}_z\right)$$

$$\left.+2\left(-\sin\frac{\pi}{3}\,\mathbf{a}_x + \cos\frac{\pi}{3}\,\mathbf{a}_y\right)\right]$$

$$= \frac{1}{4}\left[3\left(\frac{\sqrt{3}}{4}\,\mathbf{a}_x + \frac{3}{4}\,\mathbf{a}_y + \frac{1}{2}\,\mathbf{a}_z\right)\right.$$

$$+\sqrt{3}\left(\frac{1}{4}\,\mathbf{a}_x + \frac{\sqrt{3}}{4}\,\mathbf{a}_y - \frac{\sqrt{3}}{2}\,\mathbf{a}_z\right)$$

$$\left.+2\left(-\frac{\sqrt{3}}{2}\,\mathbf{a}_x + \frac{1}{2}\,\mathbf{a}_y\right)\right]$$

$$= \mathbf{a}_y$$

$\therefore$ Shortest distance from $(\sqrt{3}, 3, 2)$ to the plane determined by the tips of **A**, **B**, and **C**

$$= \frac{|2\mathbf{a}_z \bullet \mathbf{a}_x \times \mathbf{a}_y|}{|2\mathbf{a}_z \times \mathbf{a}_x + \mathbf{a}_x \times \mathbf{a}_y + \mathbf{a}_y \times 2\mathbf{a}_z|}$$

$$= \frac{2}{|2\mathbf{a}_y + \mathbf{a}_z + 2\mathbf{a}_x|}$$

$$= \frac{2}{3}$$

R1.3. Let us assume the center of the sphere of radius unity to be at the origin, one of the vertices of the equilateral tetrahedron to be on the z-axis, one of the other three vertices to be in the xz-plane, and the angle subtended at the center of the tetrahedron (the origin) by one of its edges to be α. Then the vertices of the tetrahedron are given in spherical coordinates by $(1, 0, 0)$, $(1, \alpha, 0)$, $(1, \alpha, 2\pi/3)$, and $(1, \alpha, 4\pi/3)$. In Cartesian coordinates, these are $(0, 0, 1)$, $(\sin\alpha, 0, \cos\alpha)$, $(-1/2 \sin\alpha, \sqrt{3}/2 \sin\alpha, \cos\alpha)$, and $(-1/2 \sin\alpha, -\sqrt{3}/2 \sin\alpha, \cos\alpha)$. For an equilateral tetrahedron, all edges are equal in length. Therefore, setting the distance from $(0, 0, 1)$ to $(\sin\alpha, 0, \cos\alpha)$ equal to the distance from $(-1/2 \sin\alpha, \sqrt{3}/2 \sin\alpha, \cos\alpha)$ to $(\sin\alpha, 0, \cos\alpha)$, we have

$$|\sin\alpha\, \mathbf{a}_x + (\cos\alpha - 1)\mathbf{a}_z| = \left|\frac{3}{2}\sin\alpha\, \mathbf{a}_x - \frac{\sqrt{3}}{2}\sin\alpha\, \mathbf{a}_y\right|$$

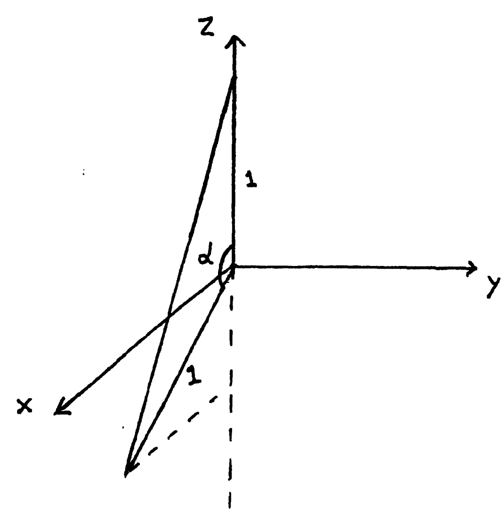
$$\sin^2\alpha + \cos^2\alpha + 1 - 2\cos\alpha = \frac{9}{4}\sin^2\alpha + \frac{3}{4}\sin^2\alpha$$

$$2 - 2\cos\alpha = 3(1 - \cos^2\alpha)$$

$$3\cos^2\alpha - 2\cos\alpha - 1 = 0$$

$$\cos\alpha = \frac{2 \pm \sqrt{4+12}}{6} = 1 \text{ or } -\frac{1}{3}$$

Ruling out 1, we get $\cos\alpha = -\dfrac{1}{3}$



$$\text{Edge length } = \sqrt{2 - 2\cos\alpha}$$

$$= \sqrt{2 + \frac{2}{3}} = \sqrt{\frac{8}{3}}$$

Radius of the largest sphere that can be fit inside the tetrahedron
$= $ Distance from the origin to one of the sides of the tetrahedron
$= 1 \cos(180^\circ - \alpha)$
$= -\cos\alpha$
$= \dfrac{1}{3}$

R1.3. (continued)

Note that the expression in part (a) of Problem R1.2 can also be used by considering three vectors from the origin to three of the vertices, say, (1, 0, 0), (1, α, 0), and (1, α, $2\pi/3$), of the tetrahedron.

Thus, let

$$\mathbf{A} = \mathbf{a}_z$$

$$\mathbf{B} = \sin\alpha\,\mathbf{a}_x + \cos\alpha\,\mathbf{a}_z = \frac{\sqrt{8}}{3}\,\mathbf{a}_x - \frac{1}{3}\,\mathbf{a}_z$$

$$\mathbf{C} = -\frac{1}{2}\sin\alpha\,\mathbf{a}_x + \frac{\sqrt{3}}{2}\sin\alpha\,\mathbf{a}_y + \cos\alpha\,\mathbf{a}_z$$

$$= -\frac{\sqrt{2}}{3}\mathbf{a}_x + \frac{\sqrt{6}}{3}\mathbf{a}_y - \frac{1}{3}\mathbf{a}_z$$

Then

$$\mathbf{A} \times \mathbf{B} = \frac{\sqrt{8}}{3}\,\mathbf{a}_y$$

$$\mathbf{B} \times \mathbf{C} = \frac{\sqrt{6}}{9}\mathbf{a}_x + \left(\frac{\sqrt{2}}{9} + \frac{\sqrt{8}}{9}\right)\mathbf{a}_y + \frac{\sqrt{48}}{9}\mathbf{a}_z$$

$$\mathbf{C} \times \mathbf{A} = \frac{\sqrt{6}}{3}\mathbf{a}_x + \frac{\sqrt{2}}{3}\mathbf{a}_y$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \frac{\sqrt{48}}{9} = 4\frac{\sqrt{3}}{9}$$

$$|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}|$$

$$= \sqrt{\frac{96}{81} + \frac{288}{81} + \frac{48}{81}}$$

$$= 12\,\frac{\sqrt{3}}{9}$$

$$\frac{|\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}|}{|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}|} = \frac{1}{3}$$

same as the answer obtained above.

R1.4. (a) The direction of movement of the observer is always along $\mathbf{a}_\theta + \mathbf{a}_\phi$, because $\mathbf{a}_\theta$ is southward and $\mathbf{a}_\phi$ is eastward, and hence $\mathbf{a}_\theta + \mathbf{a}_\phi$, which bisects $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ is southeastward. Therefore, the direction line is given by

$$\frac{a\,d\theta}{1} = \frac{a \sin\theta\, d\phi}{1}$$

$$\frac{d\theta}{\sin\theta} = d\phi$$

$$\ln \tan\frac{\theta}{2} = \phi + C$$

Since the starting point is $\left(a, \frac{\pi}{2}, 0\right)$,

$$\ln \tan\frac{\pi}{4} = 0 + C, \text{ or, } C = 0$$

$\therefore$ The direction line is given by

$$\ln \tan\frac{\theta}{2} = \phi\,,\ r = a$$

(b) $\theta = 2\tan^{-1} e^{\phi}$

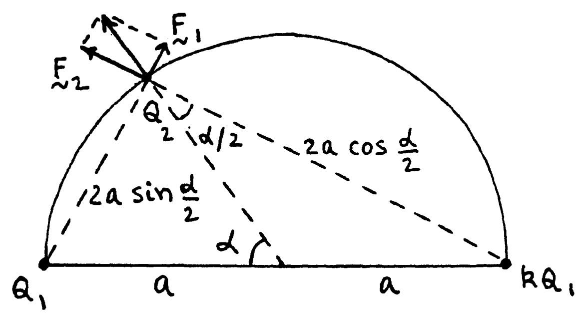
For $\phi = 2\pi$, $\theta = 2\tan^{-1} e^{2\pi} = 179.786°$

For $\phi = 4\pi$, $\theta = 2\tan^{-1} e^{4\pi} = 179.9996°$

$\therefore$ The first two values of the south latitude are 89.786° and 89.9996°.

(c) The observer will never reach the south pole because for θ to be 180°, $\phi = \ln\tan\frac{\pi}{2} = \infty$.

R1.5.



(a) For equilibrium, the force on Q_2 must be along the line from the center of the circle to Q_2. Thus

$$\tan\frac{\alpha}{2} = \frac{F_1}{F_2} = \frac{Q_1Q_2\Big/\left[4\pi\varepsilon_0\left(2a\sin\frac{\alpha}{2}\right)^2\right]}{kQ_1Q_2\Big/\left[4\pi\varepsilon_0\left(2a\cos\frac{\alpha}{2}\right)^2\right]}$$

$$= \frac{1}{k\tan^2\frac{\alpha}{2}}$$

$$\tan^3\frac{\alpha}{2} = \frac{1}{k}$$

$$\alpha = 2\tan^{-1}\left(\frac{1}{k^{1/3}}\right)$$

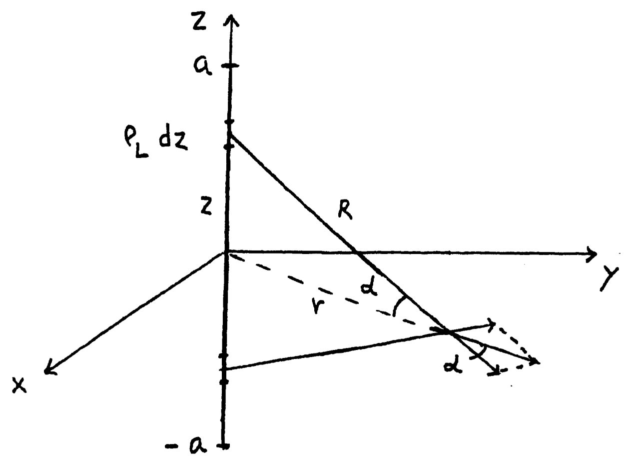
(b) For $k = 8$,

$$\alpha = 2\tan^{-1}\left(\frac{1}{8^{1/3}}\right)$$

$$= 2\tan^{-1}\frac{1}{2}$$

$$= 53.13^\circ$$

R1.6. (a)



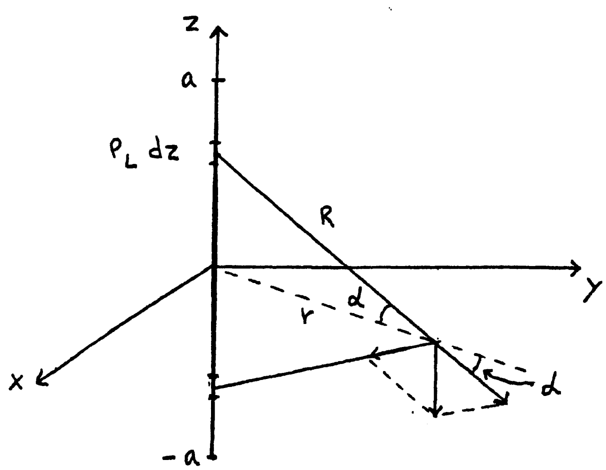
For an even function, $\rho_L(-z) = \rho_L(z)$. Therefore for any two infinitesimal segments of the line charge equidistant and on either side of the origin, the electric field intensities at $(r, \phi, 0)$ are equal in magnitude and directed as shown in the figure making equal angles with the xy-plane, so that the resultant has only an r-component. Thus

$$\mathbf{E}(r, \phi, 0) = E_r(r, \phi, 0)\mathbf{a}_r$$

where

$$\begin{aligned} E_r(r, \phi, 0) &= 2\int_{z=0}^{a} \frac{\rho_L(z)\,dz}{4\pi\varepsilon_0 R^2}\cos\alpha \\ &= \frac{1}{2\pi\varepsilon_0}\int_{z=0}^{a} \frac{\rho_L(z)\,r}{(z^2+r^2)^{3/2}}\,dz \end{aligned}$$

(b)



R1.6. (continued)

For an odd function, $\rho_L(-z) = -\rho_L(z)$. The two electric field intensities, which are equal in magnitude, now make equal angles with the vertical as shown in the figure, so that the resultant has only a z-component. Thus

$$\mathbf{E}(r, \phi, 0) = E_z(r, \phi, 0)\mathbf{a}_z$$

where

$$\begin{aligned} E_z(r, \phi, 0) &= -2\int_{z=0}^{a} \frac{\rho_L(z)\, dz}{4\pi\varepsilon_0 R^2} \sin\alpha \\ &= -\frac{1}{2\pi\varepsilon_0}\int_{z=0}^{a} \frac{\rho_L(z)\, z}{(z^2 + r^2)^{3/2}}\, dz \end{aligned}$$

(c)

$$f(z) = f_1(z) + f_2(z) \qquad (1)$$

$$\begin{aligned} f(-z) &= f_1(-z) + f_2(-z) \\ &= f_1(z) - f_2(z) \qquad (2) \end{aligned}$$

From (1) and (2), we get

$$f_1(z) = \frac{1}{2}\,[\, f(z) + f(-z)]$$

$$f_2(z) = \frac{1}{2}\,[\, f(z) - f(-z)]$$

For

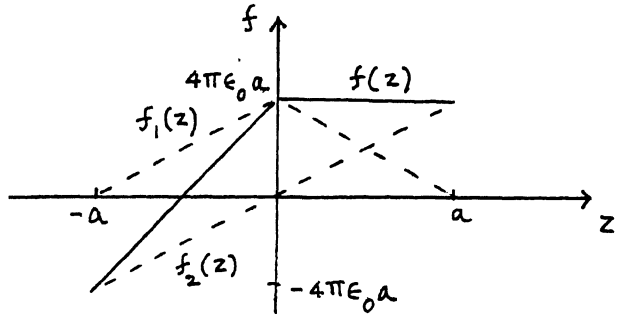
$$f(z) = \begin{cases} 4\pi\varepsilon_0(a + 2z) & \text{for } -a \le z \le 0 \\ 4\pi\varepsilon_0 a & \text{for } 0 \le z \le a \end{cases}$$

$$f(-z) = \begin{cases} 4\pi\varepsilon_0 a & \text{for } -a \le z \le 0 \\ 4\pi\varepsilon_0(a - 2z) & \text{for } 0 \le z \le a \end{cases}$$

$$f_1(z) = \begin{cases} 4\pi\varepsilon_0(a + z) & \text{for } -a \le z < 0 \\ 4\pi\varepsilon_0(a - z) & \text{for } 0 \le z \le a \end{cases}$$

$$f_2(z) = 4\pi\varepsilon_0 z \qquad \text{for } -a \le z \le a$$

R1.6. (continued)



$$E_r(r, \phi, 0) = \frac{1}{2\pi\varepsilon_0}\int_{z=0}^{a} \frac{4\pi\varepsilon_0(a-z)r}{(z^2+r^2)^{3/2}}\,dz$$

$$= 2r\left[\frac{az}{r^2\sqrt{z^2+r^2}} + \frac{1}{\sqrt{z^2+r^2}}\right]_{z=0}^{a}$$

$$= 2r\left[\frac{a^2}{r^2\sqrt{a^2+r^2}} + \frac{1}{\sqrt{a^2+r^2}} - \frac{1}{r}\right]$$

$$= 2\left(\frac{\sqrt{a^2+r^2}}{r} - 1\right)$$

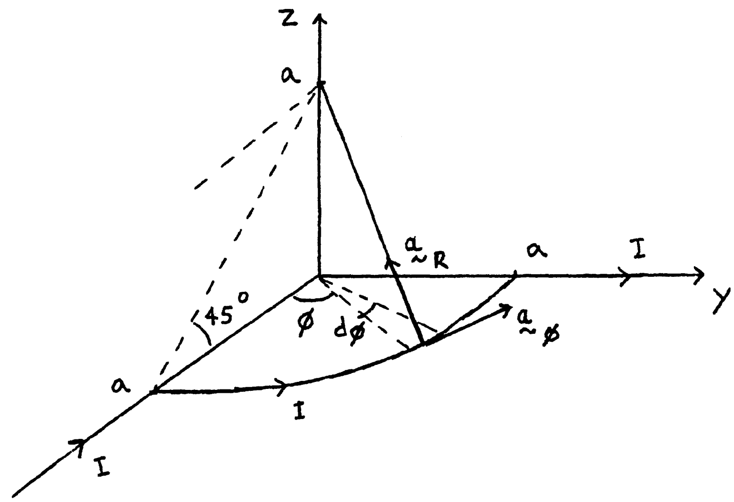
$$E_z(r, \phi, 0) = -\frac{1}{2\pi\varepsilon_0}\int_{z=0}^{a} \frac{4\pi\varepsilon_0 z^2}{(z^2+r^2)^{3/2}}\,dz$$

$$= -2\left[-\frac{z}{\sqrt{z^2+r^2}} + \ln\left(z+\sqrt{z^2+r^2}\right)\right]_{z=0}^{a}$$

$$= -2\left[-\frac{a}{\sqrt{a^2+r^2}} + \ln\left(a+\sqrt{a^2+r^2}\right) - \ln r\right]$$

$$= 2\left(\frac{a}{\sqrt{a^2+r^2}} - \ln\frac{a+\sqrt{a^2+r^2}}{r}\right)$$

R1.7.



We find contributions to the three segments separately, and add them.

For the segment from ∞ to a on the x-axis,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi a}(\cos 0° - \cos 45°)\,\mathbf{a}_y$$

$$= \frac{0.2929\mu_0 I}{4\pi a}\,\mathbf{a}_y$$

For the circular segment,

$$\int_{\phi=0}^{\pi/2} \frac{\mu_0 Ia\, d\phi\, \mathbf{a}_\phi \times \mathbf{a}_R}{4\pi(\sqrt{2}a)^2}$$

$$= \frac{\mu_0 Ia}{8\pi a^2}\int_{\phi=0}^{\pi/2} d\phi\,(-\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y) \times \frac{1}{\sqrt{2}}(\cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y + \mathbf{a}_z)$$

$$= \frac{\mu_0 I}{8\sqrt{2}\pi a}\int_{\phi=0}^{\pi/2}(\cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y - \mathbf{a}_z)\,d\phi$$

$$= \frac{0.3536\mu_0 I}{4\pi a}\left(\mathbf{a}_x + \mathbf{a}_y - \frac{\pi}{2}\,\mathbf{a}_z\right)$$

For the segment from a to ∞ on the y-axis,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi a}(\cos 135° - \cos 180°)\,\mathbf{a}_x$$

$$= \frac{0.2929\mu_0 I}{4\pi a}\,\mathbf{a}_x$$

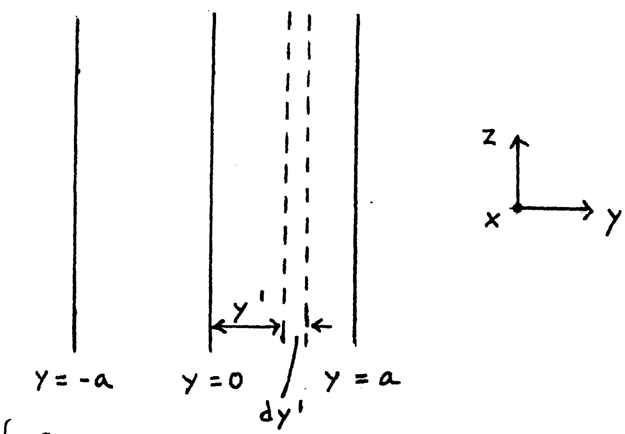
R1.7. (continued)

Thus, for the entire wire,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi a}\left(0.9394\mathbf{a}_x + 0.9394\mathbf{a}_y - \frac{\pi}{2}\mathbf{a}_z\right)$$

R1.8.

$$\mathbf{J} = \begin{cases} J_0(y/a)\,\mathbf{a}_z & \text{for } -a < 0 < a \\ \mathbf{0} & \text{otherwise} \end{cases}$$



$$\mathbf{B} = \begin{cases} \displaystyle\int_{y'=-a}^{a} \frac{\mu_0 J_0 y'\,dy'}{2a}\,\mathbf{a}_z \times (-\mathbf{a}_y) & \text{for } y < -a \\ \displaystyle\int_{y'=-a}^{y} \frac{\mu_0 J_0 y'\,dy'}{2a}\,\mathbf{a}_z \times (\mathbf{a}_y) + \int_{y'=y}^{a} \frac{\mu_0 J_0 y'\,dy'}{2a}\,\mathbf{a}_z \times (-\mathbf{a}_y) & \text{for } -a < y < a \\ \displaystyle\int_{y'=-a}^{a} \frac{\mu_0 J_0 y'\,dy'}{2a}\,\mathbf{a}_z \times (\mathbf{a}_y) & \text{for } y > a \end{cases}$$

$$= \begin{cases} \dfrac{\mu_0 J_0}{2a}\left[\dfrac{(y')^2}{2}\right]_{-a}^{a} \mathbf{a}_x & \text{for } y < -a \\ \dfrac{\mu_0 J_0}{2a}\left[\dfrac{(y')^2}{2}\right]_{-a}^{y} (-\mathbf{a}_x) + \left[\dfrac{(y')^2}{2}\right]_{y}^{a} \mathbf{a}_x & \text{for } -a < y < a \\ \dfrac{\mu_0 J_0}{2a}\left[\dfrac{(y')^2}{2}\right]_{-a}^{a} (-\mathbf{a}_x) & \text{for } y > a \end{cases}$$

$$= \begin{cases} \mathbf{0} & \text{for } y < -a \\ \dfrac{\mu_0 J_0}{2a}(a^2 - y^2)\,\mathbf{a}_x & \text{for } -a < y < a \\ \mathbf{0} & \text{for } y > a \end{cases}$$

R1.9. For the test charge to move with constant velocity,

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = \mathbf{0}$$

$$\therefore\ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}$$

Substituting $\mathbf{E} = E_0(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$, $\mathbf{v} = v_0(2\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z)$, and $\mathbf{B} = B_{x0}\mathbf{a}_x + B_{y0}\mathbf{a}_y + B_{z0}\mathbf{a}_z$, we have

$$E_0(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) + v_0 \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -2 & 1 \\ B_{x0} & B_{y0} & B_{z0} \end{vmatrix} = \mathbf{0}$$

or,

$$2E_0 + v_0(-2B_{z0} - B_{y0}) = 0$$

$$E_0 + v_0(B_{x0} - 2B_{z0}) = 0$$

$$-2E_0 + v_0(2B_{y0} + 2B_{x0}) = 0$$

or,

$$B_{y0} + 2B_{z0} = \frac{2E_0}{v_0} \qquad (1)$$

$$B_{x0} - 2B_{z0} = -\frac{E_0}{v_0} \qquad (2)$$

$$B_{x0} + B_{y0} = \frac{E_0}{v_0} \qquad (3)$$

We note, however, Eq. (3) is not independent of Eqs. (1) and (2), since adding (1) and (2) gives (3). Therefore, the given information is not sufficient to find uniquely B_{x0}, B_{y0} and B_{z0}, and we need another independent equation, which is provided by the additional condition that **v** is perpendicular to **B**. Thus

$$\mathbf{v} \bullet \mathbf{B} = v_0(2\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z) \bullet (B_{x0}\mathbf{a}_x + B_{y0}\mathbf{a}_y + B_{z0}\mathbf{a}_z) = 0$$

or,

$$2B_{x0} - 2B_{y0} + B_{z0} = 0 \qquad (4)$$

Solving (1), (2), and (4), we obtain

$$B_{x0} = \frac{E_0}{3v_0}$$

$$B_{y0} = \frac{2E_0}{3v_0}$$

$$B_{z0} = \frac{2E_0}{3v_0}$$