CHAPTER 2: ANALYTIC GEOMETRY OF SPACE, VECTOR FUNCTIONS

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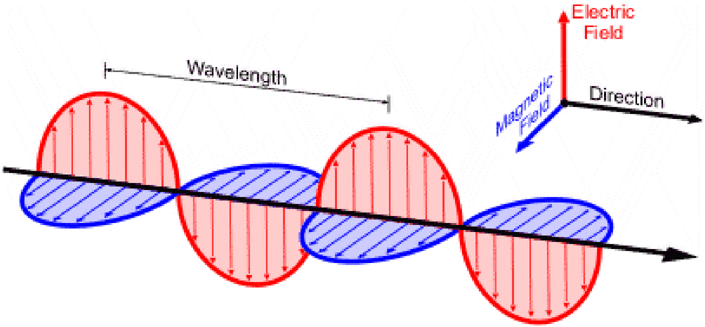

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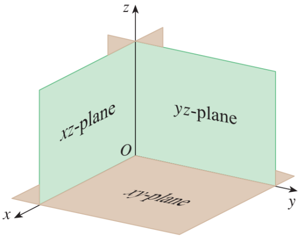
Three-Dimensional (3D) Coordinate

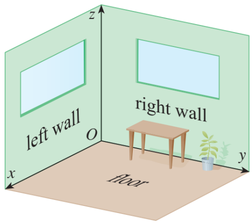
- We will introduce vectors and coordinate systems for 3D space. This will be the setting for our study of the calculus of functions of two variables

- We will see that vectors provide particularly simple descriptions of lines and planes in space.

- Reference for Chapter 2: Chapters 12-13 of the textbook by J. Stewart.



(a) Coordinate planes

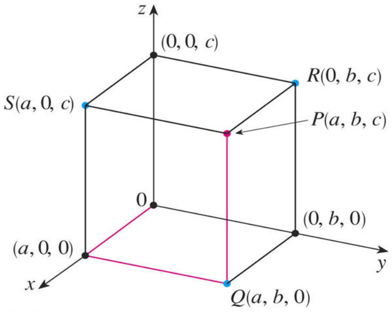


(b)

Three-Dimensional (3D) Coordinate

- The Cartesian **coordinates** (a, b, c) of a point $P(a, b, c)$ in space are the numbers at which the planes through P perpendicular to the axes cut the axes. The value a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate.

- If we drop a perpendicular from $P(a, b, c)$ to the xy-plane, we get a point $Q(a, b, 0)$ called the **projection** of P on the xy-plane.

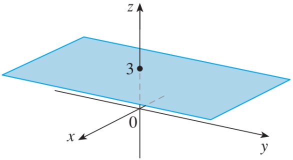


Three-Dimensional (3D) Coordinate

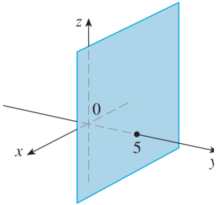
- The Cartesian product
$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) |\ x, y, z \in \mathbb{R}\}$$
is denoted by $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**.

- In three-dimensional analytic geometry, an equation in x, y, and z represents a **surface** in $\mathbb{R}^3$.

- The equation $z = 3$ represents the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3. The right figure is the plane $y = 5$.



(a) $z = 3$, a plane in $\mathbb{R}^3$



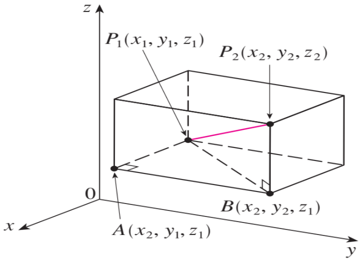
(b) $y = 5$, a plane in $\mathbb{R}^3$

Three-Dimensional (3D) Coordinate

Distance between two points

The distance $|P_1P_2|$ *between* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Example The distance between $P_1(2, -1, 7)$ and $P_2(1, -3, 5)$ is

$$|P_1P_2| = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2} = 3.$$

Three-Dimensional (3D) Coordinate

Example An equation of a sphere with center $C(a, b, c)$ and radius r is

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2}$$

In particular, if the center is the origin O, then an equation of the sphere is $\boxed{x^2 + y^2 + z^2 = r^2}$

Example Show that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

is the equation of a sphere, and find its center and radius.

Solution We have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$
$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4}$$

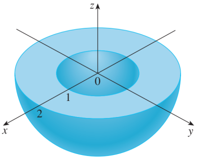
It is the equation of a sphere with center $(-3/2, 0, 2)$ and radius $\sqrt{21}/2$.

Three-Dimensional (3D) Coordinate

Example

Equations/inequalities	Description
a) $x^2 + y^2 + z^2 \leq 4$	The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$.
b) $x^2 + y^2 + z^2 = 4$, $z \leq 0$	The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane.

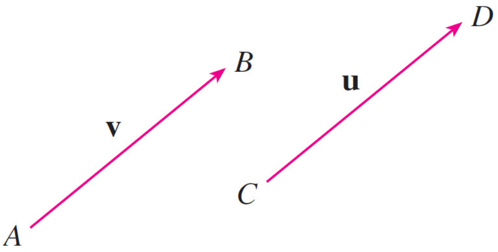
Example What region in $\mathbb{R}^3$ is represented by $1 \leq x^2 + y^2 + z^2 \leq 4$, $z \leq 0$?



Answer: Between (or on) the spheres and beneath (or on) the $xy-$plane.

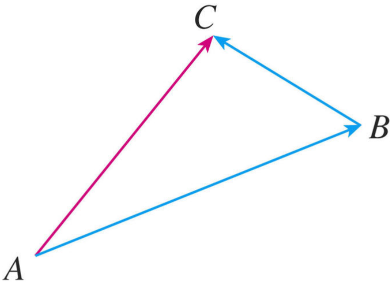
Combining Vectors

- The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.
- A vector is often represented by an arrow or a directed line segment.
- For example, a particle moves along a line segment from point A to point B. One can describe this moving by the **displacement vector** $\mathbf{v} = \overrightarrow{AB}$.



Combining Vectors

- Two vectors are **equivalent** or **equal** if they have the same length and direction.
- The zero vector, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.
- Suppose a particle moves from A to B, and changes direction and moves from B to C. The resulting displacement vector $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

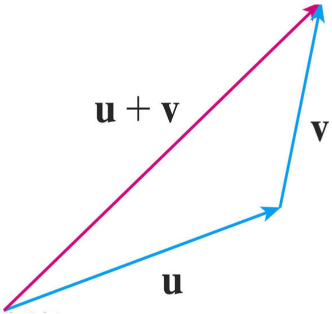


Combining Vectors

Definition

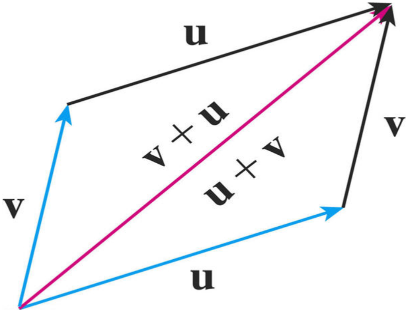
If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

This definition is sometimes called the **Triangle Law**.



Combining Vectors

If we place $\mathbf{u}$ and $\mathbf{v}$ so they start at the same point, then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as sides. This is called the **Parallelogram Law**.

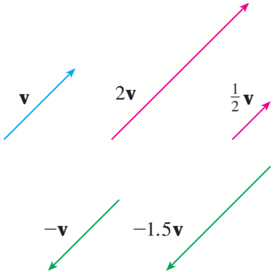


The Parallelogram Law

Combining Vectors

Definition

If c is a scalar (a real number) and $\mathbf{v}$ is a vector, then scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.



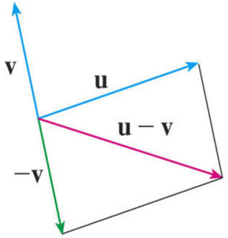
Note: Two nonzero vectors are **parallel** if they are scalar multiples of one another. Also, we call $-\mathbf{v}$ the **negative** of $\mathbf{v}$.

Combining Vectors

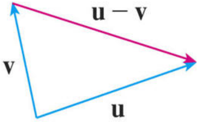
By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

So we can construct $\mathbf{u} - \mathbf{v}$ by first drawing the negative of $\mathbf{v}$, $-\mathbf{v}$, and then adding it to $\mathbf{u}$ by the Parallelogram Law (Fig. (a) below). Alternatively, since $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$ the vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$. So we could construct $\mathbf{u} - \mathbf{v}$ by means of the Triangle Law as in Fig. (b).



(a)



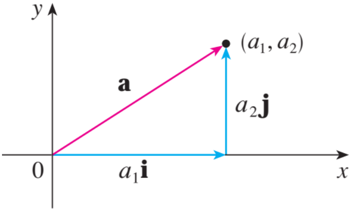
(b)

Components of a vector

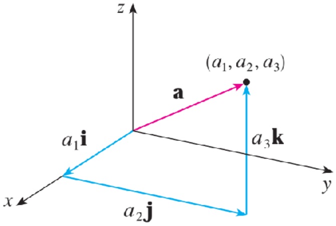
- If we place the initial point of a vector **a** at the origin, then the terminal point of **a** has coordinates of the form $(a_1, a_2) \in \mathbb{R}^2$ or $(a_1, a_2, a_3) \in \mathbb{R}^3$.
- These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle.$$

- The vector $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are the **basic vectors**.

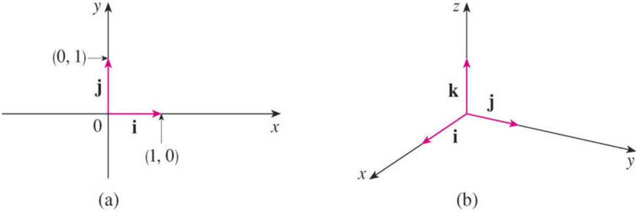


(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$

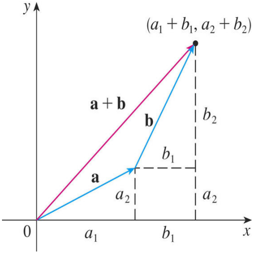


(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Components of a vector



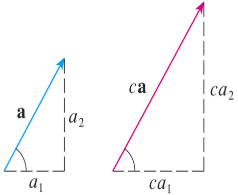
If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive. So



Components of a vector

Similarly, *to subtract vectors we subtract components*.

From the similar triangles, we see that the components of $c\mathbf{a}$ are ca_1 and ca_2. So *to multiply a vector by a scalar we multiply each component by that scalar*.



If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} \pm \mathbf{b} = \langle a_1 \pm b_1, a_2 \pm b_2 \rangle$$
$$c\mathbf{a} = \langle ca_1, ca_2 \rangle.$$

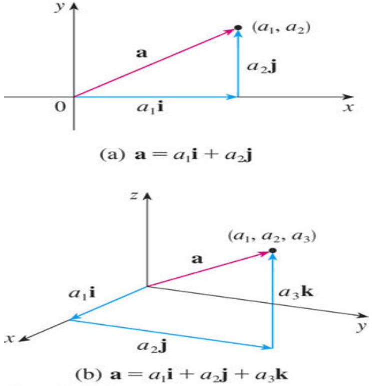
Since $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, we have $\boxed{\langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}}$

Components of a vector

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle \pm \langle b_1, b_2, b_3 \rangle = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$$
$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Since $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, we have $\boxed{\langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}}$



Components of a vector

- Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in $\mathbb{R}^3$, then

$$\boxed{\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle}$$

- Similarly, in two dimensions, the vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\boxed{\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle}$$

- The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}\,.$$

- The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\,.$$

Example The length of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is $\sqrt{14}$.

Components of a vector

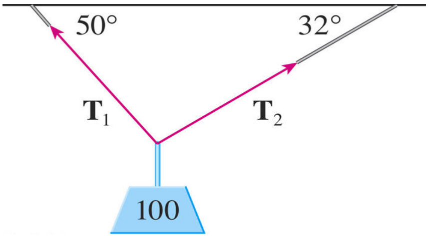
- Any vector whose length is 1 is a **unit vector**.
- For instance, the vector $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors.
- If $\mathbf{v} \neq \mathbf{0}$, $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector, called the **direction** of $\mathbf{v}$ or the **unit vector in the direction of** $\mathbf{v}$.
- Any nonzero vector can be expressed as a product of its length and direction:

$$\boxed{\mathbf{v} = |\mathbf{v}| \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (\text{length of } \mathbf{v}) \cdot (\text{direction of } \mathbf{v})}$$

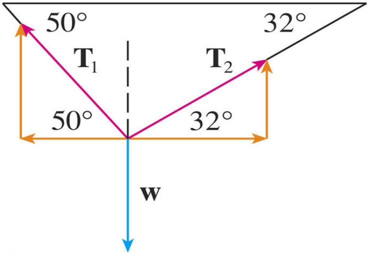
- A force is represented by a vector because it has both a magnitude and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Components of a vector

Example A 100-lb weight hangs from two wires as shown in the figure below. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and their magnitudes.



Components of a vector



Solution We see that

$$T_1 = -|T_1|\cos 50°\mathbf{i} + |T_1|\sin 50°\mathbf{j}$$
$$T_2 = |T_2|\cos 32°\mathbf{i} + |T_2|\sin 32°\mathbf{j}$$

The resultant $T_1 + T_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have $T_1 + T_2 = -\mathbf{w} = 100\mathbf{j}$:

$$\big(-|T_1|\cos 50° + |T_2|\cos 32°\big)\mathbf{i} + \big(|T_1|\sin 50° + |T_2|\sin 32°\big)\mathbf{j} = 100\mathbf{j}.$$

Components of a vector

Equating components, we get

$$\begin{aligned} -|T_1|\cos 50^\circ + |T_2|\cos 32^\circ &= 0 \\ |T_1|\sin 50^\circ + |T_2|\sin 32^\circ &= 100. \end{aligned}$$

Solving gives

$$|T_1|\sin 50^\circ + \frac{|T_1|\cos 50^\circ}{\cos 32^\circ}\sin 32^\circ = 100.$$

So

$$\begin{aligned} |T_1| &= \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb} \\ |T_2| &= \frac{|T_1|\cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb} \end{aligned}$$

Hence the tension vectors are

$$T_1 \approx -55.05\mathbf{i} + 65.60\mathbf{j} \quad \text{and} \quad T_2 \approx 55.05\mathbf{i} + 34.40\mathbf{j}.$$

The dot product

Definition

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Similarly, if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

The dot product is sometimes called the **scalar product** (or **inner product**).

The dot product

Theorem

If $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors and* λ *is a scalar, then*

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$;
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$;
4. $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda\mathbf{b})$;
5. $\mathbf{0} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{0} = 0$.

Theorem

If θ *is the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$, *then*

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cdot \cos\theta}$$

The dot product

Corollary

If θ is the angle between the nonzero vectors **a** *and* **b**, *then*

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}}$$

Example Find the angle between the vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. **Solution**

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= 1\times 6 + (-2)\times 3 + (-2)\times 2 = -4 \\
|\mathbf{a}| &= \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \\
|\mathbf{b}| &= \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7 \\
\theta &= \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \cos^{-1}\left(\frac{-4}{3\times 7}\right) \approx 1.76 \text{ rad}.
\end{aligned}$$

The dot product

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Thus,

$$\boxed{\text{Two vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal} \iff \mathbf{a} \cdot \mathbf{b} = 0}$$

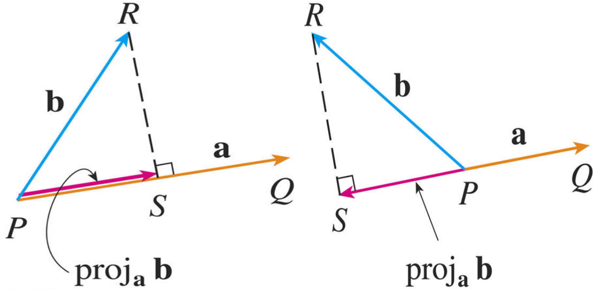
Example

$\mathbf{a} = \langle 3, -2, 1 \rangle$ and $\mathbf{b} = \langle 0, 2, 4 \rangle$ are orthogonal because

$$\mathbf{a} \cdot \mathbf{b} = (3)(0) + (-2)(2) + (1)(4) = 0.$$

The dot product

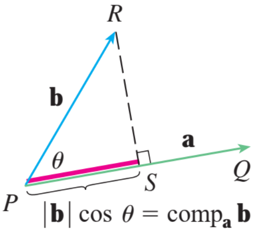
Projection Suppose that $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$.



The vector projection of **b** onto **a**

The dot product

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the **component of b along a**) is defined to be the number $|\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$.



$$\boxed{\begin{aligned} \text{comp}_{\mathbf{a}}\,\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} \\ \text{proj}_{\mathbf{a}}\,\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} \end{aligned}}$$

The dot product

Example

Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

Solution Since $\mathbf{a} = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$,

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2) \times 1 + 3 \times 1 + 1 \times 2}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

Thus

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

The dot product

One use of projections occurs in physics in calculating work. If the force moves the object from P to Q, then the **displacement vector** is $\overrightarrow{PQ}$.

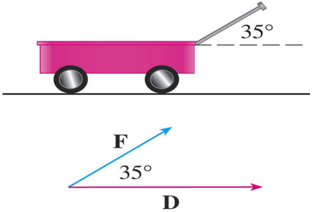
Definition

The **work** done by a constant force $\boldsymbol{F}$ acting through a displacement $\overrightarrow{PQ}$ is

$$\text{Work} = \mathbf{F} \cdot \overrightarrow{PQ} = |\boldsymbol{F}||\overrightarrow{PQ}| \cos\theta.$$

The dot product

Example A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.



Solution If **F** and **D** are the force and displacement vectors, then the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos 35^\circ$$
$$= (70)(100) \cos 35^\circ \approx 5734 \ \mathrm{N} \cdot \mathrm{m} = 5734 \ \mathrm{J}.$$

The cross product

Definition

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Cross product is also called the **vector product**.

$\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are three-dimensional vectors.

The cross product

A **determinant of order** 2 is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

A **determinant of order** 3 can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

The cross product

Then the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \quad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example

Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in $\mathbb{R}^3$.

Solution If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_2 a_3 - a_3 a_2)\mathbf{i} + (a_3 a_1 - a_1 a_3)\mathbf{j} + (a_1 a_2 - a_2 a_1)\mathbf{k} \quad = \mathbf{0}.$$

The cross product

Example Show that

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \end{array}$$

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

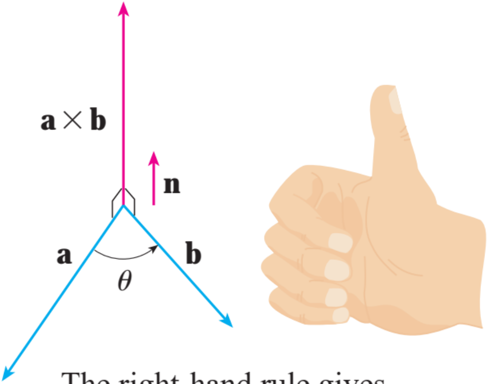
Proof Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Then

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= (a_2 b_3 - a_3 b_2) a_1 + (a_3 b_1 - a_1 b_3) a_2 \\ &\quad + (a_1 b_2 - a_2 b_1) a_3 \\ &= 0 \end{aligned}$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.

The cross product

The direction of $\mathbf{a} \times \mathbf{b}$ is given by the right-hand rule: If the curled fingers of the right hand are rotated from the direction of $\mathbf{a}$ to the direction of $\mathbf{b}$, the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.



The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

The cross product

Example Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} \qquad = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} \quad = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

Thus,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

The cross product

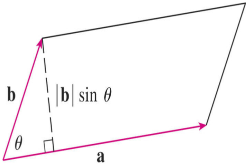
Theorem

If θ is the angle between **a** *and* **b** *(so $0 \le \theta \le \pi$), then*

$$\boxed{|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta}$$

Thus,

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.



$|\mathbf{a} \times \mathbf{b}|$ =area of parallelogram

The cross product

Corollary

Two nonzero vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if* $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Example Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}| = 6\sqrt{2}.$$

The triangle's area is half of this, $3\sqrt{2}$.

The cross product. Properties

Theorem

If $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{b}$ *are vectors and* λ *is a scalar, then*

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{b}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{b} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{b}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{b})$ that occurs in Property 6 is called the **vector triple product** of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{b}$.

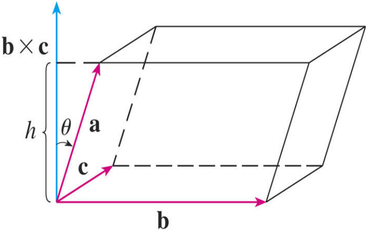
The cross product. Triple Products

- The product that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. It can be shown that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

- The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$



The cross product

Example

Find the volume of the box (parallelepiped) determined by $\mathbf{a} = \langle 1, 2, -1 \rangle$, $\mathbf{b} = \langle -2, 0, 3 \rangle$, and $\mathbf{c} = \langle 0, 7, -4 \rangle$.

Solution

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}$$

$$= -21 - 16 + 14 = -23.$$

The volume is $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 23$.

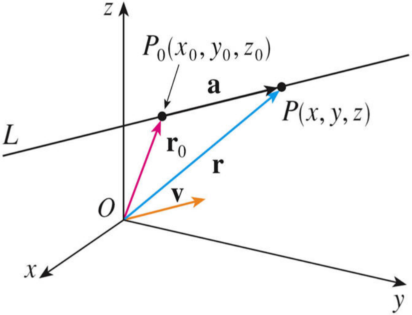
Note that if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, then the vectors must lie in the same plane; that is, they are **coplanar**.

Equations of Lines and Planes

Equations for Lines Suppose L is a line in three-dimensional space that passes a point $P_0(x_0, y_0, z_0)$. Let $\mathbf{v}$ be a vector parallel to L, $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$, respectively. Then

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

which is a **vector equation** of L.



Equations of Lines

Suppose $\mathbf{v} = \langle a, b, c \rangle$, then we have the three scalar equations:

$$\boxed{x = x_0 + ta, \quad y = y_0 + tb \quad z = z_0 + tc, \quad t \in \mathbb{R}} \qquad (1)$$

These equations are called **parametric equations** of the line through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$.

Note The vector equation and parametric equations of a line are not unique.

Equations of Lines

Example

(a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(b) Find two other points on the line.

Solution (a) The vector equation is

$$\begin{aligned}\mathbf{r} &= (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &= (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}.\end{aligned}$$

Parametric equations are
$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t, \quad t \in \mathbb{R}.$
(b) Choosing the parameter value $t = 1$ gives $x = 6,\ y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$.

Equations of Lines

If none of a, b, or c is 0, we can solve each of Equations (1) for t, equate the results, and obtain

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These equations are called **symmetric equations** of L. If $a = 0$, we can write the equations of L as

$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Equations for Linesegments *The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation*

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1, \quad 0 \le t \le 1}$$

Equations of Lines

Example Show that the lines L_1 and L_2 with parametric equations

$$\begin{array}{lll} x = 1 + t & y = -2 + 3t & z = 4 - t \\ x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

Solution The lines are not parallel because the corresponding vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

These equations have no solution, so L_1 and L_2 do not intersect. Thus L_1 and L_2 are skew lines.

Equations of Lines

Example

Show that the midpoint of the line segment joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

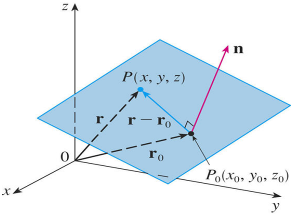
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Solution

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \overrightarrow{P_1M} = \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} \\ &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

Equations of Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**. The plane consists of all points $P(x, y, z)$ for which $\overrightarrow{P_0P} = \langle x - x_0, x - y_0, x - z_0 \rangle$ is orthogonal to $\mathbf{n}$.



We have **vector equation** of the plane:

$$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0} \qquad (2)$$

which can be rewritten as

$$\boxed{\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0} \qquad (3)$$

Equations of Planes

- Suppose $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation (2) becomes

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \tag{4}$$

  Equation (4) is the **scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector** $\mathbf{n} = \langle a, b, c \rangle$.

- We can rewrite the equation of a plane as

$$\boxed{ax + by + cz + d = 0} \tag{5}$$

  where $d = -(ax_0 + by_0 + cz_0)$. Equation (5) is called a **linear equation** in x, y, and z.

Equations of Planes

Example Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

Solution Since both $\overrightarrow{PQ} = \langle 2,-4,4\rangle$ and $\overrightarrow{PR} = \langle 4,-1,-2\rangle$ lie in the plane, $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is a normal vector of the plane. Thus

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}.$$

An equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

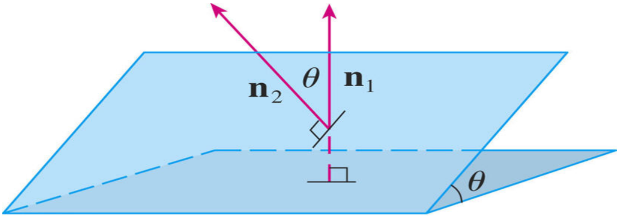
or

$$6x + 10y + 7z = 50.$$

Equations of Planes

Angles Between Planes Two planes are parallel if their normal vectors are parallel.

If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.



The angle between planes

Equations of Planes

Example Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

Solution The normal vectors of these planes are $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3 \rangle$ and so, if θ is the angle between the planes, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1 \cdot 1 + 1(-2) + 1 \cdot 3}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ.$$

Equations of Planes

Distance from a Point to a Plane

Example Find a formula for the distance from a point $P_1(x_1, y_1, z_1)$ to the plane

$$ax + by + cz + d = 0.$$

Solution Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b} = \overrightarrow{P_0P_1}$. Then $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$. The distance from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$.

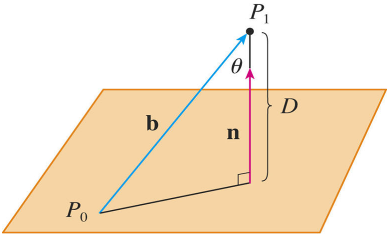
$$\begin{aligned} D &= |\operatorname{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Equations of Planes

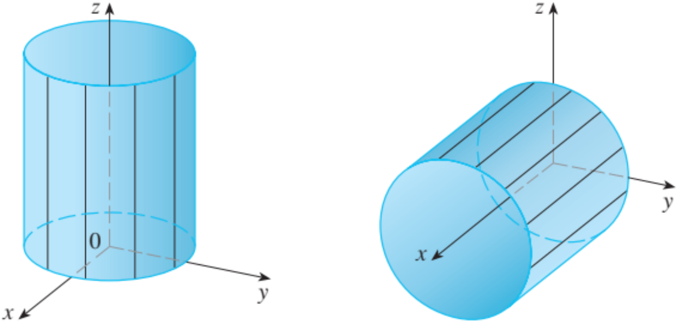
$$D = \frac{\left|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)\right|}{\sqrt{a^2 + b^2 + c^2}}$$

Since P_0 lies in the plane, $ax_0 + by_0 + cz_0 + d = 0$. Thus

$$\boxed{D = \frac{\left|ax_1 + by_1 + cz_1 + d\right|}{\sqrt{a^2 + b^2 + c^2}}}$$



Equations of Cylinders

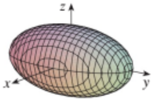
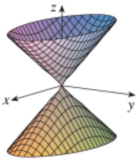


When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ (left) or $y^2 + z^2 = 1$ (right) represents a cylinder and not a circle.

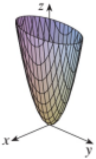
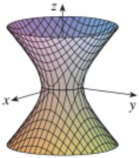
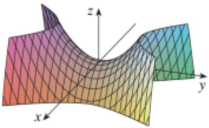
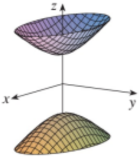
Equations of Quadric Surfaces

Quadric Surfaces

A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ where $A, B, C, \ldots, J$ are constants.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

Equations of Quadric Surfaces

Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Vector Functions and Space Curves

Definitions

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I. \tag{6}$$

The points $(x, y, z) = \big(f(t), g(t), h(t)\big)$, $t \in I$, make up the **curve** in space that we call the particle's **path**. The equation and interval in (6) **parametrize** the curve.

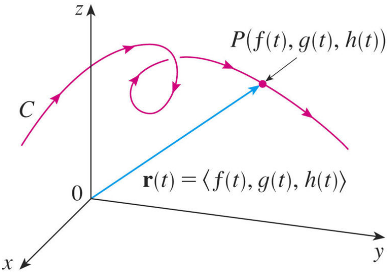
The vector $\mathbf{r}(t) = \overrightarrow{OP} = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ from the origin to the particle's **position** $P = \big(f(t), g(t), h(t)\big)$ at time t is the particle's **position vector**. The functions f, g, and h are the **components** or **coordinate functions** of the position vector.

Vector Functions and Space Curves

Definitions (cont'd)

More generally, a **vector-valued function** or **vector function** is a function whose range is a set of vectors. The vector function's domain to be the intersection of the domains of its component functions.

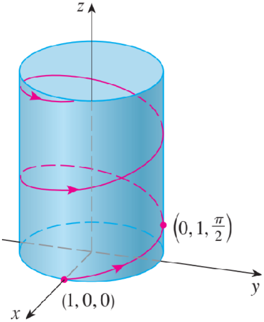
When we need to distinguish real-valued functions from vector functions, we refer to real-valued functions as **scalar functions**.



Vector Functions and Space Curves

Example: Space Curves Sketch the curve whose vector equation is $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. **Solution** We have

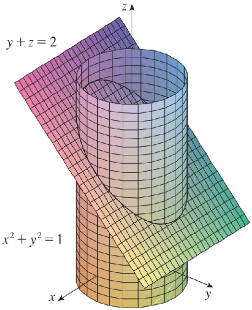
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Thus, the curve must lies on the circular cylinder $x^2 + y^2 = 1$. The curve spirals upward around the cylinder as $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The curve is called a **helix**.

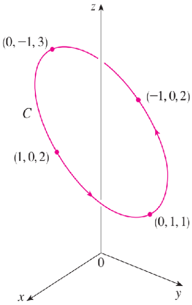


Vector Functions and Space Curves

Example

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$





Answer: $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2 - \sin t)\mathbf{k}$, $0 \le t \le 2\pi$.

Vector Functions. Limits and Continuity

Definition

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Example If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned} \lim_{t \to \pi/4} \mathbf{r}(t) &= \left(\lim_{t \to \pi/4} \cos t \right)\mathbf{i} + \left(\lim_{t \to \pi/4} \sin t \right)\mathbf{j} + \left(\lim_{t \to \pi/4} t \right)\mathbf{k} \\ &= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}. \end{aligned}$$

Vector Functions. Continuity

Definition

A vector function $\mathbf{r}(t)$ is **continuous** at a if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

The function is **continuous** if it is continuous at every point in its domain.

A vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at a if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous at a.

Example The function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ is continuous.

Derivatives of vector functions

Definition

The derivative of $\mathbf{r}(t)$ is the limit of the difference quotient

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}}$$

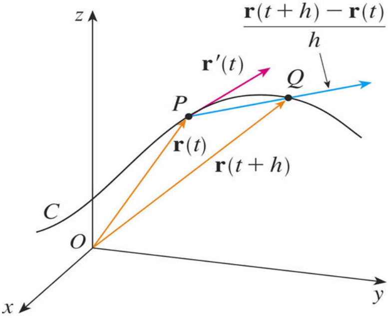
if this limit exists.

The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}(t)$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The **unit tangent vector** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}(t)|}.$$

Derivatives of vector functions



Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Derivatives of vector functions

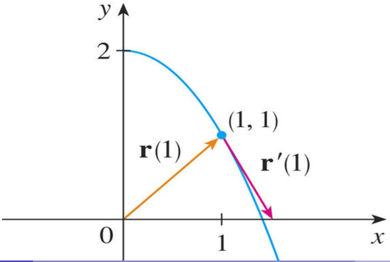
Example For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2-t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$. Find the corresponding unit tangent vector.

Solution

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}.$$

The unit tangent vector at the point where $t = 1$ is

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\frac{1}{2}\mathbf{i} - \mathbf{j}}{\sqrt{5}/2} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}.$$



Derivatives of vector functions

Definition

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is **differentiable** at $t = a$ if f, g, and h are differentiable at a. Also, $\mathbf{r}$ is said to be **differentiable** if it is differentiable at every point of its domain. The curve traced by $\mathbf{r}$ is **smooth** if $d\mathbf{r}/dt$ is continuous and never equal to $\mathbf{0}$, i.e., if f, g, and h have first derivatives that are not simultaneously 0.

- A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion so that the initial point of one curve is the terminal point of the immediately preceding one is called **piecewise smooth**.

- The second derivative of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

Derivatives of vector functions

Definition

If $\mathbf{r}(t)$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative

$$\mathbf{a} = d\mathbf{v}/dt,$$

when it exists, is the particle's **acceleration vector**.

Note

$$\boxed{\text{Velocity} = |\mathbf{v}| \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (\text{Speed}) \cdot (\text{Direction})}$$

Derivatives of vector functions

Example The vector $\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$ gives the position of a moving body at time t. Find the body's speed and direction when $t = 2$. At what times, if any, are the body's velocity and acceleration orthogonal?

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k},$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}.$$

At $t = 2$, the body's speed and direction are $|\mathbf{v}(2)| = 5$ and

$$\frac{\mathbf{v}(2)}{|\mathbf{v}(2)|} = \Big(-\frac{3}{5}\sin 2\Big)\mathbf{i} + \Big(\frac{3}{5}\cos 2\Big)\mathbf{j} + \frac{4}{5}\mathbf{k},$$

The body's velocity and acceleration are orthogonal when

$$\mathbf{v}\cdot\mathbf{a} = 9\sin t\cos t - 9\cos t\sin t + 4t = 4t = 0.$$

Therefore, $t = 0$.

Derivatives of vector functions

Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$.
4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\dfrac{d}{dt}[\mathbf{u}\big(f(t)\big)] = f'(t)\mathbf{u}'\big(f(t)\big)$.

Derivatives of vector functions

Example If $\mathbf{r}(t)$ is a differentiable vector function of constant length, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$:

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0.$$

Solution Since $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$ is constant,

$$0 = \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 2\mathbf{r} \cdot \mathbf{r}'.$$

Thus, $\mathbf{r} \cdot \mathbf{r}' = 0$.

Integrals of vector functions

Definition

If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over the interval $a \le t \le b$, then $\mathbf{r}$ is **integrable** over $[a, b]$ and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)dt = \Big(\int_a^b f(t)dt\Big)\mathbf{i} + \Big(\int_a^b g(t)dt\Big)\mathbf{j} + \Big(\int_a^b k(t)dt\Big)\mathbf{k}.$$

For example,

$$\int_0^\pi \langle 1, t, \sin t\rangle dt = \Big\langle \int_0^\pi 1dt, \int_0^\pi tdt, \int_0^\pi \sin tdt\Big\rangle \ = \Big\langle \pi, \frac{1}{2}\pi^2, 2\Big\rangle.$$

Integrals of vector functions

- An **antiderivative** of $\mathbf{r}(t)$ on an interval I is a vector function $\mathbf{R}(t)$ such that $\mathbf{R}'(t) = \mathbf{r}(t)$ at each point of I.

- If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on I, it can be shown that every antiderivative of $\mathbf{r}(t)$ on I has the form $\mathbf{R}(t) + \mathbf{C}$ for some constant $\mathbf{C}$.

- The set of all antiderivatives of $\mathbf{r}$ on I is the **indefinite integral** of $\mathbf{r}$ on I and denoted by $\int \mathbf{r}(t)dt$.

- Thus, if $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$$\boxed{\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}}$$

Integrals of vector functions

Example The velocity of a particle moving in the space is

$$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k}.$$

Find the particle's position as a function of t if $\mathbf{r} = 2\mathbf{i} + \mathbf{k}$ when $t = 0$.

Solution

$$\mathbf{r}(t) = \int \mathbf{r}'(t)dt = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k} + \mathbf{C}.$$

To determine $\mathbf{C}$, we use the initial condition $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{k}$:

$$\begin{aligned}(\sin 0)\mathbf{i} + (\cos 0)\mathbf{j} + 0\mathbf{k} + \mathbf{C} &= 2\mathbf{i} + \mathbf{k} \\ \mathbf{C} &= 2\mathbf{i} - \mathbf{j} + \mathbf{k}.\end{aligned}$$

The particle's position as a function of t is

$$\mathbf{r}(t) = (\sin t + 2)\mathbf{i} + (\cos t - 1)\mathbf{j} + (t + 1)\mathbf{k}.$$

Length of space curves

Arc Length Suppose $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t), y = g(t), z = h(t)$, where f', g', and h' are continuous.

If the curve is traversed exactly once as increases from $t = a$ to $t = b$, then it can be shown that its **length** is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

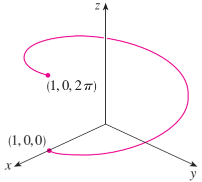
That is,

$$L = \int_a^b |\mathbf{r}'(\mathbf{t})| dt$$

Length of space curves

Example

Find the length of the arc of the helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1,0,0)$ to the point $(1,0,2\pi)$.



Solution We have $\mathbf{r}'(\mathbf{t}) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 1\mathbf{k}$, so $|\mathbf{r}'(\mathbf{t})| = \sqrt{2}$. The arc length is

$$L = \int_0^{2\pi} |\mathbf{r}'(\mathbf{t})| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$

–END OF CHAPTER 2–