VIETNAM NATIONAL UNIVERSITY-HCMC
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Lecture Notes/Slides for

APPLIED LINEAR ALGEBRA

Chapter 3B. Subspaces, Span and Linear independence

Definition (Recall: Linear Combination)

Let $\vec{u}_1, \cdots, \vec{u}_n, \vec{v}$ be vectors. Then $\vec{v}$ is said to be a **linear combination** of the vectors $\vec{u}_1, \cdots, \vec{u}_n$ if there exist scalars, $a_1, \cdots, a_n$ such that

$$\vec{v} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$$

Definition (Span of a Set of Vectors)

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is known as the span of these vectors and is written as $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Additional Terminology. If $U = \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$, then

- U is spanned by the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$.
- the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ span U.
- the set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a spanning set for U.

Problem

Let $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ *and* $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$. *Show that* $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$ *is in span* $\{\vec{u}, \vec{v}\}$.

Solution

For a vector to be in span $\{\vec{u}, \vec{v}\}$, it must be a linear combination of these vectors. If $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$, we must be able to find scalars a, b such that

$$\vec{w} = a\vec{u} + b\vec{v}$$

$$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

This is equivalent to the following system of equations

$$\begin{aligned} a + 3b &= 4 \\ a + 2b &= 5 \end{aligned}$$

Solution (continued)

We solving this system the usual way, constructing the augmented matrix and row reducing to find the reduced row-echelon form .

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 2 & 5 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]$$

The solution is $a = 7, b = -1$. This means that

$$\vec{w} = 7\vec{u} - \vec{v}$$

Therefore we can say that $\vec{w}$ is in span $\{\vec{u}, \vec{v}\}$.

Span of a Set of Vectors

Example

Let $\vec{x} \in \mathbb{R}^3$ be a nonzero vector. Then $\text{span}\{\vec{x}\} = \{k\vec{x} \mid k \in \mathbb{R}\}$ is a line through the origin having direction vector $\vec{x}$.

Problem

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.

Solution

Notice that any linear combination of the vectors $\vec{u}$ and $\vec{v}$ yields a vector $\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$ in the YZ-plane.

Suppose we take an arbitrary vector $\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$ in the YZ-plane. It turns out we can write any such vector as a linear combination of $\vec{u}$ and $\vec{v}$.

$$\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} = (-3y + 2z) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (2y - z) \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $\text{span}\{\vec{u}, \vec{v}\}$ is the YZ-plane.

Span of a Set of Vectors

Consider the previous example where the span of $\vec{u}$ and $\vec{v}$ was the YZ-plane. Suppose we add another vector $\vec{w}$, and consider the span of $\vec{u}, \vec{v}$, and $\vec{w}$. What would happen to the span?

Scenario 1 Suppose $\vec{w}$ is a vector in the YZ-plane. For example, $\vec{w} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. Then $\vec{w}$ is in the span of $\vec{u}, \vec{v}$. Adding $\vec{w}$ to the set doesn't change the span at all.

$$\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v}\}$$

Span of a Set of Vectors

Scenario 2 Suppose $\vec{w}$ is not in the YZ-plane. For example, suppose $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Notice that now, the three vectors span $\mathbb{R}^3$. Any vector in $\mathbb{R}^3$ can be written as a linear combination of $\vec{u}, \vec{v}, \vec{w}$ as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (-4x + 5y + 2z) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (2x + 2y - z) \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + (x) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

You can see that the span of these three vectors depended on whether $\vec{w}$ was in $\text{span}\{\vec{u}, \vec{v}\}$ or not. In the next section, we will examine the distinction between these two scenarios using the concept of linear independence.

Linearly Independent Set of Vectors

Definition

Let $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. This set is linearly independent if no vector in the set is in the span of the other vectors of that set.

If a set of vectors is not linearly independent, we call it linearly dependent.

A Linearly Dependent Set

Problem

Consider the vectors $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. Is the set $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

Solution

Notice that we can write $\vec{w}$ as a linear combination of $\vec{u}, \vec{v}$ as follows:

$$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = (-10)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (7)\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $\vec{w}$ is in $\text{span}\{\vec{u}, \vec{v}\}$. By the definition, this set is not linearly independent (it is linearly dependent).

A Linearly Independent Set

Problem

Consider the vectors

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Is the set $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

Solution

We cannot write any of the three vectors as a linear combination of the other two. (We will see how to show this soon.) Therefore the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Linear Independence as a Linear Combination

The following theorem provides a familiar way to check if a set of vectors is linearly independent.

Theorem

The collection of vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is linearly independent if and only if whenever

$$\sum_{i=1}^{n} a_i \vec{u}_i = \vec{0}$$

it follows that each $a_i = 0$.
Thus $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is linearly independent exactly when the system of linear equations $AX = 0$ has only the trivial solution, where A is the $n \times k$ matrix having these vectors as columns.

Linear Independence

We can state the conclusion of this theorem in another way: The set of vectors $\{\vec{u}_1, ..., \vec{u}_k\}$ is linearly independent if and only if there is no nontrivial linear combination which equals zero. If a linear combination of the vectors equals zero, then all the coefficients of the combination are zero.

If the set is linearly independent, then

$$a_1\vec{u}_1 + ... + a_k\vec{u}_k = 0$$

implies that

$$a_1 = a_2 = ... = a_k = 0$$

Linear Independence

We can use the reduced row-echelon form of the matrix to determine if the columns form a linearly independent set of vectors.

Problem

Determine whether the following set of vectors are linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution

Construct the 3 x 3 matrix A having these vectors as columns:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

By the above theorem, the set of vectors is linearly independent if the system $AX = 0$ has only the trivial solution. We can see this from the reduced row-echelon form of the matrix A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since all columns are pivot columns (and the rank of A is 3), the vectors are linearly independent.

Problem

Determine whether the following vectors are linearly independent. If they are linearly dependent, write one of the vectors as a linear combination of the others.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 17 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix} \right\}$$

Solution

Construct the matrix A using these vectors as columns.

$$A = \begin{bmatrix} 1 & 2 & 0 & 8 \\ 2 & 0 & 1 & 5 \\ 4 & 0 & 3 & 11 \\ 1 & 3 & 0 & 11 \end{bmatrix}$$

Solution (continued)

The reduced row-echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank of A is $3 < 4$, the vectors are linearly dependent. Therefore, there are infinitely many solutions to $AX = 0$, one of which is

$$\begin{bmatrix} -2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

Solution (continued)

Therefore we can write:

$$-2\begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can rewrite this as:

$$2\begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix}$$

This shows that one of the vectors can be written as a linear combination of the other three vectors. While here we chose the fourth vector, we could have chosen any of the vectors to isolate.

Subspaces

Definition

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace if whenever a and b are scalars and $\vec{u}$ and $\vec{v}$ are vectors in V, $a\vec{u} + b\vec{v}$ is also in V.

Subspaces are closely related to the span of a set of vectors which we discussed earlier.

Theorem

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if there exist vectors $\{\vec{u}_1, ..., \vec{u}_k\}$ in V such that

$$V = \operatorname{span}\{\vec{u}_1, ..., \vec{u}_k\}$$

Subspaces

Subspaces are also related to the property of linear independence.

Theorem

If V is a subspace of $\mathbb{R}^n$, then there exist linearly independent vectors $\{\vec{u}_1, ..., \vec{u}_k\}$ of V such that

$$V = \operatorname{span}\{\vec{u}_1, ..., \vec{u}_k\}$$

In other words, subspaces of $\mathbb{R}^n$ consist of spans of finite, linearly independent collections of vectors in $\mathbb{R}^n$.

Basis of a Subspace

Definition

Let V be a subspace of $\mathbb{R}^n$. Then $\{\vec{u}_1, ..., \vec{u}_k\}$ is called a basis for V if the following conditions hold:

- $\text{span}\{\vec{u}_1, ..., \vec{u}_k\} = V$
- $\{\vec{u}_1, ..., \vec{u}_k\}$ is linearly independent.

The following theorem claims that any two bases of a subspace must be of the same size.

Theorem

Let V be a subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, ..., \vec{u}_k\}$ and $\{\vec{v}_1, ..., \vec{v}_m\}$ are two bases for V. Then $k = m$.

Dimension

The previous theorem shows than all bases of a subspace will have the same size. This size is called the dimension of the subspace.

Definition

Let V be a subspace of $\mathbb{R}^n$. Then the dimension of V is the number of a vectors in a basis of V.

Properties of $\mathbb{R}^n$

Note that the dimension of $\mathbb{R}^n$ is n.

There are some other important properties of vectors in $\mathbb{R}^n$.

Theorem

- If $\{\vec{u}_1, ..., \vec{u}_n\}$ is a linearly independent set of a vectors in $\mathbb{R}^n$, then $\{\vec{u}_1, ..., \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.
- Suppose $\{\vec{u}_1, ..., \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.
- If $\{\vec{u}_1, ..., \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, ..., \vec{u}_n\}$ is linearly independent.
- If $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_k\}$ is a set of vectors in $\mathbb{R}^n$ with $k > n$, then the set is linearly dependent.

It follows then that a basis is a minimal spanning set. If a subspace has dimension d, then any spanning set has size at least d, and any spanning set of size d must be a basis (and is therefore independent).

Row and Column Space

Definition

Let A be an $m \times n$ matrix. The column space of A is the span of the columns of A. The row space of A is the span of the rows of A.

Problem

Find the rank of the matrix A and describe the column and row spaces efficiently.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

Example: Column Space

Solution

To find the column space, we first find the reduced row-echelon form of A:

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $\text{rank}(A) = 2$.
Note the first two columns are the pivot columns. All columns of the above reduced row-echelon matrix are in

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Example: Column Space

Solution (continued)

To construct the column space, we use the pivot columns of the original matrix - in this case, the first and second columns. Therefore the column space of A is

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

Example: Row Space

Solution (continued)

To find the row space of A we again look at the reduced row-echelon form of the matrix.

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The row space of A is the span of the non-zero rows of the above matrix:

$$\operatorname{span}\left\{\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & -3 & 0 \end{bmatrix}\right\}$$

Notice that the vectors used in the description of the column space are from the original matrix, while those in the row space are from the reduced row-echelon form of the original matrix.

Null Space

Definition

The null space of A, or kernel of A is defined as:

$$\ker(A) = \{X : AX = 0\}$$

We also speak of the image of A, $\operatorname{Im}(A)$, which is all vectors of the form AX where X is in $\mathbb{R}^n$.
To find $\ker(A)$, we solve the system of equations $AX = 0$.

Problem

Find $\ker(A)$ for the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Null Space

Solution

The first step is to set up the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right]$$

Place this matrix in reduced row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Null Space

Solution (continued)

The solution to this system of equations is

$$\left\{ \begin{bmatrix} 3t \\ t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

Therefore the null space of A is the span of this vector:

$$\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Nullity

Definition

The dimension of the null space of a matrix is called the nullity, denoted null(A).

Theorem

Let A be an $m \times n$ matrix. Then,

$$\text{rank}(A) + \text{null}(A) = n$$

For instance, in the last example, A was a 3×3 matrix. The rank was 2 and the nullity was 1 (since the null space had dimension 1).

$$\text{rank}(A) + \text{null}(A) = 2 + 1 = 3 = n$$