

Calculus 2

Fall 2023

Homework

Chapter 1

Week 1

1. Consider the following sequence $\{a_n\}$ with the few first terms given as

$$\left\{-\frac{1}{2}, \frac{16}{3}, -\frac{81}{4}, \frac{256}{5}, -\frac{625}{6}, \dots\right\}$$

Find a formula for the general term a_n .

2. If \$600 is invested at 4% interest, compounded annually. Find the size of investment after 7 years.

3. Determine the limits of the following sequences

a) $a_n = \frac{3n^3}{n^3+1}$

b) $b_n = \left(\frac{5+n}{n}\right)^n$

c) $c_n = n^{1/n}$

d) $d_n = \ln(n^3 + 1) - \ln(3n^3 + 10n)$

Solution

- a) Consider function $f(x) = \frac{3x^3}{x^3+1}$, we see that $f(n) = a_n$ and $\lim_{x \rightarrow \infty} f(x) = 3$. Hence $\lim_{n \rightarrow \infty} a_n = 3$

- b) Consider $\lim_{n \rightarrow \infty} \ln(b_n) = \lim_{n \rightarrow \infty} n \ln\left(\frac{5+n}{n}\right) = \lim_{n \rightarrow \infty} \frac{\ln\left(\frac{5+n}{n}\right)}{1/n}$. Using L'Hopital we get $\lim_{n \rightarrow \infty} \ln(b_n) = \lim_{n \rightarrow \infty} \frac{5}{1+5/n} = 5$. So $\lim_{n \rightarrow \infty} b_n = e^5$

- c) Similarly to b) we have $\lim_{n \rightarrow \infty} c_n = 1$

- d) Note that $d_n = \ln(n^3 + 1) - \ln(3n^3 + 10n) = \ln\left(\frac{n^3+1}{3n^3+10n}\right)$. So $\lim_{n \rightarrow \infty} d_n = -\ln(3)$

4. Using squeeze Theorem find the limit of the sequence

$$a_n = \frac{\sin(2n)}{2^n}$$

Solution Since $-1 \leq \sin(2n) \leq 1$ for all n , hence, $-1/2^n \leq \frac{\sin(2n)}{2^n} \leq 1/2^n$. Using squeeze theorem we have $\lim_{n \rightarrow \infty} a_n = 0$

Week 2

1. Check the following series if the series is convergent or divergent

a) $\sum_{n=0}^{\infty} 2^{1-3n} 3^{n+2}$

b) $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$

Solution

a) We have

$$\sum_{n=0}^{\infty} 2^{1-3n} 3^{n+2} = 18 \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n$$

Since $r = 3/8 < 1$ then $\sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n$ is convergent, so $\sum_{n=0}^{\infty} 2^{1-3n} 3^{n+2}$ is convergent.

b) Note that

$$\frac{3}{n^2 + 7n + 12} = 3\left(\frac{1}{n+3} - \frac{1}{n+4}\right)$$

So

$$S_n = \sum_{k=1}^n \frac{3}{k^2 + 7k + 12} = 3\left[\sum_{k=1}^n \left(\frac{1}{k+3} - \frac{1}{k+4}\right)\right] = \frac{3}{4} - \frac{3}{n+4}$$

We get,

$$\lim_{n \rightarrow \infty} S_n = \frac{3}{4}$$

Hence, the series is convergent and $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12} = \frac{3}{4}$

2. Using integral test to check the following series if the series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{1}{n^\pi}$

b) $\sum_{n=0}^{\infty} \frac{2}{5n+3}$

c) $\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$

d) $\sum_{n=0}^{\infty} \frac{1}{n^2+4}$

Solution

- a) Since $p = \pi > 1$, hence the series converges
- b) we have $a_n > 0$ and $a_n > a_{n+1}$. Now compute the integral for the test.

$$\int_1^{\infty} \frac{2}{5x+3} dx = \infty$$

So the series diverges.

- c) Consider the function $f(x) = \frac{x^2}{x^3+1}$, hence, $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$. So $f(x)$ is decreasing if $x > \sqrt[3]{2}$. Now compute the integral for the test

$$\int_1^{\infty} \frac{x^2}{x^3+1} dx = \infty$$

So the series diverges.

3. Using the Divergence Test to determine if the following series diverges or conclude that the Divergence Test is inconclusive

- a) $\sum_{n=1}^{\infty} \frac{3^n+1}{2^n}$
- b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$
- c) $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$

Solution

- c) Since $\lim_{n \rightarrow \infty} a_n = \infty$ hence the series diverges

4. Using comparison test or limit comparison test to determine if the following series diverges or converges

- a) $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + 1\right)^2$
- b) $\sum_{n=1}^{\infty} \frac{4}{n^2-2n-3}$
- c) $\sum_{n=1}^{\infty} \frac{n^3}{2n^4-1}$
- d) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$

Solution

a) We see that for all $n \geq 1$, it holds

$$\left(\frac{1}{n^2} + 1\right)^2 = \frac{1}{n^2} + \frac{2}{n} + 1 < \frac{3}{n} + 1$$

However, the series

$$\sum_{i=1}^n \frac{3}{n}$$

is divergent and also

$$\sum_{i=1}^n 1 = \infty$$

So the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + 1\right)^2$ also diverges.

Alternative. We check

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + 1\right)^2 = 1 \neq 0$$

So the series is divergent.

b) We see that for $n \geq 7$, it holds $n^2 - 2n - 3 > 0$. Therefore, the series terms are positive, decreasing. We have $\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{4n^2}{n^2 - 2n - 3} = 1 < \infty$$

So $\sum_{n=1}^{\infty} \frac{4}{n^2 - 2n - 3}$ also converges.

c) we have $\frac{n^3}{2n^4 - 1} > \frac{1}{2n}$. But $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges. So the series diverges.

d) Note that for all $n \geq 2$, we have $\ln(n) < n$. So

$$\frac{\ln(n)}{n^3} < \frac{1}{n^2}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, hence the series converges.

5. Determine whether the following series converges or diverges

a) $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n}}$

b) $\sum_{n=0}^{\infty} \frac{10}{n^2 + 9}$

c) $\sum_{n=2}^{\infty} \frac{4}{n \ln^2(n)}$

Solution

- a) Since $\lim_{n \rightarrow \infty} a_n = 1 \neq 0$ hence the series diverges
- b) Using integration test the series is convergent.
- c) Using integration test $\int_2^{\infty} \frac{4}{x \ln^2(x)} dx = \frac{4}{\ln(2)}$ hence the series is convergent.

6 Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^p(n)}$$

where p is a real number.

- i) Using the integral test to determine the value of p for which the series converges
- ii) Does the series converge faster for $p = 2$ or $p = 3$? Explain.

Solution

- i) In order to the series converges, the integral

$$\int_2^{\infty} \frac{1}{x \ln^p(x)} dx$$

must exist. We have

$$\int \frac{1}{x \ln^p(x)} dx = \frac{1}{1-p} \ln^{1-p}(x)$$

So we have $1 - p < 0$ hence $p > 1$

- ii) The series converges faster for $p = 3$ since the term of the series get smaller faster.

Chapter 2**Week 4**

1. Determine the dot product of vectors a and b given as follows

- i) $a = (9, 5, -4, 2); b = (-3, -2, 7, -1)$
- (ii) $a = (0, 4, -2), b = 2i - j + 7k$
- (iii) $\|a\| = 5, \|b\| = 3/7$ and the angle between the two vectors is $\theta = \pi/12$

2. Determine the angle between the following two vectors

i) $u = (1, 0, 3); v = (1, -4, 2)$

ii) $a = i + 3j - 2k; b = (-9, 1, -5)$

3. Cross product

i) Find the cross product of vectors $a = (3, -1, 5); b = (0, 4, -2)$

ii) Find a vector such that it is orthogonal to the plane containing the points

$$P = (3, 0, 1); Q = (4, -2, 1); R = (5, 3, -1)$$

iii) Check if the vectors $u = (1, 2, -4); v = (-5, 3, -7); w = (-1, 4, 2)$ are in the same plane?

4. For the given vectors u and v , calculate $proj_v u$, and $comp_v u$

a) $u = \langle 13, 0, 26 \rangle$, and $v = \langle 4, -1, -3 \rangle$

b) $u = \langle -8, 0, 2 \rangle$, and $v = \langle 1, 3, -3 \rangle$

c) $u = 5i + j - 5k$, and $v = -i + j - 2k$

5. Find the area of the parallelogram that has two adjacent sides u and v .

a) $u = 2i - j - 2k, v = 3i + 2j - k$

b) $u = 8i + 2j - 3k, v = 2i + 4j - 4k$

6. For the given points A, B , and C , find the area of the triangle with vertices A, B , and C

$$A = (1, 2, 3), B = (5, 1, 5), C = (2, 3, 3)$$

7. Find equations of the following lines.

a) the line through $(1, -3, 4)$ that is parallel to the line $r(t) = \langle 3 + 4t, 5 - t, 7 \rangle$

b) The line through $(-3, 4, 2)$ that is perpendicular to both $u = \langle 1, 1, -5 \rangle$, and $v = \langle 0, 4, 0 \rangle$

8. Find the equation of the line segment between $P_0(3, -1, 4)$, and $P_1 = (0, 5, 2)$

9. Find an equation of the plane.

a) The plane through the point $(5, 3, 5)$ and with normal vector $2i + j - k$

b) The plane through the point $(2, 0, 1)$ and perpendicular to the

$$x = 3t, y = 2 - t, z = 3 + 4t$$

10. Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?

11. The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 5$

12. The plane that contains the line $x = 1 + t, y = 2 - t, z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$.

13. The plane through the points $(2, 1, 2)$, $(3, -8, 6)$, and $(-2, -3, 1)$

14. Find the domain of the vector function

$$r(t) = \cos(t)i + \ln(t)j + \frac{1}{t-2}k$$

15. Let

$$r(t) = \langle te^{-t}, \frac{t^3 + t}{2t^3 - 1}, t \sin(1/t) \rangle.$$

Find $\lim_{t \rightarrow \infty} r(t)$.

16. Find the length of the curve

a) $r(t) = \langle t, 3 \cos(t), 3 \sin(t) \rangle, \quad -5 \leq t \leq 5$

b) $r(t) = \sqrt{2t}i + e^t j + e^{-t}k, \quad 0 \leq t \leq 1$

Chapter 3

Week 5

1. Find the domain of the given functions.

a) $f(x, y) = \sqrt{x^2 - 2y}$

b) $f(x, y) = \ln(2x - 3y + 1)$

2. Find limits of the following functions

(a) $\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x - 4y}{6y + 7x}$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^6}{xy^3}$

3. Find the first order partial derivatives of the following functions

a) $f(x, y, z) = 4x^3y^2 - e^zy^4 + \frac{z^3}{x^2} + 4y - x^{16}$

b) $f(x, y) = \frac{x^2}{y^2 + 1} - \frac{y^2}{x^2 + y}$

Week 6

1. Find an equation of the plane tangent to the following at the given point

a) $x^2 + y + z = 1; P(1, 1, 1)$

b) $x^2 + y^3 + z^4 = 2; Q(1, 0, 1)$

c) $xy + xz + yz - 12 = 0; R(2, 0, 6)$

2. Find the linear approximation to the function f at the given point and estimate the given function value.

a) $f(x, y) = xy + x - y; P(2, 3); \text{estimate } f(2.1, 2.99)$

b) $f(x, y) = \sqrt{x^2 + y^2}; Q(3, -4); \text{estimate } f(3.06, -3.92)$

c) $f(x, y, z) = \ln(1 + x + y + 2z); R(0, 0, 0); \text{estimate } f(0.1, -0.2, 0.2)$

3. Find the following derivatives.

a. $z = x^2y, -xy^2, x = t^2, y = t^{-2}$. Find dz/dt

b. $z = e^{x+y}, x = st, y = s + t$. Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$

c. $y \ln(x^2 + y^2 + 4) = 0$, find dy/dx

Week 7

1. Consider the function $f(x, y) = 8 = \frac{x^2}{2} - y^2$. Find the directional derivative at $(2, 0)$ in the corresponding directions by the unit vectors $u = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$, $v = \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle$, and $w = \langle -\sqrt{2}/2, -\sqrt{2}/2 \rangle$.

2. Use the second derivatives test to show that function f has a local minimum or maximum

a) $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$

b) $f(x, y) = xy(x - 2)(y + 3)$

c) $f(x, y) = 2xye^{-x^2-y^2}$

3. Find the absolute maximum and minimum values of the function over the given region R

a) $f(x, y) = 4 + 2x^2 + y^2, \quad R = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$

b) $f(x, y) = xy - 8x - y^2 + 12y + 160, \quad R = \{(x, y) : 0 \leq x \leq 15, 0 \leq y \leq 15 - x\}$

c) $f(x, y) = x^2 + y^2 - 2y + 1, \quad R = \{(x, y) : x^2 + y^2 \leq 4\}$

4. Lagrange multipliers: Find the absolute maximum and minimum values of the functions

a) $f(x, y) = xy^2$ subject to $g(x, y) = x^2 + xy + y^2 - 4$

b) $f(x, y) = x + 2y$ subject to $g(x, y) = x^2 + y^2 = 4$