Lecture notes: Differential Equations for ISE (MA029IU)

Week 7 *

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*This note is taken from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl

1 Higher order linear ODEs

We briefly study higher order equations. Equations appearing in applications tend to be second order. Higher order equations do appear from time to time, but generally the world around us is "second order."

The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by n. The important concept of linear independence is somewhat more complicated when more than two functions are involved. For higher order constant coefficient ODEs, the methods developed are also somewhat harder to apply, but we will not dwell on these complications. It is also possible to use the methods for systems of linear equations from chapter **??** to solve higher order constant coefficient equations.

Let us start with a general homogeneous linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \qquad (1)$$

Theorem 1.1 (Superposition). *Suppose y_1, y_2, …, y_n are solutions of the homogeneous equation* (1). *Then*

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

also solves (1) *for arbitrary constants $C_1, C_2, \ldots, C_n$.*

In other words, a *linear combination* of solutions to (1) is also a solution to (1). We also have the existence and uniqueness theorem for nonhomogeneous linear equations.

Theorem 1.2 (Existence and uniqueness). *Suppose p_0 through p_{n-1}, and f are continuous functions on some interval I, a is a number in I, and $b_0, b_1, \ldots, b_{n-1}$ are constants. The equation*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

1.1 Linear independence

When we had two functions y_1 and y_2 we said they were linearly independent if one was not the multiple of the other. Same idea holds for n functions. In this case it is easier to state as follows. The functions $y_1, y_2, \ldots, y_n$ are *linearly independent* if the equation

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$, where the equation must hold for all x. If we can solve equation with some constants where for example $c_1 \neq 0$, then we can solve for y_1 as a linear combination of the others. If the functions are not linearly independent, they are *linearly dependent*.

Example 1.1: Show that e^x, e^{2x}, e^{3x} are linearly independent.

Let us give several ways to show this fact.

Let us write down

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

We use rules of exponentials and write $z = e^x$. Hence $z^2 = e^{2x}$ and $z^3 = e^{3x}$. Then we have

$$c_1 z + c_2 z^2 + c_3 z^3 = 0.$$

The left-hand side is a third degree polynomial in z. It is either identically zero, or it has at most 3 zeros. Therefore, it is identically zero, $c_1 = c_2 = c_3 = 0$, and the functions are linearly independent.

Let us try another way. As before we write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

This equation has to hold for all x. We divide through by e^{3x} to get

$$c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0.$$

As the equation is true for all x, let $x \longrightarrow \infty$. After taking the limit we see that $c_3 = 0$. Hence our equation becomes

$$c_1 e^x + c_2 e^{2x} = 0.$$

Rinse, repeat!

How about yet another way. We again write

$$c_1e^x + c_2e^{2x} + c_3e^{3x} = 0.$$

We can evaluate the equation and its derivatives at different values of x to obtain equations for c_1, c_2, and c_3. Let us first divide by e^x for simplicity.

$$c_1 + c_2e^x + c_3e^{2x} = 0.$$

We set $x = 0$ to get the equation $c_1 + c_2 + c_3 = 0$. Now differentiate both sides

$$c_2e^x + 2c_3e^{2x} = 0.$$

We set $x = 0$ to get $c_2 + 2c_3 = 0$. We divide by e^x again and differentiate to get $2c_3e^x = 0$. It is clear that c_3 is zero. Then c_2 must be zero as $c_2 = -2c_3$, and c_1 must be zero because $c_1 + c_2 + c_3 = 0$.

There is no one best way to do it. All of these methods are perfectly valid. The important thing is to understand why the functions are linearly independent.

Example 1.2: On the other hand, the functions e^x, e^{-x}, and $\cosh x$ are linearly dependent. Simply apply definition of the hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{or} \qquad 2\cosh x - e^x - e^{-x} = 0.$$

1.2 Constant coefficient higher order ODEs

When we have a higher order constant coefficient homogeneous linear equation, the song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is n^{th} order, we need to find n linearly independent solutions. It is best seen by example.

Example 1.3: Find the general solution to

$$y''' - 3y'' - y' + 3y = 0. \tag{2}$$

Try: $y = e^{rx}$. We plug in and get

$$\underbrace{r^3 e^{rx}}_{y'''} - 3\underbrace{r^2 e^{rx}}_{y''} - \underbrace{r e^{rx}}_{y'} + 3\underbrace{e^{rx}}_{y} = 0.$$

We divide through by e^{rx}. Then

$$r^3 - 3r^2 - r + 3 = 0.$$

The trick now is to find the roots. There is a formula for the roots of degree 3 and 4 polynomials but it is very complicated. There is no formula for higher degree polynomials. That does not mean that the roots do not exist. There are always n roots for an n^{th} degree polynomial. They may be repeated and they may be complex. Computers are pretty good at finding roots approximately for reasonable size polynomials.

A good place to start is to plot the polynomial and check where it is zero. We can also simply try plugging in. We just start plugging in numbers $r = -2, -1, 0, 1, 2, \ldots$ and see if we get a hit (we can also try complex numbers). Even if we do not get a hit, we may get an indication of where the root is. For example, we plug $r = -2$ into our polynomial and get -15; we plug in $r = 0$ and get 3. That means there is a root between $r = -2$ and $r = 0$, because the sign changed. If we find one root, say r_1, then we know $(r - r_1)$ is a factor of our polynomial. Polynomial long division can then be used.

A good strategy is to begin with $r = 0$, 1, or -1. These are easy to compute. Our polynomial has two such roots, $r_1 = -1$ and $r_2 = 1$. There should be 3 roots and the last root is reasonably easy to find. The constant term in a monic* polynomial such as this is the multiple of the negations of all the roots because $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$. So

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3.$$

You should check that $r_3 = 3$ really is a root. Hence e^{-x}, e^x and e^{3x} are solutions to (2). They are linearly independent as can easily be checked, and there are 3 of them, which happens to be exactly the number we need. So the general solution is

$$y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x}.$$

Suppose we were given some initial conditions $y(0) = 1$, $y'(0) = 2$,

*The word monic means that the coefficient of the top degree r^d, in our case r^3, is 1.

and $y''(0) = 3$. Then

$$\begin{aligned} 1 = y(0) &= C_1 + C_2 + C_3, \\ 2 = y'(0) &= -C_1 + C_2 + 3C_3, \\ 3 = y''(0) &= C_1 + C_2 + 9C_3. \end{aligned}$$

It is possible to find the solution by high school algebra, but it would be a pain. The sensible way to solve a system of equations such as this is to use matrix algebra. For now we note that the solution is $C_1 = -1/4$, $C_2 = 1$, and $C_3 = 1/4$. The specific solution to the ODE is

$$y = \frac{-1}{4} e^{-x} + e^x + \frac{1}{4} e^{3x}.$$

Next, suppose that we have real roots, but they are repeated. Let us say we have a root r repeated k times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, \quad xe^{rx}, \quad x^2e^{rx}, \quad \ldots, \quad x^{k-1}e^{rx}.$$

We take a linear combination of these solutions to find the general solution.

Example 1.4: Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0.$$

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0.$$

By inspection we note that $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3$. Hence the roots given with multiplicity are $r = 0, 1, 1, 1$. Thus the general solution is

$$y = \underbrace{(C_1 + C_2 x + C_3 x^2)\, e^x}_{\text{terms coming from } r=1} + \underbrace{C_4}_{\text{from } r=0}.$$

The case of complex roots is similar to second order equations. Complex roots always come in pairs $r = \alpha \pm i\beta$. Suppose we have two such complex roots, each repeated k times. The corresponding solution is

$$(C_0 + C_1 x + \cdots + C_{k-1} x^{k-1})\, e^{\alpha x} \cos(\beta x) + (D_0 + D_1 x + \cdots + D_{k-1} x^{k-1})\, e^{\alpha x} \sin(\beta x).$$

where $C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}$ are arbitrary constants.

Example 1.5: Solve

$$y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0.$$

The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 8r^2 - 8r + 4 &= 0, \\ (r^2 - 2r + 2)^2 &= 0, \\ \big((r-1)^2 + 1\big)^2 &= 0. \end{aligned}$$

Hence the roots are $1 \pm i$, both with multiplicity 2. Hence the general solution to the ODE is

$$y = (C_1 + C_2 x)\, e^x \cos x + (C_3 + C_4 x)\, e^x \sin x.$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots.

1.3 Exercises

***Exercise* 1.1:** *Find the general solution for* $y''' - y'' + y' - y = 0$.

***Exercise* 1.2:** *Find the general solution for* $y^{(4)} - 5y''' + 6y'' = 0$.

***Exercise* 1.3:** *Find the general solution for* $y''' + 2y'' + 2y' = 0$.

***Exercise* 1.4:** *Suppose the characteristic equation for an ODE is* $(r-1)^2(r-2)^2 = 0$.

a) *Find such a differential equation.*

b) *Find its general solution.*

***Exercise* 1.5:** *Suppose that a fourth order equation has a solution* $y = 2e^{4x} x \cos x$.

a) *Find such an equation.*

b) *Find the initial conditions that the given solution satisfies.*

***Exercise* 1.6:** *Find the general solution for the equation of Exercise 1.5.*

***Exercise* 1.7:** *Let* $f(x) = e^x - \cos x$, $g(x) = e^x + \cos x$, *and* $h(x) = \cos x$. *Are* $f(x)$, $g(x)$, *and* $h(x)$ *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 1.8:** *Let* $f(x) = 0$, $g(x) = \cos x$, *and* $h(x) = \sin x$. *Are* $f(x)$, $g(x)$, *and* $h(x)$ *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 1.9:** *Are* x, x^2, *and* x^4 *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 1.10:** *Are* e^x, xe^x, *and* x^2e^x *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 1.11:** *Find an equation such that* $y = xe^{-2x}\sin(3x)$ *is a solution.*

***Exercise* 1.101:** *Find the general solution of* $y^{(5)} - y^{(4)} = 0$.

***Exercise* 1.102:** *Suppose that the characteristic equation of a third order differential equation has roots* $\pm 2i$ *and* 3.

a) *What is the characteristic equation?*

b) *Find the corresponding differential equation.*

c) *Find the general solution.*

***Exercise* 1.103:** *Solve* $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

***Exercise* 1.104:** *Are* e^x, e^{x+1}, e^{2x}, $\sin(x)$ *linearly independent? If so, show it, if not find a linear combination that works.*

***Exercise* 1.105:** *Are* $\sin(x)$, x, $x\sin(x)$ *linearly independent? If so, show it, if not find a linear combination that works.*

***Exercise* 1.106:** *Find an equation such that* $y = \cos(x)$, $y = \sin(x)$, $y = e^x$ *are solutions.*