Cal 3 2017/11

Q1.

a)

$$\frac{5+j}{3-4j}+\frac{20}{4+3j}=\frac{91}{25}-\frac{37}{25}j$$

b)

Let $z_1 = x_1 + jy_1, z_2 = x_2 + jy_2$, it holds that $|z_1|^2 = x_1^2 + y_1^2, |z_2|^2 = x_2^2 + y_2^2$

$LHS = |z_1 + z_2|^2 + |z_1 - z_2|^2 = |x_1 + jy_1 + x_2 + jy_2|^2 + |x_1 + jy_1 - (x_2 + jy_2)|^2$

$= |(x_1 + x_2) + j(y_1 + y_2)|^2 + |(x_1 - x_2) + j(y_1 - y_2)|^2$

$= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= 2(x_1^2 + y_1^2) + 2(x_1^2 + y_1^2) = 2(|z_1|^2 + |z_2|^2) = RHS$

Therefore, $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Q2.

a)

$$f(z) = \overline{z} = x - yi = u(x,y) + jv(x,y)$$

Clearly,

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}\ (1 \neq -1)$$

The given complex function is not satisfied first equation of Cauchy-Riemann equation which implies nowhere differentiable.

b)

Assume that $u(x,y)$ and $v(x,y)$ are already satisfied Cauchy-Riemann equation:

- $u_x = e^{-x}(y\cos y - x\sin y + \sin y) = v_y$

  $\to v = \int v_y\,dy = e^{-x}\big(y\sin y + x\cos y + h(x)\big)$

  $\to v_x = e^{-x}\big(\cos y - y\sin y - x\cos y - h'(x) + h(x)\big)$ (1)
- $u_y = e^{-x}(x\cos y + y\sin y - \cos y) = -v_x$

  $\to v_x = e^{-x}(\cos y - x\cos y - y\sin y)$ (2)

Comparing (1) and (2), we obtain: $h'(x) - h(x) = 0$

$$\to \frac{h'(x)}{h(x)} = 1 \to \ln\big(h(x)\big) = x + C \to h(x) = e^{x+C}$$

Therefore, $v(x,y) = e^{-x}(y\sin y + x\cos y + e^{x+C}) = e^{-x}(y\sin y + x\cos y) + e^C$

Thus, $v(x,y) = e^{-x}(y\sin y + x\cos y) + e^C$, with C be any constant, is complex conjugate of function $u(x,y)$ such that $f(z) = u(x,y) + jv(x,y)$ is an analytic function

Q3.

a)

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

b)

$$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+1)+3\times\frac{1}{3}}{(s+1)^2+3^2}\right\} = \left(2e^{-t}\cos 3t + \frac{1}{3}e^{-t}\sin 3t\right)u(t)$$

Q4.

Given that:

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = te^{2t} + e^{3t} \ \ (*), \qquad y(0) = 0, \quad y'(0) = 1$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$
$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s) - 1] - 5[sY(s)] + 6Y(s) = \frac{1}{(s-2)^2} + \frac{1}{s-3}$$
$$\leftrightarrow Y(s)(s^2 - 5s + 6) = \frac{1}{(s-2)^2} + \frac{1}{s-3} + 1$$
$$\leftrightarrow Y(s) = \frac{\frac{1}{(s-2)^2} + \frac{1}{s-3} + 1}{s^2 - 5s + 6}$$
$$\leftrightarrow Y(s) = \frac{1}{(s-3)^2} + \frac{1}{s-3} - \frac{1}{(s-2)^3} - \frac{1}{(s-2)^2} - \frac{1}{s-2}$$

$$\to y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left(te^{3t} + e^{3t} - \frac{1}{2}t^2e^{2t} - te^{2t} - e^{2t}\right)u(t)$$

Thus, the solution of the given differential equation is:

$$y(t) = \left(te^{3t} + e^{3t} - \frac{1}{2}t^2e^{2t} - te^{2t} - e^{2t}\right)u(t)$$