Cal 3 2019/04

Q1.

a)

$$(1-3j)^2 + \frac{5-2j}{3-2j} = -\frac{85}{13} - \frac{74}{13}j$$

b) Let $P = \left(e^{j\theta} + e^{-j\theta}\right)^2$

Since we know that:

$$\begin{cases} \cos\theta = \dfrac{e^{j\theta}+e^{-j\theta}}{2} \\ \cos 2\theta = \dfrac{e^{j2\theta}+e^{-j2\theta}}{2} \end{cases}$$

Therefore, $P = (2\cos\theta)^2$

However, $P = e^{j2\theta} + e^{-j2\theta} + 2 = 2\cos 2\theta + 2$

Hence, $(2\cos\theta)^2 = 2\cos 2\theta + 2$

Or it equivalent with

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

Q2.

a)

1. For any $z_0 \in \mathbb{C}$, we have:

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \overline{z} = \overline{z_0}$$

The limit exists which leads to $f(z)$ is continuous at any z_0 or everywhere

2. $f(z) = \overline{z} = x - yi = u(x,y) + jv(x,y)$

Clearly,

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ (1 \neq -1)$$

The given complex function is not satisfied first equation of Cauchy-Riemann equation which implies nowhere differentiable.

From both reasons above, $f(z)$ is continuous at everywhere but nowhere differentiable.

b)

$$f(z) = \frac{z+2}{z^2-5z+4} = -\frac{1}{z-1} + \frac{2}{z-4}$$

Apply power series for analyzing this problem:

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

We have:

$$f(z) = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\frac{1}{1-\frac{z}{4}}$$

With $1 < |z| \leftrightarrow \frac{1}{|z|} < 1$, it holds that:

$$\frac{1}{1-\frac{1}{z}} = \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^n$$

With $|z| < 4 \leftrightarrow \left|\frac{z}{4}\right| < 1$, it holds that:

$$\frac{1}{1-\frac{z}{4}} = \sum_{n=0}^{+\infty} \left(\frac{z}{4}\right)^n$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{1}{z}\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{4}\right)^n \\ &= -\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}} - \frac{1}{2}\sum_{n=0}^{+\infty}\frac{1}{4^n}z^n \\ &= -\sum_{n=0}^{+\infty}\left(\frac{1}{z^{n+1}} + \frac{1}{2^{2n+1}}z^n\right) \end{aligned}$$

Q3.

a)

$$\mathcal{L}\{5\sin 3t - 7e^{-2t} + t^3\} = \frac{15}{s^2+3^2} - \frac{7}{s+2} + \frac{6}{s^4}$$

b)

Since we have:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2t)\,u(t)$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}e^{-3s}\right\} = \cos\big(2(t-3)\big)\,u(t-3)$$

Q4.

Given that:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 4t^2 \ \ (*), \qquad y(0) = 1, \quad y'(0) = 4$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$$
$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 4$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s) - s - 4] - [sY(s) - 1] - 2Y(s) = \frac{8}{s^3}$$

$$\leftrightarrow Y(s)(s^2 - s - 2) = \frac{8}{s^3} + s + 3$$

$$\leftrightarrow Y(s) = \frac{\frac{8}{s^3} + s + 3}{s^2 - s - 2}$$

$$\leftrightarrow Y(s) = -\frac{4}{s^3} + \frac{2}{s^2} - \frac{3}{s} + \frac{2}{s-2} + \frac{2}{s+1}$$

$\rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = (-2t^2 + 2t - 3 + 2e^{2t} + 2e^{-t})u(t)$

Thus, the solution of the given differential equation is:

$$y(t) = (-2t^2 + 2t - 3 + 2e^{2t} + 2e^{-t})u(t)$$