Cal 3 2016/08

Q1.

Rewrite $f(t)$ as unit step function we have:

$$f(t) = t[u(t) - u(t-1)]$$

$$\rightarrow F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2} + \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}$$

$(\mathcal{L}\{tu(t-1)\} = \mathcal{L}\{(t-1+1)u(t-1)\} = \mathcal{L}\{(t+1)u(t)\}e^{-s})$

Q2.

Rewrite $f(t)$ as unit step function we have:

$$f(t) = (t-4)u(t) + (2t^2 - t - 4)u(t-3) - 2t^2u(t-5)$$

Q3.

$$\mathcal{L}^{-1}\left\{\frac{s+10}{(s^2+4)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-s+3}{s^2+4} + \frac{1}{s+2}\right\} = -\cos 2t + \frac{3}{2}\sin 2t + e^{-2t}$$

Q4.

Given that:

$$y_{k+2} - 5y_{k+1} + 6y_k = 4^n \ \ (*), \qquad y_0 = 0, \quad y_1 = 1$$

Let $Y(z) = \mathcal{Z}\{y_k\}$, it holds that:

$$\mathcal{Z}\{y_{k+1}\} = zY(z) - zy_0 = zY(z)$$

$$\mathcal{Z}\{y_{k+2}\} = z^2Y(z) - z^2y_0 - zy_1 = z^2Y(z) - z$$

Taking $\mathcal{Z}$-transform both side of $(*)$, we obtain:

$$[z^2Y(z) - z] - 5[zY(z)] + 6[Y(z)] = \frac{z}{z-4}$$

$$\leftrightarrow Y(z)(z^2 - 5z + 6) = \frac{z}{z-4} + z$$

$$\rightarrow \frac{Y(z)}{z} = \frac{1}{(z-4)(z^2-5z+6)} + \frac{1}{z^2-5z+6}$$

$$\leftrightarrow \frac{Y(z)}{z} = \frac{1}{2}\left(\frac{1}{z-4} - \frac{1}{z-2}\right)$$

$$\rightarrow Y(z) = \frac{1}{2}\frac{1}{z-4} - \frac{1}{2}\frac{1}{z-2}$$

a)

$$Y(z) = \frac{1}{2}\frac{1}{z-4} - \frac{1}{2}\frac{1}{z-2}$$

b)

$\rightarrow y_k = \mathcal{Z}^{-1}\{Y(z)\} = \frac{1}{2}.4^k - \frac{1}{2}.2^k$

Thus, the solution of the given system difference equations is:

$$y_k = \frac{1}{2}.4^k - \frac{1}{2}.2^k$$

Q5.

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-1}^{1}(1-|t|)e^{-j\omega t}dt$$

$$= \int_0^1 (1-t)e^{-j\omega t}dt = \left[\frac{t-1}{j\omega}e^{-j\omega t} + \frac{1}{(j\omega)^2}e^{-j\omega t}\right]\Bigg|_0^1$$

$$= \frac{1}{(j\omega)^2}e^{-j\omega} - \frac{1}{(j\omega)^2} + \frac{1}{j\omega} = \frac{1}{\omega^2}\big(-\cos\omega + 1 + j(\sin\omega - \omega)\big)$$

Q6.

We have:

$$f(t)=\mathcal{F}^{-1}\{F(\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{2\sin\omega}{\omega}e^{-j\omega t}dt$$
$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin\omega}{\omega}(\cos\omega t+j\sin\omega t)dt$$
$$=\frac{1}{\pi}\left[\int_{-\infty}^{+\infty}\frac{\sin\omega\cos\omega t}{\omega}dt+j\int_{-\infty}^{+\infty}\frac{\sin\omega\sin\omega t}{\omega}dt\right]$$

Since we have: $\frac{\sin\omega\sin\omega t}{\omega}$ is an odd function with respect to ω, which leads to:

$$\int_{-\infty}^{+\infty}\frac{\sin\omega\sin\omega t}{\omega}dt=0$$

Therefore,

$$f(t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin\omega\cos\omega t}{\omega}dt$$

Since $f(t)=0$ for $t>1$

Thus,

$$\int_{-\infty}^{+\infty}\frac{\sin\omega\cos\omega t}{\omega}dt=0,\qquad\forall t>1$$

Q7.

Given that: $\quad f(t)=1-|t|,\quad -1\le t\le 1,\quad T=2\to\omega=\frac{2\pi}{T}=\pi$

a)

Since, $f(t)$ is an even function, so:

- $a_0=\frac{4}{T}\int_0^{T/2}f(t)dt=\frac{4}{2}\int_0^1(1-t)dt=1$

b)

- $a_n=\frac{4}{T}\int_0^{T/2}f(t)\cos(n\omega t)\,dt=\frac{4}{2}\int_0^1(1-t)\cos(n\pi t)\,dt$

$$=2\left[\frac{1-t}{n\pi}\sin(n\pi t)-\frac{1}{(n\pi)^2}\cos(n\pi t)\right]\Bigg|_0^1$$
$$=\frac{2}{n^2\pi^2}(1-(-1)^n)$$

For any even integer n: $a_n=0$

c)

For any odd integer $n=2k-1, k\ge 1$: $a_n=4/n^2\pi^2$

d)

Due to the given function is even, therefore, $b_n=0$

e)

By Parseval's identity we obtain:

$$\frac{1}{2}\int_{-1}^{1}(1-|t|)^2dt=\frac{1}{4}1^2+\frac{1}{2}\sum_{n=1}^{+\infty}\left[\frac{2}{n^2\pi^2}(1-(-1)^n)\right]^2$$
$$\leftrightarrow\frac{1}{3}=\frac{1}{4}+\frac{1}{2}\sum_{k=1}^{+\infty}\left[\frac{4}{(2k-1)^2\pi^2}\right]^2\leftrightarrow\sum_{k=1}^{+\infty}\frac{1}{(2k-1)^4}=\frac{\pi^4}{96}$$