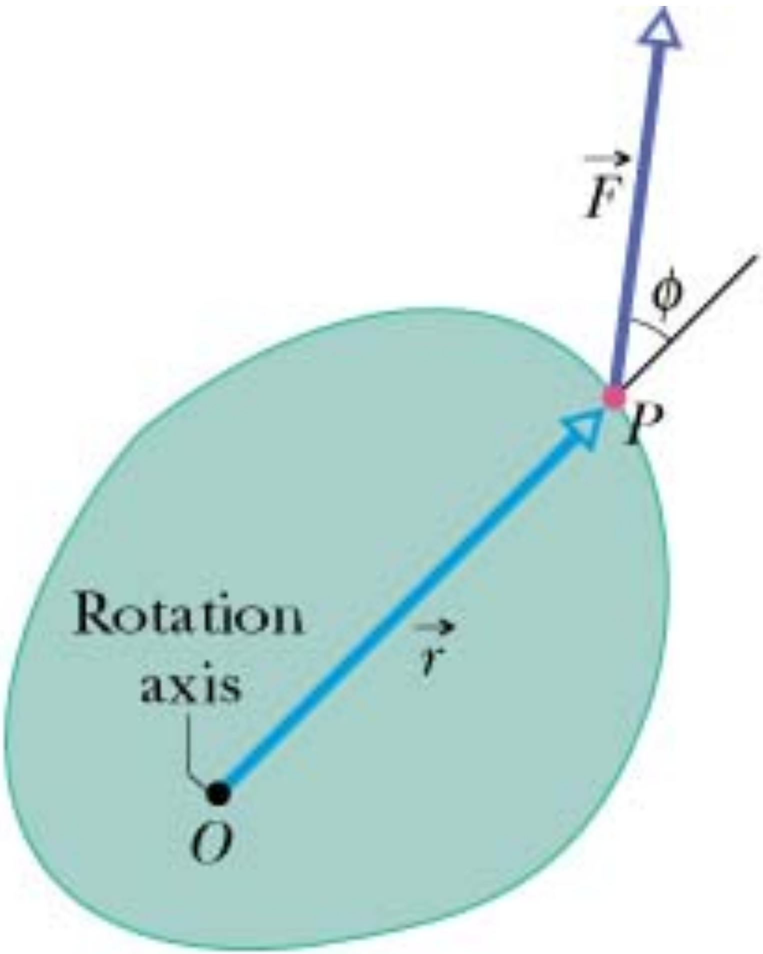
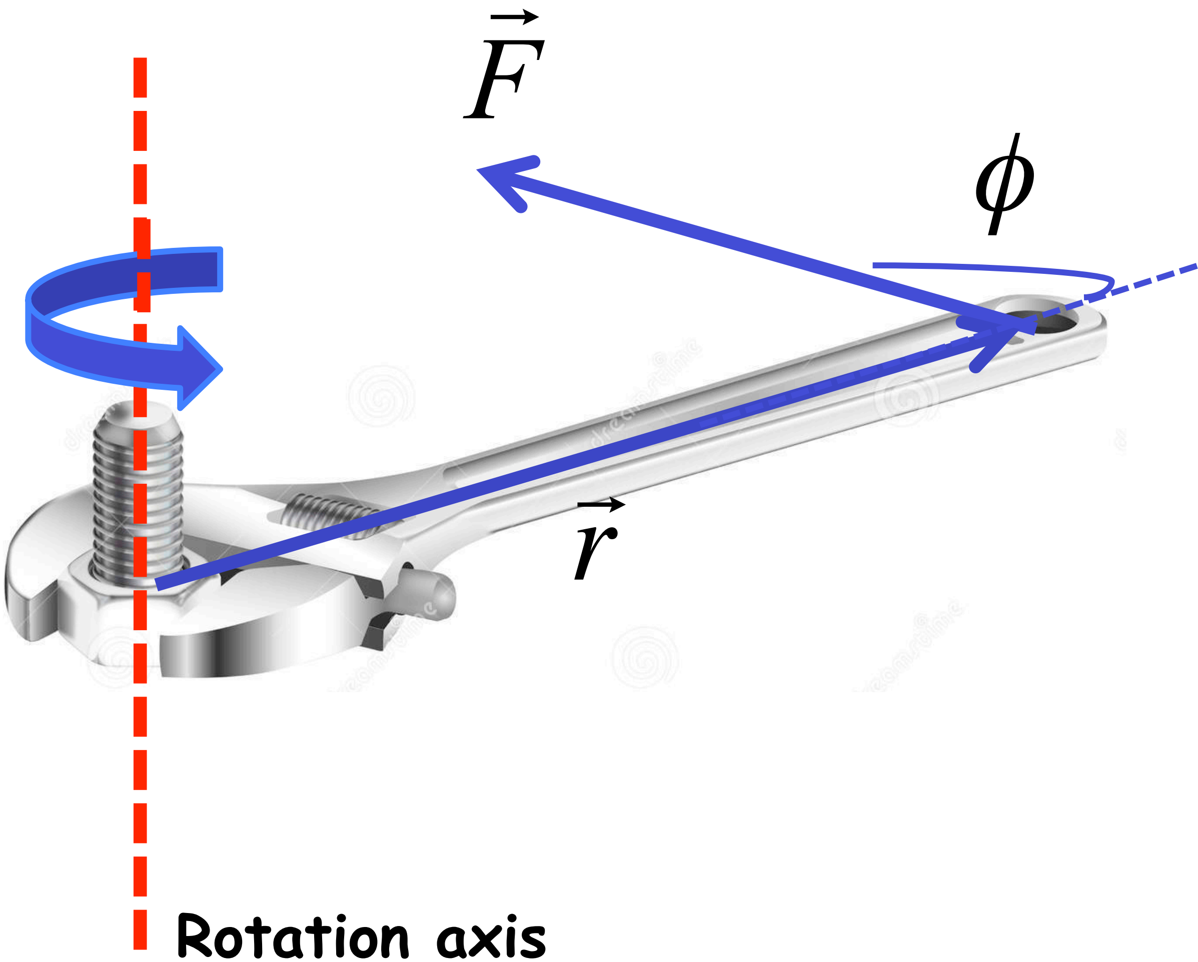
# Part C Dynamics and Statics of Rigid Body
# Chapter 5 Rotation of a Rigid Body About a Fixed Axis

## 5.4. Torque, and Newton's Second Law for Rotation

### a. Torque ("to twist"):

•A force $\vec{F}$ applied at point P of a body that is free to rotate about an axis through O. The force $\vec{F}$ has no component parallel to the rotation axis.



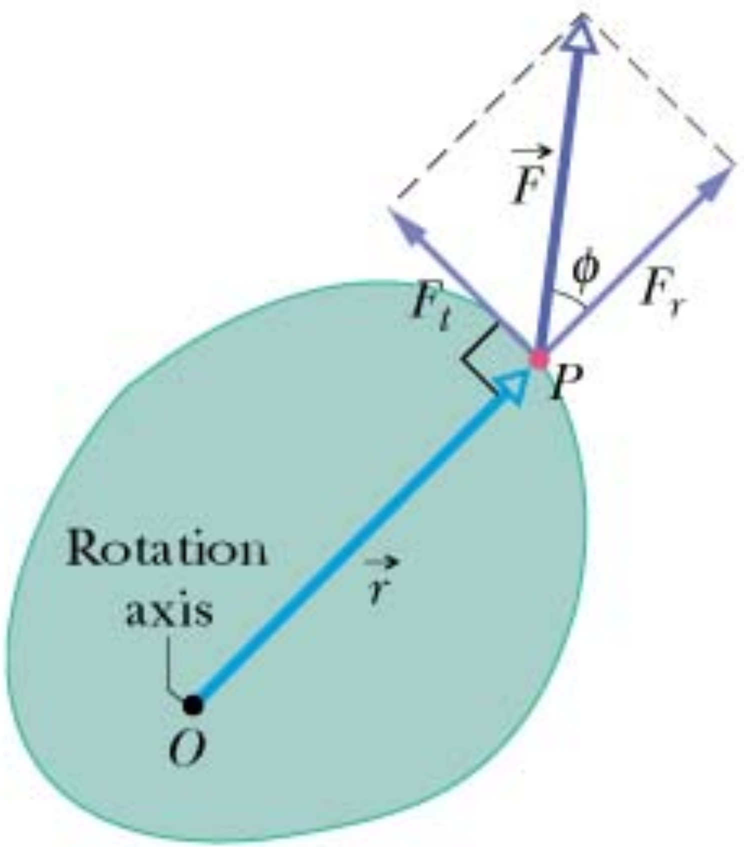



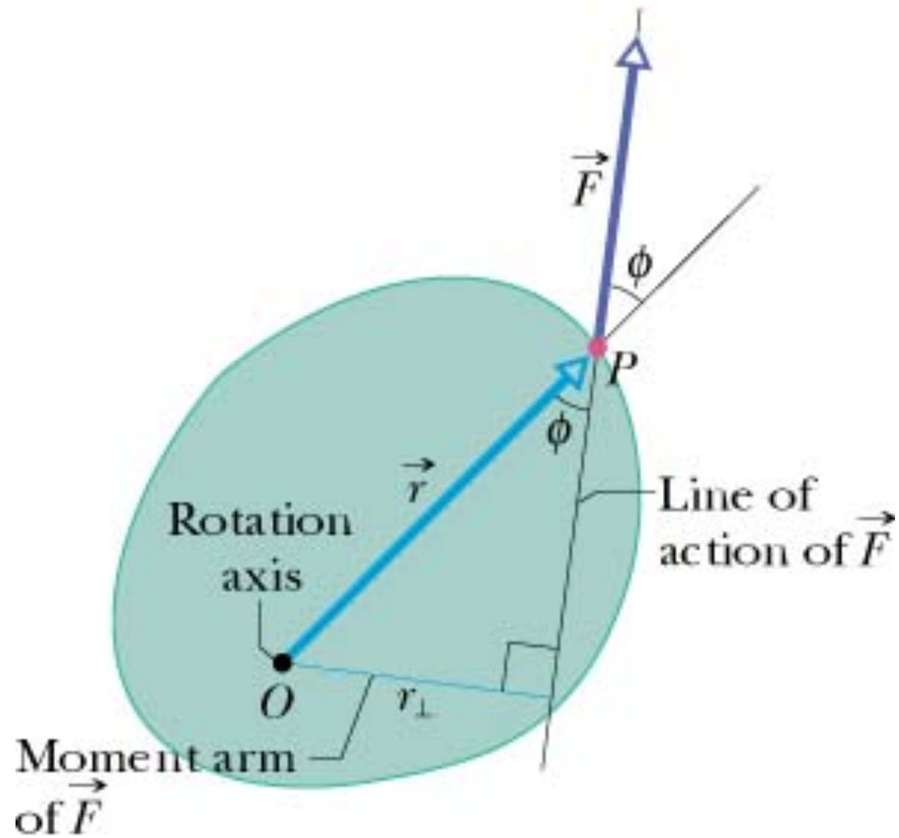
- Resolve $\vec{F}$ into 2 components: $F_t$ (tangential) and $F_r$ (radial).
- The ability of $\vec{F}$ to rotate the body is defined by torque τ:

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$ (Unit: N.m)

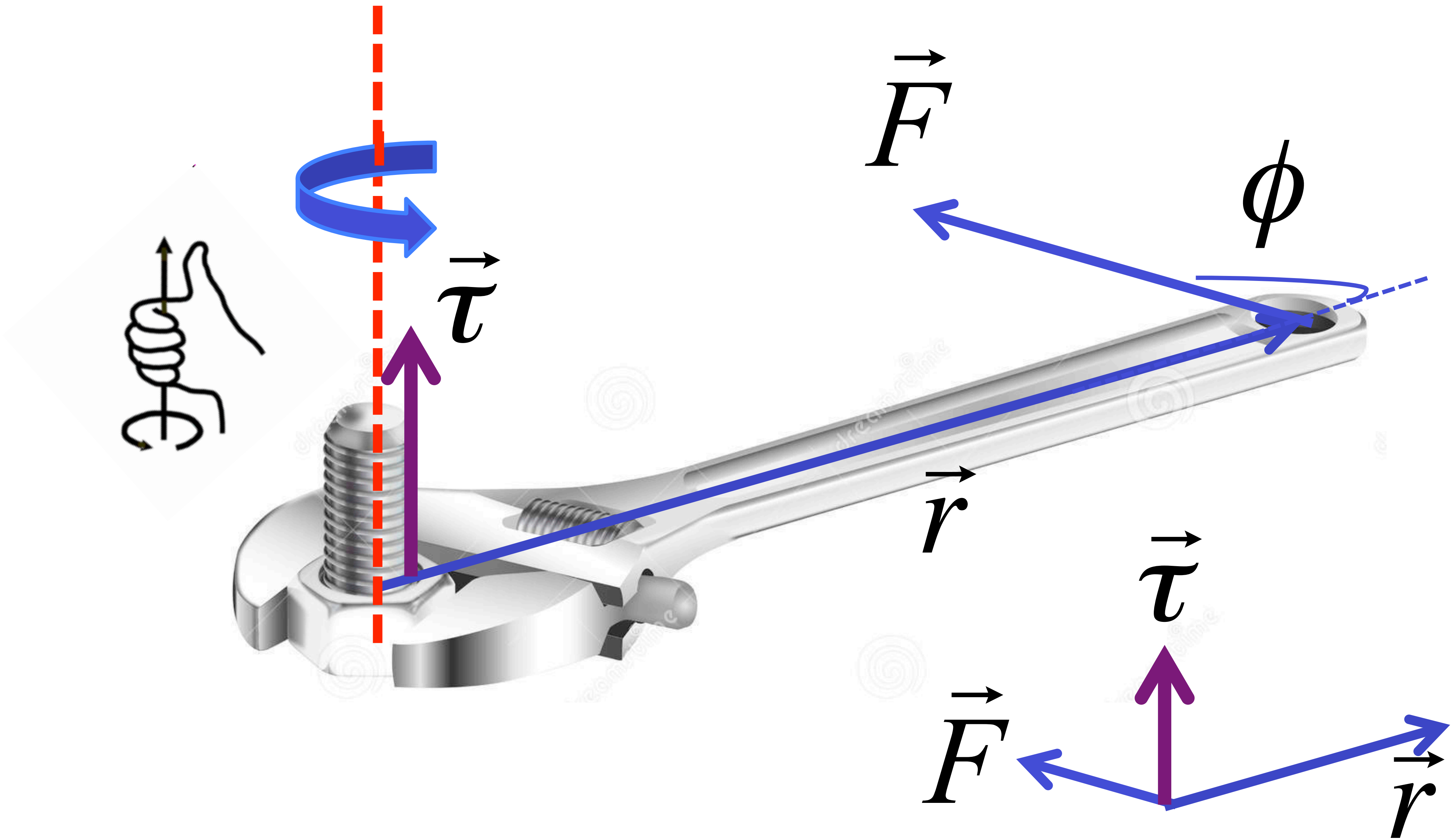
$$\boxed{\begin{aligned} \tau &= r(F\sin\phi) = rF_t \\ \tau &= (r\sin\phi)F = r_\perp F \end{aligned}}$$

$r_\perp$ : the moment arm of $\vec{\mathrm{F}}$

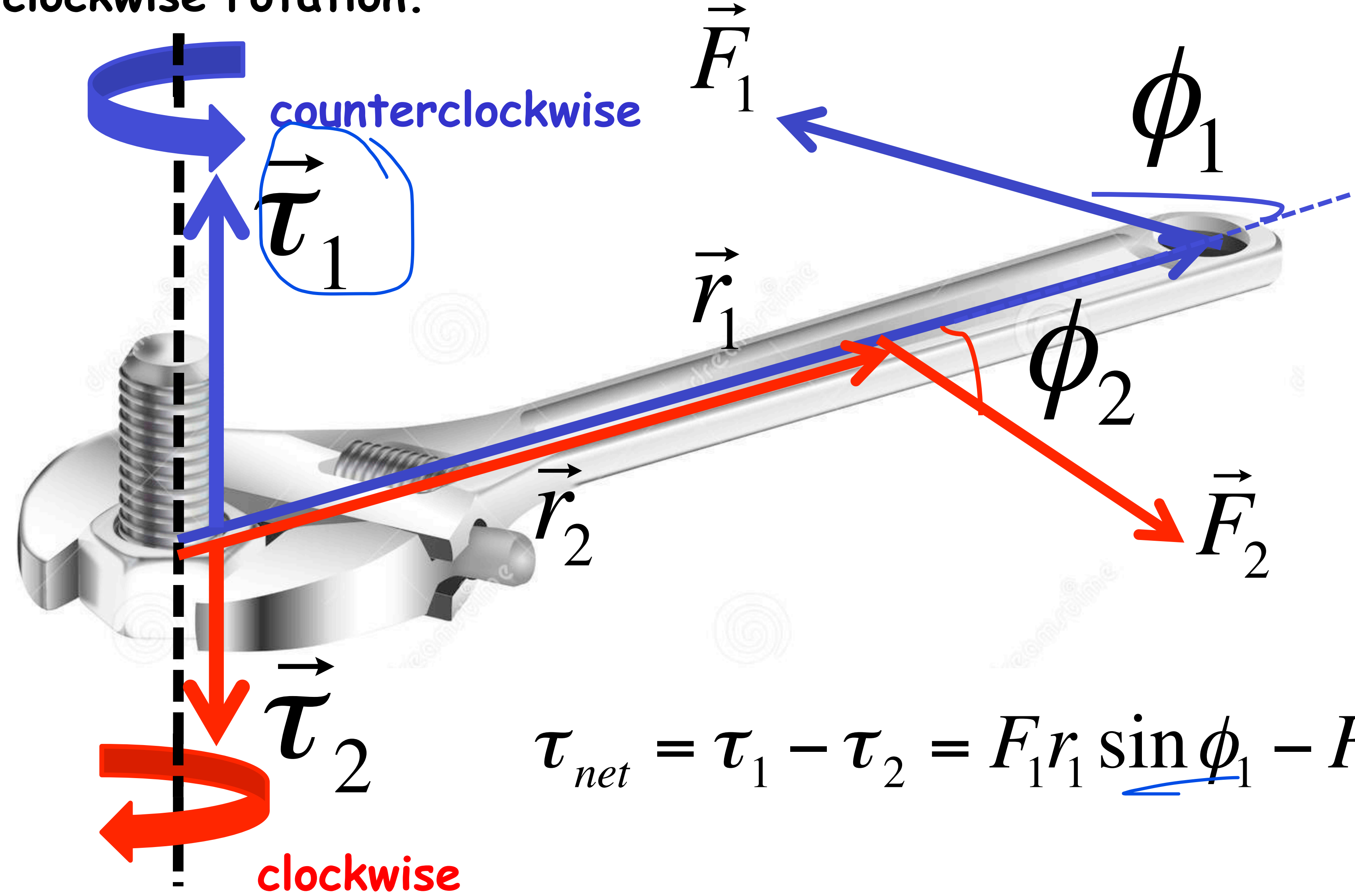


**Direction of torque:** Use the right hand rule to determine

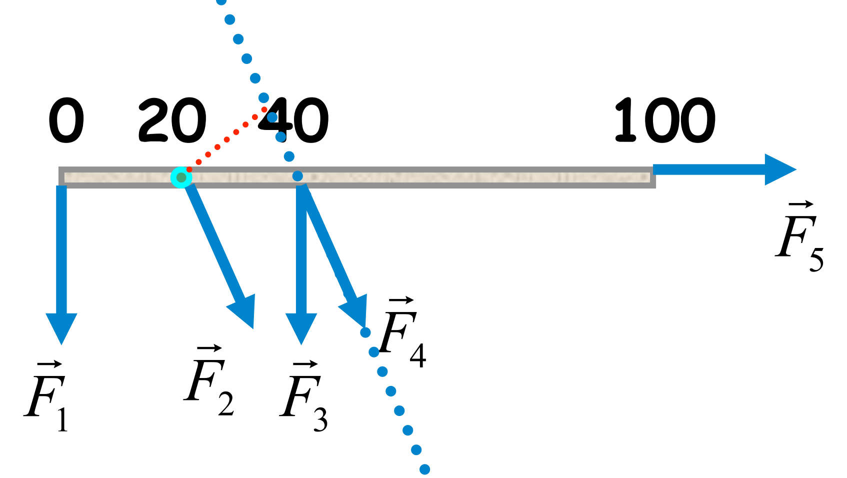


**Important Note:** Torque is a vector quantity, however, because we consider only rotation around a single axis, we therefore do not need vector notation. Instead, a torque is a positive value if it would produce a counterclockwise rotation and a negative value for a clockwise rotation.




$$\tau_{net} = \tau_1 - \tau_2 = F_1 r_1 \sin\phi_1 - F_2 r_2 \sin\phi_2$$

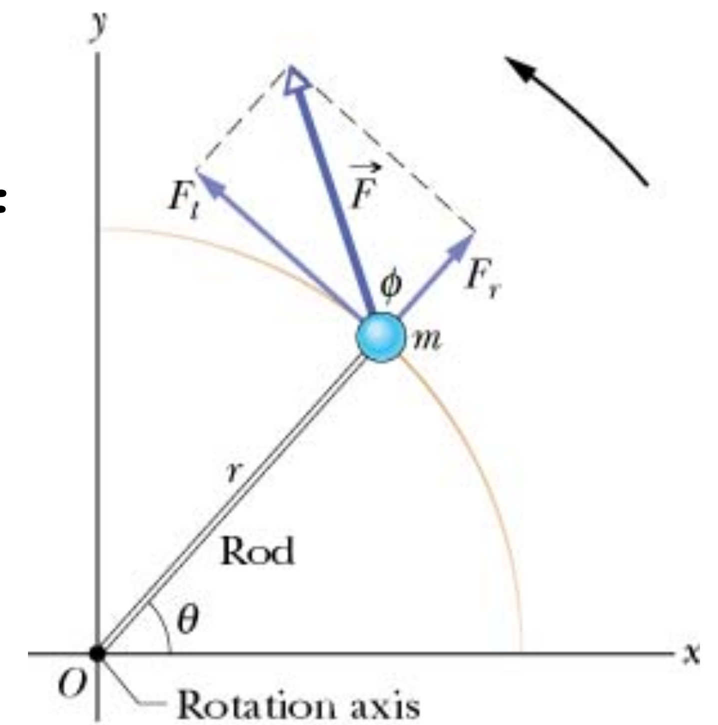
**Checkpoint 6 (p. 259):** A meter stick can pivot about the dot at the position marked 20 (cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to their torque magnitude, greatest first.




F1-F3, F4, F2-F5 (0)

## b. Newton's Second Law for Rotation

- We consider a simple problem, the rotation of a rigid body consisting of a particle of mass m on one end of a massless rod.




$$F_t = ma_t$$ ($a_t$: tangential acceleration)

The torque acting on the particle:

$$\tau = F_t r = ma_t r$$

$$\tau = m(\alpha r)r = (mr^2)\alpha$$

α: angular acceleration

$$\tau = \overbrace{(mr^2)}^{I}\alpha = I\alpha \text{ (radian measure)}$$

if more than one force applied to the particle:

$$\boxed{\tau_{\text{net}} = I\alpha}$$

This equation (Newton's second law for rotation) is valid for any rigid body rotating about a fixed axis.

## 5.5. Work and Rotational Kinetic Energy

•We consider a torque of a force F accelerates a rigid body in rotation about a fixed axis:

The work-kinetic theorem applied for rotation of a rigid body:

$$\boxed{\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W}$$

The work done by the torque:

$$\Delta K = \int_{\theta_i}^{\theta_f} \tau d\theta$$

If $\tau$ is a constant:

$$W = \tau(\theta_f - \theta_i)$$

The power:

$$\boxed{P = \frac{dW}{dt} = \tau\omega}$$

$\tau$ $F$

**Problem 50 (p. 270):** If a 40.0 N.m torque on a wheel causes angular acceleration 25.0 rad/s², what is the wheel's rotational inertia?

$\alpha$

$$\boxed{\tau = I\alpha}$$

(N.m) (rad/s²)

$$\tau = 40.0\text{N m}; \alpha = 25.0\,\text{rad/s}^2 \Rightarrow I = \frac{\tau}{\alpha} = \frac{40.0}{25.0} = 1.6\,(\text{kg m}^2)$$

**Problem 61 (p. 271):** A 32.0 kg wheel, essentially a thin hoop with radius 1.2 m, is rotating at 280 rev/min. It must be brought to a stop in 15.0 s. (a) How much work must be done to stop it?
(b) What is the required average power?

- The rotational inertia of a wheel (a thin hoop) about central axis:

$$I = MR^2 = 32.0 \times 1.2^2 = 46.1\,(\mathrm{kg\,m^2})$$

- To stop the wheel, ω=0:

$$\omega_0 = 280\,\mathrm{rev/min} = \frac{280 \times 2\pi\,(\mathrm{rad})}{60\,(\mathrm{s})} = 29.3\,(\mathrm{rad/s})$$

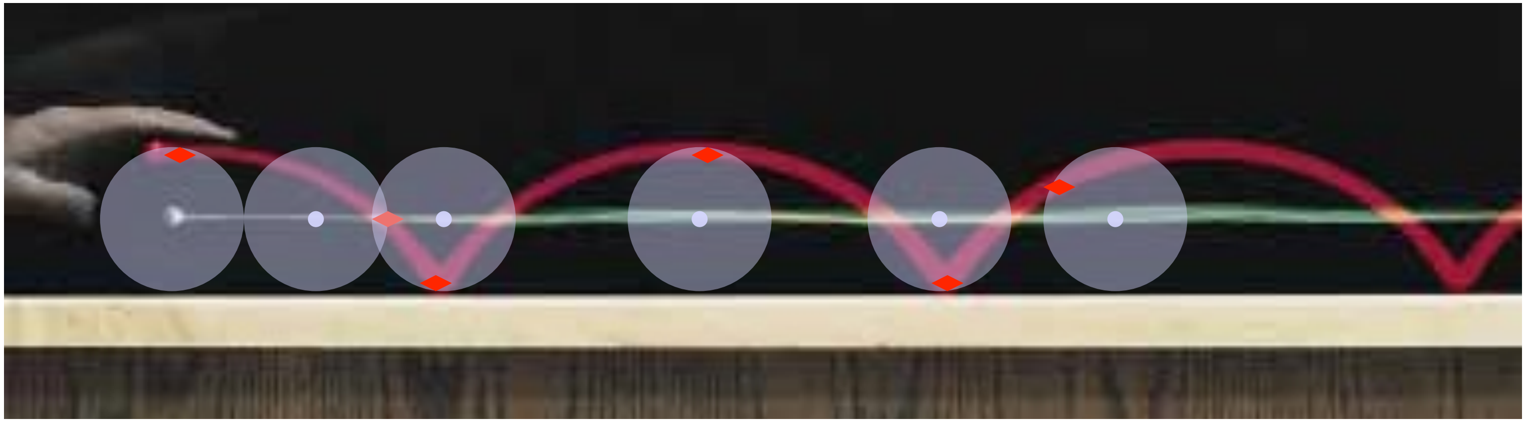
(a) The work is needed to stop the wheel:

$$W = K_f - K_i = 0 - \frac{1}{2} I\omega_0^2 = -\frac{1}{2} 46.1 \times 29.3^2 \approx -19788\,(\mathrm{J}) \text{ or } \approx -19.8\,(\mathrm{kJ})$$

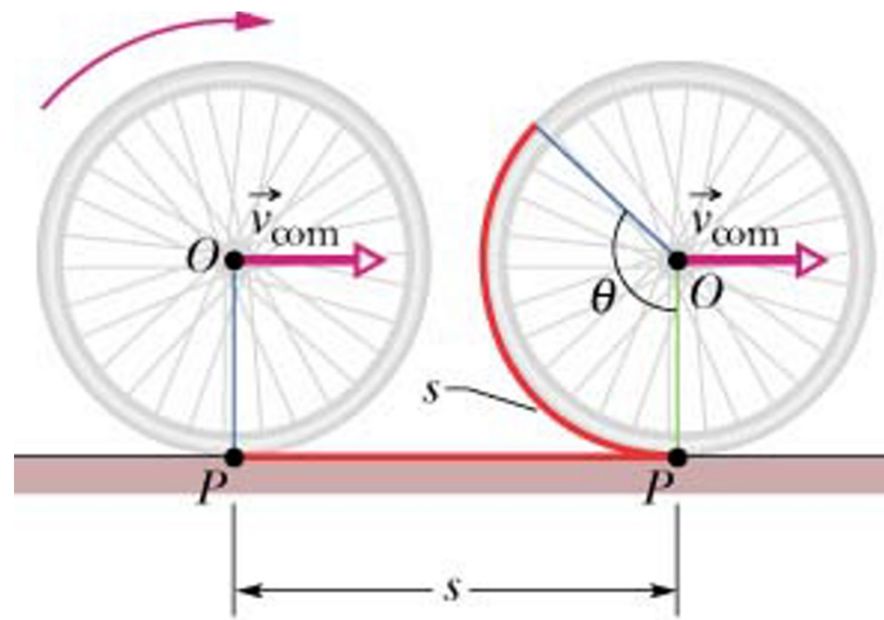
(W<0: energy transferred from the wheel)

(b) The average power: $P = \frac{|W|}{\Delta t} = \frac{19788}{15} \approx 1319\,(\mathrm{W}) \text{ or } \approx 1.32\,(\mathrm{kW})$

# 5.6. Rolling Motion of a Rigid Body



•We consider a rigid body smoothly rolling along a surface. Its motion consists of two motions: translation of the center of mass and rotation of the rest of the body around that center.




Example: A bike wheel is rolling along a street. During a time interval t, both O (the center of mass) and P (the contact point between the wheel and the street) move by a distance s:

$$\boxed{s = \theta R}$$

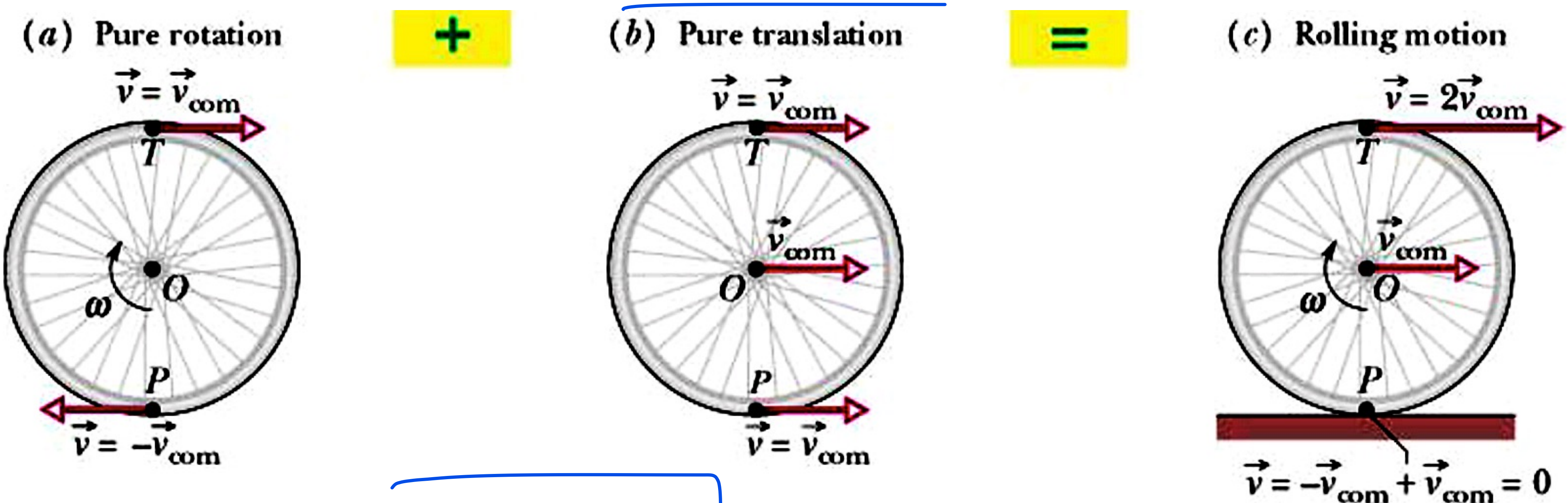
where R is the radius of the wheel.

•The speed of the center of the wheel:

$$\boxed{v_{com} = \omega R}$$

•The linear acceleration of the center of the wheel:

$$\boxed{a_{com} = \alpha R}$$

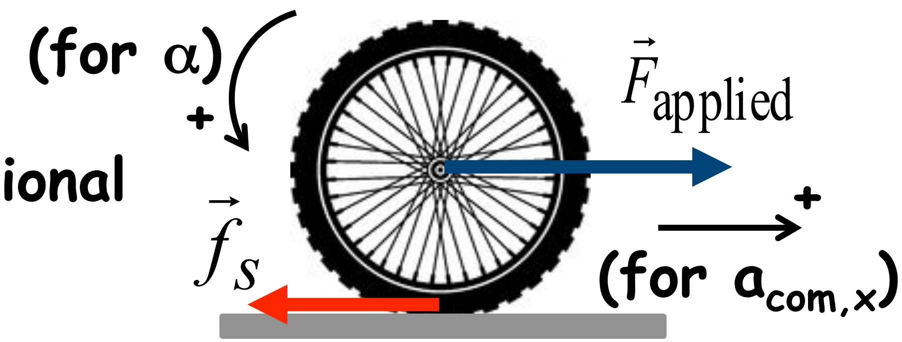



## The Kinetic Energy of Rolling

$$\boxed{K = \underbrace{\frac{1}{2} I_{com}\omega^2}_{\text{Rotational KE}} + \underbrace{\frac{1}{2} M v^2_{com}}_{\text{Translational KE}}}$$

## Examples:

**1. Rolling on a horizontal surface**: A 2.0 kg wheel, rolling smoothly on a horizontal surface, has a rotational inertia about its axis I = $MR^2/2$, where M is its mass and R is its radius. A horizontal force is applied to the axle so that the center of mass has an acceleration of 4.0 m/s$^2$. What is the magnitude of the frictional force of the surface acting on the wheel? (Final exam, June 2014)




•Applying Newton's second law for rotational motion:

$$\tau_{net} = -Rf_s = I_{com}\alpha \, (1)$$

$$a_{com,x} > 0 \text{ (in the positive direction of the x axis)}$$

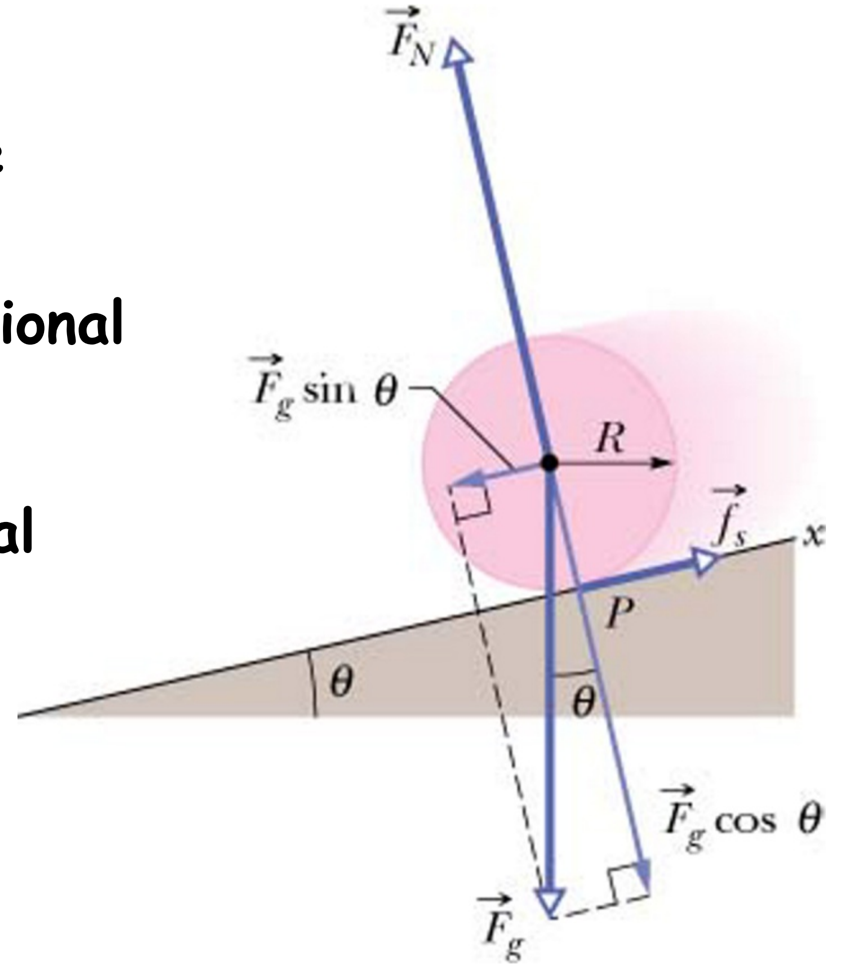
$$\alpha < 0 \text{ (clockwise)}$$

$$a_{com,x} = -\alpha R \, (2)$$

(1), (2):

$$f_s = -I_{com}\frac{\alpha}{R} = \frac{MR^2}{2}\frac{a_{com,x}}{R^2} = \frac{Ma_{com,x}}{2} = \frac{2 \times 4.0}{2} = 4.0 \text{ (N)}$$

**2. Rolling Down a Ramp:** We consider a rigid body smoothly rolling down a ramp.




Force analysis: $F_g$, $F_N$, and $f_s$ (opposing the sliding of the body, so the force is up the ramp)

- Applying Newton's second law for translational motion:

$$f_s - Mg\sin\theta = Ma_{com,x} \quad (1)$$

- Applying Newton's second law for rotational motion:

$$\tau_{net} = Rf_s = I_{com}\alpha \quad (2)$$

$a_{com,x} < 0$ (in the negative direction of the x axis)

$\alpha > 0$ (counterclockwise)

$$a_{com,x} = -\alpha R \quad (3)$$

(1), (2), (3):

$$\boxed{\Rightarrow a_{com,x} = -\frac{g\sin\theta}{1 + \dfrac{I_{com}}{MR^2}}}$$

## 3. The Yo-Yo: (homework)

For translation motion:

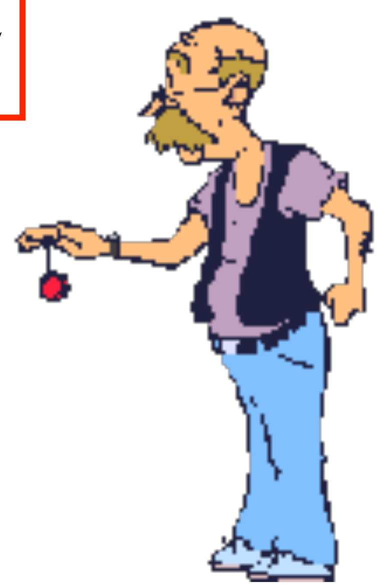
$$T - Mg = Ma_{com,y} \quad (1)$$

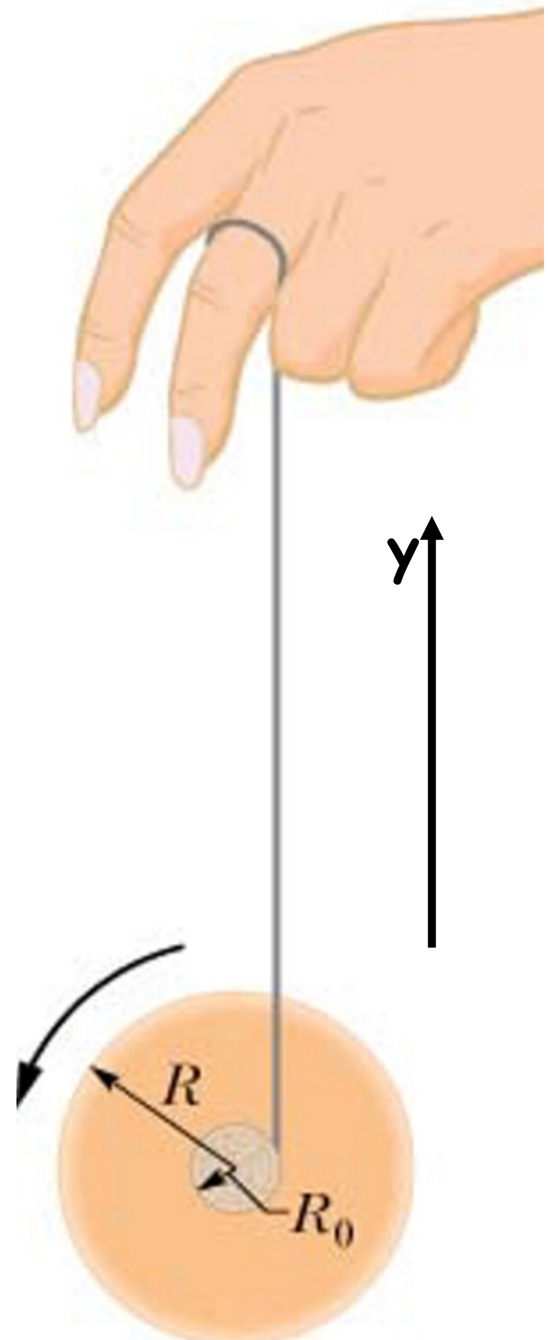
For rotational motion:

$$\tau = R_0 T = I_{com}\alpha \quad (2)$$

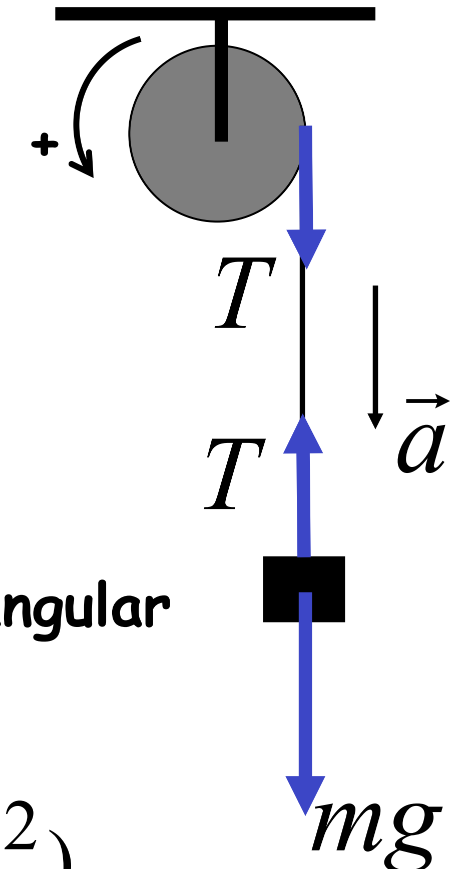
$$a_{com,y} = -\alpha R_0 \quad (3)$$

$$\boxed{\Rightarrow a_{com,y} = -\frac{g}{1 + \dfrac{I_{com}}{MR_0^2}}}$$






**4. Pulley:** A 10.0 kg block hangs from a cord which is wrapped around the rim of a frictionless pulley. Calculate the acceleration, *a*, of the block as it moves down? (The rotational inertia of the pulley is 0.50 kg m$^2$ and its radius is 0.10 m). (Final exam, January 2014)




- For translation motion:

$$mg - T = ma \,(1)$$

- For rotational motion:

$$\tau = -R_0 T = I_{\text{pulley}}\alpha \,(2)$$

$$a = -\alpha R_0 \,(3)$$

(Note: a is the acceleration of the block, $\alpha$ is the angular acceleration of the pulley)

$$\Rightarrow a = \frac{mg}{m + \dfrac{I_{\text{pulley}}}{R^2}} = \frac{10 \times 9.8}{10 + \dfrac{0.5}{0.1^2}} \approx 1.63\,(\text{m/s}^2)$$

**Example:** (The Kinetic Energy of Rolling) A 10 kg cylinder rolls without slipping. When its translational speed is 10 m/s, What is its translational kinetic energy, its rotational kinetic energy, and its total kinetic energy?

- The rotational inertia of a cylinder about central axis:

$$I = \frac{1}{2}MR^2$$

$v = \omega R$

- Translational kinetic energy:

$$K_k = \frac{1}{2}Mv^2 = \frac{1}{2}10 \times 10^2 = 500\,(\mathrm{J}) \quad (1)$$

- Rotational kinetic energy:

$$K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{4}Mv^2$$

The Kinetic Energy of Rolling

$$K_r = 250\,(\mathrm{J}) \quad (2)$$

- Total kinetic energy:

$$(1)(2) \Rightarrow K = K_k + K_r = 750\,(\mathrm{J})$$

# 5.7. Angular Momentum of a Rotating Rigid Body

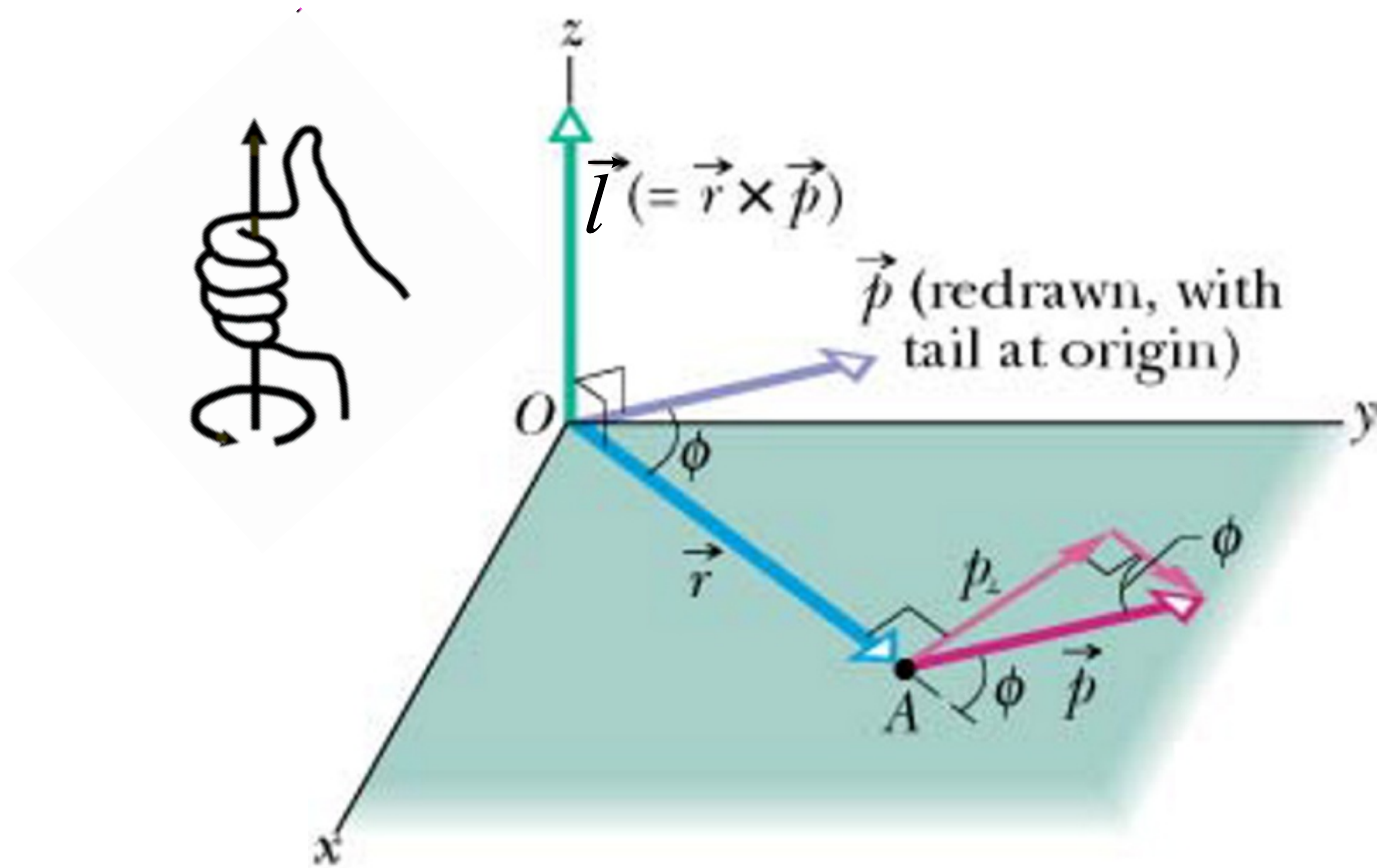
## a. Angular Momentum of a Particle:

$$\boxed{\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})}$$

$\vec{r}$ : the position vector of the particle with respect to O

$\vec{p}$ : the linear momentum of the particle

•The direction of $\vec{l}$ determined by the right-hand rule

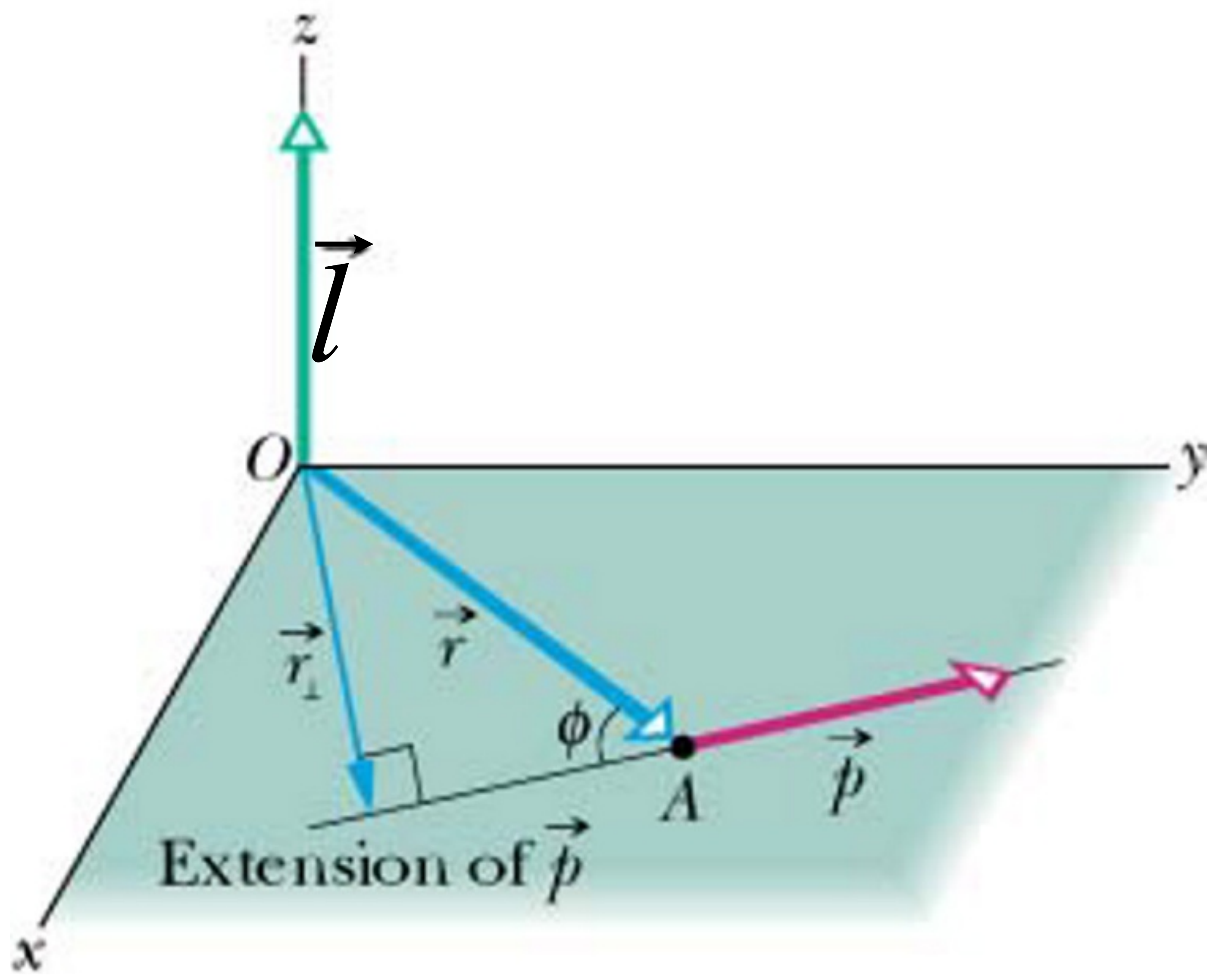


$$\boxed{l = rmv\sin\phi}$$

(Unit: kg m² s⁻¹)

or

$$l = r_{\perp}p = rp_{\perp}$$



## Newton's Second Law in Angular Form for a Particle:

For translational motions: $\vec{F}_{net} = \frac{d\vec{p}}{dt}$

$$\underbrace{\vec{r} \times \vec{F}_{net}}_{} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\vec{\tau}_{net} = m(\vec{r} \times \frac{d\vec{v}}{dt} + \vec{v} \times \vec{v})$$

$$\vec{\tau}_{net} = m(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}) = m\frac{d(\vec{r} \times \vec{v})}{dt} = \frac{d\vec{l}}{dt}$$

So, Newton's Second Law in Angular Form: $\boxed{\vec{\tau}_{net} = \frac{d\vec{l}}{dt}}$

Note: The torques $\vec{\tau}$ and the angular momentum $\vec{l}$ must be defined with respect to the same origin O.

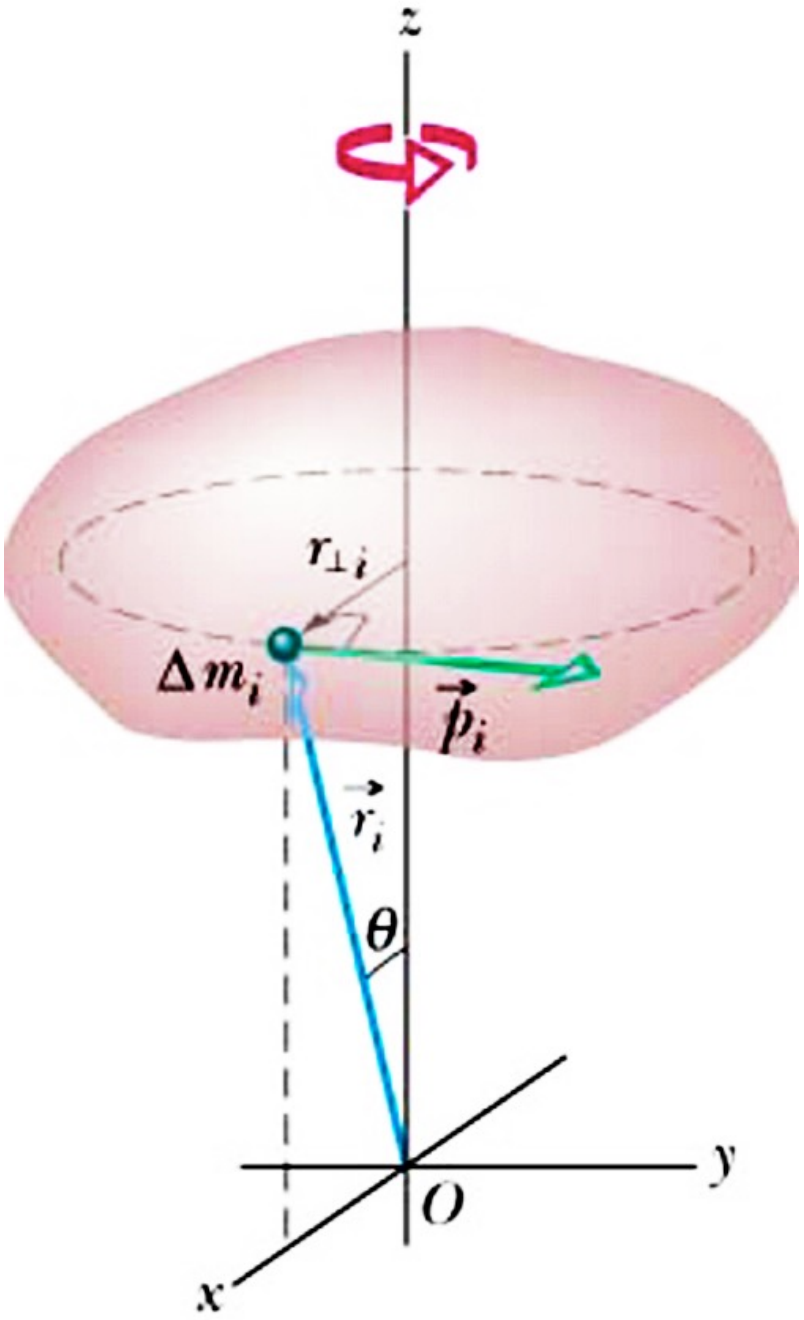
## b. Angular Momentum of a System of Particles:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \ldots + \vec{l}_n = \sum^{n} \vec{l}_i$$

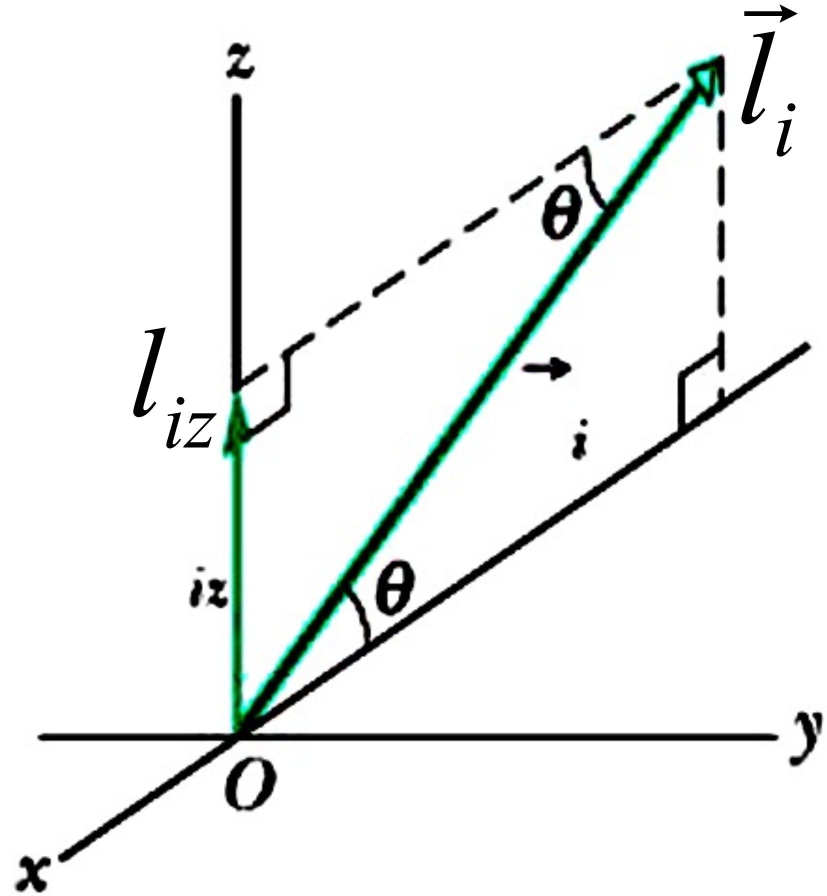
**Newton's Second Law in Angular Form:**

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}}$$

## c. Angular Momentum of a Rotating Rigid Body:




**Method:** To calculate the angular momentum of a body rotating about a fixed axis (here the z axis), we evaluate the angular momentum of a system of particles (mass elements) that form the body.




$\vec{l}_i$ **is the angular momentum of element i of mass $\Delta m_i$:**

$$l_i = r_i p_i \sin 90^0 = r_i \Delta m_i v_i$$

$$l_{i,z} = l_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i) = r_{\perp i} \Delta m_i v_i$$

$$L_z = \sum_{i=1}^{n} l_{i,z} = \sum_{i=1}^{n} \Delta m_i v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i}$$

$$L_z = \omega \underbrace{\sum_{i=1}^{n} \Delta m_i r_{\perp i}^2}$$

$I_z$ : rotational inertia of the body about the z axis

$$L_z = I_z \omega$$

**we drop the subscript z:**

$$\boxed{L = I\omega}$$

**Note:** **L and I are the angular momentum and the rotational inertia of a body rotating about the same axis.**

# 5.8. Conservation of Angular Momentum

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}}$$

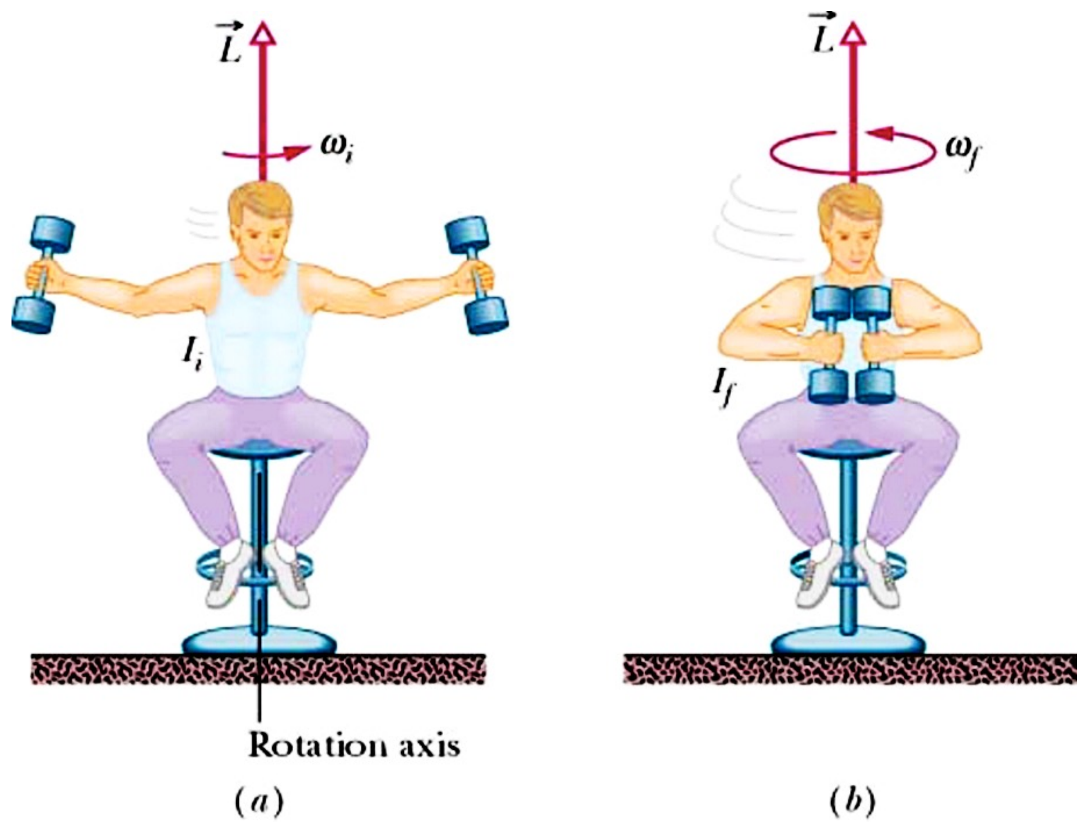
**If no net external torque acts on the system:** $\tau_{net} = 0$

$$\vec{L} = \text{constant}$$

$$\vec{L}_i = \vec{L}_f$$

$$\boxed{I_i \omega_i = I_f \omega_f}$$

$L = L'$



**Example:** You stand on a frictionless platform that is rotating at an angular speed of 1.5 rev/s. Your arms are outstretched, and you hold a heavy weight in each hand. The moment of inertia of you, the extended weights, and the platform is 6.0 kg.m². When you pull the weights in toward your body, the moment of inertia decreases to 1.8 kg.m². **(a)** What is the resulting angular speed of the platform? **(b)** What is the change in kinetic energy of the system? **(c)** Where did this increase in energy come from?

(a)No net torque acting on the rotating system (platform+you+weights):

$$I\omega = \text{constant} \Rightarrow \mathrm{I}_1\omega_1 = \mathrm{I}_2\omega_2 \Rightarrow \omega_2 = \frac{\mathrm{I}_1\omega_1}{\mathrm{I}_2} = \frac{6.0\times 1.5}{1.8} = 5\,(\text{rev/s})$$

(b) The change in kinetic energy:

$$\Delta K = \frac{1}{2}I_2\omega_2^2 - \frac{1}{2}I_1\omega_1^2 = \frac{1}{2}\left[1.8\times(5\times 2\pi)^2 - 6.0\times(1.5\times 2\pi)^2\right] \approx 621\,(\text{J})$$

(c) Because no external agent does work on the system, so the increase in kinetic energy comes from your internal energy (biochemical energy in your muscles).

# More Corresponding Variables and Relations for Translational and Rotational

| Translational | | Rotational | |
|---|---|---|---|
| Force | $\vec{F}$ | Torque | $\vec{\tau} = \vec{r} \times \vec{F}$ |
| Translational momentum | $\vec{p}_{\text{sys}}$ | Rotational momentum | $\vec{\ell} = \vec{r} \times \vec{p}$ |
| Translational momentum[a] | $\vec{p}_{\text{sys}} = \Sigma \vec{p}_i$ | Rotational momentum[a] | $\vec{L} = \Sigma \vec{\ell}_i$ |
| Translational momentum[a] | $\vec{p}_{\text{sys}} = M\vec{v}_{\text{com}}$ | Rotational momentum[b] | $L = I\omega$ |
| Newton's Second Law[a] | $\Sigma \vec{F}_{\text{ext}} = \dfrac{d\vec{p}_{\text{sys}}}{dt}$ | Newton's Second Law[a] | $\Sigma \vec{\tau}_{\text{ext}} = \dfrac{d\vec{L}}{dt}$ |
| Conservation law[c] | $\vec{p}_{\text{sys}} =$ a constant | Conservation law[c] | $\vec{L} =$ a constant |

[a] For systems of particles, including rigid bodies.

[b] For a rigid body about a fixed axis, with $L$ being the component along that axis.

[c] For a closed, isolated system ($\vec{F}_{\text{net}} = 0$, $\vec{\tau}_{\text{net}} = 0$).

Homework: 53, 56 (p. 270-271); 26, 38 (p. 299-300)

Final exam: Chapter 7, 8, 9, 10 (textbook)

**Assignment:**

+ A group of 5 people
+ Each group submits ONLY 1 report