DE AY1819 S2

Q1.

Given that:

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P)\ (*), \qquad P(0) = 5000$$

$$(*) \to \frac{10^7 dP}{P(10^6 - P)} = dt \leftrightarrow 10\left(\frac{1}{P} + \frac{1}{10^6 - P}\right)dP = dt$$

$$\leftrightarrow 10\ln\left(\frac{P}{10^6 - P}\right) = t + C$$

At time $t = 0, P = 5000 \to C = 10\ln\left(\frac{5000}{10^6 - 5000}\right) = -52.933$

With this value, solve for P, we obtain:

$$P(t) = \frac{10^6 e^{\frac{t-52.933}{10}}}{e^{\frac{t-52.933}{10}} + 1}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{10^6 e^{\frac{t-52.933}{10}}}{e^{\frac{t-52.933}{10}} + 1} = 10^6$$

Therefore the limit of the population is 10^6

At the time the population is one-half of the limit is:

$$P(t) = \frac{1}{2}.10^6 \leftrightarrow \frac{10^6 e^{\frac{t-52.933}{10}}}{e^{\frac{t-52.933}{10}} + 1} = 5.10^5$$

Solve for t, we get: $t = 52.933$

Thus the limit of the population model is 10^6 and it takes $t = 52.933$ to reach the one-half of its limit.

Q2.

Given that: $(e^{2y} - y\sin(xy))dx + (2xe^{2y} - x\sin(xy) + 2y)dy = 0\ (*)$

$$\leftrightarrow M(x,y)dx + N(x,y)dy = 0$$

Where: $\begin{cases} M(x,y) = e^{2y} - y\sin(xy) \\ N(x,y) = 2xe^{2y} - x\sin(xy) + 2y \end{cases}$

And: $\begin{cases} \frac{\partial M}{\partial y} = 2e^{2y} - \sin(xy) - xy\cos(xy) \\ \frac{\partial N}{\partial x} = 2e^{2y} - \sin(xy) - xy\cos(xy) \end{cases}$

$$\to \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore the given differential equation is exact.

Solve the given differential equation:

$$(*) \leftrightarrow e^{2y}dx - y\sin(xy)\,dx + 2xe^{2y} - x\sin(xy) + 2ydy = 0$$
$$\leftrightarrow e^{2y}dx - \sin(xy)\,(ydx + xdy) + xd(e^{2y}) + d(y^2) = 0$$
$$\leftrightarrow e^{2y}dx + xd(e^{2y}) - \sin(xy)\,d(xy) + d(y^2) = 0$$
$$\leftrightarrow d(xe^{2y}) + d(\cos(xy)) + d(y^2) = 0$$
$$\leftrightarrow d(xe^{2y} + \cos(xy) + y^2) = 0$$

Integrating both sides we obtain the final result:

$$\leftrightarrow xe^{2y} + \cos(xy) + y^2 + C = 0$$

Q3.

Given that: $xy' + (x+1)y = e^{2019x}$ $(*)$, $\quad y(1) = 2020$

$$(*) \leftrightarrow xe^x y' + (x+1)e^x y = e^{2020x}$$

(Multiply both sides with e^x)

$$\leftrightarrow xe^x \frac{dy}{dx} + \frac{d(xe^x)}{dx} y = e^{2020x}$$

$$\leftrightarrow \frac{d(xe^x y)}{dx} = e^{2020x}$$

$$\leftrightarrow d(xe^x y) = e^{2020x} dx$$

Integrating both sides we obtain:

$$\leftrightarrow xe^x y = \frac{1}{2020} e^{2020x} + C$$

With the initial condition: $y(1) = 2020$, it leads to:

$$1.e.2020 = \frac{1}{2020} e^{2020} + C \leftrightarrow C = 2020e - \frac{1}{2020} e^{2020}$$

Hence, the solution of the equation is:

$$xe^x y = \frac{1}{2020} e^{2020x} + 2020e - \frac{1}{2020} e^{2020}$$

Or:

$$y = \frac{1}{2020x} e^{2019x} + \frac{1}{xe^x}\left(2020e - \frac{1}{2020} e^{2020}\right)$$

Q4.

a) Given that: $\quad y'' - 4y' + 4y = e^{2x}(x^3 + 1) + e^x(x+1)\sin x$

$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y'' - 4y' + 4y \\ g_1(x) = e^{2x}(x^3+1) \\ g_2(x) = e^x(x+1)\sin x \end{cases}$

Characteristic equation of the given ODE: $r^2 - 4r + 4 = 0$

$$\to r_1 = r_2 = 2$$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two term: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore y_{p1} from: $\mathrm{L}[y_{p1}] = g_1(x) \leftrightarrow y_{p1}'' - 4y_{p1}' + 4y_{p1} = e^{2x}(x^3+1)$ $(\alpha = 2)$

Since we have: $\alpha = 2 \equiv r_1 \equiv r_2$ (double roots)

Hence: $y_{p1} = x^2 e^{2x}(Ax^3 + Bx^2 + Cx + D)$

Solve fore y_{p2} from: $\mathrm{L}[y_{p2}] = g_2(x) \leftrightarrow y_{p2}'' - 4y_{p2}' + 4y_{p2} = e^x(x+1)\sin x$

$$(\alpha + i\beta = 1 + i)$$

Since, we have: $\alpha + i\beta = 1 + i$ is not a root of the characteristic equation.

Hence: $y_{p2} = e^x(Ex + F)\sin x + e^x(Gx + H)\cos x$

So: $\quad y_p = y_{p1} + y_{p2}$

$$= x^2 e^{2x}(Ax^3 + Bx^2 + Cx + D) + e^x(Ex+F)\sin x + e^x(Gx+H)\cos x$$

b) Given that: $\quad y'' - 4y' + 3y = xe^x$

Characteristic equation of the given ODE: $r^2 - 4r + 3 = 0$

$$\leftrightarrow \begin{cases} r_1 = 1 \\ r_2 = 3 \end{cases}$$

Hence, the complement solution is: $y_c = C_1 e^x + C_2 e^{3x}$

Since, the right hand side xe^x with $\alpha = 1 \equiv r_1$ is a single root of the characteristic equation.

Therefore the particular solution has the following form:

$$y_p = xe^x(Ax + B) = e^x(Ax^2 + Bx)$$

$$\to y_p' = e^x(Ax^2 + (B + 2A)x + B)$$

$$\to y_p'' = e^x(Ax^2 + (B + 4A)x + 2B + 2A)$$

Substituting back into the given equation we obtain:

$$e^x\big((-4A)x - 2B + 2A\big) = xe^x$$

$$\to \begin{cases} -4A = 1 \\ -2B + 2A = 0 \end{cases} \leftrightarrow \begin{cases} A = -\dfrac{1}{4} \\ B = -\dfrac{1}{4} \end{cases}$$

$$\to y_p = -\frac{1}{4}e^x(x^2 + 1)$$

Thus the general solution of the equation is:

$$y_G = y_c + y_p = C_1 e^x + C_2 e^{3x} - \frac{1}{4}e^x(x^2 + 1)$$

Q5.

a) Given that: $(1 - 2x - x^2)y'' + 2(x + 1)y' - 2y = 0 \ (*)$

We have: $y_1 = ax + b; \to y_1' = a \to y_1'' = 0.$

We know that y_1 is a solution of $(*)$, therefore substituting y_1 into $(*)$, we get:

$$(1 - 2x - x^2).0 + 2(x + 1).a - 2(ax + b) = 0$$

$$\leftrightarrow 0.ax + 2a - 2b = 0$$

$$\to \begin{cases} b = a \\ a \in R \end{cases}$$

Thus, with any constant a and $b = a$, $y_1 = ax + b$ is a solution of $(*)$

b) To find the general solution of $(*)$, we rewire $(*)$ in the following form:

$$y'' + \frac{2(x + 1)}{1 - 2x - x^2}y' - \frac{2}{1 - 2x - x^2}y = 0$$

$$(y'' + p(x)y' + q(x) = 0)$$

The Wronskian determinant for the equation is:

$$\mathrm{W}[y_1, y_2] = C_1 e^{-\int p(x)dx} = C_1 e^{-\int \frac{2(x+1)}{1-2x-x^2}dx}$$

$$\to \mathrm{W}[y_1, y_2] = C_1(x^2 + 2x - 1)$$

Hence:

$$y_2 = y_1\left[\int \frac{\mathrm{W}[y_1, y_2]}{y_1^2}dx + C_2\right]$$

Choose: $a = b = 1$ for y_1, it leads to:

$$y_2 = (x + 1)\left[\int \frac{C_1(x^2 + 2x - 1)}{(x + 1)^2}dx + C_2\right]$$

$$\to y_2 = (x+1)\left[C_1\left(x+\frac{2}{x+1}\right)+C_2\right]$$
$$\to y_2 = C_1(x^2+x+2)+C_2(x+1)$$

Choose $C_1 = 1, C_2 = 0 \to y_2 = x^2 + x + 2$

Since, the Wronskian determinant different from 0 for some x, therefore y_1 and y_2 are linearly independence solution of the equation.

Thus, the general solution of the equation is:

$$y_G = C_1 y_1 + C_2 y_2 = C_1(x+1) + C_2(x^2+x+2)$$