INTERNATIONAL UNIVERSITY-VNUHCM

FINAL EXAMINATION

Semester 3, Academic Year 2015-2016

Duration: 120 minutes

SUBJECT: **Calculus 2**	
Chair of Department of Mathematics	Lecturers:
Signature:	Signature:
Full name: Assoc.Prof. Nguyen Dinh	Full names: Assoc.Prof. Mai Duc Thanh

- **Each student is allowed a maximum of two double-sided sheets of reference material (of size A4 or similar) and a scientific calculator. All other documents and electronic devices are forbidden.**
- **Each question carries 20 marks.**

Question 1. a) Find the first partial derivatives of the function $f(x,y) = e^{4x-y^2}$.

b) Find the directional derivative $D_{\mathbf{u}}f(x,y)$ of the function $f(x,y) = e^{4x-y^2}$ at the point $(1,2)$ in the direction of the vector $\mathbf{u} =< 1/\sqrt{2}, -1/\sqrt{2} >$.

Question 2. Find the local maximum and minimum values and saddles point(s) of the function

$$f(x,y) = e^x(x^2 - y^2).$$

Question 3. a) Evaluate the double integral

$$I = \iint_D 2y\ dA, \quad D = \{(x,y) \mid \quad 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}.$$

b) Find the volume of the solid under the surface $z = 1 + 2xy$ and above the region in the xy-plane bounded by $y = 1 - x$ and $y = 1 - x^2$.

Question 4. a) Evaluate the triple integral

$$\iiint_E x\ dV, \quad E = \{(x,y,z) \mid \quad 0 \le x \le 1,\ 0 \le y \le x, 0 \le z \le x + 2y\}.$$

b) Evaluate the line integral

$$\int_C (2x^2 - y)\ dx + 2y\ dy$$

where C is the arc of the curve $y = x^2$ from $(0,0)$ to $(2,4)$.

Question 5. Let a vector field $\mathbf{F}(x,y,z) = xy\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be given.

a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C has the parametric equations $x = t, y = t^2, z = e^t, 0 \le t \le 1$.

b) Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the part of the paraboloid $z = 16 - 3x^2 - 3y^2$ that lies above the rectangle $0 \le x \le 2, 0 \le y \le 1$.

SOLUTIONS OF FINAL EXAM

Subject: CALCULUS 2

Question 1. a) Find the first partial derivatives of the function $f(x, y) = e^{4x-y^2}$.

$$f_x(x, y) = 4e^{4x-y^2} \quad f_y(x, y) = -2ye^{4x-y^2}.$$

b) Find the directional derivative of the function $f(x, y) = e^{4x-y^2}$ at the point $(1, 2)$ in the direction of the vector $\mathbf{u} = (1/\sqrt{2}, -1/\sqrt{2})$.

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} =< 4, -4 > \cdot < 1/\sqrt{2}, -1/\sqrt{2} >= 4\sqrt{2}$$

Question 2. Find the local maximum and minimum values and saddles point(s) of the function

$$f(x, y) = e^x(x^2 - y^2).$$

Partial derivatives

$$f_x(x, y) = e^x(x^2 - y^2 + 2x) \quad f_y(x, y) = -2ye^x$$

Critical points:

$$f_x(x, y) = e^x(x^2 - y^2 + 2x) = 0 \quad f_y(x, y) = -2ye^x = 0$$

or

$$y = 0, \quad x(x + 2) = 0$$

so that $x = 0$ or $x = -2$ and $y = 0$. So, there are two critical points $(0, 0)$ and $(-2, 0)$.

Second partial derivatives:

$$f_{xx}(x, y) = e^x(x^2 - y^2 + 4x + 2), \quad f_{xy}(x, y) = -2ye^x, \quad f_{yy}(x, y) = -2e^x.$$

Consider

$$D = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2.$$

At $M_1(0, 0)$, it holds

$$D = f_{xx}(0, 0)f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = 2(-2) - 0 = -4 < 0,$$

so (0,0) is a saddle point.

At $M_2(-2, 0)$ it holds

$$D = f_{xx}(-2, 0)f_{yy}(-2, 0) - (f_{xy}(-2, 0))^2 = e^{-4}(-6)(-2) - 0 = 12e^{-4} > 0$$

and $f_{xx}(-2, 0) = -6e^{-4} < 0$. So $f(-2, 0) = 4e^{-2}$ is a local maximum value.

Question 3. a)

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 2y \; dy \; dx = \int_0^1 y^2 \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (x - x^4)dx = (x^2/2 - x^5/5)\Big|_0^1 = 1/2 - 1/5 = 3/10.$$

b) The volume is given by

$$V = \iint_D (1+2xy)\ dA, \quad D = \{(x,y)|0 \le x \le 1, 1-x \le y \le 1-x^2\},$$

Therefore

$$\begin{aligned}
V &= \int_0^1 \int_{1-x}^{1-x^2} (1+2xy)dydx \\
&= \int_0^1 (y+xy^2)\Big|_{y=1-x}^{y=1-x^2} dx \\
&= \int_0^1 [1-x^2+x(1-x^2)^2-(1-x)-x(1-x)^2]dx \\
&= \int_0^1 [1-x^2+x(1-2x^2+x^4)-(1-x)-x(1-2x+x^2)] \\
&= \int_0^1 (x^5-3x^3+x^2+x)dx = (x^5/5-3x^4/4+x^3/3+x^2/2)\Big|_0^1 = 17/60 = 0.2833.
\end{aligned}$$

Question 4. a) Evaluate the triple integral

$$\iiint_E x\ dV, \quad E = \{(x,y,z) \mid \quad 0 \le x \le 1,\ 0 \le y \le x, 0 \le z \le x+2y\}.$$

$$\begin{aligned}
\iiint_E xdV &= \int_0^1 \int_0^x \int 0^{x+2y} xdzdydx \\
&= \int_0^1 \int_0^x x(x+2y)dydx = \int_0^1 \int_0^x (x^2+2xy)dydx \\
&= \int_0^1 (x^2y+xy^2)\Big|_0^x dx = \int_0^1 2x^3dx \\
&= x^4/2\Big|_0^1 = 1/2
\end{aligned}$$

b) Evaluate the line integral

$$\int_C (2x^2-y)\ dx + 2y\ dy$$

where C is the arc of the curve $y=x^2$ from $(0,0)$ to $(2,4)$.

We have $y' = 2x, 0 \le x \le 2$. Thus,

$$\begin{aligned}
\int_C (2x^2-y)\ dx + 2y\ dy &= \int_0^2 \ [(2x^2-x^2)+2x^2 2x]\ dx \\
&= \int_0^2 \ (4x^3+x^2)\ dx = (x^4+x^3/3)\Big|_0^2 = 2^3(2+1/3) = 56/3 = 18.67.
\end{aligned}$$

Question 5. Let a vector field $\mathbf{F}(x,y,z) = xy\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be given.

a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C has the parametric equations $x=t, y=t^2, z=e^t, 0 \le t \le 1$.

Vector equation of C: $\mathbf{r}(t) =< t, t^2, e^t >, 0 \leq t \leq 1$. We have $\mathbf{r}'(t) =< 1, 2t, e^t >$, and

$$\mathbf{F}(\mathbf{r}(t)) =< t^3, t^2, e^t > .$$

Thus

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\
&= \int_0^1 (t^3 + 2t^3 + e^{2t}) dt = \int_0^1 (3t^3 + e^{2t}) dt \\
&= (3t^4/4 + e^{2t}/2)\Big|_0^1 = 3/4 + e^2/2 - 1/2 = e^2/2 + 1/4 = 3.9445
\end{aligned}$$

b) Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the part of the paraboloid $z = 16 - 3x^2 - 3y^2$ that lies above the rectangle $0 \leq x \leq 2, 0 \leq y \leq 1$.

Set $\mathbf{F}(x, y, z) =< P, Q, R >$, where $P = xy, Q = y, R = z$, and S is given by $z = g(x, y) = 16 - 3x^2 - 3y^2$. Applying the formula

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D (-Pg_x - Qg_y + R)\ dA \\
&= \iint_D [-xy(-6x) - y(-6y) + (16 - 3x^2 - 3y^2)]\ dA \\
&= \iint_D (6x^2y + 3y^2 - 3x^2 + 16)\ dA \\
&= \int_0^2 \int_0^1 (6x^2y + 3y^2 - 3x^2 + 16) dy dx \\
&= \int_0^2 (3x^2y^2 + y^3 - 3x^2y + 16y)\Big|_0^1) dx \\
&= \int_0^2 (3x^2 + 1 - 3x^2 + 16) dx = \int_0^2 17 dx = 34
\end{aligned}$$