PROBLEMS

In each of Problems 1 through 8 find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $y'' + 3y' + 2y = 0$
3. $6y'' - y' - y = 0$
4. $2y'' - 3y' + y = 0$
5. $y'' + 5y' = 0$
6. $4y'' - 9y = 0$
7. $y'' - 9y' + 9y = 0$
8. $y'' - 2y' - 2y = 0$

In each of Problems 9 through 16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

9. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$
10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
11. $6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0$
12. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$
13. $y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
14. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
15. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$
16. $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$
17. Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.
18. Find a differential equation whose general solution is $y = c_1e^{-t/2} + c_2e^{-2t}$.
19. Find the solution of the initial value problem

$$y'' - y = 0, \quad y(0) = \tfrac{5}{4}, \quad y'(0) = -\tfrac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

20. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \tfrac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

21. Solve the initial value problem $y'' - y' - 2y = 0, y(0) = \alpha, y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.
22. Solve the initial value problem $4y'' - y = 0, y(0) = 2, y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

In each of Problems 23 and 24 determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

23. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
24. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$
25. Consider the initial value problem

$$2y'' + 3y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -\beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Plot the solution when $\beta = 1$. Find the coordinates (t_0, y_0) of the minimum point of the solution in this case.

(c) Find the smallest value of β for which the solution has no minimum point.

26. Consider the initial value problem (see Example 4)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.

(c) Determine the smallest value of β for which $y_m \geq 4$.

(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

27. Consider the equation $ay'' + by' + cy = d$, where a, b, c, and d are constants.

(a) Find all equilibrium, or constant, solutions of this differential equation.

(b) Let y_e denote an equilibrium solution, and let $Y = y - y_e$. Thus Y is the deviation of a solution y from an equilibrium solution. Find the differential equation satisfied by Y.

28. Consider the equation $ay'' + by' + cy = 0$, where a, b, and c are constants with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic equation are:

(a) real, different, and negative.

(b) real with opposite signs.

(c) real, different, and positive.

3.2 Fundamental Solutions of Linear Homogeneous Equations

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In turn, this understanding will assist us in finding the solutions of other problems that we will encounter later.

In developing the theory of linear differential equations, it is helpful to introduce a differential operator notation. Let p and q be continuous functions on an open interval I, that is, for $\alpha < t < \beta$. The cases $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

Note that $L[\phi]$ is a function on I. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin 3t$, then

$$\begin{aligned} L[\phi](t) &= (\sin 3t)'' + t^2(\sin 3t)' + (1 + t)\sin 3t \\ &= -9\sin 3t + 3t^2\cos 3t + (1 + t)\sin 3t. \end{aligned}$$

Wronskian W of y_1 and y_2 is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions and the general solution is

$$y = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a point in $\alpha < t < \beta$ where $W \neq 0$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

PROBLEMS

In each of Problems 1 through 6 find the Wronskian of the given pair of functions.

1. $e^{2t}, \quad e^{-3t/2}$
2. $\cos t, \quad \sin t$
3. $e^{-2t}, \quad te^{-2t}$
4. $x, \quad xe^x$
5. $e^t \sin t, \quad e^t \cos t$
6. $\cos^2\theta, \quad 1 + \cos 2\theta$

In each of Problems 7 through 12 determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7. $ty'' + 3y = t, \quad y(1) = 1, \quad y'(1) = 2$
8. $(t-1)y'' - 3ty' + 4y = \sin t, \quad y(-2) = 2, \quad y'(-2) = 1$
9. $t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, \quad y'(3) = -1$
10. $y'' + (\cos t)y' + 3(\ln|t|)y = 0, \quad y(2) = 3, \quad y'(2) = 1$
11. $(x-3)y'' + xy' + (\ln|x|)y = 0, \quad y(1) = 0, \quad y'(1) = 1$
12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2$
13. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.
14. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.
15. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, then $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.
16. Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients? Explain your answer.
17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.
18. If the Wronskian W of f and g is t^2e^t, and if $f(t) = t$, find $g(t)$.
19. If $W(f, g)$ is the Wronskian of f and g, and if $u = 2f - g$, $v = f + 2g$, find the Wronskian $W(u, v)$ of u and v in terms of $W(f, g)$.
20. If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g$, $v = f - g$, find the Wronskian of u and v.

In each of Problems 21 and 22 find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

21. $y'' + y' - 2y = 0, \quad t_0 = 0$
22. $y'' + 4y' + 3y = 0, \quad t_0 = 1$

In each of Problems 23 through 26 verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

23. $y'' + 4y = 0; \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
24. $y'' - 2y' + y = 0; \quad y_1(t) = e^t, \quad y_2(t) = te^t$
25. $x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0; \quad y_1(x) = x, \quad y_2(x) = xe^x$
26. $(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi; \quad y_1(x) = x, \quad y_2(x) = \sin x$
27. Consider the equation $y'' - y' - 2y = 0$.
    (a) Show that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions.
    (b) Let $y_3(t) = -2e^{2t}$, $y_4(t) = y_1(t) + 2y_2(t)$, and $y_5(t) = 2y_1(t) - 2y_3(t)$. Are $y_3(t)$, $y_4(t)$, and $y_5(t)$ also solutions of the given differential equation?
    (c) Determine whether each of the following pairs form a fundamental set of solutions: $[y_1(t), y_3(t)]$; $[y_2(t), y_3(t)]$; $[y_1(t), y_4(t)]$; $[y_4(t), y_5(t)]$.
28. **Exact Equations.** The equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be exact if it can be written in the form $[P(x)y']' + [f(x)y]' = 0$, where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated once immediately, resulting in a first order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is $P''(x) - Q'(x) + R(x) = 0$. It can be shown that this is also a sufficient condition.

In each of Problems 29 through 32 use the result of Problem 28 to determine whether the given equation is exact. If so, solve the equation.

29. $y'' + xy' + y = 0$
30. $y'' + 3x^2y' + xy = 0$
31. $xy'' - (\cos x)y' + (\sin x)y = 0, \quad x > 0$
32. $x^2y'' + xy' - y = 0, \quad x > 0$
33. **The Adjoint Equation.** If a second order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that $\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$ can be written in the form $[\mu(x)P(x)y']' + [f(x)y]' = 0$. By equating coefficients in these two equations and eliminating $f(x)$, show that the function μ must satisfy

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

    This equation is known as the adjoint of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second order equation.

In each of Problems 34 through 36 use the result of Problem 33 to find the adjoint of the given differential equation.

34. $x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad$ Bessel's equation
35. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad$ Legendre's equation
36. $y'' - xy = 0, \quad$ Airy's equation
37. For the second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$, show that the adjoint of the adjoint equation is the original equation.
38. A second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be self-adjoint if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 34 through 36 is self-adjoint.

PROBLEMS

In each of Problems 1 through 8 determine whether the given pair of functions is linearly independent or linearly dependent.

1. $f(t) = t^2 + 5t, \quad g(t) = t^2 - 5t$
2. $f(\theta) = \cos 2\theta - 2\cos^2\theta, \quad g(\theta) = \cos 2\theta + 2\sin^2\theta$
3. $f(t) = e^{\lambda t}\cos\mu t, \quad g(t) = e^{\lambda t}\sin\mu t, \quad \mu \neq 0$
4. $f(x) = e^{3x}, \quad g(x) = e^{3(x-1)}$
5. $f(t) = 3t - 5, \quad g(t) = 9t - 15$
6. $f(t) = t, \quad g(t) = t^{-1}$
7. $f(t) = 3t, \quad g(t) = |t|$
8. $f(x) = x^3, \quad g(x) = |x|^3$
9. The Wronskian of two functions is $W(t) = t\sin^2 t$. Are the functions linearly independent or linearly dependent? Why?
10. The Wronskian of two functions is $W(t) = t^2 - 4$. Are the functions linearly independent or linearly dependent? Why?
11. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that c_1y_1 and c_2y_2 are also linearly independent solutions, provided that neither c_1 nor c_2 is zero.
12. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a linearly independent set of solutions. Conversely, if y_3 and y_4 are linearly independent solutions of the differential equation, show that y_1 and y_2 are also.
13. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, determine under what conditions the functions $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$ also form a linearly independent set of solutions.
14. (a) Prove that any two-dimensional vector can be written as a linear combination of $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j}$.
    (b) Prove that if the vectors $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$ and $\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j}$ are linearly independent, then any vector $\mathbf{z} = z_1\mathbf{i} + z_2\mathbf{j}$ can be expressed as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. Note that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then $x_1y_2 - x_2y_1 \neq 0$. Why?

In each of Problems 15 through 18 find the Wronskian of two solutions of the given differential equation without solving the equation.

15. $t^2y'' - t(t+2)y' + (t+2)y = 0$
16. $(\cos t)y'' + (\sin t)y' - ty = 0$
17. $x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad$ Bessel's equation
18. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad$ Legendre's equation
19. Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = c/p(t)$, where c is a constant.
20. If y_1 and y_2 are linearly independent solutions of $ty'' + 2y' + te^ty = 0$ and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.
21. If y_1 and y_2 are linearly independent solutions of $t^2y'' - 2y' + (3 + t)y = 0$ and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(4)$.
22. If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients p and q?
23. If f, g, and h are differentiable functions, show that $W(fg, fh) = f^2W(g, h)$.

In Problems 24 through 26 assume that p and q are continuous and that the functions y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I.

24. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.

25. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.
26. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.
27. Show that t and t^2 are linearly independent on $-1 < t < 1$; indeed, they are linearly independent on every interval. Show also that $W(t, t^2)$ is zero at $t = 0$. What can you conclude from this about the possibility that t and t^2 are solutions of a differential equation $y'' + p(t)y' + q(t)y = 0$? Verify that t and t^2 are solutions of the equation $t^2y'' - 2ty' + 2y = 0$. Does this contradict your conclusion? Does the behavior of the Wronskian of t and t^2 contradict Theorem 3.3.2?
28. Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$ but are linearly independent on $-1 < t < 1$. Although f and g are linearly independent there, show that $W(f, g)$ is zero for all t in $-1 < t < 1$. Hence f and g cannot be solutions of an equation $y'' + p(t)y' + q(t)y = 0$ with p and q continuous on $-1 < t < 1$.

3.4 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

If the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of Eq. (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq. (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]. \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1, \mu = 2$, and $t = 3$, then from Eq. (5),

$$y_1(3) = e^{-3+6i}. \tag{6}$$

What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as Euler's formula.

Euler's Formula. To assign a meaning to the expressions in Eqs. (5) we need to give a definition of the complex exponential function. Of course, we want the definition to reduce to the familiar real exponential function when the exponent is real. There

PROBLEMS

In each of Problems 1 through 6 use Euler's formula to write the given expression in the form $a + ib$.

1. $\exp(1 + 2i)$
2. $\exp(2 - 3i)$
3. $e^{i\pi}$
4. $e^{2-(\pi/2)i}$
5. 2^{1-i}
6. π^{-1+2i}

In each of Problems 7 through 16 find the general solution of the given differential equation.

7. $y'' - 2y' + 2y = 0$
8. $y'' - 2y' + 6y = 0$
9. $y'' + 2y' - 8y = 0$
10. $y'' + 2y' + 2y = 0$
11. $y'' + 6y' + 13y = 0$
12. $4y'' + 9y = 0$
13. $y'' + 2y' + 1.25y = 0$
14. $9y'' + 9y' - 4y = 0$
15. $y'' + y' + 1.25y = 0$
16. $y'' + 4y' + 6.25y = 0$

In each of Problems 17 through 22 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

17. $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
18. $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$
19. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$
20. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$
21. $y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$
22. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$
23. Consider the initial value problem
$$3u'' - u' + 2u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$
(a) Find the solution $u(t)$ of this problem.
(b) Find the first time at which $|u(t)| = 10$.
24. Consider the initial value problem
$$5u'' + 2u' + 7u = 0, \qquad u(0) = 2, \quad u'(0) = 1.$$
(a) Find the solution $u(t)$ of this problem.
(b) Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.
25. Consider the initial value problem
$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$
(a) Find the solution $y(t)$ of this problem.
(b) Find α so that $y = 0$ when $t = 1$.
(c) Find, as a function of α, the smallest positive value of t for which $y = 0$.
(d) Determine the limit of the expression found in part (c) as $\alpha \to \infty$.
26. Consider the initial value problem
$$y'' + 2ay' + (a^2 + 1)y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$
(a) Find the solution $y(t)$ of this problem.
(b) For $a = 1$ find the smallest T such that $|y(t)| < 0.1$ for $t > T$.
(c) Repeat part (b) for $a = 1/4, 1/2$, and 2.
(d) Using the results of parts (b) and (c), plot T versus a and describe the relation between T and a.

27. Show that $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}$.
28. In this problem we outline a different derivation of Euler's formula.
(a) Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.
(b) Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore,

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{i}$$

for some constants c_1 and c_2. Why is this so?
(c) Set $t = 0$ in Eq. (i) to show that $c_1 = 1$.
(d) Assuming that Eq. (14) is true, differentiate Eq. (i) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in Eq. (i) to arrive at Euler's formula.
29. Using Euler's formula, show that

$$\cos t = (e^{it} + e^{-it})/2, \qquad \sin t = (e^{it} - e^{-it})/2i.$$

30. If e^{rt} is given by Eq. (13), show that $e^{(r_1+r_2)t} = e^{r_1 t}e^{r_2 t}$ for any complex numbers r_1 and r_2.
31. If e^{rt} is given by Eq. (13), show that

$$\frac{d}{dt}e^{rt} = re^{rt}$$

for any complex number r.
32. Let the real-valued functions p and q be continuous on the open interval I, and let $y = \phi(t) = u(t) + iv(t)$ be a complex-valued solution of

$$y'' + p(t)y' + q(t)y = 0, \tag{i}$$

where u and v are real-valued functions. Show that u and v are also solutions of Eq. (i).
Hint: Substitute $y = \phi(t)$ in Eq. (i) and separate into real and imaginary parts.
33. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, show that between consecutive zeros of y_1 there is one and only one zero of y_2. Note that this result is illustrated by the solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of the equation $y'' + y = 0$.
Hint: Suppose that t_1 and t_2 are two zeros of y_1 between which there are no zeros of y_2. Apply Rolle's theorem to y_1/y_2 to reach a contradiction.

Change of Variables. Often a differential equation with variable coefficients,

$$y'' + p(t)y' + q(t)y = 0, \tag{i}$$

can be put in a more suitable form for finding a solution by making a change of the independent and/or dependent variables. We explore these ideas in Problems 34 through 42. In particular, in Problem 34 we determine conditions under which Eq. (i) can be transformed into a differential equation with constant coefficients and thereby becomes easily solvable. Problems 35 through 42 give specific applications of this procedure.

34. In this problem we determine conditions on p and q that enable Eq. (i) to be transformed into an equation with constant coefficients by a change of the independent variable. Let $x = u(t)$ be the new independent variable, with the relation between x and t to be specified later.
(a) Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \qquad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Show that the differential equation (i) becomes

$$\left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \tag{ii}$$

(c) In order for Eq. (ii) to have constant coefficients, the coefficients of d^2y/dx^2 and of y must be proportional. If $q(t) > 0$, then we can choose the constant of proportionality to be 1; hence

$$x = u(t) = \int [q(t)]^{1/2}\, dt. \tag{iii}$$

(d) With x chosen as in part (c), show that the coefficient of dy/dx in Eq. (ii) is also a constant, provided that the expression

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} \tag{iv}$$

is a constant. Thus Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function $(q' + 2pq)/q^{3/2}$ is a constant. How must this result be modified if $q(t) < 0$?

In each of Problems 35 through 37 try to transform the given equation into one with constant coefficients by the method of Problem 34. If this is possible, find the general solution of the given equation.

35. $y'' + ty' + e^{-t^2}y = 0, \qquad -\infty < t < \infty$
36. $y'' + 3ty' + t^2y = 0, \qquad -\infty < t < \infty$
37. $ty'' + (t^2 - 1)y' + t^3y = 0, \qquad 0 < t < \infty$
38. **Euler Equations.** An equation of the form

$$t^2y'' + \alpha ty' + \beta y = 0, \qquad t > 0,$$

where α and β are real constants, is called an Euler equation. Show that the substitution $x = \ln t$ transforms an Euler equation into an equation with constant coefficients. Euler equations are discussed in detail in Section 5.5.

In each of Problems 39 through 42 use the result of Problem 38 to solve the given equation for $t > 0$.

39. $t^2y'' + ty' + y = 0$
40. $t^2y'' + 4ty' + 2y = 0$
41. $t^2y'' + 3ty' + 1.25y = 0$
42. $t^2y'' - 4ty' - 6y = 0$

3.5 Repeated Roots; Reduction of Order

In earlier sections we showed how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

either are real and different or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case is transitional

Separating the variables in Eq. (32) and solving for $v'(t)$, we find that

$$v'(t) = ct^{1/2};$$

then

$$v(t) = \tfrac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = \tfrac{2}{3}ct^{1/2} + kt^{-1}, \tag{33}$$

where c and k are arbitrary constants. The second term on the right side of Eq. (33) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new independent solution. Neglecting the arbitrary multiplicative constant, we have $y_2(t) = t^{1/2}$.

PROBLEMS

In each of Problems 1 through 10 find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' - 4y' - 3y = 0$
4. $4y'' + 12y' + 9y = 0$
5. $y'' - 2y' + 10y = 0$
6. $y'' - 6y' + 9y = 0$
7. $4y'' + 17y' + 4y = 0$
8. $16y'' + 24y' + 9y = 0$
9. $25y'' - 20y' + 4y = 0$
10. $2y'' + 2y' + y = 0$

In each of Problems 11 through 14 solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

11. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
12. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
13. $9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2$
14. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$
15. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, \quad y'(0) = -4.$$

(a) Solve the initial value problem and plot its solution for $0 \le t \le 5$.
(b) Determine where the solution has the value zero.
(c) Determine the coordinates (t_0, y_0) of the minimum point.
(d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

16. Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b and then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

17. Consider the initial value problem

$$4y'' + 4y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

(a) Solve the initial value problem and plot the solution.
(b) Determine the coordinates (t_M, y_M) of the maximum point.
(c) Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.

(d) Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependence of t_M and y_M on b as b increases.

18. Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \qquad y(0) = a > 0, \quad y'(0) = -1.$$

(a) Solve the initial value problem.

(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

19. If the roots of the characteristic equation are real, show that a solution of $ay'' + by' + cy = 0$ can take on the value zero at most once.

Problems 20 through 22 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

20. (a) Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$, so that one solution of the equation is e^{-at}.

(b) Use Abel's formula [Eq. (8) of Section 3.3] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at},$$

where c_1 is a constant.

(c) Let $y_1(t) = e^{-at}$ and use the result of part (b) to obtain a differential equation satisfied by a second solution $y_2(t)$. By solving this equation, show that $y_2(t) = te^{-at}$.

21. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then $\exp(r_1t)$ and $\exp(r_2t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that $\phi(t; r_1, r_2) = [\exp(r_2t) - \exp(r_1t)]/(r_2 - r_1)$ is also a solution of the equation for $r_2 \neq r_1$. Then think of r_1 as fixed and use L'Hospital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the second solution in the case of equal roots.

22. (a) If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2e^{rt}. \tag{i}$$

Since the right side of Eq. (i) is zero when $r = r_1$, it follows that $\exp(r_1t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

(b) Differentiate Eq. (i) with respect to r and interchange differentiation with respect to r and with respect to t, thus showing that

$$\frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \tag{ii}$$

Since the right side of Eq. (ii) is zero when $r = r_1$, conclude that $t\exp(r_1t)$ is also a solution of $L[y] = 0$.

In each of Problems 23 through 30 use the method of reduction of order to find a second solution of the given differential equation.

23. $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \qquad y_1(t) = t^2$
24. $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \qquad y_1(t) = t$
25. $t^2y'' + 3ty' + y = 0, \quad t > 0; \qquad y_1(t) = t^{-1}$
26. $t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \qquad y_1(t) = t$
27. $xy'' - y' + 4x^3y = 0, \quad x > 0; \qquad y_1(x) = \sin x^2$
28. $(x-1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$
29. $x^2y'' - (x - 0.1875)y = 0, \quad x > 0; \qquad y_1(x) = x^{1/4}e^{2\sqrt{x}}$

30. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x$

31. The differential equation

$$xy'' - (x + N)y' + Ny = 0,$$

where N is a nonnegative integer, has been discussed by several authors.[6] One reason why it is interesting is that it has an exponential solution and a polynomial solution.

(a) Verify that one solution is $y_1(x) = e^x$.

(b) Show that a second solution has the form $y_2(x) = ce^x \int x^N e^{-x}\,dx$. Calculate $y_2(x)$ for $N = 1$ and $N = 2$; convince yourself that, with $c = -1/N!$,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}.$$

Note that $y_2(x)$ is exactly the first $N + 1$ terms in the Taylor series about $x = 0$ for e^x, that is, for $y_1(x)$.

32. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.

33. The method of Problem 20 can be extended to second order equations with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W(y_1, y_2)/y_1^2$, where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then use Abel's formula [Eq. (8) of Section 3.3] to determine y_2.

In each of Problems 34 through 37 use the method of Problem 33 to find a second independent solution of the given equation.

34. $t^2y'' + 3ty' + y = 0, \quad t > 0; \qquad y_1(t) = t^{-1}$

35. $ty'' - y' + 4t^3y = 0, \quad t > 0; \qquad y_1(t) = \sin(t^2)$

36. $(x - 1)y'' - xy' + y = 0, \quad x > 1; \qquad y_1(x) = e^x$

37. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x$

Behavior of Solutions as $t \to \infty$. Problems 38 through 40 are concerned with the behavior of solutions as $t \to \infty$.

38. If $a, b,$ and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

39. (a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 38 is no longer true, but that all solutions are bounded as $t \to \infty$.

(b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 38 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0, y'(0) = y_0'$.

40. Show that $y = \sin t$ is a solution of

$$y'' + (k\sin^2 t)y' + (1 - k\cos t\sin t)y = 0$$

[6] T. A. Newton, "On Using a Differential Equation to Generate Polynomials," *American Mathematical Monthly 81* (1974), pp. 592–601. Also see the references given there.

for any value of the constant k. If $0 < k < 2$, show that $1 - k\cos t \sin t > 0$ and $k\sin^2 t \geq 0$. Thus observe that even though the coefficients of this variable-coefficient differential equation are nonnegative (and the coefficient of y' is zero only at the points $t = 0, \pi, 2\pi, \ldots$), it has a solution that does not approach zero as $t \to \infty$. Compare this situation with the result of Problem 38. Thus we observe a not unusual situation in the theory of differential equations: equations that are apparently very similar can have quite different properties.

Euler Equations. Use the substitution introduced in Problem 38 in Section 3.4 to solve each of the equations in Problems 41 and 42.

41. $t^2y'' - 3ty' + 4y = 0, \quad t > 0$
42. $t^2y'' + 2ty' + 0.25y = 0, \quad t > 0$

3.6 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now return to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \tag{1}$$

where p, q, and g are given (continuous) functions on the open interval I. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{2}$$

in which $g(t) = 0$ and p and q are the same as in Eq. (1), is called the homogeneous equation corresponding to Eq. (1). The following two results describe the structure of solutions of the nonhomogeneous equation (1) and provide a basis for constructing its general solution.

Theorem 3.6.1 If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (1), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (2). If, in addition, y_1 and y_2 are a fundamental set of solutions of Eq. (2), then

$$Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t), \tag{3}$$

where c_1 and c_2 are certain constants.

To prove this result, note that Y_1 and Y_2 satisfy the equations

$$L[Y_1](t) = g(t), \qquad L[Y_2](t) = g(t). \tag{4}$$

Subtracting the second of these equations from the first, we have

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0. \tag{5}$$

However,

$$L[Y_1] - L[Y_2] = L[Y_1 - Y_2],$$

so Eq. (5) becomes

$$L[Y_1 - Y_2](t) = 0. \tag{6}$$

PROBLEMS

In each of Problems 1 through 12 find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' + 2y' + 5y = 3\sin 2t$
3. $y'' - 2y' - 3y = -3te^{-t}$
4. $y'' + 2y' = 3 + 4\sin 2t$
5. $y'' + 9y = t^2e^{3t} + 6$
6. $y'' + 2y' + y = 2e^{-t}$
7. $2y'' + 3y' + y = t^2 + 3\sin t$
8. $y'' + y = 3\sin 2t + t\cos 2t$
9. $u'' + \omega_0^2 u = \cos\omega t, \quad \omega^2 \neq \omega_0^2$
10. $u'' + \omega_0^2 u = \cos\omega_0 t$
11. $y'' + y' + 4y = 2\sinh t \quad$ *Hint:* $\sinh t = (e^t - e^{-t})/2$
12. $y'' - y' - 2y = \cosh 2t \quad$ *Hint:* $\cosh t = (e^t + e^{-t})/2$

In each of Problems 13 through 18 find the solution of the given initial value problem.

13. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
15. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
16. $y'' - 2y' - 3y = 3te^{2t}, \quad y(0) = 1, \quad y'(0) = 0$
17. $y'' + 4y = 3\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$
18. $y'' + 2y' + 5y = 4e^{-t}\cos 2t, \quad y(0) = 1, \quad y'(0) = 0$

In each of Problems 19 through 26:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

19. $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$
20. $y'' + y = t(1 + \sin t)$
21. $y'' - 5y' + 6y = e^t\cos 2t + e^{2t}(3t + 4)\sin t$
22. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$
23. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$
24. $y'' + 4y = t^2\sin 2t + (6t + 7)\cos 2t$
25. $y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$
26. $y'' + 2y' + 5y = 3te^{-t}\cos 2t - 2te^{-2t}\cos t$
27. Consider the equation

$$y'' - 3y' - 4y = 2e^{-t} \tag{i}$$

from Example 5. Recall that $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$ are solutions of the corresponding homogeneous equation. Adapting the method of reduction of order (Section 3.5), seek a solution of the nonhomogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$, where $v(t)$ is to be determined.

(a) Substitute $Y(t)$, $Y'(t)$, and $Y''(t)$ into Eq. (i) and show that $v(t)$ must satisfy $v'' - 5v' = 2$.

(b) Let $w(t) = v'(t)$ and show that $w(t)$ must satisfy $w' - 5w = 2$. Solve this equation for $w(t)$.

(c) Integrate $w(t)$ to find $v(t)$ and then show that

$$Y(t) = -\tfrac{2}{5}te^{-t} + \tfrac{1}{5}c_1e^{4t} + c_2e^{-t}.$$

The first term on the right side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of t and e^{-t}.

28. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t,$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \dots, N$.

29. In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution $y = \phi(t)$ of

$$y'' + y = \begin{cases} t, & 0 \leq t \leq \pi, \\ \pi e^{\pi - t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are also continuous at $t = \pi$. Plot the nonhomogeneous term and the solution as functions of time. *Hint:* First solve the initial value problem for $t \leq \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity conditions at $t = \pi$.

30. Follow the instructions in Problem 29 to solve the differential equation

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \leq t \leq \pi/2, \\ 0, & t > \pi/2 \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

Behavior of Solutions as $t \to \infty$. In Problems 31 and 32 we continue the discussion started with Problems 38 through 40 of Section 3.5. Consider the differential equation

$$ay'' + by' + cy = g(t), \tag{i}$$

where a, b, and c are positive.

31. If $Y_1(t)$ and $Y_2(t)$ are solutions of Eq. (i), show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Is this result true if $b = 0$?

32. If $g(t) = d$, a constant, show that every solution of Eq. (i) approaches d/c as $t \to \infty$. What happens if $c = 0$? What if $b = 0$ also?

33. In this problem we indicate an alternative procedure[7] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \tag{i}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

(a) Verify that Eq. (i) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

(b) Let $u = (D - r_2)y$. Then show that the solution of Eq (i) can be found by solving the following two first order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

[7]R. S. Luthar, "Another Approach to a Standard Differential Equation," *Two Year College Mathematics Journal 10* (1979), pp. 200–201; also see D. C. Sandell and F. M. Stein, "Factorization of Operators of Second Order Linear Homogeneous Ordinary Differential Equations," *Two Year College Mathematics Journal 8* (1977), pp. 132–141, for a more general discussion of factoring operators.

In each of Problems 34 through 37 use the method of Problem 33 to solve the given differential equation.

34. $y'' - 3y' - 4y = 3e^{2t}$ (see Example 1)
35. $2y'' + 3y' + y = t^2 + 3\sin t$ (see Problem 7)
36. $y'' + 2y' + y = 2e^{-t}$ (see Problem 6)
37. $y'' + 2y' = 3 + 4\sin 2t$ (see Problem 4)

3.7 Variation of Parameters

In this section we describe another method of finding a particular solution of a nonhomogeneous equation. This method, known as **variation of parameters**, is due to Lagrange and complements the method of undetermined coefficients rather well. The main advantage of variation of parameters is that it is a *general method*; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second order linear nonhomogeneous differential equation. On the other hand, the method of variation of parameters eventually requires that we evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties. Before looking at this method in the general case, we illustrate its use in an example.

EXAMPLE 1

Find a particular solution of

$$y'' + 4y = 3\csc t. \tag{1}$$

Observe that this problem is not a good candidate for the method of undetermined coefficients, as described in Section 3.6, because the nonhomogeneous term $g(t) = 3\csc t$ involves a quotient (rather than a sum or a product) of $\sin t$ or $\cos t$. Therefore, we need a different approach. Observe also that the homogeneous equation corresponding to Eq. (1) is

$$y'' + 4y = 0, \tag{2}$$

and that the general solution of Eq. (2) is

$$y_c(t) = c_1 \cos 2t + c_2 \sin 2t. \tag{3}$$

The basic idea in the method of variation of parameters is to replace the constants c_1 and c_2 in Eq. (3) by functions $u_1(t)$ and $u_2(t)$, respectively, and then to determine these functions so that the resulting expression

$$y = u_1(t)\cos 2t + u_2(t)\sin 2t \tag{4}$$

is a solution of the nonhomogeneous equation (1).

To determine u_1 and u_2 we need to substitute for y from Eq. (4) in Eq. (1). However, even without carrying out this substitution, we can anticipate that the result will be a single equation involving some combination of u_1, u_2, and their first two derivatives. Since there is only one equation and two unknown functions, we can expect that there are many possible choices of u_1 and u_2 that will meet our needs. Alternatively, we may be able to impose a second condition of our own choosing, thereby obtaining two equations for the two unknown functions u_1 and

PROBLEMS

In each of Problems 1 through 4 use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' - y' - 2y = 2e^{-t}$
3. $y'' + 2y' + y = 3e^{-t}$
4. $4y'' - 4y' + y = 16e^{t/2}$

In each of Problems 5 through 12 find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

5. $y'' + y = \tan t, \quad 0 < t < \pi/2$
6. $y'' + 9y = 9\sec^2 3t, \quad 0 < t < \pi/6$
7. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$
8. $y'' + 4y = 3\csc 2t, \quad 0 < t < \pi/2$
9. $4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$
10. $y'' - 2y' + y = e^t/(1 + t^2)$
11. $y'' - 5y' + 6y = g(t)$
12. $y'' + 4y = g(t)$

In each of Problems 13 through 20 verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

13. $t^2y'' - 2y = 3t^2 - 1, \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$
14. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0; \quad y_1(t) = t, \quad y_2(t) = te^t$
15. $ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \quad y_1(t) = 1 + t, \quad y_2(t) = e^t$
16. $(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1; \quad y_1(t) = e^t, \quad y_2(t) = t$
17. $x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0; \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$
18. $x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x, \quad x > 0;$
    $y_1(x) = x^{-1/2}\sin x, \quad y_2(x) = x^{-1/2}\cos x$
19. $(1-x)y'' + xy' - y = g(x), \quad 0 < x < 1; \quad y_1(x) = e^x, \quad y_2(x) = x$
20. $x^2y'' + xy' + (x^2 - 0.25)y = g(x), \quad x > 0; \quad y_1(x) = x^{-1/2}\sin x, \quad y_2(x) = x^{-1/2}\cos x$
21. Show that the solution of the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y'_0 \tag{i}$$

can be written as $y = u(t) + v(t)$, where u and v are solutions of the two initial value problems

$$L[u] = 0, \qquad u(t_0) = y_0, \quad u'(t_0) = y'_0, \tag{ii}$$

$$L[v] = g(t), \qquad v(t_0) = 0, \quad v'(t_0) = 0, \tag{iii}$$

respectively. In other words, the nonhomogeneities in the differential equation and in the initial conditions can be dealt with separately. Observe that u is easy to find if a fundamental set of solutions of $L[u] = 0$ is known.

22. By choosing the lower limit of integration in Eq. (28) in the text as the initial point t_0, show that $Y(t)$ becomes

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)\,ds.$$

Show that $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Thus Y can be identified with v in Problem 21.

23. (a) Use the result of Problem 22 to show that the solution of the initial value problem

$$y'' + y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0 \tag{i}$$

is

$$y = \int_{t_0}^{t} \sin(t - s)g(s)\, ds. \tag{ii}$$

(b) Use the result of Problem 21 to find the solution of the initial value problem

$$y'' + y = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'.$$

24. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D - a)(D - b)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a and b are real numbers with $a \neq b$.

25. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = [D^2 - 2\lambda D + (\lambda^2 + \mu^2)]y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Note that the roots of the characteristic equation are $\lambda \pm i\mu$.

26. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D - a)^2 y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a is any real number.

27. By combining the results of Problems 24 through 26, show that the solution of the initial value problem

$$L[y] = (D^2 + bD + c)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where b and c are constants, has the form

$$y = \phi(t) = \int_{t_0}^{t} K(t - s)g(s)\, ds. \tag{i}$$

The function K depends only on the solutions y_1 and y_2 of the corresponding homogeneous equation and is independent of the nonhomogeneous term. Once K is determined, all nonhomogeneous problems involving the same differential operator L are reduced to the evaluation of an integral. Note also that although K depends on both t and s, only the combination $t - s$ appears, so K is actually a function of a single variable. When we think of $g(t)$ as the input to the problem and of $\phi(t)$ as the output, it follows from Eq. (i) that the output depends on the input over the entire interval from the initial point t_0 to the current value t. The integral in Eq. (i) is called the **convolution** of K and g, and K is referred to as the **kernel**.

28. The method of reduction of order (Section 3.5) can also be used for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{i}$$

provided one solution y_1 of the corresponding homogeneous equation is known. Let $y = v(t)y_1(t)$ and show that y satisfies Eq. (i) if v is a solution of

$$y_1(t)v'' + [2y_1'(t) + p(t)y_1(t)]v' = g(t). \tag{ii}$$

Equation (ii) is a first order linear equation for v'. Solving this equation, integrating the result, and then multiplying by $y_1(t)$ lead to the general solution of Eq. (i).

In each of Problems 29 through 32 use the method outlined in Problem 28 to solve the given differential equation.

29. $t^2y'' - 2ty' + 2y = 4t^2$, $t > 0$; $y_1(t) = t$
30. $t^2y'' + 7ty' + 5y = t$, $t > 0$; $y_1(t) = t^{-1}$
31. $ty'' - (1+t)y' + y = t^2e^{2t}$, $t > 0$; $y_1(t) = 1 + t$ (see Problem 15)
32. $(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}$, $0 < t < 1$; $y_1(t) = e^t$ (see Problem 16)

3.8 Mechanical and Electrical Vibrations

One of the reasons why second order linear equations with constant coefficients are worth studying is that they serve as mathematical models of some important physical processes. Two important areas of application are in the fields of mechanical and electrical oscillations. For example, the motion of a mass on a vibrating spring, the torsional oscillations of a shaft with a flywheel, the flow of electric current in a simple series circuit, and many other physical problems are all described by the solution of an initial value problem of the form

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'. \tag{1}$$

This illustrates a fundamental relationship between mathematics and physics: *Many physical problems may have the same mathematical model.* Thus, once we know how to solve the initial value problem (1), it is only necessary to make appropriate interpretations of the constants a, b, and c, and of the functions y and g, to obtain solutions of different physical problems.

We will study the motion of a mass on a spring in detail because an understanding of the behavior of this simple system is the first step in the investigation of more complex vibrating systems. Further, the principles involved are common to many problems. Consider a mass m hanging on the end of a vertical spring of original length l, as shown in Figure 3.8.1. The mass causes an elongation L of the spring in the downward (positive) direction. There are two forces acting at the point where the mass is attached to the spring; see Figure 3.8.2. The gravitational force, or weight of the mass, acts downward and has magnitude mg, where g is the acceleration due to gravity. There is also a force F_s, due to the spring, that acts upward. If we assume that the elongation L of the spring is small, the spring force is very nearly proportional to L; this is known as Hooke's[8] law. Thus we write $F_s = -kL$, where the constant of proportionality k is called the spring constant, and the minus sign is due to the fact that the spring force acts in the upward (negative) direction. Since the mass is in equilibrium, the two forces balance each other, which means that

$$mg - kL = 0. \tag{2}$$

[8]Robert Hooke (1635–1703) was an English scientist with wide-ranging interests. His most important book, *Micrographia*, was published in 1665 and described a variety of microscopical observations. Hooke first published his law of elastic behavior in 1676 as *ceiiinosssttuv*; in 1678 he gave the interpretation *ut tensio sic vis*, which means, roughly, "as the force so is the displacement."