DE AY1718 S1

Q1.

Given that: $xy' = y + 2x\ln x \ (*)$, $y(1) = 0$

Observing that the equation $(*)$ valid for all $x > 0$. Dividing both sides of the equation, we get:

$$(*) \to \frac{y'}{x} - \frac{y}{x^2} = \frac{2\ln x}{x}$$
$$\leftrightarrow \frac{dy}{dx}\frac{1}{x} + y\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$$
$$\leftrightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{2\ln x}{x}$$
$$\leftrightarrow \int \frac{d}{dx}\left(\frac{y}{x}\right)dx = \int \frac{2\ln x}{x}dx$$
$$\leftrightarrow \int d\left(\frac{y}{x}\right) = 2\int \ln x\, d(\ln x)$$
$$\leftrightarrow \frac{y}{x} = \ln^2 x + C$$

With the initial condition: $y(1) = 0$, it leads to:

$$0 = 0 + C \leftrightarrow C = 0$$

Hence, the solution of the equation is:

$$\frac{y}{x} = \ln^2 x$$

Or:

$$y = x\ln^2 x$$

Q2.

Given that: $y\cos x\,dx + (2y + \sin x + 1)dy = 0 \ (*)$

$$\leftrightarrow M(x,y)dx + N(x,y)dy = 0$$

Where: $\begin{cases} M(x,y) = y\cos x \\ N(x,y) = 2y + \sin x + 1 \end{cases}$

And: $\begin{cases} \dfrac{\partial M}{\partial y} = \cos x \\ \dfrac{\partial N}{\partial x} = \cos x \end{cases}$

$$\to \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore the given differential equation is exact.

Solve the given differential equation:

$$(*) \leftrightarrow y\cos x\,dx + 2ydy + \sin x\,dy + dy = 0$$
$$\leftrightarrow yd(\sin x) + d(y^2) + \sin x\,dy + dy = 0$$
$$\leftrightarrow yd(\sin x) + \sin x\,dy + d(y^2) + dy = 0$$
$$\leftrightarrow d(y\sin x) + d(y^2) + dy = 0$$
$$\leftrightarrow d(y\sin x + y^2 + y) = 0$$

Integrating both sides we obtain the final result:

$$\leftrightarrow y\sin x + y^2 + y + C = 0$$

Q3.

Given that: $$y'' - 4y' + 20y = e^x(x+2) + xe^{2x}$$
$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y'' - 4y' + 20y \\ g_1(x) = e^x(x+2) \\ g_2(x) = xe^{2x} \end{cases}$

Characteristic equation of the given ODE: $r^2 - 4r + 20 = 0$
$$\to r_1 = 2 + 4i, r_2 = 2 - 4i$$

So, the complement solution is: $y_c = C_1 e^{2x}\cos 4x + C_2 e^{2x}\sin x$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two term: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore y_{p1} from: $\mathrm{L}\left[y_{p1}\right] = g_1(x) \leftrightarrow y_{p1}'' - 4y_{p1}' + 20y_{p1} = e^x(x+2)$ $\quad(\alpha = 1)$

Since, $\alpha = 1$ is not a root of characteristic equation.

So, y_{p1} has the following form: $y_{p1} = (Ax + B)e^x$
$$\to y_{p1}' = (Ax + B + A)e^x$$
$$\to y_{p1}'' = (Ax + B + 2A)e^x$$

Substituting into the equation we obtain:
$$e^x(17A + 17B - 2A) = e^x(x+2)$$
$$\to \begin{cases} 17A = 1 \\ 17B - 2A = 2 \end{cases} \to \begin{cases} A = \dfrac{1}{17} \\ B = \dfrac{36}{289} \end{cases}$$

Therefore: $y_{p1} = \left(\frac{1}{17}x + \frac{36}{289}\right)e^x$

Solve fore y_{p2} from: $\mathrm{L}\left[y_{p2}\right] = g_2(x) \leftrightarrow y_{p2}'' - 4y_{p2}' + 20y_{p2} = xe^{2x}$ $\qquad(\alpha = 2)$

Since, $\alpha = 2$ is not a root of characteristic equation.

So, y_{p2} has the following form: $y_{p2} = (Ax + B)e^{2x}$
$$\to y_{p2}' = (2Ax + 2B + A)e^{2x}$$
$$\to y_{p2}'' = (4Ax + 4B + 4A)e^x$$

Substituting into the equation we obtain:
$$e^{2x}(16Ax + 16B) = xe^{2x}$$
$$\to \begin{cases} 16A = 1 \\ 16B = 0 \end{cases} \leftrightarrow \begin{cases} A = \dfrac{1}{16} \\ B = 0 \end{cases}$$

Therefore: $y_{p2} = \frac{1}{16}xe^{2x}$

So: $$\begin{aligned} y_p &= y_{p1} + y_{p2} \\ &= \left(\frac{1}{17}x + \frac{36}{289}\right)e^x + \frac{1}{16}xe^{2x} \end{aligned}$$

Thus, the general solution of the given differential equation is:
$$\begin{aligned} y_G &= y_c + y_p \\ &= C_1 e^{2x}\cos 4x + C_2 e^{2x}\sin x + \left(\frac{1}{17}x + \frac{36}{289}\right)e^x + \frac{1}{16}xe^{2x} \end{aligned}$$

Q4.

Given that: $(x-2017)^2y''-(x-2017)y'+y=2018\ (*),\quad x>2017$

It holds that the homogeneous equation: $(x-2017)^2y''-(x-2017)y'+y=0\ (1)$

Assume that $y_1=ax+b$ is a solution of the given homogeneous equation

We have: $y_1=ax+b; \rightarrow y_1'=a\rightarrow y_1''=0.$

We know that y_1 is a solution of (1), therefore substituting y_1 into (1), we get:

$$(x-2017)^2.0-(x-2017).a+ax+b=0$$
$$\leftrightarrow 0.ax+2017a+b=0$$
$$\rightarrow \begin{cases} b=-2017a \\ a\in R \end{cases}$$

Thus, with any constant a and $b=-2017a$, $y_1=ax+b$ is a solution of (1)

To find the general solution of $(*)$, we rewire $(*)$ in the following form:

$$y''-\frac{1}{x-2017}y'+\frac{1}{(x-2017)^2}y=\frac{2018}{(x-2017)^2}$$
$$(y''+p(x)y'+q(x)=r(x))$$

The Wronskian determinant for the equation is:

$$\mathrm{W}[y_1,y_2]=C_1e^{-\int p(x)dx}=C_1e^{\int\frac{1}{x-2017}dx}$$
$$\rightarrow \mathrm{W}[y_1,y_2]=C_1(x-2017)$$

Hence:

$$y_2=y_1\left[\int\frac{\mathrm{W}[y_1,y_2]}{y_1^2}dx+C_2\right]$$

Choose: $a=1\rightarrow b=-2017$ for y_1, it leads to:

$$y_2=(x-2017)\left[\int\frac{C_1(x-2017)}{(x-2017)^2}dx+C_2\right]$$
$$\rightarrow y_2=(x-2017)[C_1\ln(x-2017)+C_2]$$
$$\rightarrow y_2=C_1(x-2017)\ln(x-2017)+C_2(x-2017)$$

Choose $C_1=1, C_2=0\rightarrow y_2=(x-2017)\ln(x-2017)$

Since, the Wronskian determinant different from 0 for all $x>2017$, therefore y_1 and y_2 are linearly independence solutions of the homogeneous equation.

Clearly, $y_p=2018$ is a particular solution of $(*)$

Thus, the general solution of the equation is:

$$y_G=C_1y_1+C_2y_2+y_p=C_1(x-2017)+C_2(x-2017)\ln(x-2017)+2018$$

Q5.

Due to Newton's Cooling Law:

$$\frac{dT}{dt}=-k(T-T_e)\ (*)$$

Where:

$T(t)$: Temperature of a body at time t.

k: Positive constant characteristic of the system.

T_e: Environment temperature.

$$(*)\rightarrow\frac{dT}{T-T_e}=-kdt$$
$$\rightarrow \ln(T-T_e)=-kt+C\ (1)$$

With the condition given in the prolem:

$$\begin{cases} T(0) = 37 \\ T(1) = 28 \end{cases} \rightarrow \begin{cases} \ln(37 - 22) = -k.0 + C \\ \ln(28 - 22) = -k.1 + C \end{cases} \leftrightarrow \begin{cases} C = \ln 15 \\ k = \ln 2.5 \end{cases}$$

From (1), Solve for $T(t)$, we obtain:

$$T(t) = e^{-kt+C} + T_e$$

If $T(t) = 30$, Solve for t, we get $t = 0.686$ (hour) = 41 (minutes)

Therefore, the victim is killed at around 7:49 AM