Introduction to Computer for Engineers

Lecture 8
Gaussian Elimination

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Review Math

Find the solution of

$$\begin{cases} 2x_1 & + & x_2 & + & 3x_3 & = 1 \\ 2x_1 & + & 6x_2 & + & 8x_3 & = 3 \\ 6x_1 & + & 8x_2 & + & 18x_3 & = 5 \end{cases}$$

Apply the row reduction process → create an augmented matrix in the form

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{array}\right]$$

OR

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{bmatrix}$$

Review Math

Apply the row reduction process → create an augmented matrix in the form

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{array}\right] \quad \textbf{OR} \quad \begin{bmatrix} 2 & 1 & 3 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{bmatrix}$$

<u>Objective of row reduction</u> → Upper triangular matrix

$$\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}$$

Solution is obtained by substituting back to the system of equ.

Gaussian elimination algorithm

$$\begin{cases} 2x_1 & + & x_2 & + & 3x_3 & = 1 \\ 2x_1 & + & 6x_2 & + & 8x_3 & = 3 \\ 6x_1 & + & 8x_2 & + & 18x_3 & = 5 \end{cases}$$

$$\begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}$$

3 equations with 3 unknowns

→ **Input: a 3 by 4 matrix**

→ **Output: one columns vector x of dim. 3 by 1**

2 main steps

→ **transform to upper triangular matrix (forward step)**

→ **get solution by substituting back to matrix (back-substituting step)**

Gaussian elimination algorithm

Forward step

Sub-step 1 → Row 1

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{bmatrix} \xrightarrow{R1/2 \to R1} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{bmatrix}$$

Sub-step 1 → Row 2

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 5 & 5 & 2 \\ 6 & 8 & 18 & 5 \end{bmatrix}$$

Gaussian elimination algorithm

Forward step

Sub-step 1 → Row 3

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 5 & 5 & 2 \\ 6 & 8 & 18 & 5 \end{bmatrix} \xrightarrow{R3 \to R2 - 6R1} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 5 & 5 & 2 \\ 0 & 5 & 9 & 2 \end{bmatrix}$$

Sub-step 2 → Row 2

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 5 & 5 & 2 \\ 0 & 5 & 9 & 2 \end{bmatrix} \xrightarrow{R2 \to R2/5} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 5 & 9 & 2 \end{bmatrix}$$

Gaussian elimination algorithm

Forward step

Sub-step 2 → Row 3

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 5 & 9 & 2 \end{bmatrix} \xrightarrow{R3 \to R3 - 5R2} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Sub-step 3 → Row 3

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 4 & 0 \end{bmatrix} \xrightarrow{R3 \to R3/4} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Gaussian elimination algorithm

Back-substitution step

Sub-step 1 → Row 3

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{x_3 \to 0/1} \begin{bmatrix} - \\ - \\ 0 \end{bmatrix}$$

Sub-step 2 → Row 2

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{x_2 \to 2/5 - 1 \times x_3} \begin{bmatrix} - \\ 2/5 \\ 0 \end{bmatrix}$$

Gaussian elimination algorithm

<u>Back-substitution step</u>

Sub-step 3 → Row 1

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{x_1 \to 1/2 - 1/2 \times x_2 - 3/2 \times x_3} \begin{bmatrix} 3/10 \\ 2/5 \\ 0 \end{bmatrix}$$

Gaussian elimination algorithm

Roadmap to Gaussian elimination algorithm

3 equations with 3 unknowns → n = 3

Observe in Forward step, Back-substitution step

→ how many sub-steps comparing to n=3 ?

→ for each sub-step, how many rows transformation?

→relation #sub-step and #row ?

Gaussian elimination algorithm

Roadmap to Gaussian elimination algorithm

3 equations with 3 unknowns → n = 3

Forward step: 3 sub-steps, counted from 1 to 3

→ for loop, count k=1:1:n, 3 passes

→ pass 1 – 3 row operations – from 1 to 3

pass 2 - 2 row operations – from 2 to 3

pass 3 – 1 row operation – from 3 to 3

Back-substitution step: 3 sub-steps, counted from 3 to 1

(for loop, count k=3:-1:1, 3 passes)

How many for loops? Does it required nested for loop?

Gaussian elimination algorithm

Roadmap to Gaussian elimination algorithm

Formulation Forward step:

Write a script that show in the command window the sequence

1	1
1	2
1	3
2	2
2	3
3	3

Gaussian elimination algorithm

Roadmap to Gaussian elimination algorithm

Formulation the Back-substitution step:

Write a script that show in the command window the sequence

3

2

1

Gaussian elimination algorithm

Roadmap to Gaussian elimination algorithm

Formulation the Back-substitution step:

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & \boxed{1} & \boxed{0} \end{bmatrix} \xrightarrow{x_3 \to \boxed{0}/\boxed{1}} \begin{bmatrix} - \\ - \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & \boxed{1} & \boxed{2/5} \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{x_2 \to \boxed{2/5} - \boxed{1} \times x_3} \begin{bmatrix} - \\ 2/5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \boxed{1/2} & \boxed{3/2} & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{x_1 \to 1/2 - \boxed{1/2} \times x_2 - \boxed{3/2} \times x_3} \begin{bmatrix} 3/10 \\ 2/5 \\ 0 \end{bmatrix}$$

Gaussian elimination algorithm

<u>Roadmap to Gaussian elimination algorithm</u>

Formulation the Back-substitution step:

3 sub-steps, counted from 3 to 1

(for loop, count k=3:-1:1, 3 passes)

$$\begin{cases} x_3 = M(3,4) \\ x_2 = M(2,4) - M(2,3) \times x_3 \\ x_1 = M(1,4) - M(1,2) \times x_2 - M(1,3) \times x_3 \end{cases}$$

Does for loop need counting from 3 to 1 ?

Gaussian elimination algorithm

<u>Roadmap to Gaussian elimination algorithm</u>

Formulation the Back-substitution step:

$$\begin{cases} x_3 = M(3,4) \\ x_2 = M(2,4) - M(2,3) \times x_3 \\ x_1 = M(1,4) - M(1,2) \times x_2 - M(1,3) \times x_3 \end{cases}$$

No ! Just need counting from 2 to 1!

General form of x(k,1) using in the loop?

Gaussian elimination algorithm

Roadmap to Gaussian elimination algorithm

Formulation the Back-substitution step:

$$\begin{cases} x_3 = M(3,4) \\ x_2 = M(2,4) - M(2,3) \times x_3 \\ x_1 = M(1,4) - M(1,2) \times x_2 - M(1,3) \times x_3 \end{cases}$$

General form of x(k,1) using in the loop?

$$\begin{array}{lcll} x(2,1) & = & M(2,4)- & [M(2,3)].*[x(3,1)] \\ x(1,1) & = & M(1,4)- & [M(1,2) \quad M(1,3)].*[x(2,1) \quad x(3,1)] \end{array}$$

Gaussian elimination algorithm

Formulation the Back-substitution step:

General form of x(k,1) using in the loop?

$$
\begin{array}{rcl}
x(2,1) & = & M(2,4) - \qquad\qquad [M(2,3)].*[x(3,1)] \\
x(1,1) & = & M(1,4) - [M(1,2)\;\; M(1,3)].*[x(2,1)\;\; x(3,1)]
\end{array}
$$

Recall vector indexing in MATLAB

$$
M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \rightarrow \begin{array}{l} M(1,:) = [\; M_{11} \quad M_{12} \quad M_{13} \quad M_{14}] \\ M(1,2:3) = [\; M_{12} \quad M_{13}] \\ M(1,2:2) = [\; M_{12}] \end{array}
$$

Gaussian elimination algorithm

Formulation the Back-substitution step:

General form of x(k,1) using in the loop?

$$
\begin{array}{rcl}
x(2,1) & = & M(2,4) - \qquad [M(2,3)].*[x(3,1)] \\
x(1,1) & = & M(1,4) - [M(1,2) \;\; M(1,3)].*[x(2,1) \;\; x(3,1)]
\end{array}
$$

Recall vector indexing in MATLAB

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow
\begin{array}{rl}
x' & = [\; x_1 \quad x_2 \quad x_3 \;] \\
x(2:3,1)' & = [\; x_2 \quad x_3 \;] \\
x(3:3,1)' & = \; x_3
\end{array}
$$

Gaussian elimination algorithm

Implementation

```
%% forward step
disp('Forward step');
for i=1:N %sub-step
    for j=1:N %number of row correpondign to sub-step
        if (j>=i)
            % print out the #sub-step and the #row corresponding to
            % each sub-step
            fprintf('#sub-step %d, #row %d',i,j);
            if (j == i)
                A(j,:) = A(i,:)/A(i,j);
            else
                A(j,:) = A(j,:) - A(j,i)*A(i,:);
            end
        end
        A
    end
end

%% back-substitution step
disp('Back-substitution step');
x(N) = A(N,N+1);
for i=(N-1):-1:1
    x(i,1) = A(i,N+1)-sum(A(i,(i+1):N).*x((i+1):N,1)');
end
```

Gaussian elimination algorithm

Complexity analysis

Definition: g(n) has order of magnitude f(n) denoted by

$$g(n) \sim f(n)$$

If

$$\lim_{n\to\infty}\left|\frac{g(n)-f(n)}{f(n)}\right| = 0$$

or equivalently

$$\lim_{n\to\infty}\frac{g(n)}{f(n)} = 1$$

Gaussian elimination algorithm

Complexity analysis

Theorem: if m ≥ 0 then

$$\sum_{k=1}^{n} k^m \sim \frac{n^{m+1}}{m+1}$$

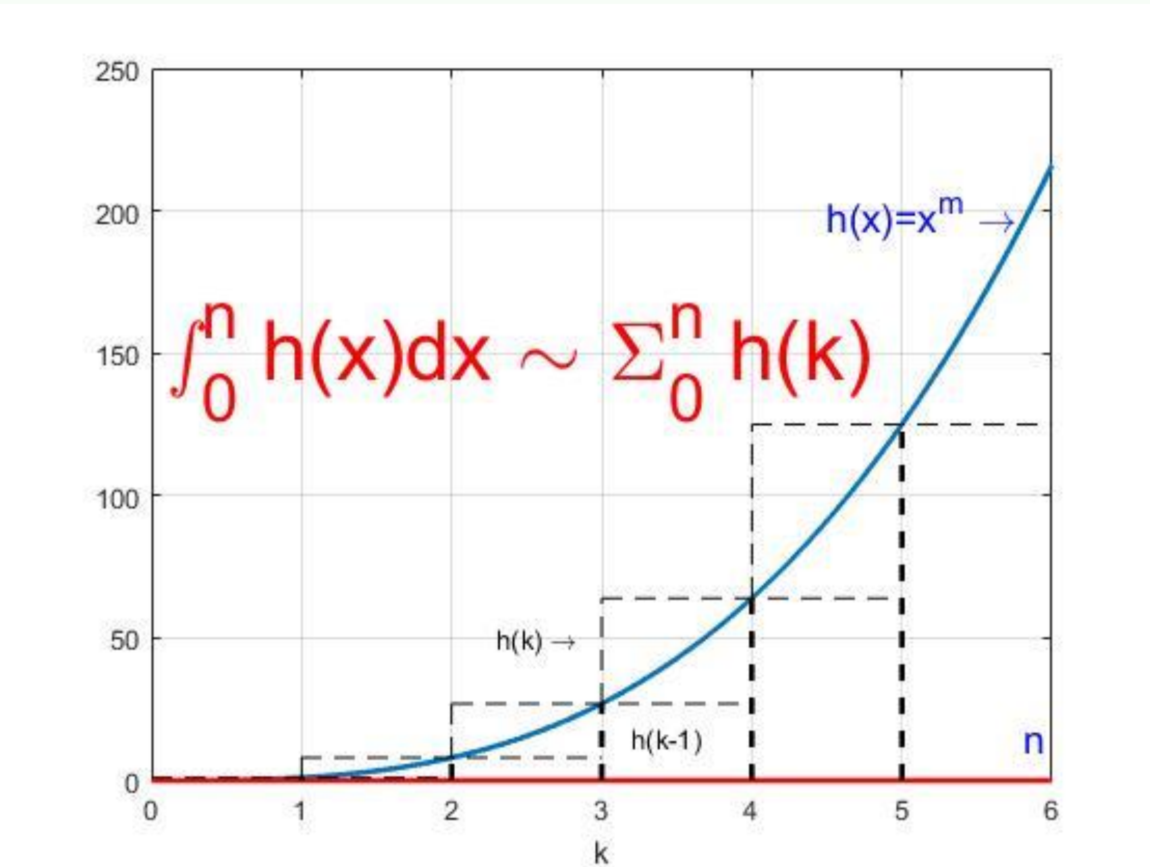
Example

$$1 + 2 + \ldots + n \sim \frac{n^2}{2}$$

$$1^2 + 2^2 + \ldots + n^2 \sim \frac{n^3}{3}$$

Gaussian elimination algorithm

Complexity analysis – Proof of theorem



Let

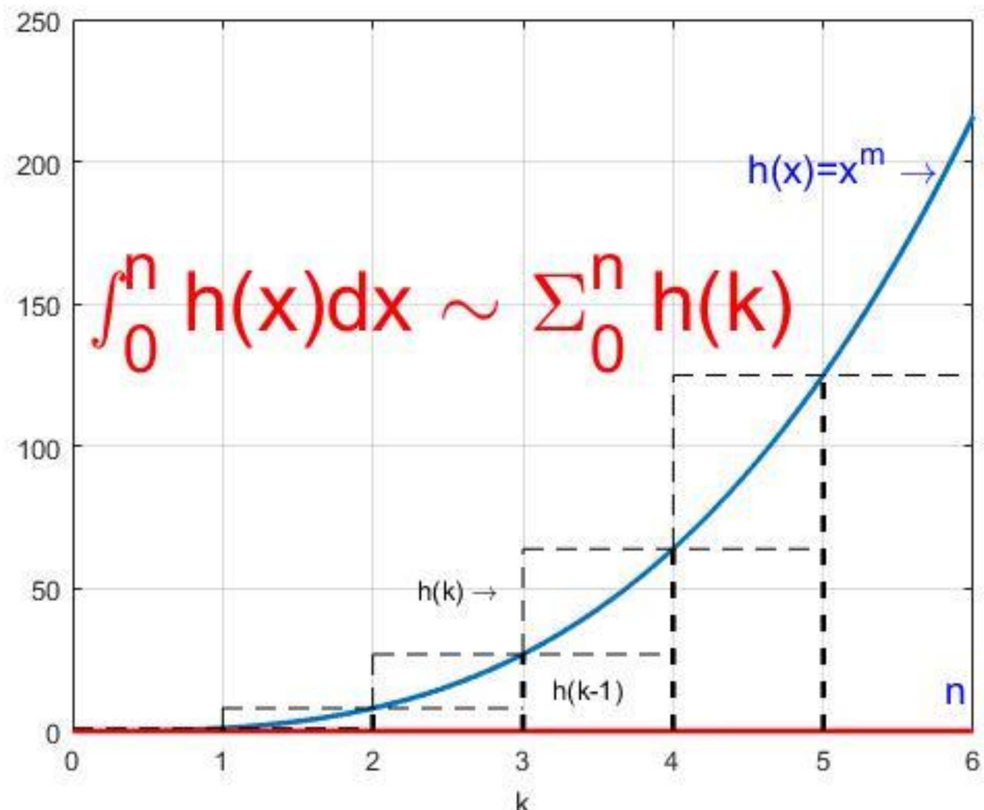
$$
\begin{aligned}
h(x) &= x^m \\
g(n) &= \textstyle\sum_{k=0}^{n} h(k)
\end{aligned}
$$

g(n) : sum of rec. area appro. of int.

at k=4 → g(n) > the integral of h(x) over x = [0,4]

Gaussian elimination algorithm

Complexity analysis – Proof of theorem



Let

$$
\begin{aligned}
h(x) &= x^m \\
g(n) &= \sum_{k=0}^{n} h(k)
\end{aligned}
$$

g(n) : sum of rec. area appro. of int.

at k=4 → we have

$$g(4) = \sum_{k=0}^{4} h(k) \geq \int_{0}^{4} h(x)dx$$

Gaussian elimination algorithm

Complexity analysis – Proof of theorem

at k=4 → we have

$$g(4) = \sum_{k=0}^{4} h(k) \geq \int_{0}^{4} h(x)dx$$

at k=n → we also have

$$g(n) = \sum_{k=0}^{n} h(k) \geq \int_{0}^{n} h(x)dx \geq \sum_{k=0}^{n-1} h(k) = g(n-1)$$

and

$$0 \geq \int_{0}^{n} h(x)dx - g(n) \geq g(n-1) - g(n)$$

Gaussian elimination algorithm

Complexity analysis – Proof of theorem

$$0 \geq \int_0^n h(x)dx - g(n) \geq g(n-1) - g(n)$$

We have

$$\int_0^n h(x)dx = \int_0^n x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_{x=0}^n = \frac{n^{m+1}}{m+1}$$

$$\begin{array}{rc} & g(n) - g(n+1) \\ = & (1^m + 2^m + \ldots + (n-1)^m + \quad n^m) \\ - & (1^m + 2^m + \ldots + (n-1)^m) \end{array} = n^m$$

Gaussian elimination algorithm

Complexity analysis – Proof of theorem

$$0 \geq \int_0^n h(x)dx - g(n) \geq g(n-1) - g(n)$$

Finally, we have:

$$0 \geq \frac{n^{m+1}}{m+1} - g(n) \geq -n^m$$

$$0 \geq f(n) - q(n) \geq -n^{m+1}$$

$$\left|\frac{q(n) - f(n)}{f(n)}\right| \geq \left|\frac{-n^m}{n^{m+1}/(m+1)}\right| = \frac{m+1}{n} \to 0 \quad (n \to \infty)$$

Gaussian elimination algorithm

Complexity analysis – Proof of theorem

Finally, we have:

$$0 \geq f(n) - q(n) \geq -n^{m+1}$$

$$\left|\frac{q(n) - f(n)}{f(n)}\right| \geq \left|\frac{-n^m}{n^{m+1}/(m+1)}\right| = \frac{m+1}{n} \to 0 \quad (n \to \infty)$$

$$q(n) = \sum_{k=0}^{n} h(k) = \sum_{k=0}^{n} k^m \sim \frac{n^{m+1}}{m+1} = f(n)$$

Gaussian elimination algorithm

Complexity analysis

Assumption 1: comparing to multiplication and division, all others operations such as addition, subtraction, assignment, if-else-end …. take negligible time

Assumption 2: matrix M can be transform to upper-triangular form. There is no zero leading coef.

Gaussian elimination algorithm

Complexity analysis

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 5 & 5 & 2 \\ 0 & 5 & 9 & 2 \end{bmatrix} \xrightarrow{R2 \to R2/5} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 5 & 9 & 2 \end{bmatrix}$$

Lemma 1: **clearing first column of a n by (n+1) matrix M**

- **take n^2-n operations on the n first columns**
- **take n operation on the last columns**

Gaussian elimination algorithm

Complexity analysis

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 5 & 5 & 2 \\ 0 & 5 & 9 & 2 \end{bmatrix} \xrightarrow{R2 \to R2/5} \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 2/5 \\ 0 & 5 & 9 & 2 \end{bmatrix}$$

Lemma 2: apply the forward step on matrix M

- take ~ ($n^3/3-n^2/2$) operations on the n first columns
- take ~ n^2 operation on the last columns

Gaussian elimination algorithm

Complexity analysis

$$\begin{cases} x_3 = M(3,4) \\ x_2 = M(2,4) - M(2,3) \times x_3 \\ x_1 = M(1,4) - M(1,2) \times x_2 - M(1,3) \times x_3 \end{cases}$$

Lemma 3: back-substitution step takes n(n+1)/2 operations

Gaussian elimination algorithm

Complexity analysis

$$\begin{cases} x_3 = M(3,4) \\ x_2 = M(2,4) - M(2,3) \times x_3 \\ x_1 = M(1,4) - M(1,2) \times x_2 - M(1,3) \times x_3 \end{cases}$$

Theorem: complexity of Gaussian Elimination

$$n^3/3+2n^2/2+n/2 \sim n^3/3$$

End of Lecture 11