Q1.

Given that: $$(3x^2y + e^y)dx + (x^3 + xe^y - 2y)dy = 0\ (*)$$

$$(*) \leftrightarrow 3x^2ydx + e^ydx + x^3dy + xe^ydy - 2ydy = 0$$
$$\leftrightarrow yd(x^3) + e^ydx + x^3dy + xd(e^y) - d(y^2) = 0$$
$$\leftrightarrow yd(x^3) + x^3dy + e^ydx + xd(e^y) - d(y^2) = 0$$
$$\leftrightarrow d(x^3y) + d(xe^y) - d(y^2) = 0$$
$$\leftrightarrow d(x^3y + xe^y - y^2) = 0$$

Integrating both sides we obtain the final result:

$$\leftrightarrow x^3y + xe^y - y^2 + C = 0$$

Q2.

Let $x(t)$ be the number of grams of C present at time t (minute). Due to the fact that 1 gram of A and 4 grams of B used to combine C, therefore, the amount of A and B used are $\frac{x(t)}{5}, \frac{4x(t)}{5}$, respectively.

The amount of remain chemical A: $50 - \frac{x(t)}{5}$

The amount of remain chemical B: $32 - \frac{4x(t)}{5}$

The problem tells us that the rate of formed chemical C depends on the proportional product of instantaneous amount of A and B not converted to C. It means that:

$$\frac{dx}{dt} = K\left(50 - \frac{x}{5}\right)\left(32 - \frac{4x}{5}\right)$$
$$\rightarrow \frac{25dx}{(250 - x)(160 - 4x)} = Kdt$$
$$\leftrightarrow \left(\frac{5}{42}\frac{1}{160 - 4x} - \frac{5}{168}\frac{1}{250 - x}\right)dx = Kdt$$

Integrating both sides we get:

$$\rightarrow \frac{5}{168}\ln\left(\frac{250 - x}{160 - 4x}\right) = Kt + C (1)$$

With the initial condition:

$$\begin{cases} x(0) = 0 \\ x(10) = 30 \end{cases} \rightarrow \begin{cases} 0.0133 = K.0 + C \\ 0.0527 = K.10 + C \end{cases} \leftrightarrow \begin{cases} C = 0.0133 \\ K = 3.7454 \times 10^{-3} \end{cases}$$

From (1) solve for $x(t)$, we get:

$$x(t) = \frac{160e^{\frac{168}{5}(Kt+C)} - 250}{4e^{\frac{168}{5}(Kt+C)} - 1}$$

Therefore: $x(20) = 37.2544$ grams

Q3.

Given that: $$y' = (\sin x)y + 2\sin x\ (*), \qquad y\left(\frac{\pi}{2}\right) = 1$$

$$(*) \leftrightarrow y'e^{\cos x} - \sin x\, e^{\cos x}y = 2\sin x\, e^{\cos x}$$
$$\leftrightarrow \frac{dy}{dx}e^{\cos x} + \frac{de^{\cos x}}{dx}y = 2\sin x\, e^{\cos x}$$
$$\leftrightarrow \frac{d(ye^{\cos x})}{dx} = 2\sin x\, e^{\cos x}$$

$$\leftrightarrow \int \frac{d(ye^{\cos x})}{dx}dx = \int 2\sin x\, e^{\cos x}dx$$

$$\leftrightarrow \int d(ye^{\cos x}) = -2\int e^{\cos x}d(\cos x)$$

$$\leftrightarrow ye^{\cos x} = -2e^{\cos x} + C$$

With the initial condition: $y\left(\frac{\pi}{2}\right) = 1$, it leads to:

$$1.e^0 = -2e^0 + C \leftrightarrow C = 3$$

Hence, the solution of the equation is:

$$ye^{\cos x} = -2e^{\cos x} + 3$$

Or:

$$y = -2 + \frac{3}{\cos x}$$

Q4.

Given that: $$y'' - 6y' + 9y = 2018e^{3x} + e^x(x+1)$$

$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y'' - 6y' + 9y \\ g_1(x) = 2018e^{3x} \\ g_2(x) = e^x(x+1) \end{cases}$

Characteristic equation of the given ODE: $r^2 - 6r + 9 = 0$

$$\rightarrow r_1 = r_2 = 3$$

So, the complement solution is: $y_c = C_1e^{-3x} + C_2xe^{3x}$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two term: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore y_{p1} from: $\mathrm{L}[y_{p1}] = g_1(x) \leftrightarrow y_{p1}'' - 6y_{p1}' + 9y_{p1} = 2018e^{3x}$ $(\alpha = 3)$

Since, $\alpha = 3$ is a double root of characteristic equation.

So, y_{p1} has the following form: $y_{p1} = Ax^2e^{3x}$

$$\rightarrow y'_{p1} = A(3x^2 + 2x)e^{3x}$$

$$\rightarrow y''_{p1} = A(9x^2 + 12x + 2)e^{3x}$$

Substituting into the equation we obtain:

$$A\big((9x^2 + 12x + 2) - 6(3x^2 + 2x) + 9x^2\big) = 2018e^{3x}$$

$$\rightarrow 2A = 2018 \leftrightarrow A = 1009$$

Therefore: $y_{p1} = 1009x^2e^{3x}$

Solve fore y_{p2} from: $\mathrm{L}[y_{p2}] = g_2(x) \leftrightarrow y_{p2}'' - 6y_{p2}' + 9y_{p2} = e^x(x+1)$ $(\alpha = 1)$

Since, $\alpha = 1$ is not a root of characteristic equation.

So, y_{p2} has the following form: $y_{p2} = (Ax + B)e^x$

$$\rightarrow y'_{p2} = (Ax + B + A)e^x$$

$$\rightarrow y''_{p2} = (Ax + B + 2A)e^x$$

Substituting into the equation we obtain:

$$e^x[4Ax + 4B - 4A] = e^x(x+1)$$

$$\rightarrow \begin{cases} 4A = 1 \\ 4B - 4A = 1 \end{cases} \leftrightarrow \begin{cases} A = \dfrac{1}{4} \\ B = \dfrac{1}{2} \end{cases}$$

Therefore: $y_{p2} = e^x\left(\frac{1}{4}x + \frac{1}{2}\right)$

So: $y_p = y_{p1} + y_{p2}$

$$= 1009x^2e^{3x} - e^x\left(\frac{1}{4}x + \frac{1}{2}\right)$$

Thus, the general solution of the given differential equation is:

$$y_G = y_c + y_p$$

$$= C_1e^{-3x} + C_2xe^{3x} - 1009x^2e^{3x} - e^x\left(\frac{1}{4}x + \frac{1}{2}\right)$$

Q5.

Given that:

$$y'' - 3y' + 2y = \frac{e^{2x}}{e^x + 1}\ (*)$$

Characteristic equation of the given DE: $r^2 - 3r + 2 = 0$

$$\rightarrow r_1 = 1;\ r_2 = 2$$

So, the complement solution is: $y_c = C_1e^x + C_2e^{2x}$ (1)

Multiply both sides of $(*)$ by e^{-x}, we get:

$$\rightarrow y''e^{-x} - e^{-x}y' - 2(y'e^{-x} - e^{-x}y) = \frac{e^x}{e^x + 1}$$

$$\leftrightarrow (y'e^{-x})' - 2(ye^{-x})' = \frac{e^x}{e^x + 1}$$

Integrating both sides, it leads to:

$$\rightarrow y'e^{-x} - 2ye^{-x} = \ln(e^x + 1) + C_1$$

Multiply both sides of $(*)$ by e^{-x} again, we get:

$$\rightarrow y'e^{-2x} - 2ye^{-2x} = e^{-x}\ln(e^x + 1) + C_1e^{-x}$$

$$\leftrightarrow (ye^{-2x})' = e^{-x}\ln(e^x + 1) + C_1e^{-x}$$

Integrating both sides, it leads to:

$$\rightarrow ye^{-2x} = -e^{-x}\ln(e^x + 1) + x - \ln(e^x + 1) - C_1e^{-x} + C_2$$

$$\leftrightarrow y = -e^x\ln(e^x + 1) + xe^{2x} - e^{2x}\ln(e^x + 1) + C_1'e^x + C_2e^{2x}\ (2)$$

Comparing (1) and (2), we obtain the particular solution:

$$y_p = -e^x\ln(e^x + 1) + xe^{2x} - e^{2x}\ln(e^x + 1)$$