An Introduction to Applied Linear Algebra

Lecture 1: Matrices and Linear Systems

by Associate Prof. Pham Huu Anh Ngoc

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Textbook:

E. Kreyszig, Advanced Engineering Mathematics, 9th edition, John Wiley & Sons, 2006 (Chapters: 7, 8)

I. Matrix and operations

Definition:

An **m x n matrix** is a rectangular array of numbers arranged in m rows (horizontal lines) and n columns (vertical lines).

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An **m x n matrix** is a rectangular array of numbers arranged in m rows (horizontal lines) and n columns (vertical lines).

Example: A matrix with 3 rows and 2 columns : a 3 x 2 matrix (read "a 3 by 2 matrix")

$$\begin{pmatrix} 0 & 1 \\ 3 & -1 \\ 0 & 0 \end{pmatrix}$$

A matrix with 3 rows and 3 columns : a 3 x 3 matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 100 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

In general an m x n matrix **A** has the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & \dots & a_{mn} \end{pmatrix}$$

Another denotation for matrix **A** is $\mathbf{A} = [a_{ij}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. We denote matrices by capital boldface letter **A, B, C,... .**

The **order** of a matrix having m rows and n columns is mn. Then a_{ij} $(1 \leq i \leq m; 1 \leq j \leq n)$ are called entries of the matrix **A**.

If $m = n$, we call **A** an $n \times n$ **square matrix** and its main diagonal entries are: $a_{11}, a_{22}, \dots, a_{nn}$.

Example

Let

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 23 & 0 & 1 \end{pmatrix}.$$

It is an 3×3 **square matrix** and its main diagonal entries are: $0, 0, 1$. The **order** of this matrix is 9.

The following

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 5 \\ 23 & 0 & 1 & 6 \end{pmatrix}$$

is not a square matrix.

Remarks

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices. Then $\mathbf{A} = \mathbf{B}$ if and only if $a_{ij} = b_{ij}$ for all i, j.

A **vector** is a matrix with only one row or one column. We denote vectors by lowercase boldface letter **a, b, c** ....

A **row vector** is of the form

$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \dots & \dots & a_{n-1} & a_n \end{pmatrix}.$$

A **column vector** is of the form

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}$$

Addition of two matrices

Only matrices of the **same number of rows and same number of columns may be added** by adding corresponding elements.
By definition:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & \dots & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \dots & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & \dots & \dots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & \dots & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & \dots & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & \dots & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

Example:

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

Example:

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

Let

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 5 & 100 \end{pmatrix}$$

Note that $\mathbf{A} + \mathbf{C}$ or $\mathbf{B} + \mathbf{C}$ is **NOT defined**.

Scalar Multiplication of a Matrix

To multiply matrix **A** of order m x n by a scalar k, we multiply each entry of **A** by k to obtain another matrix of the same order.
That is,

$$k\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \dots & \dots & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & \dots & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & \dots & \dots & ka_{mn} \end{pmatrix}$$

Example

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}.$$

Then

$$2\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 \\ -2 & -4 & 2 \end{pmatrix}.$$

$$5\mathbf{A} = \begin{pmatrix} 5 & 5 & 0 \\ -5 & -10 & 5 \end{pmatrix} \qquad \frac{1}{2}\mathbf{A} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & -1 & 1/2 \end{pmatrix}.$$

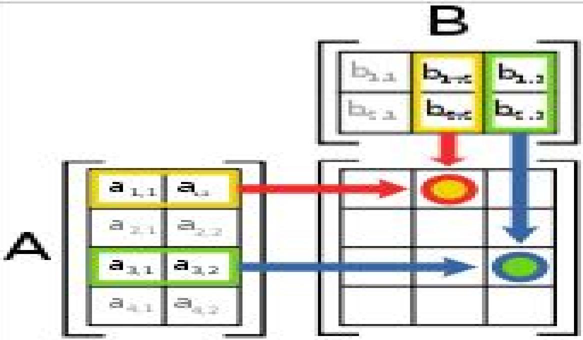
Multiplication of two Matrices

If $\mathbf{A} = [a_{ij}]$ is an **m x n**-matrix and $\mathbf{B} = [b_{ij}]$ an **n x p**-matrix, then the product $\mathbf{C} = \mathbf{AB}$ of the two matrices is an **m x p**-matrix defined by $\mathbf{C} = [c_{ij}]$ where c_{ij} is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj},$$

(the **inner product** of $(a_{i1}, a_{i2}, ..., a_{in})$ and $(b_{1j}, b_{2j}, ..., b_{nj})$).

Note: (**m x n**-matrix) (**n x p**-matrix)=(**m x p**-matrix)



Schematic depiction of the matrix product **AB** of two matrices **A** and **B**.

Example:

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

NOTE: AB $\neq$ BA.

$$\mathbf{AC} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

NOTE: We **can not do: CA**.

Ex:

$$\overbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}}^{3\times 2}\overbrace{\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}}^{2\times 4} =$$

Ex:

$$\overbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}}^{3\times 2} \overbrace{\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}}^{2\times 4} = \overbrace{\begin{pmatrix} 1 & 2 & 2 & 1 \\ & & & \\ & & & \end{pmatrix}}^{3\times 4}$$

Ex:

$$\overbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}}^{3\times 2}\overbrace{\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}}^{2\times 4} = \overbrace{\left(\begin{matrix} 1 & 2 & 2 & 1 \\ 4 & 7 & 5 & 2 \\ & & & \end{matrix} \right.}^{3\times 4}$$

Ex:

$$\overbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}}^{3\times 2} \overbrace{\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}}^{2\times 4} = \overbrace{\begin{pmatrix} 1 & 2 & 2 & 1 \\ 4 & 7 & 5 & 2 \\ 5 & 9 & 7 & 3 \end{pmatrix}}^{3\times 4}$$

Properties

$$
\begin{aligned}
\mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} \\
(\mathbf{A} + \mathbf{B}) + \mathbf{C} &= \mathbf{A} + (\mathbf{B} + \mathbf{C}) \\
\mathbf{A} + \mathbf{0} &= \mathbf{A} \\
\mathbf{A} + (-\mathbf{A}) &= \mathbf{0} \\
(\mathbf{A}\mathbf{B})\mathbf{C} &= \mathbf{A}(\mathbf{B}\mathbf{C}) \\
(\mathbf{A} + \mathbf{B})\mathbf{C} &= \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C} \\
\mathbf{C}(\mathbf{A} + \mathbf{B}) &= \mathbf{C}\mathbf{A} + \mathbf{C}\mathbf{B} \\
k(\mathbf{A}\mathbf{B}) &= (k\mathbf{A})\mathbf{B} = k(\mathbf{A}\mathbf{B}).
\end{aligned}
$$

Applications:

An ice-cream shop makes two types of ice-cream, known as light and rich. Matrix A shows the quantities of fresh eggs (in dozen), cream (in gallons) and milk (in gallons) needed to make one batch of each type of ice-cream. Matrix B shows the prices (in dollars) of a dozen of eggs, a gallon of milk and a gallon of cream if purchased from supplier X and the prices if purchased from supplier Y:

$$\begin{array}{c} \\ eggs \\ cream \\ milk \end{array} \begin{array}{c} \begin{array}{cc} light & rich \end{array} \\ \begin{pmatrix} 1.5 & 2 \\ 2.5 & 5 \\ 5.5 & 3 \end{pmatrix} \end{array} := A; \quad \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{ccc} eggs & cream & milk \end{array} \\ \begin{pmatrix} 1.25 & 3.00 & 2.75 \\ 1.15 & 3.25 & 2.60 \end{pmatrix} \end{array} := B.$$

a) Calculate the product BA and explain what it represents.
b) Every day the shop makes 6 batches of light and 10 batches of rich ice-cream. Find a matrix showing the total quantities of eggs, cream and milk used each day. Which supplier gives a lower total daily cost?

Solution: a)

$$\begin{array}{c} \\ eggs \\ cream \\ milk \end{array} \begin{array}{c} \begin{array}{cc} light & rich \end{array} \\ \begin{pmatrix} 1.5 & 2 \\ 2.5 & 5 \\ 5.5 & 3 \end{pmatrix} \end{array} := A; \quad \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{ccc} eggs & cream & milk \end{array} \\ \begin{pmatrix} 1.25 & 3.00 & 2.75 \\ 1.15 & 3.25 & 2.60 \end{pmatrix} \end{array} := B.$$

The matrix BA is given by

$$\begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} \text{total cost per batch of ''}\textit{light}\text{''} & \text{total cost per batch of ''}\textit{rich}\text{''} \end{array} \\ \begin{pmatrix} 1.25x1.5 + 3.00x2.5 + 2.75x5.5 & 1.25x2 + 3.00x5 + 2.75x3 \\ 1.15x1.5 + 3.25x2.5 + 2.60x5.5 & 1.15x2 + 3.25x5 + 2.60x3 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} \text{total cost per batch of ''}\textit{light}\text{''} & \text{total cost per batch of ''}\textit{rich}\text{''} \end{array} \\ \begin{pmatrix} 24.5 & 25.75 \\ 24.15 & 26.75 \end{pmatrix} \end{array}$$

The matrix BA represents the total cost per batch of each type of ice-creams.

$$BA = \begin{array}{c c} & \begin{array}{cc} \text{total cost per batch of } {''light''} & \text{total cost per batch of } {''rich''} \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} 24.5 & 25.75 \\ 24.15 & 26.75 \end{pmatrix} \end{array}$$

More precisely,

- The first row represents the total cost per batch of the light and rich ice-cream, respectively, when eggs, cream, milk are purchased from the supplier X.

The second row represents the total cost per batch of the light and rich ice-cream, respectively, when eggs, cream, milk are purchased from the supplier Y.

b)
Let

$$D := \begin{pmatrix} 6 \\ 10 \end{pmatrix}.$$

The required matrix is

$$AD = \begin{pmatrix} 1.5 & 2 \\ 2.5 & 5 \\ 5.5 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 10 \end{pmatrix} = \begin{array}{c} \textit{eggs} \\ \textit{cream} \\ \textit{milk} \end{array} \begin{pmatrix} 29 \\ 65 \\ 63 \end{pmatrix}$$

c) Home work!

Applications:

Computer production:

The Apple company produces two computer models PC 1 and PC 2.

Matrix A

$$\begin{array}{l} \\ \textit{Raw components} \\ \textit{Labor} \\ \textit{Miscellaneous} \end{array} \begin{array}{c} \begin{array}{cc} PC1 & PC2 \end{array} \\ \begin{pmatrix} 1.1 & 1.6 \\ 0.4 & 0.5 \\ 0.4 & 0.6 \end{pmatrix} \end{array} := A;$$

shows the cost per computer (in thousands of dollars) and the matrix B

$$\begin{array}{l} \\ PC1 \\ PC2 \end{array} \begin{array}{c} \begin{array}{ccc} Quarter1 & Quarter2 & Quarter3 \end{array} \\ \begin{pmatrix} 4 & 5 & 7 \\ 5 & 6 & 8 \end{pmatrix} \end{array} := B$$

gives the production figures for the year 2012 (in multiples of 10.000 units).

Find a matrix C that shows the shareholders the cost per quarter (in million of dollars) for raw material, labor and miscellaneous.

Identity matrix

The **identity matrix** or **unit matrix** of size n is the n-by-n square matrix with ones on the main diagonal and zeros elsewhere. It is denoted by $\mathbf{I}_n$.
For example

$$\mathbf{I}_1 = [1] \qquad \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \dots$$

$\mathbf{I}_n$ is called the $n \times n$ **identity matrix**.
Then, it is easy to see that

$$\mathbf{A}\mathbf{I}_n = \mathbf{A}, \qquad \textit{for any } m \times n \textit{ matrix } \mathbf{A},$$

$$\mathbf{I}_n\mathbf{B} = \mathbf{B} \qquad \textit{for any } n \times p \textit{ matrix } \mathbf{B}.$$

II. Systems of Linear Equations

DEFINITION: (i) A **linear system of m equations in n unknowns** $x_1, x_2, ..., x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ .... &= ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

II. Systems of Linear Equations

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(ii) A solution of the system (1) is a set of numbers $x_1, x_2, ..., x_n$ that satisfies all m equations.

Example

$$\begin{cases} x + 2y = 0 \\ 2x + y = 1 \end{cases} \qquad \textit{and} \qquad \begin{cases} x + 2y + z = 2 \\ 2x + y + z = 1 \end{cases}$$

are linear systems.

The following is not a linear system

$$\begin{cases} x + 2xy = 0 \\ 2x + y = 1 \end{cases}$$

2. Systems of Linear Equations

DEFINITION:

The matrix form of the system (1) is

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{pmatrix}$$

The matrix $\mathbf{A}$ is called the coefficient matrix of the system (1).

The matrix

$$\tilde{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} & b_m \end{pmatrix}$$

is called the **augmented matrix** of the system (1).

Example

The matrix form of the system

$$\begin{cases} x + 2y = 0 \\ 2x + y = 1 \end{cases}$$

is

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Furthermore, $x = \frac{2}{3}; y = -\frac{1}{3}$ is a solution of the given system.

DEFINITION:

We say that a matrix is in **row echelon form** if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeros, and

- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is **always strictly to the right of the leading coefficient of the row above it**.

Example

The following matrices are in the row echelon form

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -18 & 36 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

Example

The following matrices are in the row echelon form

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -18 & 36 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

However the matrix below is not in the row echelon form.

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 6 & 0 & 36 \end{pmatrix}$$

as the leading coefficient of row 3 (that is 6) is not strictly to the right of the leading coefficient of row 2 (that is -1).

DEFINITION

A system of linear equations is said to be in row echelon form if its augmented matrix is in the row echelon form.

Ex: The system

$$\begin{aligned} x_1 - 3x_2 + x_3 \quad &= 4 \\ -x_2 + 3x_3 \quad &= -5 \\ 2x_3 \quad &= 2 \end{aligned}$$

is in the row echelon form because its augmented matrix is

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

Remark: It is very easy to solve a linear system whose augmented matrix is in the row echelon form.

Solving linear systems: Gaussian Elimination

Elementary Operations on a linear system

(a) Add a multiple of one equation to another
(b) Interchange two equations
(c) Multiply an equation by a nonzero constant.

Elementary operations on a linear system correspond to the following

Elementary row operations on a matrix

(a) Add a multiple of one row to another
(b) Interchange two rows
(c) Multiply a row by a nonzero constant.

Example

Linear system

$$\begin{cases} x - 3y & = 4 \\ 2x - 8y & = -2 \end{cases}$$

Associated augmented matrix

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -8 & -2 \end{pmatrix}$$

Example

Linear system

$$\begin{cases} x - 3y & = 4 \\ 2x - 8y & = -2 \end{cases}$$

Associated augmented matrix

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -8 & -2 \end{pmatrix}$$

Adding -2 times the first equation to the second equation $\rightleftarrows$ Adding -2 times the first row to the second row

$$\begin{cases} x - 3y & = 4 \\ 0x - 2y & = -10 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 4 \\ 0 & -2 & -10 \end{pmatrix}$$

The second equation gives $y = 5$ and replacing y with 5 into the first equation, we get $x = 19$.

Gaussian Elimination

Gaussian elimination is an algorithm for solving systems of linear equations.

Algorithm overview:
The process of Gaussian elimination has two parts:

1. Reduce a given system to the **row echelon form** (using of elementary row operations).
(Or equivalently, we reduce an augmented matrix to the row echelon form using elementary row operations)

2. Use **back substitution** to find solutions of the given system.

Solving linear systems

Linear system

$$\begin{cases} x - 3y & = 4 \\ 2x - 8y & = -2 \end{cases}$$

Associated augmented matrix

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -8 & -2 \end{pmatrix}$$

Solving linear systems

Linear system

$$\begin{cases} x - 3y &= 4 \\ 2x - 8y &= -2 \end{cases}$$

Associated augmented matrix

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -8 & -2 \end{pmatrix}$$

Adding -2 times the first equation to the second equation $\rightleftarrows$ Adding -2 times the first row to the second row

$$\begin{cases} x - 3y &= 4 \\ 0x - 2y &= -10 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 4 \\ 0 & -2 & -10 \end{pmatrix}$$

Solving linear systems

Linear system

$$\begin{cases} x - 3y & = 4 \\ 2x - 8y & = -2 \end{cases}$$

Associated augmented matrix

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Adding -2 times the first equation to the second equation $\rightleftarrows$ Adding -2 times the first row to the second row

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$$\begin{pmatrix} 1 & -3 & 4 \\ 0 & -2 & -10 \end{pmatrix}$$

The second equation gives $y = 5$ and replacing y with 5 into the first equation, we get $x = 19$.

Linear system

$$\begin{cases} x - 3y + z = 4 \\ 2x - 8y + 8z = -2 \\ -6x + 3y - 15z = 9 \end{cases}$$

Associated augmented matrix

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{pmatrix}$$

Linear system

Associated augmented matrix

$$\begin{cases} x - 3y + z = 4 \\ 2x - 8y + 8z = -2 \\ -6x + 3y - 15z = 9 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{pmatrix}$$

Adding -2 times the first equation to the second equation $\rightleftarrows$ Adding -2 times the first row to the second row

$$\begin{cases} x - 3y + z = 4 \\ 0x - 2y + 6z = -10 \\ -6x + 3y - 15z = 9 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{pmatrix}$$

Linear system

$$\begin{cases} x - 3y + z = 4 \\ 2x - 8y + 8z = -2 \\ -6x + 3y - 15z = 9 \end{cases}$$

Associated augmented matrix

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{pmatrix}$$

Adding -2 times the first equation to the second equation $\rightleftarrows$ Adding -2 times the first row to the second row

$$\begin{cases} x - 3y + z = 4 \\ 0x - 2y + 6z = -10 \\ -6x + 3y - 15z = 9 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{pmatrix}$$

Adding 6 times the first equation to the third equation $\rightleftarrows$ 6 times the first row the third row

$$\begin{cases} x - 3y + z = 4 \\ 0x - 2y + 6z = -10 \\ -0x - 15y - 9z = 33 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{pmatrix}$$

Multiplying the second equation by $\frac{1}{2}$ $\rightleftarrows$ Multiplying the second row by $\frac{1}{2}$

$$\begin{cases} x - 3y + z = 4 \\ 0x - y + 3z = -5 \\ -0x - 15y - 9z = 33 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & -15 & -9 & 33 \end{pmatrix}$$

Multiplying the second equation by $\frac{1}{2}$ $\rightleftarrows$ Multiplying the second row by $\frac{1}{2}$

$$\begin{cases} x - 3y + z = 4 \\ 0x - y + 3z = -5 \\ -0x - 15y - 9z = 33 \end{cases} \qquad \begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & -15 & -9 & 33 \end{pmatrix}$$

Multiplying the second equation by $\frac{1}{3}$ $\rightleftarrows$ Multiplying the second row by $\frac{1}{3}$

$$\begin{cases} x - 3y + z = 4 \\ 0x - y + 3z = -5 \\ -0x - 5y - 3z = 11 \end{cases} \qquad \begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & -5 & -3 & 11 \end{pmatrix}$$

Adding -5 times the second equation to the third equation $\rightleftarrows$ Adding -5 times the second row to the third row

$$\begin{cases} x - 3y + z = 4 \\ 0x - y + 3z = -5 \\ -0x - 0y - 18z = 36 \end{cases} \qquad \begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -18 & 36 \end{pmatrix}$$

Adding -5 times the second equation $\rightleftarrows$ Adding -5 times the second row
to the third equation to the third row

$$\begin{cases} x - 3y + z = 4 \\ 0x - y + 3z = -5 \\ -0x - 0y - 18z = 36 \end{cases}$$

$$\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -18 & 36 \end{pmatrix}$$

From the third equation, we now get $z = -2$. Substitute $z = -2$ into the second equation, we get $y = -1$. Substitute $z = -2, y = -1$ into the first equation we get $x = 5$.