Chapter 5: APPLICATIONS OF INTEGRATION

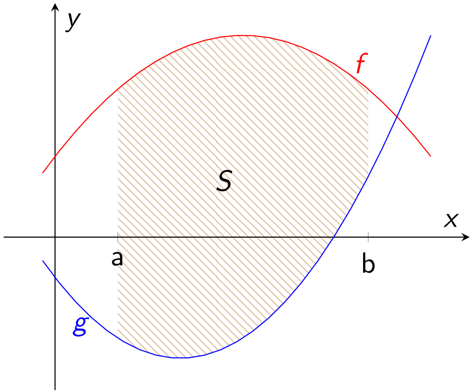
Duong T. PHAM

CALCULUS I

Outline

Chapter 5 (Applications of Integration):
Areas between curves,
Volumes of Solid by revolution (Disk/washer/cylindrical shell methods),
Arc length,
Average value of a function,
Applications of Integration to Engineering, Economics, and Science.

Area between curves

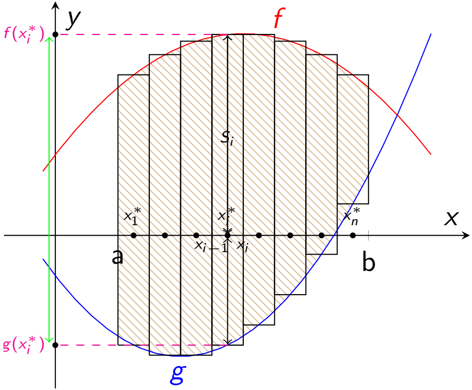


Here, $g(x) \leq f(x)$ for any $a \leq x \leq b$.

$$S = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ g(x) \leq y \leq f(x).\}$$

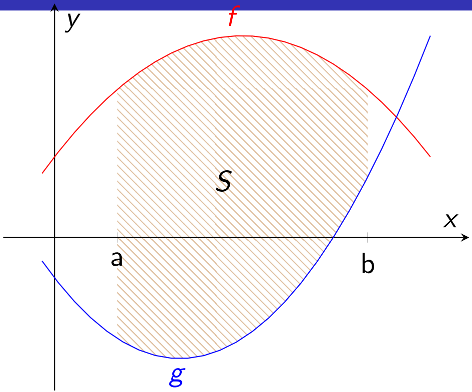
What is the area(S)?

Area between curves



$$Area(S) = \lim_{n\to\infty} Area(S_i) = \lim_{n\to\infty}[f(x_i^*) - g(x_i^*)]\Delta x = \boxed{\int_a^b [f(x) - g(x)]\ dx}$$

Area between curves



Here, $g(x) \leq f(x)$ for any $a \leq x \leq b$.

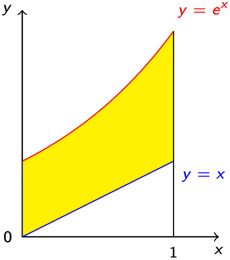
$$S = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ g(x) \leq y \leq f(x).\}$$

The area of S is

$$\boxed{\text{Area}(S) = \int_a^b [f(x) - g(x)]\, dx}$$

Ex: Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Ans:



Denote by A the area of the considered region. Then

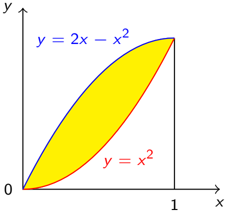
$$A = \int_0^1 [e^x - x]dx = \left(e^x - \frac{x^2}{2}\right)\Big|_0^1$$
$$= e - \frac{1}{2} - 1 = e - \frac{3}{2}$$

Ex: Find the area of the region enclosed by $y = x^2$, bounded below by $y = 2x - x^2$.

Ans: The intersection of these two parabolas satisfies

$$\begin{cases} y = x^2 \\ y = 2x - x^2 \end{cases} \iff \begin{cases} x^2 = 2x - x^2 \\ y = x^2 \end{cases} \iff \begin{cases} x^2 = x \\ y = x^2 \end{cases} \iff \begin{cases} x = 0 \vee x = 1 \\ y = x^2 \end{cases}$$

The two parabolas intersect at $(0,0)$ and $(1,1)$.



Denote by A the area of the considered region. Then

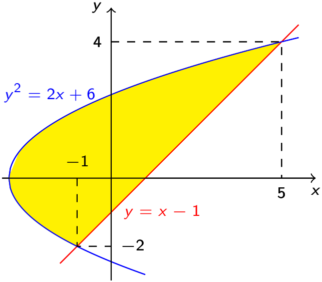
$$A = \int_0^1 [2x - x^2 - x^2]dx = 2\int_0^1 [x - x^2]dx$$
$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

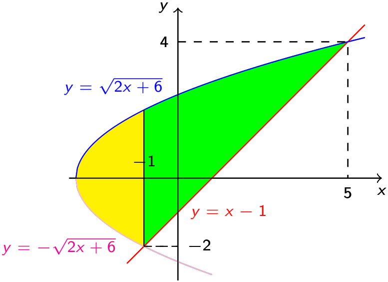
Ex: Find the area of the region enclosed by the straight line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Ans: The intersection of the straight line and the parabola satisfies

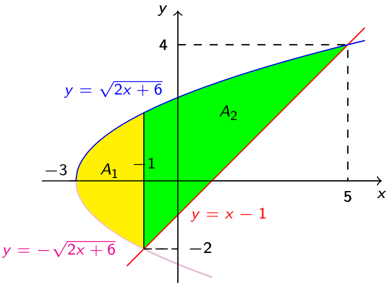
$$\begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases} \iff \begin{cases} y = x - 1 \\ (x-1)^2 = 2x + 6 \end{cases} \iff \begin{cases} y = x - 1 \\ x^2 - 4x - 5 = 0 \end{cases} \iff \begin{cases} y = x - 1 \\ x = -1 \vee x = 5 \end{cases}$$

The intersection is $(-1, -2)$ and $(5, 4)$.





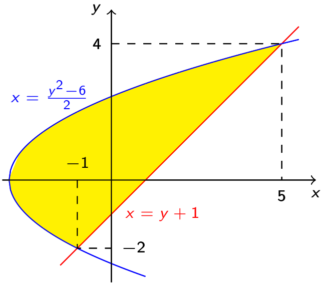
Area between curves



$$A_1 = \int_{-3}^{-1} \left[\sqrt{2x+6} - (-\sqrt{2x+6}) \right] dx$$

$$A_2 = \int_{-1}^{5} \left[\sqrt{2x+6} - (x-1) \right] dx$$

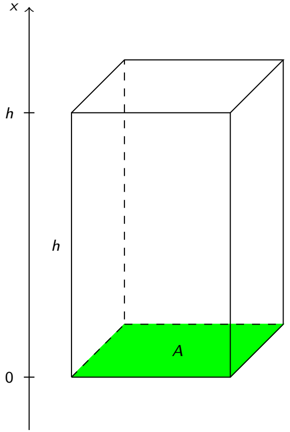
Area between curves



Denote by A the area of the considered region. Then

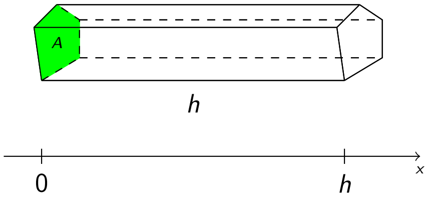
$$A = \int_{-2}^{4} \left[(y+1) - \frac{y^2-6}{2}\right] dy = \int_{-2}^{4} [-\frac{y^2}{2}+y+4] dy$$
$$= \left(-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right) \Big|_{-2}^{4} = 18$$

Volumes

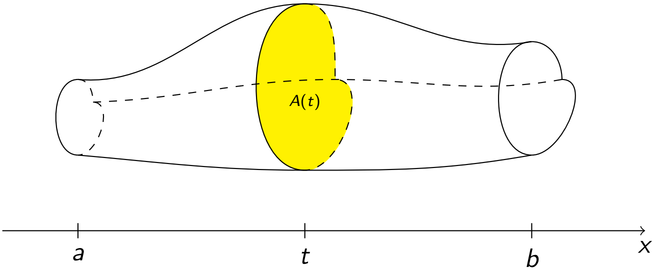


$$V = A \cdot h \; = \int_0^h A\,dx$$

$$V = \; A \cdot h \; = \int_0^h A\,dx$$



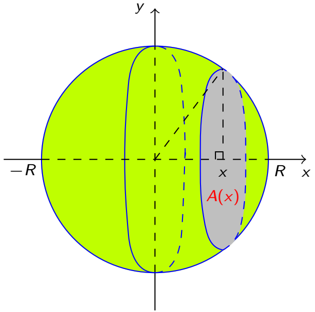
Volumes



$$\boxed{V := \int_a^b A(x)\,dx}$$

Ex: Show that volume of a sphere of radius R is $\frac{4}{3}\pi R^3$

Ans:



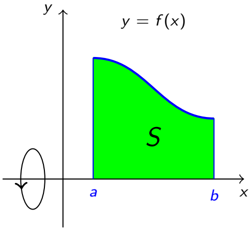
$$V = \int_{-R}^{R} A(x)\,dx, \text{ where}$$

$$\begin{aligned} A(x) &= \text{area of a circle of radius } \sqrt{R^2 - x^2} \\ &= \pi(R^2 - x^2) \end{aligned}$$

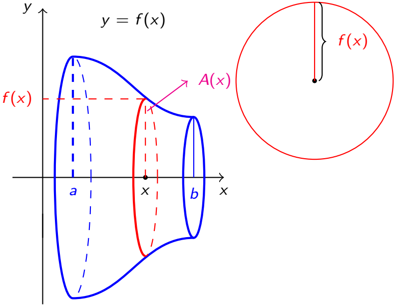
Hence,

$$\begin{aligned} V &= \int_{-R}^{R} \pi(R^2 - x^2)\,dx = \pi \left(R^2 x - \frac{x^3}{3} \right) \Bigg|_{-R}^{R} \\ &= \frac{4}{3}\pi R^3 \end{aligned}$$

Solid created by rotating about the x-axis



$f(x) \geq 0 \quad \forall x \in [a, b]$

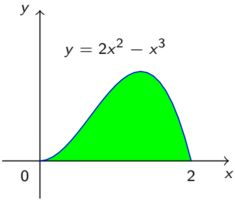


The volume of the solid created by rotating S about the x-axis is

$$\boxed{V_x = \int_a^b A(x)\, dx = \int_a^b \pi [f(x)]^2\, dx}$$

Ex: Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

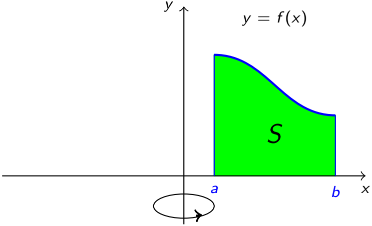
Ans:

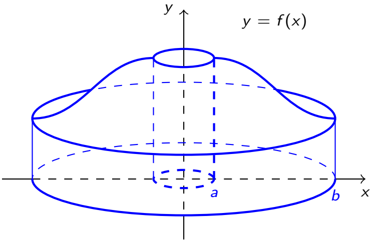


We have

$$V_x = \int_0^2 \pi(2x^2 - x^3)^2 dx = \pi \int_0^2 (4x^4 - 4x^5 + x^6) dx$$
$$= \pi \left(\frac{4x^5}{5} - \frac{4x^6}{6} + \frac{x^7}{7} \right) \Big|_0^2$$
$$= \frac{\pi 2^7}{105}$$

Solid created by rotating about the y-axis





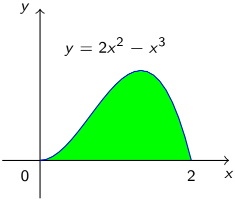
$f(x) \geq 0 \quad \forall x \in [a, b]$

The volume of the solid created by rotating S about the y-axis is

$$\boxed{V_y = \int_a^b 2\pi x f(x)\, dx}$$

Ex: Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

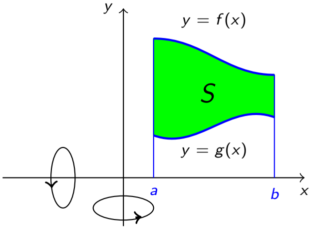
Ans:



Applying the shell method, we have

$$V_y = \int_0^2 2\pi x(2x^2 - x^3)\,dx = 2\pi \int_0^2 (2x^3 - x^4)\,dx$$
$$= 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5} \right)\Bigg|_0^2$$
$$= \frac{16\pi}{5}$$

Solid created by rotating about the x and y-axes



$$f(x) \geq g(x) \geq 0 \quad \forall x \in [a, b]$$

The volume of the solid created by rotating S about the x-axis is

$$\boxed{V_x = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2\right)\, dx}$$

The volume of the solid created by rotating S about the y-axis is

$$\boxed{V_y = \int_a^b 2\pi\, x\, [f(x) - g(x)]\, dx}$$

Average value of a function

- Given n numbers $y_1, y_2, \ldots, y_n$, the average of these numbers is

$$\boxed{y_{\text{ave}} = \frac{y_1 + y_2 + \ldots + y_n}{n}}$$

- Given a fuction $f : [a, b] \to \mathbb{R}$, the average value of f on $[a, b]$ is defined by

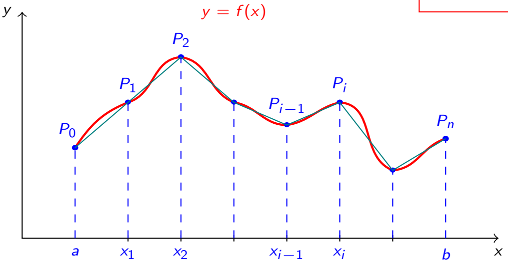
$$\boxed{f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx}$$

Mean value theorem for integrals: If f is continuous on $[a, b]$, then exists $c \in [a, b]$ such that

$$\boxed{f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx}$$

Arc length

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$



$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(x_i - x_{i-1})^2 + [f'(x_i^*)(x_i - x_{i-1})]^2}$$

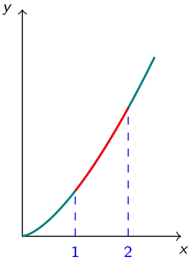
$$= \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

The arc length formular: If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$ from $x = a$ to b is

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

Ex: Find the length of the curve $y = \sqrt{x^3}$ from $x = 1$ to $x = 2$.



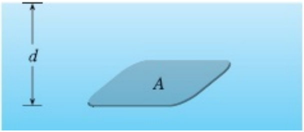
The derivative $y' = \frac{3}{2}x^{1/2}$. The length of the parabola from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + [f'(x)]^2}\, dx = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_1^2 = \frac{1}{27}(80\sqrt{10} - 13\sqrt{3})$$

Hydrostatic force and pressure

- Deep-sea divers realize that water pressure increases as they dive deeper. This is because the weight of the water above them increases.



- Suppose that a thin horizontal plate with area A square meters is submerged in a fluid of density ρ kilograms per cubic meter at a depth d meters below the surface of the fluid

- The fluid directly above the plate has volume $V = Ad$. Its mass is $m = \rho V = \rho Ad$. The force exerted by the fluid on the plate is therefore

$$F = mg = \rho g A d, \text{ where } g \text{ is the acceleration due to gravity.}$$

  The pressure P on the plate is defined to be the force per unit area:

$$P = \frac{F}{A} = \rho g d.$$

- The SI unit for measuring pressure is newtons per square meter, which is called a pascal (abbreviation: $1\ \mathrm{N/m^2} = 1$ Pa). 1 kilopascal (kPa) $= 1000$ Pa.

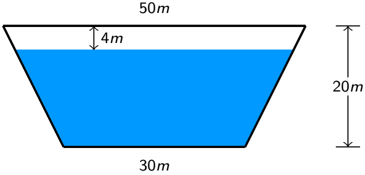
Hydrostatic force and pressure

Son La hydroelectric power station:

- Concrete Gravity Dam $138\mathrm{m}$ high, $90\mathrm{m}$ wide at the base, length $> 1,000\mathrm{m}$
- Capacity: $2,400 Mw$, annual generation: $10,246 Gwh$. The cost was US$2 bil.

Hydrostatic force and pressure

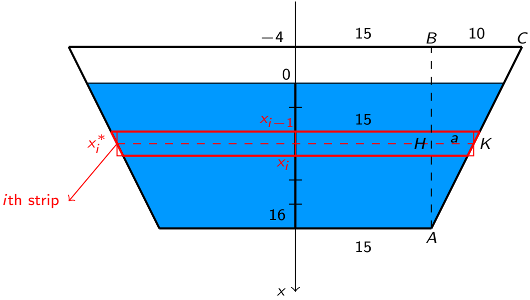
Example: A dam has the shape of the trapezoid shown in the Figure. The height is $20\mathrm{m}$, and the width is $50\mathrm{m}$ at the top and $30\mathrm{m}$ at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is $4\mathrm{m}$ from the top of the dam.



Why it is so important to find the total force on the dam?

- Decides demanded strength of the dam.
- Decides cost, safety level and endurance of the dam.
- etc.

Hydrostatic force and pressure



- Divide $[0, 16]$ into subintervals $[x_{i-1}, x_i]$ for $i = 1, \dots, n$.
- Choose $x_i^* \in [x_{i-1}, x_i]$

- The ith horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i.
- Then $\dfrac{a}{16 - x_i^*} = \dfrac{10}{20} \Longrightarrow a = 8 - \dfrac{x_i^*}{2}$. The width: $w_i = 2(15 + a) = 46 - x_i^*$
- If A_i is the area of the ith strip, then $A_i \approx w_i \Delta x = (46 - x_i^*)\Delta_x$
- Δ_x: small $\Rightarrow$ the pressure P_i on the ith strip is almost constant

$$P_i = 1000\, g\, x_i^*.$$

Hydrostatic force and pressure

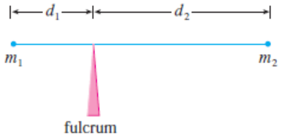
Hydrostatic force F_i on the ith strip is

$$F_i = P_i * A_i \approx 1000\, g\, x_i^*(46 - x_i^*)\, \Delta x.$$

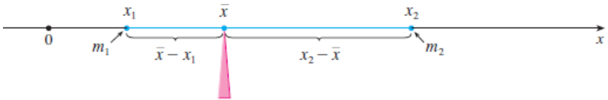
The total force on the dam:

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} F_i = \lim_{n\to\infty} \sum_{i=1}^{n} 1000\, g\, x_i^*(46 - x_i^*)\, \Delta x \\ &= \int_0^{16} 1000\, g\, x(46 - x)\, dx \\ &= 1000(9.8) \int_0^{16} (46x - x^2)\, dx \\ &= 1000(9.8) \left[23x^2 - \frac{x^3}{3} \right]_{x=0}^{16} \\ &\approx 4.43 \times 10^7 N. \end{aligned}$$

Moments and ~~Centers~~ of Mass   Ko Thi



- Two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum
- The rod will balance if $m_1 d_1 = m_2 d_2$.
- Suppose that the rod lies along the x-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\bar{x}$.



We then have

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \quad \Longrightarrow \quad \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Moments and Centers of Mass

- In general, if we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown similarly that the center of mass of the system is located at

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

  where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system.

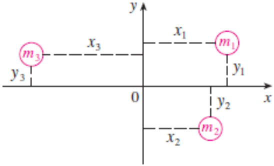
- The sum of the individual moments $M = \sum_{i=1}^{n} m_i x_i$ is called the moment of the system about the origin .

- We then have

$$m\bar{x} = M.$$

  This means that if the total mass m were considered as being concentrated at the center of mass $\bar{x}$, then its moment would be the same as the moment of the system.

Moments and Centers of Mass



- A system of particles with masses m_1, m_2, ..., m_n located at the points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) in the xy-plane

- We define the moment of the system about the y-axis and x-axis to be

$$M_y = \sum_{i=1}^{n} m_i x_i \quad \text{and} \quad M_x = \sum_{i=1}^{n} m_i y_i$$

- Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis.
- The center of mass are given in terms of the moments by the formulas

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass.

Moments and Centers of Mass ko thi

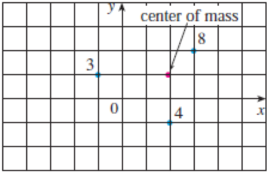
Ex. Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1)$, $(2, -1)$, and $(3, 2)$, respectively.

Ans. The moments:

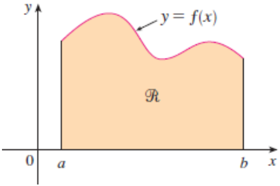
$$M_y = 3 \times (-1) + 4 \times 2 + 8 \times 3 = 29$$
$$M_x = 3 \times 1 + 4 \times (-1) + 8 \times 2 = 15$$

The total mass is $m = 3 + 4 + 8 = 15$. The center of mass is

$$\bar{x} = \frac{M_y}{m} = \frac{29}{15} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1.$$



Moments and Centers of Mass



- Suppose that $\mathcal{R}$ is a flat plate with uniform density ρ. Here, $\mathcal{R}$ lies between the lines $x = a$ and $x = b$, above the x-axis, and beneath the graph of $y = f(x)$, where f is a continuous function.
- The moment of $\mathcal{R}$ about the y- and x-axes:

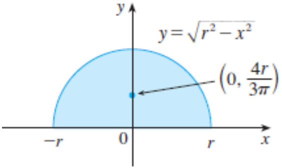
$$M_y = \rho \int_a^b x f(x) dx \quad \text{and} \quad M_x = \rho \int_a^b \frac{1}{2}[f(x)]^2 dx.$$

- The mass of the plate: $m = \rho \int_a^b f(x) dx = A$, where A is the area of $\mathcal{R}$.
- The center of mass:

$$\bar{x} = \frac{1}{A} \int_a^b x f(x) dx \quad \text{and} \quad \bar{y} = \frac{1}{A} \int_a^b \frac{1}{2}[f(x)]^2 dx$$

Ex. Find the center of mass of a semicircular plate of radius r.

Ans.



- The plate is above the x-axis, below the graph $y = \sqrt{r^2 - x^2}$, and between the two lines $x = -r$ and $x = r$. The area of the plate is

$$A = \frac{\pi r^2}{2}$$

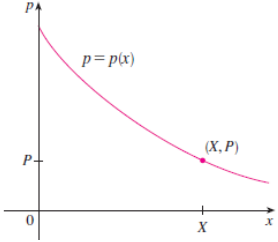
- The center of mass: $\bar{x} = \frac{1}{A}\int_{-r}^{r} x\sqrt{r^2 - x^2}dx = 0$ (Why?) and

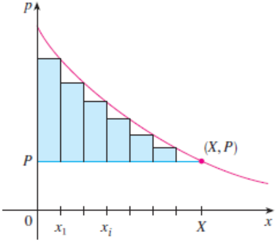
$$\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}[f(x)]^2 dx = \frac{1}{\frac{1}{2}\pi r^2}\int_{-r}^{r} \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx = \frac{1}{\pi r^2}\int_{-r}^{r} (r^2 - x^2)dx$$

$$= \frac{1}{\pi r^2}\left(r^2 x - \frac{x^3}{3}\right)\Big|_{x=-r}^{r} = \frac{4r}{3\pi}.$$

Hence, the center of mass is located at the point $\left(0, \frac{4r}{3\pi}\right)$.

Consumer Surplus





- Demand function $p(x)$ is the price that a company has to charge in order to sell x units of a commodity
- Usually, $p(x)$ is a decreasing function
- If X is the amount of the commodity that is currently available, then $p(X)$ is the current selling price
- Divide $[0, X]$ into n subintervals, each of length X/n, and let $x_i^* = x_i$.
- If, after the first x_{i-1} units were sold, a total of only x_i units had been available and the price per unit had been set at $p(x_i)$ dollars, then the additional Δx units could have been sold (but no more).
- The consumers who would have paid $p(x_i)$ dollars placed a high value on the product; they would have paid what it was worth to them.

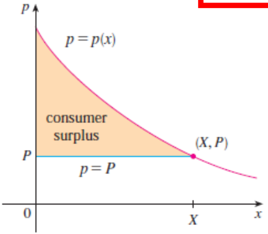
Consumer Surplus

- So, in paying only P dollars they have saved an amount of

$$(\text{savings per unit})(\text{number of units}) = (p(x_i) - P)\Delta x$$

- The total savings: $\displaystyle\sum_{i=1}^{n} [p(x_i) - P]\,\Delta x$
- The consumer surplus of the commodity is

$$\boxed{\int_0^X [p(x) - P]\;dx = \lim_{n\to\infty} \sum_{i=1}^{n} [p(x_i) - P]\,\Delta x}$$



- The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X.

Consumer Surplus

Ex. The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2.$$

Find the consumer surplus when the sales level is 500.

Ans. Since the number of products sold is $X = 500$, the corresponding price is

$$P = 1200 - 0.2 \times 500 - 0.0001 \times 500^2 = 1075.$$

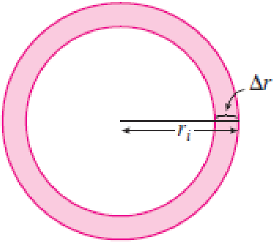
The consumer surplus is

$$\begin{aligned}\int_0^{500} [p(x) - P]\, dx &= \int_0^{500} (1200 - 0.2x - 0.0001x^2 - 1075) dx \\ &= \int_0^{500} (125 - 0.2x - 0.0001x^2) dx \\ &= 125x - 0.1x^2 - 0.0001\frac{x^3}{3}\Bigg|_0^{500} = \$33,333.33\end{aligned}$$

Blood Flow

- The law of laminar flow: $\boxed{v(r) = \frac{P}{4\eta\ell}(R^2 - r^2)}$
  which gives the velocity of blood that flows along a blood vessel with radius R and length ℓ at a distance r from the central axis, where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood
- We want to compute the rate of blood flow, or flux (volume per unit time)
- We divide the radius $[0, R$ into smaller equally spaced radii

$$0 < r_1 < r_2 < \cdots < r_{n-1} < r_n = R.$$



The area of the ring with inner radius r_{i-1} and outer radius r_i is

$$\pi(r_i^2 - r_{i-1}^2) = \pi(r_i + r_{i-1})(r_i - r_{i-1}) \approx 2\pi r_i \Delta r.$$

If Δr is small, then the velocity is almost constant throughout this ring and can be approximated by $v(r_i)$

Blood Flow

- Thus the volume of blood per unit time that flows across the ring is approximately $(2\pi r_i \Delta r)v(r_i) = 2\pi r_i v(r_i)\Delta r$.
- The total volume of blood that flows across a cross-section per unit time is approximately
$$\sum_{i=1}^{n} 2\pi r_i v(r_i)\Delta r.$$
- The volume of blood that passes a cross-section per unit time (the flux) is
$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i v(r_i)\Delta r = \int_0^R 2\pi r v(r)\, dr = \int_0^R 2\pi r \frac{P}{4\eta\ell}(R^2 - r^2)\, dr$$
$$= \frac{\pi P}{2\eta\ell}\int_0^R (R^2 r - r^3)\, dr = \frac{\pi P}{2\eta\ell}\left(R^2\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^R = \frac{\pi P R^4}{8\eta\ell}$$
- The equation
$$\boxed{F = \frac{\pi P R^4}{8\eta\ell}}$$
is called Poiseuille's Law; it shows that the flux is proportional to the fourth power of the radius of the blood vessel.

Probability

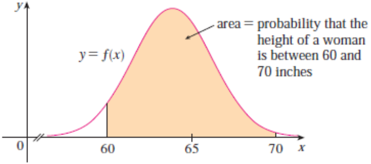
- In reality, we are often interested in questions like the probability that a blood cholesterol level is greater than 250, or the probability that the height of an adult female is between 60 and 70 inches, or the probability that the battery we are buying lasts between 100 and 200 hours.
- If X represents the lifetime of that type of battery, we denote this last probability as follows

$$P(100 \leq X \leq 200)$$

- Every continuous random variable X has a probability density function f . This means that the probability that X lies between a and b is found by integrating f from a to b:

$$\boxed{P(a \leq X \leq b) = \int_a^b f(x)dx}$$

Ex. The below figure shows the graph of a model for the probability density function f for a random variable X defined to be the height in inches of an adult female in the United States (according to data from the National Health Survey)



- The probability that the height of a woman chosen at random from this population is between 60 and 70 inches is equal to the area under the graph of f from 60 to 70.

In general, the probability density function f of a random variable X satisfies the condition $f(x) \geq 0$ for all x. Because probabilities are measured on a scale from 0 to 1, it follows that

$$\boxed{\int_{-\infty}^{\infty} f(x)dx = 1.}$$

Ex. Let $f(x) = 0.006x(10 - x)$ for $0 \le x \le 10$ and $f(x) = 0$ otherwise.

(a) Verify: f is a probability density. (b) Find $P(4 \le x \le 8)$.

Ans.

(a) We first have $f(x) = 0.006x(10 - x) \ge 0$ for all $0 \le x \le 10$. Thus, $f(x) \ge 0$ for all x. Furthermore,

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{10} 0.006x(10 - x)dx = 0.006 \int_0^{10} (10x - x^2)dx$$

$$= 0.006 \left(5x^2 - \frac{x^3}{3}\right) \Big|_0^{10} = 0.006 \left(500 - \frac{1000}{3}\right) = 1.$$

Hence, f is a probability density.

(b) The probability that X lies between 4 and 8 is

$$P(4 \le x \le 8) = \int_4^8 f(x)dx = 0.006 \int_4^8 (10x - x^2)dx$$

$$= 0.006 \left(5x^2 - \frac{x^3}{3}\right) \Big|_4^8 = 0.544.$$

Average Values

- In general, the mean of any probability density function f is defined to be

$$\boxed{\mu = \int_{-\infty}^{\infty} x\, f(x) dx}$$

  The mean can be interpreted as the long-run average value of the random variable X. It can also be interpreted as a measure of centrality of the probability density function.

- Phenomena such as waiting times and equipment failure times are commonly modeled by exponentially decreasing probability density functions

$$\boxed{f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{\mu} e^{-\frac{t}{\mu}} & \text{if } t \geq 0, \end{cases}}$$

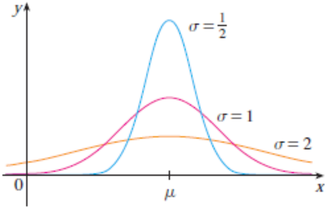
  where μ is the mean value of the probability density function f.

Normal Distribution

- Many important random phenomena such as test scores on aptitude tests, heights and weights of individuals from a homogeneous population, annual rainfall in a given location are modeled by a normal distribution. This means that the probability density function of the random variable X is a member of the family of functions

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

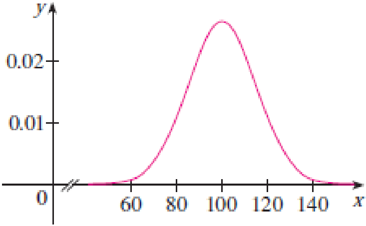
- The mean value for this f is μ (can be verified).
- The positive constant σ is called the standard deviation; it measures how spread out the values of X are.



Ex. Intelligence Quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15. (Figure below shows the corresponding probability density function.)

(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?



- Since IQ scores are normally distributed, we use the probability density function given by

$$f(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2\times 15^2}}$$