Lecture notes: Differential Equations for ISE (MA029IU)

Week 13-14 *

May 18, 2022

1 PDEs, separation of variables, and the heat equation

Let us recall that a *partial differential equation* or *PDE* is an equation containing the partial derivatives with respect to *several* independent variables. Solving PDEs will be our main application of Fourier series.

A PDE is said to be *linear* if the dependent variable and its derivatives appear at most to the first power and in no functions. We will only talk about linear PDEs. Together with a PDE, we usually specify some *boundary conditions*, where the value of the solution or its derivatives is given along the boundary of a region, and/or some *initial conditions* where the value of the solution or its derivatives is given for some initial time. Sometimes such conditions are mixed together and we will refer to them simply as *side conditions*.

We will study three specific partial differential equations, each one representing a general class of equations. First, we will study the *heat equation*, which is an example of a *parabolic PDE*. Next, we will study the *wave equation*, which is an example of a *hyperbolic PDE* (skipped!). Finally, we will study the *Laplace equation*, which is an example of an *elliptic PDE*. Each of our examples will illustrate behavior that is typical for the whole class.

1.1 Heat on an insulated wire

Let us start with the heat equation. Consider a wire (or a thin metal rod) of length L that is insulated except at the endpoints. Let x denote the position along the wire and let t denote time. See Figure 1.

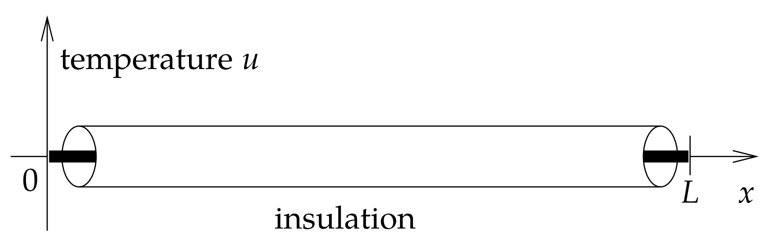


Figure 1: Insulated wire.

Let $u(x,t)$ denote the temperature at point x at time t. The equation governing this setup is the so-called *one-dimensional heat equation*:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},}$$

*This note is taken from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl where $k > 0$ is a constant (the *thermal conductivity* of the material). That is, the change in heat at a specific point is proportional to the second derivative of the heat along the wire. This makes sense; if at a fixed t the graph of the heat distribution has a maximum (the graph is concave down), then heat flows away from the maximum. And vice versa.

We generally use a more convenient notation for partial derivatives. We write u_t instead of $\frac{\partial u}{\partial t}$, and we write u_{xx} instead of $\frac{\partial^2 u}{\partial x^2}$. With this notation the heat equation becomes

$$u_t = k u_{xx}.$$

For the heat equation, we must also have some boundary conditions. We assume that the ends of the wire are either exposed and touching some body of constant heat, or the ends are insulated. If the ends of the wire are kept at temperature 0, then the conditions are

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

If, on the other hand, the ends are also insulated, the conditions are

$$u_x(0,t) = 0 \qquad \text{and} \qquad u_x(L,t) = 0.$$

Let us see why that is so. If u_x is positive at some point x_0, then at a particular time, u is smaller to the left of x_0, and higher to the right of x_0. Heat is flowing from high heat to low heat, that is to the left. On the other hand if u_x is negative then heat is again flowing from high heat to low heat, that is to the right. So when u_x is zero, that is a point through which heat is not flowing. In other words, $u_x(0,t) = 0$ means no heat is flowing in or out of the wire at the point $x = 0$.

We have two conditions along the x-axis as there are two derivatives in the x direction. These side conditions are said to be *homogeneous* (i.e., u or a derivative of u is set to zero).

We also need an initial condition—the temperature distribution at time $t = 0$. That is,

$$u(x,0) = f(x),$$

for some known function $f(x)$. This initial condition is not a homogeneous side condition.

1.2 Separation of variables

The heat equation is linear as u and its derivatives do not appear to any powers or in any functions. Thus the principle of superposition still applies for the heat equation (without side conditions): If u_1 and u_2 are solutions and c_1, c_2 are constants, then $u = c_1 u_1 + c_2 u_2$ is also a solution.

Exercise* 1.1*: *Verify the principle of superposition for the heat equation.*

Superposition preserves some of the side conditions. In particular, if u_1 and u_2 are solutions that satisfy $u(0,t) = 0$ and $u(L,t) = 0$, and c_1, c_2 are constants, then $u = c_1 u_1 + c_2 u_2$ is still a solution that satisfies $u(0,t) = 0$ and $u(L,t) = 0$. Similarly for the side conditions $u_x(0,t) = 0$ and $u_x(L,t) = 0$. In general, superposition preserves all homogeneous side conditions.

The method of *separation of variables* is to try to find solutions that are products of functions of one variable. For the heat equation, we try to find solutions of the form

$$u(x,t) = X(x)T(t).$$

That the desired solution we are looking for is of this form is too much to hope for. What is perfectly reasonable to ask, however, is to find enough "building-block" solutions of the form $u(x,t) = X(x)T(t)$ using this procedure so that the desired solution to the PDE is somehow constructed from these building blocks by the use of superposition.

Let us try to solve the heat equation

$$u_t = ku_{xx} \qquad \text{with} \quad u(0,t) = 0, \qquad u(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

We guess $u(x,t) = X(x)T(t)$. We will try to make this guess satisfy the differential equation, $u_t = ku_{xx}$, and the homogeneous side conditions, $u(0,t) = 0$ and $u(L,t) = 0$. Then, as superposition preserves the differential equation and the homogeneous side conditions, we will try to build up a solution from these building blocks to solve the nonhomogeneous initial condition $u(x,0) = f(x)$.

First we plug $u(x,t) = X(x)T(t)$ into the heat equation to obtain

$$X(x)T'(t) = kX''(x)T(t).$$

We rewrite as

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

This equation must hold for all x and all t. But the left-hand side does not depend on x and the right-hand side does not depend on t. Hence, each side must be a constant. Let us call this constant $-\lambda$ (the minus sign is for convenience later). We obtain the two equations

$$\frac{T'(t)}{kT(t)} = -\lambda = \frac{X''(x)}{X(x)}.$$

In other words,

$$X''(x) + \lambda X(x) = 0,$$
$$T'(t) + \lambda kT(t) = 0.$$

The boundary condition $u(0,t) = 0$ implies $X(0)T(t) = 0$. We are looking for a nontrivial solution and so we can assume that $T(t)$ is not identically zero. Hence $X(0) = 0$. Similarly, $u(L,t) = 0$ implies $X(L) = 0$. We are looking for nontrivial solutions X of the eigenvalue problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We have previously found that the only eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$, for integers $n \geq 1$, where eigenfunctions are $\sin\left(\frac{n\pi}{L}x\right)$. Hence, let us pick the solutions

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding T_n must satisfy the equation

$$T_n'(t) + \frac{n^2\pi^2}{L^2}kT_n(t) = 0.$$

This is one of our fundamental equations, and the solution is just an exponential:

$$T_n(t) = e^{\frac{-n^2\pi^2}{L^2}kt}.$$

It will be useful to note that $T_n(0) = 1$. Our building-block solutions are

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2}{L^2}kt}.$$

We note that $u_n(x,0) = \sin\left(\frac{n\pi}{L}x\right)$. Let us write $f(x)$ as the sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

That is, we find the Fourier series of the odd periodic extension of $f(x)$. We used the sine series as it corresponds to the eigenvalue problem for $X(x)$ above. Finally, we use superposition to write the solution as

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2}{L^2}kt}.}$$

Why does this solution work? First note that it is a solution to the heat equation by superposition. It satisfies $u(0,t) = 0$ and $u(L,t) = 0$, because $x = 0$ or $x = L$ makes all the sines vanish. Finally, plugging in $t = 0$, we notice that $T_n(0) = 1$ and so

$$u(x,0) = \sum_{n=1}^{\infty} b_n u_n(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x).$$

Example 1.1: Consider an insulated wire of length 1 whose ends are embedded in ice (temperature 0). Let $k = 0.003$ and let the initial heat distribution be $u(x,0) = 50\,x\,(1-x)$. See Figure 2. Suppose we want to find the temperature function $u(x,t)$. Let us also suppose we want to find when (at what t) does the maximum temperature in the wire drop to one half of the initial maximum of 12.5.

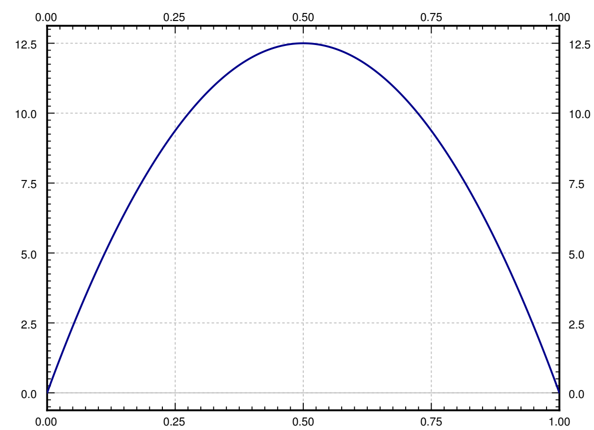


Figure 2: Initial distribution of temperature in the wire.

We are solving the following PDE problem:

$$\begin{aligned} u_t &= 0.003\,u_{xx}, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= 50\,x\,(1-x) \qquad \text{for } \ 0 < x < 1. \end{aligned}$$

We write $f(x) = 50\,x\,(1-x)$ for $0 < x < 1$ as a sine series. That is, $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$, where

$$b_n = 2\int_0^1 50\,x\,(1-x)\sin(n\pi x)\,dx = \frac{200}{\pi^3 n^3} - \frac{200\,(-1)^n}{\pi^3 n^3} = \begin{cases} 0 & \text{if } n \text{ even,} \\ \frac{400}{\pi^3 n^3} & \text{if } n \text{ odd.} \end{cases}$$

The solution $u(x,t)$, plotted in Figure 3 on the following page for $0 \le t \le 100$, is given by the series:

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{400}{\pi^3 n^3} \sin(n\pi x)\, e^{-n^2\pi^2\, 0.003\, t}.$$

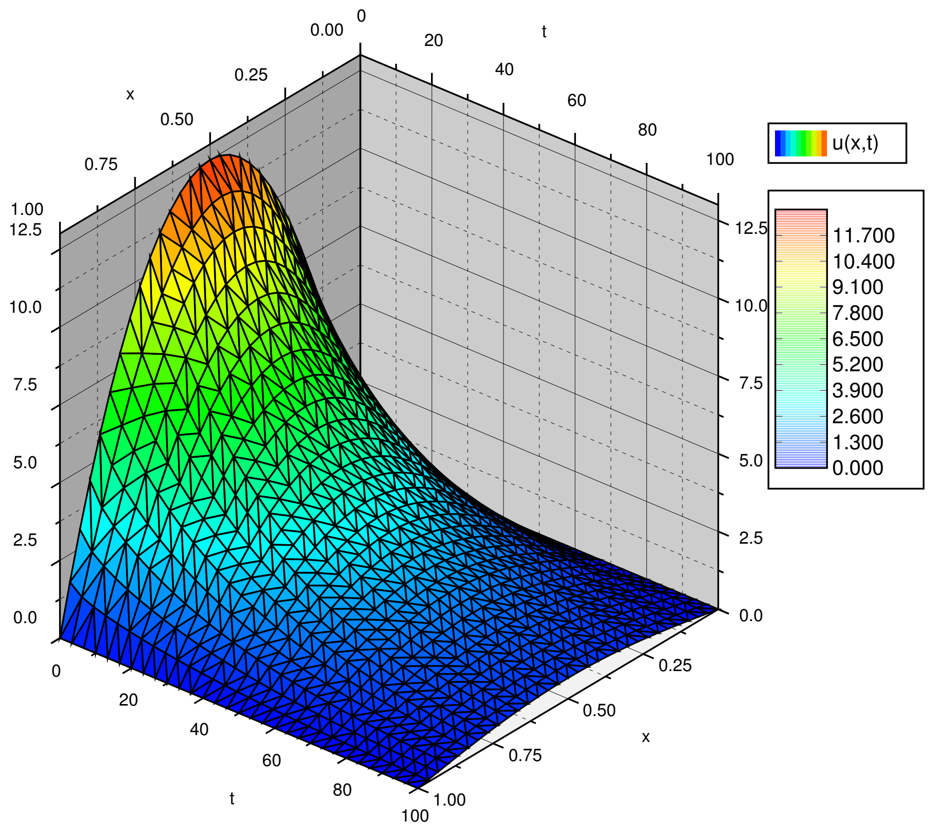


Figure 3: Plot of the temperature of the wire at position x at time t.

Finally, let us answer the question about the maximum temperature. It is relatively easy to see that the maximum temperature at any fixed time is always at $x = 0.5$, in the middle of the wire. The plot of $u(x,t)$ confirms this intuition. If we plug in $x = 0.5$, we get

$$u(0.5,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{400}{\pi^3 n^3} \sin(n\pi\, 0.5)\, e^{-n^2\pi^2\, 0.003\, t}.$$

For $n = 3$ and higher (remember n is only odd), the terms of the series are insignificant compared to the first term. The first term in the series is already a very good approximation of the function. Hence

$$u(0.5,t) \approx \frac{400}{\pi^3}\, e^{-\pi^2\, 0.003\, t}.$$

The approximation gets better and better as t gets larger as the other terms decay much faster. Let us plot the function $u(0.5,t)$, the temperature at the midpoint of the wire at time t, in Figure 4 on the next page. The figure also plots the approximation by the first term.

After $t = 5$ or so it would be hard to tell the difference between the first term of the series for $u(x,t)$ and the real solution $u(x,t)$. This behavior is a general feature of solving the heat equation. If you are interested in behavior for large enough t, only the first one or two terms may be necessary.

Let us get back to the question of when is the maximum temperature one half of the initial maximum temperature. That is, when is the temperature at the midpoint $12.5/2 = 6.25$. We notice on the graph that if we

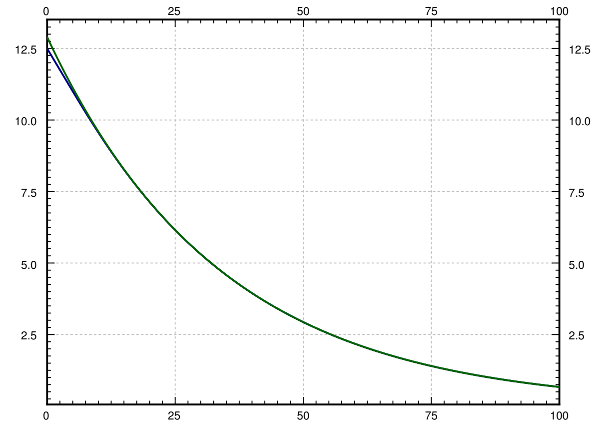


Figure 4: Temperature at the midpoint of the wire (the bottom curve), and the approximation of this temperature by using only the first term in the series (top curve).

use the approximation by the first term we will be close enough. We solve

$$6.25 = \frac{400}{\pi^3} e^{-\pi^2 \, 0.003 \, t}.$$

That is,

$$t = \frac{\ln \frac{6.25 \, \pi^3}{400}}{-\pi^2 0.003} \approx 24.5.$$

So the maximum temperature drops to half at about $t = 24.5$.

We mention an interesting behavior of the solution to the heat equation. The heat equation "smoothes" out the function $f(x)$ as t grows. For a fixed t, the solution is a Fourier series with coefficients $b_n e^{\frac{-n^2\pi^2}{L^2}kt}$. If $t > 0$, then these coefficients go to zero faster than any $\frac{1}{n^p}$ for any power p. In other words, the Fourier series has infinitely many derivatives everywhere. Thus even if the function $f(x)$ has jumps and corners, then for a fixed $t > 0$, the solution $u(x,t)$ as a function of x is as smooth as we want it to be.

Example 1.2: When the initial condition is already a sine series, then there is no need to compute anything, you just need to plug in. Consider

$$u_t = 0.3 \, u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = 0.1 \sin(\pi t) + \sin(2\pi t).$$

The solution is then

$$u(x,t) = 0.1 \sin(\pi t) e^{-0.3\pi^2 t} + \sin(2\pi t) e^{-1.2\pi^2 t}.$$

1.3 Insulated ends

Now suppose the ends of the wire are insulated. In this case, we are solving the equation

$$u_t = k u_{xx} \qquad \text{with} \quad u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Yet again we try a solution of the form $u(x,t) = X(x)T(t)$. By the same procedure as before we plug into the heat equation and arrive at the following two equations

$$X''(x) + \lambda X(x) = 0,$$
$$T'(t) + \lambda k T(t) = 0.$$

At this point the story changes slightly. The boundary condition $u_x(0,t) = 0$ implies $X'(0)T(t) = 0$. Hence $X'(0) = 0$. Similarly, $u_x(L,t) = 0$ implies $X'(L) = 0$. We are looking for nontrivial solutions X of the eigenvalue problem $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(L) = 0$. We have previously found that the only eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$, for integers $n \geq 0$, where eigenfunctions are $\cos\left(\frac{n\pi}{L}x\right)$ (we include the constant eigenfunction). Hence, let us pick solutions

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right) \qquad \text{and} \qquad X_0(x) = 1.$$

The corresponding T_n must satisfy the equation

$$T_n'(t) + \frac{n^2\pi^2}{L^2} k T_n(t) = 0.$$

For $n \geq 1$, as before,

$$T_n(t) = e^{\frac{-n^2\pi^2}{L^2}kt}.$$

For $n = 0$, we have $T_0'(t) = 0$ and hence $T_0(t) = 1$. Our building-block solutions are

$$u_n(x,t) = X_n(x)T_n(t) = \cos\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2}{L^2}kt},$$

and

$$u_0(x,t) = 1.$$

We note that $u_n(x,0) = \cos\left(\frac{n\pi}{L}x\right)$. Let us write f using the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right).$$

That is, we find the Fourier series of the even periodic extension of $f(x)$.

We use superposition to write the solution as

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2}{L^2}kt}.}$$

Example 1.3: Let us try the same equation as before, but for insulated ends. We are solving the following PDE problem

$$\begin{aligned} u_t &= 0.003\, u_{xx}, \\ u_x(0,t) &= u_x(1,t) = 0, \\ u(x,0) &= 50\, x\,(1-x) \qquad \text{for } \ 0 < x < 1. \end{aligned}$$

For this problem, we must find the cosine series of $u(x,0)$. For $0 < x < 1$ we have

$$50\, x\,(1-x) = \frac{25}{3} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left(\frac{-200}{\pi^2 n^2}\right) \cos(n\pi x).$$

The calculation is left to the reader. Hence, the solution to the PDE problem, plotted in Figure 5 on the following page, is given by the series

$$u(x,t) = \frac{25}{3} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left(\frac{-200}{\pi^2 n^2}\right) \cos(n\pi x)\, e^{-n^2\pi^2\, 0.003\, t}.$$

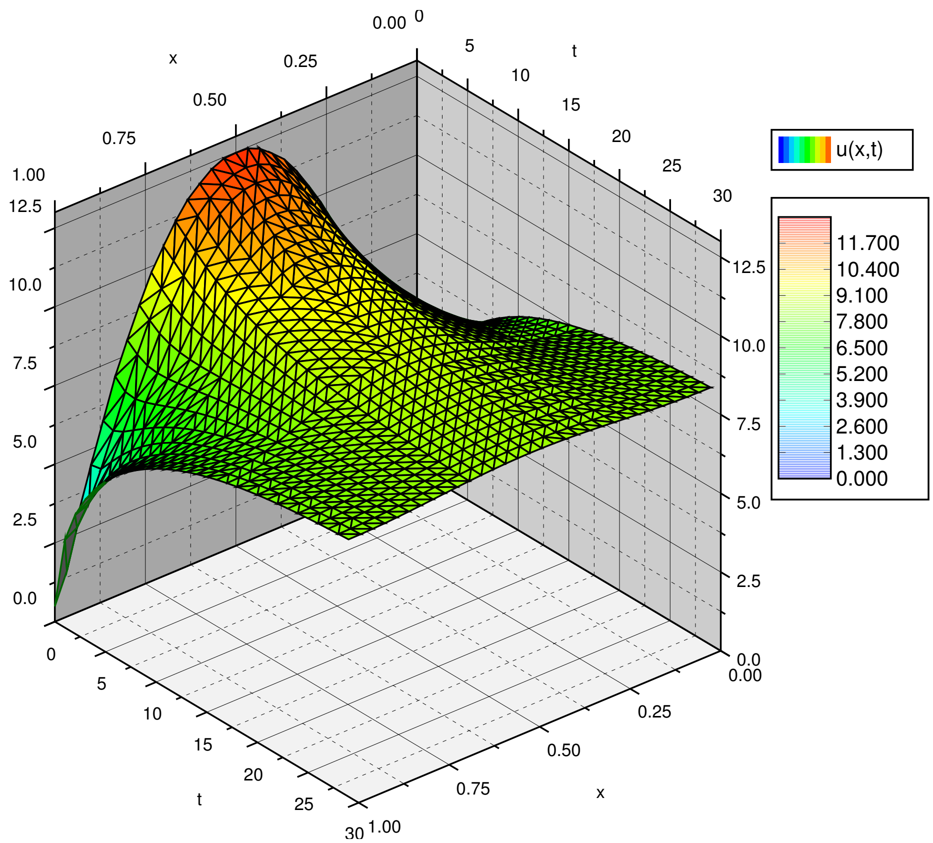


Figure 5: Plot of the temperature of the insulated wire at position x at time t.

Note in the graph that as time goes on, the temperature evens out across the wire. Eventually, all the terms except the constant die out, and you will be left with a uniform temperature of $\frac{25}{3} \approx 8.33$ along the entire length of the wire.

Let us expand on the last point. The constant term in the series is

$$\frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx.$$

In other words, $\frac{a_0}{2}$ is the average value of $f(x)$, that is, the average of the initial temperature. As the wire is insulated everywhere, no heat can get out, no heat can get in. So the temperature tries to distribute evenly over time, and the average temperature must always be the same, in particular it is always $\frac{a_0}{2}$. As time goes to infinity, the temperature goes to the constant $\frac{a_0}{2}$ everywhere.

1.4 Exercises

***Exercise* 1.2:** *Consider a wire of length* 2*, with* $k = 0.001$ *and an initial temperature distribution* $u(x, 0) = 50x$*. Both ends are embedded in ice* (*temperature* 0)*. Find the solution as a series.*

Exercise* 1.3*: *Find a series solution of*

$$\begin{aligned} &u_t = u_{xx}, \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = 100 \qquad \textit{for } 0 < x < 1. \end{aligned}$$

Exercise* 1.4*: *Find a series solution of*

$$\begin{aligned} &u_t = u_{xx}, \\ &u_x(0,t) = u_x(\pi,t) = 0, \\ &u(x,0) = 3\cos(x) + \cos(3x) \qquad \textit{for } 0 < x < \pi. \end{aligned}$$

Exercise* 1.5*: *Find a series solution of*

$$\begin{aligned} &u_t = \frac{1}{3}u_{xx}, \\ &u_x(0,t) = u_x(\pi,t) = 0, \\ &u(x,0) = \frac{10x}{\pi} \qquad \textit{for } 0 < x < \pi. \end{aligned}$$

Exercise* 1.6*: *Find a series solution of*

$$\begin{aligned} &u_t = u_{xx}, \\ &u(0,t) = 0, \quad u(1,t) = 100, \\ &u(x,0) = \sin(\pi x) \qquad \textit{for } 0 < x < 1. \end{aligned}$$

Hint: *Use the fact that* $u(x,t) = 100x$ *is a solution satisfying* $u_t = u_{xx}$, $u(0,t) = 0$, $u(1,t) = 100$. *Then use superposition.*

Exercise* 1.7*: *Find the* steady state temperature *solution as a function of x alone, by letting $t \to \infty$ in the solution from exercises* 1.5 *and* 1.6. *Verify that it satisfies the equation* $u_{xx} = 0$.

Exercise* 1.8*: *Use separation variables to find a nontrivial solution to* $u_{xx} + u_{yy} = 0$, *where* $u(x,0) = 0$ *and* $u(0,y) = 0$. *Hint*: *Try* $u(x,y) = X(x)Y(y)$.

Exercise* 1.9** (challenging):*** *Suppose that one end of the wire is insulated (say at $x = 0$) and the other end is kept at zero temperature. That is, find a series solution of*

$$\begin{aligned} &u_t = k u_{xx}, \\ &u_x(0,t) = u(L,t) = 0, \\ &u(x,0) = f(x) \qquad \textit{for } 0 < x < L. \end{aligned}$$

Express any coefficients in the series by integrals of $f(x)$.

Exercise* 1.10** (challenging):*** *Suppose that the wire is circular and insulated, so there are no ends. You can think of this as simply connecting the two ends and making sure the solution matches up at the ends. That is, find a series solution of*

$$\begin{aligned} &u_t = k u_{xx}, \\ &u(0,t) = u(L,t), \qquad u_x(0,t) = u_x(L,t), \\ &u(x,0) = f(x) \qquad \textit{for } 0 < x < L. \end{aligned}$$

Express any coefficients in the series by integrals of $f(x)$.

Exercise* 1.11*: *Consider a wire insulated on both ends,* $L = 1$, $k = 1$, *and* $u(x,0) = \cos^2(\pi x)$.

a) *Find the solution* $u(x, t)$. *Hint*: *a trig identity.*

b) *Find the average temperature.*

c) *Initially the temperature variation is* 1 (*maximum minus the minimum*). *Find the time when the variation is* 1/2.

Exercise* 1.101*: *Find a series solution of*

$$\begin{aligned} &u_t = 3u_{xx}, \\ &u(0,t) = u(\pi,t) = 0, \\ &u(x,0) = 5\sin(x) + 2\sin(5x) \qquad \textit{for } 0 < x < \pi. \end{aligned}$$

Exercise* 1.102*: *Find a series solution of*

$$\begin{aligned} &u_t = 0.1u_{xx}, \\ &u_x(0,t) = u_x(\pi,t) = 0, \\ &u(x,0) = 1 + 2\cos(x) \qquad \textit{for } 0 < x < \pi. \end{aligned}$$

Exercise* 1.103*: *Use separation of variables to find a nontrivial solution to* $u_{xt} = u_{xx}$.

Exercise* 1.104*: *Use separation of variables to find a nontrivial solution to* $u_x + u_t = u$. *Hint*: *Try* $u(x, t) = X(x) + T(t)$.

Exercise* 1.105*: *Suppose that the temperature on the wire is fixed at* 0 *at the ends,* $L = 1$, $k = 1$, *and* $u(x, 0) = 100\sin(2\pi x)$.

a) *What is the temperature at* $x = 1/2$ *at any time.*

b) *What is the maximum and the minimum temperature on the wire at* $t = 0$.

c) *At what time is the maximum temperature on the wire exactly one half of the initial maximum at* $t = 0$.

2 Steady state temperature and the Laplacian

Consider an insulated wire, a plate, or a 3-dimensional object. We apply certain fixed temperatures on the ends of the wire, the edges of the plate, or on all sides of the 3-dimensional object. We wish to find out what is the *steady state temperature* distribution. That is, we wish to know what will be the temperature after long enough period of time.

We are really looking for a solution to the heat equation that is not dependent on time. Let us first solve the problem in one space variable. We are looking for a function u that satisfies

$$u_t = k u_{xx},$$

but such that $u_t = 0$ for all x and t. Hence, we are looking for a function of x alone that satisfies $u_{xx} = 0$. It is easy to solve this equation by integration and we see that $u = Ax + B$ for some constants A and B.

Consider an insulated wire where we apply constant temperature T_1 at one end (say where $x = 0$) and T_2 on the other end (at $x = L$ where L is the length of the wire). Our steady state solution is

$$u(x) = \frac{T_2 - T_1}{L} x + T_1.$$

This solution agrees with our common sense intuition with how the heat should be distributed in the wire. So in one dimension, the steady state solutions are basically just straight lines.

Things are more complicated in two or more space dimensions. Let us restrict to two space dimensions for simplicity. The heat equation in two space variables is

$$u_t = k(u_{xx} + u_{yy}), \tag{1}$$

or more commonly written as $u_t = k\Delta u$ or $u_t = k\nabla^2 u$. Here the Δ and ∇^2 symbols mean $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We will use Δ from now on. The reason for using such a notation is that you can define Δ to be the right thing for any number of space dimensions and then the heat equation is always $u_t = k\Delta u$. The operator Δ is called the *Laplacian*.

OK, now that we have notation out of the way, let us see what does an equation for the steady state solution look like. We are looking for a solution to (1) that does not depend on t, or in other words $u_t = 0$. Hence we are looking for a function $u(x, y)$ such that

$$\boxed{\Delta u = u_{xx} + u_{yy} = 0.}$$

This equation is called the *Laplace equation**, and is an example of an elliptic equation. Solutions to the Laplace equation are called *harmonic functions* and have many nice properties and applications far beyond the steady state heat problem.

Harmonic functions in two variables are no longer just linear (plane graphs). For example, you can check that the functions $x^2 - y^2$ and xy are harmonic. However, note that if u_{xx} is positive, u is concave up in the x direction, then u_{yy} must be negative and u must be concave down in the y direction. A harmonic function can never have any “hilltop” or “valley” on the graph. This observation is consistent with our intuitive idea of steady state heat distribution; the hottest or coldest spot will not be inside.

Commonly the Laplace equation is part of a so-called *Dirichlet problem*†. That is, we have a region in the xy-plane and we specify certain values along the boundaries of the region. We then try to find a solution u to the Laplace equation defined on this region such that u agrees with the values we specified on the boundary.

In this section we consider a rectangular region. For simplicity we specify boundary values to be zero at 3 of the four edges and only specify an arbitrary function at one edge. As we still have the principle of superposition, we can use this simpler solution to derive the general solution for arbitrary boundary values

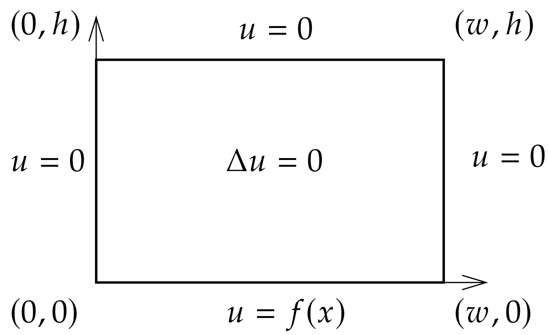
*Named after the French mathematician Pierre-Simon, marquis de Laplace (1749–1827).

†Named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859).

by solving 4 different problems, one for each edge, and adding those solutions together. This setup is left as an exercise.

We wish to solve the following problem. Let h and w be the height and width of our rectangle, with one corner at the origin and lying in the first quadrant.

$$\Delta u = 0, \tag{2}$$
$$u(0,y) = 0 \quad \text{for } 0 < y < h, \tag{3}$$
$$u(x,h) = 0 \quad \text{for } 0 < x < w, \tag{4}$$
$$u(w,y) = 0 \quad \text{for } 0 < y < h, \tag{5}$$
$$u(x,0) = f(x) \quad \text{for } 0 < x < w. \tag{6}$$



The method we apply is separation of variables. Again, we will come up with enough building-block solutions satisfying all the homogeneous boundary conditions (all conditions except (6)). We notice that superposition still works for the equation and all the homogeneous conditions. Therefore, we can use the Fourier series for $f(x)$ to solve the problem as before.

We try $u(x,y) = X(x)Y(y)$. We plug u into the equation to get

$$X''Y + XY'' = 0.$$

We put the Xs on one side and the Ys on the other to get

$$-\frac{X''}{X} = \frac{Y''}{Y}.$$

The left-hand side only depends on x and the right-hand side only depends on y. Therefore, there is some constant λ such that $\lambda = \frac{-X''}{X} = \frac{Y''}{Y}$. And we get two equations

$$X'' + \lambda X = 0,$$
$$Y'' - \lambda Y = 0.$$

Furthermore, the homogeneous boundary conditions imply that $X(0) = X(w) = 0$ and $Y(h) = 0$. Taking the equation for X we have already seen that we have a nontrivial solution if and only if $\lambda = \lambda_n = \frac{n^2\pi^2}{w^2}$ and the solution is a multiple of

$$X_n(x) = \sin\left(\frac{n\pi}{w}x\right).$$

For these given λ_n, the general solution for Y (one for each n) is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{w}y\right) + B_n \sinh\left(\frac{n\pi}{w}y\right). \tag{7}$$

We only have one condition on Y_n and hence we can pick one of A_n or B_n to be something convenient. It will be useful to have $Y_n(0) = 1$, so we let $A_n = 1$. Setting $Y_n(h) = 0$ and solving for B_n we get that

$$B_n = \frac{-\cosh\left(\frac{n\pi h}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}.$$

After we plug the A_n and B_n we into (7) and simplify by using the identity $\sinh(\alpha - \beta) = \sinh(\alpha)\cosh(\beta) - \cosh(\alpha)\sinh(\beta)$, we find

$$Y_n(y) = \frac{\sinh\left(\frac{n\pi(h-y)}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}.$$

We define $u_n(x,y) = X_n(x)Y_n(y)$. And note that u_n satisfies (2)–(5).

Observe that

$$u_n(x,0) = X_n(x)Y_n(0) = \sin\left(\frac{n\pi}{w}x\right).$$

Suppose

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{w}\right).$$

Then we get a solution of (2)–(6) of the following form.

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n u_n(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{w}x\right)\left(\frac{\sinh\left(\frac{n\pi(h-y)}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}\right).}$$

As u_n satisfies (2)–(5) and any linear combination (finite or infinite) of u_n also satisfies (2)–(5), then u satisfies (2)–(5). By plugging in $y = 0$, we see u satisfies (6) as well.

Example 2.1: Take $w = h = \pi$ and let $f(x) = \pi$. Let us compute the sine series for the function π (same as the series for the square wave). For $0 < x < \pi$, we have

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n}\sin(nx).$$

Therefore the solution $u(x,y)$, see Figure 6 on the next page, to the corresponding Dirichlet problem is given as

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n}\sin(nx)\left(\frac{\sinh\big(n(\pi-y)\big)}{\sinh(n\pi)}\right).$$

This scenario corresponds to the steady state temperature on a square plate of width π with 3 sides held at 0 degrees and one side held at π degrees. If we have arbitrary initial data on all sides, then we solve four problems, each using one piece of nonhomogeneous data. Then we use the principle of superposition to add up all four solutions to have a solution to the original problem.

A different way to visualize solutions of the Laplace equation is to take a wire and bend it so that it corresponds to the graph of the temperature above the boundary of your region. Cut a rubber sheet in the shape of your region—a square in our case—and stretch it fixing the edges of the sheet to the wire. The rubber sheet is a good approximation of the graph of the solution to the Laplace equation with the given boundary data.

2.1 Exercises

Exercise 2.1: *Let R be the region described by $0 < x < \pi$ and $0 < y < \pi$. Solve the problem*

$$\Delta u = 0, \quad u(x,0) = \sin x, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u(\pi,y) = 0.$$

Exercise 2.2: *Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem*

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(x,0) = \sin(\pi x) - \sin(2\pi x), \quad u(x,1) = 0, \\ &u(0,y) = 0, \quad u(1,y) = 0. \end{aligned}$$

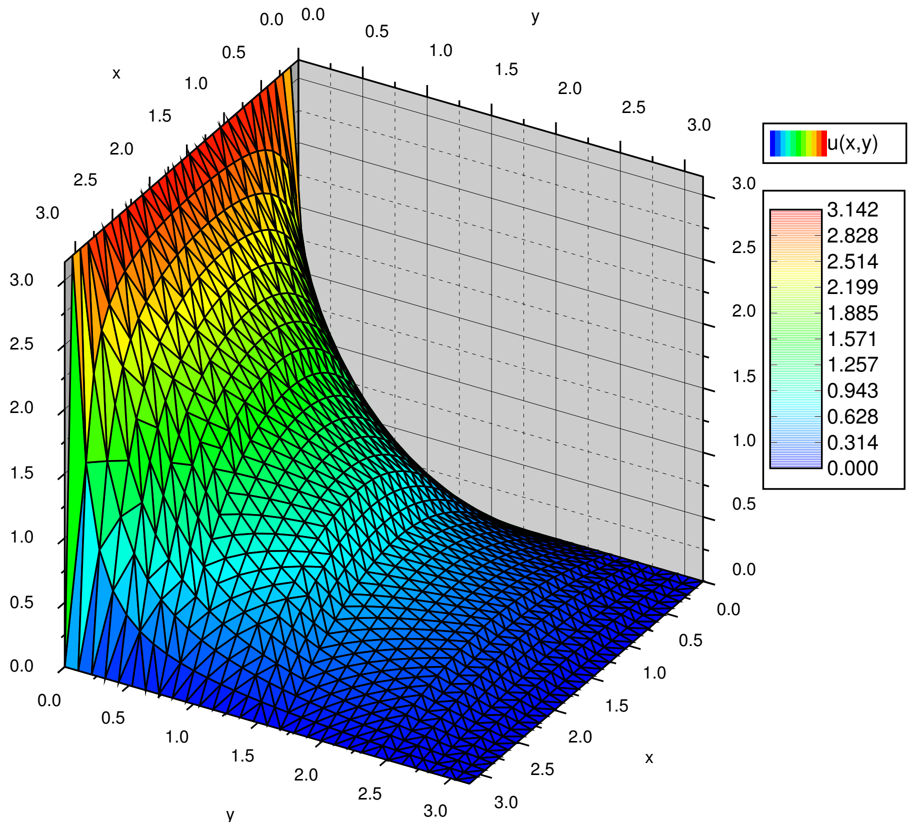


Figure 6: Steady state temperature of a square plate, three sides held at zero and one side held at π.

***Exercise* 2.3:** *Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem*

$$u_{xx} + u_{yy} = 0,$$
$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = C.$$

for some constant C. Hint: Guess, then check your intuition.

***Exercise* 2.4:** *Let R be the region described by $0 < x < \pi$ and $0 < y < \pi$. Solve*

$$\Delta u = 0, \quad u(x,0) = 0, \quad u(x,\pi) = \pi, \quad u(0,y) = y, \quad u(\pi,y) = y.$$

Hint: Try a solution of the form $u(x,y) = X(x) + Y(y)$ (different separation of variables).

***Exercise* 2.5:** *Use the solution of Exercise 2.4 to solve*

$$\Delta u = 0, \quad u(x,0) = \sin x, \quad u(x,\pi) = \pi, \quad u(0,y) = y, \quad u(\pi,y) = y.$$

Hint: Use superposition.

***Exercise* 2.6:** *Let R be the region described by $0 < x < w$ and $0 < y < h$. Solve the problem*

$$u_{xx} + u_{yy} = 0,$$
$$u(x,0) = 0, \quad u(x,h) = f(x),$$
$$u(0,y) = 0, \quad u(w,y) = 0.$$

The solution should be in series form using the Fourier series coefficients of $f(x)$.

Exercise* 2.7*: *Let R be the region described by $0 < x < w$ and $0 < y < h$. Solve the problem*

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(x,0) = 0, \quad u(x,h) = 0, \\ &u(0,y) = f(y), \quad u(w,y) = 0. \end{aligned}$$

The solution should be in series form using the Fourier series coefficients of $f(y)$.

Exercise* 2.8*: *Let R be the region described by $0 < x < w$ and $0 < y < h$. Solve the problem*

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(x,0) = 0, \quad u(x,h) = 0, \\ &u(0,y) = 0, \quad u(w,y) = f(y). \end{aligned}$$

The solution should be in series form using the Fourier series coefficients of $f(y)$.

Exercise* 2.9*: *Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem*

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(x,0) = \sin(9\pi x), \quad u(x,1) = \sin(2\pi x), \\ &u(0,y) = 0, \quad u(1,y) = 0. \end{aligned}$$

Hint: *Use superposition.*

Exercise* 2.10*: *Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem*

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(x,0) = \sin(\pi x), \quad u(x,1) = \sin(\pi x), \\ &u(0,y) = \sin(\pi y), \quad u(1,y) = \sin(\pi y). \end{aligned}$$

Hint: *Use superposition.*

Exercise* 2.11** (challenging):*** *Using only your intuition find $u(1/2, 1/2)$, for the problem $\Delta u = 0$, where $u(0,y) = u(1,y) = 100$ for $0 < y < 1$, and $u(x,0) = u(x,1) = 0$ for $0 < x < 1$. Explain.*

Exercise* 2.101*: *Let R be the region described by $0 < x < 1$ and $0 < y < 1$. Solve the problem*

$$\Delta u = 0, \quad u(x,0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x), \quad u(x,1) = 0, \quad u(0,y) = 0, \quad u(1,y) = 0.$$

Exercise* 2.102*: *Let R be the region described by $0 < x < 1$ and $0 < y < 2$. Solve the problem*

$$\Delta u = 0, \quad u(x,0) = 0.1\sin(\pi x), \quad u(x,2) = 0, \quad u(0,y) = 0, \quad u(1,y) = 0.$$