Chapter 1: Signal

1. Energy and Power of Signal

	Continuous time - CT	Discrete time - DT
Periodic	$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \lvert x(t)\rvert^2 dt$	$P_x = \frac{1}{N_0}\sum_{n=0}^{N_0-1} \lvert x[n]\rvert^2$
	$E_x = \lim_{k\to\infty} kT_0P_x$	$E_x = \lim_{k\to\infty} kT_0P_x$
Aperiodic	$E_x = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$	$E_x = \sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2$
	$P_x = \lim_{T_0\to\infty} \frac{1}{2T_0}\int_{-T_0}^{+T_0} \lvert x(t)\rvert^2 dt$	$P_x = \lim_{N_0\to\infty} \frac{1}{2N_0+1}\sum_{n=-N_0}^{N_0} \lvert x[n]\rvert^2$

Note that:

1. If $E_x < M, M$ is finite, then the signal is called as energy signal. If $P_x < M, M$ is finite, then the signal is called as power signal.
2. An aperiodic signal can be energy signal with zero average power. A periodic signal can be power signal with infinite total energy.
3. If a signal is summation of sine with amplitude $A_i, i = 0, .., n$ and cosine with amplitude $B_i, i = 0, \dots, m$, then the power of this signal is given by
$$P_x = \sum_{i=0}^{n} \frac{A_i^2}{2} + \sum_{i=0}^{m} \frac{B_i^2}{2}$$
4. A signal is called as **causal signal** if and only if $x(t) = 0, \forall t < 0$ for CT signal or $x[n] = 0, \forall n < 0$ for DT signal, respectively.

2. Basic Signal Functions

	Continuous time - CT	Discrete time - DT
Impulse	$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$	$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$
Unit step	$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Chapter 2: System

1. Properties of System

1. 1. Causality

A causal system is a system where the output y only depends on present and past values of input x but not the future inputs.

1. 2. Linearity

A system $\mathbf{H}$ is called linear system if and only if it satisfies the condition

$$\mathbf{H}\{a_1x_1 + a_2x_2\} = a_1y_1 + a_2y_2$$

Check for linearity:

- Step 1: Calculate the output for linear combinations of input

$$x = a_1x_1 + a_2x_2 \xrightarrow{\mathbf{H}} y = \mathbf{H}\{a_1x_1 + a_2x_2\} = ?\,(1)$$

- Step 2: Calculate linear combination of output for independent inputs

$$\begin{cases} x_1 \xrightarrow{\mathbf{H}} y_1 = \mathbf{H}\{x_1\} \\ x_2 \xrightarrow{\mathbf{H}} y_2 = \mathbf{H}\{x_2\} \end{cases} \rightarrow a_1y_1 + a_2y_2 = ?\,(2)$$

- Step 3: Compare (1) and (2), if it equals, conclude that the system is linear.

1. 3. Time Invariant

If a time-delay on the input directly equates to a time-delay of the output function, the system will be considered time-invariant.

Check for time invariant:

- Step 1: Delay input, calculate its output

$$x_T(t) = x(t-T) \xrightarrow{\mathbf{H}} y_T(t) = \mathbf{H}\{x(t-T)\} = ?\,(1)$$

- Step 2: Calculate the delaying output for the normal input.

$$x(t) \xrightarrow{\mathbf{H}} y(t) = \mathbf{H}\{x\} \rightarrow y(t-T)\ (2)$$

- Step 3:
  Compare (1) and (2), if it equals, conclude that the system is time invariant.
  (Similarly for discrete system, calculate: $y[n-N], y_N = \mathbf{H}\{x[n-N]\}$ and compare)

1. 4. Bounded-input Bounded-output (BIBO) Stable

If the system has bounded for all input ($|x(t)| < M, M$ is finite) which leads to all output is bounded ($|y(t)| < N, N$ is finite) then the system is said to be BIBO system.

Check for BIBO system:

- Assume that $|x(t)| < M, \forall t, M$ is finite. Calculate $|y(t)|$, if we can prove that $|y(t)| < N, N$ is finite, then the system is BIBO system.

Continuous time BIBO System

- If the impulse response of the continuous time system $h(t)$ is absolutely integrable, the system is said to be BIBO stable.

$$\int_{-\infty}^{+\infty} |h(t)|dt < M, M \text{ is finite}$$

Discrete time BIBO System

- If the impulse response of the discrete time system $h[n]$ is absolutely integrable, the system is said to be BIBO stable.

$$\sum_{n=-\infty}^{+\infty} |h[n]| < M, M \text{ is finite}$$

1. 5. Memory

A system is said to be memoryless if for all value of t_0, the output $y(t_0)$ only depends on value of input at t_0, i.e. $x(t_0)$.

A system which is not memoryless is considered to have memory.

1. 6. Invertibility

If there is exists a system **S** for the given system **H** such that

$$\mathbf{S}\{\mathbf{H}\{x\}\} = x$$

the system **S** is said to be the inverse of system **H**, denoted by $\mathbf{H}^{-1}$.

Check for non-invertible system:

- If there are exists two different inputs x_1 and x_2 that produce the same output $\mathbf{H}\{x_1\} = \mathbf{H}\{x_2\}$, the given system is non-invertible.

2. Block Diagram Represent for Systems

	Block Diagram
Cascade	
Parallel	
Feedback	

Chapter 3: Convolution

1. Convolution

Continuous time	Discrete time
$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$	$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$

2. Convolution properties

Name	Formula
Linearity	$(a_1 f + a_2 g) * h = a_1(f * h) + a_2(g * h)$
Associativity	$(f * g) * h = f * (g * h)$
Identity	$f * \delta = f$ $f(t) * \delta(t-a) = f(t-a)$

Chapter 4: Transformations

1. Summary of Transformations

		Time domain	
		Continuous	**Discrete**
Fourier transform (Frequency domain)	**Periodic Discrete**	$X[k] = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}dt$ $x(t) = \sum_{k=-\infty}^{+\infty} X[k]e^{jk\omega_0 t}$	$X[k] = \sum_{k=0}^{n-1} x[n]e^{-j2\pi nk/N_0}$ $x[n] = \frac{1}{N_0}\sum_{k=0}^{n-1} X[k]e^{j2\pi nk/N_0}$
	Non-Periodic Continuous	$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$	$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\omega$
Laplace and z-transform		Laplace transform: $X(s) = \mathcal{L}\{x(t)\} = \int_{0^-}^{+\infty} x(t)e^{-st}dt$	z-transform: $X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{+\infty} x[n]z^{-n}$
Convolution		$y(t) = x(t) * h(t)$ $\begin{cases} Y(s) = X(s).H(s) \\ X(\omega) = X(\omega).H(\omega) \\ Y[k] = X[k].H[k] \end{cases}$	$y[n] = x[n] * h[n]$ $\begin{cases} Y(z) = X(z).H(z) \\ X(\omega) = X(\omega).H(\omega) \\ Y[k] = X[k].H[k] \end{cases}$

2. Laplace Transform

2. 1. Definition

If $f(t)$ is continuous and there are positive numbers M, a such that $|f(t)| < Me^{at}$, for all $t \geq c$. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

2. 2. Properties

$\boldsymbol{f(t)}$	$\boldsymbol{\mathcal{L}\{f(t)u(t)\}}$	$\boldsymbol{f(t)}$	$\boldsymbol{\mathcal{L}\{f(t)u(t)\}}$
$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	$e^{-at}f(t)$	$F(s+a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$f(t-a)u(t-a)$	$e^{-as}F(s)$
$f'(t)$	$sF(s) - f(0)$	$(f * g)(t)$	$F(s).G(s)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\frac{f(t)}{t}$	$\int_s^{+\infty} F(\tau)d\tau$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$	$\int_0^t f(\tau)d\tau = u(t) * f(t)$	$\frac{1}{s}F(s)$

2. 3. Formulas

$f(t)$	$\mathcal{L}\{f(t)u(t)\}$	$f(t)$	$\mathcal{L}\{f(t)u(t)\}$
1	$\frac{1}{s}$	$\delta(t-a)$	e^{-as}
t^n	$\frac{n!}{s^{n+1}}$	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$tf(t)$	$-F'(s)$

2. 4. Initial and Final Value Theorem

Initial-value theorem

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = f(0^+)$$

Final-value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

2. 5. Convolution

Solving a convolution: Find $x(t) * h(t)$ or $(x * h)(t)$

Let: $$y(t) = x(t) * h(t)$$

$$\to Y(s) = \mathcal{L}\{x(t) * h(t)\} = X(s).H(s)$$

Taking inverse Laplace transform to find the result of the convolution:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

3. z-transform

3. 1. Definition

Causal sequence: $\{x_n\}_0^\infty = \{x_0, x_1, x_2, \ldots\}$

Infinite sequence: $\{x_n\}_{-\infty}^\infty = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$

The z-transform of an **infinite sequence** is defined whenever the sum exists and where z is a complex variable

$$\mathcal{Z}\{x_n\}_{-\infty}^{\infty} = X(z) = \sum_{n=-\infty}^{\infty} \frac{x_n}{z^n}$$

The z-transform of a **causal sequence**:

$$\mathcal{Z}\{x_n\}_0^{\infty} = X(z) = \sum_{n=0}^{\infty} \frac{x_n}{z^n}$$

Where: $\mathcal{Z}$ is the z-Transform operator, $\{x_k\} - X(z)$: is a z-transform pair.

3. 2. Properties

x_n	$\mathcal{Z}\{x_n\}$	x_n	$\mathcal{Z}\{x_n\}$
$a^n x_n$	$X\left(\frac{z}{a}\right)$	$n^m x_n$	$-z^m \frac{d^m}{dz^m} X(z)$
x_{-n}	$X\left(\frac{1}{z}\right)$	x_{n-1}	$\frac{X(z)}{z}$
x_{n+1}	$zX(z) - zx_0$	x_{n+2}	$z^2X(z) - z^2x_0 - zx_1$

3. 3. Formulas

x_n	$\mathcal{Z}\{x_n\}$	x_n	$\mathcal{Z}\{x_n\}$
δ_{n-n_0}	z^{-n_0}	1	$\frac{z}{z-1}$
a^n	$\frac{z}{z-a}$	n	$\frac{z}{(z-1)^2}$
na^{n-1}	$\frac{z}{(z-a)^2}$	e^{-nT}	$\frac{z}{z-e^{-T}}$
$a^n \cos(n\omega T)$	$\frac{z(z-\cos\omega T)}{z^2 - 2za\cos\omega T + a^2}$	$a^n \sin(n\omega T)$	$\frac{z\sin\omega T}{z^2 - 2za\cos\omega T + a^2}$

3. 4. Initial and Final Value Theorem

Initial-value theorem

$$\lim_{n\to 0} x_n = \lim_{z\to\infty} X(z) = x_0$$

Final-value theorem

$$\lim_{n\to\infty} x_n = \lim_{z\to 1}\left(1 - \frac{1}{z}\right) X(z)$$

Chapter 5: Fourier Series

1. Full Range Series

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{+\infty} b_k \sin(k\omega_0 t)$$

Where:

$$a_0 = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)dt\,;\ a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\omega_0 t)\,dt\,;\ b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\omega_0 t)\,dt$$

Odd function: $a_0 = a_k = 0$, and

$$b_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin(k\omega_0 t)\,dt$$

Even function: $b_k = 0$, and

$$a_0 = \frac{4}{T_0}\int_0^{T_0/2} x(t)dt\,;\ a_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos(k\omega_0 t)\,dt$$

Parseval's identity:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{k=1}^{+\infty}(a_k^2 + b_k^2)$$

2. Half Range Series

2. 1. Half Range Sine Series:

$$x(t) = \sum_{k=1}^{+\infty} b_k \sin\left(\frac{k\pi t}{L}\right);\ b_n = \frac{2}{L}\int_0^L x(t)\sin\left(\frac{k\pi t}{L}\right)dt$$

2. 2. Half Range Cosine Series:

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty} a_k \cos\left(\frac{k\pi t}{L}\right);\ \ a_0 = \frac{2}{L}\int_0^L x(t)dt\,;\ \ a_k = \frac{2}{L}\int_0^L x(t)\cos\left(\frac{k\pi t}{L}\right)dt$$

3. Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

where:

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}dt$$

4. Frequently Used Formulas

Euler's formula

$$\cos x = \frac{e^{ix} + e^{-ix}}{2};\quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Sine, cosine of odd number of pi

$\cos \pi n = (-1)^n$

$\sin \pi n = 0$

Helpful integration

$$I_1 = \int (at+b)\sin ct\, dt = -\frac{at+b}{c}\cos ct + \frac{a}{c^2}\sin ct$$

$$I_2 = \int (at+b)\cos ct\, dt = \frac{at+b}{c}\sin ct + \frac{a}{c^2}\cos ct$$

$$I_3 = \int \sin(at+b)\sin(ct+d)\, dt = \frac{1}{2}\left(\frac{\sin(t(a-c)+b-d)}{a-c} - \frac{\sin(t(a+c)+b-d)}{a+c}\right)$$

$$I_4 = \int \cos(at+b)\cos(ct+d)\, dt = \frac{1}{2}\left(\frac{\sin(t(a-c)+b-d)}{a-c} + \frac{\sin(t(a+c)+b-d)}{a+c}\right)$$

$$I_5 = \int \sin(at+b)\cos(ct+d)\, dt = -\frac{1}{2}\left(\frac{\cos(t(a-c)+b-d)}{a-c} + \frac{\cos(t(a+c)+b-d)}{a+c}\right)$$