Chapter 3
Continuous time convolution

LTI SYSTEMS

- In engineering, linear time-invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

- The (CT) **convolution** of the functions x and h, denoted $x * h$, is defined as the function

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- The convolution result $x * h$ evaluated at the point t is simply a weighted average of the function x, where the weighting is given by h time reversed and shifted by t.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in systems theory.
- In particular, convolution has a special significance in the context of LTI systems.

PRACTICAL CONVOLUTION COMPUTATION

- To compute the convolution

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau,$$

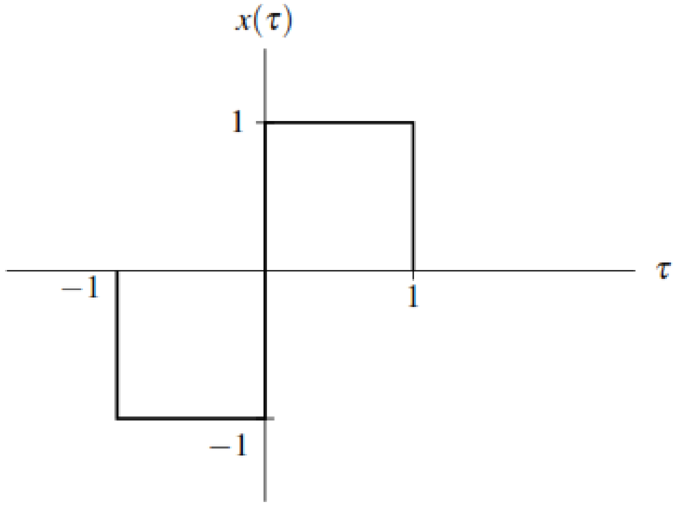
we proceed as follows:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $x * h(t)$.
4. Increase t gradually until the expression for $x * h(t)$ changes form. Record the interval over which the expression for $x * h(t)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $x * h(t)$ for all t.

EXAMPLE

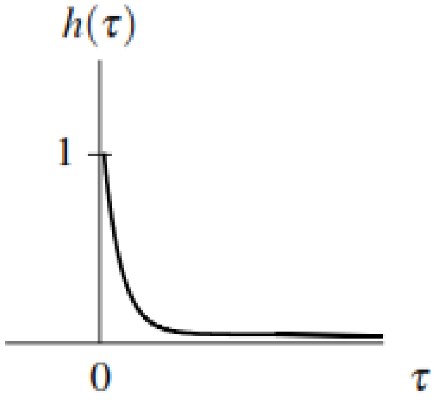
Compute the convolution $x * h$ where

$$x(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$



(a)

Solution. We begin by plotting the functions x and h



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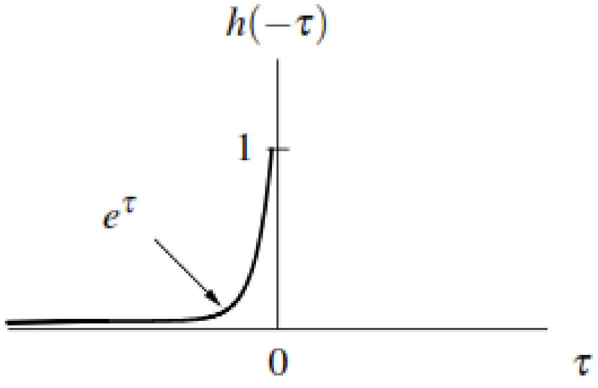

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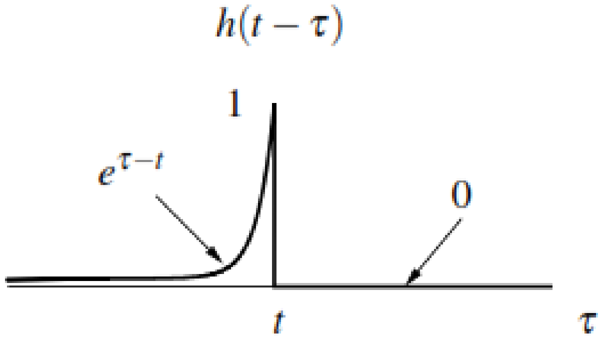
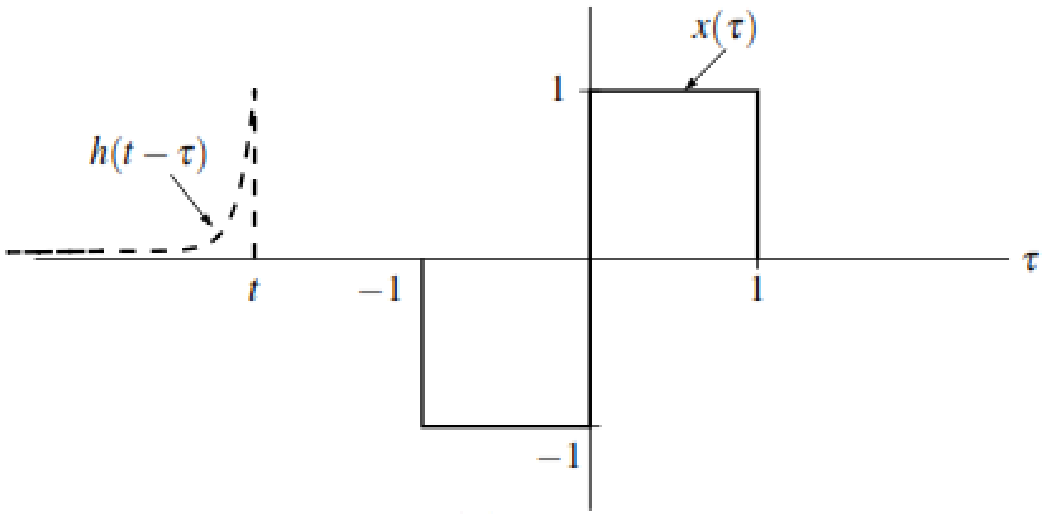


(d)

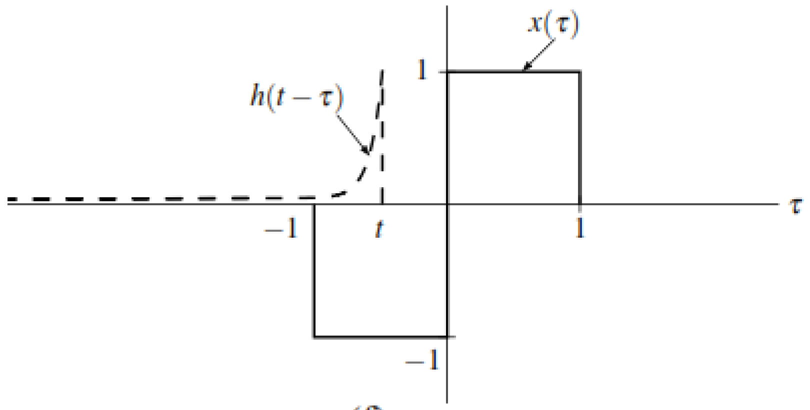
First, we consider the case of $t < -1$.



(e)

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0.$$

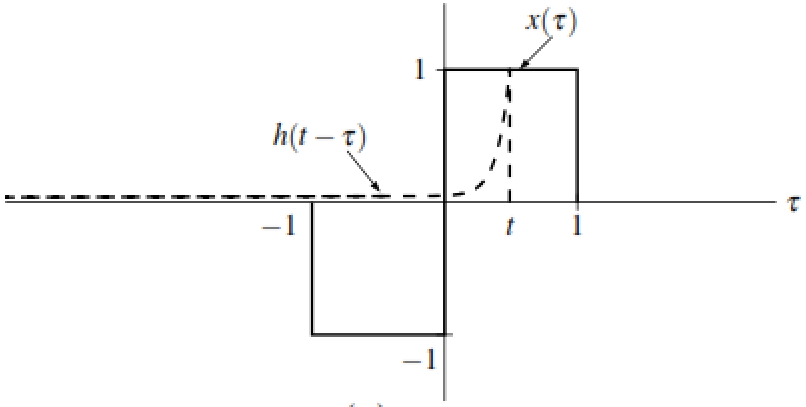
Second, we consider the case of $-1 \leq t < 0$.



(f)

$$
\begin{aligned}
x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t} d\tau \\
&= -e^{-t} \int_{-1}^{t} e^{\tau} d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{t} \\
&= -e^{-t}[e^{t} - e^{-1}] \\
&= e^{-t-1} - 1.
\end{aligned}
$$

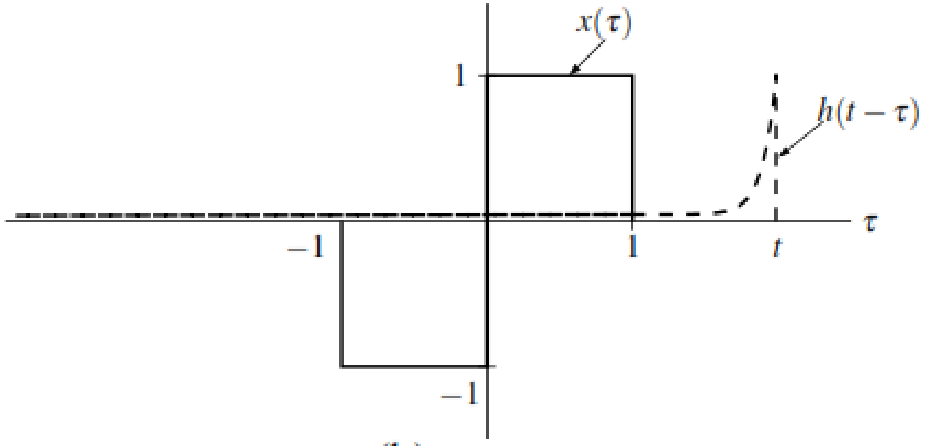
Third, we consider the case of $0 \le t < 1$.



(g)

$$
\begin{aligned}
x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau - t} d\tau + \int_{0}^{t} e^{\tau - t} d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau} d\tau + e^{-t}\int_{0}^{t} e^{\tau} d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\
&= -e^{-t}[1 - e^{-1}] + e^{-t}[e^{t} - 1] \\
&= e^{-t}[e^{-1} - 1 + e^{t} - 1] \\
&= 1 + (e^{-1} - 2)e^{-t}.
\end{aligned}
$$

Fourth, we consider the case of $t \geq 1$.



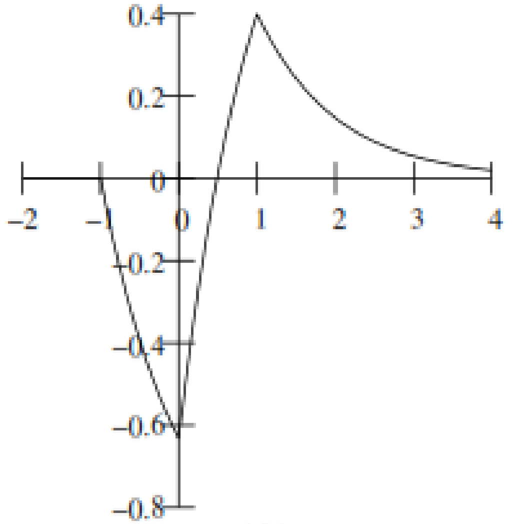
(h)

$$
\begin{aligned}
x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\
&= e^{-t}[e^{-1} - 1 + e - 1] \\
&= (e - 2 + e^{-1})e^{-t}.
\end{aligned}
$$

Combining the results

$$x * h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1} - 1 & -1 \le t < 0 \\ (e^{-1} - 2)e^{-t} + 1 & 0 \le t < 1 \\ (e - 2 + e^{-1})e^{-t} & 1 \le t. \end{cases}$$

$x * h$



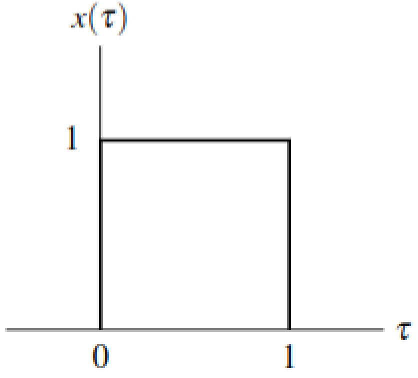
(i)

EXAMPLE

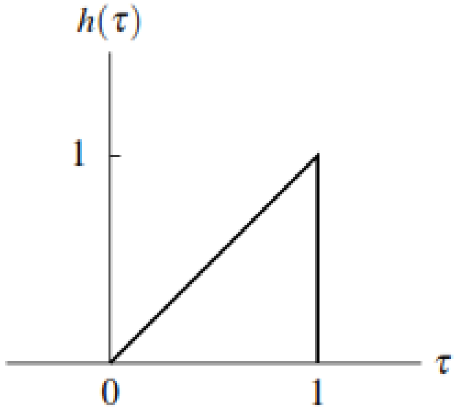
Compute the convolution $x * h$, where

$$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. We begin by plotting the functions x and h



(a)



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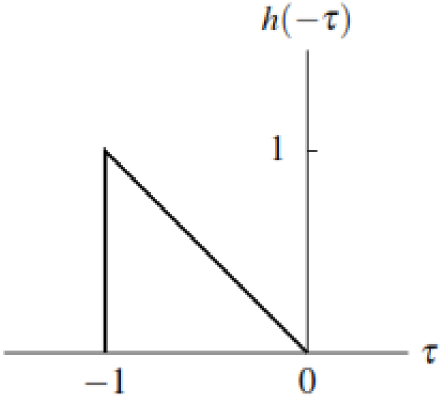

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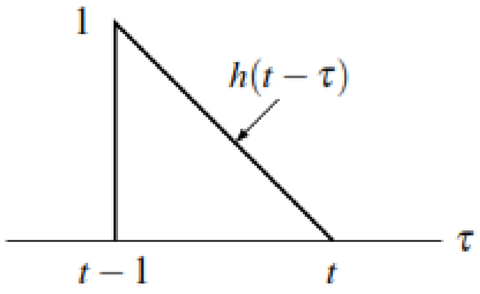
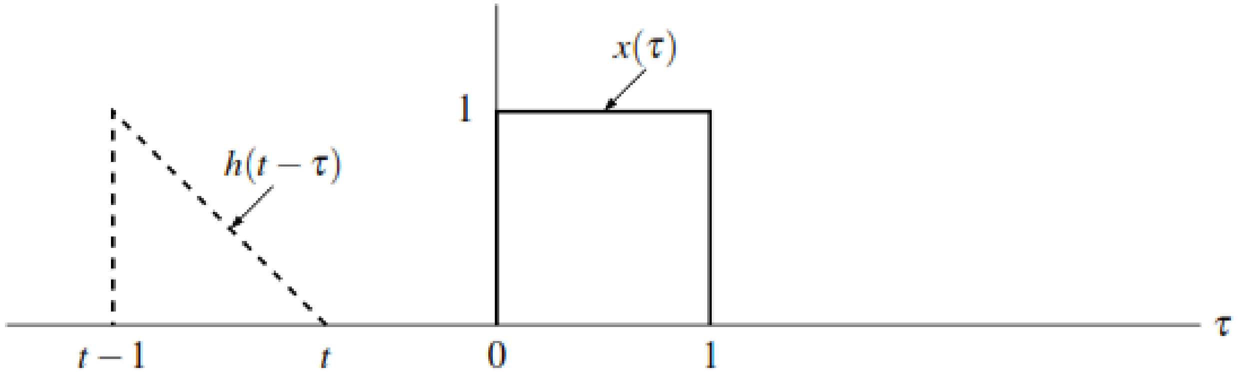


(d)

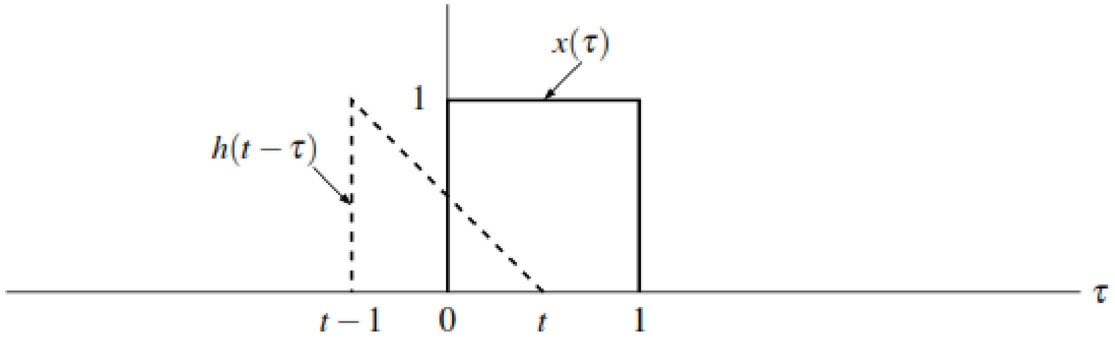
First, we consider the case of $t < 0$.

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0.$$



(e)

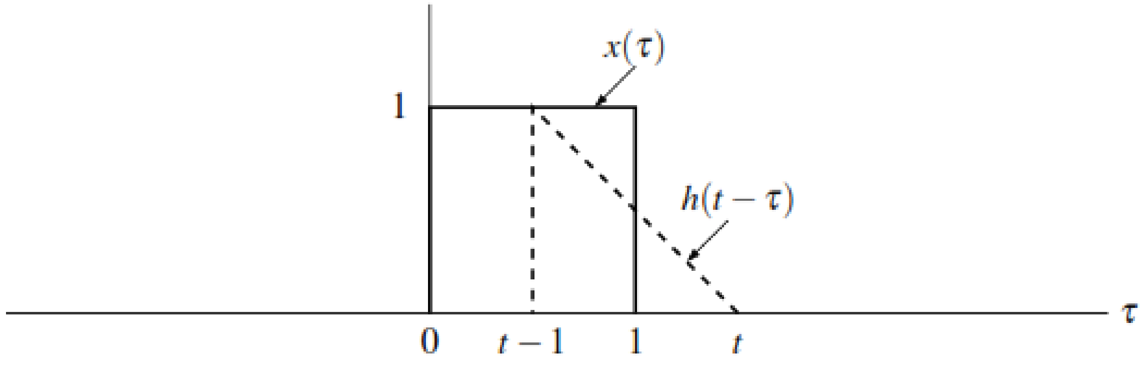
Second, we consider the case of $0 \le t < 1$.



(f)

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^t (t-\tau)d\tau$$
$$= [t\tau - \tfrac{1}{2}\tau^2]|_0^t$$
$$= t^2 - \tfrac{1}{2}t^2$$
$$= \tfrac{1}{2}t^2.$$

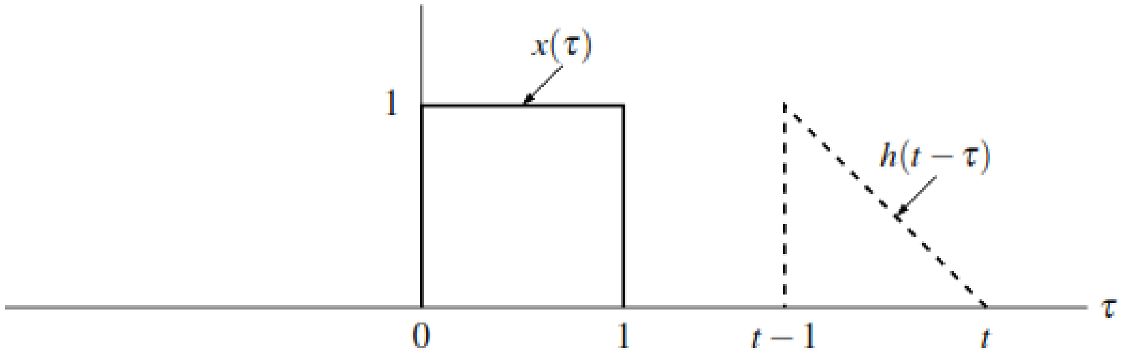
Third, we consider the case of $1 \leq t < 2$.



(g)

$$
\begin{aligned}
x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} (t-\tau)d\tau \\
&= [t\tau - \tfrac{1}{2}\tau^2]_{t-1}^{1} \\
&= t - \tfrac{1}{2}(1)^2 - [t(t-1) - \tfrac{1}{2}(t-1)^2] \\
&= t - \tfrac{1}{2} - [t^2 - t - \tfrac{1}{2}(t^2 - 2t + 1)] \\
&= -\tfrac{1}{2}t^2 + t.
\end{aligned}
$$

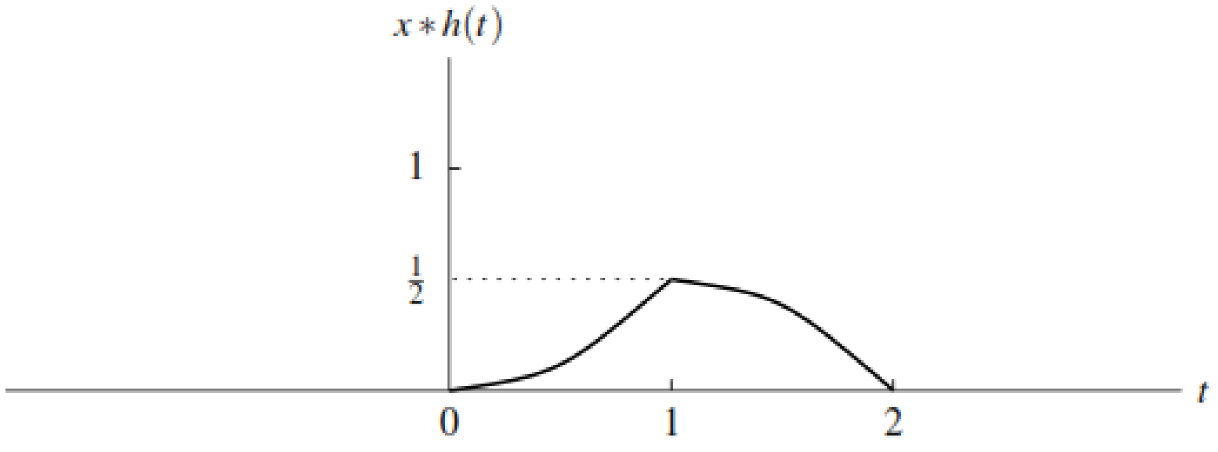
Fourth, we consider the case of $t \geq 2$.



(h)

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0.$$

Combining the results



(i)

$$x * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \le t < 1 \\ -\frac{1}{2}t^2 + t & 1 \le t < 2 \\ 0 & t \ge 2. \end{cases}$$

PROPERTIES OF CONVOLUTION

- The convolution operation is ***commutative***. That is, for any two functions x and h,

$$x * h = h * x.$$

- The convolution operation is ***associative***. That is, for any functions x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any functions x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

PROPERTIES OF CONVOLUTION

Proof. We now provide a proof of the commutative property stated above.

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Next, we perform a change of variable. Let $v = t - \tau$ which implies that $\tau = t - v$ and $d\tau = -dv$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned}
x * h(t) &= \int_{t+\infty}^{t-\infty} x(t-v)h(v)(-dv) \\
&= \int_{\infty}^{-\infty} x(t-v)h(v)(-dv) \\
&= \int_{-\infty}^{\infty} x(t-v)h(v)dv \\
&= \int_{-\infty}^{\infty} h(v)x(t-v)dv \\
&= h * x(t).
\end{aligned}$$

(Note that, above, we used the fact that, for any function f, $\int_a^b f(x)dx = -\int_b^a f(x)dx$.) Thus, we have proven that convolution is commutative. ■

PROPERTIES OF CONVOLUTION

Proof. *Convolution is associative.*

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

$$\begin{aligned}([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} [x * h_1(v)] h_2(t-v) dv \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) h_1(v-\tau) d\tau \right) h_2(t-v) dv.\end{aligned}$$

Now, we change the order of integration to obtain

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h_1(v-\tau) h_2(t-v) dv d\tau.$$

Pulling the factor of $x(\tau)$ out of the inner integral yields

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(v-\tau) h_2(t-v) dv d\tau.$$

PROPERTIES OF CONVOLUTION

Next, we perform a change of variable. Let $\lambda = v - \tau$ which implies that $v = \lambda + \tau$ and $d\lambda = dv$. Using this change of variable, we can write

$$\begin{aligned}
([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty-\tau}^{\infty-\tau} h_1(\lambda) h_2(t - \lambda - \tau) d\lambda d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(\lambda) h_2(t - \lambda - \tau) d\lambda d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h_1(\lambda) h_2([t - \tau] - \lambda) d\lambda \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left[h_1 * h_2(t - \tau) \right] d\tau \\
&= (x * [h_1 * h_2])(t).
\end{aligned}$$

Thus, we have proven that convolution is associative. ∎

PROPERTIES OF CONVOLUTION

Proof. Convolution is distributive.

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

$$\begin{aligned}
(x * [h_1 + h_2])(t) &= \int_{-\infty}^{\infty} x(\tau)[h_1(t-\tau) + h_2(t-\tau)]d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau \\
&= x * h_1(t) + x * h_2(t).
\end{aligned}$$

Thus, we have shown that convolution is distributive.

REPRESENTATION OF FUNCTIONS USING IMPULSES

- For any function x,

$$x * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t).$$

- Thus, any function x can be written in terms of an expression involving δ.
- Moreover, δ is the ***convolutional identity***. That is, for any function x,

$$x * \delta = x.$$

REPRESENTATION OF FUNCTIONS USING IMPULSES

Proof. Suppose that we have an arbitrary function x. From the definition of convolution, we can write

$$x * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau.$$

Let $\lambda = -\tau$ so that $\tau = -\lambda$ and $d\tau = -d\lambda$. Applying the change of variable,

$$\begin{aligned} x * \delta(t) &= \int_{-(-\infty)}^{-(\infty)} x(-\lambda)\delta(t+\lambda)(-1)\,d\lambda \\ &= \int_{\infty}^{-\infty} x(-\lambda)\delta(t+\lambda)(-1)\,d\lambda \\ &= \int_{-\infty}^{\infty} x(-\lambda)\delta(\lambda+t)\,d\lambda. \end{aligned}$$

REPRESENTATION OF FUNCTIONS USING IMPULSES

From the equivalence property of δ, we can rewrite the preceding equation as

$$x * \delta(t) = \int_{-\infty}^{\infty} x(-[-t])\delta(\lambda + t)d\lambda$$

$$= \int_{-\infty}^{\infty} x(t)\delta(\lambda + t)d\lambda.$$

Factoring $x(t)$ out of the integral, we obtain

$$x * \delta(t) = x(t)\int_{-\infty}^{\infty} \delta(\lambda + t)d\lambda.$$

Since $\int_{-\infty}^{\infty} \delta(\lambda)d\lambda = 1$ implies that $\int_{-\infty}^{\infty} \delta(\lambda + t)d\lambda = 1$, we have

$$x * \delta(t) = x(t).$$

Thus, δ is the convolutional identity (i.e., $x * \delta = x$).

PERIODIC CONVOLUTION

- The convolution of two periodic functions is usually not well defined.
- This motivates an alternative notion of convolution for periodic functions known as periodic convolution.
- The **periodic convolution** of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)d\tau,$$

where $\int_T$ denotes integration over an interval of length T.

- The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as follows:

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

(i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere).

IMPULSE RESPONSE

- The response h of a system $\mathcal{H}$ to the input δ is called the **impulse response** of the system (i.e., $h = \mathcal{H}\delta$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply ***computes a convolution***.
- Furthermore, a LTI system is ***completely characterized*** by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.
- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.
- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.

STEP RESPONSE

- The response s of a system $\mathcal{H}$ to the input u is called the **step response** of the system (i.e., $s = \mathcal{H}u$).
- The impulse response h and step response s of a LTI system are related as

$$h(t) = \frac{ds(t)}{dt}.$$

- Therefore, the impulse response of a system can be determined from its step response by differentiation.
- The step response provides a practical means for determining the impulse response of a system.

EXAMPLE

Consider a LTI system $\mathcal{H}$ with impulse response

$$h(t) = u(t).$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Solution. Since the system is LTI, we have that

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t). \\
&= x * u(t) \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)u(t-\tau)d\tau + \int_{t^+}^{\infty} x(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)d\tau.
\end{aligned}$$

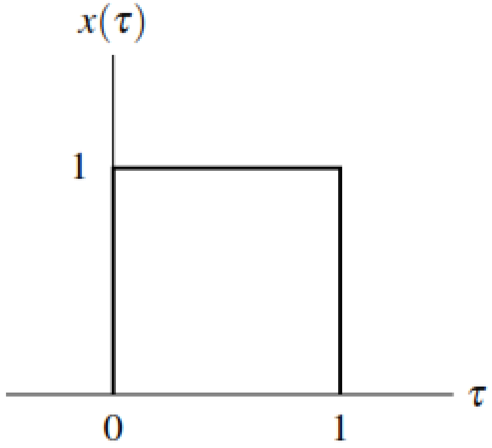
EXAMPLE

Consider a LTI system $\mathcal{H}$ with impulse response h, where

$$h(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response y of the system to the input x given by

$$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$



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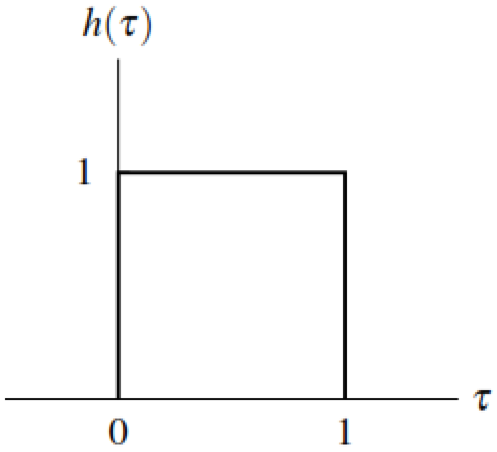

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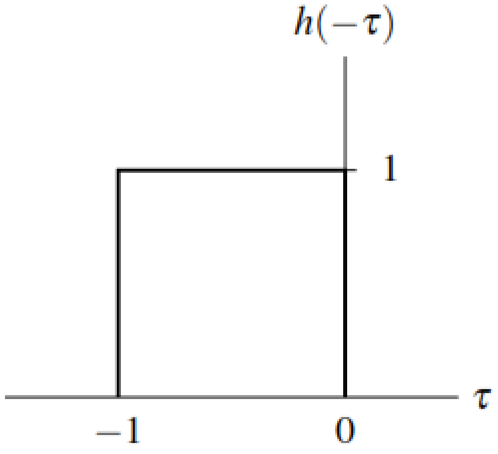

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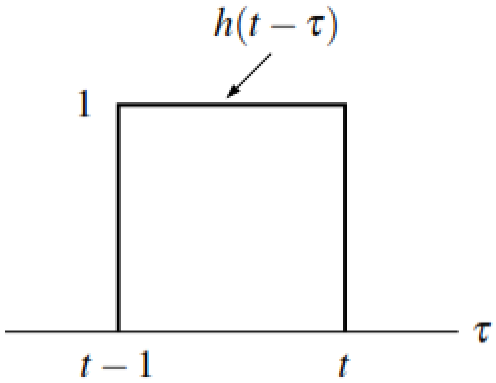


(d)

EXAMPLE

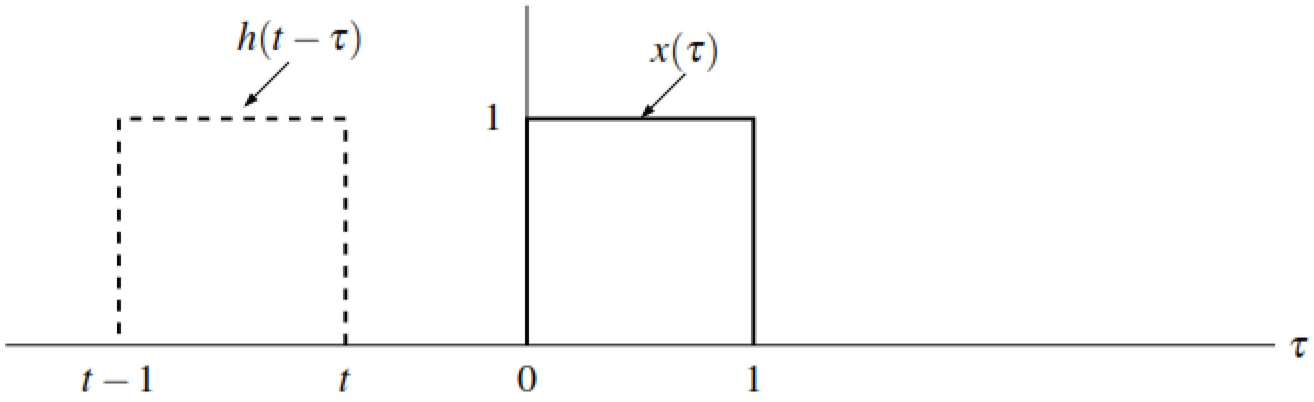
Solution.

Since the system is LTI, we know that

$$y(t) = x * h(t).$$

First, we consider the case of $t < 0$.

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0.$$

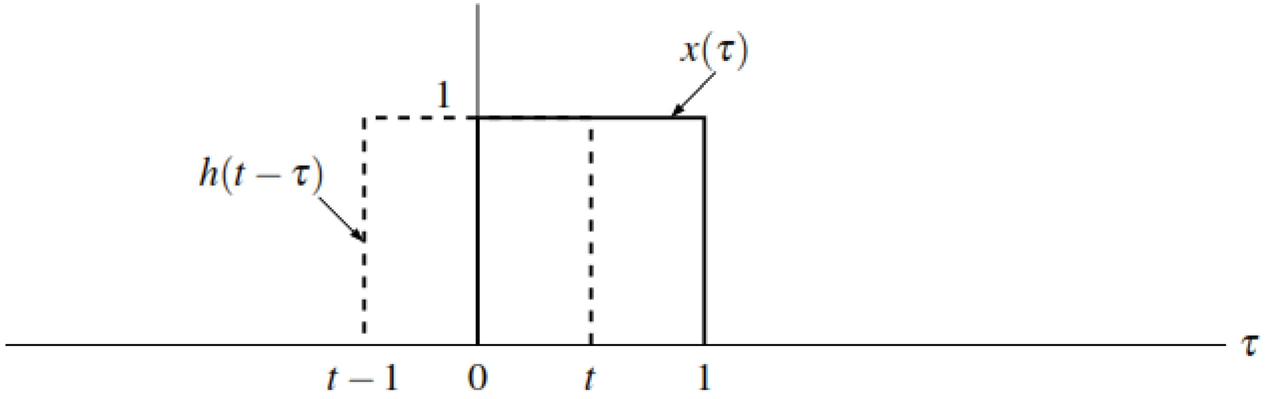


(e)

EXAMPLE

Second, we consider the case of $0 \le t < 1$.

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^t d\tau$$
$$= [\tau]|_0^t$$
$$= t.$$

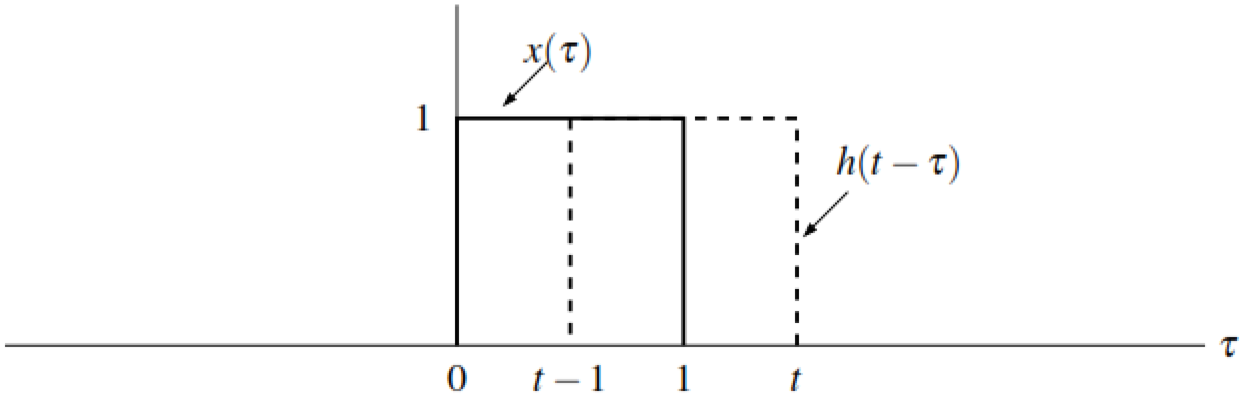


(f)

EXAMPLE

Third, we consider the case of $1 \leq t < 2$.

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{1} d\tau$$
$$= [\tau]|_{t-1}^{1}$$
$$= 1 - (t-1)$$
$$= 2 - t.$$

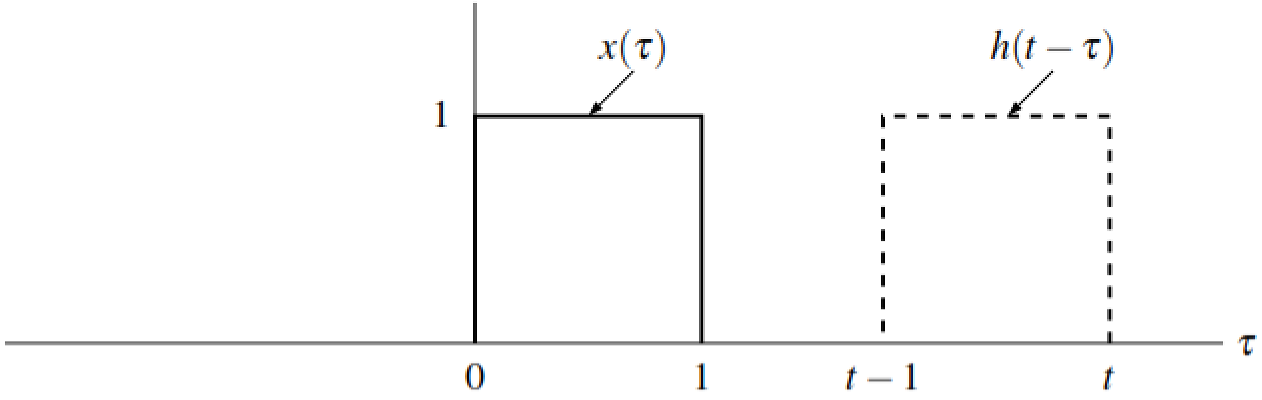


(g)

EXAMPLE

Fourth, we consider the case of $t \geq 2$.

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0.$$

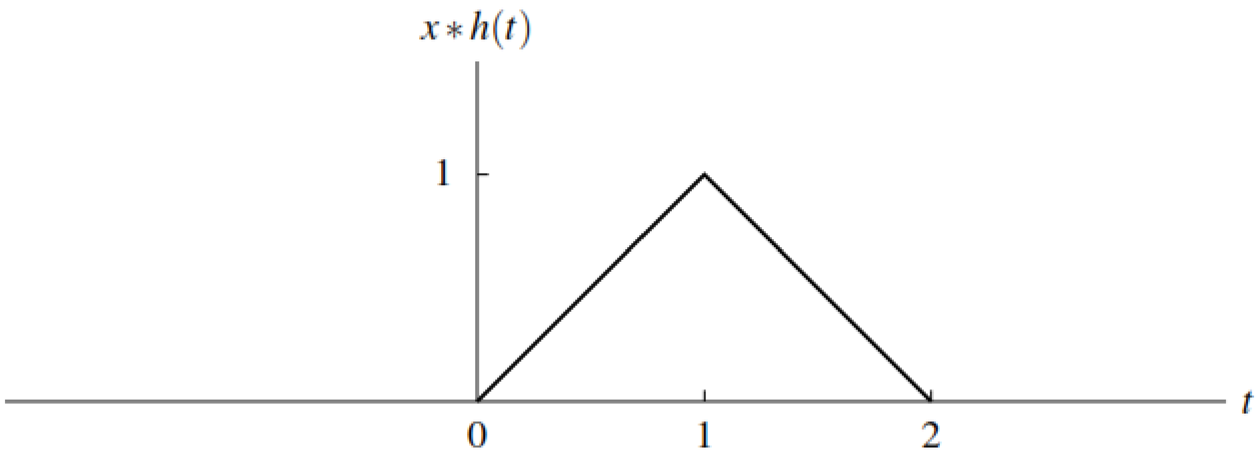


(h)

EXAMPLE

Combining the results

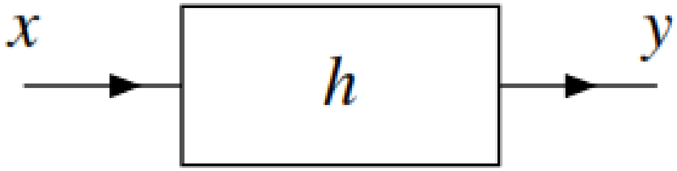
$$x * h(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & t \geq 2. \end{cases}$$



(i)

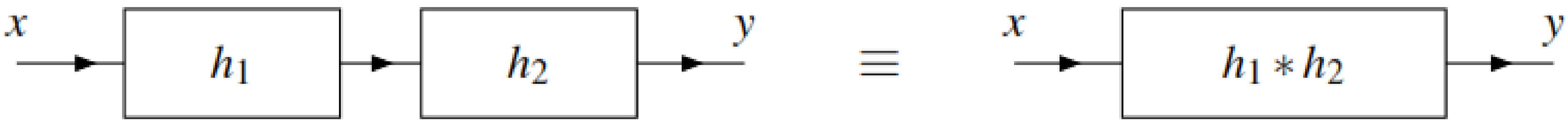
BLOCK DIAGRAM REPRESENTATION OF LTI SYSTEMS

- Often, it is convenient to represent a (CT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.

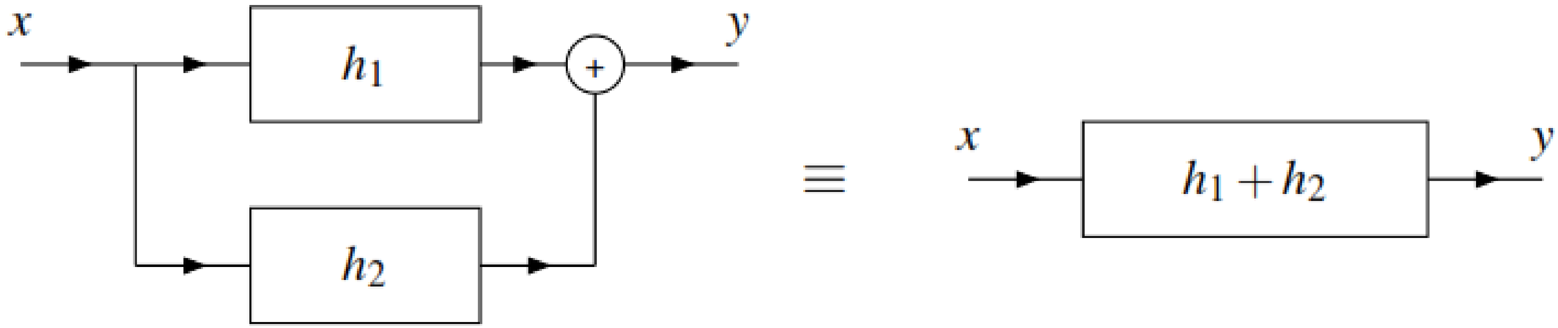


INTERCONNECTION OF LTI SYSTEMS

- The ***series*** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h_1 * h_2$. That is, we have the equivalence shown below.



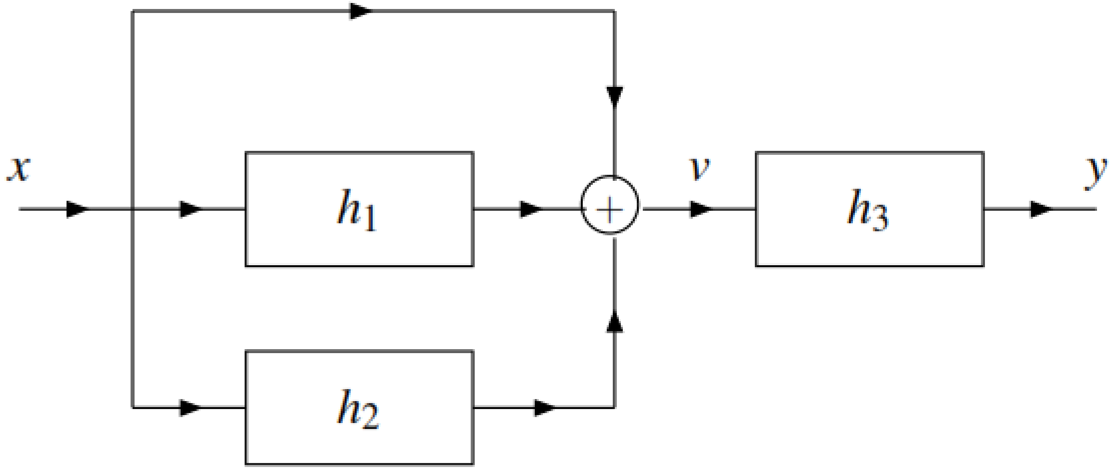
- The ***parallel*** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h_1 + h_2$. That is, we have the equivalence shown below.



EXAMPLE

Consider the system with input x, output y, and impulse response h

Find h.



Solution. From the left half of the block diagram, we can write

$$\begin{aligned} v(t) &= x(t) + x * h_1(t) + x * h_2(t) \\ &= x * \delta(t) + x * h_1(t) + x * h_2(t) \\ &= (x * [\delta + h_1 + h_2])(t). \end{aligned}$$

EXAMPLE

Similarly, from the right half of the block diagram, we can write

$$y(t) = v * h_3(t).$$

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y(t) &= v * h_3(t) \\ &= (x * [\delta + h_1 + h_2]) * h_3(t) \\ &= x * [h_3 + h_1 * h_3 + h_2 * h_3](t). \end{aligned}$$

Thus, the impulse response h of the overall system is

$$h(t) = h_3(t) + h_1 * h_3(t) + h_2 * h_3(t).$$

PROPERTIES OF LTI SYSTEMS - MEMORY

- A LTI system with impulse response h is memoryless if and only if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(t) = K\delta(t),$$

  where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

  (i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

PROPERTIES OF LTI SYSTEMS – CAUSALITY

- A LTI system with impulse response h is causal if and only if

$$h(t) = 0 \quad \text{for all } t < 0$$

  (i.e., h is a causal function).

- It is due to the above relationship that we call a function x, satisfying

$$x(t) = 0 \quad \text{for all } t < 0,$$

  a causal function.

EXAMPLE

Consider the LTI system with the impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system has memory.

Determine whether this system is causal.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$).

Clearly, $h(t) = 0$ for $t < 0$ (due to the $u(t)$ factor in the expression for $h(t)$).

PROPERTIES OF LTI SYSTEMS - INVERTIBILITY

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{\text{inv}} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a function h_{inv} such that

$$h * h_{\text{inv}} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

EXAMPLE

Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(t) = A\delta(t - t_0),$$

where A and t_0 are real constants and $A \neq 0$.

Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv}

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\text{inv}} = \delta.$$

$$\begin{aligned}
& h * h_{\text{inv}}(t) = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} h(\tau) h_{\text{inv}}(t - \tau) d\tau = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} A\delta(\tau - t_0) h_{\text{inv}}(t - \tau) d\tau = \delta(t) \\
\Rightarrow \quad & \int_{-\infty}^{\infty} \delta(\tau - t_0) h_{\text{inv}}(t - \tau) d\tau = \tfrac{1}{A}\delta(t).
\end{aligned}$$

EXAMPLE

Using the sifting property of the unit-impulse function,

$$h_{\text{inv}}(t-\tau)\big|_{\tau=t_0} = \tfrac{1}{A}\delta(t)$$

$$\Rightarrow \quad h_{\text{inv}}(t-t_0) = \tfrac{1}{A}\delta(t).$$

Substituting $t+t_0$ for t in the preceding equation yields

$$h_{\text{inv}}([t+t_0]-t_0) = \tfrac{1}{A}\delta(t+t_0) \quad \Leftrightarrow$$

$$h_{\text{inv}}(t) = \tfrac{1}{A}\delta(t+t_0).$$

Since $A \neq 0$, the function h_{inv} is always well defined.

Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible.

PROPERTIES OF LTI SYSTEMS – BIBO STABILITY

- A LTI system with impulse response h is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

(i.e., h is ***absolutely integrable***).

EXAMPLE

Consider the LTI system with impulse response h given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

EXAMPLE

Solution. We need to determine for what values of a the impulse response h is absolutely integrable.

$$
\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\, dt &= \int_{-\infty}^{\infty} \left|e^{at} u(t)\right| dt \\
&= \int_{-\infty}^{0} 0\, dt + \int_{0}^{\infty} e^{at}\, dt \\
&= \int_{0}^{\infty} e^{at}\, dt \\
&= \begin{cases} \int_0^\infty e^{at}\, dt & a \neq 0 \\ \int_0^\infty 1\, dt & a = 0 \end{cases} \\
&= \begin{cases} \left[\frac{1}{a} e^{at}\right]\Big|_0^\infty & a \neq 0 \\ [t]\big|_0^\infty & a = 0. \end{cases}
\end{aligned}
$$

Suppose that $a \neq 0$. We have

$$
\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\, dt &= \left[\tfrac{1}{a} e^{at}\right]\Big|_0^\infty \\
&= \tfrac{1}{a}\left(e^{a\infty} - 1\right).
\end{aligned}
$$

EXAMPLE

Suppose now that $a = 0$. In this case, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\, dt &= [t]|_0^{\infty} \\ &= \infty.\end{aligned}$$

Thus, we have shown that

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases}$$

In other words, the impulse response h is absolutely integrable if and only if $a < 0$.

Consequently, the system is BIBO stable if and only if $a < 0$.

EXAMPLE

Consider the LTI system with input x and output y defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Determine whether this system is BIBO stable.

EXAMPLE

Solution. First, we find the impulse response h of the system. We have

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)\,d\tau \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= u(t). \end{aligned}$$

Using this expression for h, we now check to see if h is absolutely integrable. We have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} |u(t)|\,dt \\ &= \int_{0}^{\infty} 1\,dt \\ &= \infty. \end{aligned}$$

Thus, h is not absolutely integrable. Therefore, the system is not BIBO stable.

EIGENFUNCTIONS OF LTI SYSTEMS

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{e^{st}\}(t) = H(s)e^{st},$$

where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.
- We refer to H as the **system function** (or **transfer function**) of the system $\mathcal{H}$.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

REPRESENTATIONS OF FUNCTIONS USING EIGENFUNCTIONS

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials
$$x(t) = \sum_k a_k e^{s_k t},$$
where the a_k and s_k are complex constants.
- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude
$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$
- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as a linear combination of the *same* complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.