Elements of Probability

June 8, 2023

1 Sample space - Events

1. A box contains three marbles — one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box, then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat for the case in which the second marble is drawn without first replacing the first marble.

2. An experiment involves tossing a pair of dice, one green and one red, and recording the numbers that come up. If x equals the outcome on the green die and y the outcome on the red die

   (a) describe the sample space S

   (b) ist the elements corresponding to the event A that the sum is greater than 8;

   (c) list the elements corresponding to the event B that a 2 occurs on either die;

   (d) list the elements corresponding to the event C that a number greater than 4 comes up on the green die;

   (e) list the elements corresponding to the event AC;

   (f) list the elements corresponding to the event AB;

   (g) list the elements corresponding to the event BC;

   (h) construct a Venn diagram to illustrate the intersections and unions of the events A, B, and C.

3. The rise time of a reactor is measured in minutes (and fractions of minutes). Let the sample space be positive, real numbers. Define the events A and B as follows: $A = \{x|x < 72.5\}$ and $B = \{x|x > 52.5\}$. Describe the following event

   $$a.\, A^c \qquad b.\, B^c \qquad c.\, AB \qquad d.\, A \cup B$$

4. A system is composed of four components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector (x_1, x_2, x_3, x_4) where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

   (a) How many outcomes are in the sample space of this experiment?

   (b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working. Specify all the outcomes in the event that the system works.

   (c) Let E be the event that components 1 and 3 are both failed. How many outcomes are contained in event E ?

2 Axiom of Probability

1. A decision maker subjectively assigned the following probabilities to the four outcomes of an experiment: $P(E_1) = .10$, $P(E_2) = .15$, $P(E_3) = .40$, and $P(E_4) = .20$. Are these probability assignments valid? Explain.

2. The National Highway Traffic Safety Administration (NHTSA) conducted a survey to learn about how drivers throughout the United States are using seat belts (Associated Press, August 25, 2003). Sample data consistent with the NHTSA survey are as follows.

	Driver Using Seat Belt?	
Region	**Yes**	**No**
Northeast	148	52
Midwest	162	54
South	296	74
West	252	48
Total	858	228

(a) For the United States, what is the probability that a driver is using a seat belt?

(b) The seat belt usage probability for a U.S. driver a year earlier was .75. NHTSA chief Dr. Jeffrey Runge had hoped for a .78 probability in 2003. Would he have been pleased with the 2003 survey results?

(c) What is the probability of seat belt usage by region of the country? What region has the highest seat belt usage?

(d) What proportion of the drivers in the sample came from each region of the country? What region had the most drivers selected? What region had the second most drivers selected?

(e) Assuming the total number of drivers in each region is the same, do you see any reason why the probability estimate in part (a) might be too high? Explain

3. A box contains 500 envelopes, of which 75 contain \$100 in cash, 150 contain \$25, and 275 contain \$10. An envelope may be purchased for \$25. What is the sample space for the different amounts of money? Assign probabilities to the sample points and then find the probability that the first envelope purchased contains less than \$100.

4. If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, what is the probability that

(a) the dictionary is selected?

(b) 2 novels and 1 book of poems are selected?

3 Additive Rule

1. If $P(A) = 0.3$, $P(B) = 0.2$ and $P(AB) = 0.1$, determine the following probabilities

$$a., P(A^c) \qquad b. P(AB^c) \qquad c. P(A^cB) \qquad d. P[(A \cup B)^c] \qquad e. P(A \cup B^c).$$

2. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized below:

		Shock resistance	
		high	*low*
scratch	*high*	70	9
resistance	*low*	16	5

Let A denote the event that a disk has high shock resistance, and let B denote the event that a disk has high scratch resistance. Compute probability of AB, A^c and $A \cup B$.

3. If each coded item in a catalog begins with 3 distinct letters followed by 4 distinct nonzero digits, find the probability of randomly selecting one of these coded items with the first letter a vowel and the last digit even.

4. It is common in many industrial areas to use a filling machine to fill boxes full of product. This occurs in the food industry as well as other areas in which the product is used in the home, for example, detergent. These machines are not perfect, and indeed they may A, fill to specification, B, underfill, and C, overfill. Generally, the practice of underfilling is that which one hopes to avoid. Let $P(B) = 0.001$ while $P(A) = 0.990$.

   (a) Give $P(C)$.

   (b) What is the probability that the machine does not underfill?

   (c) What is the probability that the machine either overfills or underfills?

4 Conditional probability - Multiplication rule

1. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized below:

		Shock resistance	
		high	*low*
scratch	*high*	70	9
resistance	*low*	16	5

   (a) If a disk is selected at random, what is the probability that its scratch resistance is high and its shock resistance is high?

   (b) If a disk is selected at random, what is the probability that its scratch resistance is high given that its shock resistance is high?

2. A total of 600 of the 1,000 people in a retirement community classify themselves as Republicans, while the others classify themselves as Democrats. In a local election in which everyone voted, 60 Republicans voted for the Democratic candidate, and 50 Democrats voted for the Republican candidate. If a randomly chosen community member voted for the Republican, what is the probability that she or he is a Democrat?

3. We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

   (a) Find the probability that doubles are rolled.

   (b) Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.

   (c) Find the probability that at least one die roll is a 6.

   (d) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

4. Toss a fair 6-sided dice twice. Let X and Y be the result of the first and second toss. Find $P(A|B)$ where

$$A = \{\max(X, Y) = 5\}$$

   and

$$B = \{\min(X, Y) = 3\}.$$

5. Suppose there are two electrical components. The chance that the first component fails is 10%. If the first component fails, the chance that the second component fails is 20%. But if the first component works, the chance that the second component fails is 5%. Calculate the probabilities of the following events:

   (a) at least one of the components works;

   (b) exactly one of the components works;

   (c) the second component works.

6. **Radar detection.** If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of false alarm (a false indication of aircraft presence), and the probability of missed detection (nothing registers, even though an aircraft is present)?

5 Independence

1. A batch of 500 containers for frozen orange juice contains five that are defective. Two are selected, at random, without replacement, from the batch. Let A and B denote the events that the first and second containers selected are defective, respectively. Are A and B independent events?

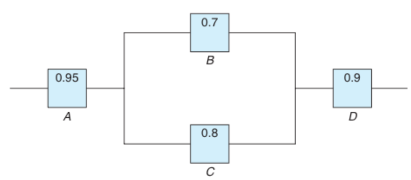
2. The probability that a lab specimen contains high levels of contamination is 0.10. Five samples are checked, and the samples are independent.

   (a) What is the probability that none contains high levels of contamination?

   (b) What is the probability that exactly one contains high levels of contamination?

   (c) What is the probability that at least one contains high levels of contamination?

3. A circuit system is given in the following figure



   Assume the components fail independently.

   (a) What is the probability that the entire system works?

   (b) Given that the system works, what is the probability that the component B is not working?

6 Total probability - Bayes's formula

1. Suppose $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$. What is $P(A)$?

2. A paint-store chain produces and sells latex and semigloss paint. Based on long-range sales, the probability that a customer will purchase latex paint is 0.75. Of those that purchase latex paint, 60% also purchase rollers. But only 30% of semigloss paint buyers purchase rollers. A randomly selected buyer purchases a roller and a can of paint. What is the probability that the paint is latex?

3. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

4. Suppose that there are two boxes, labeled odd and even. The odd box contains three balls numbered 1, 3, 5. The even box contains two balls labeled 2, 4. One of the boxes is picked at random by tossing a fair coin. Then a ball is picked at random from this box. What is the probability that the ball drawn is ball 3?

5. **(False positive)** Suppose that a laboratory test on a blood sample yields one of two results, positive or negative. It is found that 95% of people with a particular disease produce a positive result. But 2% of people without the disease will also produce a positive result (a false positive). Suppose that 1% of the population actually has the disease. What is the probability that a person chosen at random from the population will have the disease, given that the person's blood yields a positive result?