Chapter 1: Introduction to Digital Signal Processing

1. Signals

Continuous time (CT) signal: $x(t)$.

Discrete time (DT) signal: $x[n]$.

Causal signals are signals (CT) which only have value when $t \geq 0$, or

$$x(t) = 0, \forall t < 0$$

Causal signals for DT are similar to CT.

2. Energy and Power

	Continuous time	Discrete time
Periodic	$E_x = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$ $P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \lvert x(t)\rvert^2 dt$	$E_x = \sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2$ $P_x = \frac{1}{N_0}\sum_{n=0}^{N_0-1} \lvert x[n]\rvert^2$
Aperiodic	$E_x = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$ $P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \lvert x(t)\rvert^2 dt$	$E_x = \sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2$ $P_x = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} \lvert x[n]\rvert^2$

Note that:

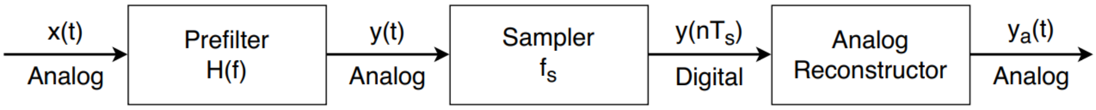
1. A non-periodic signal maybe energy signal. If $E_x < M, M$ is finite, then the signal is called as energy signal.
2. A periodic signal maybe power signal. If $P_x < M, M$ is finite, then the signal is called as power signal.
3. If a signal is summation of sine signal with amplitude $A_i, i = 0, .., n$ and cosine signal with amplitude $B_i, i = 0, \dots, m$, then the power of this signal is given by

$$P_x = \sum_{i=0}^{n} \frac{A_i^2}{2} + \sum_{i=0}^{m} \frac{B_i^2}{2}$$

Chapter 2: Sampling and Reconstruction

1. Overview

Given the input signal if form of summation of its frequency components $x(t) = x_1(t) + \cdots + x_i(t) \ldots + x_3(t)$. Then the sampling and reconstruction procress follows the below figure



2. Prefiter Process

Normally, there are 3 types of filter which are used in prefilter process

1. No prefilter: $y(t) = x(t)$
2. Ideal low pass filter with cut off frequency f_c:

$$\begin{cases} y_i(t) = x_i(t), & \text{if } f_i \le f_c \\ y_i(t) = 0, & \text{if } f_i > f_c \end{cases}$$

3. Practical low pass filter with cut off frequency f_c(To easier we make the assumption $y_i(t) = x_i(t),\ f_i \le f_c$):

$$\begin{cases} y_i(t) = x_i(t), & \text{if } f_i \le f_c \\ y_i(t) = \dfrac{x_i(t)}{A_i}, & \text{if } f_i > f_c \end{cases}$$

Where $A_i = n_{oc} \times A_{oc} = n_{od} \times A_{od}$ is the attenuation of signal at i-th frequency component. In more detail:

- $n_{oc} = \log_2(f_i/f_c)$ (octave) is the number of octave from f_i to f_c.
- A_{oc} (dB/octave) is the attenuation of the filter after cut off frequency.
- $n_{od} = \log_{10}(f_i/f_c)$ (decade) is the number of decade from f_i to f_c.
- A_{od} (dB/decade) is the attenuation of the filter after cut off frequency.

3. Sampling Process

The process of sampling the CT signal at rate f_s is the process of taking value of original signal each period time of $T_s = 1/f_s$ or

$$t = nT_s \to x[n] = x(nT_s) = x(t)$$

To fully reconstruct the signal $x(t)$ must be band limited signal, the sampling ratef_s should choose follow the Nyquist theorem, that is, $f_s \ge 2f_{\max}$.

4. Reconstruction Process

The Nyquist interval (NI):

$$\mathrm{NI} = \left[-\frac{f_s}{2}, \frac{f_s}{2}\right]$$

If the i-th frequency component of the signal **belongs to NI** then the analog reconstructed frequency is $f_{ia} = f_i$. If the i-th frequency component of the signal is **beyond the NI** then the analog reconstruct ted frequency is $f_{ia} = f_i \bmod(f_s)$.

Chapter 3: Quantization Process

1. Parameters

Analog signal range $[x_{\min}, x_{\max}]$

$$R = x_{\max} - x_{\min}$$

Quantization bit

$$B \to 2^B \text{ posible values}$$

Quantization resolution

$$Q = \frac{R}{2^B}$$

Mean error (Expectation error) $(E(e))$

$$\bar{e} = \int_{-\infty}^{+\infty} ep(e)de$$

Second moment error $(E(e^2))$

$$\overline{e^2} = \int_{-\infty}^{+\infty} e^2 p(e)de$$

RMS error

$$e_{\text{rms}} = \sqrt{\overline{e^2}}$$

Noise variance or average noise power

$$\sigma_e^2 = \overline{e^2} - \bar{e}^2$$

Normalized Signal to Noise Ratio

$$\text{SNR} = 20 \log_{10}\left(\frac{R}{Q}\right) = 6.02 \times B \qquad \text{(dB)}$$

Non Normalized Signal to Quantization Noise Ratio

$$\text{SQNR} = 6.02 \times B + 4.81 - 20 \log_{10}\left(\frac{X_{\max}}{\sigma_x}\right) \qquad \text{(dB)}$$

2. Over-Sampling and Noise Shaping

2. 1. Over-Sampling without Using Noise Shaping

Over-sampling ratio

$$L = \frac{f_s'}{f_s} = 2^{2(B-B')}$$

Bit reduce

$$\Delta B = 0.5 \log_2 L$$

2. 2. Over-Sampling with p-th order Noise Shaping Filter

Bit reduce

$$\Delta B = (p + 0.5)\log_2 L - 0.5\log_2\left(\frac{\pi^{2p}}{2p+1}\right)$$

3. DAC/ADC

Given a sequence with B bits input, that is, $b = [b_1, b_2, \dots, b_B]$ the DAC will convert this sequence to quantized signal x_Q.

3. 1. Conversion Types

Type	**Relationship**
Natural Binary	$x_Q = R(b_1 2^{-1} + b_2 2^{-2} + \cdots + b_B 2^{-B})$
Offset Binary	$x_Q = R(b_1 2^{-1} + b_2 2^{-2} + \cdots + b_B 2^{-B} - 0.5)$
2's Complement	$x_Q = R(\overline{b}_1 2^{-1} + b_2 2^{-2} + \cdots + b_B 2^{-B} - 0.5)$

3. 2. Conversion Code Table

Let m be the decimal value corresponding with a binary value $m = b_1 2^{B-1} + b_2 2^{B-2} + \cdots + b_B$ and $m' = m - 2^{B-1}$. Then the conversion code table is built as follows

$b_1 b_2 \dots b_B$	Natural		Offset		2's C
	m	x_Q	m'	x_Q	$b_1 b_2 \dots b_B$
$-$	2^B	R	2^{B-1}	$R/2$	$-$
$11 \dots 1$	$2^B - 1$	Qm	$2^{B-1} - 1$	Qm'	$01 \dots 1$
$11 \dots 0$	$2^B - 2$	Qm	$2^{B-1} - 2$	Qm'	$01 \dots .0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$00 \dots 1$	1	Qm	$-2^{B-1} + 1$	Qm'	$10 \dots 1$
$00 \dots 0$	0	0	-2^{B-1}	$-R/2$	$10 \dots 0$

Chapter 4: Analysis of LTI Systems

1. System Classification by Energy

Type	Relationship
Passive system	$E_y < E_x$
Lossless system	$E_y = E_x$
Active system	$E_y > E_x$

2. Properties of LTI System

2. 1. Causality

A causal system is a system that output $y(t)$ only depends on present and past value of input $x(t)$.

2. 2. Linearity

A system $\boldsymbol{S}$ is called linear system if and only if it satisfies the condition

$$\boldsymbol{S}\{a_1x_1 + a_2x_2\} = a_1y_1 + a_2y_2$$

Check for linearity:

- Step 1:

$$\begin{cases} x_1 \xrightarrow{s} y_1 = \boldsymbol{S}\{x_1\} \\ x_2 \xrightarrow{s} y_2 = \boldsymbol{S}\{x_2\} \end{cases} \to a_1y_1 + a_2y_2 = ?\,(1)$$

- Step 2:

$$x = a_1x_1 + a_2x_2 \xrightarrow{s} y = \boldsymbol{S}\{a_1x_1 + a_2x_2\} = ?\,(2)$$

- Step 3:

Compare (1) and (2), if it equals, conclude that the system is linear.

2. 3. Time Invariant

A time-varying system is one whose parameters vary with time.
Check for time invariant:

- Step 1:

$$x[n] \xrightarrow{s} [n] = \boldsymbol{S}\{x\}$$

Calculate: $y[n-D]$ (1) (delay the ouput).

- Step 2:

$$x_D[n] = x[n-D] \xrightarrow{s} y_D[n] = \boldsymbol{S}\{x[n-D]\} = ?\,(2)$$

- Step 3:

Compare (1) and (2), if it equals, conclude that the system is time invariant.

The checking process is similar for CT system
Bounded-input Bounded-output (BIBO) Stable

2. 4. BIBO Stable System (Stability)

If the system has bounded for all input ($|x(t)| < M, M$ is finite) which leads to all output is bounded ($|y(t)| < N, N$ is finite) then the system is said to be BIBO system.

Check for BIBO system: Assume that $|x(t)| < M, \forall t, M$ is finite. Calculate $|y(t)|$, if we can prove that $|y(t)| < N, N$ is finite, we can conclude that the system is BIBO system.

Discrete time BIBO System: If the impulse response of the discrete time system $h[n]$ is absolutely integrable, the system is said to be BIBO stable.

$$\sum_{n=-\infty}^{+\infty} |h[n]| < M, M \text{ is finite}$$

3. I/O Relationship

3. 1. Impulse Response

When $x[n] = \delta[n]$, the output or the response of the system is called impulse response or $y[n] = h[n]$.

3. 2. Difference Equation

$$y_n + \sum_{i=1}^{k} a_i y_{n-i} = \sum_{j=1}^{l} b_j x_{n-j}, \qquad k \geq l$$

3. 3. Block Diagram