SOLUTIONS OF FINAL EXAM

Subject: CALCULUS 2

Question 1. $f(x,y) = xe^{x+y}$

a)

$$f_x = e^{x+y} + xe^{x+y}, \quad f_y = xe^{x+y}, \quad f_x(1,-1) = 2, \quad f_y(1,-1) = 1$$

Equation of tangent plane to the graph of $f(x,y) = xe^{x+y}$ at the point $(1,-1,1)$:

$$z = 1 + 2(x-1) + (y+1) = 2x + y$$

b) It follows from the part a) that $D_{\mathbf{u}}f(1,-1) =< 2,1 >$. So

$$D_{\mathbf{u}}f(1,-1) =< 2,1 > \cdot < 1/\sqrt{2}, 1/\sqrt{2} >= \frac{3}{\sqrt{2}}$$

Question 2. Set $g(x,y,z) = x^2 + y^2 + z^2$. Then

$$f_x = 1, f_y = 1, f_z = -1, g_x = 2x, g_y = 2y, g_z = 2z.$$

Method of Lagrange multipliers read

$$\nabla f = \lambda \nabla g, \quad g(x,y,z) = 1,$$

or

$$1 = 2x\lambda, \quad 1 = 2y\lambda, \quad -1 = 2z\lambda, \quad x^2 + y^2 + z^2 = 1.$$

This implies $\lambda \neq 0$ and

$$x = y = -z$$

so that

$$x^2 + y^2 + z^2 = 1 = 3x^2.$$

This yields $x = \pm 1/\sqrt{3}$, and this step gives us two points

$$A(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}), B(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}).$$

and $f(A) = \sqrt{3}$: max, $\quad f(B) = -\sqrt{3}$: min.

Question 3. a) Let $f(x,y) = xe^{y^2/10}$. The upper right corner points: $(x_i, y_j) = (2i, 2j), i = 1,2,3; j = 1,2.$

$$\Delta x = 2, \Delta y = 2, \Delta V = 4$$

The volume of the solid can be estimated as

$$\begin{aligned} V &\approx \sum_{i=1}^{3}, \sum_{j=1}^{2} f(x_i, y_j)\Delta V = [f(2,2) + f(4,2) + f(6,2) + f(2,4) + f(4,4) + f(6,4)]4 \\ &= 4(2e^{4/10} + 4e^{4/10} + 6e^{4/10} + 2e^{16/10} + 4e^{16/10} + 6e^{16/10}) \\ &= 48(e^{2/5} + e^{8/5}) = 309.3531 \end{aligned}$$

b) Since the solid lies above xy-plane, $z = 2x^2y \geq 0$. This implies $y \geq 0$ and the solid is in fact lies above the domain D which can be expressed as a region of type II:

$$D = \{(x, y)|0 \leq y \leq 1, 0 \leq x \leq \sqrt{1 - y^2}\}$$

This is a quarter-circular region in the xy-plane and it is convenient to change into polar coordinates:

$$x = r\cos t, \quad y = r\sin t, \quad 0 \leq r \leq 1, \quad 0 \leq t \leq \pi/2.$$

Thus, the volume is given by

$$\begin{aligned} V &= \iint_D (2x^2y)dA = \int_0^1 \int_0^{\pi/2} (2(r\cos t)^2(r\sin t)rdtdr \\ &= \int_0^1 (2/3)r^4(-\cos^3)t\Big|_0^{\pi/2} dr \\ &= \int_0^1 (2/3)r^4 dr = (2/3)r^5/5\Big|_0^1 = 2/15. \end{aligned}$$

Question 4. a) $\mathbf{F}(x, y) =< P, Q >=< x + y^2, 2xy >$ is defined on the entire $\boldsymbol{R}^2$. It holds that $P_y = 2y = Q_x$. So $\mathbf{F}(x, y)$ is conservative.

b) Let $\mathbf{F}(x, y) =< x + y^2, 2xy >= \nabla f(x, y)$. Then

$$f_x = x + y^2, f_y = 2xy$$

Integrate the first equation with respect to x to get

$$f(x, y) = \int (x + y^2)dx = x^2/2 + xy^2 + g(y).$$

Differentiating the last function with respect to y, and comparing the result with $f_y = 2xy$ gives us

$$f_y = 2xy + g'(y) = 2xy.$$

This yields $g'(y) = 0$ and so $g(y) = C$, where C is constant.

Thus $f(x, y) = x^2/2 + xy^2 + C$

c)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = 1/2 + 1 = 3/2$$

Question 5. a) $\mathbf{F}(x, y, z) = e^{x+2y}\mathbf{i} + (x - y)\mathbf{j} + (y + 3z)\mathbf{k}$

$$\begin{aligned} \operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+2y} & (x - y) & (y + 3z) \end{vmatrix} \\ &=< 1, 0, 1 - 2e^{x+2y} > \end{aligned}$$

and

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = e^{x+2y} - 1 + 3 = e^{x+2y} + 2$$

b) $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = 2x\mathbf{i} + \mathbf{j} + z\mathbf{k}$, and S is the surface $g(x, y) = (x + 1)e^y, 0 \leq x \leq 1, 0 \leq y \leq 1$ with upward orientation.

$$g_x = e^y, g_y = (x+1)e^y.$$

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D (-Pg_x - Qg_y + R)dA \\
&= \iint_D (-2xe^y - (x+1)e^y + (x+1)e^y)dA \\
&= \int_0^1 \int_0^1 (-2xe^y)dxdy \\
&= \int_0^1 (-x^2e^y)\Big|_0^1 dy \\
&= \int_0^1 (-e^y)dy \\
&= (-e^y)\Big|_0^1 = 1 - e
\end{aligned}$$