Nonhomogeneous systems of linear differential equations

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1. General solution of nonhomogeneous systems

Theorem: Let X_p be a particular solution to the nonhomogeneous system

$$X'(t) = A(t)X(t) + F(t) \tag{0.1}$$

on the interval I, and let $\{X_1, X_2, .., X_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $X'(t) = A(t)X(t)$. Then every solution to (0.1) on I can be expressed in the form

$$c_1X_1(t) + c_2X_2(t) + \cdots + c_nX_n(t) + X_p(t), \tag{0.2}$$

where $c_1, c_2, ..., c_n$ are constants.

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where $c_1, c_2, ..., c_n$ are constants.

The linear combination of $X_1, X_2, .., X_n, X_p$:

$$X(t) = c_1X_1(t) + c_2X_2(t) + \cdots + c_nX_n(t) + X_p(t),$$

with arbitrary constants $c_1, c_2, ..., c_n$ is called the **general solution** of (0.1).

2. Particular solution of nonhomogeneous linear differential systems

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How to find a particular solution of the nonhomogeneous linear differential system (0.3)?

1. **Method of variation of parameters**.

This method can be applied to any nonhomogeneous linear differential systems.

2. **Method of undetermined coefficients**

This method can be applied to a nonhomogeneous linear differential system of the form:

$$X'(t) = AX(t) + F(t) \tag{0.4}$$

where $A \in \mathbb{R}^{n\times n}$ is a constant matrix and the entries of $F(t)$ are constants, polynomials, exponential functions, sines, cosines, or finite sums and products of these functions.

2.1 Method of variation of parameters

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Let $\{X_1(t), X_2(t), .., X_n(t)\}$ be a fundamental solution set on $I = (a, b)$ for the corresponding homogeneous system

$$X'(t) = A(t)X(t). \tag{0.6}$$

Define $M(t) := [X_1(t)\ X_2(t)\ \dots\ X_n(t)]$. Then $M(t)$ is said to be a fundamental matrix solution of (0.6). The general solution of (0.6) is now given by

$$X(t) = M(t)C,\ \ t \in (a, b), \qquad C := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n. \tag{0.7}$$

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We seek a solution of (0.5) in the form $X_p(t) = M(t)C(t)$, where $C(t)$ is now a vector function of t.

2.1 Method of variation of parameters

Let $X_p(t) = M(t)C(t)$. Note that $X_p'(t) = M'(t)C(t) + M(t)C'(t)$.
Since $X_p(t)$ is a solution of

$$X'(t) = A(t)X(t) + F(t) \tag{0.8}$$

it follows that $X_p'(t) = A(t)X_p(t) + F(t)$. That is,

$$A(t)M(t)C(t) + M(t)C'(t) = A(t)M(t)C(t) + F(t).$$

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Therefore, $M(t)C'(t) = F(t)$.

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Therefore, $M(t)C'(t) = F(t)$. Since $M(t)$ is a fundamental matrix solution, it follows that $\det M(t) \neq 0, \forall t \in (a, b)$. Thus,

$$C'(t) = M(t)^{-1}F(t).$$

Then, we have

$$C(t) = \int M(t)^{-1}F(t)dt,$$

and

$$\boxed{X_p(t) = M(t)\int M(t)^{-1}F(t)dt}.$$

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Thus, the **general solution** of the nonhomogeneous system

$$X(t) = c_1X_1(t) + c_2X_2(t) + \cdots + c_nX_n(t) + X_p(t),$$

now becomes

$$\boxed{X(t) = M(t)C + M(t)\int M(t)^{-1}F(t)dt},$$

where $C := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$, is an arbitrary constant vector.

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Remark: The solution of the initial value problem:

$$X'(t) = A(t)X(t) + F(t), \qquad X(t_0) = X_0,$$

is given by

$$\boxed{X(t) = M(t)M(t_0)^{-1}X_0 + M(t)\int_{t_0}^{t} M(\tau)^{-1}F(\tau)d\tau}.$$

This equation is called the variation of parameters formula for linear differential systems.

Example:

Solve the system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & 1 \\ 0 & \frac{2}{t} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

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Solution: First, we find a fundamental solution set for the corresponding homogeneous system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & 1 \\ 0 & \frac{2}{t} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \tag{0.9}$$

Solving (0.9) yields

$$x_1(t) = c_1 t + c_2\left(\frac{t^3}{2} - \frac{t}{2}\right); \quad x_2(t) = 0c_1 + c_2 t^2.$$

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Solving (0.9) yields

$$x_1(t) = c_1 t + c_2(\frac{t^3}{2} - \frac{t}{2}); \quad x_2(t) = 0c_1 + c_2 t^2.$$

In other words, $\{X_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}; \quad X_2(t) = \begin{pmatrix} \frac{t^3}{2} - \frac{t}{2} \\ t^2 \end{pmatrix}\}$ forms a fundamental solution set of (0.9). So the fundamental matrix solution of (0.9) is given by:

$$M(t) := [X_1(t)\ X_2(t)] = \begin{pmatrix} t & \frac{t^3}{2} - \frac{t}{2} \\ 0 & t^2 \end{pmatrix}$$

Let $F(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix}$. To find the general solution,

$$X(t) = M(t)C + M(t)\int M(t)^{-1}F(t)dt,$$

we calculate $M(t)^{-1}$. Recall that $M(t) := \begin{pmatrix} t & \frac{t^3}{2} - \frac{t}{2} \\ 0 & t^2 \end{pmatrix}$. Then we have

$$M(t)^{-1} = \begin{pmatrix} \frac{1}{t} & \frac{1}{2t^2} - \frac{1}{2} \\ 0 & \frac{1}{t^2} \end{pmatrix}.$$

Thus,

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} =$$

$$\begin{pmatrix} t & \frac{t^3}{2} - \frac{t}{2} \\ 0 & t^2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} t & \frac{t^3}{2} - \frac{t}{2} \\ 0 & t^2 \end{pmatrix}\int \begin{pmatrix} \frac{1}{t} & \frac{1}{2t^2} - \frac{1}{2} \\ 0 & \frac{1}{t^2} \end{pmatrix}\begin{pmatrix} t \\ t^2 \end{pmatrix} dt.$$

Example:

Find the solution of the initial value problem

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & 1 \\ 0 & \frac{2}{t} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix}, \quad \begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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Solution: The solution of the initial value problem is given by

$$X(t) = M(t)M(t_0)^{-1}X_0 + M(t)\int_{t_0}^{t} M(\tau)^{-1}F(\tau)d\tau,$$

where $t_0 := 2; X_0 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $F(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix}$.

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$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & 1 \\ 0 & \frac{2}{t} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix}, \quad \begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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where $t_0 := 2; X_0 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $F(t) := \begin{pmatrix} t \\ t^2 \end{pmatrix}$. Recall that $M(t) := \begin{pmatrix} t & \frac{t^3}{2} - \frac{t}{2} \\ 0 & t^2 \end{pmatrix}$ and $M(t)^{-1} = \begin{pmatrix} \frac{1}{t} & \frac{1}{2t^2} - \frac{1}{2} \\ 0 & \frac{1}{t^2} \end{pmatrix}$. Thus

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{t}{2} & \frac{t^3}{8} - \frac{t}{2} \\ 0 & \frac{t^2}{4} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \int_2^t \begin{pmatrix} \frac{t}{\tau} & \frac{t^3}{2\tau^2} - \frac{t}{2} \\ 0 & \frac{t^2}{\tau^2} \end{pmatrix}\begin{pmatrix} \tau \\ \tau^2 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} \frac{t^4}{3} - \frac{7t^3}{8} + t^2 - \frac{2t}{3} \\ t^3 - \frac{9t^2}{4} \end{pmatrix}.$$

2.2 Method of undetermined coefficients

Consider the nonhomogeneous linear time-invariant system

$$X'(t) = AX(t) + F(t) \tag{0.10}$$

where $A \in \mathbb{R}^{n\times n}$ is a constant matrix and $F(t)$ is a vector-valued function whose entries are constants, polynomials, exponential functions, sines, cosines, or finite sums and products of these functions.

We consider two special cases:

a) $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix (i.e. 0 is not a root of the characteristic equation) and $F(t)$ is a vector of polynomials in t:

$$F(t) := t^k X_k + t^{k-1} X_{k-1} + \dots + X_0,$$

where $X_0, X_1, \dots, X_k \in \mathbb{R}^n$ are constant vectors.

b) $F(t) := e^{rt} B_0$, where $B_0 \in \mathbb{R}^n$ is a constant vector and r is not a root of the characteristic equation:

$$\det(rI_n - A) \neq 0.$$

2.2 Method of undetermined coefficients

Consider the nonhomogeneous linear time-invariant system

$$X'(t) = AX(t) + F(t) \tag{0.11}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix (i.e. 0 is not a root of the characteristic equation) and $F(t)$ is a vector of polynomials in t:

$$F(t) := t^k X_k + t^{k-1} X_{k-1} + ... + X_0,$$

where $X_0, X_1, ..., X_k \in \mathbb{R}^n$ are constant vectors.

In this case, we can find a particular solution of (0.11) in the form

$$X_p(t) := t^k C_k + t^{k-1} C_{k-1} + ... + C_0,$$

where $C_0, C_1, ..., C_k \in \mathbb{R}^n$ are constant vectors.

Example:

Find a particular solution of

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

Example:

Find a particular solution of

$$\frac{d}{dt}\left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right) = \left(\begin{array}{cc} 1 & 1 \\ -1 & 2 \end{array}\right)\left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right) + \left(\begin{array}{c} t \\ t^2 \end{array}\right).$$

Solution: Let $F(t) := \left(\begin{array}{c} t \\ t^2 \end{array}\right)$. Then $F(t)$ can be represented in the form

$$F(t) = t^2\left(\begin{array}{c} 0 \\ 1 \end{array}\right) + t\left(\begin{array}{c} 1 \\ 0 \end{array}\right).$$

Since 0 is not a root of the characteristic equation, we seek a particular solution in the form

$$X_p(t) := \left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array}\right) = t^2\left(\begin{array}{c} a_1 \\ a_2 \end{array}\right) + t\left(\begin{array}{c} b_1 \\ b_2 \end{array}\right) + \left(\begin{array}{c} c_1 \\ c_2 \end{array}\right)$$

If $X_p(t)$ is a solution of the given system then

$$2t\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\left[t^2\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right] + t^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

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$$t^2\left[\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + t\left[\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

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$$+\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

This implies

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \quad \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Solving the above system, we get

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 \\ -1 \end{pmatrix}; \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore,

$$X_p(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{t^2}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{t}{3}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{1}{9}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2.2 Method of undetermined coefficients

Consider the nonhomogeneous linear time-invariant system

$$X'(t) = AX(t) + e^{rt}B_0, \tag{0.12}$$

where $A \in \mathbb{R}^{n\times n}$ is a constant matrix such that r is not a root of the characteristic equation (i.e. $\det(rI_n - A) \neq 0$) and $B_0 \in \mathbb{R}^n$ is a (given) constant vector.

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We seek a solution of (0.12) the form:

$$X_p(t) = e^{rt}C,$$

where $C \in \mathbb{R}^n$ is a constant vector. If $X_p(t)$ is a solution of (0.12) then

$$re^{rt}C = Ae^{rt}C + e^{rt}B_0.$$

This gives

$$(rI_n - A)C = B_0.$$

Since $rI_n - A$ is invertible, it follows that

$$C = (rI_n - A)^{-1}B_0.$$

Thus,

$$\boxed{X_p(t) = e^{rt}(rI_n - A)^{-1}B_0}.$$

Example:

Find a particular solution of the system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + e^t\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

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Solution: Let $A := \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$, $B_0 := \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $r = 1$. The characteristic equation is given by

$$\det(zI_2 - A) = \det\begin{pmatrix} z-1 & -1 \\ 1 & z-2 \end{pmatrix} = 0.$$

This gives $z = \frac{3 \pm \mathbf{i}\sqrt{3}}{2}$. Since $r = 1$ is not a root of the characteristic equation, a particular solution is given by

$$X_p(t) = e^t(I_n - A)^{-1}B_0 = e^t\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

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Example:

Find a particular solution of the system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix}. \tag{0.13}$$

Solution: We seek particular solutions of the following systems

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.14}$$

and

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{0.15}$$

Let $X_1(t)$ and $X_2(t)$ be solutions of (0.14) and (0.15), respectively. Then $X_1(t) + X_2(t)$ is a particular solution of (0.13).

Do it!

Example:

Determine the form of a particular solution for the system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -2e^{-t} + 1 \\ e^{-t} - 5t + 7 \end{pmatrix}. \quad (0.16)$$

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Solution: Let $F(t) := \begin{pmatrix} -2e^{-t}+1 \\ e^{-t}-5t+7 \end{pmatrix}$. Then $F(t)$ can be rewritten as

$$F(t) := e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

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Since -1 is not a root of the characteristic equation, the form of a particular solution of

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (0.17)$$

is

$$X_1(t) = e^{-t}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Since 0 is not a root of the characteristic equation, the form of a particular solution of

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + t\begin{pmatrix} 0 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (0.18)$$

is

$$X_2(t) := t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

By the super-position principle, the form of a particular solution for the system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

is

$$X_p(t) = e^{-t}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$