DE AY1516 S2

Q1.

Given that:

$$y'' + 2y' + y = e^{-x} \ln x \ \ (*)$$

Characteristic equation of the given DE: $r^2 + 2r + 1 = 0$

$$\rightarrow r_1 = r_2 = -1$$

So, the complement solution is: $y_c = C_1 e^{-x} + C_2 x e^{-x}$ (1)

Multiply both sides of $(*)$ by e^x, we get:

$$\rightarrow y'' e^x + e^x y' + e^x y' + e^{-x} y = \ln x$$

$$\leftrightarrow (y' e^x)' + (y e^x)' = \ln x$$

Integrating both sides, it leads to:

$$\rightarrow y' e^x + y e^x = x \ln x - x + C_1$$

$$\leftrightarrow (y e^x)' = x \ln x - x + C_1$$

Integrating both sides, it leads to:

$$\rightarrow y e^x = \frac{x^2 \ln x}{2} - \frac{3x^2}{4} + C_1 x + C_2$$

$$\leftrightarrow y = \left(\frac{x^2 \ln x}{2} - \frac{3x^2}{4}\right) e^{-x} + C_1 x e^{-x} + C_2 e^{-x} (2)$$

Comparing (1) and (2), we obtain the particular solution:

$$y_p = \left(\frac{x^2 \ln x}{2} - \frac{3x^2}{4}\right) e^{-x}$$

Q2.

Given that: $\qquad x^2 y'' + 10xy' + 8y = 0$ (1), $\qquad x > 0$

Proof that $y_1 = x^{-1}$ is a solution of (1):

We have: $y_1 = x^{-1}; \rightarrow y_1' = -x^{-2} \rightarrow y_1'' = 2x^{-3}$.

We know that y_1 is a solution of $(*)$, therefore substituting y_1 into $(*)$, we get:

$$x^2 . 2x^{-3} + 10x(-x^{-2}) + 8x^{-1} = 0$$

$$\leftrightarrow 0 . x^{-1} = 0 \text{ (Valid)}$$

Thus, $y_1 = x^{-1}$ is a solution of (1)

Given that: $\qquad x^2 y'' + 10xy' + 8y = x^2 + 1$ (2), $\qquad x > 0$

Find the general solution of (2)

To find the another solution of (1), we rewire (1) in the following form:

$$y'' + \frac{10}{x} y' + \frac{8}{x^2} y = 0$$

$$(y'' + p(x) y' + q(x) = 0)$$

The Wronskian determinant for the equation is:

$$\mathrm{W}[y_1, y_2] = C_1 e^{-\int p(x) dx} = C_1 e^{-\int \frac{10}{x} dx}$$

$$\rightarrow \mathrm{W}[y_1, y_2] = C_1 x^{-10}$$

Hence:

$$y_2 = y_1 \left[\int \frac{\mathrm{W}[y_1, y_2]}{y_1^2} dx + C_2\right]$$

$$y_2 = x^{-1} \left[\int \frac{C_1 x^{-10}}{(x^{-1})^2} dx + C_2\right]$$

$$\to y_2 = x^{-1}\left[C_1\frac{x^{-7}}{-7} + C_2\right]$$

$$\to y_2 = C_1\frac{x^{-8}}{-7} + C_2x^{-1}$$

Choose $C_1 = -7, C_2 = 0 \to y_2 = x^{-8}$

Since, the Wronskian determinant different from 0 for all $x > 0$, therefore y_1 and y_2 are linearly independence solution of the homogeneous equation or equation (1).

Thus, the complement solution of the equation (2) is:

$$y_C = C_1y_1 + C_2y_2 = C_1x^{-1} + C_2x^{-8}$$

Assume that y_p has the following form: $y_p = Ax^2 + Bx + C$, y_p must be satisfied equation (2). It holds that:

$$(2) \to x^2.2A + 10x(2Ax + B) + 8(Ax^2 + Bx + C) = x^2 + 1$$

$$\leftrightarrow 30Ax^2 + 18Bx + 8C = x^2 + 1$$

$$\to \begin{cases} 30A = 1 \\ 18B = 0 \\ 8C = 1 \end{cases} \leftrightarrow \begin{cases} A = \frac{1}{30} \\ B = 0 \\ C = \frac{1}{8} \end{cases}$$

Hence, $y_p = \frac{1}{30}x^2 + \frac{1}{8}$

Thus, the general solution of the given differential equation is:

$$\begin{aligned} y_G &= y_c + y_p \\ &= C_1x^{-1} + C_2x^{-8} + \frac{1}{30}x^2 + \frac{1}{8} \end{aligned}$$

Q3.

a) Given that: $y^{(5)} + y^{(4)} - y' - y = x^2 - (x^2 + 1)e^{-x} + 5\sin x$

$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x) + g_3(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y^{(5)} + y^{(4)} - y' - y \\ g_1(x) = x^2 \\ g_2(x) = -(x^2 + 1)e^{-x} \\ g_3(x) = 5\sin x \end{cases}$

Characteristic equation of the given ODE: $r^5 + r^4 - r - 1 = 0$

$$\leftrightarrow (r^2 + i)(r + 1)^2(r - 1) = 0$$

$$\leftrightarrow r_1 = i;\ r_2 = -i;\ r_3 = r_4 = -1;\ r_5 = 1$$

Since the right hand side of the given equation has three terms $g_1(x)$, $g_2(x)$ and $g_3(x)$, therefore the particular solution also has three terms: $y_p = y_{p1} + y_{p2} + y_{p3}$, respectively.

Solve fore y_{p1} from:

$$\mathrm{L}[y_{p1}] = g_1(x) \leftrightarrow {y_{p1}}^{(5)} + {y_{p1}}^{(4)} - {y_{p1}}' - y_{p1} = x^2 \ \ (\alpha = 0)$$

Since, $\alpha = 0$ is not a root of characteristic equation.

Hence, y_{p1} has the following form: $y_{p1} = Ax^2 + Bx + C$

Solve fore y_{p2} from:

$$\mathrm{L}[y_{p2}] = g_2(x) \leftrightarrow {y_{p2}}^{(5)} + {y_{p2}}^{(4)} - {y_{p2}}' - y_{p2} = -(x^2 + 1)e^{-x} \quad (\alpha = -1)$$

Since, $\alpha = -1$ is double root of characteristic equation.

Hence, y_{p2} has the following form: $y_{p2} = x^2e^x(Dx^2 + Ex + F)$

Solve fore y_{p3} from:

$$L[y_{p3}] = g_3(x) \leftrightarrow {y_{p3}}^{(5)} + {y_{p3}}^{(4)} - {y_{p3}}' - y_{p3} = 5\sin x \qquad (\alpha + i\beta = 0 + 1i = i)$$

Since, $\alpha + i\beta = i$ is a single root of characteristic equation.

Hence, y_{p3} has the following form: $y_{p2} = (G\sin x + H\cos x)x$

So: $\quad y_p = y_{p1} + y_{p2} + y_{p3}$

$$= Ax^2 + Bx + C + x^2e^x(Dx^2 + Ex + F) + (G\sin x + H\cos x)x$$

b) Given that: $\quad y''' - y'' + 2y' - 2y = e^x + x + 1$

$$\leftrightarrow L[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} L[y] = y''' - y'' + 2y' - 2y \\ g_1(x) = e^x \\ g_2(x) = x + 1 \end{cases}$

Characteristic equation of the given ODE: $r^3 - r^2 + 2r - 2 = 0$

$$\rightarrow r_1 = 1;\ r_2 = i\sqrt{2};\ r_3 = -i\sqrt{2}$$

So, the complement solution is: $y_c = C_1e^x + C_2\sin\sqrt{2}x + C_3\cos\sqrt{2}x$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two terms: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore y_{p1} from: $L[y_{p1}] = g_1(x) \leftrightarrow {y_{p1}}''' - {y_{p1}}' + 2y'_{p1} - 2y_{p1} = e^x \qquad (\alpha = 1)$

Since, $\alpha = 1$ is a single root of characteristic equation.

So, y_{p1} has the following form: $y_{p1} = xAe^x$

$$\rightarrow y'_{p1} = A(x+1)e^x$$
$$\rightarrow y''_{p1} = A(x+2)e^x$$
$$\rightarrow y'''_{p1} = A(x+3)e^x$$

Substituting into the equation we obtain:

$$3Ae^x = e^x$$
$$\rightarrow 3A = 1 \leftrightarrow A = \frac{1}{3}$$

Therefore: $y_{p1} = \frac{1}{3}xe^x$

Solve fore y_{p2} from: $L[y_{p2}] = g_2(x) \leftrightarrow {y_{p2}}''' - {y_{p2}}' + 2y'_{p2} - 2y_{p2} = x + 1 \quad (\alpha = 0)$

Since, $\alpha = 0$ is not a root of characteristic equation.

So, y_{p2} has the following form: $y_{p2} = Ax + B$

$$\rightarrow y'_{p2} = A$$
$$\rightarrow y''_{p2} = 0$$
$$\rightarrow y''_{p2} = 0$$

Substituting into the equation we obtain:

$$0 - 0 + 2A - 2(Ax + B) = x + 1$$
$$\rightarrow \begin{cases} -2A = 1 \\ -2B + 2A = 1 \end{cases} \leftrightarrow \begin{cases} A = -\frac{1}{2} \\ B = -1 \end{cases}$$

Therefore: $y_{p2} = -\frac{1}{2}x - 1$

So: $y_p = y_{p1} + y_{p2}$

$$= \frac{1}{3}xe^x - \frac{1}{2}x - 1$$

Thus, the general solution of the given differential equation is:

$$y_G = y_c + y_p$$
$$= C_1e^x + C_2\sin\sqrt{2}x + C_3\cos\sqrt{2}x + \frac{1}{3}xe^x - \frac{1}{2}x - 1$$

Q4.

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \quad (1) \\ \dfrac{dy}{dt} = 4x + 3y \ (2) \end{cases}$$

Differentiating both sides of (1), we get: $x'' = x' + 2y'$ (3).

Taking $2 \times (2) - 3 \times (1)$, we obtain: $2y' - 3x' = 5x \leftrightarrow 2y' = 3x' + 5x$ (4)

Substituting (4) into (3), it leads to:

$$x'' = x' + 3x' + 5x \leftrightarrow x'' - 4x' - 5x = 0$$

Characteristic equation: $r^2 - 3r - 4 = 0 \to r_1 = -1; r_2 = 5$

Therefore: $x(t) = C_1e^{-t} + C_2e^{5t}$

$$\to x'(t) = -C_1e^{-t} + 5C_2e^{5t}$$

From (1): $y(t) = \frac{1}{2}\left(x'(t) - x(t)\right) = C_1e^{-t} - 2C_2e^{5t}$

Thus, the solution of the given system of differential equations is:

$$\begin{cases} x(t) = C_1e^{-t} + C_2e^{5t} \\ y(t) = C_1e^{-t} - 2C_2e^{5t} \end{cases}$$

Q5.

Let $x(t)$ be the number of grams of C present at time t (minute). Due to the fact that 1 gram of A and 4 grams of B used to combine C, therefore, the amount of A and B used are $\frac{x(t)}{5}, \frac{4x(t)}{5}$, respectively.

The amount of remain chemical A: $60 - \frac{x(t)}{5}$

The amount of remain chemical B: $40 - \frac{4x(t)}{5}$

The problem tells us that the rate of formed chemical C depends on the proportional product of instantaneous amount of A and B not converted to C. It means that:

$$\frac{dx}{dt} = K\left(60 - \frac{x}{5}\right)\left(40 - \frac{4x}{5}\right)$$
$$\to \frac{25dx}{(300 - x)(200 - 4x)} = Kdt$$
$$\leftrightarrow \left(\frac{1}{10}\frac{1}{200 - 4x} - \frac{1}{40}\frac{1}{300 - x}\right)dx = Kdt$$

Integrating both sides we get:

$$\to \frac{1}{40}\ln\left(\frac{300 - x}{200 - 4x}\right) = Kt + C (1)$$

With the initial condition:

$$\begin{cases} x(0) = 0 \\ x(8) = 20 \end{cases} \rightarrow \begin{cases} 0.0101 = K.0 + C \\ 0.0138 = K.20 + C \end{cases} \leftrightarrow \begin{cases} C = 0.0101 \\ K = 0.1842 \times 10^{-3} \end{cases}$$

From (1) solve for $x(t)$, we get:

$$x(t) = \frac{200e^{40(Kt+C)} - 300}{4e^{40(Kt+C)} - 1}$$

Therefore: $x(20) = 49.993$ grams

And:

$$\lim_{t\to\infty} x(t) = \frac{200}{4} = 50 \text{ grams}$$