Midterm: March, 2014

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Question 1

a)

The component of $\mathbf{E}$ along $\mathbf{F}$ is

$$\text{comp}_{\mathbf{F}}\mathbf{E} = \frac{\mathbf{E}\cdot\mathbf{F}}{F} = \frac{0\times 4 + 3\times(-10) + 4\times 5}{\sqrt{4^2+(-10)^2+5^2}} = -\frac{10\sqrt{141}}{141} \approx -0.842$$

b)

The common normal vector of $\mathbf{E}$ along $\mathbf{F}$ is

$$\mathbf{n} = \mathbf{E}\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = 55\hat{\mathbf{x}} + 16\hat{\mathbf{y}} - 12\hat{\mathbf{z}}$$

Then the unit normal vector is given by

$$\mathbf{a_n} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{55\hat{\mathbf{x}} + 16\hat{\mathbf{y}} - 12\hat{\mathbf{z}}}{5\sqrt{137}}$$

Question 2

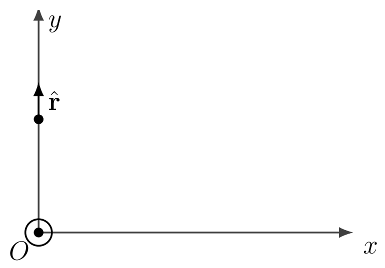
We have $\mathbf{R} = (2-1)\hat{\mathbf{x}} + (2-1)\hat{\mathbf{y}} + (2-1)\hat{\mathbf{z}} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$
And $Id\mathbf{l} = I\hat{\mathbf{l}}dl = 4\times 10^{-3}\hat{\mathbf{x}}$
The magnetic field due to the current element is given by

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{Id\mathbf{l}\times\hat{\mathbf{r}}}{R^2} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{l}\times\mathbf{R}}{R^3} \\ &= \frac{4\pi\times 10^{-7}}{4\pi}\frac{4\times 10^{-3}\hat{\mathbf{x}}\times(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})}{(\sqrt{1^2+1^2+1^2})^3} \\ &= \frac{4\sqrt{3}}{9}\times 10^{-10}(\hat{\mathbf{z}}-\hat{\mathbf{y}})\ (\text{Wb})\end{aligned}$$

Question 3

a)

The given given point and line charge separate each other by distance $R = 3$ m.



Displacement flux of a long line charge at (0,3,0) is

$$
\begin{aligned}
\mathbf{D} = \varepsilon_0 \mathbf{E} &= \frac{\rho_l}{2\pi R}\hat{\mathbf{r}} \\
&= \frac{8}{2\pi \times 3}\hat{\mathbf{y}} = \frac{4}{3\pi}\hat{\mathbf{y}}\ (\text{nC/m}^2)
\end{aligned}
$$

b)

By Gauss's law, the total charge is

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{S_1} \mathbf{D} \cdot d\mathbf{S_1} + \int_{S_2} \mathbf{D} \cdot d\mathbf{S_2} + \int_{S_3} \mathbf{D} \cdot d\mathbf{S_3}$$

Where S is closed cylindrical surface radius of R with axes as line charge and S_1, S_2, S_2 are top, bottom, surrounding surface of the cylindrical. This leads to

$$\int_{S_1} \mathbf{D} \cdot d\mathbf{S_1} = \int_{S_2} \mathbf{D} \cdot d\mathbf{S_2} = 0$$

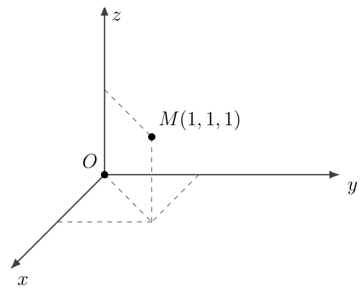
($d\mathbf{S_1}$ and $d\mathbf{S_2}$ are perpendicular to $\mathbf{D}$)
We have

$$\int_{S_3} \mathbf{D} \cdot d\mathbf{S_3} = \int_{S_3} \frac{\rho_l}{2\pi R}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\ dS_3 = \frac{\rho_l}{2\pi R}\int_{S_3} dS_3 = \frac{\rho_l}{2\pi R} 2\pi R l = \rho_l l$$

Therefore, the total charge of 5 m length line charge is

$$Q = \rho_l \times l = 8 \times 5 = 40\ (\text{nC})$$

Question 4



For the plane $x = 0$ the normal vector point to (1,1,1) is $\mathbf{a_n} = \hat{\mathbf{x}}$

$$\mathbf{B}_{[x=0]} = \frac{\mu_0}{2}\mathbf{J_S} \times \mathbf{a_n} = \frac{\mu_0}{2}(-J_{S0}\hat{\mathbf{z}}) \times \hat{\mathbf{x}} = -\frac{\mu_0}{2}J_{S0}\hat{\mathbf{y}}$$

For the plane $y = 0$ the normal vector point to (1,1,1) is $\mathbf{a_n} = \hat{\mathbf{y}}$

$$\mathbf{B}_{[y=0]} = \frac{\mu_0}{2}\mathbf{J_S} \times \mathbf{a_n} = \frac{\mu_0}{2}(2J_{S0}\hat{\mathbf{x}}) \times \hat{\mathbf{y}} = \mu_0 J_{S0}\hat{\mathbf{z}}$$

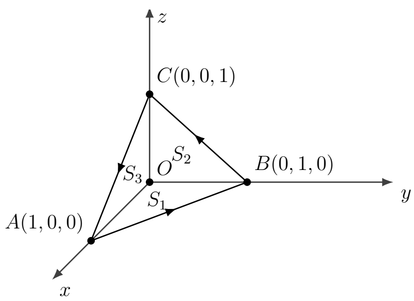
For the plane $z = 0$ the normal vector point to (1,1,1) is $\mathbf{a_n} = \hat{\mathbf{z}}$

$$\mathbf{B}_{[z=0]} = \frac{\mu_0}{2}\mathbf{J_S} \times \mathbf{a_n} = \frac{\mu_0}{2}(-J_{S0}\hat{\mathbf{y}}) \times \hat{\mathbf{z}} = -\frac{\mu_0}{2}J_{S0}\hat{\mathbf{x}}$$

The resulting magnetic flux density at (1,1,1) is

$$\begin{aligned}\mathbf{B} &= \mathbf{B}_{[x=0]} + \mathbf{B}_{[y=0]} + \mathbf{B}_{[z=0]} \\ &= -\frac{\mu_0}{2}J_{S0}(\hat{\mathbf{x}} + \mathbf{x}\hat{\mathbf{y}} - 2\hat{\mathbf{z}})\end{aligned}$$

Question 5



Solution 1:

Consider a closed surface including S_1, S_2, S_3 and $S_4 \equiv (ABC)$ as in the above figure. The Gauss's law in integral form gives us

$$0 = \oint_S \mathbf{B} \cdot d\mathbf{S} = \int_{S_1} \mathbf{B} \cdot d\mathbf{S_1} + \int_{S_2} \mathbf{B} \cdot d\mathbf{S_2} + \int_{S_3} \mathbf{B} \cdot d\mathbf{S_3} + \int_{S_4} \mathbf{B} \cdot d\mathbf{S_4} \ (*)$$

With all the normal vectors of outward direction, we have

- $\displaystyle\int_{S_1} \mathbf{B}\cdot d\mathbf{S_1} = \int_{S_1} B_0(\sin(\omega t)\hat{\mathbf{x}} - \cos(\omega t)\hat{\mathbf{y}})\cdot(-\hat{\mathbf{z}}dS_1) = 0$
- $\displaystyle\int_{S_2} \mathbf{B}\cdot d\mathbf{S_2} = \int_{S_2} B_0(\sin(\omega t)\hat{\mathbf{x}} - \cos(\omega t)\hat{\mathbf{y}})\cdot(-\hat{\mathbf{x}}dS_2) = B_0\sin(\omega t)\int_{S_2} dS_2 = -\frac{1}{2}B_0\sin(\omega t)$
- $\displaystyle\int_{S_3} \mathbf{B}\cdot d\mathbf{S_3} = \int_{S_3} B_0(\sin(\omega t)\hat{\mathbf{x}} - \cos(\omega t)\hat{\mathbf{y}})\cdot(-\hat{\mathbf{y}}dS_3) = B_0\cos(\omega t)\int_{S_3} dS_3 = \frac{1}{2}B_0\cos(\omega t)$

From $(*)$, we have

$$\begin{aligned}\int_{S_4} \mathbf{B}\cdot d\mathbf{S_4} &= -\left(\int_{S_1} \mathbf{B}\cdot d\mathbf{S_1} + \int_{S_2} \mathbf{B}\cdot d\mathbf{S_2} + \int_{S_3} \mathbf{B}\cdot d\mathbf{S_3}\right)\\ &= \frac{1}{2}B_0(\sin(\omega t) - \cos(\omega t))\end{aligned}$$

And,

$$\Psi = \int_{S_4} \mathbf{B}\cdot d\mathbf{S_4} = \frac{1}{2}B_0(\sin(\omega t) - \cos(\omega t))$$

Then, the induced electromotive force is given by

$$\text{emf} = -\frac{d\Psi}{dt} = -\frac{1}{2}\omega B_0(\cos(\omega t) + \sin(\omega t))$$

Solution 2:

The equation for the plane (ABC) is given by $x + y + z = 1$ corresponding to the normal vector $\mathbf{n} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$. Then, the unit normal vector for the plane is

$$\mathbf{a_n} = \frac{1}{\sqrt{3}}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$$

Applying the right hand rule for the given path, there is one differential surface vector point outward, that is, $d\mathbf{S} = +\mathbf{a_n}dS$. Therefore,

$$\begin{aligned}\Psi = \int_S \mathbf{B}\cdot d\mathbf{S} &= \int_S B_0(\sin(\omega t)\hat{\mathbf{x}} - \cos(\omega t)\hat{\mathbf{y}})\cdot\frac{1}{\sqrt{3}}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})dS\\ &= \frac{1}{\sqrt{3}}B_0(\sin(\omega t) - \cos(\omega t))\int_S dS\\ &= \frac{1}{\sqrt{3}}B_0(\sin(\omega t) - \cos(\omega t))\frac{(\sqrt{2})^2\sqrt{3}}{4}\\ &= \frac{1}{2}B_0(\sin(\omega t) - \cos(\omega t))\end{aligned}$$

(The area of equilateral triangle ABC side of $\sqrt{2}$ is $(\sqrt{2})^2\sqrt{3}/4$)

Then, the induced electromotive force is given by

$$\text{emf} = -\frac{d\Psi}{dt} = -\frac{1}{2}\omega B_0(\cos(\omega t) + \sin(\omega t))$$

Question 6

We have $\mathbf{D} = \langle 2xy, x^2, 0\rangle \Rightarrow \nabla\cdot\mathbf{D} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle 2xy, x^2, 0\rangle = 2y$ Applying the Gauss's law integral form and divergence theorem, immediately give us the total charge

$$\begin{aligned} Q = \oint_S \mathbf{D} d\mathbf{S} &= \int_V (\nabla\cdot\mathbf{D}) dV = \int_V 2y dV \\ &= \int_0^3 \int_0^1 \int_0^2 2y \; dydxdz = \cdots \\ &= 12 \text{ (C)} \end{aligned}$$

Question 8

Denote that $\mathbf{F} = 2x\hat{\mathbf{y}} + 3y\hat{\mathbf{z}}$, we have

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 2x & 3y \end{vmatrix} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{z}}$$

The two given path are lying in the xy-plane which give us $d\mathbf{S} = \pm\hat{\mathbf{z}} dS$. Now, applying the Stokes's theorem for the closed path with the given vector field yields

$$\begin{aligned} L = \oint_C \mathbf{F}\cdot d\mathbf{l} &= \int_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} \\ &= \int_S (3\hat{\mathbf{x}} + 2\hat{\mathbf{z}})\cdot(\pm\hat{\mathbf{z}}) dS \\ &= \pm 2\int_S dS \end{aligned}$$

a)

For a square path, the area enclosed by the path is $S = 1$ (m^2). Therefore the absolute value for the line integral is

$$|L| = 2S = 2$$

b)

For a circular path, the area enclosed by the path is $S = \pi$ (m^2). Therefore the absolute value for the line integral is

$$|L| = 2S = 2\pi$$