Q1.

a)

$$\mathcal{L}\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

b)

Let $f(z) = u(x,y) + jv(x,y)$, where $u(x,y) = x^2, v(x,y) = y^2$

$f'(z)$ exists at a point if and only if at this point it satisfies the Cauchy-Riemann equation:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \leftrightarrow \begin{cases} 2x = 2y \\ 0 = -0 \end{cases} \leftrightarrow y = x$$

Thus, for all points which belong to the line $y = x$ in the z-plane lead to existence of $f'(z)$

Q2.

Given that:

$$\frac{dy}{dt} + 3y = 13\sin 2t \quad (*), \qquad y(0) = 6$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 6$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4}$$

$$\leftrightarrow Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)}$$

$$\leftrightarrow Y(s) = \frac{8}{s+3} + \frac{-2s + 3\times 2}{s^2+2^2}$$

$\rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = (8e^{-3t} - 2\cos 2t + 3\sin 2t)u(t)$

Thus, the solution of the given differential equation is:

$$y(t) = (8e^{-3t} - 2\cos 2t + 3\sin 2t)u(t)$$

Q3.

a)

Let: $z = -27$

$$\rightarrow \begin{cases} r = |z| = \sqrt{0^2+1^2} = 27 \\ \theta = \pi - \tan^{-1} 0 = \pi \end{cases}$$

Since, we know that:

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + j\sin\frac{\theta+2k\pi}{n}\right), \quad k = 0,1,\dots,n-1$$

Therefore, there is exist 3 cubic roots of z as follows:

$$w_0 = \sqrt[3]{27}\left(\cos\frac{\pi+0}{3} + j\sin\frac{\pi+0}{3}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}j$$

$$w_1 = \sqrt[3]{27}\left(\cos\frac{\pi+2\pi}{3} + j\sin\frac{\pi+2\pi}{3}\right) = -3$$

$$w_2 = \sqrt[3]{27}\left(\cos\frac{\pi+4\pi}{3} + j\sin\frac{\pi+4\pi}{3}\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}j$$

b)

Let the current flow to capacitor which depend on time t is $i(t)$, it leads to the charge of the capacitor is given by $q(t) = \int_{-\infty}^{t} i(x)dx$

Applying Kirchhoff voltage law for the circuit, we obtain the first order differential equation:

$$\frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau + Ri(t) = e(t) \quad (*)$$

The problem gives us: $i(0) = 0, q(0) = 0, e(t) = Eu(t), E$is constant.

Taking Laplace transforms both sides of $(*)$, we get:

$$\frac{I(s)}{Cs} + RI(s) = \frac{E}{s}$$

$$\leftrightarrow I(s) = \frac{E}{Rs + 1/C} = \frac{E}{R}\frac{1}{s + 1/RC}$$

Therefore, $i(t) = \mathcal{L}^{-1}\{I(s)\} = \frac{E}{R}e^{-\frac{1}{RC}t}u(t)$

Thus, the charge on the capacitor is:

$$q(t) = \int_{-\infty}^{t} \frac{E}{R}e^{-\frac{1}{RC}\tau}u(\tau)d\tau = \frac{E}{R}\int_{0}^{t} e^{-\frac{1}{RC}\tau}d\tau = EC\left(1 - e^{-\frac{1}{RC}t}\right)$$

Q4.

a)

$$j\left(\frac{1+3j}{1-2j}\right)^2 = 2$$

b)

$$\mathcal{L}^{-1}\left\{\frac{s^4 + 5s^2 + 2}{s^3(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-2s}{s^2+1} + \frac{2}{s^3} + \frac{3}{s}\right\} = -2\cos t + t^2 + 3$$

Q5.

a)

Since, we know that: $\cos 3\omega t = 4\cos^3 \omega t - 3\cos \omega t$

Therefore, $\cos^3 \omega t = \frac{1}{4}(\cos 3\omega t + 3\cos \omega t)$

Hence,

$$\mathcal{L}\{\cos^3 \omega t\} = \mathcal{L}\left\{\frac{1}{4}(\cos 3\omega t + 3\cos \omega t)\right\} = \frac{1}{4}\left(\frac{s}{s^2+9} + \frac{3s}{s^2+1}\right)$$

b)

$$f(z) = \frac{1}{2 + (z-1)}$$

Apply power series for analyzing this problem:

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

Case 1:

$$f(z) = \frac{1}{2+(z-1)} = \frac{1}{2}\frac{1}{1 + \frac{z-1}{2}} = \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z-1}{2}\right)^n$$

This series hold for $\left|\frac{z-1}{2}\right| < 1 \leftrightarrow |z-1| < 2$, according to the power series

Case 2:

$$f(z) = \frac{1}{z-1}\frac{1}{1+\dfrac{2}{z-1}} = \frac{1}{z-1}\sum_{n=0}^{+\infty}\left(\frac{2}{z-1}\right)^n$$

This series hold for $\left|\frac{2}{z-1}\right| < 1 \leftrightarrow |z-1| > 2$, according to the power series

Therefore,

$$f(z) = \begin{cases} \displaystyle\sum_{n=0}^{+\infty}\frac{1}{2^{n+1}}(z-1)^n, & |z-1| < 2 \\ \displaystyle\sum_{n=0}^{+\infty} 2^n(z-1)^{-(n+1)}, & |z-1| > 2 \end{cases}$$