Q1.

a)

Rewrite $f(t)$ as unit step function we have:

$$f(t) = \cos(\pi t)\, u(t-1) - \cos(\pi t)\, u(t-4)$$

- $\mathcal{L}\{\cos(\pi t)\, u(t-1)\} = \mathcal{L}\{-\cos(\pi(t-1))\, u(t-1)\} = -\dfrac{s}{s^2+\pi^2} e^{-s}$
- $\mathcal{L}\{\cos(\pi t)\, u(t-4)\} = \mathcal{L}\{\cos(\pi(t-4))\, u(t-4)\} = \dfrac{s}{s^2+\pi^2} e^{-4s}$

Therefore,

$$F(s) = \mathcal{L}\{f(t)\} = \frac{-s}{s^2+\pi^2}\left(e^{-s} + e^{-4s}\right)$$

b)

$$\mathcal{Z}^{-1}\left\{\frac{z}{z^2+2z-3}\right\} = \frac{1}{4}\mathcal{Z}^{-1}\left\{\frac{z}{z-1} - \frac{z}{z+3}\right\} = \frac{1}{4} - \frac{1}{4}(-3)^n$$

Q2.

a)

Given that:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 2\frac{du}{dt} + 4u \ (*)$$

To find transfer function, we set all initial condition to 0.

Taking Laplace transform both sides of $(*)$, we obtain:

$$s^2X(s) + 6sX(s) + 13X(s) = 2sU(s) + 4U(s)$$

Therefore, the transfer function is:

$$H(s) = \frac{X(s)}{U(s)} = \frac{2s+4}{s^2+6s+13}$$

From the transfer function we obtain 1 zero and 2 poles, which are: $z_1 = -2, p_1 = -3-2j$, $p_2 = -3+2j$. (put those three point in the complex system coordinates we obtain the pole-zero plot, reader plot by yourself).

b)

Given that:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = \delta(t-2\pi)\ \ (*),\qquad x(0) = 0,\quad x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$, it holds that:

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0) = sX(s)$$

$$\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0) = s^2X(s)$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$s^2X(s) + 6sX(s) + 13X(s) = e^{-2\pi s}$$

$$\leftrightarrow X(s)(s^2+6s+13) = e^{-2\pi s}$$

$$\leftrightarrow X(s) = \frac{1}{2}\frac{2}{(s+3)^2+2^2}e^{-2\pi s}$$

$$\rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2}\sin(2t)\, e^{-3t}u(t-2\pi)$$

Thus, the solution of the given differential equation is:

$$x(t) = \frac{1}{2}\sin(2t)\, e^{-3t}u(t-2\pi)$$

Q3.

Given that:

$$2y_{k+2} - 3y_{k+1} - 2y_k = 15k + 15 \ \ (*), \qquad y_0 = 1, \quad y_1 = 2$$

Let $Y(z) = \mathcal{Z}\{y_k\}$, it holds that:

$$\mathcal{Z}\{y_{k+1}\} = zY(z) - zy_0 = zY(z) - z$$
$$\mathcal{Z}\{y_{k+2}\} = z^2Y(z) - z^2y_0 - zy_1 = z^2Y(z) - z^2 - 2z$$

Taking $\mathcal{Z}$-transform both side of $(*)$, we obtain:

$$2[z^2Y(z) - z^2 - 2z] - 3[zY(z) - z] - 2[Y(z)] = \frac{15z}{(z-1)^2} + \frac{15z}{z-1}$$
$$\leftrightarrow Y(z)(2z^2 - 3z - 2) = \frac{15z}{(z-1)^2} + \frac{15z}{z-1} + 2z^2 + z$$
$$\rightarrow \frac{Y(z)}{z} = \frac{\frac{15}{(z-1)^2} + \frac{15}{z-1} + 2z + 1}{2z^2 - 3z - 2}$$
$$\leftrightarrow \frac{Y(z)}{z} = \frac{-5}{(z-1)^2} - \frac{20/3}{z-1} + \frac{7}{z-2} + \frac{2/3}{z+1/2}$$
$$\rightarrow Y(z) = \frac{-5z}{(z-1)^2} - \frac{20}{3}\frac{z}{z-1} + \frac{7}{z-2} + \frac{2}{3}\frac{z}{z+1/2}$$

$$\rightarrow y_k = \mathcal{Z}^{-1}\{Y(z)\} = -5k - \frac{20}{3} + 7.2^k + \frac{2}{3}\left(-\frac{1}{2}\right)^k$$

Thus, the solution of the given system difference equations is:

$$y_k = -5k - \frac{20}{3} + 7.2^k + \frac{2}{3}\left(-\frac{1}{2}\right)^k$$

Q4.

Given that: $\qquad f(t) = \begin{cases} 1, & -\pi < t \le 0 \\ -1, & 0 < t \le \pi \end{cases}, \qquad T = 2\pi \rightarrow \omega = \frac{2\pi}{T} = 1$

a)

Due to odd function, we obtain:

- $a_0 = a_n = 0$
- $b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega t)\,dt = \frac{4}{2\pi}\int_0^{\pi}(-1)\sin(nt)\,dt$

$$= \frac{2}{\pi n}[\cos(nt)]|_0^{\pi}$$
$$= \frac{2}{\pi n}((-1)^n - 1)$$

The Fourier series is given by:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n\cos(n\omega t) + \sum_{n=1}^{+\infty} b_n\sin(n\omega t)$$
$$= \sum_{n=1}^{+\infty}\frac{2}{\pi n}((-1)^n - 1)\sin(nt)$$
$$= -\frac{4}{\pi}\sum_{k=1}^{+\infty}\frac{\sin((2k-1)t)}{2k-1}, \quad n = 2k-1, k \ge 1$$

b)

By Parseval's identity we obtain:

$$\frac{1}{T}\int_{t_0}^{t_0+T}|f(t)|^2dt = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{+\infty}(a_n^2+b_n^2)$$

$$\leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}(1)^2dt = \frac{1}{4}0^2 + \frac{1}{2}\sum_{k=1}^{+\infty}\left(\frac{1}{2k-1}\right)^2$$

$$\leftrightarrow \sum_{k=1}^{+\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

Q5.

Given that: $V(t) = \begin{cases} 6+60t, & -0.1 < t \le 0 \\ 6-60t, & 0 < t \le 0.1 \end{cases}$, $\quad T = 0.2 \to \omega = \frac{2\pi}{T} = 10\pi$

Due to odd function, we obtain:

- $b_n = 0$
- $a_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)dt = \frac{4}{0.2}\int_0^{0.1}(6-60t)dx = 6$
- $a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega t)\,dt = \frac{4}{0.2}\int_0^{0.1}(6-60t)\cos(10n\pi t)\,dt$

$$= 20\left[\frac{6-60t}{10n\pi}\sin(10n\pi t) + \frac{-60}{(10n\pi)^2}\cos(10n\pi t)\right]\Bigg|_0^{0.1}$$

$$= \frac{12}{\pi^2n^2}(1-(-1)^n)$$

The Fourier series is given by:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty}a_n\cos(n\omega t) + \sum_{n=1}^{+\infty}b_n\sin(n\omega t)$$

$$= 3 + \sum_{n=1}^{+\infty}\frac{12}{\pi^2n^2}(1-(-1)^n)\cos(nt)$$

Q6.

a)

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-1}^{1} 1e^{-j\omega t}dt$$

$$= \frac{-1}{j\omega}\left(e^{-j\omega t}\right)\Big|_{-1}^{1} = \frac{1}{j\omega}\left(e^{j\omega} - e^{-j\omega}\right)$$

$$= \frac{2\sin\omega}{\omega}$$

b)

$$G(\omega) = \mathcal{F}\{g(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-\pi}^{\pi}\sin t\, e^{-j\omega t}dt$$

$$= \int_{-\pi}^{\pi}\sin t\,(\cos(\omega t) - j\sin(\omega t))dt = \int_{-\pi}^{\pi}[\sin t\cos(\omega t) - j\sin t\sin(\omega t)]dt$$

$$= -2j\int_0^{\pi}\sin t\sin(\omega t)\,dt$$

$$= -j\left[\frac{\sin\big((\omega-1)t\big)}{\omega-1} - \frac{\sin\big((\omega+1)t\big)}{\omega+1}\right]\Bigg|_0^{\pi} = -j\left(-\frac{\sin(\omega n)}{\omega-1} + \frac{\sin(\omega n)}{\omega+1}\right)$$

$$= \frac{2j\sin(\omega n)}{\omega^2-1}$$

$$\left(\int_{-\pi}^{\pi} \sin t \cos(\omega t)\, dt = 0, \text{integral from} - a \text{ to } a \text{ of odd function}\right)$$