$$(y - 3x^2)\,dx + (x - 1)\,dy = 0$$

$$M(x;y) = y - 3x^2$$
$$N(x;y) = x - 1$$

$$M_y = \frac{dM}{dy} = 1 \qquad N_x = \frac{dN}{dx} = 1$$

So, the given equation is exact.

$$F_x = M(x;y) = y - 3x^2$$
$$F_y = N(x;y) = x - 1 \qquad (1)$$

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$$\int F_x\,dx = \int y - 3x^2\,dx$$

So, the given equation is exact.

$$F_x = M(x;y) = y - 3x^2$$
$$F_y = N(x;y) = x - 1 \qquad (1)$$

$$\int F_x\,dx = \int y - 3x^2\,dx$$

$$F(x;y) = xy - x^3 + C(y)$$
$$F_y = x + C'(y) \qquad (2)$$

$$F_y = N(x;y) = x - 1 \quad (1)$$

$$\int F_x \, dx = \int y - 3x^2 \, dx$$

$$F(x;y) = xy - x^3 + C(y)$$

$$F_y = x + C'(y) \quad (2)$$

$$(1),(2) \Rightarrow x - 1 = x + C'(y)$$

$$\Rightarrow C'(y) = -1$$

$$\int C'(y)\, dy = \int -1 \, dy$$

$$C(y) = -y$$

So, we have: $F(x;y) = xy - x^3 - y$

Finally, the general solution is: $xy - x^3 - y = C$

$$x^2 y''(x) - 2x y'(x) + 2y(x) = 0 \quad (1)$$

$y = x^\alpha \Rightarrow y' = \alpha x^{\alpha-1}$ $\qquad x \in (0; +\infty)$

$\Rightarrow y'' = \alpha(\alpha-1) x^{\alpha-2}$

If $y = x^\alpha$ is a solution of (1)

Then: $x^2 \alpha(\alpha-1) x^{\alpha-2} - 2x.\alpha x^{\alpha-1} + 2x^\alpha = 0$

$$(\alpha^2 - \alpha) x^\alpha - 2\alpha x^\alpha + 2x^\alpha = 0$$

$$x^\alpha (\alpha^2 - 3\alpha + 2) = 0$$

$$\Leftrightarrow \left[\begin{array}{l} \alpha_1 = 1 \\ \alpha_2 = 2 \end{array} \right.$$

$$x^2 y''(x) - 2x\, y'(x) + 2y(x) = 0 \quad (1)$$

$y = x^\alpha \Rightarrow y' = \alpha x^{\alpha-1}$ $\qquad x \in (0; +\infty)$

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If $y = x^\alpha$ is a solution of (1)

Then: $x^2 \alpha(\alpha-1) x^{\alpha-2} - 2x.\alpha x^{\alpha-1} + 2x^\alpha = 0$

$(\alpha^2-\alpha)\, x^\alpha - 2\alpha x^\alpha + 2x^\alpha = 0$

$x^\alpha(\alpha^2 - 3\alpha + 2) = 0$

$\Leftrightarrow \left[\begin{array}{l}\alpha_1 = 1 \\ \alpha_2 = 2\end{array}\right.$

$\Rightarrow y_1(x) = x$ and $y_2(x) = x^2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0 \quad (x \in (0; +\infty))$$

So, $y_1(x)$ and $y_2(x)$ are linearly independent

General solution: $y(x) = C_1.y_1 + C_2 y_2 = C_1 x + C_2.x^2$

$$x^2 y'' - x y'(x) + y(x) = 0 \quad , \quad x \in (0; +\infty) \quad (1)$$

Assume: $y = x^\alpha \Rightarrow y' = \alpha x^{\alpha - 1}$

$\Rightarrow y'' = \alpha(\alpha - 1) x^{\alpha - 2}$ $\qquad x \in (0; +\infty)$

If $y = x^\alpha$ is a solution of (1)

Then: $x^2 . \alpha(\alpha - 1) x^{\alpha - 2} - x . \alpha x^{\alpha - 1} + x^\alpha = 0$

$(\alpha^2 - \alpha) x^\alpha - \alpha x^\alpha + x^\alpha = 0$

$x^\alpha (\alpha^2 - 2\alpha + 1) = 0$ $\qquad \alpha$

$\Rightarrow \alpha_1 = \alpha_2 = -1$

So $y_1(x) = x \quad , \quad x \in (0; +\infty)$ is a solution of (1)

$$y_2(x) = y_1(x) . \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx$$

$$x^2 y'' - x y'(x) + y(x) = 0 \quad (1)$$

Standard form of (1) is given by: $y'' - \frac{1}{x} y'(x) + \frac{1}{x^2} y(x) = 0$

$\Rightarrow p(x) = -\frac{1}{x}$

$$y_2(x) = x . \int \frac{e^{\int \frac{1}{x} dx}}{x^2}\,dx$$

$$= x . \int \frac{\cancel{e^{\ln x}}}{x^2}\,dx \qquad e^{\ln a} = a$$

$$= x . \int \frac{x}{x^2}\,dx$$

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$$= x . \int \frac{1}{x}\,dx = x \ln x$$

$\Rightarrow \; p(x) = -\frac{1}{x}$

$$y_2(x) = x \cdot \int \frac{e^{\int \frac{1}{x} dx}}{x^2} \, dx$$

$$= x \cdot \int \frac{e^{\ln x}}{x^2} \, dx$$

$$= x \cdot \int \frac{x}{x^2} \, dx$$

$$= x \cdot \int \frac{1}{x} \, dx = x \ln x$$

$y_1 = x$

$y_2 = x \ln x$

So $y_1(x) = x$, $x \in (0; +\infty)$ is a solution of (1)

$$y_2(x) = y_1(x) \cdot \int \frac{e^{-\int p(x) \, dx}}{y_1^2(x)} \, dx$$

$$x^2 y'' - x y'(x) + y(x) = 0 \quad (1)$$

Standard form of (1) is given by: $y'' - \frac{1}{x} y'(x) + \frac{1}{x^2} y(x) = 0$

$\Rightarrow \; p(x) = -\frac{1}{x}$

$$y_2(x) = x \cdot \int \frac{e^{\int \frac{1}{x} dx}}{x^2} \, dx$$

$$= x \cdot \int \frac{e^{\ln x}}{x^2} \, dx$$

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$$W(y_1; y_2) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix}$$

$$= x(\ln x + 1) - x\ln x$$

$$= x\ln x + x - x\ln x = \boxed{x \neq 0} \quad x \in (0; +\infty)$$

Thus, $y_1(x)$ and $y_2(x)$ are linearly independent

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General solution : $y(x) = C_1 x + C_2 x\ln x$

Find a particular solution

$$\underline{y'' + 2y' + y} = e^{-x} \ln x \quad (1)$$

The homogeneous:

$$y'' + 2y' + y = 0 \quad (2)$$

The characteristic equation is given by:

$$r^2 + 2r + 1 = 0$$

$$\Rightarrow \quad r_1 = r_2 = -1$$

The general solution is given by:

$$y(x) = C_1 \underbrace{e^{-x}}_{y_1(x)} + C_2 \underbrace{e^{-x} . x}_{y_2(x)}$$

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$$y_p(x) = \underline{C_1(x) . e^{-x} + C_2(x) . e^{-x} x}$$

The characteristic equation is given by:

$$r^2 + 2r + 1 = 0$$

$$\Rightarrow \quad r_1 = r_2 = -1$$

The general solution is given by:

$$y(x) = C_1 \underbrace{e^{-x}}_{y_1(x)} + C_2 \underbrace{e^{-x}.x}_{y_2(x)}$$

$$y_p(x) = C_1(x).e^{-x} + C_2(x).e^{-2}x$$

Here $(C_1'; C_2') = (X; Y)$ is the solution of the linear system:

$$\begin{cases} y_1(x)\,X + y_2(x)\,Y = 0 \\ y_1'(x)\,X + y_2'(x)\,Y = g(x) \end{cases} \Leftrightarrow \begin{cases} e^{-x}.X + e^{-x}.x\,Y = 0 \\ -e^{-x}.X + \underbrace{(-e^{-x}.x + e^{-x})}_{-e^{-x}(x-1)}.Y = e^{-2} \end{cases}$$

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$$r^2 + 2r + 1 = 0$$

$$\Rightarrow \quad r_1 = r_2 = -1$$

The general solution is given by:

$$y(x) = C_1 \underbrace{e^{-x}}_{y_1(x)} + C_2 \underbrace{e^{-x}.x}_{y_2(x)}$$

$$y_p(x) = C_1(x).e^{-x} + C_2(x).e^{-2}x$$

Here $(C_1'; C_2') = (X; Y)$ is the solution of the linear system:

$$\begin{cases} y_1(x)\,X + y_2(x)\,Y = 0 \\ y_1'(x)\,X + y_2'(x)\,Y = g(x) \end{cases} \Leftrightarrow \begin{cases} e^{-x}.X + e^{-x}.x\,Y = 0 \\ -e^{-x}.X + \underbrace{(-e^{-x}.x + e^{-x})}_{-e^{-x}(x-1)}.Y = e^{-2}\ln x \end{cases}$$

$$\Leftrightarrow \begin{cases} X + xY = 0 \\ X + (x-1)Y = -\ln x \end{cases} \qquad \Leftrightarrow \begin{cases} X = -xY & (1) \\ -xY + (x-1)Y = -\ln x & (2) \end{cases}$$

Here $(C_1'; C_2') = (X; Y)$ is the solution of the linear system:

$$\begin{cases} y_1(x)\,X + y_2(x)\,Y = 0 \\ y_1'(x)\,X + y_2'(x)\,Y = g(x) \end{cases} \Leftrightarrow \begin{cases} e^{-x}.X + e^{-x}.x\,Y = 0 \\ -e^{-x}.X + \underbrace{(-e^{-x}.x + e^{-x})}_{-e^{-x}(x-1)}.Y = \underbrace{e^{-x}\ln x}_{-e^{-x}} \end{cases}$$

$$\begin{cases} X + xY = 0 \\ X + (x-1)Y = -\ln x \end{cases} \Leftrightarrow \begin{cases} X = -xY & (1) \\ -xY + (x-1)Y = -\ln x & (2) \end{cases}$$

$$(2) \Rightarrow -xY + xY - Y = -\ln x$$
$$\Leftrightarrow \quad Y = \ln x \quad (3)$$

$$\begin{cases} y_1(x)\,X + y_2(x)\,Y = 0 \\ y_1'(x)\,X + y_2'(x)\,Y = g(x) \end{cases} \Leftrightarrow \begin{cases} e^{-x}.X + e^{-x}.x\,Y = 0 \\ -e^{-x}.X + \underbrace{(-e^{-x}.x + e^{-x})}_{-e^{-x}(x-1)}.Y = e^{-x}\ln x \end{cases}$$

$$\Leftrightarrow \begin{cases} X + xY = 0 \\ X + (x-1)Y = -\ln x \end{cases} \Leftrightarrow \begin{cases} X = -xY & (1) \\ -xY + (x-1)Y = -\ln x & (2) \end{cases}$$

$$(2) \Rightarrow -xY + xY - Y = -\ln x$$
$$\Leftrightarrow \quad Y = \ln x \quad (3)$$

(3) into (1) $\Rightarrow$ $X = -x\ln x$

Thus, $C_1'(x) = -x\ln x$

$C_2'(x) = \ln x$

Therefore: $C_1(x) = \int C_1'(x)\,dx = \int -x\ln x\,dx$

$$\begin{cases} u = \ln x \\ dv = -x\,dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\,dx \\ v = \frac{-x^2}{2} \end{cases}$$

$$\int -x\ln x\,dx = uv - \int v\,du = \frac{-x^2}{2}\ln x - \int \frac{-x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{-x^2}{2}\ln x + \int \frac{1}{2}x\,dx$$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

Therefore: $C_1(x) = \int C_1'(x)\,dx = \int -x\ln x\,dx$

$$\begin{cases} u = \ln x \\ dv = -x\,dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\,dx \\ v = \frac{-x^2}{2} \end{cases}$$

$$\int -x\ln x\,dx = uv - \int v\,du = \frac{-x^2}{2}\ln x - \int \frac{-x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{-x^2}{2}\ln x + \int \frac{1}{2}x\,dx$$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

2 - Đa

3 - Lượng

4 - Mũ

$$C_2(x) = \int C_2'(x)\,dx = \int \ln x\,dx = x\ln x - x$$

$$y(x) = C_1(x)\,e^{-x} + C_2(x)\cdot xe^{-x}$$

$$= \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^{-x} + (x\ln x - x)\cdot xe^{-x}$$

$$x^2 y'' - x y' + y = \ln x \quad (1) \quad x \in (0; +\infty)$$

Homogeneous equation: $x^2 y'' - x y' + y = 0 \quad (2)$

Assume: $y = x^{\alpha}$ is solution of (2)

$$y' = \alpha x^{\alpha - 1}$$

$$y'' = \alpha(\alpha - 1) x^{\alpha - 2}$$

$$x^2 y'' - x y' + y = \ln x \quad (1) \quad x \in (0; +\infty)$$

Homogeneous equation: $x^2 y'' - x y' + y = 0 \quad (2)$

Assume: $y = x^{\alpha}$ is solution of (2)

$$y' = \alpha x^{\alpha - 1}$$

$$y'' = \alpha(\alpha - 1) x^{\alpha - 2}$$

$$x^2 . \alpha(\alpha - 1) x^{\alpha - 2} - x . \alpha x^{\alpha - 1} + x^{\alpha} = 0$$

$$(\alpha^2 - \alpha) x^{\alpha} - \alpha x^{\alpha} + x^{\alpha} = 0$$

$$x^{\alpha} (\alpha^2 - 2\alpha + 1) = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = 1 \qquad y_1 = x^{\alpha_1} = x^1 \qquad \cancel{y_2}$$

Homogeneous equation: $x^2y'' - xy' + y = 0 \quad (2)$

Assume: $y = x^{\alpha}$ is solution of (2)

$y' = \alpha x^{\alpha-1}$

$y'' = \alpha(\alpha-1)x^{\alpha-2}$

$$x^2 . \alpha(\alpha-1)x^{\alpha-2} - x . \alpha x^{\alpha-1} + x^{\alpha} = 0$$

$$(\alpha^2-\alpha)x^{\alpha} - \alpha x^{\alpha} + x^{\alpha} = 0$$

$$x^{\alpha}(\alpha^2 - 2\alpha + 1) = 0$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = 1$$

So, $y_1(x) = x$, $x \in (0; +\infty)$ is a solution of (2)

$$y_2(x) = y_1(x) . \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx$$

The standard form of (2): $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\ln x}{x^2}$

The standard form of (2): $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\ln x}{x^2}$

$$y_2(x) = x . \int \frac{e^{-\int -\frac{1}{x}dx}}{x^2}\,dx$$

$$= x \int \frac{e^{\ln x}}{x^2}\,dx$$

$$= x . \int \frac{x}{x^2}\,dx$$

$$= x . \int \frac{1}{x}\,dx$$

$$= x \ln x$$

$$y(x) = C_1 . (x) + C_2 (x\ln x)$$

$$y_p(x) = C_1(x) . x + C_2(x) . (x\ln x) \qquad x \in (0; +\infty)$$

$$y_p(x) = C_1(x).x + C_2(x).(x\ln x) \qquad x \in (0;+\infty)$$

Here $(C_1';C_2') = (X;Y)$ is the solution of the linear system.

$$\begin{cases} y_1(x)\,X + y_2(x)\,Y = 0 \\ y_1'(x)\,X + y_2'(x)\,Y = g(x) \end{cases} \Leftrightarrow \begin{cases} x\,X + (x\ln x)\,Y = 0 \\ 1\,X + (\ln x + 1).\,Y = \dfrac{\ln x}{x^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} X + \ln x\,Y = 0 \\ X + Y\ln x + Y = \dfrac{\ln x}{x^2} \quad (2) \end{cases} \Rightarrow X = -Y\ln x \quad (1)$$

(1) into (2) $\Rightarrow$ $-Y\ln x + Y\ln x + Y = (\ln x/x^2)$

$\Leftrightarrow$ $Y = \dfrac{\ln x}{x^2}$

$$\begin{cases} y_1(x)\,X + y_2(x)\,Y = 0 \\ y_1'(x)\,X + y_2'(x)\,Y = g(x) \end{cases} \Leftrightarrow \begin{cases} x\,X + (x\ln x)\,Y = 0 \\ 1\,X + (\ln x + 1).\,Y = \dfrac{\ln x}{x^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} X + \ln x\,Y = 0 \\ X + Y\ln x + Y = \dfrac{\ln x}{x^2} \quad (2) \end{cases} \Rightarrow X = -Y\ln x \quad (1)$$

(1) into (2) $\Rightarrow$ $-Y\ln x + Y\ln x + Y = (\ln x/x^2)$

$\Leftrightarrow$ $Y = \dfrac{\ln x}{x^2}$

(3) into (1) $\Rightarrow$ $X = -\dfrac{\ln^2 x}{x^2}$

Therefore: $C_1'(x) = X = -\dfrac{\ln^2 x}{x^2}$

into (2) $\Rightarrow$ $-Y\ln x + Y\ln x + Y = (\ln x / x^2)$

$(\Leftrightarrow)$ $Y = \dfrac{\ln x}{x^2}$

into (1) $\Rightarrow$ $X = -\dfrac{\ln^2 x}{x^2}$

Therefore: $C_1'(x) = X = -\dfrac{\ln^2 x}{x^2}$

$\Rightarrow C_1(x) = \int C_1'(x)\,dx = \int -\dfrac{\ln^2 x}{x^2}\,dx$

$$\begin{cases} u = \ln^2 x \\ dv = \dfrac{1}{x^2}\,dx = x^{-2}\,dx \end{cases} \Rightarrow \begin{cases} du = 2\ln x \cdot \dfrac{1}{x}\,dx \\ v = \dfrac{x^{-1}}{-1} = -\dfrac{1}{x} \end{cases}$$

$$uv - \int v\,du = -\frac{1}{x}\ln^2 x - \int -\frac{1}{x} \cdot 2\ln x \cdot \frac{1}{x}\,dx$$

$$= -\left(-\frac{1}{x}\ln^2 x + 2\int \frac{1}{x^2} \cdot \ln x\,dx\right) \quad (1)$$

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$$uv - \int v\,du = -\frac{1}{x}\ln^2 x - \int -\frac{1}{x} \cdot 2\ln x \cdot \frac{1}{x}\,dx$$

$$= -\left(-\frac{1}{x}\ln^2 x + 2\int \frac{1}{x^2} \cdot \ln x\,dx\right) \quad (1)$$

$$\begin{cases} u = \ln x \\ dv = \dfrac{1}{x^2}\,dx = x^{-2}\,dx \end{cases} \Rightarrow \begin{cases} du = \dfrac{1}{x}\,dx \\ v = \dfrac{x^{-1}}{-1} = -\dfrac{1}{x} \end{cases}$$

$$uv - \int v\,du = -\frac{1}{x}\ln x - \int -\frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x + \int x^{-2}\,dx$$

$$= -\frac{1}{x}\ln x + \frac{x^{-1}}{-1} = -\frac{1}{x}\ln x - \frac{1}{x} \quad (2)$$

$$\Rightarrow C_2(x) = \int C_2'(x)\,dx = \int \frac{\ln x}{x^2}\,dx$$

$$\begin{cases} u = \ln x \\ dv = \frac{1}{x^2}\,dx = x^{-2}\,dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\,dx \\ v = \frac{x^{-1}}{-1} = -\frac{1}{x} \end{cases}$$

$$uv - \int v\,du = -\frac{1}{x}\ln x - \int -\frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x + \int x^{-2}\,dx$$

$$= -\frac{1}{x}\ln x + \frac{x^{-1}}{-1} = -\frac{1}{x}\ln x - \frac{1}{x}$$

$$y(x) = C_1(x).x + C_2(x).(x\ln x)$$

$$\Rightarrow C_2(x) = \int C_2'(x)\,dx = \int \frac{\ln x}{x^2}\,dx$$

$$\begin{cases} u = \ln x \\ dv = \frac{1}{x^2}\,dx = x^{-2}\,dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}\,dx \\ v = \frac{x^{-1}}{-1} = -\frac{1}{x} \end{cases}$$

$$uv - \int v\,du = -\frac{1}{x}\ln x - \int -\frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln x + \int x^{-2}\,dx$$

$$= -\frac{1}{x}\ln x + \frac{x^{-1}}{-1} = -\frac{1}{x}\ln x - \frac{1}{x}$$

$$y(x) = C_1(x).x + C_2(x).(x\ln x)$$

$$= \left(\frac{1}{x}\ln^2 x - 2\left(-\frac{1}{x}\ln x - \frac{1}{x}\right)\right)x + \left(-\frac{1}{x}\ln x - \frac{1}{x}\right).(x\ln x)$$