DE AY1516 S1

Q1.

Read solution of DE AY1819 S2 (Midterm)

Q2.

Given that: $x^2y'' - xy' + y = 0 \ \ (*), \qquad x > 0$

Check for solution:

With $y_1 = 2015x\ln x$, it holds that: $y_1' = 2015(\ln x + 1)$; $y_1'' = \frac{2015}{x}$. Substituting into $(*)$, we get:

$$x^2.\frac{2015}{x} - x.2015(\ln x + 1) + 2015x\ln x = 0 \text{ (valid)}$$

With $y_2 = 2016x$, it holds that: $y_2' = 2016, y_2'' = 0$. Substituting into $(*)$, we get:

$$x^2.0 - x.2016 + 2016x = \text{(valid)}$$

So, y_1, y_2 are solutions of $(*)$ (1)

Check for linearity:

$$\mathrm{W}[y_1, y_2] = \begin{bmatrix} 2015x\ln x & 2016x \\ 2015(\ln x + 1) & 2016 \end{bmatrix} = -2015 \times 2016 \neq 0, \forall x > 0$$

So, y_1, y_2 are linearly independence (2)

From (1) and (2) , y_1, y_2 are linearly independence solutions of $(*)$

Thus, the general solution of $(*)$ is:

$$y_G = C_1x\ln x + C_2x$$

Q3.

a) Given that: $y^{(5)} - y^{(4)} + y''' - y'' = x - (x^2+1)e^x + 5\sin x$

$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x) + g_3(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y^{(5)} - y^{(4)} + y''' - y'' \\ g_1(x) = x \\ g_2(x) = -(x^2+1)e^x \\ g_3(x) = 5\sin x \end{cases}$

Characteristic equation of the given ODE: $r^5 - r^4 + r^3 - r^2 = 0$

$$\leftrightarrow r^2(r^2+1)(r-1) = 0$$

$$\leftrightarrow r_1 = i;\ r_2 = -i;\ r_3 = r_4 = 0;\ r_5 = 1$$

Since the right hand side of the given equation has three terms $g_1(x)$, $g_2(x)$ and $g_3(x)$, therefore the particular solution also has three terms: $y_p = y_{p1} + y_{p2} + y_{p3}$, respectively.

Solve fore y_{p1} from:

$$\mathrm{L}\left[y_{p1}\right] = g_1(x) \leftrightarrow {y_{p1}}^{(5)} - {y_{p1}}^{(4)} + {y_{p1}}''' - {y_{p1}}'' = x \ \ (\alpha = 0)$$

Since, $\alpha = 0$ is double root of characteristic equation.

Hence, y_{p1} has the following form: $y_{p1} = x^2(Ax + B)$

Solve fore y_{p2} from:

$$\mathrm{L}\left[y_{p2}\right] = g_2(x) \leftrightarrow {y_{p2}}^{(5)} - {y_{p2}}^{(4)} + {y_{p2}}''' - {y_{p2}}'' = -(x^2+1)e^x \qquad (\alpha = 1)$$

Since, $\alpha = 1$ is single root of characteristic equation.

Hence, y_{p2} has the following form: $y_{p2} = x(Cx^2 + Dx + E)e^x$

Solve fore y_{p3} from:

$$\mathrm{L}\left[y_{p3}\right] = g_3(x) \leftrightarrow {y_{p3}}^{(5)} - {y_{p3}}^{(4)} + {y_{p3}}''' - {y_{p3}}'' = 5\sin x \qquad (\alpha + i\beta = 0 + 1i = i)$$

Since, $\alpha + i\beta = i$ is a single root of characteristic equation.

Hence, y_{p3} has the following form: $y_{p2} = x(F\sin x + G\cos x)$

So: $\quad y_p = y_{p1} + y_{p2} + y_{p3}$

$$= Ax^2(Ax + B) + x(Cx^2 + Dx + E)e^x + x(F\sin x + G\cos x)$$

b) Given that: $\quad y^{(4)} - 2y''' + y'' = e^x + 1$

$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y^{(4)} - 2y''' + y'' \\ g_1(x) = e^x \\ g_2(x) = 1 \end{cases}$

Characteristic equation of the given ODE: $r^4 - 2r^3 + r^2 = 0$

$$\rightarrow r_1 = r_2 = 0;\ r_3 = r_4 = 1$$

So, the complement solution is: $y_c = C_1 + C_2x + C_3e^x + C_4xe^x$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two terms: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore y_{p1} from: $\mathrm{L}[y_{p1}] = g_1(x) \leftrightarrow y_{p1}^{(4)} - 2y_{p1}''' + y_{p1}'' = e^x \qquad (\alpha = 1)$

Since, $\alpha = 1$ is double root of characteristic equation.

So, y_{p1} has the following form: $y_{p1} = x^2Ae^x$

$$\rightarrow y_{p1}' = A(x^2 + 2x)e^x$$
$$\rightarrow y_{p1}'' = A(x^2 + 4x + 2)e^x$$
$$\rightarrow y_{p1}''' = A(x^2 + 6x + 6)e^x$$
$$\rightarrow y_{p1}^{(4)} = A(x^2 + 8x + 12)e^x$$

Substituting into the equation we obtain:

$$2Ae^x = e^x$$
$$\rightarrow 2A = 1 \leftrightarrow A = \frac{1}{2}$$

Therefore: $y_{p1} = \frac{1}{2}x^2e^x$

Solve fore y_{p2} from: $\mathrm{L}[y_{p2}] = g_2(x) \leftrightarrow y_{p1}^{(4)} - 2y_{p1}''' + y_{p1}'' = 1 \quad (\alpha = 0)$

Since, $\alpha = 0$ is double root of characteristic equation.

So, y_{p2} has the following form: $y_{p2} = Ax^2$

$$\rightarrow y_{p2}' = 2Ax$$
$$\rightarrow y_{p2}'' = 2A$$
$$\rightarrow y_{p2}'' = 0 = y_{p1}^{(4)}$$

Substituting into the equation we obtain:

$$0 - 0 + 2A = 1$$
$$\leftrightarrow A = \frac{1}{2}$$

Therefore: $y_{p2} = \frac{1}{2}x^2$

So: $\quad y_p = y_{p1} + y_{p2}$

$$= \frac{1}{2}x^2e^x + \frac{1}{2}x^2$$

Thus, the general solution of the given differential equation is:

$$y_G = y_c + y_p$$
$$= C_1 + C_2x + C_3e^x + C_4xe^x + \frac{1}{2}x^2e^x + \frac{1}{2}x^2$$

Q4.

$$\begin{cases} \dfrac{dx}{dt} = x - 8y & (1) \\ \dfrac{dy}{dt} = x - 3y & (2) \end{cases}$$

Differentiating both sides of (1), we get: $x'' = x' - 8y'$ (3).

Taking $8 \times (2) - 3 \times (1)$, we obtain: $8y' - 3x' = 5x \leftrightarrow 8y' = 3x' + 5x$ (4)

Substituting (4) into (3), it leads to:

$$x'' = x' - (3x' + 5x) \leftrightarrow x'' + 2x' + 5x = 0$$

Characteristic equation: $r^2 + 2r + 5 = 0 \rightarrow r_1 = 1 + 2i; r_2 = 1 - 2i$

Therefore: $x(t) = e^t(C_1 \sin 2t + C_2 \cos 2t)$

$$\rightarrow x'(t) = e^t\big((C_1 - 2C_2)\sin 2t + (2C_1 + C_2)\cos 2t\big)$$

From (1): $y(t) = \frac{1}{8}\big(x(t) - x'(t)\big) = \frac{1}{8}e^t(2C_2 \sin 2t - 2C_1 \cos 2t)$

Thus, the solution of the given system of differential equations is:

$$\begin{cases} x(t) = C_1e^t \sin 2t + C_2e^t \cos 2t \\ y(t) = 2C_2e^t \sin 2t - 2C_1e^t \cos 2t \end{cases}$$

Q5.

Given that:

$$y'' - 3y' + 2y = \frac{e^{2x}}{e^x + 1} \; (*)$$

Characteristic equation of the given DE: $r^2 - 3r + 2 = 0$

$$\rightarrow r_1 = 1;\; r_2 = 2$$

So, the complement solution is: $y_c = C_1e^x + C_2e^{2x}$ (1)

Multiply both sides of $(*)$ by e^{-x}, we get:

$$\rightarrow y''e^{-x} - e^{-x}y' - 2(y'e^{-x} - e^{-x}y) = \frac{e^x}{e^x + 1}$$
$$\leftrightarrow (y'e^{-x})' - 2(ye^{-x})' = \frac{e^x}{e^x + 1}$$

Integrating both sides, it leads to:

$$\rightarrow y'e^{-x} - 2ye^{-x} = \ln(e^x + 1) + C_1$$

Multiply both sides of $(*)$ by e^{-x} again, we get:

$$\rightarrow y'e^{-2x} - 2ye^{-2x} = e^{-x}\ln(e^x + 1) + C_1e^{-x}$$
$$\leftrightarrow (ye^{-2x})' = e^{-x}\ln(e^x + 1) + C_1e^{-x}$$

Integrating both sides, it leads to:

$$\rightarrow ye^{-2x} = -e^{-x}\ln(e^x + 1) + x - \ln(e^x + 1) - C_1e^{-x} + C_2$$
$$\leftrightarrow y = -e^x\ln(e^x + 1) + xe^{2x} - e^{2x}\ln(e^x + 1) + C_1'e^x + C_2e^{2x} \; (2)$$

Comparing (1) and (2), we obtain the particular solution:

$$y_p = -e^x\ln(e^x + 1) + xe^{2x} - e^{2x}\ln(e^x + 1)$$