CHAPTER 1: SEQUENCES AND SERIES

Lecturer: Nguyen Minh Quan, PhD
quannm@hcmiu.edu.vn

CONTENTS

Introduction

- Why do we need to study sequences and series?
- Series are used in signal processing applications. Many functions are defined as sums of series. The theory of infinite series is a third branch of calculus, in addition to differential and integral calculus.
- Main textbook: J. Stewart, Calculus, 8^{th} Edition, Cengage Learning, 2016. (Chapters: 11-16)

Reference for this Chapter: Chapter 11 of the textbook by J. Stewart, 8^{th} Edition (2016).

Sequences. Definition

Definition

A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The sequence $a_1, a_2, a_3, \ldots$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

The number a_1 is called the **first term**, a_2 is the **second term**, and in general a_n is the ***n*th term**. The integer n is called the **index** of a_n and indicates where a_n occurs in the list.

Sequences. Descriptions

Common methods for descriptions of the sequence

There are three common descriptions: using the preceding notation, using the defining formula, and writing out the terms of the sequence.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \dfrac{n}{n+1}$ $\quad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dots, \dfrac{n}{n+1}, \dots\right\}$

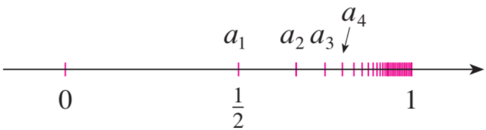
(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\quad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\quad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \dots, \dfrac{(-1)^n(n+1)}{3^n}, \dots\right\}$

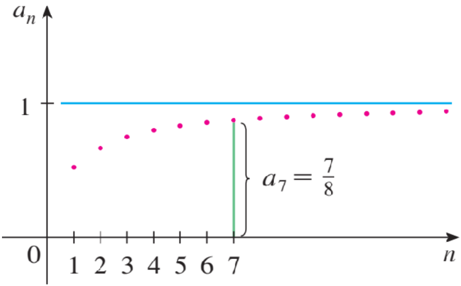
(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3},\ n \geqslant 3$ $\quad \left\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\right\}$

Note A sequence may begin with $n = 0$ or with some other integer values of n.

Sequences. Figure

A sequence can be pictured either by: Plotting its terms on a number line (upper figure) or plotting its (lower figure). For example, for the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$:





Sequences. Examples

Example

Find a formula for the general term a_n of the sequence

$$\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots\}$$

Answer: $a_n = (-1)^{(n+1)}\dfrac{n+2}{5^n}$

Example (The Fibonacci sequence)

The Fibonacci sequence does not have a simple definition. It is defined recursively by the conditions

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$$

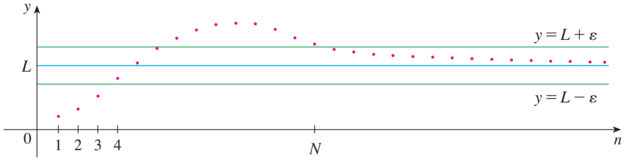
Sequences. Limit

Definition

A sequence $\{a_n\}$ has the **limit** L, and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there exists an integer N such that if $n > N$ then $|a_n - L| < \epsilon$.

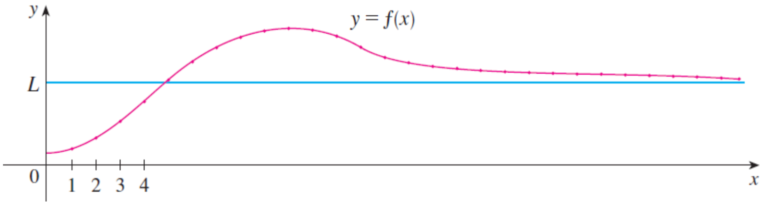


If $\lim_{n\to\infty} a_n$ exists, the sequence converges (or is convergent). Otherwise, it diverges (or is divergent).

Sequences. Convergence

Theorem

If $\lim_{x\to\infty} f(x) = L$ and define $a_n = f(n)$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.



Example

1. We have $\lim_{x\to\infty} \dfrac{x}{x+1} = 1$. Therefore, $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$.
2. By L'Hopital's rule, $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$. Therefore, $\lim_{n\to\infty} \dfrac{\ln n}{n} = 0$.

Sequences. Divergence

Example

Show that the sequence $a_n = (-1)^n$ is divergent.

Solution Write out the terms of the sequence:

$$\{-1, 1, -1, 1, -1, 1, -1, \dots\}$$

The terms oscillate between 1 and -1 infinitely often. Thus, a_n does not approach any number and $\lim_{n\to\infty}(-1)^n$ does not exist. That is, the sequence $\{(-1)^n\}$ is divergent.

Sequences. Properties

Definition

The sequence $\{a_n\}$ **diverges to infinity** if for every positive number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty.$$

Theorem

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

In particular, if $a_n \to L$ then $|a_n| \to |L|$.

Sequences. Properties

Properties

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then,

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad \text{if} \quad \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^{\alpha} = \Big[\lim_{n\to\infty} a_n\Big]^{\alpha} \quad \text{if } \alpha > 0 \text{ and } a_n > 0.$$

Example

Find $\lim_{n\to\infty} \frac{2n^2-16n+5}{4n^2+3n-1}$. (Answer: $1/2$)

Sequences. Squeeze theorem

Theorem

If

$$a_n \le b_n \le c_n \quad \textit{for } n \ge N$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Corollary

(a) *If* $|a_n| \le b_n$ *and* $b_n \to 0$*, then* $a_n \to 0$.

(b) $a_n \to 0$ *if and only if* $|a_n| \to 0$.

Sequences. Squeeze theorem

Example

For what values of r is the sequence $\{r^n\}$ convergent?

Hint: $\bullet r = 1$: Convergent (obviously); $\bullet 0 < r < 1$: Convergent (obviously); $\bullet -1 < r < 0$: Convergent by the Corollary of the Squeeze Theorem; $\bullet r \le -1$: Divergent (see slide #10 for a similar solution); $\bullet r > 1$: Divergent.
Answer: $-1 < r \le 1$.

Example

Discuss the convergence of the sequence

$$a_n = \frac{n!}{n^n}, \quad \text{where} \quad n! = 1 \cdot 2 \cdot 3 \cdots n.$$

Hint: $0 < a_n \le \dfrac{1}{n}$.

Sequences. Bounded sequences

Definition

A sequence $\{a_n\}$ is called:

- **increasing**, if $a_n < a_{n+1}$ for all n, that is, $a_1 < a_2 < a_3 < \cdots$;
- **decreasing**, if $a_n > a_{n+1}$ for all n, that is, $a_1 > a_2 > a_3 > \cdots$;
- **monotonic**, if it is either increasing or decreasing.

Definition

• A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$, for all n. (M is an **upper bound**.)

• $\{a_n\}$ is **bounded below** if there is a number m such that $a_n \geq m$ for all n. (m is a **lower bound**.)

• If it is bounded above and below, then $\{a_n\}$ is **bounded**.

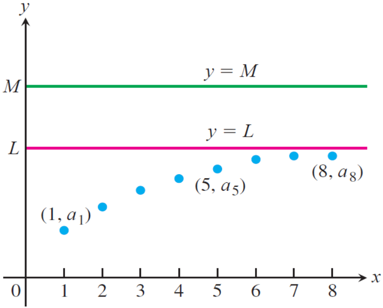
Remark $\{a_n\}$ is bounded if and only if there is a constant K such that $|a_n| \leq K$ for every n.

Sequences. Bounded and monotonic $\rightarrow$ Conv.

Theorem

- If $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \leq M$.
- If $\{a_n\}$ is decreasing and $a_n \geq m$ for all n, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \geq m$.

Thus, every bounded and monotonic sequence is convergent.



Series. Definition

Definition

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an **infinite series** (or just a **series**) and the numbers $a_1, a_2, \ldots$ are called the **terms** of the series. The number a_n is the ***n*th term** or the **general term** of the series.

For the series $\sum_{n=1}^{\infty} a_n$ above, the sum $s_n = \sum_{i=1}^{n} a_i$ is called the ***n*th partial sum** of the series. These partial sums form a new sequence $\{s_n\}$.

Q: Limit of $\{s_n\}$?

Series. Convergence/Divergence

Definition

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called **convergent** and we write:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s.$$

The number s is called the **sum** of the series. If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Thus, *the sum of a series is the limit of the sequence of partial sums:*

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Series. Convergence/Divergence

Example

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is convergent, and find its sum.

Hint: $s_n = 1 - \dfrac{1}{n+1}$. Its sum is $S = 1$.

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$ is divergent.

Hint: By induction, show that $s_{2^n} \geq 1 + \dfrac{n}{2}$.

Series. Note for the beginning index

Note

- Infinite series may begin with any index. For example,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}, \quad \sum_{n=9}^{\infty} \frac{1}{(n-8)^2}.$$

- When it does not necessary to specify the starting point, we write simply

$$\sum a_n.$$

- Any letter $i, j, k, p, m, n, \cdots$ can be used for the index. Thus we may write $a_i, a_j, a_k, a_p, a_m, a_n, \cdots$

Series. Consequence for a_n

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$.

It follows that If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

Example

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n-1}{3n+1} \quad \text{and} \quad \sum_{n=0}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$$

are convergent or divergent.

Answer: Divergent.

Series. Geometric series

Example

Geometric series are series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n, \quad a \neq 0,$$

in which a and r are fixed numbers. The constant r is called **the common ratio r**.

(a) *If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n$ is convergent, with sum $\frac{a}{1-r}$.*
(b) *If $|r| \geq 1$, and $a \neq 0$, then $\sum_{n=0}^{\infty} ar^n$ is divergent.*

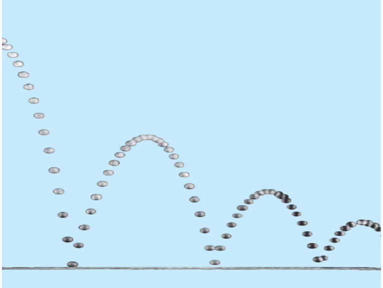
In other words, the sum of a convergent geometric series is

$$\frac{\textbf{first term}}{\mathbf{1} - \textbf{common ratio}}$$

Series. Geometric series

Example

You drop a ball from H meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down.



Answer: $$\frac{H}{1-r} + \frac{rH}{1-r} = H\frac{1+r}{1-r}.$$

Series. The role of the tail

Remark We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum.

That is, a finite number of terms doesn't affect the convergence or divergence of a series.

The convergence of a series depends only on the tail of the series.

In other words,

> $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$.

Series. Properties

Theorem

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

(i) $\sum ca_n = c \sum a_n$;

(ii) $\sum(a_n + b_n) = \sum a_n + \sum b_n$;

(iii) $\sum(a_n - b_n) = \sum a_n - \sum b_n$.

Example

Find

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{n(n+1)}\right)$$

Answer: 2

Series. Remainder

- Let us consider a convergent series

$$s = a_1 + a_2 + \cdots + a_n + \cdots$$

- The difference between the sum of the series and its nth partial sum is called the **remainder after *n* terms** or the ***n*th remainder** and is denoted by R_n:

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

- Since $s_n \to s$, $|r_n| = |s - s_n| \to 0$ as $n \to \infty$. Thus *it is always to possible to* compute an approximation to the sum of a convergent series by taking a sufficiently large number of its first terms.

Series of Nonnegative terms

1. Does the series converge? What is its sum?
2. How to control the errors involved in using the partial sums to approximate the sum of the series?

Next, we study nonnegative series, whose all terms are nonnegative.

Theorem

A series of nonnegative terms converges if and only if its partial sums S_n are bounded from above.

Example

Test the series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}2^n}$ for convergence or divergence.

Hint: $\dfrac{1}{\sqrt{n}2^n} \leq \dfrac{1}{2^n}$. Convergent.

Series of Nonnegative terms. Integral test.

Theorem (the Integral test)

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, nonincreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the improper integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

In other words,

(i) If $\int_N^{\infty} f(x)dx$ is convergent, then $\sum_{n=N}^{\infty} a_n$ is convergent;

(ii) If $\int_N^{\infty} f(x)dx$ is divergent, then $\sum_{n=N}^{\infty} a_n$ is divergent.

Series of Nonnegative terms. Integral test.

Example

Test the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

for convergence or divergence.
Hint: $\int_2^{\infty} \frac{1}{x \ln x} dx$ is divergent.

Example

Test the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

for convergence or divergence.
Hint: $\int_1^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{4}$ is convergent.

Series of Nonnegative terms

p-Series and Harmonic Series

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is a ***p*-series**, where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

is the **harmonic** series.

Series of Nonnegative terms

Theorem (*p*-series)

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \iff p > 1.$$

Remark The harmonic series $\sum \frac{1}{n}$ (the case $p = 1$ of the p-series) is on the borderline between convergence and divergence, although it diverges.

Series of Nonnegative terms. Comparison test

We consider some comparison tests. Their idea is to compare a given series with one that is known to be convergent or divergent.

Theorem (The Comparison Test)

Let $\sum a_n$ and $\sum b_n$ be series with nonnegative terms such that $a_n \leq b_n$ for all $n \geq N$.

(a) *If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.*

(b) *If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.*

The Comparison Test states that for series with nonnegative terms,

(i) Convergence of larger series forces convergence of smaller series;

(ii) Divergence of smaller series forces divergence of larger series.

Series of Nonnegative terms. Comparison test

Example

Test the given series for convergence or divergence:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}.$$

Hint: Compare with the $p-$*series* for $p = 1$ by using $\frac{\ln n}{n} \geq \frac{1}{n}$.
Divergent.

Series of Nonnegative terms. Comparison test

Theorem (The Limit Comparison Test)

Suppose that $\sum a_n$ *and* $\sum b_n$ *are series with nonnegative terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K$$

where K *is a finite number and* $K > 0$*, then either both series converge or both diverge.*

In words,

if the limit of the ratio of a_n to b_n exists and is positive, then the series $\sum a_n$ and $\sum b_n$ are simultaneously convergent or divergent.

Series of Nonnegative terms. Comparison test

Example

Test the given series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{2n^2 - 5\sqrt{n} + 1}{7n^4 - 3n^3 + 9n - 1}.$$

Hint: One can use the Limit Comparison Test with $b_n = \dfrac{2}{7n^2}$.

Example

Test the series $\sum_{n=1}^{\infty} \dfrac{1 + \sin n}{n^2}$ for convergence or divergence.

Hint: $\dfrac{1 + \sin n}{n^2} \leq \dfrac{2}{n^2}$.

Alternating series

Definition

An **alternating series** is a series whose terms are alternately positive and negative.

Two examples for alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$-1 + \frac{1}{2} - \frac{1}{2^2} + \cdots + \frac{(-1)^{n+1}}{2^n} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

Alternating Series Test

Form for alternating series: $a_n = (-1)^{n+1} b_n$ or $a_n = (-1)^n b_n$, where b_n is a positive number. In fact, $b_n = |a_n|$.

Theorem (Alternating Series Test)

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} c_n = c_1 - c_2 + c_3 - c_4 + \cdots + (-1)^{n+1} c_n + \cdots$$

satisfies

(i) $0 < c_{n+1} \leq c_n$ for all n and

(ii) $\lim_{n\to\infty} c_n = 0$,

then the series is convergent. Furthermore, the sum S of the series satisfies $0 < S < c_1$.

Alternating Series Test

Example

Show that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is convergent.

Solution: The alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

satisfies

i) $0 < c_{n+1} < c_n$ because $\frac{1}{n+1} < \frac{1}{n}$;

ii) $c_n = \frac{1}{n} \to 0$.

Thus, it is convergent by the Alternating Series Test.

Absolute convergence

Definition

- $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent.
- $\sum a_n$ is **conditionally convergent** if it is convergent but $\sum |a_n|$ is divergent.

Theorem

If a series is absolutely convergent, then it is convergent.

Example Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{3}}{n^2} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2 \cdot 2^2} + 0 - \frac{\sqrt{3}}{2 \cdot 4^2} - \frac{\sqrt{3}}{2 \cdot 5^2} + 0 + \cdots$$

is convergent or divergent.

Answer: Convergent.

Absolute convergence. Examples

Example When $p > 0$, the sequence $\left\{\frac{1}{n^p}\right\}$ is a decreasing sequence with limit zero. Therefore the **alternating *p*-series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

- For $p > 1$, the series converges absolutely.
- For $0 < p \leq 1$, the series converges but not absolutely (conditionally convergent).

The Ratio Test

Theorem (The Ratio Test)

Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

Then

(a) the series absolutely converges if $\rho < 1$,

(b) the series diverges if $\rho > 1$,

(c) the series may converge or it may diverge if $\rho = 1$.

In case (c) the test provides no information.

The Ratio Test. Examples

Example

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

Solution We compute the limit ρ. Let $a_n = 1/n!$. Then

$$\frac{a_{n+1}}{a_n} = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1}$$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

Thus the series us convergent by the Ratio Test.

The Root Test

Theorem (The Root Test)

Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

Then

(i) the series absolutely converges if $\rho < 1$,

(ii) the series diverges if $\rho > 1$,

(iii) the test is not conclusive if $\rho = 1$.

Example Determine whether $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converges.

Note If $\rho = 1$ in the Ratio Test, don't try the Root Test because ρ will again be 1.

Conclusion remarks: Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth term approaches 0? If not, the series diverges.
2. Is the series one of the special types- geometric, p-series, telescoping, or alternating?
3. Can the series be compare to one of the special types?
4. Can the Integral Test, the Ratio Test, or the Root Test be applied?

In some instances, more than one test is applicable.

Power series. Definition

Definition

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \cdots$$

is called a **power series in $(x-a)$** or a **power series centered at a** or a **power series about a**.

Power series. Domain

- For each fixed x, a power series becomes a series of constants that we can test for convergence or divergence.

- A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

  whose domain is the set of all x for which the series converges.

- The domain contains a since the power series converges for $x = a$.

Power series. Examples

Example The power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Thus,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}, \ -1 < x < 1.$$

Power series. Examples

Example

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots;$

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots;$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots;$

(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots.$

Answer: (a) $-1 < x \leq 1$ (b) $-1 \leq x \leq 1$
(c) $-\infty < x < \infty$ (d) $\{0\}$

Power series. Possibilities of Convergence

Theorem

For a given power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

there are only three possibilities:

(i) *The series converges only when $x = a$.*

(ii) *The series converges for all x.*

(iii) *There is a positive number R such that the series absolutely converges if $|x-a| < R$ and diverges if $|x-a| > R$.*

In Case (i), set $R = 0$, and in Case (ii), set $R = \infty$. We call R the **radius of convergence** *of the power series.*

The set of values x for which the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is an interval centered at $x = a$. We call this interval the **interval of convergence** of the power series.

Power series. Examples

Example

Determine the center, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}.$$

Answer. $a = -5/2$, $R = 3/2$. The interval of convergence is $-4 \leq x \leq -1$.

Important remark: If $\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \rho$ (or, $\lim_{n\to\infty} \sqrt[n]{|c_n|} = \rho$), where $0 < \rho < 1$, then the **radius of convergence is $\mathbf{R = 1/\rho}$.**

Power series. Examples

How to Test a Power Series for Convergence?

1. If the interval of convergence is finite, test for convergence or divergence at each of the two endpoints. Neither the Ratio Test nor the Root Test helps at these points. Use a Comparison Test, the Integral Test, or the alternating Series Test.

2. If the radius of convergence is R, the series diverges for $|x - a| > R$.

Power series for $f'(x)$ and $\int f(x)dx$

Theorem. If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$\int f(x)dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \cdots$$

These series have the same radius of convergence R.

Power series for $f'(x)$ and $\int f(x)dx$

Note for the theorem in slide #52

1. It follows from the first part of the previous Theorem in slide #52 that

   > *Every power series is infinitely termwise differentiable inside its interval of convergence.*

2. Although this Theorem says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same.

Power series for $f'(x)$ and $\int f(x)dx$

Example Find a power series representation for $\ln(1-x)$ and its radius of convergence.

Solution Since $f(x) = \int f'(x)dx$ and $\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$, we have

$$-\ln(1-x) = \int \frac{1}{1-x}dx = \int \left(1 + x + x^2 + \cdots\right)dx$$
$$= C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = C + \sum_{n=1}^{\infty} \frac{x^n}{n}. \qquad |x| < 1$$

Setting $x = 0$ gives $C = 0$ and hence,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad |x| < 1.$$

The radius of convergence is the same as for the original series: $R = 1$.

Power series for $f'(x)$ and $\int f(x)dx$

Example

Find a power series representation for $\frac{1}{(1-x)^2}$ and its radius of convergence.

Solution Differentiating each side of the equation term by term

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

we get the power series for $\dfrac{1}{(1-x)^2}$:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n, \quad -1 < x < 1.$$

The radius of convergence is $R = 1$.

Power series for $f'(x)$ and $\int f(x)dx$

Example Identify the function

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \; -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \\ &= \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}, \qquad -1 < x < 1, \\ f(x) &= \int f'(x)dx = \int \frac{1}{1+x^2} dx = \tan^{-1} x + C. \end{aligned}$$

The series for f is zero when $x = 0$, so $C = 0$. Hence
$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \tan^{-1} x, \quad -1 \le x \le 1.$

Taylor and Maclaurin Series

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in the interval $(a-R, a+R)$, we say that the power series is a **representation** of $f(x)$ on that interval and f is **expanded into the power series**.

In this section we investigate general problems:

1. Which functions have power series representations?
2. How can we find such representations?

Moreover, we will show how to control the errors involved in using the partial sums of the series to approximate the functions the series represent.

Taylor Series and Maclaurin Series

Suppose that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, \quad |x-a| < R.$$

Question: Can we determine c_n via $f(x)$?

Theorem

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Taylor Series and Maclaurin Series

Thus, if f has a power series expansion at a, then it must be of the form.

$$\boxed{\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}}$$

This series is called the **Taylor series of the function f at a** (or **about a** or **centered at a**).
For $a = 0$ the Taylor series is called the **Maclaurin series**:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots}$$

Taylor Series and Maclaurin Series

Thus, we have established the following result:

> *If a function $f(x)$ can be expanded in to a series in power of $x - a$, this series is uniquely determined and necessary coincides with the Taylor series of this function at a.*

Example Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Answer:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \cdots ; \quad -\infty < x < \infty$$

Taylor Series and Maclaurin Series

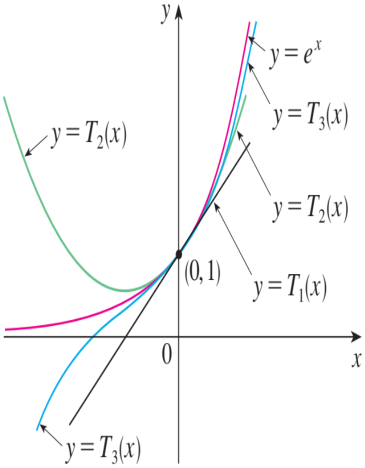
Question: When is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

Definition

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at a is the polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor Series and Maclaurin Series



$f(x) = e^x$ and its Taylor polynomials $T_1(x)$, $T_2(x)$, and $T_3(x)$.

Taylor Series and Maclaurin Series

Example

Find the Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

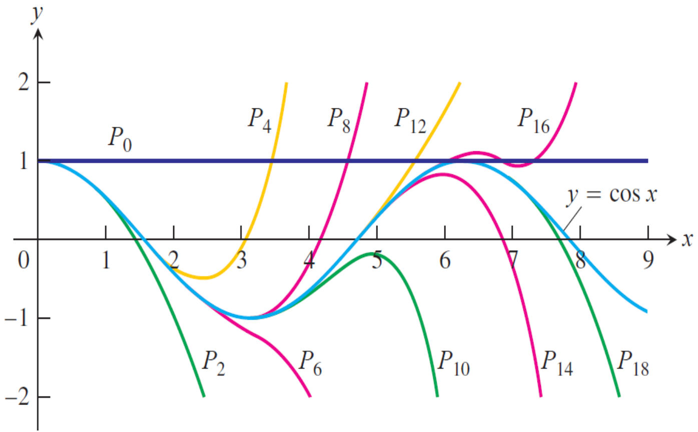
Solution The cosine and its derivatives are

$$\begin{array}{rclrcl} f(x) & = & \cos x, & f'(x) & = & -\sin x \\ f''(x) & = & -\cos x, & f''(x) & = & \sin x \\ & \vdots & & & \vdots & \\ f^{(2n)}(x) & = & (-1)^n \cos x, & f^{(2n+1)}(x) & = & (-1)^{n+1} \sin x. \end{array}$$

Therefore, $f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$
Hence the Taylor polynomials of orders $2n$ and $2n+1$ are identical:

$$T_{2n}(x) = T_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Taylor Series and Maclaurin Series



T_n or P_n approaches $f(x)$ as n is increasing

Taylor Series and Maclaurin Series

Clearly, f is the sum of its Taylor series at a if

$$f(x) = \lim_{n\to\infty} T_n(x), \qquad |x-a| < R.$$

If we let

$$R_n(x) = f(x) - T_n(x)$$

then $R_n(x)$ is called the **remainder of the Taylor series** and we have

$$f(x) = T_n(x) + R_n(x).$$

Taylor Series and Maclaurin Series

Theorem

If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the Taylor polynomial of order n of f at a and

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \text{for} \quad |x-a| < R,$$

then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Note that the value of $R_n(x)$ is exactly equal to the error appearing when the function $f(x)$ is replaced by Taylor's polynomial $T_n(x)$.

Taylor Series and Maclaurin Series

Theorem

Let $f(x)$ be an infinitely differentiable function on the open interval $I = (a - \alpha, a + \alpha)$ with $\alpha > 0$. Assume there exists a constant $K \geq 0$ such that for all n,

$$\boxed{\left|f^{(n)}(x)\right| \leq K \quad \text{for all} \quad x \in I.}$$

Then $f(x)$ is represented by its Taylor series in I:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n \quad \text{for all} \quad x \in I.$$

Taylor Series and Maclaurin Series

Example Show that the following Maclaurin expressions are valid for all x:

$$\sin x = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Solution For $f(x) = \sin x$, we have

$$f^{(2n)}(x) = (-1)^n \sin x, \qquad f^{(2n+1)}(x) = (-1)^n \cos x.$$

Therefore, $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$. The nonzero Taylor coefficients for $\sin x$ are $c_{2n+1} = \frac{(-1)^n}{(2n+1)!}$.

Similarly, for $f(x) = \cos x$,

$$f^{(2n)}(x) = (-1)^n \cos x, \qquad f^{(2n+1)}(x) = (-1)^{n+1} \sin x.$$

Taylor Series and Maclaurin Series

Therefore, $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$. The nonzero Taylor coefficients for $\cos x$ are $c_{2n} = \dfrac{(-1)^n}{(2n)!}$.

In both cases, $|f^{(n)}(x)| \leq 1$ for all x and n. Thus one can apply the theorem in slide #67 with $K = 1$ and any α to conclude that the Maclaurin series converges to $f(x)$ for $|x| < \alpha$.

Since α is arbitrary, the Maclaurin expansions hold for all x.

Taylor Series and Maclaurin Series

Example

Find the Taylor series of e^x at $x = a$.

Answer: $\sum_{k=0}^{\infty} \frac{e^a}{n!}(x-a)^n, \qquad -\infty < x < \infty.$

In particular, we have arrived at series for e^x:

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \ -\infty < x < \infty,}$$

and for e,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots, \quad -\infty < x < \infty.$$

Taylor Series and Maclaurin Series

Example

Find the Taylor series for $\ln x$ in powers of $x = 2$. Where does the series converge to $\ln x$?

Answer:

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \times 2^n}(x-2)^n, \quad 0 < x \leq 4.$$

Taylor Series and Maclaurin Series

Binomial Series

For any number α (integer or not) and nonnegative integer n, we define the **binomial coefficient**

$$\left(\begin{array}{c} k \\ n \end{array} \right) = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}, \ \left(\begin{array}{c} k \\ 0 \end{array} \right) = 1.$$

For example,

$$\left(\begin{array}{c} 6 \\ 3 \end{array} \right) = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20, \ \left(\begin{array}{c} 4/3 \\ 3 \end{array} \right) = \frac{\frac{4}{3} \cdot \frac{1}{3} \cdot (-\frac{2}{3})}{1 \cdot 2 \cdot 3} = -\frac{4}{81}.$$

Taylor Series and Maclaurin Series

The **traditional Binomial Theorem** states that for any integer number $k > 0$,

$$(a+b)^k = a^k + \binom{k}{1} a^{k-1}b + \binom{k}{2} a^{k-2}b^2 + \cdots + \binom{k}{k-1} ab^{k-1} + b^k.$$

Setting $a = 1$ and $b = x$, we obtain an expansion of $f(x) = (1+x)^k$:

$$(1+x)^k = 1 + \binom{k}{1} x + \binom{k}{2} x^2 + \cdots + \binom{k}{k-1} x^{k-1} + x^k.$$

Taylor Series and Maclaurin Series

Now, for any real number k, consider the function $f(x) = (1+x)^k$ without assuming that k is a whole number. We have

$$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$$
$$f^{(n)}(0) = k(k-1)\cdots(k-n+1).$$

- If k is a nonnegative integer, then the series stops after $k+1$ terms because the coefficients from $n = k+1$ on are zero.
- If k is not a nonnegative integer, then the series is infinite and
$$\frac{f^{(n)}(0)}{n!} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} = \binom{k}{n}.$$

Taylor Series and Maclaurin Series

Hence, the Maclaurin series of $f(x) = (1+x)^k$ is

$$\boxed{\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots}$$

This series is called the **binomial series**.

The Ratio Test shows that this series has radius of convergence $R = 1$. Furthermore, the binomial series converges to $(1+x)^k$ for $|x| < 1$.

Taylor Series and Maclaurin Series

Theorem (The Binomial Series)

If k is any real number and $|x| < 1$, then

$$\boxed{\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n\end{aligned}}$$

Taylor Series and Maclaurin Series

Example

Find the Maclaurin series for

$$\frac{1}{\sqrt{1+x}}.$$

Answer: For $-1 < x \leq 1$,

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n (n!)} x^n.$$

Example

Find the Maclaurin series for $\sin^{-1} x$.

Answer: $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40} x^5 + \cdots \qquad (-1 < x < 1).$

Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

–END OF CHAPTER 1–