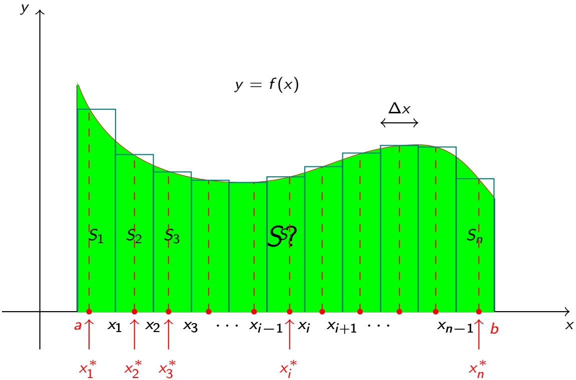
Chapter 4: INTEGRATION

Duong T. PHAM

CALCULUS I

Outline

Chapter 4 (Integration):
Riemann sums,
Definition of Definite Integral,
Properties of Definite Integral,
Fundamental Theorem of Calculus,
Techniques of integration (Substitution, integration by parts, trigonometric integrals, trigonometric substitution, partial fractions),
Numerical (approximate) integrals (Midpoint, Trapezoidal rule, perhaps can omit Simpson's rule),
Improper integrals.



- $R_n = f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_i^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x$
- $S = \lim\limits_{n\to\infty} R_n$

Definition.

Let $f : [a, b] \to \mathbb{R}$. Divide $[a, b]$ by $x_0, x_1, \ldots, x_n$ into n equal subintervals of width

$$\Delta_x = \frac{b-a}{n}$$

$(a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b)$.

- Let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** ($x_i^* \in [x_{i-1}, x_i]$)
- The **definite integral of f from a to b is**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

if this limit exists. In this case, we say f **is integrable on** $[a, b]$

- The sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is called **Riemann sum**.

Theorem.

- If f is continuous on $[a, b]$, then f is integrable on $[a, b]$
- If f is continuous on $[a, b]$, except at a finite number of points and f is bounded, then f is integrable on $[a, b]$

Theorem.

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Ex: Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be the right endpoints with $a = 0$, $b = 3$ and $n = 6$

- Evaluate $\int_0^3 (x^3 - 6x)\, dx$

Ans:

(a) For $n = 6$, the interval width is $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2} = 0.5$, and the right endpoints are

$$x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2,\ x_5 = 2.5,\ x_6 = 3.$$

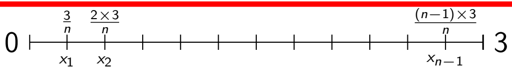
The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x = \Delta x\Big(f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)\Big)$$

$$= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9\ ) = -3.9375$$

Ex: Evaluate $\int_0^3 (x^3 - 6x)\, dx$

Ans:



With n subintervals, $\Delta x = \frac{3}{n}$, and

$$x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = 2\frac{3}{n}, \dots,\ x_i = i\frac{3}{n}, \dots,\ x_n = 3.$$

$$\int_0^3 (x^3 - 6x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \left[\frac{81}{n^4}\left[\frac{n(n+2)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{2}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right] = \frac{81}{4} - 27 = -\frac{27}{4}$$

Properties of Integrals

Properties of Integrals:

1. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ → Đổi cận
2. $\displaystyle\int_a^a f(x)\,dx = 0$
3. $\displaystyle\int_a^b c\,dx = c(b-a)$
4. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
5. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
6. $\displaystyle\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
7. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Comparison properties:

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$,
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$,
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

5.2:

- 1–4, 9–12 21–22, 27–28, 35–40, 52–54

Theorem.

If f is continuous on $[a, b]$, then the function g defined by

$$\boxed{g(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b}$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

Proof: Let x and $x + h$ in (a, b) (Suppose $h > 0$) . Then

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

So, $\dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt$. Since f is continuous on $[x, x+h]$, by the extreme value theorem, there are $u, v \in [x, x+h]$ such that

$$f(u) = \min\{f(t) : t \in [x, x+h]\} \text{ and } f(v) = \max\{f(t) : t \in [x, x+h]\}$$

(continuing) We have $f(u) \leq f(t) \leq f(v)$ for $x \leq t \leq x+h$. Hence,

$$\int_x^{x+h} f(u)dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} f(v)dt$$

$$f(u)h \leq \int_x^{x+h} f(t)\,dt \leq f(v)h \Rightarrow f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v)$$

$$\Rightarrow f(u) \leq \frac{g(x+h)-g(x)}{h} \leq f(v) \quad (\star)$$

When $h \to 0$, since $u, v \in [x, x+h]$, we have $u \to x$, $v \to x$. Note that f is continuous on $[a, b]$, thus $f(u) \to f(x)$ and $f(v) \to f(x)$. This together with $(\star)$ yields

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x).$$

This means that $g(x)$ is differentiable (and then continuous) at $x \in (a, b)$ and $g'(x) = f(x)$.

The cases $x = a$ and $x = b$ can be proved in the same manner, using one-sided limits.

Ex: Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Ans: Since $f(t) = \sqrt{1+t^2}$ is continuous, Part 1 of the Fundamental theorem of calculus gives

$$g'(x) = f(x) = \sqrt{1+x^2}.$$

Ex: Find $\dfrac{d}{dx}\displaystyle\int_1^{x^2} \sin t\,dt$

Ans: Denote $u(x) = x^2$. Using the chain rule,

$$\begin{aligned}\frac{d}{dx}\int_1^{u(x)} \sin t\,dt &= \frac{d}{du}\left(\int_1^u \sin t\,dt\right)\frac{du}{dx}\\ &= \sin u\,\frac{d}{dx}(x^2) = \sin(x^2)\,2x\\ &= 2x\sin x^2\end{aligned}$$

The fundamental theorems of calculus, $\boxed{\text{part II}}$

Theorem.

If f is continuous on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where F is any antiderivative of f.

Proof: Denote $g(x) = \int_a^x f(t)\,dt$. By part 1, $g'(x) = f(x)$. It means that g is another antiderivative of f. Thus $g(x) = F(x) + C$ for some constant C. Then

$$\begin{aligned} F(b) - F(a) &= [g(b) - C] - [g(a) - C] = g(b) - g(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt \end{aligned}$$

Ex: Evaluate $\int_1^3 e^x \, dx$

Ans: The function $f(x) = e^x$ is continuous everywhere and its antiderivative is $F(x) = e^x$. Thus

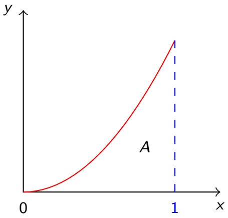
$$\int_1^3 e^x \, dx = e^3 - e^1$$

Remark: People usually the notation

$$F(x)\Big|_a^b = F(b) - F(a)$$

Ex: Find the area under the parabola $y = x^2$ from 0 to 1

Ans:



The area

$$A = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

The fundamental theorems of calculus: Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$
2. If $F' = f$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.

Remark: We note that the two above statements can be written as follows:

- $\dfrac{d}{dx} \int_a^x f(t)\, dt = f(x)$
- $\int_a^b F'(x)\, dx = F(b) - F(a)$.

Exercises

5.3:

- 2, 4, 7–18, 19–30, 41–42, 53–56

Notation: we denote $\int f(x)\,dx = F(x)$ to indicate that $F' = f$.

Ex: We can write $\int x^2\,dx = \dfrac{x^3}{3} + C$ as $\dfrac{d}{dx}\left(\dfrac{x^3}{3} + C\right) = x^2$

Ex: $\int \sec^2 x\,dx = ?\ \tan x + C$

because $\dfrac{d}{dx}(\tan x + C) = \sec^2 x$

Indefinite Integrals

Indefinite integral table:

(i) $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$

(ii) $\displaystyle\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(iii) $\displaystyle\int k\,dx = kx + c$

(iv) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \ (n \neq -1)$

(v) $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c$

(vi) $\displaystyle\int e^x\,dx = e^x + c$

(vii) $\displaystyle\int a^x\,dx = \frac{e^x}{\ln a} + c$

(viii) $\displaystyle\int \sin x\,dx = -\cos x + c$

(ix) $\displaystyle\int \cos x\,dx = \sin x + c$

(x) $\displaystyle\int \sec^2 x\,dx = \tan x + c$

(xi) $\displaystyle\int \csc^2 x\,dx = -\cot x + c$

(xii) $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

(xiii) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

Ex: Evaluate $\int_1^3 (x^3 - 6x)\,dx$

Ans:

$$\int_1^3 (x^3 - 6x)\,dx = \frac{x^4}{4} - 3x^2 \bigg|_1^3 = \left(\frac{81}{4} - 27\right) - \left(\frac{1}{4} - 3\right) = -6.75$$

Ex: Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$

Ans:

$$\begin{aligned}\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt &= \int_1^9 \left(2 + \sqrt{t} - t^{-2}\right)\,dt \\ &= 2t + \frac{2}{3}t^{3/2} + t^{-1}\bigg|_1^9 \\ &= 32\frac{4}{9}\end{aligned}$$

- Let $y = F(x)$. Then F' represents the rate of change of $y = F(x)$ w.r.t. x.
- $F(b) - F(a)$ is the change in y when x changes from a to b.

The net change theorem: The integral of a rate of change is the net change:

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So
$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$
is the change in amount of water in the reservoir between t_1 and t_2.
- If $[C](t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[C]/dt$. So
$$\int_{t_1}^{t_2} \frac{d[C]}{dt}\,dt = [C](t_2) - [C](t_1)$$
is the change in the concentration of $[C]$ from time t_1 to time t_2.

Some applications

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density $\rho(x) = m'(x)$. So
$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$
is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

- If the rate of growth of a population is dn/dt, then
$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$
is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

Exercises

5.4: 1–10, 21–30, 48, 49–52

The substitution rule

The substitution rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u) du$$

Proof: Suppose that F is an antiderivative of f, i.e., $F' = f$. Then the chain rule

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))\, g'(x).$$

This implies

$$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C.$$

Using $u = g(x)$, we obtain

$$\int F'(g(x))\, g'(x)\, dx = F(u) + C = \int F'(u)\, du = \int f(u)\, du$$

Ex: Find $\int x^3 \cos(x^4+2)\,dx$

Ans: We use a change of variables $u = x^4 + 2$. Then $du = 4x^3\,dx$. Thus $x^3\,dx = \frac{du}{4}$. Using the change of variables, we obtain

$$\begin{aligned}\int x^3 \cos(x^4+2)\,dx &= \int \cos u \frac{du}{4}\\ &= \frac{1}{4}\int \cos u\,du\\ &= \frac{1}{4}\sin u + C\\ &= \frac{1}{4}\sin(x^4+2) + C.\end{aligned}$$

The substitution rule

Ex: Find $\int \sqrt{2x+1}\,dx$

Ans: We use a change of variables $u = 2x+1$. Then $du = 2\,dx$. Thus $dx = \frac{du}{2}$. Using the change of variables, we obtain

$$\begin{aligned}
\int \sqrt{2x+1}\,dx &= \int \sqrt{u}\frac{du}{2} \\
&= \frac{1}{2}\int \sqrt{u}\,du \\
&= \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.
\end{aligned}$$

The substitution rule for definite integrals

The substitution rule for definite integrals: If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof: Let F be an antiderivative of f. Then

$$\frac{d}{dx} F(g(x)) = F'(g(x))\, g'(x) = f(g(x))\, g'(x).$$

Fundamental Theorem for Calculus (part 2) gives

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(b)) - F(g(a)).$$

Since $F' = f$, using Fund. Theo. for Cal. (II) again, we have

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).$$

Comparing the two above equalities yielding the conclusion.

Ex: Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$

Ans: Denote $u = 3-5x$. Then $du = -5\,dx \Longrightarrow dx = -\frac{du}{5}$.
The endpoints are $u(1) = -2$, $u(2) = -7$. Applying the substitution rule, we obtain

$$\begin{aligned}
\int_1^2 \frac{dx}{(3-5x)^2} &= \int_{-2}^{-7} \frac{-\frac{du}{5}}{u^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} \\
&= -\frac{1}{5}\left(-\frac{1}{u}\right)\Big|_{-2}^{-7} = \frac{1}{5}\left(\frac{1}{u}\right)\Big|_{-2}^{-7} \\
&= \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) \\
&= \frac{1}{14}
\end{aligned}$$

Integrals of symmetric functions: Let f be a continuous funtion on $[-a, a]$.

1. If f is even $(f(-x) = f(x))$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
2. If f is odd $(f(-x) = -f(x))$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof:

1. (f is even). We have $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$.
Denote $u(x) = -x$. Then $du = -dx$ and $u(-a) = a$, $u(0) = 0$.
Noting that $f(-u) = f(u)$ and using substitution rule, we obtain

$$\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-u)(-du) = -\int_a^0 f(-u)du = -\int_a^0 f(u)du$$
$$= \int_0^a f(u)du = \int_0^a f(x)dx.$$

Hence, $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

Exercises

5.5: 1–15, 51–60, 73–74, 77, 86.

The product gives

$$\frac{d}{dx}[f(x)g(x)] = f'(x)\,g(x) + f(x)g'(x).$$

Then

$$\int [f'(x)\,g(x) + f(x)g'(x)]\,dx = f(x)g(x)$$

Integration by parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Remark: Another form of the integration by parts is

$$\int u\,dv = uv - \int v\,du$$

Ex: Evaluate $\int x \sin x \, dx$

Ans: Let $u(x) = x$ and $v'(x) = \sin x$. Then $u'(x) = 1$ and $v(x) = -\cos x$. Hence,

$$\begin{aligned}\int x \sin x \, dx &= u(x)v(x) - \int v(x)u'(x) \, dx \\ &= x(-\cos x) - \int (-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C.\end{aligned}$$

Integration by parts

Ex: Evaluate $\int \ln x \, dx$

Ans: Denote $u = \ln x$ and $dv = dx$. Then $du = \dfrac{dx}{x}$ and $v = x$.
Integration by parts gives

$$\begin{aligned}
\int \ln x \, dx &= x \ln x - \int x \frac{dx}{x} \\
&= x \ln x - \int dx \\
&= x \ln x - x + C
\end{aligned}$$

Integration by parts

Ex: Evaluate $\int t^2 e^t \, dt$

Ans: Denote $u = t^2$ and $dv = e^t \, dt$. Then $du = 2t \, dt$ and $v = e^t$. Integration by parts gives

$$\int t^2 e^t \, dt = t^2 e^t - \int 2te^t \, dt = t^2 e^t - 2 \int te^t \, dt.$$

Denote $u = t$ and $dv = e^t \, dt$. Then $du = dt$ and $v = e^t$. Applying integration by parts again yields

$$\int te^t \, dt = te^t - \int e^t \, dt = te^t - e^t + C.$$

Hence,

$$\int t^2 e^t \, dt = t^2 e^t - 2(te^t - e^t + C) = t^2 e^t - 2te^t + 2e^t + C_1$$

Integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ex: Evaluate $\displaystyle\int_0^{\pi/2} e^x \sin x\,dx$

Ans: Denote $u = e^x$ and $dv = \sin x$. Then $du = e^x\,dx$ and $v = -\cos x$. We have

$$\int_0^{\pi/2} e^x \sin x\,dx = -e^x \cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} e^x(-\cos x)dx = 1 + \int_0^{\pi/2} e^x \cos x\,dx$$

Denote $u = e^x$ and $dv = \cos x$. Then $du = e^x\,dx$ and $v = \sin x$. Thus

$$\int_0^{\pi/2} e^x \cos x\,dx = e^x \sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx = e^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx$$

Hence, $\int_0^{\pi/2} e^x \sin x\,dx = 1 + e^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx$. This implies

$$\int_0^{\pi/2} e^x \sin x\,dx = \frac{1+e^{\pi/2}}{2}$$

Exercises

7.1: 1–20, 33–38, 44–45,

Ex: Evaluate $\displaystyle\int \sin^2 x \cos^3 x \, dx$

Ans: Noting that $\cos^2 x = 1 - \sin^2 x$,

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x \,(1 - \sin^2 x) \cos x \, dx.$$

Denote $t = \sin x$. Then $dt = \cos x dx$ and

$$\int \sin^2 x \cos^3 x \, dx = \int t^2(1 - t^2) \, dt = \int (t^2 - t^4) dt = \frac{t^3}{3} - \frac{t^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \cos^{2k} x \cos x \, dx = \int \sin^m x \,(1 - \sin^2 x)^k \cos x \, dx$$

$$= \int t^m (1 - t^2)^k \, dt \quad \text{(by substitution } t = \sin x)$$

Ex: Evaluate $\int \sin^3 x \cos^2 x \, dx$

Ans: Noting that $\sin^2 x = 1 - \cos^2 x$, we have

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx.$$

Denote $t = \cos x$. Then $dt = -\sin x \, dx$ and thus

$$\int \sin^3 x \cos^2 x \, dx = -\int (1-t^2)t^2 \, dt = -\int (t^2 - t^4) dt = -\left(\frac{t^3}{3} - \frac{t^5}{5} + C\right)$$

$$= -\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + C\right)$$

$$\int \sin^{2k+1} x \cos^m x \, dx = \int \sin^{2k} x \cos^m x \sin x \, dx = \int (1 - \cos^2 x)^k \cos^m x \sin x \, dx$$

$$= -\int (1 - t^2)^k t^m \, dt \quad (\text{denote } t = \cos x)$$

Ex: Evaluate $\int \sin^2 x \, dx$

Ans: Applying the following identity

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

to obtain

$$\int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) dx = \frac{1}{2} \left(x - \frac{\sin 2x}{2} + C \right)$$

$$\boxed{\int \sin^{2m} x \cos^{2n} x \, dx = \int \left(\frac{1 - \cos 2x}{2} \right)^m \left(\frac{1 + \cos 2x}{2} \right)^n dx}$$

Ex: Evaluate $\int \tan^3 x \sec^4 x\, dx$

Ans: Note that $1 + \tan^2 x = sec^2 x$ and $\frac{d \tan x}{dx} = \sec^2 x$. We have

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x (1 + \tan^2 x) \sec^2 x\, dx = \int t^3 (1 + t^2) dt$$
$$= \frac{t^4}{4} + \frac{t^6}{6} + C = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$$

$$\boxed{\begin{aligned}\int \tan^m x \sec^{2k} x\, dx &= \int \tan^m x \sec^{2k-2} x \sec^2 x dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx \\ &= \int t^m (1 + t^2)^{k-1}\, dt \quad (\text{denote } t = \tan x)\end{aligned}}$$

Ex: Evaluate $\displaystyle\int \tan^3 x \sec^3 x \, dx$

Ans: Denote $u = \sec x$. Then $du = \sec x \tan x \, dx$.

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \sec x \tan x \, dx = \int (1 - \sec^2 x) \sec^2 x \sec x \tan x \, dx$$

$$= \int (1 - u^2) u^2 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sec^3 x}{3} - \frac{\sec^5 x}{5} + C$$

$$\begin{aligned} \int \tan^{2k+1} x \sec^n x \, dx &= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x \, dx \\ &= \int (1 - \sec^2 x)^k \sec^{n-1} x \sec x \tan x \, dx \\ &= \int (1 - t^2)^k t^{n-1} \, dt \quad (\text{denote } t = \sec x) \end{aligned}$$

Exercises

7.2: 1–30

Ex: Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

Ans: Denote $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$. We then have

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\,3\cos\theta\,d\theta = \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta\,d\theta \\ &= \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta \\ &= \cot\theta - \theta + C\end{aligned}$$

Trigonometric substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < -\frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Trigonometric substitution

Ex: Evaluate $\displaystyle\int_0^2 \frac{1}{(x^2+4)\sqrt{x^2+4}}\,dx$

Ans: Denote $x = 2\tan\theta$. Then $dx = \frac{2}{\cos^2\theta}\,d\theta$. We then have

$$\begin{aligned}
\int_0^2 \frac{1}{(\sqrt{x^2+4})^3}\,dx &= \int_0^{\pi/4} \frac{2\,d\theta}{(\sqrt{4\tan^2\theta+4})^3\ \cos^2\theta} \\
&= \frac{1}{4}\int_0^{\pi/4} \frac{d\theta}{(\sqrt{\tan^2\theta+1})^3\ \cos^2\theta} \\
&= \frac{1}{4}\int_0^{\pi/4} \frac{d\theta}{\left(\sqrt{\frac{1}{\cos^2\theta}}\right)^3\ \cos^2\theta} = \frac{1}{4}\int_0^{\pi/4} \cos\theta\,d\theta \\
&= \frac{1}{4}\sin\theta\Big|_0^{\pi/4} = \frac{\sqrt{2}}{8}
\end{aligned}$$

7.3: 1–20, 31–32, 41–42

Evaluate $\int \frac{P(x)}{Q(x)}\,dx$, where P and Q are polynomials

Step 1: If $\deg(P) > \deg(Q)$, then divide P by Q to obtain

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}, \quad \deg(R) < \deg(Q)$$

Ex:

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1}\right) dx = \int \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

Step 2: Factorize $Q(x)$:

2.1: If $Q(x) = (a_1x + b_1)\cdots(a_kx + b_k)$ has no reapeated factor, then write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1)} + \cdots + \frac{A_k}{a_kx + b_k)}$$

Ex: Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$

Ex: $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$. We have $2x^3+3x^2-2x = x(2x-1)(x+2)$. Then

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A_1}{x} + \frac{A_2}{2x-1} + \frac{A_3}{x+2} \quad \forall x$$
$$\Longrightarrow x^2+2x-1 = A_1(2x-1)(x+2) + A_2x(x+2) + A_3x(2x-1) \quad \forall x$$
$$\Longleftrightarrow x^2+2x-1 = (2A_1+A_2+2A_3)x^2 + (3A_1+2A_2-A_3)x - 2A_1 \quad \forall x$$
$$\Longrightarrow \begin{cases} 2A_1+A_2+2A_3 = 1 \\ 3A_1+2A_2-A_3 = 2 \\ -2A_1 = -1 \end{cases} \Longrightarrow \begin{cases} A_1 = \frac{1}{2} \\ A_2 = \frac{1}{5} \\ A_3 = -\frac{1}{10} \end{cases}$$

Hence
$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \left(\frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}\right)dx$$
$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

2.2: Q has repeated factors, i.e., $Q(x) = (a_1x+b_1)^r(a_2x+b_2)\cdots(a_kx+b_k)$. We write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_r}{(a_1x+b_1)^r} + \frac{B_2}{a_2x+b_2} + \cdots + \frac{B_k}{a_kx+b_k}$$

Ex: Evaluate $\int \frac{4x}{(x-1)^2(x+1)}dx$. We write

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad \forall x$$

$$\Longrightarrow 4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \quad \forall x$$

$$\Longleftrightarrow 4x = (A+C)x^2 + (B-2C)x + (-A+B+C) \quad \forall x$$

$$\Longrightarrow \begin{cases} A+C=0 \\ B-2C=4 \\ -A+B+C=0 \end{cases} \Longrightarrow \begin{cases} A=1 \\ B=2 \\ C=-1 \end{cases}$$

Hence

$$\int \frac{4x}{(x-1)^2(x+1)}dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right) dx$$
$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K,$$

where K is a constant.

2.3: If $Q(x) = (a_0x^2 + b_0x + c_0)(a_1x + b_1)\cdots(a_kx + b_k)$, where $b_0^2 - 4a_0c_0 < 0$, then we write

$$\frac{R(x)}{Q(x)} = \frac{B_1x + C_1}{a_0x^2 + b_0 + c_0} + \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_k}{a_kx + b_k}$$

2.3: If $Q(x) = (a_0x^2 + b_0x + c_0)^r(a_1x + b_1)\cdots(a_kx + b_k)$, where $b_0^2 - 4a_0c_0 < 0$, then we write

$$\frac{R(x)}{Q(x)} = \frac{B_1x + C_1}{a_0x^2 + b_0 + c_0} + \cdots + \frac{B_rx + C_r}{(a_0x^2 + b_0 + c_0)^r} + \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_k}{a_kx + b_k}$$

Exercises

7.4 1–6, odd numbers from 7–38, 39–40

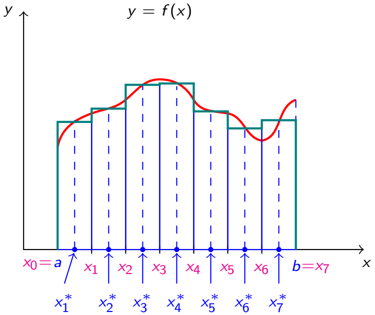
Why we need approximate integration

Many integrals can not be computed exactly, e.g.,

$$\int_0^1 e^{x^2}\, dx \qquad\qquad \int_{-1}^1 \sqrt{1+x^3}\, dx$$

Many integrals arising from science and real life do not have a closed integrands.

$\Longrightarrow$ **Approximate integration**



- $\int_a^b f(x)dx$?
- $x_i = a + i * \Delta x$ where $\Delta x = \frac{b-a}{n}$
- $\int_a^b f(x)dx \approx \sum_{i=1}^n f(x_i^*)\,\Delta x$,
  where $x_i^* \in [x_{i-1}, x_i]$
- How to choose x_i^*?
- **Midpoint rule:** $x_i^* = \frac{x_{i-1}+x_i}{2}$

Midpoint rule:

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

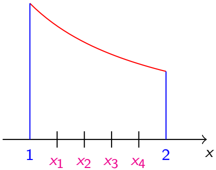
The error:

$$E_M^n := \left|\int_a^b f(x)dx - M_n\right|$$

Error bound for Midpoint rule: Suppose that $|f''(x)| \leq K$ for $a \leq x \leq b$. Then

$$E_M^n \leq \frac{K(b-a)^3}{24n^2}$$

Ex: Approximate $\displaystyle\int_1^2 \frac{1}{x}dx$ by Midpoint method with $n = 5$.



- $n = 5 \Longrightarrow \Delta x = \frac{2-1}{5} = 0.2$ and $x_0 = 1$, $x_1 = 1.2$, $x_2 = 1.4$, $x_3 = 1.6$, $x_4 = 1.8$, $x_5 = 2$
- The midpoints: $x_1^* = 1.1$, $x_2^* = 1.3$, $x_3^* = 1.5$, $x_4^* = 1.7$, $x_5^* = 1.9$

$$M_n = \sum_{i=1}^{5} f(x_i^*)\Delta x = 0.2\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) = 0.691907885715935$$

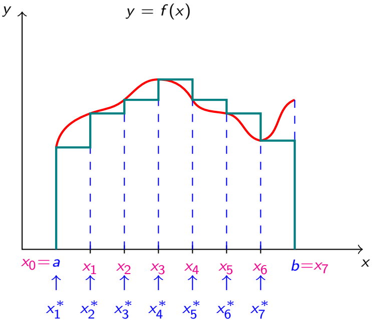
Meanwhile, $\displaystyle\int_1^2 \frac{1}{x}dx = \ln 2 \Longrightarrow E_M^n = 0.001239294844010$

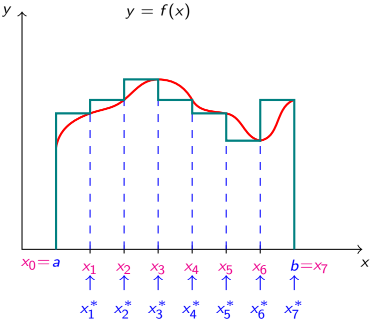
Error of Midpoint rule

We applied the Midpoint rule for different divisions:

n	M_n	E_M^n
5	0.691907885715935	0.00123929484401O
10	0.692835360409960	$3.118201499850981e-04$
20	0.693069098225587	$7.808233435824263e-05$
40	0.693127651979310	$1.952858063514196e-05$
80	0.693142297914324	$4.882645621484549e-06$
200	0.693146399314218	$7.812457272216022e-07$
1000	0.693147149309952	$3.124999337078549e-08$

Trapezoidal Rule





$x_i^* = x_{i-1}$, $L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$

$x_i^* = x_i$, $R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$

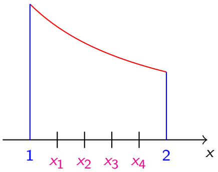
Trapezoidal rule:

$$\int_a^b f(x)dx \approx T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right)$$

Error bound for Trapezoidal rule: Suppose that $|f''(x)| \leq K$ for $a \leq x \leq b$. Then

$$E_T^n := \left| \int_a^b f(x)dx - T_n \right|, \quad E_M^n \leq \frac{K(b-a)^3}{12n^2}$$

Ex: Approximate $\displaystyle\int_1^2 \frac{1}{x}dx$ by Trapezoidal method with $n = 5$.



- $n = 5 \Longrightarrow \Delta x = \frac{2-1}{5} = 0.2$ and $x_0 = 1$, $x_1 = 1.2$, $x_2 = 1.4$, $x_3 = 1.6$, $x_4 = 1.8$, $x_5 = 2$
- $T_n = \dfrac{\Delta x}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$

$$T_n = \frac{0.2}{2}\left(\frac{1}{1} + 2\frac{1}{1.2} + 2\frac{1}{1.4} + 2\frac{1}{1.6} + 2\frac{1}{1.8} + f(2)\right) = 0.695634920634921$$

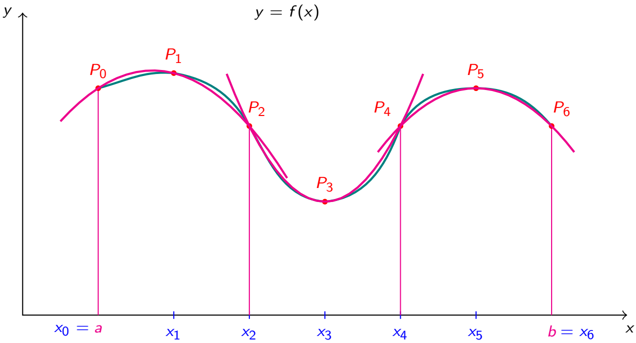
Meanwhile, $\displaystyle\int_1^2 \frac{1}{x}dx = \ln 2 \Longrightarrow E_T^n = 0.002487740074976$

Error of Trapezoidal rule

We applied the Trapezoidal rule for different divisions:

n	T_n	E_T^n
5	0.695634920634921	0.002487740074976
10	0.693771403175428	0.000624222615483
20	0.693303381792694	0.000156201232749
40	0.693186240009141	0.000039059449195
80	0.693156945994225	0.000009765434280
200	0.693148743055062	0.000001562495117
1000	0.693147243059937	0.000000062499992

Simpson Rule



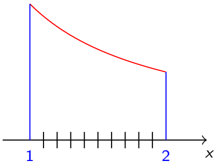
Simpson rule: Let n be even .

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots \right.$$
$$\left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

Error bound for Simpson rule: Suppose that $\left|f^{(4)}(x)\right| \leq K$ for $a \leq x \leq b$. Then

$$E_S^n \leq \frac{K(b-a)^5}{180n^4}$$

Ex: Approximate $\displaystyle\int_1^2 \frac{1}{x}dx$ by Simpson rule with $n = 10$.



- $n = 10 \Longrightarrow \Delta x = \frac{2-1}{10} = 0.1$ and $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$, $x_6 = 1.6$, $x_7 = 1.7$, $x_8 = 1.8$, $x_9 = 1.9$, $x_{10} = 2$

$$S_{10} = \frac{0.1}{3}\left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2}\right)$$

$$= 0.693150230688930 \quad \Longrightarrow \quad E_S^{10} = 0.000003050128985$$

Simpson rule

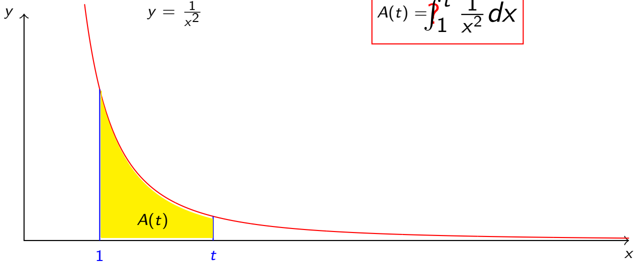
We applied the Simpson rule for different divisions:

n	S_n	E_S^n
10	0.693150230688930	0.000003050128985
20	0.693147374665116	0.000000194105171
40	0.693147192747956	0.000000012188011
80	0.693147181322587	0.000000000762642
200	0.693147180579475	0.000000000019530
1000	0.693147180559975	0.000000000000030

Exercises

7.7: 7–12, 21, 22

Improper integral of type I



$$A(t) = \int_1^t \frac{1}{x^2}\,dx$$

$$A(t) = \int_1^t \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^t = 1 - \frac{1}{t} \Longrightarrow \lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

We define: $\boxed{\int_1^{\infty} \frac{1}{x^2}dx := \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}dx}$

Improper integral of type I

Improper integrals of type I:

1. If $\int_a^t f(x)dx$ exists for all $t \geq a$, then
$$\int_a^{\infty} f(x)dx := \lim_{t\to\infty} \int_a^t f(x)dx$$
provided that the limit exists.
2. If $\int_t^b f(x)dx$ exists for all $t \leq b$, then
$$\int_{-\infty}^{b} f(x)dx := \lim_{t\to-\infty} \int_t^b f(x)dx$$
provided that the limit exists.
3. If $\int_a^{\infty} f(x)dx$ and $\int_{\infty}^{a} f(x)dx$ are convergent then
$$\int_{-\infty}^{\infty} f(x)dx := \int_a^{\infty} f(x)dx + \int_{\infty}^{a} f(x)dx$$

Ex: Determine the convergence of $\displaystyle\int_1^\infty \frac{1}{x}dx$

Ans: By definition, we have

$$\begin{aligned}
\int_1^\infty \frac{1}{x}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\,\Big|_1^t \\
&= \lim_{t\to\infty}(\ln|t| - \ln 1) \\
&= \lim_{t\to\infty} \ln|t| \\
&= \infty.
\end{aligned}$$

Hence, the $\int_1^\infty \frac{1}{x}dx$ is divergent (not convergent).

Ex: Evaluate $\int_{-\infty}^{0} xe^x dx$

Ans: By definition, we have $\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \int_t^0 xe^x dx.$
Denote $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. Integration by parts gives

$$\int_t^0 xe^x dx = xe^x\Big|_t^0 - \int_t^0 e^x dx = -te^t - e^x\Big|_t^0 = -te^t - 1 + e^t$$

Hence,

$$\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty}(-te^t - 1 + e^t) = \lim_{t\to-\infty}(-te^t) - 1 + \lim_{t\to-\infty} e^t$$

$$= \lim_{t\to-\infty} \frac{-t}{e^{-t}} - 1 \overset{\text{L'Hopital}}{=} \lim_{t\to-\infty} \frac{-1}{-e^{-t}} - 1 = 1$$

Improper integrals of type II

Ex: Determine the convergence of $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

Ans: When $p = 1$, the integral is divergent (see previous example). when $p \neq 1$, by definition, we have

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}dx &= \lim_{t\to\infty}\int_1^t x^{-p}dx = \lim_{t\to\infty}\frac{x^{-p+1}}{-p+1}\Big|_1^t \\
&= \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right) = \frac{1}{1-p}\lim_{t\to\infty} t^{1-p} - \frac{1}{1-p} \\
&= \begin{cases} -\frac{1}{1-p} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}
\end{aligned}$$

$$\boxed{\text{Hence, } \int_1^\infty \frac{1}{x^p}dx \quad \text{is} \quad \begin{cases} \text{convergent } \left(= -\frac{1}{1-p}\right) & \text{if } p > 1 \\ \text{divergent } (\infty) & \text{if } p \leq 1 \end{cases}}$$

Improper integrals of type II:

1. Let $f : [a, b) \to \mathbb{R}$ be continuous and f be discontinuous at b. Then
$$\int_a^{b} f(x)dx := \lim_{t \to b^-} \int_a^{t} f(x)dx$$
provided that the limit exists.

2. Let $f : (a, b] \to \mathbb{R}$ be continuous and f be discontinuous at a. Then
$$\int_{a}^b f(x)dx := \lim_{t \to a^+} \int_{t}^b f(x)dx$$
provided that the limit exists.

3. If f has a discontinuity at c, where $a < c < b$ and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define
$$\int_a^b f(x)dx := \int_a^c f(x)dx + \int_c^b f(x)dx$$

Improper integrals of type II

Ex: Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}}dx$

Ans: We first note that $\frac{1}{\sqrt{x-2}}$ is discontinuous at 2. By definition, we have

$$\begin{aligned}\int_2^5 \frac{1}{\sqrt{x-2}}dx &= \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}dx\\ &= \lim_{t\to 2^+}\left(2\sqrt{x-2}\Big|_t^5\right)\\ &= \lim_{t\to 2^+}\left(2\sqrt{3}-2\sqrt{t-2}\right)\\ &= 2\sqrt{3}.\end{aligned}$$

Hence, $\int_2^5 \frac{1}{\sqrt{x-2}}dx = 2\sqrt{3}$

Improper integrals of type II

Ex: Evaluate $\displaystyle\int_0^1 \ln x\,dx$

Ans: Note that $\ln x$ is not defined at 0. By definition, we have

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx.$$

Denote $u = \ln x$ and $dv = dx$. Then $du = \frac{dx}{x}$ and $v = x$. Integration by parts gives

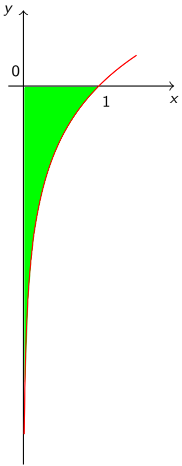
$$\int_t^1 \ln x\,dx = x\ln x\Big|_t^1 - \int_t^1 dx = -t\ln t - x\Big|_t^1 = -t\ln t - 1 + t$$

L'Hopital rule: $\displaystyle\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t}{1/t} = \lim_{t\to 0^+}\frac{1/t}{-1/t^2} = \lim_{t\to 0^+}(-t) = 0$

Hence,

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}(-t\ln t - 1 + t) = \lim_{t\to 0^+}(-t\ln t) - 1 + \lim_{t\to 0^+} t = -1$$

Improper integrals of type II



Comparison test for improper integrals

Theorem (Comparison Theorem).

Let f and g be continuous functions with

$$0 \leq f(x) \leq g(x) \text{ for } x \geq a.$$

- If $\displaystyle\int_a^\infty g(x)\,dx$ is convergent then $\displaystyle\int_a^\infty f(x)\,dx$ is convergent.
- If $\displaystyle\int_a^\infty f(x)\,dx$ is divergent then $\displaystyle\int_a^\infty g(x)\,dx$ is divergent.

Ex: Determine whether the integral $\displaystyle\int_1^\infty \frac{dx}{x^2\,e^x}$ is convergent?

Ans: When $x \geq 1$, $e^x > 1$ and hence, $\dfrac{1}{x^2\,e^x} \leq \dfrac{1}{x^2}$ for all $x \geq 1$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ is convergent (?), by Comparison Theorem, $\displaystyle\int_1^\infty \frac{dx}{x^2 e^x}$ is convergent.

Exercises

7.8: 1–2, 3, 5–30, 55, 57–59, 75