THE INTERNATIONAL UNIVERSITY – VIETNAM NATIONAL UNIVERSITY – HCMC

SAMPLE OF FINAL EXAMINATION

Semester XXX • Date: XXX • Duration: 120 minutes

INSTRUCTIONS:

Each student is allowed a scientific calculator and a maximum of TWO double-sided sheets of reference material (size A4 or similar) marked with their name and ID. All other documents and electronic devices are forbidden.

Each question carries 10 marks.

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Question 1. Show that the function $U(x, y) = e^{-x}\sin(y)$ satisfies the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

Question 2. Find the equation of the plane tangent to the surface

$$z(x, y) = x^2 e^{x-y}$$

at the point $(2, 2, 4)$.

Question 3. Find the critical points of the following function $f(x, y)$, and determine whether each critical point corresponds to a local maximum, local minimum or a saddle point?

$$f(x, y) = x^4 + 2y^2 - 4xy.$$

Question 4. Using Lagrange multipliers, find the absolute maximum and minimum values of the function

$$f(x, y) = x - y \quad \text{subject to} \quad g(x, y) = x^2 + y^2 - 3xy = 20.$$

Question 5. Evaluate the integral $\iint_D y^2 dA$ where D is bounded by: $x = 1, y = 2x+2, y = -x-1$.

Question 6. Find the volume of the solid bounded by the plane $z = 0$ and hyperboloid $z = 3 - \sqrt{1 + x^2 + y^2}$.

Please turn over...

Question 7. Evaluate the line integral $\int_C xyds$, where C is a portion of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

in the first quadrant, oriented counterclockwise.

Question 8. Find $\iiint_E z \, dV$, where E is the solid tetrahedron with vertices $(0,0,0)$, $(4,0,0)$, $(0,1,0)$, and $(0,0,5)$.

Question 9. Let D be part of the annulus $\{(x,y) : 1 \leq x^2 + y^2 \leq 9\}$ that lies in the upper half plane. Let C be the boundary of D, positively oriented. Compute

$$\int_C (xy + e^{x^2})dx + (\sin(\sqrt{y}) + x^2)dy.$$

Question 10. Let $F = \langle 2x + 3y, 3x - 2y \rangle$. Find $f(x,y)$ such that $F = \nabla f$.

—END OF QUESTION PAPER—

Short answer

Q.1

Q.2 $f_x = (2x + x^2)e^{x-y}$, $f_y = -x^2 e^{x-y}$. So

$$x = -x - y - 1$$

Q3. $f_x = 4x^3 - 4y; f_y = 4y - 4x;\ x_{xx} = 12x^2, f_{yy} = 4, f_{xy} = -4$, and $D(x, y) = 38x^2 - 16$

$D(0, 0) = -16 < 0 \to f$ has saddle point at (0,0)

$D(1, 1) = D(-1, -1) = 32 > 0; f_{xx}(1, 1) = f_{xx}(-1, -1) = 12 > 0 \Rightarrow f$ has local minimum at (1,1) and (-1,-1)

Q.4 $\nabla f(x, y) =< 1, -1 >; \nabla g(x, y) =< 2x - 3y, 2y - 3x >$. Lagrang multiplier is

$$\begin{cases} 2x - 3y = 1/\lambda \\ 2y - 3x = -1/\lambda \\ x^2 + y^2 - 3xy = 20 \end{cases}$$

$\Rightarrow (x, y) = (\pm 2, \pm 2)$. So maximum $f(x, y) = 4$ at $(2, -2)$, minimum $f(x, y) = -4$ at $(-2, 2)$

Q5. $I = \int_{-1}^{1} \int_{-x-1}^{2x+2} y^2 dy dx = 12$

Q6. $V = \int_0^{2\pi} \int_0^{2\sqrt{2}} \int_0^{3-\sqrt{1+r^2}} 1 dz dr d\theta = \frac{20\pi}{3}$

Q7. $r(t) =< 2\cos(t), 4\sin(t) >$, $0 \le t \le \pi/2$, and

$$|r'(t)| = \sqrt{4\cos^2(t)_1 6\sin^2(t)} = 2\sqrt{1 + 3\cos^2(t)}$$

$$I = \int_0^{\pi/2} 16\sin(t)\cos(t)\sqrt{1 + 3\cos^2(t)} dt = 112/9$$

Q8. The equation of plane passing given vertices is $\frac{x}{4} + y + \frac{z}{5} = 1$, and hence

$$E = \{(x, y, z) : 0 \le x \le 4, 0 \le y \le 1 - \frac{x}{4}, 0 \le z \le 5 - \frac{5x}{4} - 5y\}.$$

Thus,

$$\iiint_E z\, dV = \int_0^4 \int_0^{1-\frac{x}{4}} \int_0^{5-\frac{5x}{4}-5y} z\, dx\, dy\, dz = \frac{25}{6} \approx 4.1667.$$

Q9. By Green's theorem, it equals $\iint_D x dA = 0$.

Q10. $f(x, y) = x^2 + 3xy - y^2 + C$.