1) Derivatives

$(uv)' = u'v + v'u$; $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$(\tan u)' = \frac{u'}{\cos^2 u}$; $(\cot u)' = -\frac{u'}{\sin^2 u}$; $\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$

$(\cot u)' = -\frac{u'}{\sin^2 u}$; $(\cos^{-1}\frac{x}{a})' = -\frac{1}{\sqrt{a^2-x^2}}$; $(e^u)' = u'e^u$

$\left(\frac{1}{\cos x}\right)' = (\sec x)' = \sec x \tan x$; $(a^u)' = a^u u' \ln a$

$\left(\frac{1}{\sin x}\right)' = (\csc x)' = -\csc x \cot x$; $(\ln|u|)' = \frac{u'}{u}$

$(\sin^{-1}\frac{x}{a})' = \frac{1}{\sqrt{a^2-x^2}}$; $(\tan^{-1}\frac{x}{a})' = \frac{a}{\sqrt{a^2+x^2}}$

$(\sin u)' = u'\cos u$; $(\cos u)' = u'(-\sin u)$

$(\log_a|u|)' = \frac{u'}{u\ln a}$; $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$(\sqrt[m]{u})' = \frac{u'}{m\sqrt[m]{u^{m-1}}}$; $\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$

$(u^\alpha)' = \alpha u^{\alpha-1} u'$; $f'(u) = u' f'(u)$

2) Partial Derivatives

Def: If $z = f(x,y)$ is func of two variables, its partial derivatives are the func f_x and f_y defined

$f_x(x,y) = \frac{\partial z}{\partial x} = D_x f = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

$f_y(x,y) = \frac{\partial z}{\partial y} = D_y f = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$

Đạo hàm theo $x \to y$ là constant

Đạo hàm theo $y \to x$ là constant

Ex: Calculate f_{xxyz} if $f = \sin(3x+yz)$

$f_x = 3\cos(3x+yz)$; $f_{xx} = -9\sin(3x+yz)$

$f_{xxy} = -9z\cos(3x+yz)$; $f_{xxyz} = -9\cos(\ ) + 9yz\sin(\ )$

Laplace's equation (they play a role in problems of heat conduction, fluid flow, electric potential)

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0}$$

Ex: Show that the eq $u = e^x \sin y$ is a solu of Laplace's equation:

We first compute the needed second-order partial derivatives.

$u_x = e^x \sin y$ $u_y = e^x \cos y$

$u_{xx} = e^x \sin y$ $u_{yy} = -e^x \sin y$

So, $u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$

Therefore, u satisfy Laplace's equation

3) Tangent planes and linear approximations

Equation of tangent plane. Suppose f has cont partial derivatives. An equation of the tangent plane to the surface $z = f(x,y)$ at point $P(x_0,y_0,z_0)$

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Ex: Determine the unit normal vector to $x^2+y^2+z^2$ at the point $(2,1,1)$: $\vec{n} = \frac{\nabla F}{|\nabla F|}$

$F(x,y,z) = x^2 + y^2 + z^2 - 6 = 0$

$F_x = 2x$ $F_y = 2y$ $F_z = 2z$

$F_x = 4$ $F_y = 2$ $F_z = 2$

$\vec{n} = \frac{\langle 4,2,2\rangle}{\sqrt{16+4+4}} = \frac{1}{2\sqrt{6}}\langle 4,2,2\rangle = \langle \frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\rangle$

$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$

Linearization of f at (a,b)

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Linear approx. of f at (a,b)

$f(x,y) \approx L(a,b) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

(a,b) and cont at (a,b), then f is differentiable at (a,b)

Ex: Show that $f(x,y) = xe^{xy}$ is diff at $(1,0)$ and find its linearization there. Then approx $f(1.1,-0.1)$

The partial derivatives are

$f_x(1,0) = e^{xy} + xye^{xy} = 1$, $f_y(1,0) = x^2e^{xy} = 1$

Both f_x, f_y are cont func, so f is diff.

The linearization is

$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$= 1 + 1(x-1) + 1y = x+y$

The corresponding linear approx is

$xe^{xy} \approx x+y \Rightarrow f(1.1,-0.1) \approx 1.1 - 0.1 = 1$.

Compare this with actual value of $f(1.1,-0.1) = 1.1e^{-0.11}$

Differentials

Total differentials (dz) is defined by:

$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

If $dx = \Delta x = x - a$; $dy = \Delta y = y - b$

$\Rightarrow dz = f_x(x,y)(x-a) + f_y(x,y)(y-b)$

Ex: $z = f(x,y) = x^2 + 3xy - y^2$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare Δz and dz.

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x+3y)dx + (3x-2y)dy$

$dx = \Delta x = 0.05$, $dy = \Delta y = -0.04$, $x=2$, $y=3$

$dz = [2(2)+3(3)]0.05 + [3(2)-2(3)](-0.04)$

The increment of z is $\Delta z = f(2.05, 2.96) - f(2,3)$

$= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [\ldots] = 0.6$

Chain Rule (case 1)

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}} \to \begin{cases} z \text{ diff of } x, y \\ x = g(t) \\ y = h(t) \end{cases}$$

case 2:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Implicit Function Theorem

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Ex: $x^3 + y^3 + z^3 + 6xyz + 4 = 0$. Find $\frac{\partial z}{\partial x}$

Let $F(x,y,z) = x^3 + y^3 + z^3 + 6xyz + 4$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$

Directional Derivatives: unit vector $u = \frac{v}{|v|}$ ($\vec{v}(a,b)$)

$u = \langle a,b\rangle \Rightarrow D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$.

If u makes an angle θ with $(+)$ x-axis

$\Rightarrow u = \langle\cos\theta, \sin\theta\rangle \Rightarrow D_u f = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$

Gradient Vector:

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j$.

$\Rightarrow \boxed{D_u f(x,y) = \nabla f(x,y) \cdot u}$

D_u max when $u = \frac{\nabla f}{|\nabla f|}$

Ex: $f(x,y) = xe^y$ find rate of change of f at $P(2,0)$ in direction from P to $Q(\frac{1}{2},2)$

b) In what direction f have max rate of change? What is this max RoC.

a) $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle$

$\nabla f(2,0) = \langle 1,2\rangle$; $u = \frac{v}{|v|} = \frac{\langle -\frac{3}{2}, 2\rangle}{\sqrt{(-\frac{3}{2})^2 + 2^2}} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$

$D_u f(2,0) = \nabla f(2,0) \cdot u = \langle 1,2\rangle \cdot \langle \ldots\rangle = 1$

b) According T.15, f increase fastest in direction of gradient vector $\nabla f(2,0) = \langle 1,2\rangle$. The max

$|\nabla f(2,0)| = |\langle 1,2\rangle| = \sqrt{5}$

Tangent planes to level surface

$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(\ldots)(z-z_0) = 0$

Ex: Find the eq of tangent plane and normal line to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$ at $(-2,1,-3)$

The ellipsoid is level surface ($k=3$) of F

$F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$. Therefore

$F_x = \frac{x}{2}$, $F_y = 2y$; $F_z = \frac{2z}{9}$; $F_x(-2,1,3) = -1$

$F_y(-2,1,-3) = 2$, $F_z(-2,1,3) = -\frac{2}{3}$

The eq of tangent plane at $(-2,1,-3)$

$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$

By eq 20, symmetric eq of normal line are

$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}$

Ex Find the tangent plane to the surface $z = 2x^2 + y^2$ at the point $(1,1,3)$

The surface $z = 2x^2 + y^2$ or $F = 2x^2 + y^2 - z$ is a level surface ($k=0$). Then

$F_x = 4x$ $F_y = 2y$ $F_z = -1$

$F_x(1,1,3) = 4$ $F_y(1,1,3) = 2$ $F_z(1,1,3) = -1$

The eq ______ at $(1,1,3)$

$4(x-1) + 2(y-1) - (z-3) = 0$ which simplifies

$z = 4x + 2y - 3$.

Maximum and minimum value

Second Derivatives Test: Suppose the second partial derivatives of f are cont on a disk with center (a,b) and $f_x(a,b) = 0$, $f_y(a,b) = 0$ [(a,b) is a critical point of f] Let

$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

a) If $D > 0$ and $f_{xx}(a,b) < 0$, then f is local max

b) ______ $f_{xx}(a,b) > 0$, ______ min

c) If $D < 0$, then (a,b) is a saddle point.

Ex: Find and classify the critical points of the func $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$

$f_x = 20xy - 10x - 4x^3$ $f_y = 10x^2 - 8y - 8y^3$

To find CPs, we need to solve the equations

$2x(10y - 5 - 2x^2) = 0$ (1) $5x^2 - 4y - 4y^3 = 0$ (2)

(1) we see $x = 0$ or $10y - 5 - 2x^2 = 0$.

when $x = 0$, (2) becomes $4y(1+y^2) = 0$, so $y = 0$

we have CT $(0,0)$,

when $10y - 5 - 2x^2 = 0$, we get $x^2 = 5y - 2.5$ (3)

Putting this in Eq (2), we have $25y - 12.5 - 4y - 4y^3 = 0$

we obtain $y = -2.545$; $y = 0.6468$; $y = 1.8984$.

Putting all y to Eq (3), we have

If $y = -2.545$, then no corresponding x values

If $y = 0.6468$, then $x = \pm 0.8567$

If $y = 1.8984$, then $x = \pm 2.6442$

At $(0,0)$, $f = 0$, $f_{xx} = -10 < 0$, $D = 80 > 0 \Rightarrow$ max

At $(\pm 2.6442, 1.90)$, $f = 8.50$, $f_{xx} = -55.93 < 0$, $D = 2488 > 0 \Rightarrow$ max

At $(\pm 0.86, 0.65)$, $f = -1.48$, $f_{xx} = -5.87 < 0$, $D = -187.64 < 0 \Rightarrow$ SP

Absolute max - min

1. Find the values of f at CT points of f in D
2. Find the extreme values of f on the D.
3. The largest values from (1) and (2) is absolute max and opposite.

Method of Lagrange Multipliers

To find max/min values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$

1. find all value of x, y, z such that (and λ)

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

and $g(x,y,z) = k$.

2. Evaluate f at all points (x,y,z) that results from Step 1. The largest and smallest is max and min, respectively

Integrals

1. $\int dx = x + C$; 2. $\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$

3. $\int \frac{1}{x^2}dx = -\frac{1}{x} + C$; 4. $\int \frac{1}{x}dx = \ln|x| + C$

5. $\int e^x dx = e^x + C$; 6. $\int a^x dx = \frac{a^x}{\ln a} + C$

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8. $\int \tan x\, dx = -\ln|\cos x| + C$; $\int \frac{1}{\cos^2 x}dx = \tan x + C$

9. $\int (ax+b)^\alpha = \frac{1}{a}\frac{(ax+b)^{\alpha+1}}{\alpha+1} + C$; $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$

10. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$; $\int a^{kx+b}dx = \frac{1}{k}\frac{a^{kx+b}}{\ln a} + C$

11. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

12. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$.

$\int u\,dv = uv - \int v\,du$; nhất log, nhì đa, tam mũ, tứ lượng

Volume and Double Integrals

$R = [a,b] \times [c,d] = \{x,y \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$

If $f(x,y) \ge 0$, then the V of solid that lies above the rectangle R and below the surface $z = f(x,y)$ is: $V = \iint_R f(x,y)\,dA$

Fubini's Theorem If f is cont on the rec

$R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$ then

$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$.

11. $\iint_R g(x)h(y)\,dA = \int_a^b g(x)dx \int_c^d h(y)dy$ where $R = [a,b]\times[c,d]$

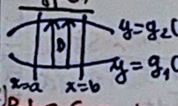
Type I: $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

then $\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.

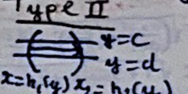
Type II: $D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

then $\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$.

Type 1

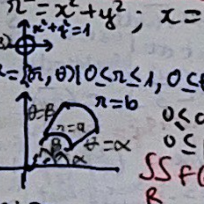


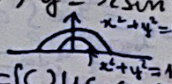
Type II



Polar Coordinates

$r^2 = x^2 + y^2$; $x = r\cos\theta$, $y = r\sin\theta$.





$R = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ $R = \{(\ ) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

$0 \le a \le r \le b$, $\alpha \le \theta \le \beta$ where $0 \le \beta - \alpha \le 2\pi$.

$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

or $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

Center Mass: Mass of a lamina:

$m = \iint_R \rho(x,y)\,dA$.

$M_x = \iint_R y\rho(x,y)\,dA$ Moment about x-axis

$M_y = \iint_R x\rho(x,y)\,dA$ Moment about y-axis

$\bar{x} = \frac{M_y}{m}$; $\bar{y} = \frac{M_x}{m}$ Center Mass of..

Surface Area

$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$

Triple Integrals

$B=[a,b]\times[c,d]\times[r,s]$

$\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$

General Regions

Type I: $a\le x\le b,\ g_1(x)\le y\le g_2(x),\ u_1(x,y)\le z\le u_2(x,y)$

$\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$

Type II: $c\le y\le d,\ h_1(y)\le x\le h_2(y),\ u_1\ldots$

$\iiint_E f(x,y,z)\,dV = \int_c^d\int_{h_1(y)}^{h_2(y)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$

To convert from cylindrical to rectangular: $x=r\cos\theta,\ y=r\sin\theta,\ z=z.$

To convert from rectangular to cylindrical: $r^2=x^2+y^2,\ \tan\theta=\frac{y}{x},\ z=z.$

Polar coordinates

$\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta,r\sin\theta)}^{u_2(r\cos\theta,r\sin\theta)} f(r\cos\theta,r\sin\theta,z)\,r\,dz\,dr\,d\theta$

Triple in Spherical Coordinates

$x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$

$\rho^2=x^2+y^2+z^2=r^2+z^2$

Exercise: Textbook

Chapter 16: Line Integral

3. $\int_C yz\cos x\,ds$, $C: x=t,\ y=3\cos t,\ z=3\sin t,\ 0\le t\le\pi$

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\sqrt{(1)^2+(-3\sin t)^2+(3\cos t)^2}\,dt$

$=\sqrt{1+9\sin^2 t+9\cos^2 t}\,dt=\sqrt{10}\,dt.$

$\int_C yz\cos x\,ds=\int_0^\pi(3\cos t)(3\sin t)\cos t\sqrt{10}\,dt$

$=9\sqrt{10}\int_0^\pi\cos^2 t\sin t\,dt$ $\begin{cases}\cos t=u\\ -\sin t\,dt=du\end{cases}$

$=-9\sqrt{10}\int_1^{-1}u^2\,du=-9\sqrt{10}\left[\frac{u^3}{3}\right]_1^{-1}=6\sqrt{10}.$

2. $\int_C y\,dx+(x+y^2)\,dy$, C is ellipse $4x^2+9y^2=36$ with counterclockwise orientation.

Rewrite the equation of the ellipse as: $\left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$. We can parametrize the ellipse as: $C: x=3\cos t,\ y=2\sin t$. t increases from 0 to 2π. $\begin{cases}dx=-3\sin t\,dt\\ dy=2\cos t\,dt\end{cases}$

Therefore, $\int_C y\,dx+(x+y^2)\,dy=\int_0^{2\pi}2\sin t(-3\sin t)\,dt+(3\cos t+4\sin^2 t)(2\cos t)\,dt.$

$=\int_0^{2\pi}-6\sin^2 t+6\cos^2 t+8\sin^2 t\cos t\,dt$

$=\int_0^{2\pi}6\cos 2t+8(1-\cos^2 t)\cos t\,dt=0$

Alternative $\int_C F\cdot dr=\int_a^b F(r(t))\,r'(t)\,dt=\int_C F\cdot T\,ds$

Rewrite: $F=yi+(x+y^2)j$ ($\int_C F\,dr=\int_C P\,dx+Q\,dy+R\,dz$)

$(yi+(x+y^2)j)\cdot(dx\,i+dy\,j)$

Note that F is a conservative field $\left[\frac{\partial y}{\partial y}=\frac{\partial(x+y^2)}{\partial x}\right]$

Line integral of a conservative field along any path is zero.

Line integrals of conservative vector fields are independent of path ($\oint_C F\cdot dr=0$)

5. $\int_C y^3\,dx+x^2\,dy$, C is the arc of the parabola $x=1-y^2$ from $(0,-1)$ to $(0,1)$. Take y as parameter.

We can parametrize the parabola $x=1-y^2$ as $C: x=1-t^2,\ y=t$. As we move from $(0,-1)$ to $(0,1)$, t increases from -1 to 1. $\begin{cases}dx=-2t\,dt\\ dy=dt\end{cases}$

$\int_C y^3\,dx+x^2\,dy=\int_{-1}^1(t^3)(-2t\,dt)+(1-t^2)^2\,dt.$

$=\int_{-1}^1(-2t^4+t^4-2t^2+1)\,dt=\frac{4}{15}.$

6. $\int_C\sqrt{xy}\,dx+e^y\,dy+xz\,dz$, C is given by $r(t)=t^4i+t^2j+t^3k$, $0\le t\le1$. Given that $C: r(t)=t^4i+t^2j+t^3k$.

Therefore, $x=t^4\Rightarrow dx=4t^3\,dt$, $y=t^2\Rightarrow dy=2t\,dt$, $z=t^3\Rightarrow dz=3t^2\,dt$. $\int_C\sqrt{xy}\,dx+e^y\,dy+xz\,dz$

$=\int_0^1\sqrt{t^4t^2}(4t^3\,dt)+e^{t^2}(2t\,dt)+t^4t^3(3t^2\,dt)$

$=\int_0^1 4t^3\sqrt{t^6}+2te^{t^2}+3t^9\,dt=\int_0^1 4t^6+3t^9\,dt+\int_0^1 2te^{t^2}\,dt$

$=e-\frac{9}{70}.$

7. $\int_C xy\,dx+y^2\,dy+yz\,dz$, C is the line segment from $(1,0,-1)$ to $(3,4,2)$. Equation of line joining

$\frac{x-1}{3-1}=\frac{y-0}{4-0}=\frac{z+1}{2-(-1)}=t$, $C: x=2t+1,\ y=4t,\ z=3t-1$

$r(t)=(1-t)r_0+t\,r_1$ (from r_0 to r_1) $(0\le t\le1)$

$r_0=(1,0,-1)$ to $r_1=(3,4,2)\Rightarrow t$ increases 0 to 1

$dx=2dt,\ dy=4dt,\ dz=3dt.$

$\int_C xy\,dx+y^2\,dy+yz\,dz$

$=\int_0^1(2t+1)(4t)(2dt)+(4t)^2(4dt)+(4t)(3t-1)(3dt)$

$=\int_0^1(16t^2+8t+64t^2+36t^2-12t)\,dt=\frac{110}{3}.$

8. $\int_C F\cdot dr$, where $F(x,y,z)=e^zi+xzj+(x+y)k$ and $r(t)=t^2i+t^3j-tk$, $0\le t\le1$.

$C: r(t)=t^2i+t^3j-tk$. Differentiate

$\frac{dr}{dt}=2ti+3t^2j-k$, $dr=(2ti+3t^2j-k)\,dt.$

$F(x,y)=e^zj+xzj+(x+y)k$

$F(r(t))=e^{-t}i+(t^2)(-t)j+(t^2+t^3)k.$

$F(r(t))=e^{-t}i-t^3j+(t^2+t^3)k.$

$\int_C F\cdot dr=\int_0^1(e^{-t}i-t^3j+(t^2+t^3)k)(2ti+3t^2j-k)\,dt$

$=\int_0^1(2te^{-t}-3t^5-t^2-t^3)\,dt$

$=\int_0^1 2te^{-t}\,dt+\int_0^1-3t^5-t^2-t^3\,dt=\frac{11e-48}{12e}$

Show that F is conservative vector field. Then find a function f such that $F=\nabla f$.

11. $F(x,y)=(1+xy)e^{xy}i+(e^y+x^2e^{xy})j$

Remember that $F=Pi+Qj$ is conservative when $P_y=Q_x$. $F(x,y)=(1+xy)e^{xy}i+(e^y+x^2e^{xy})j$

$P=(1+xy)e^{xy}\Rightarrow P_y=(x)e^{xy}+x(1+xy)e^{xy}$

$Q=e^y+x^2e^{xy}\Rightarrow Q_x=2xe^{xy}+x^2ye^{xy}=(2+xy)xe^{xy}$

$P_y=Q_x$. Therefore the given vector field F is conservative.

$F(x,y)=\nabla f$, $f_xi+f_yj=(1+xy)e^{xy}i+(e^y+x^2e^{xy})j$

Therefore, $f_x=(1+xy)e^{xy}$ (1) and $f_y=e^y+x^2e^{xy}$ (2)

Integrating (2) with respect to y, we obtain $f(x,y)=e^y+xe^{xy}+p(x)\to(3)$.

We differentiate (3) both sides respect to x: $f_x=e^{xy}+xye^{xy}+p'(x)$. (4)

Compare (1), (4), we see that $p'(x)=0$. Integrating respect to x, we have $p(x)=C$.

Putting this in (3), we have: $f(x,y)=xe^{xy}+e^y+C$

12. $F(x,y,z)=\sin y\,i+x\cos y\,j-\sin z\,k$

$F=Pi+Qj+Rk$ conser... when

$\frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}$ and $\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}$ and $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$. We have

$R_y=Q_z$ and $P_z=R_x$ and $Q_x=P_y$.

$P=\sin y\Rightarrow P_z=0$ and $P_y=\cos y$

$Q=x\cos y\Rightarrow Q_z=0$ and $Q_x=\cos y$

$R=-\sin z\Rightarrow R_x=0$ and $R_y=0$

We have $R_y=Q_z=0$ and $P_z=R_x=0$ and $Q_x=P_y=\cos y$

Therefore, the given vector field is conservative.

$F(x,y,z)=\nabla f$; $f_xi+f_yj+f_zk=\sin y\,i+x\cos y\,j-\sin z\,k$

Therefore, $f_x=\sin y$ (1), $f_y=x\cos y$ (2), $f_z=-\sin z$ (3)

Integrating (1) respect to x, $f(x,y)=x\sin y+g(y,z)$ (4)

We differentiate (4) respect to y, $f_y(x,y)=x\cos y+g_y(y,z)$ (5)

Compare (2), (5) we see that $g_y(y,z)=0$

Integrating with respect to y, $g(y,z)=h(z)$

Differentiating (4) respect to z gives, $f_z=h'(z)$

Differentiating (4) respect to z gives $f_z=g'(z)$

Compare (3), (6) we see that $f_z(x,y)=g_z(y,z)$ (6)

But we know that $g_z(y,z)=-\sin z$.

Therefore, $g'(z)=-\sin z$.

Integrating to get $g(z)=\cos z+K$.

Therefore, $g(y,z)=\cos z+K$, substitute (4)

$f(x,y,z)=x\sin y+\cos z+K$

16. Use Green's Theorem to evaluate: $\int_C\sqrt{1+x^3}\,dx+2xy\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,3)$.

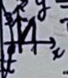
Green's Theorem: $\oint_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$

where D is the region enclosed inside the counter-clockwise oriented loop C.

$\int_C\sqrt{1+x^3}\,dx+2xy\,dy=\iint_D\left(\frac{\partial(2xy)}{\partial x}-\frac{\partial\sqrt{1+x^3}}{\partial y}\right)dA$

$=\iint_D(2y-0)\,dA$. D is the region inside the triangle with vertices $(0,0)$ $(1,0)$ $(1,3)$

The equation of line joining $(0,0)$ and $(1,3)$: $\frac{y-0}{x-0}=\frac{3-0}{1-0}$ ($y=mx+b$, $m=\text{slope}=\frac{y_2-y_1}{x_2-x_1}$, plug point after finding)



$y=3x\Rightarrow D=\{(x,y)\in D\mid 0\le y\le 3x,\ 0\le x\le1\}$

Therefore, $\iint_D(2y-0)\,dA=\int_0^1\int_0^{3x}2y\,dy\,dx$

$=\int_0^1 9x^2\,dx=3$. So,...

17. $\int_C x^2y\,dx-xy^2\,dy$ where C is the circle $x^2+y^2=4$ with counterclockwise orientation.

$\int_C x^2y\,dx-xy^2\,dy=\iint_D\left(\frac{\partial(-xy^2)}{\partial x}-\frac{\partial(x^2y)}{\partial y}\right)dA.$

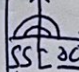
$=\iint_D(-y^2-x^2)\,dA$. D is the region inside the circle $x^2+y^2=4$. We can define D as

$(r,\theta)\in D\mid 0\le r\le2,\ 0\le\theta\le2\pi$.

$\int_0^{2\pi}\int_0^2-r^2(r)\,d\theta\,dr=-8\pi$

Ex4: $\oint_C y^2\,dx+3xy\,dy$, where C is the boundary of the semiannular region D in the upper-half plane between $x^2+y^2=1$ and $x^2+y^2=4$.

In polar coordinate, we can write $D=\{(r,\theta)\in D\mid 1\le r\le2,\ 0\le\theta\le\pi\}$



$\iint_D\left[\frac{\partial(3xy)}{\partial x}-\frac{\partial(y^2)}{\partial y}\right]dA=\iint_D y\,dA=\int_0^\pi\int_1^2(\sin\theta r)r\,dr\,d\theta$

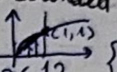
$=\int_0^\pi\sin\theta\,d\theta\int_1^2 r^2\,dr=\frac{14}{3}$ $\text{Div}F=\frac{dP}{dx}+\frac{dQ}{dy}+\frac{\partial R}{\partial z}.$

$\text{Curl}F=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)i-\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)j+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)k$

Chapter ...

Calculate the values of multiple integral.

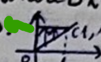
$\iint_D\frac{y}{1+x^2}\,dA$ where D is bounded by $y=\sqrt{x},\ y=0,\ x=1$



$D=\{(x,y)\mid 0\le y\le\sqrt{x},\ 0\le x\le1\}$. $\begin{cases}u=1+x^2\\ 2x\,dx=du\end{cases}$

$\iint_D\frac{y}{1+x^2}\,dA=\int_0^1\int_0^{\sqrt{x}}\frac{y}{1+x^2}\,dy\,dx=\frac{1}{2}\int_0^1\frac{x}{1+x^2}\,dx$

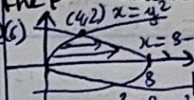
$\iint_D\frac{1}{1+x^2}\,dA$, where D is triangle with $(0,0)$ $(1,1)$ $(0,1)$.



$D=\{(\,)\mid x\le y\le1,\ 0\le x\le1\}$ $\int\frac{1}{1+x^2}\,dx=\tan^{-1}(x)+C$

$\iint_D\frac{1}{1+x^2}\,dA=\int_0^1\int_x^1\frac{1}{x^2+1}\,dy\,dx=\int_0^1\left[\frac{1}{1+x^2}-\frac{x}{1+x^2}\right]dx$

$\iint_D y\,dA$, D is first quadrant bounded by the parabolas $x=y^2$ and $x=8-y^2$.



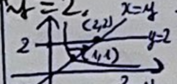
Find where $x=y^2$ and $8-y^2=x$.

$y^2=8-y^2\Rightarrow y^2=4\Rightarrow y=\pm2$

$D=\{(\,)\mid 0\le y\le2,\ y^2\le x\le8-y^2\}$

$\iint_D y\,dA=\int_0^2\int_{y^2}^{8-y^2}y\,dx\,dy=8.$

$\iint_D y\,dA$, where D is region in the first quadrant that lies above the hyperbola $xy=1$ and the line $y=x$, below the line $y=2$.

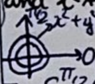


$\frac{1}{y}=y\Rightarrow y^2=1\Rightarrow y=1.$

$D=\{(\,)\mid 1\le y\le2,\ \frac{1}{y}\le x\le y\}$

$\iint_D y\,dA=\int_1^2\int_{1/y}^{y}y\,dx\,dy=\frac{4}{3}.$

$\iint_D x\,dA$, where D is the region in the first quadrant that lies b/w $x^2+y^2=1$ and $x^2+y^2=2$. In polar coordinate



$D=\{(r,\theta)\mid 1\le r\le\sqrt{2},\ 0\le\theta\le\frac{\pi}{2}\}$

$x=r\cos\theta,\ y=r\sin\theta,\ dx\,dy=r\,dr\,d\theta$

$I=\int_0^{\pi/2}\int_1^{\sqrt{2}}r\cos\theta\,r\,dr\,d\theta=\frac{2\sqrt{2}-1}{3}$

$\iiint_E xy\,dV$, where $E=\{(\,)\mid 0\le x\le3,\ 0\le y\le x,\ 0\le z\le x+y\}$

$I=\int_0^3\int_0^x\int_0^{x+y}xy\,dz\,dy\,dx=\frac{81}{2}.$

$\iiint_T xy\,dV$, where T is the solid tetrahedron with vertices $(0,0,0)$, $(\frac{1}{3},0,0)$, $(0,1,0)$, $(0,0,1)$

Eq of the plane: $\frac{x}{1/3}+\frac{y}{1}+\frac{z}{1}=1.$

$3x+y+z=1$. For xy-trace, $z=0$

$\Rightarrow y=1-3x.$

$T=\{(\,)\mid 0\le x\le\frac{1}{3},\ 0\le y\le1-3x,\ 0\le z\le1-3x-y\}$

$\int_0^{1/3}\int_0^{1-3x}\int_0^{1-y-3x}xy\,dz\,dy\,dx=\frac{1}{1080}$

xy plane: $z=0$ (plane)

yz planes: $x=0$ (plane)

xz planes: $y=0$ (plane).

31. $\iiint_E y^2z^2\,dV$, where E is bounded by paraboloids $x=1-y^2-z^2$ and the plane $x=0$.
The paraboloid is form x is : $x=y^2+z^2$
Since the given paraboloid is $x=1-(y^2+z^2)$ we can conclude paraboloid has a max value on the x-axis: $x=1$
The region in the yz plane is intersection of $x=0$ and $x=1-y^2-x^2$
$\Rightarrow y^2+z^2=1$ So the region R=1
Since the region in the yz-plane is a circle, we will use cylindrical coor
$y=r\cos\theta$, $z=r\sin\theta$, $x=x$
$dx\,dy\,dz = r\,dx\,dr\,d\theta$
$E=\{(\,)\in E \mid 0\le x\le 1-r^2, 0\le r\le 1, 0\le\theta\le 2\pi\}$
$I=\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r^2\cos^2\theta\, r^2\sin^2\theta\, r\,dx\,dr\,d\theta = \frac{\pi}{96}$

33. $\iiint_E yz\,dV$, where E lies above the plane $z=0$, below the plane $z=y$, and inside the cylinder $x^2+y^2=4$.
The plane $z=0$ is xy plane.
consider the z-axis. The limit of z are the planes $z=0$ and $z=y$ so
$0\le z\le y$
The limit of x and y are defined by the circle $x^2+y^2=4$, so $-2\le x\le 2$
$0\le y\le\sqrt{4-x^2}$
$I=\int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_0^y yz\,dz\,dy\,dx = \frac{64}{15}$

34. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV$ where H is the solid hemisphere that lies above the xy-plane and has center the origin and radius 1.
we will use the spherical coordinates
$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$
$dx\,dy\,dz=\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.
Since the radius of the hemisphere is 1, $0\le\rho\le 1$.
Since the solid is the hemisphere, ϕ are $0\le\phi\le\frac{\pi}{2}$.
Since the region in the xy-plane is a circle of radius 1, $0\le\theta\le 2\pi$.
Rewrite: $z^3\sqrt{x^2+y^2+z^2}=\rho^3\cos^3\phi\sqrt{\rho^2}$
$I=\int_0^{\pi/2}\int_0^{2\pi}\int_0^1(\rho^3\cos^3\phi)\rho\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$
$=\frac{\pi}{14}$

Find the Volume of given solid:
Above the paraboloid $z=x^2+y^2$ and below the half cone $z=\sqrt{x^2+y^2}$
$x=r\cos\theta$, $y=r\sin\theta$, $z=z$
The limit of z are: $x^2+y^2\le z\le\sqrt{x^2+y^2}$
$r^2\le z\le r$.
The region in xy plane is the intersection of $z=x^2+y^2$ and $z=\sqrt{x^2+y^2}$
$x^2+y^2=\sqrt{x^2+y^2} \Rightarrow (r^2)^2=r^2 \Rightarrow r^2=1$
so, the region in the xy-plane is a circle of radius 1, so the limit of r and θ are $0\le r\le 1$, $0\le\theta\le 2\pi$
$V=\int_0^{2\pi}\int_0^1\int_{r^2}^r r\,dz\,dr\,d\theta=\frac{\pi}{6}$.

Bounded By the cylinder $x^2+y^2=4$ and the planes $z=0$ and $y+z=3$.
The limit of z are the planes $z=0$ and $y+z=3$, so, $0\le z\le 3-y$
The region in the xy plane is the circle $x^2+y^2=2^2$ of $r=2$ at $(0,0)$
so, $0\le r\le 2$. The region is the whole circle so, $0\le\theta\le 2\pi$.
$V=\int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta=12\pi$.

Surface Area

Find the surface area of part of the surface $z=x^2+2y+2$ that lies above the triangular region $(0,0)$ $(1,0)$ $(1,1)$
$T=\{(x,y)\mid 0\le x\le 1, 0\le y\le x\}$
$f(x,y)=x^2+2y+2$.
$A=\iint_T\sqrt{(2x)^2+2^2+1}\,dA$
$=\int_0^1\int_0^x\sqrt{4x^2+5}\,dy\,dx=\frac{1}{12}(27-5\sqrt5)$

Riemann Sum

Estimate the volume of the solid that lies above the square $R=[0,2]\times[0,2]$ and below the elliptic paraboloid $z=16-x^2-2y^2$.
The paraboloid is the graph of z and the area of each square $\Delta A=1$
Approximating the volume by Rieman sum with $m=n=2$
$V\approx\sum_{i=1}^2\sum_{j=1}^2 f(x_i,y_i)\,\Delta A$.
$=f(1,1)\Delta A+f(1,2)\Delta A+f(2,1)\Delta A+f(2,2)\Delta A$
$=13+7+10+4=34$.

Volume of Solid

$z=x^2+y^2$ above xy-plane and inside $x^2+y^2=2x$
$(x-1)^2+y^2=1$
In polar coordinate, we have
$x^2+y^2=r^2$, $x=r\cos\theta$
$x^2+y^2=2x$ becomes $r^2=2r\cos\theta$
$r=2\cos\theta$. $D=\{(\,)\mid -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, 0\le r\le 2\cos\theta\}$
$V=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,r\,dr=\frac{3\pi}{2}$.

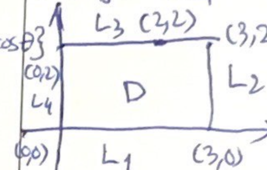
chapter 14 L4

23. If $z=xy+xe^{y/x}$, Show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z$.
$\frac{\partial z}{\partial x}=y+e^{y/x}-e^{y/x}\frac{y}{x}$
$\frac{\partial z}{\partial y}=x+e^{y/x}$
Therefore, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$
$=x(y+e^{y/x}-e^{y/x}\frac{y}{x})+y(x+e^{y/x})$
$=xy+xe^{y/x}+xy=z+xy$.

Lagrange Multipliers

Find the extreme values of the function $f(x,y)=x^2+2y^2$ on the circle $x^2+y^2=1$.
$g(x,y)=x^2+y^2=1$. Using lagrange multipliers, $\nabla f=\lambda\nabla g$ and $g(x,y)=1$
$f_x=\lambda g_x \Rightarrow 2x=2x\lambda$ (1)
$f_y=\lambda g_y \Rightarrow 4y=2y\lambda$ (2)
$g(x,y)=1 \Rightarrow x^2+y^2=1$. (3)
from (1), we have $2x(1-\lambda)=0$ so $x=0$ or $\lambda=1$. If $x=0$, then (3) gives $y=\pm1$. If $\lambda=1$, then $y=0$ from (2) and (4) gives $x=\pm1$
Therefore, f has 4 extreme values $(0,\pm1)$, $(\pm1,0)$.
$f(0,\pm1)=2$, $f(\pm1,0)=1$
Therefore, the max is 2 and min is 1.

Find the absolute max-min $f(x,y)=x^2-2xy+2y$ on the rectangle $D=\{(x,y)\mid 0\le x\le 3, 0\le y\le 2\}$
Since f is polynominal, it cont on the closed.
$f_x=2x-2y=0$, $f_y=-2x+2=0$
So the only CT point is $(1,1)$. This point is $f(1,1)=1$
We look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4. On L_1, we have $y=0$ and $f(x,0)=x^2$ $0\le x\le 3$



This is increasing function of x, so its minimum value is $f(0,0)=0$ and its max is $f(3,0)=9$. On L_2, we have $x=3$
$f(3,y)=9-4y$ $0\le y\le 2$
This is decreasing func of y, so its max is $f(3,0)=9$ and its min is $f(3,2)=1$. On L_3, we have $y=2$ and
$f(x,2)=x^2-4x+4$ $0\le x\le 3$
By method C4, $f(x,2)=(x-2)^2$ we see that min is $f(2,2)=0$ and the max is $f(0,2)=4$. Finally, L_4; we have $x=0$ and
$f(0,y)=2y$ $0\le y\le 2$
with max is $f(0,2)=4$ and min $f(0,0)=0$. Thus, on the boundary, the max is 9, the min of f is 0 and we compare these values with value $f(1,1)=1$ at the CP and conclude that the absolute max of f on D is $f(3,0)=9$ and the absolute min value is $f(0,0)=f(2,2)=0$