Second Order Differential Equation

Given the ODE (constant coefficient):

$$ay'' + by' + cy = g(x)$$

The general solution of this equation is given by:

$$y = y_c + y_p$$

(General solution = Complement solution + Particular solution)

The Characteristics Equation: $ar^2 + br + c = 0 \rightarrow r_1, r_2$

- Case 1: $g(x) = 0$ (Homogeneous)
  In this case: $y_p = 0 \rightarrow y = y_c$
  + r_1, r_2 are distinct real roots:
  $$y_c = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$
  + r_1, r_2 are complex roots: ($r_1 = \alpha + i\beta$; $r_2 = \alpha - i\beta$)
  $$y_c = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$$
  + r_1, r_2 are double roots ($r_1 = r_2 = r$)
  $$y_c = C_1 e^{rx} + C_2 x e^{rx}$$

- Case 2: $g(x) = P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
  + $c \neq 0$:
  $$y_p = Q_n(x)$$
  + $c = 0,\ b \neq 0$:
  $$y_p = xQ_n(x)$$
  + $c = b = 0$:
  $$y_p = x^2 Q_n(x)$$
- Case 3: $g(x) = P_n(x)e^{\alpha x} = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)e^{\alpha x}$
  + $\alpha \neq r_1, r_2$:
  $$y_p = Q_n(x)e^{\alpha x}$$
  + $\alpha \equiv r_1$ (or r_2) (Single root):
  $$y_p = xQ_n(x)e^{\alpha x}$$
  + $\alpha \equiv r$ (double roots):
  $$y_p = x^2 Q_n(x)e^{\alpha x}$$
- Case 4: $g(x) = P_n(x)e^{\alpha x} \times \begin{cases} \cos \beta x \\ \sin \beta x \end{cases}$
  + $\alpha + i\beta \neq r_1, r_2$:
  $$y_p = e^{\alpha x}[Q_n(x) \cos \beta x + R_n(x) \sin \beta x]$$
  + $\alpha + i\beta \equiv r_1, r_2$:
  $$y_p = xe^{\alpha x}[Q_n(x) \cos \beta x + R_n(x) \sin \beta x]$$