VIETNAM NATIONAL UNIVERSITY-HCMC
International University

Chapter 6. Eigenvalues and eigenvectors

Applied Linear Algebra

Fall 2022

Contents

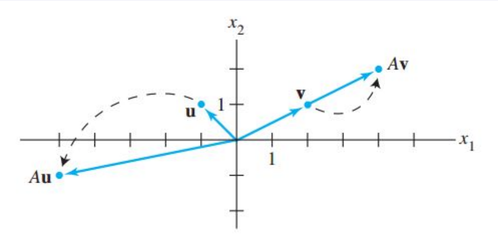
Example

Let A be a matrix and vectors u and v in $\mathbb{R}^n$ as follows.

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}; u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We have the transformations Au and Av

$$Au = \begin{pmatrix} -5 \\ -1 \end{pmatrix}; Av = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2v$$



Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. An eigenvector of A is a non-zero (column) vector X such that $AX = \lambda X$,
for some scalar (number) λ called eigenvalue.

Note: If X is an eigenvector associated with λ, then for any number $t \neq 0$, the vector tX is also an eigenvector associated with λ.

Theorem

Eigenvalues of A are roots of the equation

$$|A - \lambda I| = 0$$

and the polynomial $p(\lambda) := |A - \lambda I|$ is called the characteristics polynomial of A, where I is the $n \times n$ identity matrix

Eigenvalues and Eigenvectors

Example

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

Solution:

Eigenvalues of A are roots of the equation

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6$$

$$\lambda^2 - 5\lambda + 6 = 0 \Leftrightarrow (\lambda - 2)(\lambda - 3) = 0$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. Find eigenvectors associated with $\lambda_1 = 2$?

Find Eigenvectors

Find eigenvectors associated with $\lambda_1 = 2$. Let

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Consider $AX = 2X$:

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

That is,

$$\begin{cases} x_1 + x_2 = 2x_1 \\ -2x_1 + 4x_2 = 2x_2 \end{cases}$$

which reduces to $x_1 = x_2$.
Choose $x_2 = t$ to get, the set of eigenvectors associated with $\lambda_1 = 2$ has the form $X = (t, t)^T = t(1, 1)^T$, for any $t \neq 0$.

Find Eigenvectors

Find eigenvectors associated with $\lambda_1 = 3$.
Consider $AX = 3X$:

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

That is,

$$\begin{cases} x_1 + x_2 = 3x_1 \\ -2x_1 + 4x_2 = 3x_2 \end{cases}$$

which reduces to $2x_1 = x_2$.
Choose $x_1 = t$ to get, the set of eigenvectors associated with $\lambda_1 = 3$ has the form $X = (t, 2t)^T = t(1, 2)^T$, for any $t \neq 0$.

Eigenvalues and Eigenvectors

Example

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

Solution

Characteristic equation:

$$\det(A - \lambda \mathrm{I}) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

$$\lambda_1 = 5, \lambda_2 = -1$$

Eigenvalues and Eigenvectors

Solution

With $\lambda_1 = 5$, to find the eigenvectors, we need to solve

$$(A - \lambda_1 \mathrm{I})X = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \neq 0$$

Eigenvalues and Eigenvectors

Solution

With $\lambda_1 = -2$, we need to solve

$$(A - \lambda_2 \mathrm{I})X = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \end{bmatrix}, t \neq 0$$

Eigenvalues and Eigenvectors

Example

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Hint

$$\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 3$$

Eigenvalues and Eigenvectors

$$\lambda_1 = 1 \Rightarrow Eigenvectors = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \Rightarrow Eigenvectors = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 3 \Rightarrow Eigenvectors = t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

Exercise

Find the eigenvalues and the corresponding eigenvectors of the matrices

1. $$A = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}.$$

2. $$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{pmatrix}.$$

Hint:

1. $\lambda_1 = 5, \lambda_2 = -3.$
2. $\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -3$

Matrix Diagonalization

Definition

A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Namely, $P^{-1}AP = D$, where D is a diagonal matrix. D and A is called similar matrices.

Matrix Diagonalization

Theorem

An $n \times n$ matrix A is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors. Moreover, the elements on the main diagonal of D are the eigenvalues of A.

If a matrix A is similar to a diagonal matrix, we say that A is diagonalizable or can be diagonalized.

Matrix Diagonalization

Example

Let

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

We found that eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$ with associated eigenvectors

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

respectively. Since

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

is linearly independent, A can be diagonalized.

How?

Matrix Diagonalization

Method of Diagonalization

If λ_1, ..., λ_n are n eigenvalues of an $n \times n$ matrix A. Then we can diagonalize A by letting P be the matrix whose jth column is X_j.

$$P = \begin{bmatrix} X_1 & \dots & X_n \end{bmatrix}$$

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Therefore,

$$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Linear Systems of Differential Equations

Consider differential equations of the form

$$\frac{dx_1}{dt} = m_{11}x_1 + m_{12}x_2$$

$$\frac{dx_2}{dt} = m_{21}x_1 + m_{22}x_2$$

We can write the equations in matrix form

$$\frac{dX}{dt} = MX$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

To solve such equations we need the tools of eigenvalues and eigenvectors and a technique called matrix diagonalization.

Linear Systems of Differential Equations

Example

Solve the linear systems of differential equations

$$\frac{dX}{dt} = MX, \text{ where } M = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$$

Solution

We first find eigenvalues M. We have:

$$M - \lambda I = \begin{bmatrix} -2-\lambda & 2 \\ 1 & -3-\lambda \end{bmatrix}$$

$$\det(M - \lambda I) = (-2-\lambda)(-3-\lambda) - 2 = 0$$

$$\Leftrightarrow (\lambda + 4)(\lambda + 1) = 0$$

$$\Leftrightarrow \lambda = -4, \lambda = -1$$

Linear Systems of Differential Equations

Solution (Cont.)

We now find eigenvectors:

For $\lambda = -4$, the linear system

$$(M - \lambda I)X = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has infinite number of solutions

$$t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, t \in \mathbb{R}$$

Let us take arbitrarily one solution, which is the eigenvector corresponding to $\lambda = -4$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Linear Systems of Differential Equations

Solution (Cont.)

We now find eigenvectors:
For $\lambda = -1$, the linear system

$$(M - \lambda I)X = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has infinite number of solutions

$$t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Let us take arbitrarily one solution, which is the eigenvector corresponding to $\lambda = -1$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Linear Systems of Differential Equations

Solution (Cont.)

We form a matrix P whose columns are these eigenvectors:

$$P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

We next compute $P^{-1}MP$

$$P^{-1} = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}, \; P^{-1}MP = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$$

$P^{-1}MP$ is the diagonal matrix with eigenvalues on the diagonal!

Let $Y = P^{-1}X$ or $X = PY$. We re-write the original system as

$$\frac{dX}{dt} = MX \Leftrightarrow \frac{d(PY)}{dt} = MPY$$

Linear Systems of Differential Equations

Solution (Cont.) Thus,

$$\frac{dY}{dt} = P^{-1}MPY = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} Y$$

Or

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

which is equivalently with two separate differential equations:

$$\frac{dy_1}{dt} = -4y_1, \; \frac{dy_2}{dt} = -y_2$$

We can solve each of these two differential equations easily:

$$y_1(t) = C_1 e^{-4t}, \; y_2(t) = C_2 e^{-t}, \quad \text{where} \;\; C_1, C_2 \text{ are constants.}$$

Linear Systems of Differential Equations

Solution (Cont.)

Finally, we calculate $X = PY$ to get the general solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{-4t} \\ C_2 e^{-t} \end{bmatrix} = \begin{bmatrix} C_1 e^{-4t} + 2C_2 e^{-t} \\ -C_1 e^{-4t} + C_2 e^{-t} \end{bmatrix}$$

Remarks:

(1) Since there are infinitely many eigenvectors, there are also an infinite number of choices for the matrix P.

(2) The values of C_1, C_2 can be determined by the initial conditions.

Linear Systems of Differential Equations

Example

Solve the linear systems of differential equations

$$\frac{dX}{dt} = MX, \text{ where } M = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$$

where $x_1(0) = 5$ and $x_2(0) = 1$.

Solution

According to the previous example $x_1(t) = C_1 e^{-4t} + 2C_2 e^{-t}$ and $x_2(t) = -C_1 e^{-4t} + C_2 e^{-t}$.

$$\left.\begin{aligned} C_1 + 2C_2 &= 5 \\ -C_1 + C_2 &= 1 \end{aligned}\right\} \Leftrightarrow C_1 = 1, C_2 = 2$$

Therefore, the solution is given by

$$x_1(t) = e^{-4t} + 4e^{-t}, \; x_2(t) = -e^{-4t} + 2e^{-t}$$