Introduction to probability

Textbook

(1) Walpole et al, *Probability and Statistics for Engineers and Scientists*, 9th edition.

(2) S. Ross , *Introduction to Probability*, 9th edition

(3) R. Ross, Introduction to Probability and Statistics for Engineers and Scientists, 3th edition

Grade

- Progress score: 20%
    - Quiz: 10%
    - Homework: 5%
    - Attendance: 5%
- Midterm exam: 30%
- Final exam: 50%

3 parts of this course

- Probability
  Theory of the randomness
- Statistics
  the art of learning from data
- Random process
  Probability with time line

Chapter 2, 3, 4, 5, 6 in textbook (1)

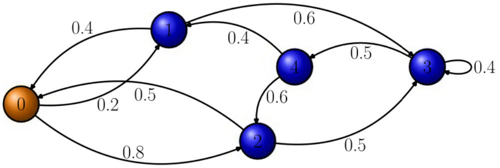
- Elements of probability
  - Probability space and event
  - Rules to compute probability (addition rule, conditional probability, multiple rule, total rule, Bayes's rule)
- Random variable
  - Probability distribution
  - Mathematical expectation

Chapter 1, 9, 10, 11 in textbook (1)

- Descriptive statistics
- Inference statistics
  - parameter estimation
  - hypothesis testing
- Linear regression
  - Least square estiamtors of coefficients
  - Inference about the estimators.

Random process

Chapter 3 in textbook (2)



Markov chain

- Transition probability
- State classification
- Stationary distribution

PART 1: Probability

What is Probability?

- Probability is the mathematics of chance - a discipline in mathematics which deals with phenomena whose outcome is affected by random events and therefore they can not be predicted with certainty
- Probability is a numerical measure of the likelihood that a specific event will occur - Measure of belief

Example

What is the chance that head occur when tossing a fair coin?

Probability that result of tossing a coin is head?

The events whose probabilities we wish to compute all arise as outcomes of experiments.

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Session Objectives

- Understand and describe sample spaces and events for random experiments with graphs, tables, lists, or tree diagrams
- Interprete and use operations on events such as unions, intersections, complement
- Interpret probabilities and use probabilities of outcomes to calculate probabilities of events in finite sample spaces

Table of contents

Experiment- Sample space

- An **experiment** is an activity with an observable result
- Each result is called an **outcome**
- The set of all possible outcomes is called the **sample space** denoted by Ω or S or U

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Examples

Sample space of experiment

- Toss a coin

$$\Omega = \{\text{Head}, \text{Tail}\}$$

- Roll a dice

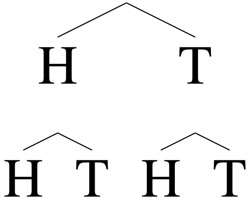
$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- Gender of a unborn baby

$$\Omega = \{\text{male}, \text{female}\}$$

Two tosses of a coin

Tree diagram of sample space



Sample space

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

Examples

- Sample space of rolling 2 dice

$$\Omega = \{(x, y) : x, y = 1, \ldots, 6\}$$

Finite sample space

- Sample space of measuring the thickness a connector

$$\Omega = (0, \infty)$$

Uncountable sample space

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Uncountable sample space

Consider the experiment of tossing a coin until "Tails" appear for the first time.
Suggest a suitable sample space that describes the outcomes of this experiment.

Different sample spaces for the same experiment

- A car store has 2 salespersons
- The store stock 2 cars for sales
- If we are interested in the number of cars which will be sold by each of the two salespersons during next week then the sample space is the set of pairs (i, j) where i and j are the number of cars sold by the first and second salesperson

- There are 2 cars available for sales $\Longrightarrow i + j \leq 2$
- Arrive at the sample space

$$\Omega_1 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$$

- if the store owner is only interested in the total number of cars sold during next week, then we could use as a sample space the set

$$\Omega_2 = \{0, 1, 2\}$$

Practice

Three students are selected at random from a chemistry class and classified as male or female. **List the elements of the sample space**

1. If we're interested in gender of each selected student. Using the letter M for male and F for female.
2. If we're interested in the number of females selected.

Events

- A set A of possible outcomes in sample space Ω of an experiment is called an **event** A
- An event is a subset of sample space
- Event A occurs or appears if the outcome is an element in A

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Example

- Toss a coin 3 times
- Observe the outcome HHT
- The event that there is exactly 1 tail

$$A = \{HHT, HTH, THH\}$$

  has occured
- But the

$$B = \{HHH, TTT\}$$

  event has not occured

Examples

- Roll 1 dice, event = having an odd face

$$A = \{1, 3, 5\}$$

- Roll 2 dice, event = sum of 2 faces is 6

$$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

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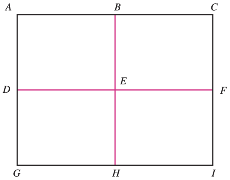
Practice - Go home

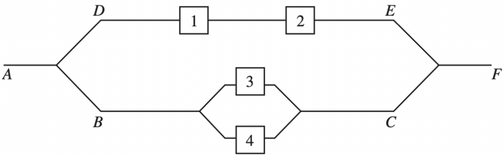
Bill just visited a friend who lives in Place A of the graph below and he wants to return home, at Place I on the graph.

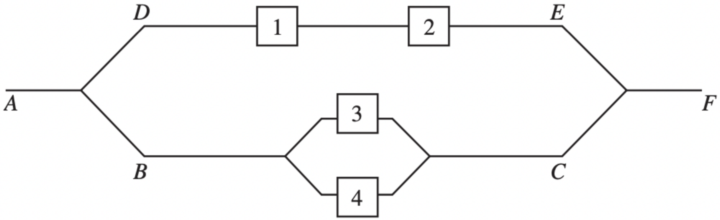


In order to minimize the distance he has to walk, he moves either downwards (e.g. from Place A to Place D) or to the right

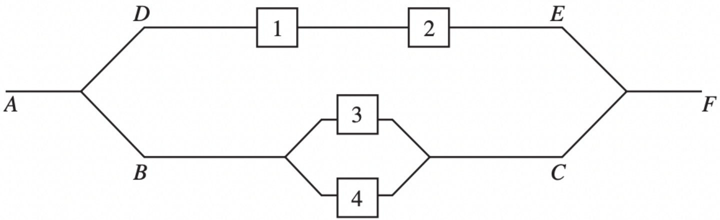
1. Give a sample space for the different routes Bill can follow to return home.
2. Write down explicitly the following events, representing them as subsets of the sample space given in (1):
    1. he passes through Place E on his way back home
    2. he does not pass through Place D

- The water is transferred from point A to point F through water tubes
- At the positions 1, 2, 3, and 4 , there are four switches which, if turned off, stop the water supply passing through the tube.





Find a sample space for the experiment which describes the positions of the four switches (ON or OFF).



Identify each of the following events

A_1 : there is water flow from D to E

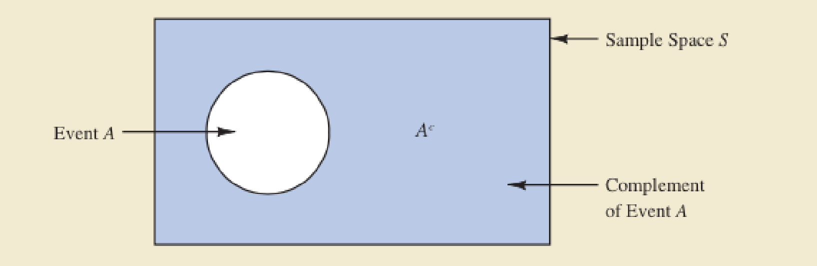
A_2 : there is water flow from B to C

Basic Relationships of Events

- Complement
- Intersection
- Mutual exclusive
- Union

Complement

The complement of event A, denoted by A^c or $\bar{A}$ or A' is the subset containing all the elements of Ω that are not in A.



$\boxed{A^c\text{: } A \textbf{ does not occur}}$

Example

- Sample space $\Omega = \{$book, cell phone, mp3, paper, stationery, laptop$\}$
- A = $\{$book, stationery, laptop, paper$\}$
- $A' = \{$cell phone, mp3$\}$

Example

Light bulb lifetime:
E = bulb last more than 3 hours,
E' = bulb last less than or equal 3 hours

Measurements of the thickness of a plastic connector might be modeled with the sample space $\Omega = R_+$ the set of positive real numbers. Let

$$A = \{x|x \geq 10\}$$

Then,

$$A' = \{x|x < 10\}$$

Example

Measurements of the thickness of a plastic connector might be modeled with the sample space $\Omega = R_+$ the set of positive real numbers. Let

$$B = \{x | 8 < x < 15\}.$$

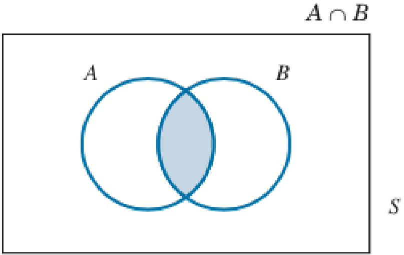
be the event that the random selected connector has thickness between 8 and 15

$$B' = ?$$

Intersection

Intersection of A and B, denoted by AB or $A \cap B$, is the subset of all elements that are in both A and B

$$\boxed{AB : \textbf{both } A \textbf{ and } B \textbf{ occurs}}$$



Example

Let A be the event that a person selected at random in a classroom is **majoring in engineering**, and let B be the event that the person is **female**. Then AB is the event of all **female engineering students** in the classroom.

Example - thickness of plastic connector

$$A = \{x | x \geq 10\}, \; B = \{x | 8 < x < 15\}.$$

then

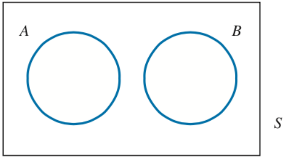
$$AB = \{x | 10 \leq x < 15\}$$

Mutually exclusive

Two events A and B are called **mutually exclusive** or **disjoint** if

$$AB = \emptyset$$

(A and B have no common element)



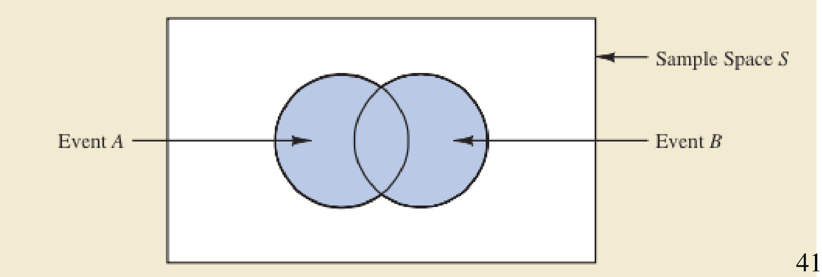
A and B **never occurs simultaneously**

Example

- Roll a 6-sided dice
- $A = \{1, 2\}$
- $B = \{4, 6\}$
- $AB = \emptyset$ so A and B are mutually exclusive

Union of A and B, denoted by $A \cup B$ or $A + B$. is the set of all elements that are in A or in B

$A \cup B$= either A or B or both occurs.



Example

Let $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$
then $A \cup B = \{a, b, c, d, e\}$

Example

Let P be the event that an employee selected at random from an oil drilling company smokes cigarettes. Let Q be the event that the employee selected drinks alcoholic beverages. Then the event $P \cup Q$ is the set of all employees who either drink or smoke or do both.

Example - thickness of plastic connector

$$A = \{x | x \geq 10\}, \; B = \{x | 8 < x < 15\}.$$

then

$$A \cup B = \{x | x > 8\}$$

Example

- $A = \{1, 3, 5\}, B = \{1, 2, 3\}$
- $AB = \{1, 3\}$ (in both A and B)
- $A \cup B = \{1, 2, 3, 5\}$ (in A or in B or in both)
- $AB' = \{5\}$ (in A but not in B)
- $BA' = \{2\}$ (in B but not in A)

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Practice

Consider the sample space

$$\Omega = \{(x, y) : -5 \leq x \leq 5, -3 \leq y \leq 7\}$$

and the events

$$A = \{(x, y) \in \Omega : x = y\},$$
$$B = \{(x, y) \in \Omega : x^2 = y^2,$$
$$C = \{(x, y) \in \Omega : x + y = 0\}\}$$

Which of the following statements are correct?

1. $B = AC$
2. $AC \subset B$
3. A and C are mutually exclusive (disjoint)
4. $A = \{(x, x) : -3 \leq x \leq 3\}$

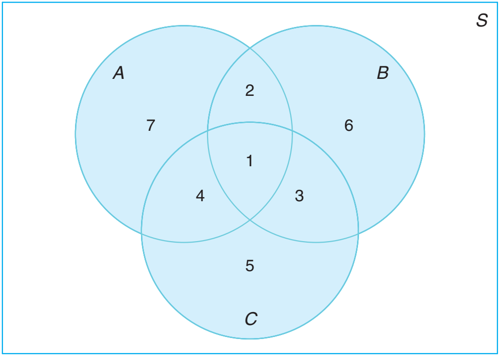
Practice

Let A, B, and C be three events in a sample space . Express each of the following events by the use of the operators (unions, intersections, complements) among sets:

1. all three events occur;
2. at least one of the three events occur;
3. A occurs, but not A and B;
4. A and C occur, but not B

Practice

Express each region in term of A, B and C



Practice

Let

$$\begin{aligned} \Omega =& \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ A =& \{1, 3, 5, 7\} \\ B =& \{2, 3, 5, 5\} \end{aligned}$$

List all elements in

1. AB
2. AB'
3. $A \cup B$

Properties for Operations

1. $A \cup B = B \cup A$, $A \cap B = B \cap A$
2. $A \cup A^c = \Omega$, $A \cap A^c = \emptyset$
3. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
4. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
5. If $A \subset B$ then $B^c \subset A^c$ and $A \cap B = A$, $A \cup B = B$

De Morgan's laws

1. $$\underbrace{(A \cup B)'}_{\text{complement of at least one event occurs}} = \underbrace{A' \cap B'}_{\text{no event occurs}}$$

2. $$\underbrace{(A \cap B)'}_{\text{complement of all events occur}} = \underbrace{A' \cup B'}_{\text{at least one event not occurs}}$$

Table of contents

Probability

- A **probability** is a numerical measure of the likelihood that a specific event will occur
- Probability of an event, denoted by **P(event)**, is a number between 0 and 1
- The larger the number, the more confident we are that the event will occur.

- Probability of an event is equal to 0: we can almost be sure that this event cannot occur

- Probabiity of an event is equal to 1: this event will occur for sure

Assign Probability Methods

1. A **logical probability** is obtained by mathematical reasoning—often, by the use of the counting techniques
2. An **empirical probability** is obtained by sampling or observation and is calculated as a relative frequency.
3. A **judgmental probability** is obtained by an educated guess.

Axiom of Probability

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$ (**Normalization**)
3. If $A_1, A_2 \ldots$ are mutually exclusive then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Example

Suppose the probability of an event A is larger by 0.5 than the probability of its complement, that is $P(A) = P(A^c) + 0.5$.
Find $P(A)$

Solution

Because A and A^c are mutually exclusive and $\Omega = A \cup A^c$, we have

$$1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$$

Combining this condition with the assumption $P(A) = P(A^c) + 0.5$, we obtain $P(A) = 0.75$

Practice

When a salesman visits a certain town, he stays in one of three available hotels, H_1, H_2, H_3. Let A_i be the event that he stays in Hotel H_i. It is known that

$$P(A_1 \cup A_2) = 3P(A_3), 3P(A_2) = 1 - \frac{P(A_3)}{2}$$

Find $P(A_i)$ - the probability that he stays in Hotel H_i , for $i = 1, 2, 3$.

Practice

For two disjoint events A and B in a sample space, suppose

$$P(A \cup B) = 0.5, 3P(A^c) + 2P(B) - 0.3$$

Find $P(A)$ and $P(B)$.

Probability law on finite sample space

- Sample space $\Omega = \{s_1, s_2, \ldots, s_n\}$
- Assign each outcome s_i with a probability $p(s_i)$ which satisfies
  - $0 \leq p(s_i) \leq 1$
  - $p(s_1) + p(s_2) + \cdots + p(s_n) = 1$
- Probability of an event

$$P(A) = \sum_{s_i \in A} p(s_i)$$

Example

Suppose there is a coin for which the chance to show head is twice more likely than the chance to show tail.

- $\Omega = \{H, T\}$ with $P(H) = 2P(T)$
- Normalization: $P(H) + P(T) = 1$
- $P(\{H\}) = 2/3, P(\{T\}) = 1/3$

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Example

Consider a sample space $\Omega = \{a, b, c, d\}$ with $p(a) = 0.1$, $p(b) = 0.5$, $p(c) = 0.3$ and $p(d) = 0.1$. Let

$$A = \{a, b, d\}$$

then

$$P(A) = p(a)+p(b)+p(d) = 0.1+0.5+0.1$$

Practice

A dice is loaded in such a way that each even number is twice as likely to occur as each odd number.

If E **is the event that a number less than 4 occurs on a single toss of the die**, find $P(E)$

Equally likely outcomes

- If $p(x)$ is the same for all x in Ω then we say that Ω has equally likely outcomes.
- If Ω has equally likely outcomes then

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

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Example

- Toss a fair coin

$$P(H) = P(T) = \frac{1}{2}$$

- Flip a fair dice

$$P(i) = \frac{1}{6}$$

for $i = 1, 2, 3, 4, 5, 6$

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Example

Roll a fair dice, event = having an odd face

$$A = \{1, 3, 5\}$$

Then

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

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Example

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is an industrial engineering major.

Solution

- I: the student chosen is an industrial engineering major
- The total number of students in the class is 53, all of whom are equally likely to be selected
- 25 of the 53 students are majoring in industrial engineering
- $n(\Omega) = 53, n(I) = 25$
- $P(I) = \frac{n(I)}{n(\Omega)} = \frac{25}{53}$

Practice

A fair coin is tossed twice. What is the probability that at least 1 head occurs?

Practice

Select randomly an emission from a set emissions which come from three suppliers and are classified for conformance to air-quality specifications.

		conforms	
		yes	no
	1	22	8
supplier	2	25	5
	3	30	10

A = emission is from supplier 1, B = emission conforms to specifications

1. Determine the number of samples in $A' \cap B$, B', and $A \cup B$

2. Compute probability of the events in the previous part

Practice

A large New York store that sells toys is going to hold a draw and the winner will receive a free one-week holiday. The store has 6000 tickets for sale to its customers during a week in the pre-Christmas season. If the tickets are numbered from 1 to 6000, what is the probability that the number on the winning ticket is a multiple of 2 or 5?