APPLIED LINEAR ALGEBRA

1. Linear Systems and Matrices

(Sections 7.1-7.3 from Kreyszig)

Contents

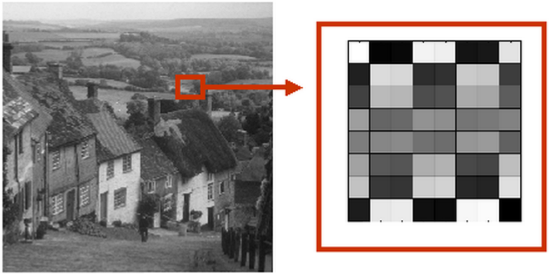
Section 1

Introduction

Introduction

Linear Algebra is a mathematical branch of foundational importance.

It has numerous practical applications in Engineering, Computer Science (Coding theory, Google, Image compression), Economics, Chemistry, Genetics,...



$$\begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}$$

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Section 2

Linear systems, matrices

A central problem of linear algebra is solving linear equations.
Example: Solve

$$\begin{cases} 2x + 3y = 5 \\ \ \ x - 4y = -3 \end{cases}$$

Here, the two *unknowns* that we need to find are x and y.

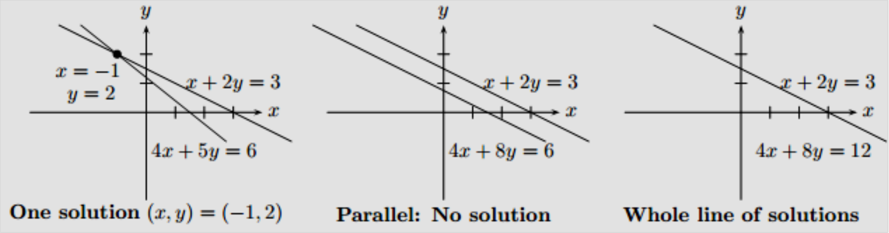
Method:

- Elimination: Substract $4 \times$ *Eq.*(1) from *Eq.* (2) to get $-3y = -6 \ \rightarrow y = 2$.
- Back-substitution: Put this value of y into the first equation to find $x = -1$.

This method is called *Gaussian elimination*. It can be easily extended to solve large systems of equations.

The Geometry of Linear Equations ($n = 2$)

- An equation
$$a_1 x + a_2 y = b$$
where a_1, a_2 and b are real constants, with a_1, a_2 not both zero, represents a line in the xy-plane.
- Thus two equations in a 2×2 system represent two lines. Depending on the relative position of these lines that the system has zero, one or infinitely many solutions.



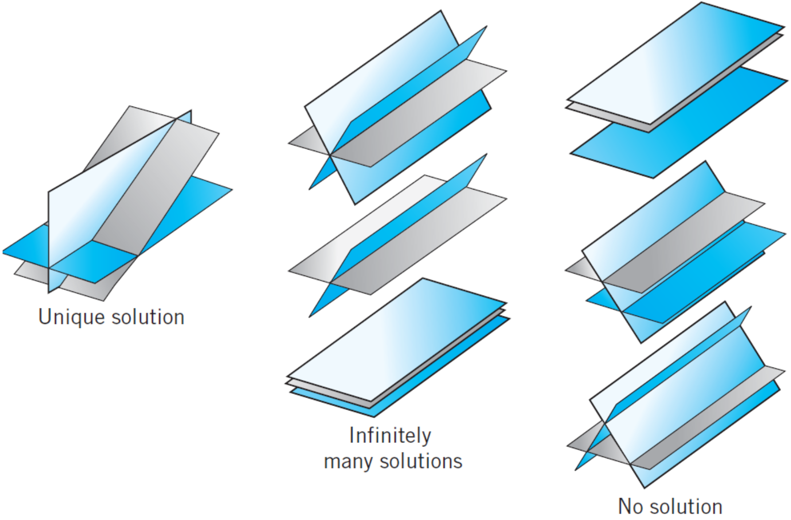
Linear systems of equations

- A system of m linear equations in n variable has the general form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

  Here the a_{ij} and b_j are real numbers. $x_1, \ldots, x_n$ are the unknowns.
- The aim is to find all solutions of the system, i.e. find all n-tuples $(x_1, \ldots, x_n)$ that satisfy these m equations simultaneously.
- A linear equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ represents a $(n-1)$-dimensional plane in an n-dimensional space. Thus the set of solutions of the system is the intersection of m such planes.

The Geometry of Linear Equations



Section 3

Gaussian elimination

Gaussian elimination

The method involves successive applications of *elementary operations* to convert a linear system into a *simpler, equivalent* system. There are three types of elementary operations:

1. Interchange any two equations.

2. Multiply an equation by a nonzero constant.

3. Add a multiple of one equation to another.

We need the concept of *matrices* to investigate the method of Gaussian Elimination.

What is a Matrix?

A matrix is a set of elements, organized into rows and columns

$$\begin{array}{cc} \begin{array}{cccc} \phantom{a_{11}} & \phantom{a_{12}} & & \textbf{Column} \end{array} & \\ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \textbf{Row} \end{array}$$

The number $a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, are called the entries (or elements) of A.
Equality: Two matrices A and B are equal if they have the same size and the corresponding entries are equal.

Example of a Matrix

Suppose that a manufacturer has four plants, each of which makes three products. If we let a_{ij} denote the number of units of product i made by plant j in one week, then we obtain a 3×4 matrix that gives the manufacturer's production in one week. In the example below, the plant 2 makes 270 units of product 3 in one week.

$$
\begin{array}{r c}
 & \begin{array}{cccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} & \text{Plant 4} \end{array} \\
\begin{array}{r} \text{Product 1} \\ \text{Product 2} \\ \text{Product 3} \end{array} & \begin{bmatrix} 560 & 360 & 380 & 0 \\ 340 & 450 & 420 & 80 \\ 280 & 270 & 210 & 380 \end{bmatrix}
\end{array}
$$

Matrices

- A matrix with m rows and n columns is called an $m \times n$ matrix (m by n matrix) or a matrix of order $m \times n$. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is a 2 by 3 matrix}$$

  The collection of all $m \times n$ matrices is denoted by $M_{m \times n}$

- If $m = n$, the matrix is called a square matrix. For example,

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 5 & 1 \\ 7 & 0 & 9 \end{bmatrix} \text{ is a (square) 3 by 3 matrix}$$

- A matrix consisting of a single column is called a *column vector* and a matrix consisting of a single row is called a *row vector*.

Coefficient and augmented matrices

The coefficient matrix of the system (1) is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example

Consider the system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \phantom{-4x_1 + {}} 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

The coefficient matrix is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Gaussian elimination

In matrix language, the method becomes: Transform the augmented matrix into a simpler one using *elementary row operations*. Denoting by R_i the i-th row of the matrix, the elementary row operations are:

1. Interchange rows R_i and R_j, denoted by I_{ij}.

2. Multiply row R_i by a nonzero constant α, denoted by αR_i.

3. Add αR_j to row R_i, denoted by $R_i + \alpha R_j$.

The simpler form that that we want to reduce the augmented matrix to is the *echelon form*.

Pivots, Echelon Form

Definition

1. A *pivot* is the leftmost nonzero entry of a row.
2. A *pivot column* is a column that contains a pivot.
3. A matrix is in *(row) echelon form* if
   a. All rows that contain only zeros are at the bottom of the matrix.

   b. For a nonzero row, its pivot appears strictly to the right of the pivot of the row above.

A matrix in row echelon has a staircase-like pattern

$$\begin{bmatrix} 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix in Row Echelon Form

Example

The following matrices are in row echelon form

$$a)\ \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$b)\ \begin{bmatrix} 1 & -3 & 2 & 7 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 2 & 9 \end{bmatrix}$$

$$c)\ \begin{bmatrix} 4 & -2 & 1 & 7 & 9 & 1 \\ 0 & 0 & -5 & 9 & 1 & 2 \\ 0 & 0 & 0 & 8 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix in Row Echelon Form

Example

The following matrices are NOT in row echelon form

$$a)\ \begin{bmatrix} 1 & -3 & 2 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$b)\ \begin{bmatrix} 1 & -3 & 2 & 7 \\ 0 & 0 & 1 & 7 \\ 0 & 2 & 2 & 9 \end{bmatrix}$$

$$c)\ \begin{bmatrix} 4 & -2 & 1 & 7 & 9 & 1 \\ 0 & 0 & -5 & 9 & 1 & 2 \\ 0 & 0 & 0 & 8 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Back-substitution

Systems whose augmented matrices are in echelon form can be easily solved by back-substitution.

Example

Solve the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 5 \\ 0 & -3 & 2 & 17 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

Variables: x, y, z

Solution The first step is to find the associated linear system

$$\begin{cases} 2x + y - 3z = 5 \\ - 3y + 2z = 17 \\ 5z = -10 \end{cases}$$

The system is in triangular form and we solve it by back-substitution: $5z = -10$ leads to $z = -2$. Thus, $y = -7, x = 3$.

Back-substitution

Example

Suppose that the augmented matrix for a linear system has been reduced to the given matrix in row echelon form and the variables are also given. Solve the system.

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 5 & 2 \\ 0 & 3 & 2 & -1 & 2 \\ 0 & 0 & -2 & -8 & 4 \end{array}\right]$$

Here the variables are a, b, c, d.

Solution: The associated linear system is

$$\begin{cases} 2a + 3b - c + 5d = 2 \\ \phantom{2a + {}} 3b + 2c - d = 2 \\ - 2c - 8d = 4 \end{cases}$$

Solution (Cont.) The variables a, b and c corresponding to the pivot columns are called the leading variables.
The remaining variables are called free variables.
The second step is to move the free variables to the right-hand side of the equations

$$\begin{cases} 2a + 3b - c = 2 - 5d \\ \phantom{2a + {}} 3b + 2c = 2 + d \\ - 2c = 4 + 8d \end{cases}$$

Now we can use back-substitution. The solution is

$$(a, b, c, d) = (-3 - 9d, 2 + 3d, -2 - 4d, d),$$

where d can take *any* value. Thus, this system has infinitely many solutions.

Back-substitution - General case

Consider a system in echelon form.

- The variables corresponding to the pivot columns are called the *leading variables*.
- The remaining variables are called the *free variables.*

One of the following three cases can happen.

1. There is a row of the form $[0 \quad 0 \quad \dots \quad 0 \,|\, b]$ where $b \neq 0$: the system has no solution.
2. There are no rows of the form $[0 \quad 0 \quad \dots \quad 0 \,|\, b]$ where $b \neq 0$:
   - If there are no free variables, the system has a unique solution.
   - If there are free variables, the system has infinitely many solutions. The free variables can take arbitrary values. By moving these variables to the right-hand side, one can find the values of the leading variables.

Reduction to Row Echelon Form

The main step of row reduction consists of three sub-steps:

1. Find the leftmost non-zero column of the matrix.
2. If necessary, apply a row exchange so that the top entry of this column is non-zero. This entry will be the pivot of the first row.
3. "Kill" (i.e. make them 0) all non-zero entries below the pivot by adding (subtracting) an appropriate multiple of the first row from the rows below.

After applying the main step, we leave the first row alone, and apply the main step to the matrix consisting of rows 2, . . . , m.

Then we leave the first two rows alone, and apply the main step to the matrix consisting of rows 3, . . . , m.

Continuing in this fashion, after at most m applications of the main step, the matrix will be in echelon form.

Examples

Example

Solve the system

$$\begin{cases} x + 4y - \ \ 4z = 3 \\ \quad\ \ 3y + \ \ 2z = 7 \\ \ \ - 9y + 20z = -8 \end{cases}$$

Solution: We transform the augmented matrix into row echelon form

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 3 \\ 0 & 3 & 2 & 7 \\ 0 & -9 & 20 & -8 \end{array}\right] \xrightarrow{R_3+3R_2} \left[\begin{array}{ccc|c} 1 & 4 & -4 & 3 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 26 & 13 \end{array}\right]$$

Solution (Cont.) The associated linear system is

$$\begin{aligned} x + 4y - 4z &= 3 \\ 3y + 2z &= 7 \\ 26z &= 13 \end{aligned}$$

This leads to the solution $(x, y, z) = (-3, 2, 1/2)$.

Examples

Example

Solve the following system

$$\begin{cases} 2x - y - z = 3 \\ -6x + 6y + 5z = -3 \\ 4x + 4y + 7z = 3 \end{cases}$$

First, we transform the augmented matrix into row echelon form

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ -6 & 6 & 5 & -3 \\ 4 & 4 & 7 & 3 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+3R_1} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ 0 & 3 & 2 & 6 \\ 0 & 6 & 9 & -3 \end{array}\right]$$

Examples

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ 0 & 3 & 2 & 6 \\ 0 & 6 & 9 & -3 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & 5 & -15 \end{array}\right]$$

The matrix is now in row echelon form with the associated system

$$\begin{cases} 2x - y - z = 3 \\ \phantom{2x -{}} 3y + 2z = 6 \\ \phantom{2x - 3y +{}} 5z = -15. \end{cases}$$

Using back-substitution we get the solution $(x, y, z) = (2, 4, -3)$.

Examples

Example

Solve the following system

$$\begin{cases} 3y + 2z = 7 \\ x + 4y - 4z = 3 \\ 3x + 3y + 8z = 1 \end{cases}$$

Hint

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 7 \\ 1 & 4 & -4 & 3 \\ 3 & 3 & 8 & 1 \end{array}\right] \xrightarrow[R_3+3R_2]{I_{12};R_3-3R_1} \left[\begin{array}{ccc|c} 1 & 4 & -4 & 3 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 26 & 13 \end{array}\right]$$

Examples

Example

Solve the following system with three equations and three unknowns:

$$\begin{cases} 2x + y + 5z = 1 \\ x - 3y + 6z = 2 \\ 3x + 5y + 4z = 0 \end{cases}$$

First, write down the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & 5 & 1 \\ 1 & -3 & 6 & 2 \\ 3 & 5 & 4 & 0 \end{array}\right] \xrightarrow[R_3-3/2R_1]{R_2-1/2R_1} \left[\begin{array}{ccc|c} 2 & 1 & 5 & 1 \\ 0 & -\,7/2 & 7/2 & 3/2 \\ 0 & 7/2 & -7/2 & -3/2 \end{array}\right]$$

Examples

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 2 & 1 & 5 & 1 \\ 0 & -7/2 & 7/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row gives us no information whatsoever, so it can be got rid of.
z is a free variable, so it can take any value a.

Substituting this into the second row, we get $y = a - 3/7$.

Substituting this into the first row, we get
$x = (1 - y - 5z)/2 = 5/7 - 3a$.

Thus, this system has an infinitely many solutions of the form $(5/7 - 3a, a - 3/7, a)$, parameterized by a.

Examples

Example

Solve the following system with three equations and four unknowns:

$$\begin{cases} x - 2y - z - w = -4 \\ 3x + y + z - 2w = 11 \\ x + 12y + 7z + w = 31 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -4 \\ 3 & 1 & 1 & -2 & 11 \\ 1 & 12 & 7 & 1 & 31 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -4 \\ 0 & 7 & 4 & 1 & 23 \\ 0 & 14 & 8 & 2 & 35 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & -1 & -4 \\ 0 & 7 & 4 & 1 & 23 \\ 0 & 0 & 0 & 0 & -11 \end{array}\right] \to \textbf{No solution!}$$

Examples

Example

Solve the following system of four equations and five unknowns:

$$\begin{cases} x + 2y \phantom{{}+ z} + w + 3t = 1 \\ - z + w \phantom{{}+ 3t} = 2 \\ 2x + 4y + z + w + 7t = 1 \\ 3x + 6y + 2z + w + 9t = -1 \end{cases}$$

Answer: $(-2a - b - 2, a, b - 2, b, 1)$, where a, b are arbitrary.

Exercises

1. Solve the following system

$$\begin{cases} 2x + 5y - z = 15 \\ x - y + 3z = 4 \\ 3x + 3y - 5z = 2 \end{cases}$$

Answer: $(x, y, z) = (1, 3, 2)$.

2. Solve the following system

$$\begin{cases} x - 2y + z = 0 \\ \phantom{-4x + {}} 2y - 8z = 8 \\ -4x + 5y + 9z = -9 \end{cases}$$

Answer: $(x, y, z) = (29, 16, 3)$.

3. Solve the following system

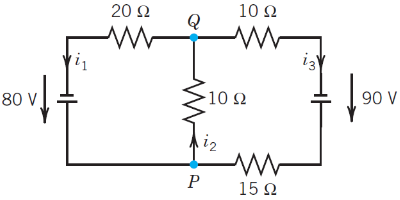
$$\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

Answer: $(x, y, z, w) = (3, -1/2, 4, 2)$.

4. Solve the following system

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80 \end{aligned}$$

Remark: This is the system for the unknown currents $x_1 = i_1$, $x_2 = i_2$, $x_3 = i_3$ in the electrical network in the following figure:



Node P:	$i_1 - i_2 + i_3 = 0$
Node Q:	$-i_1 + i_2 - i_3 = 0$
Right loop:	$10i_2 + 25i_3 = 90$
Left loop:	$20i_1 + 10i_2 = 80$

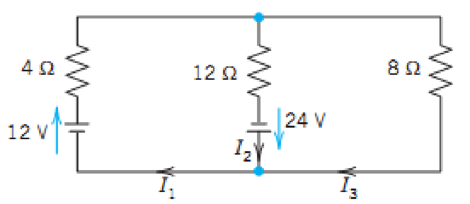
Exercises

5. Use Gaussian elimination to solve the system

$$\begin{cases} 2x + 3y - z = 7 \\ x - y + z = 1 \\ 4x - 5y + 2z = 3 \end{cases}$$

6. A certain brand of razor blades comes in packages of 6, 12, and 24 blades, costing \$2, \$3, and \$4 per package, respectively. A store sold 12 packages, containing a total of 162 razor blades and took in \$35. How many packages of each type were sold?

7. In the electrical network in the following figure, use the Kirchhoff's law to set up the linear system and solve for the currents I_1, I_2, I_3.



*The Cost of Elimination

Question: How many arithmetic operations (addition, substraction, multiplication or division) does elimination require, to solve a system of n equations in n unknowns? (we ignore swapping operations, as they are quick to perform)

Theorem

To solve an $n \times n$ system by Gaussian elemination, we need at most $\frac{1}{6}n(4n^2 + 9n - 7)$ operations.

We say the computational complexity of Gaussian elimination is $O(n^3)$. Best known algorithm to solve $n \times n$ system has complexity $O(n^{2.332})$.

Section 4

Matrix Operations

Special Matrices

- (Identity matrix) An $n \times n$ diagonal matrix whose entries on *the diagonal are all* 1 is called an *identity matrix* (or a unit matrix). It is denoted by I_n or simply I.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- (Scalar matrices) A *scalar matrix* is a diagonal matrix whose diagonal elements are equal.
- (Zero matrix) The matrix $O \in M_{m \times n}$ whose entries are all 0 is called the *$m \times n$ zero matrix*.

Basic Matrix Operations

a. **Addition and substraction:** Only define if both matrices A and B are of the same size,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

Then

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$$

Basic Matrix Operations

b. **Scalar multiplication:**

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$$

In particular, we denote

$$-A = (-1) \cdot A.$$

This is the unique matrix satisfying $A + (-A) = O$, where O is the $m \times n$ zero matrix.

Some Properties

Theorem

Let A, B, and C be $m \times n$ matrices.
(a) $A + B = B + A$.
(b) $A + (B + C) = (A + B) + C$.
(c) Let O be the zero matrix of the same size with A. Then $A + O = A$.

Theorem

If r and s are real numbers then
(a) $r(sA) = (rs)A$
(b) $(r + s)A = rA + sA$
(c) $r(A + B) = rA + rB$ if $A, B \in M_{m \times n}$,

Multiplication of Matrices

- The product AB can only be defined if the number of columns of A equals the number of rows of B. If $A \in M_{m\times n}$ and $B \in M_{n\times p}$ then $AB \in M_{m\times p}$ and

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

  for $1 \leq i \leq m$ and $1 \leq j \leq p$.
- For $k \in \mathbb{N}$, the *k-th power* of an $n \times n$ matrix A is defined by

$$A^k = A \cdot A \cdots A \qquad (k \text{ factors.})$$

  We also define $A^0 = I_n$.

Examples

$$\begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} = 10 - 2 - 12 = -4.$$

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 15 & 1 \end{bmatrix}$$

Properties of Matrix Multiplication

(a) $A(BC) = (AB)C$ if $A \in M_{m\times n}$, $B \in M_{n\times p}$ and $C \in M_{p\times q}$.
(b) $(A+B)C = AC + BC$ if $A, B \in M_{m\times n}$ and $C \in M_{n\times p}$.
(c) $C(A+B) = CA + CB$ if $A, B \in M_{n\times p}$ and $C \in M_{m\times n}$.
(d) $A(rB) = r(AB) = (rA)B$ if $A \in M_{m\times n}$ and $B \in M_{n\times p}$.
(e) $I_m A = AI_n = A$ if if $A \in M_{m\times n}$.

In the special case where A is a row matrix and B is a column matrix of the same dimension, $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}^T$, the product AB is the 1×1 matrix whose entry is

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \sum\nolimits_1^n a_i b_i.$$

This number is called the *dot product* of A and B.

Remarks

If a, b, and c are real numbers for which $ab = ac$ and $a \neq 0$ it follows that $b = c$. That is, we can cancel out the nonzero factor a. However, *the cancellation law does NOT hold for matrices*, as the following example shows.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$$

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}$$

but $B \neq C$.

Remarks

Also, unlike real numbers, for matrices A and B, $AB = 0$ does not implies that either $A = 0$ or $B = 0$.

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$$

but

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Transpose Matrix

- The *transpose* of an $m \times n$ matrix $A = [a_{ij}]$, denoted by A^T is an $n \times m$ matrix obtained by interchanging the rows and columns of A, i.e.

$$[a_{ij}]^T = [a_{ji}]$$

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 8 & 7 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -4 \\ 2 & 8 \\ 3 & 7 \end{bmatrix}$$

- A matrix A is *symmetric* if $A^T = A$. A symmetric matrix must be a square matrix. Example:

$$A = \begin{bmatrix} 1 & -1 & 4 \\ -1 & 0 & 2 \\ 4 & 2 & 6 \end{bmatrix}$$

Properties of Transpose

Theorem

If r and s are real numbers and A, B, and C are matrices of the appropriate sizes then
(a) $(A^T)^T = A$
(b) $(A + B)^T = A^T + B^T$
(c) $(AB)^T = B^T A^T$
(d) $(rA)^T = r(A^T)$.

Proof:

Example: $(AB)^T = B^T A^T$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 5 \\ 7 & -3 \end{bmatrix}, [AB]^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$$

On the other hand,

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = [AB]^T$$

Special Types of Matrices

- Let A be a square, $n \times n$, matrix. Then its (principal) diagonal consists of the entries $a_{11}, a_{22}, \ldots, a_{nn}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

- (Diagonal matrices) An $n \times n$ matrix $A = [a_{ij}]$ is called a *diagonal matrix* if $a_{ij} = 0$ for $i \neq j$.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Special Types of Matrices

- An $n \times n$ matrix $A = [a_{ij}]$ is called *upper triangular* if $a_{ij} = 0$ for $i > j$. Example

$$A = \begin{bmatrix} 1 & 7 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

- An $n \times n$ matrix $A = [a_{ij}]$ is called *lower triangular* if $a_{ij} = 0$ for $i < j$. Example:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

Section 5

Inverse matrices

Inverse Matrix

Definition

Let A be an $n \times n$ matrix. An $n \times n$ matrix B is an *inverse of A* if

$$AB = BA = I_n,$$

where I_n is the $n \times n$ identity matrix. In this case, A is said to be *invertible* or *nonsingular*.
If no such B exists then A is called *noninvertible*, or *singular*.

Note that if B is an inverse of A then A is an inverse of B.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

are inverses of each other.

Inverse Matrix

Theorem

An invertible matrix has only one inverse.

Proof: Let B and C be inverses of A. Then $AB = BA = I_n$ and $AC = CA = I_n$. We then have

$$B = BI_n = B(AC) = (BA)C = I_nC = C,$$

which proves that the inverse of a matrix, if it exists, is unique.

Definition

The inverse of an invertible matrix A is denoted by A^{-1}.

Inverse Matrix

Theorem

If A and B are both invertible $n \times n$ matrices, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:... Need to show that $(AB)(B^{-1}A^{-1}) = I_n$ and $(B^{-1}A^{-1})(AB) = I_n$.

Corollary

If A_1, A_2,..., and A_k are invertible $n \times n$ matrices, then $A_1A_2...A_k$ is invertible and

$$(A_1A_2...A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}...A_1^{-1}$$

Inverse Matrix

Theorem

Suppose that A is an invertible $n \times n$ matrix. Then

- A^{-1} is invertible and $(A^{-1})^{-1} = A$.
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
- $AB = AC$ implies $B = C$, where B and C are $n \times m$ matrices.

Proof: Taking transposes of the equation $AA^{-1} = I_n$ both sides, we get

$$\left[A^{-1}\right]^T A^T = I_n^T = I_n$$

Similarly, $A^T \left[A^{-1}\right]^T = (A^{-1}A)^T = I_n$. These equation implies that the inverse of A^T is $\left[A^{-1}\right]^T$.

Inverse Matrix

Theorem

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

An algorithm for finding A^{-1}

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. If this is the case, any sequence of elementary row operations that transforms A to I_n also transforms I_n to A^{-1}.

An algorithm for finding A^{-1}

1. Form the matrix $[A \mid I_n]$, where I_n is the identity matrix.
2. Try to find row operations that transform $[A \mid I_n]$ to a matrix of the form $[I_n \mid B]$.
3. If Step 2 is successful, B is A^{-1}. If not, A is not invertible.

An algorithm for finding A^{-1}

Details of Step 2

1. Reduce A to an echelon form A' using elementary row operations. In the process, row operations are applied to the whole matrix $[A \mid I_n]$, not only A.
2. If the last row of A' contains all zeros then A is not invertible, and we stop. Otherwise, go to next step.
3. By multiplying each row with a suitable scalar, make the diagonal entries 1.
4. Using row operations, make the entries above the diagonal 0.

An algorithm for finding A^{-1}

Example

Find the inverse of

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 6 & 7 \end{bmatrix}$$

Solution: Step 1: We form the matrix $[A|I_3]$ as

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right]$$

Solution (Cont.) Step 2: Apply Row Operations on $[A|I_3]$

$$
[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_1-3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] = [I_3|A^{-1}]
$$

Step 3: Conclusion

$$
A^{-1} = \left[\begin{array}{ccc} 10 & -3 & -3 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{array}\right]
$$

An algorithm for finding A^{-1}

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix},$$

if it exists.

Solution:

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

An algorithm for finding A^{-1}

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Since $A \sim I$, A is invertible and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

An algorithm for finding A^{-1}

Example

Find the inverse, if it exists, of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}.$$

$$\begin{aligned}
[\,A \quad I\,] &\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}
\end{aligned}$$

So A is not invertible.

Linear systems as matrix equations

The system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \tag{2}$$

can be re-written as $Ax = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{matrix} x = (x_1, x_2, \dots, x_n)^T \\ b = (b_1, b_2, \dots, b_m)^T. \end{matrix}$$

Invertibility and Solvability

Theorem

Let A be an $n \times n$ matrix. Then the following are equivalent

1. For any $b \in \mathbb{R}^n$, the equation $Ax = b$ has a solution.
2. For any $b \in \mathbb{R}^n$, the equation $Ax = b$ has a *unique* solution.
3. A is invertible.

Proof: Let A' be an echelon form of A. $Ax = b$ has solution for any b if and only if the diagonal entries of A' are non-zero. But if this is the case then the solution is also unique. Finally, A is invertible also only when the diagonal entries of A' are non-zero.

Use inverse matrix to solve linear system

Theorem

If A is invertible then for any $b \in \mathbb{R}^n$, the unique solution of $Ax = b$ is $x = A^{-1}b$.

Example

Solve the system

$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases}$$

Example

Solve the system

$$\begin{cases} \phantom{2x + {}} 2y + z = 2 \\ 2x \phantom{{}+ 2y} + z = 0 \\ x + y + 2z = -1 \end{cases}$$

Exercises

Section 7.3 (p. 280): 1-14
Section 7.2 (p. 270): 11-20