Cal 3 2019/06

Q1.

a)

Since, we have: $f(t) = tu(t) \to F(s) = \mathcal{L}\{f(t)\} = 1/s^2$

Let: $g(t) = (f * f)(t)$, it leads to:

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{(f * f)(t)\} = F(s).F(s)$$

$$\to G(s) = \frac{1}{s^4} \to g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{6}$$

Thus,

$$(f * f)(t) = \frac{t^3}{6}$$

b)

Given that:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \delta(t - \pi) \ \ (*), \qquad y(0) = 1, \quad y'(0) = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s) - s] + 2[sY(s) - 1] + 2Y(s) = e^{-\pi s}$$

$$\leftrightarrow Y(s)(s^2 + 2s + 2) = s + 2 + e^{-\pi s}$$

$$\leftrightarrow Y(s) = \frac{s + 2 + e^{-\pi s}}{s^2 + 2s + 2}$$

$$\leftrightarrow Y(s) = \frac{s + 1 + 1}{(s + 1)^2 + 1^2} + \frac{1}{(s + 1)^2 + 1^2}e^{-\pi s}$$

$$\to y(t) = \mathcal{L}^{-1}\{Y(s)\} = (e^{-t}\cos t + e^{-t}\sin t)u(t) + e^{-(t-\pi)}\sin(t - \pi)\,u(t - \pi)$$

Thus, the solution of the given differential equation is:

$$y(t) = (e^{-t}\cos t + e^{-t}\sin t)u(t) + e^{-(t-\pi)}\sin(t - \pi)\,u(t - \pi)$$

Q2.

Given that:

$$L\frac{di}{dt} + Ri = E[u(t) - u(t - a)] \ \ (*), \qquad i(0) = 0$$

Let $I(s) = \mathcal{L}\{i(t)\} \to \mathcal{L}\{i'(t)\} = sI(s) - i(0) = sI(s)$

Taking Laplace transform both sides of $(*)$, we obtain:

$$LsI(s) + RI(s) = E\frac{1}{s}(1 - e^{-as})$$

$$\leftrightarrow I(s)(Ls + R) = \frac{E}{s}(1 - e^{-as})$$

$$\leftrightarrow I(s) = \frac{E}{s(Ls + R)}(1 - e^{-as})$$

$$\leftrightarrow I(s) = \frac{E}{R}\left(\frac{1}{s} - \frac{1}{s + R/L}\right)(1 - e^{-as})$$

$$\to i(t) = \mathcal{L}^{-1}\{I(s)\} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)u(t) - \frac{E}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right)u(t - a)$$

Thus, the current with time t is:

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)u(t) - \frac{E}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right)u(t - a)$$

Q3.

Given that: $f(t) = 5\sin\frac{t}{2},\ \ 0 \le t \le 2\pi, \qquad T = 2\pi \to \omega = \frac{2\pi}{T} = 1$

a)

- $a_0 = \frac{2}{T}\int_{t_0}^{t_o+T} f(t)dt = \frac{2}{2\pi}\int_0^{2\pi} 5\sin\frac{t}{2}dt = \frac{20}{\pi}$
- $a_n = \frac{2}{T}\int_{t_0}^{t_o+T} f(t)\cos(n\omega t)\,dt = \frac{2}{2\pi}\int_0^{2\pi} 5\sin\frac{t}{2}\cos(nt)\,dt$

$$= \frac{5}{2\pi}\left[\frac{\cos[t(1/2-n)]}{1/2-n} + \frac{\cos[t(1/2+n)]}{1/2+n}\right]\Bigg|_0^{2\pi}$$

$$= \frac{20}{\pi(1-4n^2)}$$

- $b_n = \frac{2}{T}\int_{t_0}^{t_o+T} f(t)\sin(n\omega t)\,dt = \frac{2}{2\pi}\int_0^{2\pi} 5\sin\frac{t}{2}\sin(nt)\,dt$

$$= \frac{5}{2\pi}\left[\frac{\sin[t(1/2-n)]}{1/2-n} + \frac{\sin[t(1/2+n)]}{1/2+n}\right]\Bigg|_0^{2\pi}$$

$$= 0$$

The Fourier series is given by:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n\cos(n\omega t) + \sum_{n=1}^{+\infty} b_n\sin(n\omega t)$$

$$= \frac{10}{\pi} + \sum_{n=1}^{+\infty}\frac{20}{\pi(1-4n^2)}\cos(nt)$$

b)

Since we have: $f(t) = 5\sin\left(\frac{t}{2}\right) \to f(0) = 0$

Therefore,

$$f(0) = \frac{10}{\pi} + \sum_{n=1}^{+\infty}\frac{20}{\pi(1-4n^2)} = 0$$

$$\to \sum_{n=1}^{+\infty}\frac{20}{\pi(4n^2-1)} = \frac{10}{\pi}$$

$$\leftrightarrow \sum_{n=1}^{+\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

$$\to \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots = \frac{1}{2}$$

Q4.

a)

Let: $$Y(z) = \frac{z^2+z}{(z-2)^2}$$

$$\to \frac{Y(z)}{z} = \frac{z+1}{(z-2)^2} = \frac{3}{(z-2)^2} + \frac{1}{z-2}$$

$$\to Y(z) = \frac{3z}{(z-2)^2} + \frac{z}{z-2}$$

Thus,

$$\mathcal{Z}^{-1}\left\{\frac{z^2+z}{(z-2)^2}\right\} = \mathcal{Z}^{-1}\left\{\frac{3z}{(z-2)^2} + \frac{z}{z-2}\right\} = 3n2^{n-1} + 2^n$$

b)

Given that:

$$\begin{cases} x_{n+1} = x_n - 2y_n \ (1) \\ y_{n+1} = -6y_n \quad\ (2) \end{cases}, \qquad x_0 = -1, y_0 = 3$$

Let: $X(z) = \mathcal{Z}\{x_n\}, Y(z) = \mathcal{Z}\{y_n\}$

$$\to \begin{cases} \mathcal{Z}\{x_{n+1}\} = zX(z) - zx_0 = zX(z) + z \\ \mathcal{Z}\{y_{n+1}\} = zY(z) - zy_0 = zY(z) - 3z \end{cases}$$

Taking $\mathcal{Z}$-transform both side of (2), we obtain:

$$(2) \to zY(z) - 3z = -6Y(z)$$

$$\to Y(z) = \frac{3z}{z+6} \to y_n = \mathcal{Z}^{-1}\{Y(z)\} = 3(-6)^n$$

Taking $\mathcal{Z}$-transform both side of (1), we obtain:

$$(1) \to zX(z) + z = X(z) - 2Y(z)$$

$$\to X(z)(z-1) = -\frac{6z}{z+6} - z$$

$$\to \frac{X(z)}{z} = -\frac{6}{(z+6)(z-1)} - \frac{1}{z-1} = \frac{6/7}{z+6} - \frac{13/7}{z-1}$$

$$\to X(z) = \frac{6}{7}\frac{z}{z+6} - \frac{13}{7}\frac{z}{z-1} \to x_n = \mathcal{Z}^{-1}\{X(z)\} = \frac{6}{7}(-6)^n - \frac{13}{7}$$

Thus, the solution of the given system difference equations is:

$$\begin{cases} x_n = \dfrac{6}{7}(-6)^n - \dfrac{13}{7} \\ y_n = 3(-6)^n \end{cases}$$