Cal 3 2018/12

Q1.

a)

$$\frac{(2+j)(3+2j)}{1-j} = -\frac{3}{2} + \frac{11}{2}j$$

b) Let: $z = \frac{1+j}{\sqrt{2}}$

$$\rightarrow \begin{cases} r = |z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1 \\ \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4} \end{cases}$$

Therefore, in polar form, z can be expressed as: $z = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4}$

Since we know that: $z^n = r^n(\cos n\theta + j\sin n\theta)$

So $\left(\frac{1+j}{\sqrt{2}}\right)^{20} = z^{20} = 1^{20}\left(\cos\frac{4\pi}{4} + j\sin\frac{4\pi}{4}\right) = -1$

Thus,

$$\left(\frac{1+j}{\sqrt{2}}\right)^{20} = -1$$

Q2.

a)

Let: $z = 1$

$$\rightarrow \begin{cases} r = |z| = \sqrt{0^2 + 1^2} = 1 \\ \theta = \tan^{-1}\frac{0}{1} = 0 \end{cases}$$

Since, we know that:

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0,1,\dots,n-1$$

Therefore, there is exist 3 cubic roots of z as follows:

$$w_0 = \sqrt[3]{1}\left(\cos\frac{0+0}{3} + j\sin\frac{0+0}{3}\right) = 1$$

$$w_1 = \sqrt[3]{1}\left(\cos\frac{0+2\pi}{3} + j\sin\frac{0+2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$$

$$w_2 = \sqrt[3]{1}\left(\cos\frac{0+4\pi}{3} + j\sin\frac{0+4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$$

b)

1. Laplace equation for $u(x,y)$:

$$\nabla_u^2 = u_{xx} + u_{yy} = 2x^2 + 2y^2$$

Since, only one point $x = y = 0$ or $z = 0$ satisfies Laplace equation $\nabla_u^2 = 0$, but $z = 0$ is not included in the considered domain. Therefore, the function $u(x,y)$ is not satisfies the Laplace equation

2. Assume that $u(x,y)$ and $v(x,y)$ are already satisfied Cauchy-Riemann equation:

- $u_x = 2xy^2 = v_y \quad \rightarrow v = \int v_y\,dy = 2x\frac{y^3}{3} + C_1(x)$ (1)
- $u_y = 2x^2y = -v_x \rightarrow v = \int v_x\,dx = 2y\frac{x^3}{3} + C_2(y)$ (2)

From (1) and (2), we get contradiction which implies that $u(x,y)$ and $v(x,y)$ never satisfied Cauchy-Riemann equation. Therefore, there is no complex analytic function whose real part is $u(x,y) = x^2y^2$

Q3.

a)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 3t - 5e^{-2t} + te^{4t}\} = \frac{3}{s^2+3^2} - \frac{5}{s+2} + \frac{1}{(s-4)^2}$$

b)

Given that:

$$\frac{d^2x}{dt^2} + x = e^{2t} \ (*), \qquad x(0) = 0, \quad x'(0) = 1$$

Let $X(s) = \mathcal{L}\{x(t)\}$, it holds that:

$$\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0) = s^2X(s) - 1$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$s^2X(s) - 1 + X(s) = \frac{1}{s-2}$$

$$\leftrightarrow X(s)(s^2+1) = \frac{1}{s-2} + 1$$

$$\leftrightarrow X(s) = \frac{\frac{1}{s-2}+1}{s^2+1}$$

$$\leftrightarrow X(s) = \frac{1}{5}\left(\frac{-s+3}{s^2+1} + \frac{1}{s-2}\right)$$

$\rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{5}(-\cos t + 3\sin t + e^{2t})u(t)$

Thus, the solution of the given differential equation is:

$$x(t) = \frac{1}{5}(-\cos t + 3\sin t + e^{2t})u(t)$$

Q4.

$$f(z) = \frac{1}{(z-1)(z+2)} = \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right)$$

Apply power series for analyzing this problem:

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

We have:

$$f(z) = \frac{1}{3}\left(\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{2}\frac{1}{1+\frac{z}{2}}\right)$$

With $1 < |z| \leftrightarrow \frac{1}{|z|} < 1$, it holds that:

$$\frac{1}{1-\frac{1}{z}} = \sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n$$

With $|z| < 2 \leftrightarrow \left|\frac{z}{2}\right| < 1$, it holds that:

$$\frac{1}{1+\frac{z}{2}} = \sum_{n=0}^{+\infty}\left(-\frac{z}{2}\right)^n = \sum_{n=0}^{+\infty}\left(-\frac{1}{2}\right)^n z^n$$

Therefore,

$$\begin{aligned} f(z) &= \frac{1}{3}\left(\frac{1}{z}\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{+\infty}\left(-\frac{1}{2}\right)^n z^n\right) \\ &= \frac{1}{3}\left(\sum_{n=0}^{+\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{+\infty}\left(-\frac{1}{2}\right)^{n+1} z^n\right) \\ &= \frac{1}{3}\sum_{n=0}^{+\infty}\left[\frac{1}{z^{n+1}} + \left(-\frac{1}{2}\right)^{n+1} z^n\right] \end{aligned}$$