Chapter 2

First Order Differential Equations

Introduction

Any first order differential equation can be written as

$$F(x, y, y') = 0$$

by moving all nonzero terms to the left hand side of the equation. Of course, y' must appear explicitly in the expression F. Our study of first order differential equations requires an additional assumption, namely that the equation can be solved for y'. This means that we can write the equation in the form

$$y' = f(x, y). \tag{F}$$

2.1. Linear Differential Equations

A first order differential equation $y' = f(x, y)$ is a linear equation if the function f is a "linear" expression in y. That is, the equation is linear if the function f has the form

$$f(x, y) = P(x)y + q(x).$$

(c.f. The linear function $y = mx + b$.)

The solution method for linear equations is based on writing the equation as

$$y' - P(x)y = q(x) \qquad \text{which is the same as} \qquad y' + p(x)y = q(x)$$

where $p(x) = -P(x)$. The precise definition of a linear equation that we will use is:

FIRST ORDER LINEAR DIFFERENTIAL EQUATION: The first order differential equation $y' = f(x, y)$ is a *linear equation* if it can be written in the form

$$y' + p(x)y = q(x) \tag{1}$$

where p and q are continuous functions on some interval I. Differential equations that are not linear are called *nonlinear equations.*

SOLUTION METHOD:

Step 1. Identify and write the equation in the form (1).

Step 2. Calculate

$$h(x) = \int p(x) \, dx$$

(omitting the constant of integration) and form $e^{h(x)}$.

Step 3. Multiply the equation by $e^{h(x)}$ to obtain

$$e^{h(x)}y' + e^{h(x)} p(x) \, y = e^{h(x)} q(x). \qquad \text{multiply with } e^{h(x)}$$

Verify that the left side of this equation is $\left[e^{h(x)} y\right]'$.

Thus we have

$$\left[e^{h(x)} y\right]' = e^{h(x)} q(x).$$

Step 4. The equation in Step 3 implies that

$$e^{h(x)} y = \int e^{h(x)} q(x) \, dx + C$$

and

$$y = e^{-h(x)} \left[\int e^{h(x)} q(x) \, dx + C\right] = e^{-h(x)} \int e^{h(x)} q(x) \, dx + Ce^{-h(x)}.$$

Therefore, the general solution of (1) is:

$$y = e^{-h(x)} \int e^{h(x)} q(x) \, dx + Ce^{-h(x)}. \tag{2}$$

NOTES: (a) **Standard Form:** The solution method *requires* that the equation be written in the standard form (1).

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(b) **Integrating Factor:** The key step in solving $y' + p(x)y = q(x)$ is multiplication by $e^{h(x)}$ where $h(x) = \int p(x)\,dx$. It is multiplication by this factor, called an *integrating factor*, that enables us to write the left side of the equation as a derivative (the derivative of the product $e^{h(x)}y$) from which we get the general solution in Step 4. ∎

EXISTENCE AND UNIQUENESS: Obviously solutions of first order linear equations exist. It follows from Steps (3) and (4) that the general solution (2) represents all solutions of the equation (1). As you will see, if an initial condition is specified, then the constant C will be uniquely determined. Thus, a first order, linear, initial-value problem will have a unique solution.

Example 1. Find the general solution of

$$y' + 2x\,y = x.$$

SOLUTION

$$y' + p(x)\,y = q(x)$$

(1) The equation is linear; it is already in the form (1); $p(x) = 2x$, $q(x) = x$ continuous functions on $(-\infty, \infty)$.

(2) Calculate: $h(x) = \displaystyle\int 2x\,dx = x^2$ and $e^{h(x)} = e^{x^2}$.

(3) Multiply by e^{x^2}:

$p(x)$

$$e^{x^2}y' + 2x\,e^{x^2}y = x\,e^{x^2}$$

$$\left[e^{x^2}y\right]' = x\,e^{x^2} \qquad \text{(verify this)}$$

$$(uv)'$$

(4) Integrate:

$$e^{x^2}y = \int x\,e^{x^2}\,dx = \tfrac{1}{2}e^{x^2} + C$$

and

$$y = e^{-x^2}\left[\frac{1}{2}e^{x^2} + C\right] = \tfrac{1}{2} + Ce^{-x^2}.$$

Thus, $y = \tfrac{1}{2} + Ce^{-x^2}$ is the general solution of the equation. ∎

Example 2. Find the solution of the initial-value problem

$$x^2 y' - x\,y = x^4 \cos 2x, \quad y(\pi) = 2\pi.$$

$$y' + p(x)\,y = q(x) \quad \leftarrow \text{divided by } x^2$$

SOLUTION The first step is to find the general solution of the differential equation. After dividing the equation by x^2, we obtain:

$$y' - \frac{1}{x}y = x^2 \cos 2x$$ (*)

$$h(x) = \int P(x)\,dx$$

a linear equation in the standard form (1)with $p(x) = -1/x$ and $q(x) = x^2 \cos 2x$, continuous functions on $(0, \infty)$.

Set $h(x) = \int (-1/x)\,dx = -\ln x = \ln x^{-1}$. Then $e^{h(x)} = e^{\ln x^{-1}} = x^{-1}$.

$$P(x)$$

Multiplying (*) by x^{-1} we get

$$x^{-1}y' - x^{-2}y = x \cos 2x \quad \text{which is the same as} \quad \left[x^{-1}y\right]' = x \cos 2x. \quad \text{(verify this)}$$

$$(uv)'$$

It now follows that

$$x^{-1}y = \int x \cos 2x\,dx + C = \tfrac{1}{2}x \sin 2x + \tfrac{1}{4}\cos 2x + C \quad \text{(integration by parts)}$$

Thus, the general solution of the differential equation is

$$y = \tfrac{1}{2}x^2 \sin 2x + \tfrac{1}{4}x \cos 2x + Cx.$$

We now apply the initial condition:

$$y(\pi) = 2\pi \quad \text{implies} \quad \tfrac{1}{2}\pi^2 \sin 2\pi + \tfrac{1}{4}\pi \cos 2\pi + C\pi = 2\pi$$

$$\tfrac{1}{4}\pi + C\pi = 2\pi$$

$$C = \tfrac{7}{4}$$

The solution of the initial-value problem is $y = \tfrac{1}{2}x^2 \sin 2x + \tfrac{1}{4}x \cos 2x + \tfrac{7}{4}x.$ ∎

A Special Case. There is a special case of equation (1) which will be useful later. If $q(x) = 0$ for all $x \in I$, then (1) becomes

$$e^{h(x)}$$

$$y' + p(x)y = 0.$$

$$h(x) = \int p(x)\,dx$$ (3)

We multiply this equation by $e^{\int p(x)\,dx}$, to obtain

$$e^{h(x)} = e^{\int p(x)\,dx}$$

$$e^{\int p(x)\,dx}y' + p(x)\,e^{\int p(x)\,dx}y = 0$$

which is the same as

$$\left[e^{\int p(x)\,dx}y\right]' = 0.$$

$$(uv)'$$

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It follows from this that

$$e^{\int p(x)\,dx}\, y = C \qquad \text{and} \qquad y = C\, e^{-\int p(x)\,dx}.$$

Let $y = y(x)$ be a solution of (3). Since $e^{-\int p(x)\,dx} \neq 0$ for all x, we can conclude that:

(1) If $y(a) = 0$ for some $a \in I$, then $C = 0$ and $y(x) = 0$ for all $x \in I$ $(y \equiv 0)$.

(2) If $y(a) \neq 0$ for some $a \in I$, then $C \neq 0$ and $y(x) \neq 0$ for all $x \in I$. In fact, since y is continuous, $y(x) > 0$ for all x if $C > 0$; $y(x) < 0$ for all x if $C < 0$.

Final Remarks

1. The general solution (2) of a first order linear differential equation involves two integrals

$$h(x) = \int p(x)\,dx \quad \text{and} \quad \int f(x)e^{h(x)}\,dx.$$

It will not always be possible to carry out the integration steps as we did in the preceding examples. Even simple equations can lead to integrals that cannot be calculated in terms of elementary functions. In such cases you will either have to leave your answer in the integral form (2) or apply some type of numerical method. ∎

2. What does "linear" really mean? Consider the linear equation

$$y' + p(x)y = f(x).$$

We can regard the left-hand side of the equation, $L[y] = y' + p(x)y$, as an "operation" that is performed on the function y. That is, the left-hand side says "take a function y, calculate its derivative, and then add that to $p(x)$ times y." The equation asks us to find a function y such that the operation $L[y] = y' + p(x)y$ produces the function f. ∎

The operation L defined by $L[y] = y' + p(x)y$ where p is a given function, is a linear operation:

$$
\begin{aligned}
L[f(x) + g(x)] = [f(x) + g(x)]' + p(x)[f(x) + g(x)] &= f'(x) + g'(x) + p(x)f(x) + p(x)g(x) \\
&= f'(x) + p(x)f(x) + g'(x) + p(x)g(x) \\
&= L[f(x)] + L[g(x)]
\end{aligned}
$$

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and

$$L[cf(x)] = [cf(x)]' + p(x)[cf(x)] = cf'(x) + cp(x)f(x) = c[f'(x) + p(x)f(x)] = cL[f(x)].$$

Let $C^1(I)$ be the vector space of continuously differentiable functions on the interval I and let $C(I)$ be the vector space of continuous functions on I. Then

$$L: \ C^1(I) \to C(I)$$

is a linear transformation. The fact that the operation $L[y] = y' + p(x)y$ is a linear operation is the reason for calling $y' + p(x)y = f(x)$ a linear differential equation. In this context L is called a *linear differential operator*.

Exercises 2.1

Find the general solution.

1. $y' - 2xy = 2x$.

2. $xy' - 2y = -x$.

3. $xy' + 2y = \dfrac{\cos x}{x}$.

4. $(x + 1)\dfrac{dy}{dx} + 2y = (x + 1)^{5/2}$.

5. $xy' - 2y = x^4 e^x$.

6. $xy' - y = 2x \ln x$.

7. $\dfrac{dy}{dx} + y \tan x = \cos^2 x$.

8. $\dfrac{dy}{dx} + y \cot x = \csc^2 x$.

9. $xy' + y = (1 + x)e^x$.

10. $y' + 2xy = xe^{-x^2}$.

Find the solution of the initial-value problem.

11. $\dfrac{dy}{dx} + \dfrac{2y}{x} = \dfrac{4}{x}, \quad y(1) = 6$.

12. $xy' - 2y = x^3 e^x, \quad y(1) = 0.$

Bernoulli Equations: The differential equation

$$y' + p(x)y = q(x)y^n, \quad n \neq 0, \; n \neq 1, \tag{4}$$

where p and q are continuous functions on some interval I, is called a *Bernoulli equation.* To solve (4), multiply the equation by y^{-n} to obtain

$$y^{-n}y' + p(x)y^{-n+1} = q(x).$$

The substitution $v = y^{-n+1}, \; v' = (1-n)y^{-n}y'$ transforms (4) into

$$\frac{1}{1-n}v' + p(x)v = q(x) \quad \text{or} \quad v' + (1-n)p(x)v = (1-n)q(x)$$

a linear equation in v and x.

13. $y' + \dfrac{1}{x}y = 3x^2y^2.$

14. $y' - 4y = 2e^x \sqrt{y}.$

15. $2xy\,y' = 1 + y^2$

16. $3y' + 3x^{-1}y = 2x^2y^4.$

2.2. Separable Equations

A first order differential equation

$$y' = f(x, y)$$

is a *separable equation* if the function f can be expressed as the product of a function of x and a function of y. That is, the equation is separable if the function f has the form

$$f(x, y) = p(x)\,h(y).$$

where p and h are continuous functions on some interval I.

The solution method for separable equations is based on writing the equation as

$$\frac{1}{h(y)}y' = p(x)$$

or

$$q(y)\,y' = p(x) \tag{1}$$

where $q(y) = 1/h(y)$.

Of course, in dividing the equation by $h(y)$ we have to assume that $h(y) \neq 0$. Any numbers r such that $h(r) = 0$ may result in singular solutions of the form $y = r$.

If we write y' as dy/dx and interpret this symbol as "differential y" divided by "differential x," then a separable equation can be written in differential form as

$$q(y)\, dy = p(x)\, dx.$$

This is the motivation for the term "separable," the variables are separated.

Solution Method for Separable Equations:

Step 1. Identify: Can you write the equation in the form (1).

In expanded form, equation (1) is

$$q(y(x))\, y'(x) = p(x).$$

Step 2. Integrate this equation with respect to x:

$$\int q(y(x))\, y'(x)\, dx = \int p(x)\, dx + C \quad C \text{ an arbitrary constant}$$

which can be written

$$\int q(y)\, dy = \int p(x)\, dx + C$$

by setting $y = y(x)$ and $dy = y'(x)\, dx$. Now, if P is an antiderivative for p, and if Q is an antiderivative for q, then this equation is equivalent to

$$Q(y) = P(x) + C. \tag{2}$$

INTEGRAL CURVES Equation (2) is a one-parameter family of curves called the *integral curves* of equation (1). In general, the integral curves define y implicitly as a function of x. These curves are solutions of (1) since, by implicit differentiation,

$$\frac{d}{dx}[Q(y)] = \frac{d}{dx}[P(x)] + \frac{d}{dx}[C]$$

$$q(y)\, y' = p(x).$$

Example 1. The differential equation

$$y' = -\frac{x}{y}$$

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is separable since $f(x, y) = -(x/y) = (-x)(1/y)$. Writing the equation in the form

(1)
$$y\, y' = x \qquad \text{or} \qquad y\, dy = -x\, dx$$

and integrating

$$\int y\, dy = -\int x\, dx + C,$$

we get

$$\tfrac{1}{2} y^2 = -\tfrac{1}{2} x^2 + C \qquad \text{or} \qquad \tfrac{1}{2} x^2 + \tfrac{1}{2} y^2 = C,$$

which, after multiplying by 2, gives

$$x^2 + y^2 = C.$$

(Since C is an arbitrary constant, $2C$ is arbitrary and so we'll just call it C again. This "treatment" of arbitrary constants is standard in differential equations courses.)

The set of integral curves is the family of circles centered at the origin. Note that for each positive value of C, the resulting equation defines y implicitly as a function of x. ■

Remark we may or may not be able to solve the implicit relation (2) for y. This is in contrast to linear differential equations where the solutions $y = y(x)$ are given explicitly as a function of x. When we can solve (2) for y, we will.

The set of integral curves of a separable equation *may not* represent the set of all solutions of the equation and so it is not technically correct to use the term "general solution" as we did with linear equations. However for our purposes here this is a minor point and so we shall also call (2) the general solution of (1). As noted above, if $h(r) = 0$, then $y = r$ may be a singular solution of the equation; in solving separable equations, we will have to check for singular solutions.

Example 2. Show that the differential equation

$$y' = \frac{xy - y}{y + 1}$$

is separable. Then

1. Find the general solution and any singular solutions.

2. Find a solution which satisfies the initial condition $y(2) = 1$.

SOLUTION Here

$$f(x,y) = \frac{xy - y}{y + 1} = \frac{y(x - 1)}{y + 1} = (x - 1)\frac{y}{y + 1}.$$

Thus, f can be expressed as the product of a function of x and a function of y so the equation is separable.

Writing the equation in the form (1), we have

$$\frac{y + 1}{y} y' = x - 1 \quad (y \neq 0)$$

or

$$\left(1 + \frac{1}{y}\right) y' = x - 1.$$

$$\frac{dy}{dx}$$

Integrating with respect to x, we get

$$\int \left(1 + \frac{1}{y}\right) dy = \int (x - 1)\, dx + C$$

$$y + \ln y = \frac{1}{2}x^2 - x + C$$

and

$$y + \ln |y| = \frac{1}{2}x^2 - x + C$$

is the general solution. Again we have y defined implicitly as a function of x. Note that $y = 0$ is a solution of the differential equation (verify this), but this function is not included in the general solution ($\ln 0$ does not exist). Thus, $y = 0$ is a singular solution of the equation.

To find a solution that satisfies the initial condition, set $x = 2$, $y = 1$ in the general solution:

$$y(2) = 1$$

$$1 + \ln 1 = \frac{1}{2}(2)^2 - 2 + C \quad \text{which implies} \quad C = 1.$$

A particular solution that satisfies the initial condition is: $y + \ln |y| = \frac{1}{2}x^2 - x + 1$.

■

Example 3. Find the general solution and any singular solutions:

$$\frac{dy}{dx} = \frac{xy^2 - x}{y}$$

$$\frac{x(y^2 - 1)}{y} \Rightarrow \frac{dy}{dx}$$

SOLUTION The equation is separable. Separating the variables, we get

$$\frac{y}{y^2 - 1} dy = x\, dx, \quad y \neq \pm 1.$$

$$\frac{y^2 - 1}{y\,dy} = \frac{1}{x\,dx}$$

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Integrating

$$u = y^2 - 1$$
$$\rightarrow du = 2y\, dy$$
$$\rightarrow y\, dy = \frac{du}{2}$$

$$\frac{1}{2} \int \frac{du}{u}$$
$$\frac{1}{2} \ln |u| + C$$
$$\rightarrow \frac{1}{2} \ln |y^2 + 1| + C$$

$$\int \frac{y}{y^2 - 1}\, dy \;=\; \int x\, dx.$$

$$\frac{1}{2} \ln (y^2 - 1) \;=\; \frac{1}{2} x^2 + C$$

$$\ln (y^2 - 1) \;=\; x^2 + C \quad \text{(should be } 2C \text{ but } 2C \text{ is arbitrary so we call it } C.)$$

$$y^2 - 1 \;=\; e^{x^2 + C} = e^{x^2} e^{C} = C e^{x^2} \quad (C \text{ arbitrary}, \; e^{C} \text{ arbitrary, call it } C.)$$

$$y^2 \;=\; C e^{x^2} + 1 \quad \text{(the general solution)}$$

$y = 1, \; y = -1$ are not singular solutions since $C = 0$ implies $y = \pm 1$. ∎

$y^2 = C e^{x^2} + 1$; no singular solutions.

Exercises 2.2

Find the general solution and any singular solutions. If possible, express your general solution in the form $y = f(x)$.

1. $y' = \dfrac{y^2 + 1}{xy + y}$.

2. $(2xy^3 + 4x)y' = x^2 y^2 + y^2$

3. $\dfrac{dy}{dx} = x^2 y - 2x^2$

4. $y' = x e^{x+y}$.

5. $y' = xy^2 - x - y^2 + 1$.

6. $(y \ln x)y' = \dfrac{y^2 + 1}{x}$.

7. $(xy + y)y' = x - xy$.

8. $xy' + y = y^2 \ln x$.

9. $\dfrac{dy}{dx} = \dfrac{1 + y^2}{1 + x^2}$.

10. $\ln x \dfrac{dy}{dx} = \dfrac{y}{x}$.

11. $\dfrac{dy}{dx} = -\dfrac{\sin 1/x}{x^2 y \, \cos y}.$

Find a solution of the initial-value problem.

12. $\dfrac{dy}{dx} = \dfrac{e^{x-y}}{1 + e^x}, \quad y(1) = 0.$

13. $y' = \dfrac{x^2 y - y}{y + 1}, \quad y(3) = 1.$

HOMOGENEOUS EQUATIONS: A first-order differential equation $y' = f(x, y)$ is *homogeneous* if $f(\lambda x, \lambda y) = f(x, y)$ for all $\lambda \neq 0$. The change of variable defined by $y = vx$, $dy/dx = v + x \, dv/dx$ transforms a homogeneous equation into a separable equation:

$$\frac{dy}{dx} = f(x, y) \quad \text{becomes} \quad v + x\frac{dv}{dx} = f(x, vx) = f(1, v)$$

which is a separable equation in x and v.

Show that each of the following differential equations is homogeneous and find the general solution of the equation.

14. $\dfrac{dy}{dx} = \dfrac{x^2 e^{y/x} + y^2}{xy}.$

15. $y' = \dfrac{y + \sqrt{x^2 - y^2}}{x}.$

16. $y' = \dfrac{x^4 + 2y^4}{xy^3}.$

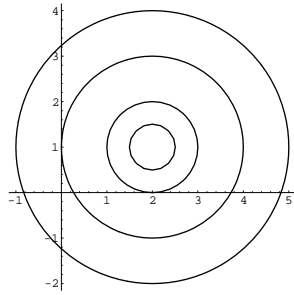
2.3. Some Applications

In this section we give some examples of applications of linear and separable differential equations.

I. Orthogonal Trajectories:

The one-parameter family of curves

$$(x - 2)^2 + (y - 1)^2 = C \qquad (C \geq 0) \tag{a}$$

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is a family of circles with center at the point $(2,1)$ and radius $\sqrt{C}$.

If we differentiate this equation with respect to x, we get
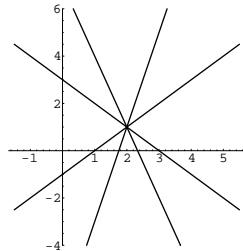
$$2(x - 2) + 2(y - 1)\,y' = 0$$

and

$$y' = -\frac{x - 2}{y - 1} \tag{b}$$

This is the differential equation of the family of circles. Note that if we choose a specific point (x_0, y_0), $y_0 \neq 1$ on one of the circles, then (b) gives the slope of the tangent line at (x_0, y_0).

Now consider the family of straight lines passing through the point $(2,1)$:

$$y - 1 = K(x - 2). \tag{c}$$

The differential equation for this family is

$$y' = \frac{y - 1}{x - 2} \qquad \text{(verify this)} \tag{d}$$

Comparing equations (b) and (d) we see that right side of (b) is the negative reciprocal of the right side of (d). We can conclude from this that if $P(x_0, y_0)$ is a point of intersection of one of the circles and one of the lines, then the line and the circle are perpendicular (orthogonal) to each other at the point P.

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A curve that intersects each member of a given family of curves at right angles (orthogonally) is called an *orthogonal trajectory* of the family. Each line in (c) is an orthogonal trajectory of the family of circles (a) [and conversely, each circle in (a) is an orthogonal trajectory of the family of lines (c)]. In general, if

$$F(x, y, c) = 0 \quad \text{and} \quad G(x, y, K) = 0$$

are one-parameter families of curves such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be *orthogonal trajectories.*

A procedure for finding a family of orthogonal trajectories $G(x, y, K) = 0$ for a given family of curves $F(x, y, C) = 0$ is as follows:

Step 1. Determine the differential equation for the given family $F(x, y, C) = 0$.

Step 2. Replace y' in that equation by $-1/y'$; the resulting equation is the differential equation for the family of orthogonal trajectories.

Step 3. Find the general solution of the new differential equation.

Example Find the orthogonal trajectories of the family of parabolas $y = Cx^2$.

SOLUTION You can verify that the differential equation for the family $y = Cx^2$ can be written as

$$y' = \frac{2y}{x}.$$

Replacing y' by $-1/y'$, we get the equation

$$-\frac{1}{y'} = \frac{2y}{x} \qquad \text{which simplifies to} \qquad y' = -\frac{x}{2y}$$
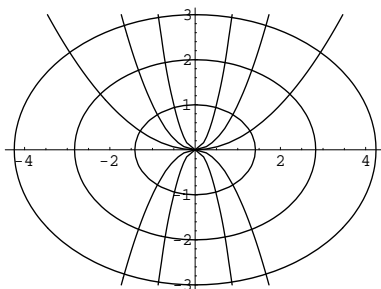
a separable equation. Separating the variables, we get

$$2y\, y' = -x \qquad \text{or} \qquad 2y\, dy = -x\, dx.$$

integrating with respect to x, we have

$$y^2 = -\frac{1}{2}x^2 + C \qquad \text{or} \qquad \frac{x^2}{2} + y^2 = C.$$

This is a family of ellipses with center at the origin and major axis on the x-axis.

∎

II. Exponential Growth and Decay

Radioactive Decay: It has been observed and verified experimentally that the rate of decay of a radioactive material at time t is proportional to the amount of material present at time t. Mathematically this says that if $A = A(t)$ is the amount of radioactive material present at time t, then

$$\frac{dA}{dt} = rA$$

where r, the constant of proportionality, is negative (since the amount of material is decreasing). To emphasize the fact that A is decreasing, this equation is often written

$$\frac{dA}{dt} = -kA, \quad k > 0 \text{ constant.}$$

The constant of proportionality k is called the *decay constant*.

The general solution of the equation is

$$A(t) = Ce^{-kt}$$

If $A_0 = A(0)$ is amount of material present at time $t = 0$, then $C = A_0$ and

$$A(t) = A_0\, e^{-kt}.$$

Note that $\lim_{t\to\infty} A(t) = 0$.

An important property of a radioactive material is the length of time T it takes to decay to one-half the initial amount. This is the so-called *half-life*. To find T we solve the equation

$$\tfrac{1}{2} A_0 = A_0\, e^{-kT}$$

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for T:

$$\frac{1}{2} A_0 = A_0 e^{-kT}$$

$$e^{-kT} = \tfrac{1}{2}$$

$$-kT = \ln(1/2) = -\ln 2$$

$$T = -\frac{\ln 2}{k}$$

Example Cobalt-60 is a radioactive element that is used in medical radiology. It has a half-life of 5.3 years. Suppose that an initial sample of cobalt-60 has a mass of 100 grams.

(a) Find the decay constant and determine an expression for the amount of the sample that will remain t years from now.

(b) How long will it take for 90% of the sample to decay?

SOLUTION (a) Since the half-life $T = (\ln 2)/k$, we have

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5.3} \cong 0.131.$$

With $A(0) = 100$, the amount of material that will remain after t years is

$$A(t) = 100\, e^{-0.131t}.$$

(b) If 90% of the material decays, then 10%, which is 10 grams, remains. Therefore, we solve the equation

$$100\, e^{-0.131t} = 10$$

for t:

$$e^{-0.131t} = 0.1, \qquad -0.131t = \ln(0.1), \qquad t = \frac{\ln(0.1)}{-0.131} \cong 17.6.$$

It will take approximately 17.6 years for 90% of the sample to decay. ∎

Population Growth; Growth of an Investment: It has been observed and verified experimentally that, under ideal conditions, a population (e.g., bacteria, fruit flies, humans, etc.) tends to increase at a rate proportional to the size of the population. Therefore, if $P = P(t)$ is the size of a population at time t, then we have

$$\frac{dP}{dt} = rP, \ r > 0 \text{ (constant)} \tag{a}$$

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In this case, the constant of proportionality r is called the *growth constant.*

Similarly, in a bank that compounds interest continuously, the rate of increase of funds at time t is proportional to the amount of funds in the account at time t. Thus equation (a) also represents the growth of funds under continuous compounding. Since the two cases are identical, we'll focus on the population growth case.

The general solution of equation (a) is

$$P(t) = Ce^{rt}.$$

If $P(0) = P_0$ is the size of the population at time $t = 0$, then

$$P(t) = P_0\, e^{rt}$$

is the size of the population at time t. Note that $\lim_{t \to \infty} P(t) = \infty$. In reality, the rate of increase of a population does not continue to be proportional to the size of the population. After some time has passed, factors such as limitations on space or food supply, introduction of diseases, and so forth affect the growth rate; the mathematical model is not valid indefinitely. In contrast, the model does hold indefinitely in the case of the growth of an investment under continuous compounding.

The analog of the half-life of a radioactive material is the so-called *doubling time,* the length of time T that it takes for a population to double in size. Using the same analysis as above, we have

$$
\begin{aligned}
2\,A_0 &= A_0\, e^{rT} \\
e^{rT} &= 2 \\
rT &= \ln 2 \\
T &= \frac{\ln 2}{r}
\end{aligned}
$$

In the banking, investment, and real estate communities there is a standard measure, called the *rule of 72,* which states that the length of time (approximately) for a principal invested at $r\%$, compounded continuously, to double in value is $72/r\%$. We know that the doubling time is

$$T = \frac{\ln 2}{r} \approx \frac{0.69}{r} = \frac{69}{r\%} \approx \frac{72}{r\%}.$$

This is the origin of the "rule of 72;" 72 is used rather than 69 because it has more divisors. ∎

Example Scientists have observed that a small colony of penguins on a remote Antarctic island obeys the population growth law. There were 2000 penguins initially and 3000 penguins 4 years later.

   (a) How many penguins will there be after 10 years?

   (b) How long will it take for the number of penguins to double?

SOLUTION Let $P(t)$ denote the number of penguins at time t. Since $P(0) = 2000$ we have

$$P(t) = 2000\, e^{rt}.$$

We use the fact that $P(4) = 3000$ to determine the growth constant r:

$$3000 = 2000\, e^{4r}, \qquad e^{4r} = 1.5, \qquad 4r = \ln 1.5,$$

and so

$$r = \frac{\ln 1.5}{4} \cong 0.101.$$

Therefore, the number of penguins in the colony at any time t is

$$P(t) = 2000\, e^{0.101t}.$$

(a) The number of penguins in the colony after 10 years is (approximately)

$$P(10) = 2000\, e^{(0.101)10} = 2000\, e^{1.01} \cong 5491.$$

(b) To find out how long it will take the number of penguins in the colony to double, we need to solve

$$2000\, e^{0.101t} = 4000$$

for t:

$$e^{0.101t} = 2, \qquad 0.101t = \ln 2, \qquad t = \frac{\ln 2}{0.101} \cong 6.86 \text{ years.}$$

Note: There is another way of expressing P that uses the exact value of r. From the equation $3000 = 2000\, e^{4r}$ we get $r = \frac{1}{4} \ln \frac{3}{2}$. Thus

$$P(t) = 2000\, e^{\frac{t}{4} \ln [3/2]} = 2000\, e^{\ln [3/2]t/4} = 2000 \left(\frac{3}{2} \right)^{t/4}. \qquad \blacksquare$$

III. Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature u of an object is proportional to the difference between u and the (constant) temperature σ of the surrounding medium (e.g., air or water), called the *ambient temperature.* The mathematical formulation of this statement is:

$$\frac{du}{dt} = m(u - \sigma), \ m \text{ constant.}$$

The constant of proportionality, m, in this model must be negative; for if the object is warmer than the ambient temperature $(u - \sigma > 0)$, then its temperature will decrease $(du/dt < 0)$, which implies $m < 0$; if the object is cooler than the ambient temperature $(u - \sigma < 0)$, then its temperature will increase $(du/dt > 0)$, which again implies $m < 0$.

To emphasize that the constant of proportionality is negative, we write Newton's Law of Cooling as

$$\frac{du}{dt} = -k(u - \sigma), \quad k > 0 \text{ constant.} \tag{5}$$

This differential equation is both linear and separable so either method can be used to solve it. As you can check, the general solution is
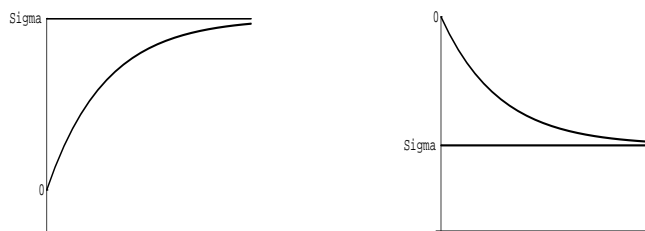
$$u(t) = \sigma + Ce^{-kt}.$$

If the initial temperature of the object is $u(0) = u_0$, then

$$u_0 = \sigma + Ce^0 = \sigma + C \quad \text{and} \quad C = u_0 - \sigma.$$

Thus, the temperature of the object at any time t is given by

$$u(t) = \sigma + [u_0 - \sigma]e^{-kt}. \tag{6}$$

The graphs of $u(t)$ in the cases $u_0 < \sigma$ and $u_0 > \sigma$ are given below. Note that $\lim_{t \to \infty} u(t) = \sigma$ in each case. In the first case, u is increasing and its graph is concave down; in the second case, u is decreasing and its graph is concave up.

Example A metal bar with initial temperature $25^o\,C$ is dropped into a container of boiling water $(100^o\,C)$. After 5 seconds, the temperature of the bar is $35^o\,C$.

(a) What will the temperature of the bar be after 1 minute?

(b) How long will it take for the temperature of the bar to be within $0.5^o\,C$ of the boiling water?

SOLUTION Applying equation (2), the temperature of the bar at any time t is

$$T(t) = 100 + (25 - 100)\,e^{-kt} = 100 - 75\,e^{-kt}.$$

The first step is to determine the constant k. Since $T(5) = 35$, we have

$$35 = 100 - 75\,e^{-5k}, \quad 75\,e^{-5k} = 65, \quad -5k = \ln(65/75), \quad k \cong 0.0286.$$

Therefore,
$$T(t) = 100 - 75\,e^{-0.0286\,t}.$$

(a) The temperature of the bar after 1 minute is, approximately:

$$T(60) = 100 - 75\,e^{-0.0286(60)} \cong 100 - 75\,e^{-1.7172} \cong 86.53^o.$$

(b) We want to calculate how long it will take for the temperature of the bar to reach 99.5^o. Thus, we solve the equation

$$99.5 = 100 - 75\,e^{-0.0286\,t}$$

for t:

$$99.5 \;=\; 100 - 75\,e^{-0.0286\,t}$$

$$-75\,e^{-0.0286\,t} \;=\; -0.5$$

$$-0.0826\,t \;=\; \ln(0.5/75), \quad t \cong 60.66 \text{ seconds.} \qquad \blacksquare$$

Exercises 2.3

In Exercises 1 - 3, find the orthogonal trajectories for the given family of curves and draw several members of each family.

1. $y^2 = Cx^3 - 2.$

2. The family of parabolas with vertical axis and vertex at the point $(1, 2)$.

3. The family of circles that pass through the origin and have their center on the x-axis.

4. A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:

   (a) An expression for the mass of the material remaining at any time t.

   (b) The mass of the material after 4 hours.

   (c) The half-life of the material.

5. The size of a certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of 0.25 square centimeters initially, and after 8 hours it occupied an area of 0.35 square centimeters.

   (a) Estimate the size of the colony t hours after the initial measurement.

   (b) What is the expected size of the colony after 12 hours?

   (c) Find the doubling time of the colony.

6. In 1980 the world population was approximately 4.5 billion and in the year 2000 it was approximately 6 billion. Assume that the world population at each time t increases at a rate proportional to the population at time t. Measure t in years after 1980.

   (a) Find the growth constant and give the world population at any time t.

   (b) How long will it take for the world population to reach 9 billion (double the 1980 population)?

   (c) The world population for 2002 was reported to be about 6.2 billion. What population does the formula in (a) predict for the year 2002?

7. It is estimated that the arable land on earth can support a maximum of 30 billion people. Extrapolate from the data given in Exercise 6 to estimate the year when the food supply becomes insufficient to support the world population.

8. A thermometer initially reading 212^oF is placed in a room where the temperature is 70^oF. After 2 minutes, the thermometer reads 125^oF.

   (a) What does the thermometer read after 4 minutes?

   (b) When will the thermometer read 75^oF?

(c) When will the thermometer read 68ºF?

Some miscellaneous exercises.

9. A 44-gallon barrel, initially full of oil, develops a leak at the bottom. Let $A(t)$ be the amount of oil in the barrel at time t. Suppose that the amount of oil is decreasing at a rate proportional to the product of the time elapsed and the amount of oil present in the barrel.

   (a) Give the mathematical model (initial-value problem) for A.

   (b) Find the general solution of the differential equation in (a).

   (c) Find the particular solution that satisfies the initial condition.

10. A 44-gallon barrel, initially full of oil, develops a leak at the bottom. Let $A(t)$ be the amount of oil in the barrel at time t. Suppose that the amount of oil is decreasing at a rate proportional to the product of the time elapsed and the square of amount of oil present in the barrel.

    (a) Give the mathematical model (initial-value problem) for A.

    (b) Find the general solution of the differential equation in (a).

    (c) Find the particular solution that satisfies the initial condition.

11. A disease is infecting a colony of 1000 penguins living on a remote island. Let $P(t)$ be the number of sick penguins t days after the outbreak. Suppose that 50 penguins had the disease initially, and suppose that the disease is spreading at a rate proportional to the product of the time elapsed and the number of penguins who do not have the disease.

    (a) Give the mathematical model (initial-value problem) for P.

    (b) Find the general solution of the differential equation in (a).

    (c) Find the particular solution that satisfies the initial condition.

2.4. Existence and Uniqueness of Solutions

The questions of existence and uniqueness of solutions of initial-value problems are of fundamental importance in the study of differential equations. We'll illustrate these concepts with some simple examples, and then we'll state an existence and uniqueness theorem for first-order initial-value problems.

Consider the differential equation

$$y' = -\frac{y^2}{x^2}$$

together with the three initial conditions:

(a) $y(0) = 1$,

(b) $y(0) = 0$,

(c) $y(1) = 1$.

Since the differential equation is separable, we can calculate the general solution.

$$
\begin{aligned}
x^2 \, y' + y^2 &= 0 \\
\frac{1}{y^2} \, y' + \frac{1}{x^2} &= 0 \\
\int \frac{1}{y^2} \, dy + \int \frac{1}{x^2} \, dx &= C \\
-\frac{1}{y} - \frac{1}{x} = C \quad &\text{or} \quad \frac{1}{y} + \frac{1}{x} = C.
\end{aligned}
$$

Solving for y we get

$$y = \frac{x}{Cx - 1}.$$

To apply the initial condition (a), we set $x = 0$, $y = 1$ in the general solution. This gives

$$1 = \frac{0}{C \cdot 0 - 1} = 0.$$

We conclude that there is no value of C such that $y(0) = 1$; there is no solution of this initial-value problem $x^2 \, y' + y^2 y = 0$, $y(0) = 1$.

Next we apply the initial condition (b) by setting $x = 0$, $y = 0$ in the general solution. In this case we obtain the equation

$$0 = \frac{0}{C \cdot 0 - 1} = 0$$

which is satisfied by all values of C. The initial-value problem $x^2 \, y' + y^2 y = 0$, $y(0) = 0$ has infinitely many solutions.

Finally, we apply the initial condition (c) by setting $x = 1$, $y = 1$ in the general solution:

$$1 = \frac{1}{C \cdot 1 - 1} \quad \text{which implies} \quad C = 2.$$

This initial-value problem $x^2 y' + y^2 y = 0$, $y(1) = 1$ has a unique solution, namely

$$y = x/(2x - 1).$$

Existence and Uniqueness Theorem Given the initial-value problem

$$y' = f(x, y) \quad y(a) = b. \tag{7}$$

If f and $\partial f/\partial y$ are continuous on a rectangle $R : a - \alpha \leq x \leq a + \alpha$, $b - \beta \leq y \leq b + \beta$, α, $\beta > 0$, then there is an interval $a - h \leq x \leq a + h$, $h \leq \alpha$ on which the initial-value problem (2) has a unique solution $y = y(x)$.

Going back to our example, note that $f(x, y) = -y^2/x^2$ is not continuous on any rectangle that contains $(0, b)$ in its interior. Thus, the existence and uniqueness theorem does not apply in the cases $y(0) = 1$ and $y(0) = 0$.

In the case of the linear differential equation

$$y' + p(x)y = q(x)$$

where p and q are continuous functions on some interval $I = [\alpha, \beta]$, we have

$$f(x, y) = q(x) - p(x)y \quad \text{and} \quad \frac{\partial f}{\partial y} = p(x)$$

and these functions are continuous on every rectangle R of the form $\alpha \leq x \leq \beta$, $-\gamma \leq y \leq \gamma$ where γ is any positive number; that is f and $\partial f/\partial y$ are continuous on the "infinite" rectangle $\alpha \leq x \leq \beta$, $-\infty < y < \infty$. Thus, every linear initial-value problem has a unique solution .