MIDTERM EXAMINATION

Semester 3, 2019-20 • Date: July 23, 2020 • Duration: 90 minutes

SUBJECT: CALCULUS II	
Department of Mathematics Chair:	Lecturer Nguyen Minh Quan, PhD

INSTRUCTIONS: Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar) marked with their name and ID. All other documents and electronic devices are forbidden.

1. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

(a) (*10 points*) $\lim\limits_{n\to\infty} \dfrac{n\sin(n^2)}{n^2+1}$ **(b)** (*5 points*) $\lim\limits_{n\to\infty} \dfrac{n^2}{e^n+n}$.

2. Determine if the following series are convergent or divergent:

(a) (*10 points*) $\sum\limits_{n=0}^{\infty} \dfrac{n+\sqrt{n}}{2n^3-n+1}$

(b) (*10 points*) $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{\sqrt{n}}{3n+2}$

(c) (*5 points*) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n+1}-\sqrt{n}}{n^{3/4}}$

3. **(a)** (*10 points*) Find the radius of convergence and interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n\,(x-1)^n}{n^2\,2^n}$$

(b) (*10 points*) Find the Maclaurin series for $\ln(3+2x)$ and find its radius of convergence.

4. Let $\mathbf{a} = \langle 1,1,2\rangle$, $\mathbf{b} = \langle -1,2,4\rangle$, and $\mathbf{c} = \langle 1,0,4\rangle$.

(a) (*10 points*) Find the volume of the box (parallelepiped) determined by the vectors **a**, **b**, and **c**.

(b) (*10 points*) Find $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ and show that the vector $(\mathbf{b} - \mathrm{proj}_{\mathbf{a}}\mathbf{b})$ is orthogonal to **a**.

5. The position of a moving robot at time t (in seconds) is determined by the vector function

$$\mathbf{r} = 2t\sqrt{t}\,\mathbf{i} + \cos(\pi t)\,\mathbf{j} + \sin(\pi t)\,\mathbf{k}, \quad t \geq 0.$$

(a) (*10 points*) Find the velocity $\mathbf{r}'(t)$ and the unit velocity vector $\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ when $t = 1$ (s).

(b) (*10 points*) Find $\int_0^1 \mathbf{r}(t)dt$ and $\int_0^1 |\mathbf{r}'(t)|dt$.

—END OF QUESTION PAPER—

ANSWERS KEY

1. **(a)** $\lim_{n\to\infty} \frac{n\sin(n^2)}{n^2+1} = 0$. Use Squeeze theorem with $|\frac{n\sin(n^2)}{n^2+1}| \leq \frac{n}{n^2+1} \to 0$ as $n \to \infty$.
(b) $\lim_{n\to\infty} \frac{n^2}{e^n+n} = 0$ by using L'Hospital's rule for $\lim_{x\to\infty} \frac{x^2}{e^x+x} = 0$.

2. **(a)** The series $\sum_{n=0}^{\infty} \frac{n+\sqrt{n}}{2n^3-n+1}$ is convergent by the Limit Comparison Test with $\sum_{n=0}^{\infty} \frac{1}{n^2}$.
(b) The series $\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n}}{3n+2}$ is convergent by applying the Alternating series test.
(c) Note $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}(\sqrt{n+1}+\sqrt{n})}$. The series is convergent by the Limit Comparison Test with $\sum_{n=0}^{\infty} \frac{1}{n^{5/4}}$.

Remark: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. However, conversely, $\lim_{n\to\infty} a_n = 0$ does NOT guarantee the convergence of $\sum_{n=1}^{\infty} a_n$. For example, $\lim_{n\to\infty} \frac{1}{n} = 0$ but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

3. **(a)** Let $a_n = \frac{(-1)^n\,(x-1)^n}{n^2\,2^n}$. Use the Ratio Test to obtain: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{|x-1|}{2} < 1$. The radius of convergence is $R = 2$. Considering $\frac{|x-1|}{2} = 1$, that is, $x = -1$ and $x = 3$, to conclude that the interval of convergence is $[-1, 3]$.
(b) $f'(x) = 2(3+2x)^{-1}$, $f''(x) = -2^2(3+2x)^{-2}$, $f^{(3)}(x) = 2^3\,2!\,(3+2x)^{-3}, \cdots, f^{(n)}(x) = (-1)^{n-1}2^n\,(n-1)!(3+2x)^{-n}$ (for $n \geq 1$).
Therefore, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \ln 3 + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^n}{3^n}\frac{x^n}{n}$.
Ratio Test for convergence gives: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{2|x|}{3} < 1 \Rightarrow R = 3/2$.

4. **(a)** The volume is $V = |a \cdot (b \times c)| = 12$.
(b) $\text{proj}_{\mathbf{a}}\mathbf{b} = \langle \frac{3}{2}, \frac{3}{2}, 3\rangle$. The vector $(\mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}) = \langle -\frac{5}{2}, \frac{1}{2}, 1\rangle$. Thus, $(\mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}) \cdot \mathbf{a} = 0$.

5. **(a)** The velocity $\mathbf{r}'(\mathbf{t}) = 3\sqrt{t}\,\mathbf{i} - \pi\sin(\pi t)\,\mathbf{j} + \pi\cos(\pi t)\,\mathbf{k}$, $\quad t \geq 0$, and the unit velocity vector when $t = 1$ is $\frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{3}{\sqrt{9+\pi^2}}\,\mathbf{i} - \frac{\pi}{\sqrt{9+\pi^2}}, \mathbf{k}$.
(b) $\int_0^1 \mathbf{r}(t)dt = \frac{4}{5}\,\mathbf{i} + \frac{2}{\pi}\,\mathbf{k}$ and $\int_0^1 |\mathbf{r}'(t)|dt = \frac{2}{27}[(9+\pi^2)^{3/2} - \pi^3] = 3.775$.