Chapter 1: Sequence and Series

1. Sequence

Given a sequence (dãy số) a_n, the sequence is said to be convergence if and only if

$$\lim_{n\to\infty} a_n = l \qquad \text{(Eq 1.1)}$$

where l is a finite number, vice versa.

An increasing sequence which is upper bounded or a decreasing sequence which is lower bounded is said to be convergent sequence.

2. Series

2. 1. Definition

Series (chuỗi số) S_n is a summation of a particular sequence a_n, symbolically it can be expressed by

$$S_n = \sum_{n=1}^{+\infty} a_n \qquad \text{(Eq 1.2)}$$

There are two fundamental convergent series which are p-series and geometric series (power series) with a given constraint of convergence as follows:

- p-series

$$\sum_{n=1}^{+\infty} \frac{1}{n^p}\,, \qquad p > 1$$

- Geometric series

$$\sum_{n=0}^{+\infty} ar^n = \frac{a}{1-r}\,, \qquad |r| < 1$$

2. 2. 7-tests for Series

i. Divergence test

$$\lim_{n\to\infty} a_n \neq 0 \qquad \text{(Eq 1.3)}$$

If a limit of a sequence a_n is not zero, or does not exists, then its sequence is divergence.

(Test này cho chúng ta biết rằng nếu tính lim ra khác 0 thì ngay lập tức series của mình divergent. Test này được dùng ĐẦU TIÊN khi thao tác với chuổi)

ii. Integral test

Given a function $f(x)$ be a positive monotonic decreasing function on the interval $[n_0, \infty)$ and $f(n) = a_n$ then

$$\sum_{n=n_0}^{+\infty} a_n \approx \int_{n_0}^{+\infty} f(x)dx \qquad \text{(Eq 1.4)}$$

It holds that

1. If $\int_{n_0}^{+\infty} f(x)dx$ is convergence then $\sum_{n=n_0}^{+\infty} a_n$ also converges.
2. If $\int_{n_0}^{+\infty} f(x)dx$ is divergence then $\sum_{n=n_0}^{+\infty} a_n$ also diverges.

(Nếu có hàm f dương, đơn điệu giảm (positive monotonic decreasing function) thể hiện cho dãy $a_n = f(n)$, khi đó integral test cho ta Eq 1.4. Khi đó nếu tích phân hội tụ thì series tương ứnghội tụ và ngược lại. Test này ÍT DÙNG)

iii. Comparison test

Suppose that we have two sequence a_n, b_n such that $0 < a_n \leq b_n$ then,

1. $\sum_{n=1}^{+\infty} a_n$ is divergent, $\sum_{n=1}^{+\infty} b_n$ aslo diverges.
2. $\sum_{n=1}^{+\infty} b_n$ is convergent, $\sum_{n=1}^{+\infty} a_n$ also converges.

(Test này nói rằng: lớn hội tụ → nhỏ hội tụ / nhỏ phân kì → lớn phân kỳ. Test này cực kì HỮU DỤNG)

iv. Limit comparison test

Suppose that we have two sequence a_n, b_n such that $a_n, b_n > 0$ if the limit of ratio of two sequence such that

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}, \qquad 0 < L < \infty \qquad \text{(Eq 1.5)}$$

then, it leads to

1. $\sum_{n=1}^{+\infty} b_n$ is divergent, $\sum_{n=1}^{+\infty} a_n$ aslo diverges.
2. $\sum_{n=1}^{+\infty} b_n$ is convergent, $\sum_{n=1}^{+\infty} a_n$ also converges.

(Khi L là số dương hữu hạng thì a_n và b_n cùng tình chất. Test này ÍT DÙNG)

v. Alternating series test

Suppose that we have a sequence $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$, where b_n is positive sequence for all n. If b_n is decreasing sequence and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} |a_n| = 0 \qquad \text{(Eq 1.6)}$$

then the series of a_n is convergence.

(Nếu ta có một dãy đan dấu mà trị tuyệt đối của nó giảm và hội tụ về 0 thi chuỗi cũa dãy đó hội tụ. Test này ÍT DÙNG)

vi & vii. Ratio test and Root test

Ratio test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \quad \text{(Eq 1.7)}$$

Root test

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{(Eq 1.8)}$$

Consider the value of L in the 3 following cases:

- $L < 1$: Absolute convergence.
- $L = 1$: No conclusion.
- $L > 1$: Divergence.

(Hai test này cực kì HỮU DỤNG và có dấu hiệu đặc trưng riêng. Có giai thừa thì ratio test, có mũ n thì root test)

2. 3. Absolute Convergence and Conditional Convergence

Given a sequence a_n, it always satisfies the following condition

$$0 \le a_n + |a_n| \le 2|a_n| \quad \text{(Eq 1.9)}$$

Consequence of comparison test give us if sequence of $|a_n|$ is convergence then, sequence of a_n also converges.

(Nếu chuổi của $|a_n|$ hội tụ thì chuỗi của a_n cũng hội tụ)

Conditional convergence	Absolute convergence
$\begin{cases} a_n & \text{converges} \\ \lvert a_n \rvert & \text{diverges} \end{cases}$	$\begin{cases} a_n & \text{converges} \\ \lvert a_n \rvert & \text{converges} \end{cases}$

<u>Consequence:</u> If a sequence is convergence due to Alternating series test, Ratio test and Root test the sequence must be absolute convergence.

(Nếu sử dụng Alternating series test, Ratio test và Root test cho ra chuối hội tụ thì chuổi đó phải là hội tụ tuyệt đối)

3. Power series

Given a power series in the following form

$$\sum_{n=0}^{+\infty} a_n (x-c)^n \quad \text{(Eq 1.10)}$$

Where c is center of expansion (expand of function f about point c)

If we found that $|x-c| < R$ such that the series is convergence then, R is called **radius of convergent**. It leads to $c - R < x < R + c$ is the open interval of convergence, we have to additionally check for convergence at the at two endpoints $x = c \pm R$ to fully find the interval of convergence.

(Nếu tìm được $|x-c| < R$ sao cho chuỗi đã cho hội tụ thì R được gọi là radius of convergent. Sau đó check thêm 2 đầu mút của khoảng để tìm interval of convergence)

(Root test và ratio test thường được sử dụng trong bài toán này)

Chapter 2: Geometry of Space

Note: All bold notations are vector, for example, $\boldsymbol{a} = \vec{a}$

1. Vector Space

1. 1. Vector Calculus

Name	Operator
Dot product	$\boldsymbol{a} \cdot \boldsymbol{b} = \lvert\boldsymbol{a}\rvert \cdot \lvert\boldsymbol{b}\rvert \cdot \cos\theta$
Cross product	$\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert = \lvert\boldsymbol{a}\rvert \cdot \lvert\boldsymbol{b}\rvert \cdot \sin\theta$
Component of vector $\boldsymbol{a}$ along vector $\boldsymbol{b}$	$\text{comp}_{\boldsymbol{b}}\boldsymbol{a} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert\boldsymbol{b}\rvert}$
Projection of $\boldsymbol{b}$ along vector $\boldsymbol{a}$	$\text{proj}_{\boldsymbol{a}}(\boldsymbol{b}) = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert\boldsymbol{a}\rvert^2}\boldsymbol{a}$
Unit vector of vector $\boldsymbol{u}$	$\boldsymbol{a_u} = \dfrac{\boldsymbol{u}}{\lvert\boldsymbol{u}\rvert}$
Del – Gradient vector	$\nabla = \dfrac{\partial}{\partial x}\boldsymbol{i} + \dfrac{\partial}{\partial y}\boldsymbol{j} + \dfrac{\partial}{\partial z}\boldsymbol{k}$

1. 2. Line Equation

Passing through point $A(x_0, y_0, z_0)$ with direction vector $\vec{u} = (a, b, c)$

(Đi qua điểm $A(x_0, y_0, z_0)$ và có vector chỉ phương $\vec{u} = (a, b, c)$)

Parametric equation	Symmetric equation
$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ (Eq 2.1)	$\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ (Eq 2.2)

1. 3. Plane Equation

Passing through point $A(x_0, y_0, z_0)$ with normal vector $\vec{n} = (a, b, c)$

(Đi qua điểm $A(x_0, y_0, z_0)$ và có vector pháp tuyến $\vec{n} = (a, b, c)$)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad \text{(Eq 2.3)}$$

or

$$ax + by + cz + d = 0 \qquad \text{(Eq 2.4)}$$

Distant from a point $M(x_0, y_0, z_0)$ to plane (P): $ax + by + cz + d = 0$

$$d_{[M,P]} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \qquad \text{(Eq 2.5)}$$

(Khoảng cách từ điểm $M(x_0, y_0, z_0)$ tới mặt phẳng (P): $ax + by + cz + d = 0$)

2. Coordinates Conversion

2. 1. Cylindrical Coordinate Systems

Coordinate conversion	Vector conversion
$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \arctan\frac{y}{x} \end{cases} \leftrightarrow \begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$	$\begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$

Differential vector:

- $d\boldsymbol{l} = dr\boldsymbol{r} + rd\phi\boldsymbol{\phi} + dz\boldsymbol{z}$
- $d\boldsymbol{S} = \pm rd\phi dz\boldsymbol{r}; \pm drdz\boldsymbol{\phi}; \pm rdrd\phi\boldsymbol{z}$
- $dv = rdrd\phi dz$

2. 2. Spherical Coordinate Systems

Coordinate conversion	Vector conversion
$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \\ \phi = \arctan\frac{y}{x} \end{cases} \leftrightarrow \begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$	$\begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{\theta} \\ \boldsymbol{\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}$

Differential vector:

- $d\boldsymbol{l} = dr\boldsymbol{r} + rd\theta\boldsymbol{\theta} + r\sin\theta\, d\phi\boldsymbol{\phi}$
- $d\boldsymbol{S} = \pm r^2\sin\theta\, d\theta d\phi\boldsymbol{r}; \pm r\sin\theta\, d\phi dr\boldsymbol{\theta}; \pm rdrd\theta\boldsymbol{\phi}$
- $dv = r^2\sin\theta\, drd\theta d\phi$

3. Vector function

Given a vector function in rectangular form

$$\boldsymbol{r}(t) = < f(t), g(t), h(t) > = f(t).\vec{\imath} + g(t).\vec{\jmath} + h(t).\vec{k} \qquad \text{(Eq 2.6)}$$

Its differential vector is given by

$$\boldsymbol{r}'(t) = < f'(t), g'(t), h'(t) > = f'(t).\vec{\imath} + g'(t).\vec{\jmath} + h'(t).\vec{k} \qquad \text{(Eq 2.7)}$$

Similarly for integration.

Arc length of a function $y = f(x)$

$$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2} dx \qquad \text{(Eq 2.8)}$$

Arc length of a vector function

$$L = \int_a^b \left|\vec{r'}(t)\right| dt \qquad \text{(Eq 2.8)}$$

Chapter 3: Partial Derivative

1. Introduction

Scalar function: $f(x, y, z)$

Vector field: $F = \ < P, Q, R > = P(x, y, z)\,\vec{\imath} + Q(x, y, z)\,\vec{\jmath} + R(x, y, z)\,\vec{k}$

Del operator

$$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} \qquad \text{(Eq 3.1)}$$

Consequence:

$$\begin{cases} \text{Del of a scalar} \rightarrow \text{gradient vector} \\ \text{Del of a vector} \rightarrow \text{Scalar (div of vetor)} \end{cases}$$

2. Chain Rule

Given that: $z = f(x, y)$, where $x = g(t), y = h(t)$, the chain rule gives us

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt} \leftrightarrow z_t = f_x.x'(t) + f_y.y'(t) \qquad \text{(Eq 3.2)}$$

Given that: $z = f(x, y)$, where $x = g(s, t), y = h(s, t)$

$$\begin{cases} z_s = z_x x_s + z_y y_s \\ z_t = z_x x_t + z_y y_t \end{cases} \qquad \text{(Eq 3.3)}$$

3. Implicit Derivative

Given that: $F(x, y) = C \rightarrow y = f(x)$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} \qquad \text{(Eq 3.4)}$$

Given that: $F(x, y, z) = C \rightarrow z = f(x, y)$

$$\begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{F_x(x, y, z)}{F_z(x, y, z)} \\ \dfrac{\partial z}{\partial y} = -\dfrac{F_y(x, y, z)}{F_z(x, y, z)} \end{cases} \qquad \text{(Eq 3.5)}$$

4. Curl and Divergence

Curl

$$\text{Curl}\,\boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \qquad \text{(Eq 3.6)}$$

Divergence

$$\text{Div}\,\boldsymbol{F} = \nabla.\boldsymbol{F} \ = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \text{(Eq 3.7)}$$

Consequence

$$\text{Curl}(\nabla.\boldsymbol{F}) = \nabla \times \nabla.\boldsymbol{F} = 0 \qquad \text{(Eq 3.8)}$$

5. Directional Derivative and Gradient Vector

Gradient vector

$$\nabla . f(x,y) = f_x(x,y)\vec{\imath} + f_y(x,y)\vec{\jmath} \qquad \text{(Eq 3.9)}$$

Given: $z = f(x,y)$, at (x_0, y_0) and **unit** directional vector $\boldsymbol{u} =< a, b >$. The directional derivative is given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \big(\nabla . f(x,y)\big) \cdot \boldsymbol{u} \qquad \text{(Eq 3.10)}$$

Maximum directional derivative

$$\max D_u f(x,y) = |\nabla . f(x,y)| \qquad \text{(Eq 3.11)}$$

6. Tangent Plane and Normal Line

Given a surface: $(S): F(x,y,z) = k$, and a point $P(x_0, y_0, z_0) \in (S)$

Tangent plane at point P of surface (S)

$$F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0 \qquad \text{(Eq 3.12)}$$

Normal line passing through point P of surface (S)

$$\frac{x - x_0}{F_x(P)} = \frac{y - y_0}{F_y(P)} = \frac{z - z_0}{F_z(P)} \qquad \text{(Eq 3.13)}$$

Chapter 4: Maximum and Minimum Problem

1. Local Min-Max Problem

Note: Critical point(s) is the point at which its first derivative equals to zero or undefined.

Consider $D = f_{xx}f_{yy} - f_{xy}^2$ at critical points (x_0, y_0)

- $D > 0, f_{xx} > 0 \to f(x_0, y_0)$ is local minimum.
- $D > 0, f_{xx} < 0 \to f(x_0, y_0)$ is local maximum.
- $D < 0 \to f(x_0, y_0)$ is a saddle point.
- $D = 0$ give nothing.

2. Lagrange Multipliers

Given that: $f(x, y, z)$, with constrains $g(x, y, z) = k$

Solve the system of equations below to find the absolute min-max

$$\begin{cases} g(x,y,z) = k \\ \nabla \cdot f(x,y,z) = m\nabla \cdot g(x,y,z) \end{cases} \qquad \text{(Eq 4.1)}$$

3. Solving a Particular Min-Max Problem

Depending on the requirements of problem, we separate a particular problem into 4 cases as follows:

- Case 1: Find local min-max points: apply (1)
- Case 2: Find absolute min-max with constrains: apply (2)
- Case 3: Find absolute min-max on the region D (simple shape): apply (1) after that substitute the boundary into f and continue to find abs min-max
- Case 4: Find absolute min-max on the region D (complex shape): apply (1) and (2).

Chapter 5: Multiple Integral

1. Line Integral

Given that: $(C): x = x(t), y = y(t), z = z(t)$ or $\boldsymbol{r}(t) = < x(t), y(t), z(t) >$

Scalar integral

$$I = \int_C f(x,y,z)ds = \int_a^b f\big(r(t)\big)|r'(t)|dt$$

$$= \int_a^b f\big(x(t),y(t),z(t)\big)\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}dt$$

Vector field integral

$$I = \int_C \boldsymbol{F}.d\boldsymbol{r} = \int_a^b f\big(\boldsymbol{r}(\boldsymbol{t})\big).\boldsymbol{r}'(\boldsymbol{t})dt$$

Arc length

$$L = \int_a^b |\boldsymbol{r}'(t)|dt$$

2. Surface Integral

Given that: $(S): \boldsymbol{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$

Simple case: $(S):\ z = g(x,y) \rightarrow \boldsymbol{r}(x,y) = x\vec{\imath} + y\vec{\jmath} + g(x,y)\vec{k}$

Scalar integral (general case):

$$I = \iint_S f(x,y,z)dS = \iint_D f\big(r(u,v)\big)|\boldsymbol{r}_u \times \boldsymbol{r}_v|dA$$

Scalar integral (simple case):

$$I = \iint_D f(x,y,z)\sqrt{g_x^2(x,y) + g_y^2(x,y) + 1}dA$$

Vector field integral (general case):

$$I = \iint_S \boldsymbol{F}.d\boldsymbol{S} = \iint_S (\boldsymbol{F}.\boldsymbol{n})dS$$

$$= \iint_S \left(\boldsymbol{F}.\frac{(\boldsymbol{r}_u \times \boldsymbol{r}_v)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}\right)dS = \iint_D (\boldsymbol{F}.(\boldsymbol{r}_u \times \boldsymbol{r}_v))dA$$

Vector field integral (simple case):

$$I = \iint_D \big(\boldsymbol{F}.(\boldsymbol{r}_x \times \boldsymbol{r}_y)\big)dA$$

$$= \iiint_E \big(-Pg_x - Qg_y + R\big)dV \text{ (up oriented)}$$

Surface area for surface $(S): \boldsymbol{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$

$$A = \iint_S d\boldsymbol{S} = \iint_D |\boldsymbol{r}_u \times \boldsymbol{r}_v|dA$$

Surface area for surface $(S):\ z = g(x,y)$

$$A = \iint_S dS = \iint_D \sqrt{g_x^2(x,y) + g_y^2(x,y) + 1}dA$$

3. Frequently Used Theorem

Stoke's theorem

$$\int_C \boldsymbol{F}.d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}).d\boldsymbol{S} \qquad \text{(Eq 5.1)}$$

(C must be a closed path)

Divergence theorem

$$\iint_S \boldsymbol{F}.d\boldsymbol{S} = \iiint_E (\nabla.\boldsymbol{F}).dV \qquad \text{(Eq 5.2)}$$

(S must be a closed surface)

Green's theorem

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad \text{(Eq 5.3)}$$

(C must be a closed path and positive oriented)

Note:

- $dA = dxdy$.
- $dV = dxdydz$.
- Unit vector $\boldsymbol{u} \to |\boldsymbol{u}| = 1$