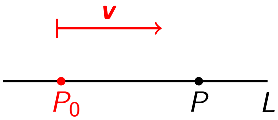
Chapter 1: Functions, Limit and Continuity

Duong T. PHAM

CALCULUS I

Type text here

Outline



$\boldsymbol{v} = (a, b)$ and $P_0 = P_0(x_0, y_0)$. Suppose $P(x, y) \in L$. Then $\overrightarrow{P_0P} // \boldsymbol{v}$. Thus, $\overrightarrow{P_0P} = t\boldsymbol{v}$, where $t \in \mathbb{R}$,

$$\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = t \begin{bmatrix} a \\ b \end{bmatrix}, \quad t \in \mathbb{R};$$

Definition.

Let L be a line which passes through a point $P_0(x_0, y_0)$ and parallel to a vector $\boldsymbol{v} = (a, b)$. Then the **parametric equation** of L is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \end{bmatrix}, \quad t \in \mathbb{R};$$

or

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}, \quad t \in \mathbb{R}.$$

Ex.

1. Find a parametric equation of the line L which passes through the point $P_0(5,1)$ and is parallel to vector $\boldsymbol{v} = (1,4)$,
2. Find two other points on L.

Ans.

1. The parametric equation of L is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad t \in \mathbb{R}.$$

2. Choose $t_1 = 1$, we have $P_1(6,5) \in L$

   Choose $t_2 = 2$, we have $P_2(7,9) \in L$

Parametric equation of L

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases} \iff \begin{cases} x - x_0 = ta \\ y - y_0 = tb \end{cases}$$

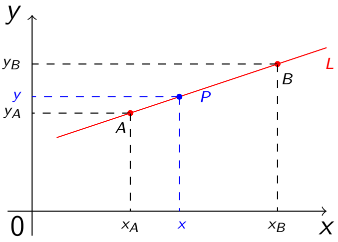
- If none of a, b is 0, we obtain

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b}}$$

  It is called **symmetric equations** of L
- If e.g. $a = 0$, then the equation of L is $x = x_0$.
- If e.g. $b = 0$, then the equation of L is $y = y_0$.
- If $b \neq 0$, the equation of L can be written in the form

$$y = \alpha x + \beta$$

  where α is the slope of L.



$\overrightarrow{AB} = (x_B - x_A, y_B - y_A)$ directional vector of L. The equation of L:

$$\begin{cases} x &= x_A + t(x_B - x_A) \\ y &= y_A + t(y_B - y_A) \end{cases}$$

If $x_A \neq x_B$, the first equation gives

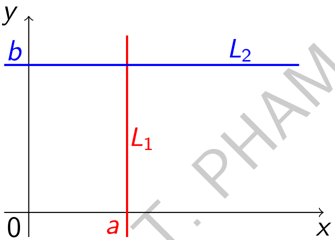
$t = \dfrac{x - x_A}{x_B - x_A}$. Substitute this into the second equation,

$$y = \frac{y_B - y_A}{x_B - x_A}(x - x_A) + y_A$$

Remark: The slope of the line connecting two points $A(x_A, y_A)$ and $B(x_B, y_B)$ is given by

$$\frac{y_B - y_A}{x_B - x_A}.$$

Lines



Equation of L_1 is:

$$x = a$$

Equation of L_2 is:

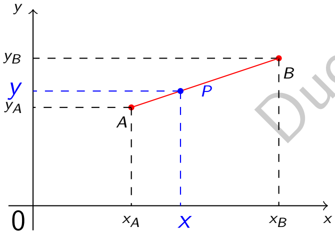
$$y = b$$

Proposition.

Given two points $A(x_A, y_A)$ and $B(x_B, y_B)$. The line segment from A to B is

$$\begin{bmatrix} x \\ y \end{bmatrix} = (1 - t)\begin{bmatrix} x_A \\ y_A \end{bmatrix} + t\begin{bmatrix} x_B \\ y_B \end{bmatrix} \quad 0 \le t \le 1$$

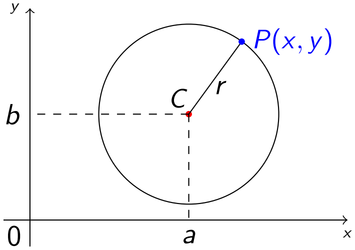
Proof.



$\overrightarrow{AB} = (x_B - x_A, y_B - y_A)$. Let $P(x, y) \in$ line segment AB. Then $\overrightarrow{AP} = t\overrightarrow{AB}$ for $0 \le t \le 1$. Here, $\overrightarrow{AP} = (x - x_A, y - y_A)$. Thus

$$\begin{bmatrix} x - x_A \\ y - y_A \end{bmatrix} = t\begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = (1 - t)\begin{bmatrix} x_A \\ y_A \end{bmatrix} + t\begin{bmatrix} x_B \\ y_B \end{bmatrix}$$



- Let $C(a, b)$. Let $P(x, y)$ belong to the circle centered at $C(a, b)$ and radius r. Then

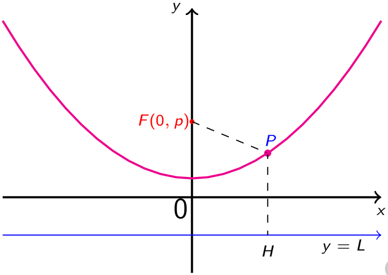
$$CP^2 = r^2$$
$$\iff (x-a)^2 + (y-b)^2 = r^2$$

- The equation of the circle with center at $C(a, b)$ and radius r is

$$(x-a)^2 + (y-b)^2 = r^2$$

- The set of all points lie inside a circle is called the interior of the circle (an open disk).
- The set of all points inside a circle together with the circle itself is said to be a closed disk (or a disk) and is represented by

$$(x-a)^2 + (y-b)^2 \le r^2$$

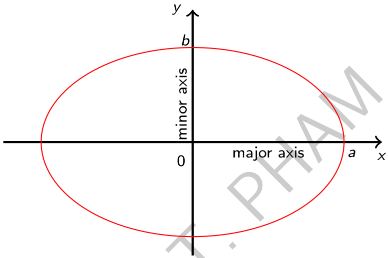


- We consider the set of points that are equidistant from the point $F(0, p)$ and the straight line $y = L$.
- If a point $P(x, y)$ satisfies the above condition then $PF = PH$.

$$x^2 + (y - p)^2 = (y - L)^2$$
$$\iff y = \frac{x^2}{2(p - L)} + \frac{p^2 - L^2}{2(p - L)}$$

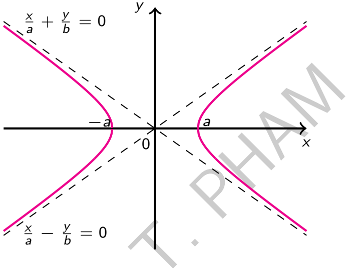
Definition.

- A parabola is a plane curve whose points are equidistant from a point F and a straight line L which does not pass the point F.
- F is the focus of the parabola. L is the directrix of the parabola. The line through F and perpendicular to L is the parabola's axis.



Definition.

- If $a, b > 0$, the equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
represents a curve (called ellipse) that lies inside the rectangle $[-a, a] \times [-b, b]$.
- The line segments connecting $(-a, 0)$ with $(a, 0)$ and $(0, -b)$ with $(0, b)$ are called principal axes of the ellipse.



Definition.

- If $a, b > 0$, the equation

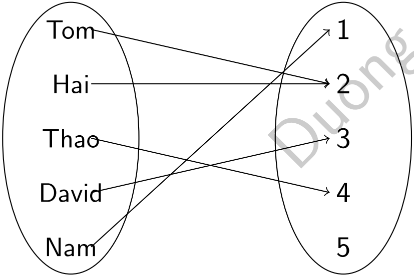
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

  represents a curve (called hyperbola) that has center at the origin and passes through $(a, 0)$ and $(-a, 0)$.
- The two asymptotes have equations $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$

Definition.

Let A and B be sets. A **function** $f : A \to B$ is an assignment of exactly one element of B to each element of A.

Ex: Let $A = \{\text{Tom, Hai, Thao, David, Nam}\}$ and $B = \{1, 2, 3, 4, 5\}$. The grades of students in A are given by

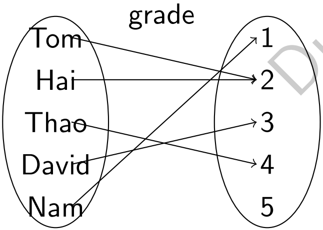


- The assigment grade of the students is a function.
- grade(Tom) $= 2$;
  grade(Hai) $= 2$;
  grade(Thao) $= 4$;
  grade(David) $= 3$;
  grade(Nam) $= 1$.

Definition.

Let $f : A \to B$ be a function. We call f maps A to B and

- The set A is called the **domain** of f. The set B is called the **codomain**
- If $f(a) = b$, then b is called the **image** of a and a is called the **pre–image** of b.
- The set $f(A) = \{f(a) |\ a \in A\}$ is called the **range** of f.



- {Tom, Hai, Thao, David, Nam} = domain
  $\{1, 2, 3, 4, 5\} =$ codomain
- range(grade) = {grade(Tom), grade(Hai), grade(Thao), grade(David), grade(Nam)}
  $= \{2, 2, 4, 3, 1\} = \{1, 2, 3, 4\}$

Domain of a function

Remark: If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number .

Ex: Find the domain of the function $f(x) = \sqrt{x+1}$.

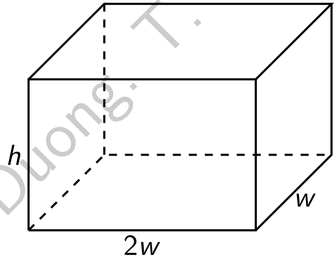
Ans: The formula $\sqrt{x+1}$ is well-defined when $x + 1 \geq 0$, which is equivalent to $x \geq -1$.

The domain of the above function is

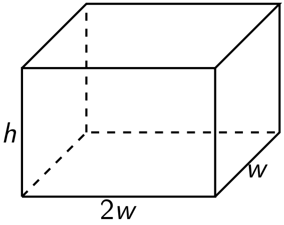
$$D = [-1, +\infty)$$

Example

Ex: We need a rectangular storage container with an open top which has a volume of $10m^3$. The length of its base is required to be twice its width. Material for the base costs $\$10/m^2$; material for the sides costs $\$6/m^2$. Express the cost of materials as a function of the width of the base.



Example



- Area of the base $= w * (2w) = 2w^2$
  $\Rightarrow$ Cost $= 10 * (2w^2) = 20w^2\$$
- Area of the front and back sides
  $= 2 * (2hw) = 4hw$
- Area of the left and right sides
  $= 2 * (hw) = 2hw$

$\Rightarrow$ Area of 4 sides $= 6hw \Rightarrow$ Cost $= 6 * (6hw) = 36hw\$$

$\Rightarrow$ Total cost $= 20w^2 + 36hw(\$)$

- Volume $= 10 \Rightarrow 2hw^2 = 10 \Rightarrow h = 5/w^2$

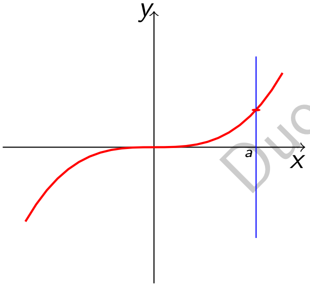
$\Rightarrow$ Total cost: $\boxed{C(w) = 20w^2 + \dfrac{180}{w}}$

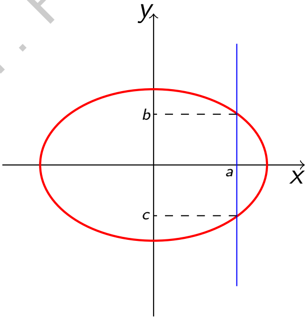
The Vertical Line Test

The graph of a function is a curve in xy-plane.

Question : Which curves in the xy-plane are graphs of functions?

Vertical Line Test: A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.



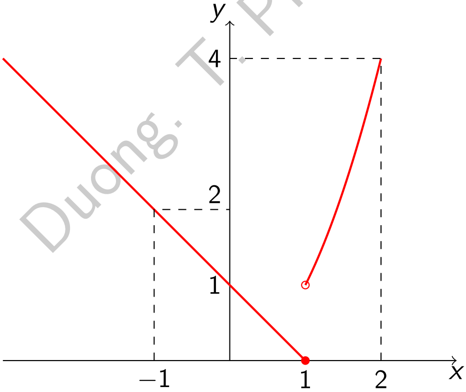


$f(a) = b$ or $f(a) = c$?

Piecewise defined functions

Ex: A function f is defined by: $$f(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1. \end{cases}$$

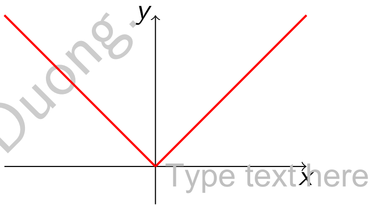
- $f(-1) = \text{?}\ 1 - (-1) = 2$
- $f(2) = ?\ 2^2 = 4$
- $f(1) = ?\ 1 - 1 = 0$



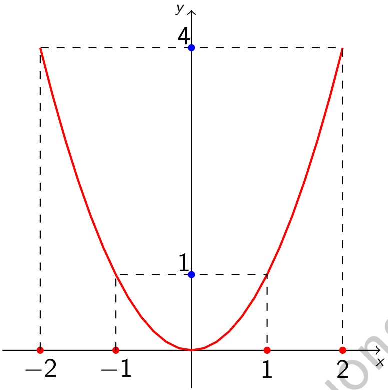
Ex: Sketch the graph of the absolute value function $f(x) = |x|$

Ans: We have

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Symmetry: Even Functions

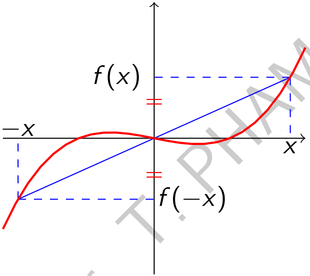


Graph of function $f(x) = x^2$

- $f(-1) =?\ (-1)^2 = 1$;
  $f(1) =?\ 1^2 = 1$;
  $\Rightarrow \quad f(-1) = f(1)$
- $f(-2) = (-2)^2 = 4$ and
  $f(2) = 2^2 = 4$
  $\Rightarrow \quad f(-2) = f(2)$
- $f(-x) = (-x)^2 = x^2$
  $f(x) = x^2 \ \Rightarrow \ \boxed{f(-x) = f(x)}$
- f is an **even function**

Definition.

A function $f : D \to \mathbb{R}$ is said to be **even** if $\boxed{f(-x) = f(x) \quad \forall x \in D}$



- $\boxed{f(-x) = -f(x) \quad \forall x \in D}$ and f is said to be an **odd** function

Definition.

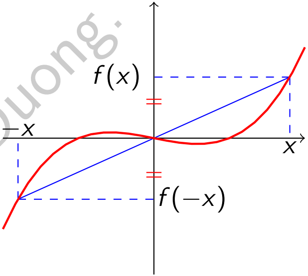
A function $f : D \to \mathbb{R}$ is said to be **odd** if $\boxed{f(-x) = -f(x) \quad \forall x \in D}$

Ex: Determine whether each of the following functions is even, odd, or neither even nor odd

$$f(x) = x^3 - x; \quad g(x) = 1 + x^2; \quad h(x) = x + 1.$$

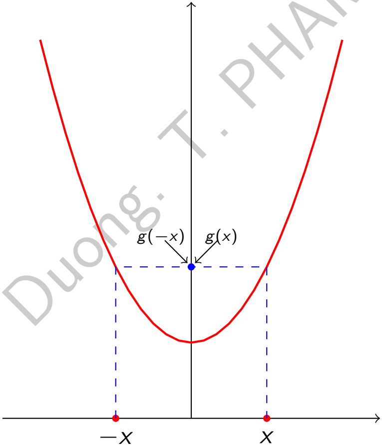
Ans:

(i) $f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$
$\Rightarrow$ f is an odd function.



Ans: $g(x) = 1 + x^2$;

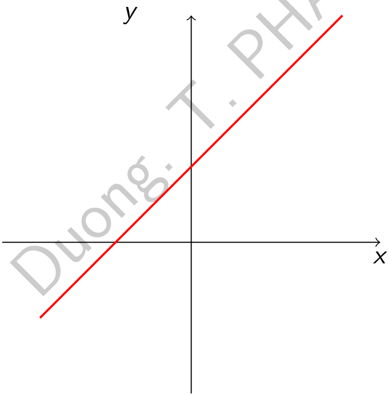
- $g(-x) = 1 + (-x)^2 = 1 + x^2 = g(x)$
  $\Rightarrow$ g is an even function.



Symmetry: Examples

Ans: $h(x) = 1 + x$;

- $h(-x) = 1 + (-x) = 1 - x \Rightarrow h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$
  $\Rightarrow$ h is NOT either even or odd.



Definition.

Let $f, g : A \to \mathbb{R}$. Then $f + g$ and fg are functions from A to $\mathbb{R}$ defined by

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$

Ex: Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and $g(x) = x - x^2$. What are the functions $f + g$ and fg?

Ans: $f + g$ and fg are functions from $\mathbb{R}$ to $\mathbb{R}$ and

$$(f+g)(x) = f(x) + g(x) = x^2 + (x - x^2) = x$$
$$(fg)(x) = f(x)g(x) = x^2(x - x^2) = x^3 - x^4.$$

Definition.

Let $f : A \to B$ be a function and $S \subset A$. The **image** of S, $f(S)$, is a subset of B given by

$$f(S) = \{f(x) |\ x \in S\}$$

Ex: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ and a function $f : A \to B$ given by

$$f(a) = 2, \qquad f(b) = 1, \qquad f(c) = 4,$$
$$f(d) = 1, \qquad f(e) = 1.$$

Find the image $f(S)$ of the subset $S = \{b, c, d\}$.

Ans:

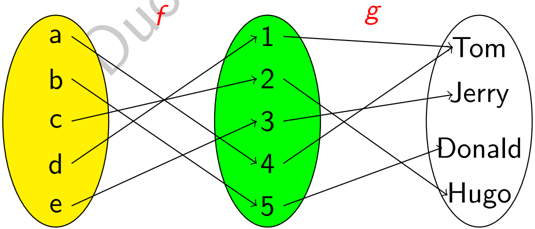
$$\begin{aligned} f(S) = f(\{b, c, d\}) &= \{f(b), f(c), f(d)\} \\ &= \{1, 4, 1\} = \{1, 4\} \end{aligned}$$

Definition.

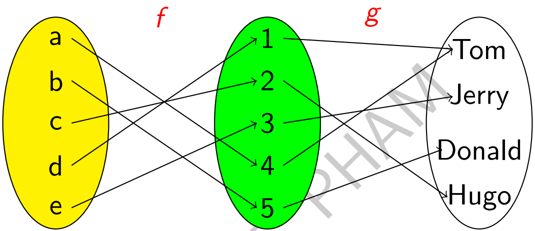
Let $f : A \to B$ and $g : B \to C$. The **composition** $g \circ f : A \to C$ is defined by
$$g \circ f(a) = g\left(f(a)\right), \quad a \in A.$$

Ex: Let $f : \{a, b, c, d, e\} \to \{1, 2, 3, 4, 5\}$ be given by
$$f(a) = 4, \; f(b) = 5, \; f(c) = 2, \; f(d) = 1, \; f(e) = 3,$$
and $g : \{1, 2, 3, 4, 5\} \to \{\text{Tom, Jerry, Donald, Hugo}\}$ be given by
$$g(1) = \text{Tom}, \; g(2) = \text{Hugo}, \; g(3) = \text{Jerry}, \; g(4) = \text{Tom}, \; g(5) = \text{Donald}.$$

Find $g \circ f$?



Composition



The composition $g \circ f : \{a, b, c, d, e\} \to \{\text{Tom, Jerry, Donald, Hugo}\}$ is defined by

$$
\begin{aligned}
g \circ f(a) &= g(f(a)) = g(4) = \text{Tom} \\
g \circ f(b) &= g(f(b)) = g(5) = \text{Donald} \\
g \circ f(c) &= g(f(c)) = g(2) = \text{Hugo} \\
g \circ f(d) &= g(f(d)) = g(1) = \text{Tom} \\
g \circ f(e) &= g(f(e)) = g(3) = \text{Jerry}
\end{aligned}
$$

Ex: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

Find $g \circ f$ and $f \circ g$.

Ans: We have $g \circ f : \mathbb{R} \to \mathbb{R}$ and

$$g \circ f(x) = g\big(f(x)\big) = g\big(2x + 3\big) = 3\big(2x + 3\big) + 2 = 6x + 11.$$

The composition $f \circ g : \mathbb{R} \to \mathbb{R}$ is given by

$$f \circ g(x) = f\big(g(x)\big) = f\big(3x + 2\big) = 2\big(3x + 2\big) + 3 = 6x + 7.$$

One–to–one Functions

Definition.

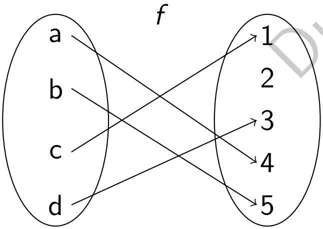
A function $f : A \to B$ is said to be **one-to-one** (or **injective**) if and only if $f(x) = f(y)$ for any $x, y \in A$ implies $x = y$

Ex: Let $f : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ be a function given by

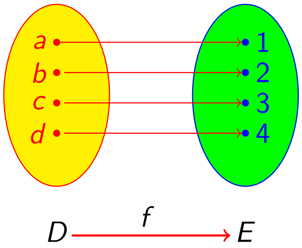
$$f(a) = 4, \ f(b) = 5, \ f(c) = 1, \ f(d) = 3.$$

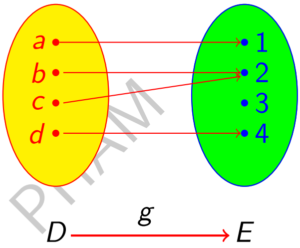
Determine if f is one-to-one.

Ans:



The function f is one-to-one since there are no two different elements in $\{a, b, c, d\}$ having the same image.





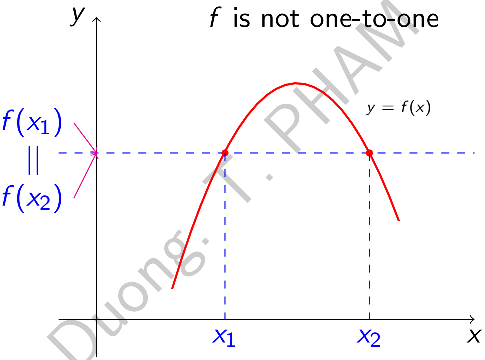
$$g(b) = g(c)$$

- f is one-to-one and g is NOT one-to-one

Remark: A function f is one-to-one if

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$$

Ex:

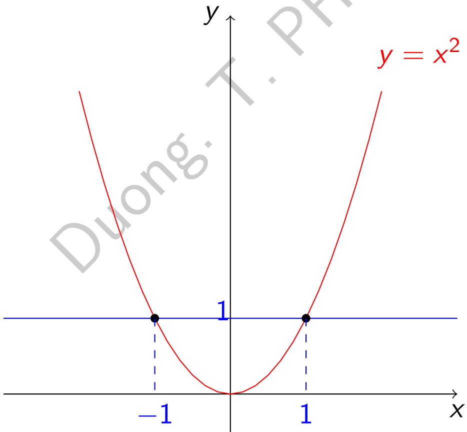


Horizontal line test: A function is one-to-one if and only if NO horizontal line intersects its graph more than once.

One-to-one

Ex: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Determine if f is one-to-one.

Ans: The function f is not one-to-one since we have $1 \neq -1$ but $f(1) = f(-1) = 1^2$.



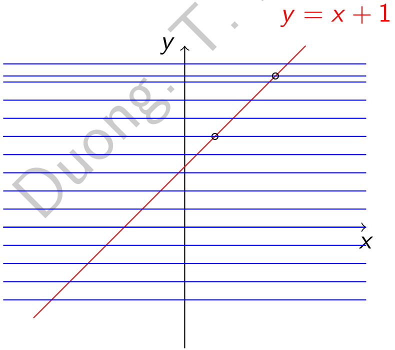
One-to-one Functions

Ex: Determine whether the function $f(x) = x + 1$ is one-to-one or not.

Ans: Suppose that $x_1, x_2 \in \mathbb{R}$ satisfy $f(x_1) = f(x_2)$. Then we have

$$\begin{aligned} f(x_1) = f(x_2) &\Longleftrightarrow x_1 + 1 = x_2 + 1 \\ &\Longrightarrow x_1 = x_2. \end{aligned}$$

Hence, $f(x) = x + 1$ is one-to-one.



Definition.

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Then

- If $\forall x, y \in A$ and $x < y$, there holds $f(x) < f(y)$, then f is said to be **strictly increasing**
- If $\forall x, y \in A$ and $x < y$, there holds $f(x) > f(y)$, then f is said to be **strictly decreasing**

Ex: Consider the function $f(x) = x^3$. Prove that f is strictly increasing.

Ans: Let $x, y \in \mathbb{R}$ and $x < y$. We have

$$f(x) - f(y) = x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$
$$= \underbrace{(x - y)}_{<0} \underbrace{\left((x + \frac{y}{2})^2 + \frac{3y^2}{4}\right)}_{>0} < 0.$$

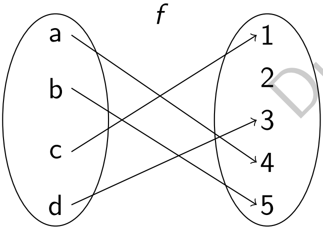
Hence, $f(x) < f(y)$ whenever $x < y$. The function $f(x) = x^3$ is increasing.

Definition.

Let $f : A \to B$ be a function. If for any $b \in B$, there is an $a \in A$ such that $f(a) = b$, then f is said to be **onto** (or **surjective**) . In this case, f is called a **surjection**.

Remark: $f : A \to B$ is surjective iff $\forall(b \in B)\ \exists(a \in A)\ (f(a) = b)$

Ex:



This function is NOT surjective since there is NO element in $\{a, b, c, d\}$ whose image is 2.

Ex: Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 1$ is surjective.

Ans: For any $y \in \mathbb{R}$, there is $x = \sqrt[3]{y-1} \in \mathbb{R}$ satisfying

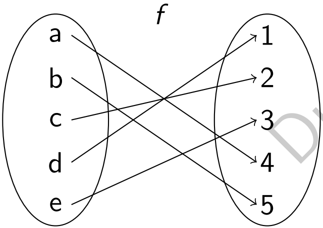
$$\begin{aligned} f(x) &= f\left(\sqrt[3]{y-1}\right) \\ &= \left(\sqrt[3]{y-1}\right)^3 + 1 \\ &= y - 1 + 1 \\ &= y. \end{aligned}$$

Hence, f is surjective.

Definition.

A function $f : A \to B$ is said to be **bijective** if it is injective and surjective.

Ex:



- The function f is injective since for all $x_1, x_2 \in \{a, b, c, d, e\}$ such that $x_1 \neq x_2$, there holds $f(x_1) \neq f(x_2)$.
- The function f is surjective since for all $y \in \{1, 2, 3, 4, 5\}$, there is a $x \in \{a, b, c, d, e\}$ such that $f(x) = y$.
- Hence, f is bijective.

Identity function and Inverse function

Definition.

Let A be a set. The **identity function** defined on A is given by

$$i_A(x) = x \quad \forall x \in A.$$

Definition.

Let $f : A \to B$ be a bijective function. The **inverse function** of f, denoted by $f^{-1} : B \to A$, is defined by

$$f^{-1}(b) := a \quad \forall b \in B \quad \text{if} \quad f(a) = b$$

A function f is said to be **invertible** if it has an inverse

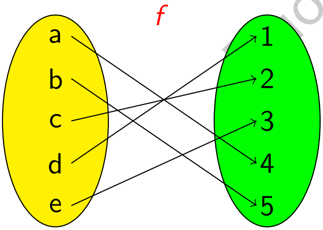
Ex: The bijective function $f : \{a, b, c, d, e\} \to \{1, 2, 3, 4, 5\}$ given by

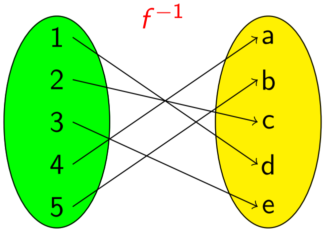
$$f(a) = 4,\ f(b) = 5,\ f(c) = 2,\ f(d) = 1,\ f(e) = 3.$$

Find f^{-1}?

Ans: The inverse $f^{-1} : \{1, 2, 3, 4, 5\} \to \{a, b, c, d, e\}$ is given by

$$f^{-1}(1) = d,\ f^{-1}(2) = c,\ f^{-1}(3) = e,$$
$$f^{-1}(4) = a,\ f^{-1}(5) = b.$$



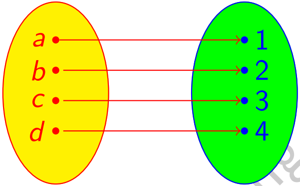


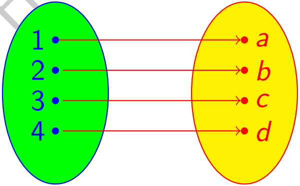
Inverse function

Remark: Let f be a bijective function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y \quad \forall y \in B.$$

Ex:





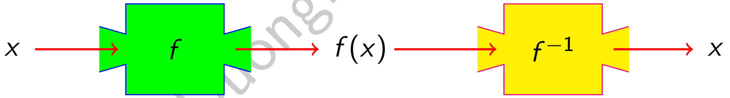
$$D \xrightarrow{f} E \qquad\qquad E \xrightarrow{f^{-1}} D$$

- $f^{-1} \circ f(a) = ?\ f^{-1}(f(a)) = f^{-1}(1) = a$
- $f \circ f^{-1}(3) = ?\ f(f^{-1}(3)) = f(c) = 3.$

Cancellation equations

Let f be a one-to-one function with domain A and range B.

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \forall x \in A$$
$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = y \quad \forall y \in B$$



Find an inverse

To find the inverse of a one-to-one function f:

Step 1: Write $y = f(x)$

Step 2: Solve this equation for x in terms of y

Step 3: To obtain f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Ex: Find the inverse of $y = 3x^3 + 5$

Ans:

- $y = 3x^3 + 5$
- $\implies 3x^3 = y - 5 \implies x = \sqrt[3]{\dfrac{y-5}{3}}$
- $y = \sqrt[3]{\dfrac{x-5}{3}}$ is the inverse of $y = 3x^3 + 5$.

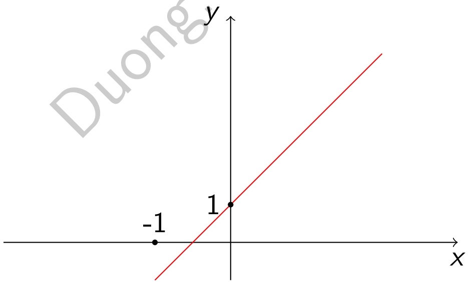
Definition.

Let $f : A \to B$ be a function. The **graph** of f is the set

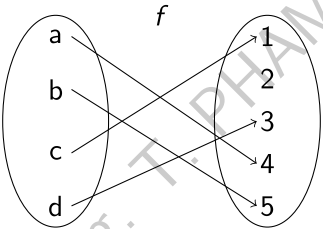
$$\{(a, b) |\ a \in A \text{ and } b = f(a)\}$$

Ex: Find the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$.

Ans: The graph of f is $\{(x, 2x + 1) \mid x \in \mathbb{R}\}$



Ex: Find the graph of the following function



Ans: The graph of f is

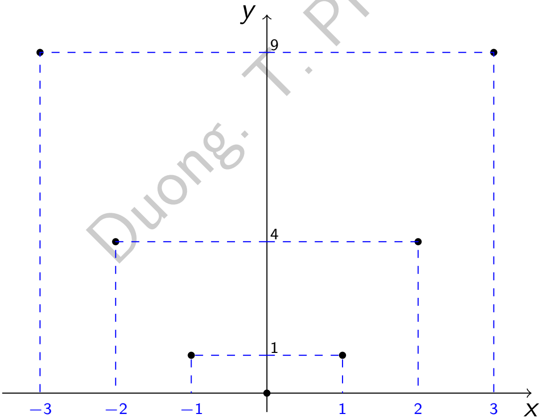
$$\{(x, f(x)) \mid x \in \{a, b, c, d\}\} = \{(a, f(a)), (b, f(b)), (c, f(c)), (d, f(d))\}$$
$$= \{(a, 4), (b, 5), (c, 1), (d, 3)\}$$

Graphs

Ex: Consider $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$. Find the graph of f.

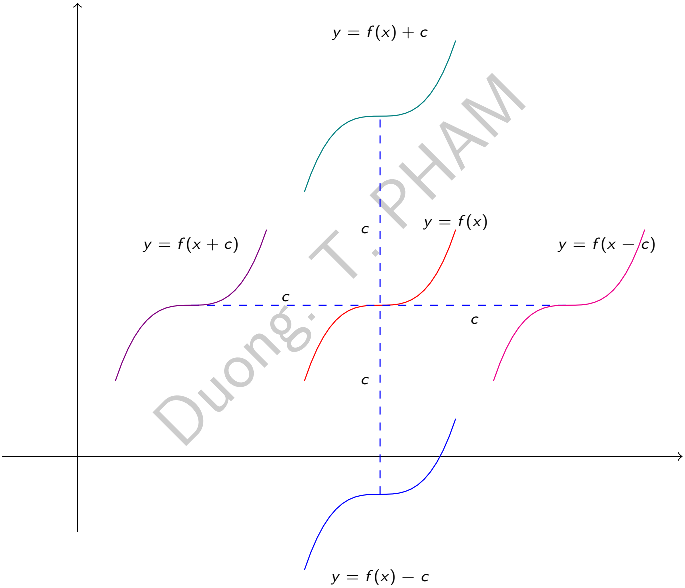
Ans: The graph of f is

$$\{(x, f(x)) \mid x \in \mathbb{Z}\} = \{(x, x^2) \mid x \in \mathbb{Z}\}$$



Vertical and horizontal shifts: Suppose $c > 0$. To obtain the graph of

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right



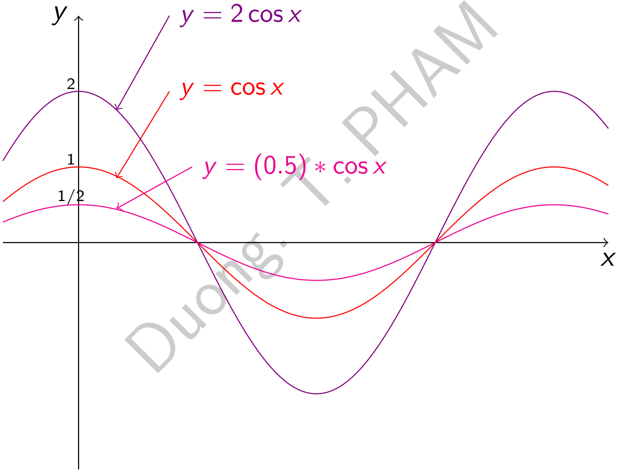
$y = f(x) + c$
c
$y = f(x)$
$y = f(x + c)$
$y = f(x - c)$
c
c
c
$y = f(x) - c$

Vertical and horizontal stretching and reflecting

Vertical and horizontal Stretching and Reflecting: Suppose $c > 1$. To obtain the graph of

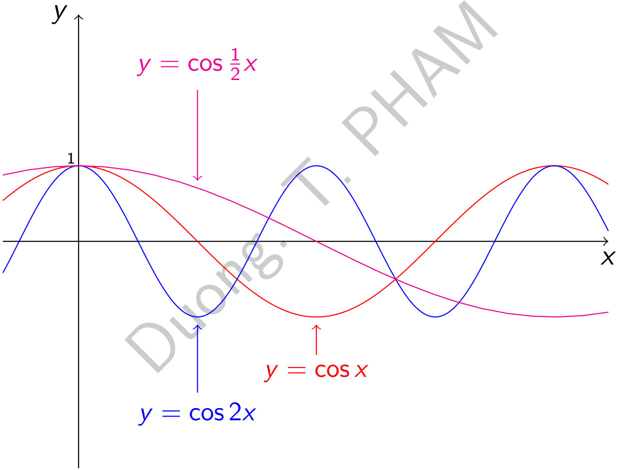
- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
- $y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
- $y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
- $y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Ex:

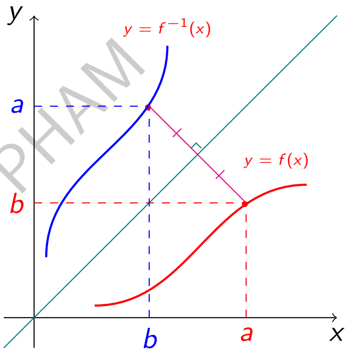


Vertical and horizontal stretching and reflecting

Ex:



- Suppose $f(a) = b \Rightarrow (a, b) \in$ graph of f
- $\Rightarrow f^{-1}(b) = a \Rightarrow (b, a) \in$ graph of f^{-1}
- The graph of f^{-1} is symmetric to that of f about the main diagonal.



The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$

Definition.

- The **floor function** assigns each real number to the largest integer that is smaller than or equal to the real number itself.
- The **ceiling function** assigns each real number to the smallest integer that is larger than or equal to the real number itself.
- The values of the floor and ceiling functions at x are denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$, resp.

Ex:

- $\lfloor 5.2 \rfloor = 5$, $\lceil 5.2 \rceil = 6$,
- $\lfloor -1.2 \rfloor = -2$, $\lceil -1.2 \rceil = -1$.

Floor and ceiling functions

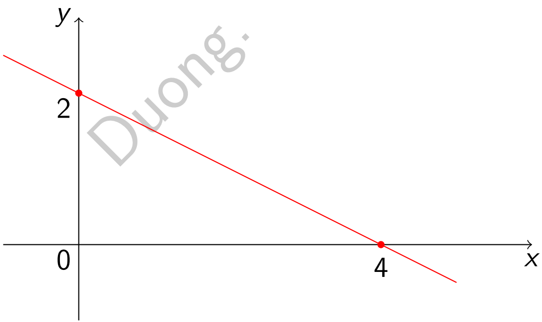
Some properties of floor and ceiling functions (n is integer)
$\lfloor x \rfloor = n$ iff $n \leq x < n+1$ $\lceil x \rceil = n$ iff $n-1 < x \leq n$ $\lfloor x \rfloor = n$ iff $x-1 < n \leq x$ $\lceil x \rceil = n$ iff $x \leq n < x+1$
$x-1 \leq \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
$\lfloor -x \rfloor = -\lceil x \rceil$ $\lceil -x \rceil = -\lfloor x \rfloor$
$\lfloor x+n \rfloor = \lfloor x \rfloor + n$ $\lceil x+n \rceil = \lceil x \rceil + n$

Def: A function $y = f(x)$ is **linear** if its graph is a straight line. The formula of a linear function has the following formula

$$y = ax + b,$$

where a is the slope of the line and b is the y-intercept.

Ex: $y = -\frac{1}{2}x + 2$



Def: A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where

- n: nonnegative integer,
- $a_0, a_1, \ldots, a_n$ are **coefficients**

The domain of P is $\mathbb{R} = (-\infty, \infty)$. If $a_n \neq 0$, the **degree** of P is n.

Ex:

- A linear function $y = ax + b$ is a polynomial of degree 1,
- A polynomial of degree 2 has the form $y = ax^2 + bx + c$ and is called a **quadratic function**,
- A polynomial of degree 3 has the form $y = ax^3 + bx^2 + cx + d$ and is called a **cubic function**.

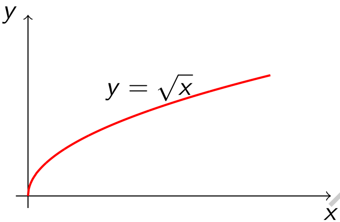
A function of the form $y = x^{\alpha}$ where α is a constant is called a **power function.**

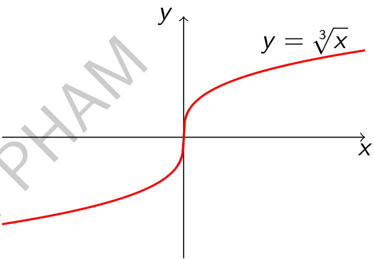
Remark: Note here that α is a real number.

- If $\alpha = n$ where n is a positive integer, then $y = x^n$ is a polynomial of degree n;
  The domain of $y = x^n$ is $\mathbb{R}$.
- If $\alpha = 1/n$ where n is a positive integer, then $y = x^{1/n}$ is called a **root function**.
  Note: $y = \sqrt[n]{x} \Rightarrow y^n = x$ and

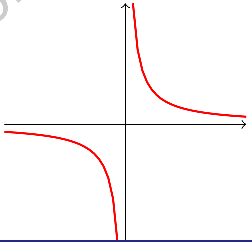
$$\text{domain of } y = \sqrt[n]{x} \text{ is } \begin{cases} \mathbb{R}_+ = [0, \infty) & \text{if } n \text{ is even} \\ \mathbb{R} & \text{if } n \text{ is odd} \end{cases}$$

- $y = \sqrt[n]{x}$





- $y = x^{-1} = \dfrac{1}{x}$;   domain is $\mathbb{R} \backslash \{0\}$



Definition.

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- Domain of f is

$$D = \{x \in \mathbb{R} : Q(x) \neq 0\}$$

Ex: $f(x) = \dfrac{x}{x^2 - 3x + 2}$ is a rational function and the domain is

$$\begin{aligned} D &= \{x \in \mathbb{R} : x^2 - 3x + 2 \neq 0\} \\ &= \{x \in \mathbb{R} : x \neq 1 \text{ and } x \neq 2\} \end{aligned}$$

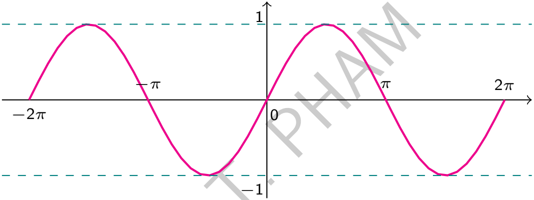
Definition.

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials

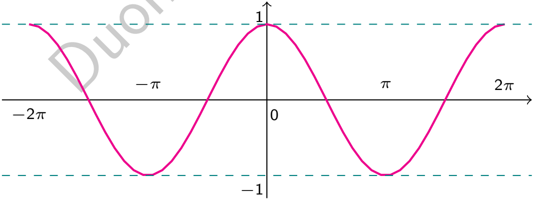
Ex:

- $f(x) = a_n x^n + \ldots + a_1 x + a_0$ and $g(x) = b_m x^m + \ldots + b_1 x + b_0$ are algebraic functions
- $f + g$, $f - g$, $f * g$, f/g and $\sqrt[k]{f}$ are algebraic functions
- $h(x) = \dfrac{1}{\sqrt[k]{x^n + 1}}$ is an algebraic function

- $y = \sin x$



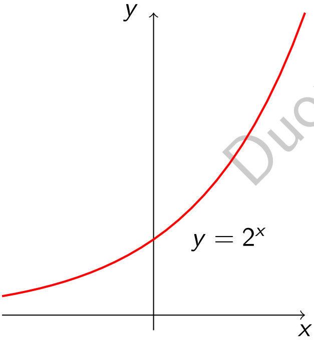
- $y = \cos x$

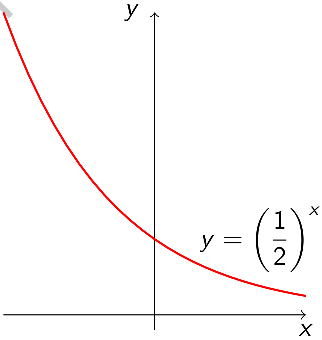


Definition.

Exponential functions are the functions of the form $f(x) = a^x$ where $a > 0$.

- The domain $= \mathbb{R}$
- The range $= \{$positive numbers$\}$

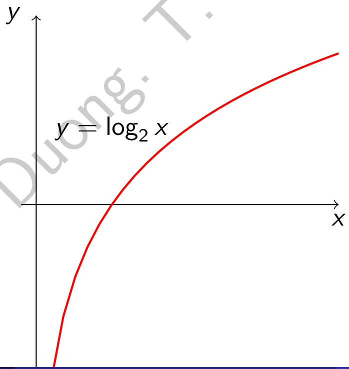




Definition.

The logarithmic functions $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions.

- Domain $= (0, \infty)$
- Range $= (-\infty, \infty)$



Definition.

Transcendental functions are functions that are NOT algbraic functions.

Ex:

- Trigonometric functions and their inverses are transcendental functions,
- Exponential and logarithmic functions are transcendental functions.

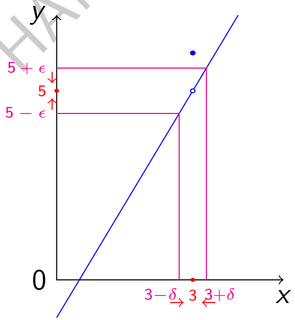
Ex: Classify the following functions as one of the types of functions that we have discussed.

1. $3^x \longrightarrow$ exponential function,
2. $x^5 \longrightarrow$ power function (or polynomial of degree 5)
3. $\frac{1+x}{1-\sqrt{x}} \longrightarrow$ algebraic function,
4. $1 - x - x^4 \longrightarrow$ polynomial of degree 4.

Limit of a function

- Consider $f(x) = \begin{cases} 2x-1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$

x	$f(x)$	$\lvert 5-f(x)\rvert$
2.99	4.98	0.02
2.999	4.998	0.002
2.9999	4.9998	0.0002
2.99999	4.99998	0.00002
2.999999	4.999998	0.000002
2.9999999	4.9999998	0.0000002



- We say: f converges to 5 as x goes to 3, and we write

$$\boxed{\lim_{x\to 3} f(x) = 5}$$

Definition.

Let f be a function defined on some open intervals that contains a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon.$$

Ex: Prove that $\lim_{x\to 3}(4x-5)=7$

Ans:

- *Guessing δ:* given an arbitrary $\epsilon>0$, we need to find $\delta>0$ s.t.

$$\text{if } 0<|x-3|<\delta \text{ then } |(4x-5)-7|<\epsilon.$$

We have

$$|(4x-5)-7|<\epsilon \Leftrightarrow 4|x-3|<\epsilon.$$

We choose $\delta=\epsilon/4$.

- *Proof:* For any $\epsilon>0$, there is $\delta=\epsilon/4$ satisfying that if $0<|x-3|<\delta$, then

$$0<|x-3|<\epsilon/4 \quad \Leftrightarrow \quad |(4x-5)-7|<\epsilon.$$

This conclude that $\lim_{x\to 3}(4x-5)=7$

Ex: Prove that $\lim_{x\to 1}(x^2-1)=0$

Ans:

- *Guessing δ:* given an arbitrary $\epsilon>0$, we need to find $\delta>0$ s.t.

$$\text{if } 0<|x-1|<\delta \text{ then } \left|(x^2-1)-0\right|<\epsilon.$$

  We have $\left|(x^2-1)-0\right| = |x-1|\,|x+1| < \epsilon$ (we want) .
  We may choose $\delta=\min\{1/4,\epsilon/3\}$.

- *Proof:* For any $\epsilon>0$, there is $\delta=\min\{1/4,\epsilon/3\}$ satisfying that if $0<|x-1|<\delta$, then $|x-1|<\delta\le 1/4 \Rightarrow 3/4<x<5/4 \Rightarrow 7/4<x+1<9/4 \Rightarrow 7/4<|x+1|<9/4$.

  Hence, if $0<|x-1|<\delta=\min\{1/4,\epsilon/3\}$, we have

$$\left|(x^2-1)-0\right| = |x-1|\,|x+1| < \delta\,|x+1| < \frac{\epsilon}{3}\frac{9}{4} < \epsilon.$$

  This conclude that $\lim_{x\to 1}(x^2-1)=0$

Limit Laws: Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and let $c \in \mathbb{R}$. Then

(i) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

(iv) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(v) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Proof:

(i) Denote $L := \lim_{x\to a} f(x)$ and $M := \lim_{x\to a} g(x)$.

Let $\epsilon > 0$ be an arbitrarily small positive number .

Since $L := \lim_{x\to a} f(x)$, there is a $\delta_1 > 0$ s.t.

$$\text{if } 0 < |x-a| < \delta_1 \text{ then } |f(x) - L| < \epsilon/2. \tag{1}$$

Since $M := \lim_{x\to a} g(x)$, there is a $\delta_2 > 0$ s.t.

$$\text{if } 0 < |x-a| < \delta_2 \text{ then } |g(x) - M| < \epsilon/2. \tag{2}$$

We choose $\delta = \min\{\delta_1, \delta_2\}$. From (1) and (2), there holds:
if $0 < |x-a| < \delta$ then

$$\begin{aligned}|(f(x)+g(x)) - (L+M)| &\le |f(x)-L| + |g(x)-M| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Hence, $\lim_{x\to a}[f(x)+g(x)] = L + M$.

Proof:

(ii) Let $\epsilon > 0$ be an arbitrarily small positive number .

Since $L := \lim_{x\to a} f(x)$, there is a $\delta_1 > 0$ s.t.

$$\text{if } 0 < |x-a| < \delta_1 \text{ then } |f(x) - L| < \epsilon/2. \tag{3}$$

Since $M := \lim_{x\to a} g(x)$, there is a $\delta_2 > 0$ s.t.

$$\text{if } 0 < |x-a| < \delta_2 \text{ then } |g(x) - M| < \epsilon/2. \tag{4}$$

We choose $\delta = \min\{\delta_1, \delta_2\}$. From (3) and (4), there holds:
if $0 < |x-a| < \delta$ then

$$\begin{aligned} |(f(x)-g(x)) - (L-M)| &\le |(f(x)-L) - (g(x)-M)| \\ &\le |f(x)-L| + |g(x)-M| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence, $\lim_{x\to a}[f(x) - g(x)] = L - M$.

Proof:

(iii) Recall $L := \lim_{x\to a} f(x)$. Prove that $\lim_{x\to a} cf(x) = cL$.

If $c = 0$ then it is not hard to prove (iii).

Consider the case $c \neq 0$. Let $\epsilon > 0$ be an arbitrarily small positive number . Since $L := \lim_{x\to a} f(x)$, there is a $\delta_1 > 0$ s.t.

$$\text{if } 0 < |x-a| < \delta_1 \text{ then } |f(x) - L| < \frac{\epsilon}{|c|}. \tag{5}$$

We choose $\delta = \delta_1 > 0$. From (5), there holds: if $0 < |x-a| < \delta$ then

$$|cf(x) - cL| = |c|\,|f(x) - L| \qquad < |c|\,\frac{\epsilon}{|c|} = \epsilon.$$

Hence, $\lim_{x\to a}[cf(x)] = cL$.

Proof:

(iv) Recall: $L := \lim_{x\to a} f(x)$ and $M := \lim_{x\to a} g(x)$.

Let $\epsilon > 0$ be an arbitrarily small positive number .

Then there are $\delta_1 > 0$ and $\delta_2 > 0$ s.t.

$$\text{if } 0 < |x-a| < \delta_1 \text{ then } |f(x) - L| < \epsilon. \tag{6}$$

$$\text{if } 0 < |x-a| < \delta_2 \text{ then } |g(x) - M| < \epsilon. \tag{7}$$

We choose $\delta = \min\{\delta_1, \delta_2\}$. Suppose that

$$0 < |x-a| < \delta = \min\{\delta_1, \delta_2\}$$

so that (6) and (7) hold. Then

$$\begin{aligned}
|(f(x)g(x)) - (LM)| &\le |f(x)(g(x)-M) + M(f(x)-L)| \\
&\le |f(x)|\,|g(x)-M| + |M|\,|f(x)-L| \\
&\le (|f(x)| + |M|)\epsilon \le (|f(x)-L| + |L| + |M|)\epsilon \\
&\le (\epsilon + |L| + |M|)\epsilon \le (1 + |L| + |M|)\epsilon
\end{aligned}$$

This proves (iv).

Proof:

(v) Recall: $L := \lim_{x \to a} f(x)$ and $M := \lim_{x \to a} g(x)$, and $M \neq 0$.

Let $\epsilon > 0$ be an arbitrarily small positive number .

Then there are $\delta_1 > 0$ and $\delta_2 > 0$ s.t.

$$\text{if } 0 < |x - a| < \delta_1 \text{ then } |f(x) - L| < \epsilon. \tag{8}$$

$$\text{if } 0 < |x - a| < \delta_2 \text{ then } |g(x) - M| < \epsilon \text{ so that } |g(x)| > \frac{M}{2}. \tag{9}$$

Moreover, $|M|/2 > 0$, there is $\delta_3 > 0$ s.t.

$$\text{if } 0 < |x - a| < \delta_3 \text{ then } |g(x) - M| < \frac{|M|}{2}. \tag{10}$$

We choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Suppose that

$$0 < |x - a| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$$

so that (8), (9) and (10) hold.

Then

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &= \frac{|Mf(x) - Lg(x)|}{|g(x)M|} \\
&\leq \frac{2}{|M|^2}|Mf(x) - Lg(x)| \\
&\leq \frac{2}{|M|^2}|M(f(x) - L) + L(M - g(x))| \\
&\leq \frac{2}{|M|^2}(|M||f(x) - L| + |L||M - g(x)|) \\
&\leq \frac{2}{|M|^2}(|M| + |L|)\epsilon
\end{aligned}$$

This proves (v).

Some corollaries

In the rest of this course, except when being asked to use the definition of limit to prove, we can use the six Limit Laws and the following simple limits without proving:

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$
4. $\lim_{x \to a} x^n = a^n$
5. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$
6. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ if the second limit exists and if n is even we assume further that $\lim_{x \to a} f(x) \geq 0$

Proof:

Ex: Evaluate $\lim\limits_{x\to 2}(3x^3 - 2x^2 + 10)$ and $\lim\limits_{x\to 2}\dfrac{x+1}{x^2-2x+3}$

Ans:

$$\begin{aligned}\lim_{x\to 2}(3x^3 - 2x^2 + 10) &= \lim_{x\to 2}(3x^3) - \lim_{x\to 2}(2x^2) + \lim_{x\to 2} 10\\ &= 3\lim_{x\to 2} x^3 - 2\lim_{x\to 2} x^2 + 10\\ &= 3\cdot 2^3 - 2\cdot 2^2 + 10 = 26\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 2}\frac{x+1}{x^2-2x+3} &= \frac{\lim\limits_{x\to 2}(x+1)}{\lim\limits_{x\to 2}(x^2-2x+3)}\\ &= \frac{\lim\limits_{x\to 2} x + \lim\limits_{x\to 2} 1}{\lim\limits_{x\to 2} x^2 - 2\lim\limits_{x\to 2} x + \lim\limits_{x\to 2} 3}\\ &= \frac{2+1}{2^2 - 2\cdot 2 + 3} = 1\end{aligned}$$

Proposition.

If $f(x) = g(x)$ for all $x \neq a$, then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$

Ex: Find $\lim\limits_{x \to 2} f(x)$ where

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 2 \\ 10 & \text{if } x = 2 \end{cases}$$

Ans: Since $f(x) = x^2 + 1$ for all $x \neq 2$, we have

$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} (x^2 + 1) \\ &= 5 \end{aligned}$$

Ex: Evaluate $\lim\limits_{x\to 0} \dfrac{(x-2)^2-4}{x}$

Ans: We have

$$\frac{(x-2)^2-4}{x} = \frac{(x-4)x}{x} = x-4 \quad \forall x \neq 0.$$

Hence,

$$\lim_{x\to 0} \frac{(x-2)^2-4}{x} = \lim_{x\to 0}(x-4) = -4.$$

Left-hand and right-hand limits

Definition.

We say that

$$\lim_{x \to a^-} f(x) = L$$

if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \epsilon$$

Definition.

We say that

$$\lim_{x \to a^+} f(x) = L$$

if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Ex: Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$

Ans:

- *Guessing δ:* Given $\epsilon > 0$, we need to find a $\delta > 0$ satisfying

$$\text{if} \quad 0 < x < \delta \quad \text{then} \quad \left|\sqrt{x} - 0\right| = \sqrt{x} < \epsilon.$$

  We may choose $\delta = \epsilon^2$?

- For every $\epsilon > 0$, there is $\delta = \epsilon^2 > 0$ such that if $0 < x < \delta = \epsilon^2$ then

$$\left|\sqrt{x} - 0\right| = \sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon.$$

  Hence, $\lim_{x \to 0^+} \sqrt{x} = 0$

Theorem.

$$\lim_{x\to a} f(x) = L \quad \Longleftrightarrow \quad \begin{cases} \lim\limits_{x\to a^-} f(x) = L \\ \lim\limits_{x\to a^+} f(x) = L \end{cases}$$

Ex: Prove that $\lim\limits_{x\to 0} |x| = 0$

Ans: We have $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

Thus,

$$\begin{cases} \lim\limits_{x\to 0^+} |x| = \lim\limits_{x\to 0^+} x = 0 \\ \lim\limits_{x\to 0^-} |x| = \lim\limits_{x\to 0^-} (-x) = 0 \end{cases} \quad \Longleftrightarrow \quad \lim_{x\to 0} |x| = 0$$

Ex: Prove that $\lim\limits_{x\to 0}\dfrac{|x|}{x}$ does not exist

Ans: We have $\dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1 & \text{if } x > 0 \\ \dfrac{-x}{x} = -1 & \text{if } x < 0 \end{cases}$

Thus,

$$\left.\begin{aligned} \lim_{x\to 0^+}\frac{|x|}{x} &= \lim_{x\to 0^+} 1 = 1 \\ \lim_{x\to 0^-}\frac{|x|}{x} &= \lim_{x\to 0^-}(-1) = -1 \end{aligned}\right\} \Leftrightarrow \lim_{x\to 0^-}\frac{|x|}{x} \neq \lim_{x\to 0^+}\frac{|x|}{x}$$

$\Longrightarrow \quad \lim\limits_{x\to 0}\dfrac{|x|}{x}$ does not exist.

Theorem.

Let $a \in (b, c)$. There holds

$$\left.\begin{array}{l} f(x) \le g(x) \quad \forall x \in (b,c)\backslash\{a\} \\ \lim\limits_{x\to a} f(x) \text{ and } \lim\limits_{x\to a} g(x) \text{ exist} \end{array}\right\} \Longrightarrow \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Proof: Denote $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Suppose further that $M < L$ $\Longrightarrow \epsilon_0 := (L-M)/3 > 0$.

- $\lim\limits_{x\to a} f(x) = L \Rightarrow \exists \delta_1 > 0$ s.t. $\boxed{\text{if } 0 < |x-a| < \delta_1 \text{ then } |f(x) - L| < \epsilon_0}$
- $\lim\limits_{x\to a} g(x) = M \Rightarrow \exists \delta_2 > 0$ s.t. $\boxed{\text{if } 0 < |x-a| < \delta_2 \text{ then } |g(x) - M| < \epsilon_0}$
- Choose $\delta_0 = \min\{\delta_1, \delta_2\}$. Then

$$\text{if } 0 < |x-a| < \delta_0 \text{ then } \begin{cases} |f(x) - L| < \epsilon_0 \\ |g(x) - M| < \epsilon_0 \end{cases} \Longrightarrow f(x) > g(x)$$

Theorem.

Let $a \in (b, c)$. There holds

$$\left.\begin{array}{l} f(x) \le g(x) \le h(x) \quad \forall x \in (b,c)\backslash\{a\} \\ \lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L \end{array}\right\} \implies \lim_{x\to a} g(x) = L$$

Ex: Evaluate $\lim\limits_{x\to 0} x^2 \cos\dfrac{1}{x}$

Ans: We have $-1 \le \cos\frac{1}{x} \le 1 \implies -x^2 \le x^2\cos\frac{1}{x} \le x^2$.

Moreover, $\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0$

by Squeeze Theorem, $\lim\limits_{x\to 0} x^2\cos\dfrac{1}{x} = 0$.

Definition.

Let $a \in (b, c)$. Then $\lim_{x \to a} f(x) = \infty$ means that for arbitrary positive number M, there exists $\delta > 0$ satisfying

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

Ex: Prove that $\lim_{x \to 0} \dfrac{1}{|x|} = \infty$.

Ans: Let M be an arbitrary positive number . We need to find $\delta > 0$ s.t.

$$\text{if } 0 < |x| < \delta \quad \text{then} \quad \frac{1}{|x|} > M$$

But $\frac{1}{|x|} > M \iff |x| < 1/M$. Therefore, we choose $\delta = \frac{1}{M}$, then clearly we have

$$\text{if } 0 < |x| < \delta = \frac{1}{M} \quad \text{then} \quad \frac{1}{|x|} > M,$$

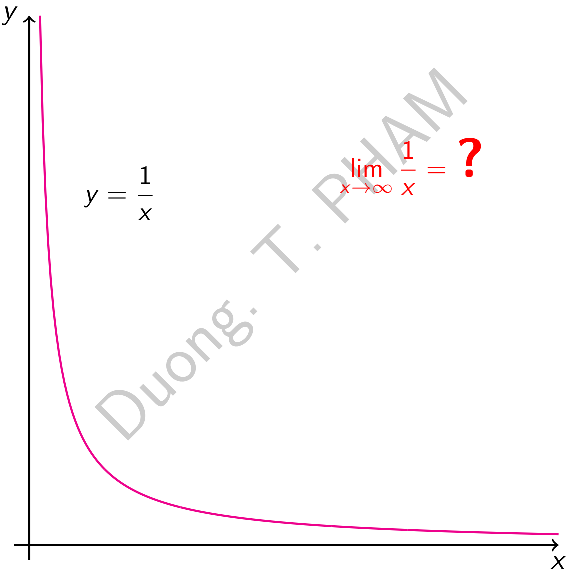
finishing the proof.

Definition.

Let $a \in (b, c)$. Then $\lim_{x \to a} f(x) = -\infty$ means that for arbitrary negative number N, there exists $\delta > 0$ satisfying

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

Limits at infinity



Definition.

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

if for all $\epsilon > 0$, there exists a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \epsilon$$

Ex: Prove that $\lim_{x\to\infty} \dfrac{1}{x} = 0$

Ans:

- Let $\epsilon > 0$ be an arbitrary positive number.
- We choose $N = \frac{1}{\epsilon} > 0$. Then

$$\text{if } x > N = \frac{1}{x} \text{ then } \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N} = \epsilon.$$

- Hence, $\lim_{x\to\infty} 1/x = 0$.

Definition.

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

if for all $\epsilon > 0$, there exists a number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \epsilon$$

Definition.

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

if for all positive number M, there exists a number N such that

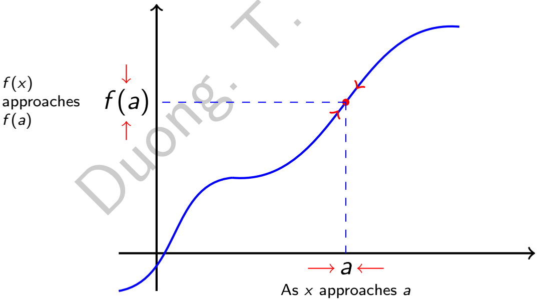
$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

Remark: Note that similar definitions apply when we replace ∞ by $-\infty$.

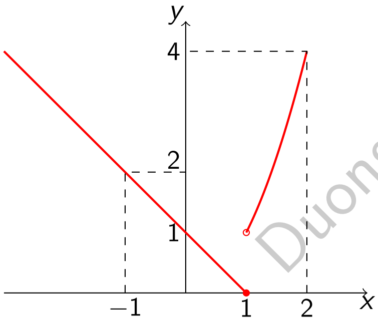
Definition.

Let $a \in (b, c)$ and let f be a function defined on (b, c). Function f is **continuous at a** if

$$\lim_{x \to a} f(x) = f(a)$$



Ex: Given $f(x) = \begin{cases} 1-x & \text{if } x \le 1 \\ x^2 & \text{if } x > 1. \end{cases}$



- $f(1) = 1 - 1 = 0$
- $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x^2 = 1$

  $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} (1-x) = 0$

  $\Longrightarrow \lim\limits_{x \to 1} f(x)$ does not exist
- f is NOT continuous at $x = 1$

Definition.

Let f be a function defined on $[a, c)$. Function f is **continuous from the right at a** if

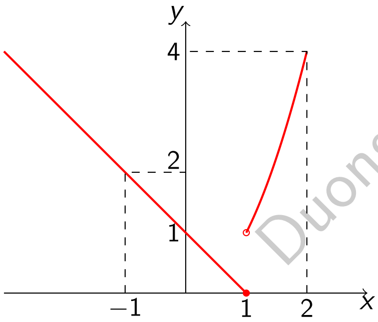
$$\lim_{x \to a^+} f(x) = f(a)$$

Definition.

Let f be a function defined on $(b, a]$. Function f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

Ex: Consider again $f(x) = \begin{cases} 1-x & \text{if } x \le 1 \\ x^2 & \text{if } x > 1. \end{cases}$



- Known: f is NOT continuous at $x = 1$
- $\lim_{x\to 1^+} f(x) = 1$; $\lim_{x\to 1^-} f(x) = 0$; and $f(1) = 0$
- $\lim_{x\to 1^-} f(x) = f(1)$ $\Rightarrow$ f is continuous from the left at $x = 1$
- $\lim_{x\to 1^+} f(x) \neq f(1)$ $\Rightarrow$ f is NOT continuous from the right at $x = 1$

Theorem.

- A function f is continuous at x_0 if and only if f is defined in an interval (a, b) containing x_0 and for each $\epsilon > 0$, there is a $\delta > 0$ such that
$$|f(x) - f(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta.$$
- A function f is continuous from the right at x_0 if and only if f is defined on an interval $[x_0, b)$ and for each $\epsilon > 0$, there is a $\delta > 0$ such that
$$|f(x) - f(x_0)| < \epsilon \text{ whenever } x_0 \leq x < x_0 + \delta.$$
- A function f is continuous from the left at x_0 if and only if f is defined on an interval $(a, x_0]$ and for each $\epsilon > 0$, there is a $\delta > 0$ such that
$$|f(x) - f(x_0)| < \epsilon \text{ whenever } x_0 - \delta < x \leq x_0.$$

Definition.

- A function f is said to be **continuous** on an open interval (a, b) if it is continuous at any point $x \in (a, b)$,
- A function f is said to be **continuous** on an closed interval $[a, b]$ if it is continuous at any point $x \in (a, b)$, and continuous from the right at a and from the left at b.
- A function f is said to be **continuous** on a half-open interval $[a, b)$ if it is continuous at any point $x \in (a, b)$, and continuous from the right at a.
- A function f is said to be **continuous** on a half-open interval $(a, b]$ if it is continuous at any point $x \in (a, b)$, and continuous from the left at b.

Ex: Show that function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

Ans:

- Let $-1 < a < 1$. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}(1 - \sqrt{1 - x^2}) = 1 - \sqrt{1 - \lim_{x\to a} x^2}$$
$$= 1 - \sqrt{1 - a^2} = f(a)$$

$\Longrightarrow$ f is continuous at a.

- Besides,

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(1-\sqrt{1-x^2}) = 1-\sqrt{1-\lim_{x\to -1^+} x^2} = 1 = f(-1)$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(1-\sqrt{1-x^2}) = 1-\sqrt{1-\lim_{x\to 1^-} x^2} = 1 = f(1)$$

$\Longrightarrow$ f is continuous at -1 from the right and at 1 from the left.

- f is continuous on $[-1, 1]$.

Theorem.

If f and g are continuous at $x = a$. Then the following functions are continuous at $x = a$:

(i) $f + g$

(ii) $f - g$

(iii) fg

(iv) $\dfrac{f}{g}$ if $g(a) \neq 0$

(v) cf where c is a constant

Proof:

(i) We have

$$\begin{aligned}
\lim_{x \to a} (f + g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\
&= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\
&= f(a) + g(a) \\
&= (f + g)(a).
\end{aligned}$$

Theorem.

1. Any polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0$ is continuous on $(-\infty, \infty)$
2. Any rational function $f(x) = \dfrac{g(x)}{h(x)}$, where g and h are polynomials, is continuous at wherever it is defined.

Proof:

Ex: Find $\lim_{x \to -2} \frac{x^2 + 2x - 3}{(x-3)(x+3)}$

Ans:

- Function $f(x) = \frac{x^2 + 2x - 3}{(x-3)(x+3)}$ is continuous at any point $x \neq -3$ and $x \neq 3$

  $\Longrightarrow$ It is continuous at $x = -2$

  $\Longrightarrow$ $\lim_{x \to -2} \frac{x^2 + 2x - 3}{(x-3)(x+3)} = f(-2) = \frac{(-2)^2 + 2 \cdot (-2) - 3}{(-2-3)(-2+3)}$

  $= \frac{3}{5}$

Theorem.

The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Composition of continuous functions

Theorem.

Let f be continuous at a and let g be continuous at $f(a)$. Then, the composition $g \circ f$ is continuous at a.

Proof: Let $\epsilon > 0$ be an arbitrarily small real number. Since g is continuous at $f(a)$, there is a $\delta_1 > 0$ such that

$$|g(y) - g[f(a)]| < \epsilon \quad \text{whenever} \quad |y - f(a)| < \delta_1. \tag{11}$$

Since f is continuous at a, there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \delta_1 \quad \text{whenever} \quad |x - a| < \delta. \tag{12}$$

Combining (11) and (12) we derive

$$g[f(x)] - g[f(a)] < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

By Theorem 97, the function $g \circ f$ is continuous at a.

Bounded Functions

Definition.

A function f is bounded below on a set S if there is a real number m such that

$$f(x) \geq m \quad \forall x \in S.$$

In this case, the set $V = \{f(x) \mid x \in S\}$ has an infimum α and we write

$$\alpha = \inf_{x \in S} f(x).$$

If there is a $x_1 \in S$ such that $f(x_1) = \alpha$, then we say that α is the minimum of f on S and we write

$$\alpha = \min_{x \in S} f(x).$$

Definition.

A function f is bounded above on a set S if there is a real number M such that

$$f(x) \leq M \quad \forall x \in S.$$

In this case, the set $V = \{f(x) \mid x \in S\}$ has a supremum β and we write

$$\beta = \sup_{x \in S} f(x).$$

If there is a $x_1 \in S$ such that $f(x_1) = \beta$, then we say that β is the maximum of f on S and we write

$$\beta = \max_{x \in S} f(x).$$

Remark.

If f is bounded above and below on S, f is said to be bounded on S .

Theorem.

If f is continuous on a finite closed interval $[a, b]$, then f is bounded on $[a, b]$.

Proof. Let $t \in [a, b]$. Since f is continuous at t, there is a $\delta_t > 0$ such that

$$|f(x) - f(t)| < 1 \quad \text{whenever} \quad x \in (t - \delta_t, t + \delta_t) \cap [a, b]. \tag{13}$$

Denote $I_t = (t - \delta_t, t + \delta_t)$. Then the collection $\mathcal{H} = \{I_t \mid t \in [a, b]\}$ is an open covering of $[a, b]$. Since $[a, b]$ is compact, Heine–Borel Theorem implies that there are finitely many points $t_1, t_2, \ldots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a, b]$. By (13), if $x \in I_{t_i}$ then

$$|f(x)| - |f(t_i)| \leq |f(x) - f(t_i)| < 1.$$

This implies that $|f(x)| < 1 + |f(t_i)|$ for all $x \in I_{t_i}$ and $i = 1, \ldots, n$. Denote $M = 1 + \max\{|f(t_i)|,\ i = 1, \ldots, n\}$. We then have

$$|f(x)| \leq M \quad \forall x \in [a, b].$$

Theorem.

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. There are $x_1, x_2 \in [a, b]$ such that

$$f(x_1) = \inf_{x\in[a,b]} f(x) \quad \text{and} \quad f(x_2) = \sup_{x\in[a,b]} f(x).$$

Proof. We shall prove the existence of x_1 (the proof for x_2 are left as an exercise). Denote $\alpha = \inf_{x\in[a,b]} f(x)$. Suppose that there is no such x_1. Then

$$f(x) > \alpha \quad \forall x \in [a, b].$$

Let $t \in [a, b]$. Then $f(t) > \alpha$, thus $f(t) > \dfrac{f(t)+\alpha}{2} > \alpha$. Since f is continuous at t, there is $\delta_t > 0$ such that

$$|f(x) - f(t)| < \frac{1}{2}\left(f(t) - \frac{f(t)+\alpha}{2}\right) \quad \forall x \in (t - \delta_t, t + \delta_t) \cap [a, b].$$

Denote $I_t = (t - \delta_t, t + \delta_t)$. This implies

$$f(x) > \frac{f(t)+\alpha}{2} \quad \forall x \in I_t \cap [a, b]. \tag{14}$$

The collection $\mathcal{H} = \{I_t \mid t \in [a,b]\}$ is an open covering of $[a,b]$. Since $[a,b]$ is compact, Heine–Borel Theorem suggests that there are finitely many $t_1, \ldots, t_n \in [a,b]$ such that the intervals $I_{t_1}, \ldots, I_{t_n}$ cover $[a,b]$. Define

$$\alpha_1 = \min_{i=1,\ldots,n} \frac{f(t_i) + \alpha}{2}.$$

Since $[a,b] \subset \bigcup_{i=1}^n I_{t_i}$, inequality (14) implies that

$$f(x) > \alpha_1 \quad \forall x \in [a,b].$$

Noting that $\alpha_1 > \alpha$, we derive

$$f(x) > \alpha_1 > \alpha \quad \forall x \in [a,b].$$

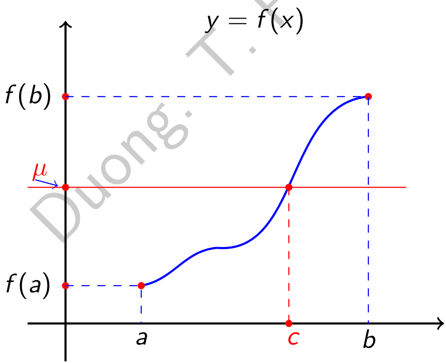
This contradicts the definition of α ($\alpha = \inf_{x\in[a,b]} f(x)$). Hence, there must be a $x_1 \in [a,b]$ such that

$$f(x_1) = \inf_{x\in[a,b]} f(x).$$

The Intermediate Value Theorem

Theorem.

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Let μ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a $c \in (a, b)$ such that $f(c) = \mu$.



The Intermediate Value Theorem

Proof. Assume that $f(a) < \mu < f(b)$. The set

$$S = \{x \mid a \le x \le b \text{ and } f(x) \le \mu\}$$

is bounded and nonempty. Let $c = \sup S$. We will show that $f(c) = \mu$.

- If $f(c) > \mu$, then $c > a$. Since f is continuous at c, there is a $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{f(c) - \mu}{2} \quad \text{whenever} \quad x \in (c - \delta, c + \delta).$$

  This implies that $f(x) > \mu$ whenever $c - \delta < x \le c$. Thus, $c - \delta/2$ is an upper bound of S. This contradicts the definition of c.

- If $f(c) < \mu$, then $c < b$. Since f is continuous at c, there is a $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{\mu - f(c)}{2} \quad \text{whenever} \quad x \in (c - \delta, c + \delta).$$

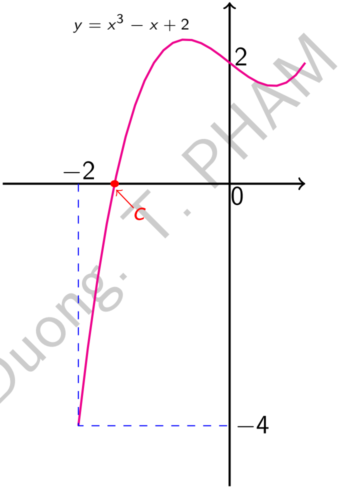
  This implies that $f(x) < \mu$ whenever $c \le x \le c + \delta$. This means that $c + \delta/2 \in S$ and thus c is not an upper bound for S. This is also a contradiction. Therefore, $f(c) = \mu$.

Ex: Prove that the equation $\boxed{x^3 - x + 2 = 0}$ has a root between -2 and 0.

Ans:

- The function $f(x) = x^3 - x + 2$ is continuous on $[-2, 0]$. Moreover, $f(-2) = -4$ and $f(0) = 2$.
- Number 0 satisfies $-2 < 0 < 2$. By using Intermediate Value Theorem, there is a $c \in (-2, 0)$ such that $f(c) = 0$.
- In other words, the equation: $x^3 - x + 2 = 0$ has a solution $c \in (-2, 0)$.

The Intermediate Value Theorem

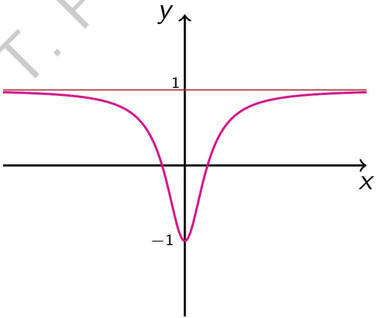


Def: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Ex: Consider $y = \dfrac{x^2-1}{x^2+1}$.

Since $\displaystyle\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1$, the line $y = 1$ is a horizontal asymptote of the curve.



Vertical Asymptote

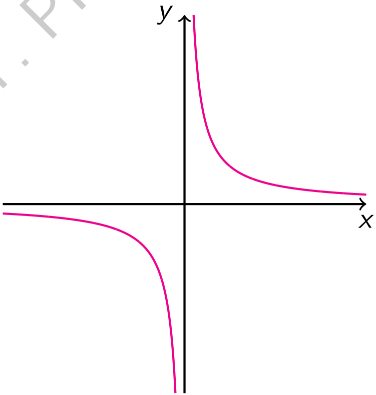
Def: The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to a^-} f(x) = \infty \text{ (or } -\infty) \quad \text{or} \quad \lim_{x \to a^+} f(x) = \infty \text{ (or } -\infty)$$

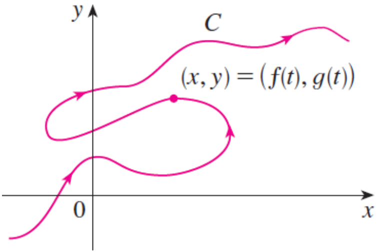
Ex: Consider $y = \dfrac{1}{x}$.

Since $\displaystyle\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\displaystyle\lim_{x \to 0^+} \frac{1}{x} = \infty$, the line $x = 0$ is a vertical asymptote of the curve.

Also, $\displaystyle\lim_{x \to \infty} \frac{1}{x} = \infty$, the line $y = 0$ is a horizontal asymptote of the curve.



Curves defined by parametric equations



- Imagine that a particle moves along the curve C shown in the figure
- Impossible to describe C by an equation of the form $y = f(x)$ as C fails the Vertical Line Test
- the x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$

Suppose that x and y are both given as functions of a third variable t

$$x = f(t), \quad y = g(t) \longrightarrow \text{parametric equations}$$

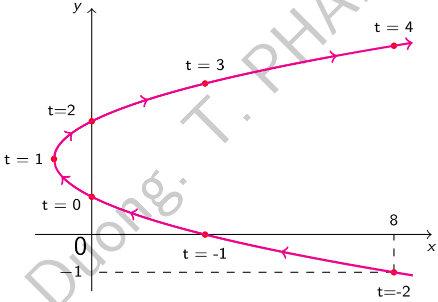
When t varies, the point $(x, y) = (f(t), g(t))$ varies and trace out a curve, which we call a parametric curve

Ex: Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t, \quad y = t + 1$$

Ans.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



Is the curve a parabola?

$$y = t + 1 \Longrightarrow t = y - 1 \Longrightarrow x = (y-1)^2 - 2(y-1)$$

$$\Longrightarrow x = y^2 - 4y + 3 \tag{15}$$

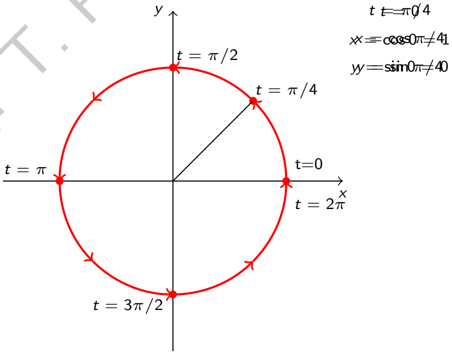
In general, the curve with parametric equations

$$x = f(t), \quad y = g(t), \quad a \le t \le b$$

has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

Ex: What curve is represented by the following parametric equations?

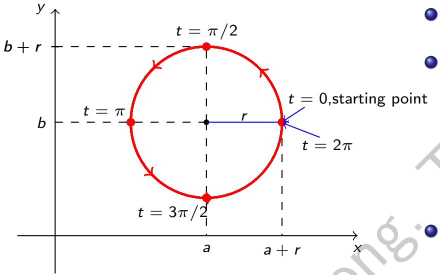
$$x = \cos t, \quad y = \sin t, \quad t \in [0, 2\pi].$$



Ans.
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$
$\longrightarrow$ It is a CIRCLE

Ex: Consider the circle centered at (a, b) and of radius r.

Ans.



- Equation: $(x - a)^2 + (y - b)^2 = r^2$
- Parametric equation:

$$\begin{cases} x = a + r\cos t \\ y = b + r\sin t, \end{cases} \quad t \in [0, 2\pi] \quad (16)$$

- $t = 0 \longrightarrow x = a + r\cos 0 = a + r$, $y = b + r\sin 0 = b \rightarrow (a + r, b)$
- $t = \pi/2 \longrightarrow x = a + r\cos\pi/2 = a$, $b = b + r\sin\pi/2 = b + r$ $\rightarrow (a, b + r)$

Equation (16) represents the circle centered at (a, b) with radius r, following the counterclockwise direction and the starting point $(a + r, b)$.

Graphing devices

Ex: Use Wolfram Alpha to graph the curve $x = y^4 - 3y^2$.
Hint: Denote $y = t$. Then $x = t^4 - 3t^2$. Type

```
plot(x = t^4-3t^2, y = t, t=[-2.5,2.5])
```

Ex: Graph the curve: $x = t + 2\sin 2t, \quad y = t + 2\cos 5t$
Hint: Type

```
plot(x=t+2*sin(2*t),  y = t + 2*cos(5*t), t=[-10,10])
```

Ex: Graph the curve:
$x = 16\sin^3 t, \quad y = 13\cos t - 5\cos(2t) - 2\cos(3t) - \cos(4t)$
Hint: Type

```
plot(x=16(sin(t))^3, y=13cos t - 5cos(2t) - 2cos(3t)-cos(4t),
t=[-5,5])
```

Ex: Graph the curve: $x = 1.5\cos t - \cos 30t, \quad y = 1.5\sin t - \sin 30t$
Hint: Type

```
plot( x=1.5*cos(t)-cos(30t), y=1.5*sin(t) - sin(30*t), t=[-10,10])
```

Ex: Graph the curve: $x = \sin(t + \cos 100t), \quad y = \cos(t + \sin 100t)$
Hint: Type