HO CHI MINH CITY INTERNATIONAL UNIVERSITY

FINAL EXAMINATION

Semester 2, Academic Year 2018-2019
Duration: 120 minutes

SUBJECT: Calculus 2	
Chair of Department of Mathematics	Lecturers:
Signature:	Signature:
	T.V. Linh, M.D. Thanh

- **Each student is allowed a maximum of two double-sided sheets of reference material (of size A4 or similar) and a scientific calculator.**
- **All other documents and electronic devices are forbidden.**

Question 1.

a.(5 points) The plane $y = 1$ intersects the surface $z = x^4 + 6xy - y^4$ in a certain curve. Find the slope of the tangent line to this curve at the point $P = (1, 1, 6)$.

b.(10 points) Find the equation of tangent plane to the graph of $f(x, y) = x^2y + xy^3$ at point $(2, 1)$, then use it to approximate the value of $f(2.1, 0.9)$

Question 2.(15 points)

Find the extreme values of the function $f(x, y, z) = 3x + 2y + 4z$ subject to the constraint $x^2 + 2y^2 + 6z^2 - 1 = 0$ using the Lagrange multiplier method.

Question 3.(15 points)

Find the critical points of the function $f(x, y) = x^3 + 2xy - 2y^2 - 10x$. Then use the Second Derivative Test to determine whether they are local minima, local maxima, or saddle points (or state that the test fails).

— PLEASE TURN OVER —

Question 4.

a)(15 points) Find the center of mass of an object occupying the region bounded by $y = 1 - x^2$ and $y = 0$ with constant density 1.

b)(10 points) Calculate the triple integral of $f(x, y, z) = x^2 + y^2$ over the unit ball $B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$.

Question 5.

a)(15 points) Given a vector field $\mathbf{F} = 3\mathbf{i} + 6y\mathbf{j}$. Show that $\mathbf{F}$ is a conservative vector field and find its potential function. Then calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ where C is a smooth path from $(1, 2)$ to $(4, 0.5)$ using the Fundamental Theorem of line integral.

b)(15 points) A satellite antenna's shape can be modeled by the part of the paraboloid $z = x^2 + y^2 + 1$ below the plane $z = 2$. The density of the antenna follows the formula $\rho(x, y, z) = \dfrac{1}{4z - 3}$. Calculate the total weight of the antenna given by the surface integral $W = \iint_S \rho(x, y, z) dS$.

———END OF QUESTIONS———

Calculus 2
Final Exam Solutions

Question 1.
a. The curve : $z = x^4 + 6x - 1$
The slope: $\dfrac{\partial z}{\partial x} = 4x^3 + 6 = 10$ when $x = 1$

b. $f(2,1) = 6,\ f_x(2,1) = 5,\ f_y(2,1) = 10$
Equation of tangent plane:

$$z = 6 + 5(x-2) + 10(y-1) = 5x + 10y - 14$$

Approximation

$$f(2.1, 0.9) \approx 6 + 5(0.1) + 10(-0.1) = 5.5$$

Question 2. Solve the equations

$$\begin{aligned} 3 &= \lambda(2x) \\ 2 &= \lambda(4y) \\ 4 &= \lambda(12z) \\ 0 &= x^2 + 2y^2 + 6z^2 - 1 \end{aligned}$$

From the first two we have $x = 3/2\lambda$, $y = 1/2\lambda$, $z = 1/3\lambda$. Plug in the third equation gives $\lambda = \pm\sqrt{123}/6$. then values at 2 critical points:

$$f\left(\frac{9}{\sqrt{123}}, \frac{3}{\sqrt{123}}, \frac{2}{\sqrt{123}}\right) = 3.7$$

$$f\left(-\frac{9}{\sqrt{123}}, -\frac{3}{\sqrt{123}}, -\frac{2}{\sqrt{123}}\right) = -3.7$$

then max is 3.7, min is -3.7.

Question 3.
Solve equations

$$\begin{aligned} 3x^2 + 2y - 10 &= 0 \\ 2x - 4y &= 0 \end{aligned}$$

to get $(-2, -1), (\frac{5}{3}, \frac{5}{6})$.

Second Derivative test:

$$D(x,y) = -24x - 4$$

Plug in values to get $(-2,-1)$ is local maximum, $(\frac{5}{3},\frac{5}{6})$ is saddle point.

Question 4. a) Total mass:

$$m = \int_{-1}^{1}\int_{0}^{1-x^2} dydx = \frac{4}{3}$$

From symmetry $\bar{x} = 0$.

$$\begin{aligned}\bar{y} &= \frac{3}{4}\int_{-1}^{1}\int_{0}^{1-x^2} ydydx \\ &= \frac{3}{8}\int_{-1}^{1}(1-x^2)^2dx \\ &= \frac{2}{5}\end{aligned}$$

Center of mass: $(0, 2/5)$

b) $f = \rho^2\sin^2\phi$, then

$$\begin{aligned}\iiint_S (x^2+y^2)dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1}(\rho^2\sin^2\phi)\rho^2\sin\phi\; d\rho\; d\phi\; d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi}\sin^3\phi\; d\phi \int_0^1 \rho^4 d\rho \\ &= \frac{8\pi}{15}\end{aligned}$$

Question 5. a) Conservative:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0$$

Find potential function f:
$\dfrac{\partial f}{\partial x} = 3$ then $f = 3x + g(y)$.
$\dfrac{\partial f}{\partial y} = g'(y) = 6y$ then $g(y) = 3y^2 + K$.
Then $f(x,y) = 3x + 3y^2 + K$ for any constant K.

Line integral:

$$\int_C \mathbf{F}\cdot d\mathbf{s} = f(4, 0.5) - f(1,2) = -\frac{9}{4}.$$

b) Domain: Unit circle D

$$\begin{aligned} W &= \iint_S \rho(x,y,z)dS \\ &= \iint_D \frac{1}{4z-3}\sqrt{1+(2x)^2+(2y)^2}dA \\ &= \int_0^{2\pi}\int_0^1 \frac{1}{4r^2+1}\sqrt{1+4r^2}\ r\ dr\ d\theta \\ &= \frac{\pi}{2}(\sqrt{5}-1) \end{aligned}$$