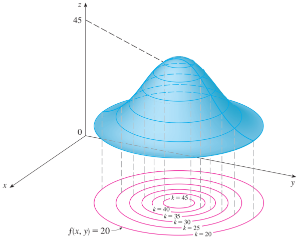
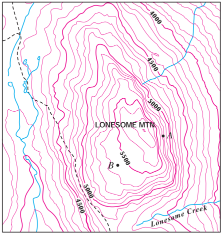
CHAPTER 3: PARTIAL DERIVATIVE





Contents

Introduction

Reference: Chapter 14, textbook by Stewart.

Example (Cobb-Douglas model)

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899-1922:

$$P(L, K) = nL^{\alpha}K^{1-\alpha},$$

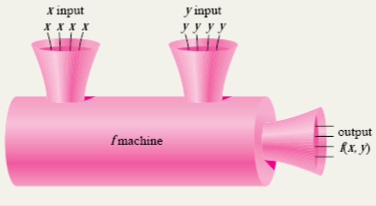
where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).
It has become known as the Cobb-Douglas production function, which is a functions of two variables.

Functions of multi-variables

Definition

A function f of n real variables is a rule that assigns to each point $(x_1, x_2, .., x_n)$ in a set D of $\mathbb{R}^n$ a unique real number denoted by $f(x_1, x_2, .., x_n)$.

The set D is the domain of f and its range is the set of values that f takes on, that is $R = \{f(x_1, x_2, .., x_n) | (x_1, x_2, .., x_n) \in D\}$.



A function of two variables as a "machine."

Functions of two variables

Example

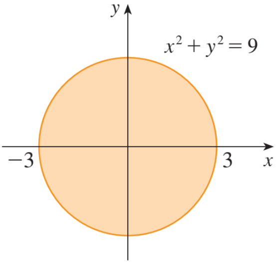
Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution

The domain is

$$D = \left\{ (x, y) \middle| \, 9 - x^2 - y^2 \geq 0 \right\} = \left\{ (x, y) \middle| \, x^2 + y^2 \leq 9 \right\}.$$

The range is $[0, 3]$ since $0 \leq g(x, y) \leq 3$ for all $(x, y) \in D$.



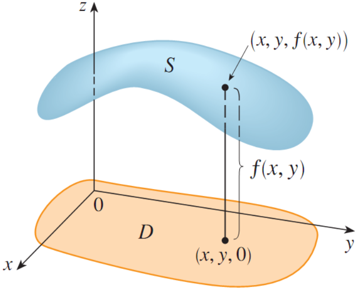
Graphs

Definition

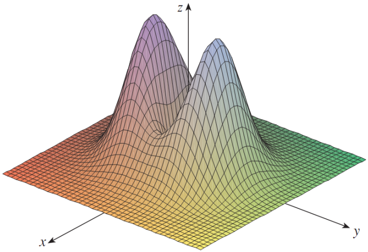
If f is a function of two variables with domain D, then the graph S of f is the set

$$S = \left\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y),\ \text{where}\ (x, y) \in D\right\}$$

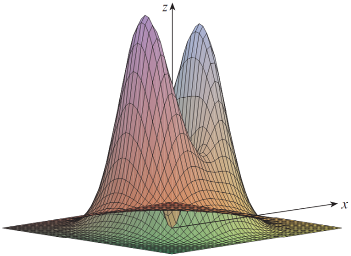
The graph of a function of two variables, $z = f(x, y)$ is called a surface in $\mathbb{R}^3$.



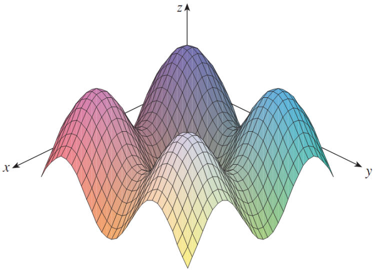
Example: Graphs



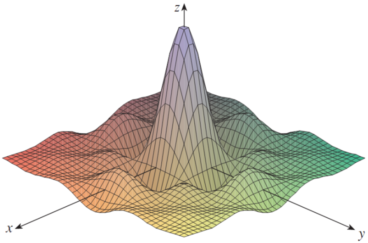
$f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



$f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$



$f(x, y) = \sin x + \sin y$



$f(x, y) = \dfrac{\sin x \, \sin y}{xy}$

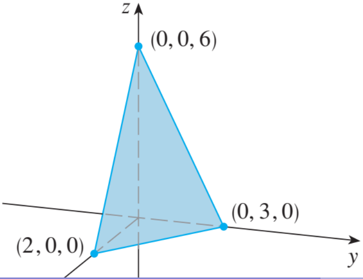
Functions of two variables

Example

Sketch the graph of $f(x, y) = 6 - 3x - 2y$.

Solution

The graph of has the equation $6 - 3x - 2y - z = 0$, which represents a plane. To graph the plane we first find the intercepts. The $x-$intercept, $y-$intercept, and $z-$intercept are $x = 2$, $y = 3$, and $z = 6$, respectively.



Graphs

- The graph of a function of three variables, $w = f(x, y, z)$ is called a three-dimensional hypersurface in the four dimensional space.

- In general, the graph of a function of n variables, $y = f(x_1, x_2, ..., x_n)$ is called a n-dimensional surface in the (n+1) dimensional space $\mathbb{R}^{n+1}$.

Domain of functions of two variables

Example

Find and sketch the domains of

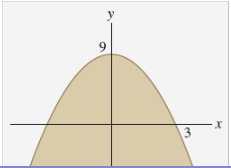
$$f(x, y) = \sqrt{9 - x^2 - y}$$

What is the range of f?

$f(x, y) = \sqrt{9 - x^2 - y}$ is defined if and only if

$$D = \left\{(x, y) : 9 - x^2 - y \geq 0\right\} = \left\{(x, y) : y \leq 9 - x^2\right\}$$

Thus, the domain consists of all points (x, y) lying below the parabola $y = 9 - x^2$. The range of f is $R = [0, \infty)$.



Domain of functions of two variables

Example

Find and sketch the domains of

$$f(x, y) = \sqrt{9 - x^2 - y^2}$$

$$g(x, y) = x \ln\left(y^2 - x\right)$$

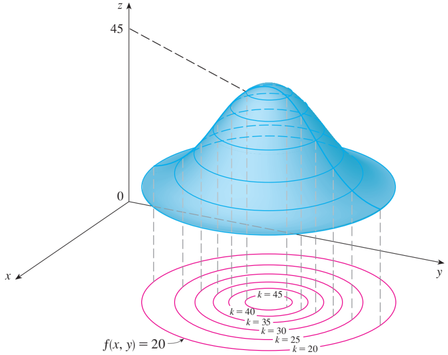
$$h(x, y) = \ln(9 - x^2 - 4y^2)$$

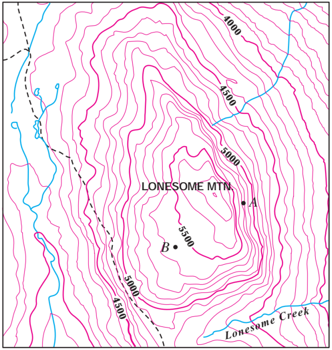
Level curves

Definition

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where is k a constant (in the range of f).

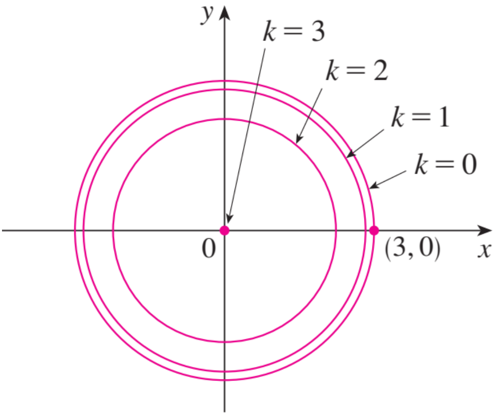
That is, the intersections of the surface with the planes $z = k$ are level curves.





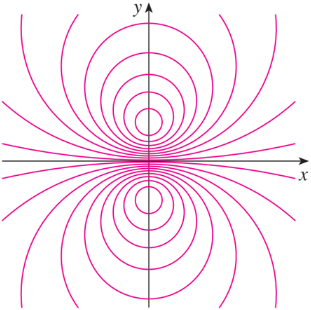
Examples: Level curves

The level curves of the function $z = f(x, y) = \sqrt{9 - x^2 - y^2}$ are concentric circles with equations: $x^2 + y^2 = 9 - k^2, \;\; 0 \leq k \leq 3$.

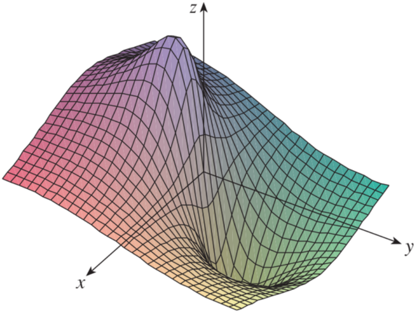


Level curves

The level curves in the figure on the left [Fig. (c)] crowd together near the origin. That corresponds to the fact that the graph in the figure on the right [Fig. (d)] is very steep near the origin.



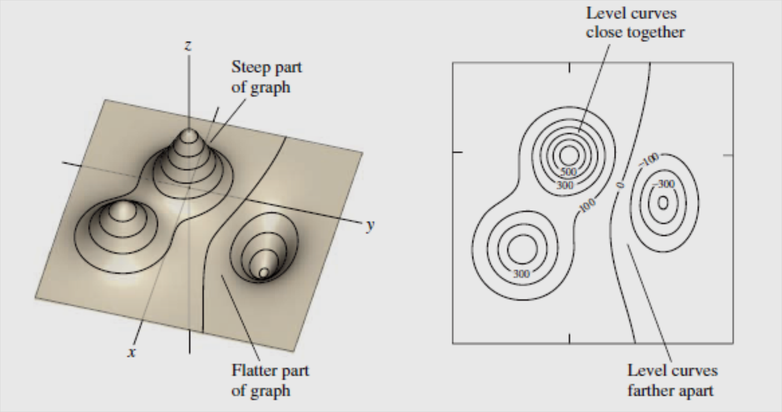
(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$



(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

Contour map

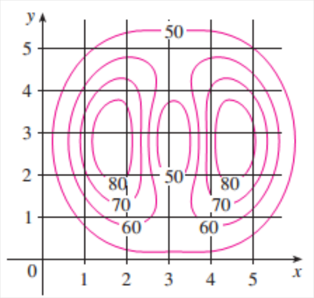
A contour map is a plot in the xy-plane that shows the level curves $f(x, y) = k$ for equally spaced values of k. The contour map represents the elevation in a topographic map.



Contour map

Examples

A contour map for a function is shown as the following figure. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.

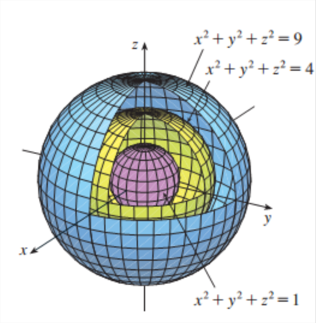


The point $(1,3)$ lies partway between the level curves with z-values 70 and 80. We estimate that $f(1,3) \approx 73$. Similarly, $f(4,5) \approx 56$.

Level surfaces

Definition

The level surfaces of a function f of three variables are the surfaces with equations $f(x, y, z) = k$, where is k a constant (in the range of f).



The level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$ form a family of concentric spheres with radius $\sqrt{k}$.

Limit of a function of two variables

Definition

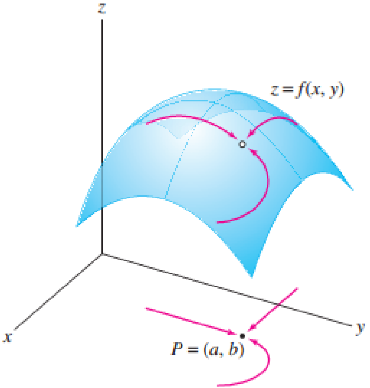
Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L, and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

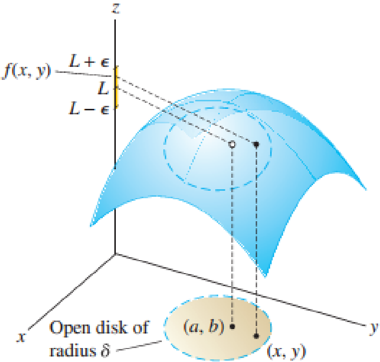
if for every positive number ϵ there exists a positive number $\delta = \delta(\epsilon)$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

Note that if a limit exists it is unique. It is not necessary that $L = f(a, b)$.

Limit of a function of two variables



(B) In two variables, (x, y) can approach $P = (a, b)$ along any direction or path.



(C) $|f(x, y) - L| < \epsilon$ for all (x, y) inside the disk

Limit of a function of two variables

Example

(a) $\lim_{(x,y)\to(2,3)} 2x - y^2 = 4 - 9 = -5$

(b) $\lim_{(x,y)\to(a,b)} x^2 y = a^2 b$

(c) $\lim_{(x,y)\to(\pi/3,2)} y\sin(x/y) = 2\sin(\pi/6) = 1$

Example

Evaluate $\lim_{(x,y)\to(3,0)} \dfrac{x^3 y}{\sin y}$.

Answer: 27.

Limit of a function of two variables

Properties

If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$ then

(i) $\lim_{(x,y)\to(a,b)} f(x,y) \pm g(x,y) = L \pm M$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM$

(iii) $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = L/M$, provided $M \neq 0$.

(iv) Also, if $F(t)$ is continuous at $t = L$, then $\lim_{(x,y)\to(a,b)} F(f(x,y)) = F(L)$.

Limit of a function of two variables

Example

Show that the function $f(x, y) = \frac{x^2 y}{x^2+y^2}$ does have a limit at the origin; specifically,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0$$

Solution 1 This function is defined everywhere except at $(0, 0)$. We have

$$|f(x, y) - L| = |f(x, y) - 0| = \left|\frac{x^2 y}{x^2 + y^2}\right| \leqslant |y| \leqslant \sqrt{x^2 + y^2}$$

which approaches zero as (x, y) approaches $(0, 0)$ (Why?). Thus, if $\epsilon > 0$, we take $\delta = \epsilon$, then $|f(x, y) - 0| < \epsilon$ whenever $\sqrt{x^2 + y^2} < \delta$. Thus, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$.

Limit of a function of two variables

Remark: The Squeeze theorem (Sandwich theorem) also holds for limit of a function of two variables.

Solution 2 Find $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$ if it exists.

We have $x^2 \leq x^2 + y^2$, so $x^2/(x^2+y^2) \leq 1$ and therefore,

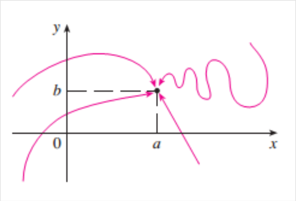
$$0 \leq \left|\frac{x^2y}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right||y| \leq |y| \to 0 \quad \text{as } (x,y) \to (0,0).$$

By the Squeeze theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

Limit of a function of two variables

When does the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ NOT exist?



If the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b). Therefore,

Two-Path Test for Discontinuity

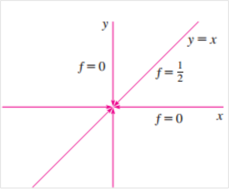
If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path $\mathcal{C}_1$ and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path $\mathcal{C}_2$, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Limit of a function of two variables

Example

If $f(x,y) = \frac{2xy}{x^2+y^2}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution



If $y = 0$, then $f(x,0) = 0$, thus $f(x,y) \to 0$ along the x-axis.
Let $(x,y) \to (0,0)$ along the line $y = x$, then $f(x,y) = f(x,x) = 1$.
Therefore, $f(x,y) \to 1$ along $y = x$. Since we have obtained different limits along different paths, the given limit does NOT exist.

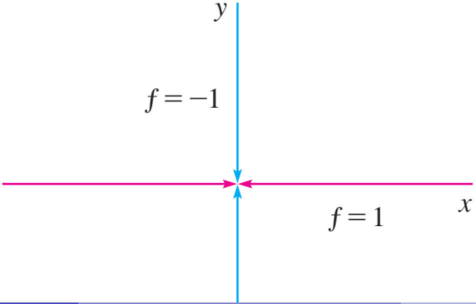
Limit of a function of two variables

Example

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution

We observe that $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.
$f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.



Continuity

Definition

The function $f(x, y)$ is continuous at the point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say f is continuous on D if is continuous at every point in D.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

Example

The function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous on its domain $D = \{(x, y) |\, (x, y) \neq (0, 0)\}$ and $f(x, y)$ is discontinuous at $(0, 0)$.

Continuity

Example

Show that function

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$.

Continuous function on $\mathbb{R}^n$

If f is defined on a subset D of $\mathbb{R}^n$. The function f is continuous at the point $a \in \mathbb{R}^n$ if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

Theorem: Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

Example

The following functions are continuous on $\mathbb{R}^2$: $\sin(x^2 e^y)$, $\ln(x^2 + e^y)$.

Rate of change

The famous triple peaks Eiger, Monch, and Jungfrau in the Swiss alps. The steepness at a point in a mountain range is measured by the gradient, a concept defined in this chapter.

Rate of change of a function f(x,y) depends on the direction! It is the concept of directional derivative.

Partial derivatives

Informal Definition

Let $z = f(x, y)$ be a function of two independent variables.

- The partial derivative of f with respect to x, denoted by $f_x(x, y)$, is the derivative of f as a function of x by considering y as a constant.
- The partial derivative of f with respect to y, denoted by $f_y(x, y)$, is the derivative of f as a function of y by considering x as a constant.

Example

Compute the partial derivatives of $f(x, y) = x^2y^5$.

$$f_x(x, y) = 2xy^5$$

$$f_y(x, y) = 5x^2y^4$$

Partial derivatives

Formal Definition

Let $z = f(x, y)$ be a function of two independent variables. Let all indicated limits exist.

Then the partial derivative of f with respect to x is:

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

The partial derivative of f with respect to y is:

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Partial derivatives

Formal Definition

Let $z = f(x, y)$ be a function of two independent variables. Let all indicated limits exist.

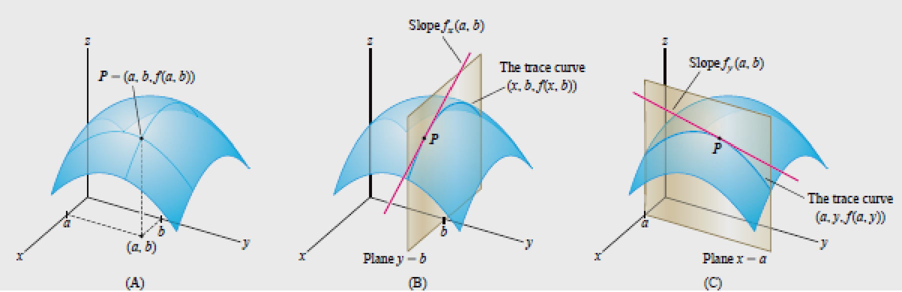
Then the partial derivative of f with respect to x is:

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

The partial derivative of f with respect to y is:

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Partial derivatives



Geometric Interpretation for $f_x(a,b)$: $f_x(a,b)$ is the slope of the tangent line to the trace curve $(x, b, f(x, b))$ [the trace curve $(x, b, f(x, b))$ is the intersection of the vertical plane $y = b$ and the surface $z = f(x, y)$].

Partial derivatives

Example

If

$$f(x,y) = 4x^2 - 9xy + 6y^3$$

then

$$f_x(x,y) = 8x - 9y$$

$$f_y(x,y) = -9x + 18y^2$$

Example

If $f(x,y) = \sin(x^2y^5)$, compute $\dfrac{\partial f}{\partial x}$

Using the chain rule

$$\frac{\partial f}{\partial x}(x,y) = 2xy^5 \cos\left(x^2y^5\right)$$

Partial derivatives

Example

If

$$f(x,y) = 2x^2 + 3xy^3 + 2y + 5$$

then

$$\frac{\partial f}{\partial x}(x,y) = 4x + 3y^3,$$

$$\frac{\partial f}{\partial x}(-1,2) = 4(-1) + 3(2)^3 = 20,$$

$$\frac{\partial f}{\partial y}(x,y) = 9xy^2 + 2,$$

$$\frac{\partial f}{\partial y}(-1,-3) = 9(-1)(-3)^2 + 2 = 20.$$

Partial derivatives

Example

If $f(x,y) = \sin(\frac{x}{1+y})$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Answer:

$$\frac{\partial f}{\partial x} = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$$

$$\frac{\partial f}{\partial y} = -\frac{x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$$

Partial derivatives

Example: Approximate the partial derivatives

The heat index I is a function of T and H: $I = f(T, H)$, where T the actual temperature and H is the relative humidity H. The following table of values of I is an excerpt from a table compiled by NWS (USA). Approximate $\dfrac{\partial I}{\partial T}(96, 70)$.

Table 1 Heat index I as a function of temperature and humidity

Actual temperature (°F) T \ Relative humidity (%) H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Answer: 3.75

Functions of more than two variables

Example

Find the partial derivatives

$$f(x, y, z) = 3x^2yz + z^3y$$

To find $f_x(x, y, z)$, we treat y and z as constants:

$$\frac{\partial}{\partial x} f(x, y, z) = f_x(x, y, z) = 6xyz$$

Similarly,

$$\frac{\partial}{\partial y} f(x, y, z) = f_y(x, y, z) = 3x^2z + z^3$$

$$\frac{\partial}{\partial z} f(x, y, z) = f_z(x, y, z) = 3x^2y + 3z^2y$$

Second-order partial derivatives

For a function of $z = f(x, y)$, if the indicated partial derivative exists, then

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = z_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = z_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = z_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = z_{yx}$$

Example

If $f(x,y) = 3x^2y + 2\sin(xy) - y^3$ then $\partial[y \; x \cos(xy)]$

$$f_x(x,y) = 6xy + 2\cos(xy)y$$

$2[\cos(xy) \; xy\sin(xy)$

$$f_y(x,y) = 3x^2 + 2\cos(xy)x - 3y^2$$

$$f_{xx}(x,y) = 6y - 2\sin(xy)y^2$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y} f_x(x,y) = 6x - 2\sin(xy)xy + 2\cos(xy)$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x} f_y(x,y) = 6x - 2\sin(xy)xy + 2\cos(xy)$$

$$f_{yy}(x,y) = -2\sin(xy)x^2 - 6y$$

Clairaut's Theorem: Equality of mixed partials

Alexis Clairaut (1713–1765)

Clairaut's Theorem

Suppose f is defined in a disk D that contains the point (a, b). If f_{xy} and f_{yx} are continuous on D, then:

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b)}$$

Note: For most functions f found in applications, f_{xy} and f_{yx} are equal.

Application: Partial differential equations

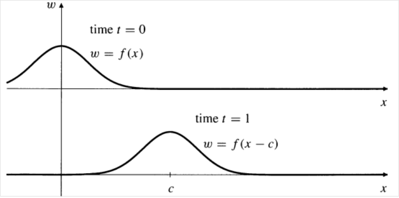
A partial differential equation (PDE) is a differential equation involving functions of several variables and their partial derivatives.

The wave equations

Show that $w = \sin(x - ct)$ satisfies the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

Hint Using the Chain Rule for functions of one variable



Application: Partial differential equations

The Laplace equation

Show that for any real number k the functions $z = e^{kx}\cos(ky)$ and $z = e^{kx}\sin(ky)$ satisfy the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

at every point in the $xy-$plane.

For $z = e^{kx}\cos(ky)$, we have

$$\frac{\partial z}{\partial x} = ke^{kx}\cos(ky), \frac{\partial^2 z}{\partial x^2} = k^2 e^{kx}\cos(ky)$$
$$\frac{\partial z}{\partial y} = -ke^{kx}\sin(ky), \frac{\partial^2 z}{\partial y^2} = -k^2 e^{kx}\cos(ky)$$

Thus, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$. The calculation for $z = e^{kx}\sin(ky)$ is similar.

Harmonic function and Laplace's equation

- A function of two variables having continuous second partial derivatives in a region of the plane is said to be harmonic there if it satisfies Laplace's equation.
- Harmonic functions play a crucial role in the theory of differentiable functions of a complex variable and are used to model various physical quantities such as steady-state temperature distributions, fluid flows, and electric and magnetic potential fields.
- A harmonic function has many interesting properties: it has derivatives of all orders, a harmonic function can achieve maximum and minimum values only on the boundary of its domain.

Differentiability

Let $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ be the increment of z.

Definition

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Differentiability

Example

Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$.

Solution:

We have

$$f_x(x, y) = e^{xy} + xye^{xy}, f_y(x, y) = x^2 e^{xy}$$

Both f_x and f_y are continuous functions, so f is differentiable.

Differentiability

Example

Where is $f(x, y) = \sqrt{x^2 + y^2}$ is differentiable?

Solution:
The function $f(x, y) = \sqrt{x^2 + y^2}$ is differentiable for $(x, y) \neq (0, 0)$ because the partial derivatives exist and are continuous except at $(0, 0)$:

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \qquad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}.$$

However, $f(x, y)$ is not differentiable at $(0, 0)$ since $f_x(0, 0)$ and $f_x(0, 0)$ do not exists.

The tangent plane to the surface

The tangent plane to the surface

Suppose has continuous partial derivatives. An equation of the tangent plane to the surface at the point (x_0, y_0, z_0) is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

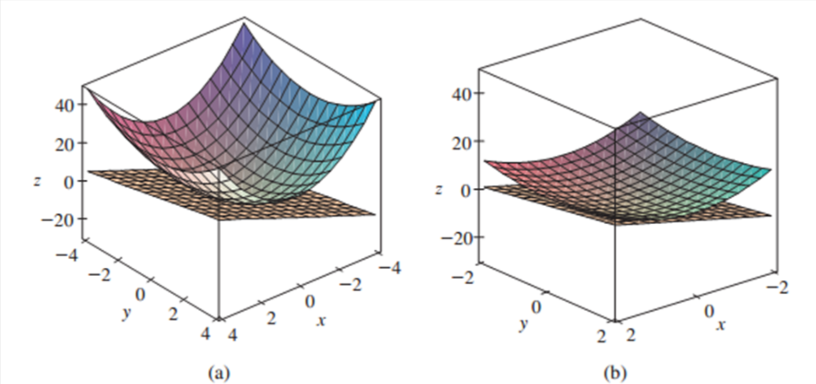
$L(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ is called the linearization of f at (x_0, y_0).

Example

The tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$ is $z = 4x + 2y - 3$.

The linearization of $f(x, y)$ at $(1, 1)$ is $L(x, y) = 4x + 2y - 3$.

The tangent plane to the surface



The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

Approximations

Formula

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b),$$

Equivalently,

$$\boxed{\Delta f \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y}$$

where $\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b)$, so called the increment of f.

Example

Approximate

$$\sqrt{2.98^2 + 4.01^2}$$

Solution

Let $f(x,y) = \sqrt{x^2+y^2}$, $(a,b) = (3,4)$, $\Delta x = -0.02, \Delta y = 0.01$.

Approximations: Example

Solution

We need to approximate $f(2.98, 4.01) = f\left(a + \Delta x, b + \Delta y\right)$.
We have

$$f\left(a + \Delta x, b + \Delta y\right) \approx f\left(a, b\right) + f_x\left(a, b\right)\Delta x + f_y\left(a, b\right)\Delta y$$

On the other hand,

$$f_x\left(x, y\right) = \frac{x}{\sqrt{x^2 + y^2}}, f_y\left(x, y\right) = \frac{y}{\sqrt{x^2 + y^2}}$$

Thus

$$f\left(2.98, 4.01\right) \approx f(3, 4) + \frac{3}{5}(-0.02) + \frac{4}{5}(0.01) = 5 - 0.004 = 4.996$$

Approximations

Example

At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment measured in units of \$1,000 and L the size of the labor force measured in worker-hours. The current capital investment is \$900,000, and 1,000 worker-hours of labor are used each day. Estimate the change in output that will result if capital investment is increased by \$1,000 and labor is increased by 2 worker-hours.

Solution $K = 900, L = 1,000, \Delta K = 1$, and $\Delta L = 2$ to get

$$\Delta Q \approx \frac{\partial Q}{\partial K}\Delta K + \frac{\partial Q}{\partial L}\Delta L$$

$$= 30K^{-1/2}L^{1/3}\Delta K + 20K^{1/2}L^{-2/3}\Delta L$$

$$= 30\left(\frac{1}{30}\right)(10)(1) + 20(30)\left(\frac{1}{100}\right)(2) = 22 \quad units$$

Total differentials

Definition

For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables. Then the total differential of z is

$$\boxed{dz = f_x(x, y)\, dx + f_y(x, y)\, dy}$$

(Sometimes dz is written as df.)

Example

Consider $z = f(x, y) = 9x^3 - 8x^2y + 4y^3$
(a) Find dz.
(b) Evaluate dz when $x = 1, y = 2, dx = 0.01, dy = -0.02$.

(a) $dz = (27x^2 - 16xy)dx + (-8x^2 + 12y^2)dy$, (b) $dz = -2.21$.

Volume of a Can of Beer

Example

A can of beer has the shape of a right circular cylinder with radius r=1 in. and height h=4 in. How sensitive is the volume to changes in the radius compared with changes in the height?

Solution

The volume of a right circular cylinder is given by $V = \pi r^2 h$.

$$dV = 2\pi rhdr + \pi r^2 dh = 8\pi dr + \pi dh$$

The factor of 8 in front of dr in this equation shows that a small change in the radius has 8 times the effect on the volume as a small change in the height.

This is the reason that beer cans are so tall and thin!

The Chain Rule

The Chain Rule: Case 1

Suppose f is a function of x and y with continuous first partial derivatives, each of which is a function of t, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example

Let $f(x, y) = x^2 y$, $x = 2t - 1, y = t^2$. Find $\frac{df}{dt}$.

Solution 1

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2xy)(2) + x^2(2t)$$

$$\frac{df}{dt} = 4t^2(2t-1) + 2t(2t-1)^2$$

Chain rule

Example

The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate (w.r.t t) at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

$$P = 8.31\frac{T}{V}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31T}{V^2}\frac{dV}{dt}$$

$$\frac{dP}{dt} = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s.

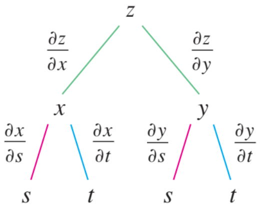
The Chain Rule

The Chain Rule: Case 2

Suppose $z = f(x, y)$ is a differentiable function of x and y with continuous first partial derivatives, each of which is a function of s and t, then

$$\frac{\partial z}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



The Chain Rule: Case 2

Example

Express $\partial z/\partial r$ and $\partial z/\partial s$ if

$$z = x^2 + y^2, \qquad x = r - s, \quad y = r + s.$$

Solution

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2x)(1) + (2y)(1) \\
&= 2(r - s) + 2(r + s) = 4r; \\
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x)(-1) + (2y)(1) \\
&= -2(r - s) + 2(r + s) = 4s.
\end{aligned}$$

The Chain Rule

The Chain Rule (General Version)

Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, and each x_j is a differentiable function of the variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for $i = 1, 2, \ldots, m$.

Implicit Functions

Suppose that the point (a, b) satisfies the equation $F(x, y) = 0$ and that F has continuous first partial derivatives (and so is differentiable) at all points near (a, b).
If there is a function $y(x)$ defined in some interval $I = (a - h, a + h)$ (where h > 0) satisfying $y(a) = b$ and such that

$$F\left(x, y\left(x\right)\right) = 0$$

By differentiating the equation $F(x, y(x)) = 0$ implicitly with respect to x, and evaluating the result at (a, b):

$$F_x\left(x, y\right) + F_y\left(x, y\right)\frac{dy}{dx} = 0$$

Therefore,

$$\boxed{\left.\frac{dy}{dx}\right|_{x=a} = -\frac{F_x\left(x, y\right)}{F_y\left(x, y\right)} \quad \text{if } F_y\left(a, b\right) \neq 0}$$

Implicit Functions

Example

Find $y'(x)$ if $x^3 + y^3 = 6xy$.

Solution

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy$$

Thus,

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Implicit Functions

Now we suppose that is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, z(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation to obtain the Implicit Function Theorem:

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}}$$

(If $\partial F/\partial z \neq 0$.)

Implicit Functions

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. By the Implicit Function Theorem:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

The Gradient vector

Definition

At any point (x, y) where the first partial derivatives of the function $f(x, y)$ exist, we define the gradient vector (or grad $f\,(x, y)$) $\nabla f\,(x, y)$ by

$$\nabla f\,(x, y) = \text{grad } f\,(x, y) = \langle f_x\,(x, y)\,, f_y\,(x, y)\rangle = f_x\,(x, y)\,\mathbf{i} + f_y\,(x, y)\,\mathbf{j}$$

The symbol ∇, called *del* or *nabla*, is a vector differential operator:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$$

In three variables,

$$\nabla f\,(x, y, z) = \langle f_x\,(x, y, z)\,, f_y\,(x, y, z)\,, f_z\,(x, y, z)\rangle\,.$$

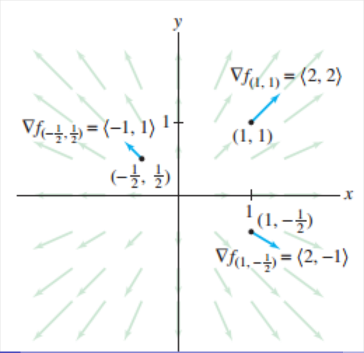
Note: One can denote $\nabla f\,(a, b, c) \equiv \nabla f_{(a,b,c)}$.

The Gradient vector

Example

Find the gradient of $f(x,y) = x^2 + y^2$ at $(1,1)$ and draw several gradient vectors.

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 2x, 2y\rangle, \quad \nabla f(1,1) = \langle 2,2\rangle .$$



The Gradient vector

Example: Gradient in Three Variables

Find the gradient of $f(x,y,z) = ze^{2x+3y}$ at $(3,-2,4)$.

Solution

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle 2ze^{2x+3y}, 3ze^{2x+3y}, e^{2x+3y} \right\rangle$$

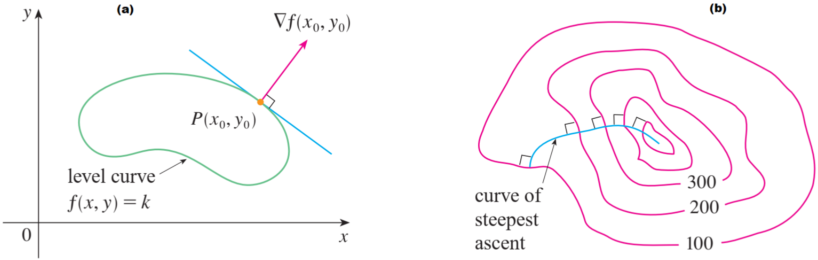
Therefore,

$$\nabla f(3,-2,4) = \langle 8, 12, 1 \rangle$$

The Gradient vector

Theorem: Gradients is normal to Level Curves

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).



(a) $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0);
(b) Applications for the method of steepest ascent

The Gradient vector

Theorem

The tangent to the level curve $f(x, y) = f(x_0, y_0)$ at the point (x_0, y_0) is the line

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Example

Find an equation for the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) on the ellipse.

Answer:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

Directional Derivatives

Definition

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

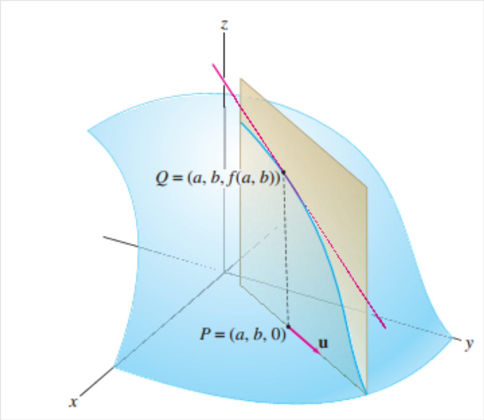
$$D_{\mathbf{u}} f\left(x_0, y_0\right) = \lim_{h \to 0} \frac{f\left(x_0 + ha, y_0 + hb\right) - f\left(x_0, y_0\right)}{h}$$

if this limit exists.

Note: The partial derivatives of f_x with respect to x is $D_{\mathbf{u}} f\left(a, b\right)$ where $\mathbf{u} = \langle 1, 0 \rangle$. That is, $f_x\left(a, b\right) \equiv D_{\mathbf{u}} f\left(a, b\right)$, f_x is therefore just a special case of the directional derivative!

Directional Derivatives

The directional derivative $D_{\mathbf{u}} f(P)$ is the slope of the tangent line at Q to the trace curve obtained when we intersect the graph with the vertical plane through P in the direction $\mathbf{u}$.



Directional Derivatives

Theorem

If f is a differentiable function of x and y, then f has a directional derivative at (x, y) in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

That is,

$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

Directional Derivatives

Example

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Solution

$$\nabla f(x, y) = 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{j}$$
$$\nabla f(2, -1) = -4\mathbf{i} + 8\mathbf{j}$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}.$$

Therefore,

$$D_{\mathbf{u}}f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = (-4)\left(\frac{2}{\sqrt{29}}\right) + 8\left(\frac{5}{\sqrt{29}}\right) = \frac{32}{\sqrt{29}}.$$

Directional Derivatives

Example

Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector make an angle $\theta = \dfrac{\pi}{6}$ with the positive $x-$axis. What is $D_{\mathbf{u}} f(1, 2)$?

Hint

Note that $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ thus

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

$$D_{\mathbf{u}} f(1, 2) = \frac{13 - 3\sqrt{3}}{2}$$

Directional Derivatives

Example

Let

$$f(x, y) = xe^{y}, \ (x_0, y_0) = (2, -1), \mathbf{v} = \langle 2, 3 \rangle$$

Calculate the directional derivative of f at $(2, -1)$ in the direction of $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

Answer

The directional derivative is

$$D_{\mathbf{u}} f(2, -1) = \frac{1}{\sqrt{13}} 8e^{-1} \approx 0.82$$

Interpretation of the Gradient

Interpretation of the Gradient

Assume that $\nabla f(x_0, y_0) \neq 0$. Let $\mathbf{u}$ be a unit vector making an angle θ with $\nabla f(x_0, y_0)$. Then

$$\boxed{D_{\mathbf{u}} f\,(x_0, y_0) = \|\nabla f(x_0, y_0)\| \cos\theta}$$

And,
(i) $\nabla f(x_0, y_0)$ points in the direction of maximum rate of increase of f at (x_0, y_0).

(ii) $-\nabla f(x_0, y_0)$ points in the direction of maximum rate of decrease at (x_0, y_0).

(ii) $\nabla f(x_0, y_0)$ is normal to the level curve (or surface) of f at (x_0, y_0).

Interpretation of the Gradient

Example

Let $f(x,y) = x^4 y^{-2}$ and $P = (2,1)$. Find the unit vector that points in direction of maximum rate of increase at P.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 4x^3 y^{-2}, -2x^4 y^{-3} \rangle \Rightarrow \nabla f(2,1) = \langle 32, -32 \rangle$$

$$\mathbf{u} = \frac{\nabla f(2,1)}{\|\nabla f(2,1)\|} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

Example

Let $f(x,y) = xe^y$ and $P = (2,0)$. In what direction does have the maximum rate of change at $P(2,0)$? What is this maximum rate of change? Answer: (a) $\nabla f(2,0) = \langle 1, 2 \rangle$ and (b) $|\nabla f(2,0)| = \sqrt{5}$

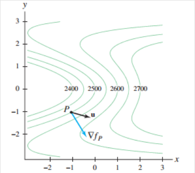
Interpretation of the Gradient

Example

The altitude of a mountain at (x, y) is

$$f(x, y) = 2500 + 100\left(x + y^2\right) e^{-0.3y^2}$$

where x, y are in units of 100 m.



(a) Find the directional derivative of f at $P = (-1, -1)$ in the direction of unit vector $\mathbf{u}$ making an angle of $\theta = \pi/4$ with the gradient.
(b) What is the interpretation of this derivative?

Interpretation of the Gradient

Solution

(a)

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 100e^{-0.3y^2}, 100y(2 - 0.6x - 0.6y^2)e^{-0.3y^2} \right\rangle$$

Therefore, at $P(-1, -1)$:

$$\nabla f(-1, -1) \approx \langle 74, -148 \rangle$$

$$D_{\mathbf{u}} f\,(P) = \|\nabla f_P\| \cos\theta \approx \sqrt{74^2 + (-148)^2} \left(\frac{\sqrt{2}}{2} \right) \approx 117.$$

(b) If you stand on the mountain at the point lying above $(-1, -1)$ and begin climbing so that your horizontal displacement is in the direction of u, then your altitude (height) increases at a rate of 117 meters per 100 meters of horizontal displacement, or 1.17 meters per meter of horizontal displacement.

Gradients and Tangents to Level Curves

Definition

The **tangent plane** at the point $P(x_0, y_0, z_0)$ on the level surface $F(x, y, z) = k$ is the plane

$$\boxed{F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0}$$

The **normal line** of the surface at $P(x_0, y_0, z_0)$ is the line

$$x = x_0 + F_x(P)t, \quad y = y_0 + F_y(P)t, \quad z = z_0 + F_x(P)t$$

In particular, the plane tangent to the surface $z = f(x, y)$ at the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is

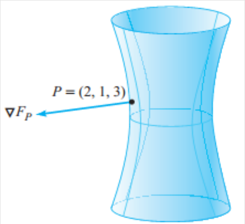
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

(which returns to the same equation as in slide # 49)

Gradients and Tangents to Level Curves

Example

Find an equation of the tangent plane to the surface $4x^2 + 9y^2 - z^2 = 16$ at $P = (2, 1, 3)$



Hint:

$$\nabla f\,(2, 1, 3) = \langle 16, 18, -6 \rangle$$

The tangent plane at P has equation:

$$16\,(x - 2) + 18\,(y - 1) - 6\,(z - 3) = 0$$

Local (relative) extremum

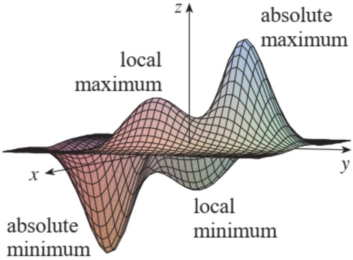
Let (a,b) be the center of a circular region D (a disk) contained in the xy-plane. Then, for a function z=f(x,y) defined on D.

- f(a,b) is a local (relative) maximum if and only if

$$f(a,b) \geqslant f(x,y),\ \forall (x,y) \in D$$

- f(a,b) is a local (relative) minimum if and only if

$$f(a,b) \leqslant f(x,y),\ \forall (x,y) \in D$$



Location of extrema

Theorem on location of extrema

Let a function $z = f(x, y)$ have a local maximum or local minimum at the point (a,b). Suppose $f_x(a, b)$ and $f_y(a, b)$ both exist. Then

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

In this case, (a, b) is a critical point of f.

Note:

a. An extremum can be a maximum, a minimum.

a. The theorem on location of extrema suggests a useful strategy for finding extrema: locate all critical points.

Location of extrema

Example

Find all critical points of

$$f(x,y) = 6x^2 + \ 6y^2 \ + \ 6xy \ + \ 36x - 5$$

Solution

To find the critical points, we solve

$$f_x(x,y) = 12x + 6y + 36 = 0$$

and

$$f_y(x,y) = 12y + 6x = 0$$

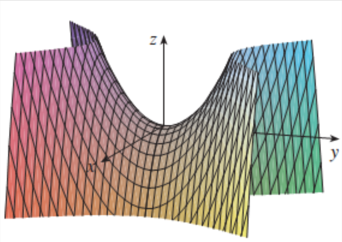
These two equations make up a system of linear equations. The solution of this system is $(x,y) = (-4,2)$.

Saddle points

Note that at a critical point the function f(x,y) may or may not attain an extremum.

Definition

If $f(x, y)$ does not attain an extremum at a critical point (a, b), then (a, b) is called a saddle point.



Second Derivative Test

How to determine whether or not a function has an extreme value at a critical point?

Theorem (Second Derivative Test)

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b).$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then (a, b) is a saddle point.

(d) If $D = 0$, then the test is inclusive at (a, b).

Extrema

Example

Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Solution

We first locate the critical points by solving

$$f_x(x, y) = 4x^3 - 4y = 0$$

$$f_y(x, y) = 4y^3 - 4x = 0$$

We substitute $y = x^3$ from the first equation into the second one. This leads to the three critical points

$$(0, 0), (1, 1), (-1, -1)$$

Extrema

Solution (Cont.)

We now apply the Second Derivative Test

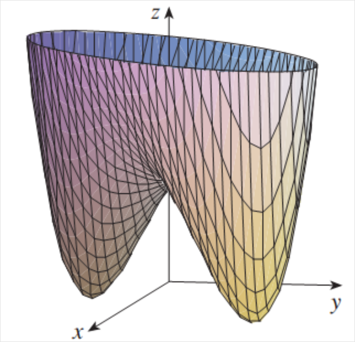
$$f_{xx} = 12x^2, f_{xy} = -4, f_{yy} = 12y^2$$

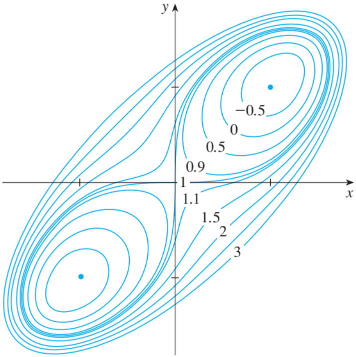
$$D(x, y) = 144x^2y^2 - 16$$

- $D(0,0) = -16 < 0$: The origin is a saddle point. That is, f has no local maximum or minimum at $(0,0)$.
- $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$: $(1,1)$ is a local minimum.
- $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$: $(-1,-1)$ is a local minimum.

Extrema

Below is the graph of $f(x, y) = x^4 + y^4 - 4xy + 1$ and its contour map





Extrema

Example

Find the local maximum and minimum values and saddle points of

$$f(x, y) = 9xy - x^3 - y^3 - 6$$

Answer $(0, 0)$: saddle point, $(3, 3)$: local maximum.

Applications: Optimization

Example

Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$

Hint Minimize $d^2 = (x-1)^2 + y^2 + (z+2)^2$ where $z = 4 - x - 2y$.

Extreme Value Theorem

Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Method for finding the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in the interior of D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Extreme Value Theorem

Example

Find the extreme values (absolute max/min) of the function

$$f(x, y) = x^2 y e^{-(x+y)}$$

on the triangular region T given by $x \geq 0, y \geq 0$ and $x + y \leq 4$.

Hint:

- Critical points: $(2, 1)$ and $(0, y)$ for any y. Only $(2, 1)$ is an interior point of T and $f(2, 1) = 4/e^3 \approx 0.199$.
- $f(x, y) = 0$ on the x-axis and y-axis.
- $f(x, y) = x^2(4 - x)e^{-4} := g(x)$, on $x + y = 4$, $0 \leq x \leq 4$.
  Critical points of g are $x = 0$ and $x = 8/3$.
  $g(0) = 0 < f(2, 1)$, $g(8/3) \approx 0.174 < f(2, 1)$.
- The absolute maximum value of f is $f(2, 1) = 4/e^3$.

Constrained Optimization

Example

The profit from the sale of x units of radiators for automobiles and y units of radiators for generators is given by

$$P(x, y) = -x^2 - y^2 + 4x + 8y$$

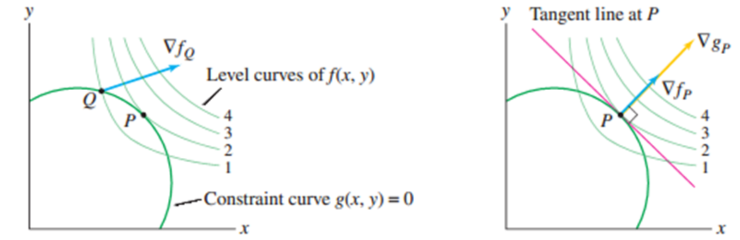
Find values of x and y that lead to a maximum profit if the firm must produce a total of 6 units of radiators.

That is, we need to maximize

$$P(x, y) = -x^2 - y^2 + 4x + 8y$$

with the constraint $x + y = 6$.
Q: How?

Constrained Optimization:Lagrange multiplier



(A) f increases as we move to the right along the constraint curve.

(B) The local maximum of f on the constraint curve occurs where ∇f_P and ∇g_P are parallel.

THEOREM **Lagrange Multipliers** Assume that $f(x, y)$ and $g(x, y)$ are differentiable functions. If $f(x, y)$ has a local minimum or a local maximum on the constraint curve $g(x, y) = 0$ at $P = (a, b)$, and if $\nabla g_P \neq \mathbf{0}$, then there is a scalar λ such that

$$\boxed{\nabla f_P = \lambda \nabla g_P}$$

Constrained Optimization

Lagrange multiplier

All relative extrema of $z = f(x, y)$, subject to a constraint $g(x, y) = 0$, will be found among those points (x,y) for which there exists a value of λ such that

$$F_x(x, y, \lambda) = 0,\ F_y(x, y, \lambda) = 0,\ F_\lambda(x, y, \lambda) = 0$$

where

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

Note: λ is call Lagrange multiplier and F is the Lagrange function.

Constrained Optimization

Example

Find the minimum value of

$$f(x, y) = 5x^2 + 6y^2 - xy$$

subject to the constraint $x + 2y = 24$.

Solution

Rewrite the constraint in the form $g(x, y) = 0$ where $g(x, y) = x + 2y - 24$. The Lagrange function is

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$F(x, y, \lambda) = 5x^2 + 6y^2 - xy - \lambda(x + 2y - 24)$$

Constrained Optimization

Example (cont.)

We now solve the system of the three equations

$$F_x(x, y, \lambda) = 0, F_y(x, y, \lambda) = 0, F_\lambda(x, y, \lambda) = 0. \text{That is}$$

$$\begin{cases} F_x(x, y, \lambda) = 10x - y - \lambda = 0 \\ F_y(x, y, \lambda) = 12y - x - 2\lambda = 0 \\ F_\lambda(x, y, \lambda) = -x - 2y + 24 = 0 \end{cases}$$

This implies $x = 6, y = 9$ (Why?).

Thus, if f has a extreme value subject to the constraint $g(x, y) = 0$ then it is $(6, 9)$.

We have $f(6, 9) = 612$ and since $f(5.8, 9.1) = 612.28 > f(6, 9)$, we conclude that $(6, 9)$ is a minimum.

Constrained Optimization

Example

The profit from the sale of x units of radiators for automobiles and y units of radiators for generators is given by

$$P(x, y) = -x^2 - y^2 + 4x + 8y$$

Find values of x and y that lead to a maximum profit if the firm must produce a total of 6 units of radiators.

Hint: We need to maximize

$$P(x, y) = -x^2 - y^2 + 4x + 8y$$

with the constraint $x + y = 6$.

Constrained Optimization

Solution

The Lagrange function is

$$F(x, y, \lambda) = -x^2 - y^2 + 4x + 8y - \lambda(x + y - 6)$$

We solve the following system of equations for (x, y):

$$\begin{cases} F_x(x, y, \lambda) = -2x + 4 - \lambda = 0 \\ F_y(x, y, \lambda) = -2y + 8 - \lambda = 0 \\ F_\lambda(x, y, \lambda) = -x - y + 6 = 0 \end{cases}$$

The first two equations implies $x = y - 2$, combine with the last one we get $(x, y) = (2, 4)$. Since $P(1, 5) = 18 < P(2, 4) = 20$, then $P(2, 4) = 20$ is the maximum value!

The firm must produce $x = 2$ units of radiators for automobiles and $y = 4$ units of radiators for generators.

Constrained Optimization

Example: Volume of a box

Find the dimensions of the closed rectangular box of maximum volume that can be produced from 6 ft^2 of material.

Hint. Maximize $V(x, y, z) = xyz$ with the constrained $2xy + 2yz + 2zx = 6$. The Lagrange function is

$$F(x, y, z, \lambda) = xyz - \lambda(xy + yz + xz - 3).$$

$$\begin{cases} F_x(x, y, z, \lambda) = yz - \lambda y - \lambda z = 0 \\ F_y(x, y, z, \lambda) = xz - \lambda x - \lambda z = 0 \\ F_z(x, y, z, \lambda) = xy - \lambda x - \lambda y = 0 \\ F_\lambda(x, y, z, \lambda) = -xy - xz - yz + 3 = 0 \end{cases}$$

$$\to \lambda = \frac{yz}{y+z}, \lambda = \frac{xz}{x+z}, \lambda = \frac{xy}{x+y}$$

$$\to x = y = z$$

Answer: $x = y = z = 1$ ft.

-END OF CHAPTER 3-