# DE AY1819 S2

## Q1.

Given that: $x^2y''' + xy'' - 4y' = 0\ (*), \qquad x > 0$

Assume that $y = x^\alpha$ is a solution of $(*)$, substituting into $(*)$, we get:

$$(y' = \alpha x^{\alpha-1} \to y'' = \alpha(\alpha-1)x^{\alpha-2} \to y'' = \alpha(\alpha-1)(\alpha-2)x^{\alpha-3})$$

$$(*) \to x^2\alpha(\alpha-1)(\alpha-2)x^{\alpha-3} + x\alpha(\alpha-1)x^{\alpha-2} - 4\alpha x^{\alpha-1} = 0$$

$$\to \alpha(\alpha-1)(\alpha-2) + \alpha(\alpha-1) - 4\alpha = 0 \ \ (x^{\alpha-1} > 0)$$

$$\leftrightarrow \alpha(\alpha+1)(\alpha-3) = 0$$

$$\leftrightarrow \left[\begin{array}{l} \alpha = 0 \\ \alpha = -1 \\ \alpha = 3 \end{array}\right. \to \left[\begin{array}{l} y_1 = 1 \\ y_2 = x^{-1} \\ y_3 = x^3 \end{array}\right.$$

Check for Wronskian determinant:

$$\mathrm{W}[y_1, y_2, y_3] = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} 1 & x^{-1} & x^3 \\ 0 & -x^{-2} & 3x^2 \\ 0 & 2x^{-3} & 6x \end{bmatrix} = -12x^{-1} \neq 0, \forall x > 0$$

So, $y_1, y_2, y_3$ are linearly independence solutions of $(*)$.

Thus, the general solution of the given differential equation is:

$$y_G = C_1y_1 + C_2y_2 + C_3y_3 = C_1 + C_2x^{-1} + C_3x^3$$

## Q2.

a) Given that: $y^{(4)} - 6y''' + 13y'' - 12y' + 4y = x^3 - (x^2+1)e^x + x^2\sin x$

$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x) + g_3(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y^{(4)} - 6y''' + 13y'' - 12y' + 4y \\ g_1(x) = x^3 \\ g_2(x) = -(x^2+1)e^x \\ g_3(x) = x^2\sin x \end{cases}$

Characteristic equation of the given ODE: $r^4 - 6r^3 + 13r^2 - 12r + 4 = 0$

$$\leftrightarrow (r-1)^2(r-2)^2 = 0$$

$$\leftrightarrow r_1 = r_2 = 1; r_3 = r_4 = 2$$

Since the right hand side of the given equation has three terms $g_1(x)$, $g_2(x)$ and $g_3(x)$, therefore the particular solution also has three term: $y_p = y_{p1} + y_{p2} + y_{p3}$, respectively.

Solve fore $y_{p1}$ from:

$$\mathrm{L}[y_{p1}] = g_1(x) \leftrightarrow {y_{p1}}^{(4)} - 6{y_{p1}}''' + 13{y_{p1}}'' - 12{y_{p1}}' + 4y_{p1} = x^3 \ \ (\alpha = 0)$$

Since, $\alpha = 0$ is not a root of characteristic equation.

Hence, $y_{p1}$ has the following form: $y_{p1} = Ax^3 + Bx^2 + Cx + D$

Solve fore $y_{p2}$ from:

$$\mathrm{L}[y_{p2}] = g_2(x) \leftrightarrow {y_{p2}}^{(4)} - 6{y_{p2}}''' + 13{y_{p2}}'' - 12{y_{p2}}' + 4y_{p2} = -(x^2+1)e^x \quad (\alpha = 1)$$

Since, $\alpha = 1$ is double root of characteristic equation.

Hence, $y_{p2}$ has the following form: $y_{p2} = x^2e^x(Ex^2 + Fx + G)$

Solve fore $y_{p3}$ from:

$$\mathrm{L}[y_{p3}] = g_3(x) \leftrightarrow {y_{p3}}^{(4)} - 6{y_{p3}}''' + 13{y_{p3}}'' - 12{y_{p3}}' + 4y_{p3} = x^2\sin x \quad (\alpha = 0)$$

Since, $\alpha = 0$ is not a root of characteristic equation.

Hence, $y_{p3}$ has the following form: $y_{p2} = (Hx^2 + Ix + J)\sin x + (Kx^2 + Mx + N)\cos x$

So: $y_p = y_{p1} + y_{p2} + y_{p3}$
$$= Ax^3 + Bx^2 + Cx + D + x^2e^x(Ex^2 + Fx + G) + (Hx^2 + Ix + J)\sin x + (Kx^2 + Mx + N)\cos x$$

b) Given that: $$y''' - 4y'' + 3y' = x + xe^{2x}$$
$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y''' - 4y'' + 3y' \\ g_1(x) = x \\ g_2(x) = xe^{2x} \end{cases}$

Characteristic equation of the given ODE: $r^3 - 4r^2 + 3r = 0$
$$\rightarrow r_1 = 0;\ r_2 = 1; r_3 = 3$$

So, the complement solution is: $y_c = C_1 + C_2e^x + C_3e^{3x}$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two term: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore $y_{p1}$ from: $\mathrm{L}\left[y_{p1}\right] = g_1(x) \leftrightarrow {y_{p1}}''' - 4{y_{p1}}' + 3y'_{p1} = x \qquad (\alpha = 0)$

Since, $\alpha = 0$ is a single root of characteristic equation.

So, $y_{p1}$ has the following form: $y_{p1} = x(Ax + B) = Ax^2 + Bx$
$$\rightarrow y'_{p1} = 2Ax + B$$
$$\rightarrow y''_{p1} = 2A$$
$$\rightarrow y'''_{p1} = 0$$

Substituting into the equation we obtain:
$$0 - 8A + 3(2Ax + B) = x$$
$$\rightarrow \begin{cases} 6A = 1 \\ 3B - 8A = 0 \end{cases} \leftrightarrow \begin{cases} A = \dfrac{1}{6} \\ B = \dfrac{4}{9} \end{cases}$$

Therefore: $y_{p1} = \frac{1}{6}x^2 + \frac{4}{9}x$

Solve fore $y_{p2}$ from: $\mathrm{L}\left[y_{p2}\right] = g_2(x) \leftrightarrow {y_{p2}}''' - 4{y_{p2}}' + 3y'_{p2} = xe^{2x} \quad (\alpha = 2)$

Since, $\alpha = 2$ is not a root of characteristic equation.

So, $y_{p2}$ has the following form: $y_{p2} = (Ax + B)e^{2x}$
$$\rightarrow y'_{p2} = (2Ax + 2B + A)e^{2x}$$
$$\rightarrow y''_{p2} = (4Ax + 4B + 4A)e^{2x}$$
$$\rightarrow y''_{p2} = (8Ax + 8B + 12A)e^{2x}$$

Substituting into the equation we obtain:
$$(-2Ax - 2B - A)e^{2x} = xe^{2x}$$
$$\rightarrow \begin{cases} -2A = 1 \\ -2B - A = 0 \end{cases} \leftrightarrow \begin{cases} A = -\dfrac{1}{2} \\ B = \dfrac{1}{4} \end{cases}$$

Therefore: $y_{p2} = \left(-\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$

So: $y_p = y_{p1} + y_{p2}$

$$= \frac{1}{6}x^2 + \frac{4}{9}x + \left(-\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

Thus, the general solution of the given differential equation is:

$$y_G = y_c + y_p$$
$$= C_1 + C_2e^x + C_3e^{3x} + \frac{1}{6}x^2 + \frac{4}{9}x + \left(-\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

## Q3.

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \quad (1) \\ \dfrac{dy}{dt} = 3x + 2y \ (2) \end{cases}$$

Differentiating both sides of (1), we get: $x'' = x' + 2y'$ (3).

Taking (2) − (1), we obtain: $y' - x' = 2x \leftrightarrow y' = x' + 2x$ (4)

Substituting (4) into (3), it leads to:

$$x'' = x' + 2(x' + 2x) \leftrightarrow x'' - 3x' - 4x = 0$$

Characteristic equation: $r^2 - 3r - 4 = 0 \to r_1 = -1; r_2 = 4$

Therefore: $$x(t) = C_1e^{-t} + C_2e^{4t}$$
$$\to x'(t) = -C_1e^{-t} + 4C_2e^{4t}$$

From (1): $y(t) = \frac{1}{2}\big(x'(t) - x(t)\big) = -C_1e^{-t} + \frac{3}{2}C_2e^{4t}$

Thus, the solution of the given system of differential equations is:

$$\begin{cases} x(t) = C_1e^{-t} + C_2e^{4t} \\ y(t) = -C_1e^{-t} + \dfrac{3}{2}C_2e^{4t} \end{cases}$$

## Q4.

Given that: $$xy'' - (1 + x)y' + y = 0 \quad (1), \qquad x > 0$$

Check for solution:

With $y_1 = e^x$, it holds that: $y_1' = y_1'' = e^x$. Substituting into (1), we get:

$$xe^x - (1 + x)e^x + e^x = 0 \text{ (valid)}$$

With $y_2 = x + 1$, it holds that: $y_2' = 1, y_2'' = 0$. Substituting into (1), we get:

$$x0 - (1 + x).1 + x + 1 = 0 \text{ (valid)}$$

So, $y_1, y_2$ are solutions of (1)

Check for linearity:

$$\mathrm{W}[y_1, y_2] = \begin{bmatrix} e^x & x + 1 \\ e^x & 1 \end{bmatrix} = -xe^x \neq 0, \forall x > 0$$

So, $y_1, y_2$ are linearly independence.

Thus, $y_1, y_2$ are linearly independence solutions of (1)

Given that: $$xy'' - (1 + x)y' + y = x^2e^x \quad (2), \qquad x > 0$$

Assume that $y_p = x(Ax + B)e^x$ is a particular solution of (2), we have to find $A$ and $B$.

$$\to y_p' = (Ax^2 + (2A + B)x + B)e^x$$
$$\to y_p'' = (Ax^2 + (4A + B)x + 2A + 2B)e^x$$

Substituting into (2), we get:

$$(2Ax^2+Bx-B)e^x = x^2e^x$$
$$\rightarrow \begin{cases} 2A = 1 \\ B = 0 \end{cases} \leftrightarrow \begin{cases} A = \dfrac{1}{2} \\ B = 0 \end{cases}$$

Thus, $y_p = \frac{1}{2}x^2e^x$

# Q5.

$$m\frac{dv}{dt} = mg - kv^2$$
$$\rightarrow \frac{dv}{\frac{mg}{k} - v^2} = \frac{m}{k}dt$$
$$\leftrightarrow \left(\frac{1}{\sqrt{\frac{mg}{k}}+v} + \frac{1}{\sqrt{\frac{mg}{k}}-v}\right)dv = \frac{2m}{k}\sqrt{\frac{mg}{k}}dt$$
$$\leftrightarrow \left(\frac{1}{\omega_c+v} + \frac{1}{\omega_c - v}\right)dv = 2\omega_c\omega_0 dt$$
$$\left(\text{Where: } \omega_c = \sqrt{\tfrac{mg}{k}};\ \ \omega_0 = \tfrac{m}{k}\right)$$
$$\leftrightarrow \ln\left(\frac{\omega_c+v}{\omega_c-v}\right) = 2\omega_c\omega_0 t + C\ (1)$$
$$\frac{\omega_c+v}{\omega_c-v} = e^{2\omega_c\omega_0 t + C}$$

Solve for $v(t)$, we get the result:

$$v(t) = \frac{\omega_c\left(e^{2\omega_c\omega_0 t + C} - 1\right)}{e^{2\omega_c\omega_0 t + C} + 1}\ (2)$$

From (1) and initial condition $v(0) = v_0$:

$$C = \ln\left(\frac{\omega_c + v_0}{\omega_c - v_0}\right)$$

i) So, the final result is:

$$v(t) = \frac{\sqrt{\frac{mg}{k}}\left(\frac{\sqrt{\frac{mg}{k}}+v_0}{\sqrt{\frac{mg}{k}}-v_0}e^{\frac{2m}{k}\sqrt{\frac{mg}{k}}t} - 1\right)}{\frac{\sqrt{\frac{mg}{k}}+v_0}{\sqrt{\frac{mg}{k}}-v_0}e^{\frac{2m}{k}\sqrt{\frac{mg}{k}}t} + 1} = \frac{\sqrt{\frac{mg}{k}}\left[\left(\sqrt{\frac{mg}{k}}+v_0\right)e^{\frac{2m}{k}\sqrt{\frac{mg}{k}}t} - \left(\sqrt{\frac{mg}{k}}-v_0\right)\right]}{\left(\sqrt{\frac{mg}{k}}+v_0\right)e^{\frac{2m}{k}\sqrt{\frac{mg}{k}}t} + \left(\sqrt{\frac{mg}{k}}-v_0\right)}$$

ii) From (2):

$$v(t) = \frac{2\omega_c e^{2\omega_c\omega_0 t + C}}{e^{2\omega_c\omega_0 t + C} + 1} - \omega_c$$

Therefore:

$$s(t) = \int v(t)dt = \int\left(\frac{2\omega_c e^{2\omega_c\omega_0 t + C}}{e^{2\omega_c\omega_0 t + C} + 1} - \omega_c\right)dt = \frac{1}{\omega_0}\ln\left(e^{2\omega_c\omega_0 t + C} + 1\right) - \omega_c + C'$$

The problem gives us $s(0) = 0$, it leads to

$$C' = \omega_c - \frac{1}{\omega_0}\ln(e^C + 1) = \omega_c - \frac{1}{\omega_0}\ln\left(\frac{2\omega_c}{\omega_c - v_0}\right)$$

So, the expression of $s(t)$ is:

$$\begin{aligned} s(t) &= \frac{1}{\omega_0}\ln\left(\frac{\omega_c + v_0}{\omega_c - v_0}e^{2\omega_c\omega_0 t} + 1\right) - \frac{1}{\omega_0}\ln\left(\frac{2\omega_c}{\omega_c - v_0}\right) \\ &= \frac{1}{\omega_0}\ln\frac{(\omega_c + v_0)e^{2\omega_c\omega_0 t} + \omega_c - v_0}{2\omega_c} \\ &= \frac{k}{m}\ln\frac{\left(\sqrt{\frac{mg}{k}} + v_0\right)e^{\frac{2m}{k}\sqrt{\frac{mg}{k}}t} + \sqrt{\frac{mg}{k}} - v_0}{2\sqrt{\frac{mg}{k}}} \end{aligned}$$

*(Too fucking long :) )*