# DE AY1819 S1

## Q1.

Given that: $\frac{dA}{dt} = k_1(M-A) - k_2A \ (*), \qquad A(0)=0$

$$(*) \leftrightarrow \frac{dA}{dt} = -(k_1+k_2)A + k_1M$$

$$\rightarrow \frac{dA}{-(k_1+k_2)A + k_1M} = dt$$

$$-\frac{1}{k_1+k_2}\ln(-(k_1+k_2)A + k_1M) = t + C$$

With the initial condition: $A(0)=0$

$$\rightarrow -\frac{1}{k_1+k_2}\ln(-(k_1+k_2)0 + k_1M) = 0 + C \leftrightarrow C = -\frac{\ln(k_1M)}{k_1+k_2}$$

Solve for $A$, we obtain the result:

$$A(t) = \frac{1}{k_1+k_2}\left(k_1M - e^{-(k_1+k_2)t+\ln(k_1M)}\right)$$

Due to the fact that $k_1>0, k_2>0$, it leads to:

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left[\frac{1}{k_1+k_2}\left(k_1M - e^{-(k_1+k_2)t+\ln(k_1M)}\right)\right] = \frac{k_1M}{k_1+k_2}$$

## Q2.

Given that: $x^2y'' + 5xy' + 4y = 0 \ (*)$

Assume that: $y_1 = x^\alpha$ is a solution of the given differential equation.

$$\rightarrow y_1' = \alpha x^{\alpha-1} \rightarrow y_1'' = \alpha(\alpha-1)x^{\alpha-2}$$

We know that $y_1$ is a solution of $(*)$, therefore substituting $y_1$ into $(*)$, we get:

$$x^2\alpha(\alpha-1)x^{\alpha-2} + 5x\alpha x^{\alpha-1} + 4x^\alpha = 0$$

$$\leftrightarrow \alpha(\alpha-1) + 5\alpha + 4 = 0 \quad (x^\alpha > 0)$$

$$\leftrightarrow \alpha^2 + 4\alpha + 4 = 0 \rightarrow \alpha = -2$$

So, $y_1 = x^{-2}$ is a solution of $(*)$

To find the general solution of $(*)$, we rewrite $(*)$ in the following form:

$$y'' + \frac{5}{x}y' + \frac{4}{x^2}y = 0$$

$$(y'' + p(x)y' + q(x) = 0)$$

The Wronskian determinant for the equation is:

$$\mathrm{W}[y_1,y_2] = C_1e^{-\int p(x)dx} = C_1e^{-\int \frac{5}{x}dx}$$

$$\rightarrow \mathrm{W}[y_1,y_2] = C_1x^{-5}$$

Hence:

$$y_2 = y_1\left[\int \frac{\mathrm{W}[y_1,y_2]}{y_1^2}dx + C_2\right]$$

Substitute $y_1$ into the above expression:

$$y_2 = x^{-2}\left[\int \frac{C_1x^{-5}}{(x^{-2})^2}dx + C_2\right]$$

$$\rightarrow y_2 = x^{-2}[C_1\ln x + C_2]$$

$$\rightarrow y_2 = C_1x^2\ln x + C_2x^{-2}$$

Choose $C_1 = 1, C_2 = 0 \rightarrow y_2 = x^2\ln x$

Since, the Wronskian determinant different from 0 for some $x$, therefore $y_1$ and $y_2$ are linearly independent solution of the equation.
Thus, the general solution of the equation is:

$$y_G = C_1 y_1 + C_2 y_2 = C_1 x^{-2} + C_2 x^2 \ln x$$

## Q3.

Given that: $$y'' + 5y' - 14y = x^2 + 1 + xe^{-7x}$$
$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y'' + 5y' - 14y \\ g_1(x) = x^2 + 1 \\ g_2(x) = xe^{-7x} \end{cases}$

Characteristic equation of the given ODE: $r^2 + 5r - 14 = 0$

$$\rightarrow r_1 = 2;\ r_2 = -7$$

So, the complement solution is: $y_c = C_1 e^{2x} + C_2 e^{-7x}$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two term: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore $y_{p1}$ from: $\mathrm{L}\left[y_{p1}\right] = g_1(x) \leftrightarrow y_{p1}'' + 5y_{p1}' - 14y_{p1} = x^2 + 1 \ \ (\alpha = 0)$

Since, $\alpha = 0$ is not a root of characteristic equation.

So, $y_{p1}$ has the following form: $y_{p1} = Ax^2 + Bx^2 + C$

$$\rightarrow y_{p1}' = 2Ax + B$$
$$\rightarrow y_{p1}'' = 2A$$

Substituting into the equation we obtain:

$$2A + 5(2Ax + B) - 14(Ax^2 + Bx + C) = x^2 + 1$$

$$\rightarrow \begin{cases} -14A = 1 \\ 10A - 14B = 0 \\ 2A + 5B - 14C = 1 \end{cases} \leftrightarrow \begin{cases} A = -\dfrac{1}{14} \\ B = -\dfrac{5}{98} \\ C = -\dfrac{137}{1372} \end{cases}$$

Therefore: $y_{p1} = -\left(\frac{1}{14}x^2 + \frac{5}{98}x + \frac{137}{1372}\right)$

Solve fore $y_{p2}$ from: $\mathrm{L}\left[y_{p2}\right] = g_2(x) \leftrightarrow y_{p2}'' + 5y_{p2}' - 14y_{p2} = xe^{-7x} \ (\alpha = -7)$

Since, $\alpha = -7$ is a single root of characteristic equation.

So, $y_{p2}$ has the following form: $y_{p2} = x(Ax + B)e^{-7x} = (Ax^2 + Bx)e^{-7x}$

$$\rightarrow y_{p2}' = (-7Ax + A - 7B)e^{-7x}$$
$$\rightarrow y_{p2}'' = (49Ax - 14A + 49B)e^{-7x}$$

Substituting into the equation we obtain:

$$e^{-7x}[-18Ax + 2A - 9B] = xe^{-7x}$$

$$\rightarrow \begin{cases} -18A = 1 \\ 2A - 9B = 0 \end{cases} \leftrightarrow \begin{cases} A = -\dfrac{1}{18} \\ B = -\dfrac{1}{81} \end{cases}$$

Therefore: $y_{p2} = -e^{-7x}\left(\frac{1}{18}x^2 + \frac{1}{81}x\right)$

So: $\quad y_p = y_{p1} + y_{p2}$

$$= -\left(\frac{1}{14}x^2 + \frac{5}{98}x + \frac{137}{1372}\right) - e^{-7x}\left(\frac{1}{18}x^2 + \frac{1}{81}x\right)$$

Thus, the general solution of the given differential equation is:

$$y_G = y_c + y_p$$
$$= C_1e^{2x} + C_2e^{-7x} - \left(\frac{1}{14}x^2 + \frac{5}{98}x + \frac{137}{1372}\right) + e^{-7x}\left(\frac{1}{18}x^2 + \frac{1}{81}x\right)$$

## Q4.

Given that: $$(3x^2y + e^y)dx + (x^3 + xe^y - 2y)dy = 0 \ (*)$$

$$(*) \leftrightarrow 3x^2ydx + e^ydx + x^3dy + xe^ydy - 2ydy = 0$$
$$\leftrightarrow yd(x^3) + e^ydx + x^3dy + xd(e^y) - d(y^2) = 0$$
$$\leftrightarrow yd(x^3) + x^3dy + e^ydx + xd(e^y) - d(y^2) = 0$$
$$\leftrightarrow d(x^3y) + d(xe^y) - d(y^2) = 0$$
$$\leftrightarrow d(x^3y + xe^y - y^2) = 0$$

Integrating both sides we obtain the final result:

$$\leftrightarrow x^3y + xe^y - y^2 + C = 0$$

## Q5.

Given that: $$xy' + (3x+1)y = e^{-3x} \ (*), \qquad y(1) = 1$$

$$(*) \leftrightarrow xe^{3x}y' + (3x+1)e^{3x}y = 1$$
$$\leftrightarrow xe^{3x}\frac{dy}{dx} + \frac{d(xe^{3x})}{dx}y = 1$$
$$\leftrightarrow \frac{d(xe^{3x}y)}{dx} = 1$$
$$\leftrightarrow d(xe^{3x}y) = dx$$
$$\leftrightarrow xe^{3x}y = x + C$$

With the initial condition: $y(1) = 1$, it leads to:

$$1.e^3.1 = 1 + C \leftrightarrow C = e^3 - 1$$

Hence, the solution of the equation is:

$$xe^{3x}y = x + e^3 - 1$$

Or:

$$y = e^{-3x} + \frac{(e^3-1)e^{-3x}}{x}$$