Chapter 7
DISCRETE TIME SIGNALS AND SYSTEMS

TIME SHIFTING (TRANSLATION)

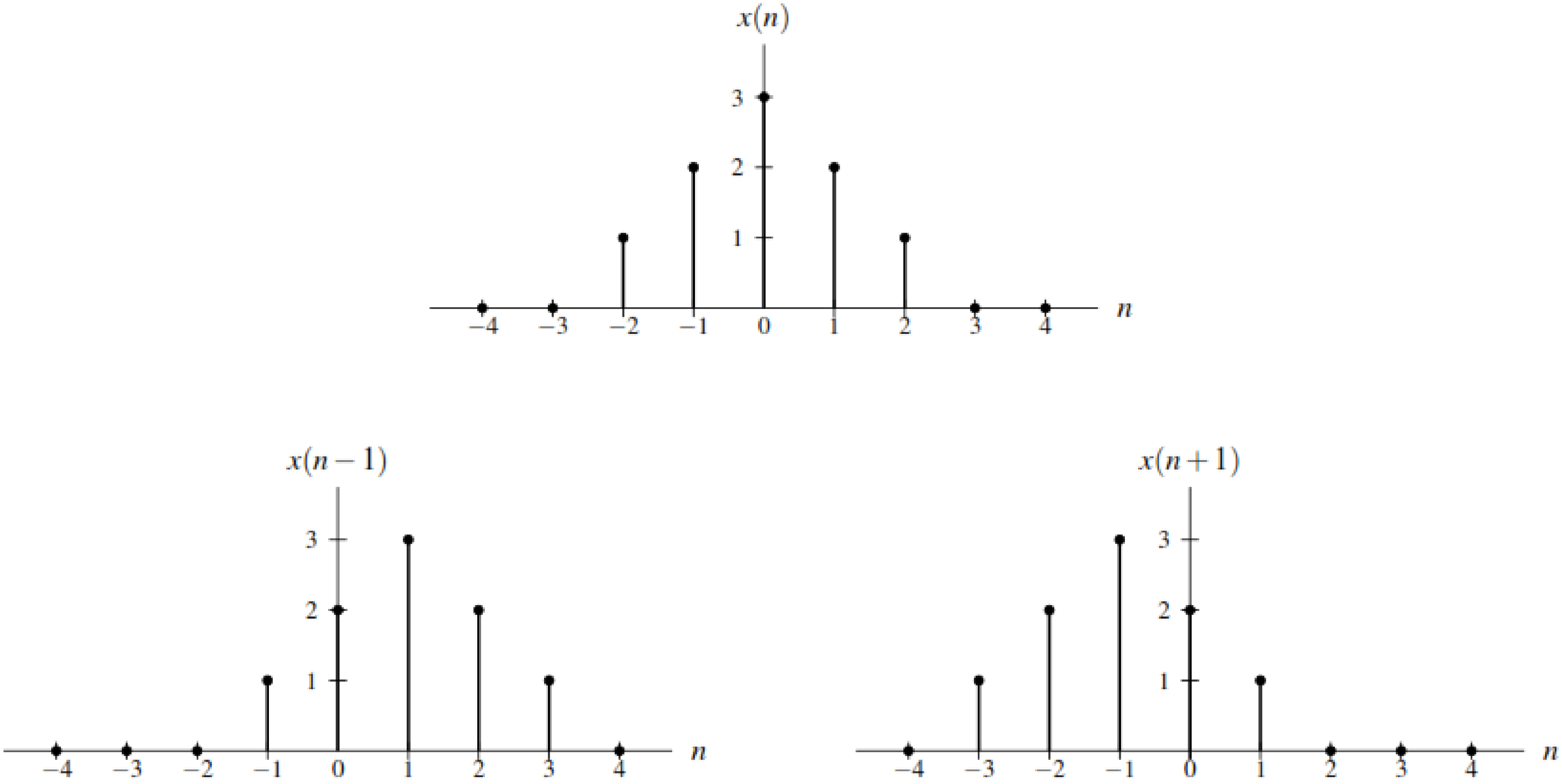
- **Time shifting** (also called **translation**) maps the input sequence x to the output sequence y as given by

$$y(n) = x(n-b),$$

where b is an integer.

- Such a transformation shifts the sequence (to the left or right) along the time axis.
- If $b > 0$, y is ***shifted to the right*** by $|b|$, relative to x (i.e., delayed in time).
- If $b < 0$, y is ***shifted to the left*** by $|b|$, relative to x (i.e., advanced in time).

TIME SHIFTING (TRANSLATION)

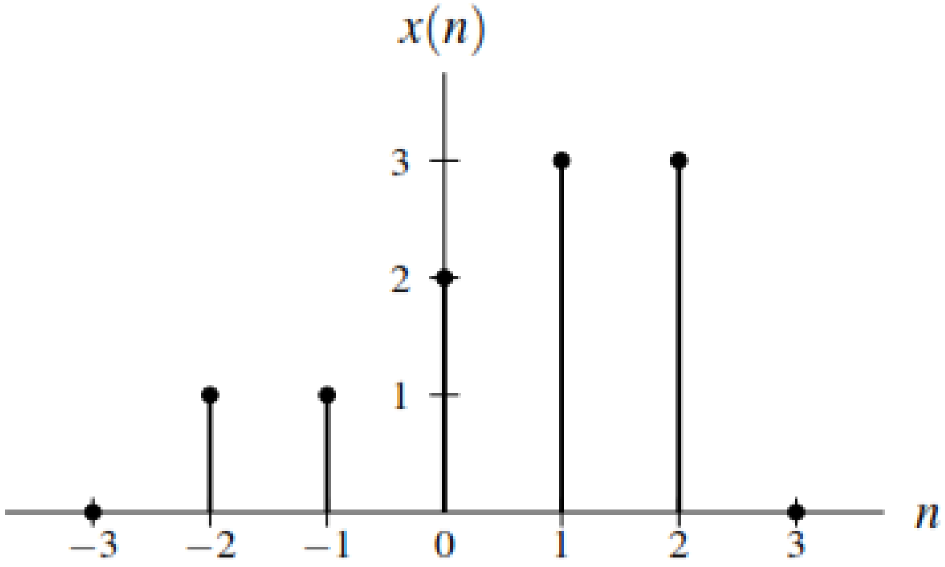


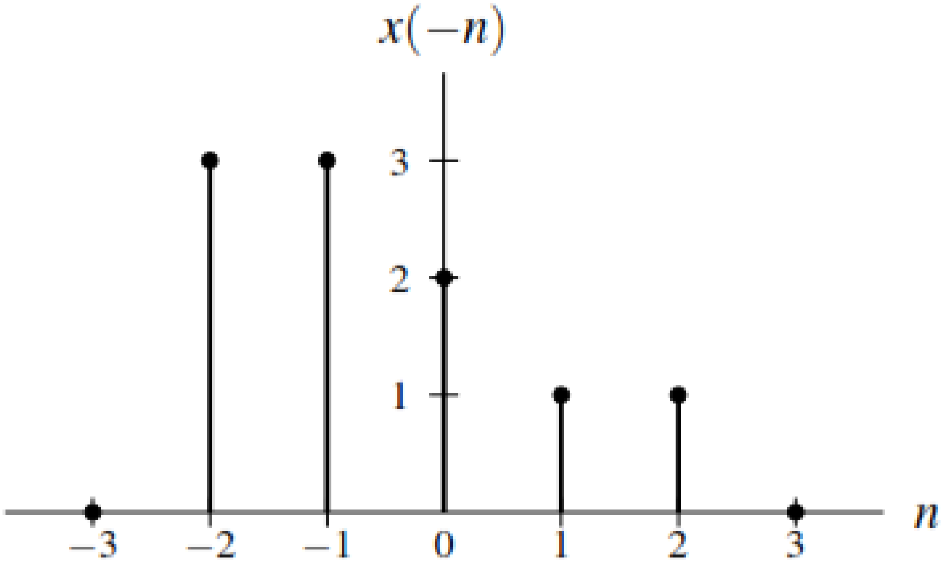
TIME REVERSAL (REFLECTION)

- **Time reversal** (also known as **reflection**) maps the input sequence x to the output sequence y as given by

$$y(n) = x(-n).$$

- Geometrically, the output sequence y is a reflection of the input sequence x about the (vertical) line $n = 0$.





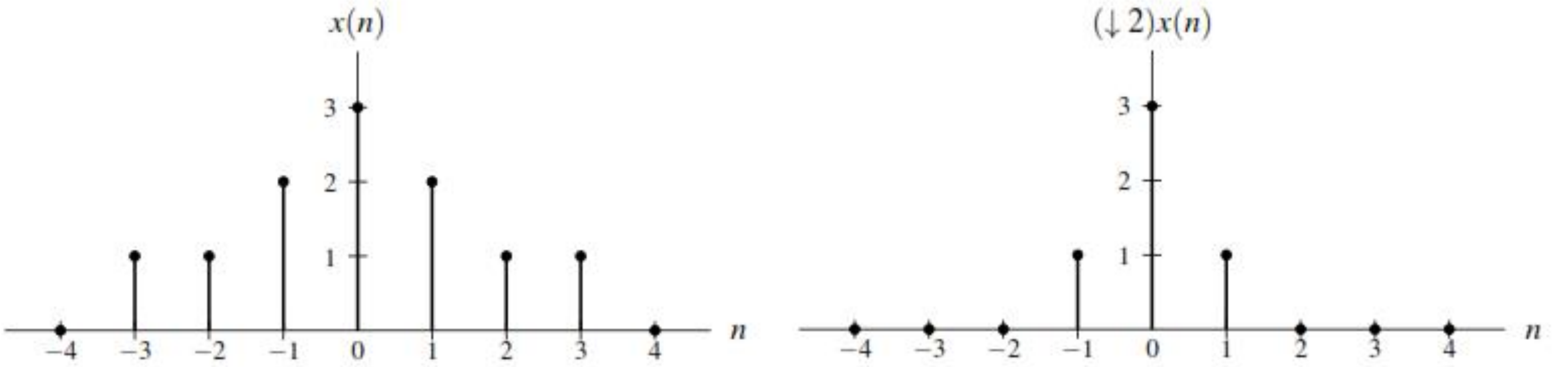
DOWNSAMPLING

- **Downsampling** maps the input sequence x to the output sequence y as given by

$$y(n) = (\downarrow a)x(n) = x(an),$$

  where a is a *strictly positive* integer.

- The output sequence y is produced from the input sequence x by keeping only every ath sample of x.



The constant a is referred to as the **downsampling factor**.

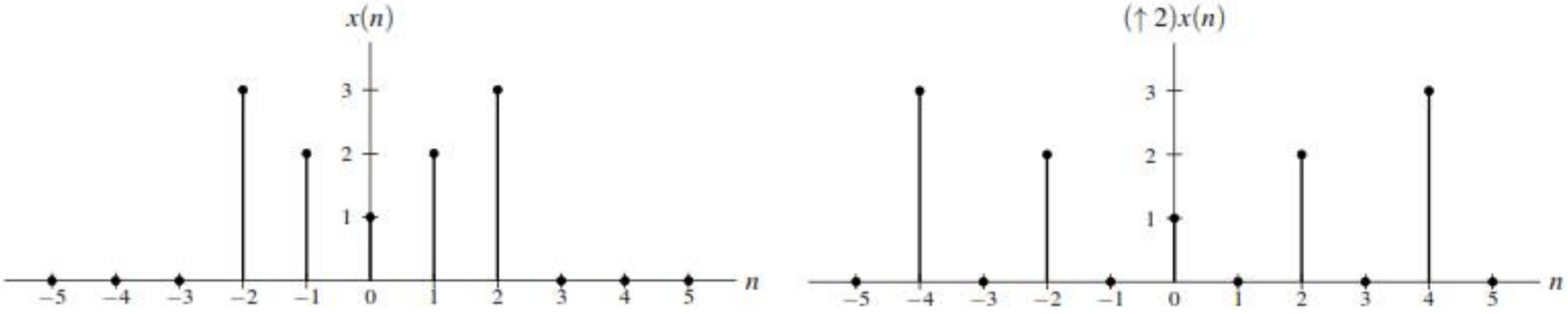
UPSAMPLING

- **Upsampling** maps the input sequence x to the output sequence y as given by

$$y(n) = (\uparrow a)x(n) = \begin{cases} x(n/a) & n/a \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$

where a is a strictly positive integer.

- The output sequence y is produced from the input sequence x by inserting $a-1$ zeros between all of the samples of x.



The constant a is referred to as the **upsampling factor**.

COMBINED INDEPENDENT-VARIABLE TRANSFORMATIONS

- Consider a transformation that maps the input sequence x to the output sequence y as given by

$$y(n) = x(an - b),$$

where a and b are integers and $a \neq 0$.

- Such a transformation is a combination of time shifting, downsampling, and time reversal operations.
- Time reversal ***commutes*** with downsampling.
- Time shifting ***does not commute*** with time reversal or downsampling.
- The above transformation is equivalent to:
  1. first, time shifting x by b;
  2. then, downsampling the result by $|a|$ and, if $a < 0$, time reversing as well.
- If $\frac{b}{a}$ is an integer, the above transformation is also equivalent to:
  1. first, downsampling x by $|a|$ and, if $a < 0$, time reversing;
  2. then, time shifting the result by $\frac{b}{a}$.
- Note that the time shift is not by the same amount in both cases.

DECOMPOSITION OF A SEQUENCE INTO EVEN AND ODD PARTS

- Every sequence x has a ***unique*** representation of the form

$$x(n) = x_{\mathrm{e}}(n) + x_{\mathrm{o}}(n),$$

  where the sequences x_{e} and x_{o} are ***even*** and ***odd***, respectively.

- In particular, the sequences x_{e} and x_{o} are given by

$$x_{\mathrm{e}}(n) = \tfrac{1}{2}\left[x(n) + x(-n)\right] \quad \text{and} \quad x_{\mathrm{o}}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

- The sequences x_{e} and x_{o} are called the **even part** and **odd part** of x, respectively.
- For convenience, the even and odd parts of x are often denoted as $\mathrm{Even}\{x\}$ and $\mathrm{Odd}\{x\}$, respectively.

THEOREM

Let x be an arbitrary N-periodic sequence x. Then, the following assertions hold:

1. *if x is even, then $x(n) = x(N - n)$ for all $n \in \mathbb{Z}$;*
2. *if x is odd, then $x(n) = -x(N - n)$ for all $n \in \mathbb{Z}$; and*
3. *if x is odd, then $x(0) = 0$ for both even and odd N, and $x\left(\frac{N}{2}\right) = 0$ for even N.*

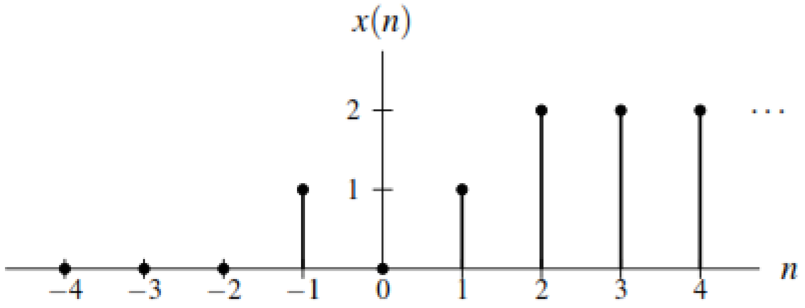
RIGHT SIDED SEQUENCES

- A sequence x is said to be **right sided** if, for some (finite) integer constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n < n_0$$

(i.e., x is ***only potentially nonzero to the right of*** n_0).

- An example of a right-sided sequence is shown below.



- A sequence x is said to be **causal** if

$$x(n) = 0 \quad \text{for all } n < 0.$$

- A causal sequence is a ***special case*** of a right-sided sequence.
- A causal sequence is not to be confused with a causal system. In these two contexts, the word "causal" has very different meanings.

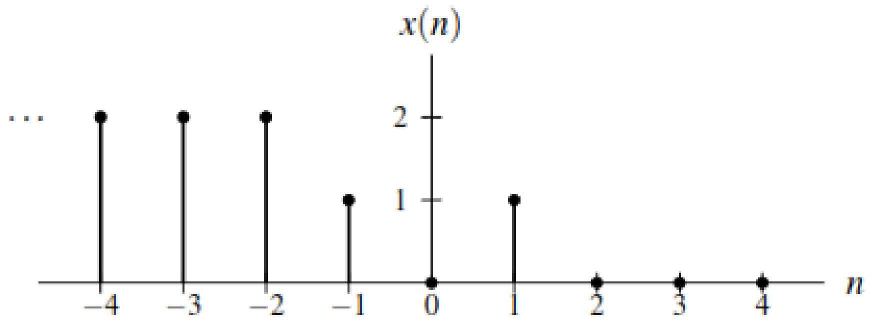
LEFT SIDED SEQUENCES

- A sequence x is said to be **left sided** if, for some (finite) integer constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n > n_0$$

(i.e., x is ***only potentially nonzero to the left of*** n_0).

- An example of a left-sided sequence is shown below.



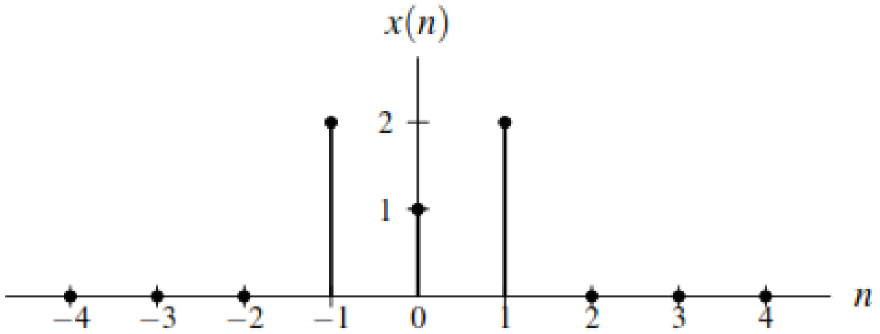
- A sequence x is said to be **anticausal** if

$$x(n) = 0 \quad \text{for all } n > 0.$$

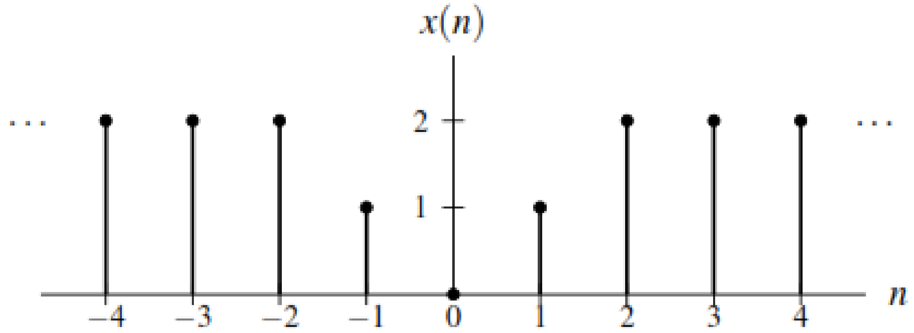
- An anticausal sequence is a ***special case*** of a left-sided sequence.
- An anticausal sequence is not to be confused with an anticausal system. In these two contexts, the word "anticausal" has very different meanings.

FINITE DURATION AND TWO SIDED SEQUENCES

- A sequence that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite-duration sequence is shown below.



- A sequence that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided sequence is shown below.



BOUNDED SEQUENCES

- A sequence x is said to be **bounded** if there exists some (*finite*) positive real constant A such that

$$|x(n)| \leq A \quad \text{for all } n$$

(i.e., $x(n)$ is ***finite*** for all n).

- Examples of bounded sequences include any constant sequence.
- Examples of unbounded sequences include any nonconstant polynomial sequence.

ENERGY OF A SEQUENCE

- The **energy** E contained in the sequence x is given by

$$E = \sum_{k=-\infty}^{\infty} |x(k)|^2 .$$

- A signal with finite energy is said to be an **energy signal**.

EXAMPLE

A sequence x has the following properties:

- $x(n) = n + 2$ for $-1 \leq n \leq 1$;
- $v_1(n) = x(n-1)$ is causal; and
- $v_2(n) = x(n+1)$ is even.

Find $x(n)$ for all n.

EXAMPLE

Solution. Since $v_1(n) = x(n-1)$ is causal, we have

$$\begin{aligned} & x(n-1) = 0 \ \text{ for } n < 0 \\ \Rightarrow \quad & x([n+1]-1) = 0 \ \text{ for } (n+1) < 0 \\ \Rightarrow \quad & x(n) = 0 \ \text{ for } n < -1. \end{aligned}$$

From this and the fact that $x(n) = n+2$ for $-1 \le n \le 1$, we have

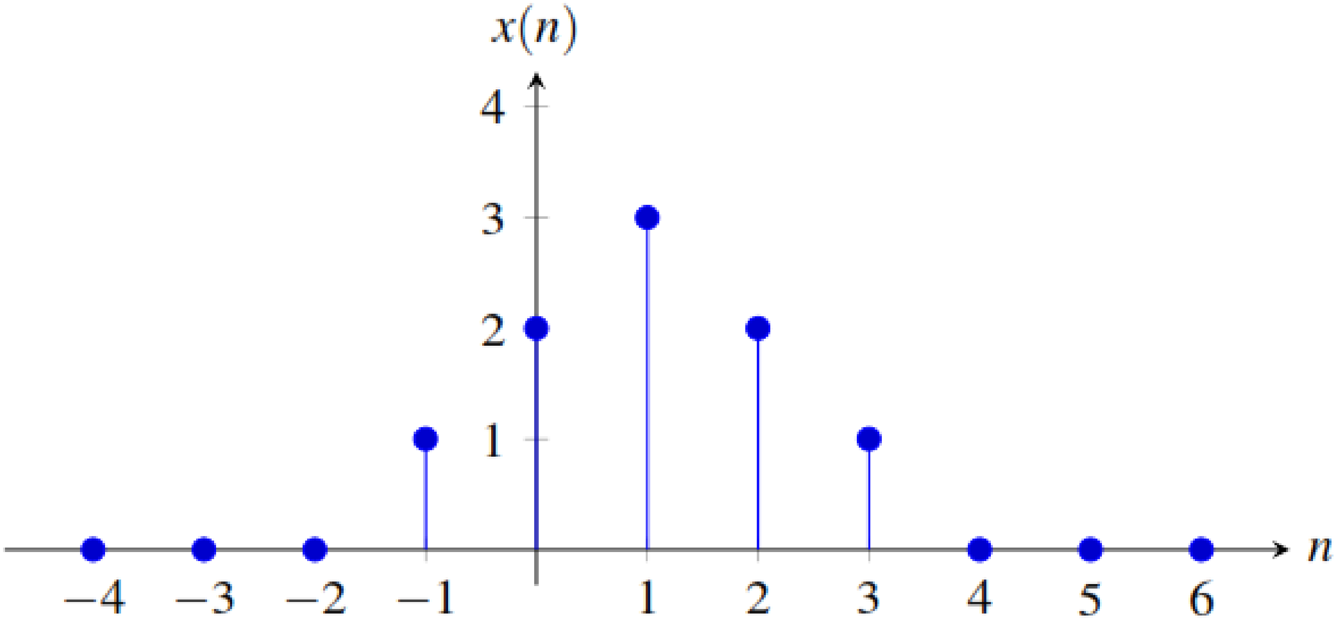
$$x(n) = \begin{cases} n+2 & -1 \le n \le 1 \\ 0 & n \le -2. \end{cases}$$

So, we only need to determine $x(n)$ for $n \ge 2$. Since $v_2(n) = x(n+1)$ is even, we have

$$\begin{aligned} & v_2(n) = v_2(-n) \\ \Rightarrow \quad & x(n+1) = x(-n+1) \\ \Rightarrow \quad & x([n-1]+1) = x(-[n-1]+1) \\ \Rightarrow \quad & x(n) = x(-n+2) \\ \Rightarrow \quad & x(n) = x(2-n). \end{aligned}$$

EXAMPLE

$$x(n) = x(2-n)$$
$$= \begin{cases} (2-n)+2 & -1 \le 2-n \le 1 \\ 0 & 2-n \le -2 \end{cases}$$
$$= \begin{cases} 4-n & -3 \le -n \le -1 \\ 0 & -n \le -4 \end{cases}$$
$$= \begin{cases} 4-n & 1 \le n \le 3 \\ 0 & n \ge 4. \end{cases}$$



Therefore, we conclude

$$x(n) = \begin{cases} 0 & n \le -2 \\ 2+n & n \in \{-1,0\} \\ 4-n & n \in \{1,2,3\} \\ 0 & n \ge 4. \end{cases}$$

The **least common multiple (LCM)** of two nonzero integers a and b, denoted $\text{lcm}(a, b)$, is the smallest positive integer that is divisible by both a and b.

The quantity $\text{lcm}(a, b)$ can be easily determined from a prime factorization of the integers a and b by taking the product of the highest power for each prime factor appearing in these factorizations.

The **greatest common divisor (GCD)** of two integers a and b, denoted $\text{gcd}(a, b)$, is the largest positive integer that divides both a and b, where at least one of a and b is nonzero.

The quantity $\text{gcd}(a, b)$ can be easily determined from a prime factorization of the integers a and b by taking the product of the lowest power for each prime factor appearing in these factorizations.

EXAMPLE

Find the LCM of each pair of integers given below.

(a) 20 and 6;
(b) 54 and 24;

Solution. (a) First, we write the prime factorizations of 20 and 6, which yields

$$20 = 2^2 \cdot 5^1 \quad \text{and} \quad 6 = 2^1 \cdot 3^1.$$

To obtain the LCM, we take the highest power of each prime factor in these two factorizations.

$$\begin{aligned} \operatorname{lcm}(20,6) &= 2^2 \cdot 3^1 \cdot 5^1 \\ &= 60. \end{aligned}$$

(b) Using a similar process as above, we have

$$\begin{aligned} \operatorname{lcm}(54,24) &= \operatorname{lcm}(2^1 \cdot 3^3, 2^3 \cdot 3^1) \\ &= 2^3 \cdot 3^3 \\ &= 216. \end{aligned}$$

EXAMPLE

Find the GCD of each pair of integers given below.

(a) 20 and 6;

(b) 54 and 24;

Solution. (a) First, we write the prime factorizations of 20 and 6, which yields

$$20 = 2^2 \cdot 5^1 \quad \text{and} \quad 6 = 2^1 \cdot 3^1.$$

To obtain the GCD, we take the lowest power of each prime factor in these two factorizations.

$$\begin{aligned} \gcd(20,6) &= 2^1 \cdot 3^0 \cdot 5^0 \\ &= 2. \end{aligned}$$

(b) Using a similar process as above, we have

$$\begin{aligned} \gcd(54,24) &= \gcd(2^1 \cdot 3^3, 2^3 \cdot 3^1) \\ &= 2^1 \cdot 3^1 \\ &= 6. \end{aligned}$$

SUM OF PERIODIC SEQUENCES

For any two periodic sequences x_1 and x_2 with periods N_1 and N_2, respectively, the sequence $x = x_1 + x_2$ is periodic with period $N = \operatorname{lcm}(N_1, N_2)$.

Proof. Since N is an integer multiple of both N_1 and N_2, we can write $N = k_1N_1$ and $N = k_2N_2$ for some positive integers k_1 and k_2. So, we can write

$$\begin{aligned}
x(n+N) &= x_1(n+N) + x_2(n+N) \\
&= x_1(n+k_1N_1) + x_2(n+k_2N_2) \\
&= x_1(n) + x_2(n) \\
&= x(n).
\end{aligned}$$

Thus, x is periodic with period N. ■

Unlike in the case of the sum of periodic functions, the sum of periodic sequences is always periodic.

EXAMPLE

The sequences $x_1(n) = \cos\left(\frac{\pi}{6}n\right)$ and $x_2(n) = \sin\left(\frac{2\pi}{45}n\right)$ have fundamental periods $N_1 = 12$ and $N_2 = 45$, Find the fundamental period N of the sequence $y = x_1 + x_2$.

EXAMPLE

The sequences $x_1(n) = \cos\left(\frac{\pi}{6}n\right)$ and $x_2(n) = \sin\left(\frac{2\pi}{45}n\right)$ have fundamental periods $N_1 = 12$ and $N_2 = 45$, Find the fundamental period N of the sequence $y = x_1 + x_2$.

Solution. We have

$$\begin{aligned} N &= \operatorname{lcm}(N_1, N_2) \\ &= \operatorname{lcm}(12, 45) \\ &= \operatorname{lcm}(2^2 \cdot 3, 3^2 \cdot 5) \\ &= 2^2 \cdot 3^2 \cdot 5 \\ &= 180. \end{aligned}$$

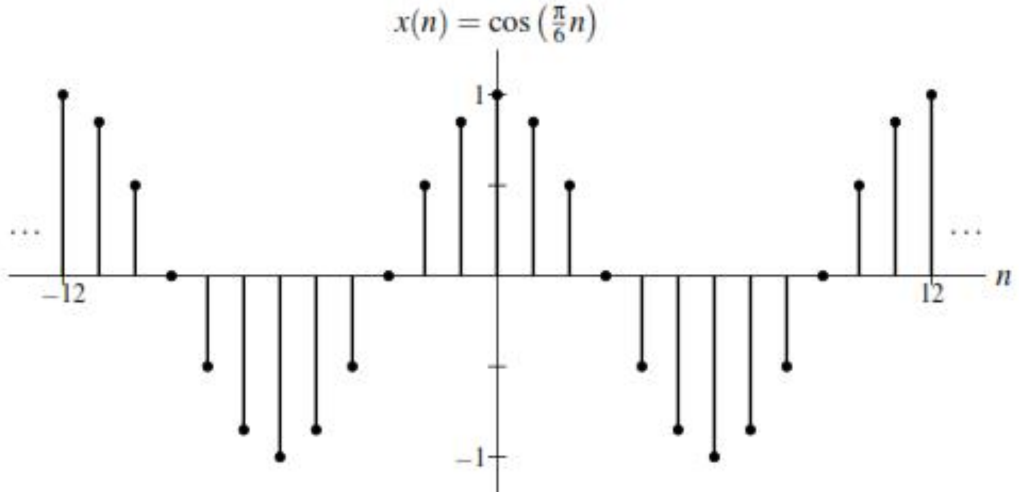
Elementary Sequences

REAL SINUSOIDAL SEQUENCES

- A **real sinusoidal sequence** is a sequence of the form

$$x(n) = A\cos(\Omega n + \theta),$$

  where A, Ω, and θ are ***real*** constants.
- A real sinusoid is ***periodic*** if and only if $\frac{\Omega}{2\pi}$ is a ***rational number***, in which case the fundamental period is the ***smallest integer*** of the form $\frac{2\pi k}{|\Omega|}$ where k is a (strictly) positive integer.
- For all integer k, $x_k(n) = A\cos([\Omega + 2\pi k]n + \theta)$ is the ***same*** sequence.
- An example of a periodic real sinusoid with fundamental period 12 is shown plotted below.



COMPLEX EXPONENTIAL SEQUENCES

- A **complex exponential sequence** is a sequence of the form

$$x(n) = ca^n,$$

where c and a are ***complex*** constants.

- Such a sequence can also be equivalently expressed in the form

$$x(n) = ce^{bn},$$

where b is a ***complex*** constant chosen as $b = \ln a$. (This this form is more similar to that presented for CT complex exponentials).

- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of the parameters c and a.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

COMMPLEX SINUSOIDAL SEQUENCES

- A complex sinusoidal sequence is a special case of a complex exponential $x(n) = ca^n$, where c and a are ***complex*** and $|a| = 1$ (i.e., a is of the form $e^{j\Omega}$ where Ω is real).
- That is, a **complex sinusoidal sequence** is a sequence of the form

$$x(n) = ce^{j\Omega n},$$

  where c is ***complex*** and Ω is ***real***.
- Using Euler's relation, we can rewrite $x(n)$ as

$$x(n) = \underbrace{|c|\cos(\Omega n + \arg c)}_{\mathrm{Re}\{x(n)\}} + j\underbrace{|c|\sin(\Omega n + \arg c)}_{\mathrm{Im}\{x(n)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are real sinusoids.
- A complex sinusoid is ***periodic*** if and only if $\frac{\Omega}{2\pi}$ is a ***rational number***, in which case the fundamental period is the ***smallest integer*** of the form $\frac{2\pi k}{|\Omega|}$ where k is a (strictly) positive integer.

COMMPLEX SINUSOIDAL SEQUENCES

$\Omega = \frac{2\pi\ell}{m}$ where ℓ and m are integers, x can be shown to have the fundamental period

$$N = \frac{m}{\gcd(\ell, m)}.$$

In the case that ℓ and m are coprime (i.e., have no common factors), $N = \frac{m}{\gcd(\ell,m)} = \frac{m}{1} = m$.

EXAMPLE

Determine if each sequence x given below is periodic, if it is, find its fundamental period.

(a) $x(n) = e^{j42n}$;
(b) $x(n) = e^{j(4\pi/11)n}$; and
(c) $x(n) = e^{j(\pi/3)n}$.

EXAMPLE

Solution. (a) Since $\frac{2\pi}{42} = \frac{\pi}{21}$ is not rational, x is not periodic.

(b) Since

$$(2\pi)/\left(\tfrac{4\pi}{11}\right) = (2\pi)\left(\tfrac{11}{4\pi}\right) = \tfrac{11}{2}$$

is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{11}{2}k$, where k is a strictly positive integer. Thus, $N = 11$ (corresponding to $k = 2$). Alternatively, the fundamental period N of $x(n) = e^{j(2\pi[2/11])n}$ is given by

$$N = \frac{11}{\gcd(11,2)} = \frac{11}{\gcd(11^1,2^1)} = \frac{11}{1} = 11.$$

(c) Since

$$(2\pi)/\left(\tfrac{\pi}{3}\right) = (2\pi)\left(\tfrac{3}{\pi}\right) = \tfrac{6}{1}$$

is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{6}{1}k$, where k is a strictly positive integer. Thus, $N = 6$ (corresponding to $k = 1$). Alternatively, the fundamental period N of $x(n) = e^{j(\pi/3)n} = e^{j(2\pi[1]/6)n}$ is given by

$$N = \frac{6}{\gcd(6,1)} = \frac{6}{\gcd(2^1 \cdot 3^1,1)} = \frac{6}{1} = 6.$$

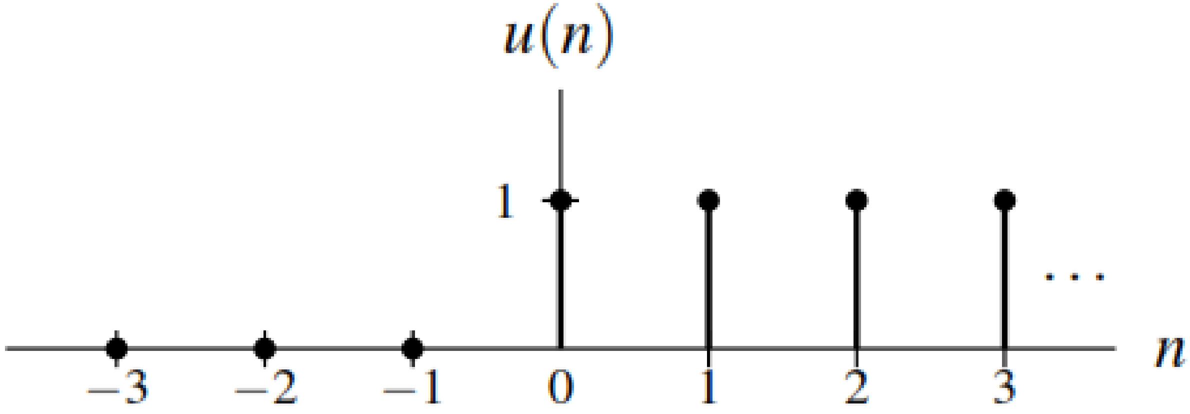
■

UNIT STEP SEQUENCE

- The **unit-step sequence**, denoted u, is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this sequence is shown below.



UNIT RECTANGULAR PULSES

- A **unit rectangular pulse** is a sequence of the form

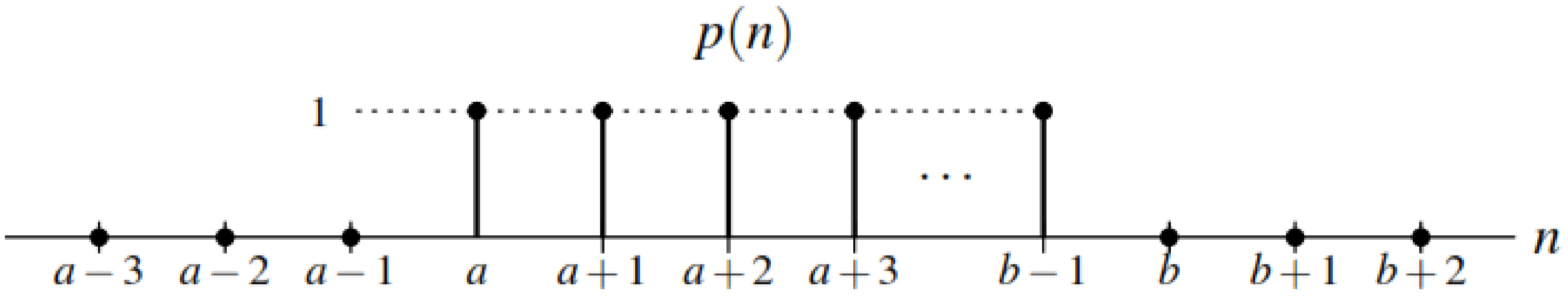
$$p(n) = \begin{cases} 1 & a \le n < b \\ 0 & \text{otherwise} \end{cases}$$

  where a and b are integer constants satisfying $a < b$.

- Such a sequence can be expressed in terms of the unit-step sequence as

$$p(n) = u(n-a) - u(n-b).$$

- The graph of a unit rectangular pulse has the general form shown below.



UNIT IMPULSE SEQUENCE

- The **unit-impulse sequence** (also known as the **delta sequence**), denoted δ, is defined as

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

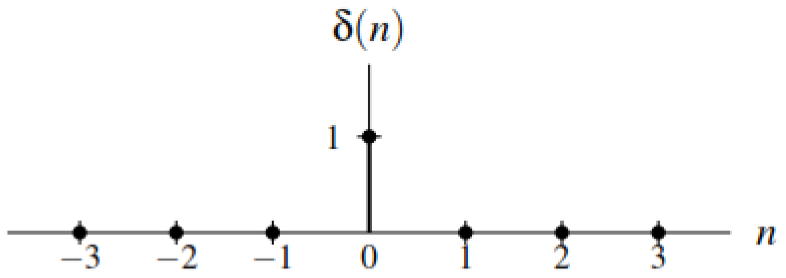
- The first-order difference of u is δ. That is,

$$\delta(n) = u(n) - u(n-1).$$

- The running sum of δ is u. That is,

$$u(n) = \sum_{k=-\infty}^{n} \delta(k).$$

- A plot of δ is shown below.



PROPERTIES OF THE UNIT IMPULSE SEQUENCES

- For any sequence x and any integer constant n_0, the following identity holds:

$$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0). \qquad \text{(Equivalence property).}$$

- For any sequence x and any integer constant n_0, the following identity holds:

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0). \qquad \text{(Sifting property).}$$

- Trivially, the sequence δ is also even.

EXAMPLE

Evaluate the summation

$$\sum_{n=-\infty}^{\infty} \sin\left(\tfrac{\pi}{2}n\right)\delta(n-1).$$

Solution. Using the sifting property of the unit impulse sequence, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \sin\left(\tfrac{\pi}{2}n\right)\delta(n-1) &= \sin\left(\tfrac{\pi}{2}n\right)\Big|_{n=1} \\ &= \sin\left(\tfrac{\pi}{2}\right) \\ &= 1.\end{aligned}$$

REPRESENTING RECTANGULAR PULSES

- For integer constants a and b where $a < b$, consider a sequence x of the form

$$x(n) = \begin{cases} 1 & a \leq n < b \\ 0 & \text{otherwise} \end{cases}$$

(i.e., x is a ***rectangular pulse*** of height one that is nonzero from a to $b-1$ inclusive).

- The sequence x can be equivalently written as

$$x(n) = u(n-a) - u(n-b)$$

(i.e., the difference of two time-shifted unit-step sequences).

- Unlike the original expression for x, this latter expression for x ***does not involve multiple cases***.
- In effect, by using unit-step sequences, we have collapsed a formula involving multiple cases into a single expression.

EXAMPLE

Consider the piecewise-linear sequence x given by

$$x(n) = \begin{cases} n+7 & -6 \leq n \leq -4 \\ 4 & -3 \leq n \leq 2 \\ 6-n & 3 \leq n \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(n)$ (involving unit-step sequences) that is valid for all n.

EXAMPLE

Solution.

$$v_1(n) = (n+7)[u(n+6) - u(n+3)].$$

$$v_2(n) = 4[u(n+3) - u(n-3)].$$

$$v_3(n) = (6-n)[u(n-3) - u(n-6)].$$

$$\begin{aligned}
x(n) &= v_1(n) + v_2(n) + v_3(n) \\
&= (n+7)[u(n+6) - u(n+3)] + 4[u(n+3) - u(n-3)] + (6-n)[u(n-3) - u(n-6)] \\
&= (n+7)u(n+6) - (n+7)u(n+3) + 4u(n+3) - 4u(n-3) + (6-n)u(n-3) - (6-n)u(n-6) \\
&= (n+7)u(n+6) + (-n-3)u(n+3) + (2-n)u(n-3) + (n-6)u(n-6).
\end{aligned}$$

Discrete-Time (DT) Systems

PROPERTIES OF DT SYSTEM

- Memory
- Causality
- Invertibility
- BIBO Stability
- Time Invariance
- Linearity

Discrete-Time Linear Time-Invariant (LTI) Systems

Convolution

DT CONVOLUTION

- The (DT) **convolution** of the sequences x and h, denoted $x * h$, is defined as the sequence

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

- The convolution $x * h$ evaluated at the point n is simply a weighted sum of elements of x, where the weighting is given by h time reversed and shifted by n.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in the theory of (DT) systems.
- In particular, convolution has a special significance in the context of (DT) LTI systems.

DT CONVOLUTION

- To compute the convolution

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

we proceed as follows:

1. Plot $x(k)$ and $h(n-k)$ as a function of k.
2. Initially, consider an arbitrarily large negative value for n. This will result in $h(n-k)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $x * h(n)$.
4. Increase n gradually until the expression for $x * h(n)$ changes form. Record the interval over which the expression for $x * h(n)$ was valid.
5. Repeat steps 3 and 4 until n is an arbitrarily large positive value. This corresponds to $h(n-k)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $x * h(n)$ for all n.

PROPERTIES OF DT CONVOLUTION

- The convolution operation is ***commutative***. That is, for any two sequences x and h,

$$x * h = h * x.$$

- The convolution operation is ***associative***. That is, for any sequences x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is ***distributive*** with respect to addition. That is, for any sequences x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

REPRESENTATION OF SEQUENCES USING IMPULSES

- For any sequence x,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = x * \delta(n).$$

- Thus, any sequence x can be written in terms of an expression involving δ.
- Moreover, δ is the ***convolutional identity***. That is, for any sequence x,

$$x * \delta = x.$$

CIRCULAR CONVOLUTION

- The convolution of two periodic sequences is usually not well defined.
- This motivates an alternative notion of convolution for periodic sequences known as circular convolution.
- The **circular convolution** (also known as the DT periodic convolution) of the N-periodic sequences x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(n) = \sum_{k=\langle N \rangle} x(k)h(n-k) = \sum_{k=0}^{N-1} x(k)h(\operatorname{mod}(n-k,N)),$$

where $\operatorname{mod}(a,b)$ is the remainder after division when a is divided by b.

- The circular convolution and (linear) convolution of the N-periodic sequences x and h are related as follows:

$$x \circledast h(n) = x_0 * h(n) \quad \text{where} \quad x(n) = \sum_{k=-\infty}^{\infty} x_0(n-kN)$$

(i.e., $x_0(n)$ equals $x(n)$ over a single period of x and is zero elsewhere).

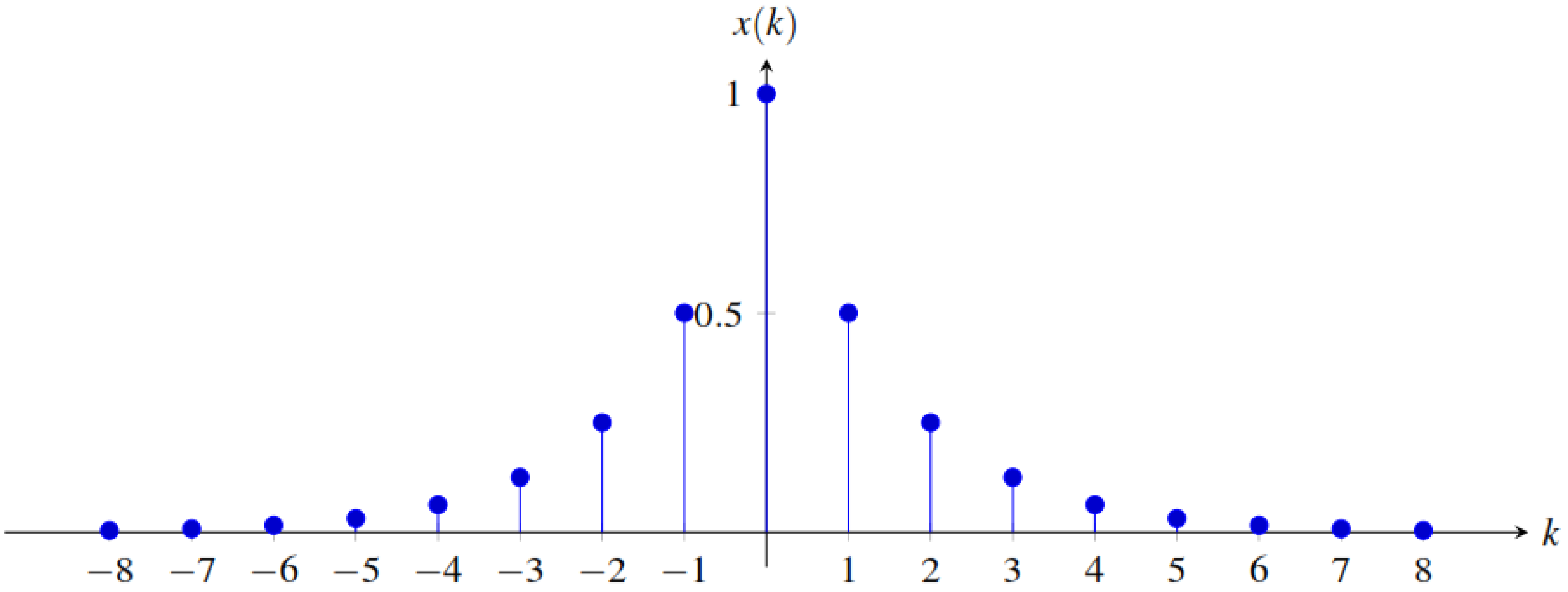
EXAMPLE

Compute $x * h$, where

$$x(n) = 2^{-|n|} \quad \text{and} \quad h(n) = u(n-2).$$

EXAMPLE

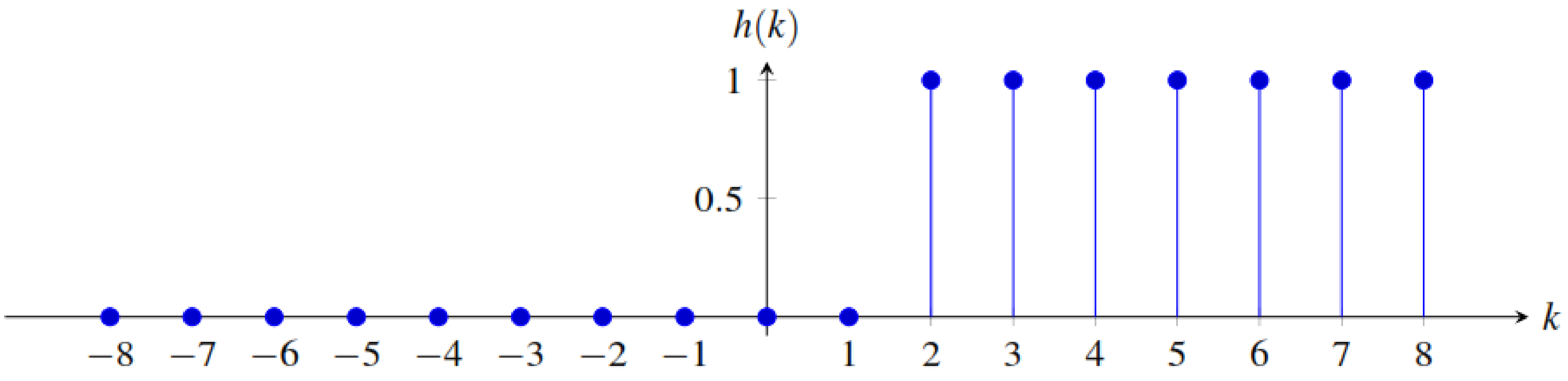
Solution.



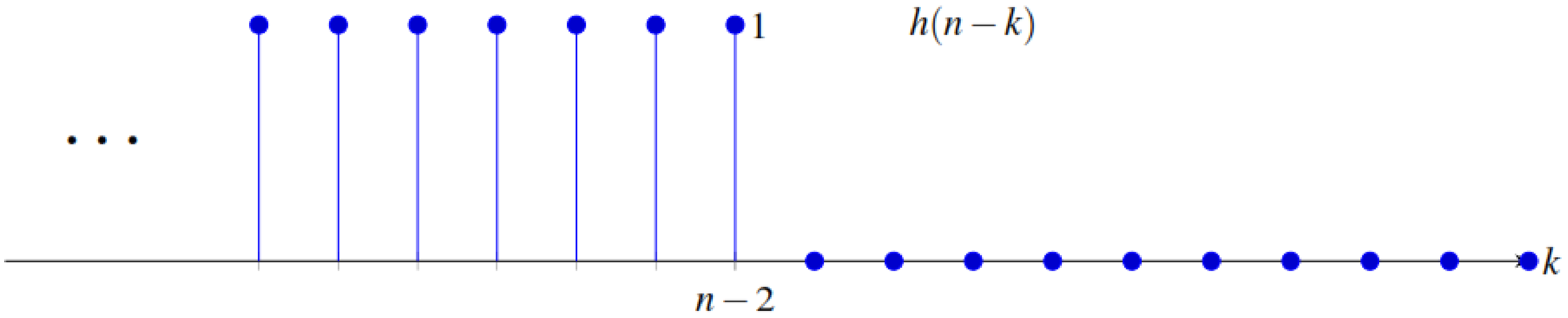
(a)

EXAMPLE

Solution.



(b)



(c)

EXAMPLE

Solution. From the definition of convolution, we have

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \sum_{k=-\infty}^{\infty} 2^{-|k|}u(n-k-2).$$

Since $u(n-k-2) = 0$ for $k > n-2$, we can write

$$x * h(n) = \sum_{k=-\infty}^{n-2} 2^{-|k|} \quad = \begin{cases} \sum_{k=-\infty}^{n-2} 2^{k} & n-2 \leq 0 \\ \sum_{k=-\infty}^{0} 2^{k} + \sum_{k=1}^{n-2} 2^{-k} & n-2 > 0 \end{cases}$$

$$= \begin{cases} \sum_{k=-\infty}^{n-2} 2^{k} & n \leq 2 \\ \sum_{k=-\infty}^{0} 2^{k} + \sum_{k=1}^{n-2} 2^{-k} & n > 2. \end{cases}$$

EXAMPLE

$$
\begin{aligned}
\sum_{k=-\infty}^{n-2} 2^k &= \sum_{k=2-n}^{\infty} \left(\tfrac{1}{2}\right)^k = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{k+2-n} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{2-n} \left(\tfrac{1}{2}\right)^k \\
&= \frac{\left(\tfrac{1}{2}\right)^{2-n}}{1-\tfrac{1}{2}} = \left(\tfrac{1}{2}\right)^{2-n} \left(\tfrac{1}{2}\right)^{-1} = \left(\tfrac{1}{2}\right)^{1-n} \\
&= 2^{n-1},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=-\infty}^{0} 2^k &= \sum_{k=0}^{\infty} 2^{-k} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \\
&= \frac{1}{1-\tfrac{1}{2}} = \frac{1}{\tfrac{1}{2}} = (1)(2) \\
&= 2, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{n-2} 2^{-k} &= \sum_{k=0}^{n-3} 2^{-(k+1)} = \sum_{k=0}^{n-3} \left(\tfrac{1}{2}\right) \left(\tfrac{1}{2}\right)^k \\
&= \left(\tfrac{1}{2}\right) \left(\frac{\left(\tfrac{1}{2}\right)^{n-2} - 1}{\tfrac{1}{2} - 1} \right) = \left(\tfrac{1}{2}\right) \left(\frac{\left(\tfrac{1}{2}\right)^{n-2} - 1}{-\tfrac{1}{2}} \right) \\
&= 1 - \left(\tfrac{1}{2}\right)^{n-2}.
\end{aligned}
$$

EXAMPLE

Substituting these simplified expressions into the earlier formula for $x * h$ yields

$$x * h(n) = \begin{cases} 2^{n-1} & n \leq 2 \\ 3 - \left(\frac{1}{2}\right)^{n-2} & n > 2. \end{cases}$$

Convolution and LTI Systems

IMPULSE RESPONSE

- The response h of a system $\mathcal{H}$ to the input δ is called the **impulse response** of the system (i.e., $h = \mathcal{H}\delta$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply ***computes a convolution***.
- Furthermore, a LTI system is ***completely characterized*** by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.

STEP RESPONSE

- The response s of a system $\mathcal{H}$ to the input u is called the **step response** of the system (i.e., $s = \mathcal{H}u$).
- The impulse response h and step response s of a system are related as

$$h(n) = s(n) - s(n-1).$$

- Therefore, the impulse response of a system can be determined from its step response by (first-order) differencing.

Properties of LTI Systems

MEMORY

- A LTI system with impulse response h is memoryless if and only if

$$h(n) = 0 \quad \text{for all } n \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(n) = K\delta(n),$$

where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

(i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

CAUSALITY

- A LTI system with impulse response h is causal if and only if

$$h(n) = 0 \quad \text{for all } n < 0$$

(i.e., h is a causal sequence).

- It is due to the above relationship that we call a sequence x, satisfying

$$x(n) = 0 \quad \text{for all } n < 0,$$

a causal sequence.

INVERTIBILITY

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{\text{inv}} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a sequence h_{inv} such that

$$h * h_{\text{inv}} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

BIBO STABILITY

- A LTI system with impulse response h is BIBO stable if and only if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

(i.e., h is ***absolutely summable***).

EIGENSEQUENCES OF LTI SYSTEMS

- As it turns out, every complex exponential is an eigensequence of all LTI systems.
- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{z^n\}(n) = H(z)z^n,$$

where z is a complex constant and

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}.$$

- That is, z^n is an eigensequence of a LTI system and $H(z)$ is the corresponding eigenvalue.
- We refer to H as the **system function** (or **transfer function**) of the system $\mathcal{H}$.
- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(z)$.

REPRESENTATION OF SEQUENCES USING EIGENSEQUENCES

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials

$$x(n) = \sum_k a_k z_k^n,$$

where the a_k and z_k are complex constants.
- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(n) = \sum_k a_k H(z_k) z_k^n.$$

- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as linear combination of the *same* complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Discrete-Time Fourier Series (DTFS)

INTRODUCTION

- The Fourier series is a representation for ***periodic*** sequences.
- With a Fourier series, a sequence is represented as a ***linear combination of complex sinusoids***.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- Perhaps, most importantly, complex sinusoids are ***eigensequences*** of (DT) LTI systems.

HARMONICALLY RELATED COMPLEX SINUSOIDS

- A set of periodic complex sinusoids is said to be **harmonically related** if there exists some constant $\frac{2\pi}{N}$ such that the fundamental frequency of each complex sinusoid is an integer multiple of $\frac{2\pi}{N}$.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(n) = e^{j(2\pi/N)kn} \quad \text{for all integer } k.$$

- In the above set $\{\phi_k\}$, only N elements are distinct, since

$$\phi_k = \phi_{k+N} \quad \text{for all integer } k.$$

- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of $\frac{2\pi}{N}$, a linear combination of these complex sinusoids must be N-periodic.

DT FOURIER SERIES

- An N-periodic complex-valued sequence x can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn},$$

  where $\sum_{k=\langle N \rangle}$ denotes summation over any N consecutive integers (e.g., $[0..N-1]$). (The summation can be taken over any N consecutive integers, due to the N-periodic nature of x and $e^{j(2\pi/N)kn}$.)

- The above representation of x is known as the (DT) **Fourier series** and the a_k are called **Fourier series coefficients**.
- The above formula for x is often called the **Fourier series synthesis equation**.
- To denote that the sequence x has the Fourier series coefficient sequence a, we write

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k.$$

- A periodic sequence x with fundamental period N has the Fourier series coefficient sequence a given by

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

(The summation can be taken over any N consecutive integers due to the N-periodic nature of x and $e^{-j(2\pi/N)kn}$.)

- The above equation for a_k is often referred to as the **Fourier series analysis equation**.
- Due to the N-periodic nature of x and $e^{-j(2\pi/N)kn}$, the sequence a is also N-periodic.

Properties of Fourier Series

PROPERTIES OF DT FOURIER SERIES

$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k$ and $y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(n) + \beta y(n)$	$\alpha a_k + \beta b_k$
Translation	$x(n - n_0)$	$e^{-jk(2\pi/N)n_0} a_k$
Modulation	$e^{j(2\pi/N)k_0 n} x(n)$	a_{k-k_0}
Reflection	$x(-n)$	a_{-k}
Conjugation	$x^*(n)$	a^*_{-k}
Duality	a_n	$\frac{1}{N} x(-k)$
Periodic Convolution	$x \circledast y(n)$	$N a_k b_k$
Multiplication	$x(n)y(n)$	$a \circledast b_k$

Property	
Parseval's Relation	$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x(n) \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

PARSEVAL'S RELATION

- A sequence x and its Fourier series coefficient sequence a satisfy the following relationship:

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2=\sum_{k=\langle N\rangle}|a_k|^2.$$

- The above relationship is simply stating that the amount of energy in a single period of x and the amount of energy in a single period of a are equal up to a scale factor.
- In other words, the transformation between a sequence and its Fourier series coefficient sequence preserves energy (up to a scale factor).

TRIGONOMETRIC FORM OF A FOURIER SERIES

- Consider the N-periodic sequence x with Fourier series coefficient sequence a.
- If x is real, then its Fourier series can be rewritten in trigonometric form as shown below.
- The **trigonometric form** of a Fourier series has the appearance

$$x(n) = \begin{cases} \alpha_0 + \sum_{k=1}^{N/2-1} \left[\alpha_k \cos\left(\frac{2\pi kn}{N}\right) + \beta_k \sin\left(\frac{2\pi kn}{N}\right)\right] + \\ \quad \alpha_{N/2} \cos(\pi n) & N \text{ even} \\ \alpha_0 + \sum_{k=1}^{(N-1)/2} \left[\alpha_k \cos\left(\frac{2\pi kn}{N}\right) + \beta_k \sin\left(\frac{2\pi kn}{N}\right)\right] & N \text{ odd,} \end{cases}$$

where $\alpha_0 = a_0$, $\alpha_{N/2} = a_{N/2}$, $\alpha_k = 2\operatorname{Re} a_k$, and $\beta_k = -2\operatorname{Im} a_k$.

- Note that the above trigonometric form contains only ***real*** quantities.