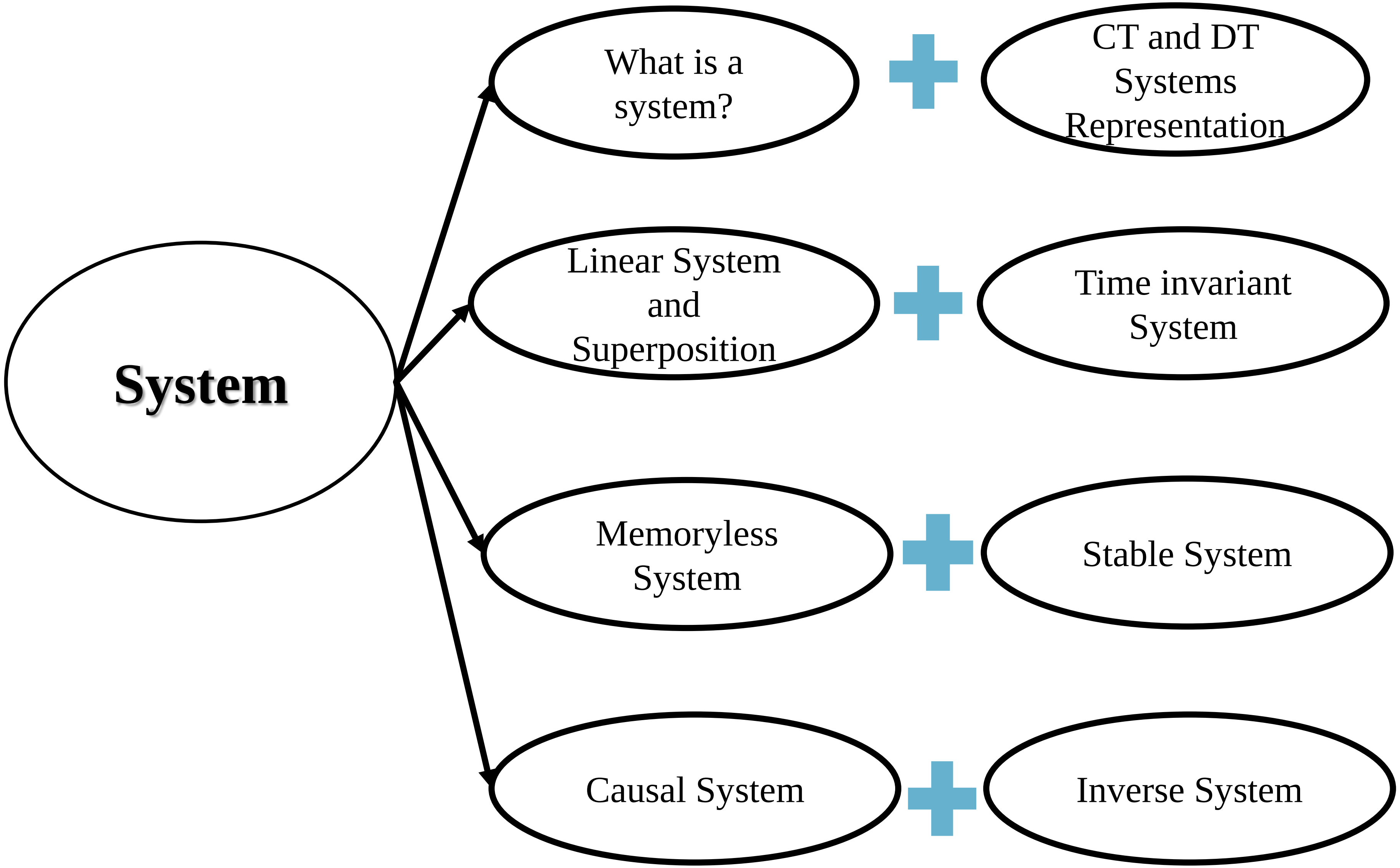
Chapter 2
System Properties

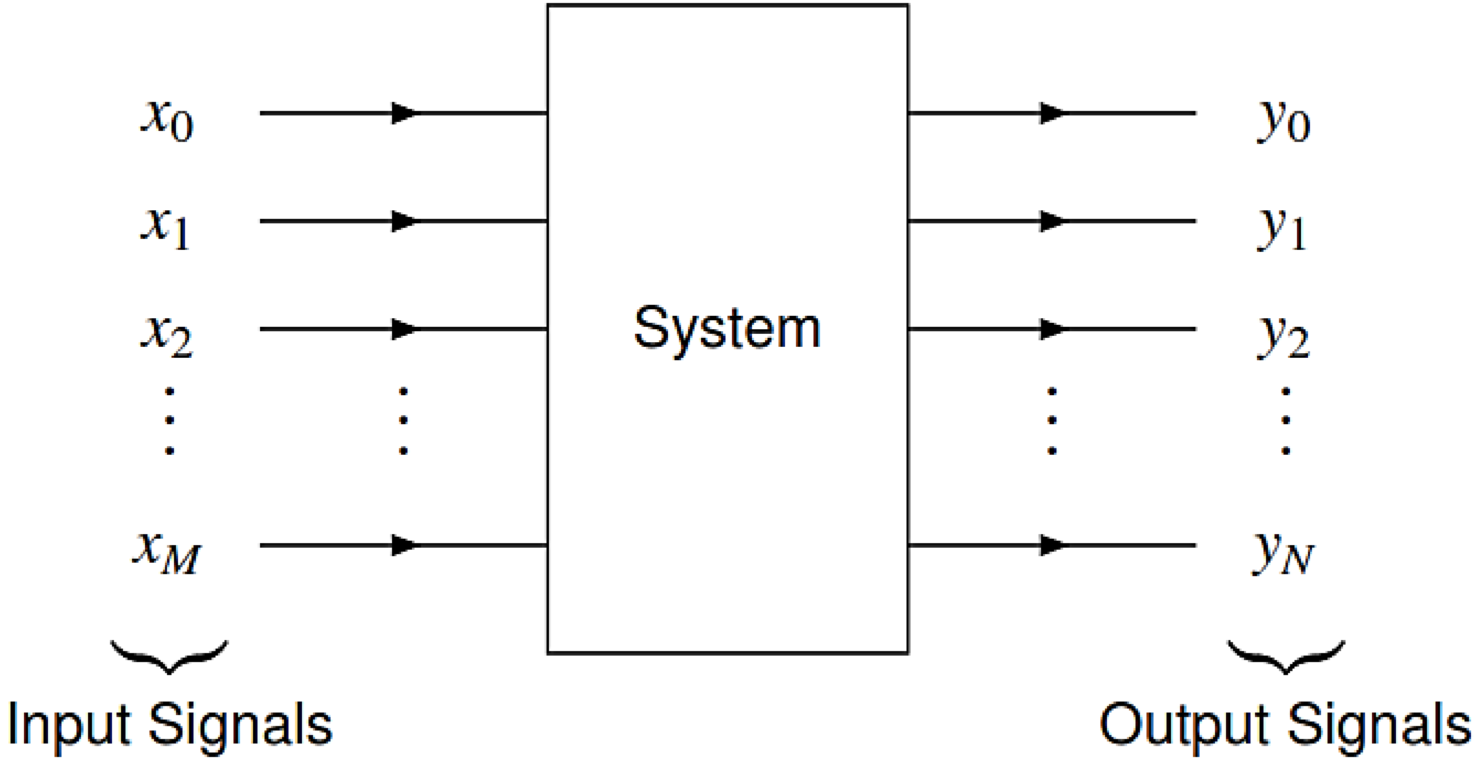
CONTENTS



System
What is a system?
CT and DT Systems Representation
Linear System and Superposition
Time invariant System
Memoryless System
Stable System
Causal System
Inverse System

CT SYSTEM

- A **system** is an entity that processes one or more input signals in order to produce one or more output signals.



CT SYSTEM

- A system with input x and output y can be described by the equation

$$y = \mathcal{H}x,$$

  where $\mathcal{H}$ denotes an operator (i.e., transformation).
- Note that the operator $\mathcal{H}$ ***maps a function to a function*** (not a number to a number).
- Alternatively, we can express the above relationship using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

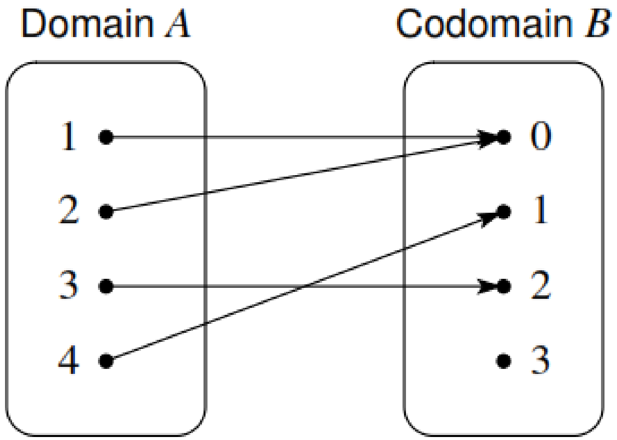
- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \rightarrow y.$$

- Note that the symbols "$\rightarrow$" and "=" have ***very different*** meanings.
- The symbol "$\rightarrow$" should be read as "***produces***" (not as "equals").

MATH REVIEW

- A **mapping** is a relationship involving two sets that associates each element in one set, called the **domain**, with an element from the other set, called the **codomain**.
- The notation $f : A \to B$ denotes a mapping f whose domain is the set A and whose codomain is the set B.
- Example:



$$f : A \to B$$
$$A = \{1, 2, 3, 4\}$$
$$B = \{0, 1, 2, 3\}$$
$$f(x) = \begin{cases} 0 & x \in \{1, 2\} \\ 1 & x = 4 \\ 2 & x = 3. \end{cases}$$

- Although many types of mappings exist, the types of most relevance to our study of signals and systems are: functions, sequences, system operators, and transforms.

MATH REVIEW

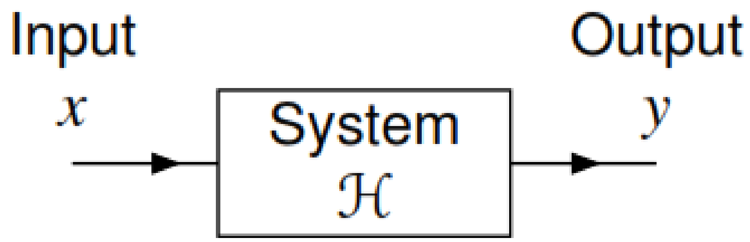
- A **system operator** is a mapping used to represent a system.
- We will focus exclusively on the case of single-input single-output systems.
- A (single-input single-output) **system operator** maps a function or sequence representing the input of a system to a function or sequence representing the output of the system.
- The domain and codomain of a system operator are sets of ***functions or sequences***, not sets of numbers.
- Example:
  - Let $\mathcal{H} : F \to F$ such that $\mathcal{H}x(t) = 2x(t)$ (for all $t \in \mathbb{R}$) and F is the set of functions mapping $\mathbb{R}$ to $\mathbb{R}$.
  - The system $\mathcal{H}$ maps a function to a function.
  - In particular, the domain and codomain are each F, which is a set of functions.
  - The system $\mathcal{H}$ multiplies its input function x by a factor of 2 in order to produce its output function $\mathcal{H}x$.
  - Note that $\mathcal{H}x$ is a function, not a number.

MATH REVIEW

- For a system operator $\mathcal{H}$ and a function x, $\mathcal{H}x$ is the function produced as the output of the system $\mathcal{H}$ when the input is the function x.
- Brackets around the operand of an operator are ***often omitted when not required*** for grouping.
- For example, for an operator $\mathcal{H}$, a function x, and a real number t, we would normally prefer to write:
  1. $\mathcal{H}x$ instead of the equivalent expression $\mathcal{H}(x)$; and
  2. $\mathcal{H}x(t)$ instead of the equivalent expression $\mathcal{H}(x)(t)$.
- Also, note that $\mathcal{H}x$ is a ***function*** and $\mathcal{H}x(t)$ is a ***number*** (namely, the value of the function $\mathcal{H}x$ evaluated at t).
- In the expression $\mathcal{H}(x_1+x_2)$, the brackets are needed for grouping, since $\mathcal{H}(x_1+x_2) \not\equiv \mathcal{H}x_1+x_2$ (where "$\not\equiv$" means "not equivalent").
- When multiple operators are applied, they group from ***right to left***.
- For example, for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$, and the function x, the expression $\mathcal{H}_2\mathcal{H}_1x$ means $\mathcal{H}_2[\mathcal{H}_1(x)]$.

BLOCK DIAGRAM REPRESENTATIONS

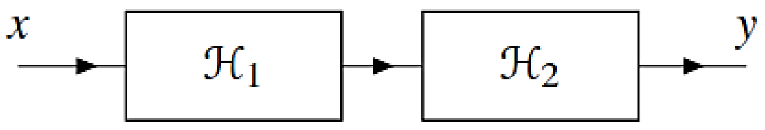
- Often, a system defined by the operator $\mathcal{H}$ and having the input x and output y is represented in the form of a ***block diagram*** as shown below.



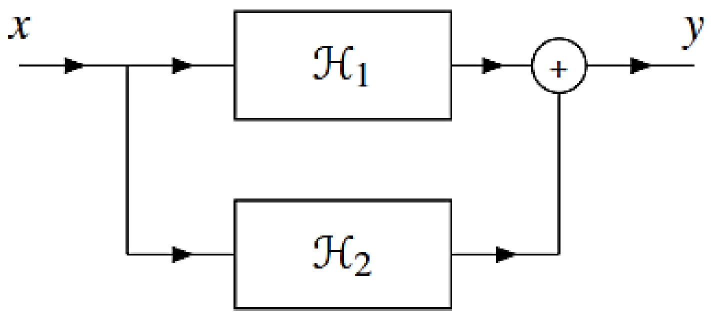
- Number of inputs:
  - A system with ***one*** input is said to be **single input (SI)**.
  - A system with ***more than one*** input is said to be **multiple input (MI)**.
- Number of outputs:
  - A system with ***one*** output is said to be **single output (SO)**.
  - A system with ***more than one*** output is said to be **multiple output (MO)**.

INTERCONNECTION OF SYSTEMS

- *Two basic ways* in which systems can be interconnected are shown below.



Series



Parallel

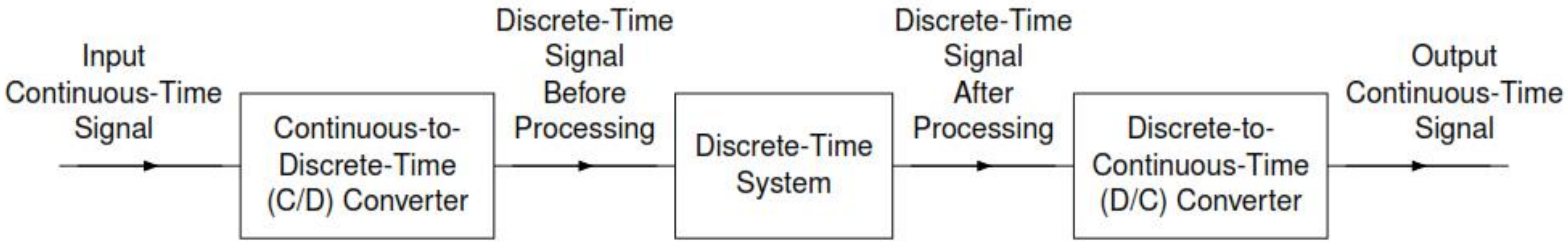
- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = \mathcal{H}_2\mathcal{H}_1 x.$$

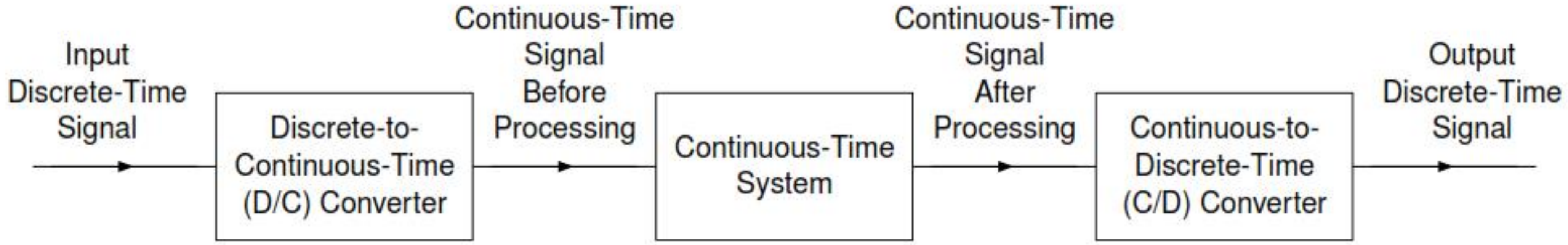
- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x.$$

SIGNAL PROCESSING SYSTEMS

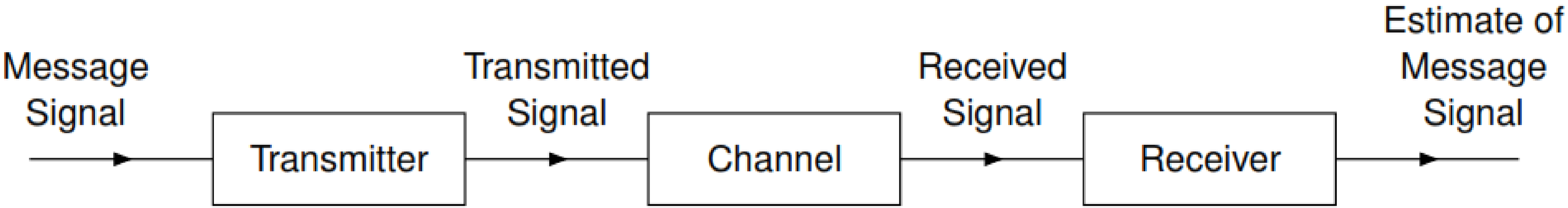


Processing a Continuous-Time Signal With a Discrete-Time System



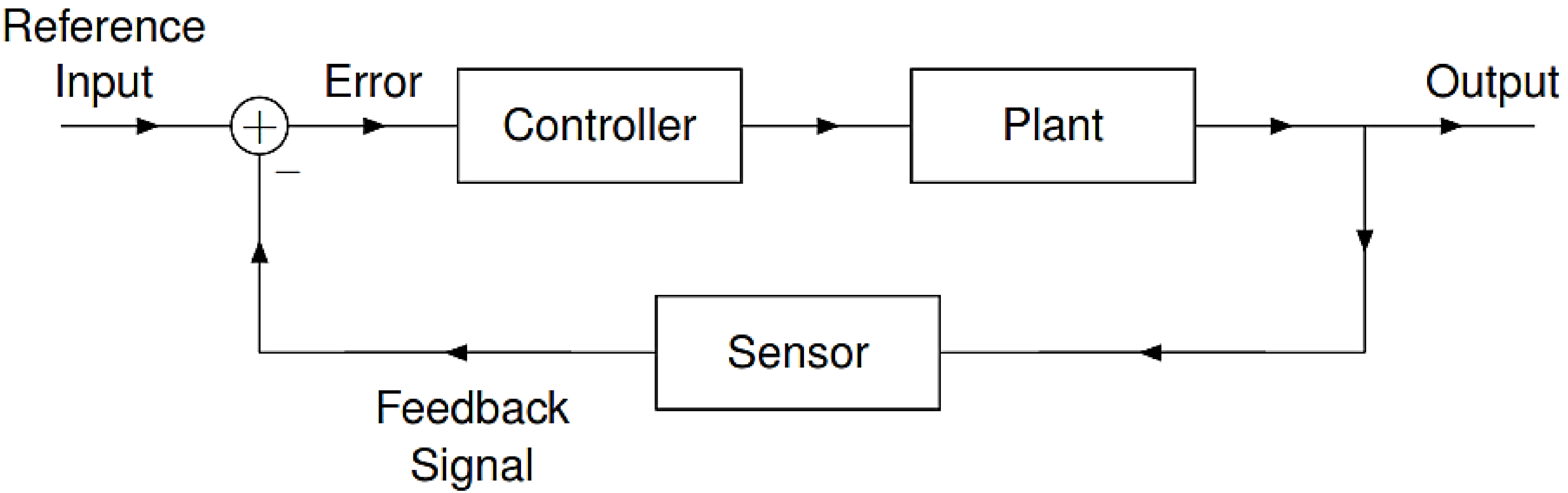
Processing a Discrete-Time Signal With a Continuous-Time System

COMMUNICATION SYSTEMS



General Structure of a Communication System

CONTROL SYSTEMS



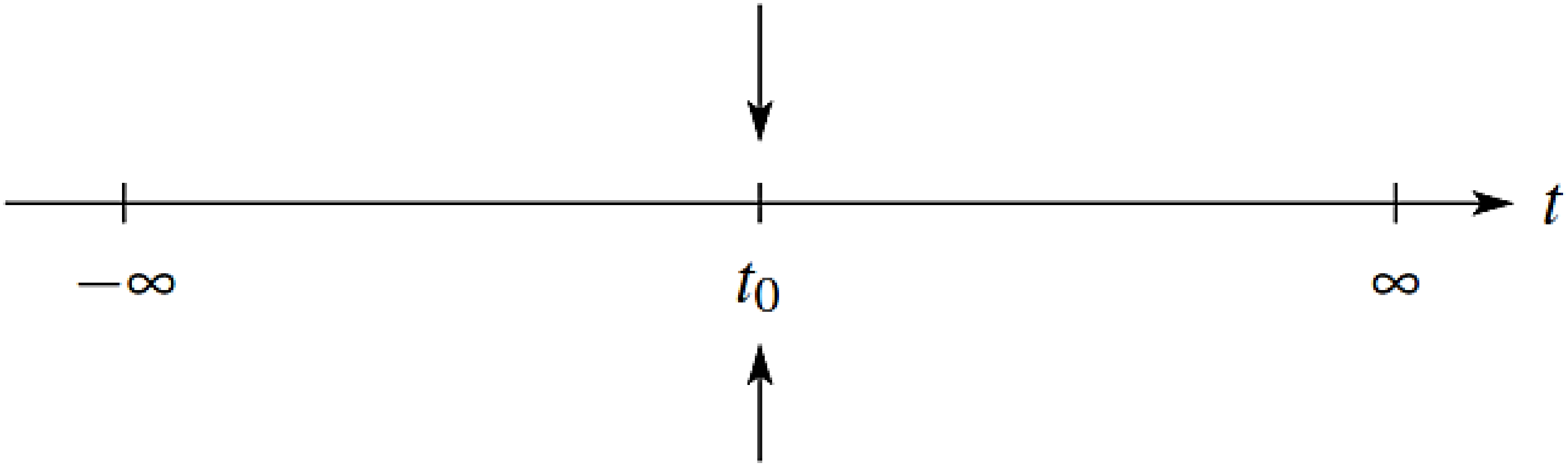
General Structure of a Feedback Control System

MEMORY

- A system $\mathcal{H}$ is said to be **memoryless** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t \neq t_0$.
- In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same*** point in time.
- A system that is not memoryless is said to have **memory**.
- Although simple, a memoryless system is ***not very flexible***, since its current output value cannot rely on past or future values of the input.

MEMORY

If the system $\mathcal{H}$ is memoryless,
the output $\mathcal{H}x$ at t_0
can depend on the input x
only at t_0.



Consider the calculation of the
output $\mathcal{H}x$ at t_0.

Example

Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = Ax(t)$$

and A is a nonzero real constant.

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = Ax(t_0).$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless.

Example

Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ for $-\infty < t \leq t_0$. So, $\mathcal{H}x(t_0)$ is dependent on $x(t)$ for some $t \neq t_0$ (e.g., $t_0 - 1$). the system has memory (i.e., is not memoryless).

Example

Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = e^{x(t)}.$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = e^{x(t_0)}.$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless.

Example

Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = \tfrac{1}{2}\left[x(t_0) - x(-t_0)\right].$$

Thus, for any x and any real t_0, we have that $\mathcal{H}x(t_0)$ depends on $x(t)$ for $t = t_0$ and $t = -t_0$.

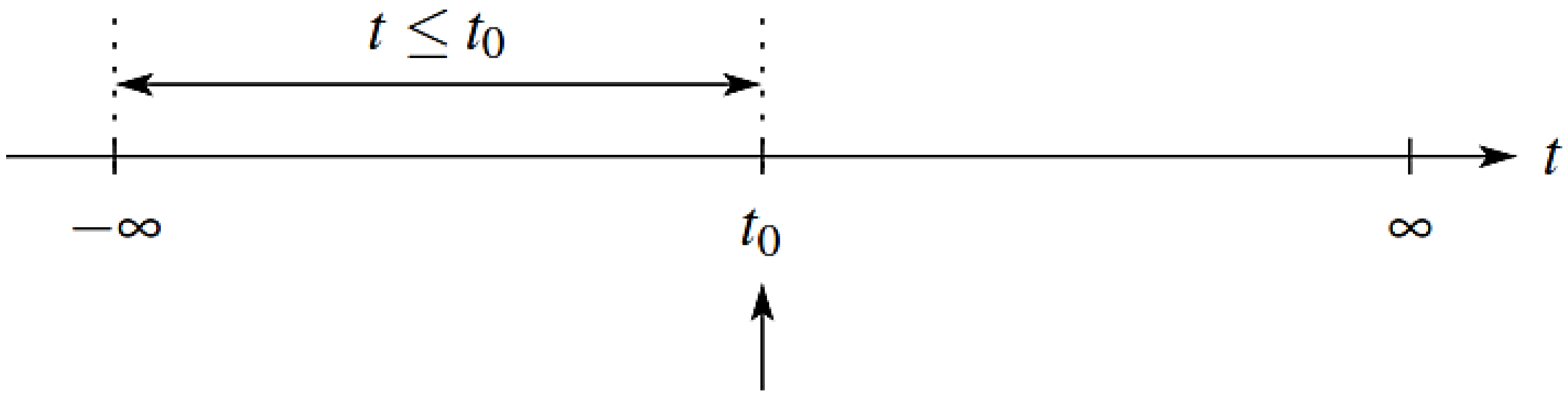
Since $\mathcal{H}x(t_0)$ depends on $x(t)$ for some $t \neq t_0$, the system has memory

CAUSALITY

- A system $\mathcal{H}$ is said to be **causal** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t > t_0$.
- In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the ***same or earlier points*** in time (i.e., ***not later points in time***).
- If the independent variable t represents time, a system must be causal in order to be ***physically realizable***.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable ***need not always represent time*** (e.g., the independent variable might represent position).
- A memoryless system is always causal, although the converse is not necessarily true.

CAUSALITY

If the system $\mathcal{H}$ is causal,
the output $\mathcal{H}x$ at t_0
can depend on the input x
only at points $t \leq t_0$.



Consider the calculation of the
output $\mathcal{H}x$ at t_0.

Example

Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $-\infty < t \leq t_0$.

Since all of the values in this interval are less than or equal to t_0, the system is causal.

Example

Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{t_0-1}^{t_0+1} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ only depends on $x(t)$ for $t_0 - 1 \leq t \leq t_0 + 1$.

Since some of the values in this interval are greater than t_0 (e.g., $t_0 + 1$), the system is not causal.

Example

Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = (t+1)e^{x(t-1)}.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = (t_0+1)e^{x(t_0-1)}.$$

Thus, we can see that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $t = t_0 - 1$.

Since $t_0 - 1 \leq t_0$, the system is causal.

Example

Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. For any x and any real constant t_0, we have that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $t = t_0$ and $t = -t_0$. Suppose that $t_0 = -1$. In this case, we have that $\mathcal{H}x(t_0)$ (i.e., $\mathcal{H}x(-1)$) depends on $x(t)$ for $t = 1$ but $t = 1 > t_0$.

Therefore, the system is not causal.

INVERTIBILITY

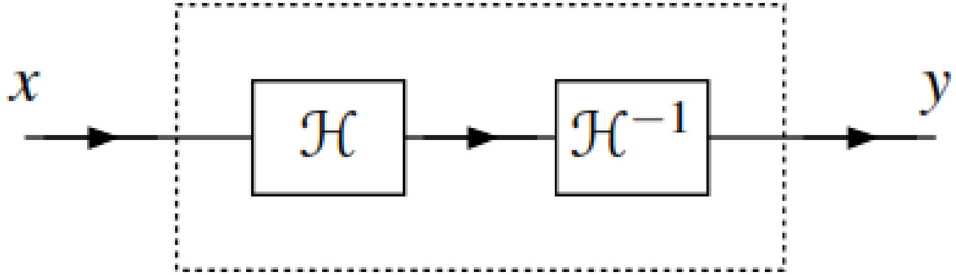
- The **inverse** of a system $\mathcal{H}$ (if it exists) is another system $\mathcal{H}^{-1}$ such that, for every function x,

$$\mathcal{H}^{-1}\mathcal{H}x = x$$

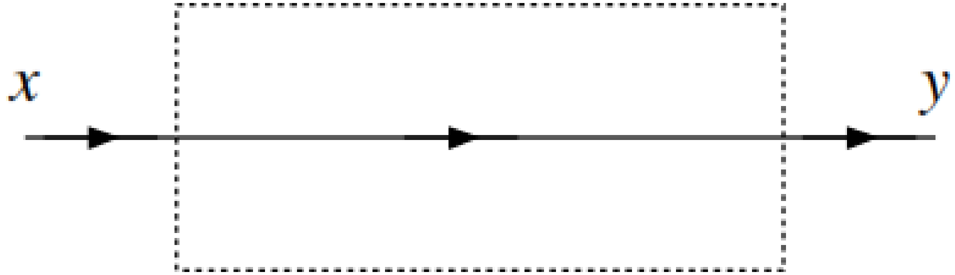
  (i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{H}^{-1}$ is a system whose input and output are equal).
- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input can always be ***uniquely*** determined from its output.
- An invertible system will always produce ***distinct outputs*** from any two ***distinct inputs*** (i.e., $x_1 \neq x_2 \Rightarrow \mathcal{H}x_1 \neq \mathcal{H}x_2$).
- To show that a system is ***invertible***, we simply find the ***inverse system***.
- To show that a system is ***not invertible***, we find ***two distinct inputs*** that result in ***identical outputs*** (i.e., $x_1 \neq x_2$ and $\mathcal{H}x_1 = \mathcal{H}x_2$).
- In practical terms, invertible systems are "nice" in the sense that their ***effects can be undone***.

INVERTIBILITY

- A system $\mathcal{H}^{-1}$ being the inverse of $\mathcal{H}$ means that the following two systems are equivalent (i.e., $\mathcal{H}^{-1}\mathcal{H}$ ***is an identity***):



System 1: $y = \mathcal{H}^{-1}\mathcal{H}x$



System 2: $y = x$

Example Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = x(t - t_0) \quad \text{and } t_0 \text{ is a real constant.}$$

Solution. Let $y = \mathcal{H}x$. By substituting $t + t_0$ for t in $y(t) = x(t - t_0)$, we obtain

$$\begin{aligned} y(t + t_0) &= x(t + t_0 - t_0) \\ &= x(t). \end{aligned}$$

Thus, we have shown that

$$x(t) = y(t + t_0).$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(t) = \mathcal{H}^{-1}y(t)$$

where

$$\mathcal{H}^{-1}y(t) = y(t + t_0).$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system $\mathcal{H}$ is invertible.

Example

Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

Solution. Consider an input of the form $x(t) = 2\pi k$ where k is an arbitrary integer.

The response $\mathcal{H}x$ to such an input

$$\begin{aligned}\mathcal{H}x(t) &= \sin[x(t)] \\ &= \sin(2\pi k) \\ &= 0.\end{aligned}$$

Thus, we have found an infinite number of distinct inputs (i.e., $x(t) = 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$)

that all result in the same output. Therefore, the system is not invertible.

Example

Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = 3x(3t+3).$$

Solution. Let $y = \mathcal{H}x$. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} & y(t) = 3x(3t+3) \\ \Rightarrow \quad & y\left(\tfrac{1}{3}t - 1\right) = 3x(t) \\ \Rightarrow \quad & x(t) = \tfrac{1}{3}y\left(\tfrac{1}{3}t - 1\right). \end{aligned}$$

In other words, $\mathcal{H}^{-1}$ is given by $\mathcal{H}^{-1}y(t) = \frac{1}{3}y\left(\frac{1}{3}t - 1\right)$.

Since we have just found $\mathcal{H}^{-1}$, $\mathcal{H}^{-1}$ exists. Therefore, the system $\mathcal{H}$ is invertible.

Example

Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Consider the response $\mathcal{H}x$ of the system to an input x of the form

$$x(t) = \alpha,$$

where α is a real constant. We have that

$$\begin{aligned}\mathcal{H}x(t) &= \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}(\alpha - \alpha) \\ &= 0.\end{aligned}$$

Therefore, any constant input yields the same zero output.

This, however, implies that distinct inputs can yield identical outputs.

Therefore, the system is not invertible.

BOUNDED INPUT BOUNDED OUTPUT (BIBO) STABILITY

- A system $\mathcal{H}$ is said to be **bounded-input bounded-output (BIBO) stable** if, for every bounded function x, $\mathcal{H}x$ is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|\mathcal{H}x(t)| < \infty$ for all t).
- In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.
- To show that a system is ***BIBO stable***, we must show that ***every bounded input*** leads to a ***bounded output***.
- To show that a system is ***not BIBO stable***, we only need to find a single ***bounded input*** that leads to an ***unbounded output***.
- In practical terms, a BIBO stable system is ***well behaved*** in the sense that, as long as the system input is finite everywhere (in its domain), the output will also be finite everywhere.
- Usually, a system that is not BIBO stable will have ***serious safety issues***.
- For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely a big lawsuit as well).

Example Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = x^2(t).$$

Solution. Suppose that the input x is bounded such that (for all t)

$$|x(t)| \leq A,$$

where A is a finite real constant. Squaring both sides of the inequality, we obtain

$$|x(t)|^2 \leq A^2.$$

Interchanging the order of the squaring and magnitude operations on the left-hand side of the inequality, we have

$$|x^2(t)| \leq A^2.$$

Using the fact that $\mathcal{H}x(t) = x^2(t)$, we can write

$$|\mathcal{H}x(t)| \leq A^2.$$

Since A is finite, A^2 is also finite. Thus, we have that $\mathcal{H}x$ is bounded

system is BIBO stable.

Example

Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Suppose that we choose the input $x = u$ (where u denotes the unit-step function).

Clearly, u is bounded (i.e., $|u(t)| \leq 1$ for all t).

Calculating the response $\mathcal{H}x$ to this input, we have

$$\begin{aligned}\mathcal{H}x(t) &= \int_{-\infty}^{t} u(\tau)d\tau \\ &= \int_{0}^{t} d\tau \\ &= [\tau]|_0^t \\ &= t.\end{aligned}$$

From this result, however, we can see that as $t \to \infty$, $\mathcal{H}x(t) \to \infty$.

Therefore, the system is not BIBO stable.

Example Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Suppose that x is bounded. Then, $x(-t)$ is also bounded.

Since the difference of two bounded functions is bounded, $x(t) - x(-t)$ is bounded.

Multiplication of a bounded function by a finite constant yields a bounded result.

So, the function $\frac{1}{2}[x(t) - x(-t)]$ is bounded.

Thus, $\mathcal{H}x(t)$ is bounded.

the system is BIBO stable.

Example Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \mathcal{D}x(t)$$

and $\mathcal{D}$ denotes the derivative operator.

Solution. Consider the input $x(t) = \sin\left(t^2\right)$.

Clearly, x is bounded, since the sine function is bounded.

$|x(t)| \leq 1$ for all real t. Now, consider the response of the system to the input x. We have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}x(t) \\ &= \mathcal{D}\left\{\sin(t^2)\right\}(t) \\ &= 2t\cos(t^2).\end{aligned}$$

Clearly, $\mathcal{H}x$ is unbounded, since $|\mathcal{H}x(t)|$ grows without bound as $|t| \to \infty$.

Therefore, the system is not BIBO stable.

TIME VARIANCE (TI)

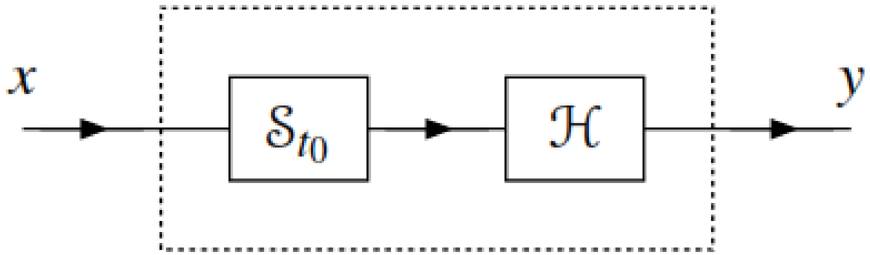
- A system $\mathcal{H}$ is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every function x and every real constant t_0, the following condition holds:

$$\mathcal{H}x(t-t_0) = \mathcal{H}x'(t) \text{ for all } t, \quad \text{where} \quad x'(t) = x(t-t_0)$$

(i.e., $\mathcal{H}$ ***commutes with time shifts***).
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an ***identical time shift*** in the output.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior ***does not change*** with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much ***easier to design and analyze***, since their behavior does not change with respect to time.

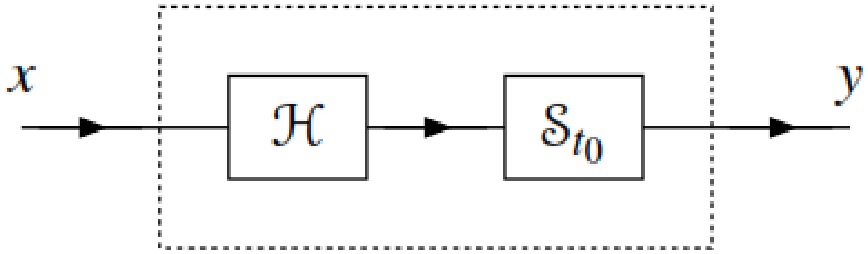
TIME VARIANCE (TI)

- Let $\mathcal{S}_{t_0}$ denote an operator that applies a ***time shift of*** t_0 to a function (i.e., $\mathcal{S}_{t_0}x(t) = x(t-t_0)$).
- A system $\mathcal{H}$ is ***time invariant*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with*** $\mathcal{S}_{t_0}$):



System 1: $y = \mathcal{H}\mathcal{S}_{t_0}x$

$$\begin{bmatrix} y(t) = \mathcal{H}x'(t) \\ x'(t) = \mathcal{S}_{t_0}x(t) = x(t-t_0) \end{bmatrix}$$



System 2: $y = \mathcal{S}_{t_0}\mathcal{H}x$

$$\left[y(t) = \mathcal{H}x(t-t_0)\right]$$

Example

Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned} \mathcal{H}x(t-t_0) &= \sin[x(t-t_0)] \quad \text{and} \\ \mathcal{H}x'(t) &= \sin[x'(t)] \\ &= \sin[x(t-t_0)]. \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ for all x and t_0, the system is time invariant. ■

Example

Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = tx(t).$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x(t-t_0) &= (t-t_0)x_1(t-t_0) \quad \text{and} \\ \mathcal{H}x'(t) &= tx'(t) \\ &= tx(t-t_0). \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant (i.e., the system is time varying). ■

Example

Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \sum_{k=-10}^{10} x(t-k).$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\mathcal{H}x(t-t_0) = \sum_{k=-10}^{10} x(t-t_0-k) \quad \text{and}$$

$$\begin{aligned} \mathcal{H}x'(t) &= \sum_{k=-10}^{10} x'(t-k) \\ &= \sum_{k=-10}^{10} x(t-k-t_0) \\ &= \sum_{k=-10}^{10} x(t-t_0-k). \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ for all x and t_0, the system is time invariant. ■

Example

Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Let $x'(t) = x(t - t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x(t - t_0) &= \tfrac{1}{2}[x(t - t_0) - x(-(t - t_0))] \\
&= \tfrac{1}{2}[x(t - t_0) - x(-t + t_0)] \quad \text{and} \\
\mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t) - x'(-t)] \\
&= \tfrac{1}{2}[x(t - t_0) - x(-t - t_0)].
\end{aligned}$$

Since $\mathcal{H}x(t - t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant.

ADDITIVITY, HOMOGENEITY, AND LINEARITY

- A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

(i.e., $\mathcal{H}$ ***commutes with addition***).
- A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

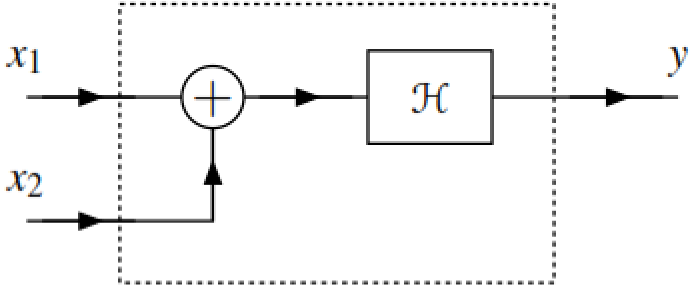
(i.e., $\mathcal{H}$ ***commutes with scalar multiplication***).
- A system that is both additive and homogeneous is said to be **linear**.
- In other words, a system $\mathcal{H}$ is ***linear***, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

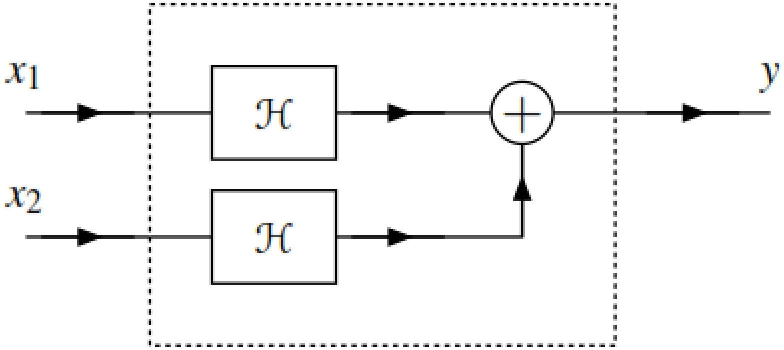
(i.e., $\mathcal{H}$ ***commutes with linear combinations***).
- The linearity property is also referred to as the **superposition** property.
- Practically speaking, linear systems are much ***easier to design and analyze*** than nonlinear systems.

ADDITIVITY, HOMOGENEITY, AND LINEARITY

- The system $\mathcal{H}$ is ***additive*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with addition***):

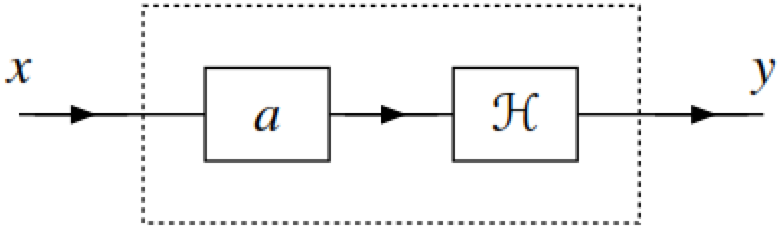


System 1: $y = \mathcal{H}(x_1 + x_2)$

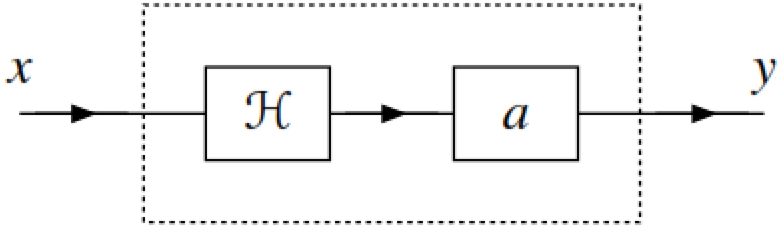


System 2: $y = \mathcal{H}x_1 + \mathcal{H}x_2$

- The system $\mathcal{H}$ is ***homogeneous*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with scalar multiplication***):



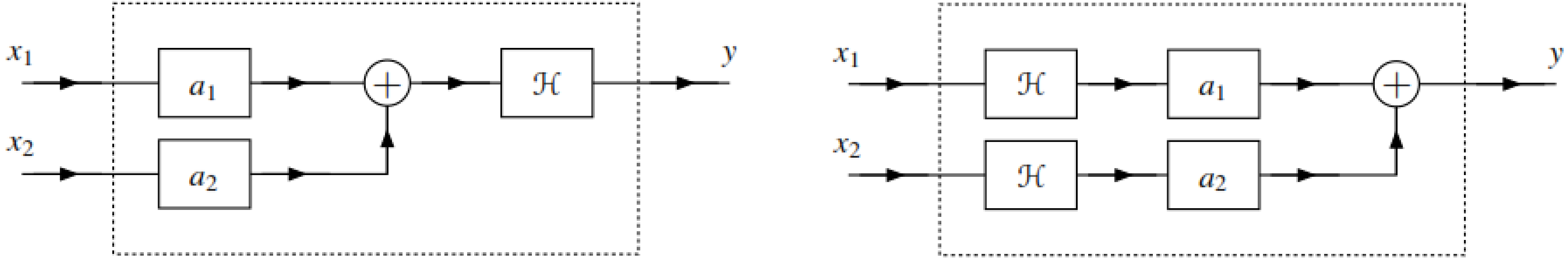
System 1: $y = \mathcal{H}(ax)$



System 2: $y = a\mathcal{H}x$

ADDITIVITY, HOMOGENEITY, AND LINEARITY

- The system $\mathcal{H}$ is ***linear*** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ ***commutes with linear combinations***):



System 1: $y = \mathcal{H}(a_1x_1 + a_2x_2)$

System 2: $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$

Example

Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = |x(t)|.$$

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1|x_1(t)| + a_2|x_2(t)| \quad \text{and} \\ \mathcal{H}x'(t) &= |x'(t)| \\ &= |a_1x_1(t) + a_2x_2(t)|. \end{aligned}$$

At this point, we recall the triangle inequality (i.e., for $a, b \in \mathbb{C}$, $|a+b| \le |a| + |b|$). Thus, $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ cannot hold for all x_1, x_2, a_1, and a_2 due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(t) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(t) = 0,$$

in which case

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = -1 \quad \text{and} \quad \mathcal{H}x'(t) = 1.$$

Therefore, the superposition property does not hold and the system is not linear. ■

Example

Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= \tfrac{1}{2}a_1[x_1(t) - x_1(-t)] + \tfrac{1}{2}a_2[x_2(t) - x_2(-t)] \quad \text{and} \\
\mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t) - x'(-t)] \\
&= \tfrac{1}{2}[a_1x_1(t) + a_2x_2(t) - [a_1x_1(-t) + a_2x_2(-t)]] \\
&= \tfrac{1}{2}[a_1x_1(t) - a_1x_1(-t) + a_2x_2(t) - a_2x_2(-t)] \\
&= \tfrac{1}{2}a_1[x_1(t) - x_1(-t)] + \tfrac{1}{2}a_2[x_2(t) - x_2(-t)].
\end{aligned}$$

Since $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the system is linear. ■

Example

Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = x(t)x(t-1).$$

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1x_1(t)x_1(t-1) + a_2x_2(t)x_2(t-1) \quad \text{and} \\
\mathcal{H}x'(t) &= x'(t)x'(t-1) \\
&= [a_1x_1(t) + a_2x_2(t)][a_1x_1(t-1) + a_2x_2(t-1)] \\
&= a_1^2x_1(t)x_1(t-1) + a_1a_2x_1(t)x_2(t-1) + a_1a_2x_1(t-1)x_2(t) + a_2^2x_2(t)x_2(t-1).
\end{aligned}$$

Clearly, the expressions for $\mathcal{H}(a_1x_1 + a_2x_2)$ and $a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ are quite different. Consequently, these expressions are not equal for many choices of a_1, a_2, x_1, and x_2 (e.g., $a_1 = 2$, $a_2 = 0$, $x_1(t) = 1$, and $x_2(t) = 0$). Therefore, the superposition property does not hold and the system is not linear. ■

Example

A system $\mathcal{H}$ is defined by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we consider the additivity property. From the definition of $\mathcal{H}$, we have

$$\mathcal{H}x_1(t) + \mathcal{H}x_2(t) = \int_{-\infty}^{t} x_1(\tau)d\tau + \int_{-\infty}^{t} x_2(\tau)d\tau \quad \text{and}$$

$$\begin{aligned}\mathcal{H}(x_1 + x_2)(t) &= \int_{-\infty}^{t} (x_1 + x_2)(\tau)d\tau \\ &= \int_{-\infty}^{t} [x_1(\tau) + x_2(\tau)]d\tau \\ &= \int_{-\infty}^{t} x_1(\tau)d\tau + \int_{-\infty}^{t} x_2(\tau)d\tau.\end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Example

Second, we consider the homogeneity property. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we can write

$$a\mathcal{H}x(t) = a\int_{-\infty}^{t} x(\tau)d\tau \quad \text{and}$$

$$\begin{aligned}\mathcal{H}(ax)(t) &= \int_{-\infty}^{t} (ax)(\tau)d\tau \\ &= \int_{-\infty}^{t} ax(\tau)d\tau \\ &= a\int_{-\infty}^{t} x(\tau)d\tau.\end{aligned}$$

Since $\mathcal{H}(ax) = a\mathcal{H}x$ for all x and a, the system is homogeneous.

Lastly, we consider the linearity property. Since the system is both additive and homogeneous, it is linear. ■

Example

A system $\mathcal{H}$ is given by

$$\mathcal{H}x(t) = \operatorname{Re}[x(t)].$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we check if the additivity property is satisfied. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x_1(t) + \mathcal{H}x_2(t) &= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)] \quad \text{and} \\
\mathcal{H}(x_1 + x_2)(t) &= \operatorname{Re}[(x_1 + x_2)(t)] \\
&= \operatorname{Re}[x_1(t) + x_2(t)] \\
&= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)].
\end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Second, we check if the homogeneity property is satisfied. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a\mathcal{H}x(t) &= a\operatorname{Re}x(t) \quad \text{and} \\
\mathcal{H}(ax)(t) &= \operatorname{Re}[(ax)(t)] \\
&= \operatorname{Re}[ax(t)].
\end{aligned}$$

Example

In order for $\mathcal{H}$ to be homogeneous, $a\mathcal{H}x(t) = \mathcal{H}(ax)(t)$ must hold for all x and all complex a. Suppose that $a = j$ and x is not identically zero (i.e., x is not the function $x(t) = 0$). In this case, we have

$$\begin{aligned} a\mathcal{H}x(t) &= j\operatorname{Re}[x(t)] \quad \text{and} \\ \mathcal{H}(ax)(t) &= \operatorname{Re}[(jx)(t)] \\ &= \operatorname{Re}[jx(t)] \\ &= \operatorname{Re}[j(\operatorname{Re}[x(t)] + j\operatorname{Im}[x(t)])] \\ &= \operatorname{Re}(-\operatorname{Im}[x(t)] + j\operatorname{Re}[x(t)]) \\ &= -\operatorname{Im}[x(t)]. \end{aligned}$$

Thus, the quantities $\mathcal{H}(ax)$ and $a\mathcal{H}x$ are clearly not equal. Therefore, the system is not homogeneous.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. ■

EIGENFUNCTIONS OF SYSTEMS

- A function x is said to be an **eigenfunction** of the system $\mathcal{H}$ with the **eigenvalue** λ if

$$\mathcal{H}x = \lambda x,$$

  where λ is a complex constant.
- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigenfunctions x, where the amplifier gain is given by the corresponding eigenvalue λ.
- Different systems have different eigenfunctions.
- Many of the mathematical tools developed for the study of CT systems have eigenfunctions as their basis.

Example

Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(t) = \mathcal{D}^2 x(t),$$

where $\mathcal{D}$ denotes the derivative operator. For each function x given below, determine if x is an eigenfunction of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(t) = \cos(2t)$; and
(b) $x(t) = t^3$.

Example

Solution. (a) We have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{\cos(2t)\}(t)\\ &= \mathcal{D}\{-2\sin(2t)\}(t)\\ &= -4\cos(2t)\\ &= -4x(t).\end{aligned}$$

Therefore, x is an eigenfunction of $\mathcal{H}$ with the eigenvalue -4.

(b) We have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{t^3\}(t)\\ &= \mathcal{D}\{3t^2\}(t)\\ &= 6t\\ &= \tfrac{6}{t^2}x(t).\end{aligned}$$

Therefore, x is not an eigenfunction of $\mathcal{H}$.

CONTINUOUS TIME LINEAR TIME INVARIANT (LTI) SYSTEMS

- In engineering, linear time-invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.