Cal 3 2013/11

Section A

Q1.

a)

Let: $z = j$

$$\rightarrow \begin{cases} r = |z| = \sqrt{0^2 + 1^2} = 1 \\ \theta = \tan^{-1}\infty = \dfrac{\pi}{2} \end{cases}$$

Since, we know that:

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0,1,\dots,n-1$$

Therefore, there is exist 3 cubic roots of z as follows:

$$w_0 = \sqrt[3]{1}\left(\cos\frac{\frac{\pi}{2} + 0}{3} + j\sin\frac{\frac{\pi}{2} + 0}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}j$$

$$w_1 = \sqrt[3]{1}\left(\cos\frac{\frac{\pi}{2} + 2\pi}{3} + j\sin\frac{\frac{\pi}{2} + 2\pi}{3}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}j$$

$$w_2 = \sqrt[3]{1}\left(\cos\frac{\frac{\pi}{2} + 4\pi}{3} + j\sin\frac{\frac{\pi}{2} + 4\pi}{3}\right) = -j$$

b)

Let: $z = 1 + \cos\theta + j\sin\theta$

$$\rightarrow \begin{cases} r = |z| = \sqrt{(1+\cos\theta)^2 + \sin^2\theta} = \sqrt{2 + 2\cos\theta} = 2\cos\dfrac{\theta}{2} \\ \theta = \tan^{-1}\dfrac{\sin\theta}{1+\cos\theta} = \tan^{-1}\dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right) = \dfrac{\theta}{2} \end{cases}$$

From Moivre's theorem, we have:

$$z^n = r^n(\cos n\theta + j\sin n\theta)$$

Therefore,

$$(1 + \cos\theta + j\sin\theta)^n = 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + j\sin\frac{n\theta}{2}\right)$$

Q2.

Given that: $f(z) = u(x,y) + jv(x,y)$, where $u(x,y) = 3x^2y^2, v(x,y) = -6x^2y^2$

Check whether or not the given function satisfied the Cauchy-Riemann equation:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \leftrightarrow \begin{cases} 6xy^2 = -12xy^2 \\ 6x^2y = 12x^2y \end{cases} \leftrightarrow \left[\begin{array}{l} x = 0 \\ y = 0 \end{array}\right.$$

Therefore, $f(z)$ is differentiable if z lies on the line $y = 0$ or $x = 0$.

Q3.

a)

$$\mathcal{L}\{e^{-t}(3\sinh 2t - 5\cosh 2t)\} = \mathcal{L}\left\{e^{-t}\left(\frac{3(e^{2t}-e^{-2t})}{2} - \frac{5(e^{2t}+e^{-2t})}{2}\right)\right\}$$

$$= \mathcal{L}\{e^{-t}(-e^{2t} - 4e^{-2t})\} = \mathcal{L}\{-e^{t} - 4e^{-3t}\} = \frac{-1}{s-1} - \frac{4}{s+3}$$

b)

$$\mathcal{L}^{-1}\left\{\frac{8}{s^2+4s}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s+4}\right\} = 2 + 2e^{-4t}$$

Q4.

a)

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

b)

Using the definition of Laplace transform, we have:

$$F(s) = \mathcal{L}\{t\sin t\} = \int_0^{+\infty}(t\sin t)e^{-st}dt = \frac{2s}{(s^2+1)^2}$$

Therefore,

$$\int_0^{+\infty}(t\sin t)e^{-3t}dt = F(3) = 0.06$$

Section B

Q1.

$$f(z) = \frac{-1}{(z-1)(z+1)} = \frac{1}{z+1}\frac{-1}{z+1-2}$$

Apply power series for analyzing this problem:

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

We have:

$$f(z) = \frac{1}{z+1}\frac{1/2}{1-\frac{z+1}{2}}$$

With $|z+1| < 2 \leftrightarrow \left|\frac{z+1}{2}\right| < 1$, it holds that:

$$\frac{1}{1-\frac{z+1}{2}} = \sum_{n=0}^{+\infty}\left(\frac{z+1}{2}\right)^n$$

Therefore,

$$f(z) = \frac{1/2}{z+1}\sum_{n=0}^{+\infty}\left(\frac{z+1}{2}\right)^n$$

$$= \sum_{n=0}^{+\infty}\frac{1}{2^{n+1}}(z+1)^{n-1}$$

Q2.

Given that:

$$\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 200q = 150u(t) \ \ (*), \qquad q(0) = 7, \quad q'(0) = 0$$

Let $Q(s) = \mathcal{L}\{q(t)\}$, it holds that:

$$\mathcal{L}\{q'(t)\} = sQ(s) - q(0) = sQ(s)$$
$$\mathcal{L}\{q''(t)\} = s^2Q(s) - sq(0) - q'(0) = s^2Q(s)$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$s^2Q(s) + 20sQ(s) + 200Q(s) = \frac{150}{s}$$
$$\leftrightarrow Q(s) = \frac{150}{s(s^2 + 20s + 200)}$$
$$\leftrightarrow Q(s) = \frac{3}{4}\left(\frac{1}{s} - \frac{s + 10 + 10}{(s + 10)^2 + 10^2}\right)$$

$$\to q(t) = \mathcal{L}^{-1}\{Q(s)\} = \frac{3}{4}(1 - e^{-10t}\cos 10t - e^{-10t}\sin 10t)u(t)$$

Thus, the current $i(t)$ is:

$$i(t) = \frac{dq(t)}{dt} = 15e^{-10t}\sin(10t)\,u(t)$$

Q3.

Given that:

$$\begin{cases} x' + y = t \\ y' + 4x = 0 \end{cases}$$

And $x(0) = 1, y(0) = -1$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{t(t)\}$, it holds that:

$$\begin{cases} \mathcal{L}\{x'(t)\} = sX(s) - x(0) = sX(s) - 1 \\ \mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) + 1 \end{cases}$$

Taking Laplace transforms both side of the whole given system equations, we obtain:

$$\begin{cases} sX(s) - 1 + Y(s) = \dfrac{1}{s^2} & (1) \\ sY(s) + 1 + 4X(s) = 0 & (2) \end{cases}$$

$$(1) \leftrightarrow Y(s) = \frac{1}{s^2} + 1 - sX(s) \qquad (3)$$

Substitute (3) into (2), we get:

$$s\left[\frac{1}{s^2} + 1 - sX(s)\right] + 1 + 4X(s) = 0$$
$$\leftrightarrow X(s)(s^2 - 4) = \frac{1}{s} + s + 1$$
$$\to X(s) = \frac{s^2 + s + 1}{s(s^2 - 4)}$$
$$\to X(s) = \frac{3}{8}\frac{1}{s + 2} + \frac{7}{8}\frac{1}{s - 2} - \frac{1}{4s} \ (4)$$

Substitute back into (3), we get:

$$Y(s) = \frac{1}{s^2} + \frac{3}{4}\frac{1}{s + 2} - \frac{7}{4}\frac{1}{s - 2} \ \ (5)$$

From (4) and (5), taking inverse Laplace transforms to get the final result:

$$\begin{cases} x(t) = \left(\dfrac{3}{8}e^{-2t} + \dfrac{7}{8}e^{2t} - \dfrac{1}{4}\right)u(t) \\ y(t) = \left(\dfrac{3}{4}e^{-2t} - \dfrac{7}{4}e^{2t} + t\right)u(t) \end{cases}$$