HO CHI MINH CITY INTERNATIONAL UNIVERSITY

MIDTERM TEST (Group 1)
Semester 2, Academic year 2019-2020
Duration: 90 minutes

SUBJECT: Calculus 2	
Chair of Department of Mathematics	Lecturer:
Signature:	Signature:
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Instructions:

- **Open-book exam.**
- **Each question carries 20 marks.**

Question 1. Find the limit if it exists, or show that the limit does not exist

$$a) \quad \lim_{n\to\infty} \big(\ln(3n^3+n^2+1) - \ln(n^3+n+2)\big) \qquad\qquad b) \quad \lim_{n\to\infty} \sin(n\pi/2)$$

Question 2. Determine whether the given series is convergent or divergent:

$$a) \quad \sum_{n=1}^{\infty} \sin\frac{1}{n^2} \qquad\qquad b) \quad \sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$$

Question 3. Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n}$$

Question 4. Determine whether the following two lines are parallel, intersecting, or skew. If they are skew, find the distance between them

$$L_1: \qquad x = 3+2t, \quad y = 4-3t, \quad z = -1-4t$$

and

$$L_2: \qquad 1+2s, \quad y = 2+s, \quad z = 3+2s$$

Question 5. (a) Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = (t+1)e^t\mathbf{i} + (t^2+2t+2)\mathbf{j} + (t^3-1)\mathbf{k}, t > -1$ at the point $(1,2,-1)$.

(b) Find the length of the curve $\mathbf{r}(t) = 2t^{3/2}\mathbf{i} - \cos 2t\mathbf{j} + \sin 2t\mathbf{k}, 0 \le t \le 1$

———END OF QUESTIONS———

CALCULUS 2

Solutions for Mid-term Test

Question 1. a)

$$\lim_{n\to\infty}\big(\ln(3n^3+n^2+1)-\ln(n^3+n+2)\big)=\lim_{n\to\infty}\ln\frac{3n^3+n^2+1}{n^3+n+2}=\ln\left(\lim_{n\to\infty}\frac{3+1/n+1/n^3}{1+1/n^2+2/n^3}\right)=\ln 3$$

b) $a_n=\sin(n\pi/2)$. If $n=2m, m\in\mathbf{N}$, then $a_n=\sin(m\pi)=0$.
If $n=4m+1, m\in\mathbf{N}$, then $a_n=\sin(\pi/2+2m\pi)=1$.
So, limit does not exist.

Question 2. a)

$$\lim_{n\to\infty}\frac{\sin 1/n^2}{1/n^2}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The series $\sum_{n=1}^{\infty}1/n^2$ is convergent, so the given series is also convergent, by Limit Comparison Test.

b) $a_n=\dfrac{n^n}{3^n n!}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L=\lim_{n\to\infty}\frac{(n+1)^{n+1}/3^{n+1}(n+1)!}{n^n/3^n n!}=\frac{1}{3}\lim_{n\to\infty}(1+1/n)^n=e/3<1$$

The series is convergent, by Ratio Test.

Question 3. Set $a_n=\dfrac{(2x-1)^n}{n}$. It holds that

$$\begin{aligned}L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|&=\lim_{n\to\infty}|2x-1|\frac{n}{(n+1)}\\&=\lim_{n\to\infty}|2x-1|\frac{1}{(1+1/n)}=|2x-1|.\end{aligned}$$

By Ratio Test, the series is convergent if $L<1$ or $|x-1/2|<1/2$, and divergent if $L>1$ or $|x-1/2|>1/2$. Thus $R=1/2$, and the series is convergent in $(0,1)$.

At $x=0$, the series becomes $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ which is convergent, by Alternating series Test.
At $x=1$, the series becomes $\sum_{n=1}^{\infty}\dfrac{1}{n}$ which is divergent.
So the interval of convergence: $0\le x<1$.

Question 4. $L_1////u_1=<2,-3,-4>, L_2//u_2=<2,1,2>$, since $2/2\neq -3/1$, u_1 is not parallel to u_2 and so L_1 is not parallel to L_2.

Let (α) be the plane containing (L_1) and parallel to (L_2). Then the distance between the lines (L_1) and (L_2) is equal to the distance from $M(1,2,3)$ on (L_2) to (α).

The normal vector n of (α) can be chosen as

$$n=(-1/2)u_1\times u_2=(-1/2)<-2,-12,8>=<1,6,-4>.$$

Hence, the plane (α) has an equation of the form

$$(x-3)+6(y-4)-4(z+1)=0$$

or

$$x+6y-4z-31=0.$$

Therefore, the distance from $M(1,2,3)$ to (α) is given by

$$d=\frac{|1+6(2)-4(3)-31|}{\sqrt{1+6^2+4^2}}=\frac{30}{\sqrt{53}}.$$

$d>0$ means the two lines do not cut. Since they are not parallel, they are skew.

Question 5. (a) $\mathbf{r}(t)=<(t+1)e^t, t^2+2t+2, t^3-1>, t>-1$. The point $(1,2,-1)$ corresponds to $t=0$.

It holds that

$$\mathbf{r}'(t)=<(t+2)e^t, 2t+2, 3t^2>, \quad \mathbf{r}'(0)=<2,2,0>.$$

Thus the tangent line has equations

$$x=1+2t, y=2+2t, z=-1.$$

(b) Find the length of the curve $\mathbf{r}(t)=2t^{3/2}\mathbf{i}-\cos 2t\mathbf{j}+\sin 2t\mathbf{k}, 0\le t\le 1$
$\mathbf{r}'(t)=3t^{1/2}\mathbf{i}+2\sin 2t\mathbf{j}+2\cos 2t\mathbf{k}, 0\le t\le 1$

$$||\mathbf{r}'(t)||=\sqrt{9t+4}$$

So

$$s=\int_0^1 ||\mathbf{r}'(t)||dt=\int_0^1 \sqrt{9t+4}dt=(2/27)(9t+4)^{3/2}\Big|_0^1=(2/27)(13^{3/2}-8)$$