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PRINCIPLES OF ELECTRICAL ENGINEERING 2

Lecture # 5 & 6: The Laplace Transform in Circuit Analysis

Chapter #13

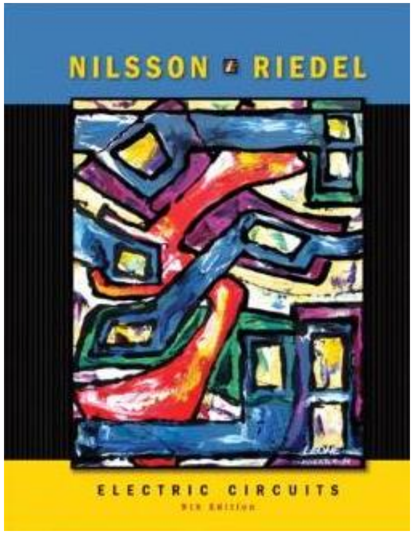
Text book: Electric Circuits

James W. Nilsson & Susan A. Riedel

9th Edition.

link: **http://blackboard.hcmiu.edu.vn/**

to download materials



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Objectives

- Be able to transform a circuit into the **s** domain using Laplace transforms. Know how to analyze a circuit in the **s** domain and be able to transform an **s** domain solution back to the time domain.

- Understand the definition and significance of the transfer function and be able to calculate the transfer function for a circuit using **s** domain techniques.

- Know how to use a circuit's transfer function to calculate the circuit's unit impulse response, its unit step response, and its steady-state response to a sinusoidal input.

The Laplace transform has two characteristics (in circuit analysis):

1. it transforms a set of linear constant coefficient differential equations into a set of linear polynomial equations which are easier to solve.
2. it automatically introduces into the polynomial equations the initial values of the current and voltage variables.

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Outline

- Circuit elements in the **s** domain
- Circuit analysis in the **s** domain
- The transfer function
- The transfer function in partial fraction expansions
- The transfer function and the steady-state sinusoidal response

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Key points

- How to represent the initial energy of L, C in the s-domain?
- Why the functional forms of natural and steady- state responses are determined by the poles of transfer function $H(s)$ and excitation source $X(s)$, respectively?

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Circuit Elements in the s domain

The procedure for developing an s-domain equivalent circuit for each circuit element:

1. Write the **time-domain** equation that relates the terminal voltage to the terminal current.
2. Do the Laplace transform of the **time-domain** equation. This step generates an algebraic relationship between the **s-domain** current and voltage.
3. Construct a circuit model that satisfies the relation ship between the **s-domain** current and voltage. We use the passive sign convention in all the derivations.

Note: dimension of a transformed voltage is volt-seconds, and a transformed current is ampere-seconds. A voltage-to-current ratio in the **s domain** caries the dimension of volts per ampere. An impedance in the **s domain** is measured in ohms, and an admittance is measured in Siemens.

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Circuit Elements in the s domain

Resistor in the s domain:

Laplace transform

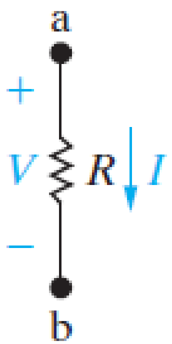
In time domain:

$v(t) = Ri(t)$

(a)

In s domain:

$V(s) = RI(s)$



(b)

where $V = L\{v\}$ ***and*** $I = L\{i\}$

The **s-domain** equivalent circuit of a resistor is simply a resistance of R ohms that caries a current of I ampere-seconds and has a terminal voltage of **V** volt-seconds.

Circuit Elements in the s domain

Inductor in the s domain:

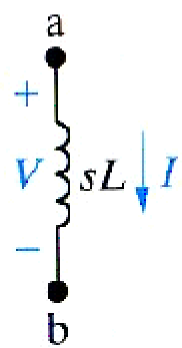
In time domain:

$$v = L\frac{d}{dt}i(t)$$

$$i = \frac{1}{L}\int_{0^-}^{t} v\,dx + I_0$$

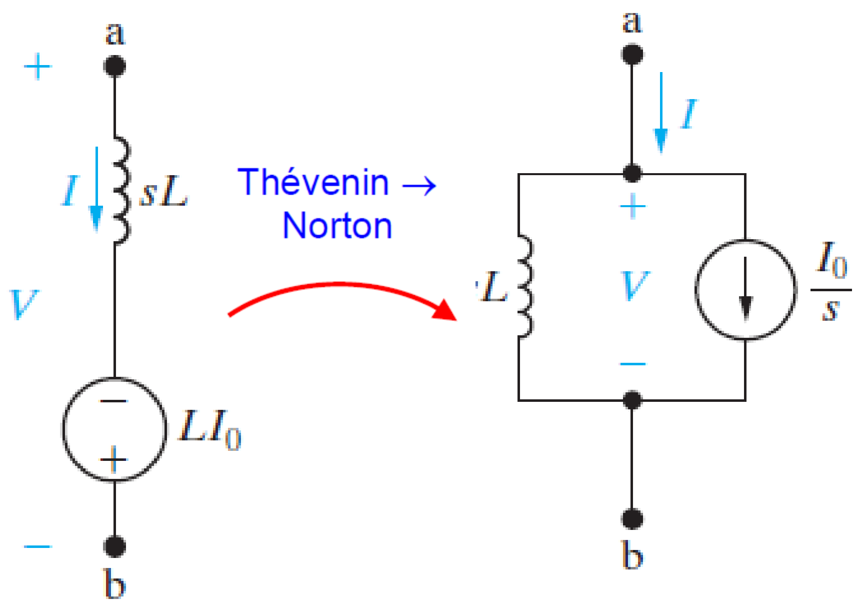
inductor carrying an initial current of I_0 amperes.

In s domain:



(initial current is 0 or initial energy stored = 0)

Two different circuit configurations satisfy



$$V = L[sI - i(0^-)] = sLI - LI_0.$$

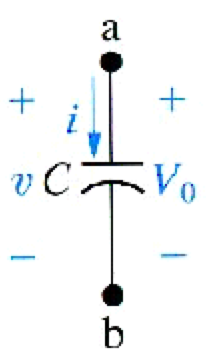
$$I = \underbrace{\frac{V + LI_0}{sL} = \frac{V}{sL} + \frac{I_0}{s}}.$$

Two other ways to get the equivalent circuits!

Circuit Elements in the s domain

Capacitor in the s domain:

In time domain:

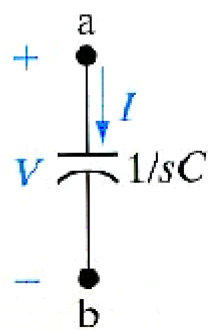


$$i = C\frac{d}{dt}v(t)$$

$$v = \frac{1}{C}\int_{0^-}^{t} i\,dx + V_0$$

capacitor initially charged to V_0 volts.

In s domain:



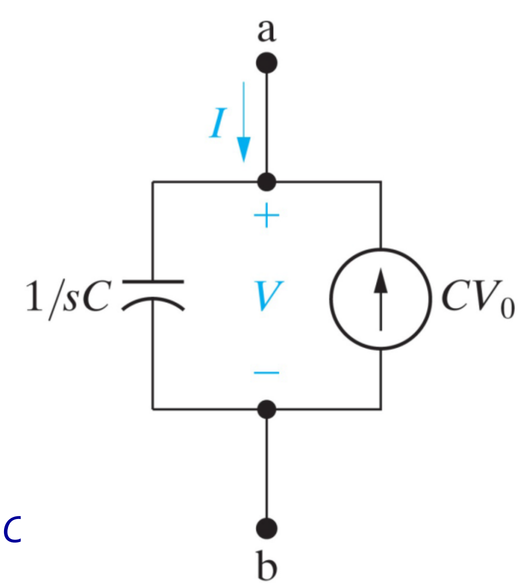
(initial voltage is 0

impedance of *1/sC* C

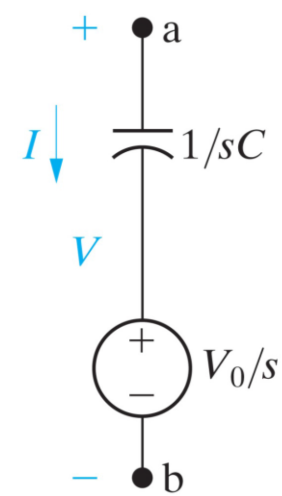
or $I = C\left[sV - v(0^-)\right]$

$$I = sCV - CV_0$$

also has two s-domain equivalent circuits.



Norton → Thévenin



$$V = \left(\frac{1}{sC}\right)I + \frac{V_0}{s}$$

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Circuit analysis in the s domain

How to analyze a circuit in the s-domain?

1. Replacing each circuit element with its s-domain equivalent. The initial energy in L or C is taken into account by adding independent source in series or parallel with the element impedance.
2. Writing & solving algebraic equations by the same circuit analysis techniques developed for resistive networks.
3. Obtaining the t-domain solutions by inverse Laplace transform.

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Circuit analysis in the s domain

Why to operate in the s-domain?

1. It is convenient in solving transient responses of linear, lumped parameter circuits, for the initial conditions have been incorporated into the equivalent circuit.
2. It is also useful for circuits with multiple essential nodes and meshes, for the simultaneous ODEs have been reduced to simultaneous algebraic equations.
3. It can correctly predict the impulsive response, which is more difficult in the t-domain.

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Circuit analysis in the s domain

Ohm's Law (*in the s-domain*):

$$V = ZI$$

Where Z refers to the s-domain impedance of the element. So:
* Resistor has impedance of *R* ohms
* Inductor has impedance of *sL* ohms
* Capacitor has impedance of *1/sC* ohms

The reciprocal of the impedance is admittance. So, in the s-domain:
* Resistor has admittance is 1/R siemens,
* Inductor has an admittance of 1/sL siemens
* Capacitor has an admittance of sC siemens

Kirchhoff's Law:

Algebraic $\Sigma I = 0$

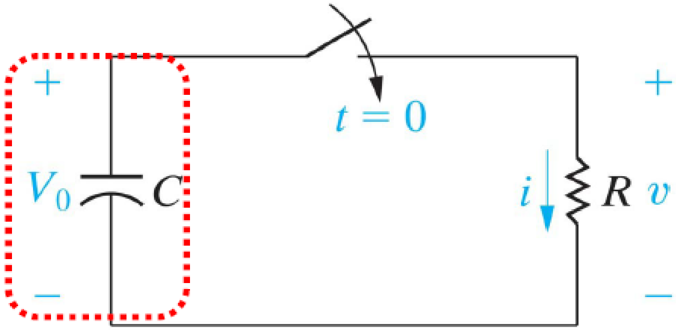
Algebraic $\Sigma V = 0$

Because the algebraic sum of the currents at a node is zero in the time domain, the algebraic sum of the transformed currents is also zero. Similarly, for the algebraic sum of the transformed voltages around a closed path is zero.

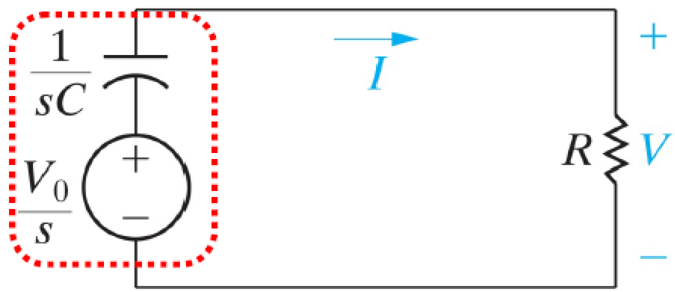
Circuit analysis in the s domain - Applications

The Natural Response of an *RC Circuit – 1st method*

- Q: $i(t),\ v(t) = ?$



The capacitor discharge circuit.



An s-domain equivalent circuit

Replacing the charged capacitor by a <u>Thévenin equivalent</u> circuit in the s-domain.

Summing the voltages around the mesh generates the expression

$$\frac{V_0}{s} = \frac{I}{sC} + IR, \Rightarrow I(s) = \frac{CV_0}{1 + RCs} = \frac{V_0/R}{s + (RC)^{-1}}$$

The t-domain solution is obtained by inverse Laplace transform:

$$i(t) = L^{-1}\left\{\frac{V_0/R}{s + (RC)^{-1}}\right\} = \frac{V_0}{R} e^{-t/(RC)} L^{-1}\left\{\frac{1}{s}\right\} = \frac{V_0}{R} e^{-t/(RC)} u(t).$$

$i(0^+) = V_0 /R$, which is true for $v_C(0^+) = v_C(0^-) = V_0$.

$i(\infty) = 0$,which is true for capacitor becomes open (no loop current) in steady state.

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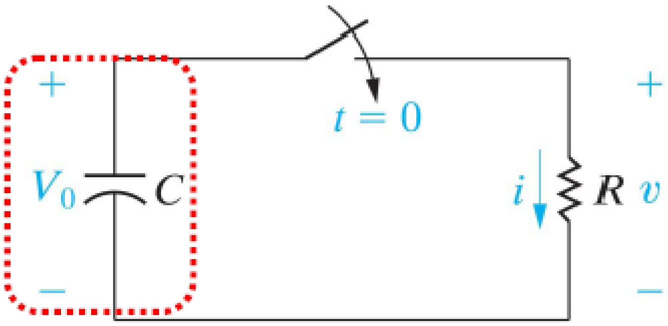
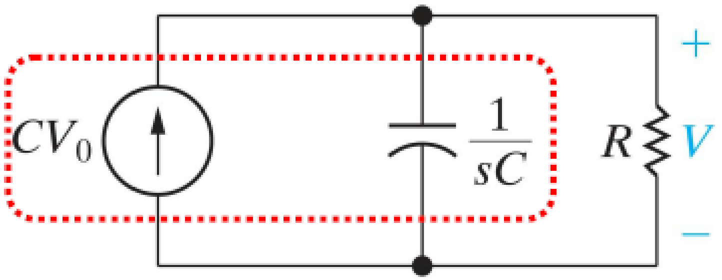
Circuit analysis in the s domain - Applications

The Natural Response of an *RC Circuit* – *2nd method*

This method, we find *v* without first finding the current *i*!



The capacitor discharge circuit.



s-domain equivalent circuit.

To directly solve $v(t)$, replacing the charged capacitor by a Norton equivalent in the s-domain.

From the original circuit, we transfer it to the s domain using the parallel equivalent circuit for the charged capacitor.

The node voltage equation is: $\frac{V}{R} + sCV = CV_0 \quad \Rightarrow \quad V = \frac{V_0}{s + (1/RC)}$

Inverse-transforming, we have:

$$v(t) = L^{-1}\left\{V_0 / \left[s + (RC)^{-1}\right]\right\} = V_0 e^{-t/(RC)} u(t) = Ri(t)$$

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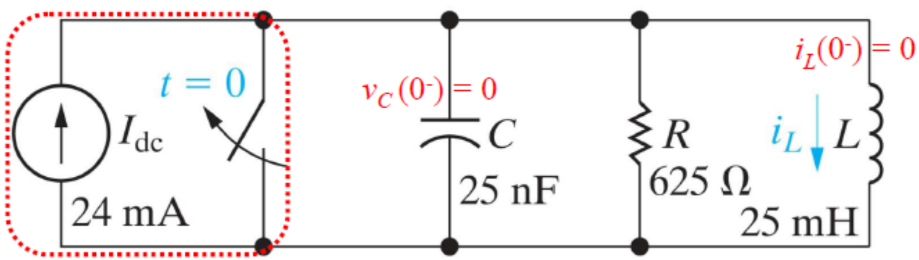
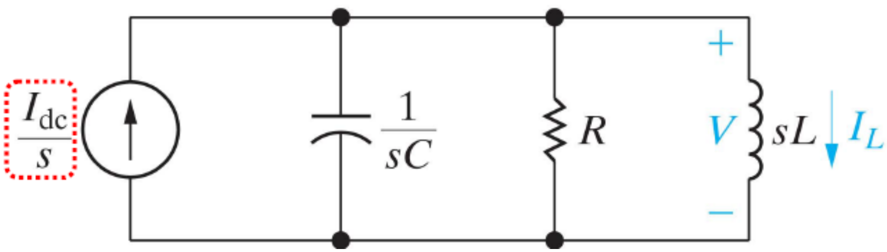
Circuit analysis in the s domain - Applications

The Step Response of a Parallel Circuit



The problem is to find the expression for i_L after the constant current source is switched across the parallel elements. The initial energy stored in the circuit is zero.



To find I_L, we first solve for V and then use

$$I_L = \frac{V}{sL}$$

to establish the s-domain expression for I_L.

Summing the currents $$sCV + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}.$$

$$\Rightarrow \quad V = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

Circuit analysis in the s domain - Applications

The Step Response of a Parallel Circuit

$$I_L = \frac{I_{dc}/LC}{s[s^2 + (1/RC)s + (1/LC)]}.$$

Substituting the numerical values of *R, L, C, and* I_{dc}

$$I_L = \frac{384 \times 10^5}{s(s^2 + 64{,}000s + 16 \times 10^8)}.$$

$$\Leftrightarrow \quad I_L = \frac{384 \times 10^5}{s(s + 32{,}000 - j24{,}000)(s + 32{,}000 + j24{,}000)}.$$

Now, we can test the, s-domain expression for I_L by checking to see whether the final-value theorem predicts the correct value for i_L *at* $t = \infty$. the final value of i_L must be *24* mA*.* The limit of sI_L as $s \rightarrow 0$ is

$$\lim_{s\to 0} sI_L = \frac{384 \times 10^5}{16 \times 10^8} = 24 \text{ mA}.$$

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Circuit analysis in the s domain - Applications

The Step Response of a Parallel Circuit

Do partial fraction expansion

$$I_L = \frac{K_1}{s} + \frac{K_2}{s + 32{,}000 - j24{,}000} + \frac{K_2^*}{s + 32{,}000 + j24{,}000}.$$

The partial fraction coefficients are $K_1 = \dfrac{384 \times 10^5}{16 \times 10^8} = 24 \times 10^{-3}$,

$$K_2 = \frac{384 \times 10^5}{(-32{,}000 + j24{,}000)(j48{,}000)} = 20 \times 10^{-3} \underline{/126.87^\circ}.$$

Substituting K_1 *and* K_2 and inverse transforming the resulting expression yields

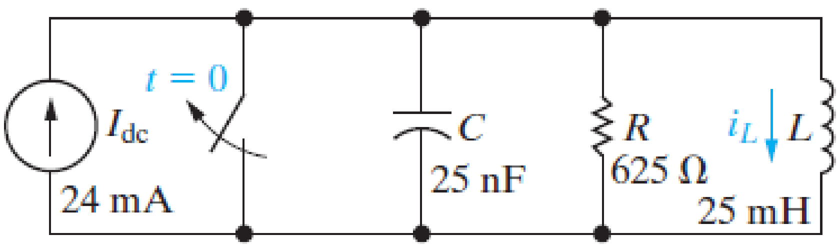
$$i_L = [24 + 40e^{-32{,}000t} \cos(24{,}000t + 126.87^\circ)]u(t)\text{mA}.$$

The answer: $40 \cos(24{,}000t + 126.87^\circ) = -24 \cos 24{,}000t - 32 \sin 24{,}000t.$

Circuit analysis in the s domain - Applications

The Transient Response of a Parallel RLC Circuit

Use the Laplace transform to find the transient behavior of a circuit shown in the figure with a sinusoidal current source: $i_g = I_m \cos\omega t$ A



where I_m = 24 mA and ω = 40,000 rad/s.

The s-domain expression for the source current is

$$I_g = \frac{sI_m}{s^2 + \omega^2}.$$

The voltage across the parallel elements is

$$V = \frac{(I_g/C)s}{s^2 + (1/RC)s + (1/LC)}.$$

$$\Rightarrow V = \frac{(I_m/C)s^2}{(s^2 + \omega^2)[s^2 + (1/RC)s + (1/LC)]}, \Rightarrow I_L = \frac{V}{sL} = \frac{(I_m/LC)s}{(s^2 + \omega^2)[s^2 + (1/RC)s + (1/LC)]}.$$

$$I_L = \frac{384 \times 10^5 s}{(s^2 + 16 \times 10^8)(s^2 + 64{,}000s + 16 \times 10^8)}. = \frac{384 \times 10^5 s}{(s - j\omega)(s + j\omega)(s + \alpha - j\beta)(s + \alpha + j\beta)},$$

where ω = 40,000, α = 32,000, and β = 24,000.

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Circuit analysis in the s domain - Applications

The Transient Response of a Parallel RLC Circuit

- Perform partial fraction expansion and inverse Laplace transform:

$$I_L(s) = \frac{K_1}{s - j\omega} + \frac{K_1^*}{s + j\omega} + \frac{K_2}{s - (-\alpha + j\beta)} + \frac{K_2^*}{s - (-\alpha - j\beta)}.$$

Driving frequency

Neper frequency

Damped frequency

$$i_L(t) = \{\underbrace{2|K_1|\cos(\omega t + \angle K_1)}_{\text{Steady-state response (source)}} + \underbrace{2|K_2|e^{-\alpha t}\cos(\beta t + \angle K_2)}_{\text{Natural response (RLC parameters)}}\}\, u(t).$$

Steady-state response (source)

Natural response (RLC parameters)

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Circuit analysis in the s domain - Applications

The Transient Response of a Parallel RLC Circuit

$$I_L = \frac{K_1}{s - j40{,}000} + \frac{K_1^*}{s + j40{,}000} + \frac{K_2}{s + 32{,}000 - j24{,}000} + \frac{K_2^*}{s + 32{,}000 + j24{,}000}.$$

$$K_1 = \frac{384 \times 10^5(j40{,}000)}{(j80{,}000)(32{,}000 + j16{,}000)(32{,}000 + j64{,}000)} = 7.5 \times 10^{-3}\underline{/-90^\circ}$$

$$K_2 = \frac{384 \times 10^5(-32{,}000 + j24{,}000)}{(-32{,}000 - j16{,}000)(-32{,}000 + j64{,}000)(j48{,}000)} = 12.5 \times 10^{-3}\underline{/90^\circ}$$

So inverse-transform the resulting expression:

$$i_L = [15\cos(40{,}000t - 90 + 25e^{-32{,}000t}\cos(24{,}000t + 90^\circ)]$$

$$= (15\sin 40{,}000t - 25e^{-32{,}000t}\sin 24{,}000t)u(t)\ \text{mA}.$$

We now test this final result of i_L to see whether it makes sense in terms of the given initial conditions and the known circuit behavior after the switch has been open for a long time. For *t = 0,* the equation predicts zero initial current, which agrees with the initial energy of zero in the circuit.

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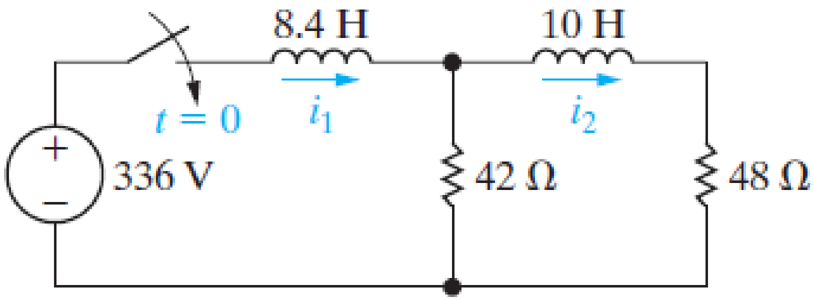
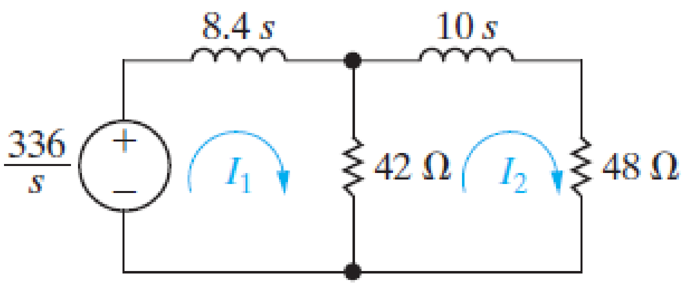
Circuit analysis in the s domain - Applications

The Step Response of a Multiple Mesh Circuit



Using Laplace techniques, we can solve a problem like the multiple-mesh circuit in the figure.

Here we want to find the branch currents i_1 and i_2 that arise when the 336 V_{dc} voltage source is applied suddenly to the circuit. Assume the initial energy stored in the circuit is zero.



s-domain equivalent circuit

Two mesh-current equations:

$$\begin{cases} 8.4sI_1 + 42(I_1 - I_2) = \dfrac{336}{s} \cdots (1) \\ 42(I_2 - I_1) + (10s + 48)I_2 = 0 \cdots (2) \end{cases}$$

$$\Rightarrow \begin{bmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 336/s \\ 0 \end{bmatrix}.$$

Using Cramer's method to solve for I_1 and I_2, we obtain

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Circuit analysis in the s domain - Applications

The Step Response of a Multiple Mesh Circuit

$$\Delta = \begin{vmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{vmatrix} = 84(s^2 + 14s + 24) = 84(s + 2)(s + 12),$$

$$N_1 = \begin{vmatrix} 336/s & -42 \\ 0 & 90 + 10s \end{vmatrix} = \frac{3360(s + 9)}{s}$$

$$N_2 = \begin{vmatrix} 42 + 8.4s & 336/s \\ -42 & 0 \end{vmatrix} = \frac{14{,}112}{s}$$

$$I_1 = \frac{N_1}{\Delta} = \frac{40(s + 9)}{s(s + 2)(s + 12)},$$

$$I_2 = \frac{N_2}{\Delta} = \frac{168}{s(s + 2)(s + 12)}$$

Expanding I_1 and I_2 into a sum of partial fractions

$$\begin{cases} I_1 = \dfrac{15}{s} - \dfrac{14}{s + 2} - \dfrac{1}{s + 12} \\ I_2 = \dfrac{7}{s} - \dfrac{8.4}{s + 2} + \dfrac{1.4}{s + 12} \end{cases}$$

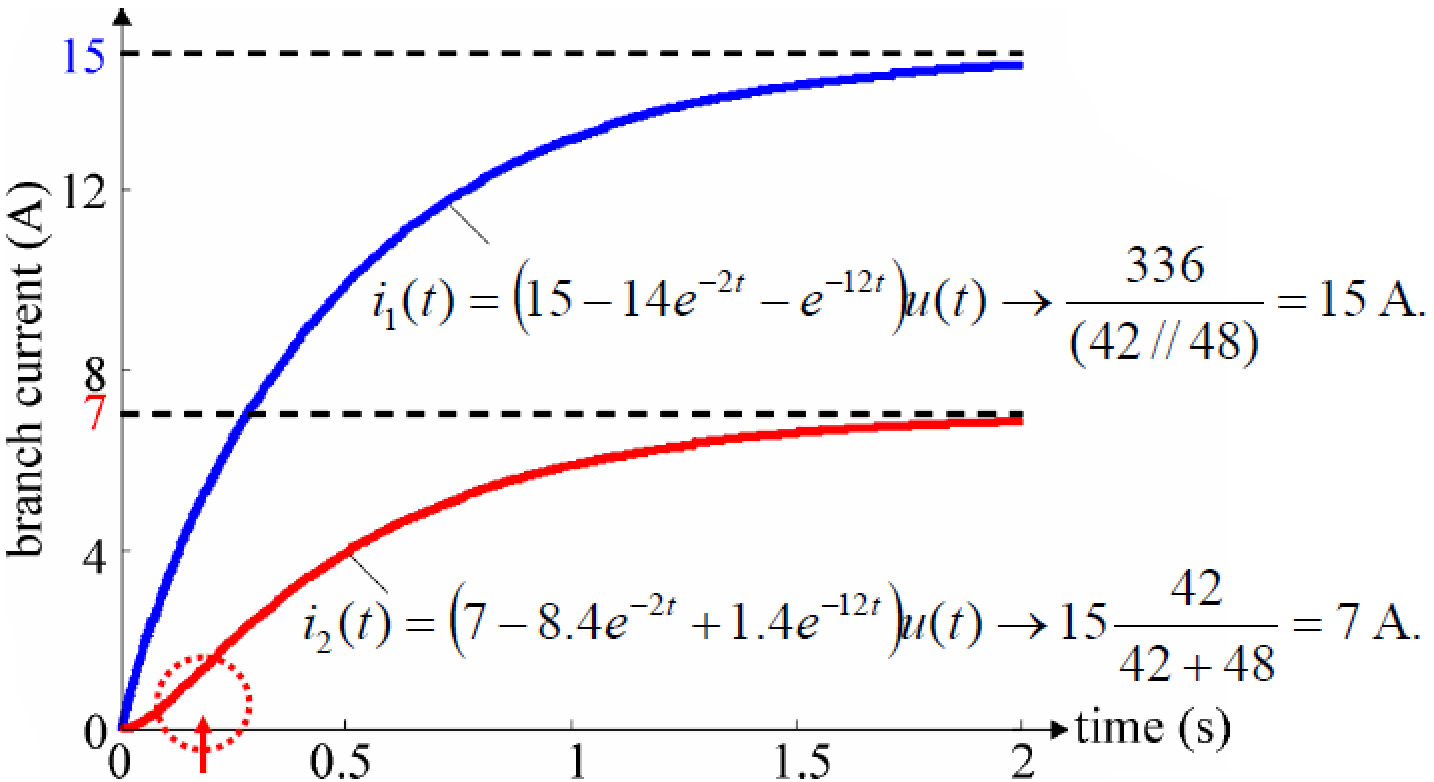
Inverse transform

$$\begin{cases} i_1 = (15 - 14e^{-2t} - e^{-12t})u(t) \text{ A}, \\ i_2 = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t) \text{ A}. \end{cases}$$

Circuit analysis in the s domain - Applications

The Step Response of a Multiple Mesh Circuit

- Perform inverse Laplace transform:



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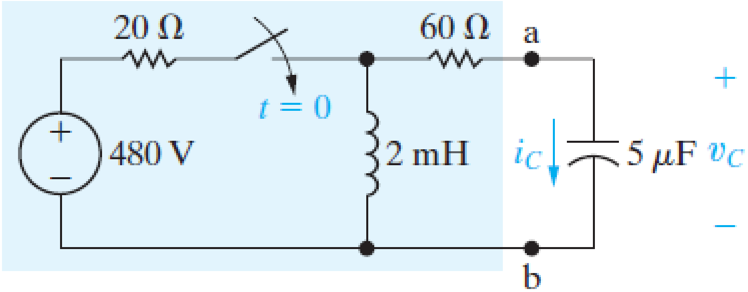
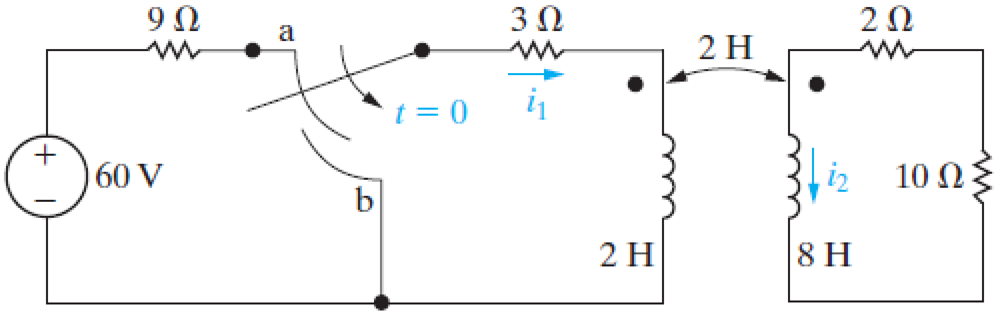
Circuit analysis in the s domain - Applications

The Use of Thevenin's Equivalent



A Circuit with Mutual Inductance



Read @ home!

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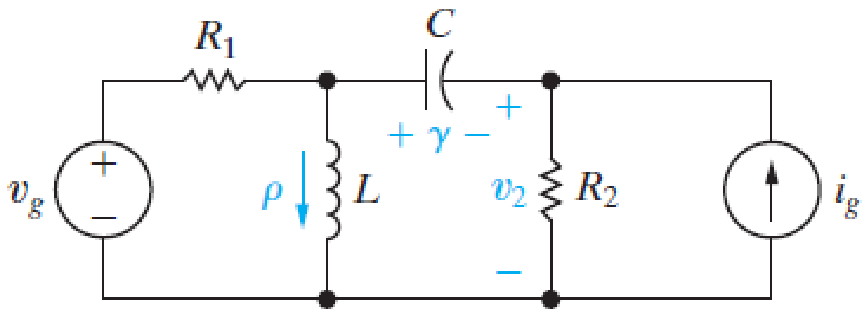
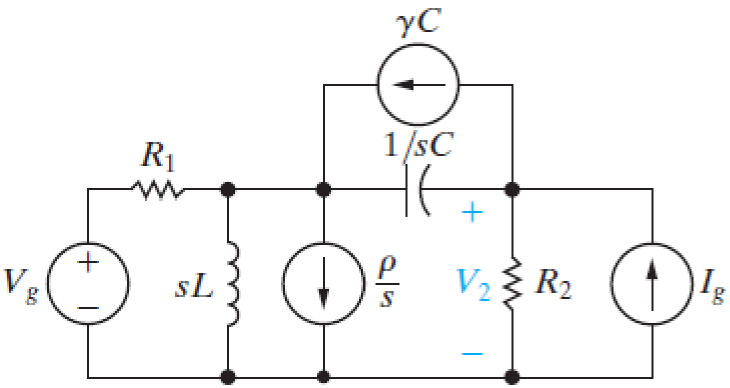
Circuit analysis in the s domain - Applications

The Use of Superposition



Use superposition to divide the response into components that can be identified with particular sources and initial conditions.



s-domain equivalent for the circuit

Assume: two sources are applied to the circuit, the inductor is carrying an initial current of ρ amperes and that the capacitor is carrying an initial voltage of γ volts. The desired response of the circuit is the voltage across the resistor R_2, *labeled* v_2.

To find V_2 *by superposition, we calculate the component of* V_2 *resulting* from each source acting alone, and then we sum the components. We begin with V_g *acting alone. Opening each of the three current sources* deactivates them.

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Circuit analysis in the s domain - Applications

The Use of Superposition

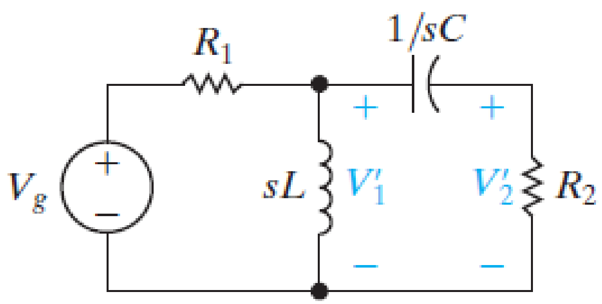


Figure shows the resulting circuit. We added the node voltage V_1' *to aid the analysis. The primes on V_1 and V_2 indicate that* they are the components of V_1 and V_2 *attributable to* V_g *acting alone.*

So, two equations that describe the circuit are

$$\left(\frac{1}{R_1} + \frac{1}{sL} + sC\right)V_1' - sCV_2' = \frac{V_g}{R_1},$$

$$-sCV_1' + \left(\frac{1}{R_2} + sC\right)V_2' = 0.$$

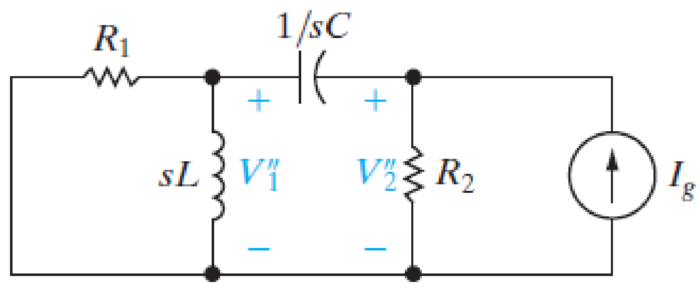
Let: $\begin{cases} Y_{11} = \dfrac{1}{R_1} + \dfrac{1}{sL} + sC; \\ Y_{12} = -sC; \\ Y_{22} = \dfrac{1}{R_2} + sC. \end{cases}$

$$\Rightarrow \begin{cases} Y_{11}V_1' + Y_{12}V_2' = V_g/R_1, \\ Y_{12}V_1' + Y_{22}V_2' = 0. \end{cases} \Rightarrow V_2' = \frac{-Y_{12}/R_1}{Y_{11}Y_{22} - Y_{12}^2}V_g.$$

Circuit analysis in the s domain - Applications

The Use of Superposition

With the current source I_g acting alone, the circuit shown

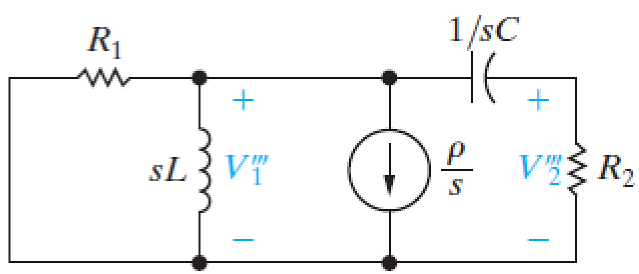


Here, V_1'' and V_2'' *are the components* of V_1 and V_2 resulting from I_g. Hence:

Two node-voltage equations that describe the circuit

$$\begin{cases} Y_{11}V_1'' + Y_{12}V_2'' = 0 \\ Y_{12}V_1'' + Y_{22}V_2'' = I_g. \end{cases} \Rightarrow \quad V_2'' = \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}^2} I_g.$$

To find the component of V_2 resulting from the initial energy stored in the inductor V_1''', we must solve the circuit shown in Fig.



$$\begin{cases} Y_{11}V_1''' + Y_{12}V_2''' = -\rho/s, \\ Y_{12}V_1''' + Y_{22}V_2''' = 0. \end{cases} \Rightarrow \quad V_2''' = \frac{Y_{12}/s}{Y_{11}Y_{22} - Y_{12}^2} \rho.$$

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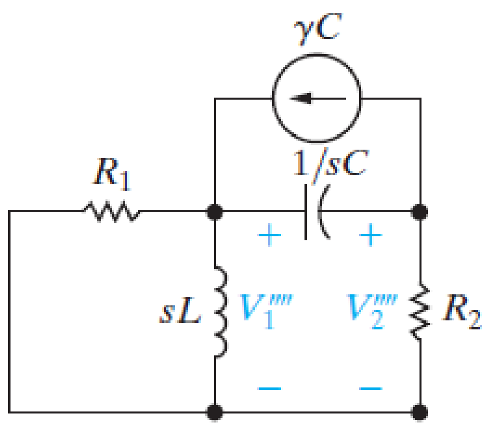
Circuit analysis in the s domain - Applications

The Use of Superposition



From the circuit shown in Fig., we find the component of V_2 (V_2'''') *resulting from the initial energy stored in the capacitor. The* node-voltage equations describing this circuit are

$$\begin{cases} Y_{11}V_1'''' + Y_{12}V_2'''' = \gamma C, \\ Y_{12}V_1'''' + Y_{22}V_2'''' = -\gamma C. \end{cases} \Rightarrow \quad V_2'''' = \frac{-(Y_{11}+Y_{12})C}{Y_{11}Y_{22}-Y_{12}^2}\gamma.$$

The expression for V_2 *is*

$$V_2 = V_2' + V_2'' + V_2''' + V_2''''$$

$$= \frac{-(Y_{12}/R_1)}{Y_{11}Y_{22}-Y_{12}^2}V_g + \frac{Y_{11}}{Y_{11}Y_{22}-Y_{12}^2}I_g + \frac{Y_{12}/s}{Y_{11}Y_{22}-Y_{12}^2}\rho + \frac{-C(Y_{11}+Y_{12})}{Y_{11}Y_{22}-Y_{12}^2}\gamma.$$

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Transfer function

The transfer function is defined as the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source):

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

The ratio of a circuit's output to its input in the s-domain

Y(s) is the Laplace transform of the output signal

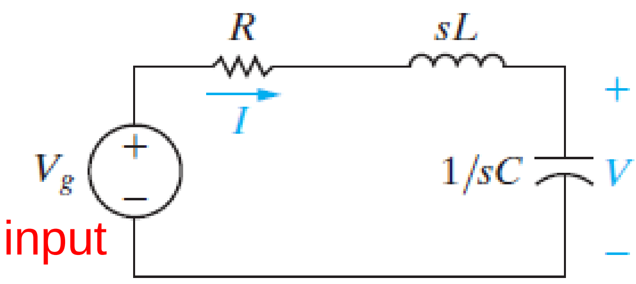
X(s) is the Laplace transform of the input signal

SEE

Transfer function

- In computing the transfer function, we restrict our attention to circuits where all initial conditions are zero.
- If a circuit has multiple independent sources, find the transfer function for each source and use superposition to find the response to all sources
- A single circuit can generate many transfer function

Example



If the current is defined as the response signal of the circuit,

$$H(s)=\frac{I}{V_g}=\frac{V_g}{R+sL+1/sC}\cdot\frac{1}{V_g}=\frac{sC}{s^2LC+sRC+1}$$

I corresponds to the output *Y(s)* & V_g *corresponds to the input X(s)*

If the voltage across the capacitor is defined as the output signal of the circuit, the transfer function is:

$$H(s)=\frac{V}{V_g}=\frac{1/sC}{R+sL+1/sC}=\frac{1}{s^2LC+sRC+1}$$

Thus, because circuits may have multiple sources and because the definition of the output signal of interest can vary, a single circuit can generate many transfer functions.

SEE

How do poles, zeros influence the solution?

❖ Since $Y(s) = H(s)\,X(s)$, ➔ the partial fraction expansion of the output $Y(s)$ yields a term $K/(s-a)$ for each pole $s = a$ of $H(s)$ or $X(s)$.

❖ The functional forms of the transient (natural) and steady-state responses $y_{tr}(t)$ and $y_{ss}(t)$ are determined by the poles of $H(s)$ and $X(s)$, respectively.

❖ The partial fraction coefficients of $Y_{tr}(s)$ and $Y_{ss}(s)$ are determined by both $H(s)$ and $X(s)$.

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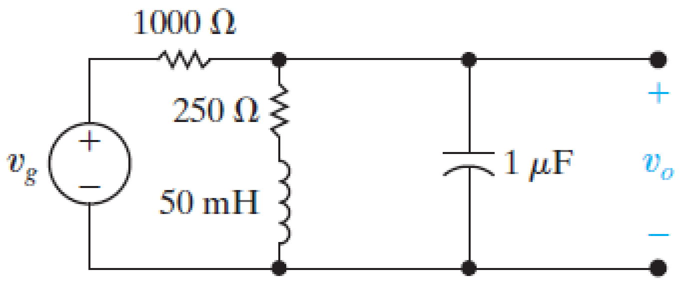
Transfer function

Example: Deriving the Transfer Function of a Circuit

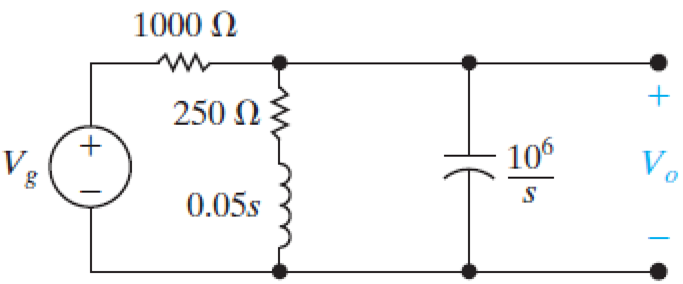


The voltage source v_g drives the circuit. The response signal is the voltage across the capacitor, v_0.

a) Calculate the numerical expression for the transfer function.

b) Calculate the numerical values for the poles and zeros of the transfer function.

Sol:



a) The first step: finding the transfer function is to construct the s-domain equivalent circuit. By definition, the transfer function is the ratio of V_0/V_g, which can be computed from a single node-voltage equation. Summing the currents away from the upper node generates

$$\frac{V_o - V_g}{1000} + \frac{V_o}{250 + 0.05s} + \frac{V_o s}{10^6} = 0. \Rightarrow V_o = \frac{1000(s + 5000)V_g}{s^2 + 6000s + 25 \times 10^6}.$$

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Transfer function

Example: Deriving the Transfer Function of a Circuit

Hence the transfer function is $H(s) = \frac{V_o}{V_g} = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$.

b) The poles of *H(s)* are the roots of the denominator polynomial (2 complex conjugate poles). So

$$-p_1 = -3000 - j4000$$
$$-p_2 = -3000 + j4000$$

The zeros of H(s) are the roots of the numerator polynomial; thus H(s) has a zero at

$$-z_1 = -5000$$

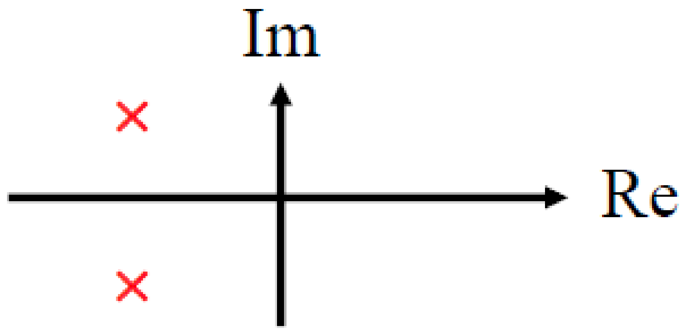
SEE

Transfer function

The location of poles and zeros of H(s)

For a linear lumped-parameter circuits:

- H(s) is always a rational function of s.
- Complex poles and zeros always appear in conjugate pairs.
- The poles must lie in the left half of the s-plane if bounded input leads to bounded output.
- The zeros of H(s) may be lie in either the right half or the left half of the s plane



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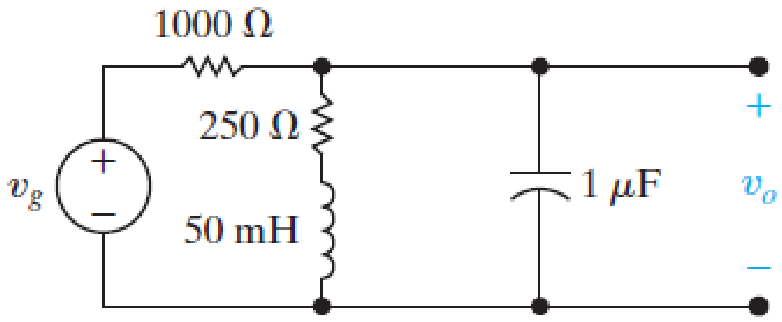
Transfer function in partial fraction expansions

Example (Linear ramp excitation)



The circuit is driven by a voltage source whose voltage increases linearly with lime: $\boldsymbol{v_g = 50t.u(t)}$.

a) Use the transfer function to find v_0.
b) Identify the transient component of the response.
c) Identify the steady-slate component of the response.
d) Sketch v_0 versus t for $0 \leq t \leq 1.5$ ms.

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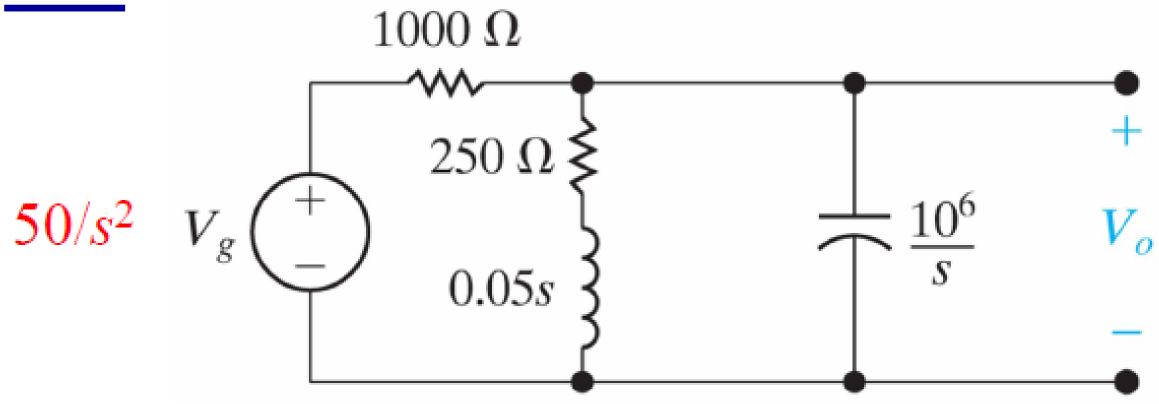
Sol:



From the previous example, we have the transfer function:

$$H(s) = \frac{V_0}{V_g} = \frac{1000(s+5000)}{s^2 + 6000s + 25\times 10^6}$$

$H(s)$ has 2 complex conjugate poles: s = - 3000 ± j4000.

$V_g(s) = 50/s^2$ has 1 repeated real pole: $s = 0^{(2)}$.

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Sol:

The transform of the driving voltage is $50/s^2$; therefore, the s-domain expression for the output voltage is

$$V_o = \frac{1000(s+5000)}{(s^2+6000s+25\times10^6)}\frac{50}{s^2} = \frac{K_1}{s+3000-j4000} + \frac{K_1^*}{s+3000+j4000} + \frac{K_2}{s^2} + \frac{K_3}{s}.$$

$$\begin{cases} K_1 = 5\sqrt{5}\times10^{-4}\angle 79.70^\circ; \quad K_1^* = 5\sqrt{5}\times10^{-4}\angle -79.70^\circ, \\ K_2 = 10, \quad K_3 = -4\times10^{-4}. \end{cases}$$

$$V_o(s) = H(s)V_g(s) = \frac{5\times10^4(s+5000)}{s^2\left(s^2+6000s+2.5\times10^7\right)} = Y_{tr} + Y_{ss}$$

expansion coefficients depend on $H(s)$ & $V_g(s)$

$$= \boxed{\frac{5\sqrt{5}\times10^{-4}\angle 80^\circ}{s+3000-j4000} + \frac{5\sqrt{5}\times10^{-4}\angle -80^\circ}{s+3000+j4000}} + \boxed{\frac{10}{s^2} - \frac{4\times10^{-4}}{s}}.$$

poles of $H(s)$: $-3k \pm j4k$ pole of $V_g(s)$: $0^{(2)}$

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Transfer function in partial fraction expansions

Sol. (cont.) The time-domain expression for v_0 is

$$v_0(t) = y_{tr} + y_{ss} =$$
$$= \left[\sqrt{5} \times 10^{-3} e^{-3000t} \cos(4000t + 80°)\right] u(t) + (10t - 4 \times 10^{-4}) u(t).$$

b) The transient component of v_0 is

$$10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.70°)$$

Note that this term is generated by the poles (-3000 + j4000) & (-3000 - j4000) of the transfer function

c) The steady-state component of the response is

$$(10t - 4 \times 10^{-4}) u(t)$$

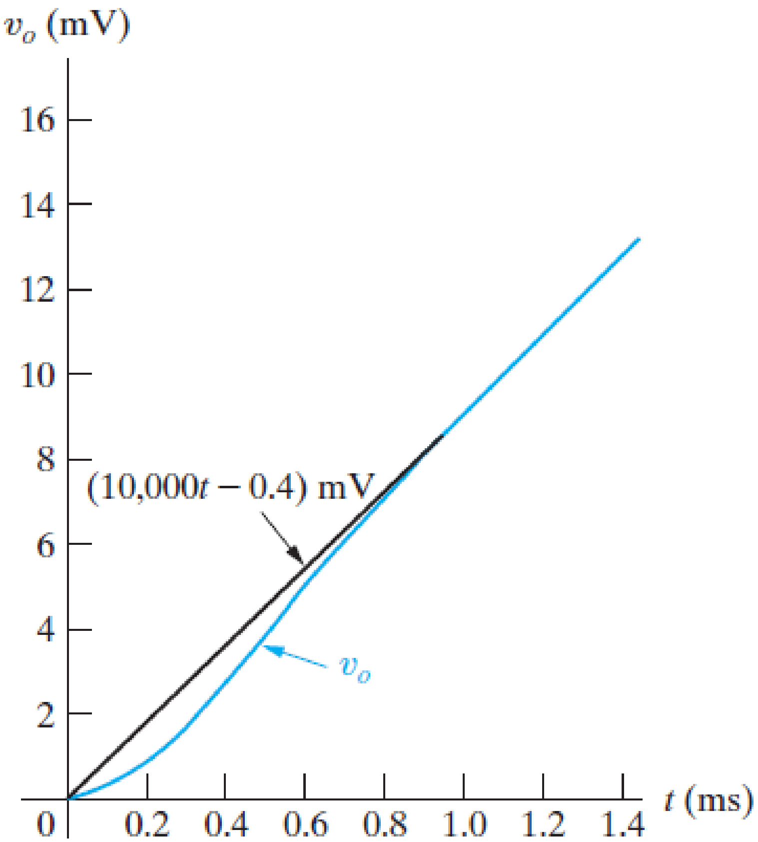
These two terms are generated by the second order pole (K/s^2) of the driving voltage.

SEE

Transfer function in partial fraction expansions

Sol. (cont.)

d) Figure shows a sketch of v_0 versus t.
Note that the deviation from the steady-state solution 10,000t - 0.4 mV is imperceptible after approximately 1 ms.



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Transfer function and the steady-state response

Once we have computed a circuit's transfer function, we no longer need to perform a separate phasor analysis of the circuit to determine its steady state response. Instead, we use the transfer function to relate the steady state response to the excitation source. First we assume that

Given sinusoidal source: $x(t) = A\cos(\omega t + \phi)$

and then we use Y(s) = H(s)X(s) to find the steady-state solution of *y(t).* To find the Laplace transform of *x(t),* we first write *x(t)* as

$$x(t) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi,$$

In s domain: $X(s) = \dfrac{(A\cos\phi)s}{s^2+\omega^2} - \dfrac{(A\sin\phi)\omega}{s^2+\omega^2} = \dfrac{A(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2}.$

The s-domain expression for the response:

The steady-state response

$$Y(s) = \frac{K_1}{s-j\omega} + \frac{K_1^*}{s+j\omega} + \sum \text{terms generated by the poles of } H(s). \quad (*)$$

Here, the first two terms result from the complex conjugate poles of the driving source. However, the terms generated by the poles of H(s) do not contribute to the steady-state response of y(t), because all these poles lie in the left half of the s plane;

SEE

Transfer function and the steady-state response

So, let find the partial fraction coefficient

$$K_1 = \frac{H(s)A(s\cos\phi - \omega\sin\phi)}{s + j\omega}\bigg|_{s=j\omega} = \frac{H(j\omega)A(j\omega\cos\phi - \omega\sin\phi)}{2j\omega}$$

$$= \frac{H(j\omega)A(\cos\phi + j\sin\phi)}{2} = \frac{1}{2}H(j\omega)Ae^{j\phi}.$$

In general, $H(j\omega)$ is a complex quantity, which we recognize by writing it in polar form, thus $H(j\omega) = H|(j\omega)|e^{j\theta(\omega)}$

$$\Rightarrow \quad K_1 = \frac{A}{2}|H(j\omega)|e^{j[\theta(\omega)+\phi]}.$$

We obtain the steady-state solution for *y(t)* by inverse-transforming **(*)**

$$y_{ss}(t) = A|H(j\omega)|\cos[\omega t + \phi + \theta(\omega)].$$

which indicates how to use the transfer function to find the steady-state sinusoidal response of a circuit. The amplitude of the response equals the amplitude of the source, A, times the magnitude of the transfer function, $|H(j\omega)|$. The phase angle of the response, $\phi + \theta(\omega)$, equals the phase angle of the source, ϕ, plus the phase angle of the transfer function, $\theta(\omega)$. We evaluate both $|H(j\omega)|$ and $\theta(\omega)$ at the frequency of the source, ω.

SEE

Transfer function and the steady-state response

Finally, the steady state response:

$$\boxed{y_{ss}(t) = A\left|H(j\omega)\right|\cos\left[\omega t + \phi + \theta(\omega)\right]}$$

- The amplitude of the response equals the amplitude of the source multiplies the magnitude of the transfer function.
- The phase angle of the response equals the phase angle of the source plus the phase angle of the transfer function.

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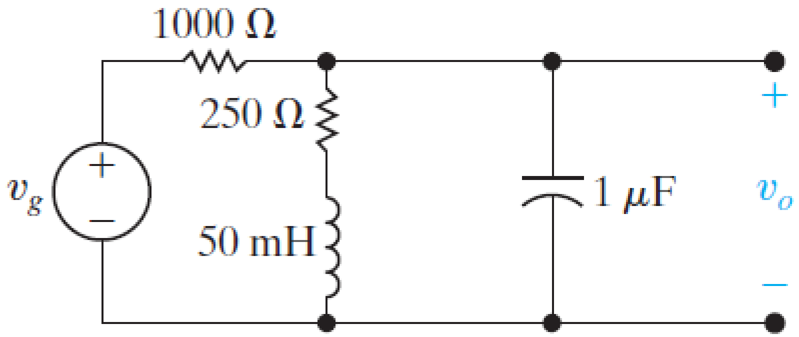
Transfer function and the steady-state response

Example: Using the Transfer Function to Find the Steady-State Sinusoidal Response



The sinusoidal source voltage is 120cos(5000t + 30^0) V. Find the steady-state expression for v_0

Sol: we have transfer function $H(s) = \dfrac{1000(s + 5000)}{s^2 + 6000s + 25 * 10^6}$.

The frequency of the voltage source is 5000 rad/s; hence we evaluate *H(s)* at H(j5000):

$$H(j5000) = \frac{1000(5000 + j5000)}{-25 * 10^6 + j5000(6000) + 25 \times 10^6} = \frac{1 + j1}{j6} = \frac{1 - j1}{6} = \frac{\sqrt{2}}{6}\angle -45^\circ.$$

$$\Rightarrow \quad v_{o_{ss}} = \frac{(120)\sqrt{2}}{6}\cos(5000t + 30^\circ - 45^\circ) = 20\sqrt{2}\cos(5000t - 15^\circ)\text{ V}.$$

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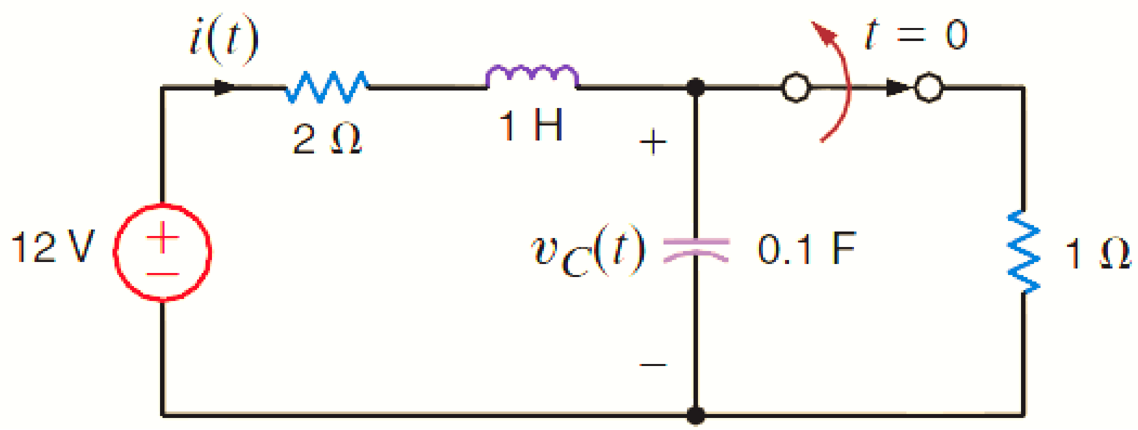
APPLICATION EXAMPLE 1

Consider the network shown in Fig. Assume that the network is in steady state prior to t = 0. Find the current *i(t)* for t > 0.



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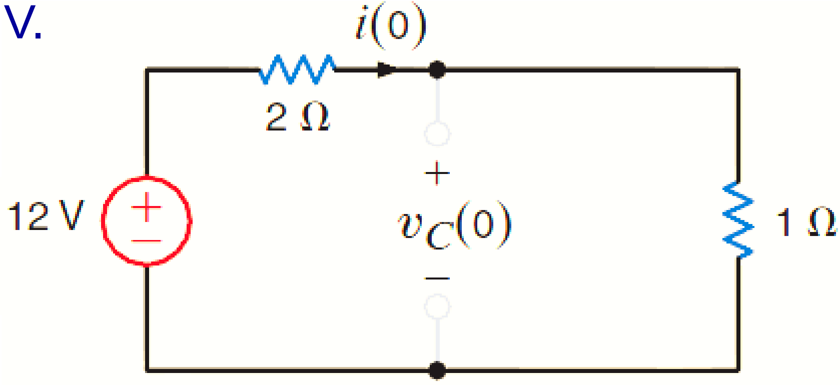
APPLICATION EXAMPLE 1 - SOLUTION

In steady state prior to t = 0, the network is as shown in Fig., since the inductor acts like a short circuit to dc and the capacitor acts like an open circuit to dc. From Fig. we note that i(0) = 4 A and $v_C(0)$ = 4 V.



For t > 0, the KVL equation for the network is

$$12u(t) = 2i(t) + 1\frac{di(t)}{dt} + \frac{1}{0.1}\int_0^t i(x)\,dx + v_C(0)$$

the transformed expression becomes

$$\frac{12}{s} = 2\mathbf{I}(s) + s\mathbf{I}(s) - i(0) + \frac{10}{s}\mathbf{I}(s) + \frac{v_C(0)}{s}$$

Using the initial conditions, we find that the equation becomes

$$\frac{12}{s} = \mathbf{I}(s)\left(2 + s + \frac{10}{s}\right) - 4 + \frac{4}{s} \Leftrightarrow \mathbf{I}(s) = \frac{4(s+2)}{s^2 + 2s + 10} = \frac{4(s+2)}{(s+1-j3)(s+1+j3)}$$

SEE

APPLICATION EXAMPLE 1 - SOLUTION

and then

$$K_1 = \left. \frac{4(s+2)}{s+1+j3} \right|_{s=-1+j3}$$

$$= 2.11 \angle{-18.4^\circ}$$

Therefore,

$$i(t) = 2(2.11)e^{-t}\cos(3t - 18.4^\circ)u(t) \text{ A}$$

Note that this expression satisfies the initial condition i(0) = 4 A.

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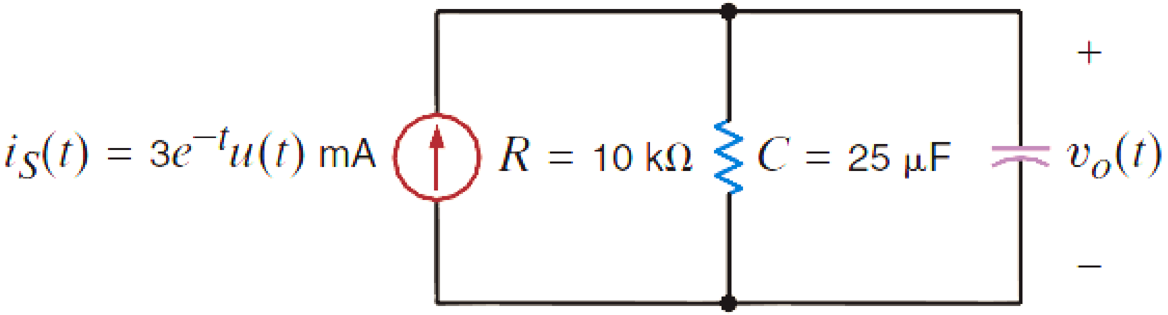
APPLICATION EXAMPLE 2

Given the network in Fig., let draw the s-domain equivalent circuit and find the output voltage in both the s and time domains.



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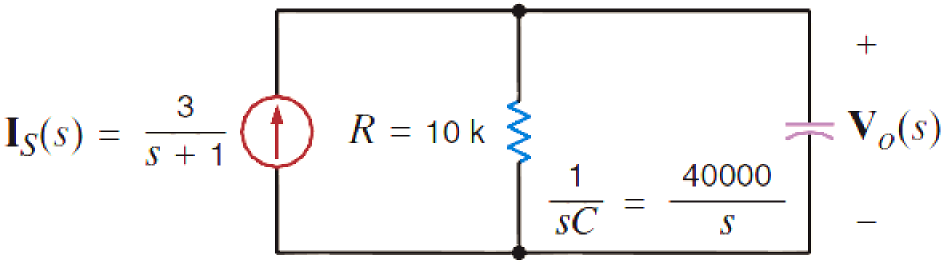
APPLICATION EXAMPLE 2 - SOLUTION

The s-domain network is shown in Fig.



We can write the output voltage as

$$V_o(s) = \left[R // \frac{1}{sC}\right] I_S(s) = \left[\frac{1/C}{s + (1/RC)}\right] I_S(s)$$

Given the element values, $V_o(s)$ becomes

$$V_o(s) = \left(\frac{40{,}000}{s+4}\right)\left(\frac{0.003}{s+1}\right) = \frac{120}{(s+4)(s+1)}$$

Expanding $V_o(s)$ into partial fractions yields

SEE

APPLICATION EXAMPLE 2 - SOLUTION

$$\mathbf{V}_o(s) = \frac{120}{(s+4)(s+1)} = \frac{40}{s+1} - \frac{40}{s+4}$$

Performing the inverse Laplace transform yields the time-domain representation

$$v_o(t) = 40\left[e^{-t} - e^{-4t}\right]u(t) \text{ V}$$

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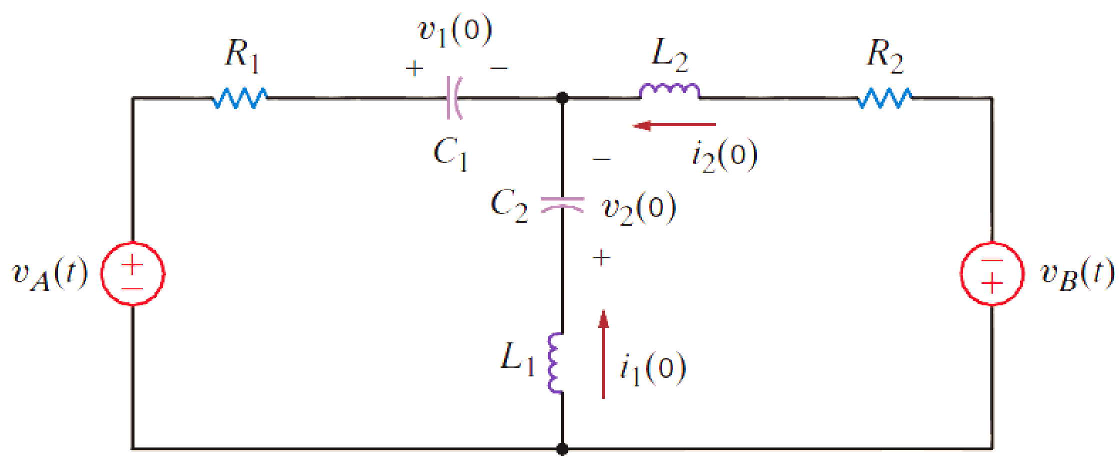
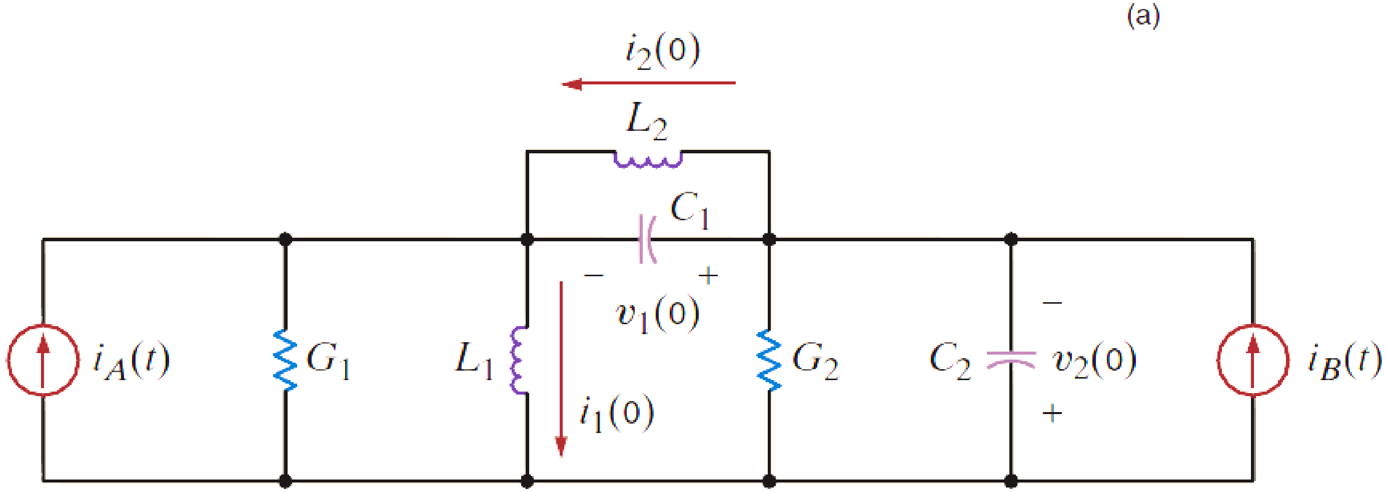
APPLICATION EXAMPLE 3

Given the circuits in Figs. a and b, we wish to write the mesh equations in the *s*-domain for the network in Fig. a and the node equations in the *s*-domain for the network in Fig. b.



(a)



(b)

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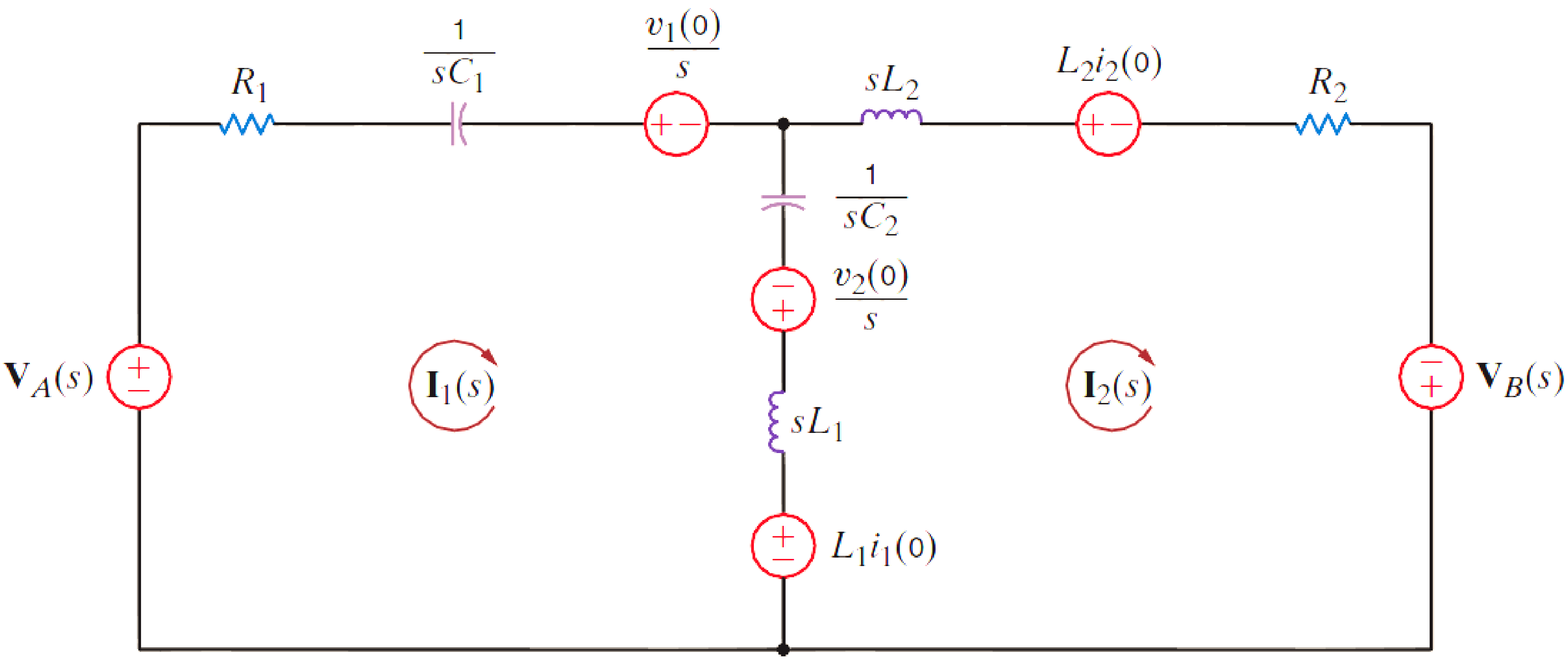
APPLICATION EXAMPLE 3 - SOLUTION

The transformed circuit for the network in Fig. a is shown in Fig. c.



(c)

The mesh equations for this network are

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APPLICATION EXAMPLE 3 - SOLUTION

$$\left(R_1 + \frac{1}{sC_1} + \frac{1}{sC_2} + sL_1\right)\mathbf{I}_1(s) - \left(\frac{1}{sC_2} + sL_1\right)\mathbf{I}_2(s) = \mathbf{V}_A(s) - \frac{v_1(0)}{s} + \frac{v_2(0)}{s} - L_1 i_1(0)$$

$$-\left(\frac{1}{sC_2} + sL_1\right)\mathbf{I}_1(s) + \left(\frac{1}{sC_2} + sL_1 + sL_2 + R_2\right)\mathbf{I}_2(s) = L_1 i_1(0) - \frac{v_2(0)}{s} - L_2 i_2(0) + \mathbf{V}_B(s)$$

The transformed circuit for the network in Fig. b is shown in Fig. d.

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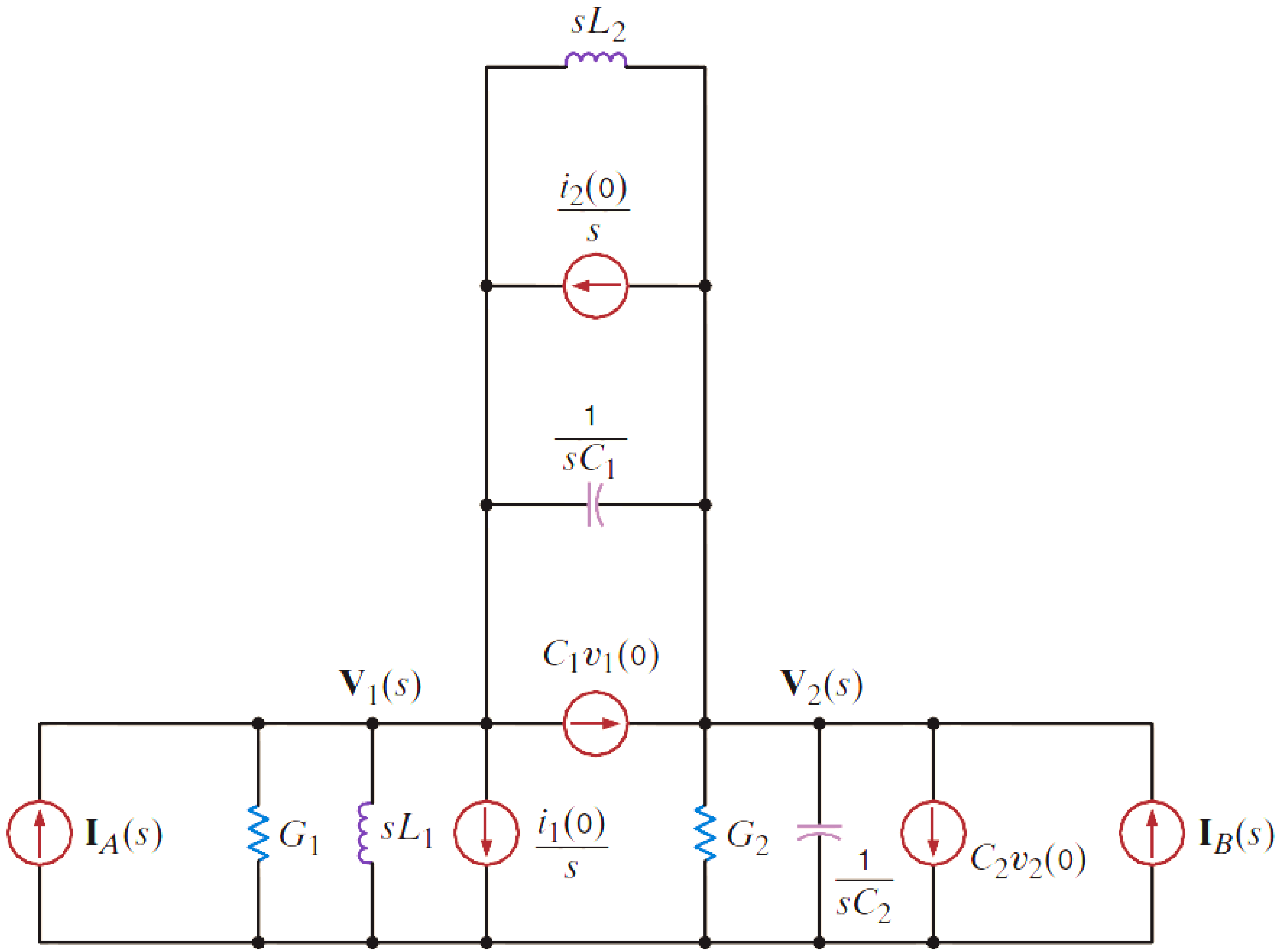
APPLICATION EXAMPLE 3 - SOLUTION



(d)

The node equations for this network are

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APPLICATION EXAMPLE 3 - SOLUTION

$$\left(G_1 + \frac{1}{sL_1} + sC_1 + \frac{1}{sL_2}\right)\mathbf{V}_1(s) - \left(\frac{1}{sL_2} + sC_1\right)\mathbf{V}_2(s)$$

$$= \mathbf{I}_A(s) - \frac{i_1(0)}{s} + \frac{i_2(0)}{s} - C_1 v_1(0)$$

$$-\left(\frac{1}{sL_2} + sC_1\right)\mathbf{V}_1(s) + \left(\frac{1}{sL_2} + sC_1 + G_2 + sC_2\right)\mathbf{V}_2(s)$$

$$= C_1 v_1(0) - \frac{i_2(0)}{s} - C_2 v_2(0) + \mathbf{I}_B(s)$$