

①

$$\begin{cases} x_1 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{cases}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[ \begin{array}{ccc|c} 4 & -8 & 12 & 1 \\ 2 & -3 & 2 & 1 \\ 1 & 0 & -4 & 8 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 4 & -8 & 12 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & 0 & 8 & 59 \end{array} \right]$$

$$\xleftarrow{R_3 = 8R_2} \left[ \begin{array}{ccc|c} 4 & -8 & 12 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & -8 & 28 & -31 \end{array} \right] \xleftarrow{-2R_3}$$

$$\left[ \begin{array}{ccc|c} 4 & -8 & 12 & 1 \\ 0 & -2 & 8 & -1 \\ 1 & 0 & -4 & 8 \end{array} \right] \xrightarrow{R_2 = R_1 - 2R_3} \left[ \begin{array}{ccc|c} 4 & -8 & 12 & 1 \\ 0 & -2 & 8 & -1 \\ 1 & 0 & -4 & 8 \end{array} \right]$$

$$\begin{cases} 4x_1 - 8x_2 + 12x_3 = 1 \\ -2x_2 + 8x_3 = -1 \\ 8x_3 = 59 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \frac{35}{4} \\ x_2 = \frac{55}{2} \\ x_3 = \frac{27}{4} \end{cases}$$

$-4R_3$

②

a) Reduced echelon form

b) ~~Echelon form~~ Reduced echelon form

c) Neither echelon nor reduced echelon form

d) Echelon form

③

$$\text{a) } \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{array} \right] \xrightarrow{R_2 = R_2 - 4R_1} \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{array} \right] \xrightarrow{R_3 = R_3 - 6R_1} \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{array} \right] \xrightarrow{R_3 = R_3 + 5R_2} \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\Rightarrow$  pivot columns is column 1 and column 2 or  $\left\{ \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \right\}$

$$\text{b) } \left[ \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array} \right] \xrightarrow{R_2 = R_2 - 3R_1} \left[ \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{array} \right] \xrightarrow{R_3 = R_3 - 5R_1} \left[ \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{array} \right] \rightarrow \left[ \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{array} \right]$$

$$\xrightarrow{R_3 = R_3 + 8R_2} \left[ \begin{array}{cccc} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{array} \right]$$

$\Rightarrow$  pivot columns is column 1, 2, 4 or  $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \\ 1 \end{bmatrix} \right\}$



$$(4) a) \begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R_1 = 5R_1 + 4R_2} \begin{bmatrix} 5 & 15 & 0 & -25 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & -5 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + 3x_2 = -5 \\ x_3 = 3 \\ x_2 = x_2 \text{ (free variable)} \end{cases} \quad (\Rightarrow) \begin{cases} x_1 = -5 - 3s \\ x_2 = s \\ x_3 = 3 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$b) \begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \xrightarrow{R_3 = R_3 + 2R_1} \begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 + 3R_1} \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 - 4x_2 + 2x_3 = 0 \quad \left| \begin{array}{l} x_2 \in \mathbb{R} \\ x_3 \in \mathbb{R} \end{array} \right.$$

$$\Rightarrow x_1 = \frac{4x_2 - 2x_3}{3}$$



$$\textcircled{5} \text{ a) } \left[ \begin{array}{ccc|c} 4 & 2 & 1 & 18 \\ 4 & -2 & -2 & 28 \\ 2 & -3 & 2 & -8 \end{array} \right] \xrightarrow{\substack{R_2 = R_1 - R_2 \\ R_3 = R_1 - 2R_2}} \left[ \begin{array}{ccc|c} 4 & 2 & 1 & 18 \\ 0 & 4 & 3 & -10 \\ 0 & 8 & -3 & 34 \end{array} \right]$$

$$\xrightarrow{R_3 = 4R_3 - 8R_2} \left[ \begin{array}{ccc|c} 4 & 2 & 1 & 18 \\ 0 & 4 & 3 & -10 \\ 0 & 0 & -36 & 216 \end{array} \right]$$

$$\Rightarrow \begin{cases} 4x_1 + 2x_2 + x_3 = 18 \\ 4x_2 + 3x_3 = -10 \\ -36x_3 = 216 \end{cases} \Rightarrow \begin{cases} x_1 = 5 \\ x_2 = 2 \\ x_3 = -6 \end{cases}$$

$$\text{b) } \left[ \begin{array}{cccc|c} 2 & 1 & 1 & 2 & -1 \\ 5 & -2 & 1 & -3 & 0 \\ -1 & 3 & 2 & 2 & 1 \\ 3 & 2 & 3 & -5 & 12 \end{array} \right] \xrightarrow{\substack{R_2 = 2R_2 - 5R_1 \\ R_3 = R_1 + 2R_2 \\ R_4 = 2R_1 - 3R_2}} \left[ \begin{array}{cccc|c} 2 & 1 & 1 & 2 & -1 \\ 0 & -9 & -3 & -16 & 5 \\ 0 & 7 & 5 & 6 & 1 \\ 0 & 1 & 3 & -16 & 27 \end{array} \right]$$

$$\xrightarrow{\substack{R_3 = 9R_3 + 7R_2 \\ R_4 = 1R_2 + 9R_4}} \left[ \begin{array}{cccc|c} 2 & 1 & 1 & 2 & -1 \\ 0 & -9 & -3 & -16 & 5 \\ 0 & 0 & 24 & -58 & 44 \\ 0 & 0 & 24 & -160 & 248 \end{array} \right] \xrightarrow{R_4 = R_4 - R_3} \left[ \begin{array}{cccc|c} 2 & 1 & 1 & 2 & -1 \\ 0 & -9 & -3 & -16 & 5 \\ 0 & 0 & 24 & -58 & 44 \\ 0 & 0 & 0 & 102 & -204 \end{array} \right]$$

$$\rightarrow \begin{cases} 2x_1 + x_2 + x_3 + 2x_4 = -1 \\ -9x_2 - 3x_3 - 16x_4 = 5 \\ 24x_3 - 58x_4 = 44 \\ 102x_4 = -204 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 4 \\ x_3 = -3 \\ x_4 = -2 \end{cases}$$

$$\textcircled{6} \text{ a) total incoming flow} = \text{total outgoing flow}$$

$$\text{at top} \Rightarrow x_2 + 200 = x_1$$

$$\text{at right} \Rightarrow x_4 = 100 + x_2$$

$$\text{at Bot} \Rightarrow x_3 = 200 + x_4$$

$$\Rightarrow \begin{cases} x_2 + 200 = x_1 \\ x_4 = 100 + x_2 \\ x_3 = 200 + x_4 \end{cases} \Leftrightarrow \begin{cases} x_2 + 200 = x_1 \\ x_4 = 100 + x_2 \\ x_3 = 300 + x_2 \end{cases} \text{ let } x = x_2 \Rightarrow \begin{cases} x_1 = 200 + x \\ x_3 = 300 + x \\ x_4 = 100 + x \end{cases}$$

$$b) x_4 = 0$$

$$x_4 = 100 + \pi \Rightarrow \pi = -100$$

$$\Rightarrow \begin{cases} x_1 = 100 \\ x_2 = -100 \\ x_3 = 200 \\ x_4 = 0 \end{cases}$$

$$c) x_4 = 100$$

$$x_4 = 100 + \pi \Rightarrow \pi = 0$$

$$\Rightarrow \begin{cases} x_1 = 200 \\ x_2 = 0 \\ x_3 = 300 \\ x_4 = 100 \end{cases}$$