CHAPTER 1

Vectors and Fields

Electromagnetics deals with the study of electric and magnetic fields. It is at once apparent that we need to familiarize ourselves with the concept of a field, and in particular with electric and magnetic fields. These fields are vector quantities and their behavior is governed by a set of laws known as Maxwell's equations. The mathematical formulation of Maxwell's equations and their subsequent application in our study of the elements of engineering electromagnetics require that we first learn the basic rules pertinent to mathematical manipulations involving vector quantities. With this goal in mind, we devote this chapter to vectors and fields in general and electric and magnetic fields in particular.

We first study certain simple rules of vector algebra without the implication of a coordinate system and then introduce the Cartesian, cylindrical, and spherical coordinate systems. After learning the vector algebraic rules, we turn our attention to a discussion of scalar and vector fields, static as well as time-varying, by means of some familiar examples. Following this general introduction to vectors and fields, we study the concepts of electric and magnetic fields by considering the experimental laws of Coulomb and Ampere, and illustrate by example the computation of electric fields due to charge distributions and magnetic fields due to current distributions. Finally, by combining the electric and magnetic field concepts, we introduce the Lorentz force equation and use it to discuss charged particle motion in electric and magnetic fields.

1.1 VECTOR ALGEBRA

Vectors versus scalars

In the study of elementary physics, we come across quantities such as mass, temperature, velocity, acceleration, force, and charge. Some of these quantities have associated with them not only a magnitude but also a direction in space, whereas others are characterized by magnitude only. The former class of quantities are known as *vectors* and the latter class of quantities are known as *scalars*. Mass, temperature, and charge are scalars, whereas velocity, acceleration, and force are vectors. Other examples are voltage and current for scalars and electric and magnetic fields for vectors.

Vector quantities are represented by symbols in boldface roman type (e.g., **A**), to distinguish them from scalar quantities, which are represented by symbols in lightface italic type (e.g., A). Graphically, a vector, say, **A**, is represented by a straight line with an arrowhead pointing in the direction of **A** and having a length proportional to the magnitude of **A**, denoted $|\mathbf{A}|$ or simply A. Figure 1.1 shows four vectors drawn to the same scale. If the top of the page represents north, then vectors **A** and **B** are directed eastward, with the magnitude of **B** being twice that of **A**. Vector **C** is directed toward the northeast and has a magnitude three times that of **A**. Vector **D** is directed toward the southwest and has a magnitude equal to that of **C**. Since **C** and **D** are equal in magnitude but opposite in direction, one is the negative of the other.

Unit vector defined

Since a vector may have in general an arbitrary orientation in three dimensions, we need to define a set of three reference directions at each and every point in space in terms of which we can describe vectors drawn at that point. It is convenient to choose these three reference directions to be mutually orthogonal, as, for example, east, north, and upward, or the three contiguous edges of a rectangular room. Thus, let us consider three mutually orthogonal reference directions and direct *unit vectors* along the three directions as shown, for example, in Fig. 1.2(a). A unit vector has magnitude *unity*. We shall represent a unit vector by the symbol **a** and use a subscript to denote its direction. We shall denote the three directions by subscripts 1, 2, and 3. We note that for a fixed orientation of $\mathbf{a}_1$, two combinations are possible for the orientations of $\mathbf{a}_2$ and $\mathbf{a}_3$, as shown in Figs. 1.2(a) and (b). If we take a right-hand screw and turn it from $\mathbf{a}_1$ to $\mathbf{a}_2$ through the 90° angle, it progresses in the direction of $\mathbf{a}_3$ in Fig. 1.2(a) but opposite to the direction of $\mathbf{a}_3$ in Fig. 1.2(b). Alternatively, a left-hand screw when turned from $\mathbf{a}_1$ to $\mathbf{a}_2$ in Fig. 1.2(b) will progress in the direction of $\mathbf{a}_3$. Hence the set of unit vectors in Fig. 1.2(a) corresponds to a right-handed system, whereas the set in Fig. 1.2(b) corresponds to a left-handed system. We shall work consistently with the right-handed system.

A vector of magnitude different from unity along any of the reference directions can be represented in terms of the unit vector along that direction. Thus $4\mathbf{a}_1$ represents a vector of magnitude 4 units in the direction of $\mathbf{a}_1$, $6\mathbf{a}_2$ represents a

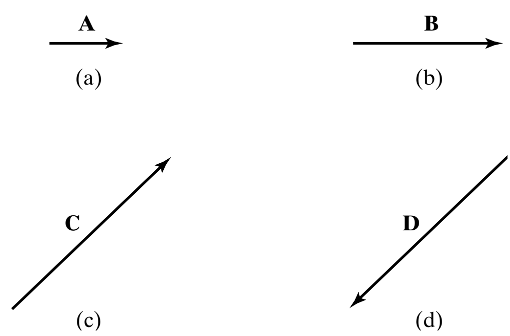

FIGURE 1.1

Graphical representation of vectors.

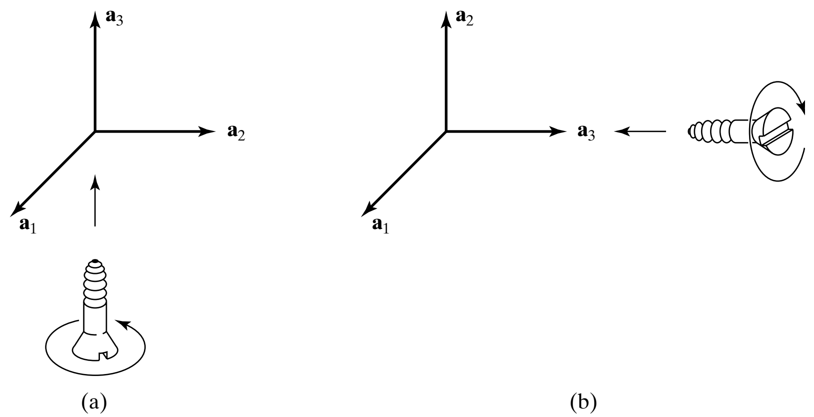

FIGURE 1.2

(a) Set of three orthogonal unit vectors in a right-handed system. (b) Set of three orthogonal unit vectors in a left-handed system.

vector of magnitude 6 units in the direction of $\mathbf{a}_2$, and $-2\mathbf{a}_3$ represents a vector of magnitude 2 units in the direction opposite to that of $\mathbf{a}_3$, as shown in Fig. 1.3. Two vectors are added by placing the beginning of the second vector at the tip of the first vector and then drawing the sum vector from the beginning of the first vector to the tip of the second vector. Thus, to add $4\mathbf{a}_1$ and $6\mathbf{a}_2$, we simply slide $6\mathbf{a}_2$ without changing its direction until its beginning coincides with the tip of $4\mathbf{a}_1$ and then draw the vector $(4\mathbf{a}_1 + 6\mathbf{a}_2)$ from the beginning of $4\mathbf{a}_1$ to the tip of $6\mathbf{a}_2$, as shown in Fig. 1.3. To see this, imagine that on the floor of an empty rectangular room, you are going from one corner to the opposite corner. Then to reach the destination, you can first walk along one edge and then along the second edge. Alternatively, you can go straight to the destination along the diagonal. By adding $-2\mathbf{a}_3$ to the vector $(4\mathbf{a}_1 + 6\mathbf{a}_2)$ in a similar manner, we obtain the vector $(4\mathbf{a}_1 + 6\mathbf{a}_2 - 2\mathbf{a}_3)$, as shown in Fig. 1.3. We note that the magnitude of $(4\mathbf{a}_1 + 6\mathbf{a}_2)$ is $\sqrt{4^2 + 6^2}$ or 7.211 and that the magnitude of $(4\mathbf{a}_1 + 6\mathbf{a}_2 - 2\mathbf{a}_3)$ is $\sqrt{4^2 + 6^2 + 2^2}$, or 7.483. Conversely to the foregoing discussion, a. vector $\mathbf{A}$ at a given point is simply the

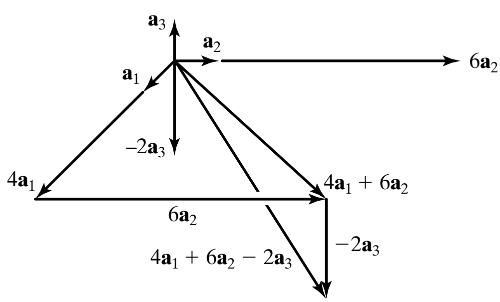

FIGURE 1.3

Graphical addition of vectors.

superposition of three vectors $A_1\mathbf{a}_1$, $A_2\mathbf{a}_2$, and $A_3\mathbf{a}_3$, which are the projections of $\mathbf{A}$ onto the reference directions at that point. A_1, A_2, and A_3 are known as the *components* of $\mathbf{A}$ along the 1, 2, and 3 directions, respectively. Thus

$$\boxed{\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3} \tag{1.1}$$

We now consider three vectors, **A, B**, and **C**, given by

$$\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3 \tag{1.2a}$$
$$\mathbf{B} = B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3 \tag{1.2b}$$
$$\mathbf{C} = C_1\mathbf{a}_1 + C_2\mathbf{a}_2 + C_3\mathbf{a}_3 \tag{1.2c}$$

at a point and discuss several algebraic operations involving vectors as follows.

Vector addition and subtraction. Since a given pair of like components of two vectors are parallel, addition of two vectors consists simply of adding the three pairs of like components of the vectors. Thus,

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) + (B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) \\ &= (A_1 + B_1)\mathbf{a}_1 + (A_2 + B_2)\mathbf{a}_2 + (A_3 + B_3)\mathbf{a}_3\end{aligned} \tag{1.3}$$

Vector subtraction is a special case of addition. Thus,

$$\begin{aligned}\mathbf{B} - \mathbf{C} &= \mathbf{B} + (-\mathbf{C}) \\ &= (B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) + (-C_1\mathbf{a}_1 - C_2\mathbf{a}_2 - C_3\mathbf{a}_3) \\ &= (B_1 - C_1)\mathbf{a}_1 + (B_2 - C_2)\mathbf{a}_2 + (B_3 - C_3)\mathbf{a}_3\end{aligned} \tag{1.4}$$

Multiplication and division by a scalar. Multiplication of a vector $\mathbf{A}$ by a scalar m is the same as repeated addition of the vector. Thus,

$$m\mathbf{A} = m(A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) = mA_1\mathbf{a}_1 + mA_2\mathbf{a}_2 + mA_3\mathbf{a}_3 \tag{1.5}$$

Division by a scalar is a special case of multiplication by a scalar. Thus,

$$\frac{\mathbf{B}}{n} = \frac{1}{n}(\mathbf{B}) = \frac{B_1}{n}\mathbf{a}_1 + \frac{B_2}{n}\mathbf{a}_2 + \frac{B_3}{n}\mathbf{a}_3 \tag{1.6}$$

Magnitude of a vector. From the construction of Fig. 1.3 and the associated discussion, we have

$$\boxed{|\mathbf{A}| = |A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3| = \sqrt{A_1^2 + A_2^2 + A_3^2}} \tag{1.7}$$

Unit vector along A. The unit vector $\mathbf{a}_A$ has a magnitude equal to unity, but its direction is the same as that of $\mathbf{A}$. Hence,

$$\boxed{\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A_1}{|\mathbf{A}|}\mathbf{a}_1 + \frac{A_2}{|\mathbf{A}|}\mathbf{a}_2 + \frac{A_3}{|\mathbf{A}|}\mathbf{a}_3} \tag{1.8}$$

Scalar or dot product of two vectors. The scalar or dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a scalar quantity equal to the product of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ and the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$. It is represented by a boldface dot between $\mathbf{A}$ and $\mathbf{B}$. Thus, if α is the angle between $\mathbf{A}$ and $\mathbf{B}$, then *Dot product*

$$\boxed{\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\alpha = AB\cos\alpha} \tag{1.9}$$

For the unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, we have

$$\mathbf{a}_1\cdot\mathbf{a}_1 = 1 \qquad \mathbf{a}_1\cdot\mathbf{a}_2 = 0 \qquad \mathbf{a}_1\cdot\mathbf{a}_3 = 0 \tag{1.10a}$$
$$\mathbf{a}_2\cdot\mathbf{a}_1 = 0 \qquad \mathbf{a}_2\cdot\mathbf{a}_2 = 1 \qquad \mathbf{a}_2\cdot\mathbf{a}_3 = 0 \tag{1.10b}$$
$$\mathbf{a}_3\cdot\mathbf{a}_1 = 0 \qquad \mathbf{a}_3\cdot\mathbf{a}_2 = 0 \qquad \mathbf{a}_3\cdot\mathbf{a}_3 = 1 \tag{1.10c}$$

By noting that $\mathbf{A}\cdot\mathbf{B} = A(B\cos\alpha) = B(A\cos\alpha)$, we observe that the dot product operation consists of multiplying the magnitude of one vector by the scalar obtained by projecting the second vector onto the first vector, as shown in Figs. 1.4(a) and (b). The dot product operation is commutative since

$$\mathbf{B}\cdot\mathbf{A} = BA\cos\alpha = AB\cos\alpha = \mathbf{A}\cdot\mathbf{B} \tag{1.11}$$

The distributive property also holds for the dot product, as can be seen from the construction of Fig. 1.4(c), which illustrates that the projection of $\mathbf{B} + \mathbf{C}$ onto $\mathbf{A}$ is equal to the sum of the projections of $\mathbf{B}$ and $\mathbf{C}$ onto $\mathbf{A}$. Thus,

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \tag{1.12}$$

Using this property, we have

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3)\cdot(B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) \\
&= A_1\mathbf{a}_1\cdot B_1\mathbf{a}_1 + A_1\mathbf{a}_1\cdot B_2\mathbf{a}_2 + A_1\mathbf{a}_1\cdot B_3\mathbf{a}_3 \\
&\quad + A_2\mathbf{a}_2\cdot B_1\mathbf{a}_1 + A_2\mathbf{a}_2\cdot B_2\mathbf{a}_2 + A_2\mathbf{a}_2\cdot B_3\mathbf{a}_3 \\
&\quad + A_3\mathbf{a}_3\cdot B_1\mathbf{a}_1 + A_3\mathbf{a}_3\cdot B_2\mathbf{a}_2 + A_3\mathbf{a}_3\cdot B_3\mathbf{a}_3
\end{aligned}$$

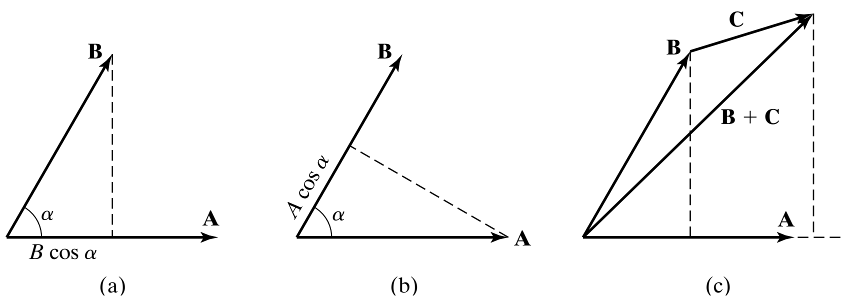

FIGURE 1.4

(a) and (b) For showing that the dot product of two vectors **A** and **B** is the product of the magnitude of one vector and the projection of the second vector onto the first vector.
(c) For proving the distributive property of the dot product operation.

Then using the relationships (1.10a)–(1.10c), we obtain

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3} \tag{1.13}$$

Thus, the dot product of two vectors is the sum of the products of the like components of the two vectors.

Finding angle between two vectors

From (1.9) and (1.13), we note that the angle between the vectors **A** and **B** is given by

$$\alpha = \cos^{-1}\frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \cos^{-1}\frac{A_1B_1 + A_2B_2 + A_3B_3}{AB} \tag{1.14}$$

Thus, the dot product operation is useful for finding the angle between two vectors. In particular, the two vectors are perpendicular if $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = 0$.

Cross product

Vector or cross product of two vectors. The vector or cross product of two vectors **A** and **B** is a vector quantity whose magnitude is equal to the product of the magnitudes of **A** and **B** and the sine of the smaller angle α between **A** and **B** and whose direction is normal to the plane containing **A** and **B** and toward the side of advance of a right-hand screw as it is turned from **A** to **B** through the angle α, as shown in Fig. 1.5. It is represented by a boldface cross between **A** and **B**. Thus, if $\mathbf{a}_N$ is the unit vector in the direction of advance of the right-hand screw, then

$$\boxed{\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\alpha\, \mathbf{a}_N = AB \sin\alpha\, \mathbf{a}_N} \tag{1.15}$$

For the unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, we have

$$\mathbf{a}_1 \times \mathbf{a}_1 = \mathbf{0} \qquad \mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{a}_3 \qquad \mathbf{a}_1 \times \mathbf{a}_3 = -\mathbf{a}_2 \tag{1.16a}$$
$$\mathbf{a}_2 \times \mathbf{a}_1 = -\mathbf{a}_3 \qquad \mathbf{a}_2 \times \mathbf{a}_2 = \mathbf{0} \qquad \mathbf{a}_2 \times \mathbf{a}_3 = \mathbf{a}_1 \tag{1.16b}$$
$$\mathbf{a}_3 \times \mathbf{a}_1 = \mathbf{a}_2 \qquad \mathbf{a}_3 \times \mathbf{a}_2 = -\mathbf{a}_1 \qquad \mathbf{a}_3 \times \mathbf{a}_3 = \mathbf{0} \tag{1.16c}$$

Note that the cross product of two identical unit vectors is the null vector **0**, that is, a vector whose components are all zero. If we arrange the unit vectors in the manner $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_2$, then going to the right, the cross product of any two successive unit vectors is the following unit vector, whereas going to the left, the cross

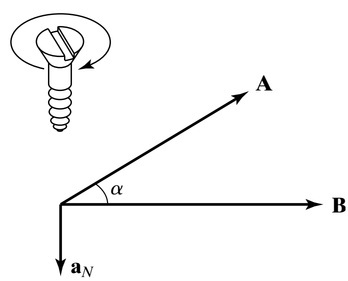

FIGURE 1.5

Cross product operation $\mathbf{A} \times \mathbf{B}$.

product of any two successive unit vectors is the negative of the following unit vector.

The cross product operation is not commutative since

$$\mathbf{B} \times \mathbf{A} = |\mathbf{B}||\mathbf{A}| \sin \alpha(-\mathbf{a}_N) = -AB \sin \alpha\, \mathbf{a}_N = -\mathbf{A} \times \mathbf{B} \tag{1.17}$$

The distributive property holds for the cross product (as we shall prove later in this section), so that

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1.18}$$

Using this property and the relationships (1.16a)–(1.16c), we obtain

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) \times (B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) \\
&= A_1\mathbf{a}_1 \times B_1\mathbf{a}_1 + A_1\mathbf{a}_1 \times B_2\mathbf{a}_2 + A_1\mathbf{a}_1 \times B_3\mathbf{a}_3 \\
&\quad + A_2\mathbf{a}_2 \times B_1\mathbf{a}_1 + A_2\mathbf{a}_2 \times B_2\mathbf{a}_2 + A_2\mathbf{a}_2 \times B_3\mathbf{a}_3 \\
&\quad + A_3\mathbf{a}_3 \times B_1\mathbf{a}_1 + A_3\mathbf{a}_3 \times B_2\mathbf{a}_2 + A_3\mathbf{a}_3 \times B_3\mathbf{a}_3 \\
&= A_1B_2\mathbf{a}_3 - A_1B_3\mathbf{a}_2 - A_2B_1\mathbf{a}_3 + A_2B_3\mathbf{a}_1 \\
&\quad + A_3B_1\mathbf{a}_2 - A_3B_2\mathbf{a}_1 \\
&= (A_2B_3 - A_3B_2)\mathbf{a}_1 + (A_3B_1 - A_1B_3)\mathbf{a}_2 \\
&\quad + (A_1B_2 - A_2B_1)\mathbf{a}_3
\end{aligned}$$

This can be expressed in determinant form in the manner

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}} \tag{1.19}$$

Finding unit vector normal to two vectors

The cross product operation is useful for obtaining the unit vector normal to two given vectors at a point. This can be seen by rearranging (1.15) in the manner

$$\boxed{\mathbf{a}_N = \frac{\mathbf{A} \times \mathbf{B}}{AB \sin \alpha} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}} \tag{1.20}$$

Triple cross product. A triple cross product involves three vectors in two cross product operations. Caution must be exercised in evaluating a triple cross product since the order of evaluation is important; that is, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is not in general equal to $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$. This can be illustrated by means of a simple example involving unit vectors. Thus, if $\mathbf{A} = \mathbf{a}_1$, $\mathbf{B} = \mathbf{a}_1$, and $\mathbf{C} = \mathbf{a}_2$, then

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{a}_1 \times (\mathbf{a}_1 \times \mathbf{a}_2) = \mathbf{a}_1 \times \mathbf{a}_3 = -\mathbf{a}_2$$

whereas

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{a}_1 \times \mathbf{a}_1) \times \mathbf{a}_2 = \mathbf{0} \times \mathbf{a}_2 = \mathbf{0}$$

Scalar triple product. The scalar triple product involves three vectors in a dot product operation and a cross product operation as, for example, $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$. It is not necessary to include parentheses since this quantity can be evaluated in only one manner, that is, by evaluating $\mathbf{B}\times\mathbf{C}$ first and then dotting the resulting vector with $\mathbf{A}$. It is meaningless to try to evaluate the dot product first since it results in a scalar quantity, and hence we cannot proceed any further. From (1.19), we have

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3)\cdot\begin{vmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

From (1.13), we then have

$$\boxed{\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}} \tag{1.21}$$

Since the value of the determinant on the right side of (1.21) remains unchanged if the rows are interchanged in a cyclical manner,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} \tag{1.22}$$

The scalar triple product has the geometrical meaning that its absolute value is the volume of the parallelepiped having the three vectors as three of its contiguous edges, as will be shown in Section 1.2.

We shall now show that the distributive law holds for the cross product operation by using (1.22). Thus, let us consider $\mathbf{A}\times(\mathbf{B}+\mathbf{C})$. Then, if $\mathbf{D}$ is any arbitrary vector, we have

$$\begin{aligned}\mathbf{D}\cdot\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= (\mathbf{B}+\mathbf{C})\cdot(\mathbf{D}\times\mathbf{A}) = \mathbf{B}\cdot(\mathbf{D}\times\mathbf{A}) + \mathbf{C}\cdot(\mathbf{D}\times\mathbf{A})\\ &= \mathbf{D}\cdot\mathbf{A}\times\mathbf{B} + \mathbf{D}\cdot\mathbf{A}\times\mathbf{C} = \mathbf{D}\cdot(\mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C})\end{aligned}$$

where we have used the distributive property of the dot product operation. Since this equality holds for any $\mathbf{D}$, it follows that

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

Example 1.1 Vector algebraic operations

Given three vectors

$$\begin{aligned}\mathbf{A} &= \mathbf{a}_1 + \mathbf{a}_2\\ \mathbf{B} &= \mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3\\ \mathbf{C} &= \mathbf{a}_2 + 2\mathbf{a}_3\end{aligned}$$

let us carry out several of the vector algebraic operations:

(a) $\mathbf{A} + \mathbf{B} = (\mathbf{a}_1 + \mathbf{a}_2) + (\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3) = 2\mathbf{a}_1 + 3\mathbf{a}_2 - 2\mathbf{a}_3$

(b) $\mathbf{B} - \mathbf{C} = (\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3) - (\mathbf{a}_2 + 2\mathbf{a}_3) = \mathbf{a}_1 + \mathbf{a}_2 - 4\mathbf{a}_3$

(c) $4\mathbf{C} = 4(\mathbf{a}_2 + 2\mathbf{a}_3) = 4\mathbf{a}_2 + 8\mathbf{a}_3$

(d) $|\mathbf{B}| = |\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = 3$

(e) $\mathbf{i}_B = \dfrac{\mathbf{B}}{|\mathbf{B}|} = \dfrac{\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3}{3} = \dfrac{1}{3}\mathbf{a}_1 + \dfrac{2}{3}\mathbf{a}_2 - \dfrac{2}{3}\mathbf{a}_3$

(f) $\mathbf{A}\cdot\mathbf{B} = (\mathbf{a}_1 + \mathbf{a}_2)\cdot(\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3) = (1)(1) + (1)(2) + (0)(-2) = 3$

(g) Angle between **A** and **B** $= \cos^{-1}\dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \cos^{-1}\dfrac{3}{(\sqrt{2})(3)} = 45°$

(h) $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ 1 & 1 & 0 \\ 1 & 2 & -2 \end{vmatrix} = (-2 - 0)\mathbf{a}_1 + (0 + 2)\mathbf{a}_2 + (2 - 1)\mathbf{a}_3$$
$$= -2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$$

(i) Unit vector normal to **A** and **B** $= \dfrac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = -\dfrac{2}{3}\mathbf{a}_1 + \dfrac{2}{3}\mathbf{a}_2 + \dfrac{1}{3}\mathbf{a}_3$

(j) $$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ -2 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 3\mathbf{a}_1 + 4\mathbf{a}_2 - 2\mathbf{a}_3$$

(k) $$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & 1 & 2 \end{vmatrix} = (1)(6) + (1)(-2) + (0)(1) = 4$$

K1.1. Scalars; Vectors; Unit vectors; Right-handed system; Components of a vector; Vector addition; Multiplication of vector by a scalar; Magnitude of a vector; Dot product; Cross product; Triple cross product; Scalar triple product.

D1.1. Vector **A** has a magnitude of 4 units and is directed toward north. Vector **B** has a magnitude of 3 units and is directed toward east. Vector **C** has a magnitude of 4 units and is directed 30° toward south from west. Find the following: **(a)** $\mathbf{A} + \mathbf{C}$; **(b)** $|\mathbf{A} - \mathbf{B}|$; **(c)** $3\mathbf{A} + 4\mathbf{B} + 3\mathbf{C}$; **(d)** $\mathbf{B}\cdot(\mathbf{A} - \mathbf{C})$; and **(e)** $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

Ans. **(a)** 4 units directed 60° west of north; **(b)** 5; **(c)** 6.212 units directed 15° east of north; **(d)** 10.392; **(e)** 24 units directed westward.

D1.2. Given three vectors

$$\mathbf{A} = 3\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$$
$$\mathbf{B} = \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3$$
$$\mathbf{C} = \mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3$$

Find the following: **(a)** $|\mathbf{A} + \mathbf{B} - 4\mathbf{C}|$; **(b)** unit vector along $(\mathbf{A} + 2\mathbf{B} - \mathbf{C})$; **(c)** $\mathbf{A}\cdot\mathbf{C}$; **(d)** $\mathbf{B}\times\mathbf{C}$; and **(e)** $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$.

Ans. **(a)** 13; **(b)** $(2\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3)/3$; **(c)** 10; **(d)** $5\mathbf{a}_1 - 4\mathbf{a}_2 + \mathbf{a}_3$; **(e)** 8.

D1.3. Three vectors **A**, **B**, and **C** are given by

$$\mathbf{A} = \mathbf{a}_1 + 2\mathbf{a}_2 + 2\mathbf{a}_3$$
$$\mathbf{B} = 2\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3$$
$$\mathbf{C} = \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3$$

Find the following: **(a)** $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$; **(b)** $\mathbf{B} \times (\mathbf{C} \times \mathbf{A})$; and **(c)** $\mathbf{C} \times (\mathbf{A} \times \mathbf{B})$.
Ans. **(a)** $2\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3$; **(b)** $\mathbf{a}_1 + 2\mathbf{a}_2 + 2\mathbf{a}_3$; **(c)** $-3\mathbf{a}_1 - 3\mathbf{a}_2$.

1.2 CARTESIAN COORDINATE SYSTEM

In the preceding section, we introduced the technique of expressing a vector at a point in space in terms of its component vectors along a set of three mutually orthogonal directions defined by three mutually orthogonal unit vectors at that point. Now to relate vectors at one point in space to vectors at another point in space, we must define the set of three reference directions at each and every point in space. To do this in a systematic manner, we need to use a coordinate system. Although there are several different coordinate systems, we shall be concerned with only three of those, namely, the Cartesian, cylindrical, and spherical coordinate systems. The *Cartesian coordinate system*, also known as the *rectangular coordinate system*, is the simplest of the three since it permits the geometry to be simple, yet sufficient to study many of the elements of engineering electromagnetics. We introduce the Cartesian coordinate system in this section and devote the next section to the cylindrical and spherical coordinate systems.

The Cartesian coordinate system is defined by a set of three mutually orthogonal planes, as shown in Fig. 1.6(a). The point at which the three planes intersect is known as the origin O. The origin is the reference point relative to which we locate any other point in space. Each pair of planes intersects in a straight line. Hence, the three planes define a set of three straight lines that

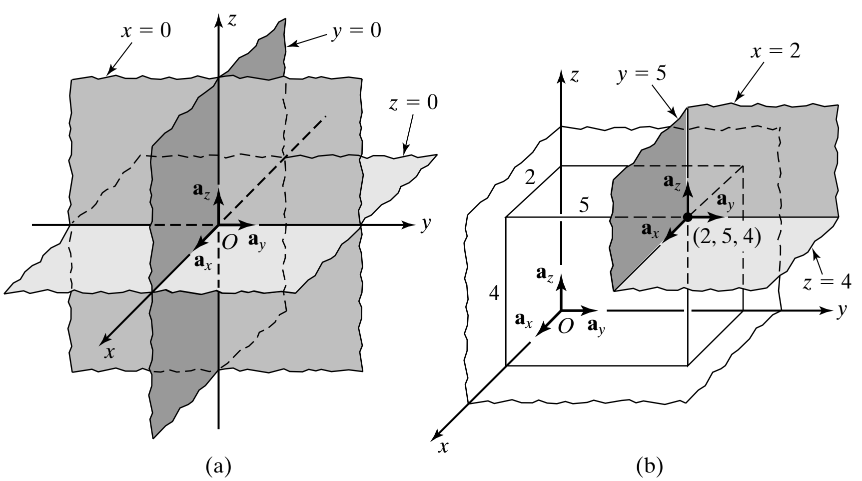

FIGURE 1.6

Cartesian coordinate system. (a) The three orthogonal planes defining the coordinate system. (b) The unit vectors in the Cartesian coordinate system are uniform.

form the coordinate axes. These coordinate axes are denoted as the x-, y-, and z-axes. Values of x, y, and z are measured from the origin; hence, the coordinates of the origin are (0, 0, 0), that is, $x = 0$, $y = 0$, and $z = 0$. Directions in which values of x, y, and z increase along the respective coordinate axes are indicated by arrowheads. The same set of three directions is used to erect a set of three unit vectors, denoted $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, as shown in Fig. 1.6(a), for the purpose of describing vectors drawn at the origin. Note that the positive x-, y-, and z-directions are chosen such that they form a right-handed system, that is, a system for which $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$.

On one of the three planes, namely, the yz-plane, the value of x is constant and equal to zero, its value at the origin, since movement on this plane does not require any movement in the x-direction. Similarly, on the zx-plane, the value of y is constant and equal to zero, and on the xy-plane, the value of z is constant and equal to zero. Any point other than the origin is now given by the intersection of three planes

$$\boxed{\begin{aligned} x &= \text{constant} \\ y &= \text{constant} \\ z &= \text{constant} \end{aligned}} \tag{1.23}$$

obtained by incrementing the values of the coordinates by appropriate amounts. For example, by displacing the $x = 0$ plane by 2 units in the positive x-direction, the $y = 0$ plane by 5 units in the positive y-direction, and the $z = 0$ plane by 4 units in the positive z-direction, we obtain the planes $x = 2$, $y = 5$, and $z = 4$, respectively, which intersect at point (2, 5, 4), as shown in Fig. 1.6(b). The intersections of pairs of these planes define three straight lines along which we can erect the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ toward the directions of increasing values of x, y, and z, respectively, for the purpose of describing vectors drawn at that point. These unit vectors are parallel to the corresponding unit vectors drawn at the origin, as can be seen from Fig. 1.6(b). The same is true for any point in space in the Cartesian coordinate system. Thus, each one of the three unit vectors in the Cartesian coordinate system has the same direction at all points, and hence it is uniform. This behavior does not, however, hold for all unit vectors in the cylindrical and spherical coordinate systems, as we shall see in the next section.

Expression for vector joining two points

It is now a simple matter to apply what we have learned in Section 1.1 to vectors in Cartesian coordinates. All we need to do is to replace the subscripts 1, 2, and 3 for the unit vectors and the components along the unit vectors by the subscripts x, y, and z, respectively, and also utilize the property that $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are uniform vectors. Thus, let us, for example, obtain the expression for the vector $\mathbf{R}_{12}$ drawn from point $P_1(x_1, y_1, z_1)$ to point $P_2(x_2, y_2, z_2)$, as shown in Fig. 1.7. To do this, we note that the position vector $\mathbf{r}_1$ drawn from the origin to the point P_1 is given by

$$\mathbf{r}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z \tag{1.24a}$$

The *position vector* is so called because it defines the position of the point in space relative to the origin. Similarly, the position vector $\mathbf{r}_2$ drawn from the origin to the point P_2 is given by

$$\mathbf{r}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z \tag{1.24b}$$

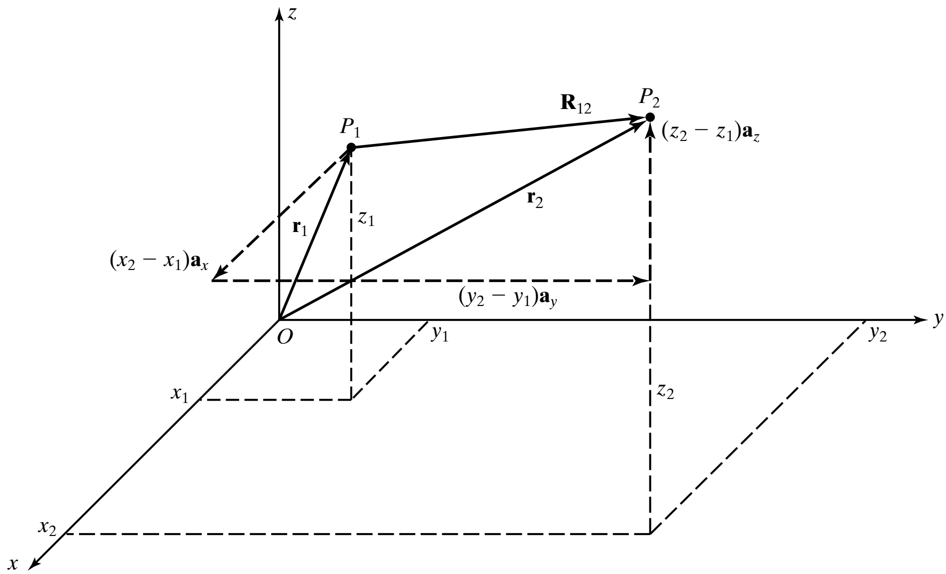

FIGURE 1.7

For obtaining the expression for the vector $\mathbf{R}_{12}$ from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$.

Since, from the rule for vector addition, $\mathbf{r}_1 + \mathbf{R}_{12} = \mathbf{r}_2$, we obtain the vector $\mathbf{R}_{12}$ to be

$$\boxed{\begin{aligned}\mathbf{R}_{12} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z\end{aligned}} \tag{1.25}$$

Thus, to find the components of the vector drawn from one point to another in the Cartesian coordinate system, we simply subtract the coordinates of the initial point from the corresponding coordinates of the final point. These components are just the distances one has to travel along the x-, y-, and z-directions, respectively, if one chooses to go from P_1 to P_2 by traveling parallel to the coordinate axes instead of traveling along the direct straight-line path.

Proceeding further, we can obtain the unit vector along the line drawn from P_1 to P_2 to be

$$\boxed{\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{(x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}}} \tag{1.26}$$

For a numerical example, if P_1 is $(1, -2, 0)$ and P_2 is $(4, 2, 5)$, then

$$\begin{aligned}\mathbf{R}_{12} &= 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z \\ \mathbf{a}_{12} &= \frac{1}{5\sqrt{2}}(3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z)\end{aligned}$$

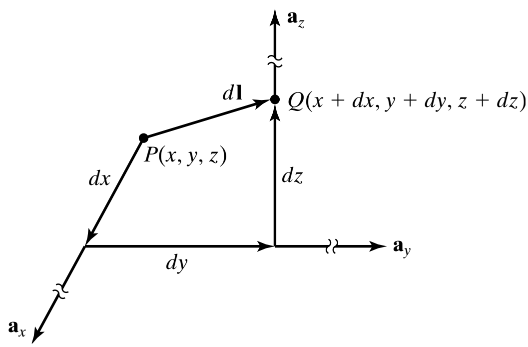

FIGURE 1.8
Differential length vector $d\mathbf{l}$.

In our study of electromagnetic fields, we have to work with line, surface, and volume integrals. These involve differential lengths, surfaces, and volumes, obtained by incrementing the coordinates by infinitesimal amounts. Since, in the Cartesian coordinate system, the three coordinates represent lengths, the differential length elements obtained by incrementing one coordinate at a time, keeping the other two coordinates constant, are $dx\,\mathbf{a}_x$, $dy\,\mathbf{a}_y$, and $dz\,\mathbf{a}_z$ for the x-, y-, and z-coordinates, respectively.

Differential length vector. The differential length vector $d\mathbf{l}$ is the vector drawn from a point $P(x, y, z)$ to a neighboring point $Q(x + dx, y + dy, z + dz)$ obtained by incrementing the coordinates of P by infinitesimal amounts. Thus, it is the vector sum of the three differential length elements, as shown in Fig. 1.8, and given by

$$\boxed{d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z} \tag{1.27}$$

The differential lengths dx, dy, and dz in (1.27) are, however, not independent of each other since in the evaluation of line integrals, the integration is performed along a specified path on which the points P and Q lie. We shall illustrate this by means of an example.

Example 1.2 Finding differential length vector along a curve

Finding differential length vector along a curve

Let us consider the curve $x = y = z^2$ and obtain the expression for the differential length vector $d\mathbf{l}$ along the curve at the point $(1, 1, 1)$ and having the projection dz on the z-axis.

The geometry pertinent to the problem is shown in Fig. 1.9. From elementary calculus, we know that for $x = y = z^2$, $dx = dy = 2z\,dz$. In particular, at the point $(1, 1, 1)$, $dx = dy = 2\,dz$. Thus,

$$\begin{aligned} d\mathbf{l} &= dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \\ &= 2\,dz\,\mathbf{a}_x + 2\,dz\,\mathbf{a}_y + dz\,\mathbf{a}_z \\ &= (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)\,dz \end{aligned}$$

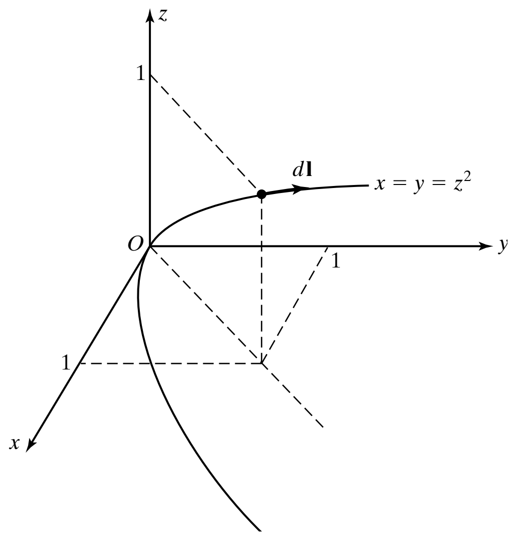

FIGURE 1.9

For finding the differential length vector along the curve $x = y = z^2$.

Note that the z-component of the $d\mathbf{l}$ vector found is dz, thereby satisfying the requirement of projection dz on the z-axis specified in the problem.

Differential length vectors are useful for finding the unit vector normal to a surface at a point on that surface. This is done by considering two differential length vectors at the point under consideration and tangential to two curves on the surface and then using (1.20). Thus, with reference to Fig. 1.10, we have

$$\boxed{\mathbf{a}_n = \frac{d\mathbf{l}_1 \times d\mathbf{l}_2}{|d\mathbf{l}_1 \times d\mathbf{l}_2|}} \tag{1.28}$$

Let us consider an example.

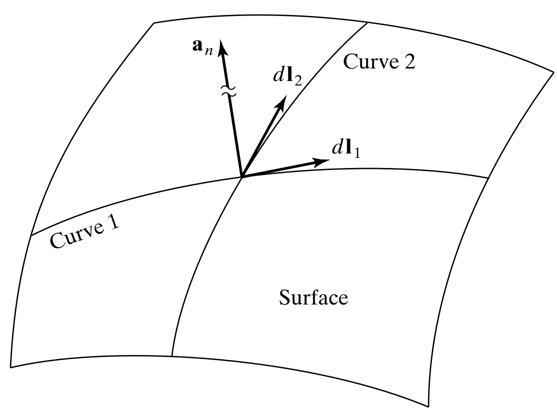

FIGURE 1.10

Finding the unit vector normal to a surface by using differential length vectors.

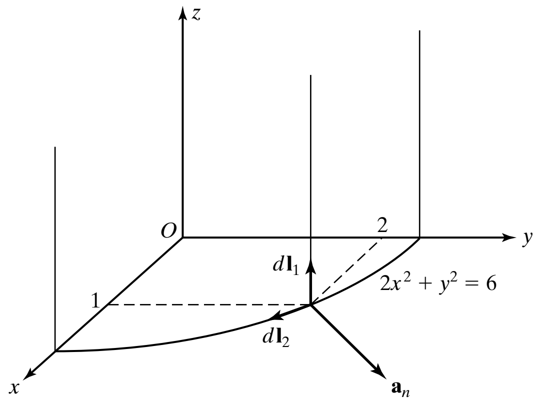

FIGURE 1.11

Example of finding the unit vector normal to a surface.

Example 1.3 Finding unit vector normal to a surface

Finding unit normal vector at a point on a surface

Find the unit vector normal to the surface $2x^2 + y^2 = 6$ at the point $(1, 2, 0)$.

With reference to the construction shown in Fig. 1.11, we consider two differential length vectors $d\mathbf{l}_1$ and $d\mathbf{l}_2$ at the point $(1, 2, 0)$. The vector $d\mathbf{l}_1$ is along the straight line $x = 1$, $y = 2$, whereas the vector $d\mathbf{l}_2$ is tangential to the curve $2x^2 + y^2 = 6$, $z = 0$. For $x = 1$ and $y = 2$, $dx = dy = 0$. Hence,

$$d\mathbf{l}_1 = dz\,\mathbf{a}_z$$

For $2x^2 + y^2 = 6$ and $z = 0$, $4x\,dx + 2y\,dy = 0$ and $dz = 0$. Specifically, at the point $(1, 2, 0)$, $dy = -dx$ and $dz = 0$. Hence,

$$d\mathbf{l}_2 = dx\,\mathbf{a}_x - dx\,\mathbf{a}_y = dx\,(\mathbf{a}_x - \mathbf{a}_y)$$

The unit normal vector is then given by

$$\begin{aligned}\mathbf{a}_n &= \frac{dz\,\mathbf{a}_z \times dx\,(\mathbf{a}_x - \mathbf{a}_y)}{|dz\,\mathbf{a}_z \times dx\,(\mathbf{a}_x - \mathbf{a}_y)|}\\ &= \frac{1}{\sqrt{2}}(\mathbf{a}_x + \mathbf{a}_y)\end{aligned}$$

Differential surface vector. Two differential length vectors $d\mathbf{l}_1$ and $d\mathbf{l}_2$ originating at a point define a differential surface whose area dS is that of the parallelogram having $d\mathbf{l}_1$ and $d\mathbf{l}_2$ as two of its adjacent sides, as shown in Fig. 1.12(a). From simple geometry and the definition of the cross product of two vectors, it can be seen that

$$dS = dl_1\,dl_2 \sin\alpha = |d\mathbf{l}_1 \times d\mathbf{l}_2| \tag{1.29}$$

In the evaluation of surface integrals, it is convenient to define a differential surface vector $d\mathbf{S}$ whose magnitude is the area dS and whose direction is normal to the differential surface. Thus, recognizing that the normal vector can be directed

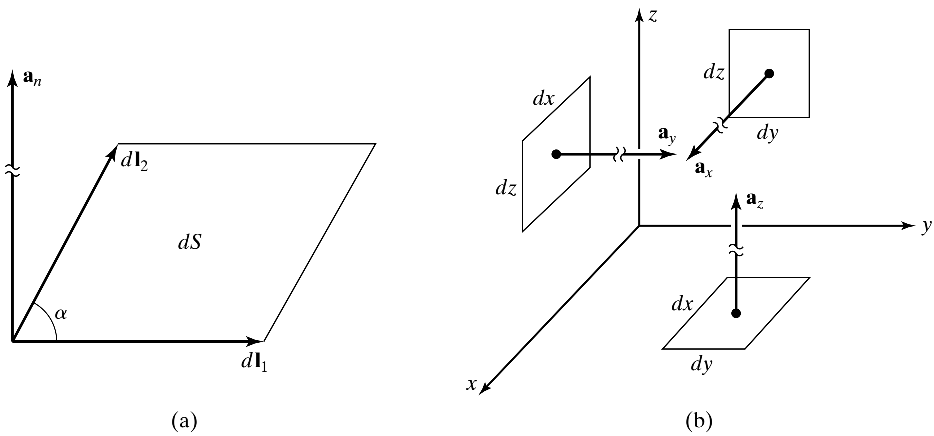

FIGURE 1.12

(a) Illustrating the differential surface vector concept. (b) Differential surface vectors in the Cartesian coordinate system.

to either side of a surface, we can write

$$d\mathbf{S} = \pm dS\,\mathbf{a}_n = \pm|d\mathbf{l}_1 \times d\mathbf{l}_2|\mathbf{a}_n$$

or

$$\boxed{d\mathbf{S} = \pm d\mathbf{l}_1 \times d\mathbf{l}_2} \tag{1.30}$$

Applying (1.30) to pairs of three differential length elements $dx\,\mathbf{a}_x$, $dy\,\mathbf{a}_y$, and $dz\,\mathbf{a}_z$, we obtain the differential surface vectors

$$\pm dy\,\mathbf{a}_y \times dz\,\mathbf{a}_z = \pm dy\,dz\,\mathbf{a}_x \tag{1.31a}$$
$$\pm dz\,\mathbf{a}_z \times dx\,\mathbf{a}_x = \pm dz\,dx\,\mathbf{a}_y \tag{1.31b}$$
$$\pm dx\,\mathbf{a}_x \times dy\,\mathbf{a}_y = \pm dx\,dy\,\mathbf{a}_z \tag{1.31c}$$

associated with the planes x = constant, y = constant, and z = constant, respectively. These are shown in Fig. 1.12(b) for the plus signs in (1.31a)–(1.31c).

Differential volume. Three differential length vectors $d\mathbf{l}_1$, $d\mathbf{l}_2$, and $d\mathbf{l}_3$ originating at a point define a differential volume dv which is that of the parallelepiped having $d\mathbf{l}_1$, $d\mathbf{l}_2$, and $d\mathbf{l}_3$ as three of its contiguous edges, as shown in Fig. 1.13(a). From simple geometry and the definitions of cross and dot products, it can be seen that

$$\begin{aligned}
dv &= \text{area of the base of the parallelepiped} \times \text{height of the parallelepiped} \\
&= |d\mathbf{l}_1 \times d\mathbf{l}_2||d\mathbf{l}_3 \cdot \mathbf{a}_n| \\
&= |d\mathbf{l}_1 \times d\mathbf{l}_2|\frac{|d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2|}{|d\mathbf{l}_1 \times d\mathbf{l}_2|} \\
&= |d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2|
\end{aligned}$$

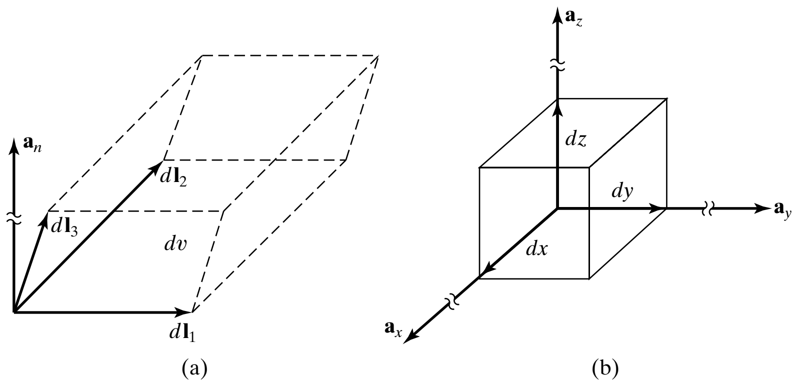

FIGURE 1.13

(a) Parallelepiped defined by three differential length vectors originating at a point.
(b) Differential volume in the Cartesian coordinate system.

or

$$\boxed{dv = |d\mathbf{l}_1 \cdot d\mathbf{l}_2 \times d\mathbf{l}_3|} \tag{1.32}$$

Thus, the scalar triple product of three vectors originating from a point has the meaning that its absolute value is the volume of the parallelepiped having the three vectors as three of its contiguous edges.

For the three differential length elements $dx\,\mathbf{a}_x$, $dy\,\mathbf{a}_y$, and $dz\,\mathbf{a}_z$ associated with the Cartesian coordinate system, we obtain the differential volume to be

$$\boxed{dv = dx\,dy\,dz} \tag{1.33}$$

which is that of the rectangular parallelepiped shown in Fig. 1.13(b).

We shall conclude this section with a brief review of some elementary analytic geometrical details that will be useful in our study of electromagnetics. An arbitrary surface is defined by an equation of the form

$$f(x, y, z) = 0 \tag{1.34}$$

In particular, the equation for a plane surface making intercepts *a, b*, and *c* on the *x*-, *y*-, and *z*-axes, respectively, is given by

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \tag{1.35}$$

Since a curve is the intersection of two surfaces, an arbitrary curve is defined by a pair of equations

$$f(x, y, z) = 0 \qquad \text{and} \qquad g(x, y, z) = 0 \tag{1.36}$$

Alternatively, a curve is specified by a set of three parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t) \tag{1.37}$$

where t is an independent parameter. For example, a straight line passing through the origin and making equal angles with the positive x-, y-, and z-axes is given by the pair of equations $y = x$ and $z = x$, or by the set of three parametric equations $x = t$, $y = t$, and $z = t$.

K1.2. Cartesian or rectangular coordinate system; Orthogonal surfaces; Unit vectors; Position vector; Vector joining two points; Differential length vector; Differential surface vector; Differential volume.

D1.4. Three points P_1, P_2, and P_3 are given by $(1, -2, 2)$, $(3, 1, 0)$, and $(5, 2, -2)$, respectively. Obtain the following: **(a)** the vector drawn from P_1 to P_2; **(b)** the straight-line distance from P_2 to P_3; and **(c)** the unit vector along the line from P_1 to P_3.

Ans. **(a)** $(2\mathbf{a}_x + 3\mathbf{a}_y - 2\mathbf{a}_z)$; **(b)** 3; **(c)** $(\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z)/\sqrt{3}$.

D1.5. For each of the following straight lines, find the differential length vector along the line and having the projection dz on the z-axis: **(a)** $x = 3$, $y = -4$; **(b)** $x + y = 0$, $y + z = 1$; and **(c)** the line passing through the points $(2, 0, 0)$ and $(0, 0, 1)$.

Ans. **(a)** $dz\,\mathbf{a}_z$; **(b)** $(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)\,dz$; **(c)** $(-2\mathbf{a}_x + \mathbf{a}_z)\,dz$.

D1.6. For each of the following pairs of points, obtain the equation for the straight line passing through the points: **(a)** $(1, 2, 0)$ and $(3, 4, 0)$; **(b)** $(0, 0, 0)$ and $(2, 2, -1)$; and **(c)** $(1, 1, 1)$ and $(3, -2, 4)$.

Ans. **(a)** $y = x + 1$, $z = 0$; **(b)** $x = y = -2z$; **(c)** $3x + 2y = 5$, $3x - 2z = 1$.

1.3 CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

Cylindrical coordinate system

In the preceding section, we learned that the Cartesian coordinate system is defined by a set of three mutually orthogonal surfaces, all of which are planes. The cylindrical and spherical coordinate systems also involve sets of three mutually orthogonal surfaces. For the cylindrical coordinate system, the three surfaces are a cylinder and two planes, as shown in Fig. 1.14(a). One of these planes is the same as the $z =$ constant plane in the Cartesian coordinate system. The second plane contains the z-axis and makes an angle ϕ with a reference plane, conveniently chosen to be the xz-plane of the Cartesian coordinate system. This plane is therefore defined by $\phi =$ constant. The cylindrical surface has the z-axis as its axis. Since the radial distance r from the z-axis to points on the cylindrical surface is a constant, this surface is defined by $r =$ constant. Thus, the three orthogonal surfaces defining the cylindrical coordinates of a point are

$$\boxed{\begin{aligned} r &= \text{constant} \\ \phi &= \text{constant} \\ z &= \text{constant} \end{aligned}} \tag{1.38}$$

Only two of these coordinates (r and z) are distances; the third coordinate (ϕ) is an angle. We note that the entire space is spanned by varying r from 0 to ∞, ϕ from 0 to 2π, and z from $-\infty$ to ∞.

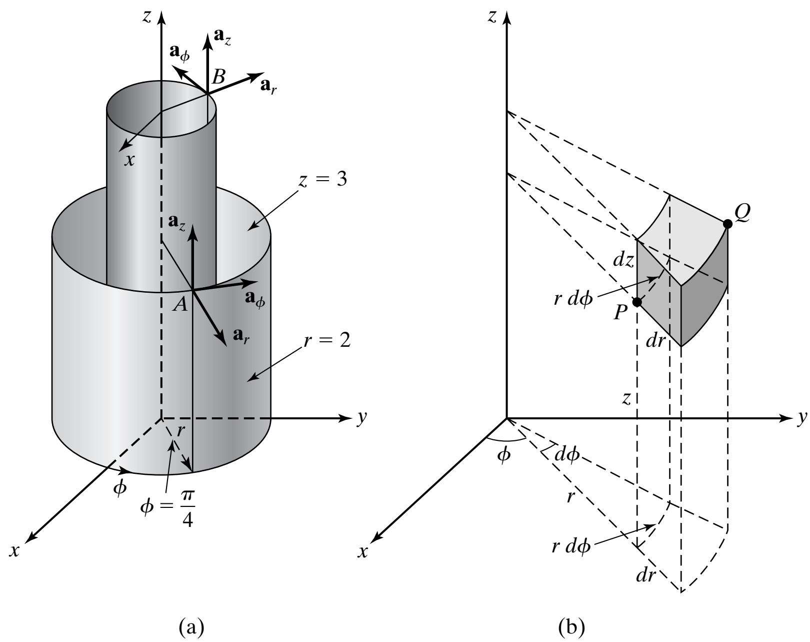

FIGURE 1.14

Cylindrical coordinate system. (a) Orthogonal surfaces and unit vectors. (b) Differential volume formed by incrementing the coordinates.

The origin is given by $r = 0$, $\phi = 0$, and $z = 0$. Any other point in space is given by the intersection of three mutually orthogonal surfaces obtained by incrementing the coordinates by appropriate amounts. For example, the intersection of the three surfaces $r = 2$, $\phi = \pi/4$, and $z = 3$ defines the point $A(2, \pi/4, 3)$, as shown in Fig. 1.14(a). These three orthogonal surfaces define three curves that are mutually perpendicular. Two of these are straight lines and the third is a circle. We draw unit vectors, $\mathbf{a}_r$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ tangential to these curves at the point A and directed toward increasing values of r, ϕ, and z, respectively. These three unit vectors form a set of mutually orthogonal unit vectors in terms of which vectors drawn at A can be described. In a similar manner, we can draw unit vectors at any other point in the cylindrical coordinate system, as shown, for example, for point $B(1, 3\pi/4, 5)$ in Fig. 1.14(a). It can now be seen that the unit vectors $\mathbf{a}_r$ and $\mathbf{a}_\phi$ at point B are not parallel to the corresponding unit vectors at point A. Thus, unlike in the Cartesian coordinate system, the unit vectors $\mathbf{a}_r$ and $\mathbf{a}_\phi$ in the cylindrical coordinate system do not have the same directions everywhere; that is, they are not uniform. Only the unit vector $\mathbf{a}_z$, which is the same as in the Cartesian coordinate system, is uniform. Finally, we note that for the choice of ϕ as in Fig. 1.14(a), that is, increasing from the positive x-axis toward the positive y-axis, the coordinate system is right-handed, that is, $\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z$.

To obtain expressions for the differential lengths, surfaces, and volumes in the cylindrical coordinate system, we now consider two points $P(r, \phi, z)$ and $Q(r + dr, \phi + d\phi, z + dz)$, where Q is obtained by incrementing infinitesimally each coordinate from its value at P, as shown in Fig. 1.14(b). The three orthogonal surfaces intersecting at P, and the three orthogonal surfaces intersecting at Q, define a box that can be considered to be rectangular since dr, $d\phi$, and dz are infinitesimally small. The three differential length elements forming the contiguous sides of this box are $dr\,\mathbf{a}_r$, $r\,d\phi\,\mathbf{a}_\phi$, and $dz\,\mathbf{a}_z$. The differential length vector $d\mathbf{l}$ from P to Q is thus given by

$$\boxed{d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z} \tag{1.39}$$

The differential surface vectors defined by pairs of the differential length elements are

$$\pm r\,d\phi\,\mathbf{a}_\phi \times dz\,\mathbf{a} = \pm r\,d\phi\,dz\,\mathbf{a}_r \tag{1.40a}$$
$$\pm dz\,\mathbf{a}_z \times dr\,\mathbf{a}_r = \pm dr\,dz\,\mathbf{a}_\phi \tag{1.40b}$$
$$\pm dr\,\mathbf{a}_r \times r\,d\phi\,\mathbf{a}_\phi = \pm r\,dr\,d\phi\,\mathbf{a}_z \tag{1.40c}$$

These are associated with the $r =$ constant, $\phi =$ constant, and $z =$ constant surfaces, respectively. Finally, the differential volume dv formed by the three differential lengths is simply the volume of the box; that is,

$$\boxed{dv = (dr)(r\,d\phi)(dz) = r\,dr\,d\phi\,dz} \tag{1.41}$$

Spherical coordinate system

For the spherical coordinate system, the three mutually orthogonal surfaces are a sphere, a cone, and a plane, as shown in Fig. 1.15(a). The plane is the same as the $\phi =$ constant plane in the cylindrical coordinate system. The sphere has the origin as its center. Since the radial distance r from the origin to points on the spherical surface is a constant, this surface is defined by $r =$ constant. The spherical coordinate r should not be confused with the cylindrical coordinate r. When these two coordinates appear in the same expression, we shall use the subscripts c and s to distinguish between cylindrical and spherical. The cone has its vertex at the origin and its surface is symmetrical about the z-axis. Since the angle θ is the angle that the conical surface makes with the z-axis, this surface is defined by $\theta =$ constant. Thus, the three orthogonal surfaces defining the spherical coordinates of a point are

$$\boxed{\begin{aligned} r &= \text{constant} \\ \theta &= \text{constant} \\ \phi &= \text{constant} \end{aligned}} \tag{1.42}$$

Only one of these coordinates (r) is distance; the other two coordinates (θ and ϕ) are angles. We note that the entire space is spanned by varying r from 0 to ∞, θ from 0 to π, and ϕ from 0 to 2π.

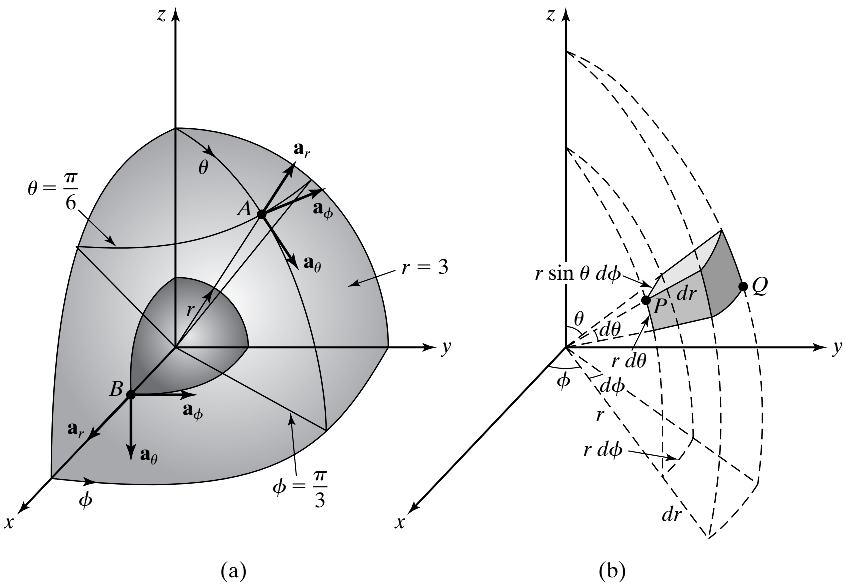

FIGURE 1.15

Spherical coordinate system. (a) Orthogonal surfaces and unit vectors. (b) Differential volume formed by incrementing the coordinates.

The origin is given by $r = 0, \theta = 0$, and $\phi = 0$. Any other point in space is given by the intersection of three mutually orthogonal surfaces obtained by incrementing the coordinates by appropriate amounts. For example, the intersection of the three surfaces $r = 3, \theta = \pi/6$, and $\phi = \pi/3$ defines the point $A(3, \pi/6, \pi/3)$, as shown in Fig. 1.15(a). These three orthogonal surfaces define three curves that are mutually perpendicular. One of these is a straight line and the other two are circles. We draw unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ tangential to these curves at point A and directed toward increasing values of r, θ, and ϕ, respectively. These three unit vectors form a set of mutually orthogonal unit vectors in terms of which vectors drawn at A can be described. In a similar manner, we can draw unit vectors at any other point in the spherical coordinate system, as shown, for example, for point $B(1, \pi/2, 0)$ in Fig. 1.15(a). It can now be seen that these unit vectors at point B are not parallel to the corresponding unit vectors at point A. Thus, in the spherical coordinate system all three unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ do not have the same directions everywhere; that is, they are not uniform. Finally, we note that for the choice of θ as in Fig. 1.15(a), that is, increasing from the positive z-axis toward the xy-plane, the coordinate system is right-handed, that is, $\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$.

To obtain expressions for the differential lengths, surfaces, and volume in the spherical coordinate system, we now consider two points $P(r, \theta, \phi)$ and $Q(r + dr, \theta + d\theta, \phi + d\phi)$, where Q is obtained by incrementing infinitesimally each coordinate from its value at P, as shown in Fig. 1.15(b). The three orthogonal surfaces intersecting at P and the three orthogonal surfaces intersecting at Q define a box that can be considered to be rectangular since dr, $d\theta$ and $d\phi$ are infinitesimally small. The three differential length elements forming the contiguous sides of this box are $dr\,\mathbf{a}_r$, $r\,d\theta\,\mathbf{a}_\theta$, and $r\sin\theta\,d\phi\,\mathbf{a}_\phi$. The differential length vector $d\mathbf{l}$ from P to Q is thus given by

$$\boxed{d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi} \tag{1.43}$$

The differential surface vectors defined by pairs of the differential length elements are

$$\pm r\,d\theta\,\mathbf{a}_\theta \times r\sin\theta\,d\phi\,\mathbf{a}_\phi = \pm r^2\sin\theta\,d\theta\,d\phi\,\mathbf{a}_r \tag{1.44a}$$
$$\pm r\sin\theta\,d\phi\,\mathbf{a}_\phi \times dr\,\mathbf{a}_r = \pm r\sin\theta\,dr\,d\phi\,\mathbf{a}_\theta \tag{1.44b}$$
$$\pm dr\,\mathbf{a}_r \times r\,d\theta\,\mathbf{a}_\theta = \pm r\,dr\,d\theta\,\mathbf{a}_\phi \tag{1.44c}$$

These are associated with the r = constant, θ = constant, and ϕ = constant surfaces, respectively. Finally, the differential volume dv formed by the three differential lengths is simply the volume of the box, that is,

$$\boxed{dv = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi} \tag{1.45}$$

Conversions between coordinate systems

In the study of electromagnetics, it is useful to be able to convert the coordinates of a point and vectors drawn at a point from one coordinate system to another, particularly from the cylindrical system to the Cartesian system and vice versa, and from the spherical system to the Cartesian system and vice versa. To derive first the relationships for the conversion of the coordinates, let us consider Fig. 1.16(a), which illustrates the geometry pertinent to the coordinates of a point P in the three different coordinate systems. Thus, from simple geometrical considerations, we have

$$x = r_c\cos\phi \qquad y = r_c\sin\phi \qquad z = z \tag{1.46a}$$
$$x = r_s\sin\theta\cos\phi \qquad y = r_s\sin\theta\sin\phi \qquad z = r_s\cos\theta \tag{1.46b}$$

Conversely, we have

$$r_c = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z \tag{1.47a}$$
$$r_s = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \qquad \phi = \tan^{-1}\frac{y}{x} \tag{1.47b}$$

Relationships (1.46a) and (1.47a) correspond to conversion from cylindrical coordinates to Cartesian coordinates, and vice versa. Relationships (1.46b) and (1.47b) correspond to conversion from spherical coordinates to Cartesian coordinates,

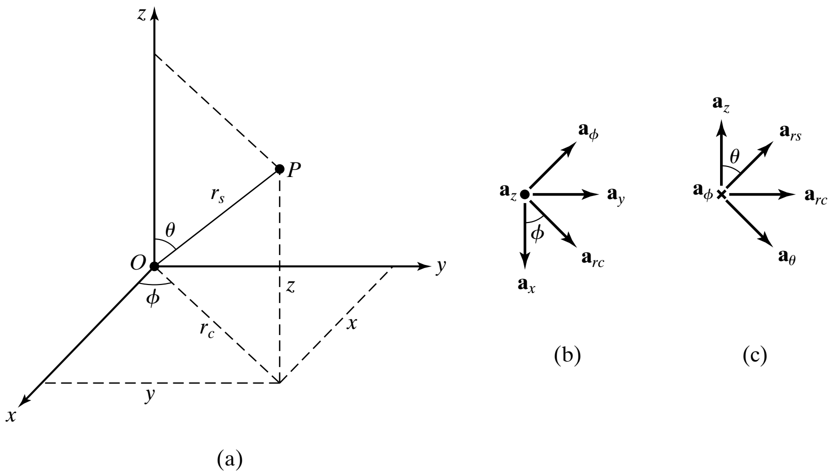

FIGURE 1.16

(a) Converting coordinates of a point from one coordinate system to another. (b) and (c) Expressing unit vectors in cylindrical and spherical coordinate systems, respectively, in terms of unit vectors in the Cartesian coordinate system.

and vice versa. It should be noted that in computing ϕ from y and x, consideration should be given to the quadrant of the xy-plane in which the projection of the point P onto the xy-plane lies.

Considering next the conversion of vectors from one coordinate system to another, we note that to do this, we need to express each of the unit vectors of the first coordinate system in terms of its components along the unit vectors in the second coordinate system. From the definition of the dot product of two vectors, the component of a unit vector along another unit vector, that is, the cosine of the angle between the unit vectors, is simply the dot product of the two unit vectors. Thus, considering the sets of unit vectors in the cylindrical and Cartesian coordinate systems, we have with the aid of Fig. 1.16(b),

$$\mathbf{a}_{rc}\cdot\mathbf{a}_x = \cos\phi \qquad \mathbf{a}_{rc}\cdot\mathbf{a}_y = \sin\phi \qquad \mathbf{a}_{rc}\cdot\mathbf{a}_z = 0 \tag{1.48a}$$
$$\mathbf{a}_{\phi}\cdot\mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_z = 0 \tag{1.48b}$$
$$\mathbf{a}_{z}\cdot\mathbf{a}_x = 0 \qquad \mathbf{a}_{z}\cdot\mathbf{a}_y = 0 \qquad \mathbf{a}_{z}\cdot\mathbf{a}_z = 1 \tag{1.48c}$$

Similarly, for the sets of unit vectors in the spherical and Cartesian coordinate systems, we obtain, with the aid of Fig. 1.16(b) and (c),

$$\mathbf{a}_{rs}\cdot\mathbf{a}_x = \sin\theta\cos\phi \qquad \mathbf{a}_{rs}\cdot\mathbf{a}_y = \sin\theta\sin\phi \qquad \mathbf{a}_{rs}\cdot\mathbf{a}_z = \cos\theta \tag{1.49a}$$
$$\mathbf{a}_{\theta}\cdot\mathbf{a}_x = \cos\theta\cos\phi \qquad \mathbf{a}_{\theta}\cdot\mathbf{a}_y = \cos\theta\sin\phi \qquad \mathbf{a}_{\theta}\cdot\mathbf{a}_z = -\sin\theta \tag{1.49b}$$
$$\mathbf{a}_{\phi}\cdot\mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_z = 0 \tag{1.49c}$$

We shall now illustrate the use of these relationships by means of an example.

Example 1.4 Conversion of a vector from Cartesian to spherical coordinates

Let us consider the vector $3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ at the point (3, 4, 5) and convert it to one in spherical coordinates.

First, from the relationships (1.47b), we obtain the spherical coordinates of the point (3, 4, 5) to be

$$r_s = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$$

$$\theta = \tan^{-1}\frac{\sqrt{3^2 + 4^2}}{5} = \tan^{-1} 1 = 45°$$

$$\phi = \tan^{-1}\tfrac{4}{3} = 53.13°$$

Then, noting from the relationships (1.49) that

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_x \cdot \mathbf{a}_{rs}) & (\mathbf{a}_x \cdot \mathbf{a}_\theta) & (\mathbf{a}_x \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_y \cdot \mathbf{a}_{rs}) & (\mathbf{a}_y \cdot \mathbf{a}_\theta) & (\mathbf{a}_y \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_z \cdot \mathbf{a}_{rs}) & (\mathbf{a}_z \cdot \mathbf{a}_\theta) & (\mathbf{a}_z \cdot \mathbf{a}_\phi) \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$
$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

we obtain at the point under consideration

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} 0.3\sqrt{2} & 0.3\sqrt{2} & -0.8 \\ 0.4\sqrt{2} & 0.4\sqrt{2} & 0.6 \\ 0.5\sqrt{2} & -0.5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$\begin{aligned} 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z &= 3(0.3\sqrt{2}\mathbf{a}_{rs} + 0.3\sqrt{2}\mathbf{a}_\theta - 0.8\mathbf{a}_\phi) \\ &\quad + 4(0.4\sqrt{2}\mathbf{a}_{rs} + 0.4\sqrt{2}\mathbf{a}_\theta + 0.6\mathbf{a}_\phi) \\ &\quad + 5(0.5\sqrt{2}\mathbf{a}_{rs} - 0.5\sqrt{2}\mathbf{a}_\theta) \\ &= 5\sqrt{2}\mathbf{a}_{rs} \end{aligned}$$

This result is to be expected since the given vector has components equal to the coordinates of the point at which it is specified. Hence, its magnitude is equal to the distance of the point from the origin, that is, the spherical coordinate r_s of the point, and its direction is along the line drawn from the origin to the point, that is, along the unit vector $\mathbf{a}_{rs}$ at that point. In fact, the given vector is a particular case of the position vector $x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z = r_s\mathbf{a}_{rs}$, which is the vector drawn from the origin to the point (x, y, z).

K1.3. Cylindrical coordinate system; Orthogonal surfaces; Unit vectors; Differential lengths, surfaces, and volume; Spherical coordinate system; Orthogonal surfaces; Unit vectors; Differential lengths, surfaces, and volume; Conversions between coordinate systems.

D1.7. Convert into Cartesian coordinates each of the following points: **(a)** $(2, 5\pi/6, 3)$ in cylindrical coordinates; **(b)** $(4, 4\pi/3, -1)$ in cylindrical coordinates; **(c)** $(4, 2\pi/3, \pi/6)$ in spherical coordinates; and **(d)** $(\sqrt{8}, \pi/4, \pi/3)$ in spherical coordinates.
Ans. **(a)** $(-\sqrt{3}, 1, 3)$; **(b)** $(-2, -2\sqrt{3}, -1)$; **(c)** $(3, \sqrt{3}, -2)$; **(d)** $(1, \sqrt{3}, 2)$.

D1.8. Convert into cylindrical coordinates the following points specified in Cartesian coordinates: **(a)** $(-2, 0, 1)$; **(b)** $(1, -\sqrt{3}, -1)$; and **(c)** $(-\sqrt{2}, -\sqrt{2}, 3)$.

Ans. **(a)** $(2, \pi, 1)$; **(b)** $(2, 5\pi/3, -1)$; **(c)** $(2, 5\pi/4, 3)$.

D1.9. Convert into spherical coordinates the following points specified in Cartesian coordinates: **(a)** $(0, -2, 0)$; **(b)** $(-3, \sqrt{3}, 2)$; and **(c)** $(-\sqrt{2}, 0, -\sqrt{2})$.

Ans. **(a)** $(2, \pi/2, 3\pi/2)$; **(b)** $(4, \pi/3, 5\pi/6)$; **(c)** $(2, 3\pi/4, \pi)$.

1.4 SCALAR AND VECTOR FIELDS

Before we take up the task of studying electromagnetic fields, we must understand what is meant by a *field*. A field is associated with a region in space, and we say that a field exists in the region if there is a physical phenomenon associated with points in that region. For example, in everyday life we are familiar with the earth's gravitational field. We do not "see" the field in the same manner as we see light rays, but we know of its existence in the sense that objects are acted upon by the gravitational force of Earth. In a broader context, we can talk of the field of any physical quantity as being a description, mathematical or graphical, of how the quantity varies from one point to another in the region of the field and with time. We can talk of scalar or vector fields depending on whether the quantity of interest is a scalar or a vector. We can talk of static or time-varying fields depending on whether the quantity of interest is independent of time or changing with it.

Scalar fields

We shall begin our discussion of fields with some simple examples of scalar fields. Thus let us consider the case of the conical pyramid shown in Fig. 1.17(a). A description of the height of the pyramidal surface versus position on its base is an example of a scalar field involving two variables. Choosing the origin to be the projection of the vertex of the cone onto the base and setting up an xy-coordinate system to locate points on the base, we obtain the height field as a function of x and y to be

$$h(x, y) = 6 - 2\sqrt{x^2 + y^2} \tag{1.50}$$

Although we are able to depict the height variation of points on the conical surface graphically by using the third coordinate for h, we will have to be content with the visualization of the height field by a set of constant-height contours on the xy-plane if only two coordinates were available, as in the case of a two-dimensional space. For the field under consideration, the constant-height contours are circles in the xy-plane centered at the origin and equally spaced for equal increments of the height value, as shown in Fig. 1.17(a).

For an example of a scalar field in three dimensions, let us consider a rectangular room and the distance field of points in the room from one corner of the room, as shown in Fig. 1.17(b). For convenience, we choose this corner to be the origin O and set up a Cartesian coordinate system with the three contiguous edges meeting at that point as the coordinate axes. Each point in the room is defined by a set of values for the three coordinates x, y, and z. The distance r from

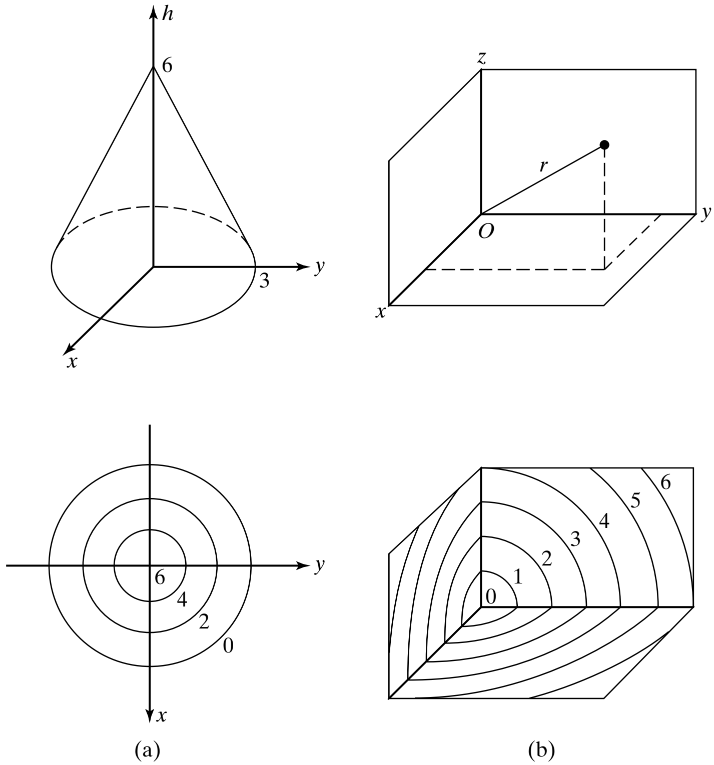

FIGURE 1.17

(a) Conical pyramid lying above the xy-plane and a set of constant-height contours for the conical surface. (b) Rectangular room and a set of constant-distance surfaces depicting the distance field of points in the room from one corner of the room.

the origin to that point is $\sqrt{x^2 + y^2 + z^2}$. Thus, the distance field of points in the room from the origin is given by

$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2} \tag{1.51}$$

Since the three coordinates are already used up for defining the points in the field region, we have to visualize the distance field by means of a set of constant-distance surfaces. A constant-distance surface is a surface for which points on it correspond to a particular constant value of r. For the case under consideration, the constant-distance surfaces are spherical surfaces centered at the origin and are equally spaced for equal increments in the value of the distance, as shown in Fig. 1.17(b).

The fields we have discussed thus far are static fields. A simple example of a time-varying scalar field is provided by the temperature field associated with points in a room, especially when it is being heated or cooled. Just as in the case of the distance field of Fig. 1.17(b), we set up a three-dimensional coordinate system and to each set of three coordinates corresponding to the location of a point in the room, we assign a number to represent the temperature T at that point. Since the temperature at that point, however, varies with time t, this number is a function of time. Thus, we describe mathematically the time-varying temperature field in the room by a function $T(x, y, z, t)$. For any given instant of time, we can visualize a set of constant-temperature or isothermal surfaces corresponding to particular values of T as representing the temperature field for that value of time. For a different instant of time, we will have a different set of isothermal surfaces for the same values of T. Thus, we can visualize the time-varying temperature field in the room by a set of isothermal surfaces continuously changing their shapes as though in a motion picture.

Vector fields

The foregoing discussion of scalar fields may now be extended to vector fields by recalling that a vector quantity has associated with it a direction in space in addition to magnitude. Hence, to describe a vector field, we attribute to each point in the field region a vector that represents the magnitude and direction of the physical quantity under consideration at that point. Since a vector at a given point can be expressed as the sum of its components along the set of unit vectors at that point, a mathematical description of the vector field involves simply the descriptions of the three component scalar fields. Thus, for a vector field $\mathbf{F}$ in the Cartesian coordinate system, we have

$$\boxed{\mathbf{F}(x, y, z, t) = F_x(x, y, z, t)\mathbf{a}_x + F_y(x, y, z, t)\mathbf{a}_y + F_z(x, y, z, t)\mathbf{a}_z} \tag{1.52}$$

Similar expressions in cylindrical and spherical coordinate systems are as follows:

$$\mathbf{F}(r, \phi, z, t) = F_r(r, \phi, z, t)\mathbf{a}_r + F_\phi(r, \phi, z, t)\mathbf{a}_\phi + F_z(r, \phi, z, t)\mathbf{a}_z \tag{1.53a}$$

$$\mathbf{F}(r, \theta, \phi, t) = F_r(r, \theta, \phi, t)\mathbf{a}_r + F_\theta(r, \theta, \phi, t)\mathbf{a}_\theta + F_\phi(r, \theta, \phi, t)\mathbf{a}_\phi \tag{1.53b}$$

We should, however, recall that the unit vectors $\mathbf{a}_r$ and $\mathbf{a}_\phi$ in (1.53a) and all three unit vectors in (1.53b) are themselves nonuniform, but known, functions of the coordinates.

Finding equations for direction lines of a vector field

A vector field is described by a set of *direction lines*, also known as *stream lines* and *flux lines.* A direction line is a curve constructed such that the field is tangential to the curve for all points on the curve. To find the equations for the direction lines for a specified vector field $\mathbf{F}$, we consider the differential length vector $d\mathbf{l}$ tangential to the curve. Then since $\mathbf{F}$ and $d\mathbf{l}$ are parallel, their components must be in the same ratio. Thus, in the Cartesian coordinate system, we obtain the differential equation

$$\boxed{\frac{dx}{F_x} = \frac{dy}{F_y} = \frac{dz}{F_z}} \tag{1.54}$$

which upon integration gives the required algebraic equation. Similar expressions in cylindrical and spherical coordinate systems are as follows:

$$\boxed{\begin{aligned}\frac{dr}{F_r} = \frac{r\,d\phi}{F_\phi} = \frac{dz}{F_z} \qquad (1.55)\\ \frac{dr}{F_r} = \frac{r\,d\theta}{F_\theta} = \frac{r\sin\theta\,d\phi}{F_\phi} \qquad (1.56)\end{aligned}}$$

We shall illustrate the concept of direction lines and the use of (1.54)–(1.56) to obtain the equations for the direction lines by means of an example.

Example 1.5 Linear velocity vector field of points on a rotating disk

Consider a circular disk of radius a rotating with constant angular velocity ω about an axis normal to the disk and passing through its center. We wish to describe the linear velocity vector field associated with points on the rotating disk.

We choose the center of the disk to be the origin and set up a two-dimensional coordinate system, as shown in Fig. 1.18(a). Note that we have a choice of the coordinates

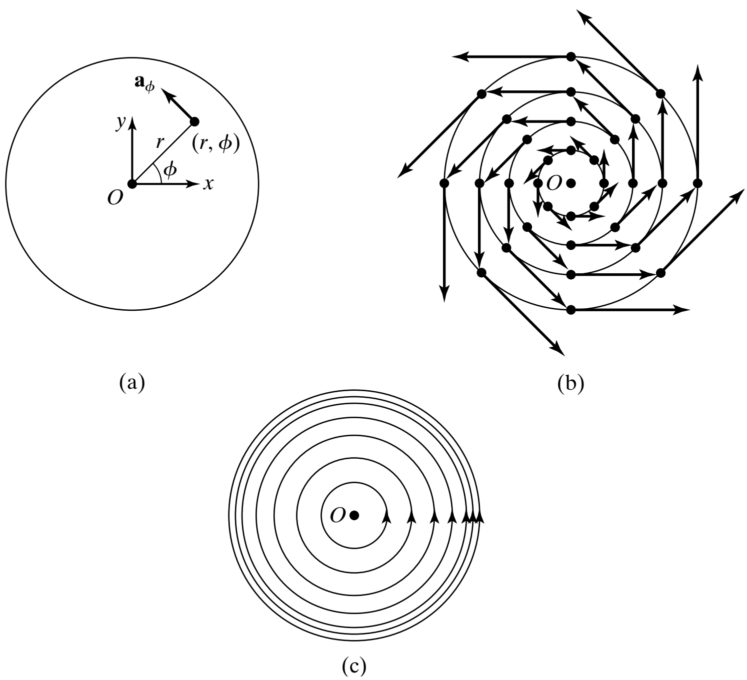

FIGURE 1.18

(a) Rotating disk. (b) Linear velocity vector field associated with points on the rotating disk. (c) Same as (b) except that the vectors are omitted and the density of direction lines is used to indicate the magnitude variation.

(x, y) or the coordinates (r, ϕ). We know that the magnitude of the linear velocity of a point on the disk is then equal to the product of the angular velocity ω and the radial distance r of the point from the center of the disk. The direction of the linear velocity is tangential to the circle drawn through that point and concentric with the disk. Hence, we may depict the linear velocity field by drawing at several points on the disk vectors that are tangential to the circles concentric with the disk and passing through those points, and whose lengths are proportional to the radii of the circles, as shown in Fig. 1.18(b), where the points are carefully selected to reveal the circular symmetry of the field with respect to the center of the disk. We find, however, that this method of representation of the vector field results in a congested sketch of vectors. Hence, we may simplify the sketch by omitting the vectors and simply placing arrowheads along the circles, thereby obtaining a set of direction lines. We note that for the field under consideration, the direction lines are also contours of constant magnitude of the velocity, and hence by increasing the density of the direction lines as r increases, we can indicate the magnitude variation, as shown in Fig. 1.18(c).

For this simple example, we have been able to obtain the direction lines without resorting to the use of mathematics. We shall now consider the mathematical determination of the direction lines and show that the same result is obtained. To do this, we note that the linear velocity vector field is given by

$$\mathbf{v}(r, \phi) = \omega r \mathbf{a}_\phi$$

Then, considering that the geometry associated with the problem is two-dimensional and using (1.55), we have

$$\frac{dr}{0} = \frac{r\,d\phi}{\omega r}$$

or

$$\begin{aligned} dr &= 0 \\ r &= \text{constant} \end{aligned}$$

which represents circles centered at the origin, as in Fig. 1.18(c).

If we wish to obtain the equations for the direction lines using Cartesian coordinates, we first write

$$\begin{aligned} \mathbf{v}(x, y) &= \omega r(\mathbf{a}_\phi \cdot \mathbf{a}_x)\mathbf{a}_x + \omega r(\mathbf{a}_\phi \cdot \mathbf{a}_y)\mathbf{a}_y \\ &= \omega r(-\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y) \\ &= \omega(-y\mathbf{a}_x + x\mathbf{a}_y) \end{aligned}$$

Then from (1.54), we have

$$\frac{dx}{-y} = \frac{dy}{x}$$

or

$$\begin{aligned} x\,dx + y\,dy &= 0 \\ x^2 + y^2 &= \text{constant} \end{aligned}$$

which again represents circles centered at the origin.

K1.4. Field; Static field, Time-varying field; Scalar field; Constant magnitude contours and surfaces; Vector field; Direction lines.

D1.10. The time-varying temperature field in a certain region of space is given by

$$T(x, y, z, t) = T_0\{[x(1 + \sin \pi t)]^2 + [2y(1 - \cos \pi t)]^2 + 4z^2\}$$

where T_0 is a constant. Find the shapes of the constant-temperature surfaces for each of the following values of t: **(a)** $t = 0$; **(b)** $t = 0.5$ s; and **(c)** $t = 1$ s.

Ans. **(a)** elliptic cylinders; **(b)** spheres; **(c)** ellipsoids.

D1.11. For the vector field $\mathbf{F} = (3x - y)\mathbf{a}_x + (x + z)\mathbf{a}_y + (2y - z)\mathbf{a}_z$, find the following: **(a)** the magnitude of **F** and the unit vector along **F** at the point (1, 1, 0); **(b)** the point at which the magnitude of **F** is 3 and the direction of **F** is along the unit vector $\frac{1}{3}(2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$; and **(c)** the point at which the magnitude of **F** is 3 and the direction of **F** is along the unit vector $\mathbf{a}_z$.

Ans. **(a)** $3, \frac{1}{3}(2\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)$; **(b)** (1, 1, 1); **(c)** (0.6, 1.8, −0.6).

D1.12. A vector field is given in cylindrical coordinates by

$$\mathbf{F} = \frac{1}{r^2}(\cos\phi\, \mathbf{a}_r + \sin\phi\, \mathbf{a}_\phi)$$

Express the vector **F** in Cartesian coordinates at each of the following points specified in Cartesian coordinates: **(a)** (1, 0, 0); **(b)** (1, −1, −3); and **(c)** $(1, \sqrt{3}, -4)$.

Ans. **(a)** $\mathbf{a}_x$; **(b)** $-\frac{1}{2}\mathbf{a}_y$; **(c)** $\frac{1}{8}(-\mathbf{a}_x + \sqrt{3}\mathbf{a}_y)$.

1.5 THE ELECTRIC FIELD

Basic to our study of the elements of engineering electromagnetics is an understanding of the concepts of the electric and magnetic fields. Hence, we devote this and the following section to an introduction of these concepts. To introduce the electric field concept, we note that, from our study of Newton's law of gravitation in elementary physics, we are familiar with the gravitational force field associated with material bodies by virtue of their physical property known as *mass*. Newton's experiments showed that the gravitational force of attraction between two bodies of masses m_1 and m_2 separated by a distance R that is very large compared with their sizes, is equal to $m_1 m_2 G/R^2$, where G is the universal constant of gravitation. In a similar manner, a force field known as the *electric field* is associated with bodies that are *charged*. A material body may be charged positively or negatively or may possess no net charge. In the International System of Units that we use throughout this book, the unit of charge is the coulomb, abbreviated C. The charge of an electron is -1.60219×10^{-19} C. Alternatively, approximately 6.24×10^{18} electrons represent a charge of one negative coulomb.

Coulomb's law

Experiments conducted by Coulomb showed that the following hold for two charged bodies that are very small in size compared to their separation so that they can be considered as *point charges*:

1. The magnitude of the force is proportional to the product of the magnitudes of the charges.

2. The magnitude of the force is inversely proportional to the square of the distance between the charges.
3. The magnitude of the force depends on the medium.
4. The direction of the force is along the line joining the charges.
5. Like charges repel; unlike charges attract.

For free space, the constant of proportionality is $1/4\pi\varepsilon_0$, where ε_0 is known as the permittivity of free space, having a value 8.854×10^{-12}, or approximately equal to $10^{-9}/36\pi$. (For convenience, we shall use a value of $10^{-9}/36\pi$ for ε_0 throughout this book.) Thus, if we consider two point charges Q_1 C and Q_2 C separated R m in free space, as shown in Fig. 1.19, then the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ experienced by Q_1 and Q_2, respectively, are given by

$$\boxed{\mathbf{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\mathbf{a}_{21}} \tag{1.57a}$$

and

$$\boxed{\mathbf{F}_2 = \frac{Q_2 Q_1}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12}} \tag{1.57b}$$

where $\mathbf{a}_{21}$ and $\mathbf{a}_{12}$ are unit vectors along the line joining Q_1 and Q_2, as shown in Fig. 1.19. Equations (1.57a) and (1.57b) represent Coulomb's law. Since the units of force are newtons, we note that ε_0 has the units (coulomb)^2 per (newton-meter^2). These are commonly known as *farads per meter*, where a farad is a (coulomb)^2 per newton-meter.

Electric field defined

In the case of the gravitational field of a material body, we define the gravitational field intensity as the force per unit mass experienced by a small test mass placed in that field. In a similar manner, the force per unit charge experienced by a small test charge placed in an electric field is known as the *electric field intensity*, denoted by the symbol **E**. Alternatively, if in a region of space, a test charge q experiences a force **F**, then the region is said to be characterized by an electric field of intensity **E** given by

$$\mathbf{E} = \frac{\mathbf{F}}{q} \tag{1.58}$$

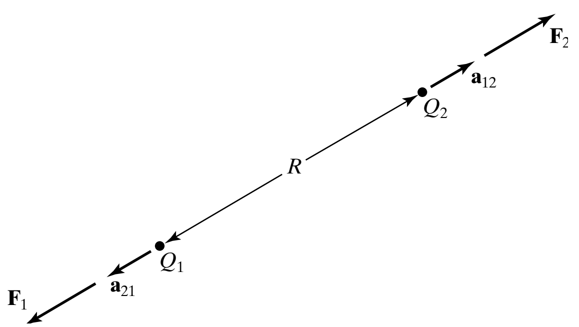

FIGURE 1.19

Forces experienced by two point charges Q_1 and Q_2.

The unit of electric field intensity is newton per coulomb, or more commonly volt per meter, where a volt is a newton-meter per coulomb. The test charge should be so small that it does not alter the electric field in which it is placed. Ideally, $\mathbf{E}$ is defined in the limit that q tends to zero; that is,

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q}} \tag{1.59}$$

Equation (1.59) is the defining equation for the electric field intensity irrespective of the source of the electric field. Just as one body by virtue of its mass is the source of a gravitational field acting on other bodies by virtue of their masses, a charged body is the source of an electric field acting on other charged bodies. We will, however, learn in Chapter 2 that there exists another source for the electric field, namely, a time-varying magnetic field.

Electrostatic separation of minerals

Equation (1.58) or (1.59) tells us that the force experienced by a charged particle placed at a point in an external electric field is in the same direction as that of the electric field if the charge is positive, but opposite to that of the electric field if the charge is negative, as shown in Fig. 1.20. This phenomenon is the basis behind *electrostatic separation*, a process widely used in industry to separate minerals.[1] An example is illustrated in Fig. 1.21. Phosphate ore composed of granules of quartz and phosphate rock is dropped through a hopper onto a vibrating feeder. The friction between the two types of granules resulting from the vibration causes the quartz particles to be positively charged and the phosphate particles to be negatively charged. The oppositely charged particles are then passed through a chute into the electric field region between two parallel plates, where they are separated and subsequently collected separately.

Cathode ray tube

There are many other devices based on the electric force on a charged particle. We shall, however, discuss only one other application, the cathode ray tube, which is used in oscilloscopes, TV receivers, computer display terminals, and so on. The schematic of a cathode ray tube is shown in Fig. 1.22. Electrons are emitted from the heated cathode and are accelerated toward the anode by an electric field directed from the anode toward the cathode. After passing through the anode, they enter a region between two orthogonal pairs of parallel plates, one pair being horizontal and the other vertical. A voltage applied to the

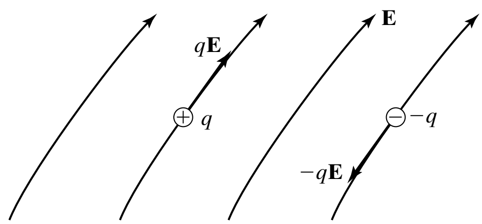

FIGURE 1.20

Forces experienced by positive and negative charges in an electric field.

[1]See, for example, A. D. Moore, ed., *Electrostatics and Its Applications* (New York: John Wiley & Sons, 1973), Chap. 10.

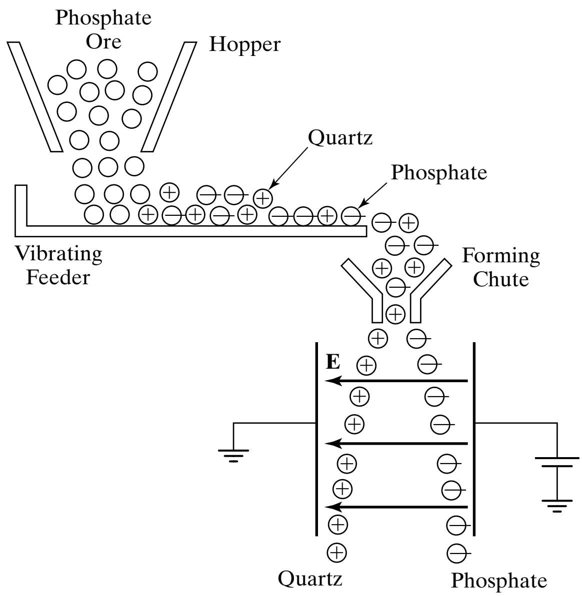

FIGURE 1.21

Example for illustrating electrostatic separation of minerals.

horizontal set of plates produces an electric field between the plates directed vertically, thereby deflecting the electrons vertically and imparting to them a vertical component of velocity as they leave the region between the plates. Similarly, a voltage applied to the vertical set of plates deflects the electrons horizontally sideways and imparts to them a sideways component of velocity as they leave the region between the plates. Thus, by varying the voltages applied to the two sets of plates, the electron beam can be made to strike the fluorescent screen and produce a bright spot at any point on the screen.

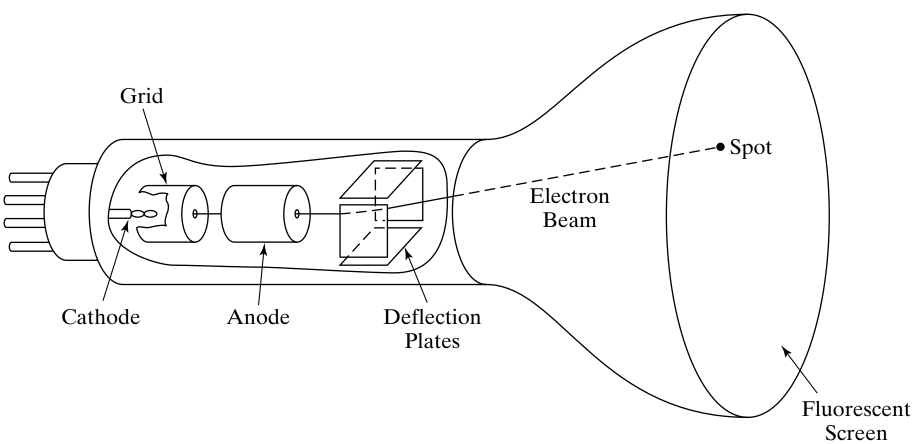

FIGURE 1.22

Schematic diagram of a cathode ray tube.

Electric field due to a point charge

Returning now to Coulomb's law and letting one of the two charges in Fig. 1.19, say, Q_2, be a small test charge q, we have

$$\mathbf{F}_2 = \frac{Q_1 q}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12} \tag{1.60}$$

The electric field intensity $\mathbf{E}_2$ at the test charge due to the point charge Q_1 is then given by

$$\mathbf{E}_2 = \frac{\mathbf{F}_2}{q} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12} \tag{1.61}$$

Generalizing this result by making R a variable, that is, by moving the test charge around in the medium, writing the expression for the force experienced by it, and dividing the force by the test charge, we obtain the electric field intensity $\mathbf{E}$ due to a point charge Q to be

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R} \tag{1.62}$$

where R is the distance from the point charge to the point at which the field intensity is to be computed and $\mathbf{a}_R$ is the unit vector along the line joining the two points under consideration and directed away from the point charge. The electric field intensity due to a point charge is thus directed everywhere radially away from the point charge and its constant-magnitude surfaces are spherical surfaces centered at the point charge, as shown by the cross-sectional view in Fig. 1.23.

Using (1.62) in conjunction with (1.25) and (1.26), we can obtain the expression for the electric field intensity at a point $P(x, y, z)$ due to a point charge Q located at a point $P'(x', y', z')$. Thus, noting that the vector $\mathbf{R}$ from P' to P is given by $[(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z]$ and the unit vector $\mathbf{a}_R$ is equal to $\mathbf{R}/R$, we obtain

$$\boxed{\begin{aligned}\mathbf{E} &= \frac{Q\mathbf{R}}{4\pi\varepsilon_0 R^3}\\ &= \frac{Q}{4\pi\varepsilon_0}\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}\end{aligned}} \tag{1.63}$$

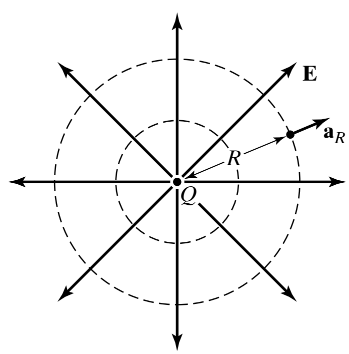

FIGURE 1.23

Direction lines and cross sections of constant-magnitude surfaces of electric field due to a point charge.

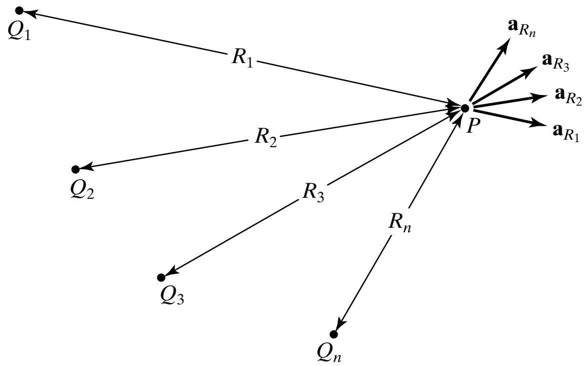

FIGURE 1.24

Collection of point charges and unit vectors along the directions of their electric fields at a point P.

For a numerical example, if P and P' are $(3, 1, 1)$ and $(1, -1, 0)$, respectively, then

$$\mathbf{E} = \frac{Q}{108\pi\varepsilon_0}(2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$$

If we now have several point charges $Q_1, Q_2, \ldots,$ as shown in Fig. 1.24, the force experienced by a test charge situated at a point P is the vector sum of the forces experienced by the test charge due to the individual charges. It then follows that the electric field intensity at point P is the superposition of the electric field intensities due to the individual charges; that is,

$$\boxed{\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\mathbf{a}_{R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\mathbf{a}_{R_2} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 R_n^2}\mathbf{a}_{R_n}} \tag{1.64}$$

We shall illustrate the application of (1.64) by means of an example involving two point charges.

Example 1.6 Electric field of two point charges

Let us consider two point charges $Q_1 = 8\pi\varepsilon_0$ C and $Q_2 = -4\pi\varepsilon_0$ C situated at $(-1, 0, 0)$ and $(1, 0, 0)$, respectively. We wish to (a) find the electric field intensity at the point $(0, 0, 1)$ and (b) discuss computer generation of the direction line of $\mathbf{E}$ passing through that point.

(a) Using (1.64) and (1.63) in conjunction with the geometry in Fig. 1.25(a), we obtain

$$\begin{aligned}[\mathbf{E}]_{(0,0,1)} &= [\mathbf{E}_1]_{(0,0,1)} + [\mathbf{E}_2]_{(0,0,1)} \\ &= \frac{8\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(\mathbf{a}_x + \mathbf{a}_z)}{2^{3/2}} - \frac{4\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(-\mathbf{a}_x + \mathbf{a}_z)}{2^{3/2}} \\ &= 1.118\left(\frac{3\mathbf{a}_x + \mathbf{a}_z}{\sqrt{10}}\right)\end{aligned} \tag{1.65}$$

Note that the direction of $\mathbf{E}$ is given by the unit vector $(3\mathbf{a}_x + \mathbf{a}_z)/\sqrt{10}$ pointing away from the positive charge Q_1. The field vectors $\mathbf{E}_1$ and $\mathbf{E}_2$, and the resultant field vector $\mathbf{E}$, are shown in Fig. 1.25(a).

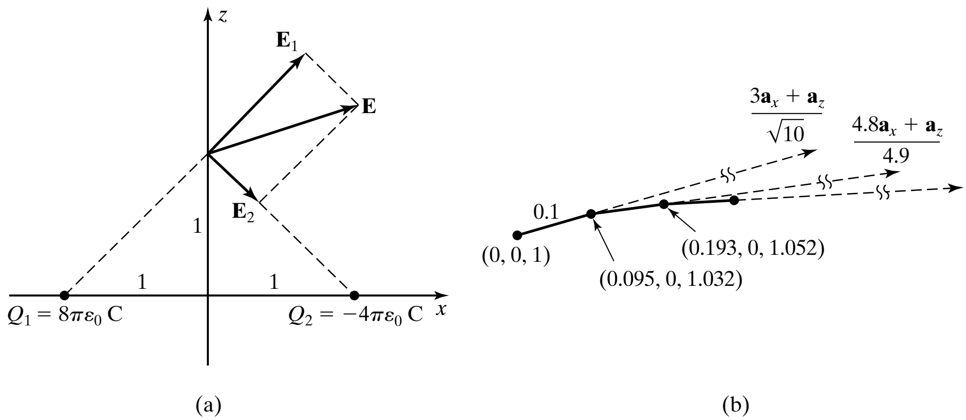

FIGURE 1.25

(a) Computation of the resultant electric field due to two point charges. (b) Generation of the direction line of the electric field of (a).

(b) To discuss the computer generation of the direction line of $\mathbf{E}$, we recall that a direction line is a curve such that at any given point on the curve, the field is tangential to the curve. For the case of the electric field, it is also the path followed by an infinitesimal test charge when released at a point on the curve. To obtain the direction line through the point (0, 0, 1), we go by an incremental distance from (0, 0, 1) along the direction of the electric field vector at that point to reach a new point, compute the field at the new point, and continue the process. Thus, choosing for the purpose of illustration an incremental distance of 0.1 m and going along the unit vector $(3\mathbf{a}_x + \mathbf{a}_z)/\sqrt{10}$ from (0, 0, 1), we obtain the new point to be (0.095, 0, 1.032), as shown in Fig. 1.25(b). The electric field at this point is

$$\begin{aligned}[\mathbf{E}]_{(0.095,\,0,\,1.032)} &= \frac{8\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(1.095\mathbf{a}_x + 1.032\mathbf{a}_z)}{(1.095^2 + 1.032^2)^{3/2}} - \frac{4\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(-0.905\mathbf{a}_x + 1.032\mathbf{a}_z)}{(0.905^2 + 1.032^2)^{3/2}} \\ &= 1.015\left(\frac{4.8\mathbf{a}_x + \mathbf{a}_z}{4.9}\right)\end{aligned} \tag{1.66}$$

Note that the direction of this electric field, which is along the unit vector $(4.8\mathbf{a}_x + \mathbf{a}_z)/4.9$, is slanted more toward the negative charge Q_2 than that of the electric field at the point (0, 0, 1), as shown in Fig. 1.25(b), indicating the swing of the direction line toward Q_2. The procedure is continued by going the incremental distance of 0.1 m from (0.095, 0, 1.032) along the unit vector $(4.8\mathbf{a}_x + \mathbf{a}_z)/4.9$ to the new point (0.193, 0, 1.052) and computing the field vector at that point, and so on, until the direction line is terminated close to the point charge Q_2. The same can be done to obtain the portion of the direction line from (0, 0, 1) toward the point charge Q_1, by moving opposite to $\mathbf{E}$. Values of the coordinates of the beginning point (X and Z), the magnitude of the electric field at that point (E), and the components of the unit vector along the electric field (UX and UZ), pertinent to the steps along the direction line computed in this manner, are listed in

TABLE 1.1 Values of Parameters Pertinent to the Steps in the Computer Generation of the Direction Line of **E** in Fig. 1.25 for (a) the Segment from (0, 0, 1) Toward the Charge Q_2 and (b) the Segment from (0, 0, 1) Back Toward the Charge Q_1.

X = 0.000	Z = 1.000	E = 1.118	UX = 0.949	UZ = 0.316
X = 0.095	Z = 1.032	E = 1.015	UX = 0.979	UZ = 0.204
X = 0.193	Z = 1.052	E = 0.942	UX = 0.997	UZ = 0.076
X = 0.292	Z = 1.060	E = 0.898	UX = 0.998	UZ = −0.065
X = 0.392	Z = 1.053	E = 0.882	UX = 0.977	UZ = −0.215
X = 0.490	Z = 1.032	E = 0.898	UX = 0.930	UZ = −0.368
X = 0.583	Z = 0.995	E = 0.951	UX = 0.858	UZ = −0.513
X = 0.669	Z = 0.944	E = 1.051	UX = 0.766	UZ = −0.643
X = 0.745	Z = 0.879	E = 1.212	UX = 0.660	UZ = −0.751
X = 0.811	Z = 0.804	E = 1.459	UX = 0.548	UZ = −0.836
X = 0.866	Z = 0.721	E = 1.837	UX = 0.439	UZ = −0.899
X = 0.910	Z = 0.631	E = 2.426	UX = 0.337	UZ = −0.942
X = 0.944	Z = 0.536	E = 3.391	UX = 0.246	UZ = −0.969
X = 0.968	Z = 0.440	E = 5.100	UX = 0.167	UZ = −0.986
X = 0.985	Z = 0.341	E = 8.537	UX = 0.101	UZ = −0.995
X = 0.995	Z = 0.241	E = 17.101	UX = 0.049	UZ = −0.999
X = 1.000	Z = 0.142	E = 49.846	UX = 0.010	UZ = −1.000
X = 1.001	Z = 0.042	E = 577.540	UX = −0.023	UZ = −1.000

Number of steps = 17

(a)

X = 0.000	Z = 1.000	E = 1.118	UX = 0.949	UZ = 0.316
X = −.095	Z = 0.968	E = 1.243	UX = 0.908	UZ = 0.420
X = −.186	Z = 0.926	E = 1.411	UX = 0.862	UZ = 0.507
X = −.272	Z = 0.876	E = 1.634	UX = 0.815	UZ = 0.580
X = −.353	Z = 0.818	E = 1.931	UX = 0.768	UZ = 0.640
X = −.430	Z = 0.754	E = 2.333	UX = 0.724	UZ = 0.689
X = −.503	Z = 0.685	E = 2.888	UX = 0.684	UZ = 0.730
X = −.571	Z = 0.612	E = 3.681	UX = 0.648	UZ = 0.762
X = −.636	Z = 0.536	E = 4.871	UX = 0.616	UZ = 0.788
X = −.697	Z = 0.457	E = 6.769	UX = 0.590	UZ = 0.808
X = −.756	Z = 0.376	E = 10.074	UX = 0.568	UZ = 0.823
X = −.813	Z = 0.294	E = 16.616	UX = 0.551	UZ = 0.835
X = −.868	Z = 0.210	E = 32.588	UX = 0.538	UZ = 0.843
X = −.922	Z = 0.126	E = 91.176	UX = 0.529	UZ = 0.849
X = −.975	Z = 0.041	E = 860.610	UX = 0.522	UZ = 0.853

Number of steps = 14

(b)

Table 1.1, parts (a) and (b), corresponding to the segments of the direction line from (0, 0, 1) toward Q_2 and Q_1, respectively. It can be seen that the test charge takes 17 steps toward Q_2 but only 14 steps back toward Q_1.

In the simple procedure employed in Example 1.6, there is a (cumulative) error associated with each step. This error can be reduced by employing a modified procedure as follows: Instead of moving the test charge by 0.1 m from its current location, say, point A, to a new location, say, point B, along the direction of the electric field at point A, it is moved by 0.1 m to a point C along a direction that bisects the directions of the fields at points A and B. Computer-plotted field maps in the xz-plane for pairs of point charges Q_1 and Q_2 located at $(-1, 0)$ and $(1, 0)$, respectively, by using this modified procedure are shown in Fig. 1.26. In each map, plotting of a direction line begins at one of the point charges and terminates when the line reaches to within a distance of 0.1 m from the second point charge, or if it goes beyond a specified rectangular region. In this manner, direction lines beginning at points around each point charge and at 30° intervals on a circle of radius 0.1 m are plotted, with the 0° angle corresponding to the $+x$-direction.

For Fig. 1.26(a), $Q_1 = 2Q$ at $(-1, 0)$ and $Q_2 = -Q$ at $(1, 0)$. The rectangular region is one having corners at $(-3, 2)$, $(3, 2)$, $(3, -2)$, and $(-3, -2)$. The direction lines beginning at each point charge either end on the second charge or go out of the boundary of the rectangular region. For Figs. 1.26(b)–(d), region of map is rectangle having corners at $(-3, 4)$, $(3, 4)$, $(3, 0)$, and $(-3, 0)$, taking advantage of the symmetry of the field map about the axis through the charges,

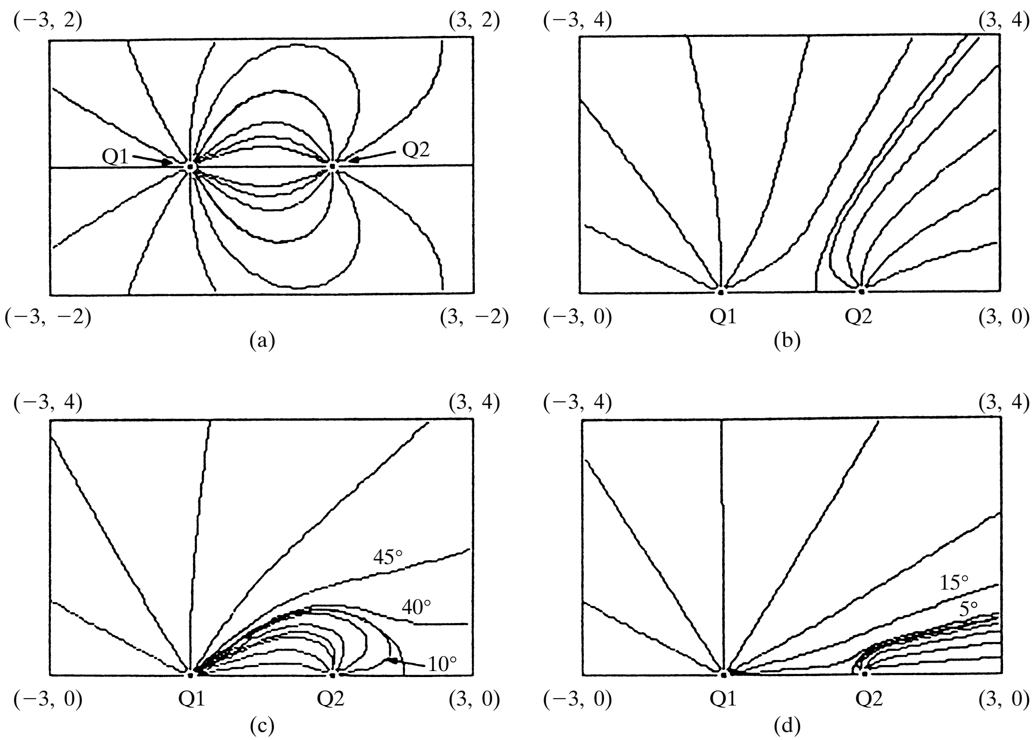

FIGURE 1.26

Computer-generated maps of direction lines of electric field for pairs of point charges Q_1 and Q_2 at $(-1, 0)$ and $(1, 0)$, respectively, in the xz-plane. (a) $Q_1 = 2Q$, $Q_2 = -Q$; (b) $Q_1 = 4Q$, $Q_2 = Q$; (c) $Q_1 = 9Q$, $Q_2 = -Q$; and (d) $Q_1 = 81Q$, $Q_2 = Q$.

illustrated in Fig. 1.26(a). For Fig. 1.26(b), $Q_1 = 4Q$ at $(-1, 0)$ and $Q_2 = Q$ at (1, 0). A zero-field point exists within the region at $(\frac{1}{3}, 0)$, between the two charges. For direction lines passing through this point, the test charge gets trapped at that point, and the procedure used is to untrap it by displacing it by 0.01 perpendicular to the axis and continue plotting the line until it terminates at a point on the boundary of the region. For Fig. 1.26(c), $Q_1 = 9Q$ at $(-1, 0)$ and $Q_2 = -Q$ at (1, 0). A zero-field point exists within the region at (2, 0), to the right of Q_2. Also, three additional field lines are shown plotted. Two of these are from Q_1 at angles of 40° and 45° and the third is from Q_2 at 10°. For Fig. 1.26(d), $Q_1 = 81Q$ at $(-1, 0)$ and $Q_2 = Q$ at $(1, 0)$. A zero-field point exists just to the left of Q_2 between the two charges. The map also includes two additional field lines originating from Q_1 at 5° and 15° angles.

Types of charge distributions

The foregoing illustration of the computation of the electric field intensity due to two point charges can be extended to the computation of the field intensity due to continuous charge distributions. Continuous charge distributions are of three types: line charges, surface charges, and volume charges, depending on whether the charge is distributed along a line like chalk powder along a thin line drawn on the blackboard, on a surface like chalk powder on the erasing surface of a blackboard eraser, or in a volume like chalk powder in the chalk itself. The corresponding charge densities are the line charge density ρ_L, the surface charge density ρ_S, and the volume charge density ρ, having the units of charge per unit length (coulombs per meter), charge per unit area (coulombs per meter squared), and charge per unit volume (coulombs per meter cubed), respectively. The technique of finding the electric field intensity due to a given charge distribution consists of dividing the region of the charge distribution into a number of differential lengths, surfaces, or volumes, depending on the type of the distribution, considering the charge in each differential element to be a point charge, and using superposition. We shall illustrate the procedure by means of three examples.

Example 1.7 Circular ring charge with uniform density

Ring charge

Charge Q C is distributed with uniform density along a circular ring of radius a lying in the xy-plane and having its center at the origin, as shown in Fig. 1.27. We wish to find the electric field intensity at a point on the z-axis.

Let us divide the ring into a large number of segments so that the charge in each segment can be considered to be a point charge located at the center of the segment. Let the segments be of equal length and numbered $1, 2, 3, \ldots, 2n$, as shown in Fig. 1.27. Then the electric field intensity at the point $(0, 0, z)$ due to the charge in the jth segment is given by

$$\mathbf{E}_j = \frac{Q_j}{4\pi\varepsilon_0 R_j^2}\mathbf{a}_{R_j}$$

where Q_j is the charge in the jth segment and R_j and $\mathbf{a}_{R_j}$ are as shown in the figure. Since the charge is uniformly distributed, Q_j is the same for all j and is equal to the charge

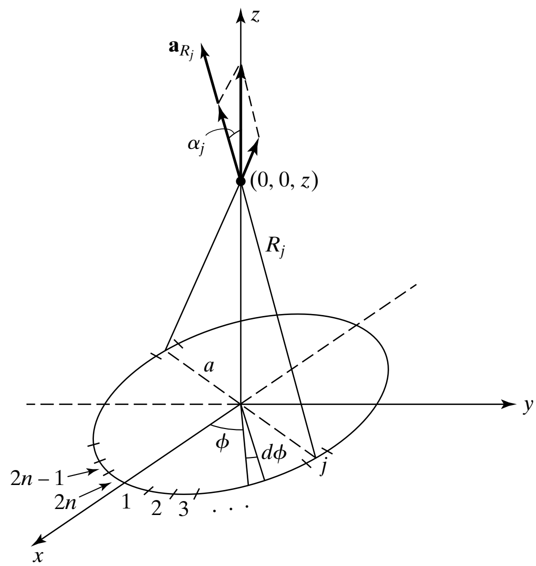

FIGURE 1.27

Determination of electric field due to a circular ring of charge of uniform density.

density times the length of the segment. Thus,

$$Q_j = \left(\frac{Q}{2\pi a}\right)\left(\frac{2\pi a}{2n}\right) = \frac{Q}{2n}$$

Furthermore, since the point $(0, 0, z)$ is along the axis of the ring, it is equidistant from all segments so that R_j is the same for all j. Hence,

$$R_j = \sqrt{z^2 + a^2}$$

Now, from symmetry considerations, we note that for every segment $1, 2, 3, \ldots, n$, there is a corresponding segment diametrically opposite to it in the other half of the ring such that the electric field intensity due to the two segments together is directed along the z-axis, as illustrated for segment j in Fig. 1.27. Hence, to find $\mathbf{E}$ due to the entire ring charge, it is sufficient if we consider the z-component of $\mathbf{E}_j$, multiply it by 2, and sum from $j = 1$ to $j = n$. Thus, we obtain the required electric field intensity to be

$$\begin{aligned}
[\mathbf{E}]_{(0,0,z)} &= \sum_{j=1}^{n} \frac{2Q_j}{4\pi\varepsilon_0 R_j^2}(\mathbf{a}_{R_j}\cdot\mathbf{a}_z)\mathbf{a}_z \\
&= \sum_{j=1}^{n} \frac{Q_j}{2\pi\varepsilon_0 R_j^2}\cos\alpha_j\,\mathbf{a}_z \\
&= \sum_{j=1}^{n} \frac{Q_j z}{2\pi\varepsilon_0 R_j^3}\mathbf{a}_z \\
&= \sum_{j=1}^{n} \frac{Qz}{4\pi\varepsilon_0 n(z^2 + a^2)^{3/2}}\mathbf{a}_z \\
&= \frac{Qz}{4\pi\varepsilon_0 (z^2 + a^2)^{3/2}}\mathbf{a}_z
\end{aligned} \tag{1.67}$$

Note that $[\mathbf{E}]_{(0,0,z)}$ is directed in the $+z$-direction above the origin $(z > 0)$ and in the $-z$-direction below the origin $(z < 0)$, as to be expected.

Alternative to the summation procedure just employed, we can obtain E_z at $(0, 0, z)$ by setting up an integral expression and evaluating it. Thus, considering a differential length $a\,d\phi$ of the ring charge at the point $(a, \phi, 0)$, as shown in Fig. 1.27, and making use of symmetry considerations as discussed in connection with the summation procedure, we obtain

$$\begin{aligned} [E_z]_{(0,0,z)} &= \int_{\phi=0}^{\pi} \frac{2(Q/2\pi a)a\,d\phi}{4\pi\varepsilon_0(a^2+z^2)} \frac{z}{(a^2+z^2)^{1/2}} \\ &= \frac{Qz}{4\pi^2\varepsilon_0(a^2+z^2)^{3/2}} \int_{\phi=0}^{\pi} d\phi \\ &= \frac{Qz}{4\pi\varepsilon_0(a^2+z^2)^{3/2}} \end{aligned} \tag{1.68}$$

For this example, the two results given by (1.67) and (1.68) are identical. In general, however, the summation procedure gives an approximate result for any finite value of n, and the integral gives the exact result, provided it can be evaluated in closed form. The summation procedure is, however, more illuminating as to the application of superposition and is convenient for computer solution.

Example 1.8 Electric field of an infinitely long line charge of uniform density

Infinitely long line charge

Let us consider an infinitely long line charge along the z-axis with uniform charge density ρ_{L0} C/m and find the electric field intensity everywhere.

Let us first consider a point $P(r, \phi, 0)$ on the xy-plane, as shown in Fig. 1.28(a). Then the solution can be carried out by dividing the line charge into a series of infinitesimal segments, considering each segment to be a point charge, and using superposition. Two such segments having lengths dz' and equidistant from the origin, located at $(0, 0, z')$ and $(0, 0, -z')$, are shown in the figure. Noting that the electric field contributions due to these two segments make equal angles α with the xy-plane and hence their superposition has only an r-component, we obtain the field due to the two segments to be

$$\begin{aligned} [d\mathbf{E}]_{(r,\phi,0)} &= 2\frac{\rho_{L0}\,dz'}{4\pi\varepsilon_0[r^2+(z')^2]}\cos\alpha\,\mathbf{a}_r \\ &= \frac{\rho_{L0} r\,dz'}{2\pi\varepsilon_0[r^2+(z')^2]^{3/2}}\mathbf{a}_r \end{aligned}$$

The electric field intensity at P due to the entire line charge is then given by

$$\begin{aligned} [\mathbf{E}]_{(r,\phi,0)} &= \int_{z'=0}^{\infty} [d\mathbf{E}]_{(r,\phi,0)} \\ &= \int_{z'=0}^{\infty} \frac{\rho_{L0} r\,dz'}{2\pi\varepsilon_0[r^2+(z')^2]^{3/2}}\mathbf{a}_r \\ &= \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\int_{\alpha=0}^{\pi/2} \cos\alpha\,d\alpha \\ &= \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\mathbf{a}_r \end{aligned}$$

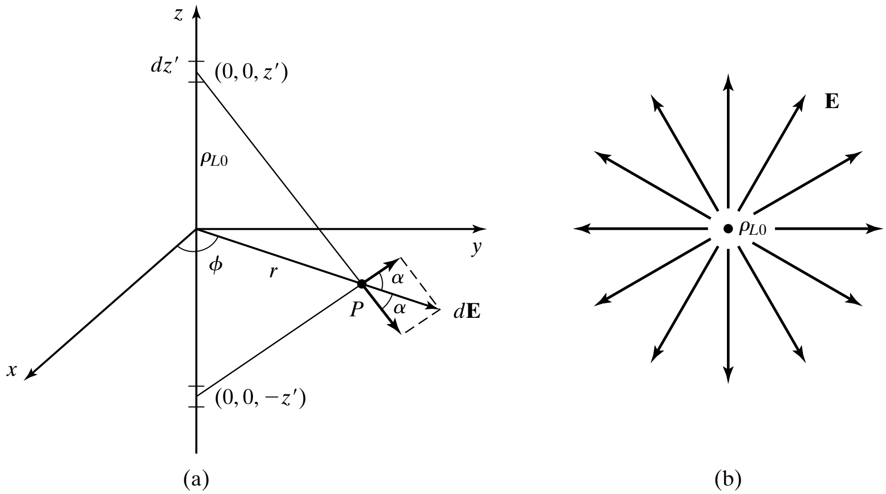

FIGURE 1.28

(a) Determination of electric field due to an infinitely long line charge of uniform charge density ρ_{L0} C/m. (b) Electric field due to the infinitely long line charge of (a).

where we have used the relationship $z' = r \tan \alpha$ to make a change of the variable of integration from z' to α.

Finally, since the charge density is uniform and the xy-plane can be chosen to be passing through any point on the line charge without changing the geometry, this result is valid for any value of z. Thus, the required electric field intensity is

$$\mathbf{E} = \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\mathbf{a}_r \tag{1.69}$$

which has the magnitude $\dfrac{\rho_{L0}}{2\pi\varepsilon_0 r}$ and is everywhere radial to the line charge as shown by the cross-sectional view in Fig. 1.28(b).

Example 1.9 Electric field of an infinite plane sheet of charge of uniform density

Infinite plane sheet of charge

Let us consider an infinite plane sheet of charge in the xy-plane with uniform surface charge density ρ_{S0} C/m^2 and find the electric field intensity due to it everywhere.

Let us first consider a point $(0, 0, z)$ on the z-axis, as shown in Fig. 1.29(a). Then the solution can be carried out by dividing the sheet into a number of infinitesimal surfaces in Cartesian coordinates and using superposition. An alternate procedure consists of using the result of Example 1.7 by dividing the sheet into concentric rings centered at the origin and each having infinitesimal width dr in the radial direction. One such ring having the arbitrary radius r and width dr is shown in Fig. 1.29(a). The charge in that ring is

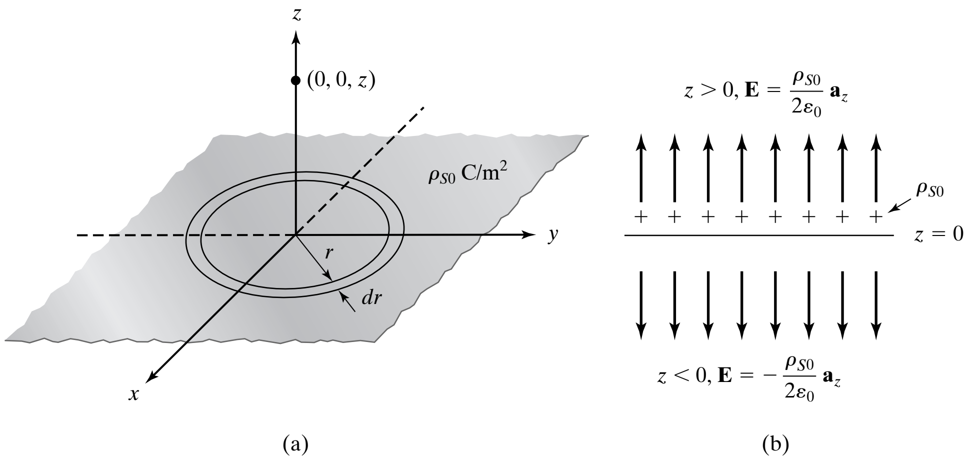

FIGURE 1.29

(a) Determination of electric field due to an infinite plane sheet of uniform surface charge density ρ_{S0} C/m^2. (b) Electric field due to the infinite plane sheet of charge of (a).

equal to $\rho_{S0}(2\pi r\,dr)$, the product of the uniform surface charge density and the area of the ring. According to the result obtained in Example 1.7, the electric field intensity at$(0, 0, z)$ due to this ring charge is given by

$$[d\mathbf{E}]_{(0,\,0,\,z)} = \frac{(\rho_{S0}2\pi r\,dr)z}{4\pi\varepsilon_0(r^2 + z^2)^{3/2}}\mathbf{a}_z$$

The electric field intensity due to the entire sheet of charge is then given by

$$\begin{aligned}
[\mathbf{E}]_{(0,\,0,\,z)} &= \int_{r=0}^{\infty}[d\mathbf{E}]_{(0,0,z)} \\
&= \int_{r=0}^{\infty}\frac{\rho_{S0}rz\,dr}{2\varepsilon_0(r^2 + z^2)^{3/2}}\mathbf{a}_z \\
&= \frac{\rho_{S0}z}{2\varepsilon_0}\left[-\frac{1}{\sqrt{r^2 + z^2}}\right]_{r=0}^{\infty}\mathbf{a}_z \\
&= \frac{\rho_{S0}z}{2\varepsilon_0|z|}\mathbf{a}_z
\end{aligned}$$

Finally, since the charge density is uniform and the origin of the coordinate system can be chosen anywhere on the infinite sheet without changing the geometry, this result is valid everywhere. Thus, the required electric field intensity is

$$\mathbf{E} = \pm\frac{\rho_{S0}}{2\varepsilon_0}\mathbf{a}_z \qquad \text{for} \quad z \gtrless 0 \tag{1.70}$$

which has the magnitude $\rho_{S0}/2\varepsilon_0$ everywhere and directed normally away from the sheet, as shown by the cross-sectional view in Fig. 1.29(b). Defining $\mathbf{a}_n$ to be the unit normal vector directed away from the sheet, that is,

$$\mathbf{a}_n = \pm\mathbf{a}_z \qquad \text{for} \quad z \gtrless 0$$

we have

$$\boxed{\mathbf{E} = \frac{\rho_{S0}}{2\varepsilon_0}\mathbf{a}_n} \tag{1.71}$$

K1.5. Coulomb's law; Electric field intensity; **E** due to a point charge; Computation of **E** due to charge distributions; **E** due to an infinitely long line charge of uniform density; **E** due to an infinite plane sheet of charge of uniform density.

D1.13. Point charges, each of value $\sqrt{4\pi\varepsilon_0}$ C, are located at the vertices of an n-sided regular polygon circumscribed by a circle of radius a. Find the electric force on each charge for **(a)** $n = 3$; **(b)** $n = 4$; and **(c)** $n = 6$.

Ans. **(a)** $0.577/a^2$ N; **(b)** $0.957/a^2$ N; **(c)** $1.827/a^2$ N; all directed away from the center of the polygon.

D1.14. In Fig. 1.25, let the point charges be $Q_1 = 8\pi\varepsilon_0$ C at $(-1, 0, 0)$ and $Q_2 = 4\pi\varepsilon_0$ C at $(1, 0, 0)$. Find the following: **(a)** **E** at $(0, 0, 1)$; **(b)** the coordinates of the point at the end of the second step; and **(c)** the unit vector along **E** at the point computed in **(b)**.

Ans. **(a)** $(0.353\mathbf{a}_x + 1.061\mathbf{a}_z)$; **(b)** $(0.060, 0, 1.191)$; **(c)** $(0.264\mathbf{a}_x + 0.965\mathbf{a}_z)$.

D1.15. In Fig. 1.27, let there be a second ring of charge $-Q$, uniformly distributed along a circle of radius a, having its center at $(0, 0, 2a)$ and lying parallel to the xy-plane. Find **E** due to the two rings of charge together at each of the following points: **(a)** $(0, 0, 0)$; **(b)** $(0, 0, a)$; and **(c)** $(0, 0, 3a)$.

Ans. **(a)** $(0.0142Q/\varepsilon_0 a^2)\mathbf{a}_z$; **(b)** $(0.0563Q/\varepsilon_0 a^2)\mathbf{a}_z$; **(c)** $(-0.0206Q/\varepsilon_0 a^2)\mathbf{a}_z$.

D1.16. Infinite plane sheets of charge lie in the $z = 0$, $z = 2$, and $z = 4$ planes with uniform surface charge densities ρ_{S1}, ρ_{S2}, and ρ_{S3}, respectively. Given that the resulting electric field intensities at the points $(3, 5, 1)$, $(1, -2, 3)$, and $(3, 4, 5)$ are $\mathbf{0}$, $6\mathbf{a}_z$, and $4\mathbf{a}_z$ V/m, respectively, find the following: **(a)** ρ_{S1}; **(b)** ρ_{S2}; **(c)** ρ_{S3}; and **(d)** **E** at $(-2, 1, -6)$.

Ans. **(a)** $4\varepsilon_0$ C/m^2; **(b)** $6\varepsilon_0$ C/m^2; **(c)** $-2\varepsilon_0$ C/m^2; **(d)** $-4\mathbf{a}_z$ V/m.

1.6 THE MAGNETIC FIELD

In the preceding section, we presented an experimental law known as Coulomb's law having to do with the electric force associated with two charged bodies, and we introduced the electric field intensity vector as the force per unit charge experienced by a test charge placed in the electric field. In this section, we present another experimental law known as *Ampère's law of force*, analogous to Coulomb's law, and use it to introduce the magnetic field concept.

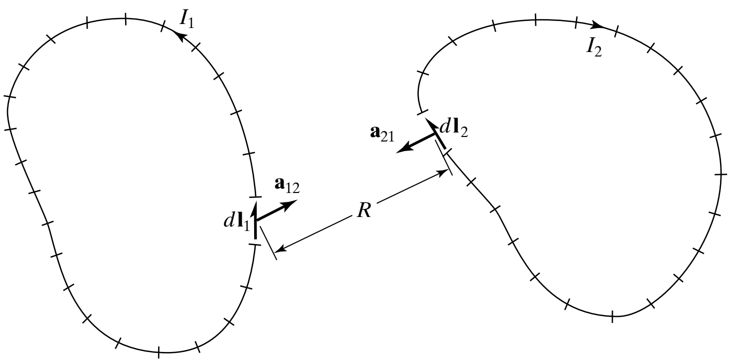

FIGURE 1.30

Two loops of wire carrying currents I_1 and I_2.

Ampère's law of force

Ampère's law of force is concerned with magnetic forces associated with two loops of wire carrying currents by virtue of motion of charges in the loops. Figure 1.30 shows two loops of wire carrying currents I_1 and I_2 and each of which is divided into a large number of elements having infinitesimal lengths. The total force experienced by a loop is the vector sum of forces experienced by the infinitesimal current elements constituting the loop. The force experienced by each of these current elements is the vector sum of the forces exerted on it by the infinitesimal current elements constituting the second loop. If the number of elements in loop 1 is m and the number of elements in loop 2 is n, then there are $m \times n$ pairs of elements. A pair of magnetic forces is associated with each pair of these elements, just as a pair of electric forces is associated with a pair of point charges. Thus, if we consider an element $d\mathbf{l}_1$ in loop 1 and an element $d\mathbf{l}_2$in loop 2, then the forces $d\mathbf{F}_1$ and $d\mathbf{F}_2$ experienced by the elements $d\mathbf{l}_1$ and $d\mathbf{l}_2$, respectively, are given by

$$d\mathbf{F}_1 = I_1\, d\mathbf{l}_1 \times \left(\frac{kI_2\, d\mathbf{l}_2 \times \mathbf{a}_{21}}{R^2}\right) \tag{1.72a}$$

$$d\mathbf{F}_2 = I_2\, d\mathbf{l}_2 \times \left(\frac{kI_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2}\right) \tag{1.72b}$$

where $\mathbf{a}_{21}$ and $\mathbf{a}_{12}$ are unit vectors along the line joining the two current elements, R is the distance between them, and k is a constant of proportionality that depends on the medium. For free space, k is equal to $\mu_0/4\pi$, where μ_0 is known as the permeability of free space, having a value $4\pi \times 10^{-7}$. From (1.72a) or (1.72b), we note that the units of μ_0 are newtons per ampere squared. These are commonly known as *henrys per meter*, where a henry is a newton-meter per ampere squared.

Equations (1.72a) and (1.72b) represent Ampère's force law as applied to a pair of current elements. Some of the features evident from these equations are as follows:

1. The magnitude of the force is proportional to the product of the two currents and to the product of the lengths of the two current elements.
2. The magnitude of the force is inversely proportional to the square of the distance between the current elements.
3. To determine the direction of the force acting on the current element $d\mathbf{l}_1$, we first find the cross product $d\mathbf{l}_2 \times \mathbf{a}_{21}$ and then cross $d\mathbf{l}_1$ into the resulting vector. Similarly, to determine the direction of the force acting on the current element $d\mathbf{l}_2$, we first find the cross product $d\mathbf{l}_1 \times \mathbf{a}_{12}$ and then cross $d\mathbf{l}_2$ into the resulting vector. For the general case of arbitrary orientations of $d\mathbf{l}_1$ and $d\mathbf{l}_2$, these operations yield $d\mathbf{F}_1$ and $d\mathbf{F}_2$, which are not equal and opposite. To illustrate by means of an example, let us consider $I_1\, d\mathbf{l}_1 = I_1\, dx\, \mathbf{a}_x$ at $(1, 0, 0)$ and $I_2\, d\mathbf{l}_2 = I_2\, dy\, \mathbf{a}_y$ at $(0, 1, 0)$. Then

$$\begin{aligned}
\mathbf{a}_{12} = -\mathbf{a}_{21} &= \frac{1}{\sqrt{2}}(-\mathbf{a}_x + \mathbf{a}_y); \quad R = \sqrt{2} \\
I_2\, d\mathbf{l}_2 \times \mathbf{a}_{21} &= (I_2\, dy\, \mathbf{a}_y) \times \frac{1}{\sqrt{2}}(\mathbf{a}_x - \mathbf{a}_y) = -\frac{I_2}{\sqrt{2}} dy\, \mathbf{a}_z \\
d\mathbf{F}_1 &= (I_1\, dx\, \mathbf{a}_x) \times \left(\frac{-kI_2\, dy\, \mathbf{a}_z}{2\sqrt{2}}\right) = \frac{kI_1I_2}{2\sqrt{2}} dx\, dy\, \mathbf{a}_y \\
I_1\, d\mathbf{l}_1 \times \mathbf{a}_{12} &= (I_1\, dx\, \mathbf{a}_x) \times \frac{1}{\sqrt{2}}(-\mathbf{a}_x + \mathbf{a}_y) = \frac{I_1}{\sqrt{2}} dx\, \mathbf{a}_z \\
d\mathbf{F}_2 &= (I_2\, dy\, \mathbf{a}_y) \times \left(\frac{kI_1\, dx\, \mathbf{a}_z}{2\sqrt{2}}\right) = \frac{kI_1I_2}{2\sqrt{2}} dx\, dy\, \mathbf{a}_x
\end{aligned}$$

Thus, $d\mathbf{F}_2 \neq -d\mathbf{F}_1$. This is not a violation of Newton's third law since isolated current elements do not exist without sources and sinks of charges at their ends. Newton's third law, however, must and does hold for complete current loops.

Magnetic flux density

The forms of (1.72a) and (1.72b) suggest that each current element is acted on by a field which is due to the other current element. By definition, this field is the magnetic field and is characterized by a quantity known as the *magnetic flux density vector*, denoted by the symbol $\mathbf{B}$. Thus, we note from (1.72b) that the magnetic flux density at the element $d\mathbf{l}_2$ due to the element $d\mathbf{l}_1$ is given by

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} \frac{I_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2} \tag{1.73}$$

and that this flux density acting on $d\mathbf{l}_2$ results in a force on it given by

$$d\mathbf{F}_2 = I_2\, d\mathbf{l}_2 \times \mathbf{B}_1 \tag{1.74}$$

Similarly, we note from (1.72a) that the magnetic flux density at the element $d\mathbf{l}_1$ due to the element $d\mathbf{l}_2$ is given by

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi}\frac{I_2\,d\mathbf{l}_2 \times \mathbf{a}_{21}}{R^2} \tag{1.75}$$

and that this flux density acting on $d\mathbf{l}_1$ results in a force on it given by

$$d\mathbf{F}_1 = I_1\,d\mathbf{l}_1 \times \mathbf{B}_2 \tag{1.76}$$

From (1.74) and (1.76), we see that the units of **B** are newtons per ampere-meter, commonly known as *webers per meter squared* (or tesla), where a weber is a newton-meter per ampere. The units of webers per unit area give the character of flux density to the quantity **B**, unlike the character of field intensity as that of **E** for the electric field case.

Generalizing (1.74) and (1.76), we say that an infinitesimal current element of length $d\mathbf{l}$ and current I placed in a magnetic field of flux density **B** experiences a force $d\mathbf{F}$ given by

$$\boxed{d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}} \tag{1.77}$$

as shown in Fig. 1.31. Alternatively, if a current element experiences a force in a region of space, then the region is said to be characterized by a magnetic field.

Principle of loudspeaker

There are many devices using the principle of magnetic force on a current-carrying wire. One such device in everyday life is the loudspeaker. As shown by the cross-sectional view in Fig. 1.32, the loudspeaker consists of a permanent magnet between the poles of which is a coil wound around a cylinder attached to the apex of a movable cone-shaped diaphragm. Current through the coil varies in accordance with the audio signal from the output stage of the hi-fi amplifier or radio receiver. A magnetic force is thus exerted on the coil, vibrating it back and forth in step with the changes in the current. Since the coil assembly is attached to the cone, the cone also vibrates, thereby producing sound waves in the air.

Magnetic field due to a current element

Returning now to (1.73) and (1.75) and generalizing, we obtain the magnetic flux density due to an infinitesimal current element of length $d\mathbf{l}$ and carrying current I to be

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l} \times \mathbf{a}_R}{R^2}} \tag{1.78}$$

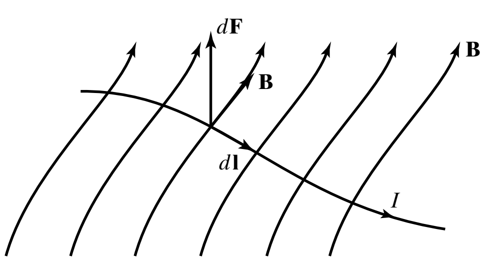

FIGURE 1.31

Force experienced by a current element in a magnetic field.

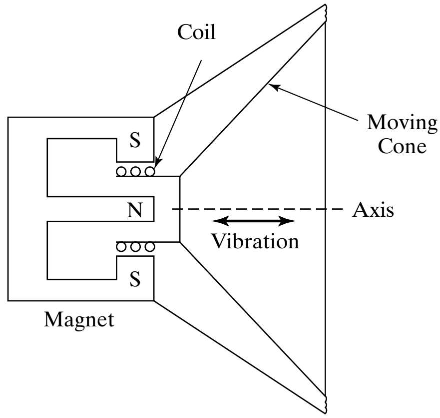

FIGURE 1.32

Cross-sectional view of a loud-speaker.

where R is the distance from the current element to the point at which the flux density is to be computed and $\mathbf{a}_R$ is the unit vector along the line joining the current element and the point under consideration and directed away from the current element, as shown in Fig. 1.33. Equation (1.78) is known as the *Biot–Savart law* and is analogous to the expression for the electric field intensity due to a point charge. The Biot–Savart law tells us that the magnitude of $\mathbf{B}$ at a point P is proportional to the current I, the element length dl, and the sine of the angle α between the current element and the line joining it to the point P, and is inversely proportional to the square of the distance from the current element to the point P. Hence, the magnetic flux density is zero at points along the axis of the current element and increases in magnitude as the point P is moved away from the axis on a spherical surface centered at the current element, becoming a maximum for α equal to 90°. This is in contrast to the behavior of the electric field intensity due to a point charge, which remains the same in magnitude at points on a spherical surface centered at the point charge. The direction of $\mathbf{B}$ at point P is normal to the plane containing the current element and the line joining the current element to P as given by the cross product operation $d\mathbf{l} \times \mathbf{a}_R$, that is, right circular to the axis of the wire. Thus, the direction

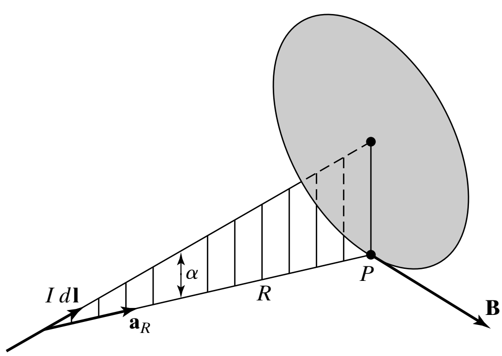

FIGURE 1.33

Magnetic flux density due to an infinitesimal current element.

lines of the magnetic flux density due to a current element are circles centered at points on the axis of the current element and lying in planes normal to the axis. This is in contrast to the direction lines of the electric field intensity due to a point charge, which are radial lines emanating from the point charge.

Example 1.10 Magnetic flux density due to a current element

Let us consider an infinitesimal length 10^{-3} m of wire located at the point (1, 0, 0) and carrying current 2 A in the direction of the unit vector $\mathbf{a}_x$. We wish to find the magnetic flux density due to the current element at the point (0, 2, 2).

Noting that the current element is given by

$$I\,d\mathbf{l} = (2)(10^{-3})\mathbf{a}_x = 0.002\mathbf{a}_x$$

and the vector $\mathbf{R}$ from the location (1, 0, 0) of the current element to the point (0, 2, 2) is given by

$$\mathbf{R} = (0-1)\mathbf{a}_x + (2-0)\mathbf{a}_y + (2-0)\mathbf{a}_z = -\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z$$

and using Biot–Savart law, we obtain

$$\begin{aligned}
[\mathbf{B}]_{(0,2,2)} &= \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{a}_R}{R^2} \\
&= \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{R}}{R^3} \\
&= \frac{\mu_0}{4\pi}\frac{0.002\mathbf{a}_x\times(-\mathbf{a}_x+2\mathbf{a}_y+2\mathbf{a}_z)}{27} \\
&= \frac{0.001\mu_0}{27\pi}(-\mathbf{a}_y+\mathbf{a}_z)\ \text{Wb/m}^2
\end{aligned}$$

The Biot–Savart law can be used to find the magnetic flux density due to a current carrying filamentary wire of any length and shape by dividing the wire into a number of infinitesimal elements and using superposition. We shall illustrate the procedure by means of an example.

Example 1.11 Magnetic field of an infinitely long straight wire of current

Infinitely long, straight wire

Let us consider an infinitely long, straight wire situated along the z-axis and carrying current I A in the $+z$-direction. We wish to find the magnetic flux density everywhere.

Let us consider a point on the xy-plane specified by the cylindrical coordinates $(r, \phi, 0)$, as shown in Fig. 1.34(a). Then the solution for the magnetic flux density at $(r, \phi, 0)$ can be obtained by considering a differential length dz of the wire at the point $(0, 0, z)$ and using superposition. Applying Biot–Savart law (1.78) to the geometry in Fig. 1.34(a), we obtain the magnetic flux density at $(r, \phi, 0)$ due to the current element

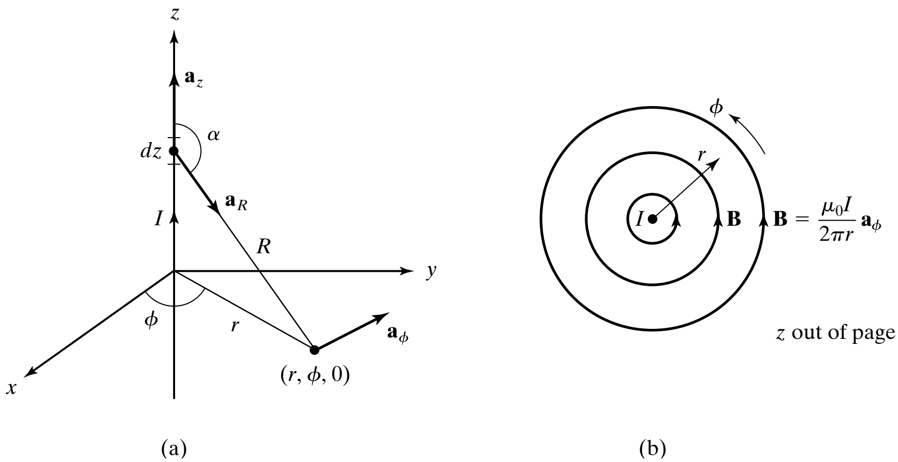

FIGURE 1.34

(a) Determination of magnetic field due to an infinitely long, straight wire of current I A.
(b) Magnetic field due to the wire of (a).

$I\,dz\,\mathbf{a}_z$ at $(0, 0, z)$ to be

$$\begin{aligned}
[d\mathbf{B}]_{(r,\phi,0)} &= \frac{\mu_0}{4\pi}\frac{I\,dz\,\mathbf{a}_z \times \mathbf{a}_R}{R^2} \\
&= \frac{\mu_0 I\,dz}{4\pi}\frac{\sin\alpha}{R^2}\mathbf{a}_\phi \\
&= \frac{\mu_0 I\,dz}{4\pi}\frac{r}{R^3}\mathbf{a}_\phi \\
&= \frac{\mu_0 Ir\,dz}{4\pi(z^2 + r^2)^{3/2}}\mathbf{a}_\phi
\end{aligned}$$

The magnetic flux density due to the entire wire is then given by

$$\begin{aligned}
[\mathbf{B}]_{(r,\phi,0)} &= \int_{z=-\infty}^{\infty} d\mathbf{B} \\
&= \int_{z=-\infty}^{\infty} \frac{\mu_0 Ir}{4\pi(z^2 + r^2)^{3/2}}\,dz\,\mathbf{a}_\phi \\
&= \frac{\mu_0 Ir}{4\pi}\left[\frac{z}{r^2\sqrt{z^2 + r^2}}\right]_{z=-\infty}^{\infty}\mathbf{a}_\phi \\
&= \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi
\end{aligned}$$

Now, since the origin can be chosen to be anywhere on the wire without changing the geometry, this result is valid everywhere. Thus, the required magnetic flux density is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi \tag{1.79}$$

which has the magnitude $\mu_0 I/2\pi r$ and surrounds the wire, as shown by the cross-sectional view in Fig. 1.34(b).

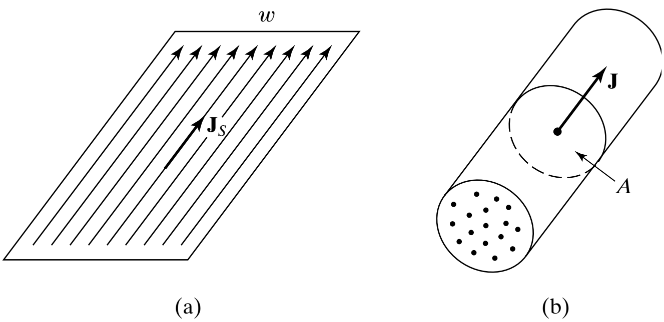

FIGURE 1.35

Determination of currents due to (a) surface current and (b) volume current distributions of uniform densities.

Types of current distributions

The magnetic field computation illustrated in Example 1.11 can be extended to current distributions. Current distributions are of two types: surface currents and volume currents, depending on whether current flows on a surface like rain water flowing down a smooth wall or in a volume like rain water flowing down a gutter downspout. The corresponding current densities are the surface current density $\mathbf{J}_S$ and the volume current density, or simply the current density $\mathbf{J}$, having the units of current crossing unit length (amperes per meter) and current crossing unit area (amperes per meter squared), respectively. Note that the current densities are vector quantities, since flow is involved. Assuming for simplicity surface current of uniform density flowing on a plane sheet, as shown in Fig. 1.35(a), one obtains the current I on the sheet by multiplying the magnitude of $\mathbf{J}_S$ by the dimension w of the sheet normal to the direction of $\mathbf{J}_S$. Similarly, for volume current of uniform density flowing in a straight wire, as shown in Fig. 1.35(b), the current I in the wire is given by the product of the magnitude of $\mathbf{J}$ and the area of cross section A of the wire normal to the direction of $\mathbf{J}$. If the current density is nonuniform, the current can be obtained by performing an appropriate integration along the width of the sheet or over the cross section of the wire, depending on the case. We shall illustrate the determination of the magnetic field due to a current distribution by means of an example.

Example 1.12 Magnetic field of an infinite plane sheet of current

Infinite plane sheet of current

Let us consider an infinite plane sheet of current in the xz-plane with uniform surface current density $\mathbf{J}_S = \mathbf{J}_{S0}\mathbf{a}_z$ A/m and find the magnetic flux density everywhere.

Let us first consider a point $(0, y, 0)$ on the positive y-axis, as shown in Fig. 1.36(a). Then the solution can be carried out by dividing the sheet into a number of thin vertical strips and using superposition. Two such strips, which are on either side of the z-axis and equidistant from it, are shown in Fig. 1.36(a). Each strip is an infinitely long filamentary wire of current $J_{S0}\,dx$. Then, applying the result of Example 1.11 to each strip and noting that the resultant magnetic flux density at $(0, y, 0)$ due to the two strips together has only an x-component, we obtain

$$d\mathbf{B} = d\mathbf{B}_1 + d\mathbf{B}_2 = -2\,dB_1 \cos\alpha\,\mathbf{a}_x$$

$$= -2\frac{\mu_0 J_{S0}\,dx}{2\pi\sqrt{x^2 + y^2}}\frac{y}{\sqrt{x^2 + y^2}}\mathbf{a}_x = -\frac{\mu_0 J_{S0} y\,dx}{\pi(x^2 + y^2)}\mathbf{a}_x$$

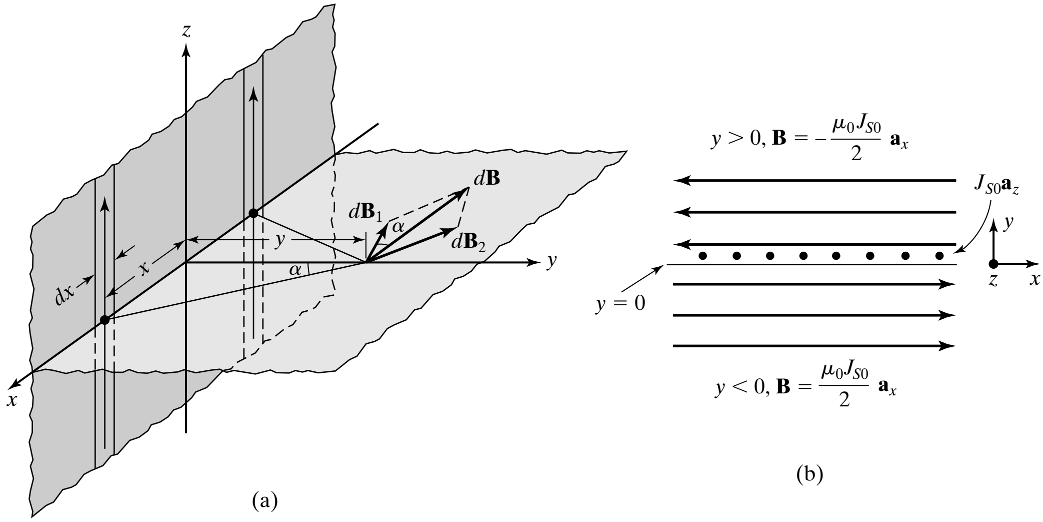

FIGURE 1.36

(a) Determination of magnetic field due to an infinite plane sheet of current density $J_{S0}\mathbf{a}_z$ A/m. (b) Magnetic field due to the current sheet of (a).

The magnetic flux density due to the entire sheet is then given by

$$\begin{aligned}[\mathbf{B}]_{(0,\,y,\,0)} &= \int_{x=0}^{\infty} d\mathbf{B} \\ &= -\int_{x=0}^{\infty} \frac{\mu_0 J_{S0} y}{\pi(x^2 + y^2)}\,dx\,\mathbf{a}_x \\ &= -\frac{\mu_0 J_{S0} y}{\pi}\left[\frac{1}{y}\tan^{-1}\frac{x}{y}\right]_{x=0}^{\infty}\mathbf{a}_x \\ &= -\frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \qquad \text{for} \quad y > 0\end{aligned}$$

Since the magnetic field due to each strip is circular to that strip, a similar result applies for a point on the negative y-axis except for $+x$-direction for the field. Thus,

$$[\mathbf{B}]_{(0,\,y,\,0)} = \frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \qquad \text{for} \quad y < 0$$

Now, since the origin can be chosen to be anywhere on the sheet without changing the geometry, the foregoing results are valid everywhere in the respective regions. Thus, the required magnetic flux density is

$$\mathbf{B} = \mp\frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \qquad \text{for} \quad y \gtrless 0 \tag{1.80}$$

which has the magnitude $\mu_0 J_{S0}/2$ everywhere and is directed in the $\mp\mathbf{a}_x$ direction for $y \gtrless 0$, as shown in Fig. 1.36(b). Defining $\mathbf{a}_n$ to be the unit normal vector directed away from the sheet, that is,

$$\mathbf{a}_n = \pm\mathbf{a}_y \qquad \text{for} \quad y \gtrless 0$$

and noting that

$$\mathbf{B} = \frac{\mu_0}{2}(J_{S0}\mathbf{a}_z) \times (\pm\mathbf{a}_y) \qquad \text{for} \quad y \gtrless 0$$

we can write

$$\boxed{\mathbf{B} = \frac{\mu_0}{2}\mathbf{J}_S \times \mathbf{a}_n} \tag{1.81}$$

Alternative to the derivation in Example 1.12, we can obtain the result given by (1.81) from analogy between the electric field due to charge distributions and the magnetic field due to current distributions. To see this, we note, with reference to Fig. 1.37(a), that **E** due to a point charge and **B** due to a current element are given by

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \qquad \leftrightarrow \qquad \mathbf{B} = \frac{\mu_0 I}{4\pi R^2}d\mathbf{l} \times \mathbf{a}_R \tag{1.82a}$$

We further note, with reference to Fig. 1.37(b), that **E** due to an infinitely long line charge of uniform density and **B** due to an infinitely long line current are given by

$$\mathbf{E} = \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\mathbf{a}_r \qquad \leftrightarrow \qquad \begin{aligned} \mathbf{B} &= \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi \\ &= \frac{\mu_0 I}{2\pi r}\mathbf{a}_z \times \mathbf{a}_r \end{aligned} \tag{1.82b}$$

Then, with reference to Fig. 1.37(c), we can write the analogy between **E** due to an infinite plane sheet charge of uniform density and **B** due to an infinite plane sheet of uniform current density as follows:

$$\mathbf{E} = \frac{\rho_{L0}}{2\varepsilon_0}\mathbf{a}_n \qquad \leftrightarrow \qquad \mathbf{B} = \frac{\mu_0}{2}\mathbf{J}_S \times \mathbf{a}_n \tag{1.82c}$$

Thus, the result given by (1.81) could have been written from this analogy, without actually carrying out the solution in Example 1.12.

Magnetic force in terms of charge

Returning now to (1.77), we can formulate the magnetic force in terms of moving charge, since current is due to flow of charges. Thus, if dt is the time taken by the charge dq contained in the length $d\mathbf{l}$ of the current element to flow with a velocity $\mathbf{v}$ across the infinitesimal cross-sectional area of the element, then $I = dq/dt$, and $d\mathbf{l} = \mathbf{v}\,dt$, so that

$$d\mathbf{F} = \frac{dq}{dt}\mathbf{v}\,dt \times \mathbf{B} = dq\,\mathbf{v} \times \mathbf{B} \tag{1.83}$$

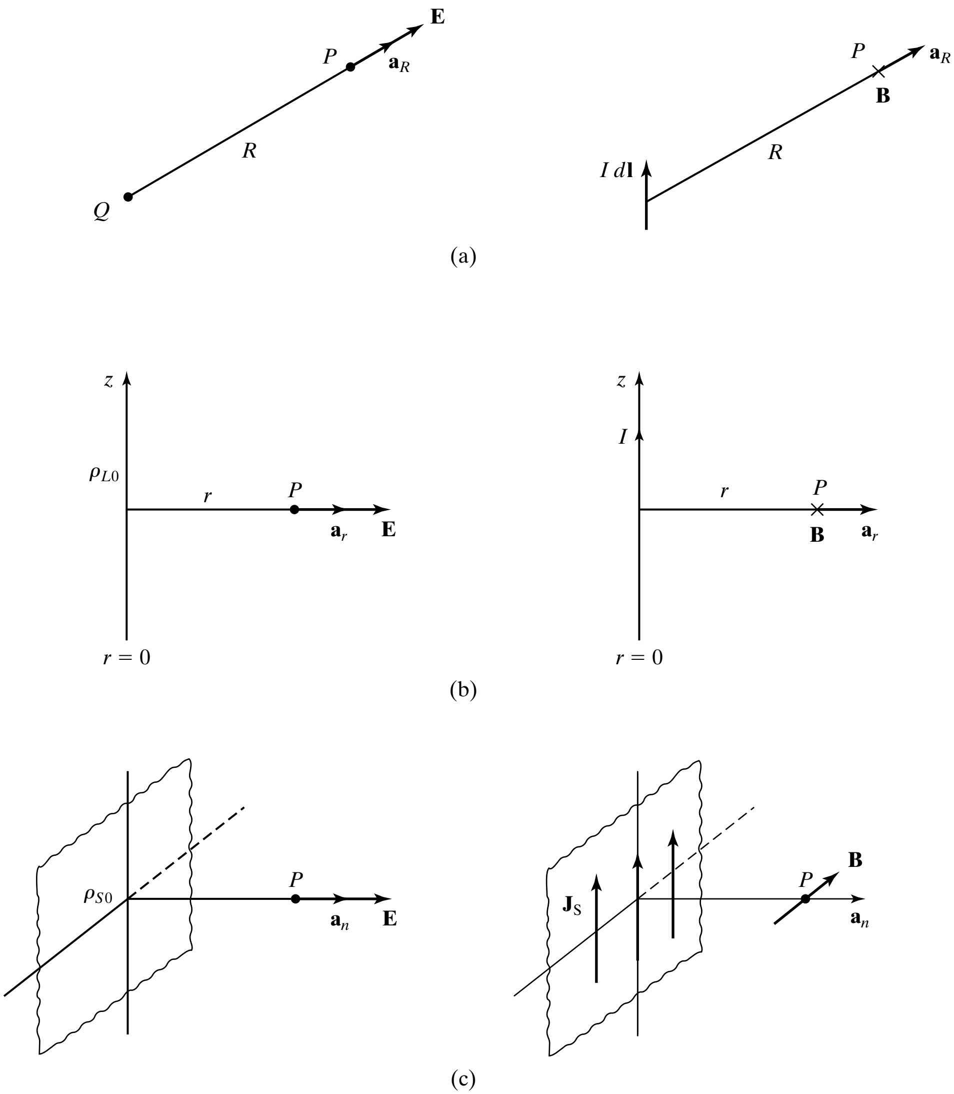

FIGURE 1.37

Analogy between electric field due to charge distributions and magnetic field due to current distributions.

It then follows that the force $\mathbf{F}$ experienced by a test charge q moving with a velocity $\mathbf{v}$ in a magnetic field of flux density $\mathbf{B}$ is given by

$$\boxed{\mathbf{F} = q\mathbf{v} \times \mathbf{B}} \tag{1.84}$$

We may now obtain a defining equation for $\mathbf{B}$ in terms of the moving test charge. To do this, we note from (1.84) that the magnetic force is directed normally to

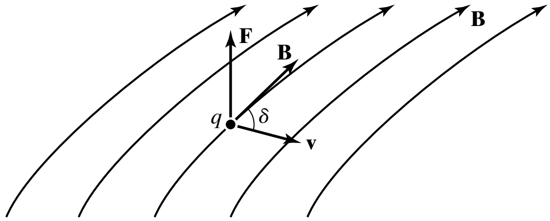

FIGURE 1.38

Force experienced by a test charge q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$.

both $\mathbf{v}$ and $\mathbf{B}$, as shown in Fig. 1.38, and that its magnitude is equal to $qvB \sin \delta$, where δ is the angle between $\mathbf{v}$ and $\mathbf{B}$. A knowledge of the force $\mathbf{F}$ acting on a test charge moving with an arbitrary velocity $\mathbf{v}$ provides only the value of $B \sin \delta$. To find $\mathbf{B}$, we must determine the maximum force qvB that occurs for δ equal to 90° by trying out several directions of $\mathbf{v}$, keeping its magnitude constant. Thus, if this maximum force is $\mathbf{F}_m$ and it occurs for a velocity $v\mathbf{a}_m$, then

$$\mathbf{B} = \frac{\mathbf{F}_m \times \mathbf{a}_m}{qv} \tag{1.85}$$

As in the case of defining the electric field intensity, we assume that the test charge does not alter the magnetic field in which it is placed. Ideally, $\mathbf{B}$ is defined in the limit that qv tends to zero; that is,

$$\boxed{\mathbf{B} = \lim_{qv \to 0} \frac{\mathbf{F}_m \times \mathbf{a}_m}{qv}} \tag{1.86}$$

Equation (1.86) is the defining equation for the magnetic flux density irrespective of the source of the magnetic field. We have learned in this section that an electric current or a charge in motion is a source of the magnetic field. We will learn in Chapter 2 that there exists another source for the magnetic field, namely, a time-varying electric field.

Charged particle motion in uniform magnetic field

There are many devices based on the magnetic force on a moving charge. Of particular interest is the motion of a charged particle in a uniform magnetic field, as shown in Fig. 1.39. In this figure, a particle of mass m and charge q entering the

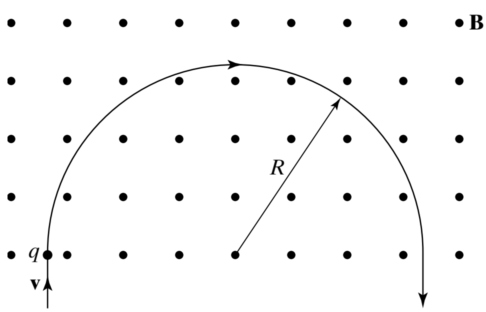

FIGURE 1.39

Circular motion of a charged particle entering a uniform magnetic field region.

magnetic field region with velocity **v** perpendicular to **B** experiences a force qvB perpendicular to **v**. Hence, the particle describes a circular path of radius R, equal to mv/qB, obtained by equating the centripetal force mv^2/R to the magnetic force qvB. The fact that the radius is equal to mv/qB is used in several different applications. In the mass spectrograph, the mass-to-charge ratio of the particles is obtained by measuring the radius of the circular orbit for known values of v and B. For ions of the same charge but of different masses, the radii of the circular paths are directly proportional to their masses and to their velocities. This forms the basis for electromagnetic separation of isotopes, two or more forms of a chemical element having the same chemical properties and the same atomic number but different atomic weights. In the cyclotron, a particle accelerator, the particle undergoes a series of semicircular orbits of successively increasing velocities and hence radii before it exits the field region with high energy.

K1.6. Ampère's law of force; Magnetic flux density; Biot-Savart law; Computation of **B** due to current distributions; **B** due to an infinitely long straight wire; **B** due to an infinite plane sheet of current of uniform density; Analogies between **E** due to charge distributions and **B** due to current distributions.

D1.17. For $I_1\, d\mathbf{l}_1 = I_1\, dy\, \mathbf{a}_y$ located at $(1, 0, 0)$ and $I_2\, d\mathbf{l}_2 = I_2\, dx\, \mathbf{a}_x$, located at $(0, 1, 0)$, find: **(a)** $d\mathbf{F}_1$ and **(b)** $d\mathbf{F}_2$.

Ans. **(a)** $-(\mu_0 I_1 I_2/8\sqrt{2}\pi)\, dx\, dy\, \mathbf{a}_x$; **(b)** $-(\mu_0 I_1 I_2/8\sqrt{2}\pi)\, dx\, dy\, \mathbf{a}_y$.

D1.18. A current I flows in a wire along the curve $x = 2y = z^2 + 2$ and in the direction of increasing z. If the wire is situated in a magnetic field $\mathbf{B} = (y\mathbf{a}_x - x\mathbf{a}_y)/(x^2 + y^2)$, find the magnetic force acting on an infinitesimal length of the wire having the projection dz on the z-axis at each of the following points: **(a)** $(2, 1, 0)$; **(b)** $(3, 1.5, 1)$; and **(c)** $(6, 3, 2)$.

Ans. **(a)** $I\, dz\, (2\mathbf{a}_x + \mathbf{a}_y)/5$; **(b)** $I\, dz\, (2\mathbf{a}_x + \mathbf{a}_y - 5\mathbf{a}_z)/7.5$; **(c)** $I\, dz\, (2\mathbf{a}_x + \mathbf{a}_y - 10\mathbf{a}_z)/15$.

D1.19. Given $\mathbf{B} = (B_0/3)(2\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z)$, find the magnitude of the magnetic force acting on a test charge q moving with velocity v_0 at the point $(2, 2, -1)$ for each of the following paths of the test charge: **(a)** $x = y = -2z$; **(b)** $4x = 4y = z + 9$; and **(c)** $x = y = 2z^2$.

Ans. **(a)** 0; **(b)** qv_0B_0; **(c)** $0.1641qv_0B_0$.

D1.20. Infinite plane sheets of current lie in the $x = 0$, $y = 0$, and $z = 0$ planes with uniform surface current densities $J_{S0}\, \mathbf{a}_z$, $2J_{S0}\, \mathbf{a}_x$, and $-J_{S0}\, \mathbf{a}_x$ A/m, respectively. Find the resulting magnetic flux densities at the following points: **(a)** $(1, 2, 2)$; **(b)** $(2, -2, -1)$; and **(c)** $(-2, 1, -2)$.

Ans. **(a)** $\mu_0 J_{S0}(\mathbf{a}_y + \mathbf{a}_z)$; **(b)** $-\mu_0 J_{S0}\, \mathbf{a}_z$; **(c)** $\mu_0 J_{S0}(-\mathbf{a}_y + \mathbf{a}_z)$.

1.7 LORENTZ FORCE EQUATION

In Section 1.5, we learned that a test charge q placed in an electric field of intensity **E** experiences a force

$$\boxed{\mathbf{F}_E = q\mathbf{E}} \tag{1.87}$$

and in Section 1.6, we learned that a test charge q moving with a velocity $\mathbf{v}$ in a magnetic field of flux density $\mathbf{B}$ experiences a force

$$\boxed{\mathbf{F}_M = q\mathbf{v} \times \mathbf{B}} \tag{1.88}$$

Combining (1.87) and (1.88), we can write the expression for the total force acting on a test charge q moving with velocity $\mathbf{v}$ in a region characterized by electric field of intensity $\mathbf{E}$ and magnetic field of flux density $\mathbf{B}$ to be

$$\boxed{\mathbf{F} = \mathbf{F}_E + \mathbf{F}_M = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})} \tag{1.89}$$

Equation (1.89) is known as the *Lorentz force equation*.

Determination of electric and magnetic fields from forces on a test charge

We observe from (1.89) that the electric and magnetic fields at a point can be determined from a knowledge of the forces experienced by a test charge at that point for several different velocities. For a given **B, E** can be found from the force for one velocity, since $\mathbf{F}_E$ acts in the direction of **E**. For a given **E, B** can be found from two forces for two noncollinear velocities, since $\mathbf{F}_M$ acts perpendicular to both **v** and **B**. Thus, to find both **E** and **B**, the knowledge of a minimum of three forces is necessary. We shall illustrate the determination of **E** and **B** from three forces by means of an example.

Example 1.13 Finding the electric and magnetic fields from forces on a test charge

The forces experienced by a test charge q for three different velocities at a point in a region of electric and magnetic fields are given by

$$\begin{aligned} \mathbf{F}_1 &= qE_0\mathbf{a}_x && \text{for} \quad \mathbf{v}_1 = v_0\mathbf{a}_x \\ \mathbf{F}_2 &= qE_0(2\mathbf{a}_x + \mathbf{a}_y) && \text{for} \quad \mathbf{v}_2 = v_0\mathbf{a}_y \\ \mathbf{F}_3 &= qE_0(\mathbf{a}_x + \mathbf{a}_y) && \text{for} \quad \mathbf{v}_3 = v_0\mathbf{a}_z \end{aligned}$$

where v_0 and E_0 are constants. We wish to find **E** and **B** at that point.

From the Lorentz force equation, we have

$$q\mathbf{E} + qv_0\mathbf{a}_x \times \mathbf{B} = qE_0\mathbf{a}_x \tag{1.90a}$$
$$q\mathbf{E} + qv_0\mathbf{a}_y \times \mathbf{B} = q(2E_0\mathbf{a}_x + E_0\mathbf{a}_y) \tag{1.90b}$$
$$q\mathbf{E} + qv_0\mathbf{a}_z \times \mathbf{B} = q(E_0\mathbf{a}_x + E_0\mathbf{a}_y) \tag{1.90c}$$

Eliminating **E** by subtracting (1.90a) from (1.90b) and (1.90c) from (1.90b), we obtain

$$v_0(\mathbf{a}_y - \mathbf{a}_x) \times \mathbf{B} = E_0(\mathbf{a}_x + \mathbf{a}_y) \tag{1.91a}$$
$$v_0(\mathbf{a}_y - \mathbf{a}_z) \times \mathbf{B} = E_0\mathbf{a}_x \tag{1.91b}$$

Since the cross product of two vectors is perpendicular to the two vectors, it follows from (1.91a) that $(\mathbf{a}_x + \mathbf{a}_y)$ is perpendicular to **B** and from (1.91b) that $\mathbf{a}_x$ is perpendicular to **B**. Thus, **B** is perpendicular to both $(\mathbf{a}_x + \mathbf{a}_y)$ and $\mathbf{a}_x$. But the cross product of $(\mathbf{a}_x + \mathbf{a}_y)$ and $\mathbf{a}_x$ is perpendicular to both of them. Therefore, **B** must be directed parallel to $(\mathbf{a}_x + \mathbf{a}_y) \times \mathbf{a}_x$. Thus, we can write

$$\mathbf{B} = C(\mathbf{a}_x + \mathbf{a}_y) \times \mathbf{a}_x = -C\mathbf{a}_z \tag{1.92}$$

where C is a proportionality constant to be determined. To do this, we substitute (1.92) into (1.91b) to obtain

$$\begin{aligned} v_0(\mathbf{a}_y - \mathbf{a}_z) \times (-C\mathbf{a}_z) &= E_0\mathbf{a}_x \\ -v_0C\mathbf{a}_x &= E_0\mathbf{a}_x \end{aligned}$$

or $C = -E_0/v_0$. Thus, we get

$$\mathbf{B} = \frac{E_0}{v_0}\mathbf{a}_z$$

Alternatively, we can obtain this result by assuming $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$, substituting in (1.91a) and (1.91b), equating the like components, and solving the resulting algebraic equations. Thus, substituting in (1.91a), we have

$$v_0\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ -1 & 1 & 0 \\ B_x & B_y & B_z \end{vmatrix} = E_0(\mathbf{a}_x + \mathbf{a}_y)$$

or

$$v_0[B_z\mathbf{a}_x + B_z\mathbf{a}_y - (B_y + B_x)\mathbf{a}_z] = E_0\mathbf{a}_x + E_0\mathbf{a}_y$$

$$B_z = \frac{E_0}{v_0} \qquad \text{and} \qquad (B_y + B_x) = 0$$

Substituting in (1.91b), we have

$$v_0\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 1 & -1 \\ B_x & B_y & B_z \end{vmatrix} = E_0\mathbf{a}_x$$

or

$$v_0[(B_z + B_y)\mathbf{a}_x - B_x\mathbf{a}_y - B_x\mathbf{a}_z] = E_0\mathbf{a}_x$$

$$B_z + B_y = \frac{E_0}{v_0} \qquad \text{and} \qquad B_x = 0$$

Thus, we obtain $B_z = E_0/v_0$, $B_x = 0$, $B_y = 0$, and, hence,

$$\mathbf{B} = \frac{E_0}{v_0}\mathbf{a}_z$$

Finally, we can find $\mathbf{E}$ by substituting the result obtained for $\mathbf{B}$ in any one of the three equations (1.90a)–(1.90c). Thus, substituting $\mathbf{B} = (E_0/v_0)\mathbf{a}_z$ in (1.90c), we obtain

$$\mathbf{E} = E_0(\mathbf{a}_x + \mathbf{a}_y)$$

Lorentz force applications

The Lorentz force equation is a fundamental equation in electromagnetics. Together with the pertinent laws of mechanics, it constitutes the starting point for the study of charged particle motion in electric and/or magnetic fields. Devices based on charged particle motion in fields are abundant in practice.

Examples, some of which we discussed in Sections 1.5 and 1.6, are cathode ray tubes, ink-jet printers, electron microscopes, mass spectrographs, particle accelerators, and microwave tubes such as klystrons, magnetrons, and traveling wave tubes. Interaction between charged particles and fields is the basis for the study of the electromagnetic properties of materials and for the study of radio-wave propagation in gaseous media such as Earth's ionosphere, in which the constituent gasses are partially ionized by the solar radiation.

Tracing of charged particle motion in electric and magnetic fields

Tracing the path of a charged particle in a region of electric and magnetic fields involves setting the mechanical force, as given by the product of the mass of the test charge and its acceleration, equal to the electromagnetic force, as given by the Lorentz force equation, and solving the resulting differential equation(s) subject to initial condition(s). For simplicity, we shall consider a two-dimensional situation in which the motion is confined to the xy-plane in a region of uniform, crossed electric and magnetic fields, $\mathbf{E} = E_0\mathbf{a}_y$ and $\mathbf{B} = B_0\mathbf{a}_z$, as shown in Fig. 1.40, where E_0 and B_0 are constants. We shall assume that a test charge q having mass m starts at $t = 0$ at the point $(x_0, y_0, 0)$ with initial velocity $\mathbf{v} = v_{x0}\mathbf{a}_x + v_{y0}\mathbf{a}_y$.

From the Lorentz force equation (1.89), the force exerted by the crossed electric and magnetic fields on the test charge is given by

$$\begin{aligned}\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ &= qE_0\mathbf{a}_y + q(v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z) \times B_0\mathbf{a}_z \\ &= qB_0v_y\mathbf{a}_x + (qE_0 - qB_0v_x)\mathbf{a}_y\end{aligned} \tag{1.93}$$

The equations of motion of the test charge can then be written as

$$\frac{dv_x}{dt} = \frac{qB_0}{m}v_y \tag{1.94a}$$

$$\frac{dv_y}{dt} = \frac{qE_0}{m} - \frac{qB_0}{m}v_x \tag{1.94b}$$

$$\frac{dv_z}{dt} = 0 \tag{1.94c}$$

Equation (1.94c), together with the initial conditions $v_z = 0$ and $z = 0$ at $t = 0$, simply tells us that the path of the test charge is confined to the $z = 0$ plane.

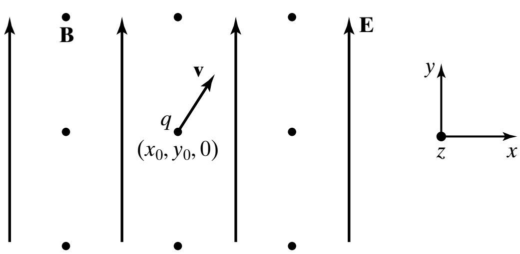

FIGURE 1.40

Test charge q in a region of crossed electric and magnetic fields.

Eliminating v_y from (1.94a) and (1.94b), we obtain

$$\frac{d^2 v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = \left(\frac{q}{m}\right)^2 B_0 E_0 \tag{1.95}$$

the solution for which is

$$v_x = \frac{E_0}{B_0} + C_1 \cos \omega_c t + C_2 \sin \omega_c t \tag{1.96a}$$

where C_1 and C_2 are constants to be determined from the initial conditions and $\omega_c = qB_0/m$. From (1.94a), the solution for v_y is then given by

$$v_y = -C_1 \sin \omega_c t + C_2 \cos \omega_c t \tag{1.96b}$$

Using initial conditions $v_x = v_{x0}$ and $v_y = v_{y0}$ at $t = 0$ to evaluate C_1 and C_2 in (1.96a) and (1.96b), we obtain

$$\boxed{v_x = \frac{E_0}{B_0} + \left(v_{x0} - \frac{E_0}{B_0}\right)\cos \omega_c t + v_{y0} \sin \omega_c t} \tag{1.97a}$$

$$\boxed{v_y = -\left(v_{x0} - \frac{E_0}{B_0}\right)\sin \omega_c t + v_{y0} \cos \omega_c t} \tag{1.97b}$$

Integrating (1.97a) and (1.97b) with respect to t and using initial conditions $x = x_0$ and $y = y_0$ at $t = 0$ to evaluate the constants of integration, we then obtain

$$\boxed{x = x_0 + \frac{E_0}{B_0}t + \frac{1}{\omega_c}\left(v_{x0} - \frac{E_0}{B_0}\right)\sin \omega_c t + \frac{v_{y0}}{\omega_c}(1 - \cos \omega_c t)} \tag{1.98a}$$

$$\boxed{y = y_0 - \frac{1}{\omega_c}\left(v_{x0} - \frac{E_0}{B_0}\right)(1 - \cos \omega_c t) + \frac{v_{y0}}{\omega_c}\sin \omega_c t} \tag{1.98b}$$

Equations (1.98a) and (1.98b) give the position of the test charge versus time, whereas (1.97a) and (1.97b) give the corresponding velocity components. For $B_0 = 0$, $\omega_c \to 0$, and the solutions reduce to

$$\boxed{x = x_0 + v_{x0}t} \tag{1.99a}$$

$$\boxed{y = y_0 + v_{y0}t + \frac{1}{2}\frac{qE_0}{m}t^2} \tag{1.99b}$$

$$\boxed{v_x = v_{x0}} \tag{1.99c}$$

$$\boxed{v_y = v_{y0} + \frac{qE_0}{m}t} \tag{1.99d}$$

These can also be obtained directly from (1.94a) and (1.94b) with B_0 set equal to zero.

The path of a test charge in the crossed electric and magnetic fields may now be traced by using (1.98a) and (1.98b) for B_0 not equal to zero, and (1.99a)

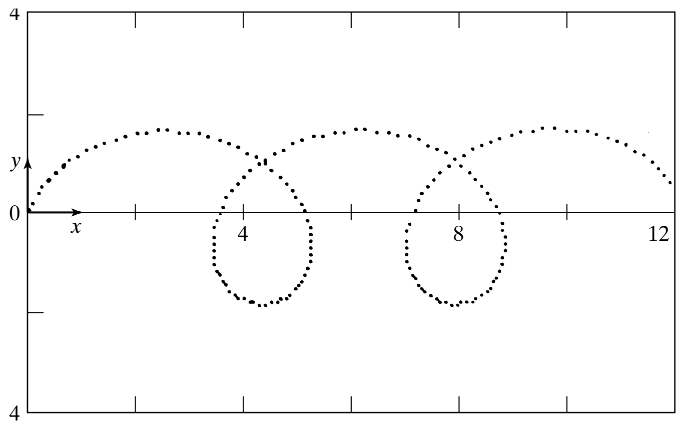

FIGURE 1.41

An example of tracing the path of an electron in crossed electric and magnetic fields.

and (1.99b) for B_0 equal to zero. For example, the path of an electron $(q/m = -1.7578 \times 10^{11}$ C/kg) for $x_0 = 0$, $y_0 = 0$, $E_0 = -10^3$ V/m, $B_0 = 10^{-4}$ Wb/m^2, $v_{x0} = 10^7$ m/s, and $v_{y0} = 3 \times 10^7$ m/s is shown in Fig. 1.41, in which the spacing between the dots corresponds to a time interval of 5×10^{-9} s.

K1.7. Lorentz force equation; Determination of **E** and **B** from forces on a test charge; Charged-particle motion in electric and magnetic fields.

D1.21. A magnetic field $\mathbf{B} = (B_0/3)(\mathbf{a}_x + 2\mathbf{a}_y - 2\mathbf{a}_z)$ exists at a point. For each of the following velocities of a test charge q, find the electric field **E** at that point for which the acceleration experienced by the test charge is zero: **(a)** $v_0(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)$; **(b)** $v_0(2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)$; and **(c)** v_0 along the line $y = -z = 2x$.

Ans. **(a)** $-v_0B_0(\mathbf{a}_y + \mathbf{a}_z)$; **(b)** $v_0B_0(2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z)$; **(c) 0**.

D1.22. In a region of uniform electric and magnetic fields $\mathbf{E} = E_0\mathbf{a}_y$ and $\mathbf{B} = B_0\mathbf{a}_z$, respectively, a test charge q of mass m moves in the manner

$$x = \frac{E_0}{\omega_c B_0}(\omega_c t - \sin \omega_c t)$$
$$y = \frac{E_0}{\omega_c B_0}(1 - \cos \omega_c t)$$
$$z = 0$$

where $\omega_c = qB_0/m$. Find the forces acting on the test charge for the following values of t: **(a)** $t = 0$; **(b)** $t = \pi/2\omega_c$; and **(c)** $t = \pi/\omega_c$.

Ans. **(a)** $qE_0\mathbf{a}_y$; **(b)** $qE_0\mathbf{a}_x$; **(c)** $-qE_0\mathbf{a}_y$.

SUMMARY

We first learned in this chapter several rules of vector algebra that are necessary for our study of the elements of engineering electromagnetics by considering vectors expressed in terms of their components along three mutually orthogonal directions. To carry out the manipulations involving vectors at different points in space in a systematic manner, we then introduced the Cartesian coordinate system and discussed the application of the vector algebraic rules to vectors in the Cartesian coordinate system. To summarize these rules, we consider three vectors

$$\begin{aligned}\mathbf{A} &= A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z\\ \mathbf{B} &= B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z\\ \mathbf{C} &= C_x\mathbf{a}_x + C_y\mathbf{a}_y + C_z\mathbf{a}_z\end{aligned}$$

in a right-handed Cartesian coordinate system, that is, with $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$. We then have

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z\\ \mathbf{B} - \mathbf{C} &= (B_x - C_x)\mathbf{a}_x + (B_y - C_y)\mathbf{a}_y + (B_z - C_z)\mathbf{a}_z\\ m\mathbf{A} &= mA_x\mathbf{a}_x + mA_y\mathbf{a}_y + mA_z\mathbf{a}_z\\ \frac{\mathbf{B}}{n} &= \frac{B_x}{n}\mathbf{a}_x + \frac{B_y}{n}\mathbf{a}_y + \frac{B_z}{n}\mathbf{a}_z\\ |\mathbf{A}| &= \sqrt{A_x^2 + A_y^2 + A_z^2}\end{aligned}$$

$$\mathbf{a}_A = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\mathbf{a}_x + \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\mathbf{a}_y + \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\mathbf{a}_z$$

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= A_xB_x + A_yB_y + A_zB_z\\ \mathbf{A} \times \mathbf{B} &= \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\\ \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} &= \begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}\end{aligned}$$

Other useful expressions are

$$\begin{aligned}d\mathbf{l} &= dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z\\ d\mathbf{S} &= \pm dy\,dz\,\mathbf{a}_x, \pm dz\,dx\,\mathbf{a}_y, \pm dx\,dy\,\mathbf{a}_z\\ dv &= dx\,dy\,dz\end{aligned}$$

We then discussed the cylindrical and spherical coordinate systems, and conversions between these coordinate systems and the Cartesian coordinate system. Relationships for carrying out the coordinate conversions are as follows:

Cylindrical to Cartesian, and vice versa

$$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z$$

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z$$

Spherical to Cartesian, and vice versa

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r\cos\theta$$
$$r = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \qquad \phi = \tan^{-1}\frac{y}{x}$$

Other useful expressions are as follows:

Cylindrical

$$\begin{aligned} d\mathbf{l} &= dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \\ d\mathbf{S} &= \pm r\,d\phi\,dz\,\mathbf{a}_r, \pm dr\,dz\,\mathbf{a}_\phi, \pm r\,dr\,d\phi\,\mathbf{a}_z \\ dv &= r\,dr\,d\phi\,dz \end{aligned}$$

Spherical

$$\begin{aligned} d\mathbf{l} &= dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \\ d\mathbf{S} &= \pm r^2\sin\theta\,d\theta\,d\phi\,\mathbf{a}_r, \pm r\sin\theta\,dr\,d\phi\,\mathbf{a}_\theta, \pm r\,dr\,d\theta\,\mathbf{a}_\phi \\ dv &= r^2\sin\theta\,dr\,d\theta\,d\phi \end{aligned}$$

Next we discussed the concepts of scalar and vector fields, static and time-varying, by means of some simple examples such as the height of points on a conical surface above its base, the temperature field of points in a room, and the velocity vector field associated with points on a disk rotating about its center. We learned about the visualization of fields by means of constant-magnitude contours or surfaces, and in addition, by means of direction lines in the case of vector fields. We also discussed the mathematical technique of obtaining the equations for the direction lines of a vector field.

Having obtained the necessary background vector algebraic tools and physical concepts, we then introduced the electric field concept from consideration of an experimental law known as Coulomb's law, having to do with the electric forces between two charges. We learned that electric force acts on charges merely by virtue of the property of charge. The electric force acting on a test charge q at a point in the field region is given by

$$\mathbf{F} = q\mathbf{E}$$

where $\mathbf{E}$ is the electric field intensity at that point. The electric field intensity due to a point charge Q in free space is given by

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R$$

where ε_0 is the permittivity of free space, R is the distance from the point charge to the point at which the field intensity is to be computed, and $\mathbf{a}_R$ is the unit vector along the line joining the two points and directed away from the point charge.

Using superposition in conjunction with the electric field due to a point charge, we discussed the computation of the electric field due to two point charges and the computer generation of the direction lines of the electric field. We then extended the determination of electric field intensity to continuous charge distributions.

Next we introduced the magnetic field concept from considerations of Ampère's law of force, having to do with the magnetic forces between two current loops. We learned that the magnetic field exerts force only on moving charges. The magnetic force acting on a test charge q moving with a velocity $\mathbf{v}$ at a point in the field region is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{B}$ is the magnetic flux density at that point. In terms of current flowing in a wire, the magnetic force acting on a current element of length $d\mathbf{l}$ and current I at a point in the field region is given by

$$\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B}$$

The magnetic flux density due to a current element $I\,d\mathbf{l}$ in free space is given by the Biot-Savart law

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l} \times \mathbf{a}_R}{R^2}$$

where μ_0 is the permeability of free space, and R and $\mathbf{a}_R$ have the same meanings as in the expression for $\mathbf{E}$ due to a point charge. Using superposition in conjunction with the Biot-Savart law, we discussed the computation of the magnetic field due to current distributions.

Combining the electric and magnetic field concepts, we then introduced the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

which gives the force acting on a test charge q moving with velocity $\mathbf{v}$ at a point in a region characterized by electric field of intensity $\mathbf{E}$ and magnetic field of flux density $\mathbf{B}$. We used the Lorentz force equation to discuss (1) the determination of $\mathbf{E}$ and $\mathbf{B}$ at a point from a knowledge of forces acting on a test charge at that point for three different velocities and (2) the tracing of charged particle motion in a region of crossed electric and magnetic fields.

REVIEW QUESTIONS

Q1.1. Give some examples of scalars.

Q1.2. Give some examples of vectors.

Q1.3. Is it necessary for the reference vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ to be an orthogonal set?

Q1.4. State whether $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ directed westward, northward, and downward, respectively, is a right-handed or a left-handed set.

Q1.5. State all conditions for which $\mathbf{A} \cdot \mathbf{B} = 0$.

Q1.6. State all conditions for which $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.

Q1.7. What is the significance of $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$?

Q1.8. What is the significance of $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$?

Q1.9. What is the particular advantageous characteristic associated with the unit vectors in the Cartesian coordinate system?

Q1.10. What is the position vector?

Q1.11. What is the total distance around the circumference of a circle of radius 1 m? What is the total vector distance around the circle?

Q1.12. Discuss the application of differential length vectors to find a unit vector normal to a surface at a point on the surface.

Q1.13. Discuss the concept of a differential surface vector.

Q1.14. What is the total surface area of a cube of sides 1 m? Assuming the normals to the surfaces to be directed outward of the cubical volume, what is the total vector surface area of the cube?

Q1.15. Describe the three orthogonal surfaces that define the cylindrical coordinates of a point.

Q1.16. Which of the unit vectors in the cylindrical coordinate system are not uniform? Explain.

Q1.17. Discuss the conversion from the cylindrical coordinates of a point to its Cartesian coordinates, and vice versa.

Q1.18. Describe the three orthogonal surfaces that define the spherical coordinates of a point.

Q1.19. Discuss the nonuniformity of the unit vectors in the spherical coordinate system.

Q1.20. Discuss the conversion from the spherical coordinates of a point to its Cartesian coordinates, and vice versa.

Q1.21. Describe briefly your concept of a scalar field and illustrate with an example.

Q1.22. Describe briefly your concept of a vector field and illustrate with an example.

Q1.23. How do you depict pictorially the gravitational field of Earth?

Q1.24. Discuss the procedure for obtaining the equations for the direction lines of a vector field.

Q1.25. State Coulomb's law. To what law in mechanics is Coulomb's law analogous?

Q1.26. What is the value of the permittivity of free space? What are its units?

Q1.27. What is the definition of electric field intensity? What are its units?

Q1.28. Discuss two applications based on the electric force on a charged particle.

Q1.29. Describe the electric field due to a point charge.

Q1.30. Discuss the computer generation of the direction lines of the electric field due to two point charges.

Q1.31. Discuss the different types of charge distributions. How do you determine the electric field intensity due to a charge distribution?

Q1.32. Describe the electric field due to an infinitely long line charge of uniform charge density.

Q1.33. Describe the electric field due to an infinite plane sheet of uniform surface charge density.

Q1.34. State Ampère's force law as applied to current elements. Why is it not necessary for Newton's third law to hold for current elements?

Q1.35. What are the units of magnetic flux density? How is magnetic flux density defined in terms of force on a current element?

Q1.36. What is the value of the permeability of free space? What are its units?

Q1.37. Describe the magnetic field due to a current element.

Q1.38. Discuss the different types of current distributions. How do you determine the magnetic flux density due to a current distribution?

Q1.39. Describe the magnetic field due to an infinite plane sheet of uniform surface current density.

Q1.40. Discuss the analogies between the electric field due to charge distributions and the magnetic field due to current distributions.

Q1.41. How is magnetic flux density defined in terms of force on a moving charge?

Q1.42. Discuss two applications based on the magnetic force on a current-carrying wire or on a moving charge.

Q1.43. State the Lorentz force equation.

Q1.44. Discuss the determination of **E** and **B** at a point from the knowledge of forces experienced by a test charge at that point for several velocities. What is the minimum required number of forces?

Q1.45. Give some examples of devices based on charged particle motion in electric and magnetic fields.

Q1.46. Discuss the tracing of the path of a charged particle in a region of crossed electric and magnetic fields.

PROBLEMS

Section 1.1

P1.1. **Geometrical computations involving conversion from rectangular to polar coordinates.** A bug starts at a point and travels 1 m northward, s m eastward, s^2 m southward, s^3 m westward, and so on, where $s < 1$, making a 90°-turn to the right and traveling in the new direction s times the distance traveled in the previous direction. Find the value of s for each of the following cases: **(a)** the total distance traveled by the bug is 1.5 m; **(b)** the straight-line distance from the initial position to the final position of the bug is 0.8 m; and **(c)** the final position of the bug relative to its initial position is 30° east of north.

P1.2. **Solution of simultaneous vector algebraic equations.** Three vectors **A**, **B**, and **C** satisfy the equations

$$\mathbf{A} + \mathbf{B} - \mathbf{C} = 2\mathbf{a}_1 + \mathbf{a}_2$$
$$\mathbf{A} + 2\mathbf{B} + 3\mathbf{C} = -2\mathbf{a}_1 + 5\mathbf{a}_2 + 5\mathbf{a}_3$$
$$2\mathbf{A} - \mathbf{B} + \mathbf{C} = \mathbf{a}_1 + 5\mathbf{a}_2$$

By writing a matrix equation for the 3×3 matrix

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

and solving it, obtain the vectors **A**, **B**, and **C**.

P1.3. Law of cosines from dot product. Two vectors $\mathbf{A}$ and $\mathbf{B}$ originate from a common point. **(a)** If $\mathbf{C} = \mathbf{B} - \mathbf{A}$ comprises the third side of the triangle, obtain using $\mathbf{C}\cdot\mathbf{C} = (\mathbf{B} - \mathbf{A})\cdot(\mathbf{B} - \mathbf{A})$ the law of cosines relating C to A, B, and the angle α between $\mathbf{A}$ and $\mathbf{B}$. **(b)** Find the expression for the distance from the common point to the side $\mathbf{C}$, in terms of $\mathbf{A}$ and $\mathbf{B}$ only.

P1.4. Using vector algebraic operations. Four vectors drawn from a common point are given as follows:

$$
\begin{aligned}
\mathbf{A} &= 2\mathbf{a}_1 - m\mathbf{a}_2 - \mathbf{a}_3 \\
\mathbf{B} &= m\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3 \\
\mathbf{C} &= \mathbf{a}_1 + m\mathbf{a}_2 + 2\mathbf{a}_3 \\
\mathbf{D} &= m^2\mathbf{a}_1 + m\mathbf{a}_2 + \mathbf{a}_3
\end{aligned}
$$

Find the value(s) of m for each of the following cases: **(a)** $\mathbf{A}$ is perpendicular to $\mathbf{B}$; **(b)** $\mathbf{B}$ is parallel to $\mathbf{C}$; **(c)** $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ lie in the same plane; and **(d)** $\mathbf{D}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

P1.5. Straight line connecting the tips of three vectors originating from a point. Show that the tips of three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ originating from a common point lie along a straight line if $\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} = \mathbf{0}$. Provide a geometric interpretation for this result.

P1.6. Plane containing the tips of four vectors originating from a point. Show that the tips of four vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ originating from a common point lie in a plane if $(\mathbf{A} - \mathbf{B})\cdot(\mathbf{A} - \mathbf{C}) \times (\mathbf{A} - \mathbf{D}) = 0$. Then determine if the tips of $\mathbf{A} = \mathbf{a}_1$, $\mathbf{B} = 2\mathbf{a}_2$, $\mathbf{C} = 2\mathbf{a}_3$, and $\mathbf{D} = \mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3$ lie in a plane.

P1.7. Some vector identities.

(a) Show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

(b) Using the result of part (a), show the following:

(i) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$

(ii) $(\mathbf{A} \times \mathbf{B})\cdot(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \times \mathbf{B}\cdot\mathbf{C})^2$

Section 1.2

P1.8. Geometrical computations in Cartesian coordinates. Three points are given by $A(12, 0, 0)$, $B(0, 15, 0)$, and $C(0, 0, -20)$. Find the following: **(a)** the distance from B to C; **(b)** the component of the vector from A to C along the vector from B to C; and **(c)** the perpendicular distance from A to the line through B and C.

P1.9. Sphere passing through four specified points in Cartesian coordinates. Consider four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4). Show that the center point (x_0, y_0, z_0) of the sphere passing through these points is given by the solution of the equation

$$2\begin{bmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_2^2 + y_2^2 + z_2^2) - (x_1^2 + y_1^2 + z_1^2) \\ (x_3^2 + y_3^2 + z_3^2) - (x_1^2 + y_1^2 + z_1^2) \\ (x_4^2 + y_4^2 + z_4^2) - (x_1^2 + y_1^2 + z_1^2) \end{bmatrix}$$

Then find the center point of the sphere and its radius if the four points are (1, 1, 4), (3, 3, 2), (2, 3, 3), and (3, 2, 3).

P1.10. Plane containing two vectors originating from a common point.

(a) Two vectors **A** and **B** originate from a common point $P(x_1, y_1, z_1)$. Show that the equation for the plane in which the two vectors lie is given by

$$\mathbf{A} \times \mathbf{B} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

where $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the position vector and $\mathbf{r}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z$ is the vector from the origin to the point P.

(b) Using the result of part (a), find the equation for the plane containing the points $(1, 1, 2)$, $(2, 2, 0)$, and $(3, 0, 1)$.

P1.11. Finding differential length vector tangential to a curve. Find the expression for the differential length vector tangential to the curve $x + y = 2$, $y = z^2$ at an arbitrary point on the curve and having the projection dz on the z-axis. Then obtain the differential length vectors tangential to the curve at the points **(a)** $(2, 0, 0)$, **(b)** $(1, 1, 1)$, and **(c)** $(-2, 4, 2)$.

P1.12. Finding unit vector normal to a curve and a line at the point of intersection. Find the expression for the unit vector normal to the curve $x = y^2 = z^3$ at the point $(1, 1, 1)$ and having no components along the line $x = y = z$.

P1.13. Finding unit vector normal to a surface. By considering two differential length vectors tangential to the surface $x^2 + y^2 + 2z^2 = 4$ at the point $(1, 1, 1)$, find the unit vector normal to the surface.

P1.14. Finding differential surface vector associated with a plane. Consider the differential surface lying on the plane $2x + y = 2$ and having as its projection on the xz-plane the rectangular differential surface of sides dx and dz in the x- and z-directions, respectively. Obtain the expression for the vector $d\mathbf{S}$ associated with that surface.

Section 1.3

P1.15. Vector algebraic operations with points in cylindrical coordinates. Three points are given in cylindrical coordinates by $A(2, \pi/3, 1)$, $B(2\sqrt{3}, \pi/6, -2)$, and $C(2, 5\pi/6, 0)$. **(a)** Find the volume of the parallelepiped having the lines from the origin to the three points as one set of its contiguous edges. **(b)** Determine if the point $D(\sqrt{3}, \pi/2, 2.5)$ in cylindrical coordinates lies in the plane containing A, B, and C.

P1.16. Vector algebraic operations with points in spherical coordinates. Four points are given in spherical coordinates by $A(1, \pi/2, 0)$, $B(\sqrt{8}, \pi/4, \pi/3)$, $C(1, 0, 0)$, and $D(\sqrt{12}, \pi/6, \pi/2)$. Show that these four points are situated at the corners of a parallelogram and find the area of the parallelogram.

P1.17. Vector algebraic operations for vectors specified in cylindrical coordinates. Three unit vectors are given in cylindrical coordinates as follows: $\mathbf{A} = \mathbf{a}_r$ at $(2, \pi/6, 0)$, $\mathbf{B} = \mathbf{a}_\phi$ at $(1, \pi/3, 2)$, and $\mathbf{C} = \mathbf{a}_\phi$ at $(3, 5\pi/6, 1)$. Find: **(a)** $\mathbf{A} \cdot \mathbf{B}$; **(b)** $\mathbf{B} \cdot \mathbf{C}$; and **(c)** $\mathbf{B} \times \mathbf{C}$.

P1.18. Vector algebraic operations for vectors specified in spherical coordinates. Three unit vectors are given in spherical coordinates as follows: $\mathbf{A} = \mathbf{a}_r$ at $(2, \pi/6, \pi/2)$, $\mathbf{B} = \mathbf{a}_\theta$ at $(1, \pi/3, 0)$, and $\mathbf{C} = \mathbf{a}_\phi$ at $(3, \pi/4, 3\pi/2)$. Find: **(a)** $\mathbf{A} \cdot \mathbf{B}$; **(b)** $\mathbf{A} \cdot \mathbf{C}$; **(c)** $\mathbf{B} \cdot \mathbf{C}$; and **(d)** $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$.

P1.19. Conversion of vector in Cartesian coordinates to one in spherical coordinates. Convert the vector $\mathbf{a}_x + \mathbf{a}_y - \sqrt{2}\,\mathbf{a}_z$ at the point $(1, 1, \sqrt{2})$ to one in spherical coordinates.

P1.20. Equality of two vectors in different coordinates. Determine if the vector $(\mathbf{a}_{rc} - \sqrt{3}\,\mathbf{a}_\phi + 3\mathbf{a}_z)$ at the point $(3, \pi/3, 5)$ in cylindrical coordinates is equal to the vector $(3\mathbf{a}_{rs} - \sqrt{3}\,\mathbf{a}_\theta - \mathbf{a}_\phi)$ at the point $(1, \pi/3, \pi/6)$ in spherical coordinates.

P1.21. Finding unit vector tangential to a curve in cylindrical coordinates. Find the expression for the unit vector tangential to the curve given in cylindrical coordinates by $r^2 \sin 2\phi = 1, z = 0$. Then obtain the unit vectors tangential to the curve at the points: **(a)** $(1, \pi/4, 0)$ and **(b)** $(\sqrt{2}, \pi/12, 0)$.

P1.22. Finding unit vector tangential to a curve in spherical coordinates. Find the expression for the unit vector tangential to the curve given in spherical coordinates by $r = 1, \phi = 2\theta, 0 \le \theta \le \pi$. Then obtain the unit vectors tangential to the curve at the points: **(a)** $(1, \pi/4, \pi/2)$ and **(b)** $(1, \pi/2, \pi)$.

Section 1.4

P1.23. Scalar field of height of a hemispherical trough in a hemispherical dome. An otherwise hemispherical dome of radius 2 m has a symmetrically situated hemispherical trough of radius 1 m, as shown by the cross-sectional view in Fig. 1.42. Assuming the origin to be at the center of the base of the dome, obtain the expression for the two-dimensional scalar field describing the height h of the dome in each of the two coordinate systems: **(a)** rectangular (x, y) and **(b)** polar (r, ϕ).

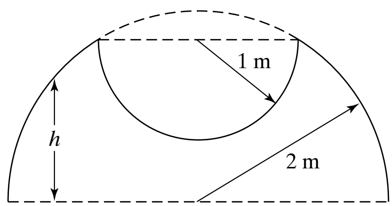

FIGURE 1.42
For Problem P1.23.

P1.24. Force field experienced by a mass in the Earth's gravitational field. Assuming the origin to be at the center of Earth and the z-axis to be passing through the poles, write vector functions for the force experienced by a mass m in the gravitational field of Earth (mass M) in each of the three coordinate systems: **(a)** Cartesian, **(b)** cylindrical, and **(c)** spherical. Describe the constant-magnitude surfaces and the direction lines.

P1.25. Field of the linear velocity of points inside the Earth. Assuming the origin to be at the center of Earth and the z-axis to be passing through the poles, write vector functions for the linear velocity of points inside Earth due to its spin motion in each of the three coordinate systems: **(a)** Cartesian; **(b)** cylindrical; and **(c)** spherical. Describe the constant-magnitude surfaces and the direction lines.

P1.26. Finding equations for the direction lines of vector fields in Cartesian coordinates. Obtain the equations for the direction lines for the following vector fields and passing through the point $(1, 2, 3)$: **(a)** $2y\mathbf{a}_x - x\mathbf{a}_y$ and **(b)** $x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$.

P1.27. Finding equation for the direction line of a vector field in cylindrical coordinates. Obtain the equation for the direction line for the vector field given in cylindrical coordinates by $(\sin\phi\,\mathbf{a}_r + \cos\phi\,\mathbf{a}_\phi)$ and passing through the point $(2, \pi/3, 1)$.

P1.28. Finding equation for the direction line of a vector field in spherical coordinates. Obtain the equation for the direction line for the vector field given in spherical coordinates by $(2\cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta)$ and passing through the point $(2, \pi/4, \pi/6)$.

Section 1.5

P1.29. Electric forces on point charges. Point charges, each of value Q, are situated at the corners of a regular tetrahedron of edge length L. Find the electric force on each point charge.

P1.30. Electric force on a test charge in the field of six point charges. Six point charges, each of value Q, are situated at $(d, 0, 0)$, $(-d, 0, 0)$, $(0, d, 0)$, $(0, -d, 0)$, $(0, 0, d)$, and $(0, 0, -d)$. A test charge q located at the origin is displaced by a distance $\Delta \ll d$ along the positive x-axis. Find an approximate expression for the electric force acting on the charge.

P1.31. Finding the point charge for specified electric field intensities. For each of the following pairs of electric field intensities, find, if possible, the location and the value of a point charge that produces both fields: **(a)** $\mathbf{E}_1 = (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$ V/m at $(2, 2, 3)$ and $\mathbf{E}_2 = (\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)$ V/m at $(-1, 0, 3)$; and **(b)** $\mathbf{E}_1 = (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$ V/m at $(1, 1, 1)$ and $\mathbf{E}_2 = (2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)$ V/m at $(1, 2, 0)$.

P1.32. Electric field intensity due to an electric dipole. Two equal and opposite point charges Q and $-Q$ are located at $(0, 0, d/2)$ and $(0, 0, -d/2)$, respectively. Such an arrangement is known as the electric dipole. Show that the electric field intensity due to the electric dipole at very large distances from the origin compared to the spacing d is given approximately by $(Qd/4\pi\varepsilon_0 r^3)(2 \cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta)$.

P1.33. Motion of a point charge along the axis of a circular ring charge. A point charge q of mass m is in static equilibrium at the origin, in the presence of a circular ring charge Q in the xy-plane and two point charges, each of value Q, on the z-axis. The ring charge is uniformly distributed, with radius a and center at the origin. The two point charges are located at $(0, 0, -a)$ and $(0, 0, a)$. All charges are of the same sign. The point charge q is constrained to move along the z-axis. It is given a slight displacement $z_0 \ll a$ and released at $t = 0$. Obtain the approximate differential equation for the motion of the charge and find the frequency of oscillation.

P1.34. Finding a circular ring charge that produces specified electric field intensities. Design an arrangement of a circular ring charge of uniform density and total charge Q equal to 1 μC that produces electric field intensities of $10^3\, \mathbf{a}_z$ V/m at the two points $(0, 0, 1)$ and $(0, 0, 2)$. If Q is not equal to 1 μC, determine, if any, the restriction on its value for a solution to exist.

P1.35. Electric field of a circular ring charge with nonuniform density. Assuming that the circular ring of Example 1.7 is coated with charge such that the charge density is given by $\rho_L = \rho_{L0} \cos\phi$ C/m, find the electric field intensity at a point on the z-axis by setting up the integral expression and evaluating it.

P1.36. Electric field of a circular disc of charge with nonuniform density. Consider a circular disc of radius a lying in the xy-plane with its center at the origin and carrying charge of density $4\pi\varepsilon_0/r$ C/m^2. Obtain the expression for the electric field intensity at the point $(0, 0, z)$ by setting up the integral and evaluating it.

P1.37. Electric field of a line charge with nonuniform density. Consider the charge distributed with density $4\pi\varepsilon_0|z|$ C/m along the line between $(0, 0, -a)$ and $(0, 0, a)$. Obtain the expression for the electric field intensity at $(r, \phi, 0)$ in cylindrical coordinates, by considering a differential length element along the line charge, setting up the field as an integral and evaluating it.

P1.38. Electric field of a slab of volume charge distributed between two planes. Consider the volume charge distributed uniformly with density ρ_0 C/m^3 between the planes $z = -a$ and $z = a$. Using superposition in conjunction with the result of

Example 1.9, show that the electric field intensity due to the slab of charge is given by

$$\mathbf{E} = \begin{cases} -(\rho_0 a/\varepsilon_0)\mathbf{a}_z & \text{for} \quad z < -a \\ (\rho_0 z/\varepsilon_0)\mathbf{a}_z & \text{for} \quad -a < z < a \\ (\rho_0 a/\varepsilon_0)\mathbf{a}_z & \text{for} \quad z > a \end{cases}$$

Section 1.6

P1.39. Magnetic forces on current elements. Three identical current elements $I\,dz\,\mathbf{a}_z$ A-m are located at equally spaced points on a circle of radius 1 m centered at the origin and lying on the xy-plane. The first point is $(1, 0, 0)$. Find the magnetic force on each current element.

P1.40. Magnetic flux density due to a current element. For the current element $I\,dx\,(\mathbf{a}_x + \mathbf{a}_y)$ A situated at the point $(1, -2, 2)$, find the magnetic flux densities at three points: **(a)** $(2, -1, 3)$, **(b)** $(2, -3, 4)$, and **(c)** $(3, 0, 2)$.

P1.41. Finding infinitely long wire for specified magnetic flux densities. For each of the following pairs of magnetic flux densities, find, if possible, the orientation of an infinitely long filamentary wire and the current in it required to produce both fields: **(a)** $\mathbf{B}_1 = 10^{-7}\,\mathbf{a}_y$ Wb/m^2 at $(3, 0, 0)$ and $\mathbf{B}_2 = -10^{-7}\,\mathbf{a}_x$ Wb/m^2 at $(0, 4, 0)$; and **(b)** $\mathbf{B}_1 = 10^{-7}(\mathbf{a}_y - \mathbf{a}_z)$ Wb/m^2 at $(\sqrt{2}, 0, 0)$ and $\mathbf{B}_2 = -10^{-7}\,\mathbf{a}_x$Wb/m^2 at $(0, \sqrt{2}, 0)$.

P1.42. Attraction between two long, horizontal filamentary wires. Two long identical rigid filamentary wires, each of length l and weight w, are suspended horizontally from the ceiling by long weightless threads, each of length L. The wires are arranged to be parallel and separated by a distance d, small compared to l and L. A current I is passed through both wires via flexible connections so as to cause the wires to be attracted to each other. **(a)** Should the currents be in the same sense or in opposite senses for attraction to occur? **(b)** If the current is gradually increased from zero, the wires will gradually approach each other. A condition may be reached when any further increase of current will cause the wires to swing and touch each other. Determine the critical value of I at which this happens.

P1.43. Magnetic field due to a circular loop of wire. A circular loop of wire of radius a is situated in the xy-plane with its center at the origin. It carries a current I in the clockwise sense as seen along the positive z-axis, that is, in the sense of increasing values of ϕ. Obtain the expression for $\mathbf{B}$ due to the current loop at a point on the z-axis.

P1.44. Magnetic field due to a finitely long straight wire of current. A straight wire along the z-axis carries current I in the positive z-direction. Consider the portion of the wire between $(0, 0, a_1)$ and $(0, 0, a_2)$, where $a_2 > a_1$. Show that the magnetic flux density at an arbitrary point $P(r, \phi, z)$ due to this portion of the wire is given by

$$\mathbf{B} = \frac{\mu_0 I}{4\pi r}(\cos\alpha_1 - \cos\alpha_2)\mathbf{a}_\phi$$

where α_1 and α_2 are the angles subtended by the lines from P to $(0, 0, a_1)$ and $(0, 0, a_2)$, respectively, with the z-axis. Verify your result in the limit $a_1 \to -\infty$ and $a_2 \to \infty$.

P1.45. Magnetic flux density due a square loop of wire. A square loop of wire lies in the xy-plane with its corners at $(1, 1, 0)$, $(-1, 1, 0)$, $(-1, -1, 0)$, and $(1, -1, 0)$. A current of 1 A flows in the loop in the sense defined by connecting the specified points in succession. Applying the result of Problem P1.44 to each side of the loop, find the magnetic flux densities at two points: **(a)** $(0, 0, 0)$ and **(b)** $(2, 0, 0)$.

P1.46. Finding a pair of infinitely long parallel wires for specified magnetic flux densities. Design an arrangement of a pair of infinitely long, straight, filamentary wires parallel to the z-direction and in a plane parallel to the xz-plane, each carrying current I equal to 1 A but in opposite directions, which produce magnetic flux densities of $10^{-7}\,\mathbf{a}_y$ Wb/m^2 and $0.5 \times 10^{-7}\,\mathbf{a}_y$ Wb/m^2 at the points $(0, 1, 0)$ and $(0, 2, 0)$, respectively. If I is not equal to 1 A, determine, if any, the restriction on its value for a solution to exist.

P1.47. Magnetic flux density due to three plane current sheets. Three infinite plane current sheets, each of a uniform current density, exist in the coordinate planes of a Cartesian coordinate system. The magnetic flux densities due to these current sheets are given at three points as follows: at $(1, 2, 3)$, $\mathbf{B} = 3B_0\mathbf{a}_x$; at $(7, -5, 6)$, $\mathbf{B} = B_0(-\mathbf{a}_x + 2\mathbf{a}_z)$; at $(8, 9, -4)$, $\mathbf{B} = B_0(\mathbf{a}_x + 2\mathbf{a}_y)$. Find the magnetic flux densities at the following points: **(a)** $(-6, -2, -3)$; **(b)** $(-4, -5, 7)$; and **(c)** $(6, -3, -5)$.

P1.48. Magnetic field for a current distribution between two planes. Consider current distribution with uniform density $J_0\mathbf{a}_z$ A/m^2 in the volume between the planes $y = -a$ and $y = 0$, and with uniform density $-J_0\mathbf{a}_z$ A/m^2 in the volume between the planes $y = 0$ and $y = a$. Using superposition in conjunction with the result of Example 1.12, show that the magnetic flux density due to the current distribution is given by

$$\mathbf{B} = \begin{cases} \mu_0 J_0(|y| - a)\mathbf{a}_x & \text{for} \quad |y| \le a \\ \mathbf{0} & \text{otherwise} \end{cases}$$

P1.49. Ratio of the radii of orbits of two charged particles in a uniform magnetic field. Show that the ratio of the radii of orbits of two charged particles of the same charge, but with different masses, entering a region of uniform magnetic field perpendicular to the field and with equal kinetic energies is equal to the ratio of the square roots of their masses.

Section 1.7

P1.50. Movement of a test charge in a region of crossed electric and magnetic fields. Show that in a region of uniform, crossed electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, respectively, a test charge released at a point in the field region with initial velocity $\mathbf{v} = (\mathbf{E} \times \mathbf{B})/B^2$ moves with constant velocity equal to the initial value. Compute $\mathbf{v}$ for $\mathbf{E}$ and $\mathbf{B}$ equal to $E_0(2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)$ and $B_0(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$, respectively.

P1.51. Finding magnetic field from forces experienced by a test charge. Let $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ be the forces experienced by a test charge q at a point in a region of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, respectively, for velocities $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, respectively, of the charge. If $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ are such that $\mathbf{A} = \mathbf{F}_1 \times \mathbf{F}_2 + \mathbf{F}_2 \times \mathbf{F}_3 + \mathbf{F}_3 \times \mathbf{F}_1 \neq 0$, that is, their tips do not lie on a straight line when drawn from a common point, show that

$$\mathbf{B} = \frac{1}{q}\left[\frac{\mathbf{F}_2 - \mathbf{F}_1}{(\mathbf{v}_2 - \mathbf{v}_1) \times \mathbf{A}}\right]\mathbf{A}$$

P1.52. Finding electric and magnetic fields from forces experienced by a test charge. The forces experienced by a test charge q at a point in a region of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, respectively, are given as follows for three different velocities of the test charge, where v_0 and E_0 are constants.

$$\begin{aligned} \mathbf{F}_1 &= qE_0(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z) && \text{for} \quad \mathbf{v}_1 = v_0\mathbf{a}_x \\ \mathbf{F}_2 &= qE_0(\mathbf{a}_x - \mathbf{a}_y - \mathbf{a}_z) && \text{for} \quad \mathbf{v}_2 = v_0\mathbf{a}_y \\ \mathbf{F}_3 &= \mathbf{0} && \text{for} \quad \mathbf{v}_3 = v_0\mathbf{a}_z \end{aligned}$$

Find **E** and **B** at that point.

P1.53. Forces experienced by a test charge in a region of electric and magnetic fields. Three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ experienced by a test charge q at a point in a region of electric and magnetic fields for three different velocities of the test charge are given as follows:

$$\begin{aligned} \mathbf{F}_1 &= \mathbf{0} && \text{for} \quad \mathbf{v}_1 = v_0\mathbf{a}_x \\ \mathbf{F}_2 &= \mathbf{0} && \text{for} \quad \mathbf{v}_2 = v_0\mathbf{a}_y \\ \mathbf{F}_3 &= qE_0\mathbf{a}_z && \text{for} \quad \mathbf{v}_3 = v_0(\mathbf{a}_x + 2\mathbf{a}_y) \end{aligned}$$

Find the forces $\mathbf{F}_4$, $\mathbf{F}_5$, and $\mathbf{F}_6$ experienced by the test charge at that point for three other velocities: **(a)** $\mathbf{v}_4 = \mathbf{0}$, **(b)** $\mathbf{v}_5 = v_0(\mathbf{a}_x + \mathbf{a}_y)$, and **(c)** $\mathbf{v}_6 = (v_0/4)(3\mathbf{a}_x + \mathbf{a}_y)$.

P1.54. Movement of a test charge in a region of uniform electric and magnetic fields. Uniform electric and magnetic fields exist in a region of space. A test charge q released with an initial velocity $\mathbf{v}_1$ or $\mathbf{v}_2$ moves with constant velocity equal to the initial value. Show that the test charge moves with constant velocity equal to the initial value when released with an initial velocity $(m\mathbf{v}_1 + n\mathbf{v}_2)/(m + n)$ for any nonzero $(m + n)$.

REVIEW PROBLEMS

R1.1. Using vector algebraic operations and equalities. Using the equality

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

show that if $\mathbf{A} \times \mathbf{F} = \mathbf{C}$ and $\mathbf{B} \times \mathbf{F} = \mathbf{D}$, then

$$\mathbf{F} = \frac{\mathbf{C} \times \mathbf{D}}{\mathbf{A} \cdot \mathbf{D}} = -\frac{\mathbf{C} \times \mathbf{D}}{\mathbf{B} \cdot \mathbf{C}}$$

Find **F** if $\mathbf{A} = (\mathbf{a}_x + \mathbf{a}_y)$, $\mathbf{B} = (\mathbf{a}_x + 2\mathbf{a}_y - 2\mathbf{a}_z)$, $\mathbf{C} = (\mathbf{a}_x - \mathbf{a}_y)$, and $\mathbf{D} = (6\mathbf{a}_x - 5\mathbf{a}_y - 2\mathbf{a}_z)$.

R1.2. Shortest distance from a point to a plane. The tips of three vectors **A, B**, and **C** originating from a common point determine a plane. **(a)** Show that the shortest distance from the common point to the plane is $|\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}|/|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}|$. **(b)** Compute its value for $\mathbf{A} = 2\mathbf{a}_z$, $\mathbf{B} = \frac{1}{2}(\mathbf{a}_{rc} - \sqrt{3}\,\mathbf{a}_\phi)$ in cylindrical coordinates, and $\mathbf{C} = \frac{1}{4}(3\mathbf{a}_{rs} + \sqrt{3}\,\mathbf{a}_\theta + 2\mathbf{a}_\phi)$ in spherical coordinates, at the point $(\sqrt{3}, 3, 2)$ in Cartesian coordinates.

R1.3. Sphere inscribed in an equilateral tetrahedron inscribed in a sphere. Find the edge length of the largest equilateral tetrahedron that can be fit inside a sphere of radius unity. Then find the radius of the largest sphere that can be fit inside that tetrahedron.

R1.4. Equation for a curve traversed on a sphere. Consider an observer always moving in the southeast direction on the surface of a spherical Earth of radius a, starting at the Greenwich meridian on the equator. **(a)** Find the equation for the curve traversed by the observer, using a spherical coordinate system with the origin at the center of Earth, the North Pole at $\theta = 0$, and $\phi = 0$, corresponding to the Greenwich meridian. **(b)** Find the first two values of the south latitude when the observer is again on the Greenwich meridian. **(c)** Does the observer ever reach the South Pole?

R1.5. Three point charges on a semicircle. Consider the arrangement of three point charges Q_1, $kQ_1 (k > 0)$, and Q_2, as shown in Fig. 1.43, where Q_1 and kQ_1 are fixed and Q_2 is constrained to move on the semicircle. **(a)** Find the value of α in terms of k for which Q_2 is in equilibrium. **(b)** Find the numerical value of α for $k = 8$.

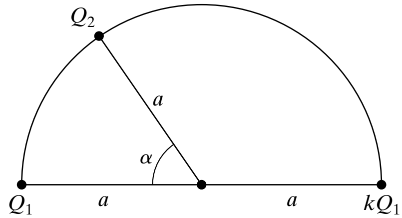

FIGURE 1.43
For Problem R1.5.

R1.6. Finitely long line charge distribution of nonuniform density. Consider a line charge distribution along the z-axis between $z = -a$ and $z = a$. **(a)** Show that, if the charge density is an even function $f_1(z)$, the electric field intensity at a point $(r, \phi, 0)$ has only an r-component, and set up the integral expression for it. **(b)** Show that, if the charge density is an odd function $f_2(z)$, the electric field intensity at $(r, \phi, 0)$ has only a z-component, and set up the integral expression for it. **(c)** Given that the charge density is

$$f(z) = \begin{cases} 4\pi\varepsilon_0(a + 2z) & \text{for} \quad -a \le z \le 0 \\ 4\pi\varepsilon_0 a & \text{for} \quad 0 \le z \le a \end{cases}$$

express $f(z)$ as the sum of even and odd functions $f_1(z)$ and $f_2(z)$, and evaluate the electric field components.

R1.7. Magnetic flux density due to a wire of current with straight and curved segments. Current I flows along a wire which is straight from ∞ to a on the x-axis, circular from $(a, 0, 0)$ to $(0, a, 0)$ and lying on the xy-plane in the sense of increasing ϕ, and then from a to ∞ on the y-axis. Find $\mathbf{B}$ at $(0, 0, a)$.

R1.8. Magnetic field due to a nonuniform current distribution between two planes. Current is distributed with density $J_0(y/a)\mathbf{a}_z$ A/m^2 in the volume between the planes $y = -a$ and $y = a$. Show that the magnetic flux density due to the current distribution is given by

$$\mathbf{B} = \begin{cases} \dfrac{\mu_0 J_0}{2a}(a^2 - y^2)\mathbf{a}_x & \text{for} \quad -a < y < a \\ \mathbf{0} & \text{otherwise} \end{cases}$$

R1.9. Movement of a test charge in a region of uniform electric and magnetic fields. Consider a test charge moving with constant velocity $\mathbf{v} = v_0(2\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z)$ in a region of a uniform electric field of intensity $\mathbf{E} = E_0(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$ and a uniform magnetic field of flux density $\mathbf{B} = B_{x0}\mathbf{a}_x + B_{y0}\mathbf{a}_y + B_{z0}\mathbf{a}_z$. Is this information sufficient to find uniquely B_{x0}, B_{y0}, and B_{z0}? If not, given that $\mathbf{v}$ is perpendicular to $\mathbf{B}$, find B_{x0}, B_{y0}, and B_{z0} in terms of E_0 and v_0.