Lecture notes: Differential Equations for ISE (MA029IU)

Week 10-11-12 [*]

May 4, 2022

1 Boundary value problems

1.1 Boundary value problems

Before we tackle the Fourier series, we study the so-called *boundary value problems* (or *endpoint problems*). Consider

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0,$$

for some constant λ, where $x(t)$ is defined for t in the interval $[a, b]$. Previously we specified the value of the solution and its derivative at a single point. Now we specify the value of the solution at two different points. As $x = 0$ is a solution, existence of solutions is not a problem. Uniqueness of solutions is another issue. The general solution to $x'' + \lambda x = 0$ has two arbitrary constants It is, therefore, natural (but wrong) to believe that requiring two conditions guarantees a unique solution.

Example 1.1: Take $\lambda = 1$, $a = 0$, $b = \pi$. That is,

$$x'' + x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

Then $x = \sin t$ is another solution (besides $x = 0$) satisfying both boundary conditions. There are more. Write down the general solution of the differential equation, which is $x = A\cos t + B\sin t$. The condition $x(0) = 0$ forces $A = 0$. Letting $x(\pi) = 0$ does not give us any more information as $x = B\sin t$ already satisfies both boundary conditions. Hence, there are infinitely many solutions of the form $x = B\sin t$, where B is an arbitrary constant.

Example 1.2: On the other hand, consider $\lambda = 2$. That is,

$$x'' + 2x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

Then the general solution is $x = A\cos(\sqrt{2}\,t) + B\sin(\sqrt{2}\,t)$. Letting $x(0) = 0$ still forces $A = 0$. We apply the second condition to find $0 = x(\pi) = B\sin(\sqrt{2}\,\pi)$. As $\sin(\sqrt{2}\,\pi) \neq 0$ we obtain $B = 0$. Therefore $x = 0$ is the unique solution to this problem.

What is going on? We will be interested in finding which constants λ allow a nonzero solution, and we will be interested in finding those solutions. This problem is an analogue of finding eigenvalues and eigenvectors of matrices.

1.2 Eigenvalue problems

For basic Fourier series theory we will need the following three eigenvalue problems. We will consider more general equations and boundary conditions.

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0, \tag{1}$$

[*]This note is taken from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl

$$x'' + \lambda x = 0, \quad x'(a) = 0, \quad x'(b) = 0, \tag{2}$$

and

$$x'' + \lambda x = 0, \quad x(a) = x(b), \quad x'(a) = x'(b). \tag{3}$$

A number λ is called an *eigenvalue* of (1) (resp. (2) or (3)) if and only if there exists a nonzero (not identically zero) solution to (1) (resp. (2) or (3)) given that specific λ. A nonzero solution is called a corresponding *eigenfunction*.

Note the similarity to eigenvalues and eigenvectors of matrices. The similarity is not just coincidental. If we think of the equations as differential operators, then we are doing the same exact thing. Think of a function $x(t)$ as a vector with infinitely many components (one for each t). Let $L = -\frac{d^2}{dt^2}$ be the linear operator. Then the eigenvalue/eigenfunction pair should be λ and nonzero x such that $Lx = \lambda x$. In other words, we are looking for nonzero functions x satisfying certain endpoint conditions that solve $(L - \lambda)x = 0$. A lot of the formalism from linear algebra still applies here, though we will not pursue this line of reasoning too far.

Example 1.3: Let us find the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

We have to handle the cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ separately. First suppose that $\lambda > 0$. Then the general solution to $x'' + \lambda x = 0$ is

$$x = A\cos(\sqrt{\lambda}\, t) + B\sin(\sqrt{\lambda}\, t).$$

The condition $x(0) = 0$ implies immediately $A = 0$. Next

$$0 = x(\pi) = B\sin(\sqrt{\lambda}\, \pi).$$

If B is zero, then x is not a nonzero solution. So to get a nonzero solution we must have that $\sin(\sqrt{\lambda}\, \pi) = 0$. Hence, $\sqrt{\lambda}\, \pi$ must be an integer multiple of π. In other words, $\sqrt{\lambda} = k$ for a positive integer k. Hence the positive eigenvalues are k^2 for all integers $k \geq 1$. Corresponding eigenfunctions can be taken as $x = \sin(kt)$. Just like for eigenvectors, constant multiples of an eigenfunction are also eigenfunctions, so we only need to pick one.

Now suppose that $\lambda = 0$. In this case the equation is $x'' = 0$, and its general solution is $x = At + B$. The condition $x(0) = 0$ implies that $B = 0$, and $x(\pi) = 0$ implies that $A = 0$. This means that $\lambda = 0$ is *not* an eigenvalue.

Finally, suppose that $\lambda < 0$. In this case we have the general solution*

$$x = A\cosh(\sqrt{-\lambda}\, t) + B\sinh(\sqrt{-\lambda}\, t).$$

Letting $x(0) = 0$ implies that $A = 0$ (recall $\cosh 0 = 1$ and $\sinh 0 = 0$). So our solution must be $x = B\sinh(\sqrt{-\lambda}\, t)$ and satisfy $x(\pi) = 0$. This is only possible if B is zero. Why? Because $\sinh \xi$ is only zero when $\xi = 0$. You should plot sinh to see this fact. We can also see this from the definition of sinh. We get $0 = \sinh \xi = \frac{e^{\xi} - e^{-\xi}}{2}$. Hence $e^{\xi} = e^{-\xi}$, which implies $\xi = -\xi$ and that is only true if $\xi = 0$. So there are no negative eigenvalues.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\lambda_k = k^2 \qquad \text{with an eigenfunction} \qquad x_k = \sin(kt) \qquad \text{for all integers } k \geq 1.$$

Example 1.4: Let us compute the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x'(0) = 0, \quad x'(\pi) = 0.$$

*Recall that $\cosh s = \frac{1}{2}(e^s + e^{-s})$ and $\sinh s = \frac{1}{2}(e^s - e^{-s})$. As an exercise try the computation with the general solution written as $x = Ae^{\sqrt{-\lambda}\, t} + Be^{-\sqrt{-\lambda}\, t}$ (for different A and B of course).

Again we have to handle the cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ separately. First suppose that $\lambda > 0$. The general solution to $x'' + \lambda x = 0$ is $x = A\cos(\sqrt{\lambda}\,t) + B\sin(\sqrt{\lambda}\,t)$. So

$$x' = -A\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,t) + B\sqrt{\lambda}\,\cos(\sqrt{\lambda}\,t).$$

The condition $x'(0) = 0$ implies immediately $B = 0$. Next

$$0 = x'(\pi) = -A\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,\pi).$$

Again A cannot be zero if λ is to be an eigenvalue, and $\sin(\sqrt{\lambda}\,\pi)$ is only zero if $\sqrt{\lambda} = k$ for a positive integer k. Hence the positive eigenvalues are again k^2 for all integers $k \geq 1$. And the corresponding eigenfunctions can be taken as $x = \cos(kt)$.

Now suppose that $\lambda = 0$. In this case the equation is $x'' = 0$ and the general solution is $x = At + B$ so $x' = A$. The condition $x'(0) = 0$ implies that $A = 0$. The condition $x'(\pi) = 0$ also implies $A = 0$. Hence B could be anything (let us take it to be 1). So $\lambda = 0$ is an eigenvalue and $x = 1$ is a corresponding eigenfunction.

Finally, let $\lambda < 0$. In this case the general solution is $x = A\cosh(\sqrt{-\lambda}\,t) + B\sinh(\sqrt{-\lambda}\,t)$ and

$$x' = A\sqrt{-\lambda}\,\sinh(\sqrt{-\lambda}\,t) + B\sqrt{-\lambda}\,\cosh(\sqrt{-\lambda}\,t).$$

We have already seen (with roles of A and B switched) that for this expression to be zero at $t = 0$ and $t = \pi$, we must have $A = B = 0$. Hence there are no negative eigenvalues.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\lambda_k = k^2 \qquad \text{with an eigenfunction} \qquad x_k = \cos(kt) \qquad \text{for all integers } k \geq 1,$$

and there is another eigenvalue

$$\lambda_0 = 0 \qquad \text{with an eigenfunction} \qquad x_0 = 1.$$

The following problem is the one that leads to the general Fourier series.

Example 1.5: Let us compute the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi).$$

We have not specified the values or the derivatives at the endpoints, but rather that they are the same at the beginning and at the end of the interval.

Let us skip $\lambda < 0$. The computations are the same as before, and again we find that there are no negative eigenvalues.

For $\lambda = 0$, the general solution is $x = At + B$. The condition $x(-\pi) = x(\pi)$ implies that $A = 0$ ($A\pi + B = -A\pi + B$ implies $A = 0$). The second condition $x'(-\pi) = x'(\pi)$ says nothing about B and hence $\lambda = 0$ is an eigenvalue with a corresponding eigenfunction $x = 1$.

For $\lambda > 0$ we get that $x = A\cos(\sqrt{\lambda}\,t) + B\sin(\sqrt{\lambda}\,t)$. Now

$$\underbrace{A\cos(-\sqrt{\lambda}\,\pi) + B\sin(-\sqrt{\lambda}\,\pi)}_{x(-\pi)} = \underbrace{A\cos(\sqrt{\lambda}\,\pi) + B\sin(\sqrt{\lambda}\,\pi)}_{x(\pi)}.$$

We remember that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. Therefore,

$$A\cos(\sqrt{\lambda}\,\pi) - B\sin(\sqrt{\lambda}\,\pi) = A\cos(\sqrt{\lambda}\,\pi) + B\sin(\sqrt{\lambda}\,\pi).$$

Hence either $B = 0$ or $\sin(\sqrt{\lambda}\,\pi) = 0$. Similarly (exercise) if we differentiate x and plug in the second condition we find that $A = 0$ or $\sin(\sqrt{\lambda}\,\pi) = 0$. Therefore, unless we want A and B to both be zero (which we do not) we must have $\sin(\sqrt{\lambda}\,\pi) = 0$. Hence, $\sqrt{\lambda}$ is an integer and the eigenvalues are yet again $\lambda = k^2$ for an integer

$k \geq 1$. In this case, however, $x = A\cos(kt) + B\sin(kt)$ is an eigenfunction for any A and any B. So we have two linearly independent eigenfunctions $\sin(kt)$ and $\cos(kt)$. Remember that for a matrix we can also have two eigenvectors corresponding to a single eigenvalue if the eigenvalue is repeated.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\begin{aligned} &\lambda_k = k^2 && \text{with eigenfunctions} && \cos(kt) \quad \text{and} \quad \sin(kt) && \text{for all integers } k \geq 1, \\ &\lambda_0 = 0 && \text{with an eigenfunction} && x_0 = 1. \end{aligned}$$

1.3 Orthogonality of eigenfunctions

Something that will be very useful in the next section is the *orthogonality* property of the eigenfunctions. This is an analogue of the following fact about eigenvectors of a matrix. A matrix is called *symmetric* if $A = A^T$ (it is equal to its transpose). *Eigenvectors for two distinct eigenvalues of a symmetric matrix are orthogonal.* The differential operators we are dealing with act much like a symmetric matrix. We, therefore, get the following theorem.

Theorem 1.1. *Suppose that $x_1(t)$ and $x_2(t)$ are two eigenfunctions of the problem* (1), (2) *or* (3) *for two different eigenvalues λ_1 and λ_2. Then they are* orthogonal *in the sense that*

$$\int_a^b x_1(t)x_2(t)\,dt = 0.$$

The terminology comes from the fact that the integral is a type of inner product. We will expand on this in the next section. The theorem has a very short, elegant, and illuminating proof so let us give it here. First, we have the following two equations.

$$x_1'' + \lambda_1 x_1 = 0 \qquad \text{and} \qquad x_2'' + \lambda_2 x_2 = 0.$$

Multiply the first by x_2 and the second by x_1 and subtract to get

$$(\lambda_1 - \lambda_2)x_1x_2 = x_2''x_1 - x_2x_1''.$$

Now integrate both sides of the equation:

$$\begin{aligned} (\lambda_1 - \lambda_2)\int_a^b x_1x_2\,dt &= \int_a^b x_2''x_1 - x_2x_1''\,dt \\ &= \int_a^b \frac{d}{dt}\left(x_2'x_1 - x_2x_1'\right)\,dt \\ &= \Big[x_2'x_1 - x_2x_1'\Big]_{t=a}^b = 0. \end{aligned}$$

The last equality holds because of the boundary conditions. For example, if we consider (1) we have $x_1(a) = x_1(b) = x_2(a) = x_2(b) = 0$ and so $x_2'x_1 - x_2x_1'$ is zero at both a and b. As $\lambda_1 \neq \lambda_2$, the theorem follows.

***Exercise* 1.1** (easy)*: Finish the proof of the theorem* (*check the last equality in the proof*) *for the cases* (2) *and* (3)*.*

The function $\sin(nt)$ is an eigenfunction for the problem $x'' + \lambda x = 0$, $x(0) = 0$, $x(\pi) = 0$. Hence for positive integers n and m we have the integrals

$$\int_0^\pi \sin(mt)\sin(nt)\,dt = 0, \quad \text{when } m \neq n.$$

Similarly,

$$\int_0^\pi \cos(mt)\cos(nt)\,dt = 0, \quad \text{when } m \neq n, \qquad \text{and} \qquad \int_0^\pi \cos(nt)\,dt = 0.$$

And finally we also get

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = 0, \quad \text{when } m \neq n, \qquad \text{and} \qquad \int_{-\pi}^{\pi} \sin(nt)\,dt = 0,$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = 0, \quad \text{when } m \neq n, \qquad \text{and} \qquad \int_{-\pi}^{\pi} \cos(nt)\,dt = 0,$$

and

$$\int_{-\pi}^{\pi} \cos(mt)\sin(nt)\,dt = 0 \qquad (\text{even if } m = n).$$

1.4 Fredholm alternative

We now touch on a very useful theorem in the theory of differential equations. The theorem holds in a more general setting than we are going to state it, but for our purposes the following statement is sufficient.

Theorem 1.2 (Fredholm alternative*)**.** *Exactly one of the following statements holds. Either*

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0 \tag{4}$$

has a nonzero solution, or

$$x'' + \lambda x = f(t), \quad x(a) = 0, \quad x(b) = 0 \tag{5}$$

has a unique solution for every function f continuous on $[a, b]$.

The theorem is also true for the other types of boundary conditions we considered. The theorem means that if λ is not an eigenvalue, the nonhomogeneous equation (5) has a unique solution for every right-hand side. On the other hand if λ is an eigenvalue, then (5) need not have a solution for every f, and furthermore, even if it happens to have a solution, the solution is not unique.

We also want to reinforce the idea here that linear differential operators have much in common with matrices. So it is no surprise that there is a finite-dimensional version of Fredholm alternative for matrices as well. Let A be an $n \times n$ matrix. The Fredholm alternative then states that either $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution, or $(A - \lambda I)\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$.

A lot of intuition from linear algebra can be applied to linear differential operators, but one must be careful of course. For example, one difference we have already seen is that in general a differential operator will have infinitely many eigenvalues, while a matrix has only finitely many.

1.5 Exercises

Hint for the following exercises: Note that when $\lambda > 0$, then $\cos\big(\sqrt{\lambda}\,(t-a)\big)$ and $\sin\big(\sqrt{\lambda}\,(t-a)\big)$ are also solutions of the homogeneous equation.

Exercise* 1.2*: *Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0,\ x(a) = 0,\ x(b) = 0$ (assume $a < b$).*

Exercise* 1.3*: *Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0,\ x'(a) = 0,\ x'(b) = 0$ (assume $a < b$).*

Exercise* 1.4*: *Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0,\ x'(a) = 0,\ x(b) = 0$ (assume $a < b$).*

Exercise* 1.5*: *Compute all eigenvalues and eigenfunctions of $x'' + \lambda x = 0,\ x(a) = x(b),\ x'(a) = x'(b)$ (assume $a < b$).*

Exercise* 1.6*: *We skipped the case of $\lambda < 0$ for the boundary value problem $x'' + \lambda x = 0,\ x(-\pi) = x(\pi),\ x'(-\pi) = x'(\pi)$. Finish the calculation and show that there are no negative eigenvalues.*

*Named after the Swedish mathematician Erik Ivar Fredholm (1866–1927).

Exercise* 1.101*: *Consider a spinning string of length* 2 *and linear density* 0.1 *and tension* 3*. Find smallest angular velocity when the string pops out.*

Exercise* 1.102*: *Suppose* $x'' + \lambda x = 0$ *and* $x(0) = 1$, $x(1) = 1$*. Find all* λ *for which there is more than one solution. Also find the corresponding solutions* (*only for the eigenvalues*)*.*

Exercise* 1.103*: *Suppose* $x'' + x = 0$ *and* $x(0) = 0$, $x'(\pi) = 1$*. Find all the solution*(*s*) *if any exist.*

Exercise* 1.104*: *Consider* $x' + \lambda x = 0$ *and* $x(0) = 0$, $x(1) = 0$*. Why does it not have any eigenvalues*? *Why does any first order equation with two endpoint conditions such as above have no eigenvalues*?

Exercise* 1.105** (challenging):*** *Suppose* $x''' + \lambda x = 0$ *and* $x(0) = 0$, $x'(0) = 0$, $x(1) = 0$*. Suppose that* $\lambda > 0$*. Find an equation that all such eigenvalues must satisfy. Hint*: *Note that* $-\sqrt[3]{\lambda}$ *is a root of* $r^3 + \lambda = 0$*.*

2 The trigonometric series

2.1 Periodic functions and motivation

As motivation for studying Fourier series, suppose we have the problem

$$x'' + \omega_0^2 x = f(t), \tag{6}$$

for some periodic function $f(t)$. We already solved

$$x'' + \omega_0^2 x = F_0 \cos(\omega t). \tag{7}$$

One way to solve (6) is to decompose $f(t)$ as a sum of cosines (and sines) and then solve many problems of the form (7). We then use the principle of superposition, to sum up all the solutions we got to get a solution to (6).

Before we proceed, let us talk a little bit more in detail about periodic functions. A function is said to be *periodic* with period P if $f(t) = f(t + P)$ for all t. For brevity we say $f(t)$ is P-periodic. Note that a P-periodic function is also $2P$-periodic, $3P$-periodic and so on. For example, $\cos(t)$ and $\sin(t)$ are 2π-periodic. So are $\cos(kt)$ and $\sin(kt)$ for all integers k. The constant functions are an extreme example. They are periodic for any period (exercise).

Normally we start with a function $f(t)$ defined on some interval $[-L, L]$, and we want to *extend* $f(t)$ *periodically* to make it a $2L$-periodic function. We do this extension by defining a new function $F(t)$ such that for t in $[-L, L]$, $F(t) = f(t)$. For t in $[L, 3L]$, we define $F(t) = f(t - 2L)$, for t in $[-3L, -L]$, $F(t) = f(t + 2L)$, and so on. To make that work we needed $f(-L) = f(L)$. We could have also started with f defined only on the half-open interval $(-L, L]$ and then define $f(-L) = f(L)$.

Example 2.1: Define $f(t) = 1 - t^2$ on $[-1, 1]$. Now extend $f(t)$ periodically to a 2-periodic function. See Figure 1.

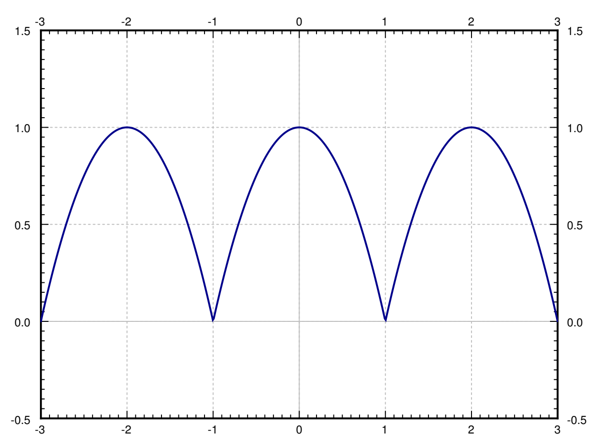


Figure 1: Periodic extension of the function $1 - t^2$.

You should be careful to distinguish between $f(t)$ and its extension. A common mistake is to assume that a formula for $f(t)$ holds for its extension. It can be confusing when the formula for $f(t)$ is periodic, but with perhaps a different period.

Exercise 2.1: *Define* $f(t) = \cos t$ *on* $[-\pi/2, \pi/2]$. *Take the* π*-periodic extension and sketch its graph. How does it compare to the graph of* $\cos t$?

2.2 Inner product and eigenvector decomposition

Suppose we have a *symmetric matrix*, that is $A^T = A$. As we remarked before, eigenvectors of A are then orthogonal. Here the word *orthogonal* means that if $\vec{v}$ and $\vec{w}$ are two eigenvectors of A for distinct eigenvalues, then $\langle \vec{v}, \vec{w} \rangle = 0$. In this case the inner product $\langle \vec{v}, \vec{w} \rangle$ is the *dot product*, which can be computed as $\vec{v}^T \vec{w}$.

To decompose a vector $\vec{v}$ in terms of mutually orthogonal vectors $\vec{w}_1$ and $\vec{w}_2$ we write

$$\vec{v} = a_1 \vec{w}_1 + a_2 \vec{w}_2.$$

Let us find the formula for a_1 and a_2. First let us compute

$$\langle \vec{v}, \vec{w}_1 \rangle = \langle a_1 \vec{w}_1 + a_2 \vec{w}_2, \vec{w}_1 \rangle = a_1 \langle \vec{w}_1, \vec{w}_1 \rangle + a_2 \underbrace{\langle \vec{w}_2, \vec{w}_1 \rangle}_{=0} = a_1 \langle \vec{w}_1, \vec{w}_1 \rangle.$$

Therefore,

$$a_1 = \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}.$$

Similarly

$$a_2 = \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}.$$

You probably remember this formula from vector calculus.

Example 2.2: Write $\vec{v} = \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right]$ as a linear combination of $\vec{w}_1 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right]$ and $\vec{w}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right]$.

First note that $\vec{w}_1$ and $\vec{w}_2$ are orthogonal as $\langle \vec{w}_1, \vec{w}_2 \rangle = 1(1) + (-1)1 = 0$. Then

$$\begin{aligned} a_1 &= \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} = \frac{2(1) + 3(-1)}{1(1) + (-1)(-1)} = \frac{-1}{2}, \\ a_2 &= \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} = \frac{2+3}{1+1} = \frac{5}{2}. \end{aligned}$$

Hence

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2.3 The trigonometric series

Instead of decomposing a vector in terms of eigenvectors of a matrix, we decompose a function in terms of eigenfunctions of a certain eigenvalue problem. The eigenvalue problem we use for the Fourier series is

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi).$$

We computed that eigenfunctions are 1, $\cos(kt)$, $\sin(kt)$. That is, we want to find a representation of a 2π-periodic function $f(t)$ as

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).}$$

This series is called the *Fourier series** or the *trigonometric series* for $f(t)$. We write the coefficient of the eigenfunction 1 as $\frac{a_0}{2}$ for convenience. We could also think of $1 = \cos(0t)$, so that we only need to look at $\cos(kt)$ and $\sin(kt)$.

*Named after the French mathematician Jean Baptiste Joseph Fourier (1768–1830).

As for matrices we want to find a *projection* of $f(t)$ onto the subspaces given by the eigenfunctions. So we want to define an *inner product of functions*. For example, to find a_n we want to compute $\langle f(t), \cos(nt) \rangle$. We define the inner product as

$$\langle f(t), g(t) \rangle \overset{\text{def}}{=} \int_{-\pi}^{\pi} f(t)\, g(t)\, dt.$$

With this definition of the inner product, we saw in the previous section that the eigenfunctions $\cos(kt)$ (including the constant eigenfunction), and $\sin(kt)$ are *orthogonal* in the sense that

$$\begin{aligned} \langle \cos(mt), \cos(nt) \rangle &= 0 \qquad \text{for } m \neq n, \\ \langle \sin(mt), \sin(nt) \rangle &= 0 \qquad \text{for } m \neq n, \\ \langle \sin(mt), \cos(nt) \rangle &= 0 \qquad \text{for all } m \text{ and } n. \end{aligned}$$

For $n = 1, 2, 3, \ldots$ we have

$$\begin{aligned} \langle \cos(nt), \cos(nt) \rangle &= \int_{-\pi}^{\pi} \cos(nt) \cos(nt)\, dt = \pi, \\ \langle \sin(nt), \sin(nt) \rangle &= \int_{-\pi}^{\pi} \sin(nt) \sin(nt)\, dt = \pi, \end{aligned}$$

by elementary calculus. For the constant we get

$$\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \cdot 1\, dt = 2\pi.$$

The coefficients are given by

$$\boxed{\begin{aligned} a_n &= \frac{\langle f(t), \cos(nt) \rangle}{\langle \cos(nt), \cos(nt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \\ b_n &= \frac{\langle f(t), \sin(nt) \rangle}{\langle \sin(nt), \sin(nt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt. \end{aligned}}$$

Compare these expressions with the finite-dimensional example. For a_0 we get a similar formula

$$\boxed{a_0 = 2\frac{\langle f(t), 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt.}$$

Let us check the formulas using the orthogonality properties. Suppose for a moment that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Then for $m \geq 1$ we have

$$\begin{aligned} \langle f(t), \cos(mt) \rangle &= \left\langle \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt), \cos(mt) \right\rangle \\ &= \frac{a_0}{2} \langle 1, \cos(mt) \rangle + \sum_{n=1}^{\infty} a_n \langle \cos(nt), \cos(mt) \rangle + b_n \langle \sin(nt), \cos(mt) \rangle \\ &= a_m \langle \cos(mt), \cos(mt) \rangle. \end{aligned}$$

And hence $a_m = \frac{\langle f(t), \cos(mt) \rangle}{\langle \cos(mt), \cos(mt) \rangle}$.

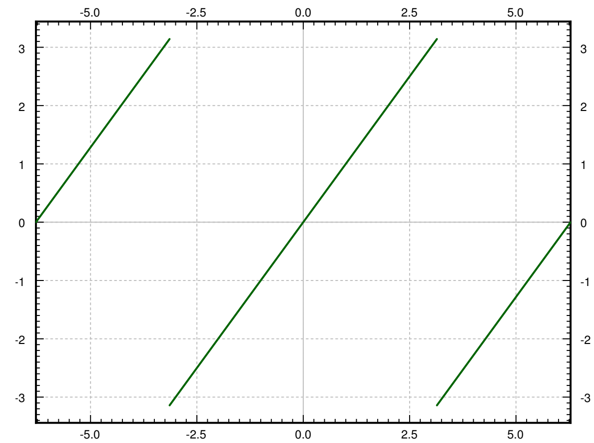


Figure 2: The graph of the sawtooth function.

***Exercise* 2.2:** *Carry out the calculation for a_0 and b_m.*

Example 2.3: Take the function

$$f(t) = t$$

for t in $(-\pi, \pi]$. Extend $f(t)$ periodically and write it as a Fourier series. This function is called the *sawtooth*.

The plot of the extended periodic function is given in Figure 2. Let us compute the coefficients. We start with a_0,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t\, dt = 0.$$

We will often use the result from calculus that says that the integral of an odd function over a symmetric interval is zero. Recall that an *odd function* is a function $\varphi(t)$ such that $\varphi(-t) = -\varphi(t)$. For example the functions t, $\sin t$, or (importantly for us) $t \cos(nt)$ are all odd functions. Thus

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos(nt)\, dt = 0.$$

Let us move to b_n. Another useful fact from calculus is that the integral of an even function over a symmetric interval is twice the integral of the same function over half the interval. Recall an *even function* is a function $\varphi(t)$ such that $\varphi(-t) = \varphi(t)$. For example $t \sin(nt)$ is even.

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin(nt)\, dt \\ &= \frac{2}{\pi} \int_{0}^{\pi} t \sin(nt)\, dt \\ &= \frac{2}{\pi} \left(\left[\frac{-t \cos(nt)}{n} \right]_{t=0}^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nt)\, dt \right) \\ &= \frac{2}{\pi} \left(\frac{-\pi \cos(n\pi)}{n} + 0 \right) \\ &= \frac{-2 \cos(n\pi)}{n} = \frac{2\,(-1)^{n+1}}{n}. \end{aligned}$$

We have used the fact that

$$\cos(n\pi) = (-1)^n = \begin{cases} 1 & \text{if } n \text{ even,} \\ -1 & \text{if } n \text{ odd.} \end{cases}$$

The series, therefore, is

$$\sum_{n=1}^{\infty} \frac{2\,(-1)^{n+1}}{n} \,\sin(nt).$$

Let us write out the first 3 harmonics of the series for $f(t)$.

$$2\,\sin(t) - \sin(2t) + \frac{2}{3}\sin(3t) + \cdots$$

The plot of these first three terms of the series, along with a plot of the first 20 terms is given in Figure 3.

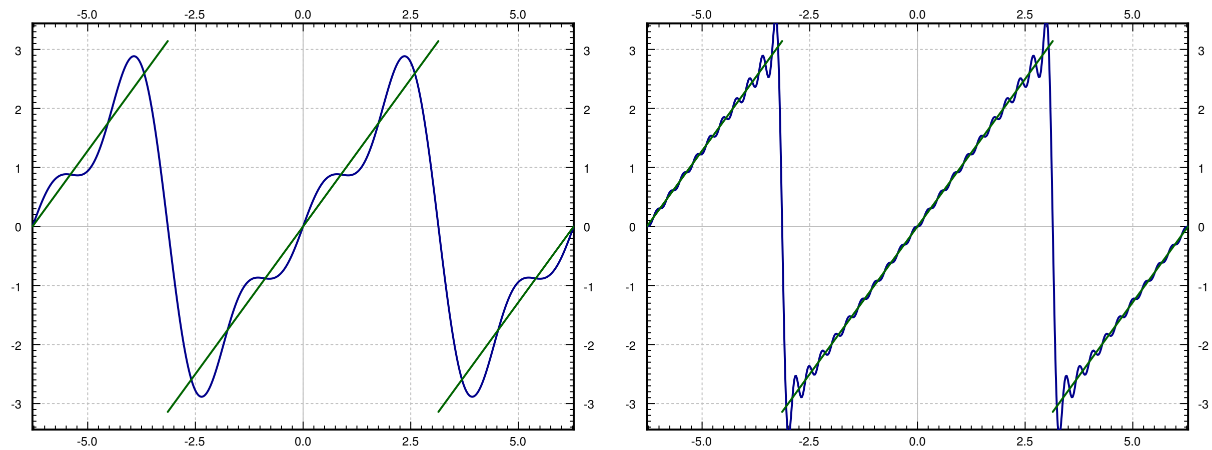


Figure 3: First 3 (left graph) and 20 (right graph) harmonics of the sawtooth function.

Example 2.4: Take the function

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \leq 0, \\ \pi & \text{if } \quad 0 < t \leq \pi. \end{cases}$$

Extend $f(t)$ periodically and write it as a Fourier series. This function or its variants appear often in applications and the function is called the *square wave*.

The plot of the extended periodic function is given in Figure 4 on the next page. Now we compute the coefficients. We start with a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} \pi\,dt = \pi.$$

Next,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_0^{\pi} \pi\cos(nt)\,dt = 0.$$

And finally,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \\ &= \frac{1}{\pi}\int_0^{\pi} \pi\sin(nt)\,dt \\ &= \left[\frac{-\cos(nt)}{n}\right]_{t=0}^{\pi} \\ &= \frac{1-\cos(\pi n)}{n} = \frac{1-(-1)^n}{n} = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

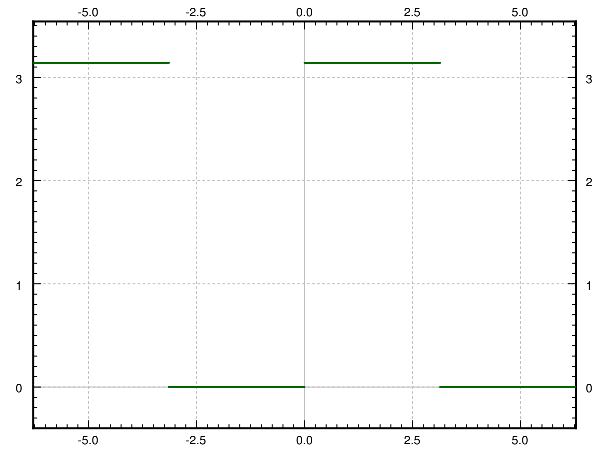


Figure 4: The graph of the square wave function.

The Fourier series is

$$\frac{\pi}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n} \sin(nt) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin\big((2k-1)\,t\big).$$

Let us write out the first 3 harmonics of the series for $f(t)$:

$$\frac{\pi}{2} + 2\,\sin(t) + \frac{2}{3}\sin(3t) + \cdots$$

The plot of these first three and also of the first 20 terms of the series is given in Figure 5.

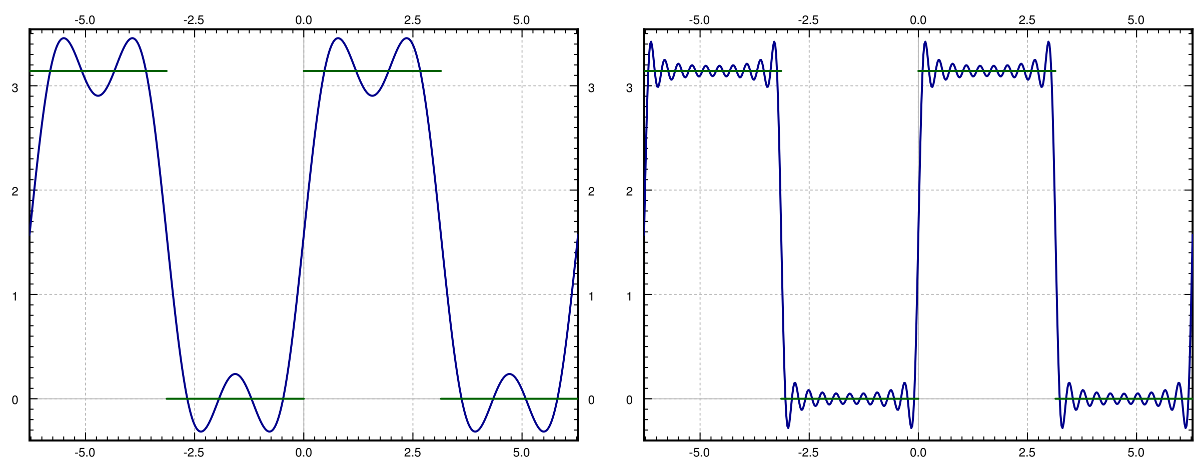


Figure 5: First 3 (left graph) and 20 (right graph) harmonics of the square wave function.

We have so far skirted the issue of convergence. For example, if $f(t)$ is the square wave function, the equation

$$f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin\big((2k-1)\,t\big).$$

is only an equality for such t where $f(t)$ is continuous. We do not get an equality for $t = -\pi, 0, \pi$ and all the other discontinuities of $f(t)$. It is not hard to see that when t is an integer multiple of π (which gives all the discontinuities), then

$$\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin\big((2k-1)\,t\big) = \frac{\pi}{2}.$$

We redefine $f(t)$ on $[-\pi, \pi]$ as

$$f(t) = \begin{cases} 0 & \text{if } \ -\pi < t < 0, \\ \pi & \text{if } \ \ 0 < t < \pi, \\ \pi/2 & \text{if } \ \ t = -\pi, t = 0, \text{ or } t = \pi, \end{cases}$$

and extend periodically. The series equals this new extended $f(t)$ everywhere, including the discontinuities. We will generally not worry about changing the function values at several (finitely many) points.

We will say more about convergence in the next section. Let us, however, briefly mention an effect of the discontinuity. Zoom in near the discontinuity in the square wave. Further, plot the first 100 harmonics, see Figure 6. While the series is a very good approximation away from the discontinuities, the error (the overshoot) near the discontinuity at $t = \pi$ does not seem to be getting any smaller as we take more and more harmonics. This behavior is known as the *Gibbs phenomenon*. The region where the error is large does get smaller, however, the more terms in the series we take.

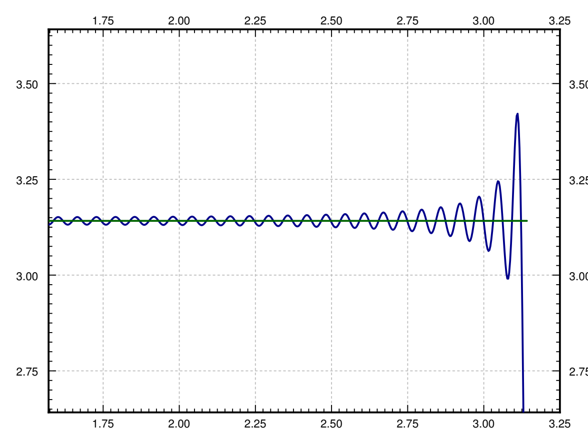


Figure 6: Gibbs phenomenon in action.

We can think of a periodic function as a "signal" being a superposition of many signals of pure frequency. For example, we could think of the square wave as a tone of certain base frequency. This base frequency is called the *fundamental frequency*. The square wave will be a superposition of many different pure tones of frequencies that are multiples of the fundamental frequency. In music, the higher frequencies are called the *overtones*. All the frequencies that appear are called the *spectrum* of the signal. On the other hand a simple sine wave is only the pure tone (no overtones). The simplest way to make sound using a computer is the square wave, and the sound is very different from a pure tone. If you ever played video games from the 1980s or so, then you heard what square waves sound like.

2.4 Exercises

***Exercise* 2.3:** *Suppose $f(t)$ is defined on $[-\pi, \pi]$ as $\sin(5t) + \cos(3t)$. Extend periodically and compute the Fourier series of $f(t)$.*

***Exercise* 2.4:** *Suppose $f(t)$ is defined on $[-\pi,\pi]$ as $|t|$. Extend periodically and compute the Fourier series of $f(t)$.*

***Exercise* 2.5:** *Suppose $f(t)$ is defined on $[-\pi,\pi]$ as $|t|^3$. Extend periodically and compute the Fourier series of $f(t)$.*

***Exercise* 2.6:** *Suppose $f(t)$ is defined on $(-\pi,\pi]$ as*

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t \leq 0, \\ 1 & \text{if } \quad 0 < t \leq \pi. \end{cases}$$

Extend periodically and compute the Fourier series of $f(t)$.

***Exercise* 2.7:** *Suppose $f(t)$ is defined on $(-\pi,\pi]$ as t^3. Extend periodically and compute the Fourier series of $f(t)$.*

***Exercise* 2.8:** *Suppose $f(t)$ is defined on $[-\pi,\pi]$ as t^2. Extend periodically and compute the Fourier series of $f(t)$.*

There is another form of the Fourier series using complex exponentials e^{nt} for $n = \ldots,-2,-1,0,1,2,\ldots$ instead of $\cos(nt)$ and $\sin(nt)$ for positive n. This form may be easier to work with sometimes. It is certainly more compact to write, and there is only one formula for the coefficients. On the downside, the coefficients are complex numbers.

***Exercise* 2.9:** *Let*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to show that there exist complex numbers c_m such that

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{imt}.$$

Note that the sum now ranges over all the integers including negative ones. Do not worry about convergence in this calculation. Hint: It may be better to start from the complex exponential form and write the series as

$$c_0 + \sum_{m=1}^{\infty} \left(c_m e^{imt} + c_{-m} e^{-imt} \right).$$

***Exercise* 2.101:** *Suppose $f(t)$ is defined on $[-\pi,\pi]$ as $f(t) = \sin(t)$. Extend periodically and compute the Fourier series.*

***Exercise* 2.102:** *Suppose $f(t)$ is defined on $(-\pi,\pi]$ as $f(t) = \sin(\pi t)$. Extend periodically and compute the Fourier series.*

***Exercise* 2.103:** *Suppose $f(t)$ is defined on $(-\pi,\pi]$ as $f(t) = \sin^2(t)$. Extend periodically and compute the Fourier series.*

***Exercise* 2.104:** *Suppose $f(t)$ is defined on $(-\pi,\pi]$ as $f(t) = t^4$. Extend periodically and compute the Fourier series.*

3 More on the Fourier series

3.1 $2L$-periodic functions

We have computed the Fourier series for a 2π-periodic function, but what about functions of different periods. Well, fear not, the computation is a simple case of change of variables. We just rescale the independent axis. Suppose we have a $2L$-periodic function $f(t)$. Then L is called the *half period*. Let $s = \frac{\pi}{L} t$. Then the function

$$g(s) = f\left(\frac{L}{\pi} s\right)$$

is 2π-periodic. We must also rescale all our sines and cosines. In the series we use $\frac{\pi}{L} t$ as the variable. That is, we want to write

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right).}$$

If we change variables to s, we see that

$$g(s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(ns) + b_n \sin(ns).$$

We compute a_n and b_n as before. After we write down the integrals, we change variables from s back to t, noting also that $ds = \frac{\pi}{L}\, dt$.

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(s)\, ds = \frac{1}{L} \int_{-L}^{L} f(t)\, dt, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(s) \cos(ns)\, ds = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L} t\right) dt, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(s) \sin(ns)\, ds = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi}{L} t\right) dt. \end{aligned}}$$

The two most common half periods that show up in examples are π and 1 because of the simplicity of the formulas. We should stress that we have done no new mathematics, we have only changed variables. If you understand the Fourier series for 2π-periodic functions, you understand it for $2L$-periodic functions. You can think of it as just using different units for time. All that we are doing is moving some constants around, but all the mathematics is the same.

Example 3.1: Let

$$f(t) = |t| \qquad \text{for } -1 < t \leq 1,$$

extended periodically. The plot of the periodic extension is given in Figure 7 on the following page. Compute the Fourier series of $f(t)$.

We want to write $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$. For $n \geq 1$ we note that $|t| \cos(n\pi t)$ is even and hence

$$\begin{aligned} a_n &= \int_{-1}^{1} f(t) \cos(n\pi t)\, dt \\ &= 2 \int_{0}^{1} t \cos(n\pi t)\, dt \\ &= 2 \left[\frac{t}{n\pi} \sin(n\pi t) \right]_{t=0}^{1} - 2 \int_{0}^{1} \frac{1}{n\pi} \sin(n\pi t)\, dt \\ &= 0 + \frac{1}{n^2\pi^2} \Big[\cos(n\pi t) \Big]_{t=0}^{1} = \frac{2\big((-1)^n - 1\big)}{n^2\pi^2} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{-4}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

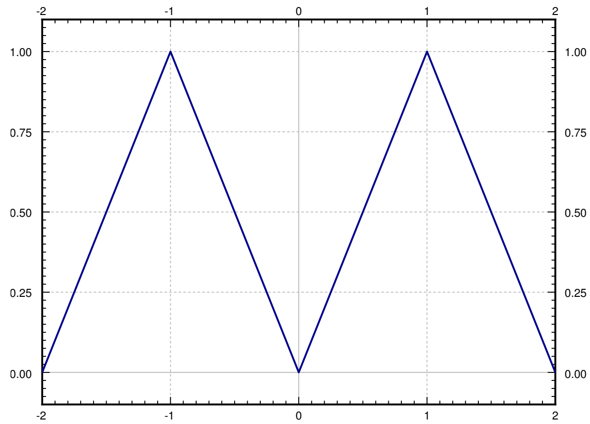


Figure 7: Periodic extension of the function $f(t)$.

Next we find a_0:

$$a_0 = \int_{-1}^{1} |t| \, dt = 1.$$

You should be able to find this integral by thinking about the integral as the area under the graph without doing any computation at all. Finally we can find b_n. Here, we notice that $|t| \sin(n\pi t)$ is odd and, therefore,

$$b_n = \int_{-1}^{1} f(t) \sin(n\pi t) \, dt = 0.$$

Hence, the series is

$$\frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^2} \cos(n\pi t).$$

Let us explicitly write down the first few terms of the series up to the 3$^{\text{rd}}$ harmonic.

$$\frac{1}{2} - \frac{4}{\pi^2} \cos(\pi t) - \frac{4}{9\pi^2} \cos(3\pi t) - \cdots$$

The plot of these few terms and also a plot up to the 20$^{\text{th}}$ harmonic is given in Figure 8 on the next page. You should notice how close the graph is to the real function. You should also notice that there is no "Gibbs phenomenon" present as there are no discontinuities.

3.2 Convergence

We will need the one sided limits of functions. We will use the following notation

$$f(c-) = \lim_{t \uparrow c} f(t), \qquad \text{and} \qquad f(c+) = \lim_{t \downarrow c} f(t).$$

If you are unfamiliar with this notation, $\lim_{t \uparrow c} f(t)$ means we are taking a limit of $f(t)$ as t approaches c from below (i.e. $t < c$) and $\lim_{t \downarrow c} f(t)$ means we are taking a limit of $f(t)$ as t approaches c from above (i.e. $t > c$). For example, for the square wave function

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \leq 0, \\ \pi & \text{if } \quad 0 < t \leq \pi, \end{cases} \tag{8}$$

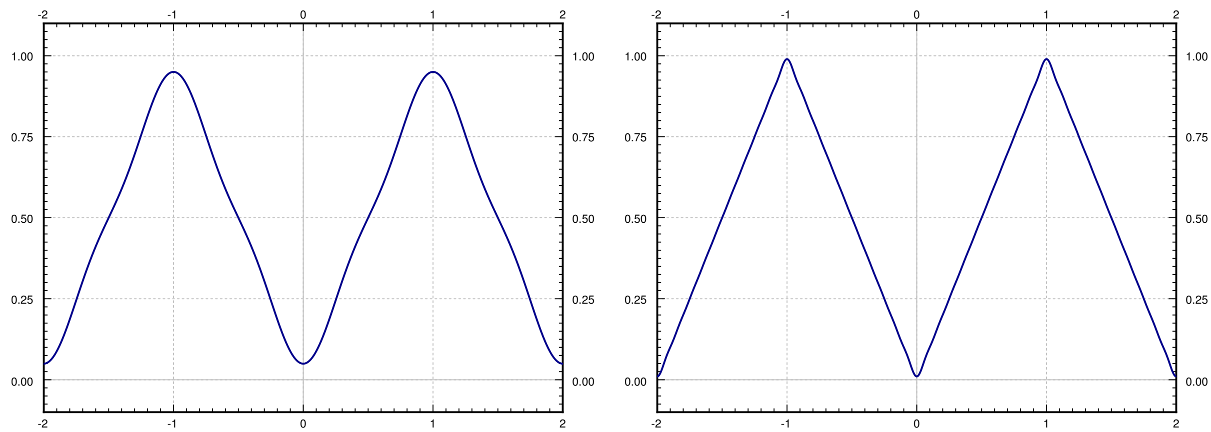


Figure 8: Fourier series of $f(t)$ up to the 3^{rd} harmonic (left graph) and up to the 20^{th} harmonic (right graph).

we have $f(0-) = 0$ and $f(0+) = \pi$.

Let $f(t)$ be a function defined on an interval $[a, b]$. Suppose that we find finitely many points $a = t_0, t_1, t_2, \ldots, t_k = b$ in the interval, such that $f(t)$ is continuous on the intervals $(t_0, t_1), (t_1, t_2), \ldots, (t_{k-1}, t_k)$. Also suppose that all the one sided limits exist, that is, all of $f(t_0+), f(t_1-), f(t_1+), f(t_2-), f(t_2+), \ldots, f(t_k-)$ exist and are finite. Then we say $f(t)$ is *piecewise continuous*.

If moreover, $f(t)$ is differentiable at all but finitely many points, and $f'(t)$ is piecewise continuous, then $f(t)$ is said to be *piecewise smooth*.

Example 3.2: The square wave function (8) is piecewise smooth on $[-\pi, \pi]$ or any other interval. In such a case we simply say that the function is piecewise smooth.

Example 3.3: The function $f(t) = |t|$ is piecewise smooth.

Example 3.4: The function $f(t) = \frac{1}{t}$ is not piecewise smooth on $[-1, 1]$ (or any other interval containing zero). In fact, it is not even piecewise continuous.

Example 3.5: The function $f(t) = \sqrt[3]{t}$ is not piecewise smooth on $[-1, 1]$ (or any other interval containing zero). $f(t)$ is continuous, but the derivative of $f(t)$ is unbounded near zero and hence not piecewise continuous.

Piecewise smooth functions have an easy answer on the convergence of the Fourier series.

Theorem 3.1. *Suppose $f(t)$ is a $2L$-periodic piecewise smooth function. Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

be the Fourier series for $f(t)$. Then the series converges for all t. If $f(t)$ is continuous at t, then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right).$$

Otherwise,

$$\frac{f(t-) + f(t+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right).$$

If we happen to have that $f(t) = \frac{f(t-)+f(t+)}{2}$ at all the discontinuities, the Fourier series converges to $f(t)$ everywhere. We can always just redefine $f(t)$ by changing the value at each discontinuity appropriately. Then we can write an equals sign between $f(t)$ and the series without any worry. We mentioned this fact briefly at the end last section.

The theorem does not say how fast the series converges. Think back to the discussion of the Gibbs phenomenon in the last section. The closer you get to the discontinuity, the more terms you need to take to get an accurate approximation to the function.

3.3 Differentiation and integration of Fourier series

Not only does Fourier series converge nicely, but it is easy to differentiate and integrate the series. We can do this just by differentiating or integrating term by term.

Theorem 3.2. *Suppose*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

is a piecewise smooth continuous function and the derivative $f'(t)$ is piecewise smooth. Then the derivative can be obtained by differentiating term by term,

$$f'(t) = \sum_{n=1}^{\infty} \frac{-a_n n\pi}{L} \sin\left(\frac{n\pi}{L}t\right) + \frac{b_n n\pi}{L} \cos\left(\frac{n\pi}{L}t\right).$$

It is important that the function is continuous. It can have corners, but no jumps. Otherwise, the differentiated series will fail to converge. For an exercise, take the series obtained for the square wave and try to differentiate the series. Similarly, we can also integrate a Fourier series.

Theorem 3.3. *Suppose*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)$$

is a piecewise smooth function. Then the antiderivative is obtained by antidifferentiating term by term and so

$$F(t) = \frac{a_0 t}{2} + C + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\left(\frac{n\pi}{L}t\right) + \frac{-b_n L}{n\pi} \cos\left(\frac{n\pi}{L}t\right),$$

where $F'(t) = f(t)$ and C is an arbitrary constant.

Note that the series for $F(t)$ is no longer a Fourier series as it contains the $\frac{a_0 t}{2}$ term. The antiderivative of a periodic function need no longer be periodic and so we should not expect a Fourier series.

3.4 Rates of convergence and smoothness

Let us do an example of a periodic function with one derivative everywhere.

Example 3.6: Take the function

$$f(t) = \begin{cases} (t+1)\,t & \text{if } -1 < t \leq 0, \\ (1-t)\,t & \text{if } \quad 0 < t \leq 1, \end{cases}$$

and extend to a 2-periodic function. The plot is given in Figure 9 on the following page.

This function has one derivative everywhere, but it does not have a second derivative whenever t is an integer.

***Exercise* 3.1:** *Compute $f''(0+)$ and $f''(0-)$.*

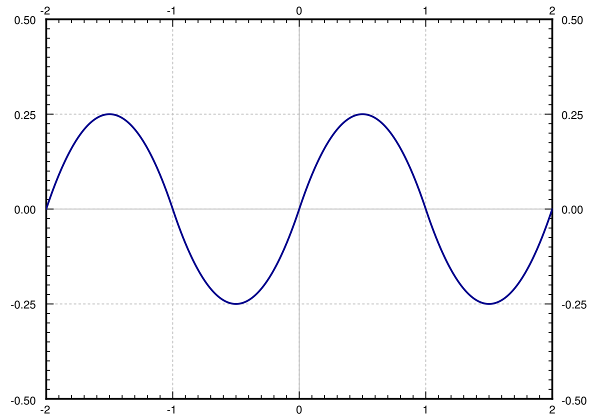


Figure 9: Smooth 2-periodic function.

Let us compute the Fourier series coefficients. The actual computation involves several integration by parts and is left to student.

$$\begin{aligned}
a_0 &= \int_{-1}^{1} f(t)\,dt = \int_{-1}^{0} (t+1)\,t\,dt + \int_{0}^{1} (1-t)\,t\,dt = 0, \\
a_n &= \int_{-1}^{1} f(t)\,\cos(n\pi t)\,dt = \int_{-1}^{0} (t+1)\,t\,\cos(n\pi t)\,dt + \int_{0}^{1} (1-t)\,t\,\cos(n\pi t)\,dt = 0, \\
b_n &= \int_{-1}^{1} f(t)\,\sin(n\pi t)\,dt = \int_{-1}^{0} (t+1)\,t\,\sin(n\pi t)\,dt + \int_{0}^{1} (1-t)\,t\,\sin(n\pi t)\,dt \\
&= \frac{4(1-(-1)^n)}{\pi^3 n^3} = \begin{cases} \frac{8}{\pi^3 n^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

That is, the series is

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{\pi^3 n^3} \sin(n\pi t).$$

This series converges very fast. If you plot up to the third harmonic, that is the function

$$\frac{8}{\pi^3} \sin(\pi t) + \frac{8}{27\pi^3} \sin(3\pi t),$$

it is almost indistinguishable from the plot of $f(t)$ in Figure 9. In fact, the coefficient $\frac{8}{27\pi^3}$ is already just 0.0096 (approximately). The reason for this behavior is the n^3 term in the denominator. The coefficients b_n in this case go to zero as fast as $1/n^3$ goes to zero.

For functions constructed piecewise from polynomials as above, it is generally true that if you have one derivative, the Fourier coefficients will go to zero approximately like $1/n^3$. If you have only a continuous function, then the Fourier coefficients will go to zero as $1/n^2$. If you have discontinuities, then the Fourier coefficients will go to zero approximately as $1/n$. For more general functions the story is somewhat more complicated but the same idea holds, the more derivatives you have, the faster the coefficients go to zero. Similar reasoning works in reverse. If the coefficients go to zero like $1/n^2$, you always obtain a continuous function. If they go to zero like $1/n^3$, you obtain an everywhere differentiable function.

To justify this behavior, take for example the function defined by the Fourier series

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nt).$$

When we differentiate term by term we notice

$$f'(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nt).$$

Therefore, the coefficients now go down like $1/n^2$, which means that we have a continuous function. The derivative of $f'(t)$ is defined at most points, but there are points where $f'(t)$ is not differentiable. It has corners, but no jumps. If we differentiate again (where we can), we find that the function $f''(t)$, now fails to be continuous (has jumps)

$$f''(t) = \sum_{n=1}^{\infty} \frac{-1}{n} \sin(nt).$$

This function is similar to the sawtooth. If we tried to differentiate the series again, we would obtain

$$\sum_{n=1}^{\infty} - \cos(nt),$$

which does not converge!

***Exercise* 3.2:** *Use a computer to plot the series we obtained for $f(t)$, $f'(t)$ and $f''(t)$. That is, plot say the first 5 harmonics of the functions. At what points does $f''(t)$ have the discontinuities?*

3.5 Exercises

***Exercise* 3.3:** *Let*

$$f(t) = \begin{cases} 0 & \text{if } -1 < t \leq 0, \\ t & \text{if } \quad 0 < t \leq 1, \end{cases}$$

extended periodically.

a) *Compute the Fourier series for $f(t)$.*

b) *Write out the series explicitly up to the 3^{rd} harmonic.*

***Exercise* 3.4:** *Let*

$$f(t) = \begin{cases} -t & \text{if } -1 < t \leq 0, \\ t^2 & \text{if } \quad 0 < t \leq 1, \end{cases}$$

extended periodically.

a) *Compute the Fourier series for $f(t)$.*

b) *Write out the series explicitly up to the 3^{rd} harmonic.*

***Exercise* 3.5:** *Let*

$$f(t) = \begin{cases} \frac{-t}{10} & \text{if } -10 < t \leq 0, \\ \frac{t}{10} & \text{if } \quad 0 < t \leq 10, \end{cases}$$

extended periodically (period is 20).

a) *Compute the Fourier series for* $f(t)$.

b) *Write out the series explicitly up to the* 3^{rd} *harmonic.*

Exercise* 3.6*: *Let* $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nt)$. *Is* $f(t)$ *continuous and differentiable everywhere? Find the derivative (if it exists everywhere) or justify why* $f(t)$ *is not differentiable everywhere.*

Exercise* 3.7*: *Let* $f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$. *Is* $f(t)$ *differentiable everywhere? Find the derivative (if it exists everywhere) or justify why* $f(t)$ *is not differentiable everywhere.*

Exercise* 3.8*: *Let*

$$f(t) = \begin{cases} 0 & \text{if } -2 < t \le 0, \\ t & \text{if } \quad 0 < t \le 1, \\ -t+2 & \text{if } \quad 1 < t \le 2, \end{cases}$$

extended periodically.

a) *Compute the Fourier series for* $f(t)$.

b) *Write out the series explicitly up to the* 3^{rd} *harmonic.*

Exercise* 3.9*: *Let*

$$f(t) = e^t \qquad \text{for } -1 < t \le 1$$

extended periodically.

a) *Compute the Fourier series for* $f(t)$.

b) *Write out the series explicitly up to the* 3^{rd} *harmonic.*

c) *What does the series converge to at* $t = 1$.

Exercise* 3.10*: *Let*

$$f(t) = t^2 \qquad \text{for } -1 < t \le 1$$

extended periodically.

a) *Compute the Fourier series for* $f(t)$.

b) *By plugging in* $t = 0$, *evaluate* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$.

c) *Now evaluate* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$.

Exercise* 3.11*: *Let*

$$f(t) = \begin{cases} 0 & \text{if } -3 < t \le 0, \\ t & \text{if } \quad 0 < t \le 3, \end{cases}$$

extended periodically. Suppose $F(t)$ *is the function given by the Fourier series of* f. *Without computing the Fourier series evaluate*

a) $F(2)$ *b*) $F(-2)$ *c*) $F(4)$

d) $F(-4)$ *e*) $F(3)$ *f*) $F(-9)$

Exercise* 3.101*: *Let*

$$f(t) = t^2 \qquad \textit{for } -2 < t \leq 2$$

extended periodically.

a) *Compute the Fourier series for* $f(t)$.

b) *Write out the series explicitly up to the* 3^{rd} *harmonic.*

Exercise* 3.102*: *Let*

$$f(t) = t \qquad \textit{for } -\lambda < t \leq \lambda \quad (\textit{for some } \lambda > 0)$$

extended periodically.

a) *Compute the Fourier series for* $f(t)$.

b) *Write out the series explicitly up to the* 3^{rd} *harmonic.*

Exercise* 3.103*: *Let*

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n(n^2+1)} \sin(n\pi t).$$

Compute $f'(t)$.

Exercise* 3.104*: *Let*

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nt).$$

a) *Find the antiderivative.*

b) *Is the antiderivative periodic*?

Exercise* 3.105*: *Let*

$$f(t) = {}^{t}\!/_{2} \qquad \textit{for } -\pi < t < \pi$$

extended periodically.

a) *Compute the Fourier series for* $f(t)$.

b) *Plug in* $t = {}^{\pi}\!/_{2}$ *to find a series representation for* ${}^{\pi}\!/_{4}$.

c) *Using the first* 4 *terms of the result from part* b) *approximate* ${}^{\pi}\!/_{4}$.

Exercise* 3.106*: *Let*

$$f(t) = \begin{cases} 0 & \textit{if } -2 < t \leq 0, \\ 2 & \textit{if } \quad 0 < t \leq 2, \end{cases}$$

extended periodically. Suppose $F(t)$ *is the function given by the Fourier series of* f. *Without computing the Fourier series evaluate*

a) $F(0)$ b) $F(-1)$ c) $F(1)$

d) $F(-2)$ e) $F(4)$ f) $F(-8)$

4 Sine and cosine series

4.1 Odd and even periodic functions

You may have noticed by now that an odd function has no cosine terms in the Fourier series and an even function has no sine terms in the Fourier series. This observation is not a coincidence. Let us look at even and odd periodic function in more detail.

Recall that a function $f(t)$ is *odd* if $f(-t) = -f(t)$. A function $f(t)$ is *even* if $f(-t) = f(t)$. For example, $\cos(nt)$ is even and $\sin(nt)$ is odd. Similarly the function t^k is even if k is even and odd when k is odd.

***Exercise* 4.1:** *Take two functions $f(t)$ and $g(t)$ and define their product $h(t) = f(t)g(t)$.*

a) *Suppose both $f(t)$ and $g(t)$ are odd. Is $h(t)$ odd or even?*

b) *Suppose one is even and one is odd. Is $h(t)$ odd or even?*

c) *Suppose both are even. Is $h(t)$ odd or even?*

If $f(t)$ and $g(t)$ are both odd, then $f(t) + g(t)$ is odd. Similarly for even functions. On the other hand, if $f(t)$ is odd and $g(t)$ even, then we cannot say anything about the sum $f(t) + g(t)$. In fact, the Fourier series of any function is a sum of an odd (the sine terms) and an even (the cosine terms) function.

In this section we consider odd and even periodic functions. We have previously defined the $2L$-periodic extension of a function defined on the interval $[-L, L]$. Sometimes we are only interested in the function on the range $[0, L]$ and it would be convenient to have an odd (resp. even) function. If the function is odd (resp. even), all the cosine (resp. sine) terms disappear. What we will do is take the odd (resp. even) extension of the function to $[-L, L]$ and then extend periodically to a $2L$-periodic function.

Take a function $f(t)$ defined on $[0, L]$. On $(-L, L]$ define the functions

$$F_{\text{odd}}(t) \overset{\text{def}}{=} \begin{cases} f(t) & \text{if} \quad 0 \le t \le L, \\ -f(-t) & \text{if} \ -L < t < 0, \end{cases}$$

$$F_{\text{even}}(t) \overset{\text{def}}{=} \begin{cases} f(t) & \text{if} \quad 0 \le t \le L, \\ f(-t) & \text{if} \ -L < t < 0. \end{cases}$$

Extend $F_{\text{odd}}(t)$ and $F_{\text{even}}(t)$ to be $2L$-periodic. Then $F_{\text{odd}}(t)$ is called the *odd periodic extension* of $f(t)$, and $F_{\text{even}}(t)$ is called the *even periodic extension* of $f(t)$. For the odd extension we generally assume that $f(0) = f(L) = 0$.

***Exercise* 4.2:** *Check that $F_{odd}(t)$ is odd and $F_{even}(t)$ is even. For F_{odd}, assume $f(0) = f(L) = 0$.*

Example 4.1: Take the function $f(t) = t\,(1-t)$ defined on $[0, 1]$. Figure 10 on the following page shows the plots of the odd and even periodic extensions of $f(t)$.

4.2 Sine and cosine series

Let $f(t)$ be an odd $2L$-periodic function. We write the Fourier series for $f(t)$. First, we compute the coefficients a_n (including $n = 0$) and get

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt = 0.$$

That is, there are no cosine terms in the Fourier series of an odd function. The integral is zero because $f(t)\cos(n\pi Lt)$ is an odd function (product of an odd and an even function is odd) and the integral of an odd function over a symmetric interval is always zero. The integral of an even function over a symmetric interval

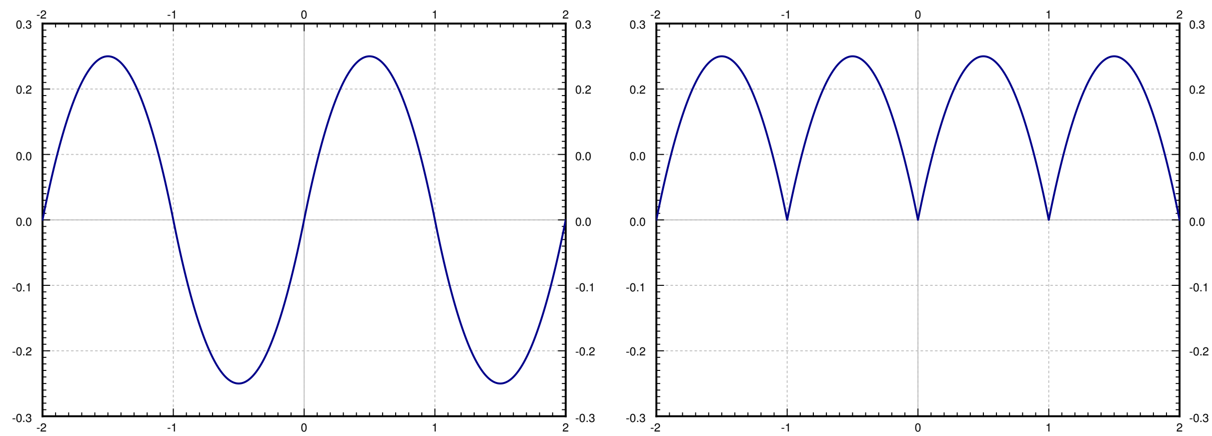


Figure 10: Odd and even 2-periodic extension of $f(t) = t(1-t)$, $0 \le t \le 1$.

$[-L, L]$ is twice the integral of the function over the interval $[0, L]$. The function $f(t)\sin\left(\frac{n\pi}{L}t\right)$ is the product of two odd functions and hence is even.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)\,dt = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi}{L}t\right)\,dt.$$

We now write the Fourier series of $f(t)$ as

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right).$$

Similarly, if $f(t)$ is an even $2L$-periodic function. For the same exact reasons as above, we find that $b_n = 0$ and

$$a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt.$$

The formula still works for $n = 0$, in which case it becomes

$$a_0 = \frac{2}{L}\int_{0}^{L} f(t)\,dt.$$

The Fourier series is then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right).$$

An interesting consequence is that the coefficients of the Fourier series of an odd (or even) function can be computed by just integrating over the half interval $[0, L]$. Therefore, we can compute the Fourier series of the odd (or even) extension of a function by computing certain integrals over the interval where the original function is defined.

Theorem 4.1. *Let $f(t)$ be a piecewise smooth function defined on $[0, L]$. Then the odd periodic extension of $f(t)$ has the Fourier series*

$$\boxed{F_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right),}$$

where

$$\boxed{b_n = \frac{2}{L}\int_0^L f(t)\,\sin\left(\frac{n\pi}{L}t\right)\,dt.}$$

The even periodic extension of $f(t)$ has the Fourier series

$$\boxed{F_{even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right),}$$

where

$$\boxed{a_n = \frac{2}{L}\int_0^L f(t)\,\cos\left(\frac{n\pi}{L}t\right)\,dt.}$$

We call the series $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}t\right)$ the *sine series* of $f(t)$ and we call the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right)$ the *cosine series* of $f(t)$. We often do not actually care what happens outside of $[0, L]$. In this case, we pick whichever series fits our problem better.

It is not necessary to start with the full Fourier series to obtain the sine and cosine series. The sine series is really the eigenfunction expansion of $f(t)$ using eigenfunctions of the eigenvalue problem $x'' + \lambda x = 0$, $x(0) = 0$, $x(L) = L$. The cosine series is the eigenfunction expansion of $f(t)$ using eigenfunctions of the eigenvalue problem $x'' + \lambda x = 0$, $x'(0) = 0$, $x'(L) = L$. We could have, therefore, gotten the same formulas by defining the inner product

$$\langle f(t), g(t)\rangle = \int_0^L f(t)g(t)\,dt,$$

and following the procedure of § 2. This point of view is useful, as we commonly use a specific series that arose because our underlying question led to a certain eigenvalue problem. If the eigenvalue problem is not one of the three we covered so far, you can still do an eigenfunction expansion, generalizing the results of this chapter. We will deal with such a generalization in chapter **??**.

Example 4.2: Find the Fourier series of the even periodic extension of the function $f(t) = t^2$ for $0 \le t \le \pi$.

We want to write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt),$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2\pi^2}{3},$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} t^2 \cos(nt)\,dt = \frac{2}{\pi}\left[t^2\frac{1}{n}\sin(nt)\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} t\sin(nt)\,dt \\ &= \frac{4}{n^2\pi}\Big[t\cos(nt)\Big]_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\cos(nt)\,dt = \frac{4(-1)^n}{n^2}. \end{aligned}$$

Note that we have "detected" the continuity of the extension since the coefficients decay as $\frac{1}{n^2}$. That is, the even periodic extension of t^2 has no jump discontinuities. It does have corners, since the derivative, which is an odd function and a sine series, has jumps; it has a Fourier series whose coefficients decay only as $\frac{1}{n}$.

Explicitly, the first few terms of the series are

$$\frac{\pi^2}{3} - 4\cos(t) + \cos(2t) - \frac{4}{9}\cos(3t) + \cdots$$

***Exercise* 4.3:**

a) *Compute the derivative of the even periodic extension of $f(t)$ above and verify it has jump discontinuities. Use the actual definition of $f(t)$, not its cosine series*!

b) *Why is it that the derivative of the even periodic extension of $f(t)$ is the odd periodic extension of $f'(t)$*?

4.3 Application

Fourier series ties in to the boundary value problems we studied earlier. Let us see this connection in an application.

Consider the boundary value problem for $0 < t < L$,

$$x''(t) + \lambda\, x(t) = f(t),$$

for the *Dirichlet boundary conditions* $x(0) = 0$, $x(L) = 0$. The Fredholm alternative (Theorem 1.2 on page 5) says that as long as λ is not an eigenvalue of the underlying homogeneous problem, there exists a unique solution. Eigenfunctions of this eigenvalue problem are the functions $\sin\left(\frac{n\pi}{L}t\right)$. Therefore, to find the solution, we first find the Fourier sine series for $f(t)$. We write x also as a sine series, but with unknown coefficients. We substitute the series for x into the equation and solve for the unknown coefficients. If we have the *Neumann boundary conditions* $x'(0) = 0$, $x'(L) = 0$, we do the same procedure using the cosine series.

Let us see how this method works on examples.

Example 4.3: Take the boundary value problem for $0 < t < 1$,

$$x''(t) + 2x(t) = f(t),$$

where $f(t) = t$ on $0 < t < 1$, and satisfying the Dirichlet boundary conditions $x(0) = 0$, $x(1) = 0$. We write $f(t)$ as a sine series

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi t).$$

Compute

$$c_n = 2\int_0^1 t \sin(n\pi t)\, dt = \frac{2\,(-1)^{n+1}}{n\pi}.$$

We write $x(t)$ as

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t).$$

We plug in to obtain

$$\begin{aligned} x''(t) + 2x(t) &= \underbrace{\sum_{n=1}^{\infty} -b_n n^2\pi^2 \sin(n\pi t)}_{x''} + 2\underbrace{\sum_{n=1}^{\infty} b_n \sin(n\pi t)}_{x} \\ &= \sum_{n=1}^{\infty} b_n (2 - n^2\pi^2) \sin(n\pi t) \\ &= f(t) = \sum_{n=1}^{\infty} \frac{2\,(-1)^{n+1}}{n\pi} \sin(n\pi t). \end{aligned}$$

Therefore,

$$b_n(2 - n^2\pi^2) = \frac{2\,(-1)^{n+1}}{n\pi}$$

or

$$b_n = \frac{2\,(-1)^{n+1}}{n\pi(2 - n^2\pi^2)}.$$

That $2 - n^2\pi^2$ is not zero for any n, and that we can solve for b_n, is precisely because 2 is not an eigenvalue of the problem. We have thus obtained a Fourier series for the solution

$$x(t) = \sum_{n=1}^{\infty} \frac{2\,(-1)^{n+1}}{n\pi\,(2 - n^2\pi^2)} \sin(n\pi t).$$

See Figure 11 for a graph of the solution. Notice that because the eigenfunctions satisfy the boundary conditions, and x is written in terms of the boundary conditions, then x satisfies the boundary conditions.

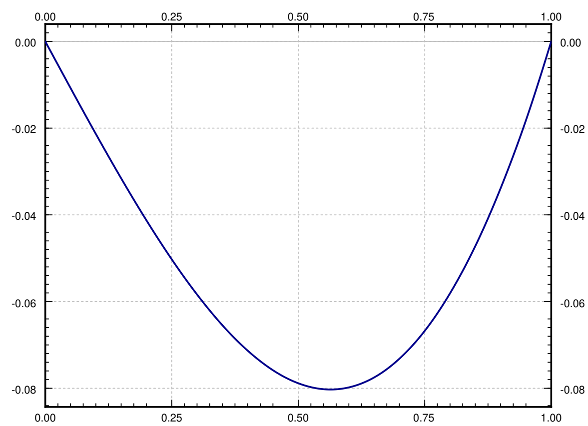


Figure 11: Plot of the solution of $x'' + 2x = t$, $x(0) = 0$, $x(1) = 0$.

Example 4.4: Similarly we handle the Neumann conditions. Take the boundary value problem for $0 < t < 1$,

$$x''(t) + 2x(t) = f(t),$$

where again $f(t) = t$ on $0 < t < 1$, but now satisfying the Neumann boundary conditions $x'(0) = 0$, $x'(1) = 0$. We write $f(t)$ as a cosine series

$$f(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi t),$$

where

$$c_0 = 2\int_0^1 t\,dt = 1,$$

and

$$c_n = 2\int_0^1 t\cos(n\pi t)\,dt = \frac{2\big((-1)^n - 1\big)}{\pi^2 n^2} = \begin{cases} \frac{-4}{\pi^2 n^2} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

We write $x(t)$ as a cosine series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t).$$

We plug in to obtain

$$
\begin{aligned}
x''(t) + 2x(t) &= \sum_{n=1}^{\infty} \left[-a_n n^2 \pi^2 \cos(n\pi t) \right] + a_0 + 2 \sum_{n=1}^{\infty} \left[a_n \cos(n\pi t) \right] \\
&= a_0 + \sum_{n=1}^{\infty} a_n (2 - n^2\pi^2) \cos(n\pi t) \\
&= f(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{\pi^2 n^2} \cos(n\pi t).
\end{aligned}
$$

Therefore, $a_0 = \frac{1}{2}$, $a_n = 0$ for n even ($n \geq 2$) and for n odd we have

$$a_n(2 - n^2\pi^2) = \frac{-4}{\pi^2 n^2},$$

or

$$a_n = \frac{-4}{n^2\pi^2(2 - n^2\pi^2)}.$$

The Fourier series for the solution $x(t)$ is

$$x(t) = \frac{1}{4} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^2(2 - n^2\pi^2)} \cos(n\pi t).$$

4.4 Exercises

***Exercise* 4.4:** *Take $f(t) = (t-1)^2$ defined on $0 \leq t \leq 1$.*

a) *Sketch the plot of the even periodic extension of f.*

b) *Sketch the plot of the odd periodic extension of f.*

***Exercise* 4.5:** *Find the Fourier series of both the odd and even periodic extension of the function $f(t) = (t-1)^2$ for $0 \leq t \leq 1$. Can you tell which extension is continuous from the Fourier series coefficients?*

***Exercise* 4.6:** *Find the Fourier series of both the odd and even periodic extension of the function $f(t) = t$ for $0 \leq t \leq \pi$.*

***Exercise* 4.7:** *Find the Fourier series of the even periodic extension of the function $f(t) = \sin t$ for $0 \leq t \leq \pi$.*

***Exercise* 4.8:** *Consider*

$$x''(t) + 4x(t) = f(t),$$

where $f(t) = 1$ on $0 < t < 1$.

a) *Solve for the Dirichlet conditions $x(0) = 0$, $x(1) = 0$.*

b) *Solve for the Neumann conditions $x'(0) = 0$, $x'(1) = 0$.*

***Exercise* 4.9:** *Consider*

$$x''(t) + 9x(t) = f(t),$$

for $f(t) = \sin(2\pi t)$ on $0 < t < 1$.

a) *Solve for the Dirichlet conditions $x(0) = 0$, $x(1) = 0$.*

b) *Solve for the Neumann conditions $x'(0) = 0$, $x'(1) = 0$.*

Exercise* 4.10*: *Consider*

$$x''(t) + 3x(t) = f(t), \quad x(0) = 0, \quad x(1) = 0,$$

where $f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$*. Write the solution* $x(t)$ *as a Fourier series, where the coefficients are given in terms of* b_n*.*

Exercise* 4.11*: *Let* $f(t) = t^2(2-t)$ *for* $0 \leq t \leq 2$*. Let* $F(t)$ *be the odd periodic extension. Compute* $F(1)$*,* $F(2)$*,* $F(3)$*,* $F(-1)$*,* $F(9/2)$*,* $F(101)$*,* $F(103)$*. Note: Do **not** compute using the sine series.*

Exercise* 4.101*: *Let* $f(t) = t/3$ *on* $0 \leq t < 3$*.*

a) Find the Fourier series of the even periodic extension.

b) Find the Fourier series of the odd periodic extension.

Exercise* 4.102*: *Let* $f(t) = \cos(2t)$ *on* $0 \leq t < \pi$*.*

a) Find the Fourier series of the even periodic extension.

b) Find the Fourier series of the odd periodic extension.

Exercise* 4.103*: *Let* $f(t)$ *be defined on* $0 \leq t < 1$*. Now take the average of the two extensions* $g(t) = \frac{F_{odd}(t)+F_{even}(t)}{2}$*.*

a) What is $g(t)$ *if* $0 \leq t < 1$ *(Justify!)* *b) What is* $g(t)$ *if* $-1 < t < 0$ *(Justify!)*

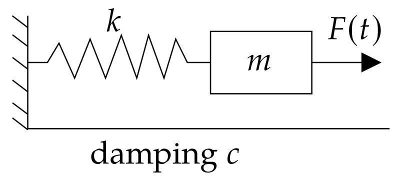
Exercise* 4.104*: *Let* $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nt)$*. Solve* $x'' - x = f(t)$ *for the Dirichlet conditions* $x(0) = 0$ *and* $x(\pi) = 0$*.*

Exercise* 4.105** (challenging):*** *Let* $f(t) = t + \sum_{n=1}^{\infty} \frac{1}{2^n} \sin(nt)$*. Solve* $x'' + \pi x = f(t)$ *for the Dirichlet conditions* $x(0) = 0$ *and* $x(\pi) = 1$*. Hint: Note that* $\frac{t}{\pi}$ *satisfies the given Dirichlet conditions.*

5 Applications of Fourier series

*Note: 2 lectures, §9.4 in [**?**], not in [**?**]*

5.1 Periodically forced oscillation



Let us return to the forced oscillations. Consider a mass-spring system as before, where we have a mass m on a spring with spring constant k, with damping c, and a force $F(t)$ applied to the mass. Suppose the forcing function $F(t)$ is $2L$-periodic for some $L > 0$. We saw this problem in chapter **??** with $F(t) = F_0 \cos(\omega t)$. The equation that governs this particular setup is

$$mx''(t) + cx'(t) + kx(t) = F(t). \tag{9}$$

The general solution of (9) consists of the complementary solution x_c, which solves the associated homogeneous equation $mx'' + cx' + kx = 0$, and a particular solution of (9) we call x_p. For $c > 0$, the complementary solution x_c will decay as time goes by. Therefore, we are mostly interested in a particular solution x_p that does not decay and is periodic with the same period as $F(t)$. We call this particular solution the *steady periodic solution* and we write it as x_{sp} as before. What is new in this section is that we consider an arbitrary forcing function $F(t)$ instead of a simple cosine.

For simplicity, suppose $c = 0$. The problem with $c > 0$ is very similar. The equation

$$mx'' + kx = 0$$

has the general solution

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. Any solution to $mx''(t) + kx(t) = F(t)$ is of the form $A\cos(\omega_0 t) + B\sin(\omega_0 t) + x_{sp}$. The steady periodic solution x_{sp} has the same period as $F(t)$.

In the spirit of the last section and the idea of undetermined coefficients we first write

$$F(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}t\right) + d_n \sin\left(\frac{n\pi}{L}t\right).$$

Then we write a proposed steady periodic solution x as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right),$$

where a_n and b_n are unknowns. We plug x into the differential equation and solve for a_n and b_n in terms of c_n and d_n. This process is perhaps best understood by example.

Example 5.1: Suppose that $k = 2$, and $m = 1$. The units are again the mks units (meters-kilograms-seconds). There is a jetpack strapped to the mass, which fires with a force of 1 newton for 1 second and then is off for 1 second, and so on. We want to find the steady periodic solution.

The equation is, therefore,

$$x'' + 2x = F(t),$$

where $F(t)$ is the step function

$$F(t) = \begin{cases} 0 & \text{if } -1 < t < 0, \\ 1 & \text{if } 0 < t < 1, \end{cases}$$

extended periodically. We write

$$F(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi t) + d_n \sin(n\pi t).$$

We compute

$$\begin{aligned}
c_n &= \int_{-1}^{1} F(t)\cos(n\pi t)\,dt = \int_{0}^{1} \cos(n\pi t)\,dt = 0 \qquad \text{for } n \geq 1, \\
c_0 &= \int_{-1}^{1} F(t)\,dt = \int_{0}^{1} dt = 1, \\
d_n &= \int_{-1}^{1} F(t)\sin(n\pi t)\,dt \\
&= \int_{0}^{1} \sin(n\pi t)\,dt \\
&= \left[\frac{-\cos(n\pi t)}{n\pi}\right]_{t=0}^{1} \\
&= \frac{1-(-1)^n}{\pi n} = \begin{cases} \frac{2}{\pi n} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}
\end{aligned}$$

So

$$F(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n} \sin(n\pi t).$$

We want to try

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t).$$

Once we plug x into the differential equation $x'' + 2x = F(t)$, it is clear that $a_n = 0$ for $n \geq 1$ as there are no corresponding terms in the series for $F(t)$. Similarly $b_n = 0$ for n even. Hence we try

$$x(t) = \frac{a_0}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \sin(n\pi t).$$

We plug into the differential equation and obtain

$$\begin{aligned}
x'' + 2x &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \Big[-b_n n^2\pi^2 \sin(n\pi t)\Big] + a_0 + 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \Big[b_n \sin(n\pi t)\Big] \\
&= a_0 + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n (2 - n^2\pi^2) \sin(n\pi t) \\
&= F(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n} \sin(n\pi t).
\end{aligned}$$

So $a_0 = \frac{1}{2}$, $b_n = 0$ for even n, and for odd n we get

$$b_n = \frac{2}{\pi n(2 - n^2\pi^2)}.$$

The steady periodic solution has the Fourier series

$$x_{sp}(t) = \frac{1}{4} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n(2 - n^2\pi^2)} \sin(n\pi t).$$

We know this is the steady periodic solution as it contains no terms of the complementary solution and it is periodic with the same period as $F(t)$ itself. See Figure 12 for the plot of this solution.

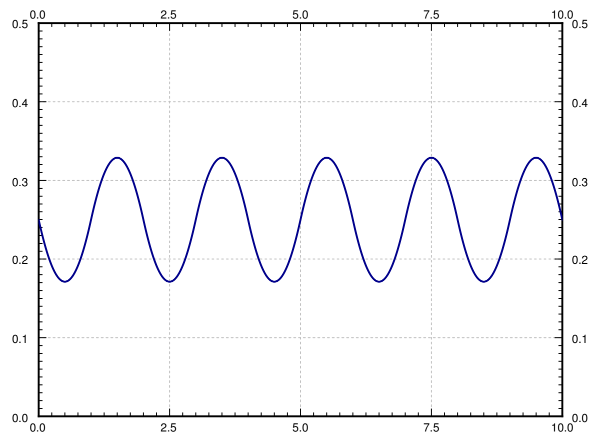


Figure 12: Plot of the steady periodic solution x_{sp} of Example 5.1.

5.2 Resonance

Just as when the forcing function was a simple cosine, we may encounter resonance. Assume $c = 0$ and let us discuss only pure resonance. Let $F(t)$ be $2L$-periodic and consider

$$mx''(t) + kx(t) = F(t).$$

When we expand $F(t)$ and find that some of its terms coincide with the complementary solution to $mx'' + kx = 0$, we cannot use those terms in the guess. Just like before, they disappear when we plug them into the left-hand side and we get a contradictory equation (such as $0 = 1$). That is, suppose

$$x_c = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

where $\omega_0 = \frac{N\pi}{L}$ for some positive integer N. We have to modify our guess and try

$$x(t) = \frac{a_0}{2} + t\left(a_N \cos\left(\frac{N\pi}{L}t\right) + b_N \sin\left(\frac{N\pi}{L}t\right)\right) + \sum_{\substack{n=1 \\ n \neq N}}^{\infty} a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right).$$

In other words, we multiply the offending term by t. From then on, we proceed as before.

Of course, the solution is not a Fourier series (it is not even periodic) since it contains these terms multiplied by t. Further, the terms $t\left(a_N \cos\left(\frac{N\pi}{L}t\right) + b_N \sin\left(\frac{N\pi}{L}t\right)\right)$ eventually dominate and lead to wild oscillations. As before, this behavior is called *pure resonance* or just *resonance*.

Note that there now may be infinitely many resonance frequencies to hit. That is, as we change the frequency of F (we change L), different terms from the Fourier series of F may interfere with the complementary solution and cause resonance. However, we should note that since everything is an approximation and in particular c is never actually zero but something very close to zero, only the first few resonance frequencies matter in real life.

Example 5.2: We want to solve the equation

$$2x'' + 18\pi^2 x = F(t), \tag{10}$$

where

$$F(t) = \begin{cases} -1 & \text{if } \; -1 < t < 0, \\ 1 & \text{if } \quad 0 < t < 1, \end{cases}$$

extended periodically. We note that

$$F(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n \pi t).$$

***Exercise* 5.1:** *Compute the Fourier series of F to verify the equation above.*

As $\sqrt{\frac{k}{m}} = \sqrt{\frac{18\pi^2}{2}} = 3\pi$, the solution to (10) is

$$x(t) = c_1 \cos(3\pi t) + c_2 \sin(3\pi t) + x_p(t)$$

for some particular solution x_p.

If we just try an x_p given as a Fourier series with $\sin(n\pi t)$ as usual, the complementary equation, $2x'' + 18\pi^2 x = 0$, eats our 3rd harmonic. That is, the term with $\sin(3\pi t)$ is already in in our complementary solution. Therefore, we pull that term out and multiply it by t. We also add a cosine term to get everything right. That is, we try

$$x_p(t) = a_3 t \cos(3\pi t) + b_3 t \sin(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} b_n \sin(n \pi t).$$

Let us compute the second derivative.

$$x_p''(t) = -6a_3\pi \; \sin(3\pi t) - 9\pi^2 a_3 \; t \; \cos(3\pi t) + 6b_3\pi \; \cos(3\pi t) - 9\pi^2 b_3 \; t \; \sin(3\pi t) \\ + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-n^2\pi^2 b_n) \; \sin(n\pi t).$$

We now plug into the left-hand side of the differential equation.

$$\begin{aligned} 2x_p'' + 18\pi^2 x_p = & -12a_3\pi \sin(3\pi t) - 18\pi^2 a_3 t \cos(3\pi t) + 12 b_3 \pi \cos(3\pi t) - 18\pi^2 b_3 t \sin(3\pi t) \\ & + 18\pi^2 a_3 t \cos(3\pi t) \qquad\qquad + 18\pi^2 b_3 t \sin(3\pi t) \\ & + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-2n^2\pi^2 b_n + 18\pi^2 b_n) \; \sin(n\pi t). \end{aligned}$$

We simplify,

$$2x_p'' + 18\pi^2 x_p = -12a_3\pi \sin(3\pi t) + 12 b_3 \pi \cos(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-2n^2\pi^2 b_n + 18\pi^2 b_n) \sin(n\pi t).$$

This series has to equal to the series for $F(t)$. We equate the coefficients and solve for a_3 and b_n.

$$\begin{aligned} a_3 &= \frac{4/(3\pi)}{-12\pi} = \frac{-1}{9\pi^2}, \\ b_3 &= 0, \\ b_n &= \frac{4}{n\pi(18\pi^2 - 2n^2\pi^2)} = \frac{2}{\pi^3 n(9-n^2)} \qquad \text{for } n \text{ odd and } n \neq 3. \end{aligned}$$

That is,

$$x_p(t) = \frac{-1}{9\pi^2}\, t\, \cos(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n\neq 3}}^{\infty} \frac{2}{\pi^3 n(9-n^2)} \sin(n\pi t).$$

When $c > 0$, you do not have to worry about pure resonance. That is, there are never any conflicts and you do not need to multiply any terms by t. There is a corresponding concept of practical resonance and it is very similar to the ideas we already explored in chapter **??**. Basically what happens in practical resonance is that one of the coefficients in the series for x_{sp} can get very big. Let us not go into details here.

5.3 Exercises

Exercise* 5.2*: *Let* $F(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi t)$*. Find the steady periodic solution to* $x'' + 2x = F(t)$*. Express your solution as a Fourier series.*

Exercise* 5.3*: *Let* $F(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin(n\pi t)$*. Find the steady periodic solution to* $x'' + x' + x = F(t)$*. Express your solution as a Fourier series.*

Exercise* 5.4*: *Let* $F(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi t)$*. Find the steady periodic solution to* $x'' + 4x = F(t)$*. Express your solution as a Fourier series.*

Exercise* 5.5*: *Let* $F(t) = t$ *for* $-1 < t < 1$ *and extended periodically. Find the steady periodic solution to* $x'' + x = F(t)$*. Express your solution as a series.*

Exercise* 5.6*: *Let* $F(t) = t$ *for* $-1 < t < 1$ *and extended periodically. Find the steady periodic solution to* $x'' + \pi^2 x = F(t)$*. Express your solution as a series.*

Exercise* 5.101*: *Let* $F(t) = \sin(2\pi t) + 0.1\cos(10\pi t)$*. Find the steady periodic solution to* $x'' + \sqrt{2}\, x = F(t)$*. Express your solution as a Fourier series.*

Exercise* 5.102*: *Let* $F(t) = \sum_{n=1}^{\infty} e^{-n} \cos(2nt)$*. Find the steady periodic solution to* $x'' + 3x = F(t)$*. Express your solution as a Fourier series.*

Exercise* 5.103*: *Let* $F(t) = |t|$ *for* $-1 \le t \le 1$ *extended periodically. Find the steady periodic solution to* $x'' + \sqrt{3}\, x = F(t)$*. Express your solution as a series.*

Exercise* 5.104*: *Let* $F(t) = |t|$ *for* $-1 \le t \le 1$ *extended periodically. Find the steady periodic solution to* $x'' + \pi^2 x = F(t)$*. Express your solution as a series.*