Solution - January, 2021

Solved by

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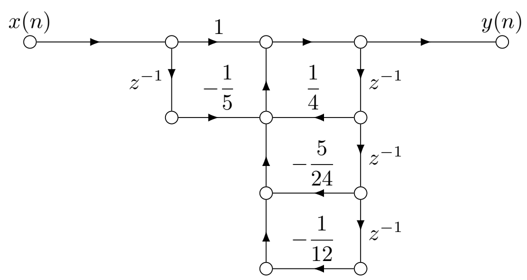
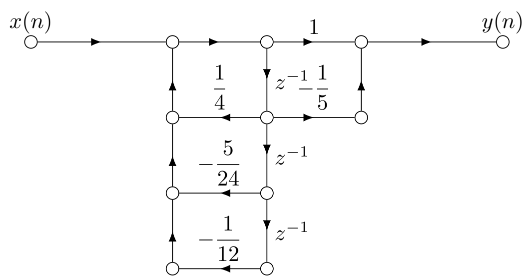
Question 1

a)

Expanding $H(z)$ gives us

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

Therefore, the signal flow graph of the filter in two form are shown in below

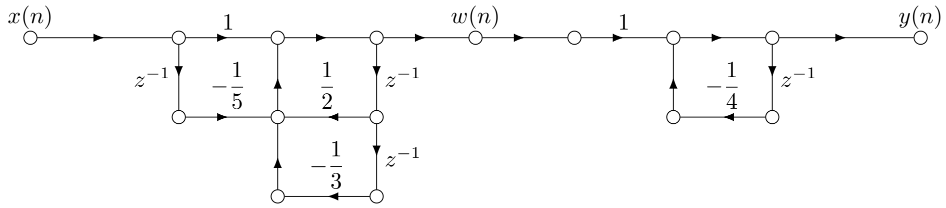
Direct Form I	**Canonical Form**
	

b)

Rewrite $H(z)$ as product of two functions as follow

$$H(z) = \underbrace{\frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}}}_{H_1(z)} \times \underbrace{\frac{1}{1 + \frac{1}{4}z^{-1}}}_{H_2(z)}$$

Then, the signal flow graph for the filter in cascade form is



c)

By partial fractions, the given filter can be separated as follow

$$H(z) = \frac{1-\dfrac{1}{5}z^{-1}}{\left(1-\dfrac{1}{2}z^{-1}+\dfrac{1}{3}z^{-2}\right)\left(1+\dfrac{1}{4}z^{-1}\right)} = \frac{A+Bz^{-1}}{1-\dfrac{1}{2}z^{-1}+\dfrac{1}{3}z^{-2}} + \frac{C}{1+\dfrac{1}{4}z^{-1}} \quad (*)$$

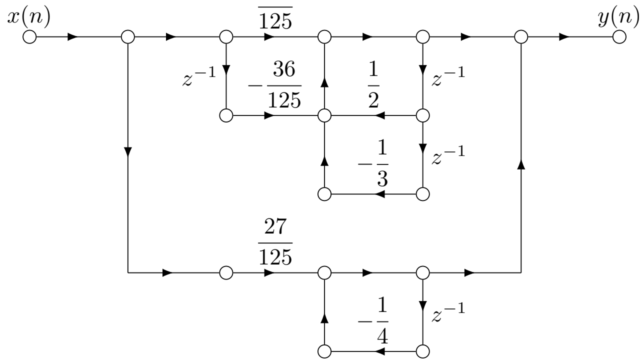
We have,

$$C = H(z)\left(1+\frac{1}{4}z^{-1}\right)\Bigg|_{z=-\frac{1}{4}} = \frac{1-\dfrac{1}{5}z^{-1}}{1-\dfrac{1}{2}z^{-1}+\dfrac{1}{3}z^{-2}}\Bigg|_{z=-\frac{1}{4}} = \frac{27}{125}$$

Chose $z=1$ and $z=-1$ give us

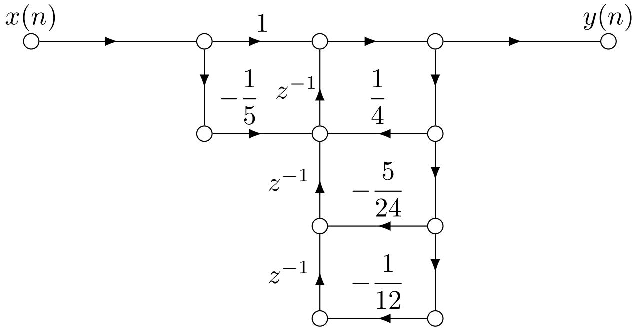
$$\begin{cases} (z=1): & H(1) = \dfrac{96}{125} = \dfrac{A+B}{1-1/2+1/3} + \dfrac{27}{125\,(1+1/4)} \\ (z=-1): & H(-1) = \dfrac{48}{55} = \dfrac{A-B}{1+1/2+1/3} + \dfrac{27}{125\,(1-1/4)} \end{cases} \Rightarrow \begin{cases} A = \dfrac{98}{125} \\ B = -\dfrac{36}{125} \end{cases}$$

Then, the signal flow graph for the filter in parallel form is



d)

The signal flow graph for the filter in transpose form is



e)

Since,

$$H(z) = \frac{1-\frac{1}{5}z^{-1}}{1-\frac{1}{4}z^{-1}+\frac{5}{24}z^{-2}+\frac{1}{12}z^{-3}} = \frac{Y(z)}{X(z)}$$

which yields,

$$Y(z)\left(1-\frac{1}{4}z^{-1}+\frac{5}{24}z^{-2}+\frac{1}{12}z^{-3}\right) = X(z)\left(1-\frac{1}{5}z^{-1}\right)$$

Taking inverse z-transform gives us the difference equation of the filter

$$y(n)-\frac{1}{4}y(n-1)+\frac{5}{24}y(n-2)+\frac{1}{12}y(n-3) = x(n)-\frac{1}{5}x(n-1)$$

Question 2

a)

By partial fractions, $H(z)$ can be rewrite as following form

$$H(z) = \frac{z^{-3}}{4\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} + k_0 + k_1 z^{-1} \ (*)$$

We have,

- $A = H(z)\left(1-\frac{1}{2}z^{-1}\right)\Big|_{z=\frac{1}{2}} = \left.\frac{z^{-3}}{1+\frac{1}{2}z^{-1}}\right|_{z=\frac{1}{2}} = 1$

- $B = H(z)\left(1+\frac{1}{2}z^{-1}\right)\Big|_{z=-\frac{1}{2}} = \left.\frac{z^{-3}}{1-\frac{1}{2}z^{-1}}\right|_{z=-\frac{1}{2}} = -1$
- $k_0 = \lim\limits_{z\to\infty}(H(z)) = 0$

Choosing $z = 1$ substitute into $(*)$, we get

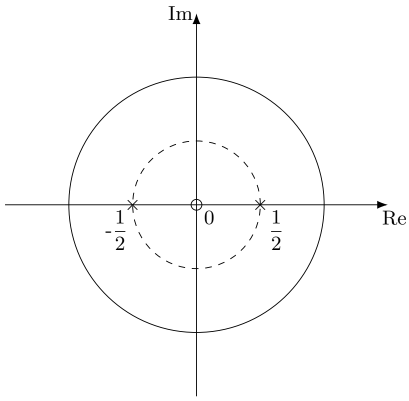
$$H(1) = \frac{1}{3} = \frac{1}{1-\frac{1}{2}} - \frac{1}{1+\frac{1}{2}} + 0 + k_1 \Rightarrow k_1 = -1$$

Thus,

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1+\frac{1}{2}z^{-1}} - z^{-1}$$

b)

Pole-Zero pattern for the filter is shown in the below figure The problem give us the filter



is causal, therefore, the region of convergence for the filter is ROC:

$$|z| > \frac{1}{2}$$

Since, this ROC include the unit circle $|z| = 1$ which leads to the filter is stable.

c)

Taking inverse z-transform for $H(z)$ directly give us the impulse response of the filter

$$h(n) = \frac{1}{2^n}u(n) - \frac{1}{(-2)^n}u(n) - \delta(n-1)$$

d)

Using transformation $z = e^{j\omega}$ and substituting into $H(z)$, we get

$$H(e^{j\omega}) = \frac{e^{-3j\omega}}{4 - e^{-2j\omega}}$$

- For $\omega = 0 \rightarrow e^{j0} = \cos(0) + j\sin(0) = 1$, then

$$H(e^{j0}) = \frac{1^{-3}}{4 - 1^{-2}} = \frac{1}{3} \rightarrow |H(e^{j0})| = \frac{1}{3}$$

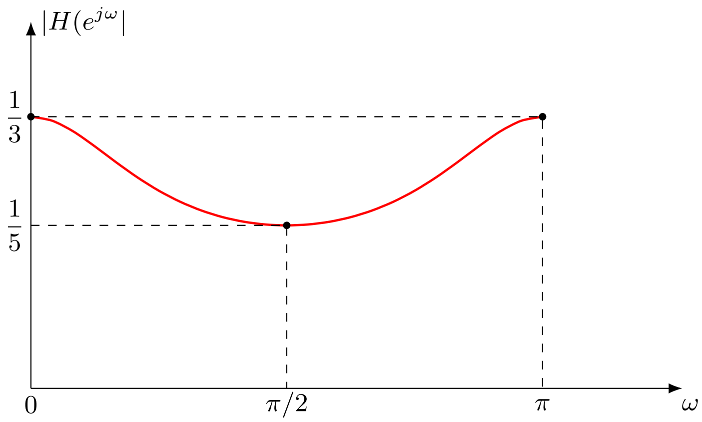
- For $\omega = \pi/2 \rightarrow e^{j\pi/2} = \cos(\pi/2) + j\sin(\pi/2) = j$

$$H(e^{j\pi/2}) = \frac{j^{-3}}{4 - j^{-2}} = \frac{j}{5} \rightarrow |H(e^{j\pi/2})| = \frac{1}{5}$$

- For $\omega = \pi \rightarrow e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1$

$$H(e^{j\pi}) = \frac{(-1)^{-3}}{4 - 1(-1)^{-2}} = -\frac{1}{3} \rightarrow |H(e^{j0})| = \frac{1}{3}$$

By these information, the magnitude response for the filter is

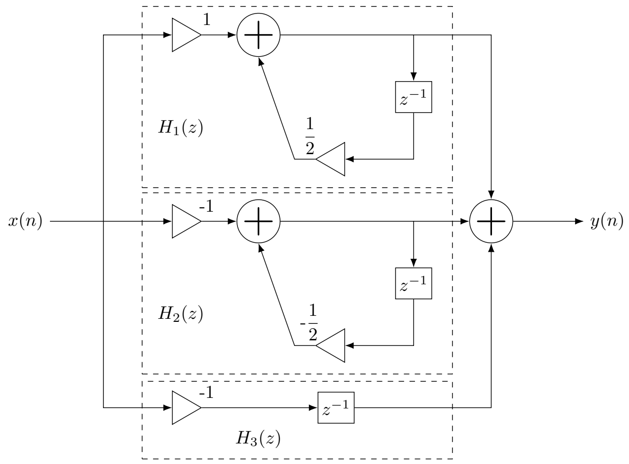


e)

Rewrite $H(z)$ as sum of three functions as follow

$$H(z) = \underbrace{\frac{1}{1 - \frac{1}{2}z^{-1}}}_{H_1(z)} + \underbrace{\frac{-1}{1 + \frac{1}{2}z^{-1}}}_{H_2(z)} + \underbrace{(-z^{-1})}_{H_3(z)}$$

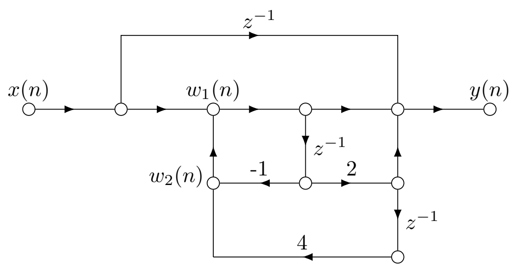
Then, the block diagram for the filter in parallel form is sketch as follow



Question 3

a)

Define some node name as the following figure



Node equations

$$Y(z) = X(z)z^{-1} + W_1(z) + 2W_1(z)z^{-1} \tag{1}$$

$$W_1(z) = X(z) + W_2(z) \tag{2}$$

$$W_2(z) = -W_1(z)z^{-1} + 8W_1(z)z^{-1} \tag{3}$$

b)

Substitute (3) into (2), we get

$$W_1(z) = X(z) - W_1(z)z^{-1} + 8W_1(z)z^{-1} \Rightarrow W_1(z) = \frac{X(z)}{1 + z^{-1} - 8z^{-2}}$$

Substituting back this result to (1) gives us

$$Y(z) = X(z)z^{-1} + (1 + 2z^{-1})\frac{X(z)}{1 + z^{-1} - 8z^{-2}}$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = z^{-1} + \frac{1 + 2z^{-1}}{1 + z^{-1} - 8z^{-2}}$$

Notice that

$$\frac{1 + 2z^{-1}}{1 - z^{-1} + 8z^{-2}} = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

Then, using partial fraction for $H(z)$ yields

$$H(z) = z^{-1} + \frac{A}{1 - p_1 z^{-1}} + \frac{B}{1 - p_2 z^{-1}}$$

where

$$p_1 = \frac{-1 + \sqrt{33}}{2}; \qquad p_2 = \frac{-1 - \sqrt{33}}{2}$$

- $A = \dfrac{1 + 2z^{-1}}{1 - +z^{-1} - 8z^{-2}}(1 - p_1 z^{-1})\Big|_{z=p_1} = \dfrac{1}{1 - p_2 z^{-1}}\Big|_{z=p_1} = \dfrac{11 + \sqrt{33}}{22}$
- $B = \dfrac{1 + 2z^{-1}}{1 + z^{-1} - 8z^{-2}}(1 - p_2 z^{-1})\Big|_{z=p_2} = \dfrac{1}{1 - p_1 z^{-1}}\Big|_{z=p_2} = \dfrac{11 - \sqrt{33}}{22}$

Thus, the impulse response is

$$h(z) = \delta(n-1) + Ap_1^n u(n) + Bp_2^n u(n)$$

Where A, B, p_1, p_2 are constant mentioned in the above.

c)

The corresponding values of $h(n)$ for $n = 1,\ 2,\ 3$ are $h(1) = 2,\ h(2) = 7,\ h(3) = 1$.

d)

Rewrite $H(z)$ as follow

$$H(z) = \frac{1 + 3z^{-1} + z^{-2} - 8z^{-3}}{1 + z^{-1} - 8z^{-2}} = \frac{Y(z)}{X(z)}$$

$$\Rightarrow Y(z)(1 + z^{-1} - 8z^{-2}) = X(z)(1 + 3z^{-1} + z^{-2} - 8z^{-3})$$

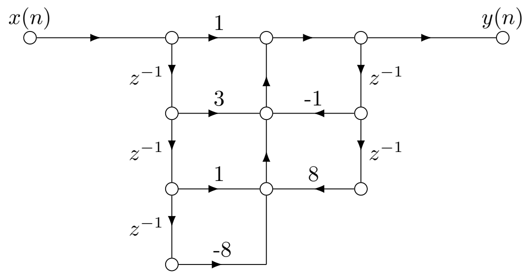
Taking inverse z-transform gives us the difference equation of the filter

$$y(n) + y(n-1) - 8y(n-2) = x(n) + 3x(n-1) + x(n-2) - 8x(n-3)$$

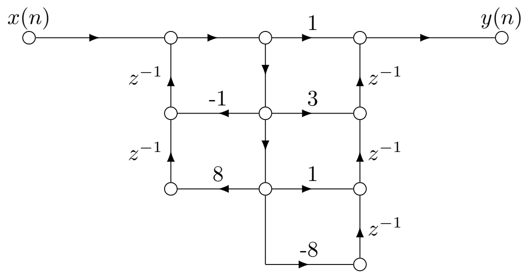
e)

The signal flow graph of the filter in two form are shown in below

Direct Form I



Canonical Form



Question 4

The given signal can be rewrite as follow

$$x(t) = \cos(24\pi t) + \sin(20\pi t) + \sin(4\pi t)$$

a)

With sampling frequency of $f_s = 8$ kHz, we have

- $f_1 = 12 > f_s/2 \rightarrow f_{1a} = 12 \mod (f_s) = 4$ kHz.
- $f_2 = 10 > f_s/2 \rightarrow f_{2a} = 10 \mod (f_s) = 2$ kHz.
- $f_3 = 2 < f_2/2 \rightarrow f_{3a} = 2$ kHz.

Thus, the aliased signal is

$$\begin{aligned} x_a(t) &= \cos(8\pi t) + \sin(4\pi t) + \sin(4\pi t) \\ &= \cos(8\pi t) + 2\sin(4\pi t) \end{aligned}$$

b)

For sampling processing, let $t = nTs = n/f_s = n/8$ substituting into the aliased signal yields

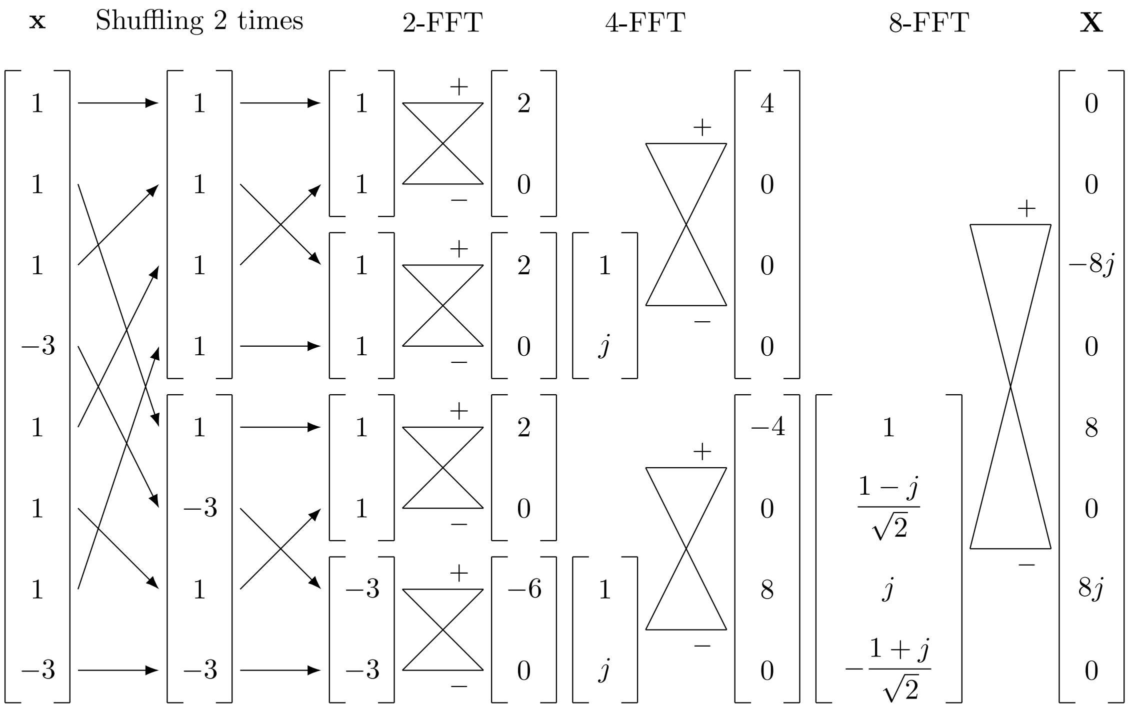
$$x(n) = \cos(\pi n) + 2\sin\left(\frac{\pi}{2}n\right)$$

Then, let n varies from 0 to 7 to calculate $x(0)$ to $x(7)$. Finally the matrix form represent for $x(0)$ to $x(7)$ is

$$\mathbf{x} = [1,\ 1,\ 1, -3,\ 1,\ 1,\ 1, -3]^T$$

c)

The 8-FFT of the signal $x(n)$ is performed as figure below



Thus, the values of 8-FFT of $x(n)$ in matrix form is

$$\mathbf{X} = [0,\ 0, -8j,\ 0,\ 8,\ 0, 8j,\ 0]^T$$

Question 5

a)

By using Euler's formulas, the signal $x(n)$ can be expanded as follows

$$\begin{aligned} x(n) &= \frac{1}{2}(e^{j\pi n} + e^{-j\pi n}) + 2\frac{1}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}) \\ &= \frac{1}{2}(e^{j\pi n} + e^{j\pi n}) - je^{j\frac{\pi}{2}n} + je^{-j\frac{\pi}{2}n} \\ &= -je^{j\frac{\pi}{2}n} + e^{j\pi n} + je^{-j\frac{\pi}{2}n} \\ &= -je^{j\omega_2 n} + e^{j\omega_4 n} + je^{j\omega_6 n} \end{aligned} \tag{1}$$

(Notice that, by periodic function's property $e^{-j\pi n} = e^{j(-\pi n + 2\pi n)} = e^{j\pi n}$)

b)

Let us recall values of $X(k)$ in previous section

$$\mathbf{X} = [0,\ 0, -8j,\ 0,\ 8,\ 0, 8j,\ 0]^T$$

Using definition of inverse 8-DFT, we have

$$\begin{aligned} x(n) &= \frac{1}{8}\sum_{k=0}^{7} X(k)e^{j\omega_k n} \qquad (2) \\ &= \frac{1}{8}(-8je^{j\omega_2 n} + 8e^{j\omega_4 n} + 8je^{j\omega_6 n}) \\ &= -je^{j\omega_2 n} + e^{j\omega_4 n} + je^{j\omega_6 n} \end{aligned}$$

c)

Expanding (2), we have

$$x(n) = \frac{1}{8}\left(X(0)e^{j\omega_0 n} + X(1)e^{j\omega_1 n} + \cdots + X(5)e^{j\omega_5 n} + X(6)e^{j\omega_6 n} + X(7)e^{j\omega_7 n}\right) \tag{3}$$

From (1), we have

$$x(n) = \frac{1}{8}\left(-8je^{j\omega_2 n} + 8e^{j\omega_4 n} + 8je^{j\omega_6 n}\right) \tag{4}$$

Comparing (3) and (4), we also get the result of $X(k)$

$$\mathbf{X} = [0,\ 0, -8j,\ 0,\ 8,\ 0, 8j,\ 0]^T$$