Lecture notes: Differential Equations for ISE (MA029IU)

Week 2 *

February 16, 2022

1 Integrals as solutions

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y),$$

or just

$$y' = f(x, y).$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x). \tag{1}$$

We could just integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)\, dx = \int f(x)\, dx + C,$$

that is

$$y(x) = \int f(x)\, dx + C.$$

This $y(x)$ is actually the general solution. So to solve (1), we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called indefinite integral. The indefinite integral is really the *antiderivative* (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)\, dx + C$ as

$$\int_{x_0}^{x} f(t)\, dt + C.$$

Hence the terminology *to integrate* when we may really mean *to antidifferentiate*. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should *always* think of the definite integral as a way to write it.

*This note is taken from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl

Example 1.1: Find the general solution of $y' = 3x^2$.

Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check by differentiating: $y' = 3x^2$. We got *precisely* our equation back.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbers x_0 and y_0 (x_0 is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x) = \int_{x_0}^{x} f(s)\, ds + y_0. \tag{2}$$

Let us check! We compute $y' = f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = \int_{x_0}^{x_0} f(x)\, dx + y_0 = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, (2) is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example 1.2: Solve

$$y' = e^{-x^2}, \qquad y(0) = 1.$$

By the preceding discussion, the solution must be

$$y(x) = \int_{0}^{x} e^{-s^2}\, ds + 1.$$

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y).$$

Let us write the equation in Leibniz notation.

$$\frac{dy}{dx} = f(y).$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point, we can simply integrate,

$$x(y) = \int \frac{1}{f(y)}\, dy + C.$$

Finally, we try to solve for y.

Example 1.3: Previously, we guessed $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We can now find the solution without guessing. First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}.$$

We integrate to obtain

$$x(y) = x = \frac{1}{k} \ln |y| + D,$$

where D is an arbitrary constant. Now we solve for y (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD} e^{kx}.$$

If we replace e^{-kD} with an arbitrary constant C, we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$.

Example 1.4: Find the general solution of $y' = y^2$.

First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}.$$

We integrate to get

$$x = \frac{-1}{y} + C.$$

We solve for $y = \frac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C - x} \qquad \text{or} \qquad y = 0.$$

Note the singularities of the solution. If for example $C = 1$, then the solution "blows up" as we approach $x = 1$. See Figure 1. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞). Usually when this happens we only consider one of these the solution. For example if we impose a condition $y(0) = 1$, then the solution is $y = \frac{1}{1-x}$, and we would consider this solution only for x on the interval $(-\infty, 1)$. In the figure, it is the left side of the graph.

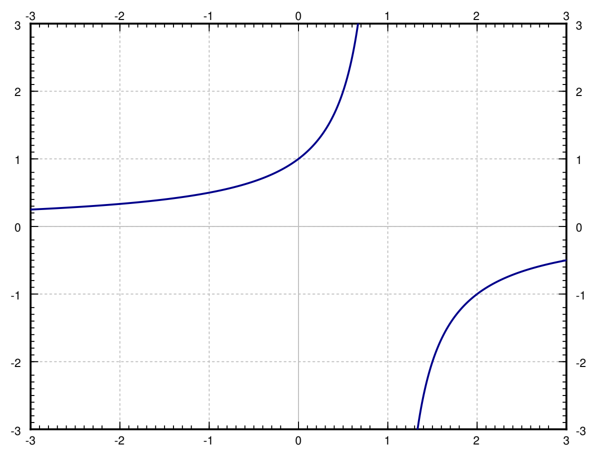


Figure 1: Plot of $y = \frac{1}{1-x}$.

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration and distance. You have surely seen these problems before in your calculus class.

Example 1.5: Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Let x denote the distance the car traveled. The equation is

$$x' = e^{t/2}.$$

We just integrate this equation to get that

$$x(t) = 2e^{t/2} + C.$$

We still need to figure out C. We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C.$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2.$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \qquad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}.$$

Example 1.6: Suppose that the car accelerates at a rate of t^2 m/s². At time $t = 0$ the car is at the 1 meter mark and is traveling at 10 m/s. Where is the car at time $t = 10$?

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \qquad x(0) = 1, \qquad x'(0) = 10.$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \qquad v(0) = 10.$$

Once we solve for v, we can integrate and find x.

Exercise* 1.1*: *Solve for v, and then solve for x. Find $x(10)$ to answer the question.*

1.1 Exercises

Exercise* 1.2*: *Solve* $\frac{dy}{dx} = x^2 + x$ *for* $y(1) = 3$.

Exercise* 1.3*: *Solve* $\frac{dy}{dx} = \sin(5x)$ *for* $y(0) = 2$.

Exercise* 1.4*: *Solve* $\frac{dy}{dx} = \frac{1}{x^2-1}$ *for* $y(0) = 0$.

Exercise* 1.5*: *Solve* $y' = y^3$ *for* $y(0) = 1$.

***Exercise* 1.6** (little harder)*:* *Solve* $y' = (y-1)(y+1)$ *for* $y(0) = 3$.

Exercise* 1.7*: *Solve* $\frac{dy}{dx} = \frac{1}{y+1}$ *for* $y(0) = 0$.

***Exercise* 1.8** (harder)*:* *Solve* $y'' = \sin x$ *for* $y(0) = 0$, $y'(0) = 2$.

Exercise* 1.9*: *A spaceship is traveling at the speed* $2t^2 + 1$ km/s (t *is time in seconds*). *It is pointing directly away from earth and at time* $t = 0$ *it is* 1000 *kilometers from earth. How far from earth is it at one minute from time* $t = 0$?

Exercise* 1.10*: *Solve* $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. *It is OK to leave your answer as a definite integral.*

Exercise* 1.11*: *A dropped ball accelerates downwards at a constant rate* 9.8 *meters per second squared. Set up the differential equation for the height above ground h in meters. Then supposing $h(0) = 100$ meters, how long does it take for the ball to hit the ground.*

Exercise* 1.12*: *Find the general solution of $y' = e^x$, and then $y' = e^y$.*

Exercise* 1.101*: *Solve $\frac{dy}{dx} = e^x + x$ and $y(0) = 10$.*

Exercise* 1.102*: *Solve $x' = \frac{1}{x^2}$, $x(1) = 1$.*

Exercise* 1.103*: *Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{4}$.*

Exercise* 1.104*: *Sid is in a car traveling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$, Sid is* 10 *miles away from Vegas. How far from Vegas is Sid* 2 *hours later*?

Exercise* 1.105*: *Solve $y' = y^n$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.*

Exercise* 1.106*: *The rate of change of the volume of a snowball that is melting is proportional to the surface area of the snowball. Suppose the snowball is perfectly spherical. The volume (in centimeters cubed) of a ball of radius r centimeters is $(4/3)\pi r^3$. The surface area is $4\pi r^2$. Set up the differential equation for how the radius r is changing. Then, suppose that at time $t = 0$ minutes, the radius is* 10 *centimeters. After* 5 *minutes, the radius is* 8 *centimeters. At what time t will the snowball be completely melted*?

Exercise* 1.107*: *Find the general solution to $y'''' = 0$. How many distinct constants do you need*?

2 Slope fields

As we said, the general first order equation we are studying looks like

$$y' = f(x, y).$$

A lot of the time, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behavior of the solutions, or find approximate solutions.

2.1 Slope fields

The equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. And this is the slope a solution $y(x)$ would have at x if its value was y. In other words, $f(x, y)$ is the slope of a solution whose graph runs through the point (x, y). At a point (x, y), we plot a short line with the slope $f(x, y)$. For example, if $f(x, y) = xy$, then at point $(2, 1.5)$ we draw a short line of slope $xy = 2 \times 1.5 = 3$. So, if $y(x)$ is a solution and $y(2) = 1.5$, then the equation mandates that $y'(2) = 3$. See Figure 2.

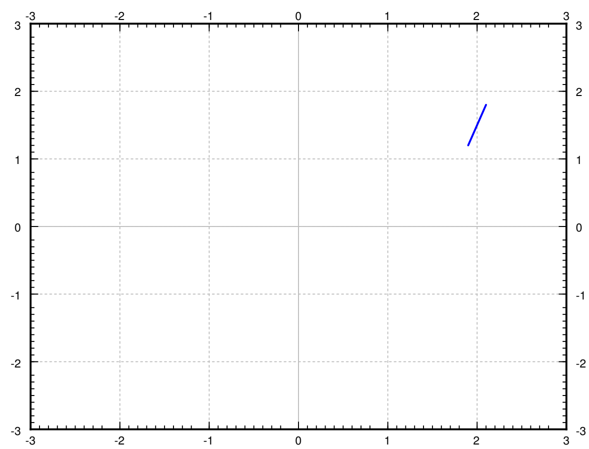


Figure 2: The slope $y' = xy$ at $(2, 1.5)$.

To get an idea of how solutions behave, we draw such lines at lots of points in the plane, not just the point $(2, 1.5)$. We would ideally want to see the slope at every point, but that is just not possible. Usually we pick a grid of points fine enough so that it shows the behavior, but not too fine so that we can still recognize the individual lines. We call this picture the *slope field* of the equation. See Figure 3 on the following page for the slope field of the equation $y' = xy$. Usually in practice, one does not do this by hand, but has a computer do the drawing.

Suppose we are given a specific initial condition $y(x_0) = y_0$. A solution, that is, the graph of the solution, would be a curve that follows the slopes we drew. For a few sample solutions, see Figure 4 on the next page. It is easy to roughly sketch (or at least imagine) possible solutions in the slope field, just from looking at the slope field itself. You simply sketch a line that roughly fits the little line segments and goes through your initial condition.

By looking at the slope field we get a lot of information about the behavior of solutions without having to solve the equation. For example, in Figure 4 on the following page we see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. A small change in the initial condition causes quite different behavior. We see this behavior just from the slope field and imagining what solutions ought to do.

We see a different behavior for the equation $y' = -y$. The slope field and a few solutions is in see Figure 5 on the next page. If we think of moving from left to right (perhaps x is time and time is usually increasing),

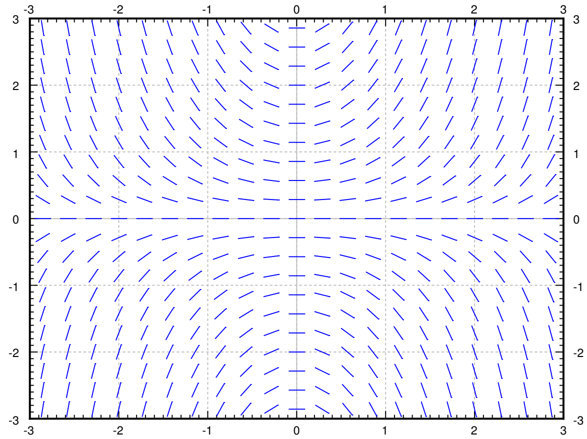


Figure 3: Slope field of $y' = xy$.

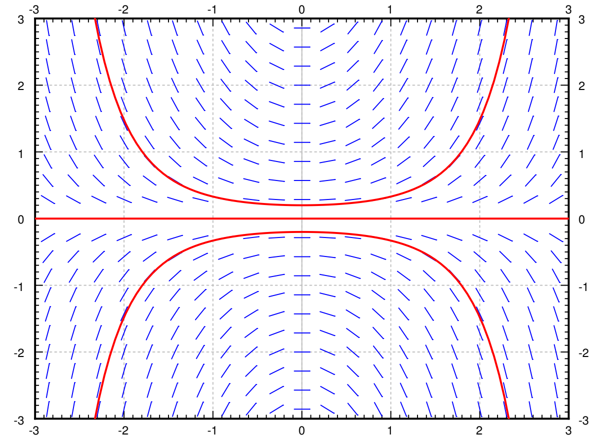


Figure 4: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

then we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. Again that behavior is clear from simply looking at the slope field itself.

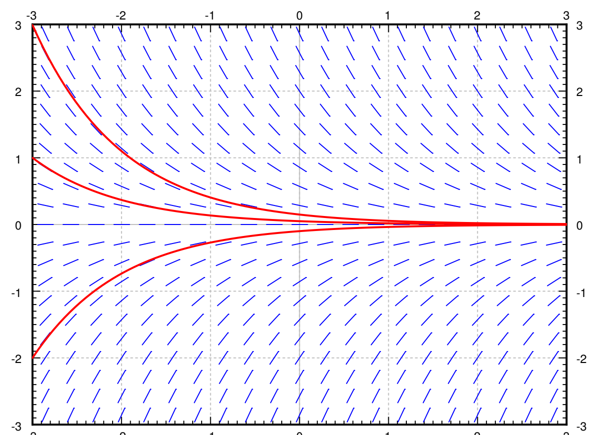


Figure 5: Slope field of $y' = -y$ with a graph of a few solutions.

2.2 Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 2.1: Attempt to solve:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

Integrate to find the general solution $y = \ln|x| + C$. The solution does not exist at $x = 0$. See Figure 6. The equation may have been written as the seemingly harmless $xy' = 1$.

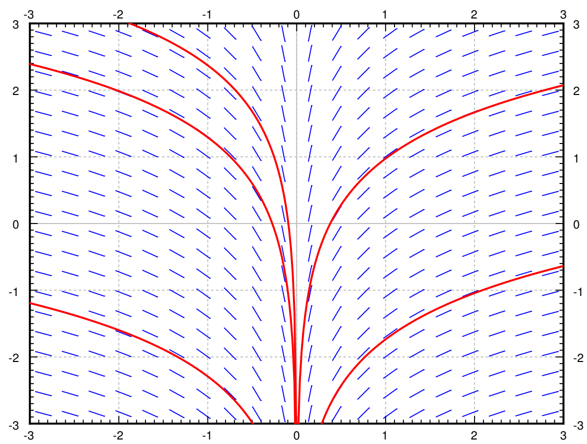


Figure 6: Slope field of $y' = 1/x$.

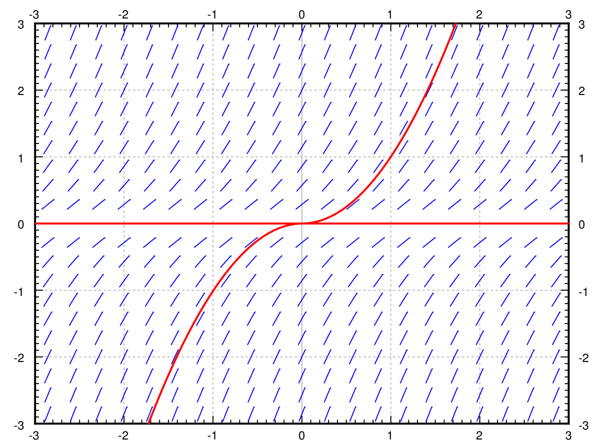


Figure 7: Slope field of $y' = 2\sqrt{|y|}$ with two solutions satisfying $y(0) = 0$.

Example 2.2: Solve:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

See Figure 7. Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as Picard's theorem*.

Theorem 2.1 (Picard's theorem on existence and uniqueness)**.** *If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to*

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (at least for x in some small interval) and is unique.

Note that the problems $y' = 1/x$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

*Named after the French mathematician Charles Émile Picard (1856–1941)

Example 2.3: For some constant A, solve:

$$y' = y^2, \qquad y(0) = A.$$

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = {}^1/y^2$, so $x = {}^{-1}/y + C$, so $y = \frac{1}{C-x}$. If $y(0) = A$, then $C = {}^1/A$ so

$$y = \frac{1}{{}^1/A - x}.$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

2.3 Exercises

***Exercise* 2.1:** *Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?*

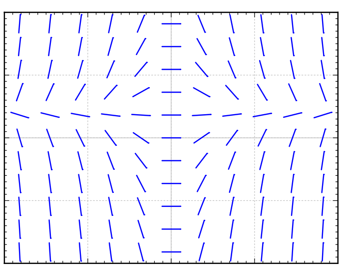
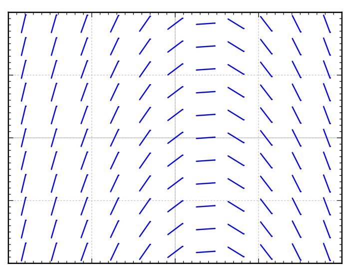
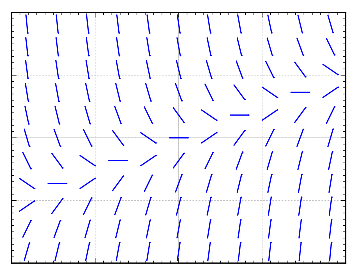
***Exercise* 2.2:** *Sketch slope field for $y' = x^2$.*

***Exercise* 2.3:** *Sketch slope field for $y' = y^2$.*

***Exercise* 2.4:** *Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.*

***Exercise* 2.5:** *Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.*

***Exercise* 2.6:** *Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1-y)$ to slope fields. Justify.*

a) 
b) 
c) 

***Exercise* 2.7** (challenging)**:** *Take $y' = f(x,y)$, $y(0) = 0$, where $f(x,y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.*

***Exercise* 2.8** (challenging)**:** *Take $(y-x)y' = 0$, $y(0) = 0$.*

a) *Find two distinct solutions.*

b) *Explain why this does not violate Picard's theorem.*

***Exercise* 2.9:** *Suppose $y' = f(x,y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x,y)$:*

a) *f does not depend on y.*

b) *f does not depend on x.*

c) *$f(t,t) = 0$ for any number t.*

d) *$f(x,0) = 0$ and $f(x,1) = 1$ for all x.*

***Exercise* 2.10:** *Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?*

***Exercise* 2.11:** *Take an equation $y' = (y-2x)g(x,y) + 2$ for some function $g(x,y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?*

Exercise* 2.12** (challenging):*** *Suppose $y' = f(x, y)$ is such that $f(x, 1) = 0$ for every x, f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y.*

a) *Guess a solution given the initial condition $y(0) = 1$.*

b) *Can graphs of two solutions of the equation for different initial conditions ever intersect?*

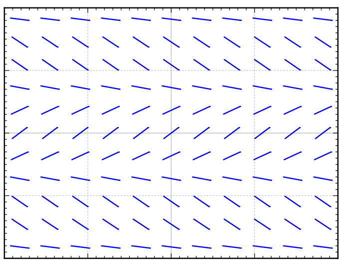
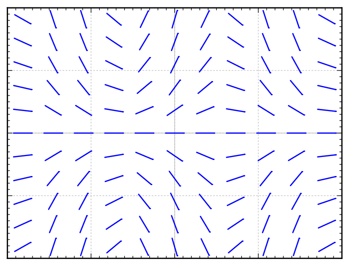
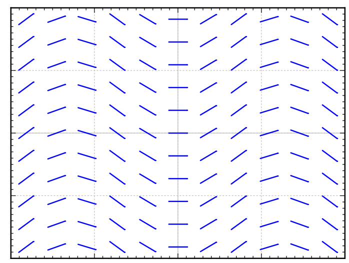
c) *Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x? Can $y(x) = 1$ for any x? Why or why not?*

Exercise* 2.101*: *Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?*

Exercise* 2.102*: *Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?*

Exercise* 2.103*: *Is it possible to solve $y' = \frac{x}{x^2-1}$ for $y(1) = 0$?*

Exercise* 2.104*: *Match equations $y' = \sin x$, $y' = \cos y$, $y' = y\cos(x)$ to slope fields. Justify.*

a) 
b) 
c) 

Exercise* 2.105** (tricky):*** *Suppose*

$$f(y) = \begin{cases} 0 & \text{if } y > 0, \\ 1 & \text{if } y \leq 0. \end{cases}$$

Does $y' = f(y)$, $y(0) = 0$ have a continuously differentiable solution? Does Picard apply? Why, or why not?

Exercise* 2.106*: *Consider an equation of the form $y' = f(x)$ for some continuous function f, and an initial condition $y(x_0) = y_0$. Does a solution exist for all x? Why or why not?*

3 Separable equations

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)\,dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)\,dx + C.$$

Notice the dependence on y in the integral.

3.1 Separable equations

We say a differential equation is *separable* if we can write it as

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example 3.1: Take the equation

$$y' = xy.$$

Note that $y = 0$ is a solution. We will remember that fact and assume $y \neq 0$ from now on, so that we can divide by y. Write the equation as $\frac{dy}{dx} = xy$. Then

$$\int \frac{dy}{y} = \int x\,dx + C.$$

We compute the antiderivatives to get

$$\ln |y| = \frac{x^2}{2} + C,$$

or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} e^C = De^{\frac{x^2}{2}},$$

where $D > 0$ is some constant. Because $y = 0$ is also a solution and because of the absolute value we can write:

$$y = De^{\frac{x^2}{2}},$$

for any number D (including zero or negative).

We check:

$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy.$$

Yay!

We should be a little bit more careful with this method. You may be worried that we integrated in two different variables. We seemingly did a different operation to each side. Let us work through this method more rigorously. Take

$$\frac{dy}{dx} = f(x)g(y).$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

We integrate both sides with respect to x:

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx + C.$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$

And we are done.

3.2 Implicit solutions

We sometimes get stuck even if we can do the integration. Consider the separable equation

$$y' = \frac{xy}{y^2+1}.$$

We separate variables,

$$\frac{y^2+1}{y}\,dy = \left(y + \frac{1}{y}\right)dy = x\,dx.$$

We integrate to get

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C,$$

or perhaps the easier looking expression (where $D = 2C$)

$$y^2 + 2\ln|y| = x^2 + D.$$

It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x, and remember that y is a function of x, to get

$$y'\left(2y + \frac{2}{y}\right) = 2x.$$

Multiply both sides by y and divide by $2(y^2+1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y, you might have to be tricky. You might get multiple solutions y for each x, so you have to pick one. Sometimes you can graph x as a function of y, and then flip your paper. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $C = 0$ if you plot all the points (x, y)

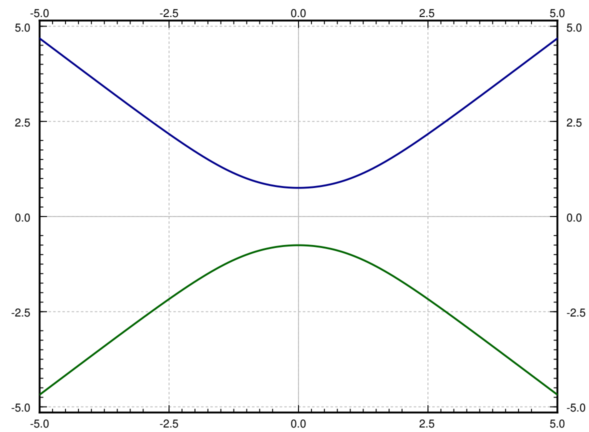


Figure 8: The implicit solution $y^2 + 2\ln|y| = x^2$ to $y' = \frac{xy}{y^2+1}$.

that are solutions to $y^2 + 2\ln|y| = x^2$, you find the two curves in Figure 8. This is not quite a graph of a function. For each x there are two choices of y. To find a function you would have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For example, the top curve satisfies the condition $y(1) = 1$. So for each C we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

The equation above also has the solution $y = 0$. So the general solution is

$$y^2 + 2\ln|y| = x^2 + C, \qquad \text{and} \qquad y = 0.$$

These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

3.3 Examples of separable equations

Example 3.2: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$, $y(1) = 0$.

Factor the right-hand side

$$x^2y' = (1 - x^2)(1 + y^2).$$

Separate variables, integrate, and solve for y:

$$\begin{aligned}
\frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\
\frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\
\arctan(y) &= \frac{-1}{x} - x + C, \\
y &= \tan\left(\frac{-1}{x} - x + C\right).
\end{aligned}$$

Solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.). The particular solution we seek is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 3.3: Bob made a cup of coffee, and Bob likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t = 0$ minutes, Bob measured the temperature and the coffee was 89 degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. Newton's law of cooling states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants):

$$\begin{aligned} \frac{1}{T-A}\frac{dT}{dt} &= -k, \\ \ln(T - A) &= -kt + C, \qquad \text{(note that } T - A > 0\text{)} \\ T - A &= D\,e^{-kt}, \\ T &= A + D\,e^{-kt}. \end{aligned}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln\frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln\frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take. See Figure 9.

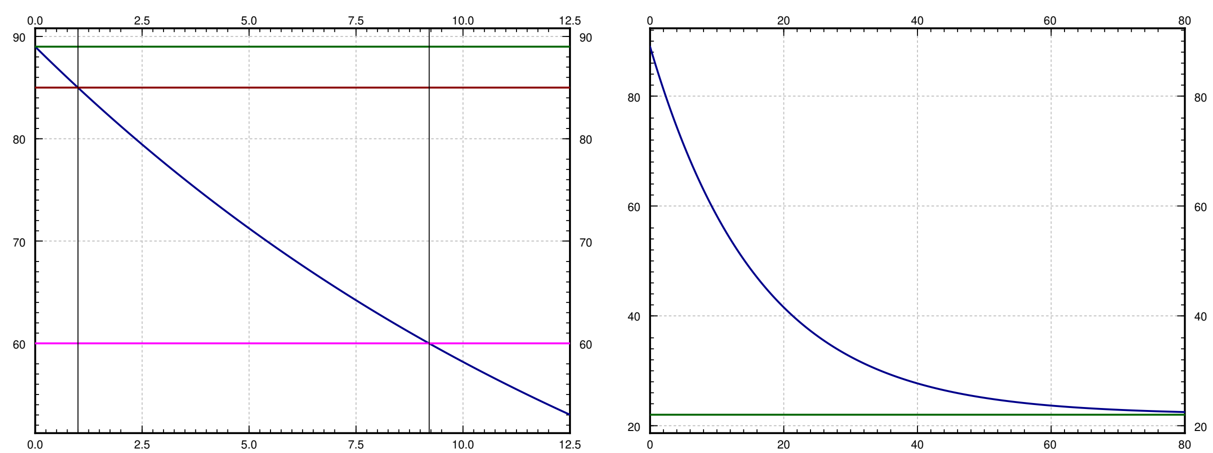


Figure 9: Graphs of the coffee temperature function $T(t)$. On the left, horizontal lines are drawn at temperatures 60, 85, and 89. Vertical lines are drawn at $t = 1$ and $t = 9.21$. Notice that the temperature of the coffee hits 85 at $t = 1$, and 60 at $t \approx 9.21$. On the right, the graph is over a longer period of time, with a horizontal line at the ambient temperature 22.

Example 3.4: Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neq 0$.

$$\frac{-3}{y^2}y' = x,$$

$$\frac{3}{y} = \frac{x^2}{2} + C,$$
$$y = \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}.$$

So the general solution is,

$$y = \frac{6}{x^2 + 2C}, \qquad \text{and} \qquad y = 0.$$

3.4 Exercises

Exercise* 3.1*: *Solve* $y' = x/y$.

Exercise* 3.2*: *Solve* $y' = x^2 y$.

Exercise* 3.3*: *Solve* $\frac{dx}{dt} = (x^2 - 1)\, t$, *for* $x(0) = 0$.

Exercise* 3.4*: *Solve* $\frac{dx}{dt} = x \sin(t)$, *for* $x(0) = 1$.

Exercise* 3.5*: *Solve* $\frac{dy}{dx} = xy + x + y + 1$. *Hint: Factor the right-hand side.*

Exercise* 3.6*: *Solve* $xy' = y + 2x^2 y$, *where* $y(1) = 1$.

Exercise* 3.7*: *Solve* $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, *for* $y(0) = 1$.

Exercise* 3.8*: *Find an implicit solution for* $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, *for* $y(0) = 1$.

Exercise* 3.9*: *Find an explicit solution for* $y' = xe^{-y}$, $y(0) = 1$.

Exercise* 3.10*: *Find an explicit solution for* $xy' = e^{-y}$, *for* $y(1) = 1$.

Exercise* 3.11*: *Find an explicit solution for* $y' = ye^{-x^2}$, $y(0) = 1$. *It is alright to leave a definite integral in your answer.*

Exercise* 3.12*: *Suppose a cup of coffee is at* 100 *degrees Celsius at time* $t = 0$, *it is at* 70 *degrees at* $t = 10$ *minutes, and it is at* 50 *degrees at* $t = 20$ *minutes. Compute the ambient temperature.*

Exercise* 3.101*: *Solve* $y' = 2xy$.

Exercise* 3.102*: *Solve* $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

Exercise* 3.103*: *Find an implicit solution for* $x' = \frac{1}{3x^2+1}$, $x(0) = 1$.

Exercise* 3.104*: *Find an explicit solution to* $xy' = y^2$, $y(1) = 1$.

Exercise* 3.105*: *Find an implicit solution to* $y' = \frac{\sin(x)}{\cos(y)}$.

Exercise* 3.106*: *Take Example* 3.3 *with the same numbers:* 89 *degrees at* $t = 0$, 85 *degrees at* $t = 1$, *and ambient temperature of* 22 *degrees. Suppose these temperatures were measured with precision of* ± 0.5 *degrees. Given this imprecision, the time it takes the coffee to cool to (exactly)* 60 *degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.*

Exercise* 3.107*: *A population* x *of rabbits on an island is modeled by* $x' = x - (1/1000)x^2$, *where the independent variable is time in months. At time* $t = 0$, *there are* 40 *rabbits on the island.*

a) *Find the solution to the equation with the initial condition.*

b) *How many rabbits are on the island in* 1 *month,* 5 *months,* 10 *months,* 15 *months (round to the nearest integer).*