INTERNATIONAL UNIVERSITY-VNUHCM

FINAL EXAMINATION

Semester 1, Academic Year 2016-2017
Duration: 120 minutes

SUBJECT: Calculus 3	
Chair of Department of Mathematics	Lecturer:
Signature:	Signature:
Full name: Assoc.Prof. Nguyen Dinh	Full names: Assoc.Prof. Mai Duc Thanh

- **Each student is allowed a maximum of two double-sided sheets of reference material (of size A4 or similar) and a scientific calculator. All other documents and electronic devices are forbidden.**
- **Each question carries 20 marks.**

Question 1. Use Laplace transform methods, obtain the solution $x(t), t \geq 0$, of the problem

$$x"(t) - 3x'(t) + 2x(t) = f(t), \qquad \text{subject to} \quad x(0) = 0,\ x'(0) = 1,$$

where $f(t)$ is the pulse function: $f(t) = \begin{cases} 2, & 0 \leq t < 3, \\ 0, & t \geq 3, \end{cases}$

Question 2. Determine the z transforms of the causal sequences

$$\text{(a)} \quad \{(\frac{1}{3})^k\} \qquad\qquad \text{(b)} \quad \{(k\frac{1}{3})^k\}$$

Question 3. Using z-transform methods, solve the difference equation

$$y_{k+2} - 3y_{k+1} - 4y_k = \frac{1}{3^k} \qquad \text{subject to} \quad y_0 = 0,\ y_1 = 1$$

Question 4. A periodic function $f(t)$ of period 2 is defined by

$$f(t) = \begin{cases} 2t, & 0 < t < 1, \\ 1, & 1 < t < 2, \end{cases}, \qquad f(t+2) = f(t), \quad -\infty < t < \infty$$

Determine a Fourier series expansion for the function $f(t)$.

Question 5. Calculate the Fourier transform of the windowed cosine function

$$f(t) = \cos 3t[H(t+1) - H(t-1)]$$

CALCULUS 3
Solutions for Final Exam

Question 1. Use Laplace transform methods, obtain the solution $x(t), t \geq 0$, of the problem

$$x"(t) - 3x'(t) + 2x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 1,$$

where $f(t)$ is the pulse function

$$f(t) = \begin{cases} 2, & 0 \leq t < 3, \\ 0, & t \geq 3, \end{cases}$$

$$f(t) = 2[H(t) - H(t-3)]$$

So, by the delaying property

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{2}{s}e^{-3s}$$

Taking Laplace transform both sides of DE $x"(t) - 3x'(t) + 2x(t) = f(t)$ gives

$$(s^2 - 3s + 2)X(s) = sx(0) + x'(0) - 3x(0) + \mathcal{L}\{f(t)\} = 1 + \frac{2}{s} - \frac{2}{s}e^{-3s}$$

That is

$$X(s) = \frac{s+2}{s(s-1)(s-2)} - e^{-3s}\frac{2}{s(s-1)(s-2)}$$

$$X(s) = \frac{1}{s} - \frac{3}{s-1} + \frac{2}{s-2} - e^{-3s}[\frac{1}{s} - \frac{2}{s-1} + \frac{1}{s-2}]$$

Taking inverse Laplace transform gives

$$x(t) = 1 - 3e^t + 2e^{2t} - (1 - 2e^{t-3} + e^{2(t-3)})H(t-3)$$

Question 2. (a)

$$\mathbf{Z}\{(1/3)^k\} = \frac{1}{1 - 1/(3z)} = \frac{3z}{3z-1} = 1 + \frac{1}{3z-1}, \quad |z| > 1/3$$

(b) By property of multiplication by k:

$$\mathbf{Z}\{(k\frac{1}{3})^k\} = -z\frac{d}{dz}X(z) = -z\frac{d}{dz}(1 + \frac{1}{3z-1})$$
$$= -z(-3(3z-1)^{-2}) = \frac{3z}{(3z-1)^2}, \quad |z| > 1/3$$

Question 3. Apply z-transform to both sides of the DE

$$y_{k+2} - 3y_{k+1} - 4y_k = \frac{1}{3^k}$$

and use advancing property to get

$$(z^2Y(z) - z^2y_0 - zy_1) - 3(zY(z) - zy_0) - 4Y(z) = \frac{3z}{3z-1}$$

Use initial conditions $y_0 = 0, y_1 = 1$, we get

$$(z^2Y(z) - z) - 3zY(z) - 4Y(z) = \frac{3z}{3z-1}$$

or

$$(z^2 - 3z - 4)Y(z) = \frac{3z}{3z-1} + z = \frac{3z^2+2z}{3z-1}$$

This yields

$$\frac{Y(z)}{z} = \frac{3z+2}{(3z-1)(z^2-3z-4)} = \frac{3z+2}{(3z-1)(z+1)(z-4)}$$

Resolving into partial fractions gives

$$\frac{Y(z)}{z} = \frac{-27}{44}\frac{1}{3z-1} - \frac{1}{20}\frac{1}{z+1} + \frac{14}{55}\frac{1}{z-4}$$

Thus

$$Y(z) = \frac{-9}{44}\frac{z}{z-1/3} - \frac{1}{20}\frac{z}{z+1} + \frac{14}{55}\frac{z}{z-4}$$

Taking inverse z-transform gives us

$$\{y_k\} = \{\frac{-9}{44}\frac{1}{3^k} - \frac{1}{20}(-1)^k + \frac{14}{55}4^k\}$$

Question 4. $T = 2$, so $\omega = 2\pi/T = \pi$.

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \int_0^1 2tdt + \int_1^2 1dt = t^2\Big|_0^1 + t\Big|_1^2 = 1 + 1 = 2,$$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega t dt = \int_0^1 2t\cos n\pi t dt + \int_1^2 \cos n\pi t dt \\
&= \int_0^1 \frac{2}{n\pi} t d\sin n\pi t + \frac{2}{n\pi}\sin n\pi t\Big|_1^2 \\
&= \frac{2}{n\pi}\sin n\pi t\Big|_0^1 - \int_0^1 \frac{2}{n\pi}\sin n\pi t dt + 0 \\
&= \frac{2}{(n\pi)^2}\cos n\pi t\Big|_0^1 \\
&= \frac{2}{(n\pi)^2}((-1)^n - 1)
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^T f(t)\sin n\omega t dt = \int_0^1 2t\sin n\pi t dt + \int_1^2 \sin n\pi t dt \\
&= \int_0^1 \frac{-2}{n\pi} t d\cos n\pi t - \frac{1}{n\pi}\cos n\pi t\Big|_1^2 \\
&= \frac{-2}{n\pi} t\cos n\pi t\Big|_0^1 + \int_0^1 \frac{2}{n\pi}\cos n\pi t dt - \frac{1}{n\pi}(\cos 2n\pi - \cos n\pi) \\
&= \frac{-2}{n\pi}\cos n\pi + \frac{2}{(n\pi)^2}\sin n\pi t\Big|_0^1 - \frac{1}{n\pi}(1 - (-1)^n) \\
&= -\frac{1}{n\pi}(1 + (-1)^n)
\end{aligned}$$

The Fourier series expansion of $f(t)$ is therefore given by

$$f(t) = 1 + \sum_{n=1}^{\infty} \left(\frac{2}{(n\pi)^2}((-1)^n - 1)\cos n\pi t - \frac{1}{n\pi}(1 + (-1)^n)\sin n\pi t \right)$$

Question 5.

$$\begin{aligned}
F(j\omega) &= \int_{-1}^{1} \cos 3t e^{-j\omega t} dt = \int_{-1}^{1} \frac{e^{j3t} + e^{-j3t}}{2} e^{-j\omega t} dt \\
&= \frac{1}{2} \int_{-1}^{1} (e^{-j(\omega-3)t} + e^{-j(\omega+3)t}) dt \\
&= \frac{1}{2} \left(\frac{e^{-j(\omega-3)t}}{-j(\omega-3)} + \frac{e^{-j(\omega+3)t}}{-j(\omega+3)} \right) \Big|_{-1}^{1} \\
&= \frac{-e^{-j(\omega-3)} + e^{j(\omega-3)}}{2j(\omega-3)} + \frac{-e^{-j(\omega+3)} + e^{j(\omega+3)}}{2j(\omega+3)} \\
&= \frac{\sin(\omega-3)}{(\omega-3)} + \frac{\sin(\omega+3)}{(\omega+3)}
\end{aligned}$$