HO CHI MINH CITY INTERNATIONAL UNIVERSITY

MIDTERM TEST

Semester 1, Academic year 2019-2020
Duration: 90 minutes

SUBJECT: **Calculus 2**	
Chair of Department of Mathematics	Lecturer:
Signature:	Signature:
Full name:	Full name: Assoc.Prof. Mai Duc Thanh

<u>**Instructions:**</u>

- **Each student is allowed a maximum of two double-sided sheets of reference material (of size A4 or similar). All other documents and electronic devices, except scientific calculators, are not allowed.**
- **Each question carries 20 marks.**

Question 1. Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a)\quad a_n = \frac{\ln(2n+1)}{\ln(n+2)} \qquad\qquad b)\quad b_n = \frac{3(-1)^n n^2 - 2}{n^2+4n+3}$$

Question 2. Determine whether the series is convergent or divergent

$$a)\quad \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}} \qquad\qquad b)\quad \sum_{n=1}^{\infty} \frac{2^{1/n}}{n}$$

.

Question 3. Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{(x-1)^n}{\sqrt{n^3+2}}$

Question 4. Recall that $\text{proj}_a b$ denotes the vector projection of b onto a.

a) Find the scalar and vector projections of $b =< 1, 3, 2 >$ onto $a =< 0, -3, 4 >$;

b) Show that the vector $c = b - \text{proj}_a b$ is orthogonal to a.

Question 5. (a) Find the limit of vector function:

$$\lim_{t\to\infty} \left\langle t \sin\frac{1}{t}, \frac{2t^2+3t+1}{t^2-1}, te^{-t} \right\rangle.$$

b) Evaluate the integral of vector function:

$$\int_0^1 \mathbf{r}(t)\ dt, \quad \text{where } \mathbf{r}(t) =< 2t+3, t\ln(t+1), 9t\sqrt{3t+1} > .$$

*** END OF QUESTIONS ***

CALCULUS 2

Solutions for Mid-term Test

Question 1. a) Applying L'Hospital rule (considering n as real variable) gives us

$$\lim_{n\to\infty} \frac{\ln(2n+1)}{\ln(n+2)} = \lim_{n\to\infty} \frac{2/(2n+1)}{1/(n+2)} = 1.$$

b) If n is even, then

$$\lim_{n\to\infty} \frac{3(-1)^n n^2 - 2}{n^2+4n+3} = \lim_{n\to\infty} \frac{3n^2-2}{n^2+4n+3} = \lim_{n\to\infty} \frac{3-2/n^2}{1+4/n+3/n^2} = 3$$

If n is odd, then

$$\lim_{n\to\infty} \frac{3(-1)^n n^2 - 2}{n^2+4n+3} = \lim_{n\to\infty} \frac{-3n^2-2}{n^2+4n+3} = \lim_{n\to\infty} \frac{-3-2/n^2}{1+4/n+3/n^2} = -3$$

So, the limit does not exists.

Question 2. a) $\sum_{n=1}^{\infty} \dfrac{1}{n\sqrt{n^2+1}}$

It holds that

$$0 < a_n = \frac{1}{n\sqrt{n^2+1}} < \frac{1}{n^2}$$

for all positive n. The 2-series is convergent, so the given series is convergent, by Comparison Test.

b) It holds that

$$\frac{2^{1/n}}{n} > \frac{1}{n}$$

for all positive n. The 1-series is divergent, so the given series is also divergent, by Comparison Test.

Question 3. Set

$$a_n = \frac{(x-1)^n}{\sqrt{n^3+2}}.$$

It holds that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| (x-1)\sqrt{\frac{n^3+2}{(n+1)^3+2}} \right| = |x-1|.$$

By Ratio Test, the series converges if $|x-1| < 1$ and diverges if $|x-1| > 1$. Thus, the radius of convergence is $R = 1$.

We now check the convergence at the endpoints $|x-1| = 1$, or $x = 0, x = 2$. When $x = 2$ the series becomes

$$\sum_{n=1}^{\infty} 1/\sqrt{n^3+2}.$$

This series is convergent, since $0 < 1/\sqrt{n^3+2} < 1/n^{3/2}$.

When $x = 0$ the series becomes

$$\sum_{n=1}^{\infty} (-1)^n/\sqrt{n^3+2}.$$

This series is absolutely convergent, as argued above. So, the interval of convergence is

$$0 \le x \le 2$$

Question 4. a) $b =< 1, 3, 2 >$ onto $a =< 0, -3, 4 >$

$$\text{comp}_a b = \frac{a \cdot b}{|a|} = \frac{0 - 9 + 8}{\sqrt{9+16}} = \frac{-1}{5}$$

and

$$\text{proj}_a b = \left(\frac{a \cdot b}{|a|}\right) \frac{a}{|a|} =< 0, \frac{3}{25}, \frac{-4}{25} >$$

b)

$$c \cdot a = (b - \text{proj}_a b) \cdot a = \left(b - \frac{a \cdot b}{|a|^2} a\right) \cdot a = a \cdot b - \frac{a \cdot b}{|a|^2} a \cdot a = 0$$

So, c is orthogonal to a.

Question 5. (a) It holds for $s = 1/t$ that

$$\lim_{t \to \infty} t \sin \frac{1}{t} = \lim_{t \to \infty} \frac{\sin 1/t}{1/t} = \lim_{s \to 0} \frac{\sin s}{s} = 1,$$

$$\lim_{t \to \infty} \frac{2t^2 + 3t + 1}{t^2 - 1} = \lim_{t \to \infty} \frac{2 + 3/t + 1/t^2}{1 - 1/t^2} = 2,$$

and

$$\lim_{t \to \infty} t e^{-t} = \lim_{t \to \infty} \frac{t}{e^t} = \lim_{t \to \infty} \frac{1}{e^t} = 0$$

So

$$\lim_{t \to \infty} \left\langle t \sin \frac{1}{t}, \frac{2t^2 + 3t + 1}{t^2 - 1}, t e^{-t} \right\rangle =< 1, 2, 0 >$$

b) Evaluate

$$\int_0^1 \mathbf{r}(t)\, dt, \quad \text{where } \mathbf{r}(t) =< 2t + 3, t \ln(t+1), 9t\sqrt{3t+1} > .$$

It holds that

$$\int_0^1 (2t + 3) dt = (t^2 + 3t)\Big|_0^1 = 4,$$

$$
\begin{aligned}
\int_0^1 t\ln(t+1)dt &= \int_0^1 \ln(t+1)dt^2/2 \\
&= (t^2/2)\ln(t+1)\Big|_0^1 - \int_0^1 t^2/2(t+1)dt \\
&= (1/2)\ln 2 - (1/2)\int_0^1 (t-1+1/(t+1)dt \\
&= (1/2)\ln 2 - (1/2)[(t^2/2-t)+\ln(t+1)]\Big|_0^1 \\
&= (1/2)\ln 2 - (1/2)[(-1/2+\ln 2] = 1/4
\end{aligned}
$$

and set $u = \sqrt{3t+1}, u^2 = 3t+1, 2udu = 3dt$, so

$$
\begin{aligned}
\int_0^1 9t\sqrt{3t+1}dt &= \int_1^2 (u^2-1)u(2u)du \\
&= \int_1^2 (2u^4-2u^2)du \\
&= 2u^5/5 - 2u^3/3\Big|_1^2 = 64/5 - 16/3 - 2/5 + 2/3 = 116/15
\end{aligned}
$$

Thus, $\int_0^1 \mathbf{r}(t)\ dt =< 4, 1/4, 116/15 >$