MIDTERM EXAMINATION
Answer PROBABILITY, STATISTICS AND RANDOM PROCESS
Semester 2, 2022-23 • March 2023 • Total duration: 90 minutes

1. Sample space $\Omega = \{\overline{a_1a_2a_3a_4a_5} : a_i = 1,2,\ldots,9\ \forall i \in \{1,\ldots,5\}\}$. Hence $n(\Omega) = 9^5$. A is the event that no digit appears more than twice. Need to compute $P(A)$

   1st approach: A^c is the event that some digit appears more than twice (three, four or five times). We have

   $$n(A^c) = C(5,3)(9)(8)(8) + C(5,4)(9)(8) + C(5,5)(9)$$

   Then
   $$P(A^c) = \frac{n(A^c)}{n(A)} \Longrightarrow P(A) = 1 - p(A^c) \approx 0.8962$$

   2nd approach: compute $n(A)$ directly by considering 3 possible cases (all selected digits are different, only one digit appears twice and two different digit appear twice)

   $$n(A) = (9)(8)(7)(6)(5) + C(5,2)(9)(8)(7)(6) + \frac{C(5,2)C(3,2)}{2}(9)(8)(7)$$

   So $P(A) = \frac{n(A)}{n(\Omega)} \approx 0.8962$

2. By total law
   P(select a red) = P(select box 1 and draw a red) + P(select box 2 and draw a red)
   where
   P(select box 1 and draw a red) = P(select box 1)P(draw a red | select box 1) = $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)$
   and
   P(select box 2 and draw a red) = P(select box 2)P(draw a red | select box 1) = $\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$
   So
   $$P((\text{select a red})) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{11}{24} \approx 0.45833$$

3. Need to compute

   $$P(\text{send 1}|\text{receive 1}) = \frac{P(\text{send 1 and receive 1})}{P(\text{receive 1})}$$

   We have

   $$\begin{aligned}
   P(\text{send 1 and receive 1}) &= P(\text{send 1})P(\text{receive 1}|\text{ send 1})\\
   &= P(\text{send 1})P(\text{not error}|\text{ send 1}) = (0.5)(1-0.1)\\
   P(\text{send 0 and receive 1}) &= P(\text{send 0})P(\text{receive 1}|\text{ send 0})\\
   &= P(\text{send 0})P(\text{ error}|\text{ send 0}) = (0.5)(0.05)\\
   P(\text{ receive 1}) &= P(\text{send 1 and receive 1}) + P(\text{send 0 and receive 1})\\
   &= (0.5)(1-0.1) + (0.5)(0.05)
   \end{aligned}$$

   So
   $$P(\text{send 1}|\text{receive 1}) = \frac{(0.5)(1-0.1)}{(0.5)(1-0.1)+(0.5)(0.05)} = \frac{18}{19} \approx 0.947$$

4. *(20 points)*

   (a) $E(X) = (0.1)(10) + (0.35) * (13) + (0.4)(16) + (0.15)(20) = 14.95$
   $E(X^2) = (0.1)(10^2) + (0.35) * (13^2) + (0.4)(16^2) + (0.15)(20^2) = 231.55$
   and $Var(X) = E(X^2) - (E(X)^2) = 231.55 - (14.95)^2 \approx 8.0475$

   (b) $P(X > 12 \mid X < 17) = \frac{P(12<X<17)}{P(X<17)} = \frac{P(X=13)+P(X=16)}{P(X+10)+P(X=13)+P(X=16)} = \frac{0.35+0.4}{0.1+0.35+0.4} = \frac{75}{85} \approx 0.8824.$

5. (a) P (A wins at least two games) = $P(W_1W_2W_3) + P(\overline{W}_1W_2W_3) + P(W_1\overline{W}_2W) + P(W_1W_2\overline{W}_3)$
   where

$$P(W_1W_2W_3) = P(W_1)P(W_2)P(W_3) = (0.4)^3$$
$$P(\overline{W}_1W_2W_3) = P(W_1\overline{W}_2W) = P(W_1W_2\overline{W}_3) = (0.4)^2(0.6)$$

   So

$$P(\text{A wins at least two games}) = (0.4)^3 + (3)(0.4)^2(0.6) \approx 0.352$$

   (b) $Range(X) = \{-3,-2,-1,0,1,2,3\}$ and $X = X_1 + X_2 + X_3$ where X_i is the point that A gets at game i.
   The p.m.f of X_i for $i = 1,2,3$ is given by

x	-1	0	1
$P(X_i = x)$	0.4	0.2	0.4

   We have $E(X_i) = (0.4)(-1) + (0.2)(0) + (0.4)(1) = 0$. So

$$E(X) = E(X_1) + E(X_2) + E(X_3) = 0$$

   You can compute p.m.f of X and then evaluate $E(X)$ directly.

6. (a) Set up $p_1 = \int_{2000}^{3000} f(x)dx$

   (b) Set up $p = \int_0^{3000} f(x)dx$ - the probability that the device need to be replaced and then we need to pay an extra \$100 for replacement.
   Set up the average of total cost

$$p(100+20) + (1-p)(20)$$

7. (a) $P(X = x, Y = y) = \frac{\binom{24}{x}\binom{48}{y}\binom{8}{3-x-y}}{\binom{80}{3}}$ for all possible pair values $x = 0,1,2,3$ and $y = 0,1,2,3$.
   You can display the joint pmf in a joint table.

   (b) Verify that $P(X = 2, Y = 2) \neq P(X = 2)P(Y = 2)$. Hence X and Y are not independent.