VIETNAM NATIONAL UNIVERSITY-HCMC
International University

Lecture Notes/Slides for

APPLIED LINEAR ALGEBRA

Chapter 2. Determinants

Determinant of a 2×2 Matrix

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the determinant of A is defined as

$$\det A = ad - bc$$

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What about the determinant of an $n \times n$ matrix for other values of n?

How do we find the determinant of an $n \times n$ matrix?

The determinant of an $n \times n$ matrix is defined recursively, using determinants of $(n-1) \times (n-1)$ submatrices, and requires some new definitions and notation.

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Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The sign of the (i, j) position is $(-1)^{i+j}$. Thus the sign is 1 if $(i + j)$ is even, and -1 if $(i + j)$ is odd.

The Minor of a Matrix

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The ij^{th} minor of A, denoted as $minor\,(A)_{ij}$, is the determinant of the $n-1 \times n-1$ matrix which results from deleting the i^{th} row and the j^{th} column of A.

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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

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For any matrix A, $minor\,(A)_{ij}$ is found by first removing the i^{th} row and j^{th} column, and taking the determinant of the remaining matrix.

The Minor of a Matrix

Example

Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Find $minor(A)_{12}$.

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$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

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Solution

First, remove the 1^{st} row and 2^{nd} column from A.

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Let

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Find $minor(A)_{12}$.

Solution

First, remove the 1^{st} row and 2^{nd} column from A.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

The resulting matrix is $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$

Solution (continued)

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Using our previous definition, we can calculate the determinant of this matrix to be

$$(2)(6) - (5)(1) = 12 - 5 = 7$$

Solution (continued)

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Using our previous definition, we can calculate the determinant of this matrix to be

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Therefore, $minor(A)_{12} = 7$.

The Cofactors of a Matrix

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Example (continued)

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Find $\mathrm{cof}(A)_{12}$.

Solution

By the definition, we know that $\text{cof}(A)_{12} = (-1)^{1+2} \textit{minor}\,(A)_{12}$

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Therefore, $\mathrm{cof}(A)_{12} = (-1)^{1+2} \mathit{minor}\,(A)_{12} = (-1)^3 7 = -7$

Cofactor Expansion

Using these definitions, we can now define the determinant of the $n \times n$ matrix A:

Definition

$\det A = a_{11}\mathrm{cof}(A)_{11} + a_{12}\mathrm{cof}(A)_{12} + a_{13}\mathrm{cof}(A)_{13} + \cdots + a_{1n}\mathrm{cof}(A)_{1n}$
This is called the cofactor expansion of $\det A$ along row 1.

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This is called the cofactor expansion of $\det A$ along row 1.

In other words,

$$\det(A) = \sum_{j=1}^{n} a_{ij}\mathrm{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\mathrm{cof}(A)_{ij}$$

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Cofactor expansion is also called Laplace Expansion.

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Solution

Using cofactor expansion along row 1,

$$\det A \quad = \quad 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A)$$

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Example

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Solution

Using cofactor expansion along row 1,

$$\begin{aligned} \det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\ &= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \end{aligned}$$

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Using cofactor expansion along row 1,

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Cofactor Expansion

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Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$. Find $\det A$.

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Using cofactor expansion along row 1,

$$\begin{aligned} \det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\ &= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \\ &= 1(24-2) - 1(12-5) + 3(4-20) \\ &= 22 - 7 + 3(-16) \end{aligned}$$

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Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$. Find $\det A$.

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$$\begin{aligned} \det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\ &= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \\ &= 1(24-2) - 1(12-5) + 3(4-20) \\ &= 22 - 7 + 3(-16) \\ &= 22 - 7 - 48 \end{aligned}$$

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Using cofactor expansion along row 1,

$$\begin{aligned} \det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\ &= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \\ &= 1(24-2) - 1(12-5) + 3(4-20) \\ &= 22 - 7 + 3(-16) \\ &= 22 - 7 - 48 \\ &= -33 \end{aligned}$$

Solution (continued)

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Now try cofactor expansion along column 2.

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$$\begin{aligned}
\det A &= 1\text{cof}_{12}(A) + 4\text{cof}_{22}(A) + 2\text{cof}_{32}(A) \\
&= 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 4(-1)^4 \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} + 2(-1)^5 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \\
&= -1(12-5) + 4(6-15) - 2(1-6) \\
&= -(7) + 4(-9) - 2(-5) \\
&= -7 - 36 + 10 \\
&= -33
\end{aligned}$$

Solution (continued)

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\det A &= 1\mathrm{cof}_{12}(A) + 4\mathrm{cof}_{22}(A) + 2\mathrm{cof}_{32}(A) \\
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\end{aligned}$$

We get the same answer!

The Determinant is Well Defined

Theorem

The determinant of an $n \times n$ matrix A can be computed using cofactor expansion along any row or column of A.

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Theorem

The determinant of an $n \times n$ matrix A can be computed using cofactor expansion along any row or column of A.

What is the significance of this theorem?

This theorem allows us to choose any row or column for computing cofactor expansion, which gives us the opportunity to save ourselves some work!

Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

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Solution

Cofactor expansion along row 1 yields

$$\det A = 0 \times \text{cof}(A)_{11} + 1 \times \text{cof}(A)_{12} + (-2) \times \text{cof}(A)_{13} + 1 \times \text{cof}(A)_{14}$$

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$$\begin{aligned} \det A &= 0 \times \mathrm{cof}(A)_{11} + 1 \times \mathrm{cof}(A)_{12} + (-2) \times \mathrm{cof}(A)_{13} + 1 \times \mathrm{cof}(A)_{14} \\ &= \mathrm{cof}(A)_{12} - 2 \times \mathrm{cof}(A)_{13} + \mathrm{cof}(A)_{14}, \end{aligned}$$

Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Solution

Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0 \times \operatorname{cof}(A)_{11} + 1 \times \operatorname{cof}(A)_{12} + (-2) \times \operatorname{cof}(A)_{13} + 1 \times \operatorname{cof}(A)_{14} \\ &= \operatorname{cof}(A)_{12} - 2 \times \operatorname{cof}(A)_{13} + \operatorname{cof}(A)_{14}, \end{aligned}$$

whereas cofactor expansion along, row 3 yields

$$\det A = 0 \times \operatorname{cof}(A)_{31} + 1 \times \operatorname{cof}(A)_{32} + (-1) \times \operatorname{cof}(A)_{33} + 0 \times \operatorname{cof}(A)_{34}$$

Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Solution

Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0 \times \mathrm{cof}(A)_{11} + 1 \times \mathrm{cof}(A)_{12} + (-2) \times \mathrm{cof}(A)_{13} + 1 \times \mathrm{cof}(A)_{14} \\ &= \mathrm{cof}(A)_{12} - 2 \times \mathrm{cof}(A)_{13} + \mathrm{cof}(A)_{14}, \end{aligned}$$

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Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

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Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0 \times \mathrm{cof}(A)_{11} + 1 \times \mathrm{cof}(A)_{12} + (-2) \times \mathrm{cof}(A)_{13} + 1 \times \mathrm{cof}(A)_{14} \\ &= \mathrm{cof}(A)_{12} - 2 \times \mathrm{cof}(A)_{13} + \mathrm{cof}(A)_{14}, \end{aligned}$$

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i.e., in the first case we must compute three cofactors, but in the second case we need only compute two cofactors.

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\det A = 1 \times \mathrm{cof}(A)_{32} + (-1) \times \mathrm{cof}(A)_{33}$$

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1 \times \operatorname{cof}(A)_{32} + (-1) \times \operatorname{cof}(A)_{33} \\ &= 1 \times (-1)^5 \begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1) \times (-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \end{aligned}$$

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1 \times \operatorname{cof}(A)_{32} + (-1) \times \operatorname{cof}(A)_{33} \\ &= 1 \times (-1)^5 \begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1) \times (-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \end{aligned}$$

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

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$$\begin{aligned} \det A &= 1 \times \mathrm{cof}(A)_{32} + (-1) \times \mathrm{cof}(A)_{33} \\ &= 1 \times (-1)^5 \begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1) \times (-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \end{aligned}$$

Each of the two determinants above can easily be evaluated using cofactor expansion along column 2.

Solution (continued)

$$\det A = -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix}$$

Solution (continued)

$$\begin{aligned}\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix}\end{aligned}$$

Solution (continued)

$$
\begin{aligned}
\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= -2(10-21) + 1(10-21)
\end{aligned}
$$

Solution (continued)

$$\begin{aligned}\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\ &= -2(10-21) + 1(10-21) \\ &= -2(-11) + (-11)\end{aligned}$$

Solution (continued)

$$
\begin{aligned}
\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= -2(10-21) + 1(10-21) \\
&= -2(-11) + (-11) \\
&= 22 - 11
\end{aligned}
$$

Solution (continued)

$$
\begin{aligned}
\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= -2(10-21) + 1(10-21) \\
&= -2(-11) + (-11) \\
&= 22 - 11 \\
&= 11.
\end{aligned}
$$

Therefore, $\det A = 11$.

A Row or Column of Zeros

Example

Let

$$A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}.$$

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By choosing column 3 for cofactor expansion, we get $\det A = 0$,

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By choosing column 3 for cofactor expansion, we get $\det A = 0$, i.e.,

$$\det A = 0 \times \text{cof}(A)_{13} + 0 \times \text{cof}(A)_{23} + 0 \times \text{cof}(A)_{33} + 0 \times \text{cof}(A)_{43}$$

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By choosing column 3 for cofactor expansion, we get $\det A = 0$, i.e.,

$$\det A = 0 \times \mathrm{cof}(A)_{13} + 0 \times \mathrm{cof}(A)_{23} + 0 \times \mathrm{cof}(A)_{33} + 0 \times \mathrm{cof}(A)_{43} = 0.$$

A Row or Column of Zeros

Example

Let

$$A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}.$$

By choosing column 3 for cofactor expansion, we get $\det A = 0$, i.e.,

$$\det A = 0 \times \mathrm{cof}(A)_{13} + 0 \times \mathrm{cof}(A)_{23} + 0 \times \mathrm{cof}(A)_{33} + 0 \times \mathrm{cof}(A)_{43} = 0.$$

Important Fact

If A is an $n \times n$ matrix with a row or column of zeros, then $\det A = 0$.

Determinants of a Triangular Matrices

Definitions

1. An $n \times n$ matrix A is called upper triangular if all entries **below** the main diagonal are zero.

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3. An $n \times n$ matrix A is called triangular if it is upper triangular or lower triangular.

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3. An $n \times n$ matrix A is called triangular if it is upper triangular or lower triangular.

Theorem

If $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\det A = a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn},$$

i.e., $\det A$ is the product of the entries of the main diagonal of A.

Example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

Example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix}$$

Example

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\
&= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix}
\end{aligned}
$$

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$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix}$$

$$= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix}$$

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Example

$$
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\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\
&= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix} \\
&= 1 \times 5 \times 9 \\
&= 45.
\end{aligned}
$$

Example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix}$$
$$= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix}$$
$$= 1 \times 5 \times 9$$
$$= 45.$$

Notice that 45 is the product of the entries on the main diagonal.

$$\begin{bmatrix} {\color{blue}1} & 2 & 3 \\ 0 & {\color{blue}5} & 6 \\ 0 & 0 & {\color{blue}9} \end{bmatrix}$$

Elementary Row Operations and Determinants

Example

Let

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

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Elementary Row Operations and Determinants

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$$A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

Computing $\det A$ by cofactor expansion along row (or column) 2 yields

$$\det A = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = 4(-1) = -4.$$

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Let B_1, B_2, and B_3 be obtained from A by interchanging rows 1 and 2, multiplying row 3 by -3, and adding twice row 1 to row 3, respectively, i.e.,

Elementary Row Operations and Determinants

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Let B_1, B_2, and B_3 be obtained from A by interchanging rows 1 and 2, multiplying row 3 by -3, and adding twice row 1 to row 3, respectively, i.e.,

$$B_1 = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 4 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{bmatrix}, B_3 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 5 & 0 & -8 \end{bmatrix}.$$

Elementary Row Operations and Determinants

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We are interested in how elementary operations affect the determinant.

Elementary Row Operations and Determinants

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Computing $\det A$ by cofactor expansion along row (or column) 2 yields

$$\det A = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = 4(-1) = -4.$$

Let B_1, B_2, and B_3 be obtained from A by interchanging rows 1 and 2, multiplying row 3 by -3, and adding twice row 1 to row 3, respectively, i.e.,

$$B_1 = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 4 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{bmatrix}, B_3 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 5 & 0 & -8 \end{bmatrix}.$$

We are interested in how elementary operations affect the determinant. Computing $\det B_1$, $\det B_2$, and $\det B_3$:

Example (continued)

$$\det B_1 \quad = \quad 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1)\det A.$$

Example (continued)

$$\det B_1 = 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1)\det A.$$

$$\det B_2 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ -3 & 6 \end{vmatrix} = 4(12-9) = 4 \times 3 = 12 = -3\det A.$$

Example (continued)

$$\det B_1 = 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1)\det A.$$

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$$\det B_3 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 5 & -8 \end{vmatrix} = 4(-16+15) = 4(-1) = -4 = \det A.$$

Example (continued)

$$\det B_1 = 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1)\det A.$$

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$$\det B_3 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 5 & -8 \end{vmatrix} = 4(-16+15) = 4(-1) = -4 = \det A.$$

The general effects of elementary row operations on the determinant are summarized in the next theorem.

Theorem

Let A be an $n \times n$ matrix and B be an $n \times n$ matrix as defined below.

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Let A be an $n \times n$ matrix and B be an $n \times n$ matrix as defined below.

1. Let B be a matrix which results from switching two rows of A. Then $\det(B) = -\det(A)$.
2. Let B be a matrix which results from multiplying some row of A by a scalar k. Then $\det(B) = k\det(A)$.
3. Let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.
4. If A contains a row which is a multiple of another row of A, then $\det(A) = 0$

Theorem

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4. If A contains a row which is a multiple of another row of A, then $\det(A) = 0$

An analogous theorem holds for elementary column operation. If A is a matrix, then an elementary column operation on A is simply the corresponding elementary row operation performed on the transpose of A, A^T.

Computing the Determinant

Example

$$\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$$

Computing the Determinant

Example

$$\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} = \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix} \quad \text{row } 1 + 3\times(\text{row } 2)$$

Computing the Determinant

Example

$$\det\begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} = \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix} \quad \text{row } 1 + 3\times(\text{row } 2)$$

$$= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 0 & -2 & -5 \end{vmatrix} \quad \text{row } 3 - 2\times(\text{row } 2)$$

Computing the Determinant

Example

$$\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} = \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix} \quad \text{row } 1 + 3\times(\text{row } 2)$$

$$= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 0 & -2 & -5 \end{vmatrix} \quad \text{row } 3 - 2\times(\text{row } 2)$$

$$= (1)(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -2 & -5 \end{vmatrix} \quad \text{cofactor expansion: column } 1$$

Computing the Determinant

Example

$$
\begin{aligned}
\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} &= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix} \quad \text{row } 1 + 3\times(\text{row } 2) \\
&= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 0 & -2 & -5 \end{vmatrix} \quad \text{row } 3 - 2\times(\text{row } 2) \\
&= (1)(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -2 & -5 \end{vmatrix} \quad \text{cofactor expansion: column } 1 \\
&= -(-10 + 6)
\end{aligned}
$$

Computing the Determinant

Example

$$
\begin{aligned}
\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} &= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix} \quad \text{row } 1 + 3\times(\text{row } 2) \\
&= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 0 & -2 & -5 \end{vmatrix} \quad \text{row } 3 - 2\times(\text{row } 2) \\
&= (1)(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -2 & -5 \end{vmatrix} \quad \text{cofactor expansion: column } 1 \\
&= -(-10+6) \\
&= 4.
\end{aligned}
$$

Example

$$\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Example

$$\det\begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} = \left|\begin{matrix} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{matrix}\right]$$

Example

$$\det\begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} = \left|\begin{matrix} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{matrix}\right]$$

$$= (-1)(-1)^8 \begin{vmatrix} 7 & 5 & 10 \\ -1 & -3 & 8 \\ 6 & 4 & 15 \end{vmatrix}$$

Example

$$
\begin{aligned}
\det\begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} &= \left|\begin{matrix} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{matrix}\right] \\
&= (-1)(-1)^8 \begin{vmatrix} 7 & 5 & 10 \\ -1 & -3 & 8 \\ 6 & 4 & 15 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & -16 & 66 \\ -1 & -3 & 8 \\ 0 & -14 & 63 \end{vmatrix}
\end{aligned}
$$

Example

$$
\begin{aligned}
\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix}
&= \left| \begin{array}{rrrr} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{array} \right] \\
&= (-1)(-1)^8 \begin{vmatrix} 7 & 5 & 10 \\ -1 & -3 & 8 \\ 6 & 4 & 15 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & -16 & 66 \\ -1 & -3 & 8 \\ 0 & -14 & 63 \end{vmatrix} \\
&= -(-1)(-1)^3 \begin{vmatrix} -16 & 66 \\ -14 & 63 \end{vmatrix}
\end{aligned}
$$

Example

$$\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} = \left| \begin{matrix} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{matrix} \right]$$

$$= (-1)(-1)^8 \begin{vmatrix} 7 & 5 & 10 \\ -1 & -3 & 8 \\ 6 & 4 & 15 \end{vmatrix}$$

$$= - \begin{vmatrix} 0 & -16 & 66 \\ -1 & -3 & 8 \\ 0 & -14 & 63 \end{vmatrix}$$

$$= -(-1)(-1)^3 \begin{vmatrix} -16 & 66 \\ -14 & 63 \end{vmatrix}$$

$$= - \begin{vmatrix} -2 & 3 \\ -14 & 63 \end{vmatrix}$$

Example

$$
\begin{aligned}
\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} &= \left| \begin{matrix} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{matrix} \right] \\
&= (-1)(-1)^8 \begin{vmatrix} 7 & 5 & 10 \\ -1 & -3 & 8 \\ 6 & 4 & 15 \end{vmatrix} \\
&= - \begin{vmatrix} 0 & -16 & 66 \\ -1 & -3 & 8 \\ 0 & -14 & 63 \end{vmatrix} \\
&= -(-1)(-1)^3 \begin{vmatrix} -16 & 66 \\ -14 & 63 \end{vmatrix} \\
&= - \begin{vmatrix} -2 & 3 \\ -14 & 63 \end{vmatrix} \\
&= -(-126 + 42)
\end{aligned}
$$

Example

$$
\begin{aligned}
\det\begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix}
&= \left|\begin{matrix} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{matrix}\right] \\
&= (-1)(-1)^8 \begin{vmatrix} 7 & 5 & 10 \\ -1 & -3 & 8 \\ 6 & 4 & 15 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & -16 & 66 \\ -1 & -3 & 8 \\ 0 & -14 & 63 \end{vmatrix} \\
&= -(-1)(-1)^3 \begin{vmatrix} -16 & 66 \\ -14 & 63 \end{vmatrix} \\
&= -\begin{vmatrix} -2 & 3 \\ -14 & 63 \end{vmatrix} \\
&= -(-126 + 42) \\
&= 84.
\end{aligned}
$$

Problem

If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

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If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix}$$

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If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{aligned}
\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} &= (-1)(3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

Problem

If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{aligned}
\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} &= (-1)(3) \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3) \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1) \times 4
\end{aligned}$$

Problem

If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{aligned}
\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} &= (-1)(3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1) \times 4 \\
&= 12.
\end{aligned}$$

Problem

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Problem

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Solution

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix}$$

Problem

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Solution

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 9 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix}$$

Problem

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Solution

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 9 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = 0.$$

Problem

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Solution

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 9 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = 0.$$

Notice:

$$\text{row2} + \text{row3} - 2(\text{row1}) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Hence the determinant equals 0.

Problem

Let

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix}.$$

Show that $\det B = 9 \det A$.

Solution

$$\det B = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

Solution

$$\det B = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} \xrightarrow{R_1+(-2)R_3} \begin{vmatrix} p-4x & q-4y & r-4z \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

Solution

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Theorem

If A is an $n \times n$ matrix and k is any scalar, then

$$\det(kA) = k^n \det A.$$

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An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

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Let $A = \begin{bmatrix} 2 & -3 \\ -6 & 4 \end{bmatrix}$.

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Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

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$$= c(10 - c^2) - c$$

Problem

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

Solution

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} \begin{aligned} &= c\begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix} \\ &= c(10 - c^2) - c \\ &= c(9 - c^2) \end{aligned}$$

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Since A is invertible when $\det(A) \neq 0$, A is invertible for all $c \neq 0, 3, -3$.

Determinants of Products and Transposes

Theorem

Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det A \times \det B.$$

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If A is an $n \times n$ matrix, then the determinant of its transpose is given by

$$\det(A^T) = \det A.$$

Problem

Suppose A, B and C are 4×4 matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$

Find $\det(2A^2(B^{-1})(C^T)^3B(A^{-1}))$.

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$$\begin{aligned} \det(2A^2(B^{-1})(C^T)^3 B(A^{-1})) &= 2^4(\det A)^2 \frac{1}{\det B}(\det C)^3(\det B)\frac{1}{\det A} \\ &= 16(\det A)(\det C)^3 \end{aligned}$$

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Therefore, $\det B = 2$.

Problem

Let A be an $n \times n$ matrix. Find all conditions that ensure $\det(-A) = \det A$.

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Let A be an $n \times n$ matrix. Find all conditions that ensure $\det(-A) = \det A$.

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Therefore, $\det(-A) = \det A$ only if $\det A = 0$ or n is even.

Using Row Operations

Problem

Let

$$A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 6 & 0 \end{bmatrix}$$

Find $\det(A)$.

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$$A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 6 & 0 \end{bmatrix}$$

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Solution

We could solve this using cofactor expansion. However, we can also use row operations to simplify A first.
First, switch rows 1 and 2 to obtain matrix B.

$$B = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & 2 \\ 2 & 6 & 0 \end{bmatrix}$$

Then, $\det(B) = -\det(A)$, which we can write as $\det(A) = -\det(B)$.

Solution (continued)

Now, multiply row 3 by $\frac{1}{2}$ to obtain matrix C.

$$C = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

Then, $\det(C) = \frac{1}{2}\det(B) = -\frac{1}{2}\det(A)$.

Solution (continued)

Now, multiply row 3 by $\frac{1}{2}$ to obtain matrix C.

$$C = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

Then, $\det(C) = \frac{1}{2}\det(B) = -\frac{1}{2}\det(A)$.
Now, subtract 5 times row 1 from row 2, and 1 times row 1 from row 3 to obtain matrix D.

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

Then, $\det(D) = \det(C) = -\frac{1}{2}\det(A)$. Hence, $\det(A) = -2\det(D)$.

Solution (continued)

Now we can use cofactor expansion to find $\det(D)$.

$$\det(D) = (1)(-1)^{1+1} \begin{vmatrix} -14 & -8 \\ 0 & -2 \end{vmatrix} = 28$$

Similarly, since D is triangular, we can find the determinant by multiplying the entries on the main diagonal.

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Similarly, since D is triangular, we can find the determinant by multiplying the entries on the main diagonal.
Then

$$\det(A) = -2\det(D) = -2(28) = -56$$

The Cofactor Matrix

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The cofactor matrix of A, is the matrix

$$[\mathrm{cof}(A)_{ij}],$$

i.e., the matrix whose (i, j)-entry is the (i, j)-cofactor of A.

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Reminder: the (i,j)-cofactor

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} \mathit{minor}\,(A)_{ij}\,,$$

where $\mathit{minor}\,(A)_{ij}$ is the determinant of the matrix obtained from A by deleting row i and column j.

Problem

Find the cofactor matrix $[\mathrm{cof}(A)_{ij}]$ of the matrix

$$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}.$$

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$$\mathrm{cof}(A)_{11} = (-1)^{1+1} \det A_{11}$$

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$$\mathrm{cof}(A)_{11} \quad = \quad (-1)^{1+1} \det A_{11} = \begin{vmatrix} 9 & 7 \\ 6 & 4 \end{vmatrix} = 9 \times 4 - 6 \times 7$$

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Find the cofactor matrix $[\mathrm{cof}(A)_{ij}]$ of the matrix

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For each i and j, $1 \le i, j \le 3$, we need to compute $\mathrm{cof}(A)_{ij}$, so there are 9 cofactors to compute.

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\mathrm{cof}(A)_{12} &= (-1)^{1+2} \det A_{12}
\end{aligned}$$

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Solution (continued)

Computing the six remaining cofactors results in the cofactor matrix

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The Adjugate

Definition

If A is an $n \times n$ matrix, then the adjugate of A is defined by

$$\text{adj } A = \left[\text{ cof}(A)_{ij} \ \right]^T,$$

where $\text{cof}(A)_{ij}$ is the (i, j)-cofactor of A, i.e., *adj A* is the transpose of the cofactor matrix.

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$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$, has cofactor matrix $\begin{bmatrix} -6 & -4 & 6 \\ 18 & 16 & -24 \\ -27 & -25 & 36 \end{bmatrix}$.

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Therefore, the adjugate of A is

$$\text{adj } A = \begin{bmatrix} -6 & -4 & 6 \\ 18 & 16 & -24 \\ -27 & -25 & 36 \end{bmatrix}^T = \begin{bmatrix} -6 & 18 & -27 \\ -4 & 16 & -25 \\ 6 & -24 & 36 \end{bmatrix}.$$

Problem

Find adj A when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

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Solution

$$\text{adj } A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}.$$

The Adjugate of a 2×2 Matrix

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Example (continued)

Observe that, regardless of the value of $\det A$,

$$A(\text{adj } A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Therefore we have $A(\text{adj } A) = (\det A)I$.

The Adjugate Formula

Theorem

If A is an $n \times n$ matrix, then

$$A(\text{adj } A) = (\det A)I = (\text{adj}A)A.$$

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Inverting a matrix using the adjugate

Except in the case of a 2×2 matrix, the adjugate formula is a very inefficient method for computing the inverse of a matrix; the matrix inversion algorithm is much more practical. However, the adjugate formula is of theoretical significance.

Proof of the Adjugate Formula

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Recall that the (i, j)-entry of $\mathrm{adj}(A)$ is equal to $\mathrm{cof}(A)_{ji}$.

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for all i. If $i \neq j$ then this matrix has its ith column equal to its jth column, and therefore

$$a_{ij} = 0 \qquad \text{if } i \neq j.$$

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You can check this by computing AA^{-1}. You could also check by using the Matrix Inversion Algorithm to find A^{-1} (though this is more work).

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Even if A is not invertible, $\det(\text{adj } A) = (\det A)^{n-1}$, but the proof is more complicated.

Cramer's Rule

If A is an $n \times n$ invertible matrix, then the solution to $AX = B$ can be given in terms of determinants of matrices.

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$$x_i = \frac{\det A_i}{\det A}$$

Example (Cramer's Rule)

Solve the following system of linear equations using Cramer's Rule.

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

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and thus $x_1 = \frac{4}{-4} = -1$.

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

and thus $x_2 = \frac{-14}{4} = \frac{7}{2}$.

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

and thus $x_2 = \frac{-14}{4} = \frac{7}{2}$. Finally, $x_3 = \frac{\det A_3}{\det A}$, where $\det A = -4$ and

$$\det A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -6,$$

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

and thus $x_2 = \frac{-14}{4} = \frac{7}{2}$. Finally, $x_3 = \frac{\det A_3}{\det A}$, where $\det A = -4$ and

$$\det A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -6,$$

and thus $x_3 = \frac{-6}{-4} = \frac{3}{2}$.

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

and thus $x_2 = \frac{-14}{4} = \frac{7}{2}$. Finally, $x_3 = \frac{\det A_3}{\det A}$, where $\det A = -4$ and

$$\det A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -6,$$

and thus $x_3 = \frac{-6}{-4} = \frac{3}{2}$. Therefore, the solution to the system is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{7}{2} \\ \frac{3}{2} \end{bmatrix}.$$

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

and thus $x_2 = \frac{-14}{4} = \frac{7}{2}$. Finally, $x_3 = \frac{\det A_3}{\det A}$, where $\det A = -4$ and

$$\det A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -6,$$

and thus $x_3 = \frac{-6}{-4} = \frac{3}{2}$. Therefore, the solution to the system is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{7}{2} \\ \frac{3}{2} \end{bmatrix}.$$

You can check this by substituting these values into the original system.