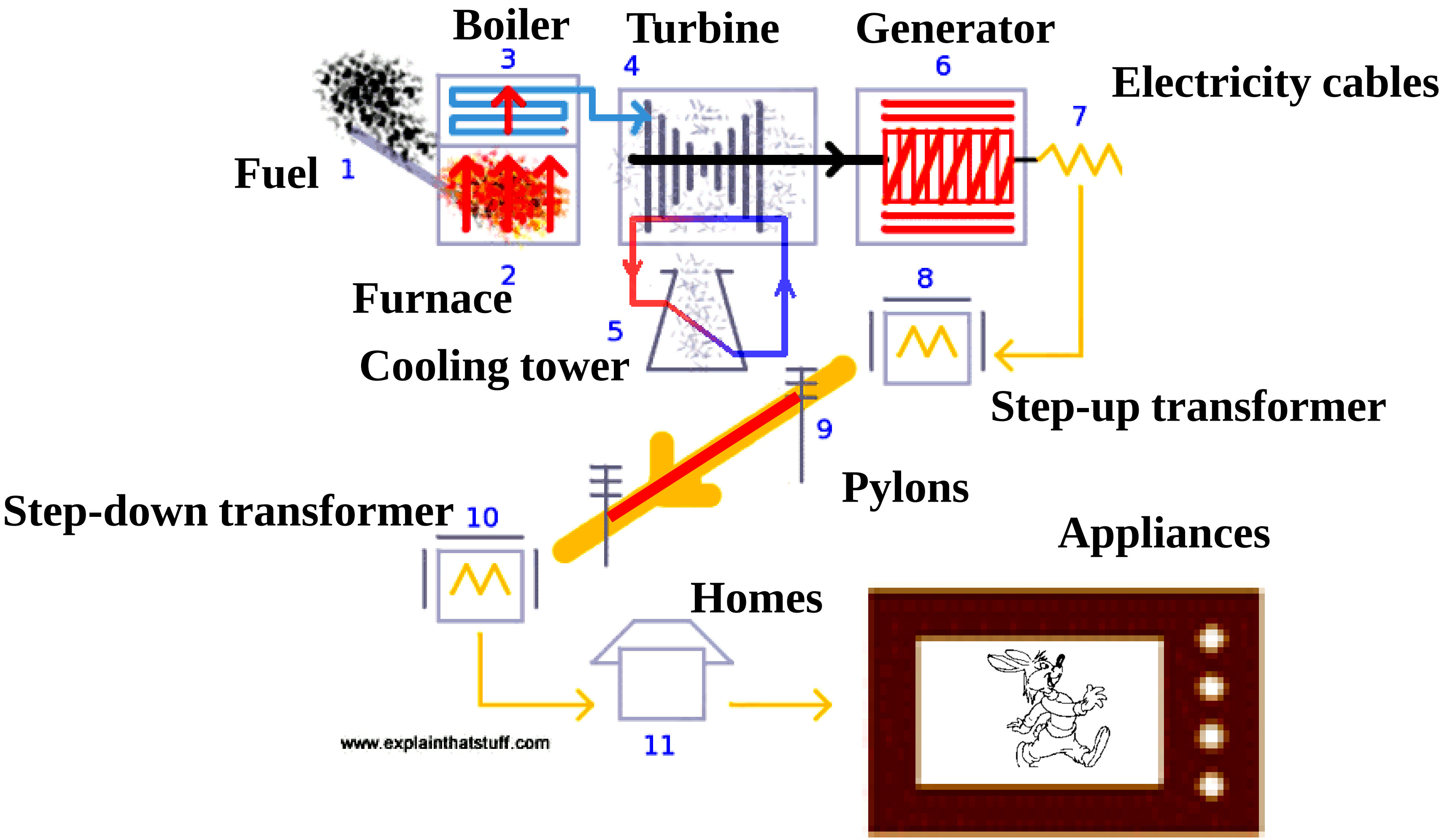
Chapter 6

Electromagnetic Oscillations and Alternating Current

Overview

• In the previous chapters, we have studied the basic physics of electric and magnetic fields and how energy can be stored in capacitors and inductors.

• In this chapter, we will study the associated applied physics, in which the energy stored in one location can be transferred to another location so that can be put to use, *e.g., energy produced at a power plant can be transferred to your home to run a computer.* The modern civilization would be impossible without this applied physics.



- In most parts of the world, electrical energy is transferred not as a direct current but as a sinusoidally oscillating current (alternating current, or ac). The challenge to us is to design ac systems that transfer energy efficiently and to build appliances that make use of that energy.

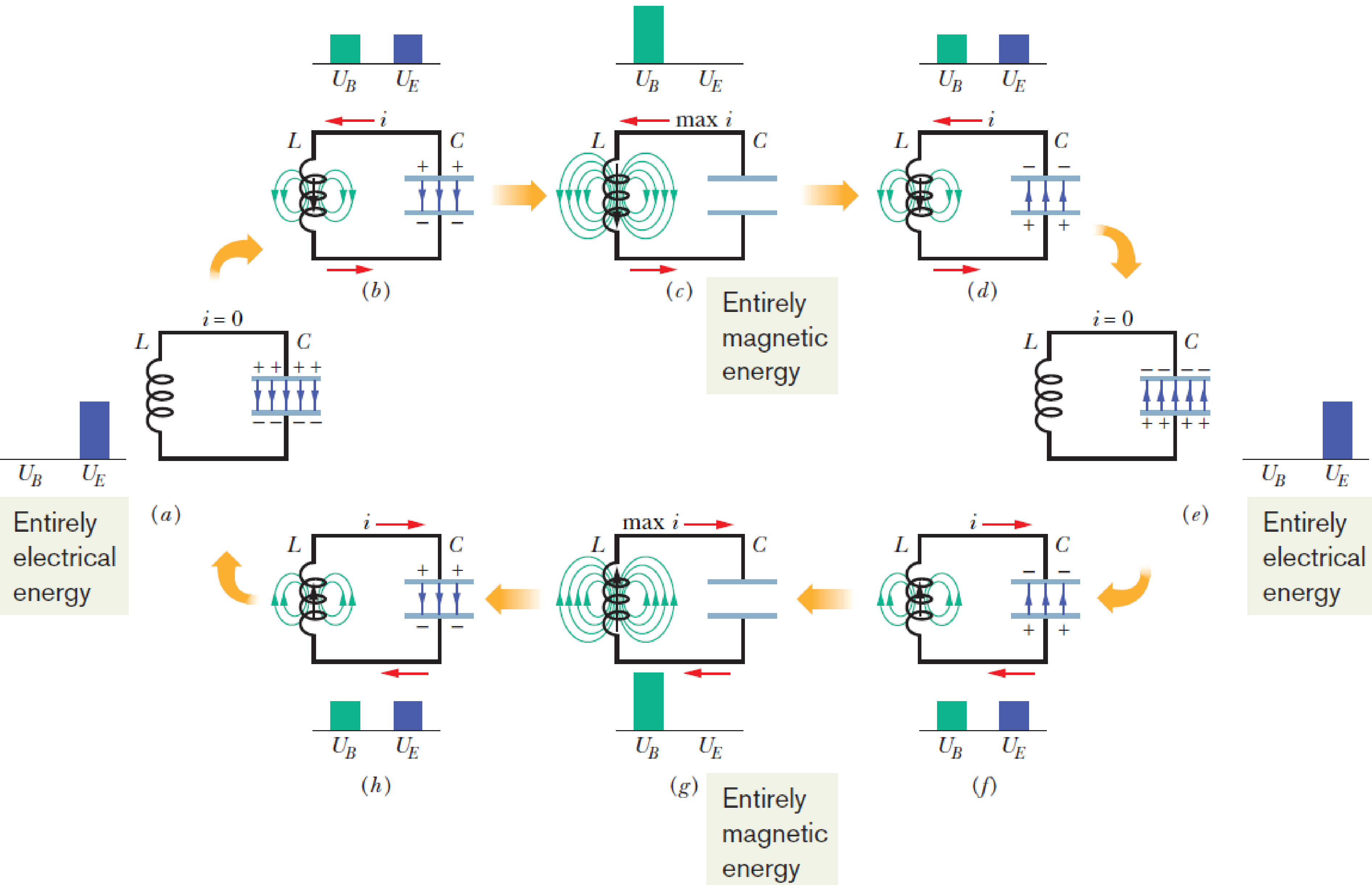
6.1. LC Oscillations:

6.1.1. LC Oscillations, Qualitatively:

- In RC and RL circuits the charge, current, and potential difference grow and decay exponentially.

- On the contrary, in an LC circuit, the charge, current, and potential difference vary sinusoidally with period T and angular frequency ω.

- The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations**.



U_B U_E
i
L C
(b)
max i
(c)
Entirely magnetic energy
(d)
$i = 0$
(a)
Entirely electrical energy
(e)
Entirely electrical energy
(h)
(g)
Entirely magnetic energy
(f)

The energy stored in the electric field of the capacitor at any time is

$$U_E = \frac{q^2}{2C}$$

where q **is the charge on the capacitor at that time.**

The energy stored in the magnetic field of the inductor at any time is

$$U_B = \frac{Li^2}{2}$$

where i **is the current through the inductor at that time.**

As the circuit oscillates, energy shifts back and forth from one type of stored energy to the other, but the total amount is conserved.

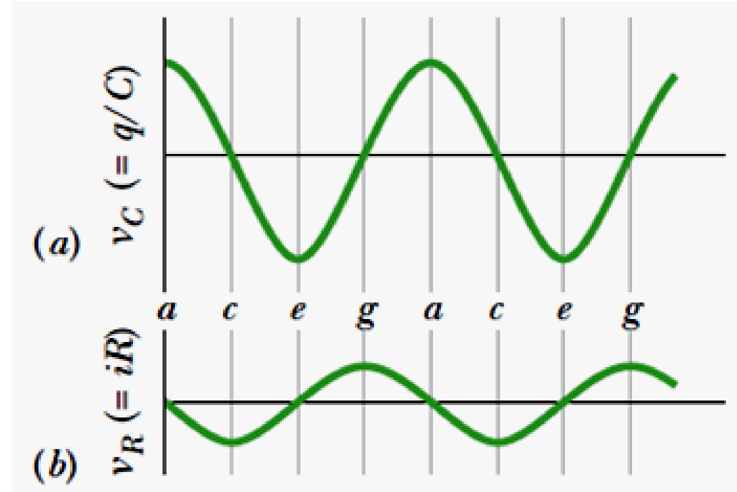
• The time-varying potential difference (or *voltage*) v_C that exists across the capacitor C is:

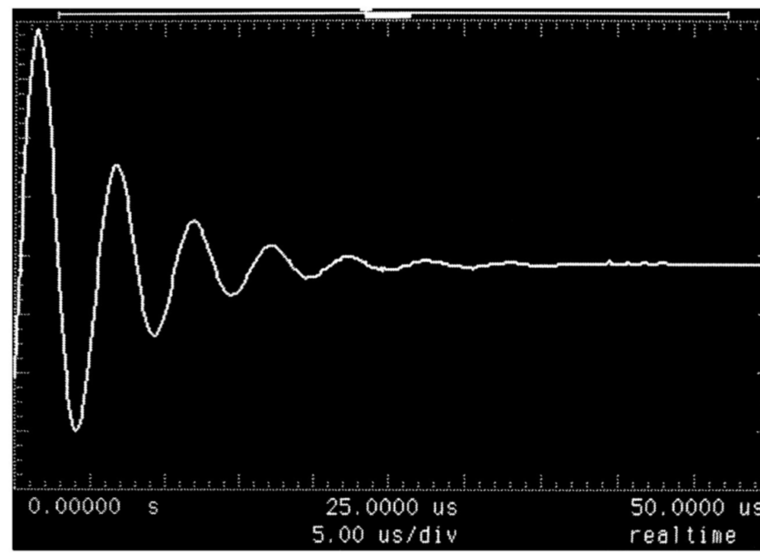
$$v_C = \left(\frac{1}{C}\right)q$$

• To measure the current, we can connect a small resistance R in series with the capacitor and inductor and measure the time-varying potential difference v_R across it:

$$v_R = iR$$

• In an actual LC circuit, the oscillations will not continue indefinitely because there is always some resistance that will dissipate electrical and magnetic energy as thermal energy





Courtesy Agilent Technologies halliday_9e_fig_31_03

6.2. The Electrical Mechanical Analogy:

• One can make an analogy between the oscillating *LC* system and an oscillating block-spring system.

• Two kinds of energy are involved in the block-spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block. Here we have the following analogies:

q corresponds to x, $1/C$ corresponds to k,

i corresponds to v, and L corresponds to m.

$$\omega = \sqrt{\frac{k}{m}}$$

Table 31-1

Comparison of the Energy in Two Oscillating Systems

Block–Spring System		*LC* Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electrical, $\frac{1}{2}(1/C)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
$v = dx/dt$		$i = dq/dt$	

The angular frequency of oscillation for an ideal (resistanceless) *LC* is:

$$\omega = \frac{1}{\sqrt{LC}} \text{ (LC circuit)}$$

The Block-Spring Oscillator:

$$U = U_b + U_s = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

$$m\frac{d^2x}{dt^2} + kx = 0 \Rightarrow x = X\cos(\omega t + \phi)\ \text{(displacement)};\ \omega = \sqrt{\frac{k}{m}}$$

ϕ: phase constant

The LC Oscillator:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0\ \text{(LC oscillations)}$$

$\Rightarrow$ $q = Q\cos(\omega t + \phi)$ (charge) $\Rightarrow$ $i = \dfrac{dq}{dt} = -\omega Q\sin(\omega t + \phi)$ (current)

The amplitude I:

$$I = \omega Q \Rightarrow i = -I\sin(\omega t + \phi)$$

Angular Frequencies:

The equation to calculate charge q is indeed the solution of the LC oscillations equation with $\omega = 1/\sqrt{LC}$, we can test:

$$\frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t + \phi)$$

We have

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \text{ (LC oscillations)}$$

$\Rightarrow$ $$-L\omega^2 Q\cos(\omega t + \phi) + \frac{1}{\mathrm{C}}Q\cos(\omega t + \phi) = 0$$

$\Rightarrow$ $$\omega = \frac{1}{\sqrt{LC}}$$

Electrical and Magnetic Energy Oscillations:

The electrical energy stored in the *LC* circuit at time *t* is,

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi)$$

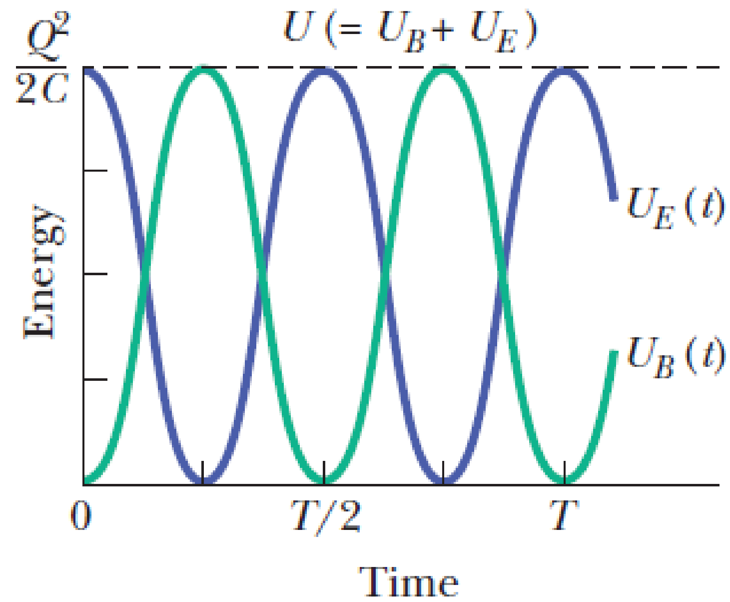
The magnetic energy is:

$$U_B = \frac{1}{2}Li^2 = \frac{1}{2}L\omega^2 Q^2 \sin^2(\omega t + \phi)$$

But $\omega = \frac{1}{\sqrt{LC}}$ (LC circuit)

Therefore $U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi)$

The electrical and magnetic energies vary but the total is constant.

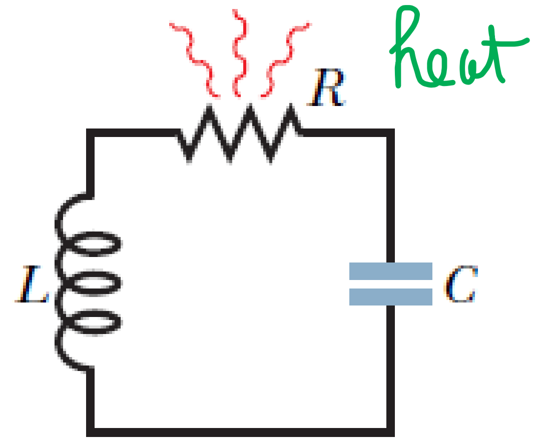


Note that:

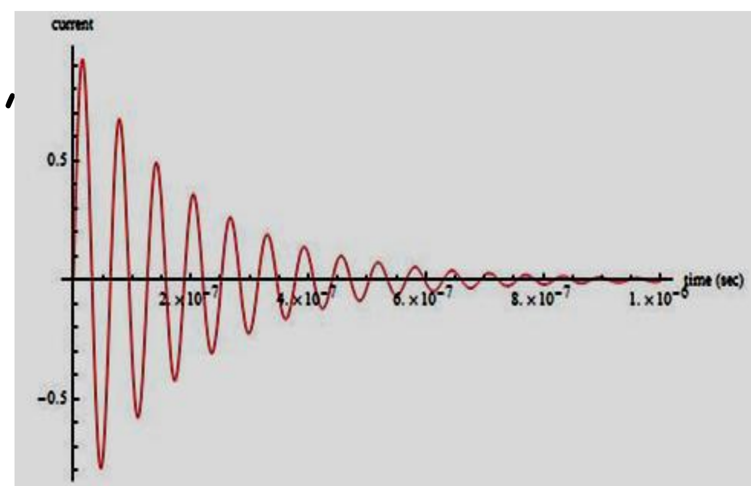
- The maximum values of U_E and U_B are both $Q^2/2C$.
- At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
- When U_E is maximum, U_B is zero, and conversely.

6.3. Damped Oscillations in an RLC Circuit:

- A circuit containing resistance, inductance, and capacitance is called an **RLC circuit**



- With R, the total energy U of the circuit (the sum of U_E and U_B) is not constant; instead, it decreases with time as energy is transferred to thermal energy in the resistance.



- The oscillations of charge, current, and potential difference continuously decrease in amplitude, and **the oscillations are said to be damped**.

Analysis:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

this total energy decreases as energy is transferred to thermal energy

$$\Rightarrow \quad \frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2 R$$

substituting dq/dt for i and d^2q/dt^2 for di/dt, we have:

$$\Rightarrow \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \text{ (RLC circuit)}$$

$$\Rightarrow \quad q = Qe^{-Rt/2L}\cos(\omega' t + \phi)$$

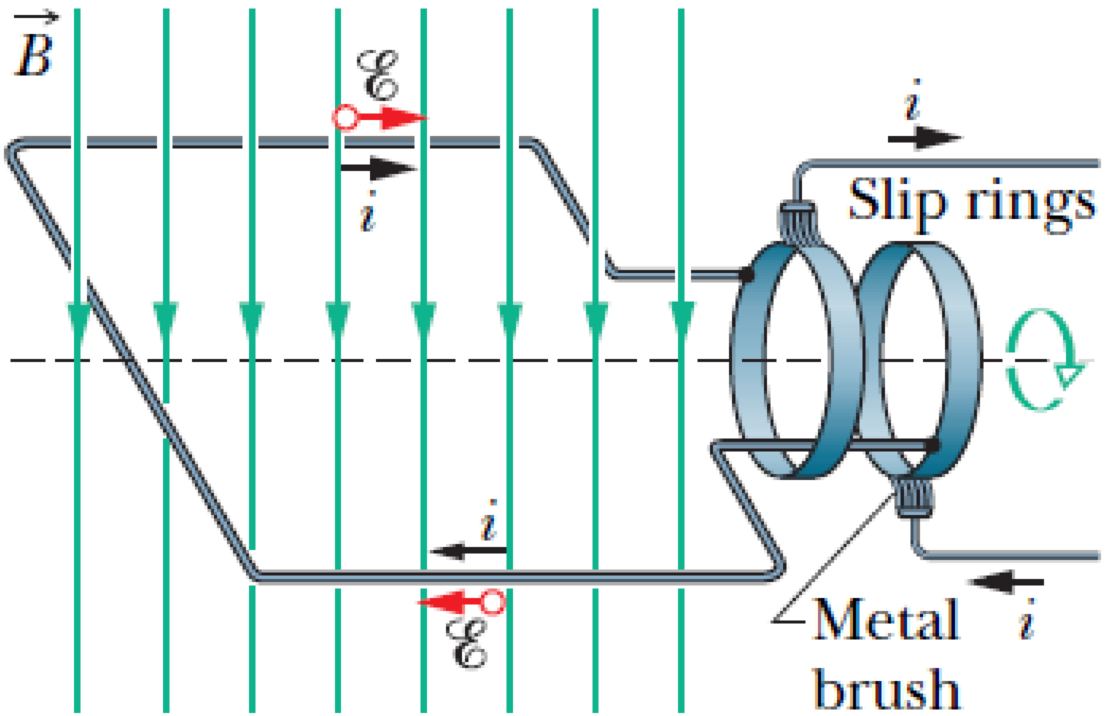
Where $\omega' = \sqrt{\omega^2 - (R/2L)^2}$

And $\omega = 1/\sqrt{LC}$

$$\Rightarrow \quad \boxed{U_E = \frac{q^2}{2C} = \frac{\left[Qe^{-Rt/2L}\cos(\omega' t + \phi)\right]^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t + \phi)}$$

6.4. Alternating Current:

- The oscillations in an RLC circuit will not damp out if an external emf device supplies enough energy to make up for the energy dissipated as thermal energy



- The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field

- The induced emf:

$$\boxed{\varepsilon = \varepsilon_m \sin \omega_d t}$$

$$\boxed{i = I \sin(\omega_d t - \phi)}$$

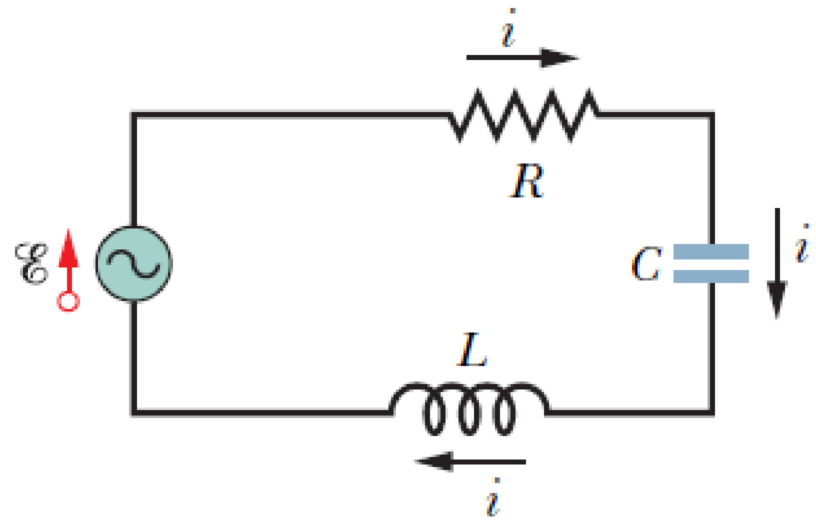
ω_d is called the driving angular frequency, and I is the amplitude of the driven current.

Forced Oscillations:

• Undamped LC circuits and damped RLC circuits oscillate at: $\omega = 1/\sqrt{LC}$ that is ***natural angular frequency***

• When the RLC circuit is connected to an external alternating emf, charge, potential difference and current oscillate with driving angular frequency ω_d, these oscillations are the so-called *driven oscillations* or ***forced oscillations***

• The amplitude I of the current in the circuit is maximum when $\omega = \omega_d$, a condition known as ***resonance***



A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and currents are indicated here at only instant.

Whatever the natural angular frequency ω of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency ω_d.

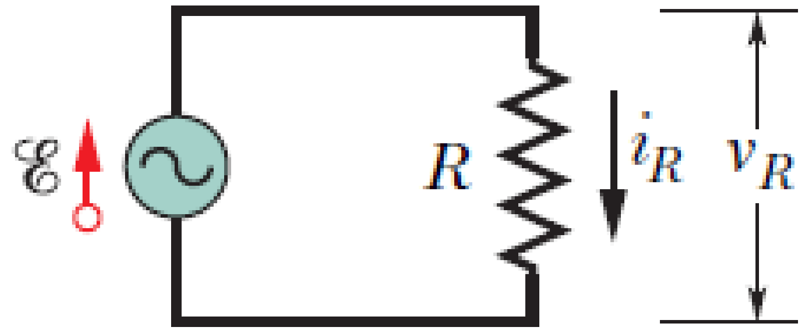
6.5. Three Simple Circuits:

6.5.1. A Resistive Load:

$$\varepsilon - v_R = 0$$

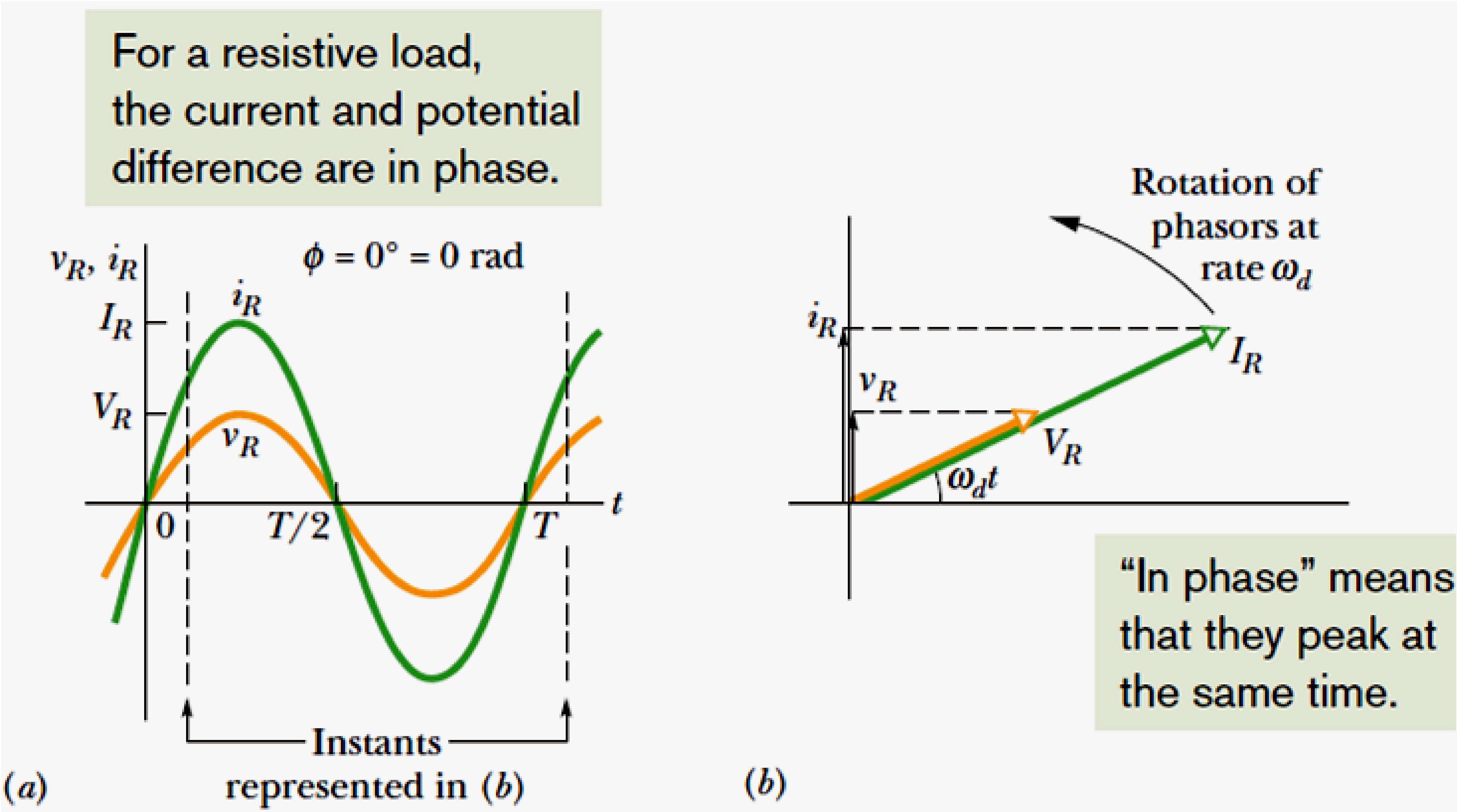
$$v_R = \varepsilon_m \sin\omega_d t = V_R \sin\omega_d t$$

$$i_R = \frac{v_R}{R} = \frac{V_R}{R}\sin\omega_d t$$



Note: for a purely resistive load the phase constant $\phi = 0°$.

• v_R (t) and i_R(t) are ***in phase*** ($\phi = 0°$), which means their corresponding maxima (and minima) occur at the same times. The time-varying quantities v_R and i_R can also be represented geometrically by ***phasors*** (vectors) with the following properties: **angular speed** (rotate counterclockwise about the origin with ω_d); **length** (representing the amplitude of V_R and I_R); **projection** (on the vertical axis, representing v_R and i_R); **rotation angle** (to be equal to the phase $\omega_d t$ at time t)

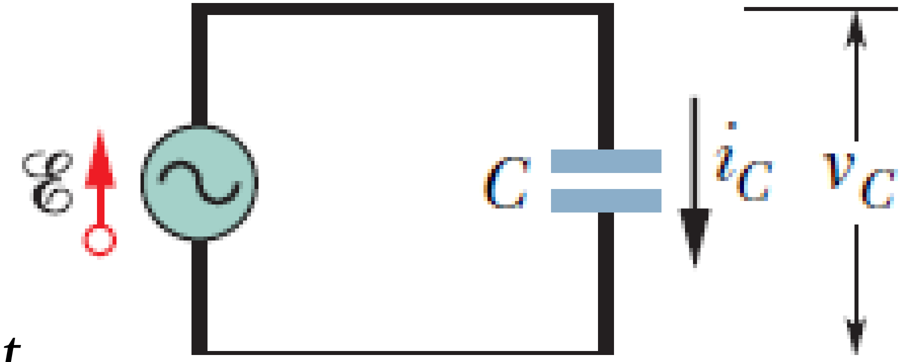

For a resistive load,
the current and potential
difference are in phase.
v_R, i_R
$\phi = 0° = 0$ rad
i_R
I_R
V_R
v_R
0
$T/2$
T
t
Instants
represented in (b)
(a)
Rotation of
phasors at
rate ω_d
i_R
I_R
v_R
V_R
$\omega_d t$
(b)
"In phase" means
that they peak at
the same time.

6.5.2. A Capacitive Load:

$$v_C = V_C \sin \omega_d t$$

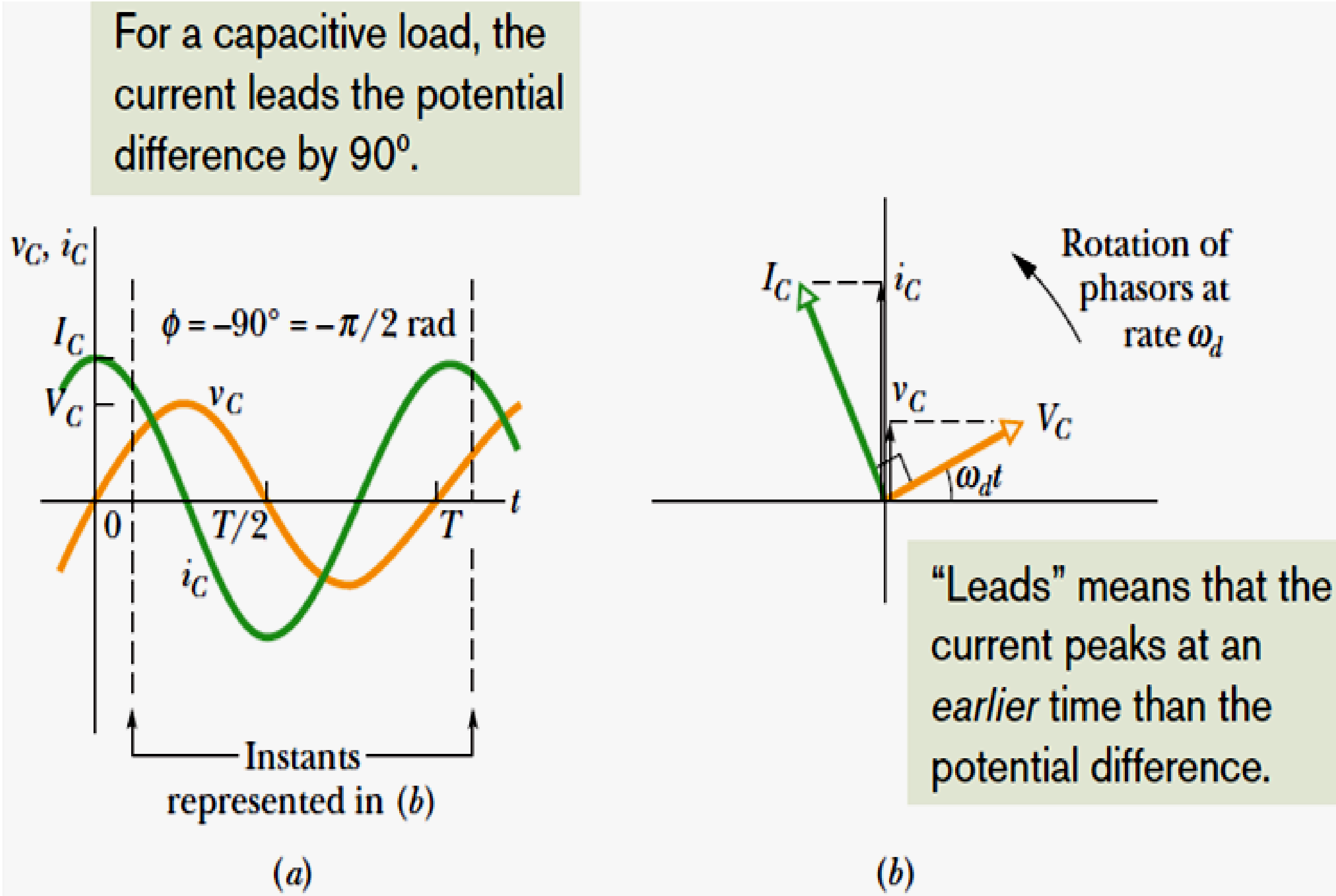
$$q_C = C v_C = C V_C \sin \omega_d t$$

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t$$



$$\boxed{X_C = \frac{1}{\omega_d C} \text{ (capacitive reactance)}}$$

X_C is called the **capacitive reactance of a capacitor**. The SI unit of X_C is the *ohm* (Ω), just as for resistance *R*.



$$\cos\omega_d t = \sin(\omega_d t + 90^0) \Rightarrow i_C = \left(\frac{V_C}{X_C}\right)\sin(\omega_d t + 90^0)$$

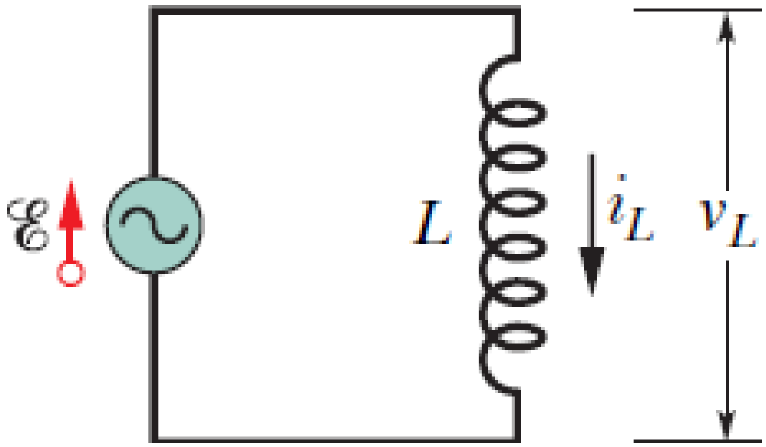
$$\Rightarrow i_C = I_C \sin(\omega_d t - \phi), \phi = -90^0$$

$$\boxed{V_C = I_C X_C \text{ (capacitor)}}$$

6.5.3. An Inductive Load:

$$v_L = V_L \sin \omega_d t$$

$$\varepsilon_L = -L\frac{di}{dt} \Rightarrow v_L = L\frac{di_L}{dt}$$



$$\frac{di_L}{dt} = \frac{V_L}{L}\sin \omega_d t$$

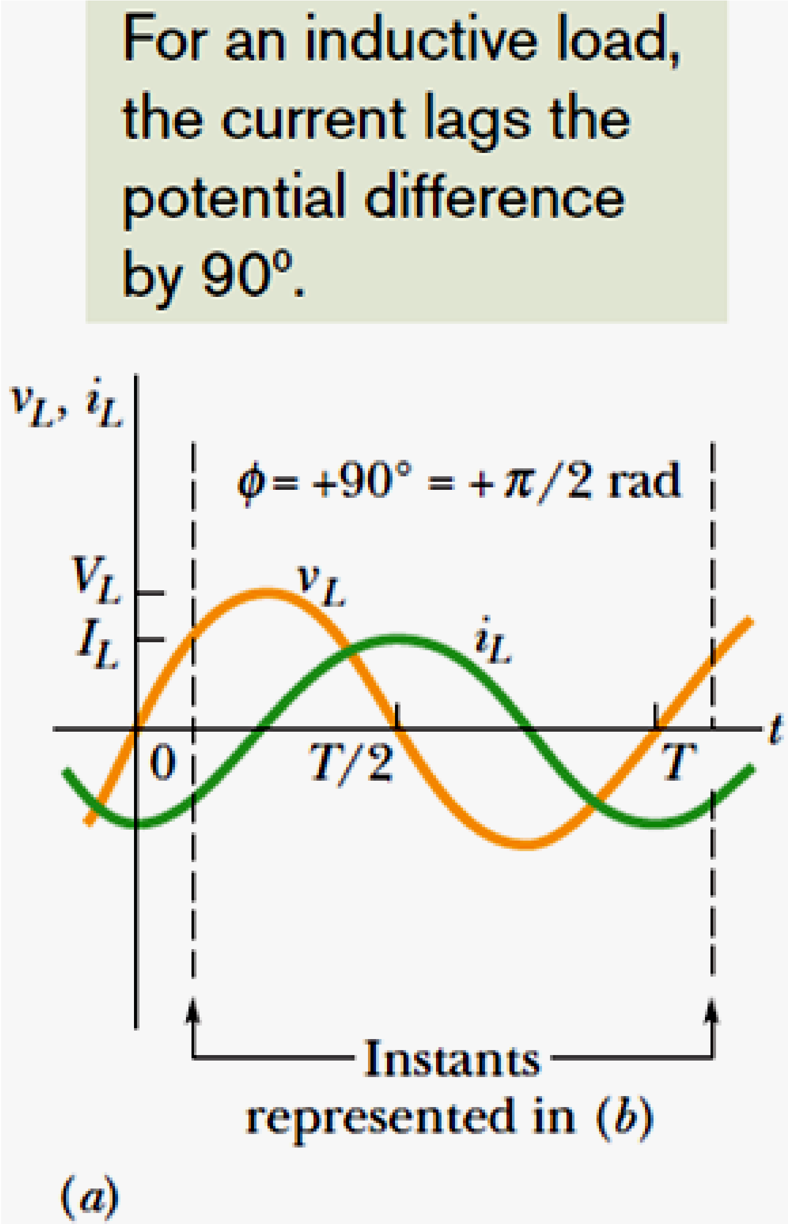
$$i_L = \int di_L = \frac{V_L}{L}\int \sin \omega_d t dt = -\left(\frac{V_L}{\omega_d L}\right)\cos \omega_d t$$

$$\boxed{X_L = \omega_d L \text{ (inductive reactance)}}$$

$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90^0); i_L = I_L \sin(\omega_d t - \phi)$$

$$\boxed{V_L = I_L X_L \text{ (inductor)}}$$

The value of X_L , **the inductive reactance**, depends on the driving angular frequency ω_d. The unit of the inductive time constant τ_L (=L/R) indicates that the SI unit of X_L is the *ohm*.



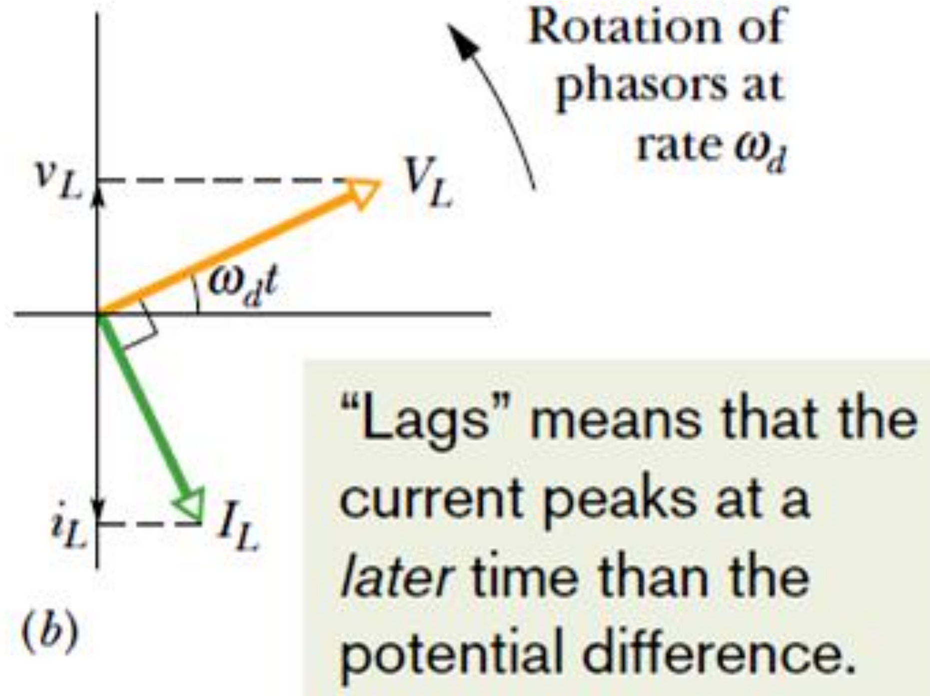


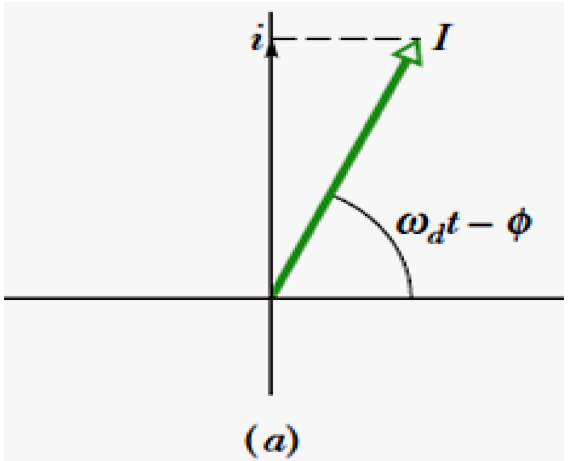
Fig. 31-13 (*a*) The current in the inductor lags the voltage by 90° ($= \pi/2$ rad). (*b*) A phasor diagram shows the same thing.

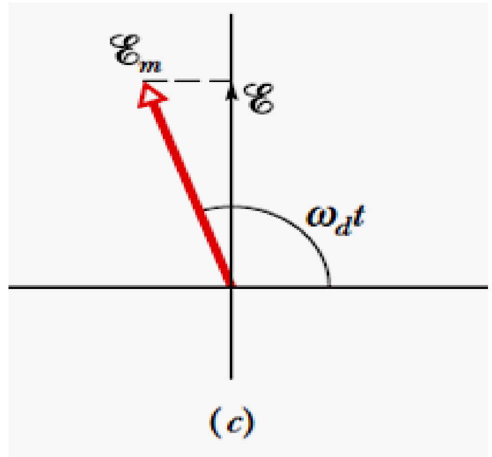
Table 31-2

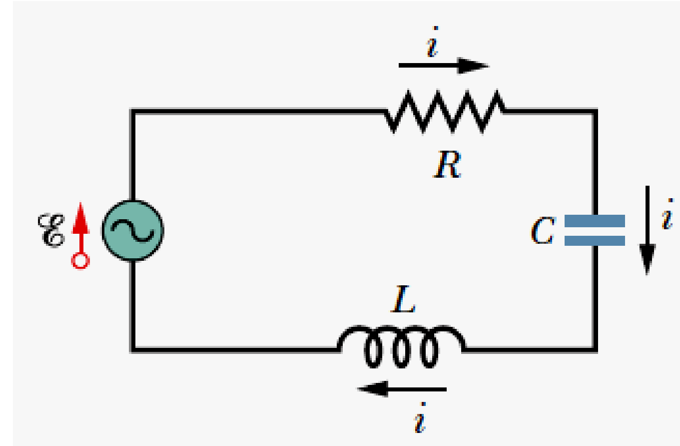
Phase and Amplitude Relations for Alternating Currents and Voltages

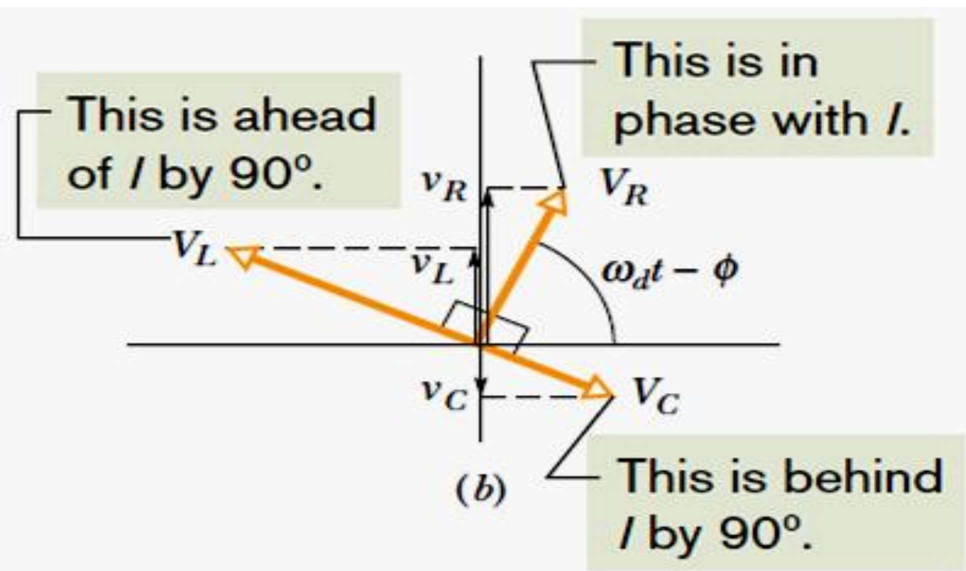
Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	$0°$ (= 0 rad)	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by $90°$ (= $\pi/2$ rad)	$-90°$ (= $-\pi/2$ rad)	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by $90°$ (= $\pi/2$ rad)	$+90°$ (= $+\pi/2$ rad)	$V_L = I_L X_L$

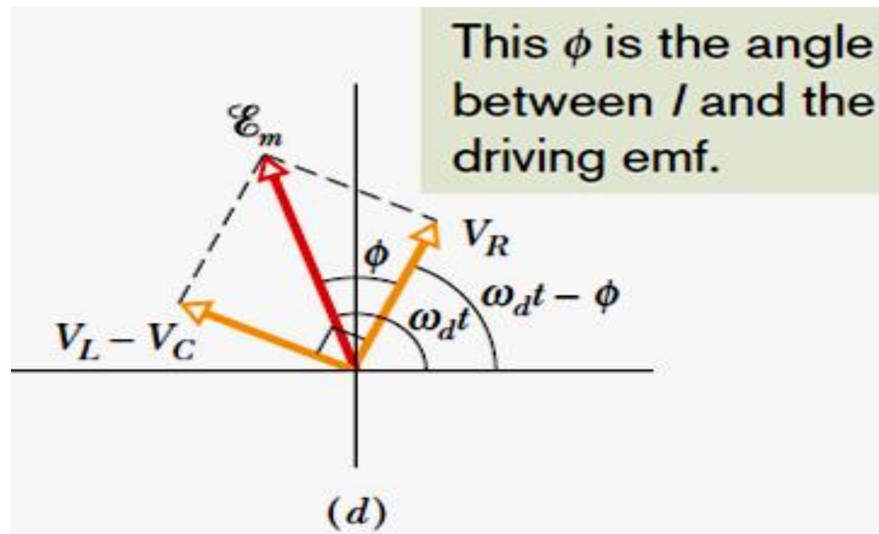
6.6. The Series RLC Circuit:





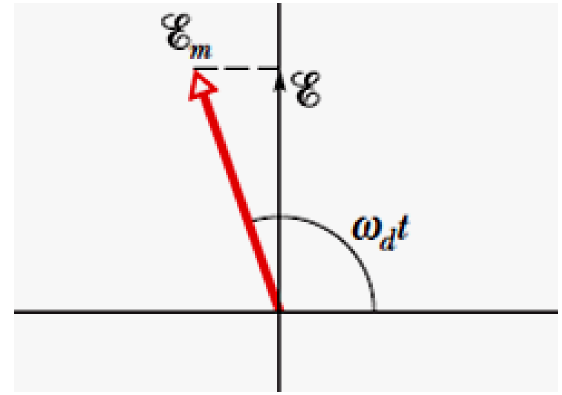






(*d)* The emf phasor is equal to the vector sum of the three voltage phasors of (b).Here, voltage phasors V_L and V_C have been added vectorially to yield their net phasor (V_L-V_C).

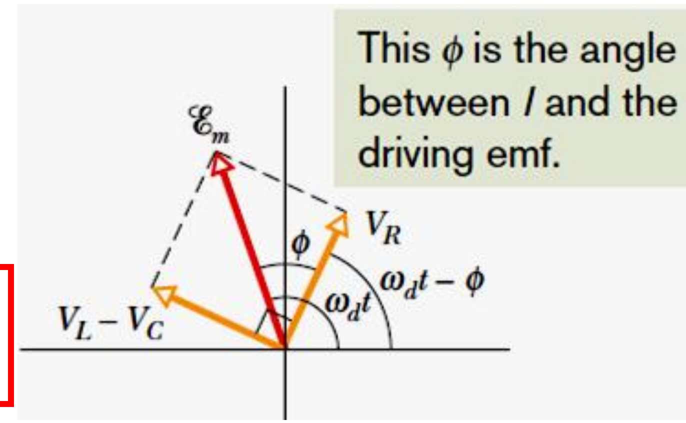
$$\varepsilon = \varepsilon_m \sin \omega_d t$$



$$i = I \sin(\omega_d t - \phi)$$

$$\varepsilon_m^2 = V_R^2 + (V_L - V_C)^2 = (IR)^2 + (IX_L - IX_C)^2$$

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

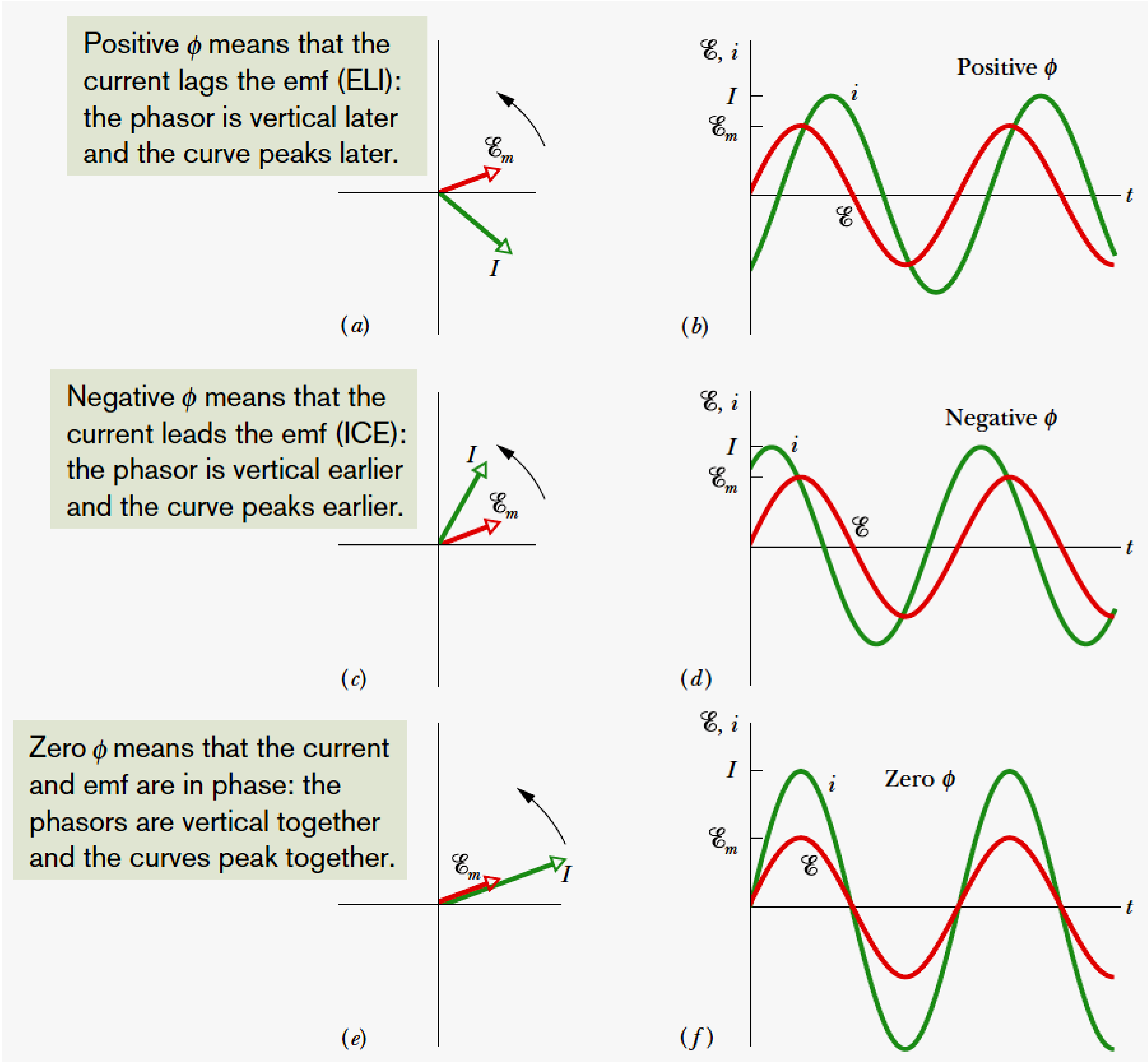


$$\boxed{Z = \sqrt{R^2 + (X_L - X_C)^2} \text{ (impedance defined)}}$$

$$\boxed{I = \frac{\varepsilon_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \text{ (current amplitude)}}$$

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR}$$

$$\boxed{\Rightarrow \tan\phi = \frac{X_L - X_C}{R} \text{ (phase constant)}}$$



Positive ϕ means that the current lags the emf (ELI): the phasor is vertical later and the curve peaks later.
ℰ_m
I
(a)
ℰ, i
I
ℰ_m
i
Positive ϕ
ℰ
t
(b)
Negative ϕ means that the current leads the emf (ICE): the phasor is vertical earlier and the curve peaks earlier.
I
ℰ_m
(c)
ℰ, i
I
ℰ_m
i
Negative ϕ
ℰ
t
(d)
Zero ϕ means that the current and emf are in phase: the phasors are vertical together and the curves peak together.
ℰ_m
I
(e)
ℰ, i
I
i
Zero ϕ
ℰ_m
ℰ
t
(f)

Resonance:

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \text{ (current amplitude)}$$

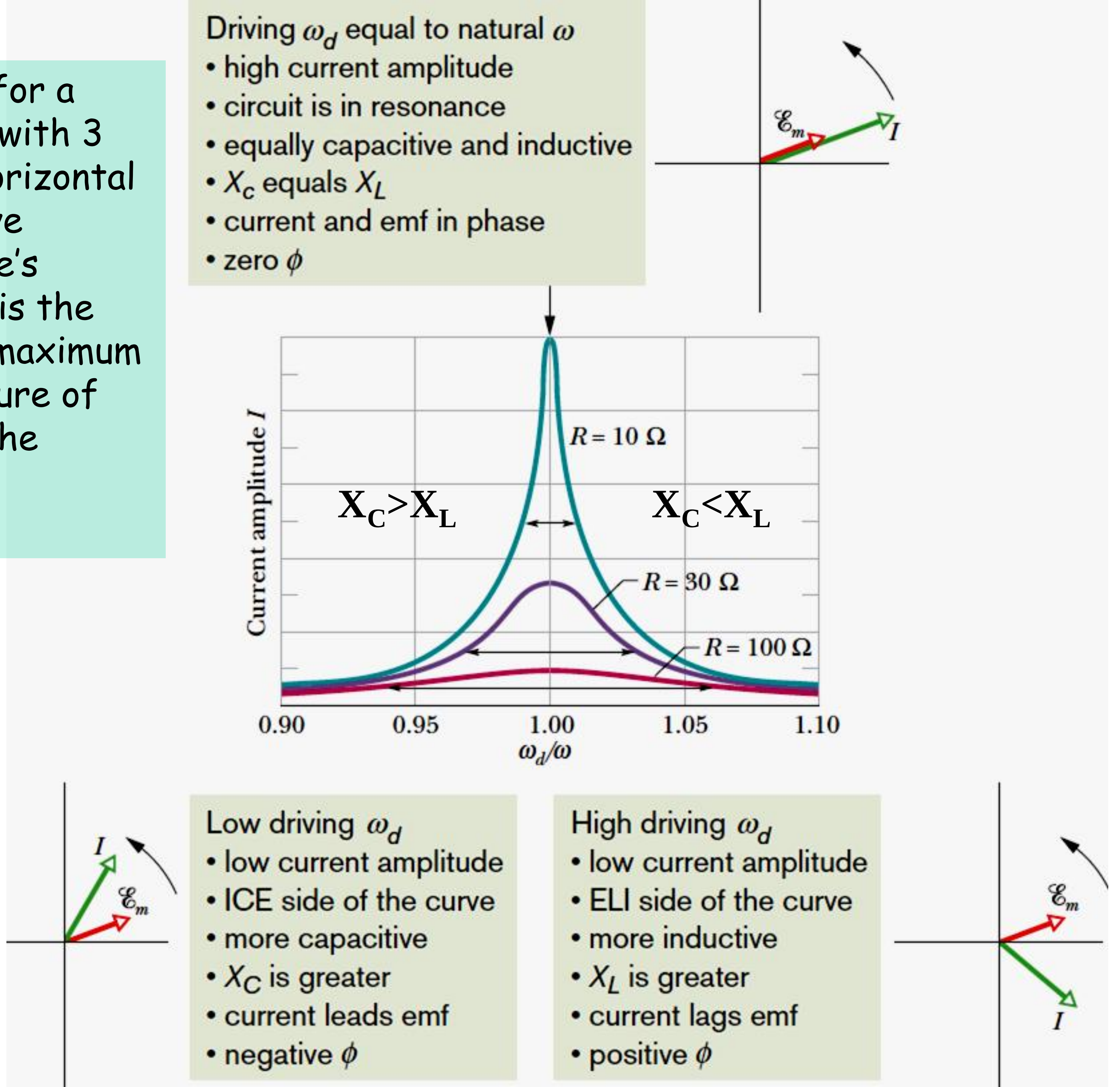
For a given resistance *R*, that amplitude is a maximum when the quantity $(\omega_d L - 1/\omega_d C)$ in the denominator is zero.

$$\Rightarrow \omega_d L = \frac{1}{\omega_d C} \qquad \Rightarrow \omega_d = \frac{1}{\sqrt{LC}} \text{ (maximum I)}$$

The maximum value of *I* occurs when the driving angular frequency matches the natural angular frequency—that is, at resonance.

$$\boxed{\omega_d = \omega = \frac{1}{\sqrt{LC}} \text{ (resonance)}}$$

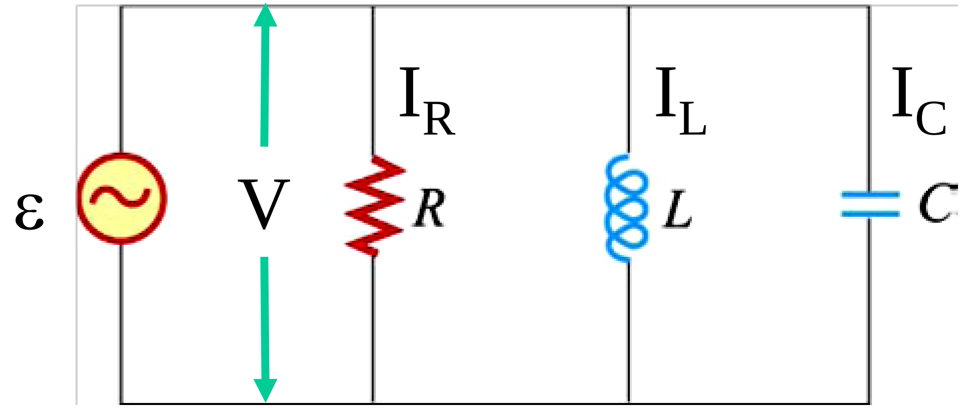
Resonance curves for a driven RLC circuit with 3 values of R. The horizontal arrow on each curve measures the curve's **half-width**, which is the width at the half-maximum level and is a measure of the sharpness of the resonance.

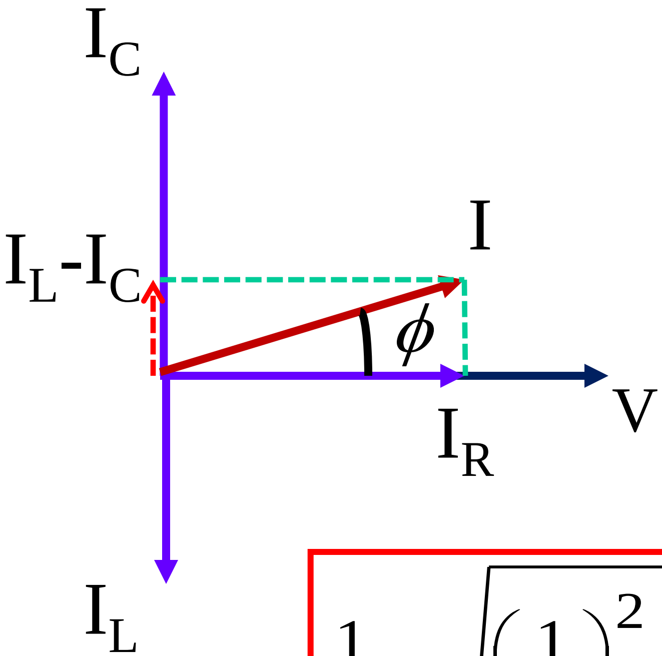


The Parallel RLC Circuit:

$$\varepsilon = \varepsilon_m \sin \omega_d t$$

$$i = I_m \sin(\omega_d t - \phi)$$





$$I = \sqrt{I_R^2 + (I_L - I_C)^2}$$

$$\frac{V}{Z} = \sqrt{\left(\frac{V}{R}\right)^2 + \left(\frac{V}{X_L} - \frac{V}{X_C}\right)^2}$$

$$\boxed{\frac{1}{Z} = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\frac{1}{\omega_d L} - \omega_d C\right)^2}}$$

$$\tan \phi = \frac{I_L - I_C}{I_R} \Rightarrow \boxed{\tan \phi = \frac{\dfrac{1}{\omega_d L} - \omega_d C}{\dfrac{1}{R}}}$$

6.7. Power in Alternating Current Circuits:

- The instantaneous rate at which energy is dissipated in the resistor:

$$P = i^2 R = [I\sin(\omega_d t - \phi)]^2 R = I^2 R \sin^2(\omega_d t - \phi)$$

- The average rate at which energy is dissipated in the resistor is the average of this over time:

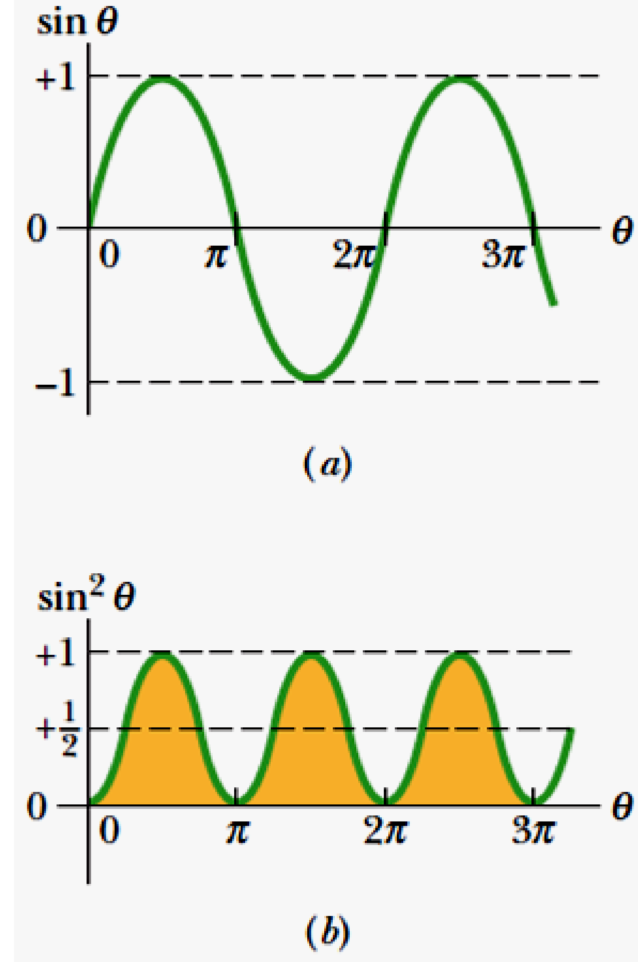
$$P_{avg} = \frac{I^2 R}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R$$

$$I_{rms} = \frac{I}{\sqrt{2}} \Rightarrow P_{avg} = I_{rms}^2 R \text{ (average power)}$$

$$V_{rms} = \frac{V}{\sqrt{2}} \text{ and } \varepsilon_{rms} = \frac{\varepsilon_m}{\sqrt{2}} \text{ (rms voltage; rms emf)}$$

$$I_{rms} = \frac{\varepsilon_{rms}}{Z} = \frac{\varepsilon_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

$$\Rightarrow P_{avg} = \frac{\varepsilon_{rms}}{Z} I_{rms} R = \varepsilon_{rms} I_{rms} \frac{R}{Z}$$



(a) A plot of sinθ versus θ. The average value over one cycle is zero; (b) A plot of sin²θ versus θ. The average value over one cycle is ½.

$$\Rightarrow \boxed{P_{avg} = \varepsilon_{rms} I_{rms} \cos\phi \text{ (average power)}} \quad \text{where } \cos\phi = \frac{VR}{\varepsilon m} = \frac{IR}{IZ} = \frac{R}{Z}$$

6.8. Transformers:

- In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory).

- Nobody wants an electric toaster or a child's electric train to operate at, say, 10 kV.

- On the other hand, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize I^2R losses (often called ohmic losses) in the transmission line.

• Example: Consider a 500 kV line used to transmit electrical energy from Hòa Bình to HCM city, 1500 km away. Suppose that the current is 500 A and the power factor is close to unity. So, energy is supplied at the average rate:

$$P_{avg} = \varepsilon I = IV = (500 \times 10^3\ V)(500A) = 250\ MW$$

The resistance of the transmission line: $r = 0.22\ \Omega/km$

So, $R = 0.22\ \Omega/km \times 1500\ km = 330\ \Omega$

Energy is dissipated due to R at a rate:

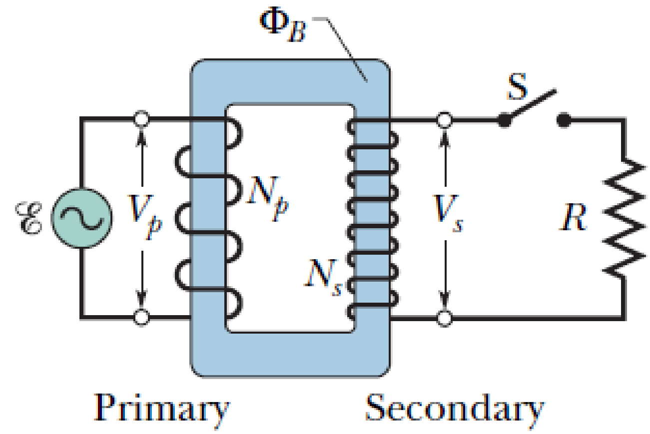
$$P'_{avg} = I^2R = 500^2 \times 330 = 82.5\ MW\text{, about } 33\%$$

If we increase the current to 625 A and reduce the voltage to 400 KV, giving the same average rate of 250 MW:

$$P'_{avg} = I^2R = 725^2 \times 330 = 128.9\ MW\text{, about } 52\%$$

Hence the general energy transmission rule: *Transmit at the highest possible voltage and the lowest possible current.*

Transformer: A device with which we can raise and lower the ac voltage in a circuit, keeping the product current voltage essentially constant.



A basic transformer circuit

- The ideal transformer consists of two coils, with different numbers of turns, wound around an iron core. In the primary coil with N_p:

$$\varepsilon = \varepsilon_m \sin \omega t$$

- The secondary coil with N_s turns. The primary current $I_{magnetizing}$ (very small) lags the primary voltage V_P by 90^0, so no power is delivered from the generator to the transformer ($\cos \phi = 0$).
- The small sinusoidally changing primary current produces a sinusoidally changing magnetic flux B in the iron core, producing an emf (dB/dt) in each turn of the secondary. This emf ε_{turn} per turn is the same in the primary and the secondary.
- Across the primary, the voltage $V_p = \varepsilon_{turn} N_p$. Similarly, across the secondary the voltage is $V_s = \varepsilon_{turn} N_s$.

$$\Rightarrow \boxed{V_S = V_P \frac{N_S}{N_P} \text{ (transformation of voltage)}}$$

- If $N_s > N_p$, the device is a **step-up** transformer because it steps the primary's voltage V_p up to a higher voltage V_s.
- If $N_s < N_p$, it is a **step-down** transformer.

• So far, switch S is open, so no energy is transferred from the generator to the rest of the circuit. If we now close S to connect the secondary to the resistive load R:

1. An alternating current I_S appears, with energy dissipation rate $I_S^2R = V_S^2/R$

2. I_S induces an opposing emf in the primary windings

3. V_P of the primary cannot change in response to this opposing emf because it must always be equal to the emf ε provided by the generator

4. To maintain V_P, the generator produces (in addition to I_{mag}) an alternating current I_P in the primary circuit. The magnitude and phase constant of I_P are just those required for the emf induced by I_P in the primary to exactly cancel the emf induced there by I_S. Because the phase constant ϕ of I_P is not 90^0, so I_P can transfer energy ($P_{avg} = \varepsilon_{rms}I_{rms}\cos\phi$) to the primary.

• If no energy is lost along the way, conservation of energy requires that

$$I_P V_P = I_S V_S \Rightarrow \boxed{I_S = I_P \frac{N_P}{N_S} \text{ (transformation of currents)}}$$

$$I_S = \frac{V_S}{R} = I_P \frac{N_P}{N_S} \Rightarrow I_P = \frac{1}{R}\frac{N_S}{N_P} V_S = \frac{1}{R}\left(\frac{N_S}{N_P}\right)^2 V_P$$

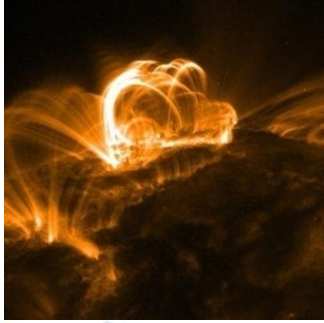
$$I_P = \frac{V_P}{R_{eq}}$$

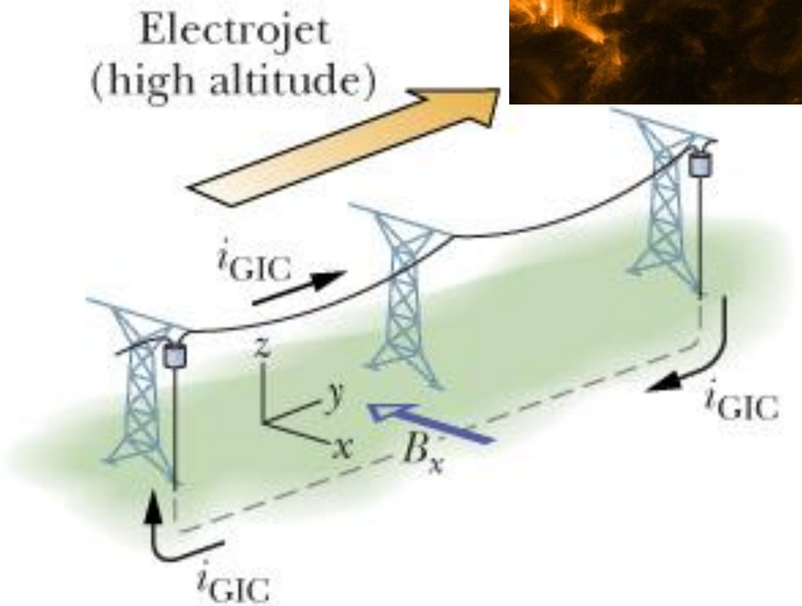
Here R_{eq} is the value of the load resistance as "seen" by the generator:

$$\boxed{R_{eq} = \left(\frac{N_P}{N_S}\right)^2 R \text{ (equivalent resistance)}}$$

Solar Activity and Power-Grid Systems:

- A solar flare: A huge loop of e- and p extends outward from the Sun's surface
- The particles produce a current, called an electrojet, setting up a B field
- The transmission line, the ground, and the wires grounding the transformers form a conducting loop. An electrojet varies in both size and location, producing an emf and a current.
- The current I_{GIC} (GIC: Geomagnetically Induced Current) saturates the iron core, this disrupts the power transmission.





Source: National Geographic

Homework: 1, 2, 7, 9, 10, 17, 23, 25, 29, 32, 38, 41, 48, 53, 57, 62 (pages 855-858)