Cal 3 2014/11

Q1.

a)

Consider the given equation as second order equation with unknown of z^2, solve for z^2 we get:

$$\left[\begin{array}{l} z^2 = -1 - j \\ z^2 = -1 + j \end{array}\right.$$

In rectangular form of complex number $z = r(\cos\theta + j\sin\theta)$, we have the formula to take the natural root of a complex number as follow:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0,1,\dots,n-1$$

With $z^2 = -1 - j$, it holds that:

$$z^2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)$$

Therefore,

$$\left[\begin{array}{l} z_1 = \sqrt{\sqrt{2}}\left(\cos\dfrac{\frac{5\pi}{4} + 0}{2} + j\sin\dfrac{\frac{5\pi}{4} + 0}{2}\right) = \sqrt[4]{2}\left(\cos\dfrac{5\pi}{8} + j\sin\dfrac{5\pi}{8}\right) \\ z_2 = \sqrt{\sqrt{2}}\left(\cos\dfrac{\frac{5\pi}{4} + 2\pi}{2} + j\sin\dfrac{\frac{5\pi}{4} + 2\pi}{2}\right) = \sqrt[4]{2}\left(\cos\dfrac{13\pi}{8} + j\sin\dfrac{13\pi}{8}\right) \end{array}\right.$$

With $z^2 = -1 + j$, it holds that:

$$z^2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}\right)$$

Therefore,

$$\left[\begin{array}{l} z_3 = \sqrt{\sqrt{2}}\left(\cos\dfrac{\frac{3\pi}{4} + 0}{2} + j\sin\dfrac{\frac{3\pi}{4} + 0}{2}\right) = \sqrt[4]{2}\left(\cos\dfrac{3\pi}{8} + j\sin\dfrac{3\pi}{8}\right) \\ z_4 = \sqrt{\sqrt{2}}\left(\cos\dfrac{\frac{3\pi}{4} + 2\pi}{2} + j\sin\dfrac{\frac{3\pi}{4} + 2\pi}{2}\right) = \sqrt[4]{2}\left(\cos\dfrac{11\pi}{8} + j\sin\dfrac{11\pi}{8}\right) \end{array}\right.$$

Thus, the given equation has 4 complex roots:

$$\left[\begin{array}{l} z_1 = \sqrt[4]{2}\left(\cos\dfrac{5\pi}{8} + j\sin\dfrac{5\pi}{8}\right) \\ z_2 = \sqrt[4]{2}\left(\cos\dfrac{13\pi}{8} + j\sin\dfrac{13\pi}{8}\right) \\ z_3 = \sqrt[4]{2}\left(\cos\dfrac{3\pi}{8} + j\sin\dfrac{3\pi}{8}\right) \\ z_4 = \sqrt[4]{2}\left(\cos\dfrac{11\pi}{8} + j\sin\dfrac{11\pi}{8}\right) \end{array}\right.$$

b)

$$z = \frac{1 + j}{j(2 + 3j)} = -\frac{1}{13} - \frac{5}{13}j$$

Q2.

a)

Given that: $f(z) = u(x,y) + jv(x,y)$, where $u(x,y) = y^3 - 3x^2y, v(x,y) = x^3 - 3xy^2 + 2$

Check whether or not the given function satisfied the Cauchy-Riemann equation:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \leftrightarrow \begin{cases} -6xy = -6xy \\ 3y^2 - 3x^2 = -(3x^2 - 3y^2) \end{cases} \text{(Valid)}$$

Therefore, $f(z)$ is an analytic function.

We have: $f(z) = j(x^3 - 3xy^2 + 3x^2jy - jy^3) + 2j$

$= j(x^3 + 3x^2(jy) + 3x(jy)^2 + (jy)^3) + 2j$

$= j(x + jy)^3 + 2j$

$= jz^3 + 2j = j(z^3 + 2)$

Thus, $f(z) = j(z^3 + 2)$ is an analytic function.

b)

$$\mathcal{L}\{t \cos 2t\} = \frac{s^2 - 4}{(s^2 + 4)^2}$$

Q3.

a)

$$\mathcal{L}^{-1}\left\{\frac{s+8}{s^2 + 4s + 13}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+2) + 2 \times 3}{(s+2)^2 + 3^2}\right\} = e^{-2t}\cos 3t + 2e^{-2t}\sin 3t$$

b)

$$\cosh z = -1 \leftrightarrow \frac{e^z + e^{-z}}{2} = -1 \leftrightarrow e^z + 2 + e^{-z} = 0$$

$$\leftrightarrow e^{2z} + 2e^z + 1 = 0 \leftrightarrow e^z = -1$$

With $e^z = -1$, solve for z in rectangular form by Euler formula:

$$e^z = -1 \leftrightarrow e^x(\cos y + j \sin y) = -1$$

$$\leftrightarrow \begin{cases} e^x \cos y = -1 \\ e^x \sin y = 0 \end{cases} \leftrightarrow \begin{cases} e^x \cos y = -1 & (1) \\ \sin y = 0 & (2) \end{cases}$$

From (2): $\left[\begin{array}{l} y = 2k\pi \\ y = \pi + 2k\pi \end{array}\right., k \in \mathbb{Z}$

With $y = 2k\pi$, it holds that: $(1) \to e^x = -1$ (contradiction)

With $y = \pi + 2k\pi$, it holds that: $(1) \to e^x = 1 \leftrightarrow x = 0$

Thus, the solution of the equation is: $z = x + jy = j(\pi + 2k\pi), \quad k \in \mathbb{Z}$

Q4.

a)

Given that:

$$\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 10y = 0 \ (*), \quad y(0) = 7, \quad y'(0) = 26$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 7$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 7s - 26$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s) - 7s - 26] - 7[sY(s) - 7] + 10Y(s) = 0$$

$$\leftrightarrow Y(s)(s^2 - 7s + 10) = 7s - 23$$

$$\leftrightarrow Y(s) = \frac{7s - 23}{s^2 - 7s + 10}$$

$$\leftrightarrow Y(s) = \frac{3}{s-2} + \frac{4}{s-5}$$

$\rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = (3e^{2t} + 4e^{5t})u(t)$

Thus, the solution of the given differential equation is:

$$y(t) = (3e^{2t} + 4e^{5t})u(t)$$

b)

Form the given information: $z^n = \cos n\theta + j\sin n\theta \quad (1)$

$$\rightarrow z^{-n} = \cos(-n\theta) + j\sin(-n\theta) = \cos n\theta - j\sin n\theta \quad (2)$$

Taking $(1) - (2) \leftrightarrow z^n - z^{-n} = 2j\sin n\theta$

$$\leftrightarrow \sin n\theta = \frac{1}{2j}\left(z^n - \frac{1}{z^n}\right) \quad \text{(proof)}$$

Q5.

a)

$$\mathcal{L}\{4 - 3t^2 + 2e^{3t} + e^{-t}\sinh 2t\} = \frac{4}{s} - \frac{6}{s^3} + \frac{2}{s-3} + \frac{2}{(s+1)^2 - 4}$$

b)

$$f(z) = \frac{2}{(z-1)(z-3)} = \frac{1}{z-3} - \frac{1}{z-1}$$

Apply power series for analyzing this problem:

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

We have:

$$f(z) = \frac{1}{1-z} - \frac{1}{3}\frac{1}{1-\frac{z}{3}}$$

With $|z| < 3 \leftrightarrow \left|\frac{z}{3}\right| < \frac{1}{3} < 1$, it holds that:

$$\frac{1}{1-\frac{z}{3}} = \sum_{n=0}^{+\infty}\left(\frac{z}{3}\right)^n$$

Therefore,

$$f(z) = \sum_{n=0}^{+\infty} z^n - \frac{1}{3}\sum_{n=0}^{+\infty}\left(\frac{z}{3}\right)^n$$

$$= \sum_{n=0}^{+\infty} z^n - \sum_{n=0}^{+\infty}\frac{z^n}{3^{n+1}}$$

$$= \sum_{n=0}^{+\infty}\left(1 - \frac{1}{3^{n+1}}\right)z^n$$

Q6. (Optional)

From the given information we obtain the system of differential equation:

$$\begin{cases} 0.5(i_1' + i_2') + 2i_1 = 6 \\ i_2' + 4i_2 - 2i_1 = 0 \end{cases}$$

Convert the system form t-domain to s-domain by Laplace transforms both sides of the system, we get:

$$\begin{cases} 0.5(sI_1 + sI_2) + 2I_1 = \dfrac{6}{s} & (1) \\ sI_2 + 4I_2 - 2I_1 = 0 & (2) \end{cases}$$

From (2) $\rightarrow I_2 = \frac{2I_1}{s+4}$, substitute into (1), we get:

$$0.5\left(sI_1 + \frac{2sI_1}{s+4}\right) + 2I_1 = \frac{6}{s}$$

$$\leftrightarrow I_1\left(0.5s + \frac{s}{s+4} + 2\right) = \frac{6}{s}$$

$$\leftrightarrow I_1 = \frac{6(s+4)}{s(0.5s^2 + 5s + 8)}$$

$$\leftrightarrow I_1 = \frac{3}{s} - \frac{2}{s+2} - \frac{1}{s+8}$$

$\rightarrow i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} = (3 - 2e^{-2t} - e^{-8t})u(t)$

Thus, the current $i_1(t)$ is:

$$i_1(t) = (3 - 2e^{-2t} - e^{-8t})u(t)$$