Fundamentals of Calculus

Chapter I: Derivative

1. Definition

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

2. Properties

Let $u(x), v(x), g(x), f(x)$ are differentiable functions and a constant k, the properties of derivative is given by:

1. $(k \times u)' = k \times u'$
2. $\left(\frac{k}{u}\right)' = -\frac{k.u'}{u^2}$
3. $(u + v)' = u' + v'$
3. $(u + v)' = u' + v'$
4. $(u \times v)' = u'v + uv'$
5. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

3. Derivative of Elementary Functions

Elementary Functions	**Elementary Functions Composition**
$C' = 0$	$(x)' = 1$
$(x^\alpha)' = \alpha x^{\alpha-1}$	$(u^\alpha)' = u'\alpha u^{\alpha-1}$
$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$	$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$
$\left(\sqrt[n]{x}\right)' = \frac{1}{n\sqrt[n]{x^{n-1}}}$	$\left(\sqrt[n]{u}\right)' = \frac{u'}{n\sqrt[n]{u^{n-1}}}$
$(\sin x)' = \cos x$	$(\sin u)' = u' \cos u$
$(\cos x)' = -\sin x$	$(\cos u)' = -u' \sin u$
$(\tan x)' = 1 + \tan^2 x = \frac{1}{\cos^2 x}$	$(\tan u)' = u'(1 + \tan^2 u) = \frac{u'}{\cos^2 u}$
$(\cot x)' = -(1 + \cot x) = -\frac{1}{\sin^2 x}$	$(\cot u)' = -u'(1 + \cot u) = -\frac{u'}{\sin^2 u}$
$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$	$(\arcsin u)' = \frac{u'}{\sqrt{1 - u^2}}$
$(\arccos x)' = \frac{-1}{\sqrt{1 - x^2}}$	$(\arccos x)' = -\frac{u'}{\sqrt{1 - u^2}}$
$(\arctan x)' = \frac{1}{x^2 + 1}$	$(\arctan u)' = \frac{u'}{u^2 + 1}$
$(e^x)' = e^x$	$(e^u)' = u'e^u$
$(a^x)' = a^x \ln a$	$(a^u)' = u'a^u \ln a$
$(\ln x)' = \frac{1}{x}$	$(\ln u)' = \frac{u'}{u}$
$(\log_a x)' = \frac{1}{x \ln a}$	$(\log_a u)' = \frac{u'}{u \ln a}$

4. Derivative of Rational Function

1. $\left(\dfrac{ax+b}{cx+d}\right)' = \dfrac{ad-bc}{(cx+d)^2}$
2. $\left(\dfrac{ax^2+bx+c}{ex+f}\right)' = \dfrac{aex^2+2afx+(bf-ce)}{(ex+f)^2}$
3. $\left(\dfrac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}\right)' = \dfrac{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} x^2 + 2\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} x + \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}}{(a_2x^2+b_2x+c_2)^2}$

5. n-th Derivative

1. $(x^m)^{(n)} = \begin{cases} \dfrac{m!}{(m-n)!}x^{m-n}, & m \geq n \\ 0, & m < n \end{cases}$
2. $(\log_a x)^{(n)} = (-1)^{n-1}\dfrac{(n-1)!}{x^n \ln a}$
3. $(\ln x)^{(n)} = (-1)^{n-1}(n-1)!\,x^{-n}$
4. $\left(e^{kx}\right)^{(n)} = k^n e^{kx}$
5. $(a^x)^{(n)} = a^x \ln^n a$
6. $(\sin ax)^{(n)} = a^n \sin\left(ax + n\dfrac{\pi}{2}\right)$
7. $(\cos ax)^{(n)} = a^n \cos\left(ax + n\dfrac{\pi}{2}\right)$
8. $\left(\dfrac{1}{ax+b}\right)^{(n)} = (-1)^n a^n \dfrac{n!}{(ax+b)^{n+1}}$

Chapter II: Anti-Derivative

1. Definition

Anti-derivative of a function $f(x)$ is a differentiable function $F(x)$ whose derivative is equal to the original function $f(x)$, symbolically it can be express as

$$F'(x) = f(x)$$

<u>Theorem:</u> $F(x)$ is an anti-derivative of $f(x)$ which leads to $F(x) + C$, C is a constant, is also an anti-derivative of $f(x)$.

2. Properties

1. $\int f(x)dx = F(x) + C$
2. $\left(\int f(x)dx\right)' = \int f'(x)dx = f(x)$
3. $\int a \cdot f(x)dx = a \int f(x)dx$
4. $\int [a \cdot f(x) \pm b \cdot g(x)]dx = a \int f(x)dx + b \int f(x)dx$

3. Anti-Derivative of Elementary Functions

With arbitrary constants a, m $(a, m \neq 0)$ and any constant b, the anti-derivative of some elementary functions and its extended are given by the below table

Elementary Functions	**Extended Elementary Functions**
$\int 0 \cdot dx = C$	$\int 1 \cdot dx = x + C$
$\int x^\alpha \cdot dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$	$\int (ax+b)^\alpha \cdot dx = \dfrac{(ax+b)^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$
$\int \dfrac{1}{x} \cdot dx = \ln\lvert x\rvert + C \quad (x \neq 0)$	$\int \dfrac{1}{ax+b} \cdot dx = \dfrac{1}{a}\ln\lvert ax+b\rvert + C \quad \left(x \neq -\dfrac{b}{a}\right)$
$\int \sin x \cdot dx = -\cos x + C$	$\int \sin(ax+b) \cdot dx = -\dfrac{1}{a}\cos(ax+b) + C$
$\int \cos x \cdot dx = \sin x + C$	$\int \cos(ax+b) \cdot dx = \dfrac{1}{a}\sin(ax+b) + C$
$\int \dfrac{1}{\cos^2 x} \cdot dx = \tan x + C$	$\int \dfrac{1}{\cos^2(ax+b)} \cdot dx = \dfrac{1}{a}\tan(ax+b) + C$
$\int \dfrac{1}{\sin^2 x} \cdot dx = -\cot x + C$	$\int \dfrac{1}{\sin^2(ax+b)} \cdot dx = -\dfrac{1}{a}\cot(ax+b) + C$
$\int \ln x \cdot dx = x\ln x - x + C$	$\int \ln(ax+b) \cdot dx = \dfrac{1}{a}(ax+b)\ln(ax+b) - \dfrac{1}{a}x + C$
$\int e^x \cdot dx = e^x + C$	$\int e^{(ax+b)} \cdot dx = \dfrac{1}{a}e^{(ax+b)} + C$
$\int m^x \cdot dx = \dfrac{m^x}{\ln m} + C \quad (m \neq 0)$	$\int m^{(ax+b)} \cdot dx = \dfrac{m^{(ax+b)}}{a\ln m} + C \quad (m \neq 0)$

4. Fundamental Theorem of Calculus

Given $f(x)$ is a continuous real-valued function defined on a closed interval $[a,b]$, let $F(x)$ be the function defined, for all x in $[a,b]$ by

$$F(x) = \int_a^x f(x)dx$$

then, $F(x)$ is uniformly continuous on $[a,b]$ and differentiable on the open interval (a,b), and

$$F'(x) = f(x)$$

for all x in (a,b).

5. Method of Solving Integrals

5. 1. Newton-Leibniz theorem

If $F(x)$ is an anti-derivative of $f(x)$ on an interval $[a,b]$, then

$$\int_a^b f(x)dx = F(a) - F(b)$$

5. 2. Substitution

If the given integral can be expressed as a composition function with a differentiation factor as follow

$$\int f(x)dx = \int g\big(u(x)\big)u'(x)dx$$

By substituting $t = u(x)$ and $dt = u'(x)dx$, the above integral becomes

$$\int g\big(u(x)\big)u'(x)dx = \int g(t)d(t)$$

Solve for integral of function $g(t)$, finally substitute back to x.

5. 3. Integral by Parts

If $u(x)$ and $v(x)$ are continuous differentiable functions on an interval I, then

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

or, it equivalents with

$$\int u(x)d[v(x)] = u(x)v(x) - \int v(x)d[u(x)]$$