Continuous random variables

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Objectives

1. Determine probabilities from probability density functions
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability density functions, and the reverse

- If $Range(X)$ is countable, we can list all possible value of random variable X between a and b to evaluate

$$P(a \leq X \leq b) = \sum_{a \leq x_i \leq b} P(X = x_i)$$

- If X is a **continuous random variable**, i.e. $Range(X)$ is uncountable, it leads to "uncountable sum"

Probability density functions of continuous R.V

Suppose $Range(X)$ is uncountable.
X is *continuous* if there is a non negative function $f(x)$ so that

$$P(a \leq X \leq b) = \int_a^b \underbrace{f(x)}_{\textit{probability density function} \text{ (p.d.f) of } X} dx$$

Interpretation of p.d.f

$$P\left(a \leq X \leq a+\epsilon\right) = \int_{a}^{a+\epsilon} f(x)dx \approx \epsilon f(a)$$

$f(a)$ is a measure of how likely it is that the random variable will be near a - "pmf per unit length"

$$f(x)dx \approx P(x < X < x + dx) \approx P(X \approx x)$$

So

$$P(a \leq X \leq b) \approx \sum_{a \leq X \leq b} P(X \approx x) = \sum_{a \leq x \leq b} f(x)dx$$

$$= \int_a^b f(x)dx$$

Suppose that the error in the reaction temperature, in $^\circ C$, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3} & -1 \leq x \leq 2 \\ 0 & elsewhere \end{cases}.$$

We have

$$f(0.3) = f(-0.3) = 0.03 < f(0.6) = 0.12$$

So the error in the reaction temperature near 0.3 $^\circ C$ is as likely as near -0.3$^\circ C$ but has less chance than near 0.12 $^\circ C$

$$f(3) = 0$$

implies that the error can not be near 3 $^\circ C$.

$$f(x) = 0 \quad \forall x \geq 2 \text{ or } x \leq -1$$

says that the error can not be greater than 2. It can not also take on values less than -1. The error can be between -1 and 2 or we say

$$Range(X) = [-1, 2]$$

which is uncountable

For example

$$P(0 \le X \le 1) = \int_0^1 f(x)dx = \int_0^1 \frac{x^2}{3}dx =$$

$$
\begin{aligned}
P(X \leq 1) = P(-\infty < X \leq 1) &= \int_{-\infty}^{1} f(x)dx \\
&= \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx \\
&= \int_{-\infty}^{-1} 0dx + \int_{-1}^{1} \frac{x^2}{3}dx = \int_{-1}^{1} \frac{x^2}{3}dx
\end{aligned}
$$

Remark that $Range(X) = [-1, 2]$ then $X \leq 1$ is equivalent to $-1 \leq X \leq 1$. So we have

$$P(X \leq 1) = P(-1 \leq X \leq 1) = \int_{-1}^{1} \frac{x^2}{3} dx$$

Example - Uniform RV

RV X takes values in an interval $[a, b]$ such that all subintervals of the same length are equally likely. X is an uniform RV with pdf

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & otherwise \end{cases}$$

Denote $X \sim Uni([a, b])$

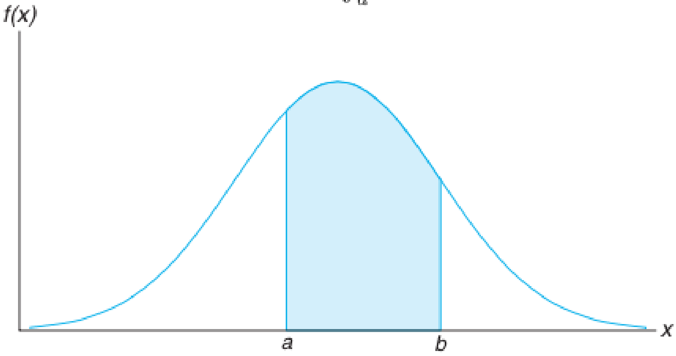
Properties of continuous RV

- $P(X = a) = 0$ for all a
- For all a, b

$$\begin{aligned} P(a \leq X \leq b) &= P(a < X \leq b) \\ &= P(a \leq X < b) \\ &= P(a < X < b) \end{aligned}$$

Probability as an Area

$$P(a < X < b) = \int_a^b f(x)\, dx.$$



Exercise

Suppose that X has p.d.f

$$f(x) = \begin{cases} \frac{3}{256}(8x - x^2) & \text{if } 0 < x < 8 \\ 0 & \text{elsewhere} \end{cases}$$

Determine

(a) $P(X = 3)$ (b) $P(2 < X < 4)$

(c) $P(X > 6)$ (d) $P(X < 5)$

Conditions of pdf

- $f(x) \geq 0$ for all x
- **(Normalization)** $P(-\infty < X < \infty) = 1$ implies that

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Example

Suppose that the error in the reaction temperature, in $\circ C$, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3} & \text{if } -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

1. Verify that $f(x)$ is a density function.
2. Find $P(0 < X \leq 1)$.

Solution

1. Obviously $f(x) \geq 0$. Need to verify the 2nd condition

$$\int_{-\infty}^{\infty} f(x)dx = 0 \text{ or } \int_{-\infty}^{-1} 0dx + \int_{-1}^{2} \frac{x^2}{3}dx + \int_{2}^{\infty} 0dx = 1$$

2. $P(0 < X \leq 1) = \int_0^1 f(x)dx = \int_0^1 \frac{x^2}{3}dx = \frac{x^3}{9}\Big|_0^1 = \frac{1}{9}$

Example

A gambler spins a wheel of fortune, continuously calibrated between 0 and 1, and observes the resulting number. Assuming that all subintervals of $[0,1]$ of the same length are equally likely. The observed number is a random variable X with pdf

$$f(x) = \begin{cases} c, & 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

The constant c is determined by

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

So

$$\int_{0}^{1} cdx = 1$$

then $c = 1$

Practice

Suppose the p.d.f of X is

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of C?

Find $P(X > 1)$?

Cumulative distribution function (cdf)

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

Properties of cdf of a continuous RV

- $F'(x) = f(x)$ for all x
- $P(a \leq X \leq b) = F(b) - F(a)$
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

Example

Suppose that the error in the reaction temperature, in $^\circ C$, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & elsewhere \end{cases}.$$

Find cdf F of X

Solution

- For $x < -1$

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{-\infty}^{x} 0du = 0$$

- For $-1 \leq x \leq 2$

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{-1}^{x} \frac{u^2}{3}du = \frac{x^3 + 1}{9}$$

- For $x > 2$

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{-1}^{2} \frac{u^2}{3} du = 1$$

- Hence the cdf of X is

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x^3+1}{9}, & -1 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

Example

Find p.d.f of X if its c.d.f is

$$F(x) = \begin{cases} 1 - e^{-2x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(x) = F'(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Use cdf to evaluate probability of a continuous random variable

- $P(X \leq b) = P(X < b) = F_X(b)$
- $P(X \geq a) = P(X > a) = 1 - F(a)$
- $P(a \leq X \leq b) = P(a < X < b) = F(b) - F(a)$

The cdf of X is

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x^3+1}{9}, & -1 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

then $P(X \le 1) = F(1) = \frac{2}{9}$

$$P(0.5 < X < 1.5) = F(1.5) - F(0.5) = \ldots$$

Practice

Consider a pdf

$$f(x) = \begin{cases} k\sqrt{x} & \text{if } 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

1. Evaluate k
2. Find the cdf $F(x)$
3. Use cdf to evaluate $P(0.3 < X < 0.6)$

Keywords

- pdf of a continuous RV
  - $f(x) \geq 0$ for all x
  - $\int_{-\infty}^{\infty} f(x)dx = 1$
  - $P(a \leq X \leq b) = \int_a^b f(x)dx$
- cdf of a continuous RV $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$

$$P(a \leq X \leq b) = F(b) - F(a)$$

- Relationship between pdf and cdf: $F'(x) = f(x)$