Cal 3 2015/10

Q1.

a)

Let: $z = \frac{1}{2} - \frac{\sqrt{3}}{2} j$

$$\rightarrow \begin{cases} r = |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1 \\ \theta = \tan^{-1} \frac{-\sqrt{3}}{1} = \frac{2\pi}{3} \end{cases}$$

Therefore, in polar form, z can be expressed as: $z = \cos \frac{2\pi}{3} + j \sin \frac{2\pi}{3}$

b)

Since, $e^{2+j\frac{\pi}{2}} = e^2 \left(\cos \frac{\pi}{2} + j \sin \frac{\pi}{2}\right) = 0 + je^2$

Therefore, the real and imaginary parts are 0 and e^2, respectively

Q2.

Let $f(z) = u(x, y) + jv(x, y)$, where $u(x, y) = (x - y)^2, v(x, y) = 2(x + y)$

$f'(z)$ exists at a point if and only if at this point it satisfies the Cauchy-Riemann equation:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \leftrightarrow \begin{cases} 2(x - y) = 2 \\ 2(y - x) = -2 \end{cases} \leftrightarrow y = x - 1$$

Thus, for all points which belong to the line $y = x - 1$ in the z-plane lead to existence of $f'(z)$

Q3.

$$f(z) = \frac{2}{(z - 1)(z - 3)} = \frac{1}{z - 3} - \frac{1}{z - 1}$$

Apply power series for analyzing this problem:

$$\frac{1}{1 - z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

We have:

$$f(z) = -\frac{1}{3} \frac{1}{1 - \frac{z}{3}} - \frac{1}{z} \frac{1}{1 - \frac{1}{z}}$$

With $1 < |z| \leftrightarrow \frac{1}{|z|} < 1$, it holds that:

$$\frac{1}{1 - \frac{1}{z}} = \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^n$$

With $|z| < 3 \leftrightarrow \left|\frac{z}{3}\right| < 1$, it holds that:

$$\frac{1}{1 - \frac{z}{3}} = \sum_{n=0}^{+\infty} \left(\frac{z}{3}\right)^n$$

Therefore,

$$f(z) = -\frac{1}{3}\sum_{n=0}^{+\infty}\left(\frac{z}{3}\right)^n - \frac{1}{z}\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n$$
$$= \frac{1}{3}\sum_{n=0}^{+\infty}\left(\frac{z}{3}\right)^n + \sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^{n+1}$$
$$= \sum_{n=0}^{+\infty}\left[\frac{z^n}{3^{n+1}} + \frac{1}{z^{n+1}}\right]$$

Q4.

$$f(z) = \frac{1}{z+j}$$

Apply power series for analyzing this problem:

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

We have:

$$f(z) = \frac{1}{1+j+z-1} = \frac{\frac{1}{1+j}}{1+\frac{z-1}{1+j}}$$
$$= \frac{1}{1+j}\sum_{n=0}^{+\infty}\left(-\frac{z-1}{1+j}\right)^n$$
$$= \left(\frac{1}{2}-\frac{1}{2}j\right)\sum_{n=0}^{+\infty}\left(-\frac{1}{2}+\frac{1}{2}j\right)^n (z-1)^n$$
$$= -\sum_{n=0}^{+\infty}\left(-\frac{1}{2}+\frac{1}{2}j\right)^{n+1} (z-1)^n$$

This series valid for all z such that $\left|\frac{z-1}{1+j}\right| < 1$

a)

The Laurent series is:

$$f(z) = -\left(-\frac{1}{2}+\frac{1}{2}j\right) + \left(-\frac{1}{2}+\frac{1}{2}j\right)(z-1) - \left(-\frac{1}{2}+\frac{1}{2}j\right)(z-1)^2 + \cdots$$
$$-\left(-\frac{1}{2}+\frac{1}{2}j\right)^{n+1}(z-1)^n$$

b)

Since we have: $\left|\frac{z-1}{1+j}\right| < 1 \leftrightarrow |z-1| < |1+j| \leftrightarrow |z-1| < \sqrt{2}$

Therefore, the radius of convergence is $R = \sqrt{2}$

Q5.

Given that:

$$x''(t) - 2x'(t) - 3x(t) = e^{-t}\cos 2t \quad (*), \qquad x(0) = 0, \quad x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$, it holds that:

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0) = sX(s)$$
$$\mathcal{L}\{x''(t)\} = s^2X(s) - sx(0) - x'(0) = s^2X(s)$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$s^2X(s) - 2sX(s) - 3X(s) = \frac{s+1}{(s+1)^2+2^2}$$

$$\leftrightarrow X(s)(s^2-2s-3) = \frac{s+1}{(s+1)^2+2^2}$$

$$\leftrightarrow X(s) = \frac{\frac{s+1}{(s+1)^2+2^2}}{s^2-2s-3}$$

$$\leftrightarrow X(s) = \frac{1}{((s+1)^2+4)(s-3)}$$

Thus,

$$X(s) = \frac{1}{((s+1)^2+4)(s-3)}$$

Q6.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+j)^3}\right\} = \frac{1}{2}e^{-jt}t^2 = \frac{1}{2}t^2(\cos t - j\sin t)$$