Cal 3 2013/07

Section A

Q1.

a)

We have: $z = 1 + j\sqrt{3}$

$$\theta_1 = \arg(z) = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

We have: $w = \dfrac{-2}{1 + j\sqrt{3}} = \dfrac{-2}{z}$

$$\theta_2 = \arg(w) = \arg(-2) - \arg(z) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

b)

Let: $z = \left(1 + j\sqrt{3}\right)^{12}$

$$\rightarrow \begin{cases} r = |z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \\ \theta = \tan^{-1}\dfrac{\sqrt{3}}{1} = \dfrac{\pi}{3} \end{cases}$$

From Moivre's theorem, we have:

$$z^n = r^n(\cos n\theta + j\sin n\theta)$$

Therefore,

$$z = 2^{12}\left(\cos\frac{12\pi}{3} + j\sin\frac{12\pi}{3}\right) = 4096$$

Q2.

a)

Let: $z = x + yj;\ x, y \in \mathbb{R}$

We have: $f(z) = z^3 + 3z - 1 = (x + yj)^3 + 3(x + yj) - 1$

$$= x^3 - 3xy^2 + 3x - 1 + j(-y^3 + 3xy^2 + 3y)$$

b)

Given that: $f(z) = u(x, y) + jv(x, y)$, where $u(x, y) = x^2, v(x, y) = y^2$

Check whether or not the given function satisfied the Cauchy-Riemann equation:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \leftrightarrow \begin{cases} 2x = 2y \\ 0 = 0 \end{cases} \leftrightarrow x = y$$

Therefore, for all $x = y$, $f'(z)$ exists. Then:

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} = 2x$$

Q3.

a)

In unit step function, $f(t)$ is rewritten as follow:

$$f(t) = (t - 1)u(t - 1) + (4 - 2t)u(t - 2) + (t - 3)u(t - 3)$$

Therefore,

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2}e^{-s} - \frac{2}{s^2}e^{-2s} + \frac{1}{s^2}e^{-3s}$$

b)

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+4^2}\right\} = \cos(4t)\,e^{-3t} - \frac{1}{2}\sin(4t)\,e^{-3t}$$

Q4.

Given that:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 6e^{-t} \ \ (*), \qquad y(0) = -2, \quad y'(0) = 8$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) + 2$$
$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + 2s - 8$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s) + 2s - 8] + 4[sY(s) + 2] + 4[Y(s)] = \frac{6}{s+1}$$
$$\leftrightarrow Y(s)(s^2+4s+4) = \frac{6}{s+1} - 2s$$
$$\leftrightarrow Y(s) = \frac{\frac{6}{s+1} - 2s}{s^2+4s+4}$$
$$\leftrightarrow Y(s) = \frac{-2}{(s+2)^2} - \frac{8}{s+2} + \frac{6}{s+1}$$

$\rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = (-2te^{-2t} - 8e^{-2t} + 6e^{-t})u(t)$

Thus, the solution of the given differential equation is:

$$y(t) = (-2te^{-2t} - 8e^{-2t} + 6e^{-t})u(t)$$

Section B

Q1.

We have: $v(t) = 200\left(1 - \frac{t^3}{3}\right)\left(u(t) - u(t-1)\right)$

$$\rightarrow V(s) = \mathcal{L}\{v(t)\} = \frac{200}{s} - \frac{400}{s^4} - \left(\frac{400}{s^4} + \frac{400}{s^3} + \frac{200}{s^2} - \frac{400}{3s}\right)e^{-s}$$

By Kirchhoff voltage law:

$$L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau + Ri = v(t)$$

Taking Laplace transform both sides, we obtain:

$$L\big(sI(s) - i(0)\big) + \frac{I(s)}{Cs} + RI(s) = V(s)$$
$$\leftrightarrow I(s)\left(s^2L + Rs + \frac{1}{C}\right) = sV(s)$$
$$\leftrightarrow I(s) = \frac{sV(s)}{s^2L + Rs + \frac{1}{C}}$$
$$\leftrightarrow I(s) = \frac{200s^3 - 400}{s^3(s^2+4)} - \frac{1}{3}\frac{400s^3 - 600s^2 - 1200s - 1200}{s^3(s^2+4)}e^{-s}$$
$$\leftrightarrow I(s) = -\frac{100}{s^3} + \frac{25}{s} + \frac{-25s+200}{s^2+4} - \frac{1}{3}\left(-\frac{300}{s^3} - \frac{300}{s^2} - \frac{75}{s} + \frac{75s+700}{s^2+4}\right)e^{-s}$$

$\rightarrow i(t) = \mathcal{L}^{-1}\{I(s)\} = (-50t^2 + 25 - 25\cos(2t) + 100\sin(2t))u(t)$

$+\left(50(t-1)^2 + 100(t-1) + 25 - 25\cos\big(2(t-1)\big) - \frac{350}{3}\sin\big(2(t-1)\big)\right)u(t-1)$

Q2.

We have:

- $\dfrac{\partial u}{\partial x} = 2e^{x^2-y^2}x\cos(2xy) \ - 2e^{x^2-y^2}y\sin(2xy)$

$\rightarrow \dfrac{\partial^2 u}{\partial x^2} = 2(2x^2 - 2y^2 + 1)e^{x^2-y^2}x^2\cos(2xy) - 8e^{x^2-y^2}xy\sin(2xy)$

- $\dfrac{\partial u}{\partial y} = -2e^{x^2-y^2}y\cos(2xy) - 2e^{x^2-y^2}x\sin(2xy)$

$\rightarrow \dfrac{\partial^2 u}{\partial y^2} = 8e^{x^2-y^2}xy\sin(2xy) - 2(2x^2 - 2y^2 + 1)e^{x^2-y^2}x^2\cos(2xy)$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, $u(x,y)$ is harmonic function.

The harmonic conjugate for $u(x,y)$ is $v(x,y) = e^{x^2-y^2}\sin(2xy)$