INTERNATIONAL UNIVERSITY-VNUHCM

FINAL EXAMINATION

Semester 3, Academic Year 2018-2019
Duration: 120 minutes

<table>
<tr><td colspan="2">SUBJECT:
Calculus 2</td></tr>
<tr><td>Chair of Department of Mathematics

Signature:

Full name:</td><td>Lecturer:

Signature:

Full name: Assoc.Prof. Mai Duc Thanh</td></tr>
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- **Each student is allowed a maximum of two double-sided sheets of reference material (of size A4 or similar) and a scientific calculator. All other documents and electronic devices are forbidden.**
- **Each question carries 20 marks.**

Question 1. a) Find the limit $\lim_{(x,y)\to(0,0)} \dfrac{4xy}{x^2+y^2}$, if it exists, or show that the limit does not exist.

b) Find the first partial derivatives of the function $f(x,y) = e^{2x-3y}$

Question 2. Find the local maximum and minimum values and saddles point(s) of the function $f(x,y) = -8x^3 + 12xy - y^3 + 2$

Question 3. Evaluate the following multiple integral

a)

$$I = \iint_D 3xy^2 \, dA, \quad D = \{(x,y) \mid \quad 0 \le x \le 1, \ 0 \le y \le 2x^2\}$$

b) $\iiint_E 27(1-xy)dV$, where E lies under the surface $z = 1 + xy$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$

Question 4. a) Evaluate the line integral $\int_C 3xyds$, where C is the line segment from $(0,0)$ to $(1,2)$.

b) Find the work done by the force field $\mathbf{F}(x,y,z) =< x + y^2, y - 2z, z - x >$ on a particle that moves along the line segment from $(1,0,0)$ to $(2,1,3)$.

Question 5. a) Find $\operatorname{curl} \mathbf{F}$ and $\operatorname{div} \mathbf{F}$ if $\mathbf{F}(x,y,z) = (x+2y)\mathbf{i} + (yz)\mathbf{j} + (xz^2)\mathbf{k}$.

b) Evaluate the surface integral $\iint_S yz \, dS$, where S is the part of the plane $z = 2 + 2x + 2y$ that lies above the triangular region in the xy-plane with the vertices $(0,0), (2,0)$ and $(0,1)$.

*** END OF QUESTIONS***

SOLUTIONS OF FINAL EXAM

Subject: CALCULUS 2

Question 1. a) Along $x = 0$: $f = 0$

Along $x = y$:

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{4x^2}{2x^2} = 2 \neq 0,$$

Thus, the limit does not exist.

b) Find the first partial derivatives of the function $f(x,y) = e^{2x-3y}$.

$$f_x(x,y) = 2e^{2x-3y}, \quad f_y(x,y) = -3e^{2x-3y}.$$

Question 2. $f(x,y) = -8x^3 + 12xy - y^3 + 2$

Critical points of f satisfy

$$f_x(x,y) = -24x^2 + 12y = 12(y - 2x^2) = 0$$
$$f_y(x,y) = 12x - 3y^2 = 3(4x - y^2) = 0.$$

This yields $y = 2x^2$ and $4x - (2x^2)^2 = 4x - 4x^4 = 4x(1 - x^3) = 0$, so that $x = 0$ or $x = 1$. Thus, there are 2 critical points $P(0,0)$ and $Q(1,2)$.

Second partial derivatives test:

$$f_{xx}(x,y) = -48x, \quad f_{xy}(x,y) = 12, \quad f_{yy}(x,y) = -6y,$$

and

$$D = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 48(6)xy - 12^2 = 12^2(2xy - 1).$$

At $P(0,0)$: $D = -12^2 < 0$, so $P(0,0)$ is a saddle point.

At $Q(1,2)$: $D = (3)12^2 > 0$, $f_{xx}(1,2) = -48 < 0$, so $f(1,2)$ is a local maximum value.

Question 3. a)

$$I = \iint_D 3xy^2 \, dA, \quad D = \{(x,y) \mid \ 0 \leq x \leq 1, \ 0 \leq y \leq 2x^2\}.$$

$$I = \int_0^1 \int_0^{2x^2} 3xy^2 \, dy \, dx = \int_0^1 xy^3 \Big|_{y=0}^{y=2x^2} dx$$
$$= \int_0^1 8x^7 dx = x^8 \Big|_0^1 = 1.$$

b) E is given by

$$E = \{(x,y,z) | 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}, 0 \leq z \leq 1 + xy\}$$

So

$$\begin{aligned}
\iiint_E 27(1-xy)dV &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+xy} 27(1-xy)dzdydx \\
&= \int_0^1 \int_0^{\sqrt{x}} 27(1-xy)z\Big|_0^{1+xy} dydx \\
&= \int_0^1 \int_0^{\sqrt{x}} 27(1-xy)z\Big|_0^{1+xy} dydx \\
&= \int_0^1 \int_0^{\sqrt{x}} 27(1-x^2y^2)dydx \\
&= \int_0^1 9(3y-x^2y^3)\Big|_0^{\sqrt{x}} dx \\
&= \int_0^1 9(3\sqrt{x}-x^{7/2})dx \\
&= (18x^{3/2}-2x^{9/2})\Big|_0^1 = 18-2 = 16.
\end{aligned}$$

Question 4.

a) $C: y = 2x, 0 \le x \le 1.$

$$\begin{aligned}
\int_C 3xy\, ds &= \int_0^1 6x^2\sqrt{1+4}dx \\
&= \sqrt{5}2x^3\Big|_0^1 = 2\sqrt{5}.
\end{aligned}$$

b) $C: \mathbf{r}(t) =< 1+t, t, 3t >, \mathbf{r}'(t) =< 1, 1, 3 >, 0 \le t \le 1.$ It holds that

$$F(\mathbf{r}(t)) =< 1+t+t^2, t-2(3t), 3t-(1+t) >=< t^2+t+1, -5t, 2t-1 >$$

and

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^2+t+1-5t+3(2t-1) = t^2+2t-2.$$

Thus, the work done is given by

$$\begin{aligned}
W = \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt \\
&= \int_0^1 (t^2+2t-2)dt \\
&= (t^3/3+t^2-2t)\Big|_0^1 = 1/3+1-2 = -2/3.
\end{aligned}$$

Question 5. a)

$$\begin{aligned}
\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & yz & xz^2 \end{vmatrix} \\
&=< 0-y, -(z^2-0), 0-2 >=< -y, -z^2, -2 >
\end{aligned}$$

and

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = 1 + z + 2xz$$

b) Let $D = \{(x, y)|0 \leq x \leq 2, 0 \leq y \leq 1 - x/2\}$. The part S can be seen as the graph of the function $z = 2 + 2x + 2y$ with the domain D. Thus,

$$\begin{aligned}
\iint_S yz dS &= \iint_D y(2 + 2x + 2y)\sqrt{2^2 + 2^2 + 1} dA \\
&= \iint_D (6y + 6xy + 6y^2) dA \\
&= \int_0^2 \int_0^{1-x/2} (6y + 6xy + 6y^2) dy dx \\
&= \int_0^2 (3y^2 + 3xy^2 + 2y^3)\Big|_0^{1-x/2} dx \\
&= \int_0^2 (1 - x/2)^2(3 + 3x + 2 - x) dx \\
&= \int_0^2 (x^3/2 - 3x^2/4 - 3x + 5) dx \\
&= (x^4/8 - x^3/4 - 3x^2/2 + 5x)\Big|_0^2 = 2 - 2 - 6 + 10 = 4
\end{aligned}$$