Midterm: April, 2017

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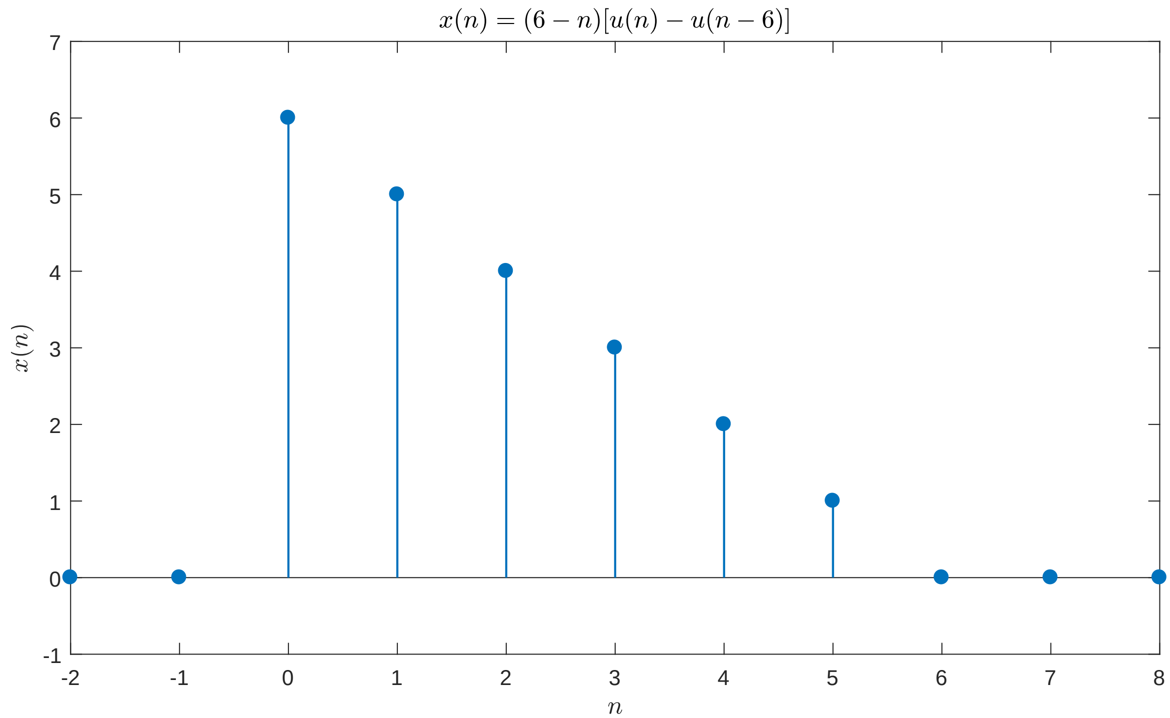
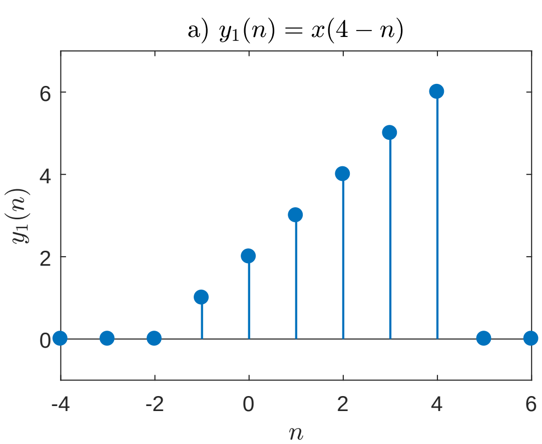
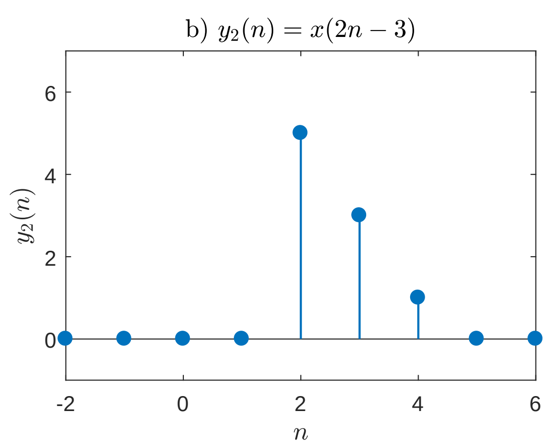
Question 1

The graph of original sequence

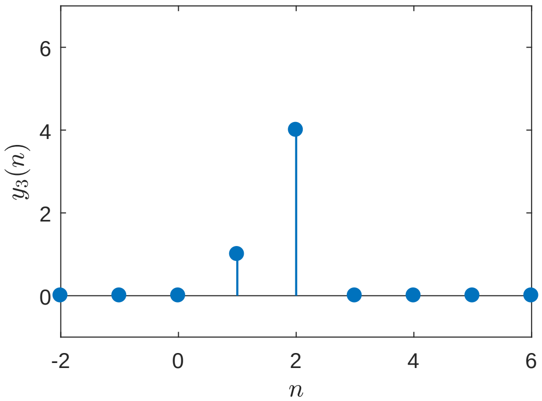


Using the properties of time shifting and time scaling, we obtain the results as follows:

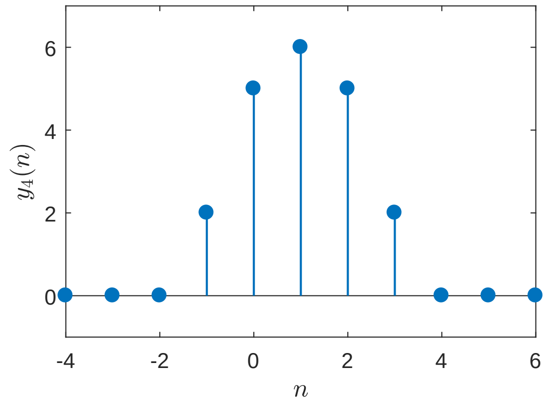




c) $y_3(n) = x(8-3n)$



d) $y_4(n) = x(n^2 - 2n + 1)$



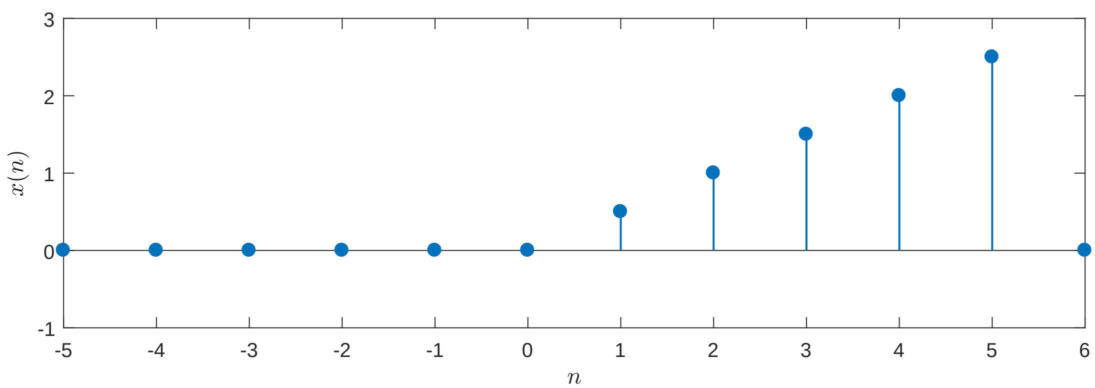
Question 2

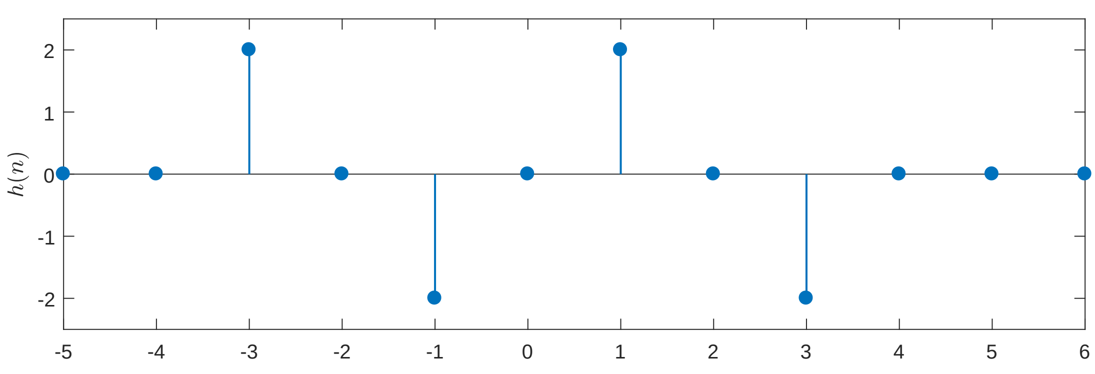
a) Value table for $x(n)$ and $h(n)$

n	0	1	2	3	4	5	6
$x(n)$	0	0.5	1.0	1.5	2.0	2.5	0

n	-4	-3	-2	-1	0	1	2	3	4
$h(n)$	0	2	0	-2	0	2	0	-2	0

The graph for $x(n)$ and $h(n)$ are sketched as follows





b) From the graph of $h(n)$, we can express $h(n)$ as sum of impulse train, that is,

$$h(n) = 2\delta(n+3) - 2\delta(n+1) + 2\delta(n-1) - 2\delta(n-3)$$

By the definition of impulse response, we obtain the I/O relation ship via the impulse response above

$$y(n) = 2x(n+3) - 2x(n+1) + 2x(n-1) - 2x(n-3)$$

Therefore, the output of the system is

$$\begin{aligned}
&+ y(-3) = 2x(0) - 2x(-2) + 2x(-4) - 2x(-6) = \cdots = 0\\
&+ y(-2) = 2x(1) - 2x(-1) + 2x(-3) - 2x(-5) = \cdots = 1\\
&+ y(-1) = 2x(2) - 2x(0) + 2x(-2) - 2x(-4) = \cdots = 2\\
&+ y(0) = 2x(3) - 2x(1) + 2x(-1) - 2x(-3) = \cdots = 2\\
&+ y(1) = 2x(4) - 2x(2) + 2x(0) - 2x(-2) = \cdots = 2\\
&+ y(2) = 2x(5) - 2x(3) + 2x(1) - 2x(-1) = \cdots = 3\\
&+ y(3) = 2x(6) - 2x(4) + 2x(2) - 2x(0) = \cdots = -2\\
&+ y(4) = 2x(7) - 2x(5) + 2x(3) - 2x(1) = \cdots = -3\\
&+ y(5) = 2x(8) - 2x(6) + 2x(4) - 2x(2) = \cdots = 2\\
&+ y(6) = 2x(9) - 2x(7) + 2x(5) - 2x(3) = \cdots = 2\\
&+ y(7) = 2x(10) - 2x(8) + 2x(6) - 2x(4) = \cdots = -4\\
&+ y(8) = 2x(11) - 2x(9) + 2x(7) - 2x(5) = \cdots = -5
\end{aligned}$$

Thus,

$$y(n) = \{0, 1, 2, 2, 2, 3, -2, -3, 2, 2, -4, -5\}$$

Question 3

Hint: The output for sinusoidal input signal frequency ω_0, that is, $x(n) = \sin(\omega_0 n + \theta)$ interacts with system by transfer function $H(e^{j\omega})$ is given by

$$y(n) = \left|H(e^{j\omega_0})\right| \sin\left(\omega_0 n + \theta + \angle H(e^{j\omega_0})\right)$$

This equality also holds for cosine signal by replace the term sine by cosine.

Question 3 (cont)

Sampling the analog input signal $x_a(t)$ at frequency $f_s = 2$ kHz we obtain the discrete time signal $x(n)$ as below

$$x(n) = x_a(nT_s) = \sin\left(1000\pi n \times \frac{1}{2000}\right) = \sin(0.5\pi n)$$

The sampled signal has digital frequency that is $\omega_0 = 0.5\pi$. The magnitude and phase of the discrete time filter at that frequency is

$$H(e^{j0.5\pi}) = 2 \angle 0.9273$$

Therefore, the response $y(n)$ of the filter after $x(n)$ passed is given by

$$y(n) = 2\sin(0.5\pi n + 0.9273)$$

Assume that the analog re-constructor is ideal, we obtain the analog reconstructed signal $y_a(t)$ by substituting back $n = t/T_s$

$$y_a(t) = 2\sin(1000\pi t + 0.9273)$$

Question 4

a) Assume that the quantization error e as a random variable which is distributed uniformly over the range $[-Q/2, Q/2]$ then having probability density

$$p(e) = \begin{cases} \dfrac{1}{Q} & \text{if} -\dfrac{Q}{2} \le e < \dfrac{Q}{2} \\ 0 & \text{otherwise} \end{cases}$$

The mean error is

$$\bar{e} = \int_{-\infty}^{+\infty} ep(e)de = \int_{-Q/2}^{Q/2} e\,\frac{1}{Q}\,de = 0$$

The second moment error is

$$\bar{e^2} = \int_{-\infty}^{+\infty} e^2 p(e)de = \int_{-Q/2}^{Q/2} e^2\,\frac{1}{Q}\,de = \frac{Q^2}{12}$$

Therefore, the average noise power is

$$\sigma_e^2 = \bar{e^2} - (\bar{e})^2 = \frac{Q^2}{12}$$

b) The non-normalized signal-to-noise ratio is given by

$$\text{SQNR} = 10\log\left(\frac{\sigma_x^2}{\sigma_e^2}\right) = 10\log\left(\frac{\sigma_x^2}{\frac{Q^2}{12}}\right) = 10\log\left(\frac{12\sigma_x^2}{Q^2}\right)$$

The quantization width Q of the quantizer is given by

$$Q = \frac{R}{2^{B+1}} = \frac{2X_{\max}}{2^{B+1}}$$

Then the SQNR becomes

$$\text{SQNR} = 10\log\left(\frac{12\sigma_x^2}{\frac{2^2X_{\max}^2}{4^{B+1}}}\right) = 10\log\left(\frac{3\cdot 4^{B+1}\sigma_x^2}{X_{\max}^2}\right)$$

Expand and simplify this equality, we obtain

$$\text{SQNR} = 6.02(B+1) + 4.77 - 20\log\left(\frac{X_{\max}}{\sigma_x}\right)$$

c) The problem gives us

$$\text{SQNR} = 6.02(B+1) + 4.77 - 20\log\left(\frac{X_{\max}}{\sigma_x}\right) \geq 90 \text{ dB}$$

With $X_{\max} = 3\sigma_x$, solving for B yields that

$$B \geq 14.74$$

Rounding B, we get $B = 15$ bits. Thus, the A/D converter needs $B + 1 = 16$ bits to ensure that SQNR at least 90 dB.

Question 7

a) Observing that the impulse function at the input can express as

$$\delta(n) = \frac{1}{2}x_1(n) - \frac{1}{2}x_2(n) + x_3(n)$$

It is known that the system is linear then the above result holds that

$$\text{L}\{\delta(n)\} = \frac{1}{2}y_1(n) - \frac{1}{2}y_2(n) + y_3(n) \quad (1)$$

Again, observing the operation of the system gives us

$$\delta(n-1) = \frac{1}{2}x_2(n) - \frac{1}{2}x_x(n)$$

It is known that the system is linear then the above result holds that

$$\mathrm{L}\{\delta(n-1)\} = \frac{1}{2}y_2(n) - \frac{1}{2}y_1(n) \quad (2)$$

From (1) and (2), it leads to the conclusion that the given system is **not time invariant**.

b) Since, we have

$$h(n) = \mathrm{L}\{\delta(n)\} = \frac{1}{2}y_1(n) - \frac{1}{2}y_2(n) + y_3(n)$$

And

$$y_1(n) = -\delta(n+1) + 3\delta(n) + 3\delta(n-1) + \delta(n-3)$$
$$y_2(n) = -\delta(n+1) + \delta(n) - 3\delta(n-1) - \delta(n-3)$$
$$y_3(n) = 2\delta(n+2) + \delta(n+1) - 3\delta(n) + 2\delta(n-2)$$

Therefore, the impulse response of the system can be

$$h(n) = 2\delta(n+2) + \delta(n+1) - 2\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$$