DIFFERENTIAL EQUATIONS

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References

William E. Boyce, Richard C. DiPrima, Elementary Differential Equations and Boundary Problems, 8th Ed., John Wiley & Sons, 2005.

Chapter 1 INTRODUCTION

Contents

1. Some Basic Mathematical Models
2. Classification of Differential Equations

1.1 SOME BASIC MATHEMATICAL MODELS

Why are differential equations important?

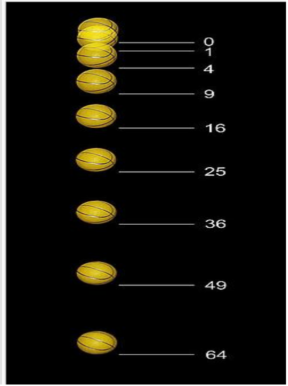
- Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen.

- When expressed in mathematical terms, the relations are equations and the rates are derivatives.

- So these mathematical models often involve an equation in which a function and its derivatives play important roles. Such equations are called **differential equations**.

A Falling Object

Suppose an object is dropped from a height at time $t = 0$.
Let $h(t)$ be the height of the object at time t;
Let $a(t)$ be the acceleration and let $v(t)$ be the velocity.



A Falling Object

The relationships between a, v and h are as follows:

$$a(t) = \frac{dv}{dt}, \quad v(t) = \frac{dh}{dt}.$$

For a falling object, $a(t)$ is constant and is equal to $g = -9.8m/s^2$ (the acceleration due to gravity).

Combining the above equations, we have the **differential equation**

$$\boxed{\frac{dh^2}{dt^2} = g}.$$

$a = v'$

$v = x'$

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$$\boxed{\frac{dh^2}{dt^2} = g}.$$

Integrate both sides of the above equation to obtain $\frac{dh}{dt} = gt + v_0$.
Integrate one more time to obtain

$$\boxed{h(t) = (1/2)gt^2 + v_0 t + h_0}$$

The above equation describes the height of a falling object, from an initial height h_0 at an initial velocity v_0, as a function of time.

Business

A large corporation starts at time $t = 0$ to invest part of its receipts at a rate of P dollars per year in a fund for future corporate expansion.

Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P,$$

where $A = 0$ when $t = 0$.

Find the amount A provided $P = \$100000$, $r = 12\%$, and $t = 5$ years.

Modeling Hybrid Selection

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. This hybrid selection model is described by

$$\frac{dy}{dt} = ky(1-y)(2-y);$$

where y represents the percent of the population that has characteristic D and t represents the time (measured in generations) and k is a constant.

At the beginning of your study ($t = 0$), you find that half the population has characteristic D.

After four generations ($t = 4$) you find that 80% of the population has characteristic D.

Find the percent of the population that will have characteristic D after 10 generations.

Radioactivity

In the case of radioactive decay, we assume that the rate of decay is proportional to the amount of radioactive substance present. This leads to the equation

$$\frac{dA}{dt} = -kA, \quad k > 0,$$

where A is the unknown amount of radioactive substance present at time t and k is the proportionality constant.

The solution of this equation is

$$A = Ce^{-kt}.$$

The constants C and k can be determined if the initial amount of radioactive substance and the half-life of the substance are given.

Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 60 grams of A are present, and after 1 hour ($t = 1$) only 10 grams of A remain unconverted. How much of A is present after 2 hours?

Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 60 grams of A are present, and after 1 hour ($t = 1$) only 10 grams of A remain unconverted. How much of A is present after 2 hours?

Solution Let y be the unconverted amount of substance A at any time t. From the given assumption about the conversion rate, you can write the differential equation as follows.

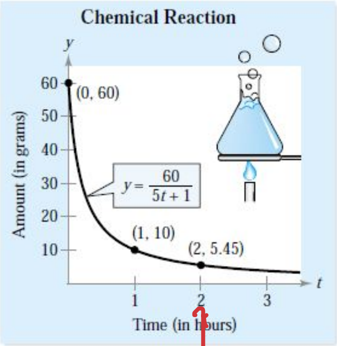
$$\underbrace{\frac{dy}{dt}}_{\text{Rate of change of } y} \underbrace{=}_{\text{is proportional to}} k\underbrace{y^2}_{\text{the square of } y.}$$

Modeling a Chemical Reaction

Solve the equation, to get

$$y(t) = \frac{60}{5t+1}.$$

So $y(2) = 5.45$



Field Mice and Owls

Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. Denote time by t and the mouse population by $p(t)$, then

$$\frac{dp}{dt} = rp,$$

where r is called the **rate constant** or **growth rate**.

The solution is $p(t) = Ce^{rt}$.

Other Examples

Example 1.1 In the study of an electric circuit consisting of resistors, inductors, and capacitors, an application of *Kirchhoff's law* leads to the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

where L is the inductance, R is the resistance, C is the capacitance, $E(t)$ is the electromotive force, $q(t)$ is the charge, and t is the time.

Other Examples

Example 1.2 In the study of the gravitational equilibrium of a star, an application of *Newton's law of gravity* and of the *Stefan-Boltzmann law* for gases leads to the equilibrium equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi\rho G,$$

where P is the sum of the gas kinetic pressure and the radiation pressure, r is the distance from the center of the star, ρ is the density of matter, and G is the gravitational constant.

Other Examples

Example 1.3 In psychology, one model of the learning of a task involves the equation

$$\frac{dy/dt}{y^{3/2}(1-y)^{3/2}} = \frac{2p}{\sqrt{n}}.$$

Here y represents the state of the learner or the leaner's skill level as a function of time t. The constants p and n depend on the individual leaner and the nature of the task.

Example 1.4 In the study of vibrating strings and the propagation of waves, we find the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

where t represents time, x the location along the string, and u the displacement of the string, which is a function of time and location, $u = u(t, x)$.

Independent and dependent variables

When an equation involves one or more derivatives with respect to a particular variable, that variable is called an **independent** variable. A variable is called **dependent** if a derivative of that variable occurs.

In the equation

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\cos\omega t \tag{0.1}$$

i is the dependent variable, t is the independent variable, and L, R, C, E, and ω are called parameters.

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The equation

$$\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{0.2}$$

has one dependent variable u and three independent variables.

Ordinary and Partial Differential Equations

Definition 2.1 A differential equation involving ordinary derivatives with respect to a *single* independent variable is called an **ordinary differential equation**.

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A differential equation involving partial derivatives with respect to more than one independent variable is a **partial differential equation**.

For instance, Equation (0.1) is an ordinary differential equation, whereas Equation (0.2) is a partial differential equation.

Order

Definition 2.2 The **order** of a differential equation is the order of the highest-order derivatives that appear in the equation.

For instance,

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) + y^5 = 0$$

is an equation of "order two" because $\frac{d^2y}{dx^2}$ is the highest order derivative present. It is also referred to as "second-order differential equation".
The differential equation

$$y^{(4)} + 2xy''' + y'' + 2y' + x^2y = x^2 + 1,$$

is a fourth order differential equation.

More generally, the equation

$$F(x, y, y', ..., y^{(n)}) = 0 \tag{0.3}$$

is called an ordinary differential equation of the nth order.

Under suitable restrictions on the function F, Equation (0.3) can be solved explicitly for $y^{(n)}$ in terms of the other variables $x, y, y', ..., y^{(n-1)}$, to obtain

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}). \tag{0.4}$$

A single equation of the form (0.3) may actually represent more than one equation of the form (0.4).

For example, the equation

$$x(y')^2 + 4y' - 6x^2 = 0$$

actually represents two different equations,

$$y' = \frac{-2 + \sqrt{4 + 6x^3}}{x} \quad \text{or} \quad y' = \frac{-2 - \sqrt{4 + 6x^3}}{x}.$$

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Similarly, the equation

$$(y')^2 - (x + y)y' + xy = 0$$

leads to the two equations

$$y' = x \qquad \text{or} \qquad y' = y.$$

Linear and Nonlinear Equations

Definition 2.3 A **linear** differential equation is any equation that can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

where $a_n(x), a_{n-1}(x), \ldots, a_0(x)$, and $g(x)$ depend only on the independent variable x, NOT on y.

If an equation is not linear, then we call it **nonlinear**.

For example, the equation

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For example, the equation

$$x^2y''' + xy' + (x^2 - 1)y = 4x^3$$

is a linear third order ordinary differential equation,whereas

$$y' + \sin y = 0$$

and

$$y'' - yy' = \cos x$$

are nonlinear because of the terms $\sin y$ and yy'.

Solutions

Definition 2.4 An **explicit solution** of the ordinary differential equation

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}) \tag{0.5}$$

on the interval $a < x < b$ is a function ϕ such that $\phi', \phi'', ..., \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f\left(x, \phi(x), \phi'(x), ..., \phi^{(n-1)}(x)\right)$$

for every x in $a < x < b$.

Example 2.1 Show that for any choice of the constant C, the function

$$\phi(x) = Ce^{2014x}$$

is a solution of the equation

$$y' = 2014y.$$

Example: Consider the differential equation

$$y'' - 2y' + y = 0.$$

Then $\phi_1(x) = e^x, x \in \mathbb{R}$ and $\phi_2(x) = xe^x, x \in \mathbb{R}$ are solutions of the given differential equation.

Furthermore, for any $c_1, c_2 \in \mathbb{R}$, the function

$$y(x) = c_1 e^x + c_2 xe^x, \quad x \in \mathbb{R}$$

is also a solution of the given differential equation.

Definition 2.5 A relation $G(x, y) = 0$ is said to be an **implicit solution** of Equation (0.5) on the interval (a, b) if it defines one or more explicit solutions on (a, b).

Example 2.2 Show that

$$y^2 - x^3 + 8 = 0$$

is an implicit solution of

$$\frac{dy}{dx} = \frac{3x^2}{2y} \tag{0.6}$$

on the interval $(2, \infty)$.

Example 2.3 Verify that

$$x + y + e^{xy} = C,$$

where C is an arbitrary constant, gives a family of implicit solutions of

$$\left(1 + xe^{xy}\right)\frac{dy}{dx} + 1 + ye^{xy} = 0 \tag{0.7}$$

and graph several of these solution curves.

For brevity, we hereafter use the term *solution* to mean either an explicit or an implicit solution.

Initial Value Problems

Definition 2.6 By an **initial value problem** for an nth order differential equation

$$F(x, y, y', ..., y^{(n)}) = 0$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0, \\ y'(x_0) &= y_1 \\ &\cdots \\ y^{(n-1)}(x_0) &= y_{n-1}, \end{aligned}$$

where $x_0 \in I$ and $y_0, y_1, ..., y_{n-1}$ are given constants.

Example 2.4 Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$y'' + y = 0; \quad y(0) = -1, \quad y'(0) = 1.$$

Example 2.5 Consider again the differential equation

$$(x^2 + 1)y' + xy = 0 \tag{0.8}$$

and the initial condition

$$y(0) = y_0. \tag{0.9}$$

If $y \neq 0$, then we can rewrite the differential equation as

$$\frac{y' dx}{y} = -\frac{x dx}{x^2 + 1}.$$

By integrating both sides, we find that

$$\ln|y| = \int -\frac{x dx}{x^2+1} = \int -\frac{1}{2} \cdot \frac{d(x^2+1)}{x^2+1} = -\frac{1}{2}\ln(x^2+1) + C_1$$

where C_1 is arbitrary.

Then taking the exponential of both sides of this equation and solving for y, we obtain

$$y = \frac{C}{\sqrt{x^2 + 1}}, \tag{0.10}$$

where $C = \pm e^{C_1}$ is also arbitrary. Observe that $C = 0$ corresponds to the solution $y = 0$.

Finally the initial condition requires that $C = y_0$, so the solution of the initial value problem (0.8)–(0.9) is

$$y = \frac{y_0}{\sqrt{x^2 + 1}}.$$

The expression (0.10) contains all possible solutions of the differential equation (0.8). So it is called the **general solution** of the differential equation (0.8).

Direction Fields

A first order equation

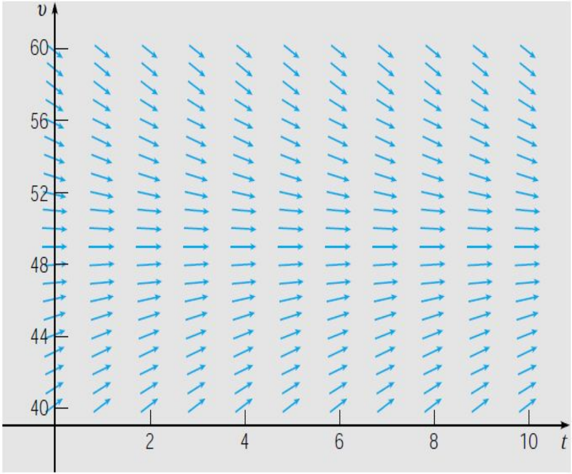
$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the plane where f is defined. In other word, it gives the direction that a solution to the equation must have at each point.

A plot of the directions associated with various points in the plane is called the **direction field** for the differential equation.

Example 2.6 The direction field for the equation

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$



Chapter 2 FIRST ORDER DIFFERENTIAL EQUATIONS

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Definition 1.1 A **linear first order** differential equation is an equation of the form

$$a_0(x)\frac{dy}{dx} + a_1(x)y = b(x), \tag{0.11}$$

where $a_0(x),\ a_1(x)$, and $b(x)$ depend only on the independent variable x, NOT on y.

We assume that the functions $a_0(x), a_1(x)$ and $b(x)$ are continuous on an interval and that $a_0(x) \neq 0$ on that interval. Then we can rewrite Equation (0.11) in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{0.12}$$

where $P(x)$ and $Q(x)$ are continuous functions on the interval.

For instance, the equation

$$(\sin x)\frac{dy}{dx} = x^2 \sin x - y \cos x$$

is linear since it can be written in the form

$$(\sin x)\frac{dy}{dx} + (\cos x)y = x^2 \sin x.$$

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$$(\sin x)\frac{dy}{dx} + (\cos x)y = x^2 \sin x.$$

However, the equation

$$yy' + x^2 y^3 = e^x$$

is not linear, because it cannot be put in the form of Equation (0.12).

2.1 SOLVING A LINEAR EQUATION:

Step 1: Rewrite the given equation as $\frac{dy}{dx} + P(x)y = Q(x)$.
Step 2: Finding an integrating factor
We multiply Equation in Step 1 by an yet undetermined function $\mu(x)$, obtaining

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x). \tag{0.13}$$

We like to have $\mu'(x) = \mu(x)P(x)$. Thus

$$\frac{1}{\mu(x)}\frac{d\mu(x)}{dx} = P(x) \qquad \text{or} \qquad \frac{d}{dx}\ln|\mu(x)| = P(x).$$

It follows that

$$\mu(x) = Ce^{\int P(x)dx},$$

where C is an arbitrary constant. Choose $C = 1$ to get:

$$\boxed{\mu(x) = \exp\left(\int P(x)dx\right) \ (\textbf{ integrating factor})}$$

Step 3: Rewrite (0.13) as

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x).$$

Thus

$$\mu(x)y = \int \mu(x)Q(x)\,dx + C,$$

where C is an arbitrary constant. Therefore

$$\boxed{y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right]}$$

This is the **general solution** of Equation (0.13).

Theorem: *Suppose $P(x)$ and $Q(x)$ are continuous on (a, b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem*

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0. \tag{0.14}$$

In fact, the solution is given by

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right],$$

where $\mu(x)$ is an integrating factor and C is a suitable constant.

Example 1.1 Solve the equation

$$\frac{dy}{dx} + \frac{2}{x}y = 8x.$$

$$y'(x) + \frac{2}{x}y(x) = 8x$$

$$\to p(x) = \frac{2}{x}$$

Example 1.1 Solve the equation

$$\frac{dy}{dx} + \frac{2}{x}y = 8x.$$

Solution: The integrating factor is given by

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = x^2.$$

Multiplying both sides of the given equation by x^2, we have

$$x^2\frac{dy}{dx} + 2xy = 8x^3,$$

or equivalently,

$$\frac{d}{dx}(x^2y) = 8x^3.$$

Thus

$$x^2y = \int 8x^3dx = 2x^4 + C, \quad \textit{or equivalently}, \; y = \left(2x^2 + \frac{C}{x^2}.\right) \times x^2$$

Example 1.2 Find the general solution to

$$\frac{dy}{dx} + y \tan x = \sin 2x.$$

Example 1.2 Find the general solution to

$$\frac{dy}{dx} + y\tan x = \sin 2x.$$

Solution: The integrating factor is defined by

$$\mu(x) = e^{\int \tan x dx} = e^{\int \frac{\sin x}{\cos x} dx} = e^{-\ln|\cos x|} = \frac{1}{|cosx|}.$$

Note that $\mu(x) = \frac{1}{\cos x}$ is also a integrating factor to the given equation. Multiplying both sides of the given equation by $\frac{1}{cosx}$, we have

$$\frac{1}{\cos x}\frac{dy}{dx} + y\frac{\sin x}{(\cos x)^2} = 2\sin x,$$

or equivalently,

$$\frac{d}{dx}\left(\frac{1}{\cos x}y\right) = 2\sin x.$$

Thus

$$\frac{1}{\cos x}y = \int 2\sin x dx = -2\cos x + C, \text{ or equivalently}, \ y = -2(\cos x)^2 + C\cos x.$$

Example 1.3 Solve the initial value problem

$$(x^2 + 1)y' + xy = -x, \quad y(0) = 1.$$

Example 1.4 A rock contains two radioactive isotopes RA_1 and RA_2 that belong to the same radioactive series; that is, RA_1 decays into RA_2 which then decays into stable atoms. Assume that the rate that RA_1 decays into RA_2 is $50e^{-10t}$kg/sec. Since the rate of decay of RA_2 is proportional to the mass $y(t)$ of RA_2 present, then the rate of change in RA_2 is

$$\begin{aligned} \frac{dy}{dt} &= \text{rate of creation} - \text{rate of decay}, \\ \frac{dy}{dt} &= 50e^{-10t} - ky, \end{aligned}$$

where $k > 0$ is the decay constant. If $k = 2$/sec and initially $y(0) = 40$ kg, find the mass of RA_2 for $t \geq 0$.

Definition 2.1 A differential equation of the form

$$N(y)dy = M(x)dx$$

is said to be **separable**.

For instance, the equation

$$\frac{dy}{dx} = \frac{2x + xy}{y^2 + 1}$$

is separable. However, the equation

$$\frac{dy}{dx} = 1 + xy$$

admits no factorization of the right-hand side and so is NOT separable.

In general, *an equation that can be written in the form*

$$\frac{dy}{dx} = g(x)h(y)$$

is a separable.

Example 2.1 Solve the equation

$$x\frac{dy}{dx} = 2y.$$

Example 2.2 Solve the initial value problem

$$\frac{dy}{dx} = 6y^2x, \quad y(0) = \frac{1}{25}.$$

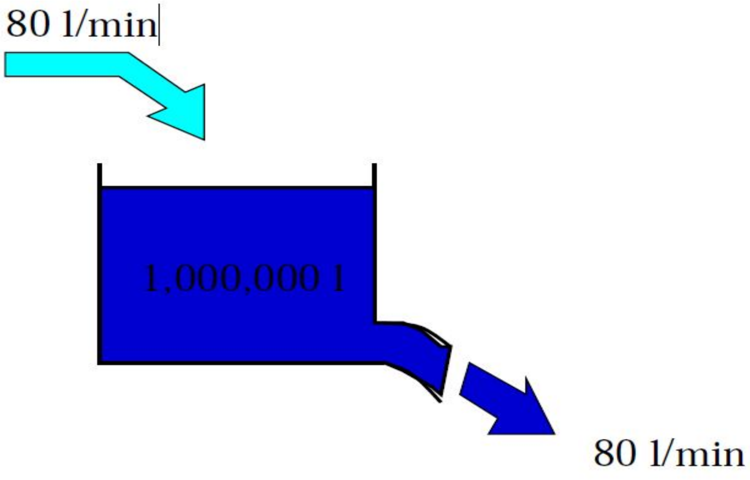
Example 2.3 Solve the initial value problem

$$2x(y+1)dx - ydy = 0, \quad y(0) = -2.$$

Mixing Problem:

Initially, a water tower contains 1 million litres of pure water. Two valves are then opened. One valve allows a solution of **water and fluoride**, with a concentration of 0.1 kg of fluoride per litre of water, to flow into the tower at a rate of 80 litres per minute. The other valve allows the solution in the tank to be drained at 80 litres per minute. Assume that the solution is mixed constantly, so that we have a homogeneous fluid in the tank, i.e., at any point in time, the concentration of fluoride in the water is uniform throughout the tank.

(a) Find an expression for the amount (in kg) of fluoride in the water tower t minutes after the valves are opened.
(b) Determine how long it will take for the concentration of fluoride in the water to reach .05 kg/l.


80 l/min
1,000,000 l
80 l/min

Solution: (a) Let $y = y(t)$ be the number of kilograms of fluoride in the tank at time t. We need to find both the input and output rates. Since water (with fluoride) is flowing in and out at 80 l/min, the volume in the tank remains constant at 1,000,000 l.

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Input Rate: We have 80 litres per minute entering the tank, with each litre containing 0.1 kg of fluoride, so each minute we have $(80)(0.1) = 8$ kg of fluoride entering the tank. Thus, we have: Input Rate $= 8$ kg/min.

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Output Rate: The entire system contains y kg of fluoride at any given time, distributed throughout the one million litres in the tank. This means that each litre of water in the tank contains $\frac{y}{1000000}$ kg of fluoride at any given moment in time. We have 80 litres per minute leaving the system, therefore in any one minute we have $(80)(\frac{y}{1000000})$ kg of fluoride leaving the tank. That is, we have: Output Rate $= (8)(\frac{y}{100000}) kg/min.$

We use the differential equation

$$\frac{dy}{dt} = (input\ rate) - (output\ rate).$$

Substituting the rates above into this formula, we get:

$$\frac{dy}{dt} = 8 - (8)(\frac{y}{100000})$$

or equivalently

$$\frac{dy}{dt} + 0.00008y = 8.$$

The general solution is

$$y = 100,000 + Ce^{-.00008t}.$$

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The general solution is

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At time $t = 0$ there was no fluoride in the tank, so we have the side condition $y(0) = 0$. It implies that $C = -100000$.
We substitute this into the general solution to get the specific solution:

$$y = 100000 - 100000e^{-0.00008t}.$$

(b) We need to determine how long it will take for the concentration to reach a level of $.05 kg/l$. This concentration of fluoride in the 1,000,000 l in the tank corresponds to having $(1000000)(.05) = 50000$ kg of fluoride in total in the tank.

Thus we see that we need to find the value of t that satisfies $y = 50000$. We do this using the formula we found in part (a).

$$50000 = 100000 - 100000e^{-0.00008t}.$$

This yields

$$t = \frac{\ln 2}{0.00008} \simeq 8664.34.$$

3.2 Newton's Law of Cooling

Experiment has shown that under certain conditions, a good approximation to the temperature of an object can be obtained by using Newton's law of cooling:

> *The temperature of a body changes at a rate that is proportional to the difference in temperature between the outside medium and the body itself.*

This means

$$\boxed{\frac{dT}{dt} = k(T - R)}$$

where T is the temperature of the object at time t, R is the temperature of the surrounding environment (constant) and k is a constant of proportionality.

Example:

A pot of liquid is put on the stove to boil. The temperature of the liquid reaches $170^o F$ and then the pot is taken off the burner and placed on a counter in the kitchen. The temperature of the air in the kitchen is $76^o F$. After two minutes the temperature of the liquid in the pot is $123^o F$. How long before the temperature of the liquid in the pot will be $84^o F$?

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Solution: Let T be the temperature of the pot of liquid at time t. By the Newton's law of cooling, we have

$$\frac{dT}{dt} = k(T - 76), \quad T(0) = 170.$$

Solve it, to get $T(t) = 76 + 94e^{kt}, t \geq 0$.

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$$\frac{dT}{dt} = k(T - 76), \quad T(0) = 170.$$

Solve it, to get $T(t) = 76 + 94e^{kt}, t \geq 0$.
"After two minutes the temperature of the liquid in the pot is $123^o F$" means

$$T(2) = 123.$$

This gives

$$123 = 76 + 94e^{2k}.$$

Thus

$$k = \frac{1}{2}\ln\frac{1}{2} \simeq -0.3465 \textit{ and } T(t) = 76 + 94e^{-0.3465t}.$$

We have to find t such that

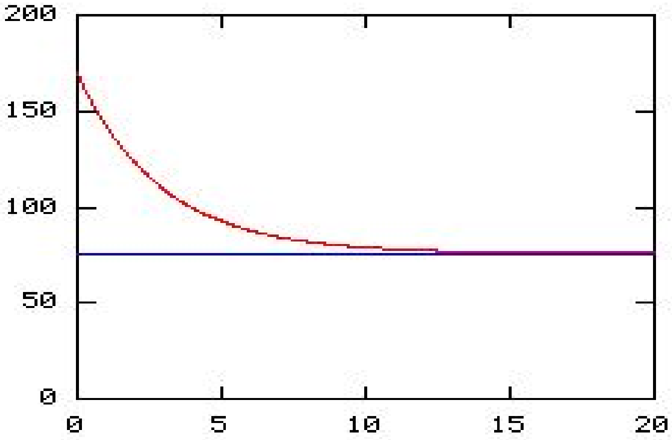
$$T(t) = 84.$$

This gives

$$84 = 76 + 94e^{-0.3465t}, \textit{ or } \ln\frac{4}{47} = -0.3465t.$$

So we have $T \simeq 7.11$ minutes.

Solution Function Graph



3.3 Population Models

Let $p(t)$ be the population of a species at time t.

As long as there are sufficient space and ample food supply for the species, we can assume that the growth and dead rates are proportional to the population present.

Hence a model for population of the species is

$$\frac{dp}{dt} = k_1 p - k_2 p, \qquad p(0) = p_0,$$

where $k_1, \ k_2$ are the proportionality constants for the growth rate and the dead rate, respectively, and p_0 is the population at time $t = 0$.

This gives the mathematical model

$$\frac{dp}{dt} = kp, \qquad p(0) = p_0,$$

which is called the **Malthusian** or **exponential law** of population growth.

Solving this initial value problem for $p(t)$, gives

$$\boxed{p(t) = p_0 e^{kt}}$$

Example 3.3 In 1790 the population of the United States was 3.93 million, and in 1890 the population was 62.95 million. Using the Malthusian model, estimate the population of the United States as a function of time.

- The Malthusian model is satisfactory as long as the population is not too large. It considers only death by natural causes and

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• A suitable choice of a competition term is $-bp^2$, where b is a constant. We have, therefore, the modified equation

$$\frac{dp}{dt} = ap - bp^2, \qquad p(0) = p_0 \tag{0.15}$$

This is called the **logistic model**.

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This is called the **logistic model**.

The equation (0.15) is separable and can be solved easily. Its solution is

$$p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-at}}. \tag{0.16}$$

The function $p(t)$ given in (0.16) is called the **logistic function**. Its graph is called the **logistic curve**.

An important property of the logistic function is that

$$\lim_{t\to\infty} p(t) = \frac{a}{b} \quad \text{if} \quad p_0 > 0.$$

The two functions $p(t) \equiv 0$ and $p(t) \equiv a/b$ are also solutions of the logistic equation (0.15). They are referred to as **equilibrium populations**.

Example 3.4 Taking the 1790 population of 3.93 million as the initial population and given the 1840 and 1890 populations of 17.07 and 62.95 million, respectively, use the logistic model to estimate the population at time t.

A comparison of the Malthusian and logistic models with U.S. census data (Population is given in millions)

Year	U.S. Cencus	Malthusian (Ex. 3.3)	Logistic (Ex. 3.4)
1790	3.93	3.93	3.93
1800	5.31	5.19	5.30
1810	7.24	6.84	7.13
1820	9.64	9.03	9.58
1830	12.87	11.92	12.82
1840	17.07	15.73	17.07
1850	23.19	20.76	22.60
1860	31.44	27.39	29.70
1870	39.82	36.15	38.65
1880	50.16	47.70	49.69
1890	62.95	62.95	62.95

A comparison of the Malthusian and logistic models with U.S. census data (Population is given in millions)

Year	U.S. Cencus	Malthusian (Ex. 3.3)	Logistic (Ex. 3.4)
1900	75.99	83.07	78.37
1910	91.97	109.63	95.64
1920	105.71	144.67	114.21
1930	122.78	190.91	133.28
1940	131.67	251.94	152.00
1950	151.33	332.47	169.56
1960	179.32	438.75	185.35
1970	203.21	579.00	199.01
1980	226.50	764.08	210.46
1990	249.63	1008.32	219.77
2000	?	1330.63	227.19

Many equations in this chapter may be written

$$\frac{dy}{dx} = f(x, y).$$

It is sometimes convenient to write the equation in the form

$$M(x, y)dx + N(x, y)dy = 0 \tag{0.17}$$

To solve Equation (0.17), it is helpful to know whether the left hand side is a total differential.

• Recall that the **total differential** $dF(x, y)$ of a function $F(x, y)$ of two variables is defined by

$$dF(x, y) = \frac{\partial F}{\partial x}(x, y)dx + \frac{\partial F}{\partial y}(x, y)dy,$$

where dx and dy are arbitrary increments.

Definition 4.1 An equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{0.18}$$

is called **exact** in a region R if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R.

In other words, Equation (0.18) is exact if we can write it in the form

$$dF(x, y) = M(x, y)dx + N(x, y)dy = 0$$

for some function $F(x, y)$.

Example 4.1 Show that the equation

$$(y - 3x^2)dx + (x - 1)dy = 0$$

is exact and find its solutions.

If the equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{0.19}$$

is exact, then by definition F exists such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y).$$

These two equations lead to

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

But, from calculus,

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x},$$

provided that these partial derivatives are continuous.

Therefore, if Equation (0.19) is exact, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{0.20}$$

Conversely, it is showed that if condition (0.20) is satisfied, (0.19) is an exact equation.

Theorem *Suppose the first partial derivatives of* $M(x, y)$ *and* $N(x, y)$ *are continuous in a region* R. *Then*

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact equation in R *if and only if*

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

for all $(x, y) \in R$.

Example 4.2 Solve

$$(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0.$$

Example 4.3 Solve

$$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0.$$

Pages	Exercises	Assignments
24–26	3, 8, 12, 19, 22, 25	4, 6, 10, 13, 14, 18, 20, 21, 24, 26, 27,30
39–41	13, 15, 30	16, 19, 20, 27, 28, 31, 33, 35, 38 ,39
47–50	3, 7, 21, 26, 30	22, 25, 31, 36, 38
59–68	3, 13	1, 4, 6, 12, 24, 29
99–101	3, 5, 15	4, 6, 9, 11, 12, 16, 20, 22