Chapter 3: APPLICATIONS of DIFFERENTIATION

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CALCULUS I

Outline

Chapter 3 (Applications of differentiation):
Related rates,
Maxima and minima,
Optimization problems,
Mean value theorem,
First and second derivative tests,
Concavity,
Shape of curves,
L'Hospital Rule,
Newton's methods,
Antiderivative/Indefinite Integral.

Applications in Optimization

Important optimization problems require differential calculus , e.g.

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle?
- What is the interest rate that the banks earn most in the market?
- What is the shape of a racing car or an aircraft to minimize drag?
- etc.

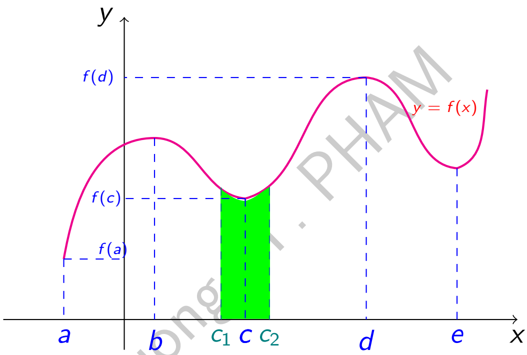
$\Longrightarrow$ finding the minimum and maximum of a function .

Definition.

Let $f : D \to \mathbb{R}$ be a function and let $c \in D$.

- f has an **absolute maximum** (global maximum) at c if
$$f(x) \leq f(c) \quad \forall x \in D.$$
Then, $f(c)$ is called the **maximum value** of f.
- f has an **absolute minimum** (global minimum) at c if
$$f(x) \geq f(c) \quad \forall x \in D.$$
Then, $f(c)$ is called the **minimum value** of f.
- The maximum and minimum values of f are called the **extreme values** of f.

Maximum and minimum of a function



- $f(a)$ is the absolute minimum of f;
- $f(d)$ is the absolute maximum of f
- $f(c) \leq f(x) \quad \forall x \in (c_1, c_2) \Longrightarrow f(c)$ is a **local minimum** of f

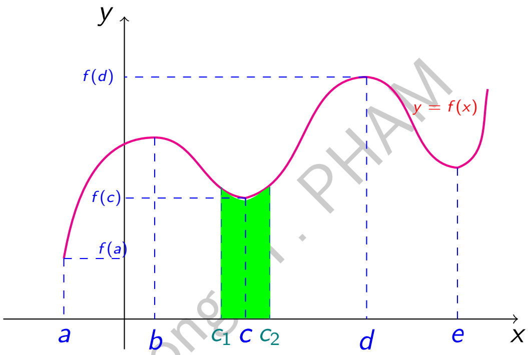
Local maximum and minimum

Definition.

Let $f : D \to \mathbb{R}$ be a function and let $c \in D$.

- f has an **local maximum** (relative maximum) at c if there exists an interval $(c_1, c_2) \subset D$ such that $c \in (c_1, c_2)$ and
$$f(x) \leq f(c) \quad \forall x \in (c_1, c_2).$$
- f has an **local minimum** (relative minimum) at c if if there exists an interval $(c_1, c_2) \subset D$ such that $c \in (c_1, c_2)$ and
$$f(x) \geq f(c) \quad \forall x \in (c_1, c_2).$$

Local maximum and local minimum



- $f(c)$ is a local minimum
- $f(b)$ is the local maximum of f;
- $f(d)$ is the local maximum of f (also an absolute maximum)
- $f(e)$ is a local minimum of f

The Extreme value Theorem

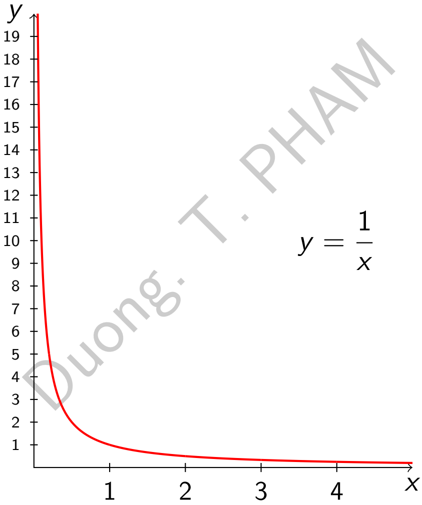
Theorem.

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

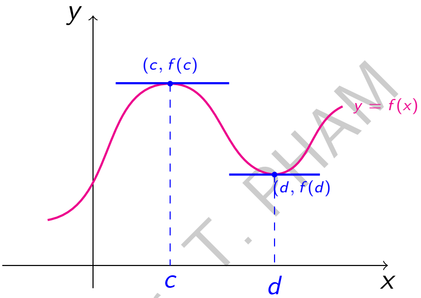
Remark: The theorem is not true if we replace the closed interval $[a, b]$ by the open interval (a, b) or other interval $(a, b]$ or $[a, b)$.

Ex: The function $f(x) = \dfrac{1}{x}$ does not attain an absolute maximum value on $(0, 5]$.

The Extreme value Theorem



The Fermat's Theorem



- f attains local maximum at c and local minimum at d $\Longrightarrow$ What is special about f at $x = c$ and $x = d$?

Theorem (The Fermat's Theorem:).

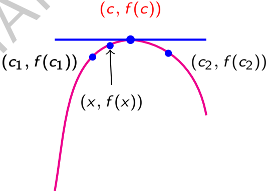
If f attains local maximum or local minimum at c and if $f'(c)$ exists, then $f'(c) = 0$

Proof of Fermat's Theorem

The Fermat's Theorem: If f attains local maximum or local minimum at c and if $f'(c)$ exists, then $f'(c) = 0$

Proof: Suppose f attains local maximum at c.

$\Longrightarrow \forall x \in (c_1, c_2) : f(x) \le f(c)$



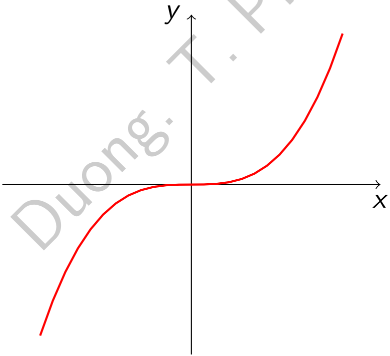
$$\Longrightarrow \begin{cases} \dfrac{f(x) - f(c)}{x - c} \ge 0 & \forall c_1 < x < c \\ \dfrac{f(x) - f(c)}{x - c} \le 0 & \forall c < x < c_2 \end{cases}$$

$$\Rightarrow \begin{cases} \lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c} \ge 0 \\ \lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \le 0 \end{cases}$$. Since $f'(c)$ exists, $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = M$ exists

$$\Longrightarrow M = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \Longrightarrow \begin{cases} M \ge 0 \\ M \le 0 \end{cases} \Longrightarrow M = 0$$

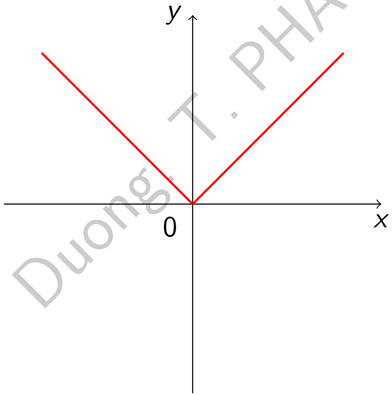
Remark: "Local maximum and local minimum at c" + existence of $f'(c)$ $\Longrightarrow f'(c) = 0$. But $(\Longleftarrow)$ is NOT true.

Ex: Let $f(x) = x^3$. We have $f'(x) = 3x^2$ and $f'(0) = 0$ but f does NOT attain local max. or local min. at 0.



Local minimum and local maximum

Ex: Function $f(x) = |x|$ attains local minimum at $x = 0$ but it is NOT differentiable at $x = 0$.



Critical number of a function

Definition.

A number c is called a **critical number** of a function f if $f'(c) = 0$ or $f'(c)$ does not exists.

$$\left.\begin{array}{l} f'(c) = 0 \\ f'(c) \text{ does NOT exist} \end{array}\right] \Longrightarrow c \text{ is a critical number of } f$$

Ex: Find critical points of $f(x) = \sqrt{x}(1-x)$.

Ans: $f'(x) = \dfrac{1-x}{2\sqrt{x}} - \sqrt{x} = \dfrac{1-x-2x}{2\sqrt{x}} = \dfrac{1-3x}{2\sqrt{x}}$

- We have $f'(x) = 0 \Longleftrightarrow \dfrac{1-3x}{2\sqrt{x}} = 0 \Longleftrightarrow x = \dfrac{1}{3}$
- f is not differentiable at $x = 0$
- The critical numbers are $\dfrac{1}{3}$ and 0

Critical numbers of a function

Corollary.

If f has a local minimum or local maximum at c then c is a critical number of f

Proof:

$$\left.\begin{array}{l} f'(c) = 0 \\ f'(c) \text{ does NOT exist} \end{array}\right] \Longrightarrow c \text{ is a critical number of } f$$

- f has local min. or local max. at $c \Longrightarrow$ There are 2 cases:

$$\left[\begin{array}{l} f'(c) \text{ does not exist} \Longrightarrow c \text{ is a critical number} \\ \\ f'(c) \text{ exists} \overset{\text{Fermat's Th.}}{\Longrightarrow} f'(c) = 0 \Longrightarrow c \text{ is a critical number} \end{array}\right.$$

Finding absolute minimum and absolute maximum

The closed interval method: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. To find the absolute max. and absolute min. of f, we follows the steps:

1. Find the values of f at critical numbers of f in (a, b);
2. Find the values $f(a)$ and $f(b)$;
3. The largest number in steps 1 and 2 is the absolute maximum value of f, and the smallest number in steps 1 and 2 is the absolute minimum of f.

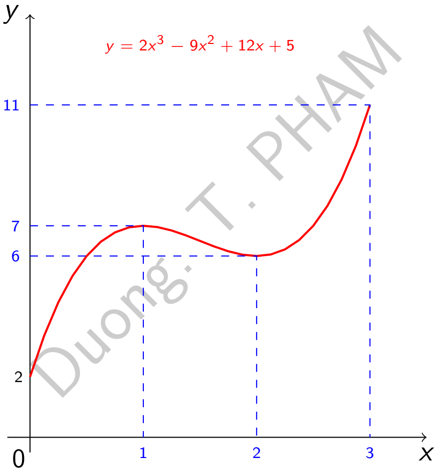
Ex: Find abs. max. and abs. min. of $f(x) = 2x^3 - 9x^2 + 12x + 2$ in $[0, 3]$

Ans: $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

1. $f'(x) = 0 \iff x = 1$ or $x = 2$, and $f(1) = 7$, $f(2) = 6$
2. $f(0) = 2$ and $f(3) = 11$
3. Comparing the 4 values in Steps 1 and 2, we conclude

$$\max_{[0,3]} f = f(3) = 11 \quad \text{and} \quad \min_{[0,3]} f = f(0) = 2$$

The closed interval method



Theorem.

Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be a function satisfying

1. f is continuous in $[a, b]$
2. f is differentiable in (a, b)
3. $f(a) = f(b)$. Then there exists a $c \in (a, b)$ such that $f'(c) = 0$

Proof: The case: $f(x) = c\ \forall x \in [a, b]$. Then clearly

$$f'(c) = 0 \quad \forall c \in (a, b).$$

The case: $\exists x \in (a, b)$ s.t. $f(x) > f(a) = f(b)$. Since f is continuous in $[a, b] \overset{Extreme Value Th.}{\Longrightarrow} \exists c \in (a, b)$ such that

$$f(c) = \max_{[a,b]} f$$

$\Longrightarrow f$ has a local max. at c + "$f'(c)$ exists" $\overset{\text{Fermat's Th.}}{\Longrightarrow} f'(c) = 0$

The case: $\exists x \in (a, b)$ s.t. $f(x) < f(a) = f(b)$. (Similarly as above)

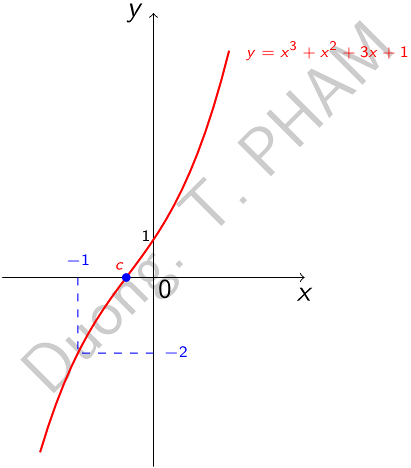
Rolle's Theorem

Ex: Prove that equation $x^3 + x^2 + 3x + 1 = 0$ has exactly one real root.

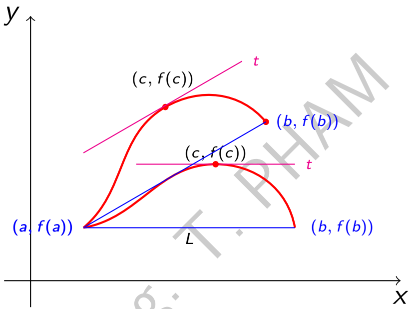
Ans: Denote $f(x) = x^3 + x^2 + 3x + 1$.

- f is a polynomial $\Longrightarrow$ f is differentiable (and thus continuous) in $(-\infty, \infty)$.
- $f(-1) = -2$ and $f(0) = 1$. Since f is continuous in $[-1, 0]$ and since $f(-1) < 0 < f(0)$, the Intermediate value Theorem $\Longrightarrow$ $\exists c \in (-1, 0)$ s.t. $f(c) = 0 \Longrightarrow$ $\underline{\text{the equation has one root } c \in (-1, 0)}$
- Suppose that $d \neq c$ is another root of the equation.
  - If $c < d$, then f is differentiable in $[c, d]$ and $f(c) = f(d) = 0 \overset{\text{Rolle's Th.}}{\Longrightarrow}$ exists $e \in (c, d)$ s.t. $f'(e) = 0$.
    On another hand, $f'(x) = 3x^2 + 2x + 3 > 0$ for all $x \in \mathbb{R}$
    $\Longrightarrow$ **Contradiction**
  - If $c > d$, similar argument $\Longrightarrow$ **Contradiction**
- The equation has exactly one root $c \in (-1, 0)$.

Rolle's Theorem



Recall the Rolle's Theorem



- The slope of secant connecting $(a, f(a))$ and $(b, f(b))$ is $\dfrac{f(b) - f(a)}{b - a} = 0$
- Rolle's Theorem: $\exists c \in (a, b)$ s.t. $f'(c) = 0$

$$\Longrightarrow f'(c) = \frac{f(b) - f(a)}{b - a} \quad \Longleftrightarrow \boxed{t // L}$$

Theorem.

Let $f : [a, b] \to \mathbb{R}$. Then

$$\begin{cases} f \text{ is continuous in } [a, b] \\ f \text{ is differentiable in } (a, b) \end{cases} \Longrightarrow \exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Denote $h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$. Then h is continuous in $[a, b]$ and h is differentiable in (a, b). Furthermore,

$$\left.\begin{aligned} h(a) &= f(a) - \tfrac{f(b)-f(a)}{b-a}(a - a) - f(a) = 0 \\ h(b) &= f(b) - \tfrac{f(b)-f(a)}{b-a}(b - a) - f(a) = 0 \end{aligned}\right\} \Longrightarrow h(a) = h(b).$$

Rolle's Theorem: there is a $c \in (a, b)$ such that $h'(c) = 0 \Longrightarrow$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem.

Let $f : (a, b) \to \mathbb{R}$ be a function which is differentiable in (a, b) and $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function.

Proof: Let $x_1, x_2 \in (a, b)$. Then f is continuous in $[x_1, x_2]$ and differentiable in (x_1, x_2). By the Mean Value Theorem, there is a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

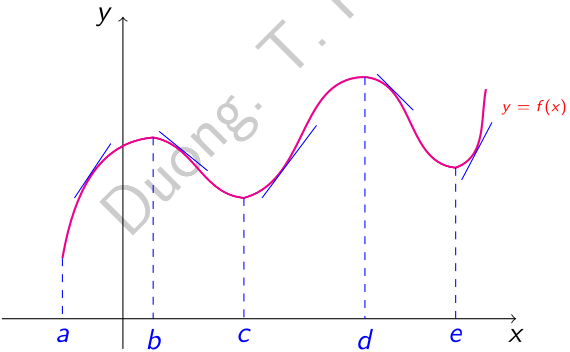
Since $f'(x) = 0$ for all $x \in (x_1, x_2)$, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0.$$

It means that $f(x_1) = f(x_2)$ and this is true for any $x_1, x_2 \in (a, b)$. Hence, f is a constant function in (a, b).

Increasing and decreasing Test: Let $f : (a, b) \to \mathbb{R}$ be a differentiable function.

1. If $f'(x) > 0, \forall x \in (a, b)$, then f is increasing in (a, b).
2. If $f'(x) < 0, \forall x \in (a, b)$, then f is decreasing in (a, b).



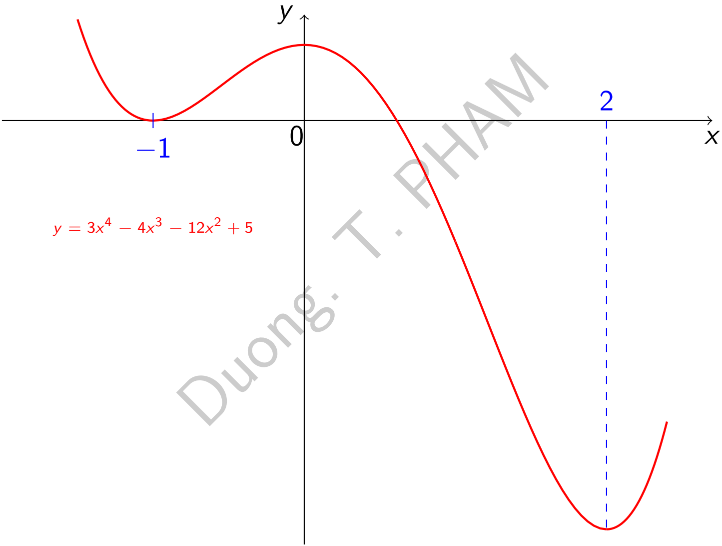
Ex: Determine when the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing.

Ans: $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$

$\Longrightarrow f'(x) = 0 \iff x = -1 \ \vee \ x = 0 \ \vee \ x = 2$

x	$-\infty$		-1		0		2		∞
$x-2$		$-$	$\vert$	$-$	$\vert$	$-$	0	$+$	
x		$-$	$\vert$	$-$	0	$+$	$\vert$	$+$	
$x+1$		$-$	0	$+$	$\vert$	$+$	$\vert$	$+$	
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		$\searrow$	0	$\nearrow$	5	$\searrow$	-27	$\nearrow$	

Increasing and decreasing Test

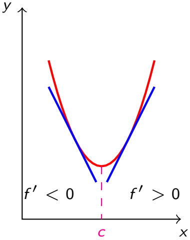


The First derivative Test: Suppose that c is a critical number of a continuous function f.

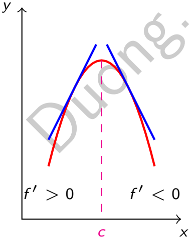
(i) If f' changes from positive to negative at c, then f has a local maximum at c.

(ii) If f' changes from negative to positive at c, then f has a local minimum at c.

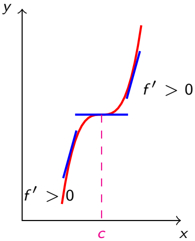
(iii) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.



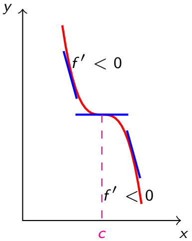
local minimum



local maximum



no local min. or max.



no local min. or max.

Ex: Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

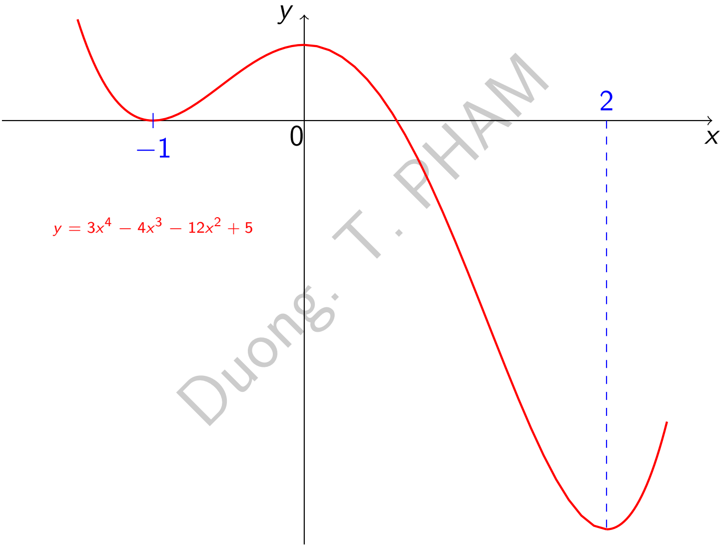
Ans: $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$

$\Longrightarrow f'(x) = 0 \iff x = -1 \ \vee \ x = 0 \ \vee \ x = 2$

x	$-\infty$		-1		0		2		∞
$x-2$		$-$	$\mid$	$-$	$\mid$	$-$	0	$+$	
x		$-$	$\mid$	$-$	0	$+$	$\mid$	$+$	
$x+1$		$-$	0	$+$	$\mid$	$+$	$\mid$	$+$	
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		$\searrow$	0	$\nearrow$	5	$\searrow$	-27	$\nearrow$	

- f attains local minimum at -1 and 2; and attains local maximum at 0.

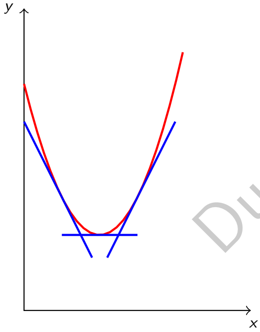
The first derivative Test



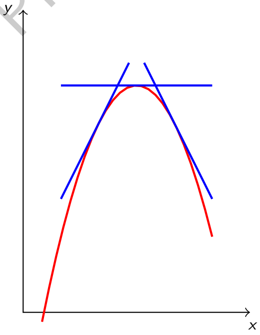
Definition.

Let $f : I \to \mathbb{R}$.

- If the graph of f lies above all of its tangent lines, then f is said to be **upward concave** on I
- If the graph of f lies below all of its tangent lines, then f is said to be **downward concave** on I



upwad concave



downward concave

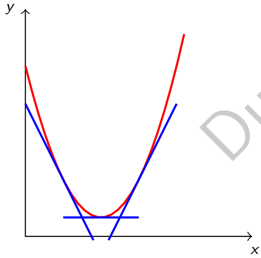
Concavity Test: Let $f : I \to \mathbb{R}$. Then

(a) If $f''(x) > 0$ $\forall x \in I$, then the graph of f is concave upward on I.

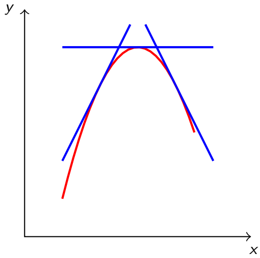
(b) If $f''(x) < 0$ $\forall x \in I$, then the graph of f is concave downward on I.

Discussion:

- $f''(x) > 0$ for all $x \in I \implies f'(x)$ is increasing on I
- $f''(x < 0)$ for all $x \in I \implies f'(x)$ is decreasing on I



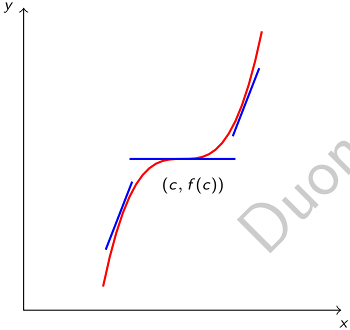
f' increases $\to$ upwad concave



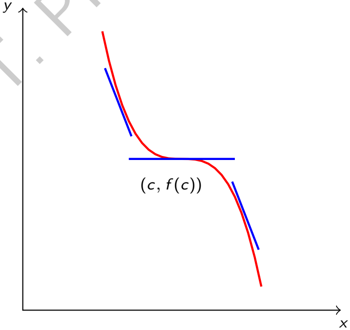
f' decreases $\to$ downward concave

Definition.

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P



$(c, f(c))$ is inflection point



$(c, f(c))$ is inflection point

The second derivative Test

The second derivative Test: Let f be a function such that f'' is continuous near c. Then

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c,

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Ex: Discuss the concavity, inflection points, local maxima and local minima of the curve $y = x^4 - 4x^3$.

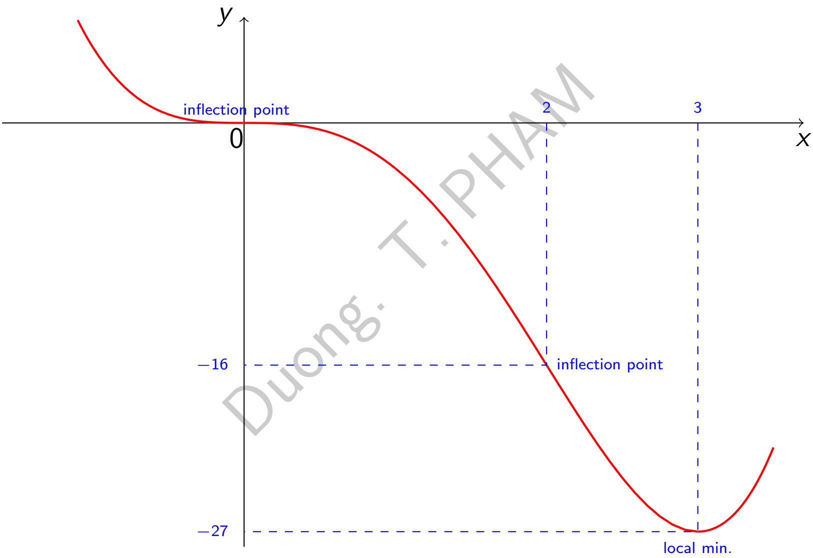
Ans: Denote $f(x) = x^4 - 4x^3$. Then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

- $f'(x) = 0 \iff x = 0 \ \vee x = 3$ and $f''(x) = 0 \iff x = 0 \ \vee x = 2$

x		0		2		3	
x^2	$+$	0	$+$	$\vert$	$+$	$\vert$	$+$
$x-3$	$-$	$\vert$	$-$	$\vert$	$-$	0	$+$
$x-2$	$-$	$\vert$	$-$	0	$+$	$\vert$	$+$
$f'(x)$	$-$	0	$-$	$\vert$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	0	$+$	$\vert$	$+$
$f(x)$	$\searrow$	0	$\searrow$	-16	$\searrow$	-27	$\nearrow$
	up.con.	infl.p.	down.con.	infl.p.	up.con.	local min.	up.con.

The second derivative Test



Indeterminate forms: consider $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$. Then

(a) If $f(x) \to 0$ and $g(x) \to 0$ when $x \to a$, then the limit is called an **indeterminate form of type** $\frac{0}{0}$

(b) If $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) when $x \to a$, then the limit is called an **indeterminate form of type** $\frac{\infty}{\infty}$

Ex: Evaluate $\lim\limits_{x\to 1} \dfrac{x^3 - x}{x - 1}$

Ans:

$$\lim_{x\to 1} \frac{x^3 - x}{x - 1} = \lim_{x\to 1} \frac{x(x^2 - 1)}{x - 1} = \lim_{x\to 1} \frac{x(x - 1)(x + 1)}{x - 1}$$
$$= \lim_{x\to 1} [x(x + 1)] = 1(1 + 1) = 2.$$

Ex: $\lim\limits_{x\to 1} \dfrac{\ln x}{x - 1} \implies$ we need a tool to evaluate this limit.

L'Hopital's Rule: Let f and g be differentiable functions on an interval (b, c), except possibly at number $x = a \in (b, c)$. Suppose further that $g'(x) \neq 0$ for all $x \in (b, c)\backslash\{a\}$. If

- $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$
- or $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$,

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right hand side exists (or is ∞ or $-\infty$)

Ex: $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1} \overset{\text{L'Hopital.}}{=} \lim\limits_{x\to 1} \dfrac{(\ln x)'}{(x-1)'} = \lim\limits_{x\to 1} \dfrac{\frac{1}{x}}{1} = 1.$

Ex:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{L'Hopital.}}{=} \lim_{x\to\infty} \frac{(e^x)'}{(x^2)'} = \lim_{x\to\infty} \frac{e^x}{2x}$$

$$\overset{\text{L'Hopital.}}{=} \lim_{x\to\infty} \frac{(e^x)'}{(2x)'} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Ex:

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{L'Hopital.}}{=} \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Indeterminate products

Intermediate products: The limit $\lim_{x \to a}[f(x)g(x)]$ in which $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$ is called an **indeterminate form of type** $0 \cdot \infty$

Ex: Evaluate $\lim_{x \to 0^+} x \ln x$

Ans: We have

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{\text{L'Hopital.}}{=} \lim_{x \to 0^+} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x \to 0^+} (-x) = 0.$$

Indeterminate difference: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}[f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$.

Ex: Compute $\lim_{x\to 0}(\sec x - \tan x)$

Ans: We have

$$\begin{aligned}\lim_{x\to(\pi/2)-}(\sec x - \tan x) &= \lim_{x\to(\pi/2)-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to(\pi/2)-}\frac{1-\sin x}{\cos x}\\ &\overset{\text{L'Hopital.}}{=} \lim_{x\to(\pi/2)-}\frac{(1-\sin x)'}{(\cos x)'} = \lim_{x\to(\pi/2)-}\frac{-\cos x}{-\sin x}\\ &= 0.\end{aligned}$$

The limit $\lim_{x\to a}[f(x)]^{g(x)}$ when f and g satisfy

(a) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ **type** 0^0

(b) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ **type** ∞^0

(c) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ **type** 1^∞ are **indeterminate powers**.

Remark: To evaluate the above limits we can either

1. write $y = [f(x)]^{g(x)}$ and thus $\ln y = g(x)\ln f(x)$, and then evaluate $\lim_{x\to a} \ln y$ first; after that, we deduce $\lim_{x\to a} y$.
2. We can write $[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$.
3. By these ways, we transfer it to the problem of finding limit of the type $0 \cdot \infty$.

Ex: Evaluate $\lim_{x\to 0^+} (1+4\sin 4x)^{\cot x}$

Ans: Denote $y = (1+4\sin 4x)^{\cot x} \implies \ln y = \cot x \ln(1+4\sin 4x)$

- Then

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \cot x \ \ln(1+4\sin 4x) = \lim_{x\to 0^+} \frac{\ln(1+4\sin 4x)}{\tan x}$$

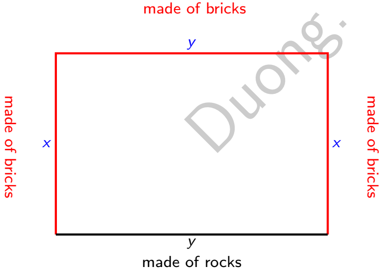
$$\overset{\text{L'Hopital.}}{=} \lim_{x\to 0^+} \frac{\dfrac{16\cos 4x}{1+4\sin 4x}}{\dfrac{1}{\cos^2 x}} = \lim_{x\to 0^+} \frac{16\cos 4x \cdot \cos^2 x}{1+4\sin 4x}$$

$$= 16$$

- Hence, $\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^{16}$.

Optimization problems

Ex: A man wants to build a rectangular fish pond in his garden and he wants to save money by using bricks left out from his house construction. The amount of bricks is enough to build 50 m of the pond banks . One side of the rectangular fish pond will be built from rocks which there are in abundance around his house. **Question:** Determine the shape of the fish pond which has the largest area and is built from the materials the man has.



- $2x + y = 50$
- $\text{Area} = x \cdot y$
- $\max \text{Area}$?

Optimization problems

Ex: Mathematical model: $\begin{cases} \text{Find } \max(xy)? \\ 2x + y = 50 \end{cases}$

Ans: We have $A = xy$ and since $2x + y = 50$, we replace $y = 50 - 2x$ in A to obtain

$$A(x) = x(50 - 2x).$$

Our task is now to find: $\max A(x)$, where $0 \leq x \leq 25$. We have $A' = 50 - 4x$, and

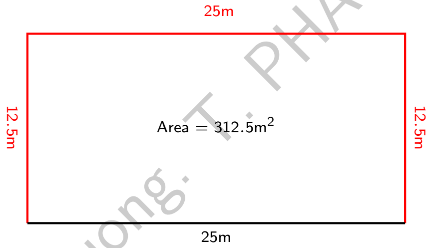
$$A' = 0 \iff 50 - 4x = 0 \iff x = 12.5.$$

x	0		12.5		25
A'	50	$+$	0	$-$	-50
A	0	$\nearrow$	312.5	$\searrow$	0

We can deduce from the table that

$$\max A = 312.5\mathrm{m}^2 \quad \text{when} \quad x = 12.5\mathrm{m}$$

Optimization problems



Applications in business and economics

- The **cost function** $C(x)$ = the cost of producing x units of a certain product.
- The **marginal cost** ($= C'(x)$) is the change of $C(x)$ w.r.t. x
- The **demand function (or price function)**, denoted by $p(x)$, is the price per unit that the company can charge if it sells x units
- If the company sells x units and the price per unit is $p(x)$, then the total revenue is denoted by the **revenue function**,

$$R(x) = xp(x)$$

- The derivative R' is the **marginal revenue function**
- If x units are sold, the total profit

$$P(x) = R(x) - C(x)$$

  and P is called the **profit function**.
- P' is called the **marginal profit function**

Ex: A store sells 200 DVD burners a week at \$350 each.

- Survey: if each \$10 rebate is offered, 20 more units will be sold every week.

Q.: Find demand and revenue functions. How large a rebate should be to maximize its revenue?

Ans: Denote by x the number of units sold every week $\Longrightarrow$ weekly increase in sales is $x - 200$.

- If the price per unit decreases by \$10, more 20 units are sold. $\Longrightarrow$ the price per unit so that the weekly sale is x units, is

$$p(x) = 350 - \frac{x - 200}{20} \cdot 10 = 450 - \frac{x}{2}.$$

- The revenue function is $R(x) = xp(x) = 450x - \dfrac{x^2}{2}$
- Our task: Find the absolute maximum of $R(x) = 450x - \dfrac{x^2}{2}$

Our task: Find the absolute maximum of $R(x) = 450x - \frac{x^2}{2}$

- $R'(x) = 450 - x$, and $R'(x) = 0 \iff x = 450.$
- consider the table

x	350		450	
$R'(x)$	100	$+$	0	$-$
$R(x)$	96250	$\nearrow$	101250	$\searrow$

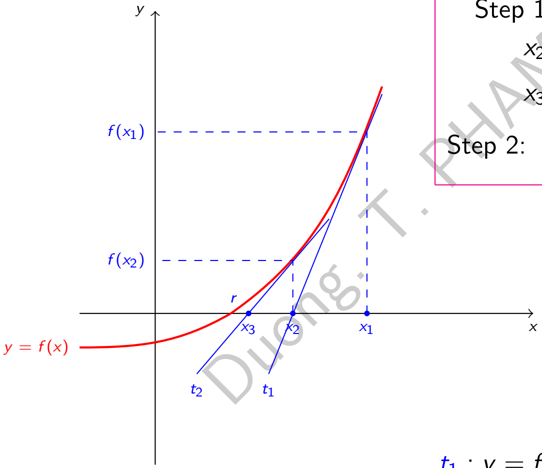
- The revenue has an absolute maximum (101250) when the number of weekly sold units is 450.
- The price per unit is then $p(450) = 450 - \frac{450}{2} = 225$
- The rebate should then be offered as $350 - 225 = 125$

Some examples:

- Solve $2x + 5 = 0 \iff x = -\frac{5}{2}$
- Solve $x^2 - 5x + 4 = 0$.
  - $\Delta = (-5)^2 - 4 \cdot 1 \cdot 4 = 9$
  - The solutions

$$\left[\begin{array}{l} x_1 = \dfrac{5 - \sqrt{\Delta}}{2} = \\ \\ x_2 = \dfrac{5 + \sqrt{\Delta}}{2} \end{array}\right. \iff \left[\begin{array}{l} x_1 = \dfrac{5 - \sqrt{9}}{2} \\ \\ x_2 = \dfrac{5 + \sqrt{9}}{2} \end{array}\right. \iff \left[\begin{array}{l} x_1 = 1 \\ \\ x_2 = 4 \end{array}\right.$$

- Solve $\cos x - x = 0$?



Step 1: Choose some x_1

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Step 2: $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

$$t_1 : y = f'(x_1)(x - x_1) + f(x_1)$$

$$t_2 : y = f'(x_2)(x - x_2) + f(x_2)$$

Newton's method

Ex: Find, correct to six decimal places, the root of the equation $\cos x = x$

Ans: The equation is equivalent to $\cos x - x = 0$.

- Denote $f(x) = \cos x - x$. Then $f'(x) = -\sin x - 1$
- Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

- Choose $x_1 = 0$. Then

$$x_2 = 0 + \frac{\cos 0 - 0}{\sin 0 + 1} = 1; \qquad x_3 = 1 + \frac{\cos 1 - 1}{\sin 1 + 1} = 0.750363868$$

$$x_4 = 0.737151911; \quad x_5 = 0.739446670$$

$$x_6 = 0.739018516; \quad x_7 = 0.739097442$$

$$x_8 = 0.739082860; \quad x_9 = 0.739085553$$

$$x_{10} = 0.739085055; \quad x_{11} = 0.739085147$$

Def: A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Ex: $F(x) = \dfrac{x^3}{3}$ is an antiderivative of $f(x) = x^2$ for any $x \in \mathbb{R}$.

Indeed, $F'(x) = \left(\dfrac{x^3}{3}\right)' = x^2 = f(x)$

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on the interval I is

$$F(x) + C,$$

where C is an arbitrary constant.

Table of antiderivatives formulas

Function	antiderivative	Function	antiderivative
x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x	$\cos x$	$\sin x$
$\sin x$	$-\cos x$	$\sec^2 x$	$\tan x$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$

Ex: Find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.

The general antiderivative is

$$f(x) = e^x + 20\tan^{-1} x + C.$$

Since $f(0) = -2$, we have $e^0 + 20\tan^{-1} 0 + C = -2 \Longrightarrow 1 + C = -2 \Longrightarrow C = -3$. Hence, $f(x) = e^x + 20\tan^{-1} x - 3$