International University

School of Electrical Engineering

PRINCIPLES OF ELECTRICAL ENGINEERING 2

Lecture #9: Fourier Series

(1768-1830)

Text book:

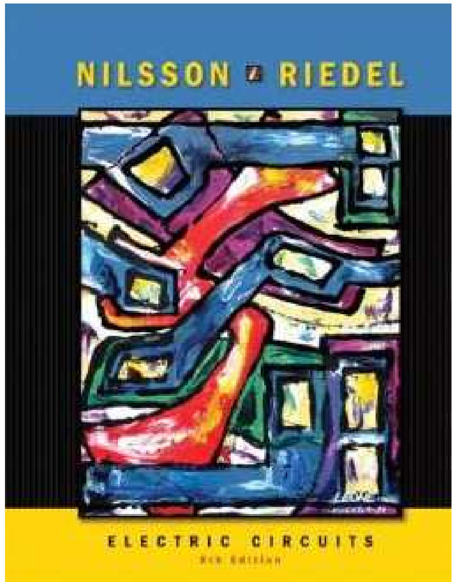
Electric Circuits

James W. Nilsson & Susan A. Riedel

8th or 9th Edition.

link: **http://blackboard.hcmiu.edu.vn/**

to download materials



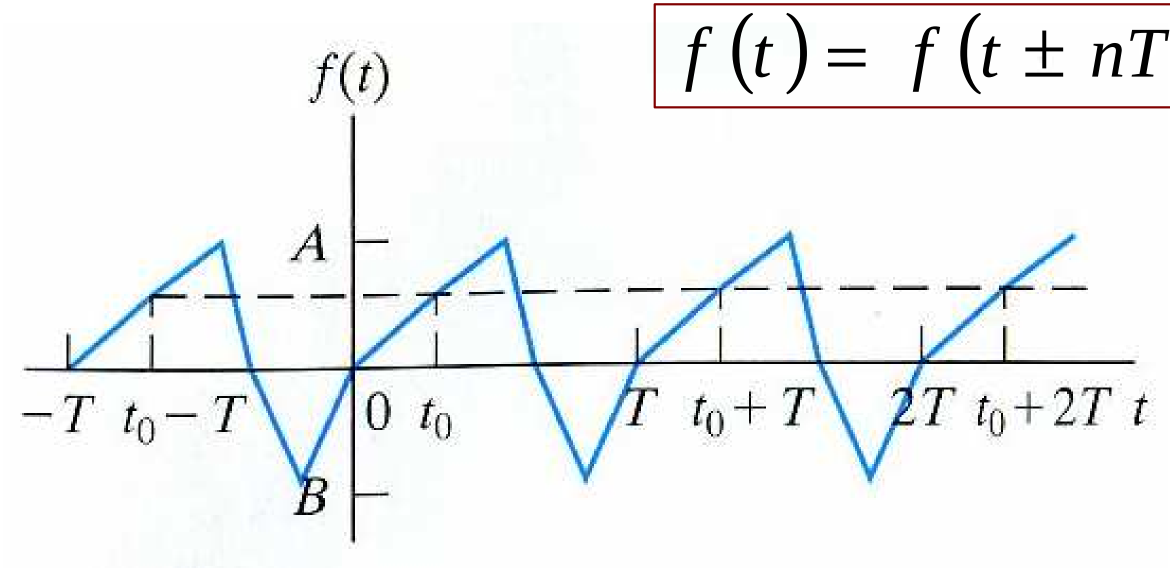
Email: mlinh@hcmiu.edu.vn

Periodic function

- A <u>periodic function</u> is a function that repeats itself every T seconds.

- A <u>period</u>, T, is the smallest time interval that a periodic function may be shifted (in either direction) to produce a function that is identical to itself.

$$\boxed{f(t) = f(t \pm nT)}$$

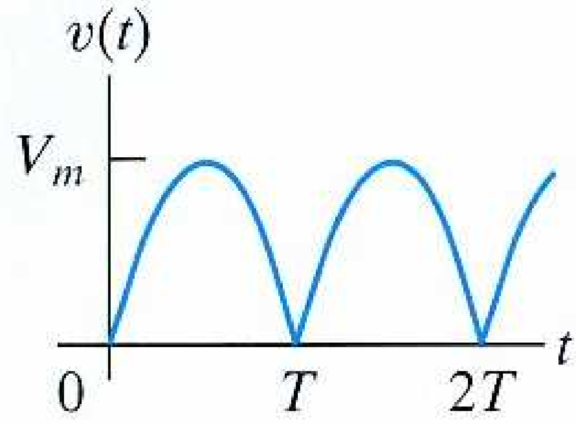


nonsinusoidal,
but periodic,
because:

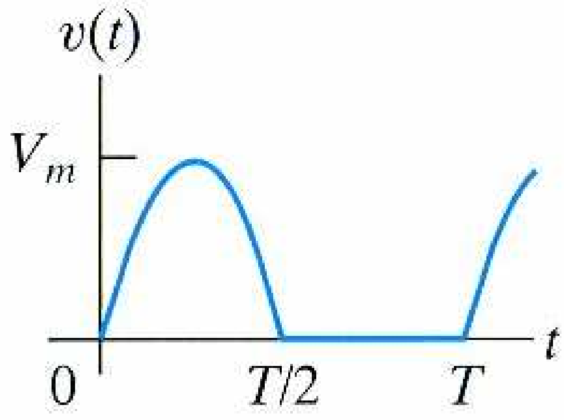
$$f(t_0) = f(t_0 - T) = f(t_0 + T) = f(t_0 + 2T) = \cdots$$

Periodic function

Rectification

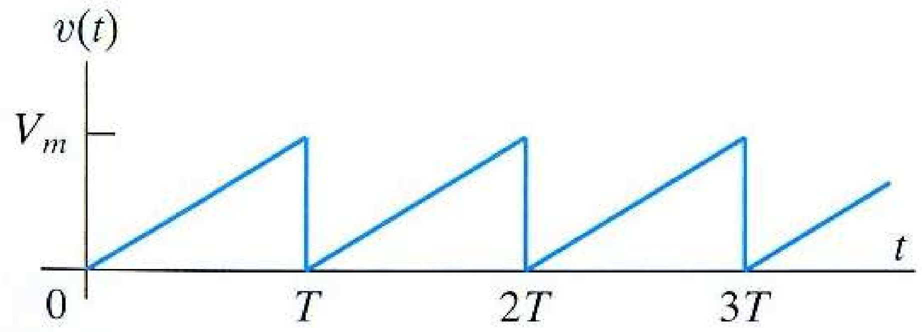


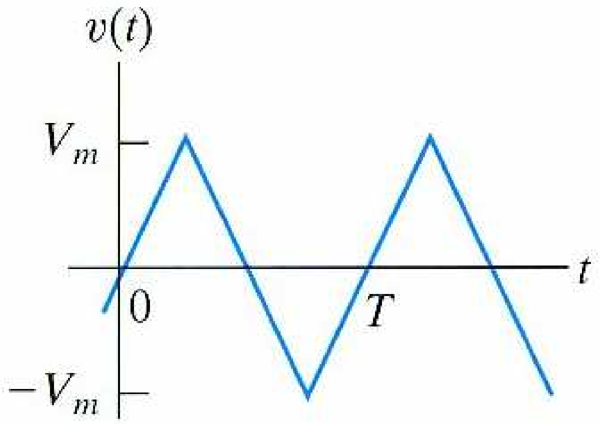
Full wave



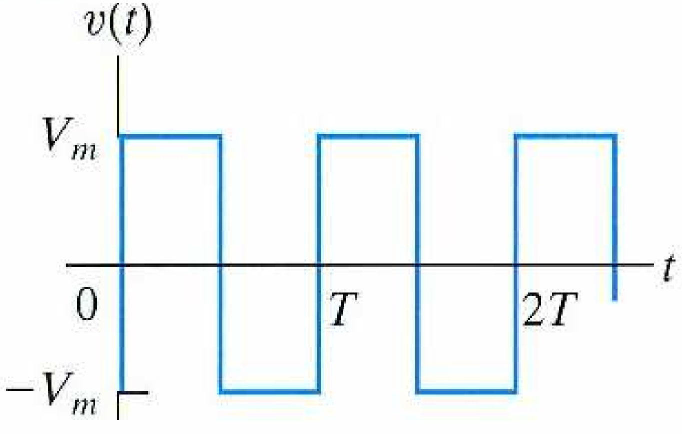
Half wave

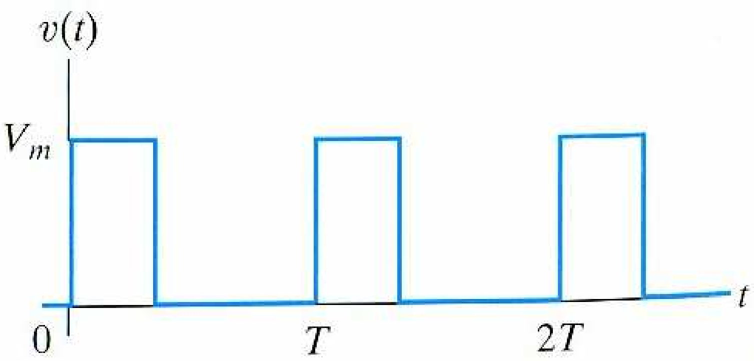
Triangular waveform





Rectangular waveform





Fourier series analysis: overview

- The ***Fourier series*** is an infinite series used to represent a periodic function. The series consists of a constant term and infinitely many harmonically related cosine and sine term.

$$\boxed{f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t}$$

n = 1, 2, 3,…

- The ***fundamental frequency*** (ω_0) is the frequency determined by the fundamental period ($f_0 = 1/T$ or $\omega_0 = 2\pi f_0$)
- The ***harmonic frequency*** is an integer multiple of the fundamental frequency such as: $2\omega_0$, $3\omega_0$, $4\omega_0$,…
- The ***Fourier coefficients*** are the constant term, $\boldsymbol{a_v}$, and the coefficient of each cosine and sine term in the series ($\boldsymbol{a_n}$, $\boldsymbol{b_n}$).

Fourier series analysis: overview

Dirichlet's conditions

- The conditions on a periodic function f(t) ensure expressing f(t) as a convergent Fourier series (known as **Dirichlet's conditions**):
  - f(t) is single-valued,
  - f(t) has a finite number of discontinuities in the periodic interval,
  - f(t) has a finite number of maxima and minima in the periodic interval,
  - the integral $\int_{t_0}^{t_0+T} |f(t)|dt$ exists.

Any periodic function generated by a physically realizable source satisfies Dirichlet's conditions. These are **sufficient** conditions, not **necessary** conditions. After we have determined **f(t)** and calculated the Fourier coefficients ($\boldsymbol{a_v}$, $\boldsymbol{a_n}$ *and* $\boldsymbol{b_n}$), *we resolve the periodic source into a dc source* ($\boldsymbol{a_n}$) *plus a* sum of sinusoidal sources ($\boldsymbol{a_n}$ *and* $\boldsymbol{b_n}$).

Fourier series analysis: overview

Periodic source is driving a linear circuit, we may use the principle of superposition to find the steady-state response.

- First calculate the response to each source generated by the Fourier series representation of **f(t),** and

- then add the individual responses to obtain the total response.

The steady-state response owing to a specific sinusoidal source is most easily found with the phasor method of analysis.

The Fourier coefficients

The Fourier coefficients are determined by the following relationships:

$$a_v = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt$$

Recall of some trigonometric functions:

$$\int_{t_0}^{t_0+T} \sin m\omega_0 t dt = 0 \ , \quad \text{for all } m$$

$$\int_{t_0}^{t_0+T} \cos m\omega_0 t dt = 0 \ , \quad \text{for all } m$$

$$\int_{t_0}^{t_0+T} \cos m\omega_0 t \sin n\omega_0 t dt = 0 \ , \quad \text{for all } m \text{ and } n$$

$$\int_{t_0}^{t_0+T} \sin m\omega_0 t \sin n\omega_0 t dt = \begin{cases} 0 & , \text{ for all } m \neq n \\ T/2 & , \text{ for all } m = n \end{cases} \ ,$$

$$\int_{t_0}^{t_0+T} \cos m\omega_0 t \cos n\omega_0 t dt = \begin{cases} 0 & , \text{ for all } m \neq n \\ T/2 & , \text{ for all } m = n \end{cases} \ ,$$

The Fourier coefficients

$$\int_{t_0}^{t_0+T} f(t)dt = \int_{t_0}^{t_0+T}\left(a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)dt$$

$$= \int_{t_0}^{t_0+T} a_v dt + \sum_{n=1}^{\infty}\int_{t_0}^{t_0+T}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)dt = a_v T + 0$$

$$\boxed{a_v = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt}$$

$$f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

first multiplying both sides by $\cos k\omega_0 t$ and then integrating each side over one period of f(t).

$$\int_{t_0}^{t_0+T} f(t)\cos k\omega_0 t dt = \int_{t_0}^{t_0+T} a_v \cos k\omega_0 t dt + \sum_{n=1}^{\infty}\int_{t_0}^{t_0+T}(a_n \cos n\omega_0 t \cos k\omega_0 t + b_n \sin n\omega_0 t \cos k\omega_0 t)dt$$

$$= 0 + a_k\left(\frac{T}{2}\right) + 0$$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt}$$

Similarly, we can derive the b_n

The Fourier coefficients

$$f(t)\sin k\omega_0 t = a_v \sin k\omega_0 t + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t \sin k\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \sin k\omega_0 t$$

Then, integrate both sides from t_0 to t_0 + T, all the integrals on the right hand side reduce to zero except in the last summation when n = k, therefore we have:

$$\int_{t_o}^{t_o+T} f(t)\sin k\omega_0 t\, dt = 0 + 0 + b_k\left(\frac{T}{2}\right)$$

$$\Rightarrow \quad b_k = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\sin k\omega_0 t\, dt$$

The Fourier coefficients

Prove: $\int_{t_0}^{t_0+T} \sin m\omega_0 t dt = 0$, for all m

$$\int_{t_o}^{t_o+T} \sin m\omega_0 t\, dt = -\frac{1}{m\omega_0} \cos m\omega_0 t \Big|_{t_o}^{t_o+T} = \frac{-1}{m\omega_0}[\cos m\omega_0(t_o+T) - \cos m\omega_0 t_o]$$

$$= \frac{-1}{m\omega_0}[\cos m\omega_0 t_o \cos m\omega_0 T - \sin m\omega_0 t_o \sin m\omega_0 T - \cos m\omega_0 t_o]$$

$$= (-1 m\omega_0)[\cos m\omega_0 t_o - 0 - \cos m\omega_0 t_o] = 0 \quad \text{for all } m,$$

Prove: $\int_{t_0}^{t_0+T} \cos m\omega_0 t dt = 0$, for all m

$$\int_{t_o}^{t_o+T} \cos m\omega_0 t_o\, dt = \frac{1}{m\omega_0}[\sin m\omega_0 t] \Big|_{t_o}^{t_o+T}$$

$$= \frac{1}{m\omega_0}[\sin m\omega_0(t_o+T) - \sin m\omega_0 t_o]$$

$$= \frac{1}{m\omega_0}[\sin m\omega_0 t_o - \sin m\omega_0 t_o] = 0 \quad \text{for all } m$$

The Fourier coefficients

Prove: $$\underbrace{\int_{t_0}^{t_0+T} \cos m\omega_0 t \sin n\omega_0 t dt}_{\mathrm{I}} = 0 \ , \quad \text{for all } m \text{ and } n$$

$$\mathrm{I} = \frac{1}{2}\int_{t_o}^{t_o+T} [\underbrace{\sin(m+n)\omega_0 t}_{=0} - \underbrace{\sin(m-n)\omega_0 t}_{=0}]\, dt$$

For all m, n

Prove: I = $$\int_{t_0}^{t_0+T} \sin m\omega_0 t \sin n\omega_0 t dt = \begin{cases} 0 & , \text{ for all } m \neq n \\ T/2 & , \text{ for all } m = n \end{cases} \ ,$$

$$\mathrm{I} = \int_{t_o}^{t_o+T} \sin m\omega_0 t \sin n\omega_0 t\, dt = \frac{1}{2}\int_{t_o}^{t_o+T} [\cos(m-n)\omega_0 t - \cos(m+n)\omega_0 t]\, dt$$

If $m \neq n$, both integrals are zero

If $m = n$, we get

$$\mathrm{I} = \frac{1}{2}\int_{t_o}^{t_o+T} dt - \frac{1}{2}\int_{t_o}^{t_o+T} \cos 2m\omega_0 t\, dt = \frac{T}{2} - 0 = \frac{T}{2}$$

The Fourier coefficients

Prove: $\mathbf{I} = \int_{t_0}^{t_0+T} \cos m\omega_0 t \cos n\omega_0 t dt = \begin{cases} 0 & , \text{ for all } m \neq n \\ T/2 & , \text{ for all } m = n \end{cases}$,

$$\mathbf{I} = \frac{1}{2}\int_{t_o}^{t_o+T} [\cos(m-n)\omega_0 t + \cos(m+n)\omega_0 t]\, dt$$

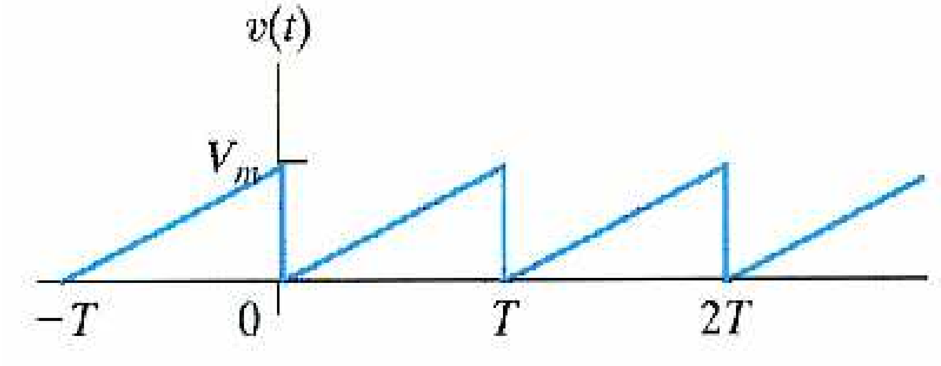
If $m \neq n$, both integrals are zero

If $m = n$, we have

$$\mathbf{I} = \frac{1}{2}\int_{t_o}^{t_o+T} dt + \frac{1}{2}\int_{t_o}^{t_o+T} \cos 2m\omega_0 t\, dt = \frac{T}{2} + 0 = \frac{T}{2}$$

The Fourier coefficients

Example: Find the Fourier series of the following periodic voltages shown in the figure:



Solution

To find a_v, a_k, and b_k, we may choose the value of t0. For the periodic voltage the best choice for t_0 is zero.

The expression for *v(t)* between 0 and T is $v(t) = \left(\frac{V_m}{T}\right)t$

$$a_v = \frac{1}{T}\int_0^T \left(\frac{V_m}{T}\right)t\,dt = \frac{1}{2}V_m.$$ This is clearly the average value of the waveform

$$a_k = \frac{2}{T}\int_0^T \left(\frac{V_m}{T}\right)t\cos k\omega_0 t\,dt = \frac{2V_m}{T^2}\left(\frac{1}{k^2\omega_0^2}\cos k\omega_0 t + \frac{t}{k\omega_0}\sin k\omega_0 t\right)\Bigg|_0^T$$

$$= \frac{2V_m}{T^2}\left[\frac{1}{k^2\omega_0^2}(\cos 2\pi k - 1)\right] = 0 \quad \text{for all } k.$$

The Fourier coefficients

Solution

$$b_k = \frac{2}{T}\int_0^T \left(\frac{V_m}{T}\right) t \sin k\omega_0 t \, dt = \frac{2V_m}{T^2}\left(\frac{1}{k^2\omega_0^2}\sin k\omega_0 t - \frac{t}{k\omega_0}\cos k\omega_0 t\right)\Bigg|_0^T$$

$$= \frac{2V_m}{T^2}\left(0 - \frac{T}{k\omega_0}\cos 2\pi k\right) = \frac{-V_m}{\pi k}.$$

The Fourier series for *v(t) is*

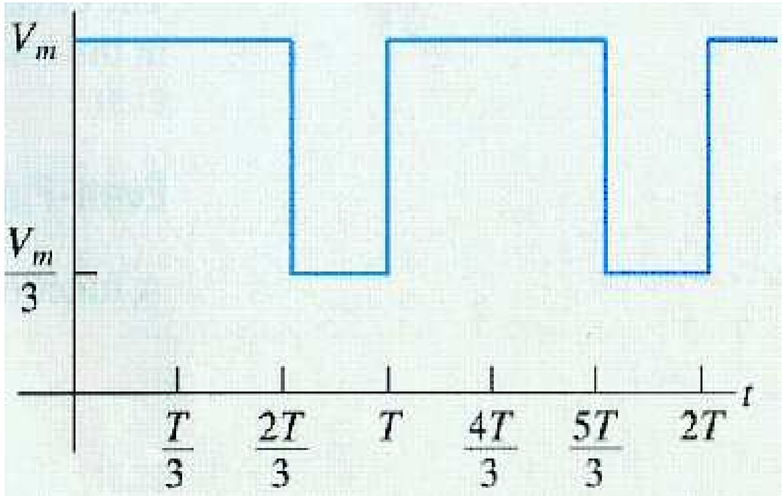
$$v(t) = \frac{V_m}{2} - \frac{V_m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\omega_0 t$$

$$= \frac{V_m}{2} - \frac{V_m}{\pi}\sin \omega_0 t - \frac{V_m}{2\pi}\sin 2\omega_0 t - \frac{V_m}{3\pi}\sin 3\omega_0 t - \cdots .$$

The Fourier coefficients

Problem 1

Derive the expressions for a_v, *a_k, and b_k for the* periodic voltage function shown if $V_m = 9\pi\,V$.



The Fourier coefficients

Solution:

The Fourier coefficients

Problem 2

Refer to Problem 1.

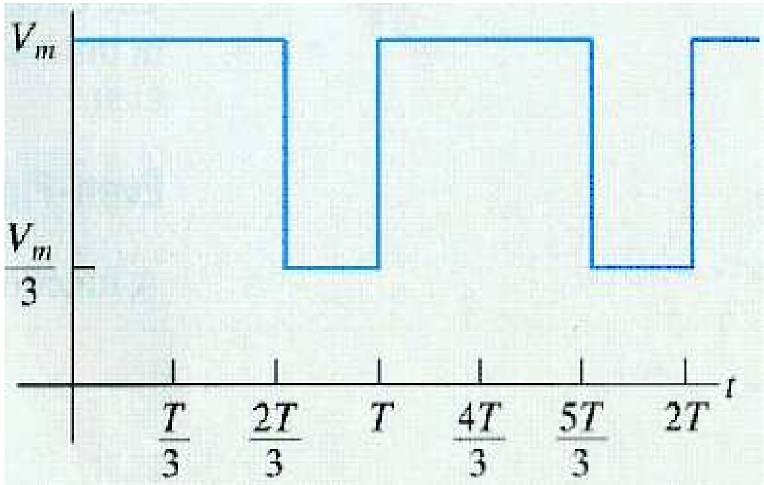
a) What is the average value of the periodic voltage?

b) Compute the numerical values of a_1 .. a_5 and b_1 .. b_5.

c) If T = *125.66 ms,* what is the fundamental frequency in radians per second?

d) What is the frequency of the third harmonic in hertz?

e) Write the Fourier series up to and including the fifth harmonic.



The Fourier coefficients

Sol:

The Effect of Symmetry on the Fourier Coefficients

Four types of symmetry may be used to simplify the task of evaluating the Fourier coefficients:

- even-function symmetry,
- odd-function symmetry,
- half-wave symmetry,
- quarter-wave symmetry.

The Effect of Symmetry on the Fourier Coefficients

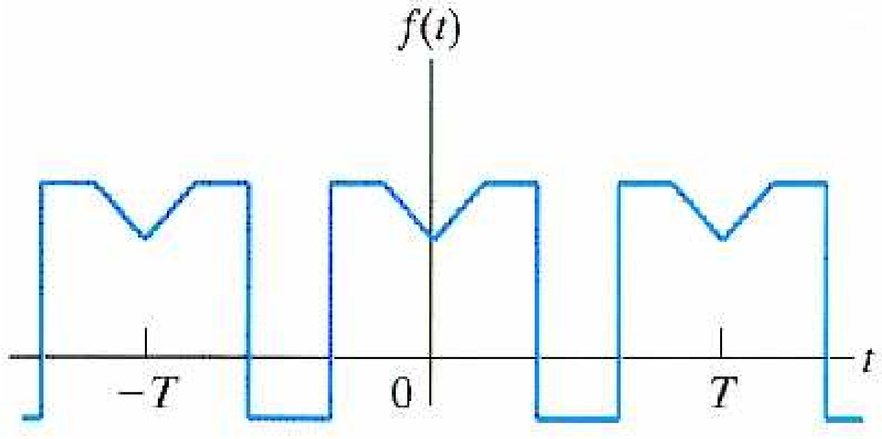
Even-function symmetry

Even-function is defined as:

$$\boxed{f(t) = f(-t)}$$

Fourier coefficients of the even periodic functions are reduced to:

$$\begin{cases} a_v = \dfrac{2}{T}\displaystyle\int_0^{T/2} f(t)dt \\ a_n = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t dt \\ b_n = 0 \quad , \quad \text{for all } n \end{cases}$$



The derivations of the even periodic functions: (select t_0 = -T/2, then break the interval of integration into the range from –T/2 to 0 and 0 to T/2)

The Effect of Symmetry on the Fourier Coefficients

Even-function symmetry

$$a_v = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\int_{-T/2}^{0} f(t)\,dt + \frac{1}{T}\int_{0}^{T/2} f(t)\,dt.$$

Let t = -x ➔ f(t) = f(-x) = f(x)

➔ x = T/2 when t = -T/2 and dt = -dx

$$\boxed{\int_{-T/2}^{0} f(t)\,dt} = \int_{T/2}^{0} f(x)(-dx) = \boxed{\int_{0}^{T/2} f(x)\,dx}$$

$$a_k = \frac{2}{T}\int_{-T/2}^{0} f(t)\cos k\omega_0 t\,dt + \frac{2}{T}\int_{0}^{T/2} f(t)\cos k\omega_0 t\,dt,$$

$$\int_{-T/2}^{0} f(t)\cos k\omega_0 t\,dt = \int_{T/2}^{0} f(x)\cos(-k\omega_0 x)(-dx) = \int_{0}^{T/2} f(x)\cos k\omega_0 x\,dx.$$

The Effect of Symmetry on the Fourier Coefficients

Even-function symmetry

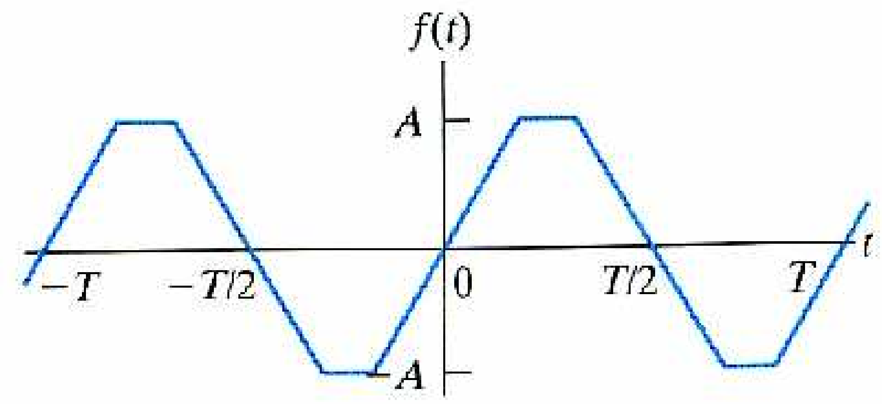
All the ***b*** coefficients are zero when ***f(t)*** is an even periodic function, because the integration from ***-T/2*** *to* ***0*** is the exact negative of the integration from **0** to **T/2**; that is,

$$\int_{-T/2}^{0} f(t)\sin k\omega_0 t\,dt = \int_{T/2}^{0} f(x)\sin(-k\omega_0 x)(-dx)$$

$$= -\int_{0}^{T/2} f(x)\sin k\omega_0 x\,dx.$$

The Effect of Symmetry on the Fourier Coefficients

Odd-function symmetry



Odd-function is defined as:

$$\boxed{f(t) = -f(-t)}$$

Fourier coefficients of the **odd periodic functions**:

$$a_v = 0$$

$$a_n = 0 \quad , \quad \text{for all } n$$

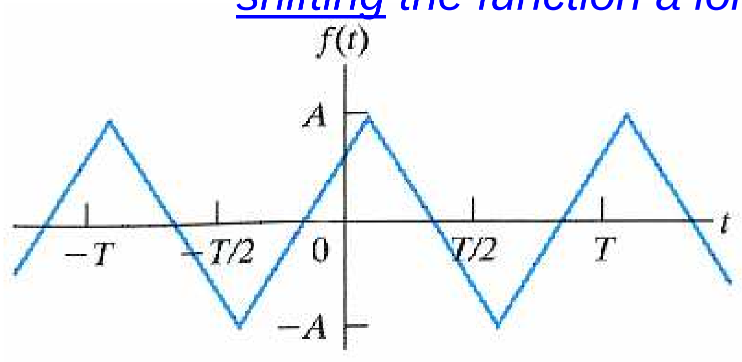
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

Prove @ home!

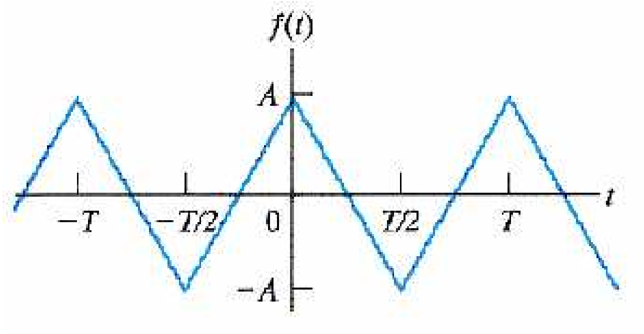
The Effect of Symmetry on the Fourier Coefficients

Half-wave symmetry

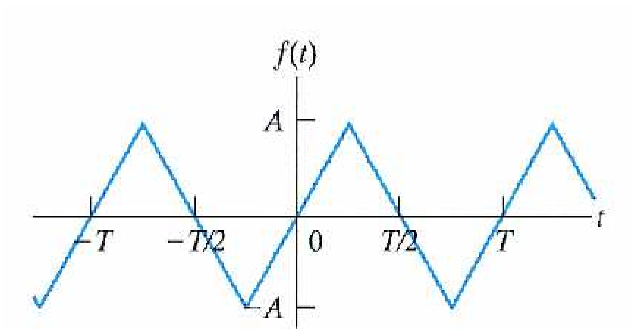
The evenness or oddness of a periodic function may be destroyed by <u>shifting</u> the function a long the <u>time axis.</u>



This triangular wave is neither even nor odd.



This triangular wave is even.



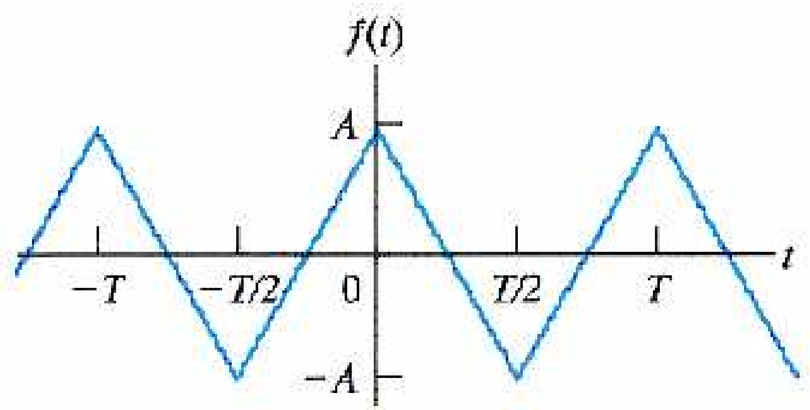
This triangular wave is odd.

Half-wave symmetry

A periodic function possesses half-wave symmetry if: $f(t) = -f(t - T/2)$

"a periodic function has half-wave symmetry if, after it is shifted one-half period and inverted, it is identical to the original function."

Half-wave, even function

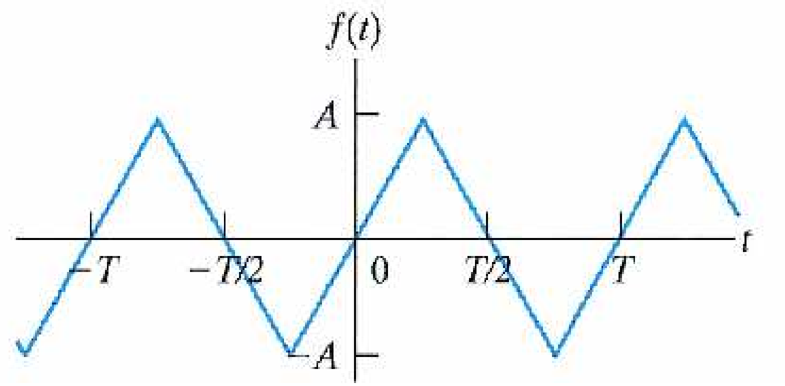


$a_v = 0 \qquad a_n = 0 \qquad b_n = 0$

"For n even"

A periodic function with half-wave symmetry has zero average, or dc, value and contains only odd harmonics

Half-wave, odd function



$$a_v = 0$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

"For n odd"

Half-wave symmetry

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos k\omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos k\omega_0 t\,dt$$

$$= \frac{2}{T}\underbrace{\int_{-T/2}^{0} f(t)\cos k\omega_0 t\,dt}_{} + \frac{2}{T}\int_{0}^{T/2} f(t)\cos k\omega_0 t\,dt.$$

Let t = x – T/2
→ x = T/2 when t = 0;
x = 0 when t = -T/2
dt = dx

$$\int_{-T/2}^{0} f(t)\cos k\omega_0 t\,dt = \int_{0}^{T/2} f(x - T/2)\cos k\omega_0(x - T/2)\,dx$$

Note that $\cos k\omega_0(x - T/2) = \cos(k\omega_0 x - k\pi) = \cos k\pi \cos k\omega_0 x$

by hypothesis f(x – T/2) = -f(x) ⇨ $\int_{-T/2}^{0} f(t)\cos k\omega_0 t\,dt = \int_{0}^{T/2} [-f(x)]\cos k\pi \cos k\omega_0 x\,dx.$

⇨ $$a_k = \frac{2}{T}(1 - \cos k\pi)\int_{0}^{T/2} f(t)\cos k\omega_0 t\,dt.$$

coskπ = 1 when k is even and -1 when k is odd

Half-wave symmetry

$$b_k = \frac{2}{T}\int_{-T/2}^{0} f(t)\sin k\omega_0 t\,dt + \frac{2}{T}\int_{0}^{T/2} f(t)\sin k\omega_0 t\,dt$$

Let t = x – T/2 → x = T/2 when t = 0; x = 0 when t = -T/2 ; dt = dx

$$\sin k\omega_0(x - T/2) = \sin(k\omega_0 x - k\pi) = \sin k\omega_0 x \cos k\pi$$

$$\Rightarrow \quad \frac{2}{T}\int_{-T/2}^{0} f(t)\sin k\omega_0 t\,dt = -\frac{2}{T}\int_{0}^{T/2} f(x)\sin k\omega_0 x \cos k\pi\,dx$$

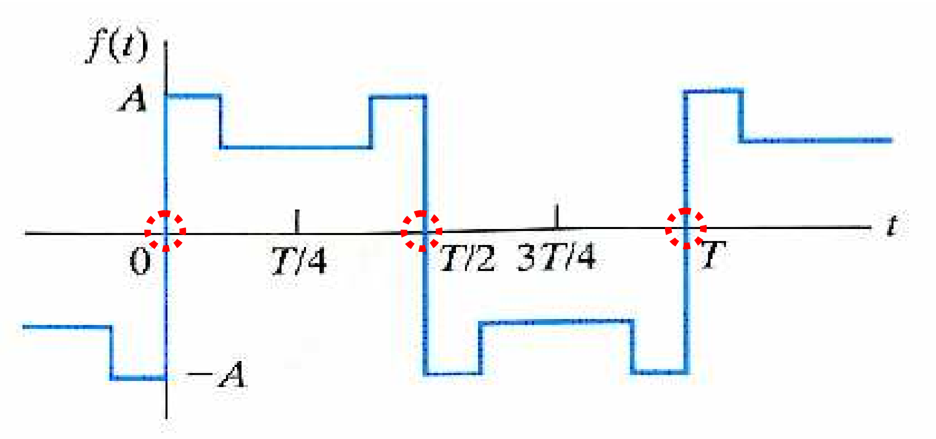
$$\Rightarrow \quad b_k = \frac{2}{T}(1 - \cos k\pi)\int_{0}^{T/2} f(x)\sin k\omega_0 t\,dt$$

1 – coskπ = 0 when k is even, and 1 - coskπ = 2 when k is odd

$$\Rightarrow \quad b_k = \frac{4}{T}\int_{0}^{T/2} f(t)\sin k\omega_0 t\,dt \quad \text{when } k \text{ is odd}$$

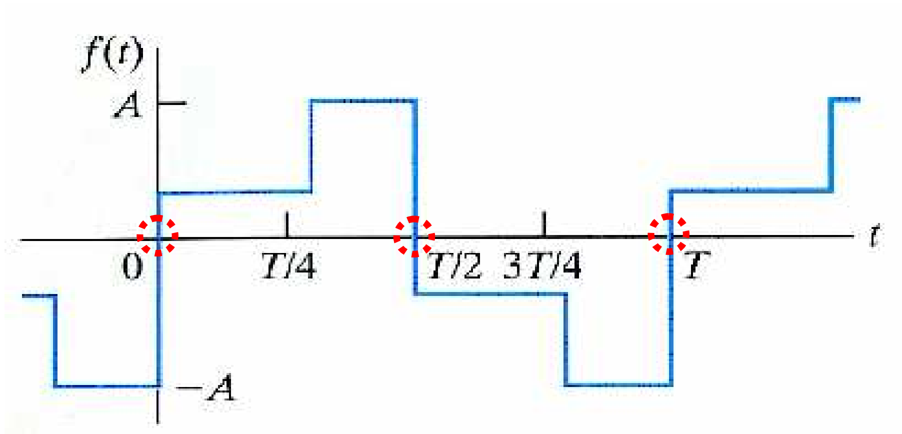
Quarter-wave symmetry

Quarter-wave symmetry describes a periodic function that has half-wave symmetry and, in addition, symmetry about the midpoint of the positive and negative half-cycles.



Quarter-wave symmetry

Function is odd and can be made even by shifting the function T/4 units either right or left along the *t axis.*



Non quarter-wave symmetry

never be made either even or odd.

Quarter-wave symmetry

To take advantage of quarter-wave symmetry in the calculation of the Fourier coefficients, you must choose the point where t = 0 to make the function either even or odd.

Quarter-wave, even function

$a_v = 0$ because of half-wave symmetry;

$$a_n = \begin{cases} 0 & \text{, for n even} \\ \dfrac{8}{T}\displaystyle\int_0^{T/4} f(t)\cos n\omega_0 t\,dt & \text{, for n odd} \end{cases}$$

$b_n = 0$ for all n, because the function is even.

Quarter-wave, odd function

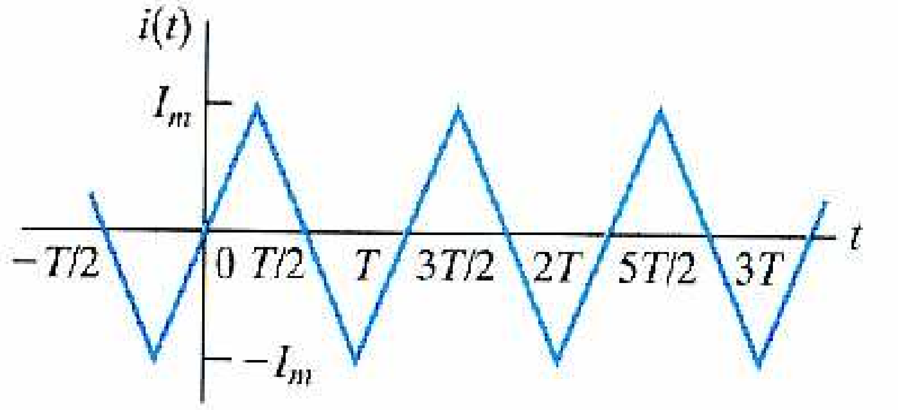
$a_v = 0$ because the function is odd;

$a_n = 0$ for all *k, because the function is odd;*

$$b_n = \begin{cases} 0 & \text{, for n even} \\ \dfrac{8}{T}\displaystyle\int_0^{T/4} f(t)\sin n\omega_0 t\,dt & \text{, for n odd} \end{cases}$$

Quarter-wave symmetry

Example: Find the Fourier series representation for the current waveform shown in Fig



Sol:

The function is odd & has half-wave, quarter-wave symmetry. Because of odd → *a_v = 0 and a_k = 0 for all k.* Because of half-wave symmetry, *b_k = 0 for even values of k. Because of* quarter-wave symmetry, the expression for b_k for odd values of *k is*

$$b_k = \frac{8}{T}\int_0^{T/4} i(t)\sin k\omega_0 t\, dt$$

In the interval $0 \le t \le T/4$, the expression for i(t) is $i(t) = \frac{4I_m}{T}t$

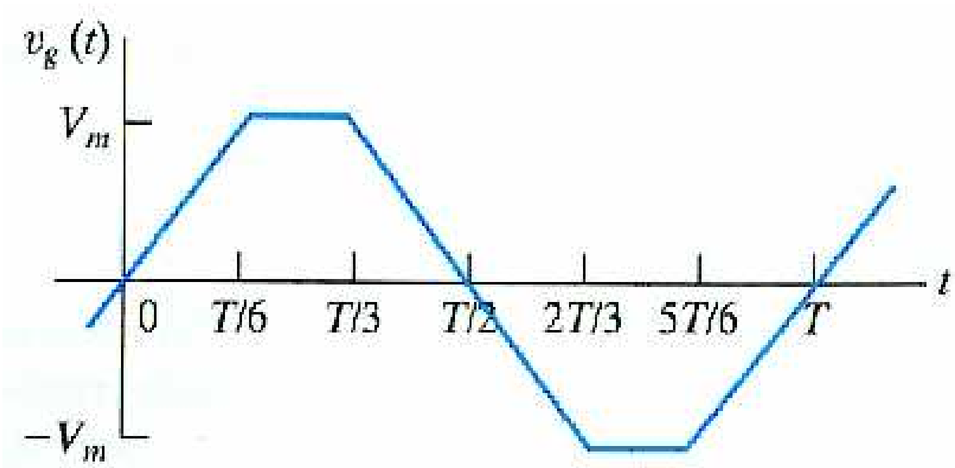
$$b_k = \frac{8}{T}\int_0^{T/4}\frac{4I_m}{T}t\sin k\omega_0 t\, dt = \frac{32I_m}{T^2}\left(\frac{\sin k\omega_0 t}{k^2\omega_0^2}\Bigg|_0^{T/4} - \frac{t\cos k\omega_0 t}{k\omega_0}\Bigg|_0^{T/4}\right) = \frac{8I_m}{\pi^2 k^2}\sin\frac{k\pi}{2} \qquad (k \text{ is odd})$$

The Fourier series representation of i(t) is

$$i(t) = \frac{8I_m}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin n\omega_0 t = \frac{8I_m}{\pi^2}\left(\sin\omega_0 t - \frac{1}{9}\sin 3\omega_0 t + \frac{1}{25}\sin 5\omega_0 t - \frac{1}{49}\sin 7\omega_0 t + \cdots\right)$$

Quarter-wave symmetry

Problem 3 Derive the Fourier series for the periodic voltage shown.



Quarter-wave symmetry

Sol:

Summary on the effect of symmetry on the Fourier coefficients

- There are five types of symmetry that are used to simplify the computation of the Fourier coefficients:
  - *Even*, in which all sine terms in the series are zero.
  - *Odd*, in which all cosine terms and the constant term are zero.
  - *Half-wave*, in which all even harmonics are zero.
  - *Quarter wave, half-wave, even*, in which the series contains only odd harmonic cosine terms.
  - *Quarter wave, half-wave, odd*, in which the series contains only odd harmonic sine terms.

Trigonometric form of Fourier series

The cosine and sine terms may be merged in either a cosine expression or a sine expression. Doing so allows the representation of each harmonic of v(t) or i(t) as a single phasor quantity. We choose the cosine expression here for the alternative form of the series.

$$f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)$$

where $a_n - jb_n = \sqrt{a_n^2 + b_n^2}\angle -\theta_n = A_n \angle -\theta_n$

from $f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$

$$\Rightarrow \quad f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \cos(n\omega_0 t - 90^\circ)$$

Adding the terms under the summation sign by using phasors gives

$$\mathcal{P}\{a_n \cos n\omega_0 t\} = a_n \underline{/0^\circ} \quad \& \quad \mathcal{P}\{b_n \cos(n\omega_0 t - 90^\circ)\} = b_n \underline{/-90^\circ} = -jb_n$$

$$\mathcal{P}\{a_n \cos(n\omega_0 t + b_n \cos(n\omega_0 t - 90^\circ)\} = a_n - jb_n = \sqrt{a_n^2 + b_n^2}\underline{/-\theta_n} = A_n\underline{/-\theta_n}.$$

inverse-transform Eq. $a_n \cos n\omega_0 t + b_n \cos(n\omega_0 t - 90^\circ) = \mathcal{P}^{-1}\{A_n \underline{/-\theta_n}\}$

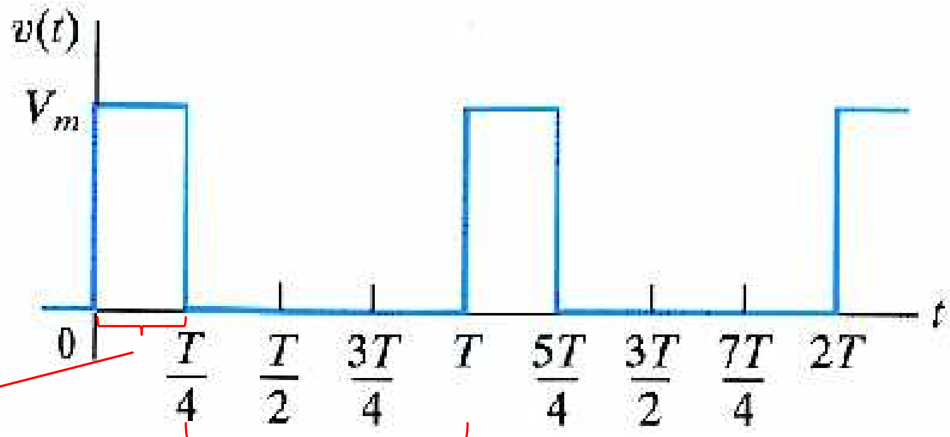
$$\boxed{= A_n \cos(n\omega_0 t - \theta_n)}$$

Trigonometric form of Fourier series

Example

a) Find the Fourier coefficients for the periodic functions shown.

b) Write the first four terms of the Fourier series representation of v(t) using the trigonometric form of the Fourier series



Sol:

a) The voltage v(t) is neither even nor odd, nor does it have half-wave symmetry. Therefore we use Eqs. to find a_k and b_k. Choosing t_0 as zero, we obtain

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} v(t)\cos k\omega_0 t dt$$

$$a_k = \frac{2}{T}\left[\int_0^{T/4} V_m \cos k\omega_0 t \, dt + \int_{T/4}^{T} (0)\cos k\omega_0 t \, dt\right] = \frac{2V_m}{T}\frac{\sin k\omega_0 t}{k\omega_0}\Bigg|_0^{T/4} = \frac{V_m}{k\pi}\sin\frac{k\pi}{2}$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} v(t)\sin k\omega_0 t dt = \frac{2}{T}\int_0^{T/4} V_m \sin k\omega_0 t \, dt = \frac{2V_m}{T}\left(\frac{-\cos k\omega_0 t}{k\omega_0}\Bigg|_0^{T/4}\right) = \frac{V_m}{k\pi}\left(1-\cos\frac{k\pi}{2}\right).$$

Trigonometric form of Fourier series

b) The average value of *v(t) is* $a_v = \dfrac{V_m(T/4)}{T} = \dfrac{V_m}{4}.$

The values of $a_k — jb_k$ for k = 1,2, and 3 are

$$a_1 - jb_1 = \frac{V_m}{\pi} - j\frac{V_m}{\pi} = \frac{\sqrt{2}V_m}{\pi} \angle -45^\circ,$$

$$a_2 - jb_2 = 0 - j\frac{V_m}{\pi} = \frac{V_m}{\pi} \angle -90^\circ,$$

$$a_3 - jb_3 = \frac{-V_m}{3\pi} - j\frac{V_m}{3\pi} = \frac{\sqrt{2}V_m}{3\pi} \angle -135^\circ.$$

Thus the first four terms in the Fourier series representation of *v(t) are*

$$v(t) = \frac{V_m}{4} + \frac{\sqrt{2}V_m}{\pi}\cos(\omega_0 t - 45^\circ) + \frac{V_m}{\pi}\cos(2\omega_0 t - 90^\circ)$$

$$+ \frac{\sqrt{2}V_m}{3\pi}\cos(3\omega_0 t - 135^\circ) + \cdots$$

Math glance

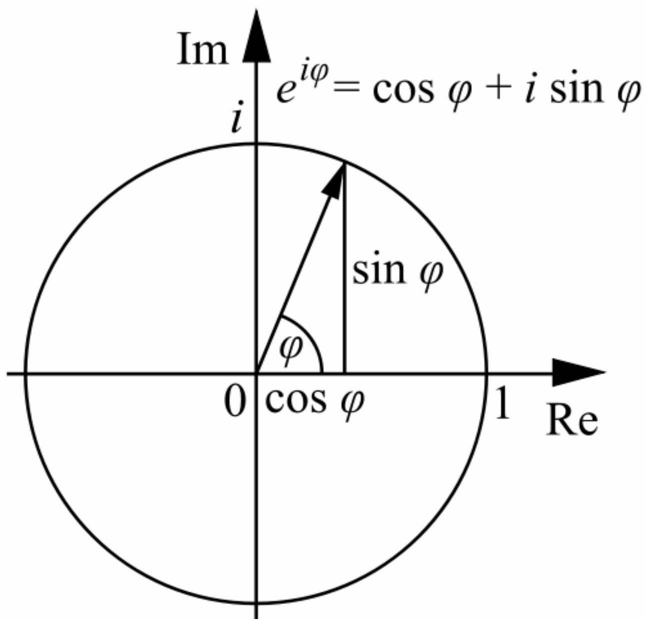
$$e^{j\beta x} = \cos\beta x + j\sin\beta x$$
$$e^{-j\beta x} = \cos\beta x - j\sin\beta x$$

$$z = a + jb = r(\cos\varphi + j\sin\varphi) = re^{j\varphi} = r\angle\varphi$$

$$r = \sqrt{a^2 + b^2}$$

$$tg\varphi = \frac{b}{a}$$

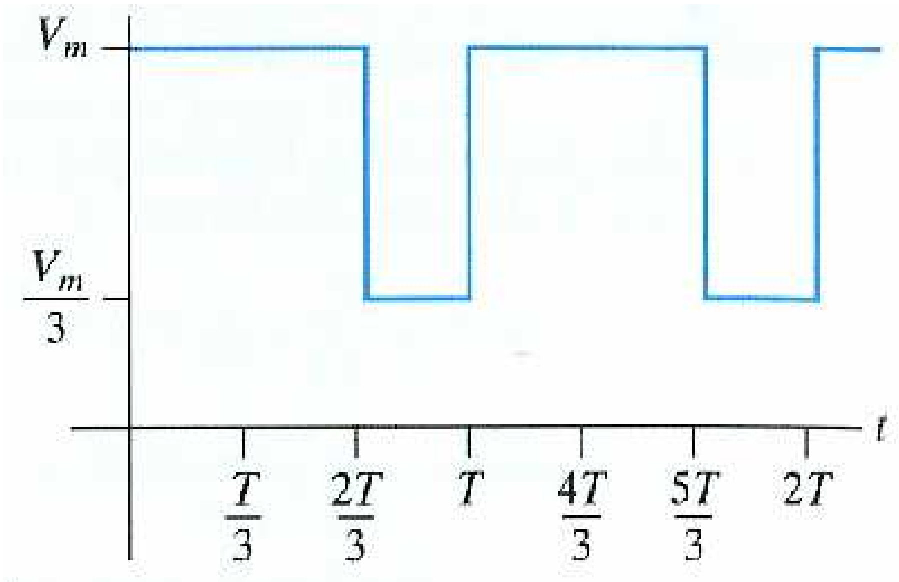
$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}} \quad ; \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}$$



Trigonometric form of Fourier series

Problem 4 a) Compute $A_1 \ldots A_5$ and $\theta_1 \ldots \theta_5$ *for the periodic* function shown if $V_m = 9\pi\ V$

b) Using eq. $f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)$ write the Fourier series for *v(t)* up to and including the fifth harmonic assuming T = 125.66 ms.

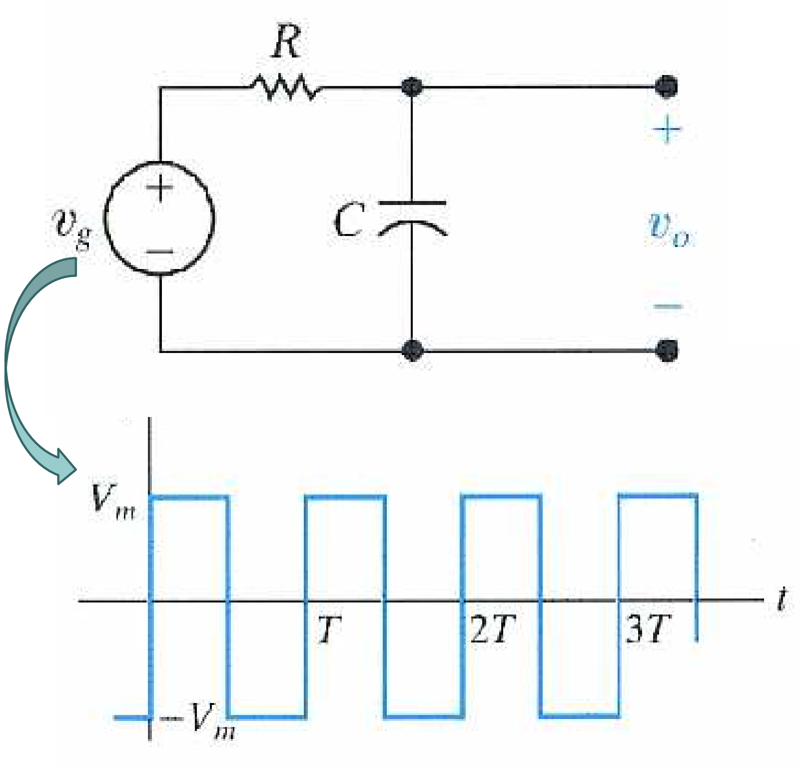


Trigonometric form of Fourier series

Sol:

An application

How to use a Fourier series for representing of a periodic excitation function, then to find the **steady-state response** of a linear circuit.



Example: An RC circuit excited by a periodic voltage as in figure shown.

Find the output voltage v_0 ?

The first step in finding the steady-state response is to represent the source v_g with its Fourier series. Here, v_g is odd, half-wave, and quarter-wave symmetry. So:

$$b_k = \frac{8}{T}\int_0^{T/4} V_m \sin k\omega_0 t \, dt = \frac{4V_m}{\pi k} \qquad (k \text{ is odd}).$$

$$v_g = \frac{4V_m}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin n\omega_0 t$$

$$\Longleftrightarrow \quad v_g = \frac{4V_m}{\pi}\sin \omega_0 t + \frac{4V_m}{3\pi}\sin 3\omega_0 t + \frac{4V_m}{5\pi}\sin 5\omega_0 t + \frac{4V_m}{7\pi}\sin 7\omega_0 t + \cdots \qquad (*)$$

An application

The v_g in Eq. **(*)** is the equivalent of infinitely many series-connected sinusoidal sources, each source having its own amplitude and frequency.

Use the principle of superposition to find the contribution of each source to the output voltage.

With the sinusoidal sources, the phasor-domain expression for the output voltage is

$$\mathbf{V}_o = \frac{\mathbf{V}_g}{1 + j\omega RC}$$

From the phasor domain back to the time domain, write sin(ωt + θ), not cos(ωt + θ)

<u>The phasor output voltage</u>

$$\boxed{\mathbf{V}_{o1} = \frac{(4V_m/\pi)\angle 0^\circ}{1 + j\omega_0 RC}}$$

⇨

<u>In polar form</u>

$$\boxed{\mathbf{V}_{o1} = \frac{(4V_m)\angle -\beta_1}{\pi\sqrt{1 + \omega_0^2 R^2 C^2}}}$$

$$\beta_1 = \tan^{-1}\omega_0 RC.$$

Time-domain expression: $v_{o1} = \dfrac{4V_m}{\pi\sqrt{1 + \omega_0^2 R^2 C^2}}\sin(\omega_0 t - \beta_1)$

The third-harmonic phasor voltage

$$\mathbf{V}_{o3} = \frac{(4V_m/3\pi)\angle 0^\circ}{1 + j3\omega_0 RC} = \frac{4V_m}{3\pi\sqrt{1 + 9\omega_0^2 R^2 C^2}}\angle -\beta_3, \qquad \beta_3 = \tan^{-1}3\omega_0 RC.$$

An application

The time-domain expression for the third-harmonic output voltage

$$v_{o3} = \frac{4V_m}{3\pi\sqrt{1 + 9\omega_0^2 R^2 C^2}} \sin(3\omega_0 t - \beta_3).$$

The expression for the k^th-harmonic component of the output voltage:

$$v_{ok} = \frac{4V_m}{k\pi\sqrt{1 + k^2\omega_0^2 R^2 C^2}} \sin(k\omega_0 t - \beta_k) \qquad (k \text{ is odd}),$$

$$\beta_k = \tan^{-1} k\omega_0 RC \qquad (k \text{ is odd})$$

The Fourier series representation of the output voltage:

$$v_o(t) = \frac{4V_m}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin(n\omega_0 t - \beta_n)}{n\sqrt{1 + (n\omega_0 RC)^2}} \qquad \textbf{(**)}$$

If C is large, $1/n\omega_0 C$ is small for the higher order harmonics. So (**) becomes:

$$v_o \approx \frac{4V_m}{\pi\omega_0 RC} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \sin(n\omega_0 t - 90^\circ) \approx \frac{-4V_m}{\pi\omega_0 RC} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos n\omega_0 t.$$

Here amplitude of the harmonic in the output is decreasing by $1/n^2$, compared with $1/n$ for the input harmonics. If C is so large, then

$$v_o(t) \approx \frac{-4V_m}{\pi\omega_0 RC} \cos \omega_0 t,$$

Fourier analysis tells us that the square-wave input is deformed into a sinusoidal output.

An application

If C → 0: The circuit shows that v_0 & v_g are the same when C = 0. Thus (**) becomes:

$$v_o = \frac{4V_m}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin n\omega_0 t.$$

This equation is identical to equation of v_g. Therefore $v_0 \rightarrow v_g$ as $C \rightarrow 0$

Thus Eq. **(**)** has proven useful because it enabled us to predict that the output will be a highly distorted replica of the input waveform if C is large, and a reasonable replica if C is small.

In the frequency domain, we look at the distortion between the steady-state input and output in terms of how the amplitude and phase of the harmonics are altered as they are transmitted through the circuit.

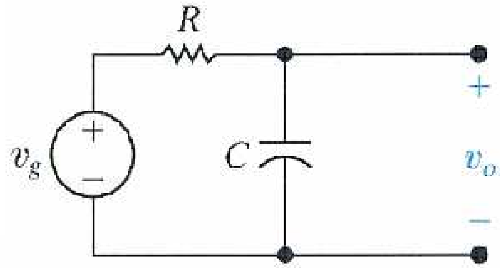
Thus, in the frequency domain, we speak of amplitude distortion and phase distortion.

For the circuit here, amplitude distortion is presented because the amplitudes of the input harmonics decrease as **1/n**, whereas the amplitudes of the output harmonics decrease as:

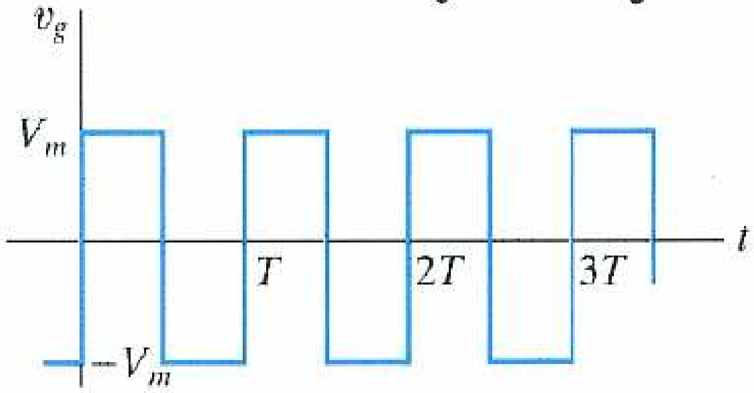
$$\frac{1}{n} \frac{1}{\sqrt{1 + (n\omega_0 RC)^2}}.$$

An application of the direct approach to the steady-state response

we can derive the expression for the steady-state response without resorting to the Fourier series representation of the excitation function.



The square-wave excitation function alternates between charging the capacitor toward $+V_m$ *and* —V_m.



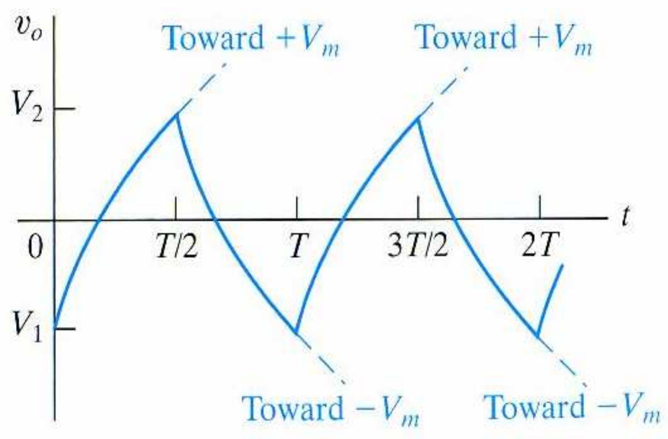
After the circuit reaches steady-state operation, this alternate charging becomes periodic.

The steady-state waveform of the voltage across the capacitor is showing

$V_0(t)$ in the time intervals

$$v_o = V_m + (V_1 - V_m)e^{-t/RC}, \qquad 0 \le t \le T/2;$$

$$v_o = -V_m + (V_2 + V_m)e^{-[t-(T/2)]/RC}, \quad T/2 \le t \le T.$$



An application of the direct approach to the steady-state response

$$\begin{cases} V_2 = V_m + (V_1 - V_m)e^{-T/2RC}, \\ V_1 = -V_m + (V_2 + V_m)e^{-T/2RC}. \end{cases}$$

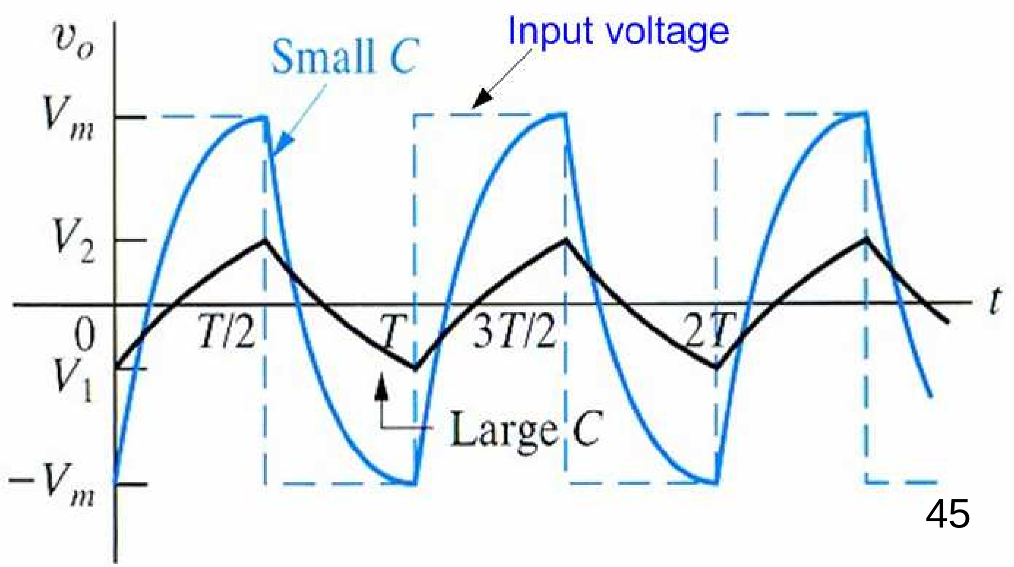
$$V_2 = -V_1 = \frac{V_m(1 - e^{-T/2RC})}{1 + e^{-T/2RC}}.$$

$$\begin{cases} v_o = V_m - \dfrac{2V_m}{1 + e^{-T/2RC}} e^{-t/RC}, \quad 0 \le t \le T/2, \\ v_o = -V_m + \dfrac{2V_m}{1 + e^{-T/2RC}} e^{-[t-(T/2)]/RC}, \quad T/2 \le t \le T. \end{cases}$$

$V_0(t)$ has half-wave symmetry → the average value of *v_0 is zero.*

V_0 → v_g as C → 0

If C is large, the output waveform becomes triangular in shape



An application of the direct approach to the steady-state response

Noted that the periodic voltage response has half-wave symmetry

$$\begin{cases} a_k = \dfrac{4}{T}\displaystyle\int_0^{T/2}\left(V_m - \dfrac{2V_m e^{-t/RC}}{1 + e^{-T/2RC}}\right)\cos k\omega_0 t\, dt = \dfrac{-8RCV_m}{T[1 + (k\omega_0 RC)^2]} \quad (k \text{ is odd}), \\ b_k = \dfrac{4}{T}\displaystyle\int_0^{T/2}\left(V_m - \dfrac{2V_m e^{-t/RC}}{1 + e^{-T/2RC}}\right)\sin k\omega_0 t\, dt = \dfrac{4V_m}{k\pi} - \dfrac{8k\omega_0 V_m R^2C^2}{T[1 + (k\omega_0 RC)^2]} \quad (k \text{ is odd}). \end{cases}$$

We can easily prove that

$$\begin{cases} \sqrt{a_k^2 + b_k^2} = \dfrac{4V_m}{k\pi}\dfrac{1}{\sqrt{1 + (k\omega_0 RC)^2}}, \\ \dfrac{a_k}{b_k} = -k\omega_0 RC. \end{cases}$$

These equations are consistent with equation **(**)**

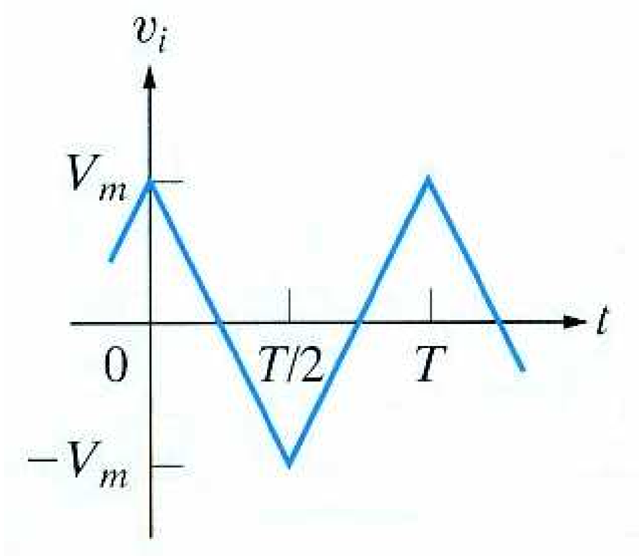
We can deduce a reasonable approximation of the steady-state response by using a finite number of appropriate terms in the Fourier series representation.

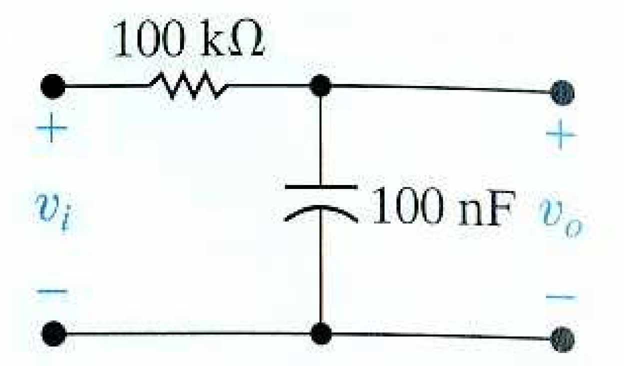
An application

Problem 5

The periodic triangle-wave voltage seen on the left is applied to the circuit shown on the right. Derive the first three nonzero terms in the Fourier series that represents the steady-state voltage v_0

Given $V_m = 281.25\,\pi^2$ (mV) and the period of the input voltage is 200π (ms)



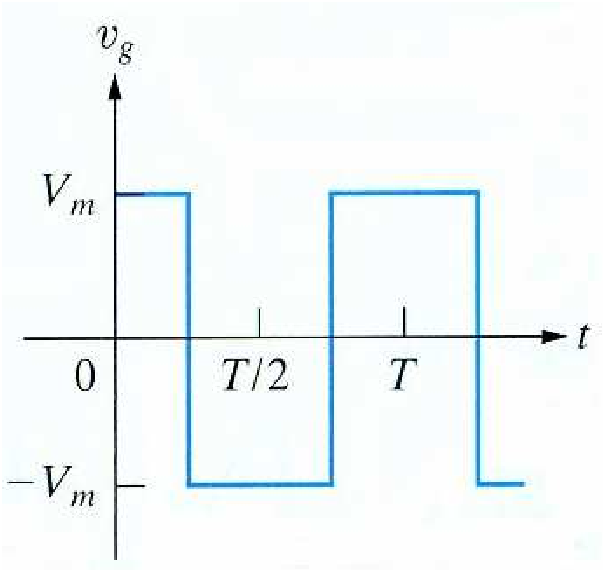


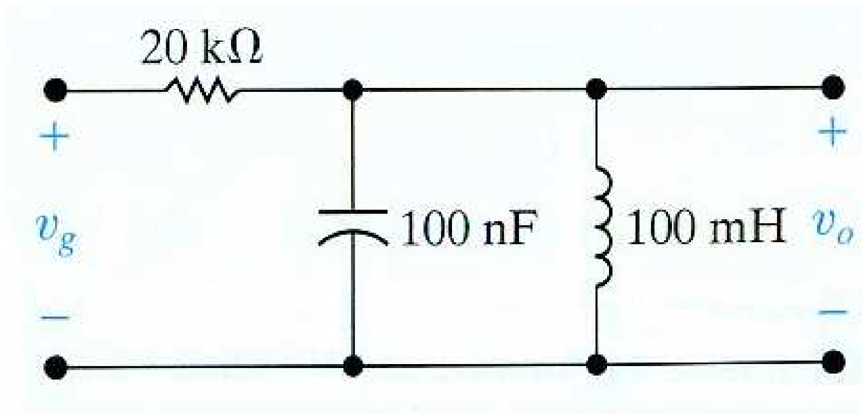
An application

Sol.:

An application

Problem 6





The periodic square-wave shown on the left is applied to the circuit shown on the right.

a) Derive the first four nonzero terms in the Fourier series that represents the steady-state voltage v_0 if $V_m = 10.5\pi$ (V) and the period of the input voltage is π (ms)

b) Which harmonic dominates the output voltage? Explain why.

An application

Sol:

Average-power calculation with periodic functions

The periodic voltage and current at the terminals of a network as trigonometric form:

$$v = V_{dc} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_{vn}),$$

$$i = I_{dc} + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t - \theta_{in}).$$

V_{dc} = the amplitude of the dc voltage component,

V_n = the amplitude of the nth-harmonic voltage,

θ_{vn} = the phase angle of the nth-harmonic voltage,

I_{dc} = *the amplitude of the dc current component*,

I_n = *the amplitude of the nth-harmonic current,*

θ_{in} = the phase angle of the nth-harmonic current.

The average power is $P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt = \frac{1}{T}\int_{t_0}^{t_0+T} vi\,dt$

$$P = \frac{1}{T} V_{dc} I_{dc} t \Big|_{t_0}^{t_0+T} + \sum_{n=1}^{\infty} \frac{1}{T} \int_{t_0}^{t_0+T} V_n I_n \cos(n\omega_0 t - \theta_{vn}) \times \cos(n\omega_0 t - \theta_{in})\, dt.$$

using the equ $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta),$

Average-power calculation with periodic functions

$$P = V_{dc}I_{dc} + \frac{1}{T}\sum_{n=1}^{\infty}\frac{V_nI_n}{2}\int_{t_0}^{t_0+T}[\cos(\theta_{vn} - \theta_{in}) + \underbrace{\cos(2n\omega_0 t - \theta_{vn} - \theta_{in})}_{0}]dt.$$

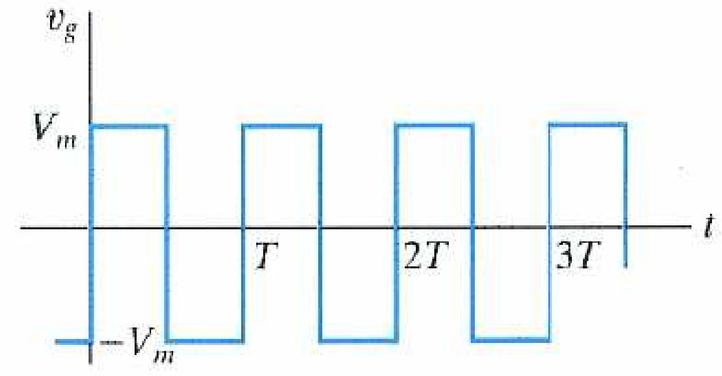
$$\Rightarrow \quad P = V_{dc}I_{dc} + \sum_{n=1}^{\infty}\frac{V_nI_n}{2}\cos(\theta_{vn} - \theta_{in}).$$

In the case of an interaction between a periodic voltage & the corresponding periodic current, the total average power P is the sum of the average powers obtained from the interaction of currents & voltages of the same frequency. Currents & voltages of different frequencies do not interact to produce average power.

Average-power calculation with periodic functions

Example

Assume that the periodic square-wave voltage in figure is applied across the terminals of a 15 Ω resistor. Assume: V_m = *60 V, T* = 5 ms.



a) Write the first five nonzero terms of the Fourier series representation of v(t). Use the trigonometric form given in Equ.: $$f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)$$

b) Calculate the average power associated with each term in (a).

c) Calculate the total average power delivered to the 15 Ω resistor.

d) What percentage of the total power is delivered by the first five terms of the Fourier series?

Average-power calculation with periodic functions

Sol: a) The dc component of *v(t) is* $a_v = \frac{(60)(T/4)}{T} = 15 \text{ V}.$

Based on the previous example $a_k = \frac{V_k}{k\pi} sin \frac{k\pi}{2}$; $b_k = \frac{V_m}{k\pi}\left(1 - cos\frac{k\pi}{2}\right)$

$A_1 = a_1 - jb_1 = \sqrt{2}\,60/\pi = 27.01 \text{ V}, \quad \theta_1 = 45^\circ,$

$A_2 = 60/\pi = 19.10 \text{ V}, \quad \theta_2 = 90^\circ,$

$A_3 = 20\sqrt{2}/\pi = 9.00 \text{ V}, \quad \theta_3 = 135^\circ,$

$A_4 = 0, \quad \theta_4 = 0^\circ,$

$A_5 = 5.40 \text{ V}, \quad \theta_5 = 45^\circ,$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi(1000)}{5} = 400\pi \text{ rad/s}.$

Thus, using the first five nonzero terms of the Fourier series,

$$v(t) = 15 + 27.01\cos(400\pi t - 45^\circ) + 19.10\cos(800\pi t - 90^\circ) + 9.00\cos(1200\pi t - 135^\circ) + 5.40\cos(2000\pi t - 45^\circ) + \cdots \text{V}.$$

Average-power calculation with periodic functions

b) The voltage is applied to the terminals of a resistor, so we can find the power associated with each term as follows:

$$P_{dc} = \frac{15^2}{15} = 15 \text{ W},$$

$$P_1 = \frac{1}{2}\frac{27.01^2}{15} = 24.32 \text{ W}, \qquad P_2 = \frac{1}{2}\frac{19.10^2}{15} = 12.16 \text{ W},$$

$$P_3 = \frac{1}{2}\frac{9^2}{15} = 2.70 \text{ W}, \qquad P_5 = \frac{1}{2}\frac{5.4^2}{15} = 0.97 \text{ W}.$$

c) To obtain the total average power delivered to the 15 Ω resistor, we first calculate the rms value of *v(t):*

$$V_{rms} = \sqrt{\frac{(60)^2(T/4)}{T}} = \sqrt{900} = 30 \text{ V}.$$

The total average power delivered to the 15 Ω resistor is $P_T = \frac{30^2}{15} = 60 \text{ W}.$

d) The total power delivered by the first five nonzero terms is

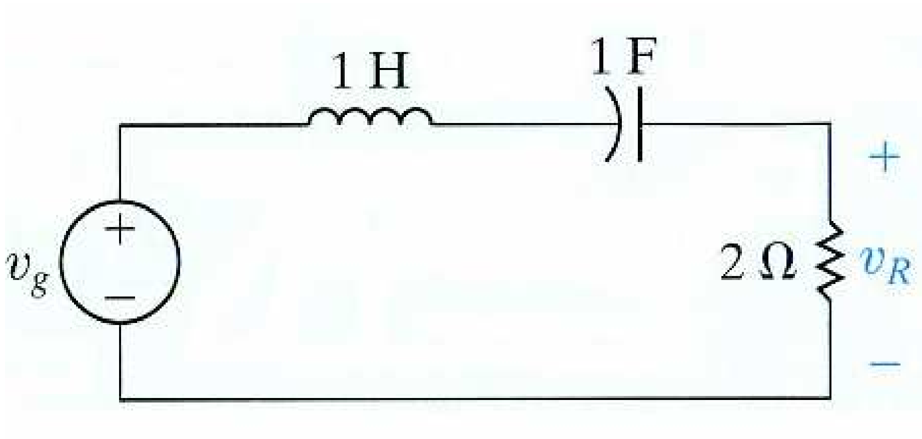
$$P = P_{dc} + P_1 + P_2 + P_3 + P_5 = 55.15 \text{ W}.$$

This is (55.15/60)(100), or 91.92% of the total.

Average-power calculation with periodic functions

Problem 7

The trapezoidal voltage function in Problem 3 is applied to the circuit shown. If $12V_m = 296.09$ *V and* $T = 2094.4ms$, *estimate* the average power delivered to the 2 Ω resistor.



Average-power calculation with periodic functions

Sol:

The rms value of a periodic function

The rms value of a periodic function can be expressed in terms of the Fourier coefficients:

$$F_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2\,dt}$$

$$\Rightarrow \quad F_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T}\left[a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)\right]^2 dt}.$$

$$F_{\text{rms}} = \sqrt{\frac{1}{T}\left(a_v^2 T + \sum_{n=1}^{\infty}\frac{T}{2}A_n^2\right)} = \sqrt{a_v^2 + \sum_{n=1}^{\infty}\frac{A_n^2}{2}}$$

$$= \sqrt{a_v^2 + \sum_{n=1}^{\infty}\left(\frac{A_n}{\sqrt{2}}\right)^2}.$$

For example, let's assume that a periodic voltage is represented by the finite series

$$v = 10 + 30\cos(\omega_0 t - \theta_1) + 20\cos(2\omega_0 t - \theta_2) + 5\cos(3\omega_0 t - \theta_3) + 2\cos(5\omega_0 t - \theta_5)$$

The rms value of this voltage is

$$V = \sqrt{10^2 + (30/\sqrt{2})^2 + (20/\sqrt{2})^2 + (5/\sqrt{2})^2 + (2/\sqrt{2})^2} = \sqrt{764.5} = 27.65\ \text{V}.$$

The rms value of a periodic function

Example: Use Eq. to estimate the rms value of the voltage in previous example

$$F_{rms} = \sqrt{a_v^2 + \sum_{n=1}^{\infty} \left(\frac{A_n}{\sqrt{2}}\right)^2}$$

Sol.: From previous example

$$v(t) = 15 + 27.01\cos(400\pi t - 45^\circ) + 19.10\cos(800\pi t - 90^\circ)$$
$$+ 9.00\cos(1200\pi t - 135^\circ) + 5.40\cos(2000\pi t - 45^\circ) + \cdots \text{V}.$$

$V_{dc} = 15$ V,

$V_1 = 27.01/\sqrt{2}$ V, the rms value of the fundamental,

$V_2 = 19.10/\sqrt{2}$ V, the rms value of the second harmonic,

$V_3 = 9.00/\sqrt{2}$ V, the rms value of the third harmonic,

$V_5 = 5.40/\sqrt{2}$ V, the rms value of the fifth harmonic.

$$V_{rms} = \sqrt{15^2 + \left(\frac{27.01}{\sqrt{2}}\right)^2 + \left(\frac{19.10}{\sqrt{2}}\right)^2 + \left(\frac{9.00}{\sqrt{2}}\right)^2 + \left(\frac{5.40}{\sqrt{2}}\right)^2} = 28.76 \text{ V}.$$

The exponential form of the Fourier series

The exponential form of the Fourier series allows us to express the series concisely.

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

We have: $\cos n\omega_0 t = \dfrac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$ $\quad \sin n\omega_0 t = \dfrac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2}$

$$f(t) = a_v + \sum_{n=1}^{\infty} \frac{a_n}{2}(e^{jn\omega_0 t} + e^{-jn\omega_0 t}) + \frac{b_n}{2j}(e^{jn\omega_0 t} - e^{-jn\omega_0 t})$$

$$= a_v + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_0 t}.$$

Define C_n as $\quad C_n = \frac{1}{2}(a_n - jb_n) = \frac{A_n}{2}\angle -\theta_n, \quad n = 1, 2, 3, \cdots$

$$C_n = \frac{1}{2}\left[\frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\, dt - j\frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\, dt\right]$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)\, dt = \frac{1}{T}\int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t}\, dt,$$

The exponential form of the Fourier series

$$\Rightarrow \quad C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt = a_v \qquad C_{-n} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{jn\omega_0 t}\,dt = C_n^* = \frac{1}{2}(a_n + jb_n).$$

Substituting
$$f(t) = C_0 + \sum_{n=1}^{\infty}(C_n e^{jn\omega_0 t} + C_n^* e^{-jn\omega_0 t}) = \sum_{n=0}^{\infty} C_n e^{jn\omega_0 t} + \underbrace{\sum_{n=1}^{\infty} C_n^* e^{-jn\omega_0 t}}.$$

$$\sum_{n=1}^{\infty} C_n^* e^{-jn\omega_0 t} = \sum_{n=-1}^{-\infty} C_n e^{jn\omega_0 t}.$$

$$f(t) = \sum_{n=0}^{\infty} C_n e^{jn\omega_0 t} + \sum_{-\infty}^{-1} C_n e^{jn\omega_0 t} = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t},$$

We may also express the rms value of a periodic function in terms of the complex Fourier coefficients.

$$F_{\mathrm{rms}} = \sqrt{a_v^2 + \sum_{n=1}^{\infty}\frac{a_n^2 + b_n^2}{2}},$$

$$|C_n| = \frac{\sqrt{a_n^2 + b_n^2}}{2},$$

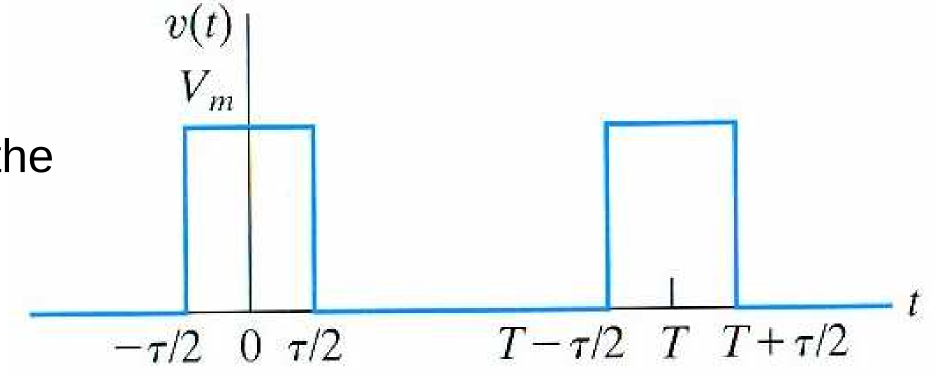
$$C_0^2 = a_v^2.$$

$$\Rightarrow \quad F_{\mathrm{rms}} = \sqrt{C_0^2 + 2\sum_{n=1}^{\infty}|C_n|^2}.$$

The exponential form of the Fourier series

Example

Find the exponential Fourier series for the periodic voltage shown in figure below



Sol.: Using - τ/2 as the starting point for the integration, we have:

$$C_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} V_m e^{-jn\omega_0 t}\,dt = \frac{V_m}{T}\left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right)\Bigg|_{-\tau/2}^{\tau/2}$$

$$= \frac{jV_m}{n\omega_0 T}\left(e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}\right)$$

$$= \frac{2V_m}{n\omega_0 T}\sin n\omega_0\tau/2.$$

The exponential form of the Fourier series

Sol.:

Here, because *v(t) has even symmetry,* $b_n = 0$ for all *n, and hence we expect* C_n *to be real.* Moreover, the amplitude of C_n *follows a (sinx)/x* distribution, as indicated when we rewrite

$$C_n = \frac{V_m \tau}{T} \frac{\sin(n\omega_0 \tau/2)}{n\omega_0 \tau/2}.$$

The exponential series representation of *v(t) is*

$$v(t) = \sum_{n=-\infty}^{\infty} \left(\frac{V_m \tau}{T}\right) \frac{\sin(n\omega_0 \tau/2)}{n\omega_0 \tau/2} e^{jn\omega_0 t}$$

$$= \left(\frac{V_m \tau}{T}\right) \sum_{n=-\infty}^{\infty} \frac{\sin(n\omega_0 \tau/2)}{n\omega_0 \tau/2} e^{jn\omega_0 t}.$$

Amplitude and phase spectra

A periodic time function is defined by its Fourier coefficients and its period.

Since we know $\boldsymbol{a_v}$, $\boldsymbol{a_n}$, $\boldsymbol{b_n}$, and $\boldsymbol{T}$, we can construct f(t).
We also know the amplitude ($\boldsymbol{A_n}$) and phase angle ($-\theta_n$) of each harmonic.

Again, we cannot, in general, visualize what the periodic function looks like in the time domain from a description of the coefficients and phase angles; nevertheless, we recognize that these quantities characterize the periodic function completely

<u>Thus</u>, with sufficient computing time, we can synthesize <u>the time-domain waveform</u> from the <u>amplitude</u> and <u>phase angle</u> data.

Also, when a periodic driving function is exciting a circuit that is <u>highly frequency selective</u>, the Fourier series of the steady-state response is dominated by <u>just a few terms</u>.
Thus the description of the response in terms of <u>amplitude</u> and <u>phase</u> may provide an understanding of the output waveform.

Amplitude and phase spectra

We can present graphically the description of a periodic function in terms of the amplitude and phase angle of each term in the Fourier series of *f(t).*

The plot of the <u>amplitude of each term versus the frequency</u> is called the **amplitude spectrum** of *f(t),* and the plot of the <u>phase angle versus the frequency</u> is called the **phase spectrum** of f(t).

Because the amplitude and phase angle data occur at discrete values of the frequency (that is, at $\omega_0, 2\omega_0, 3\omega_0$ . . .) , these plots also are referred to as **line spectra.**

Amplitude and phase spectra

Amplitude and phase spectra plots are based on either Eq.

$$f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)$$

or $$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

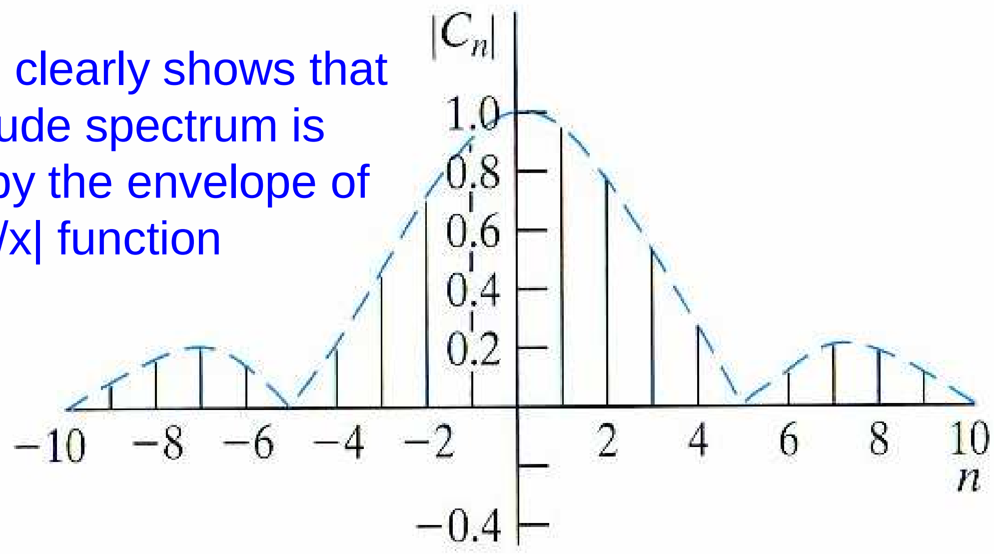
To illustrate the amplitude and phase spectra

$$C_n = \frac{V_m \tau}{T} \frac{\sin(n\omega_0 \tau/2)}{n\omega_0 \tau/2},$$

Assume that V_m = - 5 V and τ = T/5.
And using the previous example

$$C_n = 1\frac{\sin(n\pi/5)}{n\pi/5}$$

The figure clearly shows that the amplitude spectrum is bounded by the envelope of the |(sinx)/x| function



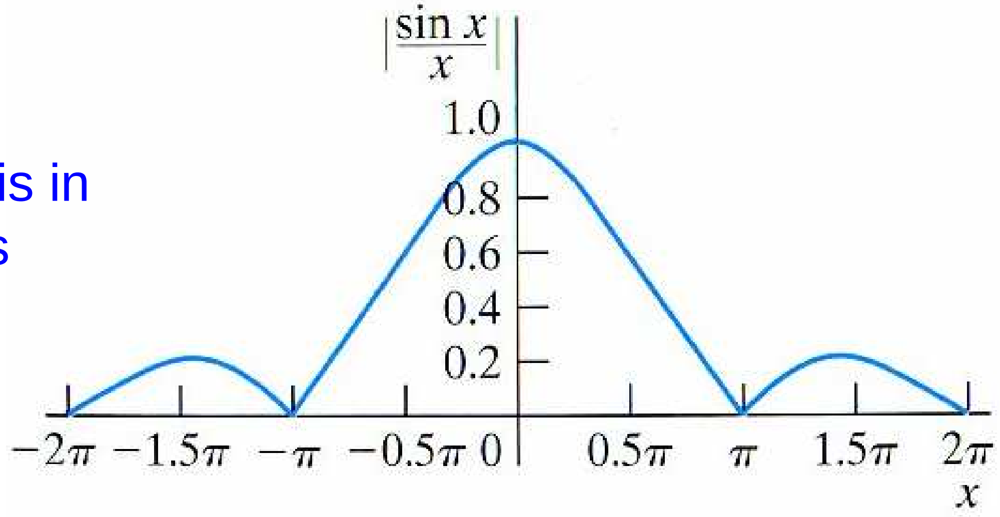
Command Window

To get started, select MATLAB Help or Demos from the Help menu.

```
>> x=-10:.01:10;
>> plot(x,abs(sin(x*pi/5)./(x*pi/5)))
>>
```

Amplitude and phase spectra

The plot of |(sinx)/x| versus *x*, where *x* is in radians. It shows that the function goes through zero whenever *x* is an integral multiple of π.



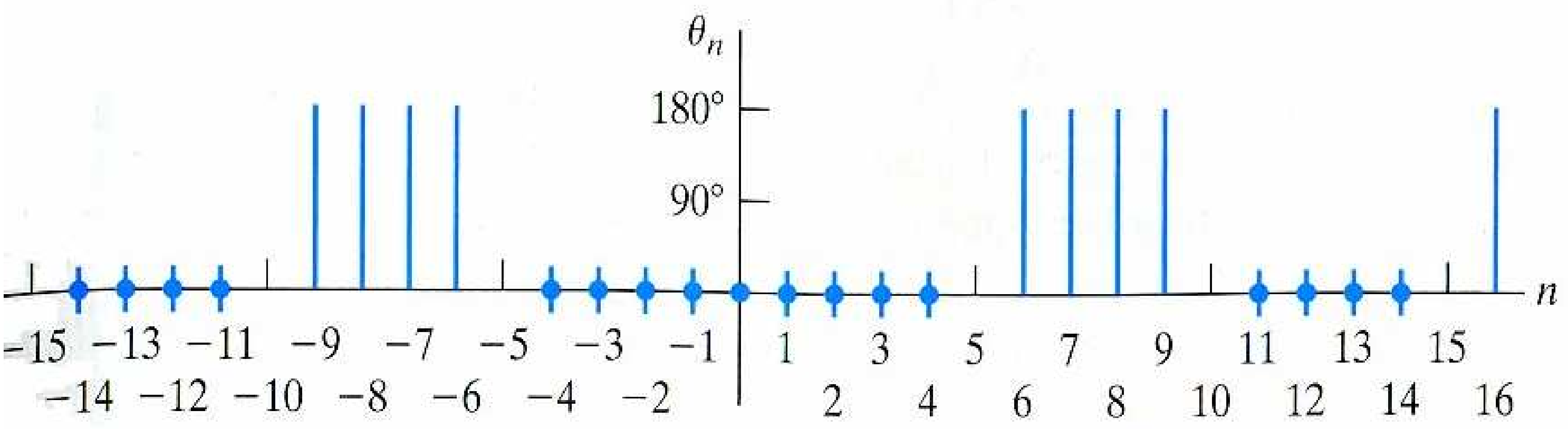
with $n\omega_0\left(\frac{\tau}{2}\right) = \frac{n\pi\,\tau}{T} = \frac{n\pi}{T/\tau}.$

➔ amplitude spectrum goes through zero whenever nτ/T is an integer

Amplitude and phase spectra

Because C_n is real for all *n,* the phase angle associated with C_n is either zero or 180°, depending on the algebraic sign of **$(\sin n\pi/5)/(n\pi/5)$**.

For example, the phase angle is zero for $n = 0, \pm1, \pm2, \pm3$, and ±4. It is not defined at $n = \pm5$, because $C_{\pm5}$ is zero.
The phase angle is 180°@ $n = \pm6, \pm7, \pm8$, and ±9, and it is not defined at ±10.
This pattern repeats itself as *n* takes on larger integer values.



The phase angle of C_n.

Amplitude and phase spectra

What happens to the amplitude and phase spectra if *f(t)* is shifted along the time axis?

Shift the periodic voltage t_0 units to the right   from $v(t)=\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

$$\Rightarrow\ v(t-t_0)=\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 (t-t_0)}=\sum_{n=-\infty}^{\infty} C_n e^{-jn\omega_0 t_0} e^{jn\omega_0 t}$$

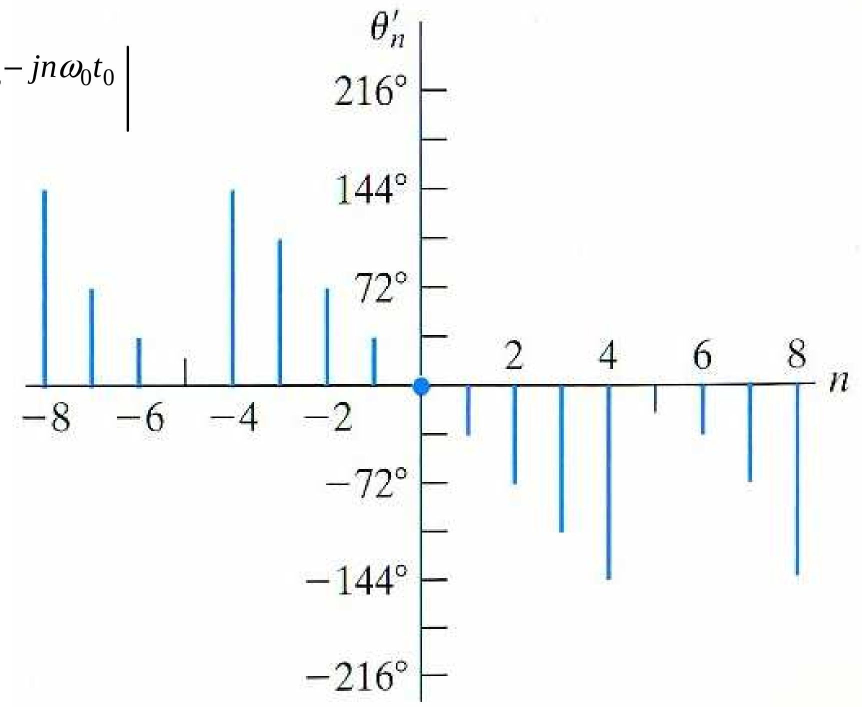
→ shifting the origin has no effect on the amplitude spectrum, because: $\left|C_n\right|=\left|C_n e^{-jn\omega_0 t_0}\right|$

with $C_n=\frac{1}{2}(a_n-jb_n)=\frac{A_n}{2}\angle-\theta_n\ ,\quad n=1,2,3,...$

the phase spectrum has changed to -(θ_n + $n\omega_0 t_0$) rads.

For example, let's shift the periodic voltage in previous Ex. τ/2 units to the right, assume τ = T/5

new phase angle $\theta_n^{'}=-(\theta_n+n\pi/5)$



The plot of θ'$_n$ versus n