

Lecture notes: Differential Equations for ISE (MA029IU)

Week 1 *

February 9, 2022

1 Introduction to differential equations

1.1 Differential equations

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. As an analogy, suppose all your classes from now on were given in Swahili. It would be important to first learn Swahili, or you would have a very tough time getting a good grade in your classes.

You saw many differential equations already without perhaps knowing about it. And you even solved simple differential equations when you took calculus. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2 \cos t. \quad (1)$$

Here x is the *dependent variable* and t is the *independent variable*. Equation (1) is a basic example of a *differential equation*. It is an example of a *first order differential equation*, since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

1.2 Solutions of differential equations

Solving the differential equation means finding x in terms of t . That is, we want to find a function of t , which we call x , such that when we plug x , t , and $\frac{dx}{dt}$ into (1), the equation holds; that is, the left hand side equals the right hand side. It is the same idea as it would be for a normal (algebraic) equation of just x and t . We claim that

$$x = x(t) = \cos t + \sin t$$

is a *solution*. How do we check? We simply plug x into equation (1)! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left-hand side of (1).

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t)}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t)}_x = 2 \cos t.$$

Yay! We got precisely the right-hand side. But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

*This note is copied from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl

We plug into the left-hand side of (1)

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t - e^{-t})}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t + e^{-t})}_x = 2 \cos t.$$

And it works yet again!

So there can be many different solutions. For this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t},$$

for some constant C . Different constants C will give different solutions, so there are really infinitely many possible solutions. See Figure 1 for the graph of a few of these solutions. We will see how we find these solutions a few lectures from now.

Solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions. And we will spend some time on understanding the equations without solving them.

Most of this book is dedicated to *ordinary differential equations* or ODEs, that is, equations with only one independent variable, where derivatives are only with respect to this one variable. If there are several independent variables, we get *partial differential equations* or PDEs.

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. When you can find exact solutions, they are usually preferable to approximate solutions. It is important to understand how such solutions are found. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may even need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you never saw before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

1.3 Differential equations in practice

So how do we use differential equations in science and engineering? First, we have some *real-world problem* we wish to understand. We make some simplifying assumptions and create a *mathematical model*. That is, we translate the real-world situation into a set of differential equations. Then we apply mathematics to get some sort of a *mathematical solution*. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real-world problem we started with.

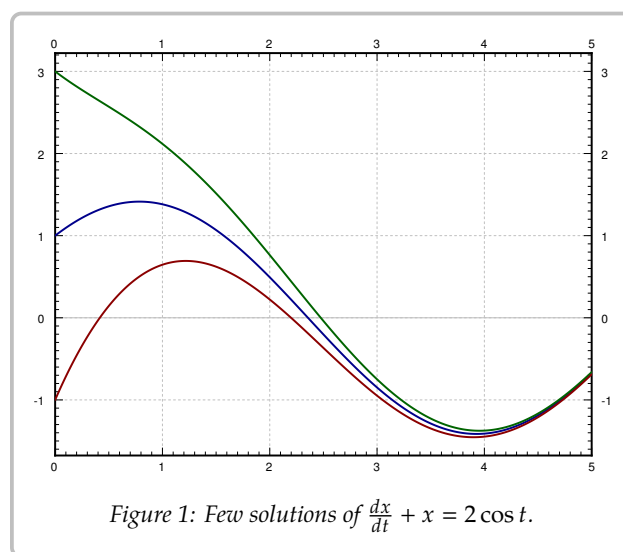
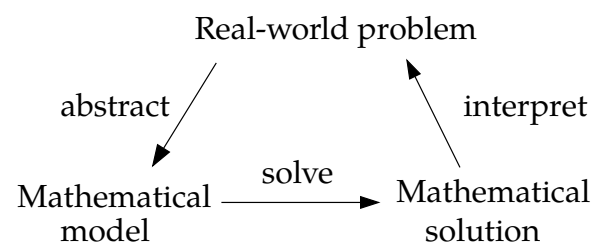


Figure 1: Few solutions of $\frac{dx}{dt} + x = 2 \cos t$.

Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course, we will focus mostly on the mathematical analysis. Sometimes we will work with simple real-world examples so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process. One of the most basic differential equations is the standard *exponential growth model*. Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive constant $k > 0$.

Example 1.1: Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

First we need to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where C is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, now what? We do not know C , and we do not know k . But we know something. We know $P(0) = 100$, and we know $P(10) = 200$. Let us plug these conditions in and see what happens.

$$100 = P(0) = Ce^{k \cdot 0} = C,$$

$$200 = P(10) = 100e^{k \cdot 10}.$$

Therefore, $2 = e^{10k}$ or $\frac{\ln 2}{10} = k \approx 0.069$. So

$$P(t) = 100e^{(\ln 2)t/10} \approx 100e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See Figure 2.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds, $P(61) \approx 6859.35$.

Normally, the k in $P' = kP$ is known, and we want to solve the equation for different *initial conditions*. What does that mean? Take $k = 1$ for simplicity. Suppose we want to solve the equation $\frac{dP}{dt} = P$ subject to $P(0) = 1000$ (the initial condition). Then the solution turns out to be (exercise)

$$P(t) = 1000e^t.$$

We call $P(t) = Ce^t$ the *general solution*, as every solution of the equation can be written in this form for some constant C . We need an initial condition to find out what C is, in order to find the *particular solution* we are looking for. Generally, when we say “particular solution,” we just mean some solution.

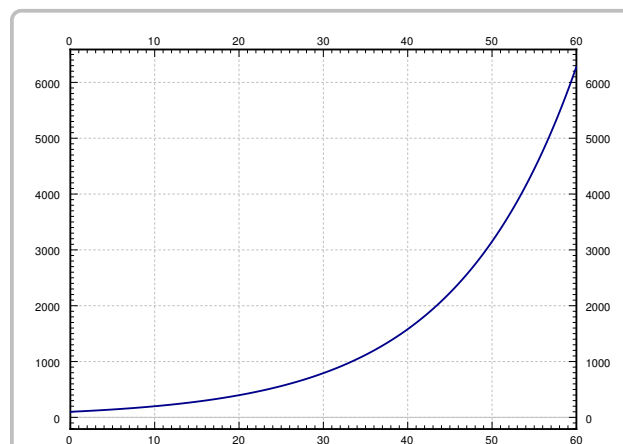


Figure 2: Bacteria growth in the first 60 seconds.

1.4 Four fundamental equations

A few equations appear often and it is useful to just memorize what their solutions are. Let us call them the four fundamental equations. Their solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. No need to wonder if you remembered the solution correctly.

First such equation is

$$\frac{dy}{dx} = ky,$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We saw above that this function is a solution, although we used different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

Exercise 1.1: Check that the y given is really a solution to the equation.

Next, take the second order differential equation

$$\frac{d^2y}{dx^2} = -k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Since the equation is a second order differential equation, we have two constants in our general solution.

Exercise 1.2: Check that the y given is really a solution to the equation.

Finally, consider the second order differential equation

$$\frac{d^2y}{dx^2} = k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 e^{kx} + C_2 e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know, \cosh and \sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

They are called the *hyperbolic cosine* and *hyperbolic sine*. These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as $\cosh 0 = 1$, $\sinh 0 = 0$, and $\frac{d}{dx} \cosh x = \sinh x$ (no that is not a typo) and $\frac{d}{dx} \sinh x = \cosh x$.

Exercise 1.3: Check that both forms of the y given are really solutions to the equation.

Example 1.2: In equations of higher order, you get more constants you must solve for to get a particular solution. The equation $\frac{d^2y}{dx^2} = 0$ has the general solution $y = C_1x + C_2$; simply integrate twice and don't forget about the constant of integration. Consider the initial conditions $y(0) = 2$ and $y'(0) = 3$. We plug in our general solution and solve for the constants:

$$2 = y(0) = C_1 \cdot 0 + C_2 = C_2, \quad 3 = y'(0) = C_1.$$

In other words, $y = 3x + 2$ is the particular solution we seek.

An interesting note about cosh: The graph of cosh is the exact shape of a hanging chain. This shape is called a *catenary*. Contrary to popular belief this is not a parabola. If you invert the graph of cosh, it is also the ideal arch for supporting its weight. For example, the gateway arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall. The formula used in the design is inscribed inside the arch:

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

1.5 Exercises

Exercise 1.4: Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise 1.5: Show that $x = e^t$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise 1.6: Is $y = \sin t$ a solution to $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$? Justify.

Exercise 1.7: Let $y'' + 2y' - 8y = 0$. Now try a solution of the form $y = e^{rx}$ for some (unknown) constant r . Is this a solution for some r ? If so, find all such r .

Exercise 1.8: Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

Exercise 1.9: Verify that $x = C_1e^{-t} + C_2e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.

Exercise 1.10: Find a solution to $(x')^2 + x^2 = 4$ using your knowledge of derivatives of functions that you know from basic calculus.

Exercise 1.11: Solve:

a) $\frac{dA}{dt} = -10A, \quad A(0) = 5$

b) $\frac{dH}{dx} = 3H, \quad H(0) = 1$

c) $\frac{d^2y}{dx^2} = 4y, \quad y(0) = 0, \quad y'(0) = 1$

d) $\frac{d^2x}{dy^2} = -9x, \quad x(0) = 1, \quad x'(0) = 0$

Exercise 1.12: Is there a solution to $y' = y$, such that $y(0) = y(1)$?

Exercise 1.13: The population of city X was 100 thousand 20 years ago, and the population of city X was 120 thousand 10 years ago. Assuming constant growth, you can use the exponential population model (like for the bacteria). What do you estimate the population is now?

Exercise 1.14: Suppose that a football coach gets a salary of one million dollars now, and a raise of 10% every year (so exponential model, like population of bacteria). Let s be the salary in millions of dollars, and t is time in years.

a) What is $s(0)$ and $s(1)$.

b) Approximately how many years will it take for the salary to be 10 million.

c) Approximately how many years will it take for the salary to be 20 million.

d) Approximately how many years will it take for the salary to be 30 million.

Note: Exercises with numbers 101 and higher have solutions in the back of the book.

Exercise 1.101: Show that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$.

Exercise 1.102: Is $y = x^2$ a solution to $x^2 y'' - 2y = 0$? Justify.

Exercise 1.103: Let $x y'' - y' = 0$. Try a solution of the form $y = x^r$. Is this a solution for some r ? If so, find all such r .

Exercise 1.104: Verify that $x = C_1 e^t + C_2$ is a solution to $x'' - x' = 0$. Find C_1 and C_2 so that x satisfies $x(0) = 10$ and $x'(0) = 100$.

Exercise 1.105: Solve $\frac{d\varphi}{ds} = 8\varphi$ and $\varphi(0) = -9$.

Exercise 1.106: Solve:

a) $\frac{dx}{dt} = -4x, \quad x(0) = 9$

b) $\frac{d^2x}{dt^2} = -4x, \quad x(0) = 1, \quad x'(0) = 2$

c) $\frac{dp}{dq} = 3p, \quad p(0) = 4$

d) $\frac{d^2T}{dx^2} = 4T, \quad T(0) = 0, \quad T'(0) = 6$

2 Classification of differential equations

There are many types of differential equations, and we classify them into different categories based on their properties. Let us quickly go over the most basic classification. We already saw the distinction between ordinary and partial differential equations:

- *Ordinary differential equations* or (ODE) are equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.
- *Partial differential equations* or (PDE) are equations that depend on partial derivatives of several variables. That is, there are several independent variables.

Let us see some examples of ordinary differential equations:

$$\begin{aligned}\frac{dy}{dt} &= ky, & (\text{Exponential growth}) \\ \frac{dy}{dt} &= k(A - y), & (\text{Newton's law of cooling}) \\ m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx &= f(t). & (\text{Mechanical vibrations})\end{aligned}$$

And of partial differential equations:

$$\begin{aligned}\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} &= 0, & (\text{Transport equation}) \\ \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & (\text{Heat equation}) \\ \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. & (\text{Wave equation in 2 dimensions})\end{aligned}$$

If there are several equations working together, we have a so-called *system of differential equations*. For example,

$$y' = x, \quad x' = y$$

is a simple system of ordinary differential equations. Maxwell's equations for electromagnetics,

$$\begin{aligned}\nabla \cdot \vec{D} &= \rho, & \nabla \cdot \vec{B} &= 0, \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, & \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t},\end{aligned}$$

are a system of partial differential equations. The divergence operator $\nabla \cdot$ and the curl operator $\nabla \times$ can be written out in partial derivatives of the functions involved in the x , y , and z variables.

The next bit of information is the *order* of the equation (or system). The order is simply the order of the largest derivative that appears. If the highest derivative that appears is the first derivative, the equation is of first order. If the highest derivative that appears is the second derivative, then the equation is of second order. For example, Newton's law of cooling above is a first order equation, while the mechanical vibrations equation is a second order equation. The equation governing transversal vibrations in a beam,

$$a^4 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0,$$

is a fourth order partial differential equation. It is fourth order as at least one derivative is the fourth derivative. It does not matter that the derivative in t is only of second order.

In the first chapter, we will start attacking first order ordinary differential equations, that is, equations of the form $\frac{dy}{dx} = f(x, y)$. In general, lower order equations are easier to work with and have simpler behavior, which is why we start with them.

We also distinguish how the dependent variables appear in the equation (or system). In particular, we say an equation is *linear* if the dependent variable (or variables) and their derivatives appear linearly, that is only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. In other words, the equation is a sum of terms, where each term is some function of the independent variables or some function of the independent variables multiplied by a dependent variable or its derivative. Otherwise, the equation is called *nonlinear*. For example, an ordinary differential equation is linear if it can be put into the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = b(x). \quad (2)$$

The functions a_0, a_1, \dots, a_n are called the *coefficients*. The equation is allowed to depend arbitrarily on the independent variable. So

$$e^x \frac{d^2 y}{dx^2} + \sin(x) \frac{dy}{dx} + x^2 y = \frac{1}{x} \quad (3)$$

is still a linear equation as y and its derivatives only appear linearly.

All the equations and systems above as examples are linear. It may not be immediately obvious for Maxwell's equations unless you write out the divergence and curl in terms of partial derivatives. Let us see some nonlinear equations. For example Burger's equation,

$$\frac{\partial y}{\partial t} + y \frac{\partial y}{\partial x} = \nu \frac{\partial^2 y}{\partial x^2},$$

is a nonlinear second order partial differential equation. It is nonlinear because y and $\frac{\partial y}{\partial x}$ are multiplied together. The equation

$$\frac{dx}{dt} = x^2 \quad (4)$$

is a nonlinear first order differential equation as there is a second power of the dependent variable x .

A linear equation may further be called *homogeneous* if all terms depend on the dependent variable. That is, if no term is a function of the independent variables alone. Otherwise, the equation is called *nonhomogeneous* or *inhomogeneous*. For example, the exponential growth equation, the wave equation, or the transport equation above are homogeneous. The mechanical vibrations equation above is nonhomogeneous as long as $f(t)$ is not the zero function. Similarly, if the ambient temperature A is nonzero, Newton's law of cooling is nonhomogeneous. A homogeneous linear ODE can be put into the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0.$$

Compare to (2) and notice there is no function $b(x)$.

If the coefficients of a linear equation are actually constant functions, then the equation is said to have *constant coefficients*. The coefficients are the functions multiplying the dependent variable(s) or one of its derivatives, not the function $b(x)$ standing alone. A constant coefficient nonhomogeneous ODE is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = b(x),$$

where a_0, a_1, \dots, a_n are all constants, but b may depend on the independent variable x . The mechanical vibrations equation above is a constant coefficient nonhomogeneous second order ODE. The same nomenclature applies to PDEs, so the transport equation, heat equation and wave equation are all examples of constant coefficient linear PDEs.

Finally, an equation (or system) is called *autonomous* if the equation does not depend on the independent variable. For autonomous ordinary differential equations, the independent variable is then thought of as time. Autonomous equation means an equation that does not change with time. For example, Newton's law of cooling is autonomous, so is equation (4). On the other hand, mechanical vibrations or (3) are not autonomous.

2.1 Exercises

Exercise 2.1: Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a) $\sin(t) \frac{d^2x}{dt^2} + \cos(t)x = t^2$

b) $\frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial y} = xy$

c) $y'' + 3y + 5x = 0, \quad x'' + x - y = 0$

d) $\frac{\partial^2 u}{\partial t^2} + u \frac{\partial^2 u}{\partial s^2} = 0$

e) $x'' + tx^2 = t$

f) $\frac{d^4x}{dt^4} = 0$

Exercise 2.2: If $\vec{u} = (u_1, u_2, u_3)$ is a vector, we have the divergence $\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$ and curl $\nabla \times \vec{u} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right)$. Notice that curl of a vector is still a vector. Write out Maxwell's equations in terms of partial derivatives and classify the system.

Exercise 2.3: Suppose F is a linear function, that is, $F(x, y) = ax + by$ for constants a and b . What is the classification of equations of the form $F(y', y) = 0$.

Exercise 2.4: Write down an explicit example of a third order, linear, nonconstant coefficient, nonautonomous, nonhomogeneous system of two ODE such that every derivative that could appear, does appear.

Exercise 2.101: Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a) $\frac{\partial^2 v}{\partial x^2} + 3 \frac{\partial^2 v}{\partial y^2} = \sin(x)$

b) $\frac{dx}{dt} + \cos(t)x = t^2 + t + 1$

c) $\frac{d^7 F}{dx^7} = 3F(x)$

d) $y'' + 8y' = 1$

e) $x'' + txy' = 0, \quad y'' + txy = 0$

f) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2} + u^2$

Exercise 2.102: Write down the general zeroth order linear ordinary differential equation. Write down the general solution.

Exercise 2.103: For which k is $\frac{dx}{dt} + x^k = t^{k+2}$ linear. Hint: there are two answers.