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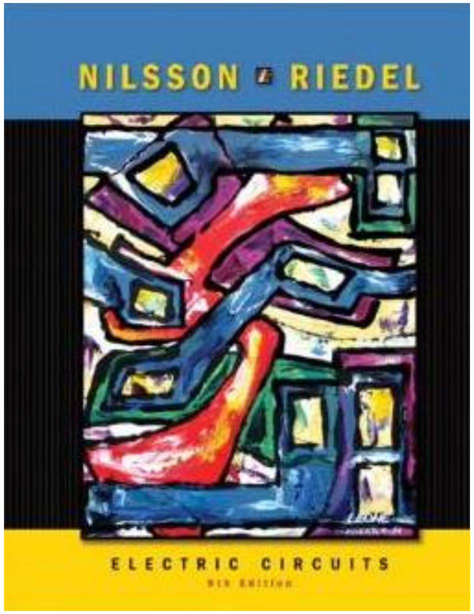
PRINCIPLES OF ELECTRICAL ENGINEERING 2

Lecture # 3 & 4: Introduction to the Laplace Transform

Chapter #12



Text book: Electric Circuits
James W. Nilsson & Susan A. Riedel
9th Edition.

link: **http://blackboard.hcmiu.edu.vn/**

to download materials

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Objectives

- Be able to calculate the Laplace transform of a function.
- Be able to calculate the inverse Laplace transform.
- Understanding and know how to use the initial value theorem and the final value theorem.

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Outline

- Definition of the Laplace transform
- The step function
- The impulse function
- Functional transforms
- Operational transforms
- Inverse transforms
- Poles and Zeros of F(s)
- Initial- and final-value theorems

Pierre-Simon Laplace
(1749–1827)

The Laplace transform is named in honor of mathematician & astronomer Pierre-Simon Laplace, who used the transform in his work on probability theory. The transform itself did not become popular until Oliver Heaviside, a famous electrical engineer, began using a variation of it to solve electrical circuits.

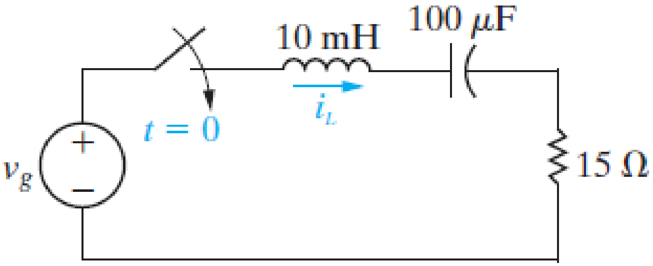
Laplace transform

In EE1, power delivered from electrical wall outlets can be modeled as a sinusoidal voltage or current source. The phasor concepts introduced in EE1 allowed us to analyze the steady-state response of a circuit to a sinusoidal source.

Response of a circuit to a sinusoidal source has: the steady-state response & the transient response.
The steady-state response is also a sinusoid whose magnitude & phase angle can be calculated using phasor circuit analysis.
The transient response depends on the components, the values of those components, and the way the components are interconnected.

The Laplace transform techniques introduced in this lecture can be used to find the complete response of a circuit to a sinusoidal source.

Exceed voltage / current rating of the circuit component?



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Laplace transform

- The Laplace transform provides a useful method of solving certain types of differential equations when certain initial conditions are given, especially when the initial values are zero.
- This is powerful analytical technique used to study the behavior of linear, lumped-parameter circuits.
- It is often easier to analyze the circuit in its Laplace form, than to form differential equations.

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Definition of the Laplace transform

What the video clip "Laplace Transform Explained and Visualized Intuitively"
https://www.youtube.com/watch?v=6MXMDrs6ZmA

The Laplace transform is a tool for converting **time-domain** equations into **frequency-domain** equations, for $t > 0$ is defined by the following integral defined over 0 to ∞

L{f(t)} is read "the Laplace transform of f(t)"

$$L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

The resulting expression is a function of *s*, which we write as F(*s*). In words we say "The Laplace Transform of *f*(*t*) equals function F of *s*" and write: L{*f*(*t*)} = F(*s*)

Similarly, the Laplace transform of a function *g*(*t*) would be written:
L{*g*(*t*)} = G(s)

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Laplace transform

- Create a new domain to make mathematical manipulations easier. After finding the unknown in the new domain, we inverse-transform it back to the original domain.
- In circuit analysis, Laplace transform is used to transform a set of differential equations from the time domain to a set of algebraic equations in the frequency domain ➔ simplify the solution.
- Some sources may not have Laplace transform.
- For *F(s)* is determined by the behavior of *f(t)* only for positive values of *t*, which is referred to as the one-sided, or unilateral Laplace transform. **F(s) is understood to be the one-side transform!**

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Laplace transform

- There are **two types** of Laplace transform:
  - ***Functional transform*** is the Laplace transform of a specific function such as $\sin(\omega t)$; t; e^{-at},…
  - ***Operational transform*** defines a general mathematical property of the Laplace transform

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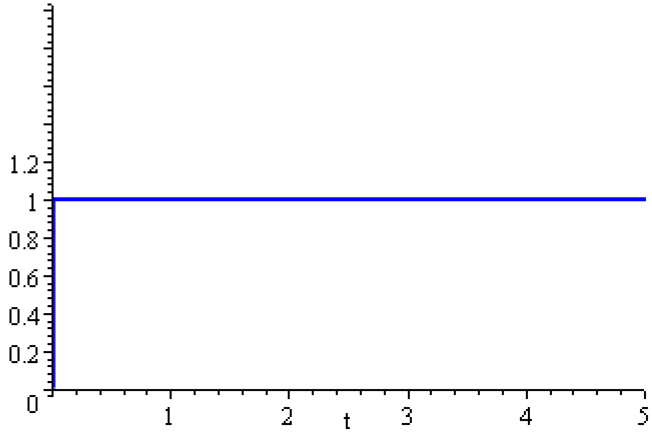
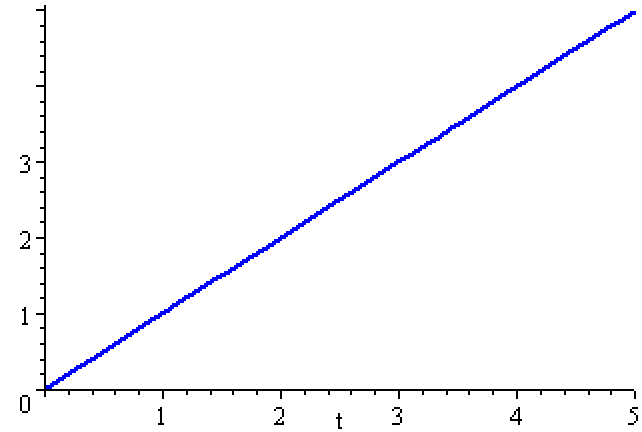
Laplace transform

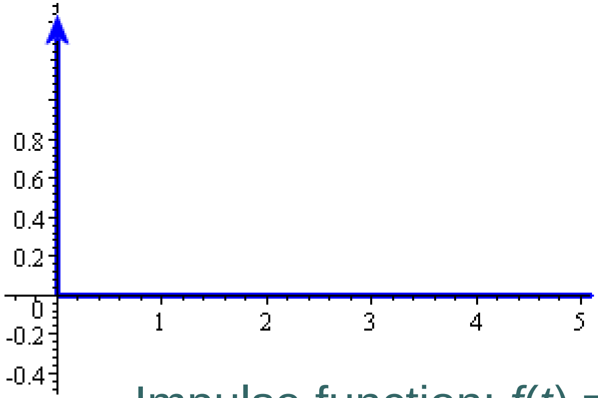
Reminder: Unit, Ramp and Impulse Functions



Unit step function: $f(t) = u(t)$



Ramp function: $f(t) = (t)$



Impulse function: $f(t) = \delta(t)$

$\delta(t)$ represents an impulse at $t = 0$ and has value 0 otherwise.

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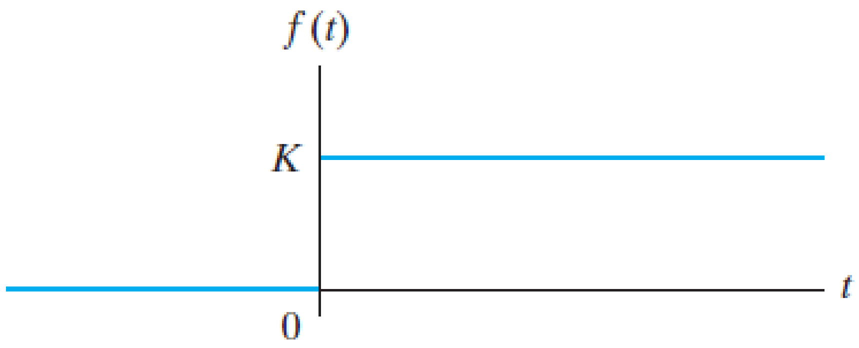
The Step Function: Ku(t)



The step function *Ku(t)* describes a function that experiences a discontinuity from one constant level to another at some point in time.

$Ku(t) = 0 \qquad , t < 0$

$Ku(t) = K \qquad , t > 0$

K is the magnitude of the jump.
If *K* = *1*, *Ku(t)* is the unit step function.

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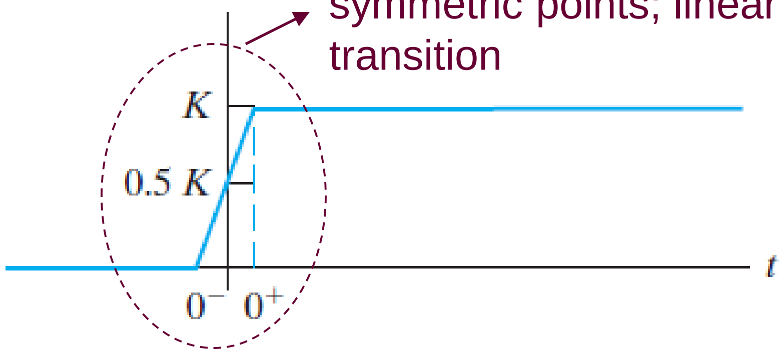
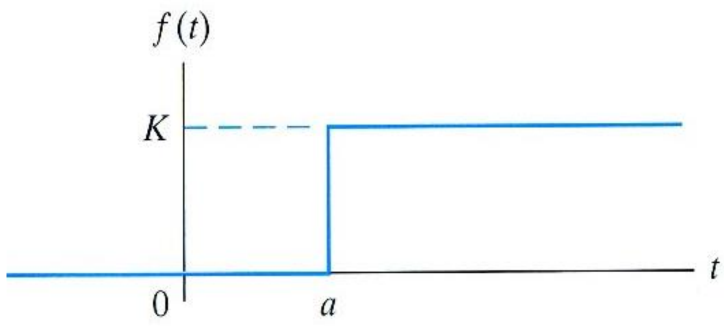
The Step Function: Ku(t)

0^- and 0^+ represent symmetric points; linear transition



When the step function is not defined at t = 0, it is necessary to define the transition between 0^- and 0^+ ,

ex.: ***Ku(0) = 0.5K***



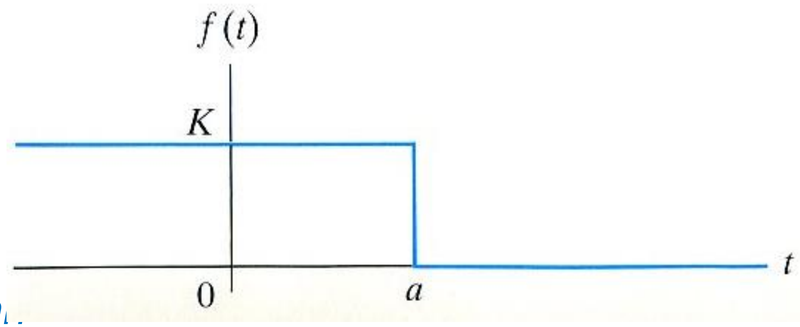
When discontinuity occur @ t ≠ 0, it is expressed:

Ku(t – a) = 0 , t < a

Ku(t – a) = K , t > a

Note: step function = 0 when t-a < 0

step function = K when t-a > 0



Ku(a – t) = K , t < a

Ku(a – t) = 0 , t > a

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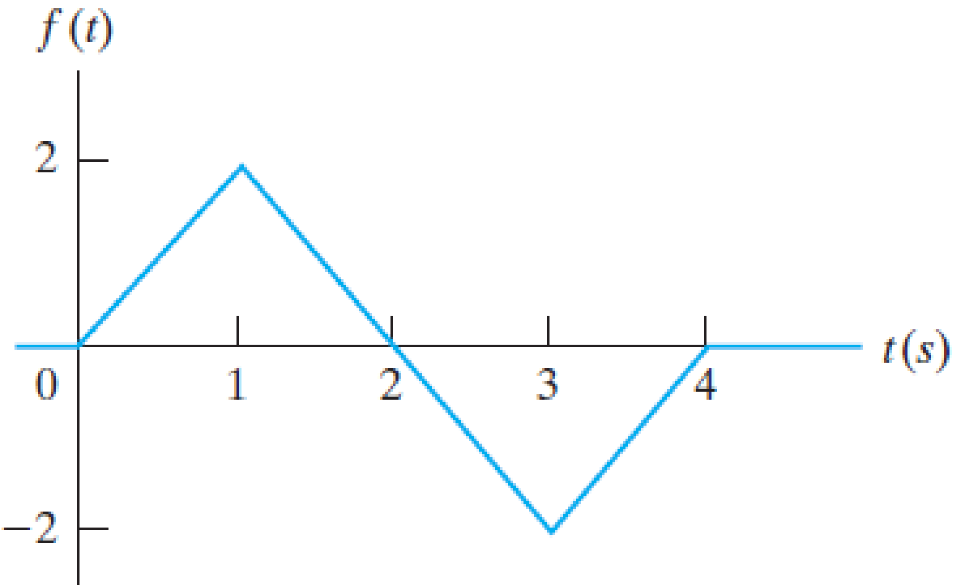
The Step Function: Ku(t)

- Example: Use step functions to write an expression for the function illustrated in Fig



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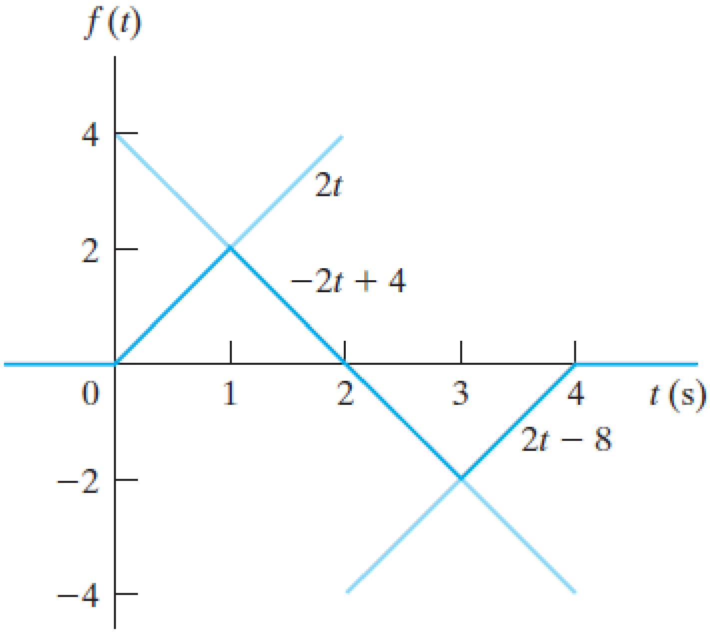
The Step Function: Ku(t)

- Solution

Linear segments have break points at 0, 1, 3, and 4 s

Use the step function to initiate & terminate these linear segments @ the proper times



Equations: (+2t) on @ t = 0, off @ = 1

(-2t + 4) on @ t = 1, off @ t = 3

(+2t – 8) on @ t = 3 , off @ t = 4

The expression for f(t) is

$f(t) = 2t[u(t) – u(t-1)] + (-2t + 4)[u(t-1) – u(t-3)] + (2t - 8)[u(t-3) – u(t-4)]$.

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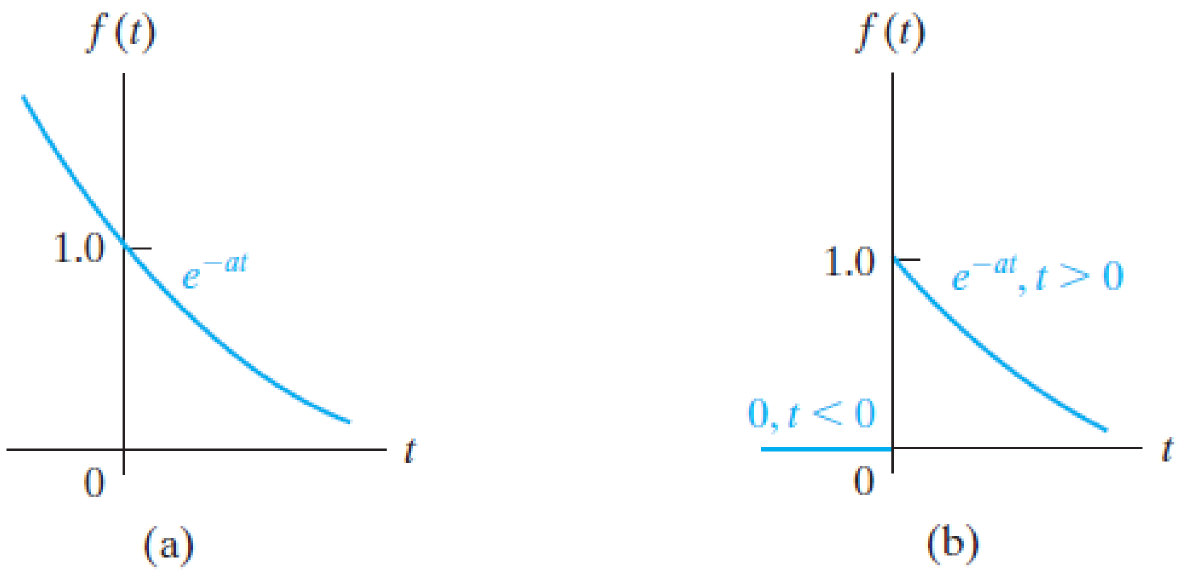
The Impulse Function

- **Continuous and discontinuous function**



f(t) is continuous @ the origin

f(t) is discontinuous @ the origin

- The concept of <u>*an impulse function*</u> enables us to define the derivative at a discontinuity, and thus to define the Laplace transform of that derivative.

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The Impulse Function

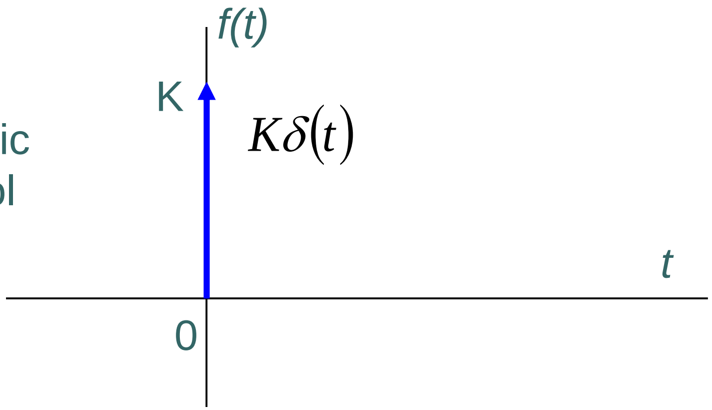
- **The impulse function Kδ(t) is defined:**

$$\int_{-\infty}^{\infty} K\delta(t)\,dt = K$$

$$\delta(t) = 0, \qquad t \neq 0$$

K is the strength of the impulse.
If ***K* = *1***, *Kδ(t)* is the **unit impulse function.**

Graphic symbol



Must has 3 characteristics:

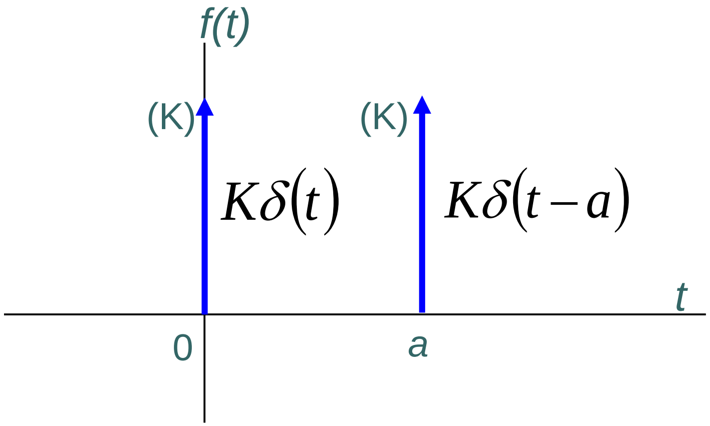
1. *An impulse is a signal of infinite amplitude and zero duration*
2. *The area under the impulse function is constant.*
3. *The impulse is zero everywhere except @ t = 0*

Impulsive voltages and currents occur in circuit analysis either because of a switching operation or because the circuit is excited by an impulsive source.

The Impulse Function

- **Sifting property:**

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)}$$



The impulse function sifts out everything except the value of f(t) at t = a

- Use the sifting property of the impulse function to find its Laplace transform:

$$L\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \int_{0^-}^{\infty} \delta(t)dt = 1$$

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Functional Transforms

- A functional transform is the Laplace transform of a specific function of t.

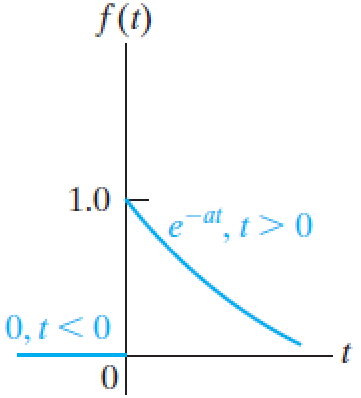
Example:

Laplace transform of the unit step function:

$$L\{u(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^+}^{\infty} 1e^{-st}dt$$
$$= \left.\frac{e^{-st}}{-s}\right|_{0^+}^{\infty} = \frac{1}{s}.$$

Laplace transform of the decaying exponential function:

$$L\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(a+s)t}\,dt = \frac{1}{s+a}.$$



A decaying exponential function.

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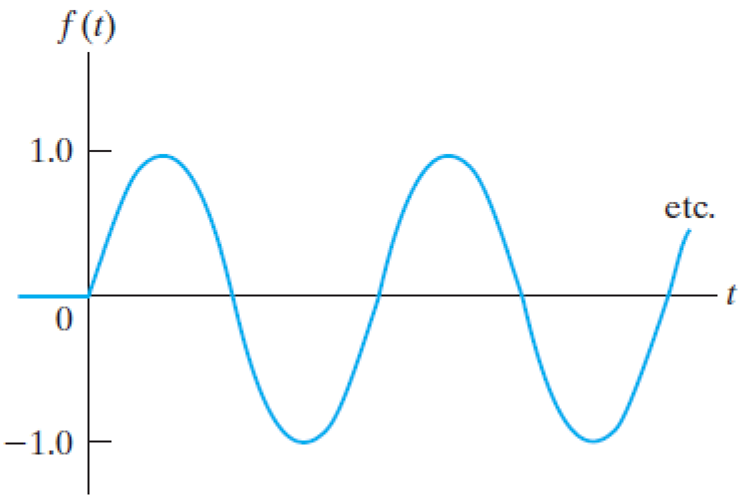
Functional Transforms

$$\mathcal{L}\{\sin(\omega t)\} = \int_{0^-}^{\infty} (\sin \omega t) e^{-st}\, dt = \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j}\, dt$$

$$= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right)$$

$$= \frac{\omega}{s^2 + \omega^2}.$$



A sinusoidal function for t > 0

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Important functional transform pairs

Type	*f(t)* $(t > 0^-)$	*F(s)*
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

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Operational Transforms

- Operational transforms define the general mathematical properties of the Laplace transform.
- The operations of primary interest include:

(1) Multiplication by a constant

*Multiplication of **f(t)** by a constant corresponds to multiplying **F(s)** by the same constant.*

$$L\{Kf(t)\} = KF(s)$$

(2) Addition (subtraction)

Addition (subtraction) in the time domain translates in to addition (subtraction) in the frequency domain.

$$L\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) - F_3(s)$$

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Operational Transforms

(3) Differentiation

*Differentiation in the time domain corresponds to multiplying **F(s)** by **s** and then subtracting the initial value of **f(t)**.*

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-),$$

*It can be seen that differentiation in the time domain reduces to an algebraic operation in the **s** domain.*

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}\,dt.$$

Letting u = e^{-st} & dv = [df(t)/dt]dt

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = e^{-st}f(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)(-se^{-st}dt).$$

The evaluation of $e^{-st}f(t)$ @ t = ∞ is 0

$$\Rightarrow -f(0^-) + s\int_{0^-}^{\infty} f(t)e^{-st}dt = sF(s) - f(0^-).$$

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Operational Transforms

(4) Integration

Integration in the time domain corresponds to dividing by s in the s domain.

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \frac{F(s)}{s},$$

Operation of integration in the time domain is transformed to the algebraic operation of multiplying by 1/s in the s domain.

Laplace transform translates a set of differential equations into a set of algebraic equations.

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)\,dx\right]e^{-st}\,dt$$

letting
$$\begin{cases} u = \int_{0^-}^{t} f(x)\,dx, \\ dv = e^{-st}\,dt. \end{cases} \Rightarrow \quad du = f(t)\,dt, \quad v = -\frac{e^{-st}}{s}.$$

integration-by-parts formula yields:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \underbrace{-\frac{e^{-st}}{s}\int_{0^-}^{t} f(x)\,dx\Big|_{0^-}^{\infty}}_{0} + \int_{0^-}^{\infty}\frac{e^{-st}}{s}f(t)\,dt = \frac{F(s)}{s}$$

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Operational Transforms

(5) Translation in the time domain (time shifting)

Translation in the time domain corresponds to multiplication by an exponential in the frequency domain.

$$L\{f(t - a)u(t - a)\} = e^{-as}F(s), \quad a > 0$$

(6) Translation in the frequency domain (frequency shifting)

Translation in the frequency domain corresponds to multiplication by an exponential in the time domain.

$$L\{e^{-at}f(t)\} = F(s + a)$$

(7) Scale changing

The scale-change property gives the relationship between f(t) and F(s) when the time variable is multiplied by a positive constant.

$$L\{f(at)\} = a^{-1}F(s/a), \quad a > 0$$

List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\dfrac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \dfrac{df(0^-)}{dt}$
nth derivative (time)	$\dfrac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dfrac{df(0^-)}{dt} - s^{n-3}\dfrac{df^2(0^-)}{dt^2} - \cdots - \dfrac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\dfrac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\dfrac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$
s integral	$\dfrac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

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Inverse Transforms

- In linear lumped-parameter circuits, F(s) is a rational function of s.
- Rational function can be expressed in the form of a ratio of two polynomials

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}.$$

  - *a and b are real constants*
  - *m and n are positive integers*

- F(s) is a *proper rational function* **if** m > n. The inverse transform is found by a partial fraction expression.
- F(s) is an *improper rational function* **if** m ≤ n. It can be inverse-transformed by first expanding it into a sum of a polynomial and a proper rational function.

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Inverse Transforms
Proper Rational Functions

- A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of D(s)

For example: $$\frac{s+6}{s(s+3)(s+1)^2}$$ → Denominator has four roots!

$$\frac{s+6}{s(s+3)(s+1)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

- Inverse transforms are found lies in recognizing the f(t) corresponding to each term in the sum of partial fractions.

$$\Rightarrow \quad L\left\{\frac{s+6}{s(s+3)(s+1)^2}\right\} = (K_1 + K_2e^{-3t} + K_3te^{-t} + K_4e^{-t})u(t)$$

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Inverse Transforms
Distinct Real Roots of *D(s)*

- To find a K associated with a term that arises because of a distinct root of D(s):
  - *multiply both sides of the identity by a factor equal to the denominator beneath the desired K.*
  - *Evaluate both sides of the identity at the root corresponding to the multiplying factor, the right-hand side is always the desired K, and the left-hand side is always its numerical value.*

Example:

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

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Inverse Transforms

Distinct Real Roots of *D(s)*

Example:

To find the value of K_1, we multiply both sides by s and then evaluate both sides at $s = 0$:

$$\left.\frac{96(s+5)(s+12)}{(s+8)(s+6)}\right|_{s=0} \equiv K_1 + \left.\frac{K_2 s}{s+8}\right|_{s=0} + \left.\frac{K_3 s}{s+6}\right|_{s=0},$$

or

$$\frac{96(5)(12)}{8(6)} \equiv K_1 = 120.$$

To find the value of K_2, we multiply both sides by $s + 8$ and then evaluate both sides at $s = -8$:

$$\left.\frac{96(s+5)(s+12)}{s(s+6)}\right|_{s=-8}$$

$$\equiv \left.\frac{K_1(s+8)}{s}\right|_{s=-8} + K_2 + \left.\frac{K_3(s+8)}{(s+6)}\right|_{s=-8},$$

or

$$\frac{96(-3)(4)}{(-8)(-2)} = K_2 = -72.$$

Inverse Transforms
Distinct Real Roots of *D(s)*

Example:

Then K_3 is

$$\left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} = K_3 = 48.$$

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}.$$

Now confident that the numerical values of the various Ks are correct, we proceed to find the inverse transform:

$$\mathscr{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = (120 + 48e^{-6t} - 72e^{-8t})u(t).$$

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Inverse Transforms

Distinct Complex Roots of *D(s)*

- The procedure is the same as for distinct real roots.
- The only difference is that the algebra in the distinct complex roots involves complex numbers.
- Note:
  - In physical realizable circuits, complex roots always appear in conjugate pairs.
  - The coefficients associated with these conjugate pairs are themselves conjugates.
  - Therefore, we just need to calculate only half the coefficients.

Example: $$F(s)=\frac{100(s+3)}{(s+6)(s^2+6s+25)}=\frac{K_1}{s+6}+\frac{K_2}{s+3-j4}+\frac{K_3}{s+3+j4}$$

Inverse Transforms

Distinct Complex Roots of *D(s)*

SEE

Example:

find the roots of the quadratic term $s^2 + 6s + 25$:

$$s^2 + 6s + 25 = (s + 3 - j4)(s + 3 + j4).$$

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} \equiv \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

To find K_1, K_2, and K_3, we use the same process as before:

$$K_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = \frac{100(-3)}{25} = -12,$$

a + jb = r(cosβ + sinβ)

$$r = (a^2 + b^2)^{1/2}$$

$e^{j\beta} = \cos\beta + j\sin\beta$

$e^{-j\beta} = \cos\beta - j\sin\beta$

$$K_2 = \left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = \frac{100(j4)}{(3+j4)(j8)}$$

$$= 6 - j8 = 10e^{-j53.13^\circ},$$

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Inverse Transforms

Distinct Complex Roots of *D(s)*

Example:

$$K_3 = \left.\frac{100(s+3)}{(s+6)(s+3-j4)}\right|_{s=-3-j4} = \frac{100(-j4)}{(3-j4)(-j8)}$$

$$= 6 + j8 = 10e^{j53.13^\circ}.$$

Finally:

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{s+3-j4} + \frac{10\angle 53.13^\circ}{s+3+j4}.$$

We now proceed to inverse-transform

$$\mathscr{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = (-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t}$$

$$+ 10e^{j53.13^\circ}e^{-(3+j4)t})u(t)$$

$$= [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t)$$

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Inverse Transforms
Repeated Real Roots of *D(s)*

- To find the coefficients associated with the terms generated by a multiple root of multiplicity *r*, multiply both sides of the identity by the multiple root raised to its *r* th power.
- *K* appearing over the factor raised to the *r* th power is found by evaluating both sides of the identity at the multiple root.
- The remaining *(r – 1)* coefficients are found by differentiating both sides of the identity *(r – 1)* times.
- At the end of each differentiation, evaluate both sides of the identity at the multiple root.
- The right-hand side is always the desired *K*, and the left-hand side is always its numerical value.

Example: $$F(s)=\frac{100(s+25)}{s(s+5)^3}=\frac{K_1}{s}+\frac{K_2}{(s+5)^3}+\frac{K_3}{(s+5)^2}+\frac{K_4}{s+5} \qquad (*)$$

SEE

We find K_1 as previously described; that is,

$$K_1 = \left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0} = \frac{100(25)}{125} = 20$$

To find K_2, we multiply both sides by $(s+5)^3$ and then evaluate both sides at -5:

$$\left.\frac{100(s+25)}{s}\right|_{s=-5} = \left.\frac{K_1(s+5)^3}{s}\right|_{s=-5} + K_2 + K_3(s+5)|_{s=-5} + \left.K_4(s+5)^2\right|_{s=-5}$$

$$\frac{100(20)}{(-5)} = K_1 \times 0 + K_2 + K_3 \times 0 + K_4 \times 0 = K_2 = -400$$

To find K_3 we first must multiply both sides of Eq.(*) by $(s+5)^3$. Next we differentiate both sides once with respect to s and then evaluate at $s=-5$:

$$\frac{d}{ds}\left[\frac{100(s+25)}{s}\right]_{s=-5} = \frac{d}{ds}\left[\frac{K_1(s+5)^3}{s}\right]_{s=-5} + \frac{d}{ds}[K_2]_{s=-5} + \frac{d}{ds}[K_3(s+5)]_{s=-5}$$

$$+ \frac{d}{ds}[K_4(s+5)^2]_{s=-5},$$

SEE

$$\Rightarrow \quad 100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5} = K_3 = -100.$$

To find K_4 we first multiply both sides of Eq.**(*)** by $(s+5)^3$. Next we differentiate both sides twice with respect to s and then evaluate both sides at $s=-5$. After simplifying the first derivative, the second derivative becomes

$$100\frac{d}{ds}\left[-\frac{25}{s^2}\right]_{s=-5} = K_1\frac{d}{ds}\left[\frac{(s+5)^2(2s-5)}{s^2}\right]_{s=-5} + 0 + \frac{d}{ds}[K_3]_{s=-5} + \frac{d}{ds}[2K_4(s+5)]_{s=-5},$$

$$\Rightarrow \quad -40 = 2K_4. \quad \Rightarrow \quad K_4 = -20.$$

$$\Rightarrow \quad \frac{100(s+25)}{s(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5}.$$

The inverse transform

$$\mathscr{L}^{-1}\left\{\frac{100(s+25)}{s(s+5)^3}\right\} = [20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t}]u(t).$$

SEE

Inverse Transforms
Repeated Complex Roots of *D(s)*

- The procedure is the same as for the repeated real roots.
- The only difference is that the algebra involves complex numbers
- Because complex roots always appear in conjugate pairs and the coefficients associated with a conjugate pair are also conjugates, only half the *K* s need to be evaluated

Example:

$$F(s)=\frac{768}{(s^2+6s+25)^2}=\frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$=\frac{K_1}{(s+3-j4)^2}+\frac{K_2}{(s+3-j4)}+\frac{K_1^*}{(s+3+j4)^2}+\frac{K_2^*}{(s+3+j4)}$$

SEE

Now we need to evaluate only K_1 and K_2, because K_1^* and K_2^* are conjugate values. The value of K_1 is

$$K_1 = \left.\frac{768}{(s+3+j4)^2}\right|_{s=-3+j4} = \frac{768}{(j8)^2} = -12$$

The value of K_2 is

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s+3+j4)^2}\right]_{s=-3+j4} = -\left.\frac{2(768)}{(s+3+j4)^3}\right|_{s=-3+j4} = -\frac{2(768)}{(j8)^3}$$

$$= -j3 = 3\angle{-90^\circ}.$$

$$K_1^* = -12, \quad K_2^* = j3 = 3\angle{90^\circ}$$

We now group the partial fraction expansion by conjugate terms to obtain

$$F(s) = \left[\frac{-12}{(s+3-j4)^2} + \frac{-12}{(s+3+j4)^2}\right] + \left(\frac{3\angle{-90^\circ}}{s+3-j4} + \frac{3\angle{90^\circ}}{s+3+j4}\right).$$

We now write the inverse transform of $F(s)$: $f(t) = [-24te^{-3t}\cos 4t + 6e^{-3t}\cos(4t - 90^\circ)]u(t).$

Note that if $F(s)$ has a real root a of multiplicity r in its denominator, the term in a partial fraction expansion is of the form

$$\frac{K}{(s+a)^r}.$$

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The inverse transform of this term is

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\} = \frac{Kt^{r-1}e^{-at}}{(r-1)!}u(t).$$

If $F(s)$ has a complex root of $\alpha + j\beta$ of multiplicity r in its denominator, the term in partial fraction expansion is the conjugate pair

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}.$$

The inverse transform of this pair is

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\} = \left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t+\theta)\right]u(t)$$

SEE

Inverse Transforms
Four Useful Transform Pairs

Pair number	Nature of roots	F(s)	f(t)
1	Distinct real	$\dfrac{K}{s+a}$	$Ke^{-at}u(t)$
2	Repeated real	$\dfrac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
3	Distinct complex	$\dfrac{K}{s+\alpha-j\beta}+\dfrac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t}cos(\beta t+\theta)u(t)$
4	Repeated complex	$\dfrac{K}{(s+\alpha-j\beta)^2}+\dfrac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-\alpha t}cos(\beta t+\theta)u(t)$

SEE

Inverse Transforms
Improper Rational Functions

- An improper rational function can always be expanded into a polynomial plus a proper rational function.
- The polynomial is inverse-transformed into impulse functions and derivatives of impulse functions.
- The proper rational function is inverse-transformed by the techniques outlined in previous section

Example:

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

$$= s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

$$= s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}$$

SEE

Poles and Zeros of *F(s)*

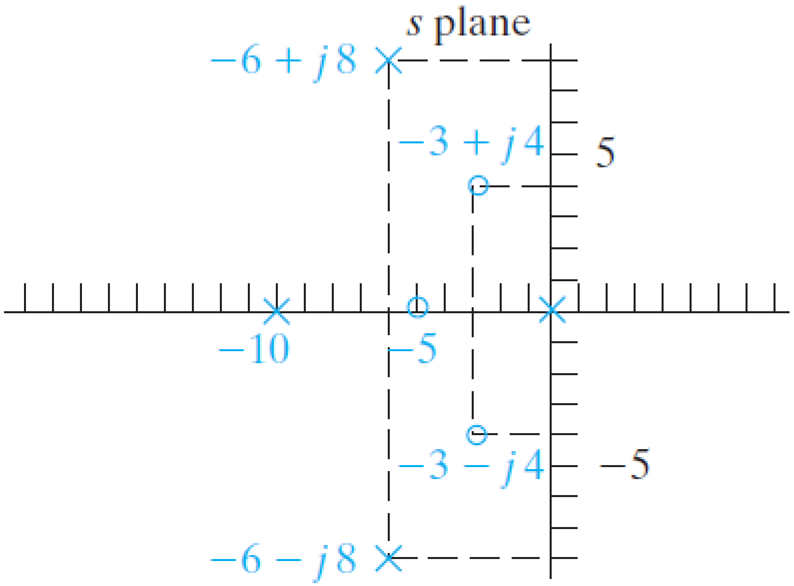
- F(s) can be expressed as the ratio of two factored polynomials.
- The roots of the denominator are called poles and are plotted as Xs on the complex s plane.
- The roots of the numerator are called zeros and are plotted as 0s on the complex s plane.

Plot the poles and zeros of F(s)

$$F(s) = \frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

The poles of $F(s)$ are at $0, -10, -6+j8$, and $-6-j8$.

The zeros are at -5, $-3+j4$, and $-3-j4$.



Plotting poles and zeros on the s plane.

SEE

Initial and Final Value Theorems

The initial- and final- value theorems are used because they enable us to determine from F(s) the behavior of f(t) at 0 and ∞

- The initial-value theorem states that:

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) \qquad (*)$$

The theorem assumes that f(t) contains no impulse functions.

- The final-value theorem states that:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \qquad (**)$$

The theorem is valid only if the poles of F(s), except for a first- order pole at the origin, lie in the left half of the s plane.

Initial and Final Value Theorems

Prove (*)

$$\mathscr{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt.$$

Now we take the limit as $s \to \infty$:

$$\lim_{s\to\infty} [sF(s) - f(0^-)] = \lim_{s\to\infty} \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt.$$

Observe that the right-hand side of Eq. 12.96 may be written as

$$\lim_{s\to\infty} \left(\int_{0^-}^{0^+} \frac{df}{dt} e^{0} dt + \int_{0^+}^{\infty} \frac{df}{dt} e^{-st}\, dt \right).$$

As $s \to \infty$, $(df/dt)e^{-st} \to 0$ The first integral reduces to $f(0^+) - f(0^-)$, which is independent of s.

$$\lim_{s\to\infty} \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt = f(0^+) - f(0^-).$$

SEE

Initial and Final Value Theorems

Prove (*)

Because $f(0^-)$ is independent of s, the left-hand side of Eq.

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}[sF(s)] - f(0^-).$$

$$\lim_{s\to\infty} sF(s) = f(0^+) = \lim_{t\to 0^+} f(t),$$

which completes the proof of the initial-value theorem.

Prove (**) we take the limit as $s \to 0$: $\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0}\left(\int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt\right)$

$$\lim_{s\to 0}\left(\int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt\right) = \int_{0^-}^{\infty}\frac{df}{dt}dt.$$

Because the upper limit on the integral is infinite, this integral may also be written as a limit process:

SEE

Initial and Final Value Theorems

Prove (**)

$$\int_{0^-}^{\infty} \frac{df}{dt} dt = \lim_{t \to \infty} \int_{0^-}^{t} \frac{df}{dy} dy,$$

where we use y as the symbol of integration to avoid confusion with the upper limit on the integral. Carrying out the integration process yields

$$\lim_{t \to \infty} [f(t) - f(0^-)] = \lim_{t \to \infty} [f(t)] - f(0^-).$$

$$\Rightarrow \quad \lim_{s \to 0} [sF(s)] - f(0^-) = \lim_{t \to \infty} [f(t)] - f(0^-).$$

$$\Rightarrow \quad \lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$$

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Initial and Final Value Theorems - Application

Consider the transform pair given by Eq.

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t).$$

The initial-value

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{100s^2[1+(3/s)]}{s^3[1+(6/s)][1+(6/s)+(25/s^2)]} = 0,$$

$$\lim_{t\to 0^+} f(t) = [-12 + 20\cos(-53.13^\circ)](1) = -12 + 12 = 0.$$

The final-value theorem gives

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{100s(s+3)}{(s+6)(s^2+6s+25)} = 0,$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t) = 0.$$