CHAPTER 6

Transmission-Line Essentials for Digital Electronics

In Chapter 3 we alluded to the fact that lumped circuit theory is based on low-frequency approximations resulting from the neglect of certain terms in one or both of Maxwell's curl equations. We further pointed out that electromagnetic wave propagation phenomena and transmission-line (distributed circuit) theory are based on the simultaneous application of the two laws, with all the terms included. We then studied wave propagation in Chapters 3 and 4. In Chapter 5 we introduced the (lumped) circuit parameters for infinitely long, parallel perfect conductor arrangements, and also extended the discussion to electric and magnetic field systems, which are low-frequency approximations of physical structures. In this chapter, we begin our study of transmission line theory. Specifically, we focus on time-domain analysis, an understanding of which is particularly essential for digital electronic systems, while being of general importance.

We introduce the transmission line by considering a uniform plane wave and placing two parallel plane, perfect, conductors such that the fields remain unchanged by satisfying the boundary conditions on the perfect conductor surfaces. The wave is then guided between and parallel to the conductors, thereby leading to the parallel-plate line. We shall learn to represent a line by the distributed parameter equivalent circuit and discuss wave propagation along the line in terms of voltage and current, as well as the computation of line parameters. We devote the remainder of the chapter to time-domain analysis using a progressive treatment, beginning with a line terminated by a resistive load and leading to interconnections between logic gates, and finally culminating in crosstalk on transmission lines.

6.1 TRANSMISSION LINE

Parallel-plate line

In Section 5.4, we considered a physical arrangement of two parallel, perfect conductors and discussed the circuit parameters, capacitance, conductance, and inductance per unit length of the structure. In the general case of time-varying fields, the situation corresponds to the structure characterized by the properties of all three of the circuit parameters, continuously and overlappingly along it. The arrangement is then called a *transmission line*. To introduce the transmission-line concept, we recall that in Section 4.5, we learned that the tangential component of the electric field intensity and the normal component of the magnetic field intensity are zero on a perfect conductor surface. Let us now consider the uniform plane electromagnetic wave propagating in the z-direction and having an x-component only of the electric field and a y-component only of the magnetic field, that is,

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x$$
$$\mathbf{H} = H_y(z, t)\mathbf{a}_y$$

and place perfectly conducting sheets in two planes $x = 0$ and $x = d$, as shown in Fig. 6.1. Since the electric field is completely normal and the magnetic field is completely tangential to the sheets, the two boundary conditions just referred to are satisfied, and, hence, the wave will simply propagate, as though the sheets were not present, being guided by the sheets. We then have a simple case of transmission line, namely, the parallel-plate transmission line. We shall assume the medium between the plates to be a perfect dielectric so that the waves propagate without attenuation; hence, the line is lossless.

According to the remaining two boundary conditions, there must be charges and currents on the conductors. The charge densities on the two plates are

$$[\rho_S]_{x=0} = [\mathbf{a}_n \cdot \mathbf{D}]_{x=0} = \mathbf{a}_x \cdot \varepsilon E_x \mathbf{a}_x = \varepsilon E_x \tag{6.1a}$$
$$[\rho_S]_{x=d} = [\mathbf{a}_n \cdot \mathbf{D}]_{x=d} = -\mathbf{a}_x \cdot \varepsilon E_x \mathbf{a}_x = -\varepsilon E_x \tag{6.1b}$$

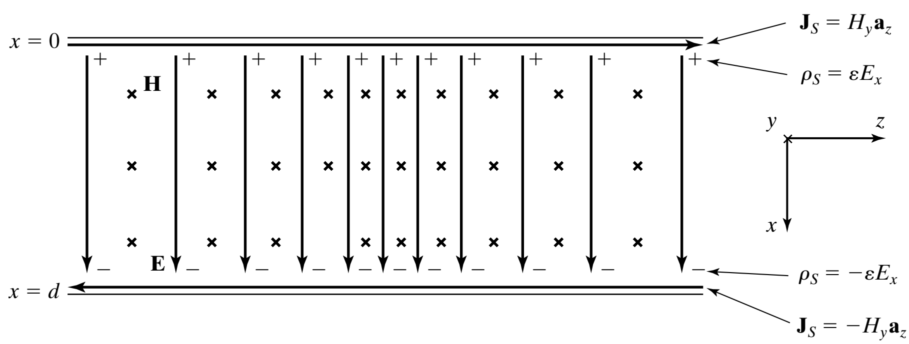


FIGURE 6.1

Uniform plane electromagnetic wave propagating between two perfectly conducting sheets, supported by charges and current on the sheets.

where ε is the permittivity of the medium between the two plates. The current densities on the two plates are

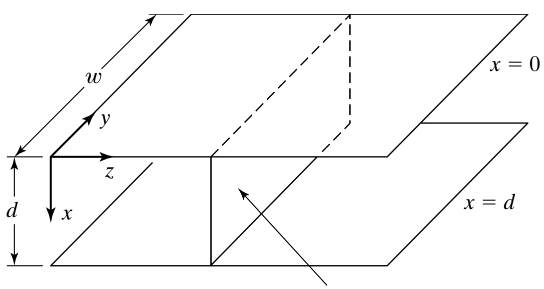
$$[\mathbf{J}_S]_{x=0} = [\mathbf{a}_n \times \mathbf{H}]_{x=0} = \mathbf{a}_x \times H_y\mathbf{a}_y = H_y\mathbf{a}_z \tag{6.2a}$$

$$[\mathbf{J}_S]_{x=d} = [\mathbf{a}_n \times \mathbf{H}]_{x=d} = -\mathbf{a}_x \times H_y\mathbf{a}_y = -H_y\mathbf{a}_z \tag{6.2b}$$

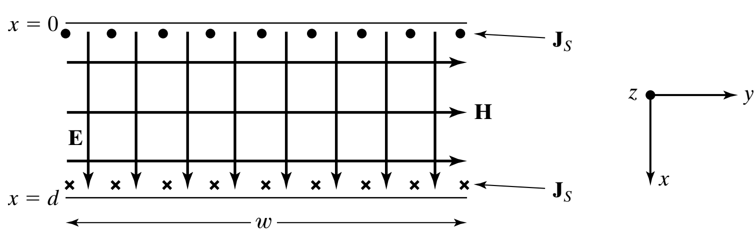
In (6.1a)–(6.2b), it is understood that the charge and current densities are functions of z and t, as are E_x and H_y. Thus, the wave propagation along the transmission line is supported by charges and currents on the plates, varying with time and distance along the line, as shown in Fig. 6.1.

Let us now consider finitely sized plates having width w in the y-direction, as shown in Fig. 6.2(a), and neglect fringing of the fields at the edges or assume that the structure is part of a much larger-sized configuration. By considering a constant-z plane, that is, a plane *transverse* to the direction of propagation of the wave, as shown in Fig. 6.2(b), we can find the voltage between the two conductors in terms of the line integral of the electric field intensity evaluated along any path in that plane between the two conductors. Since the electric field is directed in the x-direction and since it is uniform in that plane, this voltage is given by

$$V(z,t) = \int_{x=0}^{d} E_x(z,t)\,dx = E_x(z,t)\int_{x=0}^{d} dx = dE_x(z,t) \tag{6.3}$$



(a)



(b)

FIGURE 6.2

(a) Parallel-plate transmission line. (b) Transverse plane of the parallel-plate transmission line.

Thus, each transverse plane is characterized by a voltage between the two conductors, which is related simply to the electric field, as given by (6.3). Each transverse plane is also characterized by a current I flowing in the positive z-direction on the upper conductor and in the negative z-direction on the lower conductor. From Fig. 6.2(b), we can see that this current is given by

$$\begin{aligned} I(z,t) &= \int_{y=0}^{w} J_S(z,t)\,dy = \int_{y=0}^{w} H_y(z,t)\,dy = H_y(z,t)\int_{y=0}^{w} dy \\ &= wH_y(z,t) \end{aligned} \tag{6.4}$$

since H_y is uniform in the cross-sectional plane. Thus, the current crossing a given transverse plane is related simply to the magnetic field in that plane, as given by (6.4).

Proceeding further, we can find the power flow down the line by evaluating the surface integral of the Pointing vector over a given transverse plane. Thus,

$$\begin{aligned} P(z,t) &= \int_{\substack{\text{transverse}\\ \text{plane}}} (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S} \\ &= \int_{x=0}^{d}\int_{y=0}^{w} E_x(z,t)H_y(z,t)\mathbf{a}_z\cdot dxdy\,\mathbf{a}_z \\ &= \int_{x=0}^{d}\int_{y=0}^{w} \frac{V(z,t)}{d}\,\frac{I(z,t)}{w}\,dx\,dy \\ &= V(z,t)I(z,t) \end{aligned} \tag{6.5}$$

which is the familiar relationship employed in circuit theory.

Transmission-line equations

We now recall from Section 4.4 that E_x and H_y satisfy the two differential equations

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu\frac{\partial H_y}{\partial t} \tag{6.6a}$$

$$\frac{\partial H_y}{\partial z} = -\sigma E_x - \varepsilon\frac{\partial E_x}{\partial t} = -\varepsilon\frac{\partial E_x}{\partial t} \tag{6.6b}$$

where we have set $\sigma = 0$ in view of the perfect dielectric medium. From (6.3) and (6.4), however, we have

$$E_x = \frac{V}{d} \tag{6.7a}$$

$$H_y = \frac{I}{w} \tag{6.7b}$$

Substituting for E_x and H_y in (6.6a) and (6.6b) from (6.7a) and (6.7b), respectively, we now obtain two differential equations for voltage and current along the line as

$$\frac{\partial}{\partial z}\left(\frac{V}{d}\right) = -\mu\frac{\partial}{\partial t}\left(\frac{I}{w}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{I}{w}\right) = -\varepsilon\frac{\partial}{\partial t}\left(\frac{V}{d}\right)$$

or

$$\boxed{\frac{\partial V}{\partial z} = -\left(\frac{\mu d}{w}\right)\frac{\partial I}{\partial t}} \tag{6.8a}$$

$$\boxed{\frac{\partial I}{\partial z} = -\left(\frac{\varepsilon w}{d}\right)\frac{\partial V}{\partial t}} \tag{6.8b}$$

These equations are known as the *transmission-line equations*. They characterize the wave propagation along the line in terms of line voltage and line current instead of in terms of the fields.

We now denote two quantities familiarly known as the *circuit parameters*, the inductance and the capacitance per unit length of the transmission line in the z-direction by the symbols $\mathscr{L}$ and $\mathscr{C}$, respectively. We observe from Section 5.4 that the inductance per unit length, having the units henrys per meter (H/m), is the ratio of the magnetic flux per unit length at any value of z to the line current at that value of z. Noting from Fig. 6.2 that the cross-sectional area normal to the magnetic field lines and per unit length in the z-direction is $(d)(1)$, or d, we find the magnetic flux per unit length to be $B_y d$ or $\mu H_y d$. Since the line current is $H_y w$, we then have

$$\boxed{\mathscr{L} = \frac{\mu H_y d}{H_y w} = \frac{\mu d}{w}} \tag{6.9}$$

We also observe that the capacitance per unit length, having the units farads per meter (F/m), is the ratio of the magnitude of the charge per unit length on either plate at any value of z to the line voltage at that value of z. Noting from Fig 6.2 that the cross-sectional area normal to the electric field lines and per unit length in the z-direction is $(w)(1)$, or w, we find the charge per unit length to be $\rho_S w$, or $\varepsilon E_x w$. Since the line voltage is $E_x d$, we then have

$$\boxed{\mathscr{C} = \frac{\varepsilon E_x w}{E_x d} = \frac{\varepsilon w}{d}} \tag{6.10}$$

We note that $\mathscr{L}$ and $\mathscr{C}$ are purely dependent on the dimensions of the line and are independent of E_x and H_y. We further note that

$$\boxed{\mathscr{L}\mathscr{C} = \mu\varepsilon} \tag{6.11}$$

so that only one of the two parameters $\mathscr{L}$ and $\mathscr{C}$ is independent and the other can be obtained from the knowledge of ε and μ. The results given by (6.9) and (6.10) are the same as those listed in Table 5.2 for the parallel-plane conductor arrangement, whereas (6.11) is the same as given by (5.73).

Replacing now the quantities in parentheses in (6.8a) and (6.8b) by $\mathscr{L}$ and $\mathscr{C}$, respectively, we obtain the transmission-line equations in terms of these parameters as

$$\frac{\partial V}{\partial z} = -\mathscr{L}\frac{\partial I}{\partial t} \tag{6.12a}$$

$$\frac{\partial I}{\partial z} = -\mathscr{C}\frac{\partial V}{\partial t} \tag{6.12b}$$

These equations permit us to discuss wave propagation along the line in terms of circuit quantities instead of in terms of field quantities. It should, however, not be forgotten that the actual phenomenon is one of electromagnetic waves guided by the conductors of the line.

Distributed equivalent circuit

It is customary to represent a transmission line by means of its circuit equivalent, derived from the transmission-line equations (6.12a) and (6.12b). To do this, let us consider a section of infinitesimal length Δz along the line between z and $z + \Delta z$. From (6.12a), we then have

$$\lim_{\Delta z \to 0} \frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = -\mathscr{L}\frac{\partial I(z, t)}{\partial t}$$

or, for $\Delta z \to 0$,

$$V(z + \Delta z, t) - V(z, t) = -\mathscr{L}\,\Delta z\,\frac{\partial I(z, t)}{\partial t} \tag{6.13a}$$

This equation can be represented by the circuit equivalent shown in Fig. 6.3(a), since it satisfies Kirchhoff's voltage law written around the loop *abcda*. Similarly, from (6.12b), we have

$$\lim_{\Delta z \to 0} \frac{I(z + \Delta z, t) - I(z, t)}{\Delta z} = \lim_{\Delta z \to 0}\left[-\mathscr{C}\frac{\partial V(z + \Delta z, t)}{\partial t}\right]$$

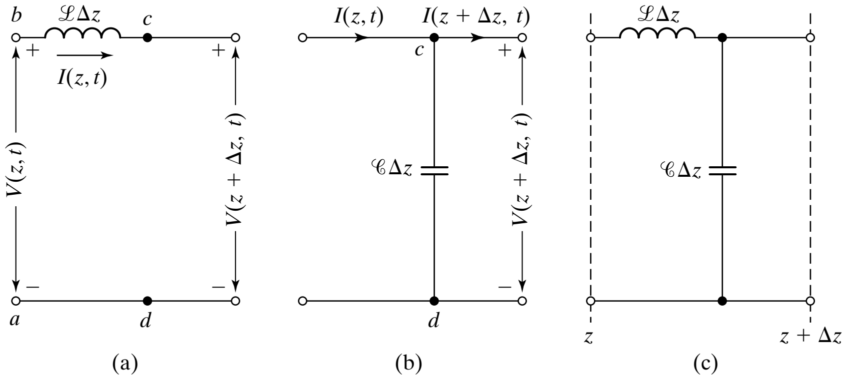


FIGURE 6.3

Development of circuit equivalent for an infinitesimal length Δz of a transmission line.

or, for $\Delta z \to 0$,

$$I(z + \Delta z, t) - I(z, t) = -\mathscr{C}\,\Delta z\,\frac{\partial V(z + \Delta z, t)}{\partial t} \tag{6.13b}$$

This equation can be represented by the circuit equivalent shown in Fig. 6.3(b), since it satisfies Kirchhoff's current law written for node c. Combining the two equations, we then obtain the equivalent circuit shown in Fig. 6.3(c) for a section Δz of the line. It then follows that the circuit representation for a portion of length l of the line consists of an infinite number of such sections in cascade, as shown in Fig. 6.4. Such a circuit is known as a *distributed circuit* as opposed to the *lumped circuits* that are familiar in circuit theory. The distributed circuit notion arises from the fact that the inductance and capacitance are distributed uniformly and overlappingly along the line.

A more physical interpretation of the distributed-circuit concept follows from energy considerations. We know that the uniform plane wave propagation between the conductors of the line is characterized by energy storage in the

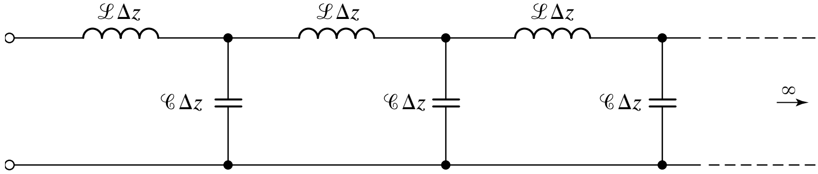


FIGURE 6.4

Distributed circuit representation of a transmission line.

electric and magnetic fields. If we consider a section Δz of the line, the energy stored in the electric field in this section is given by

$$\begin{aligned} W_e &= \frac{1}{2}\varepsilon E_x^2(\text{volume}) = \frac{1}{2}\varepsilon E_x^2(dw\,\Delta z) \\ &= \frac{1}{2}\frac{\varepsilon w}{d}(E_x d)^2\,\Delta z = \frac{1}{2}\mathscr{C}\,\Delta z\,V^2 \end{aligned} \tag{6.14a}$$

The energy stored in the magnetic field in that section is given by

$$\begin{aligned} W_m &= \frac{1}{2}\mu H_y^2(\text{volume}) = \frac{1}{2}\mu H_y^2(dw\,\Delta z) \\ &= \frac{1}{2}\frac{\mu d}{w}(H_y w)^2\,\Delta z = \frac{1}{2}\mathscr{L}\,\Delta z\,I^2 \end{aligned} \tag{6.14b}$$

Thus, we note that $\mathscr{L}$ and $\mathscr{C}$ are elements associated with energy storage in the magnetic field and energy storage in the electric field, respectively, for a given infinitesimal section of the line. Since these phenomena occur continuously and since they overlap, the inductance and capacitance must be distributed uniformly and overlappingly along the line.

TEM waves

Thus far, we have introduced the transmission-line equations and the distributed-circuit concept by considering the parallel-plate line in which the waves are uniform plane waves. In the general case of a line having conductors with arbitrary cross sections, the fields consist of both x- and y-components and are dependent on x- and y-coordinates in addition to the z-coordinate. Thus, the fields between the conductors are given by

$$\begin{aligned} \mathbf{E} &= E_x(x, y, z, t)\mathbf{a}_x + E_y(x, y, z, t)\mathbf{a}_y \\ \mathbf{H} &= H_x(x, y, z, t)\mathbf{a}_x + H_y(x, y, z, t)\mathbf{a}_y \end{aligned}$$

These fields are no longer uniform in x and y but are directed entirely transverse to the direction of propagation, that is, the z-axis, which is the axis of the transmission line. Hence, they are known as *transverse electromagnetic waves*, or *TEM waves*. The uniform plane waves are simply a special case of the transverse electromagnetic waves. The transmission-line equations (6.12a) and (6.12b) and the distributed equivalent circuit of Fig. 6.4 hold for all transmission lines made of perfect conductors and perfect dielectric, that is, for all lossless transmission lines. The quantities that differ from one line to another are the line parameters $\mathscr{L}$ and $\mathscr{C}$, which depend on the geometry of the line. Since there is no z-component of $\mathbf{H}$, the electric-field distribution in any given transverse plane at any given instant of time is the same as the static electric-field distribution resulting from the application of a potential difference between the conductors equal to the line voltage in that plane at that instant of time. Similarly, since there is no z-component of $\mathbf{E}$, the magnetic-field distribution in any given transverse plane at any given instant of time is the same as the static magnetic-field distribution resulting from current flow on the conductors equal to the line current in that plane at that instant of time. Thus, the values of $\mathcal{L}$ and $\mathcal{C}$ are the same as those obtainable from static field considerations.

General solution

Before we consider several common types of lines, we shall show that the relation (6.11) is valid in general by obtaining the general solution for the transmission-line equations (6.12a) and (6.12b). To do this, we note their analogy with the field equations (3.72a) and (3.72b) in Section 3.4, as follows:

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t} \longleftrightarrow \frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = -\varepsilon_0 \frac{\partial E_x}{\partial t} \longleftrightarrow \frac{\partial I}{\partial z} = -\mathcal{C}\frac{\partial V}{\partial t}$$

The solutions to (6.12a) and (6.12b) can therefore be written by letting

$$\begin{aligned} E_x &\longrightarrow V \\ H_y &\longrightarrow I \\ \mu_0 &\longrightarrow \mathcal{L} \\ \varepsilon_0 &\longrightarrow \mathcal{C} \end{aligned}$$

in the solutions (3.78) and (3.79) to the field equations. Thus, we obtain

$$\boxed{V(z, t) = Af(t - z\sqrt{\mathcal{L}\mathcal{C}}) + Bg(t + z\sqrt{\mathcal{L}\mathcal{C}})} \tag{6.15a}$$

$$\boxed{I(z, t) = \frac{1}{\sqrt{\mathcal{L}/\mathcal{C}}}[Af(t - z\sqrt{\mathcal{L}\mathcal{C}}) - Bg(t + z\sqrt{\mathcal{L}\mathcal{C}})]} \tag{6.15b}$$

These solutions represent voltage and current traveling waves propagating along the $+z$- and $-z$-directions with velocity

$$\boxed{v_p = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}}} \tag{6.16}$$

in view of the arguments $(t \mp z\sqrt{\mathcal{L}\mathcal{C}})$ for the functions f and g. We, however, know that the velocity of propagation in terms of the dielectric parameters is given by

$$\boxed{v_p = \frac{1}{\sqrt{\mu\varepsilon}}} \tag{6.17}$$

Therefore, it follows that

$$\boxed{\mathcal{L}\mathcal{C} = \mu\varepsilon} \tag{6.18}$$

Characteristic impedance

We now define the *characteristic impedance* of the line to be

$$\boxed{Z_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}} \tag{6.19}$$

so that (6.15a) and (6.15b) become

$$V(z,t) = Af\left(t - \frac{z}{v_p}\right) + Bg\left(t + \frac{z}{v_p}\right) \tag{6.20a}$$

$$I(z,t) = \frac{1}{Z_0}\left[Af\left(t - \frac{z}{v_p}\right) - Bg\left(t + \frac{z}{v_p}\right)\right] \tag{6.20b}$$

where we have also substituted v_p for $1/\sqrt{\mathcal{L}\mathcal{C}}$. From (6.20a) and (6.20b), it can be seen that *the characteristic impedance is the ratio of the voltage to current in the* $(+)$ *wave or the negative of the same ratio for the* $(-)$ *wave*. It is analogous to the intrinsic impedance of the dielectric medium but not necessarily equal to it. For example, for the parallel-plate line,

$$\begin{aligned} Z_0 &= \sqrt{\frac{\mu d/w}{\varepsilon w/d}} \\ &= \eta\frac{d}{w} \end{aligned} \tag{6.21}$$

is not equal to η unless d/w is equal to 1. In fact for d/w equal to 1, (6.21) is strictly not valid because fringing of the fields cannot be neglected. Note also that the characteristic impedance of a lossless line is a purely real quantity. We shall find in Section 7.6 that for a lossy line, the characteristic impedance is complex just as the intrinsic impedance of a lossy medium is complex.

Equations (6.20a) and (6.20b) are the general solutions for the voltage and current along a lossless line in terms of v_p and Z_0, the parameters that characterize the propagation along the line. Whereas v_p is dependent on the dielectric as given by (6.17), Z_0 is dependent on the dielectric as well as the geometry associated with the line, in view of (6.19). Combining (6.18) and (6.19), we note that

$$Z_0 = \frac{\sqrt{\mu\varepsilon}}{\mathcal{C}} \tag{6.22}$$

Thus, the determination of Z_0 for a given line involving a given homogeneous dielectric medium requires simply the determination of $\mathcal{C}$ of the line and then the use of (6.22). Since the dielectrics of common transmission lines are generally nonmagnetic, we can further express the propagation parameters in the manner

$$Z_0 = \frac{\sqrt{\varepsilon_r}}{c\mathscr{C}} \tag{6.23}$$

$$v_p = \frac{c}{\sqrt{\varepsilon_r}} \tag{6.24}$$

where ε_r is the relative permittivity of the dielectric, and $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the velocity of light in free space.

Microstrip line

If the cross section of a transmission line involves more than one dielectric, the situation corresponds to inhomogeneity. An example of this type of line is the microstrip line, used extensively in microwave integrated circuitry and digital systems. The basic microstrip line consists of a high-permittivity substrate material with a conductor strip applied to one side and a conducting ground plane applied to the other side, as shown by the cross-sectional view in Fig. 6.5(a). The approximate electric field distribution is shown in Fig. 6.5(b). Since it is not possible to satisfy the boundary condition of equal phase velocities parallel to the air-dielectric interface with pure TEM waves, the situation for the microstrip line does not correspond exactly to TEM wave propagation, as is the case with any other line involving multiple dielectrics.

The determination of Z_0 and v_p for the case of a line with multiple dielectrics involves a modified procedure, assuming that the inhomogeneity has no effect on $\mathscr{L}$ and the propagation is TEM. Thus, if $\mathscr{C}_0$ is the capacitance per unit length of the line with all the dielectrics replaced by free space and $\mathscr{C}$ is the capacitance per unit length of the line with the dielectrics in place and computed from static field considerations, we can write

$$Z_0 = \sqrt{\frac{\mathscr{L}}{\mathscr{C}}} = \sqrt{\frac{\mathscr{L}\mathscr{C}_0}{\mathscr{C}\mathscr{C}_0}} = \frac{1}{c\sqrt{\mathscr{C}\mathscr{C}_0}} \tag{6.25a}$$

$$v_p = \frac{1}{\sqrt{\mathscr{L}\mathscr{C}}} = \frac{1}{\sqrt{\mathscr{L}\mathscr{C}_0}}\sqrt{\frac{\mathscr{C}_0}{\mathscr{C}}} = c\sqrt{\frac{\mathscr{C}_0}{\mathscr{C}}} \tag{6.25b}$$

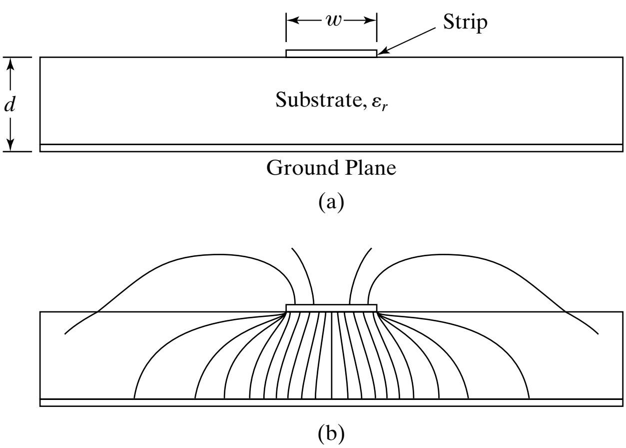


FIGURE 6.5

(a) Transverse cross-sectional view of a microstrip line. (b) Approximate electric-field distribution in the transverse plane.

where we have assumed nonmagnetic dielectrics. To express (6.25a) and (6.25b) in the form of (6.23) and (6.24), respectively, we define an effective relative permittivity $\varepsilon_{r_{\text{eff}}} = \mathscr{C}/\mathscr{C}_0$ so that

$$Z_0 = \frac{\sqrt{\varepsilon_{r_{\text{eff}}}}}{c\mathscr{C}} \tag{6.26}$$

$$v_p = \frac{c}{\sqrt{\varepsilon_{r_{\text{eff}}}}} \tag{6.27}$$

Thus, the determination of Z_0 and v_p requires the knowledge of both $\mathscr{C}$ and $\mathscr{C}_0$.

The techniques for finding $\mathscr{C}$ (and $\mathscr{C}_0$) and, hence, the propagation parameters can be broadly divided into three categories: (1) analytical, (2) numerical, and (3) graphical.

A. Analytical Techniques

Several common types of lines

The analytical techniques are based on the closed-form solution of Laplace's equation, subject to the boundary conditions, or the equivalent of such a solution. We have already discussed these techniques in Sections 5.3 and 5.4 for several configurations. Hence, without further discussion, we shall simply list in Table 6.1 the expressions for Z_0 for some common types of lines, shown by cross-sectional views in Fig. 6.6. Note that in Table 6.1, $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of the dielectric medium associated with the line.

B. Numerical Techniques

When a closed-form solution is not possible or when the approximation permitting a closed-form solution breaks down, numerical techniques can be employed. These are discussed in Chapter 11.

C. Graphical Technique

For a line with arbitrary cross section and involving a homogeneous dielectric, an approximate value of $\mathscr{C}$ and, hence, of Z_0 can be determined by constructing a *field map*, that is, a graphical sketch of the direction lines of the electric field

Table 6.1 Expressions for Characteristic Impedance for the Lines of Fig. 6.6

Description	Figure	Z_0
Coaxial cable	6.6(a)	$\frac{\eta}{2\pi} \ln \frac{b}{a}$
Parallel-wire line	6.6(b)	$\frac{\eta}{\pi} \cosh^{-1} \frac{d}{a}$
Single wire above ground plane	6.6(c)	$\frac{\eta}{2\pi} \cosh^{-1} \frac{h}{a}$
Shielded parallel-wire line	6.6(d)	$\frac{\eta}{\pi} \ln \frac{d(b^2 - d^2/4)}{a(b^2 + d^2/4)}$

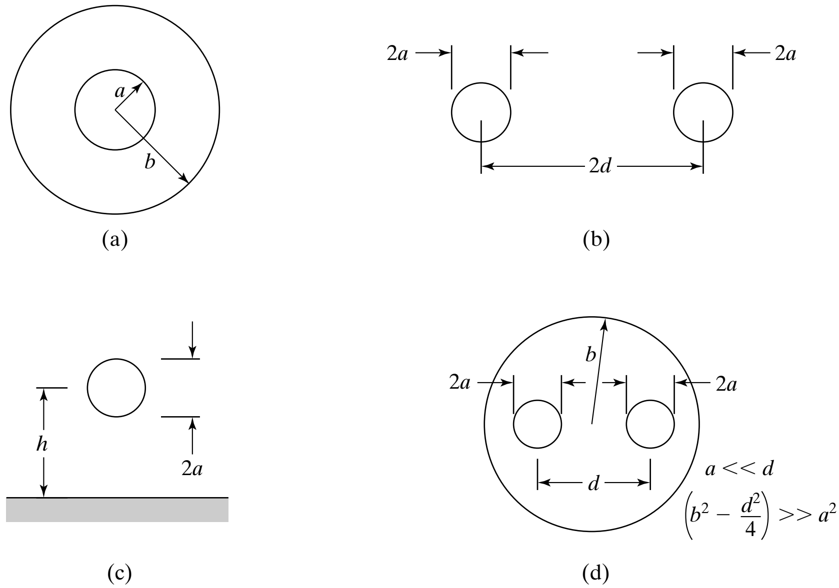


FIGURE 6.6

Cross sections of some common types of transmission lines.

and associated equipotential lines between the conductors, as illustrated, for example, in Fig. 6.7. This technique, known as the *field mapping* technique, is discussed in Section 11.5.

K6.1. Parallel-plate line; Transmission-line equations; Circuit parameters; Distributed equivalent circuit; TEM waves; Characteristic impedance Velocity of propagation; Z_0 and v_p for line with homogeneous dielectric; Z_0 and v_p for line with multiple dielectrics; Microstrip line.

D6.1. A parallel-plate transmission line is made up of perfect conductors of width $w = 0.2$ m and separation $d = 0.01$ m. The medium between the plates is a dielectric of $\varepsilon = 2.25\varepsilon_0$ and $\mu = \mu_0$. For a uniform plane wave propagating down the line, find the power crossing a given transverse plane for each of the following

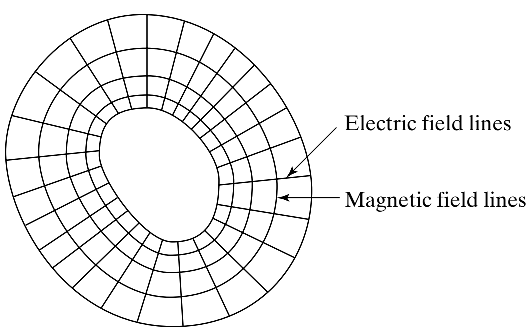


FIGURE 6.7

Example of field map for a line of arbitrary cross section.

cases at a given time in that plane: **(a)** the electric field between the plates is 300π V/m; **(b)** the magnetic field between the plates is 7.5 A/m; **(c)** the voltage across the plates is 4π V; and **(d)** the current along the plates is 0.5 A.

Ans. **(a)** 2.25π W; **(b)** 9π W; **(c)** 4π W; **(d)** π W.

D6.2. Find the following: **(a)** the ratio b/a of a coaxial cable of $Z_0 = 50\ \Omega$ if $\varepsilon = 2.56\varepsilon_0$; **(b)** the ratio b/a of a coaxial cable of $Z_0 = 75\ \Omega$ if $\varepsilon = 2.25\varepsilon_0$; and **(c)** the ratio d/a of a parallel-wire line of $Z_0 = 300\ \Omega$ if $\varepsilon = \varepsilon_0$.

Ans. **(a)** 3.794; **(b)** 6.521; **(c)** 6.132.

6.2 LINE TERMINATED BY RESISTIVE LOAD

Notation

In Section 6.1, we obtained the general solutions to the transmission-line equations for the lossless line, as given by (6.20a) and (6.20b). Since these solutions represent superpositions of (+) and (−) wave voltages and (+) and (−) wave currents, we now rewrite them as

$$V(z, t) = V^+\left(t - \frac{z}{v_p}\right) + V^-\left(t + \frac{z}{v_p}\right) \tag{6.28a}$$

$$I(z, t) = \frac{1}{Z_0}\left[V^+\left(t - \frac{z}{v_p}\right) - V^-\left(t + \frac{z}{v_p}\right)\right] \tag{6.28b}$$

or, more concisely,

$$\boxed{V = V^+ + V^-} \tag{6.29a}$$

$$\boxed{I = \frac{1}{Z_0}(V^+ - V^-)} \tag{6.29b}$$

with the understanding that V^+ is a function of $(t - z/v_p)$ and V^- is a function of $(t + z/v_p)$. In terms of (+) and (−) wave currents, the solution for the current may also be written as

$$\boxed{I = I^+ + I^-} \tag{6.30}$$

Comparing (6.29b) and (6.30), we see that

$$\boxed{I^+ = \frac{V^+}{Z_0}} \tag{6.31a}$$

$$\boxed{I^- = -\frac{V^-}{Z_0}} \tag{6.31b}$$

The minus sign in (6.31b) can be understood if we recognize that in writing (6.29a) and (6.30), we follow the notation that both V^+ and V^- have the same polarities with one conductor (say, a) positive with respect to the other conductor

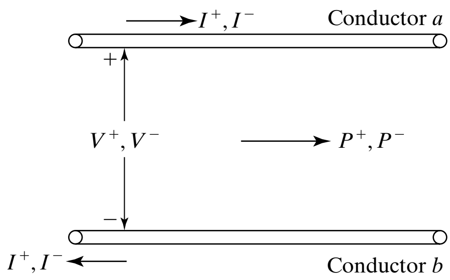


FIGURE 6.8

Polarities for voltages and currents associated with $(+)$ and $(-)$ waves.

(say, b) and that both I^+ and I^- flow in the positive z-direction along conductor a and return in the negative z-direction along conductor b, as shown in Fig. 6.8. The power flow associated with either wave, as given by the product of the corresponding voltage and current, is then directed in the positive z-direction, as shown in Fig. 6.8. Thus,

$$P^+ = V^+I^+ = V^+\left(\frac{V^+}{Z_0}\right) = \frac{(V^+)^2}{Z_0} \tag{6.32a}$$

Since $(V^+)^2$ is always positive, regardless of whether V^+ is numerically positive or negative, (6.67a) indicates that the $(+)$ wave power does actually flow in the positive z-direction, as it should. On the other hand,

$$P^- = V^-I^- = V^-\left(-\frac{V^-}{Z_0}\right) = -\frac{(V^-)^2}{Z_0} \tag{6.32b}$$

Since $(V^-)^2$ is always positive, regardless of whether V^- is numerically positive or negative, the minus sign in (6.32b) indicates that P^- is negative, and, hence, the $(-)$ wave power actually flows in the negative z-direction, as it should.

Excitation by constant voltage source

Let us now consider a line of length l terminated by a load resistance R_L and driven by a constant voltage source V_0 in series with internal resistance R_g, as shown in Fig. 6.9. Note that the conductors of the transmission line are represented by double ruled lines, whereas the connections to the conductors are single ruled lines, to be treated as lumped circuits. We assume that no voltage

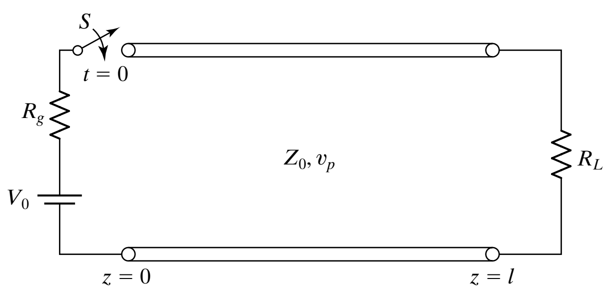


FIGURE 6.9

Transmission line terminated by a load resistance R_L and driven by a constant voltage source in series with an internal resistance R_g.

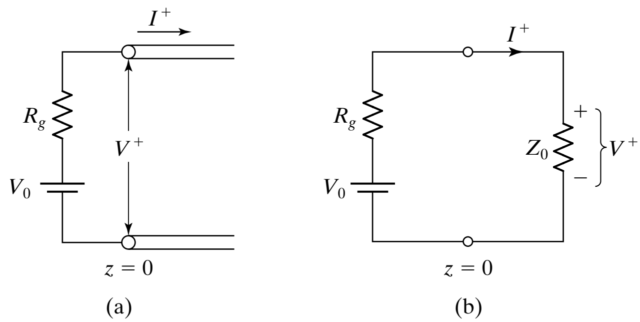


FIGURE 6.10

(a) For obtaining the $(+)$ wave voltage and current at $z = 0$ for the line of Fig. 6.9.
(b) Equivalent circuit for (a).

and current exist on the line for $t < 0$ and the switch S is closed at $t = 0$. We wish to discuss the transient wave phenomena on the line for $t > 0$. The characteristic impedance of the line and the velocity of propagation are Z_0 and v_p, respectively.

When the switch S is closed at $t = 0$, a $(+)$ wave originates at $z = 0$ and travels toward the load. Let the voltage and current of this $(+)$ wave be V^+ and I^+, respectively. Then we have the situation at $z = 0$, as shown in Fig. 6.10(a). Note that the load resistor does not come into play here since the phenomenon is one of wave propagation; hence, until the $(+)$ wave goes to the load, sets up a reflection, and the reflected wave arrives back at the source, the source does not even know of the existence of R_L. This is a fundamental distinction between ordinary (lumped-) circuit theory and transmission-line (distributed-circuit) theory. In ordinary circuit theory, no time delay is involved; the effect of a transient in one part of the circuit is felt in all branches of the circuit instantaneously. In a transmission-line system, the effect of a transient at one location on the line is felt at a different location on the line only after an interval of time that the wave takes to travel from the first location to the second. Returning now to the circuit in Fig. 6.10(a), the various quantities must satisfy the boundary condition, that is, Kirchhoff's voltage law around the loop. Thus, we have

$$V_0 - I^+R_g - V^+ = 0 \tag{6.33a}$$

We, however, know from (6.31a) that $I^+ = V^+/Z_0$. Hence, we get

$$V_0 - \frac{V^+}{Z_0}R_g - V^+ = 0 \tag{6.33b}$$

or

$$\boxed{\begin{aligned} V^+ &= V_0\frac{Z_0}{R_g + Z_0} && (6.34a) \\ I^+ &= \frac{V^+}{Z_0} = \frac{V_0}{R_g + Z_0} && (6.34b) \end{aligned}}$$

Thus, we note that the situation in Fig. 6.10(a) is equivalent to the circuit shown in Fig. 6.10(b); that is, the voltage source views a resistance equal to the characteristic impedance of the line, across $z = 0$. This is to be expected since only a $(+)$ wave exists at $z = 0$ and the ratio of the voltage to current in the $(+)$ wave is equal to Z_0.

Reflection coefficient

The $(+)$ wave travels toward the load and reaches the termination at $t = l/v_p$. It does not, however, satisfy the boundary condition there since this condition requires the voltage across the load resistance to be equal to the current through it times its value, R_L. But the voltage-to-current ratio in the $(+)$ wave is equal to Z_0. To resolve this inconsistency, there is only one possibility, which is the setting up of a $(-)$ wave, or a reflected wave. Let the voltage and current in this reflected wave be V^- and I^-, respectively. Then the total voltage across R_L is $V^+ + V^-$, and the total current through it is $I^+ + I^-$, as shown in Fig. 6.11(a). To satisfy the boundary condition, we have

$$V^+ - V^- = R_L(I^+ + I^-) \tag{6.35a}$$

But from (6.31a) and (6.31b), we know that $I^+ = V^+/Z_0$ and $I^- = -V^-/Z_0$, respectively. Hence,

$$V^+ - V^- = R_L\left(\frac{V^+}{Z_0} - \frac{V^-}{Z_0}\right) \tag{6.35b}$$

or

$$\boxed{V^- = V^+\frac{R_L - Z_0}{R_L + Z_0}} \tag{6.36}$$

We now define the *voltage reflection coefficient*, denoted by the symbol Γ, as the ratio of the reflected voltage to the incident voltage. Thus,

$$\boxed{\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_0}{R_L + Z_0}} \tag{6.37}$$

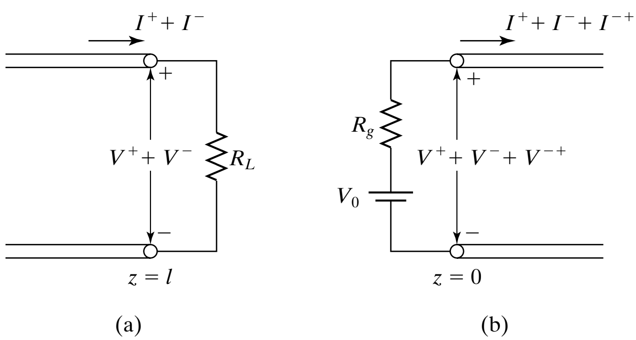


FIGURE 6.11

For obtaining the voltages and currents associated with (a) the $(-)$ wave and (b) the $(-+)$ wave, for the line of Fig. 6.9.

We then note that the *current reflection coefficient* is

$$\boxed{\frac{I^-}{I^+} = \frac{-V^-/Z_0}{V^+/Z_0} = -\frac{V^-}{V^+} = -\Gamma} \tag{6.38}$$

Now, returning to the reflected wave, we observe that this wave travels back toward the source and it reaches there at $t = 2l/v_p$. Since the boundary condition at $z = 0$, which was satisfied by the original $(+)$ wave alone, is then violated, a reflection of the reflection, or a re-reflection, will be set up and it travels toward the load. Let us assume the voltage and current in this re-reflected wave, which is a $(+)$ wave, to be V^{-+} and I^{-+}, respectively, with the superscripts denoting that the $(+)$ wave is a consequence of the $(-)$ wave. Then the total line voltage and the line current at $z = 0$ are $V^+ + V^- + V^{-+}$ and $I^+ + I^- + I^{-+}$, respectively, as shown in Fig. 6.11(b). To satisfy the boundary condition, we have

$$V^+ + V^- + V^{-+} = V_0 - R_g(I^+ + I^- + I^{-+}) \tag{6.39a}$$

But we know that $I^+ = V^+/Z_0$, $I^- = -V^-/Z_0$, and $I^{-+} = V^{-+}/Z_0$. Hence,

$$V^+ + V^- + V^{-+} = V_0 - \frac{R_g}{Z_0}(V^+ - V^- + V^{-+}) \tag{6.39b}$$

Furthermore, substituting for V^+ from (6.34a), simplifying, and rearranging, we get

$$V^{-+}\left(1 + \frac{R_g}{Z_0}\right) = V^-\left(\frac{R_g}{Z_0} - 1\right)$$

or

$$\boxed{V^{-+} = V^-\frac{R_g - Z_0}{R_g + Z_0}} \tag{6.40}$$

Reflection coefficients for some special cases

Comparing (6.40) with (6.36), we note that the reflected wave views the source with internal resistance as the internal resistance alone; that is, the voltage source is equivalent to a short circuit insofar as the $(-)$ wave is concerned. A moment's thought will reveal that superposition is at work here. The effect of the voltage source is taken into account by the constant outflow of the original $(+)$ wave from the source. Hence, for the reflection of the reflection, that is, for the $(-+)$ wave, we need only consider the internal resistance R_g. Thus, the voltage reflection coefficient formula (6.37) is a general formula and will be used repeatedly. In view of its importance, a brief discussion of the values of Γ for some special cases is in order as follows:

1. $R_L = 0$, or short-circuited line.

$$\Gamma = \frac{0 - Z_0}{0 + Z_0} = -1$$

The reflected voltage is exactly the negative of the incident voltage, thereby keeping the voltage across R_L (short circuit) always zero.

2. $R_L = \infty$, or open-circuited line.

$$\Gamma = \frac{\infty - Z_0}{\infty + Z_0} = 1$$

and the current reflection coefficient $= -\Gamma = -1$. Thus, the reflected current is exactly the negative of the incident current, thereby keeping the current through R_L (open circuit) always zero.

3. $R_L = Z_0$, or line terminated by its characteristic impedance.

$$\Gamma = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

This corresponds to no reflection, which is to be expected since $R_L(= Z_0)$ is consistent with the voltage-to-current ratio in the $(+)$ wave alone, and, hence, there is no violation of boundary condition and no need for the setting up of a reflected wave. Thus, a line terminated by its characteristic impedance is equivalent to an infinitely long line insofar as the source is concerned.

Bounce diagram

Returning to the discussion of the re-reflected wave, we note that this wave reaches the load at time $t = 3l/v_p$ and sets up another reflected wave. This process of bouncing back and forth of waves goes on indefinitely until the steady state is reached. To keep track of this transient phenomenon, we make use of the *bounce-diagram* technique. Some other names given for this diagram are *reflection diagram* and *space-time diagram*. We shall introduce the bounce diagram through a numerical example.

Example 6.1 Bounce-diagram technique for time-domain analysis of a transmission-line system

Let us consider the system shown in Fig. 6.12. Note that we have introduced a new quantity T, which is the one-way travel time along the line from $z = 0$ to $z = l$; that is, instead of specifying two quantities l and v_p, we specify $T(= l/v_p)$. Using the bounce diagram technique, we wish to obtain and plot line voltage and current versus t for fixed values z and line voltage and current versus z for fixed values t.

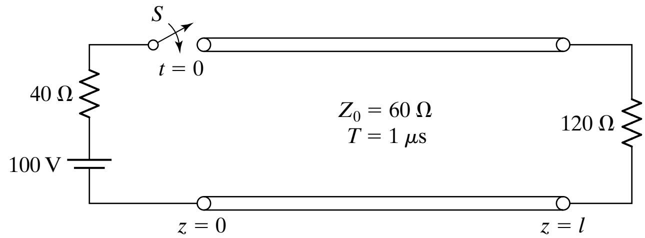


FIGURE 6.12

Transmission-line system for illustrating the bounce-diagram technique of keeping track of the transient phenomenon.

Before we construct the bounce diagram, we need to compute the following quantities:

$$\text{Voltage carried by the initial } (+) \text{ wave} = 100\frac{60}{40+60} = 60 \text{ V}$$

$$\text{Current carried by the initial } (+) \text{ wave} = \frac{60}{60} = 1 \text{ A}$$

$$\text{Voltage reflection coefficient at load, } \Gamma_R = \frac{120-60}{120+60} = \frac{1}{3}$$

$$\text{Voltage reflection coefficient at source, } \Gamma_S = \frac{40-60}{40+60} = -\frac{1}{5}$$

Construction of bounce diagrams

The bounce diagram is essentially a two-dimensional representation of the transient waves bouncing back and forth on the line. Separate bounce diagrams are drawn for voltage and current, as shown in Fig. 6.13(a) and (b), respectively. Position (z) on the line is represented horizontally and the time (t) vertically. Reflection coefficient values for the two ends are shown at the top of the diagrams for quick reference. Note that current reflection coefficients are $-\Gamma_R = -\frac{1}{3}$ and $-\Gamma_S = \frac{1}{5}$, respectively, at the load and at the source. Crisscross lines are drawn as shown in the figures to indicate the progress of the wave as a function of both z and t, with the numerical value for each leg of travel shown beside the line corresponding to that leg and approximately at the center of the line. The arrows indicate the directions of travel. Thus, for example, the first line on the voltage bounce diagram indicates that the initial $(+)$ wave of 60 V takes a time of 1 μs to reach the load end of the line. It sets up a reflected wave of 20 V, which travels back to the source, reaching there at a time of 2 μs, which then gives rise to a $(+)$ wave of voltage -4 V, and so on, with the process continuing indefinitely.

Plots of line voltage and current versus t

Now, to sketch the line voltage and/or current versus time at any value of z, we note that since the voltage source is a constant voltage source, each individual wave voltage and current, once the wave is set up at that value of z, continues to exist there forever. Thus, at any particular time, the voltage (or current) at that value of z is a superposition of all the voltages (or currents) corresponding to the crisscross lines preceding that value of time. These values are marked on the bounce diagrams for $z = 0$ and $z = l$. Noting that each value corresponds to the 2-μs time interval between adjacent crisscross lines, we now sketch the time variations of line voltage and current at $z = 0$ and $z = l$, as shown in Figs. 6.14(a) and (b), respectively. Similarly, by observing that the numbers written along the time axis for $z = 0$ are actually valid for any pair of z and t within the triangle ($\triangleright$) inside which they lie and that the numbers written along the time axis for $z = l$ are actually valid for any pair of z and t within the triangle ($\triangleleft$) inside which they lie, we can draw the sketches of line voltage and current versus time for any other value of z. This is done for $z = l/2$ in Fig. 6.14(c).

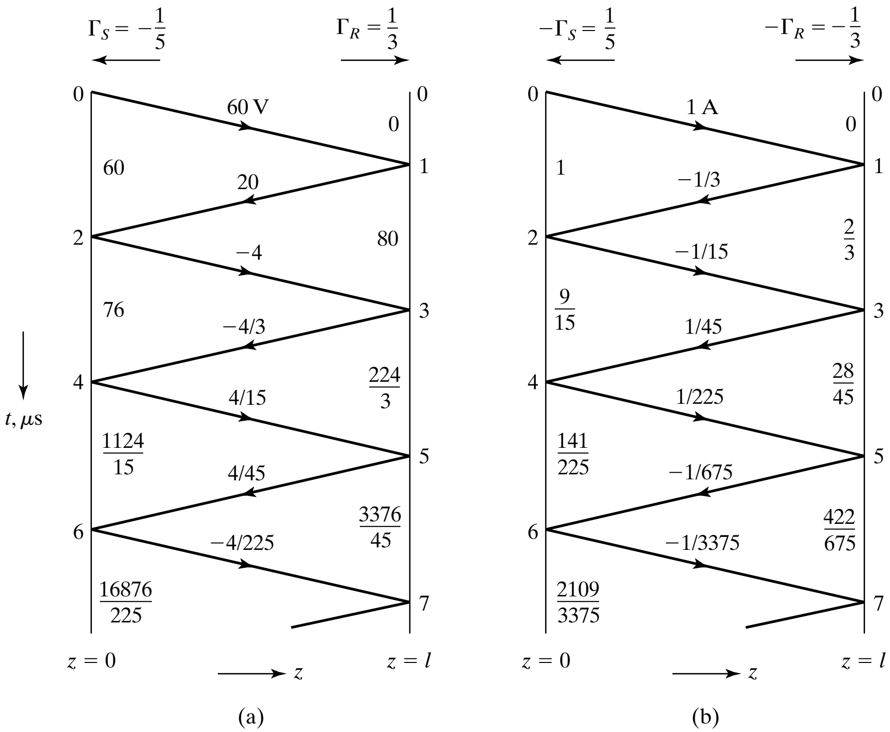


FIGURE 6.13

(a) Voltage and (b) current bounce diagrams, depicting the bouncing back and forth of the transient waves for the system of Fig. 6.12.

Steady-state situation

It can be seen from the sketches of Fig. 6.14 that as time progresses, the line voltage and current tend to converge to certain values, which we can expect to be the steady-state values. In the steady state, the situation consists of a single (+) wave, which is actually a superposition of the infinite number of transient (+) waves, and a single (−) wave, which is actually a superposition of the infinite number of transient (−) waves. Denoting the steady-state (+) wave voltage and current to be V_{SS}^{+} and I_{SS}^{+}, respectively, and the steady-state (−) wave voltage and current to be V_{SS}^{-} and I_{SS}^{-}, respectively, we obtain from the bounce diagrams

$$V_{SS}^{+} = 60 - 4 + \frac{4}{15} - \cdots = 60\left(1 - \frac{1}{15} + \frac{1}{15^2} - \cdots\right) = 56.25 \text{ V}$$

$$I_{SS}^{+} = 1 - \frac{1}{15} + \frac{1}{225} - \cdots = 1 - \frac{1}{15} + \frac{1}{15^2} - \cdots = 0.9375 \text{ A}$$

$$V_{SS}^{-} = 20 - \frac{4}{3} + \frac{4}{45} - \cdots = 20\left(1 - \frac{1}{15} + \frac{1}{15^2} - \cdots\right) = 18.75 \text{ V}$$

$$I_{SS}^{-} = -\frac{1}{3} + \frac{1}{45} - \frac{1}{675} + \cdots = -\frac{1}{3}\left(1 - \frac{1}{15} + \frac{1}{15^2} - \cdots\right) = -0.3125 \text{ A}$$

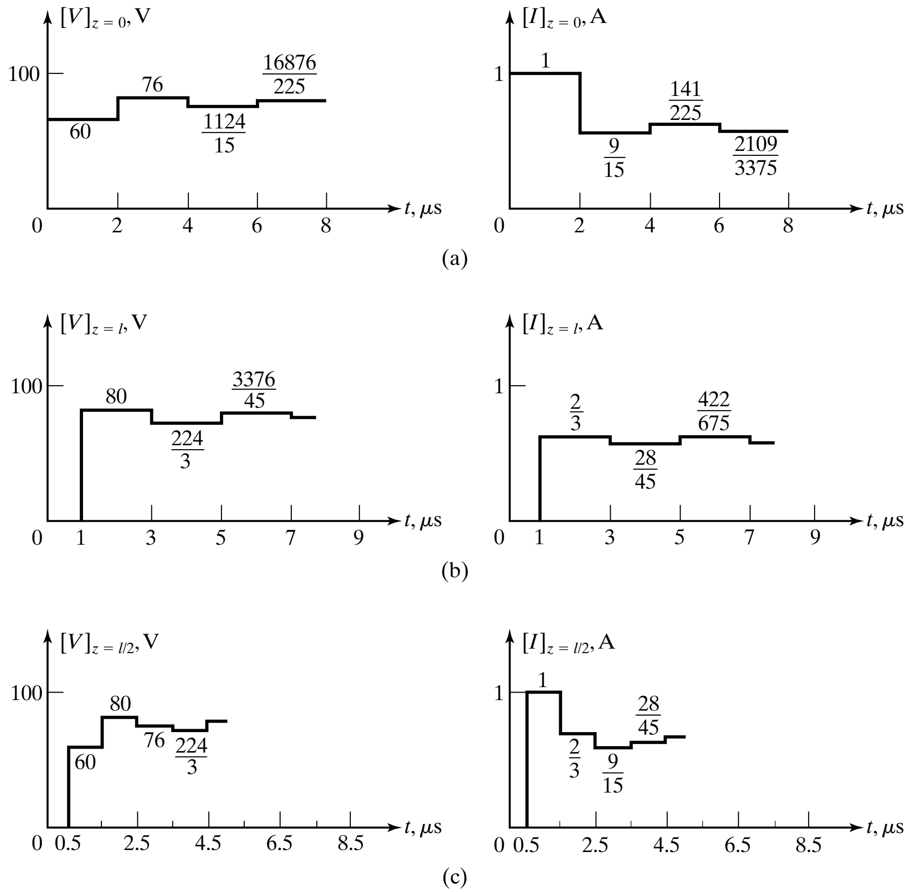


FIGURE 6.14

Time variations of line voltage and line current at (a) $z = 0$, (b) $z = l$, and (c) $z = l/2$ for the system of Fig. 6.12.

Note that $I_{SS}^+ = V_{SS}^+/Z_0$ and $I_{SS}^- = -V_{SS}^-/Z_0$, as they should be. The steady-state line voltage and current can now be obtained to be

$$V_{SS} = V_{SS}^+ + V_{SS}^- = 75 \text{ V}$$
$$I_{SS} = I_{SS}^+ + I_{SS}^- = 0.625 \text{ A}$$

These are the same as the voltage across R_L and current through R_L if the source and its internal resistance were connected directly to R_L, as shown in Fig. 6.15. This is to be expected since the series inductors and shunt capacitors of the distributed equivalent circuit behave like short circuits and open circuits, respectively, for the constant voltage source in the steady state.

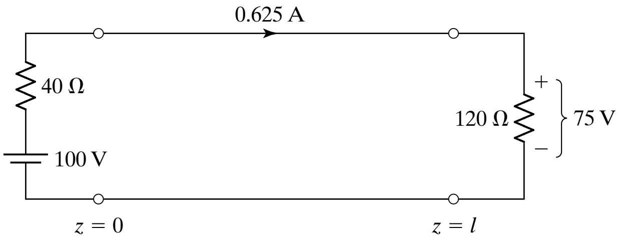


FIGURE 6.15

Steady-state equivalent for the system of Fig. 6.12.

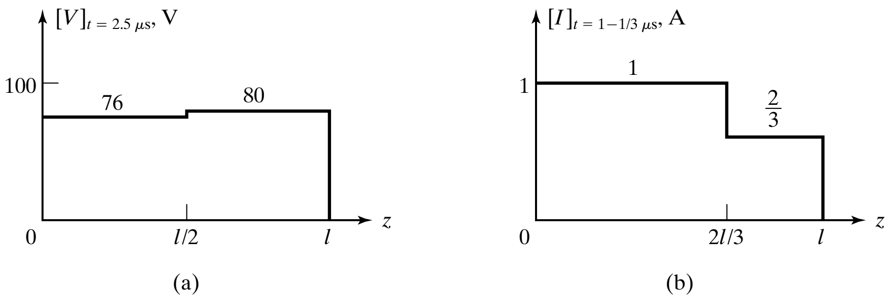


FIGURE 6.16

Variations with z of (a) line voltage for $t = 2.5\ \mu\text{s}$ and (b) line current for $t = 1\frac{1}{3}\ \mu\text{s}$, for the system of Fig. 6.12.

Plots of line voltage and current versus z

Sketches of line voltage and current as functions of distance (z) along the line for any particular time can also be drawn from considerations similar to those employed for the sketches of Fig. 6.14. For example, suppose we wish to draw the sketch of line voltage versus z for $t = 2.5\ \mu\text{s}$. Then we note from the voltage bounce diagram that for $t = 2.5\ \mu\text{s}$, the line voltage is 76 V from $z = 0$ to $z = l/2$ and 80 V from $z = l/2$ to $z = l$. This is shown in Fig. 6.16(a). Similarly, Fig. 6.16(b) shows the variation of line current versus z for $t = 1\frac{1}{3}\ \mu\text{s}$.

Excitation by pulse voltage source

In Example 6.1, we introduced the bounce-diagram technique for a constant-voltage source. The technique can also be applied if the voltage source is a pulse. In the case of a rectangular pulse, this can be done by representing the pulse as the superposition of two step functions, as shown in Fig. 6.17, and superimposing the bounce diagrams for the two sources one on another. In doing so, we should note that the bounce diagram for one source begins at a value of time greater than zero. Alternatively, the time variation for each wave can be drawn alongside the time axes beginning at the time of start of the wave. These can then be used to plot the required sketches. An example is in order to illustrate this technique, which can also be used for a pulse of arbitrary shape.

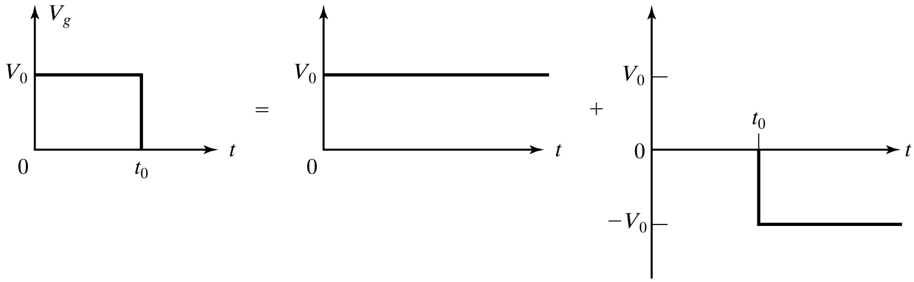


FIGURE 6.17

Representation of a rectangular pulse as the superposition of two step functions.

Example 6.2 Bounce-diagram technique for a pulse excitation

Let us assume that the voltage source in the system of Fig. 6.12 is a 100-V rectangular pulse extending from $t = 0$ to $t = 1$ μs and extend the bounce-diagram technique.

Considering, for example, the voltage bounce diagram, we reproduce in Fig. 6.18 part of the voltage bounce diagram of Fig. 6.13(a) and draw the time variations of the individual pulses alongside the time axes, as shown in the figure. Note that voltage axes are chosen such that positive values are to the left at the left end $(z = 0)$ of the diagram, but to the right at the right end $(z = l)$ of the diagram.

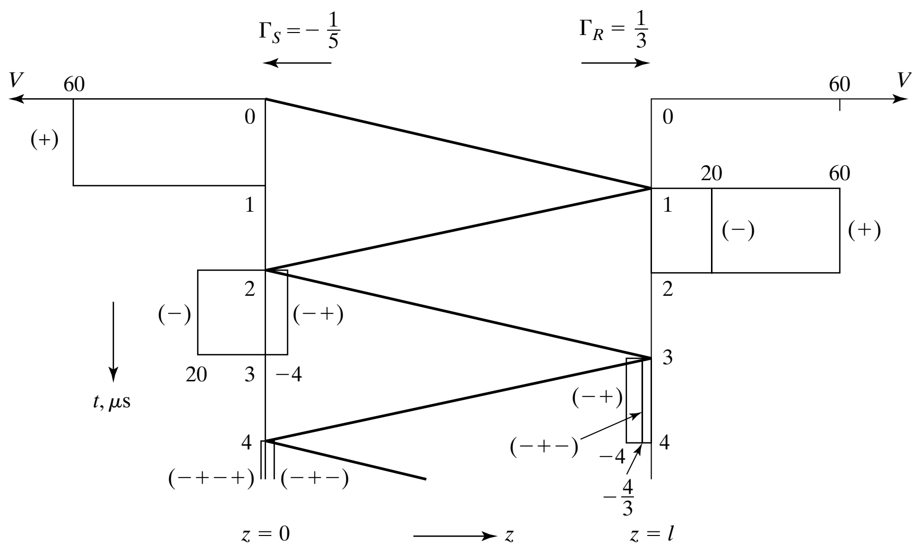


FIGURE 6.18

Voltage bounce diagram for the system of Fig. 6.12 except that the voltage source is a rectangular pulse of 1-μs duration from $t = 0$ to $t = 1$ μs.

From the voltage bounce diagram, sketches of line voltage versus time at $z = 0$ and $z = l$ can be drawn, as shown in Figs. 6.19(a) and (b), respectively. To draw the sketch of line voltage versus time for any other value of z, we note that as time progresses, the (+) wave pulses slide down the crisscross lines from left to right, whereas the (−)

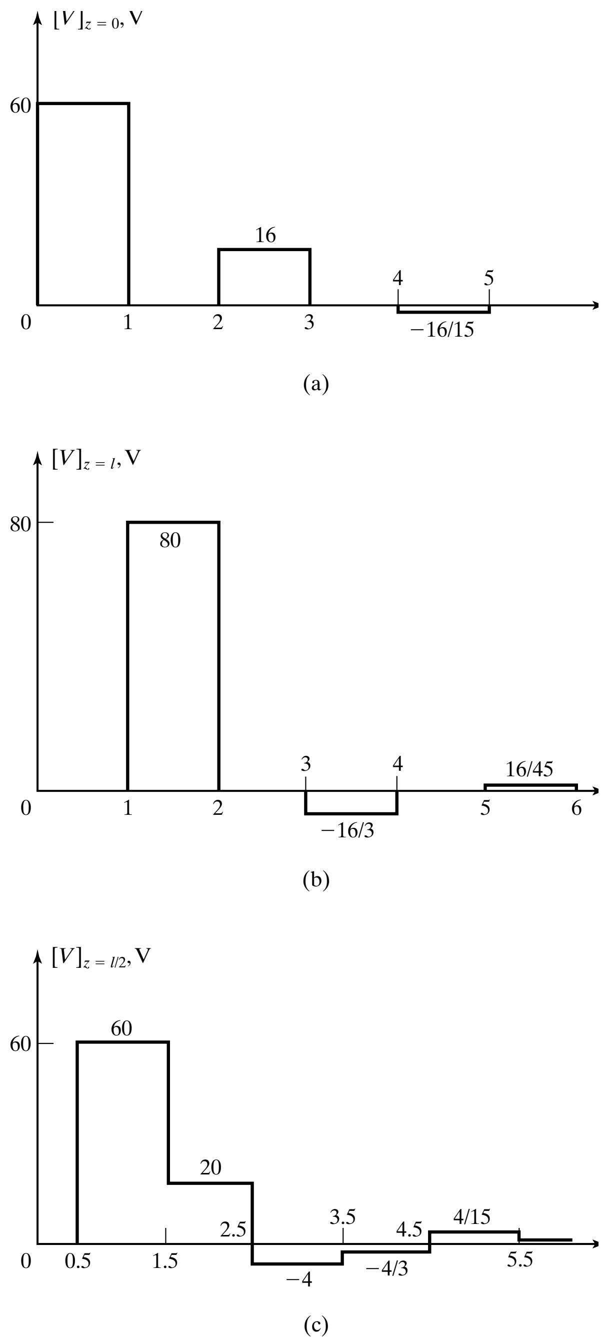


FIGURE 6.19

Time variations of line voltage at (a) $z = 0$, (b) $z = l$, and (c) $z = l/2$ for the system of Fig. 6.12, except that the voltage source is a rectangular pulse of 1-μs duration from $t = 0$ to $t = 1\ \mu$s.

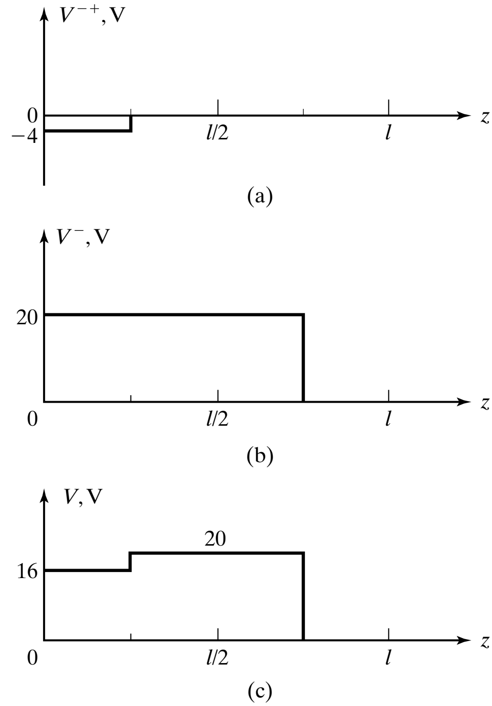


FIGURE 6.20

Variations with z of (a) the $(-+)$ wave voltage, (b) the $(-)$ wave voltage, and (c) the total line voltage, at $t = 2.25\ \mu$s for the system of Fig. 6.12, except that the voltage source is a rectangular pulse of 1-μs duration from $t = 0$ to $t = 1\ \mu$s.

wave pulses slide down from right to left. Thus, to draw the sketch for $z = l/2$, we displace the time plots of the $(+)$ waves at $z = 0$ and of the $(-)$ waves at $z = l$ forward in time by 0.5 μs, that is, delay them by 0.5 μs, and add them to obtain the plot shown in Fig. 6.19(c).

Sketches of line voltage versus distance (z) along the line for fixed values of time can also be drawn from the bounce diagram based on the phenomenon of the individual pulses sliding down the crisscross lines. Thus, if we wish to sketch $V(z)$ for $t = 2.25\ \mu$s, then we take the portion from $t = 2.25\ \mu$s back to $t = 2.25 - 1 = 1.25\ \mu$s (since the one-way travel time on the line is 1 μs) of all the $(+)$ wave pulses at $z = 0$ and lay them on the line from $z = 0$ to $z = l$, and we take the portion from $t = 2.25\ \mu$s back to $t = 2.25 - 1 = 1.25\ \mu$s of all the $(-)$ wave pulses at $z = l$ and lay them on the line from $z = l$ back to $z = 0$. In this case, we have only one $(+)$ wave pulse—that of the $(-+)$ wave—and only one $(-)$ wave pulse—that of the $(-)$ wave—as shown in Figs. 6.20(a) and (b). The line voltage is then the superposition of these two waveforms, as shown in Fig. 6.20(c).

Similar considerations apply for the current bounce diagram and plots of line current versus t for fixed values of z and line current versus z for fixed values of t.

K6.2. $(+)$ wave; $(-)$ wave; Voltage reflection coefficient; Current reflection coefficient; Transient bouncing of waves; Voltage bounce diagram; Current bounce diagram; Steady-state situation; Rectangular pulse voltage source; Superposition.

D6.3. In the system shown in Fig. 6.21, the switch S is closed at $t = 0$. Find the value of R_L for each of the following cases: **(a)** $V(0.5l, 1.7\ \mu\text{s}) = 48$ V; **(b)** $V(0.6l, 2.8\ \mu\text{s}) = 76$ V; **(c)** $I(0.3l, 4.4\ \mu\text{s}) = 1$ A; and **(d)** $I(0.4l, \infty) = 2.5$ A.

Ans. **(a)** 40 Ω; **(b)** 120 Ω; **(c)** 60 Ω; **(d)** 0 Ω.

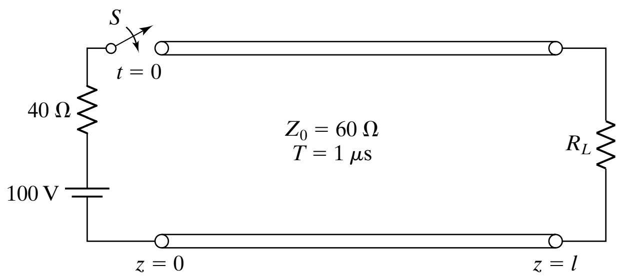


FIGURE 6.21

For Problem D6.4.

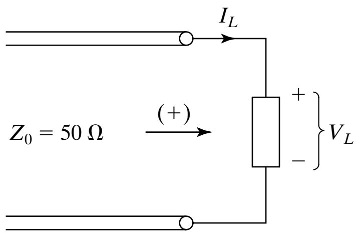


FIGURE 6.22

For Problem D6.5.

D6.4. For a line of characteristic impedance 75 Ω terminated by a resistance and driven by a constant-voltage source in series with an internal resistance, the line voltage and current in the steady state are known to be 30 V and 1.2 A, respectively. Find **(a)** the (+) wave voltage; **(b)** the (−) wave voltage; **(c)** the (+) wave current; and **(d)** the (−) wave current in the steady state.

Ans. **(a)** 60 V; **(b)** −30 V; **(c)** 0.8 A; **(d)** 0.4 A.

D6.5. In Fig 6.22, a line of characteristic impedance 50 Ω is terminated by a *passive* nonlinear element. A (+) wave of constant voltage V_0 is incident on the termination. If the volt-ampere characteristic of the nonlinear element in the region of interest is $V_L = 50\, I_L^2$, find the (−) wave voltage for each of the following values of V_0: **(a)** 36 V; **(b)** 50 V; and **(c)** 66 V.

Ans. **(a)** −4 V; **(b)** 0 V; **(c)** 6 V.

6.3 TRANSMISSION-LINE DISCONTINUITY

Junction between two lines

We now consider the case of a junction between two lines having different values for the parameters Z_0 and v_p, as shown in Fig. 6.23. Assuming that a (+) wave of voltage V^+ and current I^+ is incident on the junction from line 1, we

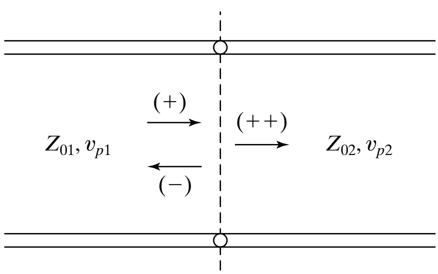


FIGURE 6.23

Transmission-line junction for illustrating reflection (−) and transmission (++) resulting from an incident (+) wave.

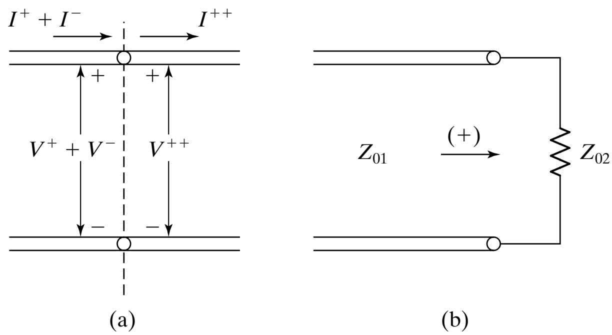


FIGURE 6.24

(a) For obtaining the reflected (−) wave and transmitted (++) wave voltages and currents for the system of Fig. 6.23.
(b) Equivalent to (a) for using the reflection coefficient concept.

find that the (+) wave alone cannot satisfy the boundary condition at the junction, since the voltage-to-current ratio for that wave is Z_{01}, whereas the characteristic impedance of line 2 is $Z_{02} \neq Z_{01}$. Hence, a reflected wave and a transmitted wave are set up such that the boundary conditions are satisfied. Let the voltages and currents in these waves be V^-, I^-, and V^{++}, I^{++}, respectively, where the superscript ++ denotes that the transmitted wave is a (+) wave resulting from the incident (+) wave. We then have the situation shown in Fig. 6.24(a).

Using the boundary conditions at the junction, we then write

$$V^+ + V^- = V^{++} \tag{6.41a}$$

$$I^+ + I^- = I^{++} \tag{6.41b}$$

But we know that $I^+ = V^+/Z_{01}$, $I^- = -V^-/Z_{01}$, and $I^{++} = V^{++}/Z_{02}$. Hence, (6.41b) becomes

$$\frac{V^+}{Z_{01}} - \frac{V^-}{Z_{01}} = \frac{V^{++}}{Z_{02}} \tag{6.42}$$

Combining (6.41a) and (6.42), we have

$$V^+ + V^- = \frac{Z_{02}}{Z_{01}}(V^+ - V^-)$$

$$V^-\left(1 + \frac{Z_{02}}{Z_{01}}\right) = V^+\left(\frac{Z_{02}}{Z_{01}} - 1\right)$$

or

$$\boxed{\Gamma = \frac{V^-}{V^+} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}} \tag{6.43}$$

Thus, to the incident wave, the transmission line to the right looks like its characteristic impedance Z_{02}, as shown in Fig. 6.24 (b). The difference between a resistive load of Z_{02} and a line of characteristic impedance Z_{02} is that, in the first case, power is dissipated in the load, whereas in the second ease, the power is transmitted into the line.

Transmission coefficient

We now define the *voltage transmission coefficient*, denoted by the symbol τ_V, as the ratio of the transmitted wave voltage to the incident wave voltage. Thus,

$$\boxed{\tau_V = \frac{V^{++}}{V^+} = \frac{V^+ + V^-}{V^+} = 1 + \frac{V^-}{V^+} = 1 + \Gamma} \tag{6.44}$$

The *current transmission coefficient*, τ_C, which is the ratio of the transmitted wave current to the incident wave current, is given by

$$\boxed{\tau_C = \frac{I^{++}}{I^+} = \frac{I^- + I^-}{I^+} = 1 + \frac{I^-}{I^+} = 1 - \Gamma} \tag{6.45}$$

At this point, one may be puzzled to note that the transmitted voltage can be greater than the incident voltage if Γ is positive. However, this is not of concern, since then the transmitted current will be less than the incident current. Similarly, the transmitted current is greater than the incident current when Γ is negative, but then the transmitted voltage is less than the incident voltage. In fact, what is important is that the transmitted power P^{++} is always less than (or equal to) the incident power P^+, since

$$\begin{aligned} P^{++} &= V^{++}I^{++} = V^+(1 + \Gamma)I^+(1 - \Gamma) \\ &= V^+I^+(1 - \Gamma^2) = (1 - \Gamma^2)P^+ \end{aligned} \tag{6.46}$$

and $(1 - \Gamma^2) \leq 1$, irrespective of whether Γ is positive or negative.

We shall illustrate the application of reflection and transmission at a junction between lines by means of an example.

Example 6.3 Unit impulse response and frequency response for a system of three lines in cascade

System of three lines

Let us consider the system of three lines in cascade, driven by a unit impulse voltage source $\delta(t)$, as shown in Fig. 6.25(a). We wish to find the output voltage V_o, thereby obtaining the unit impulse response.

To find the output voltage, we draw the voltage bounce diagram, as shown in Fig. 6.25(b). In drawing the bounce diagram, we note that since the internal resistance of the voltage source is 50 Ω, which is equal to Z_{01}, the strength of the impulse that the generator supplies to line 1 is $\frac{1}{2}$. The strengths of the various impulses propagating in the lines are then governed by the reflection and transmission coefficients indicated on the diagram. Also note that the numbers indicated beside the crisscross lines are simply the strengths of the impulses and do not represent constant voltages.

Unit impulse response

From the bounce diagram, we note that the output voltage is a series of impulses. In fact, the phenomenon can be visualized without even drawing the bounce diagram, and the strengths of the impulses can be computed. Thus, the strength of the first impulse, which occurs at $t = T_1 + T_2 + T_3 = 6\ \mu\text{s}$, is

$$1 \times \frac{50}{50 + 50} \times \left(1 + \frac{100 - 50}{100 + 50}\right) \times \left(1 + \frac{50 - 100}{50 + 100}\right) = 1 \times \frac{1}{2} \times \frac{4}{3} \times \frac{2}{3} = \frac{4}{9}$$

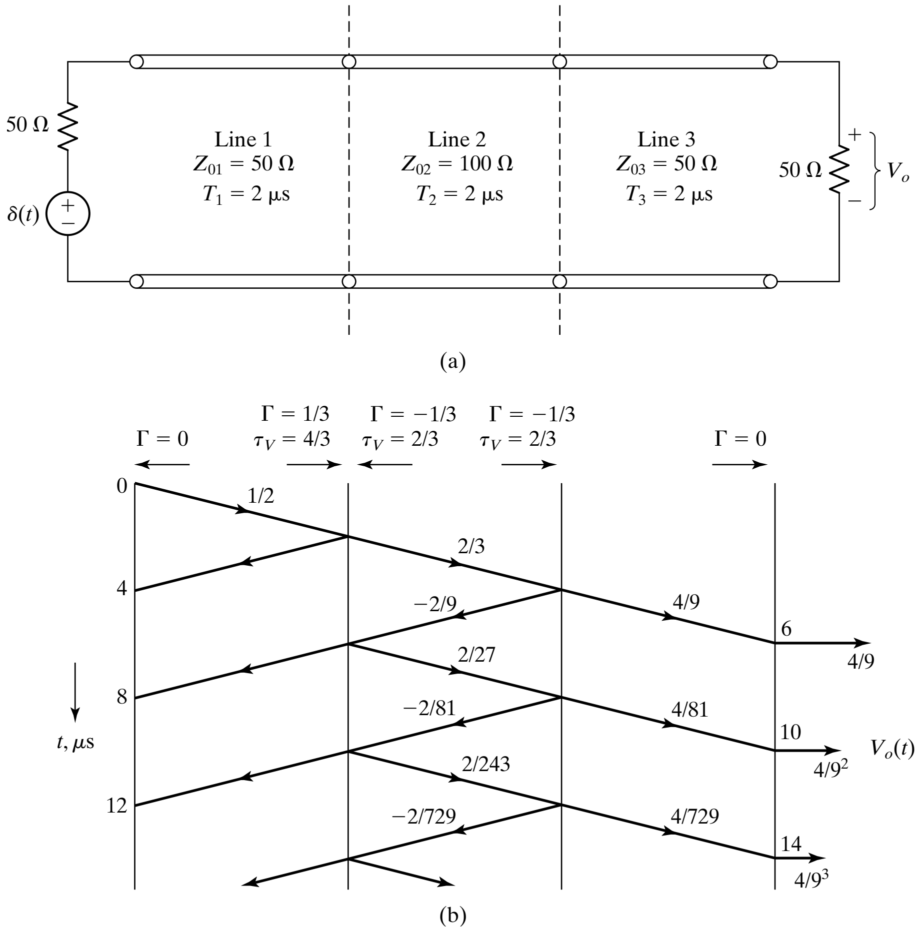


FIGURE 6.25

(a) System of three lines in cascade driven by a unit impulse voltage source. (b) Voltage bounce diagram for finding the output voltage $V_o(t)$ for the system of (a).

Each succeeding impulse is due to the additional reflection and re-reflection of the previous impulse at the right and left end, respectively, of line 2. Hence, each succeeding impulse occurs $2T_2$, or 4 μs, later than the preceding one, and its strength is

$$\left(\frac{50 - 100}{50 + 100}\right)\left(\frac{50 - 100}{50 + 100}\right) = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{1}{9}$$

times the strength of the previous impulse. We can now write the output voltage as

$$\begin{aligned} V_o(t) &= \frac{4}{9}\,\delta(t - 6 \times 10^{-6}) + \frac{4}{9^2}\,\delta(t - 10 \times 10^{-6}) \\ &\quad + \frac{4}{9^3}\,\delta(t - 14 \times 10^{-6}) + \cdots \\ &= \frac{4}{9}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n \delta(t - 4n \times 10^{-6} - 6 \times 10^{-6}) \end{aligned} \tag{6.47}$$

Note that $\frac{4}{9}$ is the strength of the first impulse and $\frac{1}{9}$ is the multiplication factor for each succeeding impulse. In terms of T_1, T_2, and T_3, we have

$$\begin{aligned} V_o(t) &= \frac{4}{9}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n \delta[t - 2nT_2 - (T_1 + T_2 + T_3)] \\ &= \frac{4}{9}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n \delta(t - 2nT_2 - T_0) \end{aligned} \tag{6.48}$$

where we have replaced $T_1 + T_2 + T_3$ by T_0.

Frequency response

Proceeding further, since the unit impulse response of the system is a series of impulses delayed in time, the response to any other excitation can be found by the superposition of time functions obtained by delaying the exciting function and multiplying by appropriate constants. In particular, by considering $V_g(t) = \cos \omega t$, we can find the frequency response of the system. Thus, assuming $V_g(t) = \cos \omega t$ and substituting the cosine function for the impulse function in (6.48), we obtain the corresponding output voltage to be

$$V_o(t) = \frac{4}{9}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n \cos \omega(t - 2nT_2 - T_0) \tag{6.49}$$

The complex voltage $\overline{V}_o(\omega)$ is then given by

$$\begin{aligned} \overline{V}_o(\omega) &= \frac{4}{9}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n e^{-j\omega(2nT_2+T_0)} \\ &= \frac{4}{9}e^{-j\omega T_0}\sum_{n=0}^{\infty}\left(\frac{1}{9}e^{-j2\omega T_2}\right)^n \\ &= \frac{(4/9)e^{-j\omega T_0}}{1 - (1/9)e^{-j2\omega T_2}} \end{aligned} \tag{6.50}$$

Without going into a detailed discussion of the result given by (6.50), we can conclude the following: maximum amplitude occurs for $2\omega T_2 = 2m\pi$, $m = 0, 1, 2, \ldots$; that is, for $\omega = m\pi/T_2$, $m = 0, 1, 2, \ldots$, and its value is $\frac{4}{9}/(1 - \frac{1}{9}) = 0.5$. Minimum amplitude occurs for $2\omega T_2 = (2m + 1)\pi$, $m = 0, 1, 2, \ldots$; that is, for $\omega = (2m + 1)\pi/2T_2$, $m = 0, 1, 2, \ldots$, and its value is $\frac{4}{9}/(1 + \frac{1}{9}) = 0.4$. The amplitude response therefore can be roughly sketched, as shown in Fig. 6.26.

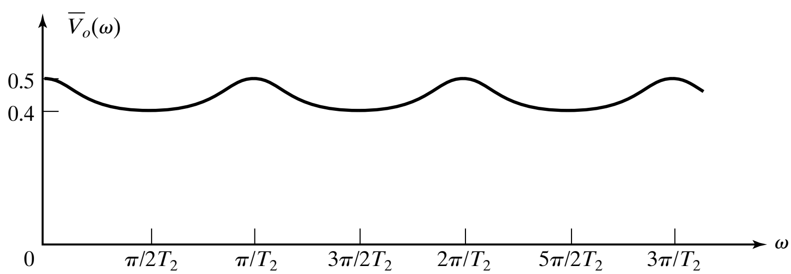


FIGURE 6.26

Rough sketch of amplitude response versus frequency for the system of Fig. 6.25(a) for sinusoidal excitation.

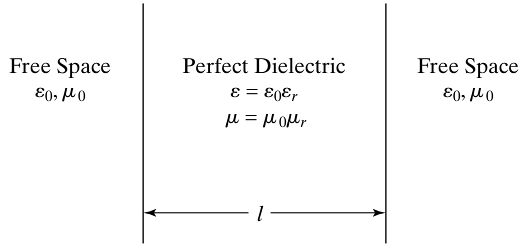


FIGURE 6.27

Perfect dielectric slab with free space on either side.

Radome

A practical situation in which the discussion of this example is applicable is in the design of a radome, which is an enclosure for protecting an antenna from the weather while allowing transparency for electromagnetic waves. A simple, idealized, planar version of the radome is a dielectric slab with free space on either side of it, as shown in Fig. 6.27. For reflection and transmission of uniform plane waves incident normally onto the dielectric slab, the arrangement is equivalent to three lines in cascade, with the characteristic impedances equal to the intrinsic impedances of the media and the velocities of propagation equal to those in the media. Thus, the amplitude versus frequency response is of the same form as that in Fig. 6.26, where T_2 is the one-way travel time in the dielectric slab and the maximum is 1 instead of 0.5 (the factor of 0.5 in Fig. 6.26 is due to voltage drop across the internal resistance of the source in the transmission-line system). Hence, the lowest frequency for which the dielectric slab is completely transparent is $\omega = \pi/T_2 = \pi c/l\sqrt{\varepsilon_r\mu_r}$ or $f = c/2l\sqrt{\varepsilon_r\mu_r}$. Conversely, for a given frequency f, the minimum thickness for which the slab is transparent is $l = c/2f\sqrt{\varepsilon_r\mu_r} = \lambda/2$, where λ is the wavelength in the dielectric, corresponding to f.

Time-domain reflectometry

We shall now discuss *time-domain reflectometry*, abbreviated TDR, a technique by means of which discontinuities in transmission-line systems can be located by making measurements with pulses. The block diagram of a typical TDR system is shown in Fig. 6.28. It consists of a pulse generator whose output is connected to the system under test through a matched attenuator. Voltage pulses are propagated down the transmission-line system, and the incident and reflected pulses are monitored by the display scope using a high-impedance probe. The matched attenuator serves the purpose of absorbing the pulses arriving back from the system so that reflections of those pulses are not produced. We shall illustrate the application of a TDR system by means of an example.

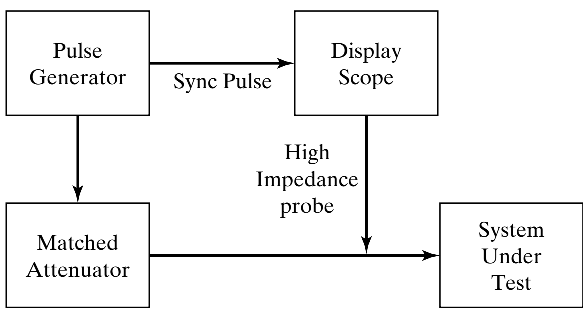


FIGURE 6.28

Block diagram of a typical time-domain reflectometer.

Example 6.4 Application of a time-domain reflectometer system for analyzing a line discontinuity

Let us consider a transmission line under test, as shown in Fig. 6.29, in which a discontinuity exists at $z = 4$ m and the line is short-circuited at the far end. We shall first analyze the system to obtain the waveform measured by a TDR system connected at the input end $z = 0$, assuming the TDR pulses to be of amplitude 1 V, duration 10 ns, and repetition rate 10^5 Hz. We shall then discuss how one can deduce the information about the discontinuity from the TDR measurement.

Assuming that a pulse from the TDR system is incident on the input of the system under test at $t = 0$, we draw the voltage-bounce diagram, as shown in Fig. 6.30. Note that for a pulse incident on the discontinuity from either side, the resistance viewed is the parallel combination of 120 Ω and Z_0 $(= 60\ \Omega)$ of the line, or 40 Ω. Hence, the reflection and transmission coefficients for the voltage are given, respectively, by

$$\Gamma = \frac{40 - 60}{40 + 60} = -0.2$$
$$\tau_V = 1 + \Gamma = 0.8$$

From the bounce diagram, the voltage pulses that would be viewed on the display scope of the TDR system up to $t = 200$ ns are shown in Fig. 6.31. Subsequent pulses become smaller and smaller in amplitude as time progresses and diminish to insignificant values well before $t = 10\ \mu$s, which is the period of the TDR pulses.

Now, to discuss how one can deduce information about the discontinuity from the TDR display of Fig. 6.31, without a prior knowledge of the discontinuity but knowing the values of Z_0 and v_p of the line and that the line is short-circuited at the far end of unknown distance from the input, we proceed in the following manner:

The first pulse is the outgoing pulse from the TDR system. The second pulse arriving at the input of the system under test at $t = 40$ ns is due to reflection from a discontinuity, since if there is no discontinuity, the voltage of the second pulse should be -1 V and there should be no subsequent pulses. From the voltage of the second pulse, we know that the reflection coefficient at the discontinuity is -0.2. The effective resistance R_L seen by the incident pulse is therefore given by the solution of

$$\frac{R_L - 60}{R_L + 60} = -0.2$$

which is 40 Ω. Since this value is less than the Z_0 of the line, the discontinuity must be due entirely to a resistance in parallel with the line or due to a combination of series and

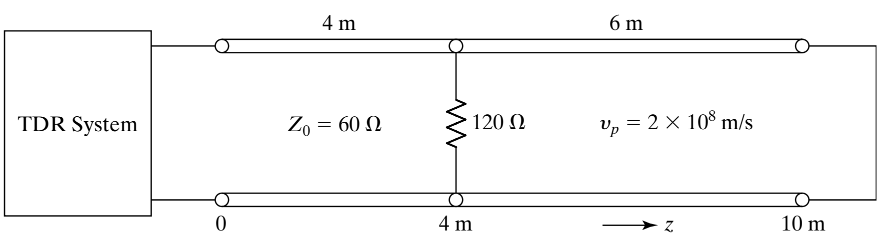


FIGURE 6.29

Transmission line with discontinuity under test by a TDR system.

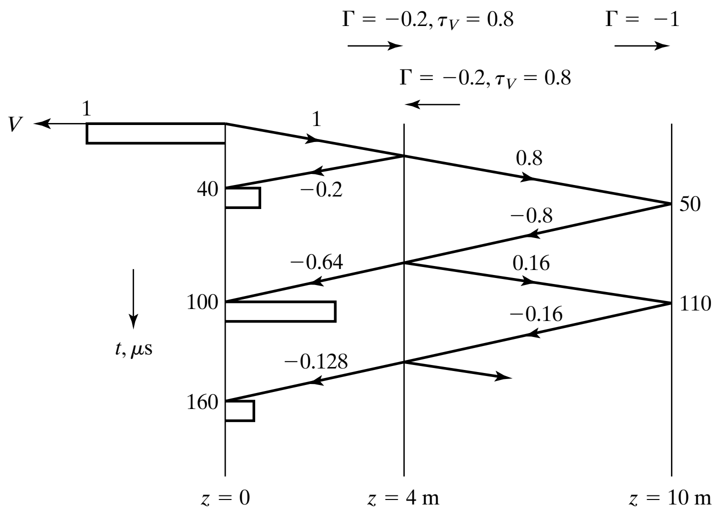


FIGURE 6.30

Voltage-bounce diagram for the system of Fig. 6.29, for TDR pulses of amplitude 1 V.

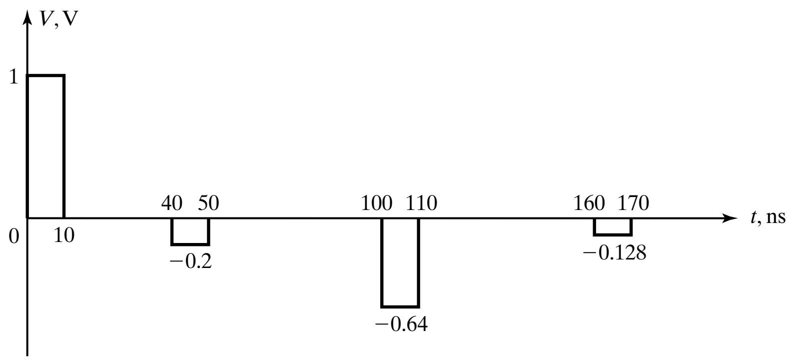


FIGURE 6.31

Voltage versus time at the input of the transmission line of Fig. 6.39, as displayed by the TDR system.

parallel resistors; it cannot be due entirely to a resistance in series with the line. Let us proceed with the assumption of a parallel resistor alone. Then the value of this resistance R must be such that

$$\frac{60R}{60 + R} = 40$$

solving which we obtain $R = 120\ \Omega$. The location of the discontinuity can be deduced by multiplying v_p by 20 ns, which is one-half of the time interval between the first and second pulses. Thus, the location is $2 \times 10^8 \times 20 \times 10^{-9} = 4$ m.

Continuing, let us postulate that the third pulse of -0.64 V at $t = 100$ ns is due to reflection occurring at a second discontinuity located at $z = 4 + 2 \times 10^8 \times (60/2) \times 10^{-9} = 10$ m. In terms of the reflection coefficient at the second discontinuity, denoted Γ_2, the voltage of the third pulse would be $\tau_{VR}\Gamma_2\tau_{VL}$, where τ_{VR} and τ_{VL} are the voltage transmission coefficients at $z = 4$ m, for pulses incident from the right and from the left, respectively. Since τ_{VR} and τ_{VL} are both equal to 0.8, we then have $0.64\,\Gamma_2 = -0.64$ or $\Gamma_2 = -1$, which corresponds to a short circuit, which would then give a fourth pulse of -0.128 V at $t = 160$ ns, and so on. From these reasonings, we confirm the assumption of a parallel resistor of 120 Ω for the discontinuity at $z = 4$ m and also conclude that the short circuit is at $z = 10$ m and that no discontinuities exist between $z = 4$ m and the short circuit. If the value of Γ_2 comes out to be different from -1, then further reasonings are necessary to deduce the information. It should also be noted that the line of reasoning depends on which of the line parameters are known.

K6.3. Voltage transmission coefficient; Current transmission coefficient; Unit impulse response; Frequency response; Time-domain reflectometry.

D6.6. Consider a (+) wave incident from line 1 onto the junction between lines 1 and 2 having characteristic impedances Z_{01} and Z_{02}, respectively. Find the value of Z_{02}/Z_{01} for each of the following cases: **(a)** the reflected wave voltage is $\frac{1}{5}$ times the incident wave voltage: **(b)** the transmitted wave voltage is $\frac{1}{5}$ times the incident wave voltage; **(c)** the reflected wave voltage is $\frac{1}{5}$ times the transmitted wave voltage; and **(d)** the reflected wave current is $\frac{1}{5}$ times the transmitted wave current.
Ans. **(a)** 1.5; **(b)** $\frac{1}{9}$; **(c)** $\frac{5}{3}$; **(d)** $\frac{3}{5}$.

D6.7. The output voltage $V_o(t)$ for a system of three lines in cascade is given by

$$V_o(t) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n \delta(t - 2 \times 10^{-6}\,n - 3 \times 10^{-6})$$

when the input voltage $V_i(t) = \delta(t)$. If $V_i(t) = \cos \omega t$, find the amplitude of $V_o(t)$ for each of the following values of ω: **(a)** $10^6\pi$; **(b)** $1.25 \times 10^6\pi$; and **(c)** $1.5 \times 10^6\pi$.

Ans. **(a)** 0.375; **(b)** 0.2372; **(c)** 0.1875.

D6.8. Consider $(n + 1)$ lines, each of characteristic impedance Z_0, emanating from a common junction, as shown in Fig. 6.32 for $n = 2$. For a wave carrying power P

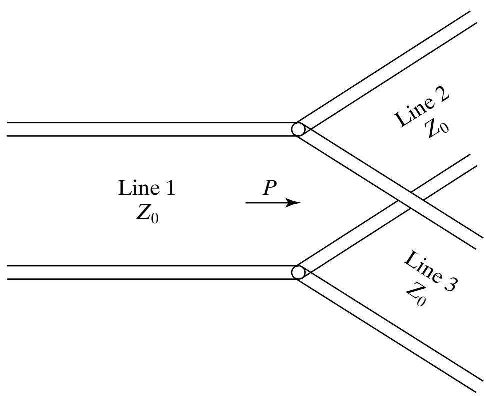


FIGURE 6.32
For Problem D.6.8.

incident on the junction from one of the lines, find the power reflected into that line and the power transmitted into each of the remaining n lines for the following cases: **(a)** $n = 2$; **(b)** $n = 3$; and **(c)** $n = 9$.

Ans. **(a)** $\frac{1}{9}P, \frac{4}{9}P$; **(b)** $\frac{1}{4}P, \frac{1}{4}P$; **(c)** $0.64P, 0.04P$.

6.4 LINES WITH REACTIVE TERMINATIONS AND DISCONTINUES

Inductive termination

Thus far, we have been concerned with purely resistive terminations and discontinuities. Now, we shall consider examples of lines terminated by reactive elements and lines having reactive discontinuities. Let us first consider the system shown in Fig. 6.33, in which a line of length l is terminated by an inductor L with zero initial current and a constant-voltage source with internal resistance equal to the characteristic impedance of the line is connected to the line at $t = 0$. The internal resistance is chosen to be equal to Z_0 so that no reflection takes place at the source end. The moment the switch S is closed at $t = 0$, a $(+)$ wave originates at $z = 0$ with voltage $V^+(= V_0/2)$ and current $I^+(= V_0/2Z_0)$ and travels down the line to reach the load end at time T. Since the inductor current cannot change instantaneously from zero to $V_0/2Z_0$, the boundary condition at $z = l$ is violated, and hence a $(-)$ wave is set up. Let the voltage and current in this $(-)$ wave be $V^-(t)$ and $I^-(t)$, respectively. Then the total voltage across L and the total current through L are $(V_0/2) + V^-$ and $(V_0/2Z_0) - (V^-/Z_0)$, respectively, as shown in Fig. 6.34. To satisfy the boundary condition at $z = l$, we then have

$$\frac{V_0}{2} + V^- = L\frac{d}{dt}\left(\frac{V_0}{2Z_0} - \frac{V^-}{Z_0}\right) \tag{6.51}$$

Noting that V_0 is a constant and hence that dV_0/dt is zero, and rearranging, we obtain

$$\boxed{\frac{L}{Z_0}\frac{dV^-}{dt} + V^- = -\frac{V_0}{2}} \tag{6.52}$$

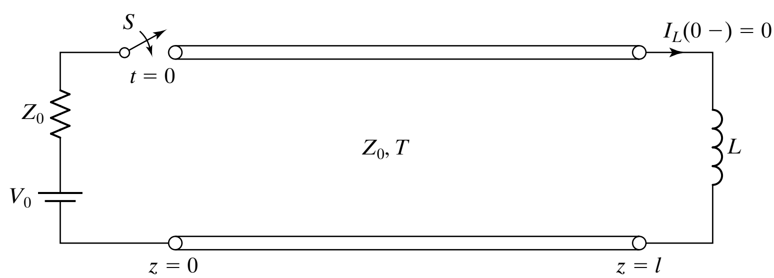


FIGURE 6.33

Line terminated by an inductor with zero initial current and driven by a constant-voltage source in series with internal resistance equal to Z_0 of the line.

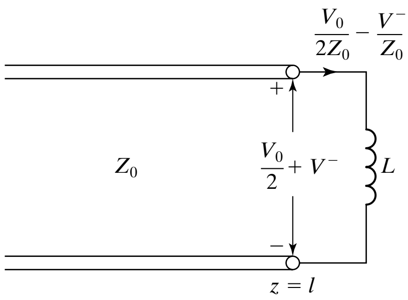


FIGURE 6.34

For obtaining the (−) wave voltage and current for the system of Fig. 6.33

This differential equation for $[V^-]_{z=l}$ has to be solved, subject to the initial condition. This initial condition is that the current through the inductor is zero at $t = T$; that is, the inductor behaves initially like an open circuit. Thus, at $z = l$,

$$\boxed{\left[\frac{V_0}{2Z_0} - \frac{V^-}{Z_0}\right]_{t=T} = 0}$$

or

$$\boxed{[V^-]_{t=T} = \frac{V_0}{2}} \tag{6.53}$$

The general solution for the differential equation can be written as

$$V^- = -\frac{V_0}{2} + Ae^{-(Z_0/L)t} \tag{6.54}$$

where A is an arbitrary constant to be evaluated using (6.53). Thus, we have

$$\frac{V_0}{2} = -\frac{V_0}{2} + Ae^{-(Z_0/L)T}$$

or

$$A = V_0 e^{(Z_0/L)T} \tag{6.55}$$

Substituting this result in (6.54), we obtain the solution for $[V^-]_{z=l}$ as

$$\boxed{V^-(l, t) = -\frac{V_0}{2} + V_0 e^{-(Z_0/L)(t-T)} \qquad \text{for} \quad t > T} \tag{6.56}$$

The corresponding solution for the (−) wave current is given by

$$\boxed{I^-(l, t) = -\frac{V^-(l, t)}{Z_0} = \frac{V_0}{2Z_0} - \frac{V_0}{Z_0} e^{-(Z_0/L)(t-T)} \qquad \text{for} \quad t > T} \tag{6.57}$$

The (−) wave, characterized by V^- and I^- as given by (6.56) and (6.57), respectively, travels back toward the source, and it does not set up a reflected wave, since the reflection coefficient at that end is zero. At this point, it can be seen that unlike in the case of linear resistive terminations and discontinuities, the concept of the reflection coefficient is not useful for studying transient behavior when reactive elements are involved. In fact, we note from (6.56) and (6.57) that the ratios of reflected voltage and current to the incident voltage and current, respectively, are no longer constants as in the resistive case.

We may now write the expressions for the total voltage across the inductor and the total current through the inductor as follows:

$$\boxed{\begin{aligned} V(l,t) &= \frac{V_0}{2} + V^-(l,t) \\ &= \begin{cases} 0 & \text{for} \quad t < T \\ V_0 e^{-(Z_0/L)(t-T)} & \text{for} \quad t > T \end{cases} \\[2ex] I(l,t) &= \frac{V_0}{2Z_0} + I^-(l,t) \\ &= \begin{cases} 0 & \text{for} \quad t < T \\ (V_0/Z_0)[1 - e^{-(Z_0/L)(t-T)}] & \text{for} \quad t > T \end{cases} \end{aligned}}$$

(6.58)

(6.59)

These quantities are shown sketched in Figs. 6.35 (a) and (b), respectively. It may be seen from these sketches that in the steady state, the voltage goes to

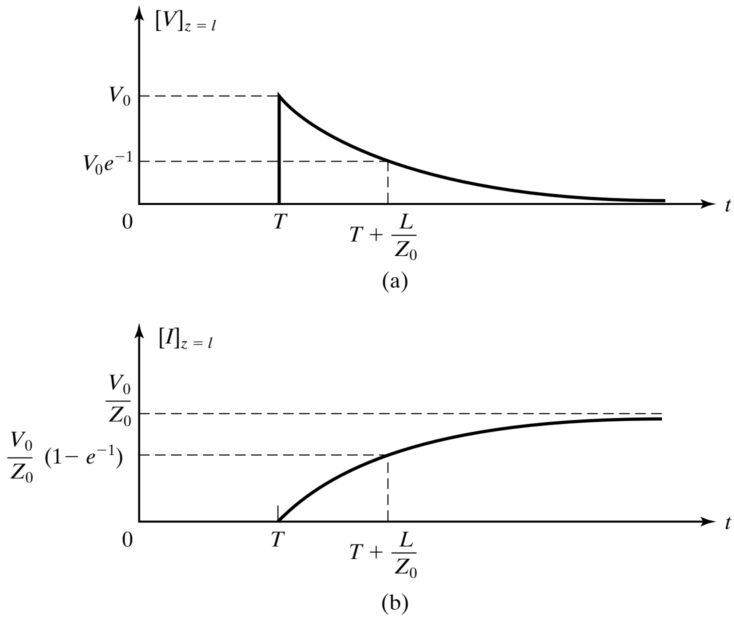


FIGURE 6.35

Time variations of (a) voltage across the inductor and (b) current through the inductor, for the system of Fig. 6.43.

zero and the current goes to V_0/Z_0. This is consistent with the fact that the inductor behaves like a short circuit for the dc voltage in the steady state, and hence the situation in the steady state is the same as that for a short-circuited line. Note also that the variations of the voltage and current from $t = T$ to $t = \infty$ are governed by the time constant L/Z_0, which is that of the inductor L in series with Z_0 of the line. In fact, we can obtain the voltage and current sketches from considerations of initial and final behaviors of the reactive element and the time constant without formally going through the process of setting up the differential equation and solving it. We shall illustrate this procedure by means of an example.

Example 6.5 A transmission-line system with a capacitive discontinuity

Capacitive discontinuity

Let us consider the system shown in Fig. 6.36 consisting of a series capacitor of value 10 pF at the junction between the two lines. Note that line 2 is terminated by its own characteristic impedance, whereas the internal resistance of the voltage source is equal to the characteristic impedance of line 1, so that no reflections occur at the two ends of the system. We shall assume that the capacitor is initially uncharged and obtain the plots of line voltage and line current at the input $z = 0$ from considerations of initial and final behaviors of the capacitor.

Plots of line voltage and line current at $z = 0$ versus time are shown in Figs. 6.37(a) and (b), respectively. We shall explain the several features in these plots as follows: When the switch S is closed at $t = 0$, a $(+)$ wave of voltage 10 V and current 0.2 A goes down the line. Since the voltage across a capacitor cannot change instantaneously, the initially uncharged capacitor behaves like a short circuit when the $(+)$ wave impinges on the junction aa' at $t = 1$ ns. Therefore, the $(+)$ wave then sees a resistance of $Z_{02}(= 150\ \Omega)$ across aa' and produces a $(-)$ wave of initial voltage 5 V and initial current -0.1 A. The $(-)$ wave arrives initially at $z = 0$ at $t = 2$ ns, thereby changing the line voltage and line current there to 15 V and 0.1 A, as shown in Figs. 6.37(a) and (b), respectively. In the steady state, the capacitor behaves like an open circuit, which explains the steady-state values of 20 V and 0 A in these plots. Between $t = 2$ ns and $t = \infty$, the voltage and current vary exponentially with a time constant of $10^{-11} \times 200 = 2 \times 10^{-9}\ \text{s} = 2$ ns, which is that of $C(= 10\ \text{pF})$ in series with $(Z_{01} + Z_{02})$, or 200 Ω. Hence, the voltage and current values at $t = 4$ ns are $15 + 5(1 - e^{-1}) = 18.16$ V and $0.1 - 0.1(1 - e^{-1}) = 0.037$ A, respectively.

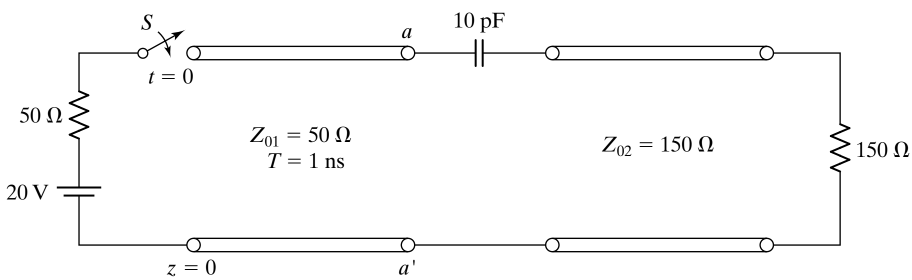


FIGURE 6.36

Transmission-line system with a capacitive discontinuity.

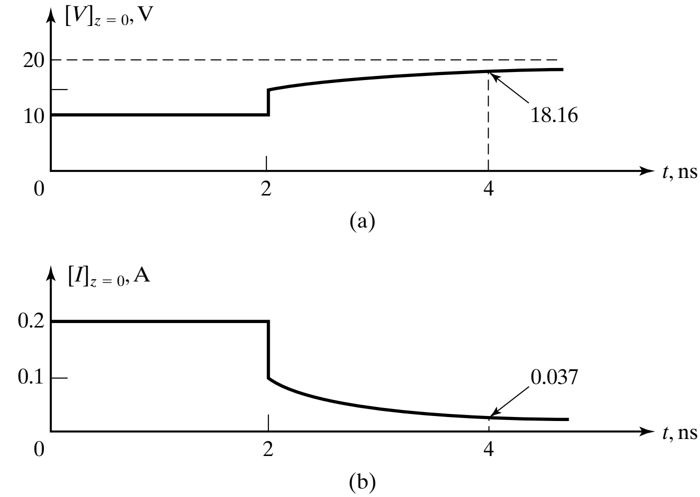


FIGURE 6.37

Plots of (a) line voltage and (b) line current at $z = 0$ for the system of Fig. 6.36.

Finally, the arguments that we have employed to explain the features in Fig. 6.37 can be used to deduce information about the nature of the discontinuity if the plots represent measurements by a time-domain reflectometer.

K6.4. Inductive termination; Capacitive discontinuity.

D6.9. In the system of Fig. 6.33, assume that $V_0 = 20$ V, $Z_0 = 50\ \Omega$, and $T = 1\ \mu$s. Find the value of the voltage across the inductor at $t = 2\ \mu$s for each of the following cases: **(a)** $L = 0.1$ mH, $I_L(0-) = 0$ A; **(b)** $L = 0.1$ mH, $I_L(0-) = 0.05$ A; **(c)** $L = 0.05$ mH; $I_L(0-) = 0.1$ A.

Ans. **(a)** 12.13 V; **(b)** 10.61 V; **(c)** 5.52 V.

D6.10. In the system shown in Fig. 6.38, the capacitor is initially uncharged. Find the values of the line voltage at $z = 0$ at the following times: **(a)** $t = 2$ ns+; **(b)** $t = \infty$; and **(c)** $t = 3$ ns.

Ans. **(a)** 0 V; **(b)** 15 V; **(c)** 7.2987 V.

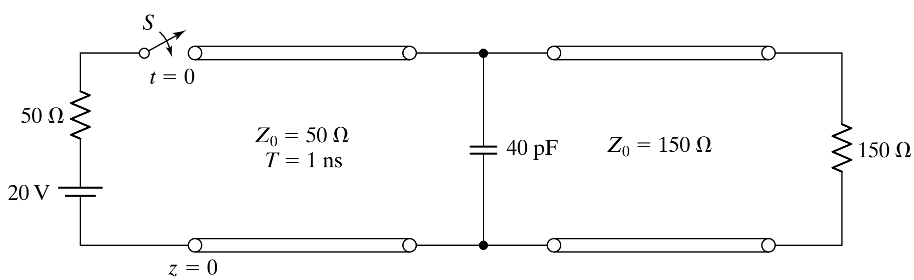


FIGURE 6.38

For Problem D6.10.

6.5 LINES WITH INITIAL CONDITIONS

Arbitrary initial distribution

Thus far, we have considered lines with quiescent initial conditions, that is, with no initial voltages and currents on them. As a prelude to the discussion of analysis of interconnections between logic gates, we shall now consider lines with nonzero initial conditions. We discuss first the general case of arbitrary initial voltage and current distributions by decomposing them into (+) and (−) wave voltages and currents. To do this, we consider the example shown in Fig. 6.39, in which a line open-circuited at both ends is charged initially, say, at $t = 0$, to the voltage and current distributions shown in the figure.

Writing the line voltage and current distributions as sums of (+) and (−) wave voltages and currents, we have

$$V^+(z, 0) + V^-(z, 0) = V(z, 0) \tag{6.60a}$$

$$I^+(z, 0) + I^-(z, 0) = I(z, 0) \tag{6.60b}$$

But we know that $I^+ = V^+/Z_0$ and $I^- = -V^-/Z_0$. Substituting these into (6.60b) and multiplying by Z_0, we get

$$V^+(z, 0) - V^-(z, 0) = Z_0 I(z, 0) \tag{6.61}$$

Solving (6.60a) and (6.61), we obtain

$$\boxed{V^+(z, 0) = \tfrac{1}{2}[V(z, 0) + Z_0 I(z, 0)]} \tag{6.62a}$$

$$\boxed{V^-(z, 0) = \tfrac{1}{2}[V(z, 0) - Z_0 I(z, 0)]} \tag{6.62b}$$

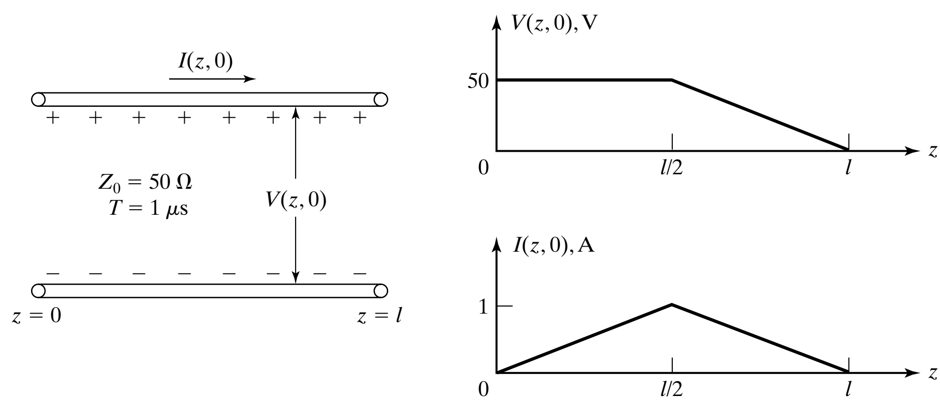


FIGURE 6.39

Line open-circuited at both ends and initially charged to the voltage and current distributions $V(z, 0)$ and $I(z, 0)$, respectively.

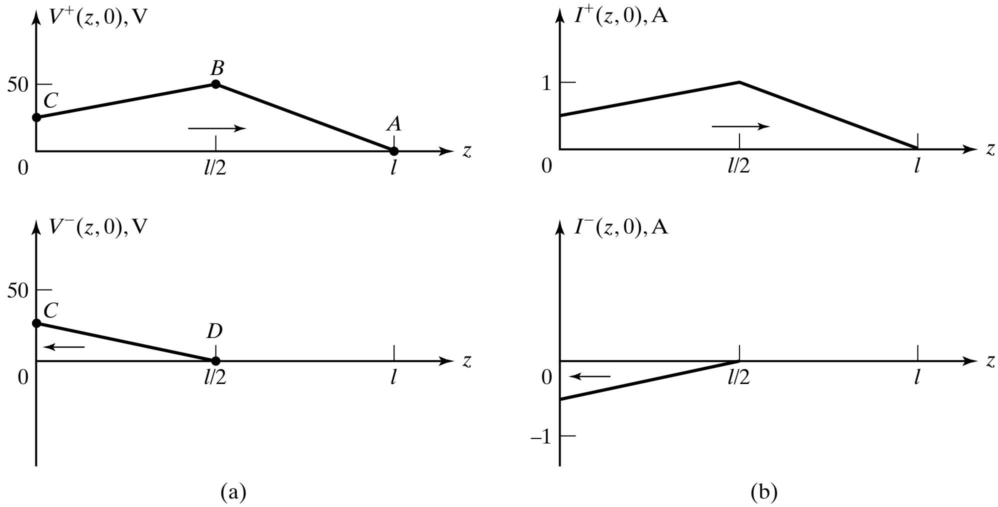


FIGURE 6.40

Distributions of (a) voltage and (b) current in the $(+)$ and $(-)$ waves obtained by decomposing the voltage and current distributions of Fig. 6.39.

Thus, for the distributions $V(z, 0)$ and $I(z, 0)$ given in Fig. 6.39, we obtain the distributions of $V^+(z, 0)$ and $V^-(z, 0)$ as shown by Fig. 6.40(a), and hence of $I^+(z, 0)$ and $I^-(z, 0)$, as shown by Fig. 6.40(b).

Suppose that we wish to find the voltage and current distributions at some later value of time, say, $t = 0.5\ \mu\text{s}$. Then, we note that as the $(+)$ and $(-)$ waves propagate and impinge on the open circuits at $z = l$ and $z = 0$, respectively, they produce the $(-)$ and $(+)$ waves, respectively, consistent with a voltage reflection coefficient of 1 and current reflection coefficient of -1 at both ends. Hence, at $t = 0.5\ \mu\text{s}$, the $(+)$ and $(-)$ wave voltage and current distributions and their sum distributions are as shown in Fig. 6.41, in which the points *A, B, C*, and *D* correspond to the points *A, B, C*, and *D*, respectively, in Fig. 6.40. Proceeding in this manner, one can obtain the voltage and current distributions for any value of time.

Suppose that we connect a resistor of value Z_0 at the end $z = l$ at $t = 0$ instead of keeping it open-circuited. Then the reflection coefficient at that end becomes zero thereafter, and the $(+)$ wave, as it impinges on the resistor, gets absorbed in it instead of producing the $(-)$ wave. The line therefore completely discharges into the resistor by the time $t = 1.5\ \mu\text{s}$, with the resulting time variation of voltage across R_L, as shown in Fig. 6.42, where the points *A, B, C*, and *D* correspond to the points *A, B, C*, and *D*, respectively, in Fig. 6.40.

Uniform initial distribution

For a line with uniform initial voltage and current distributions, the analysis can be performed in the same manner as for arbitrary initial voltage and current distributions. Alternatively, and more conveniently, the analysis can be carried out

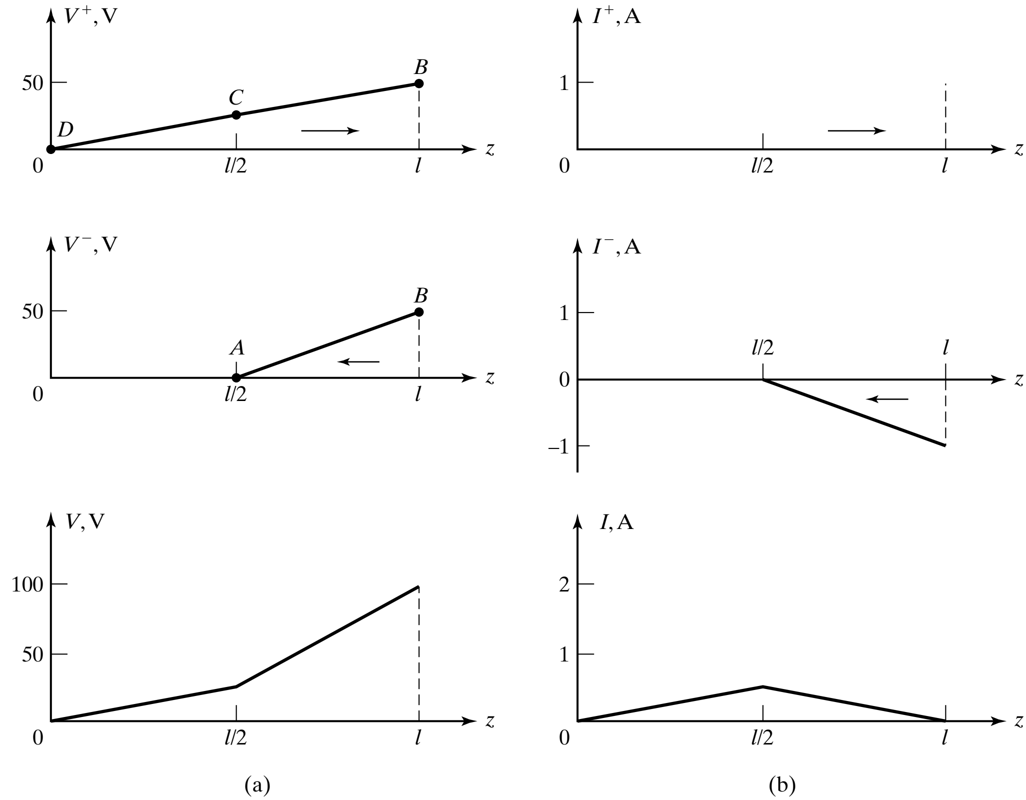


FIGURE 6.41

Distributions of (a) voltage and (b) current in the $(+)$ and $(-)$ waves and their sum for $t = 0.5\ \mu$s for the initially charged line of Fig. 6.39.

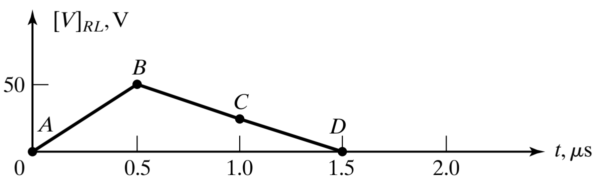


FIGURE 6.42

Voltage across $R_L(= Z_0 = 50\ \Omega)$ resulting from connecting it at $t = 0$ to the end $z = l$ of the line of Fig. 6.39.

with the aid of superposition and bounce diagrams. The basis behind this method lies in the fact that the uniform distribution corresponds to a situation in which the line voltage and current remain constant with time at all points on the line until a change is made at some point on the line. The boundary condition is then violated at that point, and a transient wave of constant voltage and current is set up, to be superimposed on the initial distribution. We shall illustrate this technique of analysis by means of an example.

Example 6.6 Bounce-diagram technique and checking energy balance for an initially charged line

Let us consider a line of $Z_0 = 50\ \Omega$ and $T = 1\ \mu$s initially charged to uniform voltage $V_0 = 100$ V and zero current. A resistor $R_L = 150\ \Omega$ is connected at $t = 0$ to the end $z = 0$ of the line, as shown in Fig. 6.43(a). We wish to obtain the time variation of the voltage across R_L for $t > 0$.

Since the change is made at $z = 0$ by connecting R_L to the line, a $(+)$ wave originates at $z = 0$, so that the total line voltage at that point is $V_0 + V^+$ and the total line current is $0 + I^+$, or I^+, as shown in Fig. 6.43(b). To satisfy the boundary condition at $z = 0$, we then write

$$V_0 + V^+ = -R_L I^+ \tag{6.63}$$

But we know that $I^+ = V^+/Z_0$. Hence, we have

$$V_0 + V^+ = -\frac{R_L}{Z_0}V^+ \tag{6.64}$$

or

$$V^+ = -V_0 \frac{Z_0}{R_L + Z_0} \tag{6.65a}$$

$$I^+ = -V_0 \frac{1}{R_L + Z_0} \tag{6.65b}$$

For $V_0 = 100$ V, $Z_0 = 50\ \Omega$, and $R_L = 150\ \Omega$, we obtain $V^+ = -25$ V and $I^+ = -0.5$ A.

We may now draw the voltage and current bounce diagrams, as shown in Fig. 6.44 We note that in these bounce diagrams, the initial conditions are accounted for by the horizontal lines drawn at the top, with the numerical values of voltage and current indicated on

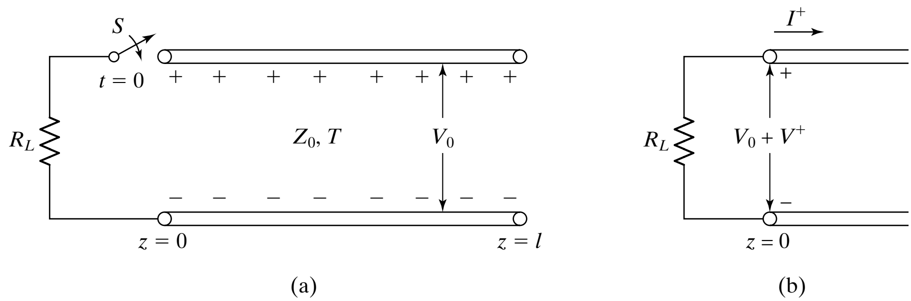


FIGURE 6.43

(a) Transmission line charged initially to uniform voltage V_0. (b) For obtaining the voltage and current associated with the transient $(+)$ wave resulting from the closure of the switch in (a).

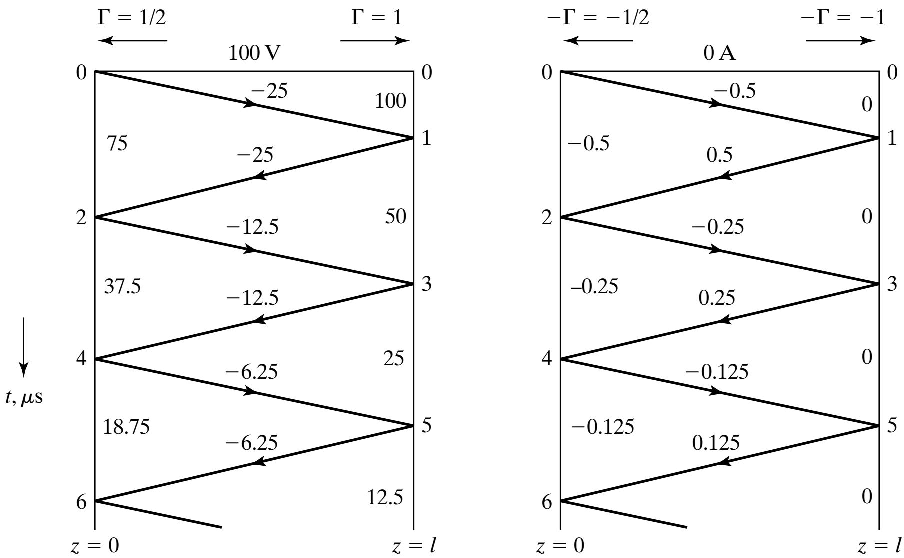


FIGURE 6.44

Voltage-and current-bounce diagrams depicting the transient phenomenon for $t > 0$ for the line of Fig. 6.43 (a), for $V_0 = 100$ V, $Z_0 = 50\ \Omega$, $R_L = 150\ \Omega$, and $T = 1\ \mu$s.

them. Sketches of line voltage and current versus z for fixed values of t can be drawn from these bounce diagrams in the usual manner. Sketches of line voltage and current versus t for any fixed value of z also can be drawn from the bounce diagrams in the usual manner. Of particular interest is the voltage across R_L, which illustrates how the line discharges into the resistor. The time variation of this voltage is shown in Fig. 6.45.

Energy balance

It is also instructive to check the energy balance, that is, to verify that the energy dissipated in the 150-Ω resistor for $t > 0$ is indeed equal to the energy stored in the line at $t = 0-$, since the line is lossless. To do this, we note that, in general, energy is stored in both electric and magnetic fields in the line, with energy densities $\frac{1}{2}\mathscr{C}V^2$ and $\frac{1}{2}\mathscr{L}I^2$, respectively. Thus, for a line charged uniformly to voltage V_0 and current I_0, the total electric and magnetic stored energies are given, respectively, by

$$\begin{aligned} W_e &= \tfrac{1}{2}\mathscr{C}V_0^2 l = \tfrac{1}{2}\mathscr{C}V_0^2 v_p T \\ &= \frac{1}{2}\mathscr{C}V_0^2 \frac{1}{\sqrt{\mathscr{L}\mathscr{C}}} T = \frac{1}{2}\frac{V_0^2}{Z_0} T \end{aligned} \tag{6.66a}$$

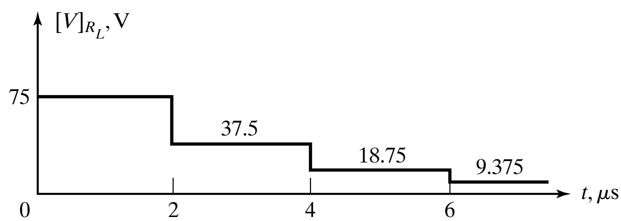


FIGURE 6.45

Time variation of voltage across R_L for $t > 0$ in Fig. 6.43(a) for $V_0 = 100$ V, $Z_0 = 50\ \Omega$, $R_L = 150\ \Omega$, and $T = 1\ \mu$s.

and

$$W_m = \frac{1}{2}\mathscr{L}I_0^2 l = \frac{1}{2}\mathscr{L}I_0^2 v_p T$$
$$= \frac{1}{2}\mathscr{L}I_0^2 \frac{1}{\sqrt{\mathscr{L}\mathscr{C}}}T = \frac{1}{2}I_0^2 Z_0 T \tag{6.66b}$$

Since for the example under consideration, $V_0 = 100$ V, $I_0 = 0$, and $T = 1\ \mu$s, $W_e = 10^{-4}$ J and $W_m = 0$. Thus, the total initial stored energy in the line is 10^{-4} J. Now, denoting the power dissipated in the resistor to be P_d, we obtain the energy dissipated in the resistor to be

$$W_d = \int_{t=0}^{\infty} P_d\, dt$$
$$= \int_0^{2\times10^{-6}} \frac{75^2}{150}\, dt + \int_{2\times10^{-6}}^{4\times10^{-6}} \frac{37.5^2}{150}\, dt + \int_{4\times10^{-6}}^{6\times10^{-6}} \frac{18.75^2}{150}\, dt + \cdots$$
$$= \frac{2\times10^{-6}}{150} \times 75^2\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) = 10^{-4}\ \text{J}$$

which is exactly the same as the initial stored energy in the line, thereby satisfying the energy balance.

K6.5. Initial conditions; Arbitrary distribution; Uniform distribution; Bounce-diagram technique.

D6.11. For the line of Fig. 6.39 with the initial voltage and current distributions as given in the figure, find: **(a)** $V(l/2, 0.25\ \mu\text{s})$; **(b)** $I(l/2, 0.25\ \mu\text{s})$; **(c)** $V(l/4, 1\ \mu\text{s})$; and **(d)** $I(l/4, 1\ \mu\text{s})$.
Ans. **(a)** 37.5 V; **(b)** 0.75 A; **(c)** 25 V; **(d)** −0.5 A.

D6.12. In the system shown in Fig. 6.46, a line of characteristic impedance 75 Ω and charged to 10 V is connected at $t = 0$ to another line of characteristic impedance 50 Ω and charged to 5 V. The one-way travel time T is equal to 1 μs for both lines. Find **(a)** the value of the voltage at the instant of time when both lines are charged to the same voltage throughout their lengths; **(b)** the value of the current to which the lines are charged at that instant of time; and **(c)** the energy stored in the system at any instant of time.
Ans. **(a)** 7 V; **(b)** 0.04 A; **(c)** 11/12 μJ.

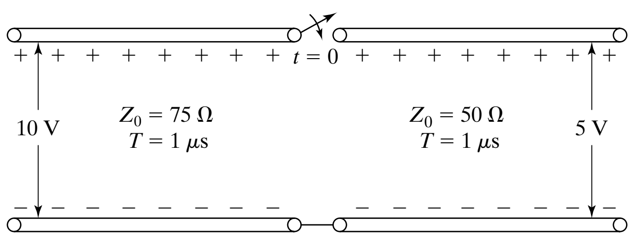


FIGURE 6.46
For Problem D6.12.

6.6 INTERCONNECTIONS BETWEEN LOGIC GATES

Thus far we have been concerned with time-domain analysis for lines with terminations and discontinuities made up of linear circuit elements. Logic gates present nonlinear resistive terminations to the interconnecting transmission lines in digital circuits. The analysis is then made convenient by a graphical technique known as the *load-line* technique. We shall first introduce this technique by means of an example.

Example 6.7 Load-line technique of analysis for a line terminated by a nonlinear element

Load-line technique

Let us consider the transmission-line system shown in Fig. 6.47, in which the line is terminated by a passive nonlinear element having the indicated *V-I* relationship. We wish to obtain the time variations of the voltages V_S and V_L at the source and load ends, respectively, following the closure of the switch *S* at $t = 0$, using the load-line technique.

With reference to the notation shown in Fig. 6.47, we can write the following equations pertinent to $t = 0+$ at $z = 0$:

$$50 = 200I_S + V_S \tag{6.67a}$$

$$V_S = V^+$$

$$I_S = I^+ = \frac{V^+}{Z_0} = \frac{V_S}{50} \tag{6.67b}$$

where V^+ and I^+ are the voltage and current, respectively, of the $(+)$ wave set up immediately after closure of the switch. The two equations (6.67a) and (6.67b) can be solved graphically by constructing the straight lines representing them, as shown in Fig. 6.48, and obtaining the point of intersection A, which gives the values of V_S and I_S. Note in particular that (6.67b) is a straight line of slope 1/50 and passing through the origin.

When the $(+)$ wave reaches the load end $z = l$ at $t = T$, a $(-)$ wave is set up. We can then write the following equations pertinent to $t = T+$ at $z = l$:

$$V_L = 50I_L|I_L| \tag{6.68a}$$

$$V_L = V^+ + V^-$$

$$\begin{aligned} I_L &= I^+ + I^- = \frac{V^+ - V^-}{Z_0} \\ &= \frac{V^+ - (V_L - V^+)}{50} = \frac{2V^+ - V_L}{50} \end{aligned} \tag{6.68b}$$

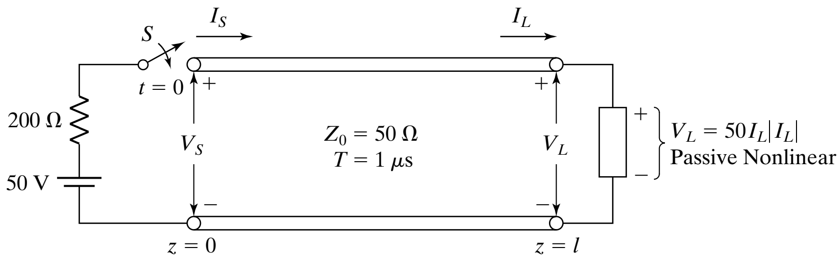


FIGURE 6.47

Line terminated by a passive nonlinear element and driven by a constant-voltage source in series with internal resistance.

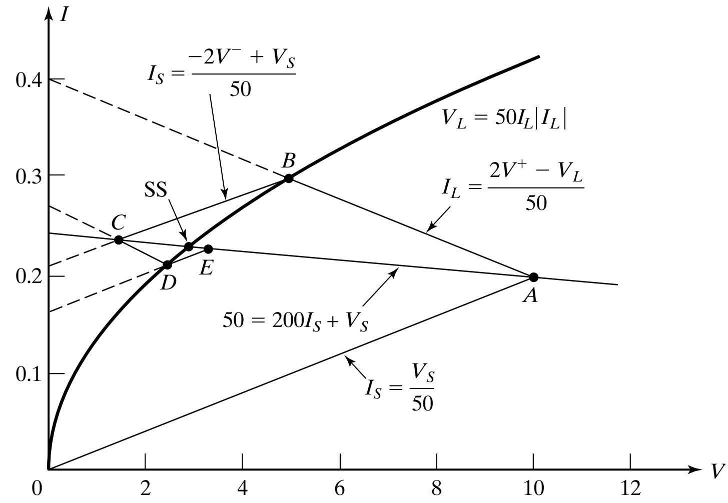


FIGURE 6.48

Graphical solution for obtaining time variations of V_S and V_L for $t > 0$ in the transmission-line system of Fig. 6.47.

where V^- and I^- are the $(-)$ wave voltage and current, respectively. The solution for V_L and I_L is then given by the intersection of the nonlinear curve representing (6.68a) and the straight line of slope $-1/50$ corresponding to (6.68b). Noting from (6.68b) that for $V_L = V^+$, $I_L = V^+/50$, we see that the straight line passes through point A. Thus, the solution of (6.68a) and (6.68b) is given by point B in Fig. 6.48.

When the $(-)$ wave reaches the source end $z = 0$ at $t = 2T$, it sets up a reflection. Denoting this to be the $(-+)$ wave, we can then write the following equations pertinent to $t = 2T+$ at $z = 0$:

$$50 = 200I_S + V_S \tag{6.69a}$$

$$V_S = V^+ + V^- + V^{-+}$$

$$\begin{aligned} I_S &= I^+ + I^- + I^{-+} = \frac{V^+ - V^- + V^{-+}}{Z_0} \\ &= \frac{V^+ - V^- + (V_S - V^+ - V^-)}{50} = \frac{-2V^- + V_S}{50} \end{aligned} \tag{6.69b}$$

where V^{-+} and I^{-+} are the $(-+)$ wave voltage and current, respectively. Noting from (6.69a) that for $V_S = V^+ + V^-$, $I_S = (V^+ - V^-)/50$, we see that (6.69b) represents a straight line of slope 1/50 passing through B. Thus, the solution of (6.69a) and (6.69b) is given by point C in Fig. 6.48.

Continuing in this manner, we observe that the solution consists of obtaining the points of intersection on the source and load *V-I* characteristics by drawing successively straight lines of slope $1/Z_0$ and $-1/Z_0$, beginning at the origin (the initial state) and with each straight line originating at the previous point of intersection, as shown in Fig. 6.48. The points A, C, $E, \ldots,$ give the voltage and current at the source end for $0 < t < 2T, 2T < t < 4T, 4T < t < 6T, \ldots,$ whereas the points B, D, $\ldots$, give the voltage and current at the load end for $T < t < 3T, 3T < t < 5T, \ldots$. Thus, for example,

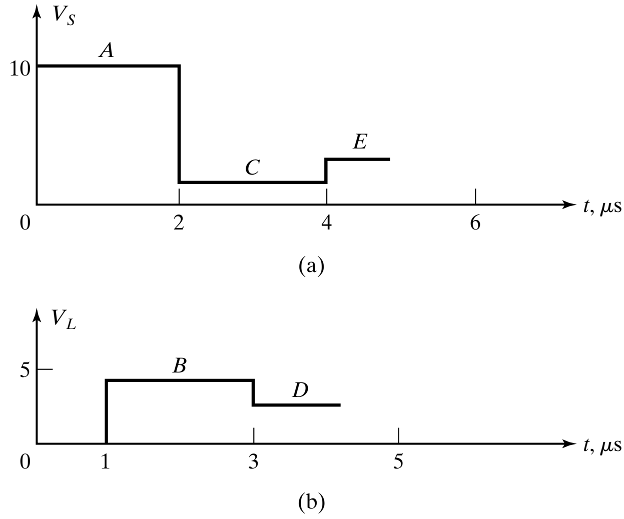


FIGURE 6.49

Time variations of (a) V_S and (b) V_L, for the transmission-line system of Fig. 6.47. The voltage levels *A, B, C*, ... correspond to those in Fig. 6.48.

the time variations of V_S and V_L are shown in Figs. 6.49(a) and (b), respectively. Finally, it can be seen from Fig. 6.48 that the steady-state values of line voltage and current are reached at the point of intersection (denoted SS) of the source and load *V-I* characteristics.

Now, going back to Example 6.6, the behavior of the system for the uniformly charged line can be analyzed by using the load-line technique, as an alternative to the solution using the bounce diagram technique. Thus, noting that the terminal voltage-current characteristics at the ends $z = 0$ and $z = l$ of the system in Fig. 6.43 are given by $V = -IR_L = -150I$ and $I = 0$, respectively, and that the characteristic impedance of the line is 50 Ω, we can carry out the load-line construction, as shown in Fig. 6.50, beginning at the point A (100 V, 0 A), and

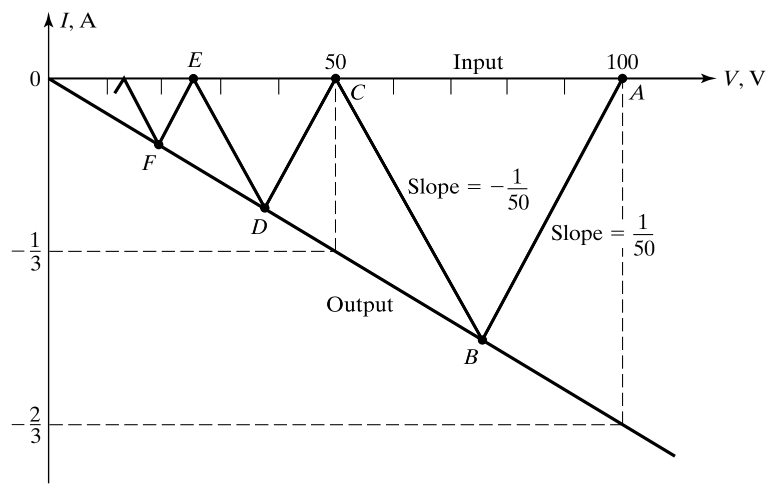


FIGURE 6.50

Load-line construction for the analysis of the system of Fig 6.43(a).

drawing alternately straight lines of slope 1/50 and −1/50 to obtain the points of intersection *B*, *C*, *D*, ... The points *B*, *D*, *F*, ... give the line voltage and current values at the end $z = 0$ for intervals of 2 μs beginning at $t = 0$ μs, 2 μs, 4 μs, ..., whereas the points *C*, *E*, ... give the line voltage and current values at the end $z = l$ for intervals of 2 μs beginning at $t = 1$ μs, 3 μs, ... For example, the time variation of the line voltage at $z = 0$ provided by the load-line construction is the same as in Fig. 6.45.

Interconnection between logic gates

We shall now apply the procedure for the use of the load-line technique for a line with uniform initial distribution, just illustrated, to the analysis of the system in Fig. 6.51(a) in which two transistor-transistor logic (TTL) inverters are interconnected by using a transmission line of characteristic impedance Z_0 and one-way travel time T. As the name inverter implies, the gate has an output that is the inverse of the input. Thus, if the input is in the HIGH (logic 1) range, the output will be in the LOW (logic 0) range, and vice versa. Typical *V-I* characteristics for a TTL inverter are shown in Fig. 6.51(b). As shown in this figure, when the system is in the steady state with the output of the first inverter in the 0 state, the

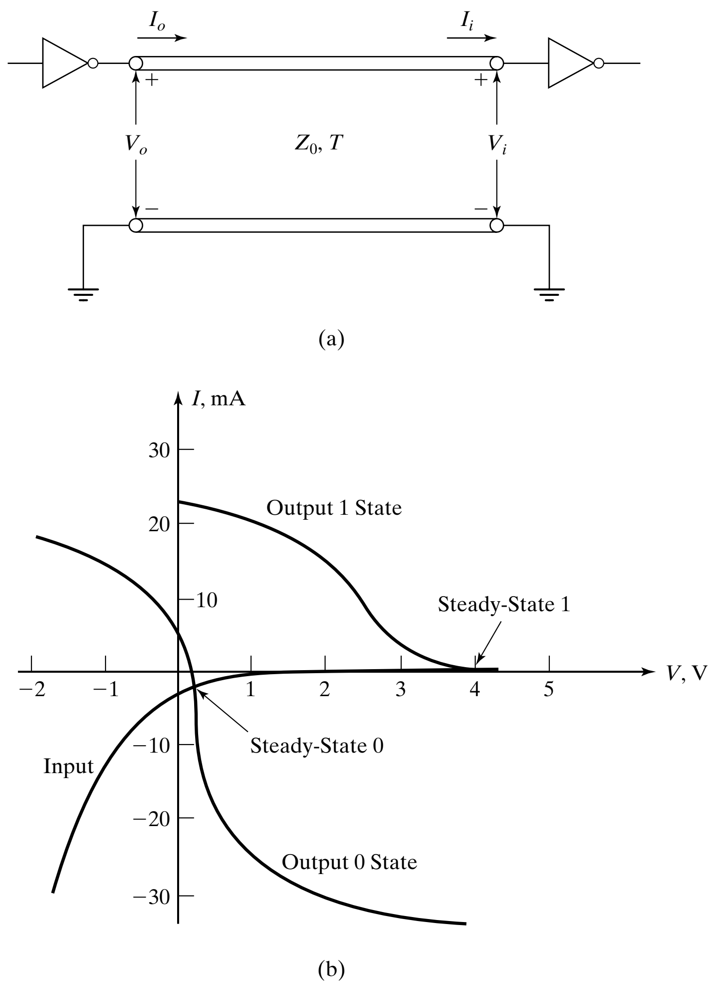


FIGURE 6.51

(a) Transmission-line interconnection between two logic gates. (b) Typical *V-I* characteristics for the logic gates.

voltage and current along the line are given by the intersection of the output 0 characteristic and the input characteristic; when the system is in the steady state with the output of the first inverter in the 1 state, the voltage and current along the line are given by the intersection of the output 1 characteristic and the input characteristic. Thus, the line is charged to 0.2 V for the steady-state 0 condition and to 4 V for the steady-state 1 condition. We wish to study the transient phenomena corresponding to the transition when the output of the first gate switches from the 0 to the 1 state, and vice versa, assuming Z_0 of the line to be 30 Ω.

Analysis of 0-to-1 transition

Considering first the transition from the 0 state to the 1 state, and following the line of argument in Example 6.7, we carry out the construction shown in Fig. 6.52(a). This construction consists of beginning at the point corresponding to the steady-state 0 (the initial state) and drawing a straight line of slope 1/30 to intersect with the output 1 characteristic at point A, then drawing from point A

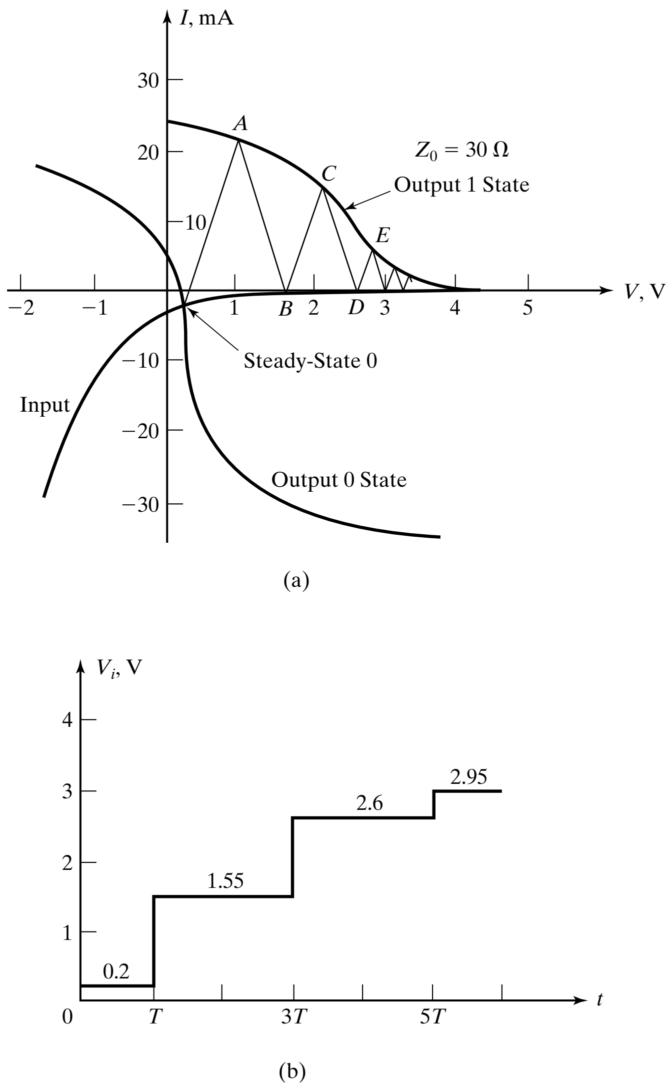


FIGURE 6.52

(a) Construction based on the load-line technique for analysis of the 0-to-1 transition for the system of Fig 6.51(a). (b) Plot of V_i versus t obtained from the construction in (a).

a straight line of slope $-1/30$ to intersect the input characteristic at point B, and so on. From this construction, the variation of the voltage V_i at the input of the second gate can be sketched as shown in Fig. 6.52(b), in which the voltage levels correspond to the points 0, B, D, . . . , in Fig. 6.52(a). The effect of the transients on the performance of the system may now be seen by noting from Fig. 6.52(b) that depending on the value of the minimum gate voltage that will reliably be recognized as logic 1, a time delay in excess of T may be involved in the transition from 0 to 1. Thus, if this minimum voltage is 2 V, the interconnecting line will result in an extra time delay of $2T$ for the input of the second gate to switch from 0 to 1, since V_i does not exceed 2 V until $t = 3T+$.

Analysis of 1-to-0 transition

Considering next the transition from the 1 state to the 0 state, we carry out the construction shown in Fig. 6.53(a), with the crisscross lines beginning at the

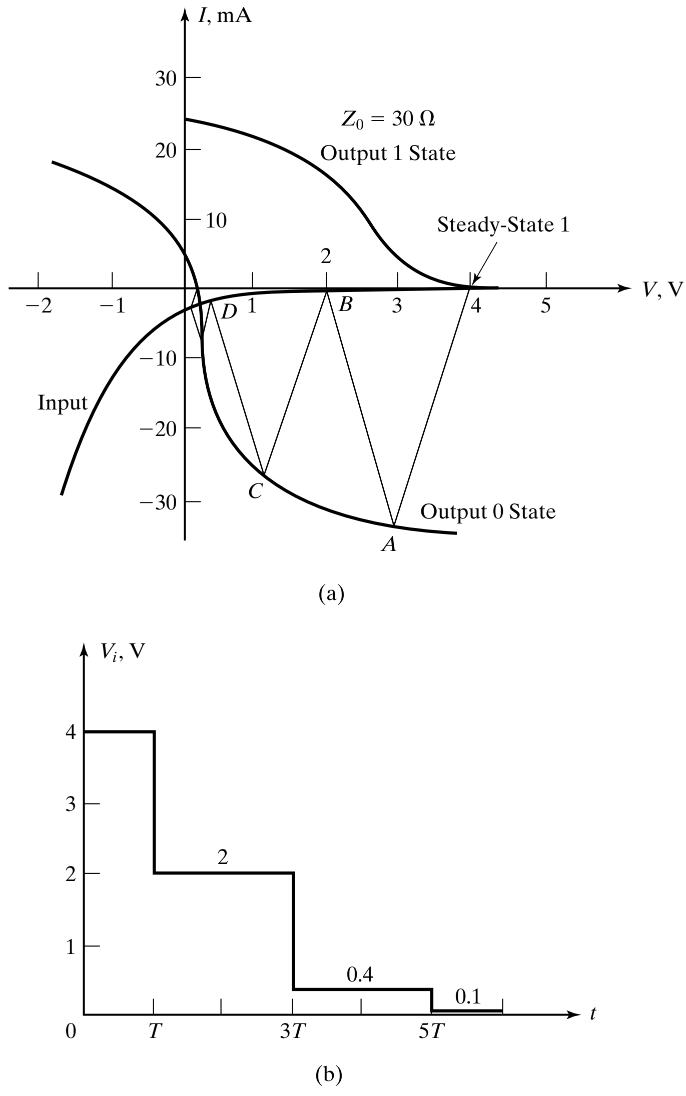


FIGURE 6.53

(a) Construction based on the load-line technique for analysis of the 1-to-0 transition for the system of Fig. 6.51(a). (b) Plot of V_i versus t obtained from the construction in (a).

point corresponding to the steady-state 1. From this construction, we obtain the plot of V_i versus t, as shown in Fig. 6.53(b), in which the voltage levels correspond to the points 1, B, D, . . . , in Fig. 6.53(a). If we assume a maximum gate input voltage that can be readily recognized as logic 0 to be 1 V, it can once again be seen that an extra time delay of $2T$ is involved in the switching of the input of the second gate from 1 to 0, since V_i does not drop below 1 V until $t = 3T+$.

K6.6. Load-line technique; Interconnection between logic gates.

D6.13. Assume that in the system of Fig. 6.47 the values of the voltage source and its internal resistance are 12 V and 10 Ω, respectively, and that Z_0 of the line is 100 Ω. By using the load-line technique, find the approximate values of: **(a)** V_L at $t = 2\ \mu$s; **(b)** V_S at $t = 3\ \mu$s; **(c)** V_L at $t = 4\ \mu$s; and **(d)** V_L at $t = \infty$.

Ans. **(a)** 2 V; **(b)** 9.3 V; **(c)** 5 V; **(d)** 8 V.

6.7 CROSSTALK ON TRANSMISSION LINES

Crosstalk explained

When two or more transmission lines are in the vicinity of one another, a wave propagating along one line, which we shall call the primary line, can induce a wave on another line, the secondary line, due to capacitive (electric field) and inductive (magnetic field) coupling between the two lines, resulting in the undesirable phenomenon of crosstalk between the lines. An example is illustrated by the arrangement of Fig. 6.54(a), which is a printed-circuit board (PCB) representation of two closely spaced transmission lines. Figure 6.54(b) represents the distributed circuit equivalent, where $\mathscr{C}_m$ and $\mathscr{L}_m$ are the coupling capacitance and coupling inductance, respectively, per unit length of the arrangement.

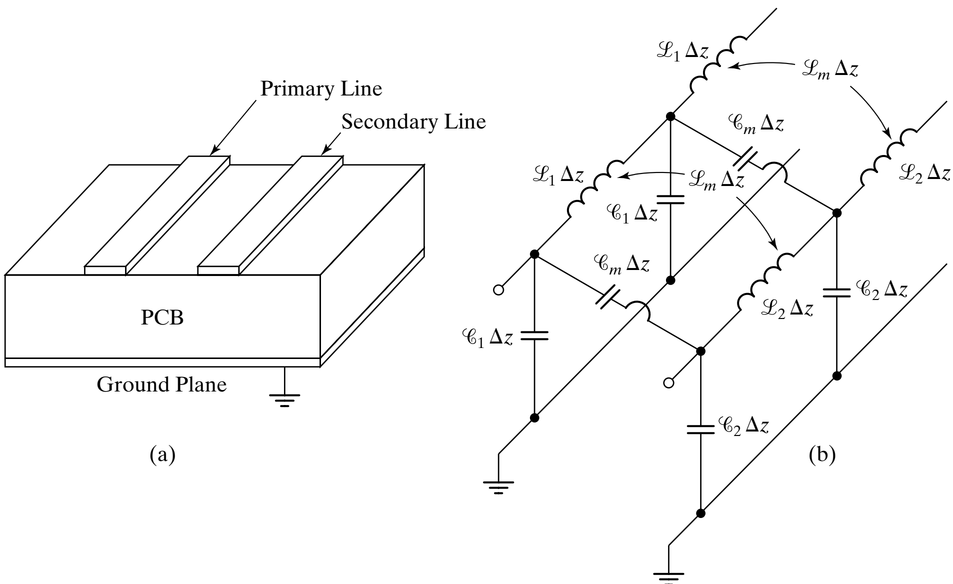


FIGURE 6.54

(a) PCB representation of two closely spaced transmission lines. (b) Distributed equivalent circuit for (a).

Weak coupling analysis

In this section, we shall analyze a pair of coupled transmission lines for the determination of induced waves on the secondary line for a given wave on the primary line. To keep the analysis simple, we shall consider both lines to be of the same characteristic impedance, velocity of propagation, and length, and terminated by their characteristic impedances, so that no reflections occur from the ends of either line. It is also convenient to assume the coupling to be weak, so that the effects on the primary line of waves induced in the secondary line can be neglected. Thus, we shall be concerned only with the crosstalk from the primary line to the secondary line and not vice versa. Briefly, as the $(+)$ wave propagates on the primary line from source toward load, each infinitesimal length of that line induces voltage and current in the adjacent infinitesimal length of the secondary line, which set up $(+)$ and $(-)$ waves on that line. The contributions due to the infinitesimal lengths add up to give the induced voltage and current at a given location on the secondary line.

We shall represent the coupled-line pair, as shown in Fig. 6.55, with the primary line as line 1 and the secondary line as line 2. Then, when the switch S is closed at $t = 0$, a $(+)$ wave originates at $z = 0$ on line 1 and propagates toward the load. Let us consider a differential length $d\xi$ at the location $z = \xi$ of line 1 charged to the $(+)$ wave voltage and current and obtain its contributions to the induced voltages and currents in line 2.

Modeling for capacitive coupling

The capacitive coupling induces a differential crosstalk current ΔI_{c2}, flowing into the nongrounded conductor of line 2, given by

$$\boxed{\Delta I_{c2}(\xi, t) = \mathscr{C}_m \, \Delta\xi \, \frac{\partial V_1(\xi, t)}{\partial t}} \tag{6.70a}$$

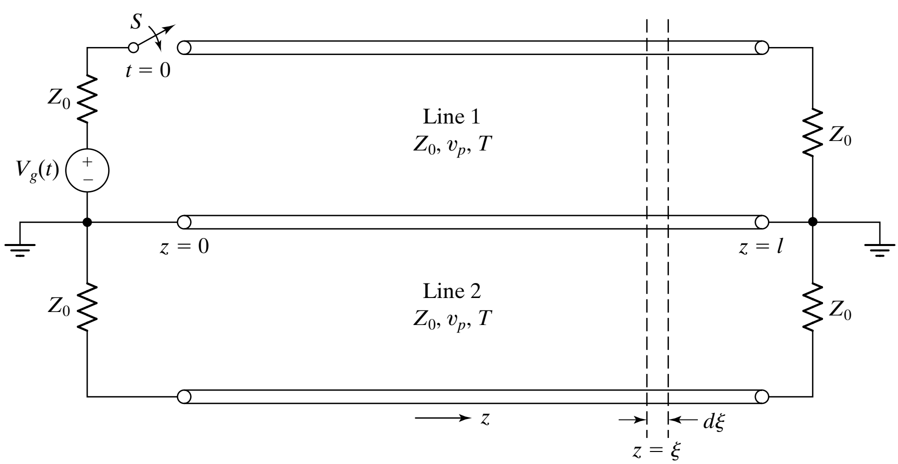


FIGURE 6.55

Coupled transmission-line pair for analysis of crosstalk.

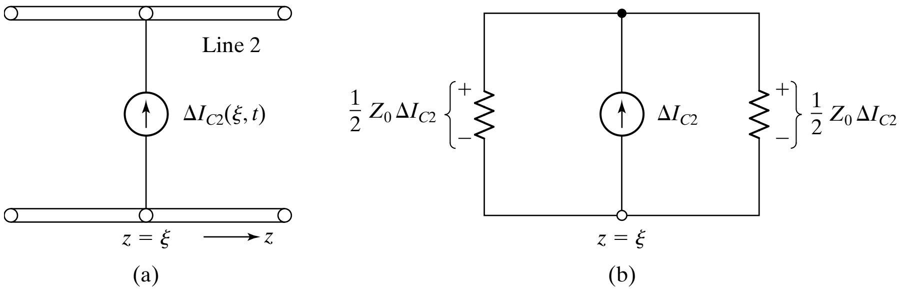


FIGURE 6.56

(a) Modeling for capacitive coupling in crosstalk analysis. (b) Equivalent circuit for (a).

where $V_1(\xi, t)$ is the line-1 voltage. This induced current is modeled by an ideal current source, connected in parallel with line 2 at $z = \xi$ on that line, as shown in Fig. 6.56(a). The current source views the characteristic impedance of the line to either side of $z = \xi$, so that the equivalent circuit is as shown in Fig. 6.56(b). Thus, voltages of $\frac{1}{2} Z_0 \, \Delta I_{c2}$ are produced to the right and left of $z = \xi$ and propagate as forward-crosstalk and backward-crosstalk voltages, respectively, on line 2.

Modeling for inductive coupling

The inductive coupling induces a differential crosstalk voltage, ΔV_{c2}, which is given by

$$\boxed{\Delta V_{c2}(\xi, t) = \mathscr{L}_m \, \Delta\xi \frac{\partial I_1(\xi, t)}{\partial t}} \tag{6.70b}$$

This induced voltage is modeled by an ideal voltage source in series with line 2 at $z = \xi$ on that line, as shown in Fig. 6.57(a). The polarity of the voltage source is such that the current due to it in line 2 produces a magnetic flux, which opposes the change in the flux due to the current in line 1, in accordance with Lenz's law. The voltage source views the characteristic impedance of the line to either side of it, so that the equivalent circuit is as shown in Fig. 6.57(b). Thus,

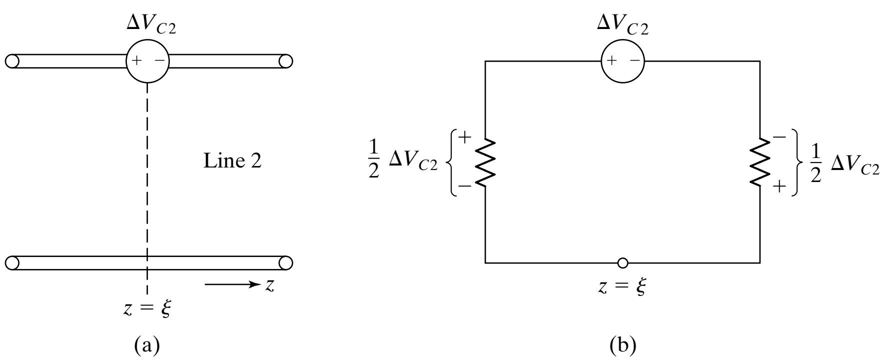


FIGURE 6.57

(a) Modeling for inductive coupling in crosstalk analysis. (b) Equivalent circuit for (a).

voltages of $\frac{1}{2}\Delta V_{c2}$ and $-\frac{1}{2}\Delta V_{c2}$ are produced to the left and right of $z = \xi$, respectively, and propagate as backward-crosstalk and forward-crosstalk voltages, respectively, on line 2.

Combining the contributions due to capacitive coupling and inductive coupling, we obtain the total differential voltages produced to the right and left of $z = \xi$ to be

$$\Delta V_2^+ = \tfrac{1}{2}Z_0\,\Delta I_{c2} - \tfrac{1}{2}\Delta V_{c2} \tag{6.71a}$$

$$\Delta V_2^- = \tfrac{1}{2}Z_0\,\Delta I_{c2} + \tfrac{1}{2}\Delta V_{c2} \tag{6.71b}$$

respectively. Substituting (6.70a) and (6.70b) into (6.71a) and (6.71b), we obtain

$$\begin{aligned}\Delta V_2^+(\xi,t) &= \left[\frac{1}{2}\mathscr{C}_m Z_0 \frac{\partial V_1(\xi,t)}{\partial t} - \frac{1}{2}\mathscr{L}_m \frac{\partial I_1(\xi,t)}{\partial t}\right]\Delta\xi \\ &= \frac{1}{2}\left(\mathscr{C}_m Z_0 - \frac{\mathscr{L}_m}{Z_0}\right)\frac{\partial V_1(\xi,t)}{\partial t}\Delta\xi\end{aligned} \tag{6.72a}$$

$$\Delta V_2^-(\xi,t) = \frac{1}{2}\left(\mathscr{C}_m Z_0 + \frac{\mathscr{L}_m}{Z_0}\right)\frac{\partial V_1(\xi,t)}{\partial t}\Delta\xi \tag{6.72b}$$

where we have substituted $I_1 = V_1/Z_0$, in accordance with the relationship between the voltage and current of a $(+)$ wave.

Forward-crosstalk voltage and coefficient

We are now ready to apply (6.72a) and (6.72b) in conjunction with superposition to obtain the $(+)$ and $(-)$ wave voltages at any location on line 2, due to a $(+)$ wave of voltage $V_1(t - z/v_p)$ on line 1. Thus, noting that the effect of V_1 at $z = \xi$ at a given time t is felt at a location $z > \xi$ on line 2 at time $t + (z - \xi)/v_p$, we can write

$$\begin{aligned}V_2^+(z,t) &= \int_0^z \frac{1}{2}\left(\mathscr{C}_m Z_0 - \frac{\mathscr{L}_m}{Z_0}\right)\frac{\partial}{\partial t}\left[V_1\left(t - \frac{\xi}{v_p} - \frac{z-\xi}{v_p}\right)\right]d\xi \\ &= \frac{1}{2}\left(\mathscr{C}_m Z_0 - \frac{\mathscr{L}_m}{Z_0}\right)\int_0^z \frac{\partial V_1(t - z/v_p)}{\partial t}d\xi\end{aligned} \tag{6.73}$$

or

$$\boxed{V_2^+(z,t) = zK_f V_1'(t - z/v_p)} \tag{6.74}$$

where we have defined

$$\boxed{K_f = \frac{1}{2}\left(\mathscr{C}_m Z_0 - \frac{\mathscr{L}_m}{Z_0}\right)} \tag{6.75}$$

and the prime associated with V_1 denotes differentiation with time. The quantity K_f is called the *forward-crosstalk coefficient*. Note that the upper limit in the integral in (6.108) is z, because the line-1 voltage to the right of a given location z

on that line does not contribute to the forward-crosstalk voltage on line 2 at that same location. The result given by (6.74) tells us that the forward-crosstalk voltage is proportional to z and the time derivative of the primary line voltage.

Backward-crosstalk voltage and coefficient

To obtain $V_2^-(z, t)$, we note that the effect of V_1 at $z = \xi$ at a given time t is felt at a location $z < \xi$ on line 2 at time $t + (\xi - z)/v_p$. Hence,

$$
\begin{aligned}
V_2^-(z,t) &= \int_z^l \frac{1}{2}\left(\mathscr{C}_m Z_0 + \frac{\mathscr{L}_m}{Z_0}\right)\frac{\partial}{\partial t}\left[V_1\left(t - \frac{\xi}{v_p} - \frac{\xi - z}{v_p}\right)\right]d\xi \\
&= \frac{1}{2}\left(\mathscr{C}_m Z_0 + \frac{\mathscr{L}_m}{Z_0}\right)\int_z^l \frac{\partial}{\partial t}\left[V_1\left(t + \frac{z}{v_p} - \frac{2\xi}{v_p}\right)\right]d\xi \\
&= -\frac{1}{4}v_p\left(\mathscr{C}_m Z_0 + \frac{\mathscr{L}_m}{Z_0}\right)\int_z^l \frac{\partial}{\partial \xi}\left[V_1\left(t + \frac{z}{v_p} - \frac{2\xi}{v_p}\right)\right]d\xi \\
&= -\frac{1}{4}v_p\left(\mathscr{C}_m Z_0 + \frac{\mathscr{L}_m}{Z_0}\right)\left[V_1\left(t + \frac{z}{v_p} - \frac{2\xi}{v_p}\right)\right]_{\xi=z}^{l}
\end{aligned}
\tag{6.76}
$$

or

$$
\boxed{V_2^-(z,t) = K_b\left[V_1\left(t - \frac{z}{v_p}\right) - V_1\left(t - \frac{2l}{v_p} + \frac{z}{v_p}\right)\right]}
\tag{6.77}
$$

where we have defined the *backward-crosstalk coefficient*

$$
\boxed{K_b = \frac{1}{4}v_p\left(\mathscr{C}_m Z_0 + \frac{\mathscr{L}_m}{Z_0}\right)}
\tag{6.78}
$$

Note that the lower limit in the integral in (6.76) is z, because the line-1 voltage to the left of a given location z on that line does not contribute to the backward-crosstalk voltage on line 2 at that same location.

We shall now consider an example to illustrate the application of (6.74) and (6.77) for a specified voltage $V_g(t)$ in Fig. 6.55.

Example 6.8 Determination of induced wave voltages in the secondary line of a coupled pair of lines

Let $V_g(t)$ in Fig. 6.55 be the function shown in Fig. 6.58, where $T_0 < T(= l/v_p)$. We wish to determine the $(+)$ and $(-)$ wave voltages on line 2.

Noting that

$$
V_1(t) = \frac{1}{2}V_g(t) = \begin{cases} (V_0/T_0)t & \text{for} \quad 0 < t < T_0 \\ V_0 & \text{for} \quad t > T_0 \end{cases}
$$

and hence

$$
V_1'(t) = \begin{cases} V_0/T_0 & \text{for} \quad 0 < t < T_0 \\ 0 & \text{for} \quad t > T_0 \end{cases}
$$

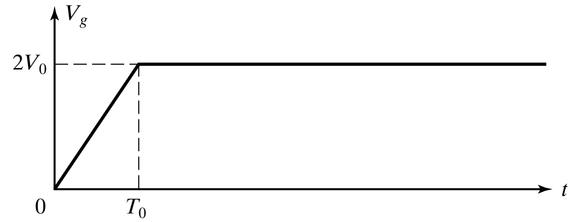


FIGURE 6.58

Source voltage for the system of Fig. 6.65 for Example 6.8.

and using (6.74), we can write the (+) wave voltage on line 2 as

$$\begin{aligned}
V_2^+(z,t) &= zK_f V_1'(t - z/v_p) \\
&= \begin{cases} zK_f V_0/T_0 & \text{for} \quad 0 < (t - z/v_p) < T_0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} zK_f V_0/T_0 & \text{for} \quad (z/v_p) < t < (z/v_p + T_0) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} zK_f V_0/T_0 & \text{for} \quad (z/l)T < t < [(z/l)t + T_0] \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

This is shown in the three-dimensional plot of Fig 6.59, in which the cross section in any constant-z plane is a pulse of voltage $zK_f V_0/T_0$ for $(z/l)T < t < (z/l)T + T_0$. Note that the pulse voltage is shown to be negative. This is because normally the effect of inductive coupling dominates that of the capacitive coupling, so that K_f is negative.

Using (6.77), the (−) wave voltage can be written as

$$V_2^-(z,t) = K_b[V_1(t - z/v_p) - V_1(t - 2l/v_p + z/v_p)]$$

where

$$\begin{aligned}
V_1\left(t - \frac{z}{v_p}\right) &= \begin{cases} \dfrac{V_0}{T_0}\left(t - \dfrac{z}{v_p}\right) & \text{for } 0 < \left(t - \dfrac{z}{v_p}\right) < T_0 \\ 0 & \text{for } \left(t - \dfrac{z}{v_p}\right) > T_0 \end{cases} \\
&= \begin{cases} \dfrac{V_0}{T_0}\left(t - \dfrac{z}{l}T\right) & \text{for } \dfrac{z}{l}T < t < \left(\dfrac{z}{l}T + T_0\right) \\ 0 & \text{for } t > \left(\dfrac{z}{l}T + T_0\right) \end{cases}
\end{aligned}$$

$$\begin{aligned}
V_1\left(t - \frac{2l}{v_p} + \frac{z}{v_p}\right) &= \begin{cases} \dfrac{V_0}{T_0}\left(t - \dfrac{2l}{v_p} + \dfrac{z}{v_p}\right) & \text{for } 0 < \left(t - \dfrac{2l}{v_p} + \dfrac{z}{v_p}\right) < T_0 \\ 0 & \text{for } \left(t - \dfrac{2l}{v_p} + \dfrac{z}{v_p}\right) > T_0 \end{cases} \\
&= \begin{cases} \dfrac{V_0}{T_0}\left(t - 2T + \dfrac{z}{l}T\right) & \text{for } \left(2T - \dfrac{z}{l}T\right) < t < \left(2T - \dfrac{z}{l}T + T_0\right) \\ 0 & \text{for } t > \left(2T - \dfrac{z}{l}T + T_0\right) \end{cases}
\end{aligned}$$

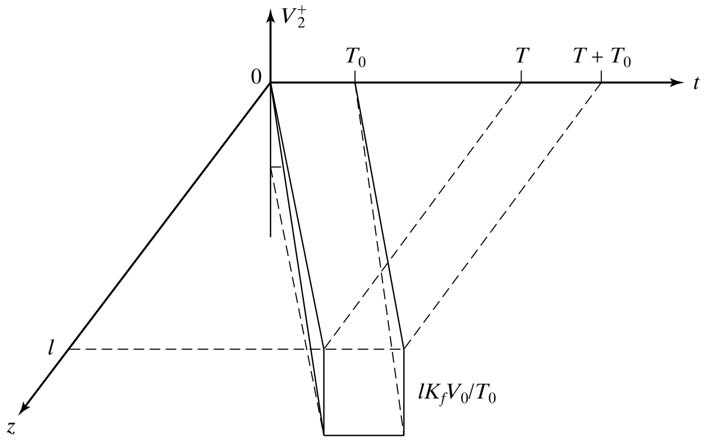


FIGURE 6.59

Three-dimensional depiction of forward-crosstalk voltage for the system of Fig. 6.55, with $V_g(t)$ as in Fig. 6.58.

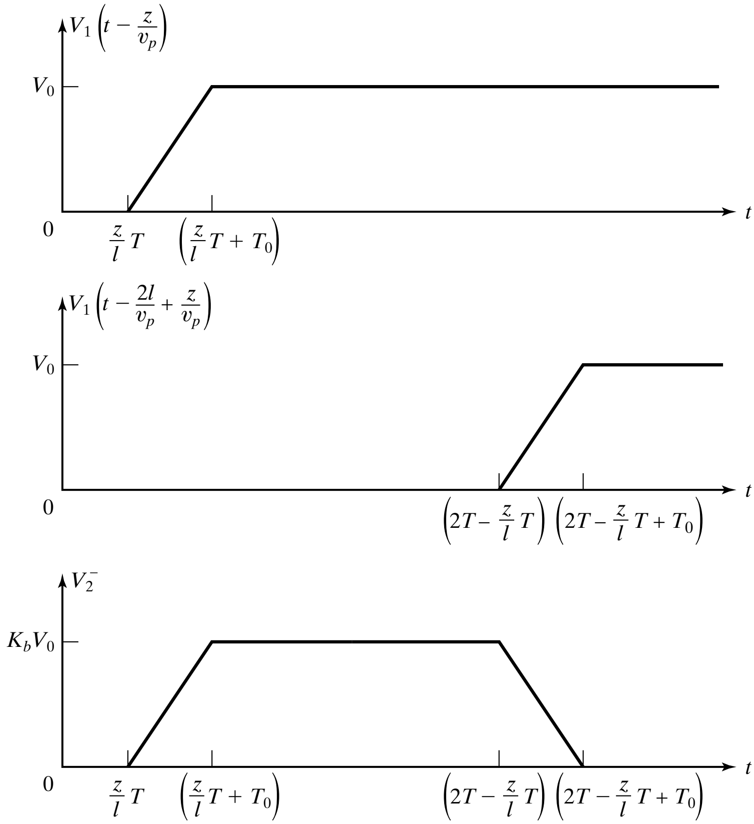


FIGURE 6.60

Determination of backward-crosstalk voltage for the system of Fig. 6.55, with $V_g(t)$ as in Fig. 6.58.

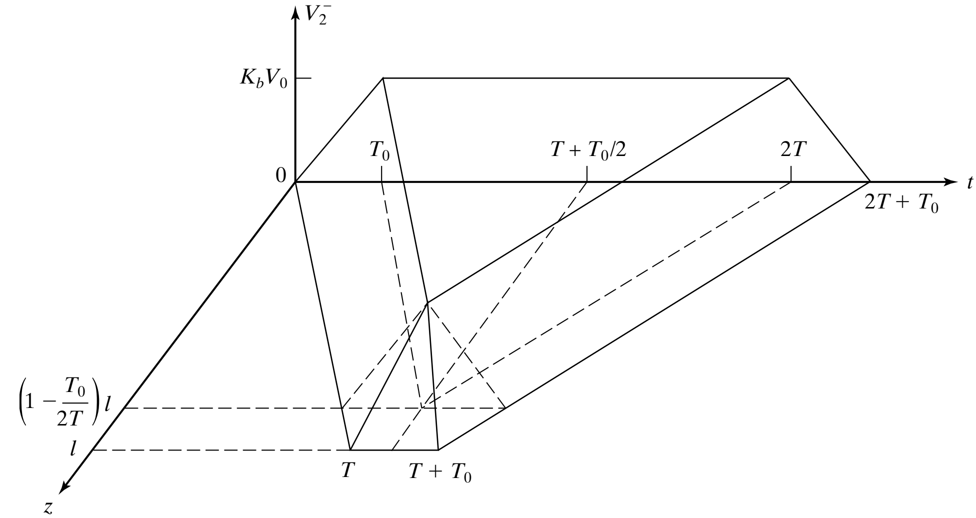


FIGURE 6.61

Three-dimensional depiction of backward-crosstalk voltage for the system of Fig. 6.55, with $V_g(t)$ as in Fig. 6.58.

These two voltages and the $(-)$ wave voltage for a value of z for which $(z/l)T + T_0 < 2T - (z/l)T$ are shown in Fig. 6.60. Figure 6.61 shows the three-dimensional plot of $V_2^-(z, t)$, in which the cross section in any given constant-z plane gives the time variation of V_2^- for that value of z. Note that as z varies from zero to l, the shape of V_2^- changes from a trapezoidal pulse with a height of K_bV_0 at $z = 0$ to a triangular pulse of height K_bV_0 and width $2T_0$ at $z = (1 - T_0/2T)l$ and then changes to a trapezoidal pulse again but with a height continuously decreasing from K_bV_0 to zero at $z = l$.

K6.7. Weak coupling analysis; Capacitive coupling; Inductive coupling; Forward crosstalk; Backward crosstalk.

D6.14. In Example 6.8, assume that $\mathscr{L} = 0.9\ \mu$H/m, $\mathscr{C} = 40$ pF/m, $\mathscr{L}_m = 0.093\ \mu$H/m, and $\mathscr{C}_m = 4$ pF/m for the line parameters in Fig. 6.55, and $T_0 = 0.2T$ for $V_g(t)$ in Fig. 6.58. Find the following: **(a)** the forward-crosstalk coefficient; **(b)** the backward-crosstalk coefficient; **(c)** $V_2(0, 0.01T)$; **(d)** $V_2(l, 1.1T)$; and **(e)** $V_2(0.5l, 0.6T)$.

Ans. **(a)** -0.01 ns/m; **(b)** 0.0508; **(c)** $0.0025V_0$; **(d)** $-0.0083V_0$; **(e)** $0.0212V_0$.

SUMMARY

In this chapter we introduced the parallel-plate transmission line by considering a uniform plane wave propagating between two parallel perfectly conducting plates. We showed that wave propagation on a transmission line can be discussed in terms of voltage and current, which are related to the electric and magnetic field, respectively, by deriving the transmission-line equations

$$\frac{\partial V}{\partial z} = -\mathcal{L}\frac{\partial I}{\partial t} \tag{6.79a}$$

$$\frac{\partial I}{\partial z} = -\mathcal{C}\frac{\partial V}{\partial t} \tag{6.79b}$$

which then led us to the concept of the distributed circuit. We learned that propagation along a transmission line in the general case is characterized by transverse electromagnetic waves, with the parameters $\mathcal{L}$ and $\mathcal{C}$ differing from one line to another and derivable from static-field considerations. The solutions to the transmission-line equations are

$$V(z, t) = Af\left(t - \frac{z}{v_p}\right) + Bg\left(t + \frac{z}{v_p}\right) \tag{6.80a}$$

$$I(z, t) = \frac{1}{Z_0}\left[Af\left(t - \frac{z}{v_p}\right) - Bg\left(t + \frac{z}{v_p}\right)\right] \tag{6.80b}$$

where $Z_0 = \sqrt{\mathcal{L}/\mathcal{C}}$ is the characteristic impedance of the line, and $v_p = 1/\sqrt{\mathcal{L}\mathcal{C}}$ is the velocity of propagation on the line.

We discussed the determination of Z_0 and v_p for the case of a line with homogeneous dielectric, as well as for the case of a line involving more than one dielectric, an example being the microstrip line. For the former case,

$$Z_0 = \frac{\sqrt{\varepsilon_r}}{c\mathcal{C}} \tag{6.81a}$$

$$v_p = \frac{c}{\sqrt{\varepsilon_r}} \tag{6.81b}$$

where ε_r, c, and $\mathcal{C}$ are the relative permittivity of the dielectric, the velocity of light in free space, and the capacitance per unit length of the line computed from static field considerations, respectively. For the latter case, assuming nonmagnetic dielectrics,

$$Z_0 = \frac{1}{c\sqrt{\mathcal{C}\mathcal{C}_0}} \tag{6.82a}$$

$$v_p = c\sqrt{\frac{\mathcal{C}_0}{\mathcal{C}}} \tag{6.82b}$$

where $\mathcal{C}$ is the capacitance per unit length of the line with the dielectrics in place, and $\mathcal{C}_0$ is the capacitance per unit length with all dielectrics replaced by free space, both computed from static field considerations. Note that (6.82a) and (6.82b) reduce to (6.81a) and (6.81b), respectively, if all dielectrics are the same, since then $\mathscr{C} = \mathscr{C}_0\varepsilon_r$. Based on (6.81a) and (6.81b), and using closed form solutions obtained in Sections 5.3 and 5.4 for the capacitance per unit length, we presented the analytical expressions for Z_0 for some common types of lines.

We then discussed time-domain analysis of a transmission line terminated by a load resistance R_L and excited by a constant voltage source V_0 in series with internal resistance R_g. Writing the general solutions (6.80a) and (6.80b) concisely in the manner

$$V = V^+ + V^- \tag{6.83a}$$
$$I = I^+ + I^- \tag{6.83b}$$

where

$$I^+ = \frac{V^+}{Z_0} \tag{6.84a}$$
$$I^- = -\frac{V^-}{Z_0} \tag{6.84b}$$

we found that the situation consists of the bouncing back and forth of transient (+) and (−) waves between the two ends of the line. The initial (+) wave voltage is $V^+Z_0/(R_g + Z_0)$. All other waves are governed by the reflection coefficients at the two ends of the line, given for the voltage by

$$\Gamma_R = \frac{R_L - Z_0}{R_L + Z_0} \tag{6.85a}$$

and

$$\Gamma_S = \frac{R_g - Z_0}{R_g + Z_0} \tag{6.85b}$$

for the load and source ends, respectively. In the steady state, the situation is the superposition of all the transient waves, equivalent to the sum of a single (+) wave and a single (−) wave. We discussed the bounce-diagram technique of keeping track of the transient phenomenon and extended it to a pulse voltage source.

We learned that when a wave is incident from, say, line 1 onto a junction with line 2, reflection occurs just as though line 1 is terminated by a load resistor equal to the characteristic impedance of line 2. A transmitted wave goes into line 2 in accordance with the voltage and current transmission coefficients

$$\tau_V = 1 + \Gamma \tag{6.86a}$$

and

$$\tau_C = 1 - \Gamma \tag{6.86b}$$

respectively, where Γ is the voltage reflection coefficient. Applying this to a system of three lines in cascade, we showed how to obtain the unit impulse response of the system and from it obtain the frequency response. We then extended the analysis to lines with discontinuities to discuss and illustrate by means of an example the application of time-domain reflectometry, an important experimental technique.

We then considered lines with reactive terminations and discontinuities, where we learned that the reflection coefficient concept is not useful to study the transient behavior. It is necessary to write the differential equations pertinent to the boundary conditions at the terminations and/or discontinuities, and solve them subject to the appropriate initial conditions; alternatively, the required voltages and currents can be obtained from considerations of initial and final behaviors of the reactive element(s), and associated time constant(s).

As a prelude to the consideration of interconnections between logic gates, we discussed time-domain analysis of lines with nonzero initial conditions. For the general case, the initial voltage and current distributions $V(z, 0)$ and $I(z, 0)$ are decomposed into $(+)$ and $(-)$ wave voltages and currents as given by

$$\begin{aligned}
V^+(z, 0) &= \tfrac{1}{2}[V(z, 0) + Z_0 I(z, 0)] \\
V^-(z, 0) &= \tfrac{1}{2}[V(z, 0) - Z_0 I(z, 0)] \\
I^+(z, 0) &= \frac{1}{Z_0} V^+(z, 0) \\
I^-(z, 0) &= -\frac{1}{Z_0} V^-(z, 0)
\end{aligned}$$

The voltage and current distributions for $t > 0$ are then obtained by keeping track of the bouncing of these waves at the two ends of the line. For the special case of uniform distribution, the analysis can be performed more conveniently by considering the situation as one in which a transient wave is superimposed on the initial distribution and using the bounce-diagram technique. We then introduced the load-line technique of time-domain analysis, and applied it to the analysis of transmission-line interconnection between logic gates.

Finally, we studied the topic of crosstalk on transmission lines, by considering the case of weak coupling between two lines. We learned that for a given wave on the primary line, the crosstalk consists of two waves, forward and backward, induced on the secondary line and governed by the forward-crosstalk coefficient and the backward-crosstalk coefficient, respectively. We illustrated by means of an example the determination of crosstalk voltages for a specified excitation for the primary line.

REVIEW QUESTIONS

Q6.1. Describe the phenomenon of guiding of a uniform plane wave by a pair of parallel, plane, perfectly conducting sheets.

Q6.2. Discuss the derivation of the transmission-line equations from the field equations by considering the parallel-plate line.

Q6.3. Discuss the concept of the distributed circuit as compared to a lumped circuit.

Q6.4. Discuss the physical interpretation of the distributed circuit concept from energy considerations.

Q6.5. What is a transverse electromagnetic wave? Discuss the electric and magnetic field distributions associated with a transverse electromagnetic wave.

Q6.6. Discuss the analogy between uniform plane wave parameters and transmission-line parameters.

Q6.7. Explain why the product of $\mathscr{L}$ and $\mathscr{C}$ of a line is equal to the product of μ and ε of the dielectric of the line.

Q6.8. What is the significance of the characteristic impedance of a line? Why is it not in general equal to the intrinsic impedance of the medium between the conductors of the line?

Q6.9. Discuss the geometry associated with the microstrip line and the determination of its characteristic impedance and velocity of propagation.

Q6.10. Discuss the general solutions for the line voltage and current and the notation associated with their representation in concise form.

Q6.11. What is the fundamental distinction between the occurrence of the response in one branch of a lumped circuit to the application of an excitation in a different branch of the circuit and the occurrence of the response at one location on a transmission line to the application of an excitation at a different location on the line?

Q6.12. Describe the phenomenon of the bouncing back and forth of transient waves on a transmission line excited by a constant voltage source in series with internal resistance and terminated by a resistance.

Q6.13. What is the nature of the formula for the voltage reflection coefficient? Discuss its values for some special cases.

Q6.14. What is the steady-state equivalent of a line excited by a constant voltage source? What is the actual situation in the steady state?

Q6.15. Discuss the bounce-diagram technique of keeping track of the bouncing back and forth of the transient waves on a transmission line for a constant voltage source.

Q6.16. Discuss the bounce-diagram technique of keeping track of the bouncing back and forth of the transient waves on a transmission line for a pulse voltage source.

Q6.17. How are the voltage and current transmission coefficients at the junction between two lines related to the voltage reflection coefficient?

Q6.18. Explain how it is possible for the transmitted voltage or current at a junction between two lines to exceed the incident voltage or current.

Q6.19. Discuss the determination of the unit impulse response of a system of three lines in cascade.

Q6.20. Outline the procedure for the determination of the frequency response of a system of three lines in cascade from its unit impulse response.

Q6.21. What is a radome? How is it analyzed by using transmission-line equivalent?

Q6.22. Describe the technique of locating discontinuities in a transmission-line system by using a time-domain reflectometer.

Q6.23. Discuss the transient analysis of a line driven by a constant voltage source in series with a resistance equal to the Z_0 of the line and terminated by an inductor.

Q6.24. Why is the concept of reflection coefficient not useful for studying the transient behavior of lines with reactive terminations and discontinuities?

Q6.25. Discuss the determination of the transient behavior of lines with reactive terminations and discontinuities without formally setting up the differential equations and solving them.

Q6.26. Discuss the determination of the voltage and current distributions on an initially charged line for any given time from the knowledge of the initial voltage and current distributions.

Q6.27. Discuss with the aid of an example the discharging of an initially charged line into a resistor.

Q6.28. Discuss the bounce-diagram technique of transient analysis of a line with uniform initial voltage and current distributions.

Q6.29. How do you check the energy balance for the case of a line with initial voltage and/or current distribution(s) and discharged into a resistor?

Q6.30. Discuss the load-line technique of obtaining the time variations of the voltages and currents at the source and load ends of a line from a knowledge of the terminal *V-I* characteristics.

Q6.31. Discuss the analysis of transmission-line interconnection between two logic gates.

Q6.32. Discuss briefly the weak-coupling analysis for crosstalk between two transmission lines.

Q6.33. Discuss the modeling of capacitive and inductive couplings for crosstalk on transmission lines.

Q6.34. Discuss and distinguish between the dependence of the forward- and backward-crosstalk coefficients on the line parameters.

Q6.35. Outline the determination of the forward- and backward-crosstalk voltages induced on a secondary line for a given excitation for the primary line.

PROBLEMS

Section 6.1

P6.1. Finding fields and power flow for a parallel-plate line for specified voltage along the line. A parallel-plate transmission line is made up of perfect conductors of width $w = 0.1$ m and lying in the planes $x = 0$ and $x = 0.01$ m. The medium between the conductors is a nonmagnetic ($\mu = \mu_0$), perfect dielectric. For a uniform plane wave propagating along the line, the voltage along the line is given by

$$V(z, t) = 10 \cos (3\pi \times 10^8 t - 2\pi z) \text{ V}$$

Neglecting fringing of fields, find: (**a**) the electric field intensity $E_x(z, t)$ of the wave; (**b**) the magnetic field intensity $H_y(z, t)$ of the wave; (**c**) the current $I(z, t)$ along the line; and (**d**) the power flow $P(z, t)$ down the line.

P6.2. Computation of parameters for a parallel-plate line with two dielectrics in parallel. A parallel-plate transmission line consists of an arrangement of two perfect dielectrics, as shown by the transverse cross section in Fig. 6.62. Note that $\mu_1\varepsilon_1 = \mu_2\varepsilon_2$, so that the TEM waves propagating in the two dielectrics are in phase at all points along the interface between the dielectrics. Neglect fringing of fields and compute the values of $\mathscr{L}$, $\mathscr{C}$, and Z_0 of the line.

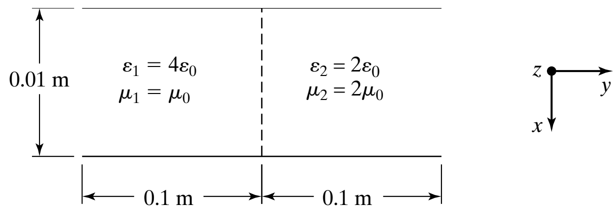


FIGURE 6.62
For Problem P6.2.

P6.3. Computation of parameters for a parallel-plate line with two dielectrics in series. Repeat Problem P6.2 for a parallel-plate line having the cross section shown in Fig. 6.63.

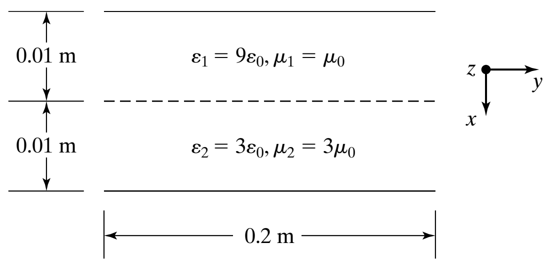


FIGURE 6.63
For Problem P6.3.

P6.4. Transmission-line equations and power flow from the geometry of a coaxial cable. Derive the transmission-line equations by considering the special case of two infinitely long coaxial cylindrical conductors. Also show that the power flow along the line is equal to the product of the voltage between the conductors and current along the conductors.

Section 6.2

P6.5. A transmission-line system involving two lines. In the system shown in Fig. 6.64, assume that V_g is a constant voltage source of 100 V and the switch S is closed at $t = 0$. Find and sketch: (**a**) the line voltage versus z for $t = 0.2\ \mu$s; (**b**) the line current versus z for $t = 0.4\ \mu$s; (**c**) the line voltage versus t for $z = 30$ m; and (**d**) the line current versus t for $z = -40$ m.

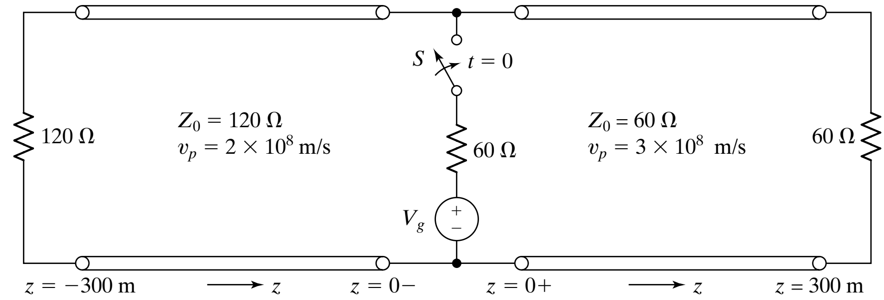


FIGURE 6.64

For Problem P6.5.

P6.6. Finding several quantities in a transmission-line system from given observations. In the system shown in Fig. 6.65(a), the switch S is closed at $t = 0$. The line voltage variations with time at $z = 0$ and $z = l$ for the first 5 μs are observed to be as shown in Fig. 6.65(b) and (c), respectively. Find the values of V_0, R_g, R_L, and T.

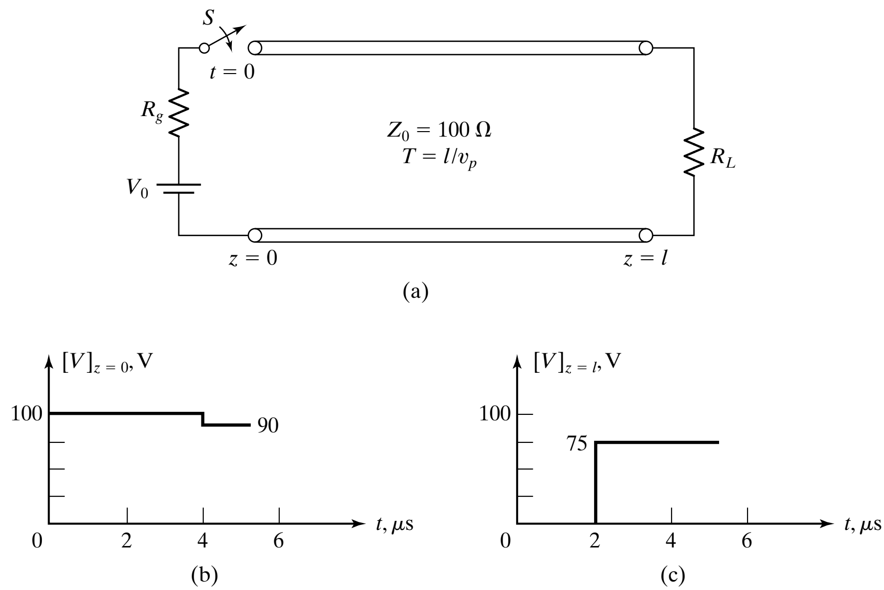


FIGURE 6.65

For Problem P6.6.

P6.7. Expressing the steady-state situation on a line as superposition of (+) and (−) waves. The system shown in Fig. 6.66 is in steady state. Find (**a**) the line voltage and current, (**b**) the (+) wave voltage and current, and (**c**) the (−) wave voltage and current.

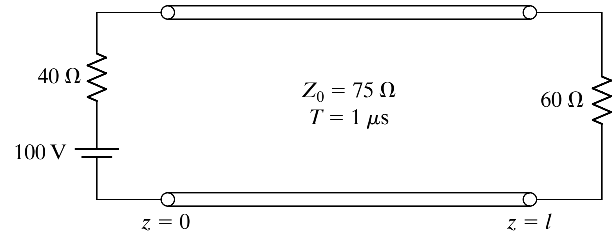


FIGURE 6.66

For Problem P6.7.

P6.8. Time-domain analysis of a transmission-line system using the bounce-diagram technique. In the system shown in Fig. 6.67, the switch S is closed at $t = 0$. Assume $V_g(t)$ to be a direct voltage of 90 V and draw the voltage and current bounce diagrams. From these bounce diagrams, sketch: **(a)** the line voltage and line current versus t (up to $t = 7.25\ \mu s$) at $z = 0$, $z = l$, and $z = l/2$; and **(b)** the line voltage and line current versus z for $t = 1.2\ \mu s$ and $t = 3.5\ \mu s$.

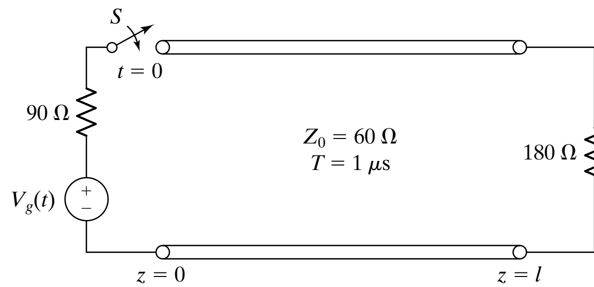


FIGURE 6.67

For Problem P6.8.

P6.9. Time-domain analysis of a transmission-line system for a triangular pulse excitation. Repeat Problem P6.5 assuming V_g to be a triangular pulse, as shown in Fig. 6.68.

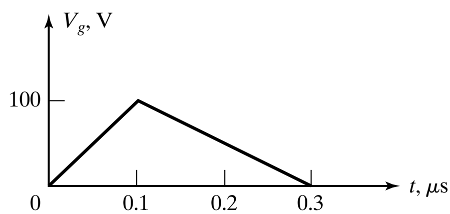
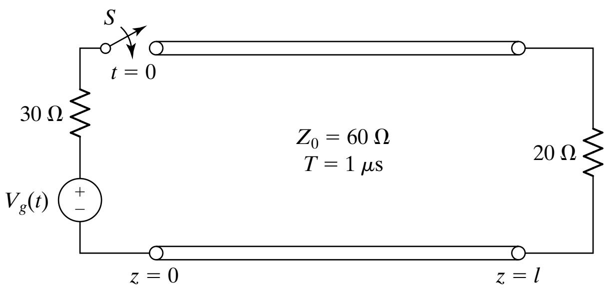


FIGURE 6.68

For Problem P6.9.

P6.10. Time-domain analysis of a transmission-line system for a rectangular pulse excitation. For the system of Problem P6.8, assume that the voltage source is of 0.3 μs duration instead of being of infinite duration. Find and sketch the line voltage and line current versus z for $t = 1.2\ \mu s$ and $t = 3.5\ \mu s$.

P6.11. Time-domain analysis of a transmission-line system for a triangular pulse excitation. In the system shown in Fig. 6.69, the switch S is closed at $t = 0$. Find and sketch: **(a)** the line voltage versus z for $t = 2\frac{1}{2}\ \mu s$; **(b)** the line current versus z for $t = 2\frac{1}{2}\ \mu s$; and **(c)** the line voltage at $z = l$ versus t up to $t = 4\ \mu s$.



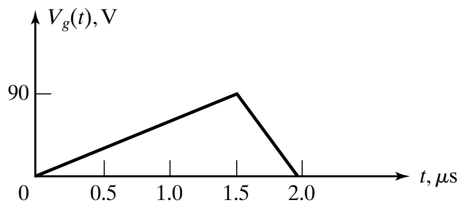


FIGURE 6.69

For Problem P6.11.

P6.12. Time-domain analysis of a transmission-line system for a sinusoidal excitation. In the system shown in Fig. 6.70, the switch S is closed at $t = 0$. Draw the voltage and current-bounce diagrams and sketch **(a)** the line voltage and line current versus t for $z = 0$ and $z = l$ and **(b)** the line voltage and line current versus z for $t = 2$, 9/4, 5/2, 11/4, and 3 μs. Note that the period of the source voltage is 2 μs, which is equal to the two-way travel time on the line.

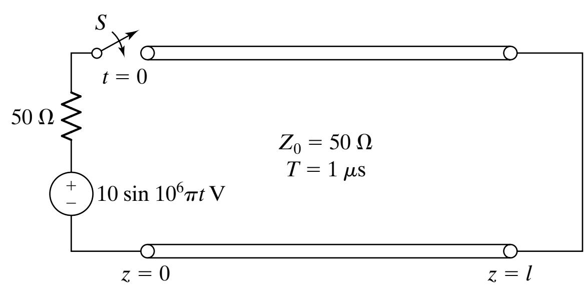


FIGURE 6.70

For Problem P6.12.

Section 6.3

P6.13. Reflection and transmission at a transmission-line discontinuity. In the system shown in Fig. 6.71, an incident wave of voltage V^+ strikes the discontinuity from the left, that is, from line 1. Find the reflected wave voltage and current into line 1 and the transmitted wave voltage and current into line 2.

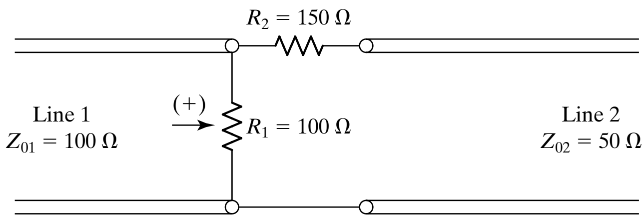


FIGURE 6.71

For Problem P6.13.

P6.14. Unit impulse response and frequency response for a system of three lines in cascade. In the system shown in Fig. 6.72: **(a)** find the output voltage V_o across the 300-Ω resistor for $V_g(t) = \delta(t)$; and **(b)** find and sketch the amplitude of $V_o(t)$ versus ω for $V_g(t) = \cos \omega t$.

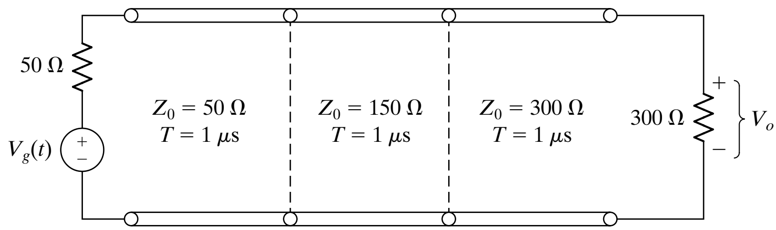


FIGURE 6.72

For Problem P6.14.

P6.15. Finding unknown parameters for a system of three media from unit impulse response. In Fig. 6.73 (a), the plane I is the input plane from which a uniform

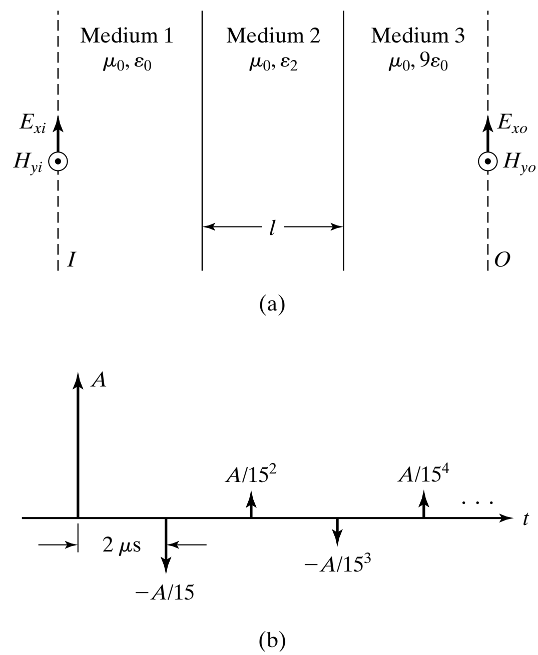


FIGURE 6.73

For Problem P6.15.

plane wave is incident normally on the interface between medium 1 and medium 2, and the plane O is the output plane in which the response of the system is observed. For an incident wave of $E_{xi}(t) = \delta(t)$, find the minimum value of the thickness l and the corresponding value of the permittivity ε_2 of medium 2 required to obtain the electric field $E_{xo}(t)$ in the output plane, as shown in Fig. 6.84(b), in which the interval between successive impulses is 2 μs. Then find the value of A, and sketch the reflected wave electric field in the plane I.

P6.16. Computing reflected and transmitted powers at a junction involving three lines. In Fig. 6.74, a (+) wave carrying power P is incident on the junction $a-a'$ from line 1. Find (**a**) the power reflected into line 1; (**b**) the power transmitted into line 2; and (**c**) the power transmitted into line 3.

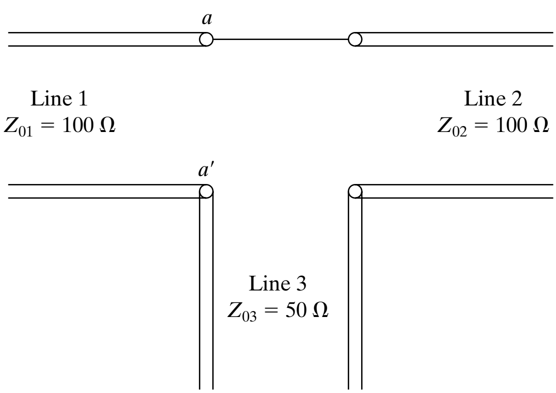


FIGURE 6.74
For Problem P6.16.

P6.17. Time-domain reflectometer system observations for a line with a given discontinuity. In the system of Fig. 6.29, assume that the discontinuity at $z = 4$ m is a resistor of value 40 Ω in series with the line, instead of the 120-Ω parallel resistor. Find and sketch the waveform that the TDR system would measure up to $t = 200$ ns.

Section 6.4

P6.18. Line terminated with an inductive load. In the system shown in Fig. 6.75, the switch S is closed at $t = 0$ with no current in the relay coil and with the relay in position 1. When the relay coil current I_L reaches 1.73 A, the relay switches to position 2; when the current drops to 0.636 A, the relay switches back to position 1. **(a)** Find the time t_1 at which the relay switches to position 2. **(b)** Find the time t_2 at which the relay switches back to position 1.

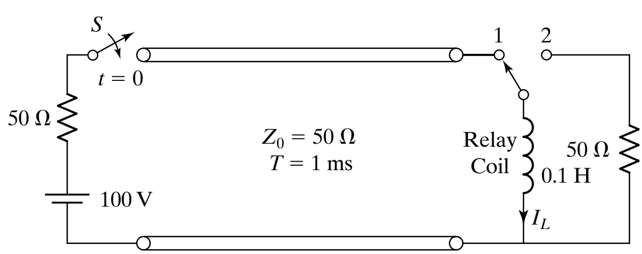


FIGURE 6.75
For Problem P6.18.

P6.19. Line terminated with a capacitive load. In the system shown in Fig. 6.76, the switch S is closed at $t = 0$, with the voltage across the capacitor equal to zero. **(a)** Obtain the differential equation for V^- at $z = l$. **(b)** Find the solution for $V^-(l, t)$.

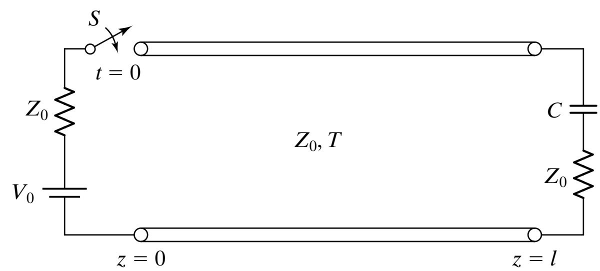


FIGURE 6.76
For Problem P6.19.

P6.20. A transmission-line system with inductive discontinuity. In the system shown in Fig. 6.77, the switch S is closed at $t = 0$, with the lines uncharged and with zero current in the inductor. Obtain the solution for the line voltage versus time at $z = l+$.

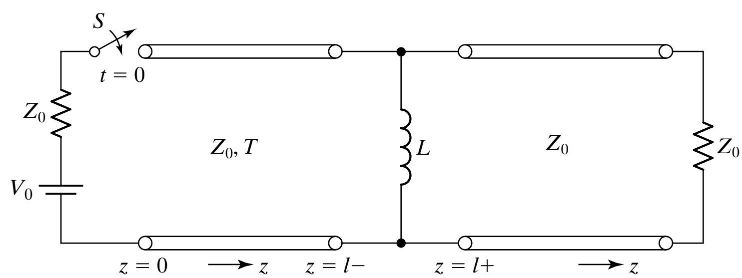


FIGURE 6.77
For Problem P6.20.

P6.21. Using observations to find the parameters for a transmission line with a discontinuity. In the system shown in Fig. 6.78(a), the network N consists of a single circuit element (R, L, or C). The system is initially uncharged. The switch S is closed at $t = 0$, and the line voltage at $z = 0$ is observed to be as shown in Fig. 6.78(b). **(a)** Determine whether the circuit element is R, L, or C. **(b)** Find the value of Z_{02}/Z_{01}.

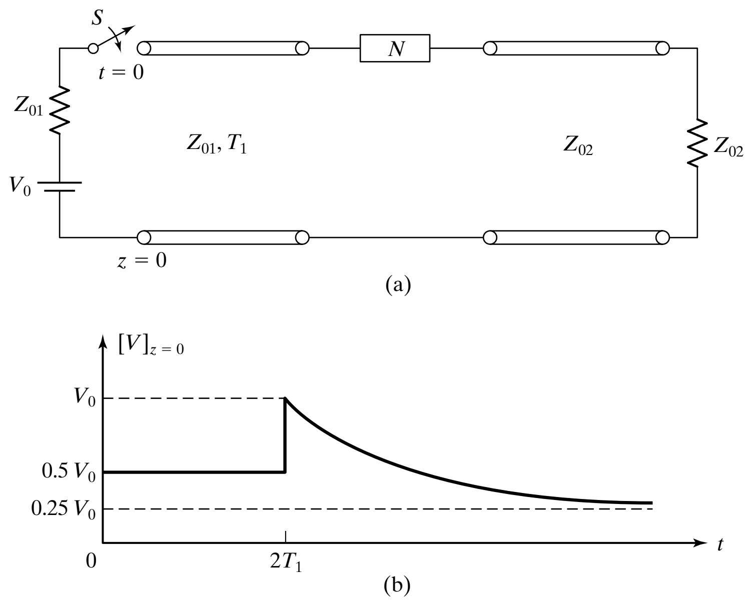


FIGURE 6.78

For Problem P6.21.

Section 6.5

P6.22. Discharging of an initially charged line into a passive nonlinear element. In the system shown in Fig. 6.79, a passive nonlinear element having the indicated volt-ampere characteristic is connected to an initially charged line at $t = 0$. Find the voltage across the nonlinear element immediately after closure of the switch.

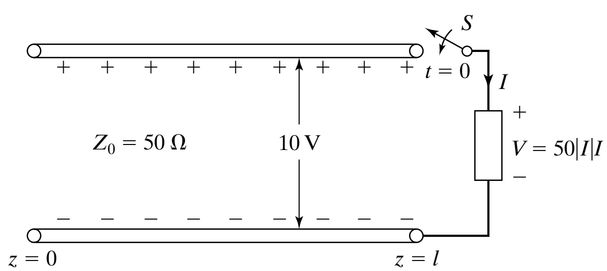


FIGURE 6.79

For Problem P6.22.

P6.23. Bounce-diagram technique and checking energy balance for an initially charged line. In the system shown in Fig. 6.80, steady-state conditions are established with the switch S closed. At $t = 0$, the switch is opened. **(a)** Find the sketch the voltage across the 150-Ω resistor for $t \geq 0$, with the aid of a bounce diagram. **(b)** Show that the total energy dissipated in the 150-Ω resistor after opening the switch is exactly the same as the energy stored in the line before opening the switch.

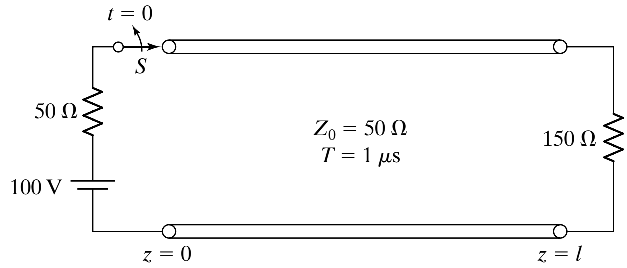


FIGURE 6.80
For Problem P6.23.

P6.24. An initially charged transmission-line system. In the system shown in Fig. 6.81, steady-state conditions are established with the switch S closed. At $t = 0$, the switch is opened. **(a)** Sketch the voltage and current along the system for $t = 0-$. **(b)** Find the total energy stored in the lines for $t = 0-$. **(c)** Find and sketch the voltages across the two resistors for $t > 0$. **(d)** From your sketches of part (c), find the total energy dissipated in the resistors for $t > 0$.

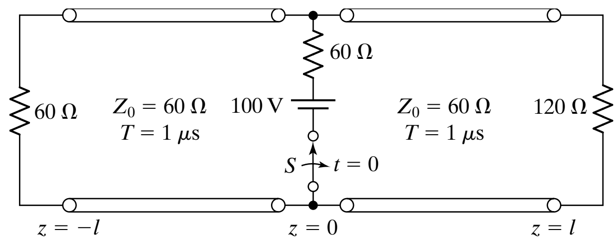


FIGURE 6.81
For Problem P6.24.

P6.25. An initially charged line connected to an inductor. In the system shown in Fig. 6.82, steady-state conditions are established with the switch S open and no current in the inductor. At $t = 0$, the switch is closed. **(a)** Obtain the expressions for the line voltage and current versus t at $z = l$. **(b)** Sketch the line voltage and current versus z for $t = T/2$.

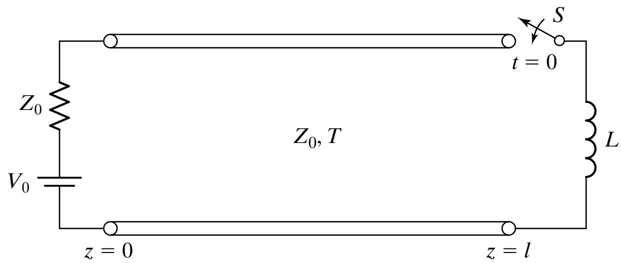


FIGURE 6.82
For Problem P6.25.

Section 6.6

P6.26. Application of load-line technique for a line with linear resistive terminations. For the system of Problem P6.8, use the load-line technique to obtain and plot line voltage and line current versus t (up to $t = 5.25\ \mu s$) at $z = 0$ and $z = l$. Also obtain the steady-state values of line voltage and current from the load-line construction.

P6.27. Application of load-line technique for an initially charged line. For the system of Problem P6.22, use the load-line technique to obtain and plot line voltage versus t from $t = 0$ up to $t = 7l/v_p$ at $z = 0$ and $z = l$.

P6.28. Analysis of transmission-line interconnection between two logic gates. For the example of interconnection between logic gates of Fig. 6.51(a), repeat the load-line constructions for $Z_0 = 50\ \Omega$ and draw graphs of V_i versus t for both 0-to-1 and 1-to-0 transitions.

P6.29. Analysis of transmission-line interconnection between two logic gates. For the example of interconnection between logic gates of Fig. 6.51(a), find **(a)** the minimum value of Z_0 such that for the transition from 0 to 1, the voltage V_i reaches 2 V at $t = T+$ and **(b)** the minimum value of Z_0 such that for the transition from 1 to 0, the voltage V_i reaches 1 V at $t = T+$.

Section 6.7

P6.30. Determination of induced wave voltages in the secondary line of a coupled pair of lines. In Example 6.8, assume that $V_g(t)$ is the function shown in Fig. 6.83, instead of as in Fig. 6.58. Find and sketch the following: **(a)** $V_2^+(l, t)$; **(b)** $V_2^-(0, t)$; and **(c)** $V_2^-(0.8l, t)$.

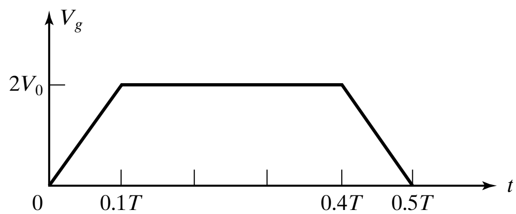


FIGURE 6.83
For Problem P6.30.

P6.31. Determination of induced wave voltages in the secondary line of a coupled pair of lines. In Example 6.8, assume that

$$V_g(t) = \begin{cases} 2V_0 \sin^2 \pi t/T & \text{for} \quad 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

Find and sketch the following: **(a)** $V_2^+(l, t)$; **(b)** $V_2^-(0, t)$; **(c)** $V_2^-(0.75l, t)$.

P6.32. Determination of induced wave voltages in the secondary line of a coupled pair of lines. In Example 6.8, assume that $K_b/K_f = -25v_p$ and $T_0 = 0.2T$. Find and sketch the following: **(a)** $V_2^+(z, 1.1T)$; **(b)** $V_2^-(z, 1.1T)$; and **(c)** $V_2(z, 1.1T)$.

REVIEW PROBLEMS

R6.1. Circuit equivalents for transmission-line equations. Show that two alternative representations of the circuit equivalent of the transmission line equations (6.12a) and (6.12b) are as shown in Figs. 6.84(a) and 6.84 (b).

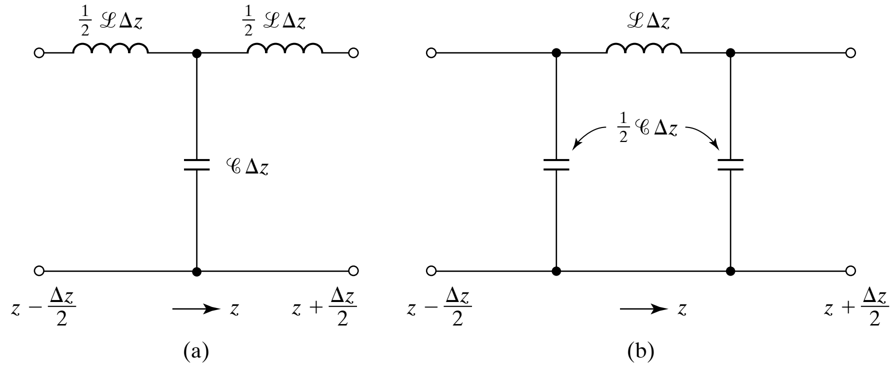


FIGURE 6.84
For Problem R6.1.

R6.2. A system of three transmission lines excited by a pulse voltage source. In the system shown in Fig. 6.85, the voltage source V_g is a pulse of amplitude 10 V and duration 1 μs from $t = 0$ to $t = 1\ \mu$s, and the switch S is closed at $t = 0$. **(a)** Find and sketch the voltage across the load resistor R_L as a function of time for $t > 0$. **(b)** Find and sketch the voltage across the internal resistance R_g of the voltage source as a function of time for $t > 0$. **(c)** Show that the total energy supplied by the voltage source is equal to the sum of the total energy dissipated in R_L and the total energy dissipated in R_g.

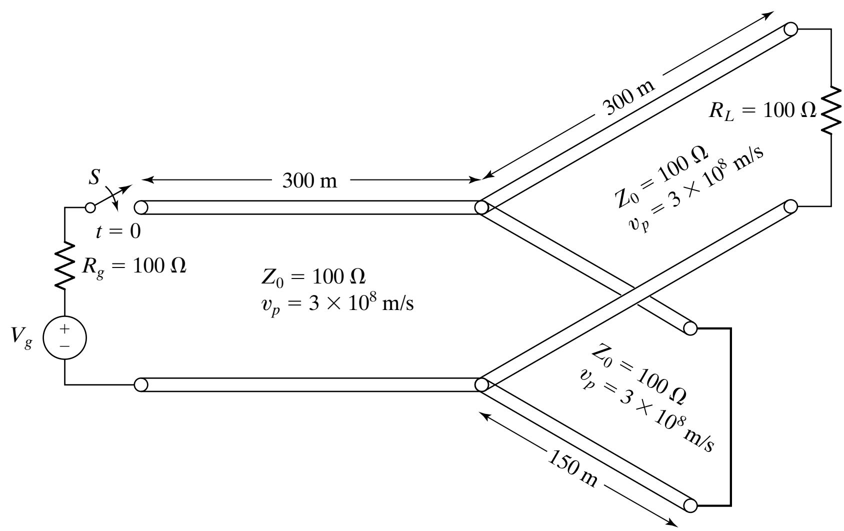


FIGURE 6.85
For Problem R6.2.

R6.3. Uniform plane-wave impulse response and transparency of a dielectric slab. In Fig. 6.86 the plane I is the input plane from which a uniform plane wave is incident normally on the interface between medium 1 and medium 2, and the plane O is the output plane in which the response of the system is observed. **(a)** For an incident wave of $E_{xi}(t) = \delta(t)$, a unit impulse, find and sketch the sequence of impulses for the electric field $E_{xo}(t)$ in the output plane. **(b)** If $E_{xi}(t)$ consists of a periodic sequence of unit impulses of frequency f, show that there exists a minimum value of f for which $E_{xo}(t)$ consists of a periodic sequence of unit impulses of the same frequency f, and find that value of f.

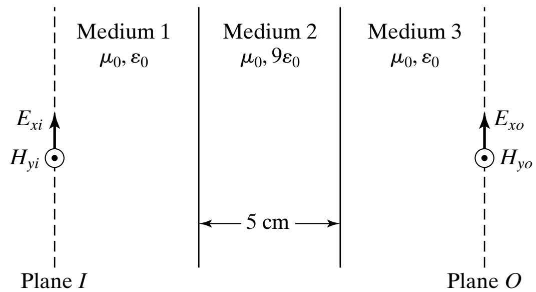


FIGURE 6.86

For Problem R6.3.

R6.4. A system of three lines with a resistive network at the junction. In the system shown in Fig. 6.87, a (+) wave carrying power P is incident on the junction $a{-}a'$ from line 1. **(a)** Find the value of R for which there is no reflected wave into line 1. **(b)** For the value of R found in (a), find the power transmitted into each of lines 2 and 3.

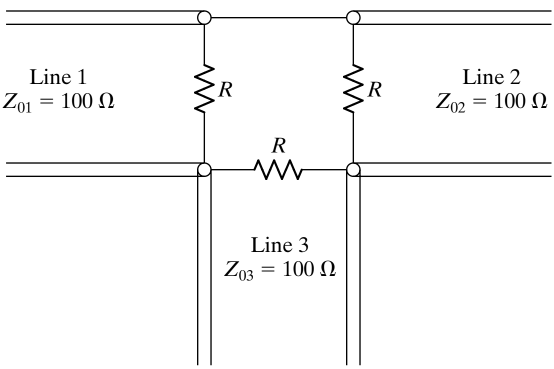


FIGURE 6.87

For Problem R6.4.

R6.5. An initially charged transmission-line system with capacitive discontinuity. In the system shown in Fig. 6.88, steady-state conditions are established with the switch S closed. At $t = 0$, the switch S is opened. **(a)** Find the energy stored in the system at $t = 0-$. **(b)** Obtain the solutions for the voltages across R_{L1} and R_{L2} for $t > 0$. **(c)** Show that the total energy dissipated in R_{L1} and R_{L2} for $t > 0$ is equal to the energy stored in the system at $t = 0-$.

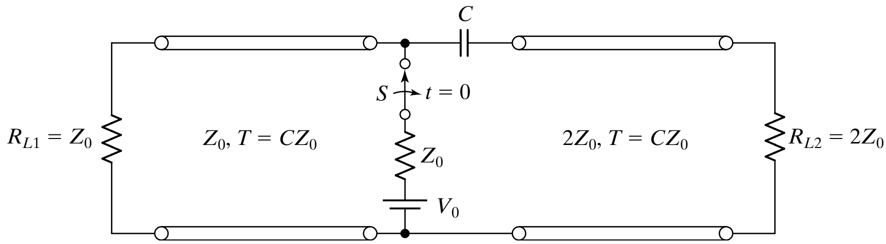


FIGURE 6.88

For Problem R6.5.

R6.6. Transmission-line interconnection between two logic gates. For the example of interconnection between logic gates in Section 6.6, find the value of Z_0 for which the voltage reached at $t = T+$ for the transition from 0 to 1 is the same as that reached for the transition from 1 to 0. What is the value of this voltage?

R6.7. A coupled line pair excited by a sinusoidal source in the primary line. In example 6.8, assume that $V_g(t) = V_0 \cos 2\pi f t$. Show that for $f = 1/4T$, the ratio of the amplitude of $V_2(0, t)$ to V_0, the amplitude of $V_g(t)$, is equal to K_b, and the ratio of the amplitude of $V_2(l, t)$ to V_0 is equal to $\pi f l |K_f|$.