APPLIED LINEAR ALGEBRA

2. Determinants

(Section 7.7 of Kreyszig)

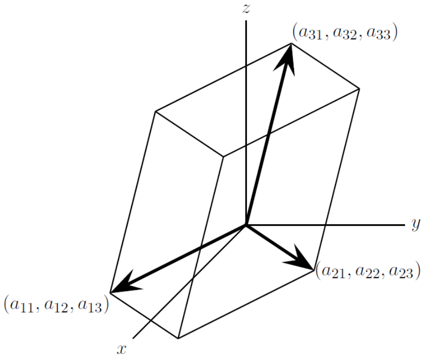
Contents

Section 1

Determinants

Introduction

1. Determinants arise naturally when we want to find volumes of parallelepipeds in *n*-dimensional space.



2. They can also be used to compute A^{-1} in terms of the entries of A.

Determinants

- Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbb{R}^n$. We denote by $\det(v_1, v_2, \ldots, v_n)$ a function whose absolute value is the volume of the parallelepiped in $\mathbb{R}^n$ whose edges are $v_1, v_2, \ldots, v_n$.
- If A is the matrix with columns $v_1, v_2, \ldots, v_n$, we define

$$|A| = \det(A) = \det(v_1, v_2, \ldots, v_n).$$

- By examining the scaling and additive properties of volumes, we will deduce several properties that det need to satisfy.

Determinants

1. $\det(v_1, \ldots, \alpha v_k, \ldots, v_n) = \alpha \det(v_1, \ldots, v_k, \ldots, v_n)$ for any $\alpha \in \mathbb{R}$.
2. $\det(v_1, \ldots, u_k + w_k, \ldots, v_n) = \det(v_1, \ldots, u_k, \ldots, v_n)$ $+\det(v_1, \ldots, w_k, \ldots, v_n)$
3. $\det(v_1, \ldots, v_n) = 0$ if for some $j \neq k$, $v_j = v_k$.
4. For the standard basis $e_1, \ldots, e_n$, we have

   $$\det(e_1, \ldots, e_n) = 1.$$

   Here e_j is the vector whose j-th entry is 1, other entries are 0.
5. If A' is the matrix obtained from A by interchanging the j-th and k-th columns then

   $$\det(A') = -\det(A).$$

Determinants of 2×2 matrices

From the Linearity Properties 1 and 2, we get

$$\begin{aligned}
\det\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) &= a\det\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) + c\det\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) \\
&= ab\det\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + ad\det\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&\quad + cb\det\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + cd\det\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)
\end{aligned}$$

By Property 3,

$$\det\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 0.$$

Determinants of 2×2 matrices

By Property 5,

$$\det\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right) = -\det\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = -1.$$

Thus,

$$\det\left(\begin{bmatrix}a\\c\end{bmatrix}, \begin{bmatrix}b\\d\end{bmatrix}\right) = ad - bc.$$

Determinants of 3×3 matrices

Similarly, we have

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}a_{12}a_{13}\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$
$$+ a_{11}a_{12}a_{23}\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$
$$+ a_{11}a_{12}a_{33}\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) + \cdots$$

There are 27 terms on the right hand side, each is the product of n entries, one from each column, and the determinant of the corresponding 0-1 matrix.

Determinants of 3×3 matrices

Most of these terms are 0, by Property 3. A term can be non zero only if there are no two entries on the same row. Thus there are 6 terms left:

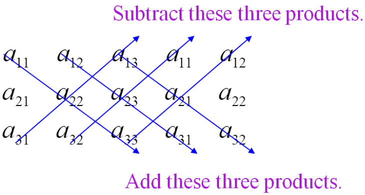
$$
\begin{aligned}
& a_{11}a_{22}a_{33}\det(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}) + a_{11}a_{32}a_{23}\det(\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}) \\
+ & a_{21}a_{12}a_{33}\det(\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}) + a_{21}a_{32}a_{13}\det(\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}) \\
+ & a_{31}a_{12}a_{23}\det(\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}) + a_{31}a_{22}a_{13}\det(\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix})
\end{aligned}
$$

Determinants of 3×3 matrices

The determinants of the 0-1 matrices can be computed using Properties 4 and 5. They are 1 or -1 depending on whether the number of column interchanges needed to bring the matrix to the identity matrix I_3 is even or odd. In the end, we get

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$



Examples

Example

$$\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} = 0 \times 2 - 1 \times 5 = -5$$

Example

$$\begin{array}{rcc} & & -4 \quad 0 \quad 6 \\ A = & \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} & \begin{matrix} 0 & 2 \\ 3 & -1 \\ 4 & -4 \end{matrix} \\ & & 0 \quad 16 \quad -12 \end{array}$$

So

$$|A| = 0 + 16 - 12 - (-4) - 0 - 6 = 2$$

Determinants of $n \times n$ matrices

In general, we have

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{\sigma(1),\sigma(2),\ldots,\sigma(n)} a_{\sigma(1),1} a_{\sigma(2),2} \ldots a_{\sigma(n),n} \det P_\sigma$$

- Here P_σ is the matrix $[e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}]$ and the sum is taken over all n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ of pairwise distinct numbers taking values in $\{1, 2, \ldots n\}$, i.e. , $(\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a reordering of $(1, 2, \ldots, n)$.
- In other words, the maps $\sigma : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$ is one-to-one.

Permutations

- We call such σ a **permutation** of $\{1, 2, \dots n\}$ and the matrix P_σ is called the **permutation matrix** corresponding to σ.
- The set of all permutations of $\{1, 2, \dots n\}$ is denoted by S_n. E.g.
$$(4, 3, 1, 2), (3, 2, 4, 1) \in S_4.$$
- Let $\text{sign}(\sigma) = 1$ or -1 depending on whether the number of position interchanges that convert $(\sigma(1), \sigma(2), \dots, \sigma(n))$ into $(1, 2, \dots, n)$ is even or odd. E.g.
$$\text{sign}(4, 3, 1, 2) = -1, \text{sign}(3, 2, 4, 1) = 1.$$
- As before, using Property 3, we see that $\det P_\sigma = \text{sign}(\sigma)$.

Determinants

Definition

An inversion of a permutation σ is a pair $i < j$ such that $\sigma(i) > \sigma(j)$.
The number of inversions of σ is denote by $I(\sigma)$.

Example:
$I(2,3,1) = 2$.
$I(4,3,1,2) = 5$.
$I(3,2,4,1) = 4$.

Theorem

For any permutation σ,

$$\mathrm{sign}(\sigma) = (-1)^{I(\sigma)}.$$

Determinants

Definition: Determinants

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \ldots a_{\sigma(n),n}$$

Cofactors

Let A_{ij} be the matrix obtained by removing the i-th row and j-th column of the matrix A

$$
|A_{ij}| = \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1j} & \cdots & a_{1n} \\
\vdots & & & \vdots & \vdots & & \\
a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)j} & \cdots & a_{(i-1)n} \\
a_{i1} & & \cdots & a_{i(j-1)} & a_{ij} & \cdots & a_{in} \\
\vdots & & & \vdots & \vdots & & \vdots \\
a_{n1} & & \cdots & a_{n(j-1)} & a_{nj} & \cdots & a_{nn}
\end{vmatrix}
$$

$C_{ij} = (-1)^{i+j}\det(A_{ij})$, $1 \leq i, j \leq n$ are called **cofactors** of A

Cofactor Matrix

Definition

The **matrix of cofactors** of A is

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

Cofactor Expansions

Theorem

- For any i,

$$\det(A) = \sum_{l=1}^{n} a_{il} C_{il} = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in}.$$

  (Cofactor expansion along the i-th row)

- For any j,

$$\det(A) = \sum_{l=1}^{n} a_{lj} C_{lj} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

  (Cofactor expansion along the j-th column)

Examples

Determinants of 2 × 2 matrices

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Determinant of a 3 × 3 matrix

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

Determinants of a matrix of order 3

Example

Compute $det(A)$ where

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$$

Answer: $det(A) = 15$

Exercises

Find the determinant

$$\begin{vmatrix} a & 0 & 0 \\ 1 & b & 0 \\ 3 & -5 & c \end{vmatrix}$$

Answer: abc.

Cofactor expansions

Example

Compute

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}\,2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

$$= 2\,(-1)^{2+1}\,(-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

Cofactor expansions

Example

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$$

Solution:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) = 3C_{13}$$

$$= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = 3 \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

Cofactor expansions

Solution (Cont.)

$$\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}$$

$$+(3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + (2)(1)(-4) + (3)(-1)(-7) = 13$$

Thus, $\det(A) = 39$.

Section 2

Properties of Determinants

Triangular matrices

Theorem

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$$

Example

If

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 10 & 4 & 0 & 0 & 0 \\ 5 & 0 & 2 & 0 & 0 \\ 7 & 6 & 0 & 5 & 0 \\ 9 & 2 & 4 & 1 & -1 \end{bmatrix}$$

then $\det(A) = (-2)(4)(2)(5)(-1) = 80$

Determinant of Transpose

Theorem

If A is a $n \times n$ matrix A then $\det(A^T) = \det(A)$.

Proof: Let $A = [a_{ij}], A^T = [b_{ij}]$. Then

$$\det\left(A^T\right) = \sum_{\sigma} \operatorname{sign}(\sigma) b_{\sigma(1),1} b_{\sigma(2),2} \dots b_{\sigma(n),n}$$

We have

$$\begin{aligned} b_{\sigma(1),1} b_{\sigma(2),2} \dots b_{\sigma(n),n} &= a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{n,\sigma(n)} \\ &= a_{\sigma^{-1}(1),1} a_{\sigma^{-1}(2),2} \dots a_{\sigma^{-1}(n),n} \end{aligned}$$

where σ^{-1} is the inverse permutation of σ.
Since $\operatorname{sign}(\sigma^{-1}) = \operatorname{sign}(\sigma)$, we get

$$\operatorname{sign}(\sigma) b_{\sigma(1),1} b_{\sigma(2),2} \dots b_{\sigma(n),n} = \operatorname{sign}(\sigma^{-1}) a_{\sigma^{-1}(1),1} a_{\sigma^{-1}(2),2} \dots a_{\sigma^{-1}(n),n}.$$

Thus the terms in $\det(A^T)$ and $\det A$ are equal, hence $\det(A^T) = \det(A)$.

Example

If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}$$

and

$$|A| = 6 = \left|A^T\right|$$

Determinant and elementary row operations

Theorem

Let A be a square matrix.

1. If B is obtained from A by interchanging two rows then $\det(B) = -\det(A)$.
2. If B is obtained from A by multipling on row of A with α then $\det(B) = \alpha \det(A)$.
3. If B is obtained from A by adding multiple of one row to another then $\det(B) = \det(A)$.

Corollary

1. If two rows (columns) of A are equal, then $\det(A) = 0$.
2. If a row (column) of A consists entirely of zeros, then $\det(A) = 0$.
3. $\det(\alpha A) = \alpha^n \det(A)$, where α is a constant.

Example

- Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

  A_1 is obtained from A by $R_2 - 2R_1$. We have $\det(A_1) = \det(A) = -2$.

- Let

$$A_2 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}.$$

  Then $\det(A_2) = 4\det(A) = -8$.

Example

Evaluate

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \overset{R_2+2R_1}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix}$$

$$= 5 \begin{vmatrix} 1 & -4 \\ -1 & 7 \end{vmatrix} = 15.$$

Example

Compute $\det(A)$ where

$$A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

Solution:
Add 2 times row 1 to row 3 to obtain

$$\det(A) = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix} = 0$$

Example

Compute

$$D_n = \begin{vmatrix} 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 0 & 1 & .. & 1 \\ . & . & . & . & . & . \\ 1 & 1 & .. & 1 & 0 & 1 \\ 1 & 1 & 1 & .. & 1 & 0 \end{vmatrix}$$

Solution: Employ the row operations: $R_k - R_1$, for $k = 2, 3, .., n$.

$$D_n = \begin{vmatrix} 1 & 1 & 1 & 1 & .. & 1 \\ 0 & -1 & 0 & 0 & .. & 0 \\ 0 & 0 & -1 & 0 & .. & 0 \\ . & . & . & . & . & . \\ 0 & 0 & .. & 0 & -1 & 0 \\ 0 & 0 & 0 & .. & 0 & -1 \end{vmatrix} = (-1)^{n-1}$$

Example

Compute

$$E_n = \begin{vmatrix} 0 & 1 & 1 & 1 & .. & 1 \\ 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 0 & 1 & .. & 1 \\ . & . & . & . & . & . \\ 1 & 1 & .. & 1 & 0 & 1 \\ 1 & 1 & 1 & .. & 1 & 0 \end{vmatrix}$$

Exercises

Find the determinant of the following matrices

1.

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{pmatrix}$$

2.

$$V_3 = \begin{bmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \end{bmatrix}$$

3*.

$$V_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & \dots & x_{n-1} \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_{n-1}^2 \\ . & . & . & . & . \\ x_0^{n-1} & x_1^{n-1} & x_2^{n-1} & \dots & x_{n-1}^{n-1} \end{bmatrix}$$

Determinants of Products

Definition

An $n \times n$ elementary matrix of type I, type II, or type III is a matrix obtained from the identity matrix I_n by performing a single elementary row (or elementary column) operation of type I, type II or type III, respectively.

Example

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then E_1 is an elementary matrix of type I and E_2 is an elementary matrix of type II.

Determinants of Products

Theorem

If E is an elementary matrix, then

$$\det(EA) = \det(AE) = \det(E)\det(A).$$

Theorem

If A and B are $n \times n$ matrices then

$$\det(AB) = \det(A)\det(B).$$

Corollary

If $\det(A) \neq 0$ then A^{-1} exists and

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Determinants and Invertibility

Equivalent conditions for invertibility

If A is an $n \times n$ matrix, then the following statements are equivalent

1. A is invertible.
2. $Ax = b$ has a unique solution for every $n \times 1$ matrix b.
3. $Ax = 0$ has only the trivial solution.
4. $\det(A) \neq 0$.

Corollary

If A is a square matrix, then $Ax = 0$ has a nontrivial solution if and only if $\det(A) = 0$.

Examples

Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Compute $\det(A^{-1})$, $\det(2A)$.

Solution:

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4.$$

Thus

$$\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

and

$$|2A| = 2^3|A| = 32$$

Exercises

1. Let A and B be 4×4 matrices, with $\det(A) = -1$ and $\det(B) = 2$. Compute
a. $\det(AB)$, b. $\det(B^5)$, c. $\det(2A)$, d. $\det(A^T A)$.

2. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det(A)$.

3. Suppose that A is a square matrix such that $det(A^4) = 0$. Explain why A can not be invertible.

Section 3

Cramer's Rule

Cramer's Rule

Consider a linear system of n equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

or in matrix notation $Ax = b$, where $A = (a_{ij})_{n \times n}$.

Cramer's Rule

1. If $D = |A| \neq 0$, the system of linear equations has a unique solution
$$x_k = \frac{D_k}{D}, k = 1, 2, ..., n$$
where D_k is the determinant of the matrix obtained by substituting the k-th column of the matrix A by the column
$$b = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}^T$$
For example:
$$D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$
2. If $D = 0$ and at least one of the D_k's is non-zero, then the system has no solution.

Example

Use Cramer's rule to solve the system

$$\begin{cases} 3x - 2y = 6 \\ -5x + 4y = 8 \end{cases}$$

We have

$$D = \det(A) = \begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix} = 2,$$

$$D_1 = \begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix} = 40, \qquad D_2 = \begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix} = 54.$$

Therefore,

$$x = \frac{D_1}{D} = \frac{40}{2} = 20, y = \frac{D_2}{D} = \frac{54}{2} = 27.$$

Example

Use Cramer's rule to solve the system

$$\begin{cases} x + \ \ y + \ \ z = -2 \\ 3x - \ \ y + 2z = 4 \\ 4x + 2y + \ \ z = -8 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 4 & 2 & 1 \end{vmatrix} = 10, \quad D_1 = \begin{vmatrix} -2 & 1 & 1 \\ 4 & -1 & 2 \\ -8 & 2 & 1 \end{vmatrix} = -10$$

$$D_2 = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 4 & 2 \\ 4 & -8 & 1 \end{vmatrix} = -30, \quad D_3 = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 4 \\ 4 & 2 & -8 \end{vmatrix} = 20$$

Thus,

$$x = \frac{D_1}{D} = -1, y = \frac{D_2}{D} = -3, z = \frac{D_3}{D} = 2$$

Example

Use Cramer's rule to solve the system

$$\begin{cases} -2x_1 + 3x_2 - x_3 = 1 \\ \quad x_1 + 2x_2 - x_3 = 4 \\ -2x_1 - \quad x_2 + x_3 = -3 \end{cases}$$

$$D = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2, \quad D_1 = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ 3 & -1 & 1 \end{vmatrix} = -4$$

$$D_2 = \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & 3 & 1 \end{vmatrix} = -6, \quad D_3 = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & 3 \end{vmatrix} = -8$$

Thus, $x_1 = 2, x_2 = 3, x_3 = 4$.

Exercises

Solve the following system using both Gaussian elimination and Cramer's rule.

1. $$\begin{cases} 2x + 5y - z = 15 \\ x - y + 3z = 4 \\ 3x + 3y - 5z = 2 \end{cases}$$

2. $$\begin{cases} 2x + 3y - z = 7 \\ x - y + z = 1 \\ 4x - 5y + 2z = 3 \end{cases}$$

Adjoint Matrix and Inverse Formula

Definition

The adjoint matrix of A, denoted by adj(A), is the transpose of the matrix of cofactors of A,

$$\operatorname{adj}(A) = C^T.$$

A formula for A^{-1}

If A is an $n \times n$ matrix, then

$$A \operatorname{adj}(A) = (\det A)\, I_n$$

Thus if $\det A \neq 0$ then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

Example

Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 0 & -2 \\ 3 & -1 & -3 \end{bmatrix}$$

Thus, $\det A = -26$ and the the matrix of cofactors is

$$C = \begin{bmatrix} -2 & 6 & -4 \\ -10 & -9 & -7 \\ 6 & 8 & 12 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A) = \frac{-1}{26} \begin{bmatrix} -2 & -10 & 6 \\ 6 & -9 & 8 \\ -4 & -7 & 12 \end{bmatrix}$$

Homework

Section 7.7 (p. 300): 7-15, 20-25
Section 7.8 (p. 308): 1-10