Lecture notes: Differential Equations for ISE (MA029IU)

Week 3 *

February 22, 2022

1 Linear equations and the integrating factor

One of the most important types of equations we will learn how to solve are the so-called *linear equations*. In fact, the majority of the course is about linear equations. In this section we focus on the *first order linear equation*. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{1}$$

The word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

The trick is to rewrite the left-hand side of (1) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}\Big[r(x)y\Big].$$

This is the left-hand side of (1) multiplied by $r(x)$. So if we multiply (1) by $r(x)$, we obtain

$$\frac{d}{dx}\Big[r(x)y\Big] = r(x)f(x).$$

Now we integrate both sides. The right-hand side does not depend on y and the left-hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the *integrating factor* and the method is called the *integrating factor method*.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)\,dx}.$$

We compute:

$$\begin{aligned}
y' + p(x)y &= f(x), \\
e^{\int p(x)\,dx}y' + e^{\int p(x)\,dx}p(x)y &= e^{\int p(x)\,dx}f(x), \\
\frac{d}{dx}\left[e^{\int p(x)\,dx}y\right] &= e^{\int p(x)\,dx}f(x), \\
e^{\int p(x)\,dx}y &= \int e^{\int p(x)\,dx}f(x)\,dx + C, \\
y &= e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx}f(x)\,dx + C\right).
\end{aligned}$$

*This note is taken from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example 1.1: Solve

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = -1.$$

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)\,dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$e^{x^2}y' + 2xe^{x^2}y = e^{x-x^2}e^{x^2},$$
$$\frac{d}{dx}\left[e^{x^2}y\right] = e^x.$$

We integrate

$$e^{x^2}y = e^x + C,$$
$$y = e^{x-x^2} + Ce^{-x^2}.$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$\boxed{y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt + y_0\right).} \tag{2}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

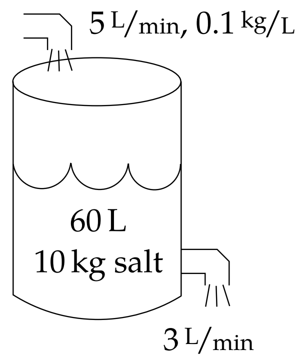
Exercise* 1.1*: *Check that $y(x_0) = y_0$ in formula (2).*

Exercise* 1.2*: *Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.*

$$y' + y = e^{x^2-x}, \qquad y(0) = 10.$$

Remark 1.1: Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula (2) for the solution. Second, it follows from the formula (2) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b).

Example 1.2: Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).



A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?

Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$, we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out}).$$

In our example,

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}. \end{aligned}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3 \frac{x}{60+2t}\right).$$

Or in the form (1)

$$\frac{dx}{dt} + \frac{3}{60+2t} x = 0.5.$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t} dt\right) = \exp\left(\frac{3}{2}\ln(60+2t)\right) = (60+2t)^{3/2}.$$

We multiply both sides of the equation to get

$$\begin{aligned} (60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x &= 0.5(60+2t)^{3/2}, \\ \frac{d}{dt}\left[(60+2t)^{3/2}x\right] &= 0.5(60+2t)^{3/2}, \\ (60+2t)^{3/2}x &= \int 0.5(60+2t)^{3/2}\,dt + C, \\ x &= (60+2t)^{-3/2}\int \frac{(60+2t)^{3/2}}{2}\,dt + C(60+2t)^{-3/2}, \\ x &= (60+2t)^{-3/2}\frac{1}{10}(60+2t)^{5/2} + C(60+2t)^{-3/2}, \\ x &= \frac{60+2t}{10} + C(60+2t)^{-3/2}. \end{aligned}$$

We need to find C. We know that at $t = 0$, $x = 10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2},$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. The tank is full when $60 + 2t = 100$, or when $t = 20$. So

$$x(20) = \frac{60+40}{10} + C(60+40)^{-3/2} \approx 10 + 1859.03(100)^{-3/2} \approx 11.86.$$

See Figure 1 for the graph of x over t.

The concentration when the tank is full is approximately 0.1186 kg/liter, and we started with $1/6$ or 0.167 kg/liter.

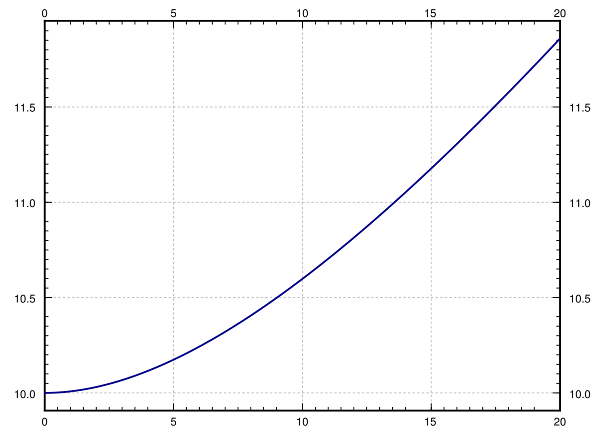


Figure 1: Graph of the solution x kilograms of salt in the tank at time t.

1.1 Exercises

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

***Exercise* 1.3:** *Solve $y' + xy = x$.*

***Exercise* 1.4:** *Solve $y' + 6y = e^x$.*

***Exercise* 1.5:** *Solve $y' + 3x^2y = \sin(x)\, e^{-x^3}$, with $y(0) = 1$.*

***Exercise* 1.6:** *Solve $y' + \cos(x)y = \cos(x)$.*

***Exercise* 1.7:** *Solve $\frac{1}{x^2+1}\, y' + xy = 3$, with $y(0) = 0$.*

***Exercise* 1.8:** *Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.*

a) Find the concentration of toxic substance as a function of time in both lakes.

b) When will the concentration in the first lake be below 0.001 kg per liter?

c) When will the concentration in the second lake be maximal?

***Exercise* 1.9:** *Newton's law of cooling states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω. That is, the ambient temperature oscillates (for example night and day temperatures).*

a) Find the general solution.

b) In the long term, will the initial conditions make much of a difference? Why or why not?

***Exercise* 1.10:** *Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 20 grams of salt in the tank?*

Exercise* 1.11*: *Initially a tank contains* 10 *liters of pure water. Brine of unknown* (*but constant*) *concentration of salt is flowing in at* 1 *liter per minute. The water is mixed well and drained at* 1 *liter per minute. In* 20 *minutes there are* 15 *grams of salt in the tank. What is the concentration of salt in the incoming brine*?

Exercise* 1.101*: *Solve* $y' + 3x^2 y = x^2$.

Exercise* 1.102*: *Solve* $y' + 2\sin(2x)y = 2\sin(2x)$, $y(\pi/2) = 3$.

Exercise* 1.103*: *Suppose a water tank is being pumped out at* $3\,{}^{L}\!/_{min}$. *The water tank starts at* $10\,L$ *of clean water. Water with toxic substance is flowing into the tank at* $2\,{}^{L}\!/_{min}$, *with concentration* $20t\,{}^{g}\!/_{L}$ *at time* t. *When the tank is half empty, how many grams of toxic substance are in the tank* (*assuming perfect mixing*)?

Exercise* 1.104*: *Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that* $\frac{dP}{dt} = (2 - 0.1\,t)P$. *If* $P(0) = 1000$, *find the population at* $t = 5$.

Exercise* 1.105*: *A cylindrical water tank has water flowing in at* I *cubic meters per second. Let* A *be the area of the cross section of the tank in square meters. Suppose water is flowing out from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for* h, *the height of the water, introducing and naming constants that you need. You should also give the units for your constants.*

2 Substitution

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

2.1 Substitution

The equation

$$y' = (x - y + 1)^2$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v, which we treat as a function of x. Let us try

$$v = x - y + 1.$$

We need to figure out y' in terms of v', v and x. We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get

$$1 - v' = v^2.$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:

$$\frac{1}{1 - v^2}\, dv = dx.$$

So

$$\frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| = x + C, \qquad \text{or} \qquad \left| \frac{v+1}{v-1} \right| = e^{2x+2C}, \qquad \text{or} \qquad \frac{v+1}{v-1} = De^{2x},$$

for some constant D. Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to "unsubstitute" to obtain

$$\frac{x - y + 2}{x - y} = De^{2x},$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y.

$$\begin{aligned} x - y + 2 &= (x - y)De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y\left(-1 + De^{2x}\right) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned}$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$.

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general patterns to look for. We summarize a few of these in a table.

When you see	Try substituting
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Usually you try to substitute in the "most complicated" part of the equation with the hopes of simplifying it. The table above is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

2.2 Bernoulli equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called *Bernoulli equation**:

$$y' + p(x)y = q(x)y^n.$$

This equation looks a lot like a linear equation except for the y^n. If $n = 0$ or $n = 1$, then the equation is linear and we can solve it. Otherwise, the substitution $v = y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

Example 2.1: Solve

$$xy' + y(x+1) + xy^5 = 0, \qquad y(1) = 1.$$

First, the equation is Bernoulli ($p(x) = (x+1)/x$ and $q(x) = -1$). We substitute

$$v = y^{1-5} = y^{-4}, \qquad v' = -4y^{-5}y'.$$

In other words, $(-1/4)\, y^5 v' = y'$. So

$$\begin{aligned} xy' + y(x+1) + xy^5 &= 0, \\ \frac{-xy^5}{4}v' + y(x+1) + xy^5 &= 0, \\ \frac{-x}{4}v' + y^{-4}(x+1) + x &= 0, \\ \frac{-x}{4}v' + v(x+1) + x &= 0, \end{aligned}$$

and finally

$$v' - \frac{4(x+1)}{x}v = 4.$$

The equation is now linear. We can use the integrating factor method. In particular, we use formula (2). Let us assume that $x > 0$ so $|x| = x$. This assumption is OK, as our initial condition is $x = 1$. Let us compute the integrating factor. Here $p(s)$ from formula (2) is $\frac{-4(s+1)}{s}$.

$$\begin{aligned} e^{\int_1^x p(s)\,ds} &= \exp\left(\int_1^x \frac{-4(s+1)}{s}\,ds\right) = e^{-4x-4\ln(x)+4} = e^{-4x+4}x^{-4} = \frac{e^{-4x+4}}{x^4}, \\ e^{-\int_1^x p(s)\,ds} &= e^{4x+4\ln(x)-4} = e^{4x-4}x^4. \end{aligned}$$

We now plug in to (2)

$$\begin{aligned} v(x) &= e^{-\int_1^x p(s)\,ds}\left(\int_1^x e^{\int_1^t p(s)\,ds}4\,dt + 1\right) \\ &= e^{4x-4}x^4\left(\int_1^x 4\frac{e^{-4t+4}}{t^4}\,dt + 1\right). \end{aligned}$$

*There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for Jacob Bernoulli (1654–1705).

The integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now "unsubstitute"

$$y^{-4} = e^{4x-4}x^4 \left(4 \int_1^x \frac{e^{-4t+4}}{t^4}\, dt + 1 \right),$$

$$y = \frac{e^{-x+1}}{x\left(4 \int_1^x \frac{e^{-4t+4}}{t^4}\, dt + 1 \right)^{1/4}}.$$

2.3 Homogeneous equations

Another type of equations we can solve by substitution are the so-called *homogeneous equations*. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right).$$

Here we try the substitutions

$$v = \frac{y}{x} \qquad \text{and therefore} \qquad y' = v + xv'.$$

We note that the equation is transformed into

$$v + xv' = F(v) \qquad \text{or} \qquad xv' = F(v) - v \qquad \text{or} \qquad \frac{v'}{F(v) - v} = \frac{1}{x}.$$

Hence an implicit solution is

$$\int \frac{1}{F(v) - v}\, dv = \ln |x| + C.$$

Example 2.2: Solve

$$x^2y' = y^2 + xy, \qquad y(1) = 1.$$

We put the equation into the form $y' = (y/x)^2 + y/x$. We substitute $v = y/x$ to get the separable equation

$$xv' = v^2 + v - v = v^2,$$

which has a solution

$$\int \frac{1}{v^2}\, dv = \ln |x| + C,$$

$$\frac{-1}{v} = \ln |x| + C,$$

$$v = \frac{-1}{\ln |x| + C}.$$

We unsubstitute

$$\frac{y}{x} = \frac{-1}{\ln |x| + C},$$

$$y = \frac{-x}{\ln |x| + C}.$$

We want $y(1) = 1$, so

$$1 = y(1) = \frac{-1}{\ln |1| + C} = \frac{-1}{C}.$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{\ln |x| - 1}.$$

2.4 Exercises

Hint: Answers need not always be in closed form.

Exercise* 2.1*: *Solve* $y' + y(x^2 - 1) + xy^6 = 0$, *with* $y(1) = 1$.

Exercise* 2.2*: *Solve* $2yy' + 1 = y^2 + x$, *with* $y(0) = 1$.

Exercise* 2.3*: *Solve* $y' + xy = y^4$, *with* $y(0) = 1$.

Exercise* 2.4*: *Solve* $yy' + x = \sqrt{x^2 + y^2}$.

Exercise* 2.5*: *Solve* $y' = (x + y - 1)^2$.

Exercise* 2.6*: *Solve* $y' = \frac{x^2 - y^2}{xy}$, *with* $y(1) = 2$.

Exercise* 2.101*: *Solve* $xy' + y + y^2 = 0$, $y(1) = 2$.

Exercise* 2.102*: *Solve* $xy' + y + x = 0$, $y(1) = 1$.

Exercise* 2.103*: *Solve* $y^2 y' = y^3 - 3x$, $y(0) = 2$.

Exercise* 2.104*: *Solve* $2yy' = e^{y^2 - x^2} + 2x$.