An Introduction to Applied Linear Algebra

Lecture 2: Descriptions of the Product of two Matrices

Associate Professor Pham Huu Anh Ngoc
Department of Mathematics
International university

2013

Other description of the product of a matrix and a vector

Example: Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Normally,

$$\mathbf{Ax} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Other description of the product of a matrix and a vector

Example: Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Normally,

$$\mathbf{Ax} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (1 \times 1) + (0 \times 0) + (1 \times 1) \\ (2 \times 1) + (2 \times 0) + (0 \times 1) \\ (1 \times 1) + (0 \times 0) + (1 \times 1) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Other description of the product of a matrix and a vector

Example: Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}; \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Normally,

$$\mathbf{Ax} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (1 \times 1) + (0 \times 0) + (1 \times 1) \\ (2 \times 1) + (2 \times 0) + (0 \times 1) \\ (1 \times 1) + (0 \times 0) + (1 \times 1) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Another way to get the result is as follows:

$$\mathbf{Ax} = 1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Theorem

If $\mathbf{A}$ *is* $m \times n$ *and* $\mathbf{x}$ *is* $n \times 1$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix}; \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

then

$$\mathbf{A}\mathbf{x} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ . \\ . \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ a_{mn} \end{pmatrix}.$$

In other words, we break up $\mathbf{A}$ into columns so that $\mathbf{A} = [\mathbf{a_1 a_2 ... a_n}]$ and then multiply the i-th column by x_i and add them up. That is

$$\mathbf{Ax} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + .. + x_n\mathbf{a_n}.$$

In other words, we break up $\mathbf{A}$ into columns so that $\mathbf{A} = [\mathbf{a_1 a_2 ... a_n}]$ and then multiply the i-th column by x_i and add them up. That is

$$\mathbf{Ax} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + .. + x_n\mathbf{a_n}.$$

Problem: Find the product of $\mathbf{A}$ and $\mathbf{x}$ in two different ways:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Other description of the product of two matrices

Theorem

Let $\mathbf{A}$ *be an* $m \times n$ *matrix and let* $\mathbf{B} = [\mathbf{b_1 b_2 ... b_p}]$ *be an* $n \times p$ *matrix* ($\mathbf{b_1}, \mathbf{b_2}, ..., \mathbf{b_p}$ *are columns of* $\mathbf{B}$). *Then*

$$\mathbf{AB} = [\mathbf{Ab_1 Ab_2 ... Ab_p}]$$

Ex:

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

Other description of the product of two matrices

Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{B} = [\mathbf{b_1}\mathbf{b_2}...\mathbf{b_p}]$ be an $n \times p$ matrix ($\mathbf{b_1}, \mathbf{b_2}, ..., \mathbf{b_p}$ are columns of $\mathbf{B}$). Then

$$\mathbf{AB} = [\mathbf{Ab_1}\mathbf{Ab_2}...\mathbf{Ab_p}]$$

Ex:

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right)$$

$$\left(\begin{array}{c|c} 1 & 0 \\ 6 & 4 \\ 1 & 0 \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 6 & 4 \\ 1 & 0 \end{pmatrix}$$

Problem: Find the product of **A** and **B** in two different ways:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Definition

Let $\mathbf{a}$ *be an* $m \times 1$ *vector and let* $\mathbf{b}$ *be a* $1 \times n$ *vector. Then the product* $\mathbf{ab}$ *(an* $m \times n$ *matrix) is called the* *outer product* *of* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Example:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$$

Definition

Let $\mathbf{a}$ *be an* $m \times 1$ *vector and let* $\mathbf{b}$ *be a* $1 \times n$ *vector. Then the product* $\mathbf{ab}$ *(an* $m \times n$ *matrix) is called the outer product of* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Example:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 4 & 6 \\ 3 & 6 & 6 & 9 \\ 1 & 2 & 2 & 3 \end{pmatrix}$$

Other description of the product

Theorem

Let **A** *be an* $m \times n$ *matrix broken up in terms of its n columns and let* **B** *be an* $n \times p$ *matrix broken up in terms of its n rows,*

$$\mathbf{A} = [\mathbf{a_1 a_2 \ldots a_n}] \qquad \mathbf{B} = \begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ . \\ . \\ \mathbf{b_n} \end{pmatrix}$$

Then the product $\mathbf{C} = \mathbf{AB}$ *is the sum of n outer products, or*

$$\mathbf{C} = \mathbf{a_1 b_1} + \mathbf{a_2 b_2} + \ldots + \mathbf{a_n b_n}$$

Example

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 1 & -4 & 9 \end{pmatrix} \qquad \mathbf{AB} = \mathbf{C}?$$

Example

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 1 & -4 & 9 \end{pmatrix} \qquad \mathbf{AB} = \mathbf{C}?$$

Now $\mathbf{a_i}$ (Each column of $\mathbf{A}$) is a 2×1 matrix, $\mathbf{b}_i$ (each row of $\mathbf{B}$) is a 1×3 matrix and the product of $\mathbf{a_i b_i}$ is a 2×3 matrix. We have

$$\mathbf{a_1 b_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{pmatrix}$$

$$\mathbf{a_2 b_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{a_3 b_3} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & -4 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & -12 & 27 \end{pmatrix}$$

$$\mathbf{a_1 b_1} + \mathbf{a_2 b_2} + \mathbf{a_3 b_3} = \begin{pmatrix} 1 & 3 & 7 \\ 5 & -14 & 29 \end{pmatrix} = \mathbf{AB}$$

Assignment:

Evaluate $\mathbf{AB}$ in three different ways:

(a)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}; \qquad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & -4 & 9 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

(b)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}; \qquad \mathbf{B} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & -4 & 9 & 1 \end{pmatrix}$$