Homework Week 6

Exercise 1:

1. Using cofactors, find inverse of the following matrices

a)

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 6 & -2 & 5 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

a)

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 6 & -2 & 5 \end{pmatrix} \qquad A^{-1} = \frac{[\mathrm{cof}(A)]^T}{\det(A)}$$

$$\begin{array}{ccccc|} 2 & 4 & -1 & 2 & 4 \\ 0 & 3 & 1 & 0 & 3 \\ 6 & -2 & 5 & 6 & -2 \end{array}$$

$\Rightarrow \det(A) = 2\times3\times5 + 4\times1\times6 + (-1)\times0\times(-2)$
$\quad - 6\times3\times(-1) - (-2)\times1\times2 - 5\times0\times4$

$\Rightarrow \det(A) = 76$

$a_{1,1} = 3\times5 - (-2)\times1 = 17$

$a_{1,2} = -[0\times5 - 6\times1] = +6$

$a_{1,3} = 0\times(-2) - 6\times3 = -18$

$a_{2,1} = -4\times5 - (-2)(-1) = -18$

$a_{2,2} = 2\times5 - 6\times(-1) = 16$

$a_{2,3} = -2\times-2 - 6\times4 = -20$

$a_{3,1} = 4\times1 - 3\times(-1) = 7$

$a_{3,2} = -2\times1 - 0\times(-1) = -2$

$a_{3,3} = 2\times3 - 0\times4 = 6$

$$\Rightarrow \mathrm{cof}(A) = \begin{pmatrix} 17 & 6 & -18 \\ -18 & 16 & -20 \\ 7 & -2 & 6 \end{pmatrix} \Rightarrow [\mathrm{cof}(A)]^T = \begin{pmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & -20 & 6 \end{pmatrix}$$

$$A^{-1} = \frac{[\mathrm{cof}(A)]^T}{\det(A)} = \frac{1}{76}\begin{pmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & -20 & 6 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\begin{array}{ccccc|} 1 & 2 & 0 & 1 & 2 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & 0 & 3 \end{array}$$

$\det(B) = 1\times0\times1 + 2\times1\times0 + 0\times(-1)\times3$
$\quad - 0\times0\times0 - 3\times1\times1 - 1\times(-1)\times2$

$\Rightarrow \det(B) = -1$

$b_{1,1} = 0\times1 - 3\times1 = -3$

$b_{2,1} = -(2\times1 - 3\times0) = -2$

$b_{3,1} = 2\times1 - 0\times0 = 2$

$b_{1,2} = -(-1\times1 - 0\times1) = 1$

$b_{2,2} = 1\times1 - 0\times0 = 1$

$b_{3,2} = -(1\times1 - (-1)\times0) = -1$

$b_{1,3} = -1\times3 - 0\times0 = -3$

$b_{2,3} = -1\times3 - 0\times2 = -3$

$b_{3,3} = 1\times0 - (-1)\times2 = 2$

$$\Rightarrow \operatorname{cof}(B) = \begin{pmatrix} -3 & 1 & -3 \\ -2 & 1 & -3 \\ 2 & -1 & 2 \end{pmatrix} \Rightarrow [\operatorname{cof}(B)]^T = \begin{pmatrix} -3 & -2 & 2 \\ 1 & 1 & -1 \\ -3 & -3 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{[\operatorname{cof}(B)]^T}{\det(B)} = \frac{1}{-1} \begin{pmatrix} -3 & -2 & 2 \\ 1 & 1 & -1 \\ -3 & -3 & 2 \end{pmatrix}$$

Exercise 2:

a) Using Gaussian elimination method:

$$\left(\begin{array}{ccc|c} 7 & 1 & -4 & 3 \\ -6 & -4 & 1 & 0 \\ 4 & -1 & -2 & 6 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_3 \times 3/2 + R_2 \\ R_3 \leftarrow R_1 \times \frac{-4}{7} + R_3 \end{array}$$

$$-11/2 \times x + -11/7 = 0$$

$$\left(\begin{array}{ccc|c} 7 & 1 & -4 & 3 \\ 0 & -11/2 & -2 & 9 \\ 0 & \frac{-11}{7} & \frac{2}{7} & \frac{30}{7} \end{array}\right) \quad R_3 \leftarrow R_2 \times \frac{-2}{7} + R_3$$

$$\left(\begin{array}{ccc|c} 7 & 1 & -4 & 3 \\ 0 & -11/2 & -2 & 9 \\ 0 & 0 & \frac{6}{7} & \frac{12}{7} \end{array}\right)$$

2. Use Gaussian elimination and Cramer's Rule to solve the systems.

a)

$$\begin{cases} 7x_1 + x_2 - 4x_3 = 3 \\ -6x_1 - 4x_2 + x_3 = 0 \\ 4x_1 - x_2 - 2x_3 = 6 \end{cases}$$

b)

$$\begin{cases} 2x_1 + 3x_2 - 5x_3 = 2 \\ 3x_1 - x_2 + 2x_3 = 1 \\ 5x_1 + 4x_2 - 6x_3 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} 7x_1 + x_2 - 4x_3 = 3 \\ -\frac{11}{2}x_2 - 2x_3 = 9 \\ \frac{6}{7}x_3 = \frac{12}{7} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \frac{21}{11} \\ x_2 = \frac{-26}{11} \\ x_3 = 2 \end{cases}$$

b) Using Cramer's Rule

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & 2 \\ 3 & -1 & 2 & 1 \\ 5 & 4 & -6 & 3 \end{array}\right)$$

$$\left.\begin{array}{ccccc} 2 & 3 & -5 & 2 & 3 \\ 3 & -1 & 2 & 3 & -1 \\ 5 & 4 & -6 & 5 & 4 \end{array}\right| \Rightarrow D = 2\times(-1)\times(-6) + 3\times 2\times 5 + (-5)\times 3\times 4 - 5\times(-1)\times(-5) - 4\times 2\times 2 + 6\times 3\times 3$$

$$\Rightarrow D = -5$$

$$\left.\begin{array}{ccccc} 2 & 3 & -5 & 2 & 3 \\ 1 & -1 & 2 & 1 & -1 \\ 3 & 4 & -6 & 3 & 4 \end{array}\right| \Rightarrow D_x = 2\times(-1)\times(-6) + 3\times 2\times 3 + (-5)\times 1\times 4 - 3\times(-1)\times(-5) - 4\times 2\times 2 + 6\times 1\times 3$$

$$\Rightarrow D_x = -3$$

$$\left.\begin{array}{ccccc} 2 & 2 & -5 & 2 & 2 \\ 3 & 1 & 2 & 3 & 1 \\ 5 & 3 & -6 & 5 & 3 \end{array}\right| \Rightarrow D_y = 2\times 1\times(-6) + 2\times 2\times 5 - 5\times 3\times 3 - 5\times 1\times(-5) - 3\times 2\times 2 + 6\times 3\times 2$$

$$\Rightarrow D_y = 12$$

$$\left.\begin{array}{ccccc} 2 & 3 & 2 & 2 & 3 \\ 3 & -1 & 1 & 3 & -1 \\ 5 & 4 & 3 & 5 & 4 \end{array}\right| \Rightarrow D_z = 2\times(-1)\times 3 + 3\times 1\times 5 + 2\times 3\times 4 - 5\times(-1)\times 2 - 4\times 1\times 2 - 3\times 3\times 3$$

$$\Rightarrow D_z = 8$$

$$\begin{cases} x = \dfrac{D_x}{D} = -3/-5 = 3/5 \\ y = \dfrac{D_y}{D} = -12/5 \\ z = \dfrac{D_z}{D} = -8/5 \end{cases}$$

Exercise 3

$$B = \begin{pmatrix} 2 & 1 & -1 \\ -4 & 3 & 3 \\ 6 & 8 & -3 \end{pmatrix} \begin{array}{l} \\ 2R_1 + R_2 \\ -3R_1 + R_3 \end{array}$$

Obtain U

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 5 & 0 \end{pmatrix} \quad R_3 \leftarrow (-1)R_2 + R_3$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Build L

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

3 Find LU-decomposition of the following matrix

$$B = \begin{pmatrix} 2 & 1 & -1 \\ -4 & 3 & 3 \\ 6 & 8 & -3 \end{pmatrix}$$