PART A

Q1.

a) Given that: $z = 1 + j\sqrt{3}$

$$\rightarrow \begin{cases} r = |z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \\ \theta = \tan^{-1}\dfrac{\sqrt{3}}{1} = \dfrac{\pi}{3} \end{cases}$$

Therefore, in polar form, z can be expressed as: $z = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)$

b) Since we know that: $z^n = r^n(\cos n\theta + j\sin n\theta)$

So $\left(1 + j\sqrt{3}\right)^4 = z^4 = 2^4\left(\cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3}\right) = -8 - j8\sqrt{3}$

Q2.

a)

$$\mathcal{L}\{e^{-2t}\sin 3t\} = \frac{3}{(s+2)^2 + 3^2}$$

b)

$$\mathcal{L}^{-1}\left\{\frac{4s-5}{s^2-s-2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s+1} + \frac{1}{s-2}\right\} = (3e^{-t} + e^{2t})u(t)$$

Q3.

a) Let $z = a + bi$; $a, b \in R$, It holds that:

$$f(z) = f(a+bi) = (a-bi)a + (a+bi)^2 + b$$

$$\leftrightarrow f(z) = 2a^2 - b^2 + b + abi$$

Thus, the real and imaginary parts of the given function are:

$$\mathrm{Re}\{f(z)\} = 2a^2 - b^2 + b$$

$$\mathrm{Im}\{f(z)\} = ab$$

b) The given function is harmonic if and only if it satisfies the Laplace equation:

$$\nabla^2 = \phi_{xx} + \phi_{yy} = 0$$

$$\leftrightarrow 2a + 2c = 0 \leftrightarrow c = -a$$

Thus, with all real value of a, b and $c = -a$, the given function is harmonic.

Q4.

Given that:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0 \ \ (*), \qquad y(0) = 3, \quad y'(0) = 7$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 3$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3s - 7$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s) - 3s - 7] + 6[sY(s) - 3] + 13Y(s) = 0$$

$$\leftrightarrow Y(s) = \frac{3s+25}{s^2+6s+13} = \frac{3(s+3) + 8\times 2}{(s+3)^2 + 2^2}$$

$\rightarrow y(t) = (3e^{-3t}\cos 2t + 8e^{-3t}\sin 2t)u(t)$

Thus, the solution of the given differential equation is:

$$y(t) = (3e^{-3t}\cos 2t + 8e^{-3t}\sin 2t)u(t)$$

PART B

Q1.

a)

$$\frac{1+2j}{3-4j}+\frac{2-j}{5j}=-\frac{2}{5}+0j$$

b)

$$g(z)=e^{y}\cos x+je^{y}\sin x=u(x,y)+jv(x,y)$$

- $\frac{\partial u}{\partial x}=-e^{y}\sin x \quad (1)$
- $\frac{\partial v}{\partial y}= \quad e^{y}\sin x \quad (2)$

From (1) and (2), $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$ which does not satisfies first equation of the Cauchy-Riemann equation. So, $g(z)$ is nowhere differentiable.

Q2.

Given that:

$$\begin{cases}\frac{dy}{dt}+\frac{dz}{dt}+y(t)+z(t)=1\\ \frac{dy}{dt}+z(t)=e^{t}\end{cases}$$

And $y(0)=-1, z(0)=2$

Let $Y(s)=\mathcal{L}\{y(t)\}$ and $Z(s)=\mathcal{L}\{z(t)\}$, it holds that:

$$\begin{cases}\mathcal{L}\{y'(t)\}=sY(s)-y(0)=sY(s)+1\\ \mathcal{L}\{z'(t)\}=sZ(s)-z(0)=sZ(s)-2\end{cases}$$

Taking Laplace transforms both side of the whole given system equations, we obtain:

$$\begin{cases}sY(s)+1+sZ(s)-2+Y(s)+Z(s)=\frac{1}{s}\\ sY(s)+1+Z(s)=\frac{1}{s-1}\end{cases}$$

$$\leftrightarrow\begin{cases}sZ(s)-2+Y(s)=\frac{1}{s}-\frac{1}{s-1} & (1)\\ sY(s)+1+Z(s)=\frac{1}{s-1} & (2)\end{cases}$$

$$(2)\leftrightarrow Z(s)=\frac{1}{s-1}-1-sY(s) \qquad (3)$$

Substitute (3) into (1), we get:

$$s\left[\frac{1}{s-1}-1-sY(s)\right]-2+Y(s)=\frac{1}{s}-\frac{1}{s-1}$$

$$\leftrightarrow Y(s)(s^{2}-1)=\frac{s+1}{s-1}-s-1-\frac{1}{s}-1$$

$$\rightarrow Y(s)(s-1)=\frac{1}{s-1}-1-\frac{1}{s}$$

$$\rightarrow Y(s)=\frac{1}{(s-1)^{2}}-\frac{2}{s-1}+\frac{1}{s} \quad (4)$$

Substitute back into (3), we get:

$$Z(s)=-\frac{1}{(s-1)^{2}}+\frac{2}{s-1} \quad (5)$$

From (4) and (5), taking inverse Laplace transforms to get the final result:

$$\begin{cases} y(t) = (te^t - 2e^t + 1)u(t) \\ z(t) = (-te^t + 2e^t)u(t) \end{cases}$$

Q3.

a) $z = -1 + j$

$$\rightarrow \begin{cases} r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \\ \theta = \tan^{-1}\dfrac{1}{-1} = \dfrac{3\pi}{4} \end{cases}$$

Since, we know that:

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0,1,\dots,n-1$$

Therefore, there is exists 3 cubic roots of z as follows:

$$w_0 = \sqrt[6]{2}\left(\cos\frac{\frac{3\pi}{4} + 0}{3} + j\sin\frac{\frac{3\pi}{4} + 0}{3}\right) = \sqrt[6]{2}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)$$

$$w_1 = \sqrt[6]{2}\left(\cos\frac{\frac{3\pi}{4} + 2\pi}{3} + j\sin\frac{\frac{3\pi}{4} + 2\pi}{3}\right) = \sqrt[6]{2}\left(-\frac{\sqrt{6} + \sqrt{2}}{4} + j\frac{\sqrt{6} - \sqrt{2}}{4}\right)$$

$$w_2 = \sqrt[6]{2}\left(\cos\frac{\frac{3\pi}{4} + 4\pi}{3} + j\sin\frac{\frac{3\pi}{4} + 4\pi}{3}\right) = \sqrt[6]{2}\left(\frac{\sqrt{6} - \sqrt{2}}{4} - j\frac{\sqrt{6} + \sqrt{2}}{4}\right)$$

b)

Let:

$$f(z) = \frac{z}{(z-1)(z-3)} = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{z-3}$$

Apply power series for analyzing this problem:

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

For $\left|\frac{z-1}{2}\right| < 1$:

$$\begin{aligned} f(z) &= -\frac{1}{2}\frac{1}{z-1} - \frac{3}{2}\frac{1}{2-(z-1)} \\ &= -\frac{1}{2}\frac{1}{z-1} - \frac{3}{4}\frac{1}{1 - \left(\frac{z-1}{2}\right)} \\ &= -\frac{1}{2}\frac{1}{z-1} - \frac{3}{4}\sum_{n=0}^{+\infty}\left(\frac{z-1}{2}\right)^n \\ &= -\frac{1}{2}\frac{1}{z-1} - \frac{3}{4}\sum_{n=0}^{+\infty}\frac{1}{2^n}(z-1)^n \end{aligned}$$

For $\left|\frac{z-1}{2}\right| > 1$:

$$f(z) = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{z-1-2}$$

$$= -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{\frac{1}{z-1}}{1-\frac{2}{z-1}}$$
$$= -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2(z-1)}\sum_{n=0}^{+\infty}\left(\frac{2}{z-1}\right)^n$$
$$= -\frac{1}{2}\frac{1}{z-1} + \frac{3}{4}\sum_{n=0}^{+\infty}\frac{2^{n+1}}{(z-1)^{n+1}}$$

Thus, the Laurent expansion series for the given function around the point $z = 1$ are:

$$f(z) = -\frac{1}{2}\frac{1}{z-1} - \frac{3}{4}\sum_{n=0}^{+\infty}\frac{1}{2^n}(z-1)^n, \quad |z-1| < 2$$
$$f(z) = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{4}\sum_{n=0}^{+\infty}\frac{2^{n+1}}{(z-1)^{n+1}}, \quad |z-1| > 2$$