Calculus 3

Chapter 1: Complex Analysis

1. Cauchy-Riemann and Laplace Equation

Given that: $f(z) = u + vi$, where u, v are functions of x, y or r, θ.
Cauchy-Riemann equation

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{cases} \qquad \text{(Eq 1.1)}$$

If u, v are differentiable and satisfy equation Eq 1.1 then:

- f is **differentiable** at z_0 and $f'(z) = u_x(x,y) + j\, v_x(x,y)$
- f is an **analytic function**

If $f(z) = u(x,y) + jv(x,y)$ is analytic in domain D, then both $u(x,y)$ and $v(x,y)$ satisfy the Laplace equations:

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = 0 \\ \nabla^2 v = v_{xx} + v_{yy} = 0 \end{cases} \qquad \text{(Eq 1.2)}$$

2. Basic formulas

Conversion of complex number between rectangular form and polar form

$$z = x + yi = r(\cos\theta + i\sin\theta)$$

$$\begin{cases} x = r\cos\theta \\ y = r\, sin\theta \end{cases} \leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\dfrac{y}{x} \end{cases}$$

Complex number in exponential form

$$e^z = e^x(\cos y + i\sin y)$$

Euler's formula

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Complex n-th exponential

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Complex n-th roots

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$
$$(k = 0,1, \dots, n-1)$$

3. Laurent Series

Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-c)^n$$

Power series

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n, \qquad |z| < 1$$

Consequences of power series

1. $\dfrac{1}{1-az} = \displaystyle\sum_{n=0}^{+\infty} (az)^n \qquad \left(|z| < \dfrac{1}{a}\right)$
2. $\dfrac{1}{1-\frac{a}{z}} = \displaystyle\sum_{n=0}^{+\infty} \left(\dfrac{a}{z}\right)^n \qquad (|z| > a)$
3. $\dfrac{1}{1-(az+b)} = \displaystyle\sum_{n=0}^{+\infty} (az+b)^n \qquad (|az+b| < 1)$
4. $\dfrac{1}{1-\frac{1}{az+b}} = \displaystyle\sum_{n=0}^{+\infty} \left(\dfrac{1}{az+b}\right)^n \qquad (|az+b| > 1)$

Chapter 2: Laplace Transform

1. Definition

If $f(t)$ is continuous and there are positive numbers M, a such that $|f(t)| < Me^{at}$, for all $t \geq c$. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt \quad \text{(Eq 2.1)}$$

2. Properties

$\boldsymbol{f(t)}$	$\boldsymbol{\mathcal{L}\{f(t)u(t)\}}$	$\boldsymbol{f(t)}$	$\boldsymbol{\mathcal{L}\{f(t)u(t)\}}$
$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	$e^{-at}f(t)$	$F(s+a)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$f(t-a)u(t-a)$	$e^{-as}F(s)$
$f'(t)$	$sF(s) - f(0)$	$(f * g)(t)$	$F(s).G(s)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	$\frac{f(t)}{t}$	$\int_s^{+\infty} F(\tau)d\tau$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$	$\int_0^t f(\tau)d\tau = u(t) * f(t)$	$\frac{1}{s}F(s)$

3. Formulas

$\boldsymbol{f(t)}$	$\boldsymbol{\mathcal{L}\{f(t)u(t)\}}$	$\boldsymbol{f(t)}$	$\boldsymbol{\mathcal{L}\{f(t)u(t)\}}$
1	$\frac{1}{s}$	$\delta(t-a)$	e^{-as}
t^n	$\frac{n!}{s^{n+1}}$	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$tf(t)$	$-F'(s)$

4. Initial and Final Value Theorem

Initial-value theorem

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = f(0^+) \quad \text{(Eq 2.2)}$$

Final-value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{(Eq 2.3)}$$

5. Heaviside – Unit step function

Given a piecewise-continuous function

$$f(t) = \begin{cases} f_1(t) & (t_1 \le t < t_2) \\ f_2(t) & (t_2 \le t < t_3) \\ f_3(t) & (t_3 \le t) \\ \dots \end{cases}$$

1. Express the piecewise-continuous function using the **unit step function**:

$$f(t) = f_1(t)u(t - t_1) + [f_2(t) - f_1(t)]u(t - t_2) + [f_3(t) - f_2(t)]u(t - t_3)$$

2. Express the piecewise-continuous function using the **top hat function**:

$$f(t) = f_1(t)[u(t - t_1) - u(t - t_2)] + f_2(t)[u(t - t_2) - u(t - t_3)] + f_3(t)u(t - t_3)$$

6. Convolution

Definition

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau \qquad \text{(Eq 2.4)}$$

Solving a convolution: Find $x(t) * h(t)$ or $(x * h)(t)$

Let: $$y(t) = x(t) * h(t)$$

$$\to Y(s) = \mathcal{L}\{x(t) * h(t)\} = X(s).H(s)$$

Taking inverse Laplace transform to find the result of the convolution

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Chapter 3: z-transform

1. Definition

Causal sequence: $\{x_n\}_0^\infty = \{x_0, x_1, x_2, \dots\}$

Infinite sequence: $\{x_n\}_{-\infty}^\infty = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$

The z-transform of an **infinite sequence** is defined whenever the sum exists and where z is a complex variable

$$\mathcal{Z}\{x_n\}_{-\infty}^{\infty} = X(z) = \sum_{n=-\infty}^{\infty} \frac{x_n}{z^n} \qquad \text{(Eq 3.1)}$$

The Z Transform of a **causal sequence**:

$$\mathcal{Z}\{x_n\}_0^{\infty} = X(z) = \sum_{n=0}^{\infty} \frac{x_n}{z^n} \qquad \text{(Eq 3.2)}$$

Where: $\mathcal{Z}$ is the z-Transform operator, $\{x_k\} - X(z)$: is a z-Transform pair.

2. Properties

x_n	$\mathcal{Z}\{x_n\}$	x_n	$\mathcal{Z}\{x_n\}$
$a^n x_n$	$X\left(\frac{z}{a}\right)$	$n^m x_n$	$-z^m \frac{d^m}{dz^m} X(z)$
x_{-n}	$X\left(\frac{1}{z}\right)$	x_{n-1}	$\frac{X(z)}{z}$
x_{n+1}	$zX(z) - zx_0$	x_{n+2}	$z^2X(z) - z^2x_0 - zx_1$

3. Formulas

x_n	$\mathcal{Z}\{x_n\}$	x_n	$\mathcal{Z}\{x_n\}$
δ_{n-n_0}	z^{-n_0}	1	$\frac{z}{z-1}$
a^n	$\frac{z}{z-a}$	n	$\frac{z}{(z-1)^2}$
na^{n-1}	$\frac{z}{(z-a)^2}$	e^{-nT}	$\frac{z}{z-e^{-T}}$
$a^n \cos(n\omega T)$	$\frac{z(z-\cos\omega T)}{z^2 - 2za\cos\omega T + a^2}$	$a^n \sin(n\omega T)$	$\frac{z\sin\omega T}{z^2 - 2za\cos\omega T + a^2}$

4. Initial and Final Value Theorem

Initial-value theorem

$$\lim_{n\to 0} x_n = \lim_{z\to\infty} X(z) = x_0 \qquad \text{(Eq 3.3)}$$

Final-value theorem

$$\lim_{n\to\infty} x_n = \lim_{z\to 1}\left(1 - \frac{1}{z}\right) X(z) \qquad \text{(Eq 3.4)}$$

Chapter 4: Fourier Series

1. Full Range Series

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{+\infty} b_k \sin(k\omega_0 t) \qquad \text{(Eq 4.1)}$$

Where:

$$a_0 = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)dt\,;\ a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\omega_0 t)\,dt\,;\ b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\omega_0 t)\,dt$$

Odd function: $a_0 = a_k = 0$, and

$$b_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin(k\omega_0 t)\,dt$$

Even function: $b_k = 0$, and

$$a_0 = \frac{4}{T_0}\int_0^{T_0/2} x(t)dt\,;\ a_k = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos(k\omega_0 t)\,dt$$

Parseval's identity:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{k=1}^{+\infty}(a_k^2 + b_k^2) \qquad \text{(Eq 4.2)}$$

2. Half Range Series

2. 1. Half Range Sine Series:

$$x(t) = \sum_{k=1}^{+\infty} b_k \sin\left(\frac{k\pi t}{L}\right);\ b_n = \frac{2}{L}\int_0^L x(t)\sin\left(\frac{k\pi t}{L}\right)dt$$

2. 2. Half Range Cosine Series:

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{+\infty} a_k \cos\left(\frac{k\pi t}{L}\right);\ \ a_0 = \frac{2}{L}\int_0^L x(t)dt\,;\ \ a_k = \frac{2}{L}\int_0^L x(t)\cos\left(\frac{k\pi t}{L}\right)dt$$

3. Frequently Used Formulas

$$I_1 = \int (at+b)\sin ct\,dt = -\frac{at+b}{c}\cos ct + \frac{a}{c^2}\sin ct$$

$$I_2 = \int (at+b)\cos ct\,dt = \frac{at+b}{c}\sin ct + \frac{a}{c^2}\cos ct$$

$$I_3 = \int \sin(at+b)\sin(ct+d)\,dt = \frac{1}{2}\left(\frac{\sin(t(a-c)+b-d)}{a-c} - \frac{\sin(t(a+c)+b-d)}{a+c}\right)$$

$$I_4 = \int \cos(at+b)\cos(ct+d)\,dt = \frac{1}{2}\left(\frac{\sin(t(a-c)+b-d)}{a-c} + \frac{\sin(t(a+c)+b-d)}{a+c}\right)$$

$$I_5 = \int \sin(at+b)\cos(ct+d)\,dt = -\frac{1}{2}\left(\frac{\cos(t(a-c)+b-d)}{a-c} + \frac{\cos(t(a+c)+b-d)}{a+c}\right)$$

$\cos \pi n = (-1)^n$

$\sin \pi n = 0$