Computing Probability

Objectives

1. Use counting technique to calculate probability of an event in a sample space with equally likely outcomes
2. Calculate the probabilities of joint events such as unions and intersections from the probabilities of individual events
3. Interpret and calculate conditional probabilities of events

4. Determine the independence of events and use independence to calculate probabilities
5. Use multiple law to compute probabilities of joint events
6. Use total law to compute probability of event with divide - conquer strategy
7. Use Bayes' theorem to calculate conditional probabilities

Table of contents

Experiments with Equally Likely Outcomes

Let S be a sample space consisting of finite equally likely outcomes

- Probability of each outcome is $\frac{1}{\text{number of outcomes in } \Omega}$
- Probability of event E

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } \Omega}$$

Example

An urn contains eight white balls and two green balls. A sample of three balls is selected at random. What is the probability of selecting only white balls?

Solution

- The experiment consists of selecting 3 balls from the
- The total number of outcomes is $n(\Omega) = C(10, 3)$,
- E = "all three balls selected are white."
- the number of different samples in which all are white is $n(E) = C(8, 3)$
- $$P(E) = \frac{n(E)}{n(\Omega)} = \frac{C(8,3)}{C(10,3)} = \frac{7}{15}$$

Practice

A toy manufacturer inspects boxes of toys before shipment. Each box contains 10 toys. The inspection procedure consists of randomly selecting three toys from the box. If any are defective, the box is not shipped. Suppose that a given box has two defective toys. What is the probability that it will be shipped?

Practice

A batch of 140 semiconductor chips is inspected by choosing a sample of five chips. Assume 10 of the chips do not conform to customer requirements.

Find probability that a sample of five contain exactly one nonconforming chip

Table of contents

Inclusion - Exclusion formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive ($AB = \emptyset$) then

$$P(A \cup B) = P(A) + P(B)$$

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General Inclusion - Exclusion Formula

- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$
- $$P(A_1 \cup A_2 \cup \ldots \cup A_n)$$
  $$= P(A_1) + P(A_2) + \cdots + P(A_n)$$
  $$- \sum_{i_1 < i_j} P(A_i A)j + \sum_{i<j<k} P(A_i A_j A_k)$$
  $$+ \cdots + (-1)^{n-1} P(A_1 A_2 \ldots A_n)$$

Example

After being interviewed at two companies he likes, John assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, **what is the probability** that he will get at least one offer from these two companies?

Solution

Denote

- A: he gets offer from company A
- B: he gets offer from company B

We have

$$P(A) = 0.8, P(B) = 0.6, P(AB) = 0.5$$

We need to compute

$$P(A \cup B)$$

Applying inclusion - exclusion formula, we obtain

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

so

$$P(A \cup B) = 0.8 + 0.6 - 0.5 = 0.9$$

Complement rule

$$P(A') = 1 - P(A)$$

Example

A group of five people is to be selected at random. What is the probability that two or more of them have the same birthday? For simplicity, we ignore February 29.

Solution

- Pick out five people, and observe their birthdays. The outcomes of this experiment are strings of five dates, corresponding to the birthdays. For example, one outcome of the experiment is (June 2, April 6, Dec. 20, Feb. 12, Aug. 5).
- the total number of possible outcomes 365^5

Solution

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- E: two or more of 5 selected people have the same birthday

- E': all 5 people have different birthdays

- Total number of outcomes in E': 365.364.363.362.361

- $P(E') = \frac{365.364.363.362.361}{365^5} \approx ...3$
  $\Rightarrow P(E) = 1 - P(E') = ...$

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Practice

An urn contains eight white balls and two green balls. A sample of three balls is selected at random. What is the probability that the sample contains at least one green ball?

Exercise

A customer will invest in tax-free bonds with a probability of 0.6, will invest in mutual funds with a probability of 0.3, and will invest in both tax-free bonds and mutual funds with a probability of 0.15. Find the probability that a customer will invest

1. in either tax-free bonds or mutual funds;
2. in neither tax-free bonds nor mutual funds.

Table of contents

Conditional probability provides us with a way to reason about the outcome of an experiment, based on **partial information**.

Example

In an experiment involving two successive rolls of a fair die, you are told that the sum of the two rolls is 9. How likely is it that the first roll was a 6?

Conditional probability provides us with a way to reason about the outcome of an experiment, based on **partial information**.

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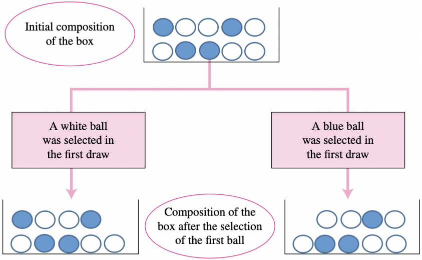
		First roll					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Second	3	4	5	6	7	8	9
roll	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

addition information

- Restrict possible outcomes on new sample space $\{(3,6),(4,5),(5,4),(6,3)\}$
- First roll was $6 = \{(6,3)\}$
- the probabililty is equal to $\frac{1}{4} = 0.25$

Example

Select randomly one out of the 10 balls and then, without returning this to the box, we take another one.



What is the prob that the second ball is blue if the first ball is white? $\frac{4}{9}$

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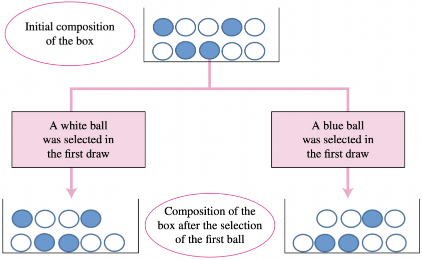

What is the prob that the second ball is blue if the first ball is white? $\frac{4}{9}$

Example

A class contains 26 students. Of these, 14 are economics majors, 15 are first-year students, and 7 are neither. A person is selected at random from the class.

1. What is the probability that the person is both an economics major and a first- year student?
2. **Given that** the person selected is a first-year student. What is the probability that he or she is also an economics major?

Solution

1. $\frac{5}{13}$
2. $\frac{2}{3}$

- 2 events A and B
- *If we know for sure that A happens, how does the likelihood of B change*?

seek to construct a new probability law, which takes into account this knowledge and which, for any event A, gives us the conditional probability of B given A, denoted by $P(B|A)$

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Conditional Probability

The conditional probability of B given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(AB)}{P(A)}$$

if $P(A) > 0$

Measure the likelihood of B in the new sample space A

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Meaning

Conditional probability provides the capability of reevaluating the idea of probability of an event in light of additional information, that is, when it is known that another event has occurred. **The probability $P(A|B)$ is an updating of $P(A)$ based on the knowledge that event B has occurred.**

Properties

1. Complement rule

$$P(B^c|A) = 1 - P(B|A)$$

2. Additive rule

$$P(B \cup C|A) = P(B|A) + P(C|A) - P(BC|A)$$

Example

Twenty percent of the employees of Acme Steel Company are college graduates. Of all of its employees, 20% are college graduate and 15% are college graduates earning more than $50,00[0]
What is the **probability that an employee selected at random earns more than $50,000 per year**, given that he or she is a college graduate?

Solution

- H = “earns more than \$50,000 per year”
- C = “college graduate.”
- We need to compute $P(H|C)$
- Given data $P(C) = 0.2$, $P(H \cap C) = 0.15$
- $$P(H|C) = \frac{P(H \cap C)}{P(C)} = \frac{0.15}{0.2} = 0.75$$

Practice

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(A \cap D) = 0.78$. Find the probability that a plane

1. arrives on time, given that it departed on time
2. departed on time, given that it has arrived on time.

Interpretation

In the group of employees which are college graduate, there are 75% earn more than $50,000 per year

Example

Education	Male	Female
Elementary	38	45
Secondary	28	50
College	22	17

If a person is picked at random from this group, find the probability that the person is a male, given that the person has a secondary education.

Question: find the probability that $\underbrace{\text{the person is a male}}_{\text{A: interested event}}$,

given that

$\underbrace{\text{the person has a secondary education}}_{\text{B: addition information - condition or new sample space}}$

Find

$$P(A|B) = \frac{P(AB)}{P(B)}$$

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Find

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Solution - 1st approach

- Sample size: number of ways to pick a person randomly is

$$|\Omega| = 38+45+28+50+22+17 = 200$$

- Convert data into probability

Education	Male	Female	Sum
Elementary	$\frac{38}{200} = .19$	.225	
Secondary	.14	.25	.39
College	.11	.085	.195
Sum	.44	.56	1

Education	Male	Female	Sum
Elementary	.19	.225	
Secondary	.14 $= P(AB)$	.25	.39 $= P(B)$
College	.11	.085	.195
Sum	.44	.56	1

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{.14}{.39}$$

Solution - 2nd approach

- New sample space is B with 78 elements
- In new sample space, the number of ways to pick a male is 28
- The probability that the person is a male, given that the person has a secondary education is

$$\frac{28}{78} \approx 0.36$$

Intepretation

Among all the person with secondary education, the fraction of male is 36%

Practice

400 parts classified by surface flaws and as (functionally) defective

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	28
	No	30	342	372
	Total	40	360	400

Select randomly a part. Find the probability that the selected part is defective given that

1. the part with surface flaws
2. the part without surface flaws

Example

A fair coin is flipped twice. what is the conditional probability that both flips land on heads, given that
(a) the first flip lands on heads?
(b) at least one flip lands on heads?

Solution for (a)

- $A = \{HH\}$ (both head)
- $F = \{HH, HT\}$ (first is head)

$$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(\{HH\})}{P(\{HH, HT\})}$$
$$= \frac{1/4}{2/4} = \frac{1}{2}$$

Solution for (b)

- $B = \{HH, HT, TH\}$ (at least one head)

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{P(\{HH\})}{P(\{HH, HT, TH\})} = \frac{1}{3}$$

Practice

Toss a fair coin twice. Compute

$$P(\text{2nd toss is head})$$

and

$$P(\text{2nd toss is head} \mid \text{1st toss is head})$$

Comment

$$P(A) = \frac{1}{4} \text{ while } P(A|B) = \frac{1}{3} \neq P(A)$$

indicates that A depends on B

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Idea

- Usually $P(A|B) \neq P(A)$.
- If $P(A|B) = P(A)$, B has no effect on A or knowing B does not change the probability that A happens then
- A and B have no relation

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Table of contents

Independent events

A and B are independent if

$$P(A|B) = P(A)$$

or

$$P(AB) = P(A)P(B).$$

Complement

If A is independent of B then it is independent of B^c.

Example

Two successive rolls of a fair 6-sided die

A: the 1st roll results in 2

B: the 2nd roll results in 4

Are A and B independent?

Solution

- $P(A) = \frac{1}{6}$
- $P(B) = \frac{1}{6}$
- $P(AB) = \frac{1}{36} = P(A)P(B)$
- A and B are independent

Practice

Suppose that $P(A|B) = 0.4$, $P(B) = 0.8$ and $P(A) = 0.5$. Are A and B are independent?

Practice

400 parts classified by surface flaws and as (functionally) defective

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	28
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	Total	40	360	400

Select randomly a part. Let

$$D = \text{"the part is defective"}$$

$$F = \text{"the part has surface flaw"}$$

Are D and F independent?

Independence of a set of events

A set of events is said to be independent if, for each collection of events chosen from them, say, $E_1, E_2, \ldots, E_n$, we have

$$P(E_1 \cap \cdots \cap E_n) = P(E_1) \ldots P(E_n)$$

Example

Three events A, B, and C are independent: $P(A) = .5$, $P(B) = .3$, and $P(C) = .2$.

1. Calculate $P(A \cap B \cap C)$.
2. Calculate $P(A \cap C)$.

Example

A company manufactures stereo components. Experience shows that defects in manufacture are independent of one another. Quality-control studies reveal that

2% of CD players are defective,
3% of amplifiers are defective,
7% of speakers are defective.

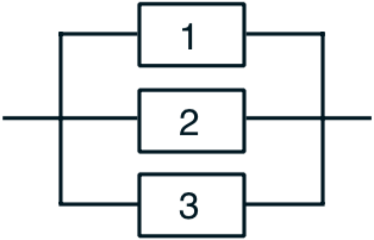
A system consists of a CD player, an amplifier, and two speakers. What is the probability that the system is not defective?

Example

- A series system is up if all of its component is up
- Components operate independently
- p_i: prob that component i is up
- Prob that series system is up: $p_1 p_2 \dots p_n$

Example

A parallel system is up if any one of its component is up



Assume that all component operates independently

- p_i: probability that component i is up
- Probability that the parallel system is down:

$$(1-p_1)(1-p_2)...(1-p_n)$$

- Probability that the parallel system is up:

$$1-(1-p_1)(1-p_2)...(1-p_n)$$

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- p_i: probability that component i is up
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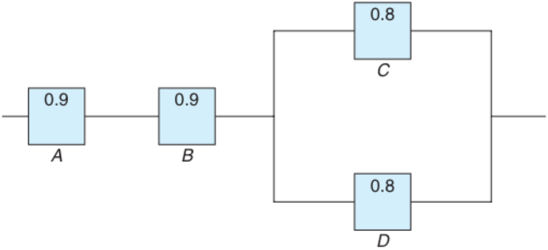
$$(1 - p_1)(1 - p_2)...(1 - p_n)$$

- Probability that the parallel system is up:

$$1 - (1 - p_1)(1 - p_2)...(1 - p_n)$$

An electronic system consists of 4 independent components. Find the probability that entire system works.



Solution

- Probability that the subsystem CD in parallel is up

$$\begin{aligned} p_{CD} &= 1 - (1 - p_C)(1 - p_D) \\ &= 1 - (1 - .8)(1 - .8) = .96 \end{aligned}$$

- Probability that the whole system is up

$$p_A p_B p_{CD} = (.9)(.9)(.96) =$$

Table of contents

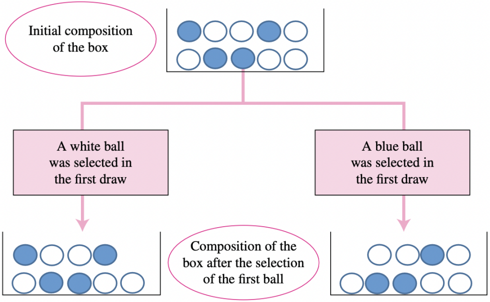
Multiplication Formula

$$P(AB) = P(B|A)P(A).$$

Think of AB as event with 2 steps, then probability equals probability of first step multiply with the conditional probability of second step given first step

Example

Draw 2 balls without replacement from the box.



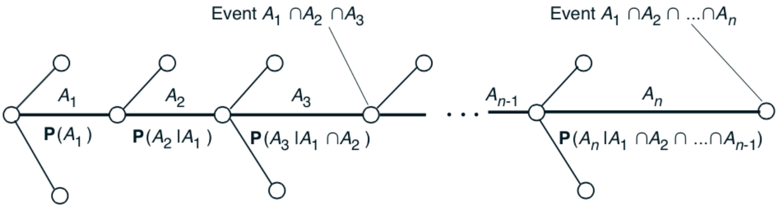
What is the probability that both balls are white?

Solution

- A_i: the ith draw is white
- $P(A_1) = \frac{6}{10}$
- $P(A_2|A_1) = \frac{5}{9}$
- $P(A_1 A_2) = P(A_1)P(A_2|A_1) = \left(\frac{6}{10}\right)\left(\frac{5}{9}\right) = \frac{1}{3}$

Sequence of events $A_1, A_2, \ldots, A_k$

$$P(A_1A_2\ldots A_k) = P(A_1) \times P(A_2|A_1) \times P($$
$$\times P(A_k|A_1\ldots A_{k-1})$$



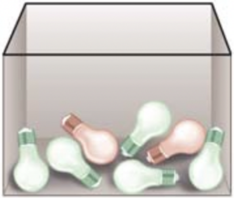
Example - Qualify control

A box contains five good lightbulbs and two defective ones.
Bulbs are selected one at a time (without replacement) until a good bulb is found. Find the **probability** that **the number of bulbs selected** is

(i) one, (ii) two, (iii) three.

Solution

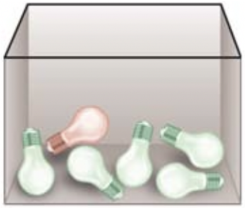
Initial situation



A bulb selected at random will be

- good (G) with probability $\frac{5}{7}$
- defective (D) with probability $\frac{2}{7}$

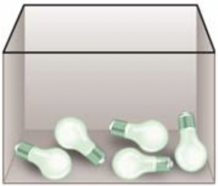
Situation if the first selected bulb is defective



The second bulb selected at random will be

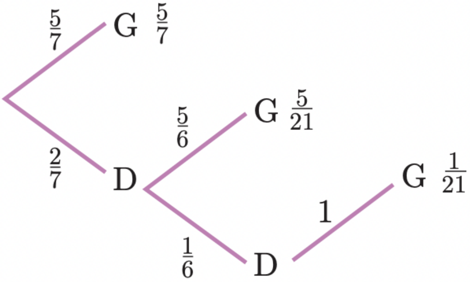
- good (G) with probability $\frac{5}{6}$
- defective (D) with probability $\frac{1}{6}$

Situation if the first and second selected bulbs are defective



The third bulb selected at random will be

- good (G) with probability 1
- defective (D) with probability 0



Each of the three paths leading to a G has a different length.

1. $P(1) = \frac{5}{7}$
2. $P(2) = \frac{2}{7} \cdot \frac{5}{6} = \frac{5}{21}$
3. $P(3) = \frac{2}{7} \cdot \frac{1}{6} \cdot 1 = \frac{1}{21}$

Practice

A lot of 100 semiconductor chips contains 20 that are defective.

Three are selected, at random, without replacement, from the lot. Determine the probability that all are defective.

Practice

Suppose that $P(A|B) = 0.4$, $P(B) = 0.5$. Determine

1. $P(A \cap B)$
2. $P(A' \cap B)$

Practice

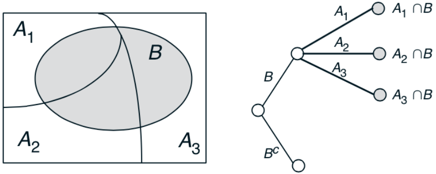
The probability that the head of a household is home when a telemarketing representative calls is 0.4. Given that the head of the house is home, the probability that goods will be bought from the company is 0.3. Find the probability that the head of the house is home and goods are bought from the company.

Table of contents

Partition

$A_1, \ldots, A_n$ is a partition of Ω if

- mutually exclusive: $A_i A_j = \emptyset$ for $i \neq j$
- $A_1 \cup A_2 \cup \ldots A_n = \Omega$



$BA_1, \ldots, BA_n$ is a partition of B

$$B = BA_1 \cup BA_2 \cup \ldots \cup BA_n$$

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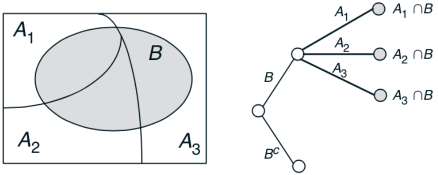


$BA_1, \ldots, BA_n$ is a partition of B

$$B = BA_1 \cup BA_2 \cup \ldots \cup BA_n$$

Total probability formula - divide - and - conquer

- Partition sample space into A_1, $A_2, \ldots, A_n$
- Know $P(B|A_i)$ for every i
- Compute $P(B)$

$$P(B) = \sum_{i=1}^{n} P(BA_i)$$

$$= \sum^{n} P(B|A_i)P(A_i)$$

Example

Select randomly one out of the a box of 6 blue balls and 4 green balls. Then without returning this to the box, we take another one.

What is the probability that the second ball is blue?

Solution

Denote

- B_2: the second ball is blue
- G_1: the first ball is green
- B_1: the first ball is blue

We need to compute $P(B_2)$.

There are two possible cases that the 2nd ball is blue

$$B_2 = (B_1G_1) \cup (B_2G_2)$$

By total law,

$$P(B_2) = P(B_2 \cap G_1) + P(B_2 \cap B_1)$$
$$= P(G_1)P(B_2|G_1) + P(B_1)P(B_2|B_1)$$
$$= \frac{4}{10} \times \frac{6}{9} + \frac{6}{10} \times \frac{5}{9} = \frac{54}{90} = 0.6$$

Example

You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

Solution

- A_i: your opponent is of type i
- W: you win
- The event that you win can be divided into three cases according to the type of your opponent

$$W = (A_1 W) \cup (A_2 W) \cup (A_2 W)$$

$P(A_i)$	$P(W\|A_i)$	$P(A_iW)$
$\frac{1}{2}$	0.3	$\frac{1}{2}(0.3) = 0.15$
$\frac{1}{4}$	0.4	$\frac{1}{4}(0.4) = 0.1$
$\frac{1}{4}$	0.5	$\frac{1}{4}(0.5) = 0.125$

$$P(W) = 0.15 + 0.16 + 0.2 = 0.375$$

Example

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Solution

Denote

- A: the selected product is defective
- B_i: the selected product is made by machine B_i

The event that the selected product is defective can be divided into three cases according to which machine made it

$$A = (AB_1) \cup (AB_2) \cup (AB_3)$$

- $P(AB_1) = P(B_1)P(A|B_1) = (.3)(.02) = .006$
- $P(AB_2) = P(B_2)P(A|B_2) = (.45)(.03) = .0135$
- $P(AB_3) = P(B_1)P(A|B_1) = (.25)(.02) = .005$

So $P(A) = P(AB_1)+P(AB_2)+P(AB_3) = 0.0245$

Re-evaluate

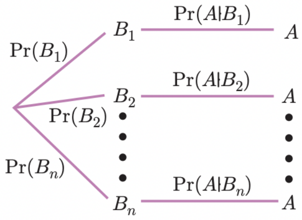
if a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

Solution

$$
\begin{aligned}
P(B_3|A) &= \frac{P(B_3A)}{P(A)} \\
&= \frac{0.005}{0.0245} = \frac{10}{49}
\end{aligned}
$$

Table of contents

Bayes' Theorem



- $B_1, B_2, \ldots, B_n$ are mutually exclusive
- $B_1 \cup B_2 \cup \cdots \cup B_n = S$

$$P(B_k|A) = \frac{P(B_k)P(A|B_k)}{P(B_1)P(A|B_1) + \cdots + P(B_n)P(A|B_n)}$$

for $k = 1, 2, \ldots, n$

Meaning

- Prior probability $P(B_i)$ - initial belief
- Know $P(A|B_i)$ for each i
- Given A occurs, wish to revise (update) ”belief” $P(B_k|A)$

$$\boxed{P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}}$$

Bayes' rule is often used for **inference**. There are a number of "causes" that may result in a certain "effect." We observe the effect, and we wish to infer the cause

Exercise

There is 0.25% of the general population suffer from Covid. To diagnose whether someone suffers from Covid, there is a medical examination which has a probability 1% of giving a false result if someone has Covid and 2% if someone does not have Covid. If we select at random a person from the general population and he/she tests positive for Covid, what is the probability that this person actually suffers from Covid?

Exericse

A contestant on a television show has to answer multiple choice questions with four possible answers. The probability that the contestant knows the answer to a question is 75%. If the contestant does not know the answer to a particular question, she gives an answer at random. If she has answered the first question correctly, what is the probability that she knew the answer?

Example

A plane is missing and it was equally likely to have gone down in any of three possible regions. Let $1 - \alpha_i$ denote the probability the plane will be found upon a search of the i-th region when the plane is, in fact, in that region, $i = 1, 2, 3$. What is the conditional probability that the plane is in the i-th region, given that a search of region 1 is unsuccessful, $i = 1, 2, 3$?

Solution

- A_i={the plane is in region i}
- B={search of region 1 was unsuccessful}
- Need $P(A_i|B) = ?$

Solution

- A_i={the plane is in region i}
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Solution

- A_i={the plane is in region i}
- B={search of region 1 was unsuccessful}
- Need $P(A_i|B) = ?$

Solution

Need to find

$$P(A_iB) \text{ and } P(B)$$

with information

- $P(A_i) = \frac{1}{3}$
- $P(\text{ plane is found in region } i|A_i) = 1 - \alpha$

Solution

$$P(A_1B) = ?$$

A_1B means that

- Plane is in region 1
- Search in region 1 was unsuccessful = plane was not found in region 1

$$P(A_1B) = P(A_1)P(B|A_1) = \frac{1}{3} * \alpha_1 = \frac{\alpha_1}{3}$$

Solution

$$P(A_1B) = ?$$

A_1B means that

- Plane is in region 1
- Search in region 1 was unsuccessful = plane was not found in region 1

$$P(A_1B) = P(A_1)P(B|A_1) = \frac{1}{3} * \alpha_1 = \frac{\alpha_1}{3}$$

Solution

$$P(A_1B) = ?$$

A_1B means that

- Plane is in region 1
- Search in region 1 was unsuccessful = plane was not found in region 1

$$P(A_1B) = P(A_1)P(B|A_1) = \frac{1}{3} * \alpha_1 = \frac{\alpha_1}{3}$$

A_2B means that

- Plane is in region 2
- Search in region 1 was unsuccessful = plane was not found in region 1

$$P(A_2B) = P(A_2)P(B|A_2) = \frac{1}{3} * 1 = \frac{1}{3}$$

Solution

A_2B means that

- Plane is in region 2
- Search in region 1 was unsuccessful = plane was not found in region 1

$$P(A_2B) = P(A_2)P(B|A_2) = \frac{1}{3} * 1 = \frac{1}{3}$$

Solution

A_3B means that

- Plane is in region 3
- Search in region 1 was unsuccessful = plane was not found in region 1

$$P(A_3B) = P(A_3)P(B|A_3) = \frac{1}{3} * 1 = \frac{1}{3}$$

Solution

A_3B means that

- Plane is in region 3
- Search in region 1 was unsuccessful = plane was not found in region 1

$$P(A_3B) = P(A_3)P(B|A_3) = \frac{1}{3} * 1 = \frac{1}{3}$$

Solution

$$
\begin{aligned}
P(B) =& P(A_1B) + P(A_2B) + P(A_3B) \\
=& \alpha_1 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3} \\
=& \frac{\alpha_1 + 2}{3}
\end{aligned}
$$

Solution

$$
\begin{aligned}
P(B) =& P(A_1B) + P(A_2B) + P(A_3B) \\
=& \alpha_1 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3} \\
=& \frac{\alpha_1 + 2}{3}
\end{aligned}
$$

Solution

$$P(A_1|B) = \frac{P(A_1B)}{P(B)} = \frac{\alpha_1}{\alpha_1 + 2}$$

$$P(A_2|B) = \frac{P(A_2B)}{P(B)} = \frac{1}{\alpha_1 + 2}$$

$$P(A_3|B) = \frac{P(A_3B)}{P(B)} = \frac{1}{\alpha_1 + 2}$$

Solution

$$P(A_1|B) = \frac{P(A_1B)}{P(B)} = \frac{\alpha_1}{\alpha_1 + 2}$$

$$P(A_2|B) = \frac{P(A_2B)}{P(B)} = \frac{1}{\alpha_1 + 2}$$

$$P(A_3|B) = \frac{P(A_3B)}{P(B)} = \frac{1}{\alpha_1 + 2}$$

Solution

$$P(A_1|B) = \frac{P(A_1B)}{P(B)} = \frac{\alpha_1}{\alpha_1 + 2}$$

$$P(A_2|B) = \frac{P(A_2B)}{P(B)} = \frac{1}{\alpha_1 + 2}$$

$$P(A_3|B) = \frac{P(A_3B)}{P(B)} = \frac{1}{\alpha_1 + 2}$$