ENGINEERING PROBABILITY & STATISTICS – SEMESTER 1 – 2023-2024

MIDTERM ANSWER KEY

Q1. (15pts)

Given the following sample data set:

A) Find mean and standard deviation of this data set? (5pts)

Mean:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \boxed{36.25}$$

Standard deviation:

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} = \boxed{7.99}$$

B) Find the value of 85 percentile? (5pts)

Position i of 85$^{\text{th}}$ percentile:

$$i = \frac{(n+1) \times X}{100} = \frac{(12+1) \times 85}{100} = 11.05$$

We will have $m = 11 \rightarrow V(11) = 45, V(12) = 47$, the value of 85$^{\text{th}}$ percentile could be calculated by this interpolation formula:

$$V = V(m) + (i - m)[V(m+1) - V(m)] = V(11) + (11.05 - 11)[V(12) - V(11)]$$
$$= 45 + 0.05(47 - 45) = \boxed{45.1}$$

C) Given value of 35 find its corresponding percentile? (5pts)

We have: $34 < 35 < 37 \rightarrow V(5) < 35 < V(6) \rightarrow i = 5$

Percentile at $i = 5$:

$$P(i) = \frac{100 \times i}{n+1} = \frac{100 \times 5}{13} = 38.46$$

Percentile at $i + 1 = 6$:

$$P(i+1) = \frac{100 \times i}{n+1} = \frac{100 \times 6}{13} = 46.15$$

Thus,

ENGINEERING PROBABILITY & STATISTICS – SEMESTER 1 – 2023-2024

MIDTERM ANSWER KEY

$$P = P(i) + (P(i+1) - P(i)) \times \frac{(V - V(i))}{(V(i+1) - V(i))} = 38.46 + (46.15 - 38.46) \times \frac{35 - 34}{37 - 34}$$
$$= \boxed{41.02}$$

Q2. (10pts)

A box contains 4 red and 4 blue balls. Two balls are withdrawn randomly. If they are the same color, then you win \$10; if they are different colors, then you win -\$8. (That is, you lose -\$8).

(a) The probability that you win in one round (5pts)

Win in this case means that you withdraw balls with identical color. There are 2 cases: you draw 2 red balls **OR** 2 blue balls.

Probability that withdrawing 2 red balls:

$$P(2\ red) = \frac{C(4,2)}{C(8,2)} = \frac{3}{14}$$

Probability that withdrawing 2 blue balls:

$$P(2\ blue) = \frac{C(4,2)}{C(8,2)} = \frac{3}{14}$$

Therefore,

$$P(win) = P(2\ red) + P(2\ blue) = \frac{3}{14} + \frac{3}{14} = \boxed{0.429}$$

(b) The expected value of the amount you win in one round (5pts)

Expected value of the amount you win:

$$\boxed{P(lose) = 1 - P(win)}$$

$$E(X) = P(win) \times Reward_{win} + P(lose) \times Reward_{lose}$$
$$= 0.429 \times \$10 + (1 - 0.429) \times (-\$8) = \boxed{-\$0.278}$$

→ On average, a player would expect to lost \$0.278 per round played.

Q3. (10pts)

An order from customer requires a dimension must meet the specification 10 ± 0.2. Due to technology limitation, your manufacturing process can only produce the dimension follow $N_D = (10, 0.15^2)$

ENGINEERING PROBABILITY & STATISTICS – SEMESTER 1 – 2023-2024

MIDTERM ANSWER KEY

(a) The proportion that your manufacturing items meet the customer specification (5pts)

Acceptable range for a dimention to meet the specification is 10 ± 0.2 which means that the dimention should lie in the interval $[10 - 0.2, 10 + 0.2] = [9.8, 10.2]$.

Denote X is the dimention that you manufactured and $X \sim N(10, 0.15^2)$

Your items meet the specification $\Leftrightarrow 9.8 \leq X \leq 10.2$

Probability that your items meet the specification:

Use the z-transform formula: $\boxed{z = \frac{x-\mu}{\sigma}}$

$$P(9.8 \leq X \leq 10.2) = P\left(\frac{9.8 - 10}{0.15} \leq Z \leq \frac{10.2 - 10}{0.15}\right) = P(-1.33 \leq Z \leq 1.33)$$
$$= P(Z \leq 1.33) - P(Z \leq -1.33) = 0.9081 - 0.0918 = \boxed{0.8163}$$

(b) One passed item can create the profit of $1, one unpassed item create profit -$2 (i.e. loss $2). Should you receive this order? (5pts)

We have the probability of passed item is:

$$P(pass) = P(9.8 \leq X \leq 10.2) = 0.8163$$

Probability of unpassed item:

$$P(unpass) = P(X < 9.8 \cup X > 10.2) = 1 - P(pass) = 1 - 0.8163 = 0.1837$$

The variable should be determined is the expected profit that the manufacturer gain:

$$E(profit) = P(pass) \times Profit(pass) + P(unpass) \times Profit(unpass)$$
$$= 0.8163 \times \$1 + 0.1837 \times (-\$2) = \boxed{\$0.4489}$$

→ As the manufacturer could make expected profit of $0.4489 per item so that they could receive this order.

Q4. (10pts)

Assumed that the height of people in a country follows normal distribution $N(165, 10^2)$. The furniture company want to design a bed, so as its length can cover the height at least 98% population. Find the minimum value of the length of the bed.

Denote X is the height of people $\rightarrow X \sim N(165, 10^2)$

The question corresponds to the height below which 98% of the population's heights fall.

ENGINEERING PROBABILITY & STATISTICS – SEMESTER 1 – 2023-2024

MIDTERM ANSWER KEY

The bed's length can cover the height of at least 98% of population $\Leftrightarrow P(X \le x) \ge 0.98$ where x is the minimum value of the length of the bed.

It holds that

$$P(X \le x) = P\left(Z \le \frac{x - 165}{10}\right) \ge 0.98$$

$$\Leftrightarrow \frac{x - 165}{10} \ge 2.054 \Leftrightarrow \boxed{x \ge 185.57}$$

→ The minimum value of the length of the bed to cover the height of at least 98% of population is 185.57.

Q5. (15pts)

A and B play a following game. First A writes down one of 3 options: Rock, Paper, or Scissors on the paper with probability: $P_A(\text{Rock}) = 0.3, P_A(\text{Paper}) = 0.4, P_A(\text{Scissors}) = 0.3$. And B does the same things with probability: $P_B(\text{Rock}) = 0.2, P_B(\text{Paper}) = 0.6, P_B(\text{Scissors}) = 0.2$. It is known that Rock wins Scissors, Paper wins Rock, Scissors wins Paper, and same options create draw results.

A\B	Rock (0.2)	Paper (0.6)	Scissors (0.2)
Rock (0.3)	Tie	B wins	A wins
Paper (0.4)	A wins	Tie	B wins
Scissors (0.3)	B wins	A wins	Tie

(a) Find probability that A wins (7pts)

That event of A wins would occur when 2 sub-events intersect each other, i.e. A chooses Rock, B must choose Scissors and so on.

There are 3 cases that A wins represented in the above table.

$$P(\text{A wins}) = 0.3 \times 0.2 + 0.4 \times 0.2 + 0.3 \times 0.6 = \boxed{0.32}$$

(b) Find probability that B wins (8pts)

Conduct the same procedure with question (a) but choose the index showed "B wins"

$$P(\text{B wins}) = 0.2 \times 0.3 + 0.3 \times 0.6 + 0.4 \times 0.2 = \boxed{0.32}$$

Q6. (15pts)

ENGINEERING PROBABILITY & STATISTICS – SEMESTER 1 – 2023-2024

MIDTERM ANSWER KEY

There are 2 box A and B. Box A has 3 balls. Box B has 4 balls. A fair coin is tossed. If the result is Head, one ball is released from box A without replacement. If the result is Tail, one ball is released from box B without replacement. If some boxes are empty, the coin is still continue being tossed in this case the number of ball released from the box is 0.

(a) Find probability that in 3 tosses, there 2 balls are drawn from A, and one ball is drawn from B (7pts)

For this to happen, there are **two Heads** and **one Tail** in any order during three times that the coin is tossed. As the coin is fair, the probability of each coin toss result is 0.5 for Head and 0.5 for Tail.

$$\mathrm{p}(\mathrm{Head}) = \mathrm{p}(\mathrm{Tail}) = 0.5$$

Denote X is the **number of Head** that the result of tossing the coin is. It means that $\boxed{X \sim Bino(3, 0.5)}$. The total number of trials is 3 times and the probability to toss a Head is 0.5.

Thus the probability that 2 balls from A and 1 ball from B are drawn in 3 tosses is:

$$\mathrm{P}(2\mathrm{A}, 1\mathrm{B}) = \mathrm{C}(3,2) \times 0.5^2 \times 0.5^{3-2} = 3 \times 0.5^3 = \boxed{0.375}$$

(b) Find probability after the drawn of the 5 tosses, box A has been ALREADY empty. (8pts)

For the box A to have been ALREADY empty after 5 tosses, **it is necessary to have at least 3 Heads within the 5 tosses**.

$$\boxed{X \sim Bino(5, 0.5)}$$

It holds that

$$\mathrm{P}(\text{box A empty after 5 tosses}) = \mathrm{P}(\mathrm{X} \geq 3 \text{ in 5 tosses}) = \mathrm{P}(\mathrm{X}=3) + \mathrm{P}(\mathrm{X}=4) + \mathrm{P}(\mathrm{X}=5)$$

$$= \sum_{i=3}^{5} C(5,i) \times 0.5^i \times 0.5^{5-i} = \boxed{0.5}$$

Q7. (25pts)

There are three automobiles namely A, B and C where weights of A, B, C follows normal distribution with $N_A = (30, 5^2), N_B = (40, 8^2), N_C = (30, 6^2)$.

A) Find the probability that the total weight of 3 automobiles does not exceed 110 (9pts)

Denote T is the total weight of 3 automobiles → $\boxed{\mathrm{T} = \mathrm{A} + \mathrm{B} + \mathrm{C}}$

ENGINEERING PROBABILITY & STATISTICS – SEMESTER 1 – 2023-2024

MIDTERM ANSWER KEY

Recaculate T's distribution attribute based on the central limit theorem:

$$E(Z) = E(X_1 \pm X_2 \pm \cdots \pm X_n) = E(X_1) \pm E(X_2) \pm \cdots \pm E(X_n)$$

$$Var(Z) = Var(X_1 \pm X_2 \pm \cdots \pm X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n)$$

$$\mu_T = \mu_A + \mu_B + \mu_C = 30 + 40 + 30 = 100$$

$$\sigma_T^2 = \sigma_A^2 + \sigma_B^2 + \sigma_C^2 = 5^2 + 8^2 + 6^2 = 125$$

Therefore, T follows normal distribution with $N_T = (100, 125)$

Probability that the total weight of 3 automobiles does not exceed 110:

$$P(T \le 110) = P\left(Z \le \frac{110 - 100}{\sqrt{125}}\right) = P(Z \le 0.894) = \boxed{0.8143}$$

B) A bridge is designed so that its strength follow $N_S = (95, 10^2)$. Find the probability that the bridge does not collapse when all 3 automobiles park on it (8pts)

That the bridge does not collapse when all 3 automobiles park on it means that the strenght of the bridge must be greater or equal to the total weight of 3 automobiles → $S \ge T$.

It means that $S - T \ge 0$. Denote $Y = S - T$ and we will have the attribute of distribution of Y thanks to the central limit theorem which is:

$$E(Z) = E(X_1 \pm X_2 \pm \cdots \pm X_n) = E(X_1) \pm E(X_2) \pm \cdots \pm E(X_n)$$

$$Var(Z) = Var(X_1 \pm X_2 \pm \cdots \pm X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n)$$

$$\mu_Y = \mu_S - \mu_T = 95 - 100 = -5$$

$$\sigma_Y^2 = \sigma_S^2 + \sigma_T^2 = 10^2 + 125 = 225 = 15^2$$

Therefore, Y follows normal distribution with $N_Y = (-5, 15^2)$

The probability that the bridge does not collapse when all 3 automobiles park on it:

$$P(Y \ge 0) = P\left(Z \ge \frac{0 - (-5)}{15}\right) = P(Z \ge 0.333) = 1 - P(Z \le 0.333) = 1 - 0.6304$$
$$= \boxed{0.3696}$$

C) If A and B always have the same weight and follow the distribution of B. Find the probability that the total weight does not exceed 120. (8pts)

A and B always have the same weight implies that the total weight of A and B is a double of B's weight. Denote Z is the total weight of 3 automobiles in this case → $Z = 2B + C$

ENGINEERING PROBABILITY & STATISTICS – SEMESTER 1 – 2023-2024

MIDTERM ANSWER KEY

Recaculate Z's distribution attribute based on the central limit theorem:

$$E(aX) = aE(X)$$

$$Var(aX) = a^2Var(X)$$

$$\mu_Z = 2 \times \mu_B + \mu_C = 2 \times 40 + 30 = 110$$

$$\sigma_Z^2 = 2^2 \times \sigma_B^2 + \sigma_C^2 = 4 \times 8^2 + 6^2 = 292$$

Therefore, Z follows normal distribution with $N_Z = (110, 292)$

Probability that the total weight of 3 automobiles does not exceed 120:

$$P(Z \leq 120) = P\left(Z \leq \frac{120 - 110}{\sqrt{292}}\right) = P(Z \leq 0.5852) = \boxed{0.7208}$$

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