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Lecture #2

Natural and Step Responses of *RLC* Circuits

Chapter #8

Text book:

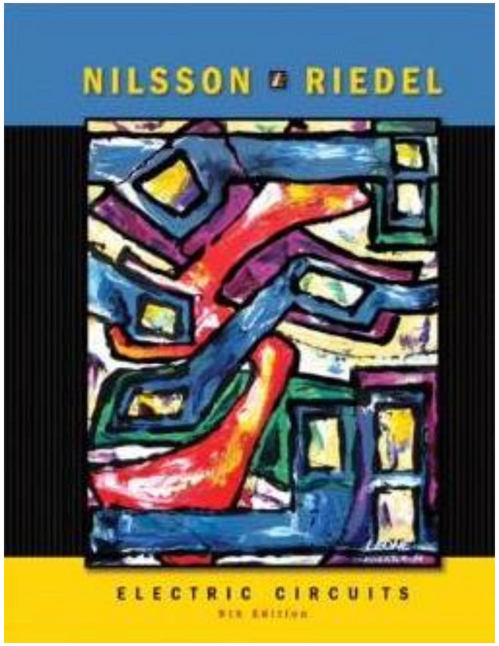
Electric Circuits

James W. Nilsson & Susan A. Riedel

9th Edition.

link: **http://blackboard.hcmiu.edu.vn/**

to download materials



Objectives

- Be able to determine the natural response and the step response of parallel RLC circuits.
- Be able to determine the natural response and the step response of series RLC circuits.

Outline

- The natural and step response of a parallel RLC circuit
- The natural and step response of a series RLC circuit

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General

The natural response:

The response that arise when stored energy in an inductor or capacitor is suddenly released.

The step response:

The response that arise when energy is being acquired by an inductor or capacitor due to sudden application of a dc voltage or current source.

Second order circuits (*RLC* circuits):

Circuits where voltages and currents are described by second-order differential equations.

- A **second-order circuit** is characterized by a *second-order differential equation*.
  - A second order circuit consists of resistors and the equivalent of two energy storage elements.
  - Two initial conditions (boundary conditions) are needed for solving a second-order differential equation, and those initial conditions can often be obtained via circuit analysis.

Finding Initial Values

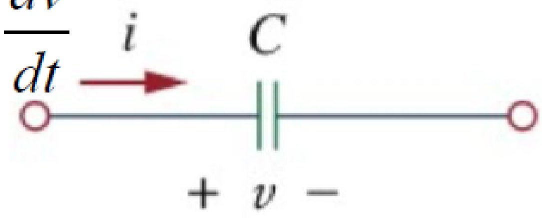
- The **initial values** of the voltage across a capacitor and the current through an inductor can be determined from circuit analysis by using the following continuous properties.
  - ✓ The capacitor voltage is always continuous.

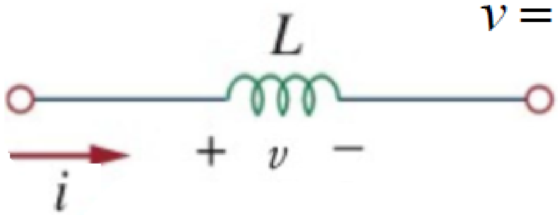
$$v_c(0^+) = v_c(0^-)$$

  - ✓ The inductor current is always continuous.

$$i_L(0^+) = i_L(0^-)$$

$$i = C\frac{dv}{dt}$$





$$v = L\frac{di}{dt}$$

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Finding Initial Values (*cont.*)

- The **initial derivatives** of the voltage across a capacitor and the current through an inductor can be determined from the initial values of voltage and current using the *i*-*v* characteristics of capacitor and inductor.

$$C\frac{dv_c(0^+)}{dt} = i_c(0^+)$$

$$L\frac{di_L(0^+)}{dt} = v_L(0^+)$$

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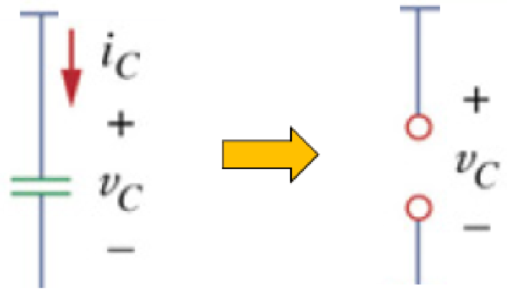
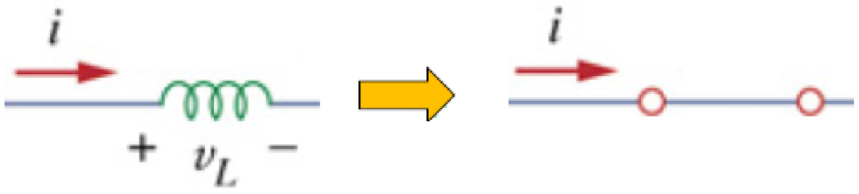
Finding Final Values

- The **final values** of voltage $v(\infty)$ and current $i(\infty)$ are obtained from circuit analysis by using the following steady-state properties.
  - ✓ The capacitor is treated as an open circuit.



  - ✓ The inductor is treated as a short circuit.



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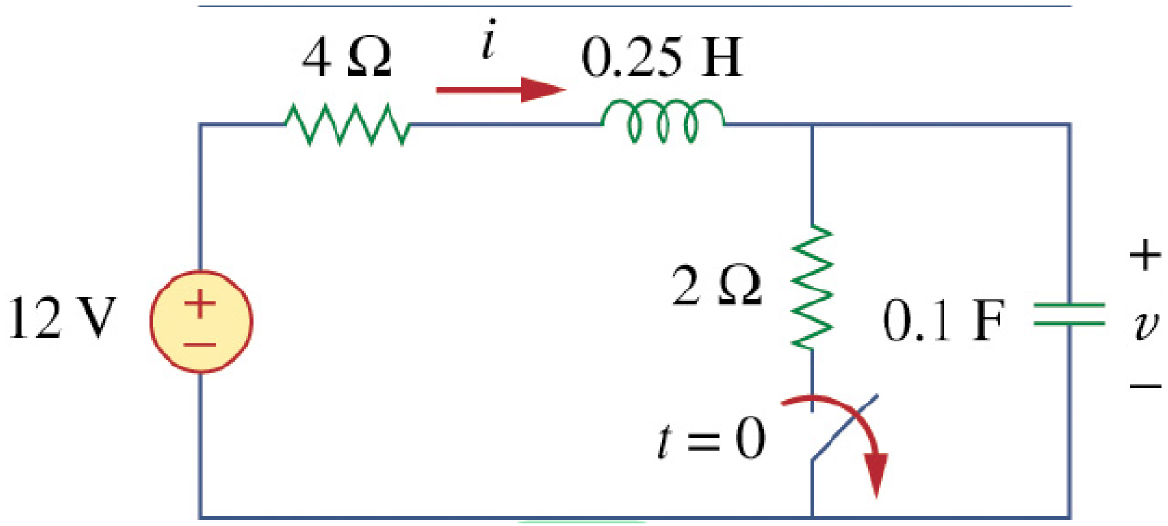
Example 1

Q: The switch in the circuit has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$



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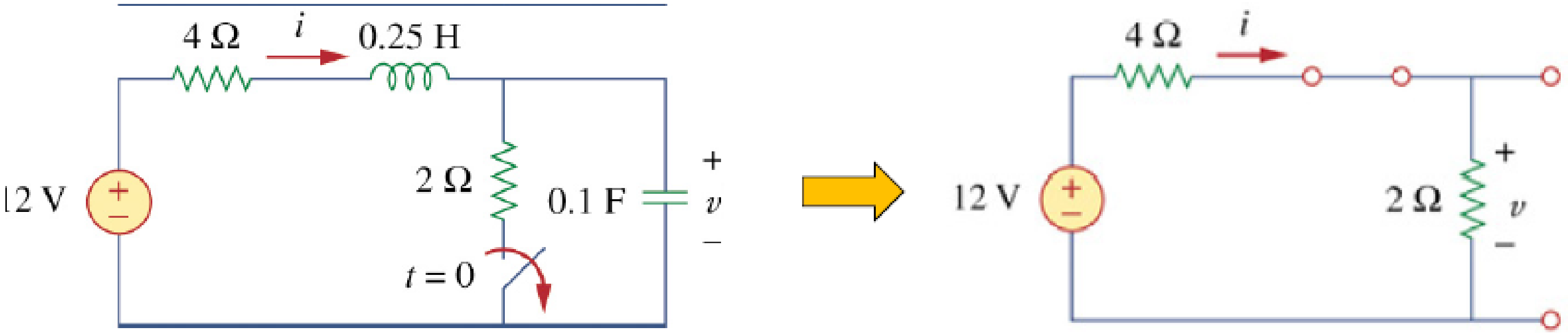
Example 1 – Sol.

(a) At dc steady state, the inductor acts like a short circuit and the capacitor like an open circuit.



1. The i is continuous (inductor current).

2. The v is continuous (capacitor voltage)

$$i(0^-) = \frac{12}{4+2} = 2\ A, \qquad v(0^-) = 2i(0^-) = 4\ V$$

$$i(0^+) = i(0^-) = 2\ A \qquad v(0^+) = v(0^-) = 4\ V$$

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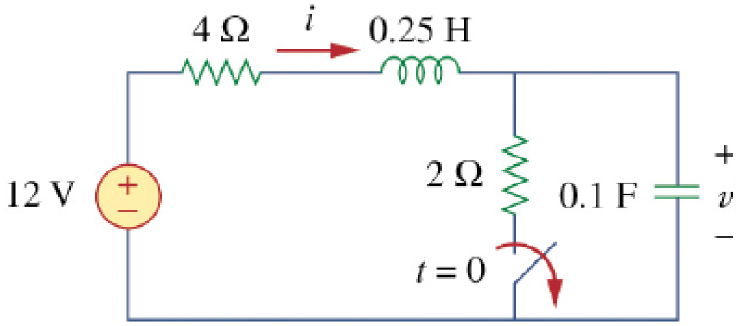
Example 1 – Sol.

(b) At $t = 0^+$, the switch is open

$$i_C(0^+) = i(0^+) = 2\text{A}$$

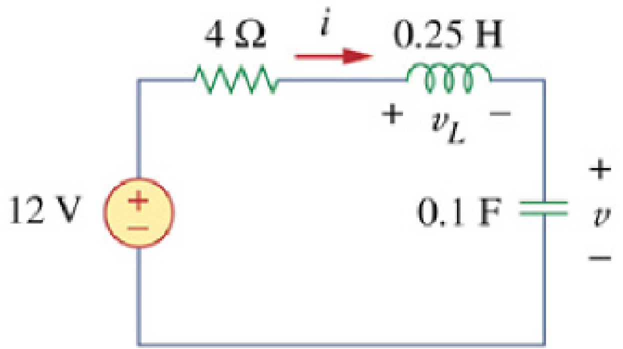
$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\text{V/s}$$



By KVL: $-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$

$$\Rightarrow v_L(0^+) = 12 - 8 - 4 = 0$$

$$\Rightarrow \frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\text{A/s}$$



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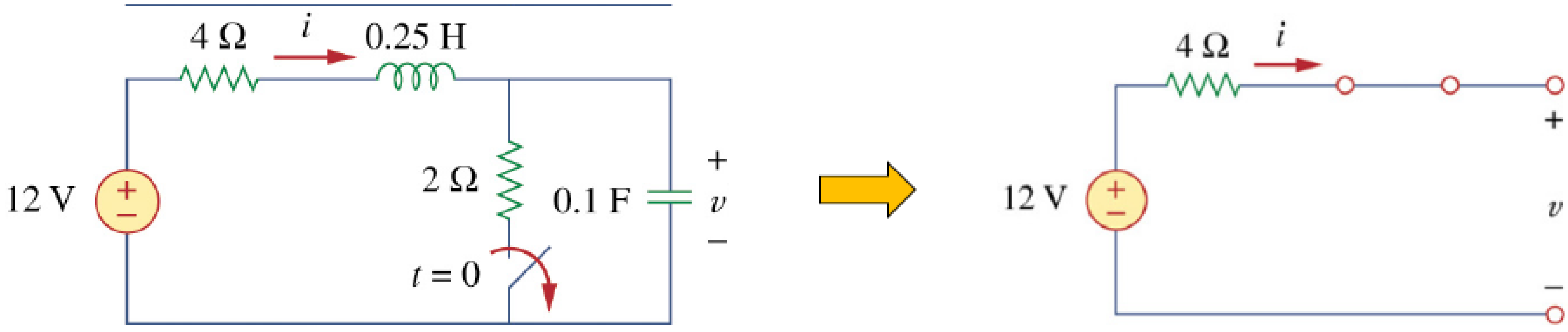
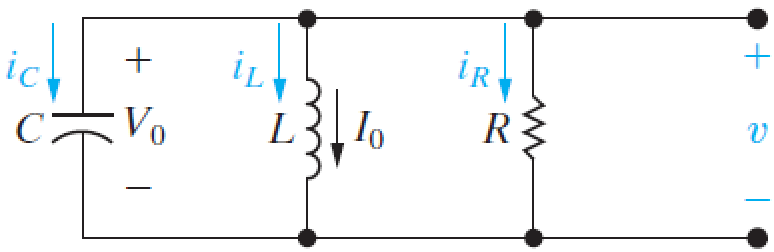
Example 1 – Sol.

(c) For $t>0$, the circuit undergoes transience. As $t \to \infty$, the circuit reaches steady state again.



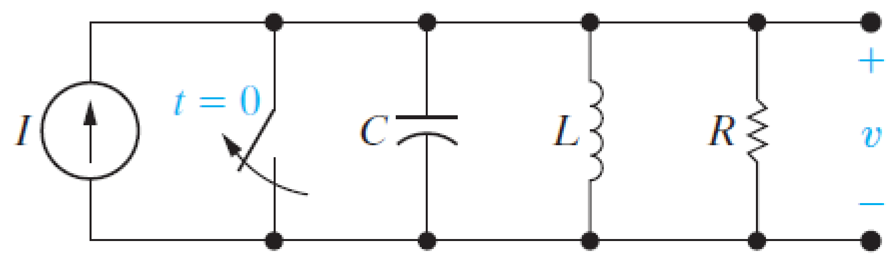
$$\begin{cases} i(\infty) = 0\text{A} \\ v(\infty) = 12\text{V} \end{cases}$$

Natural and step responses of *RLC* circuits



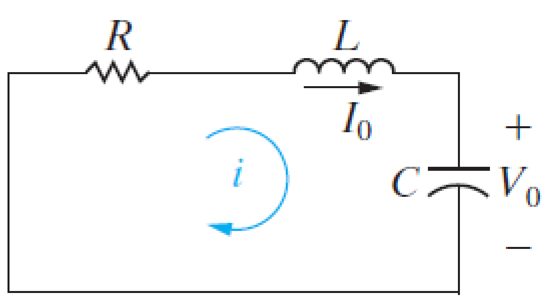
Natural response

of a parallel RLC circuit.



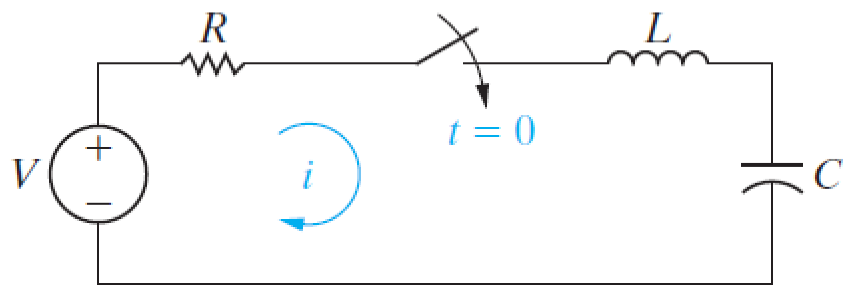
Step response

of a parallel RLC circuit.



Natural response

of a series RLC circuit.



Step response

of a series RLC circuit.

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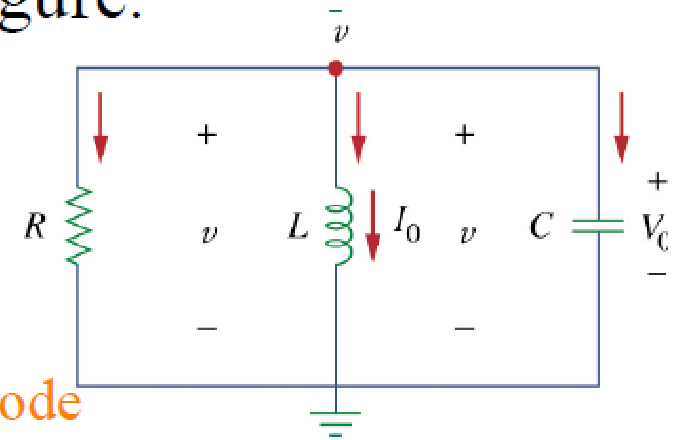
Natural responses of a parallel *RLC* circuit

Or the Source-Free Parallel *RLC* Circuit

- The parallel RLC circuit shown in Figure.

Initial Conditions



$$\text{Initial inductor current: } i(0) = \frac{1}{L}\int_{-\infty}^{0} v\,dt = V_0$$

$$\text{Initial capacitor voltage: } v(0) = V_0$$

For analyzing the circuit, take KCL at the top node

$$C\frac{dv}{dt} + \frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt = 0 \cdots (a)$$

Differentiate both sides with respect to t

$$\Rightarrow \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

Take i as the variable

$$v = L\frac{di}{dt} \Rightarrow (a) \text{ becomes}$$

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = 0$$

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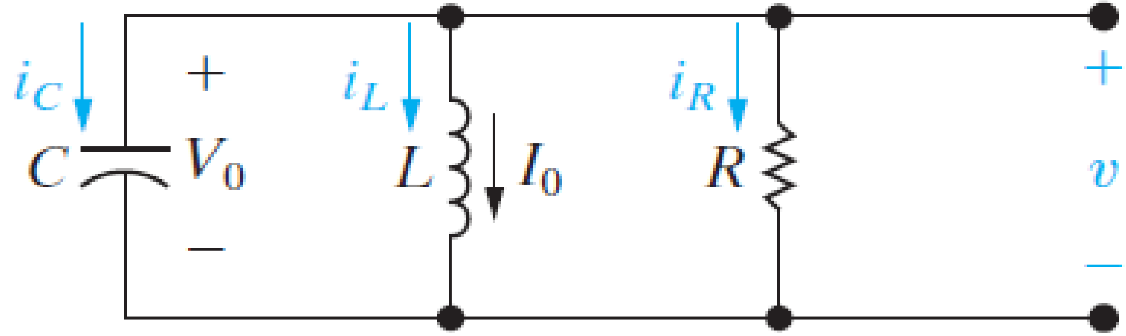
Natural responses of a parallel *RLC* circuit

$$\frac{v}{R}+\frac{1}{L}\int_0^t v\,dt+I_o+C\frac{dv}{dt}=0$$



Differentiate once

$$\frac{1}{R}\frac{dv}{dt}+\frac{v}{L}+C\frac{d^2v}{dt^2}=0$$

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$

Assume solution in the form $v = Ae^{st}$, where *A and s are unknown* constants

$$As^2e^{st}+\frac{1}{RC}Ase^{st}+\frac{Ae^{st}}{LC}=0 \rightarrow Ae^{st}\left(s^2+\frac{s}{RC}+\frac{1}{LC}\right)=0 \quad \boxed{\left(s^2+\frac{s}{RC}+\frac{1}{LC}\right)=0}$$

Characteristic equation

$$s_1=-\frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

$$s_2=-\frac{1}{2RC}-\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

$$\left.\begin{aligned} v&=A_1e^{s_1t}\\ v&=A_2e^{s_2t}\end{aligned}\right\} v=A_1e^{s_1t}+A_2e^{s_2t}$$

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Natural responses of a parallel *RLC* circuit

Parameter	Terminology	Value In Natural Response
s_1, s_2	Characteristic roots	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
α	Neper frequency	$\alpha = \dfrac{1}{2RC}$
ω_0	Resonant radian frequency	$\omega_0 = \dfrac{1}{\sqrt{LC}}$

If	$\omega_o^2 < \alpha^2$	s_1, s_2	Will be real and distinct	Overdamped
If	$\omega_o^2 > \alpha^2$	s_1, s_2	Will be complex and conjugate to each other	Underdamped
If	$\omega_o^2 = \alpha^2$	s_1, s_2	Real and equal	Critically damped

Damping affects the way the voltage response reaches its final (steady-state) value.

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Natural responses of a parallel *RLC* circuit

The over-damped response

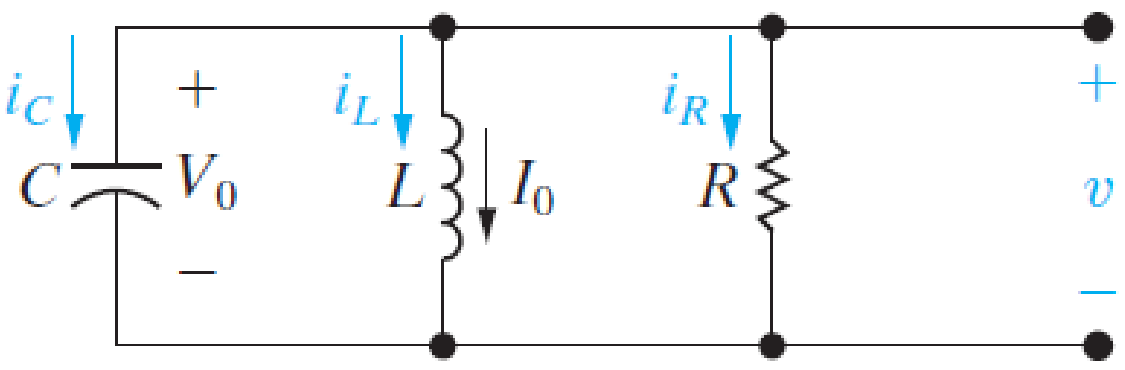
$\omega_o^2 < \alpha^2$ s_1, s_2 Will be real and distinct

$$\boxed{v = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

To solve for A_1 and A_2 $\boxed{v(0^+) = A_1 + A_2}$ $\boxed{\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2}$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C}$$

$$i_C(0^+) = -\frac{V_o}{R} - I_o$$



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Natural responses of a parallel *RLC* circuit

The over-damped response

Procedure for finding the over-damped response v(t):

(1) Find the roots of the characteristic equation, s_1 and s_2, using the values of R, L, and C.

(2) Find $v(0^+)$ and $dv(0^+)/dt$ using circuit analysis.

(3) Find the values of A_1 and A_2 by solving simultaneously equations:

$$v(0^+) = A_1 + A_2$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = s_1 A_1 + s_2 A_2$$

(4) Substitute the values for s_1, s_2, A_1, and A_2 into equation

$v = A_1 e^{s1t} + A_2 e^{s2t}$, to determine the expression for v(t) for $t \geq 0$.

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Natural responses of a parallel *RLC* circuit

The under-damped response

$\omega_o^2 > \alpha^2$ $\quad s_1, s_2$ $\quad$ Will be complex

$$s_1 = -\alpha + \sqrt{-\left(\omega_o^2 - \alpha^2\right)} \rightarrow s_1 = -\alpha + j\sqrt{\left(\omega_o^2 - \alpha^2\right)} \rightarrow \boxed{s_1 = -\alpha + j\omega_d}$$

$$\boxed{\omega_d = \sqrt{\omega_o^2 - \alpha^2}}$$ Damped Radian Frequency

$$\boxed{s_2 = -\alpha - j\omega_d}$$

$$v = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \rightarrow v = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$v = A_1 e^{-\alpha t}\left(\cos\omega_d t + j\sin\omega_d t\right) + A_2 e^{-\alpha t}\left(\cos\omega_d t - j\sin\omega_d t\right)$$

$$v = e^{-\alpha t}\left((A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t\right)$$

$$v = e^{-\alpha t}\left(B_1\cos\omega_d t + B_2\sin\omega_d t\right)$$

The under-damped response: $$\boxed{v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t}$$

$$\boxed{v(0^+) = V_0 = B_1}$$

$$\boxed{\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -\alpha B_1 + \omega_d B_2}$$

The process for finding the under-damped response is the same as that for the over-damped response, although the response equations and the simultaneous equations used to find the constants are slightly different.

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Natural responses of a parallel *RLC* circuit

The critically damped response

$\omega_o^2 = \alpha^2 \quad s_1, s_2$ Real and equal

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha = s_2$$

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

$$v = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_0 e^{-\alpha t}$$

(Not a valid solution since we have two initial conditions to satisfy and only one constant)

When a circuit is critically damped, the response is on the verge of oscillating.

It rarely happens in practical.

Possible solution $\boxed{v = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}$

$$\boxed{v(0^+) = V_0 = D_2}$$

$$\boxed{\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2}$$

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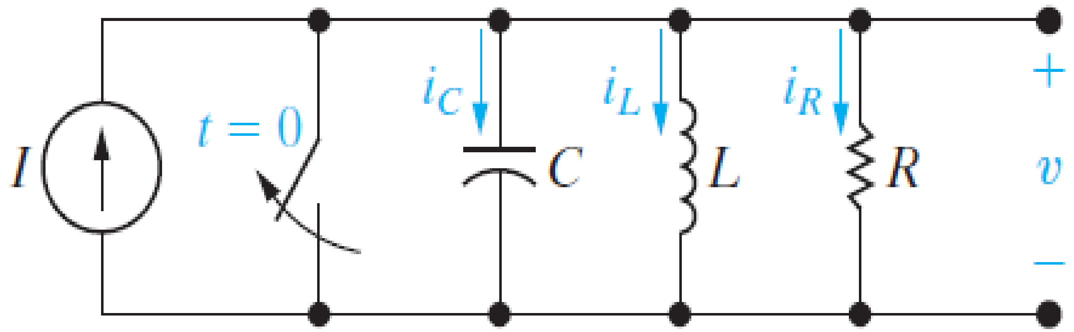
The step response of a parallel *RLC* circuit



$$i_L + i_R + i_C = I$$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

$$i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t}, \quad \text{(overdamped)}$$

$$i_L = I + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t, \quad \text{(underdamped)}$$

$$i_L = I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, \quad \text{(critically damped)}$$

$$i = I_f + \begin{Bmatrix} \text{function of the same form} \\ \text{as the natural response} \end{Bmatrix},$$

$$v = V_f + \begin{Bmatrix} \text{function of the same form} \\ \text{as the natural response} \end{Bmatrix},$$

where A_1', A_2', B_1', B_2', D_1', and D_2', are arbitrary constants.

I_f & V_f represent the final value of the response function.

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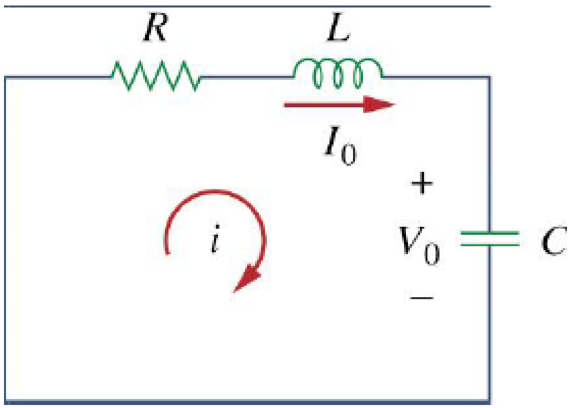
The natural response of a series *RLC* circuit

- The series RLC circuit shown in Figure.



Initial Conditions

Initial capacitor voltage: $v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0$

Initial inductor current: $i(0) = I_0$

For analyzing the circuit, take KVL around the loop

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,d\rho = 0 \cdots (a)$$

Differentiate both sides with respect to t

$$\Rightarrow \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

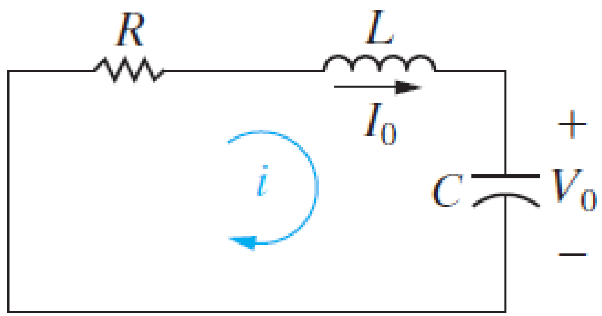
Take v as the variable

$$i = C\frac{dv}{dt} \Rightarrow (a) \text{ becomes}$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = 0$$

The natural response of a series *RLC* circuit

The procedure for finding the natural or step responses of a series RLC circuit are the same as those used to find the natural or step responses of a parallel RLC circuit.



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + V_o = 0$$

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

Characteristic equation

$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

The roots

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The neper frequency

$$\alpha = \frac{R}{2L} \text{ rad/s,}$$

Resonant radian frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s.}$$

$$i(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad \text{(overdamped)},$$

$$i(t) = B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t \quad \text{(underdamped)},$$

$$i(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t} \quad \text{(critically damped)}.$$

The current response will be overdamped, underdamped, or critically damped according to whether

$$\omega_0^2 < \alpha^2, \omega_0^2 > \alpha^2, \text{ or } \omega_0^2 = \alpha^2.$$

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The step response of a series *RLC* circuit

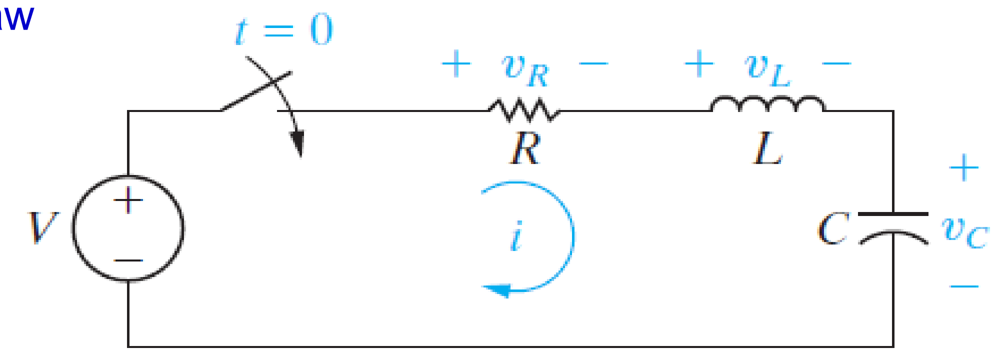
Applying Kirchhoff's voltage law

$$V = Ri + L\frac{di}{dt} + v_c$$

$$i = C\frac{dv_c}{dt}$$

$$\frac{di}{dt} = C\frac{d^2v_c}{dt^2}$$

$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_c}{LC} = \frac{V}{LC}$$



Capacitor voltage step response forms in series *RLC circuits*

3 possible solutions for v_C

$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t} \text{ (overdamped)},$$

$$v_C = V_f + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t \text{ (underdamped)},$$

$$v_C = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \text{ (critically damped)},$$

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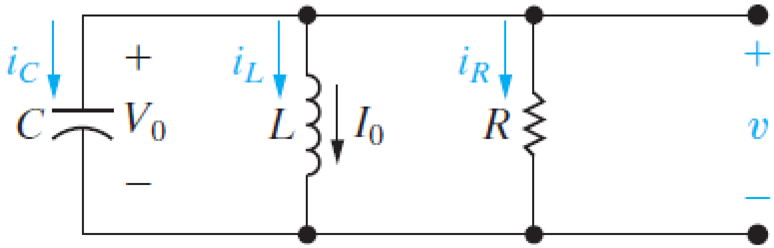
Example 2 Finding the Roots of the Characteristic Equation of a Parallel *RLC Circuit*



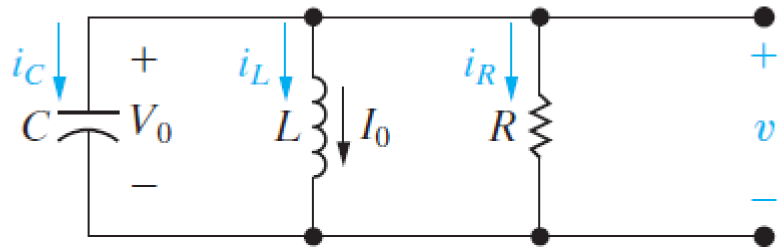
a) Find the roots of the characteristic equation that governs the transient behavior of the voltage shown in Fig. 8.5 if $R = 200\ \Omega$, $L = 50$ mH, and $C = 0.2\ \mu$F.

b) Will the response be overdamped, underdamped, or critically damped?

c) Repeat (a) and (b) for $R = 312.5\ \Omega$.

d) What value of R causes the response to be critically damped?

Example 2: Solution



a) For the given values of R, L, and C,

$$\alpha = \frac{1}{2RC} = \frac{10^6}{(400)(0.2)} = 1.25 \times 10^4 \text{ rad/s},$$

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{(50)(0.2)} = 10^8 \text{ rad}^2/\text{s}^2.$$

The roots of the characteristic equation:

$$s_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8}$$

$$= -12{,}500 + 7500 = -5000 \text{ rad/s},$$

$$s_2 = -1.25 \times 10^4 - \sqrt{1.5625 \times 10^8 - 10^8}$$

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Example 2: Solution

b) The voltage response is overdamped because $\omega_0^2 < \alpha^2$.

c) For $R = 312.5\ \Omega$,

$$\alpha = \frac{10^6}{(625)(0.2)} = 8000 \text{ rad/s},$$

$$\alpha^2 = 64 \times 10^6 = 0.64 \times 10^8 \text{ rad}^2/\text{s}^2.$$

As ω_0^2 remains at 10^8 rad^2/s^2,

$$s_1 = -8000 + j6000 \text{ rad/s},$$

$$s_2 = -8000 - j6000 \text{ rad/s}.$$

(In electrical engineering, the imaginary number $\sqrt{-1}$ is represented by the letter j, because the letter i represents current.)

In this case, the voltage response is underdamped since $\omega_0^2 > \alpha^2$.

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Example 2: Solution

d) For critical damping, $\alpha^2 = \omega_0^2$, so

$$\left(\frac{1}{2RC}\right)^2 = \frac{1}{LC} = 10^8,$$

or

$$\frac{1}{2RC} = 10^4,$$

and

$$R = \frac{10^6}{(2 \times 10^4)(0.2)} = 250\ \Omega.$$

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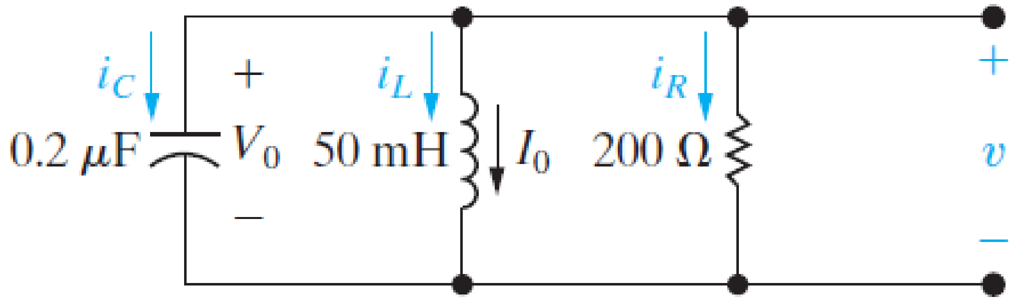
Example 3

Finding the Overdamped Natural Response of a Parallel *RLC Circuit*



For the circuit, $v(0^+) = 12V$, $i_L(0^+) = 30mA$

a) Find the initial current in each branch of the circuit.

b) Find the initial value of dv/dt.

c) Find the expression for v(t).

d) Sketch v(t) in the interval $0 \le t \le 250$ ms.

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Example 3: Solution

a) Find the initial current in each branch of the circuit.

$$i_L(0^-) = i_L(0) = i_L(0^+) = 30\ mA$$

$$i_R(0^+) = \frac{v(0^+)}{R} = \frac{12}{200} = 60\ mA$$

$$i_C(0^+) = -i_L(0^+) - i_R(0^+) = -90\ mA$$

b) Find the initial value of dv/dt.

$$i_C = C\frac{dv}{dt}$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-90\times10^{-3}}{0.2\times10^{-6}} = -450\ \mathrm{kV/s}$$

c) Find the expression for v(t).

$$\alpha = \frac{1}{2RC} = \frac{1}{2(200)(0.2\times10^{-6})} = 12500$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.2\times10^{-6})\times(50\times10^{-3})}} = 10000$$

$\omega_o^2 < \alpha^2$ (overdamped)

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -12500 + 7500 = -5000$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -12500 - 7500 = -20000$$

SEE

Example 3: Solution

Because the roots are real and distinct, we know that the response is overdamped. We find the co-efficients A_1 & A_2

$$12 = A_1 + A_2,$$

$$-450 \times 10^3 = -5000A_1 - 20{,}000A_2.$$

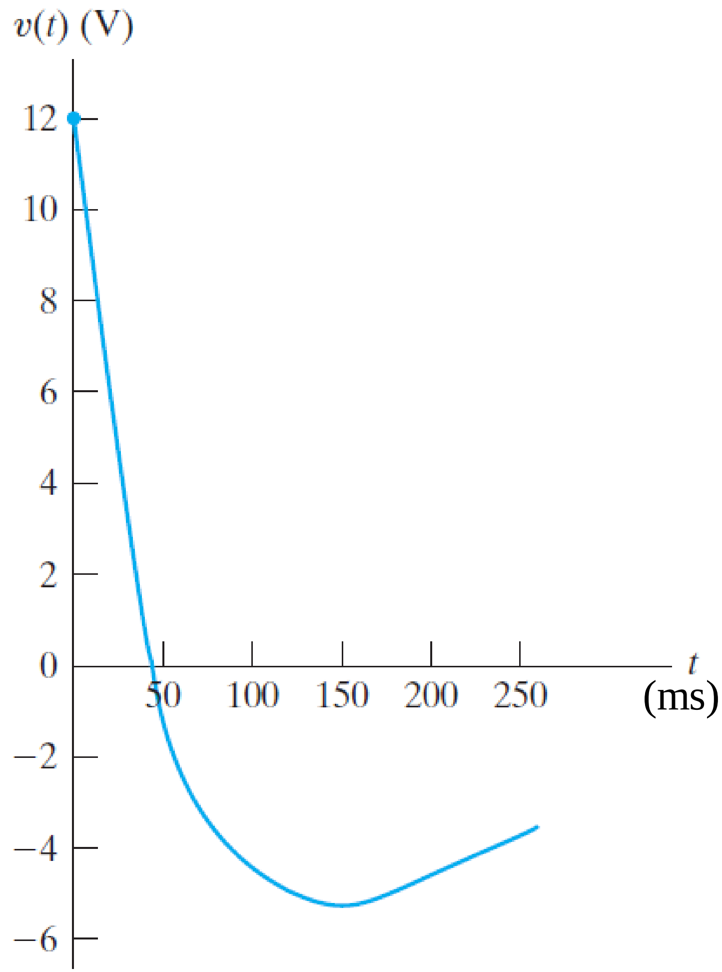
$$A_1 = -14 \text{ V and } A_2 = 26 \text{ V}.$$

➔The overdamped voltage response:

$$v(t) = (-14e^{-5000t} + 26e^{-20{,}000t}) \text{ V}, \quad t \geq 0.$$

As a check on these calculations, we note that the solution yields $v(0) = 12$ V and $dv(0^+)/dt = -450{,}000$ V/s.

d) plot of $v(t)$ versus t over the interval $0 \leq t \leq 250$ ms

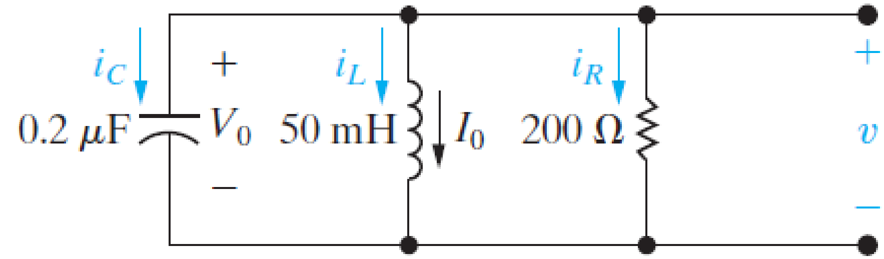


Example 4

Derive branch currents i_R, i_L, and i_C

Solution:

$$v(t) = -14e^{-5000t} + 26e^{-20,000t}$$



$$i_R = \frac{v(t)}{R} = \frac{-14e^{-5000t} + 26e^{-20,000t}}{200} = \left(-70e^{-5000t} + 130e^{-20,000t}\right)\text{mA} \qquad t \geq 0$$

$$i_C = C\frac{dv}{dt} = \left(14e^{-5000t} - 104e^{-20,000t}\right)\text{mA} \qquad t \geq 0^+$$

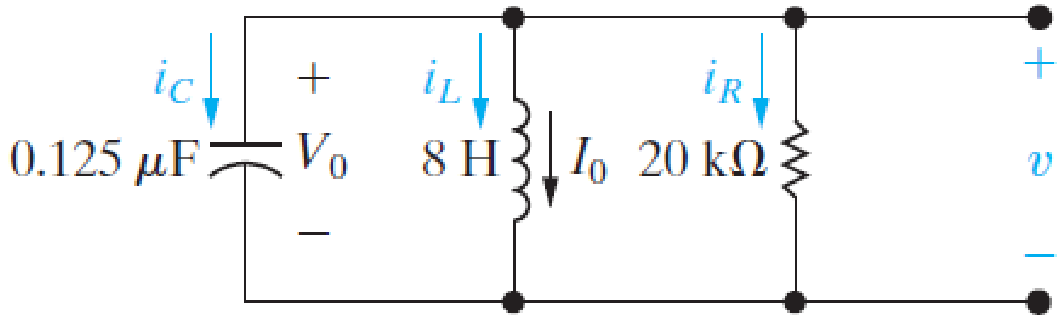
$$i_L = -i_R(t) - i_C(t) = \left(56e^{-5000t} - 26e^{-20,000t}\right)\text{mA} \qquad t \geq 0$$

$$i_L(t) = \frac{1}{L}\int_0^t v\,dt + I_o$$

SEE

Example 5

Finding the Underdamped Natural Response of a Parallel *RLC Circuit*



For the circuit, $V_0 = 0$ V, $I_0 = -12.25$ mA

a) Calculate the roots of the characteristic equation.

b) Calculate v and dv/dt at $t = 0^+$.

c) Calculate the voltage response for $t \geq 0$.

d) Plot v(t) for the time interval $0 \leq t \leq 11$ms.

SEE

Example 5: Solution

a) Because

$$\begin{cases} \alpha = \dfrac{1}{2RC} = \dfrac{10^6}{2(20)10^3(0.125)} = 200 \text{ rad/s}, \\ \omega_0 = \dfrac{1}{\sqrt{LC}} = \sqrt{\dfrac{10^6}{(8)(0.125)}} = 10^3 \text{ rad/s}, \end{cases} \Rightarrow \omega_0^2 > \alpha^2.$$

Therefore, the response is underdamped

$$\Rightarrow \omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - 4 \times 10^4} = 100\sqrt{96}$$

$$= 979.80 \text{ rad/s},$$

$$\left.\begin{aligned} s_1 &= -\alpha + j\omega_d = -200 + j979.80 \text{ rad/s}, \\ s_2 &= -\alpha - j\omega_d = -200 - j979.80 \text{ rad/s}. \end{aligned}\right\} \text{complex frequencies}$$

b) Because v is the voltage across the terminals of a capacitor, we have $v(0) = v(0^+) = V_0 = 0$. Because $v(0^+) = 0$, the current in the resistive branch is zero at $t = 0^+$. Hence the current in the capacitor at $t = 0^+$ is the negative of the inductor current: $i_C(0^+) = -(-12.25) = 12.25$ mA. Therefore the initial value of the derivative is

$$\frac{dv(0^+)}{dt} = \frac{(12.25)(10^{-3})}{(0.125)(10^{-6})} = 98{,}000 \text{ V/s}.$$

SEE

Example 5: Solution

c) Calculate the voltage response for $t \geq 0$.

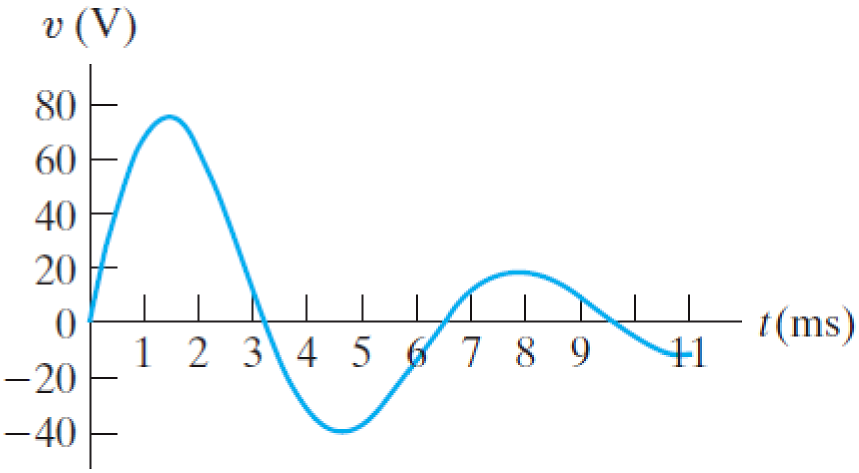
$$v = B_1 e^{-\alpha t}\cos\omega_d t + jB_2 e^{-\alpha t}\sin\omega_d t$$

$$v(0^+) = 0 = B_1$$

$$\frac{dv(0^+)}{dt} = 98{,}000 = 979.90 B_2 \dashrightarrow B_2 \approx 100$$

$$v = B_2 e^{-\alpha t}\sin\omega_d t = \boxed{100e^{-200t}\sin 979.80t} \dashrightarrow t \geq 0$$

d) The figure shows the plot of versus *t* for the first 11 ms after the stored energy is released. It clearly indicates the damped oscillatory nature of the underdamped response. The voltage approaches its final value, alternating between values that are greater than and less than the final value. Furthermore, these swings about the final value <u>decrease exponentially with time</u>.



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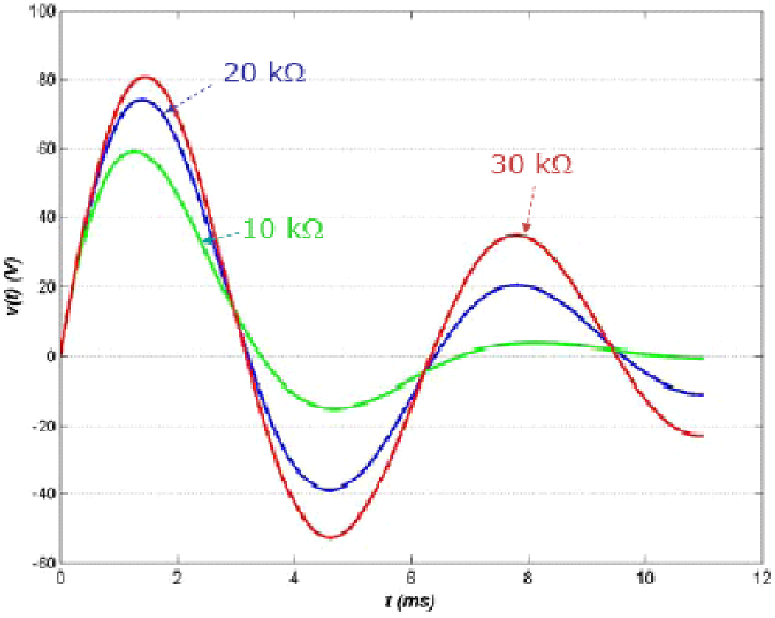
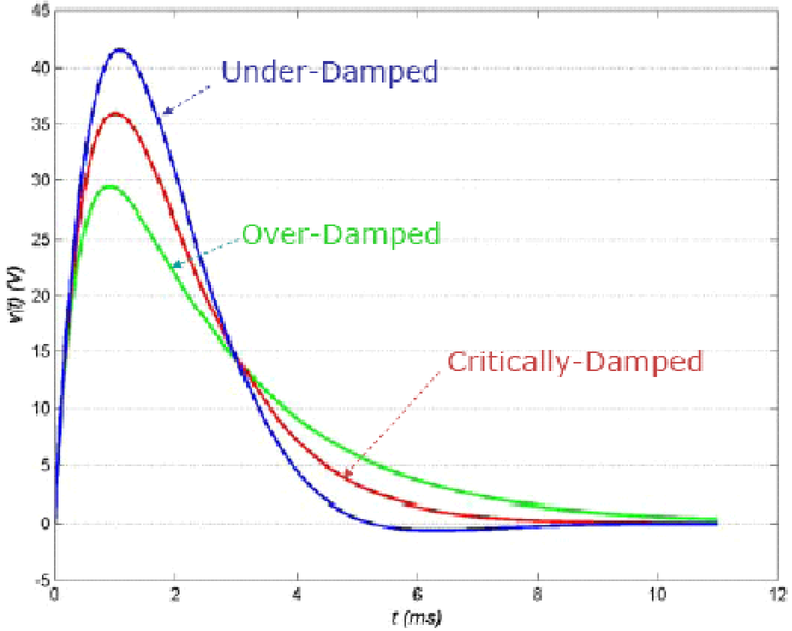
Example


20 kΩ
30 kΩ
10 kΩ
v(t) (V)
t (ms)


Under-Damped
Over-Damped
Critically-Damped
v(t) (V)
t (ms)

SEE

Example 6 Finding the Overdamped Step Response of a Parallel *RLC Circuit*

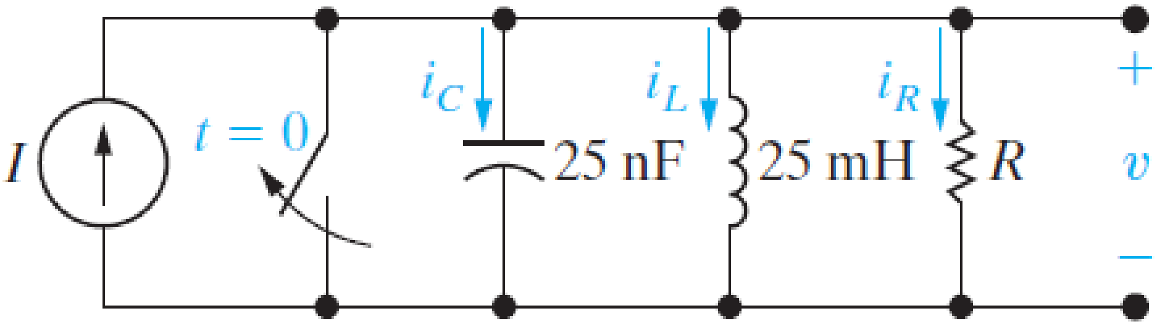
The initial energy stored in the circuit in Fig. is zero. At t = 0 a dc current source of 24 mA is applied to the circuit. The value of the resistor is 400 Ω.

a) What is the initial value of i_L?

b) What is the initial value of di_L/dt?

c) What are the roots of the characteristic equation?

d) What is the numerical expression for $i_L(t)$ when $t \geq 0$?



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Example 6 Solution

a) No energy is stored in the circuit prior to the application of the dc current source, so the initial current in the inductor is zero. The inductor prohibits an instantaneous change in inductor current; therefore $i_L(0) = 0$ immediately after the switch has been opened.

b) The initial voltage on the capacitor is zero before the switch has been opened; therefore it will be zero immediately after. Now, because $v = L di_L/dt$,

$$\frac{di_L}{dt}(0^+) = 0.$$

c) From the circuit elements, we obtain

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8,$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(400)(25)} = 5 \times 10^4 \text{ rad/s}, \rightarrow \alpha^2 = 25 \times 10^8.$$

$\omega_0^2 < \alpha^2$,

roots of the characteristic equation are real and distinct

$$s_1 = -5 \times 10^4 + 3 \times 10^4 = -20{,}000 \text{ rad/s},$$

$$s_2 = -5 \times 10^4 - 3 \times 10^4 = -80{,}000 \text{ rad/s}.$$

SEE

Example 6 Solution

d) Because the roots of the characteristic equation are real and distinct, the inductor current response will be overdamped. Thus $i_L(t)$ takes the form of

$$i_L = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}.$$

Inductor current in overdamped parallel *RLC* circuit step response

Hence, from this solution, the two simultaneous equations that determine A'_1 and A'_2 are

$$i_L(0) = I_f + A_1' + A_2' = 0,$$

$$\frac{di_L}{dt}(0) = s_1 A_1' + s_2 A_2' = 0.$$

$$\Longrightarrow \quad A_1' = -32\,\text{mA} \quad \text{and} \quad A_2' = 8\,\text{mA}.$$

The numerical solution for $i_L(t)$ is

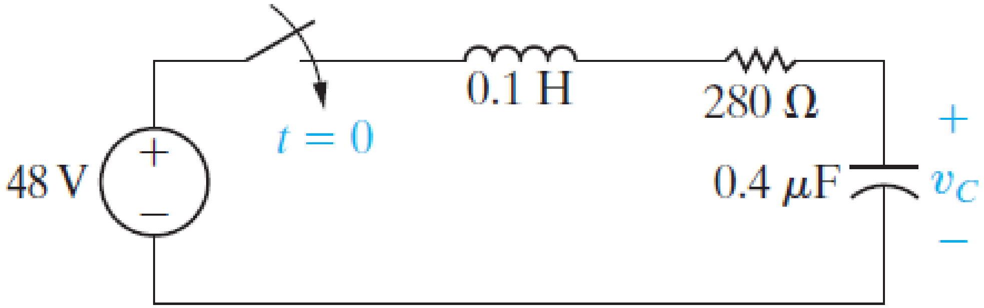
$$i_L(t) = (24 - 32e^{-20{,}000t} + 8e^{-80{,}000t})\,\text{mA}, \quad t \geq 0.$$

SEE

Example 7

Finding the Underdamped Step Response of a Series *RLC Circuit*

No energy is stored in the 100 mH inductor or the 0.4 μF capacitor when the switch in the circuit shown in Fig. is closed. Find $v_c(t)$ *for* $t \geq 0$.



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Example 7: Solution

The roots of the characteristic equation are

$$s_1 = -\frac{280}{0.2} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$

$$= (-1400 + j4800) \text{ rad/s},$$

$$s_2 = (-1400 - j4800) \text{ rad/s}.$$

The roots are complex, so the voltage response is underdamped. Thus

$$v_C(t) = 48 + B_1' e^{-1400t} \cos 4800t$$

$$+ B_2' e^{-1400t} \sin 4800t, \quad t \geq 0.$$

No energy is stored in the circuit initially, so both $v_C(0)$ and $dv_C(0^+)/dt$ are zero. Then,

SEE

Example 7: Solution

$$v_C(0) = 0 = 48 + B_1',$$

$$\frac{dv_C(0^+)}{dt} = 0 = 4800B_2' - 1400B_1'.$$

Solving for B_1' and B_2' yields

$$B_1' = -48 \text{ V},$$

$$B_2' = -14 \text{ V}.$$

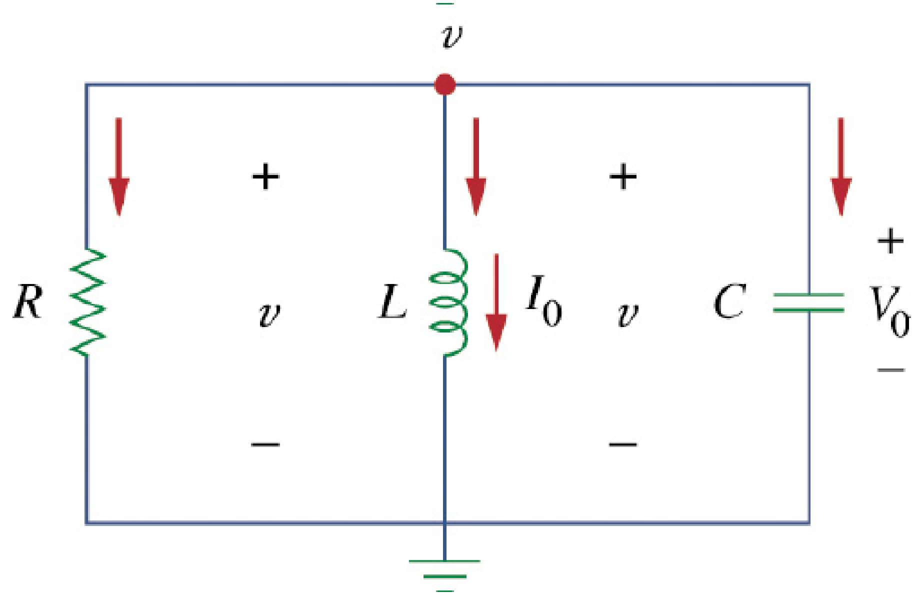
Therefore, the solution for $v_C(t)$ is

$$v_C(t) = (48 - 48e^{-1400t}\cos 4800t - 14e^{-1400t}\sin 4800t) \text{ V}, \quad t \geq 0.$$

SEE

Example 8

Q: In the following parallel circuit, find $v(t)$ for $t>0$, $v(0) = 5V$, $i(0)=0$, $L=1H$, and $C=10mF$. $R=1.923\Omega$, $R=5\Omega$, and $R=6.25\Omega$



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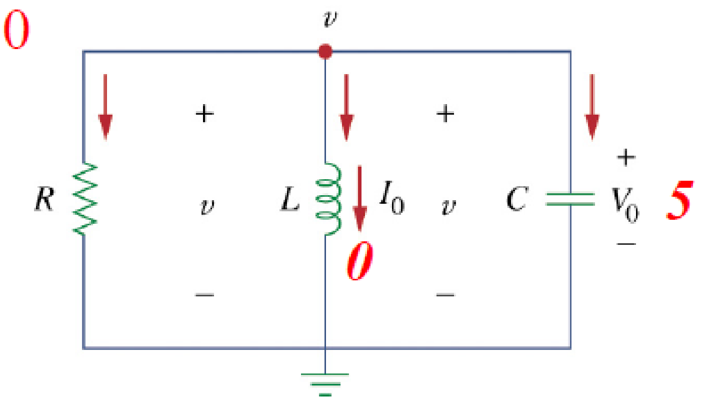
Example 8: Sol.

Characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$

where $\alpha = \dfrac{1}{2RC}$, $\omega_0 = \dfrac{1}{\sqrt{LC}}$

$$\frac{dv(0)}{dt} = -\frac{V_0 + RI_0}{RC}$$



R	1.923	5	6.25
α	26	10	8
ω_0	10	10	10
	$\alpha > \omega_0$	$\alpha = \omega_0$	$\alpha < \omega_0$
damped	*overdamped*	*critically damped*	*underdamped*
$s_{1,2}$	$-2,\ -50$	$-10,\ -10$	$-8 \pm j6$
$\frac{dv(0)}{dt}$	-260	-100	-80

Example 8: Sol.

- $R = 1.923\Omega$ *(overdamped)*

$s_{1,2} = -2, -50$

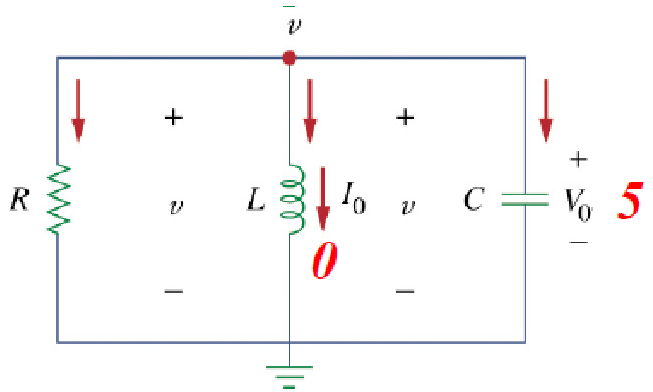
$$\Rightarrow v(t) = A_1 e^{-2t} + A_2 e^{-50t}$$

$$v(0) = 5 = A_1 + A_2 \cdots (a_1)$$

$$\frac{dv(0)}{dt} = -260 = -2A_1 e^{-2t} - 50A_2 e^{-50t}\Big|_{t=0}$$

$$= -2A_1 - 50A_2 \cdots (b_1)$$

$$\Rightarrow A_1 = -0.2083,\ A_2 = 5.208$$



$$v(t) = -0.2083e^{-t} + 5.208e^{-50t}$$

Example 8: Sol.

- $R=5\Omega$ (*critically damped*)

$s_{1,2}=-10,\ -10,\ \alpha=10$

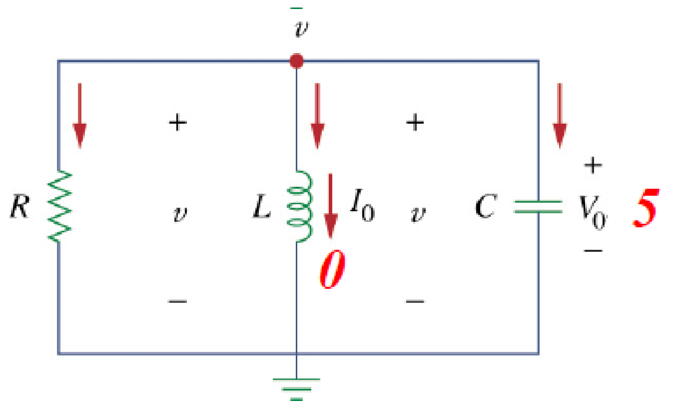
$$\Rightarrow v(t)=\left(A_1+A_2t\right)e^{-10t}$$

$$v(0)=5=A_1\cdots(a_2)$$

$$\frac{dv(0)}{dt}=-100=\left(-10A_1-10A_2t+A_2\right)e^{-10t}\Big|_{t=0}$$

$$=-10A_1+A_2\cdots(b_2)$$

$$\Rightarrow A_1=5,\ A_2=-50$$



$$v(t) = (5 - 50t)e^{-10t}$$

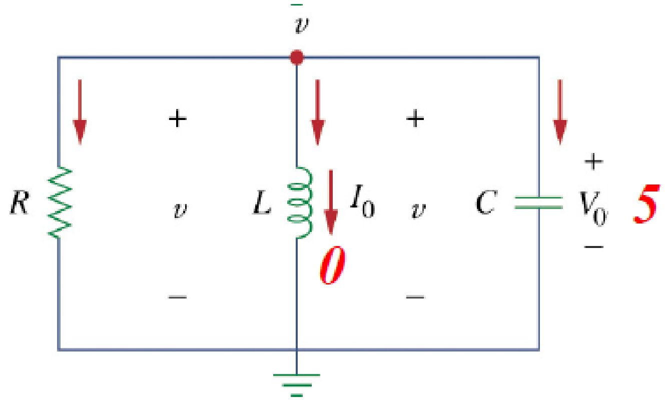
SEE

Example 8: Sol.

- R=6.25Ω (*underdamped*)

$s_{1,2}$=-8±j6, α=8

$$\Rightarrow v(t) = \left(A_1 \cos 6t + A_2 \sin 6t\right) e^{-8t}$$



$$v(0) = 5 = A_1 \cdots (a_3)$$

$$\frac{dv(0)}{dt} = -80 = \left(\left(-8A_1 + 6A_2\right)\cos 6t + \left(-6A_1 - 8A_2\right)\sin 6t\right)e^{-8t}\Big|_{t=0}$$

$$= -8A_1 + 6A_2 \cdots (b_3)$$

$$\Rightarrow A_1 = 5,\ A_2 = -6.667$$

v(t) = (5cos6t – 6.667sin6t)e^{-8t}

SEE

Example 8: Sol.