DIFFERENTIAL EQUATIONS

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Chapter 5 SYSTEMS OF DIFFERENTIAL EQUATIONS

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Chapter 5 SYSTEMS OF DIFFERENTIAL EQUATIONS

5.1 BASIC THEORY OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

5.1.1 Introduction

In this chapter we are interested in finding a solution to a *system* of first-order differential equations of the form

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(t, x_1, \ldots, x_n) \\ \frac{dx_2}{dt} &= f_2(t, x_1, \ldots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, \ldots, x_n) \end{aligned} \tag{0.1}$$

Here we denote the independent variable by t, and let $x_1, x_2, \ldots, x_n$ represent dependent variables that are functions of t.

5.1.1 INTRODUCTION

A **solution** of (0.1) is n functions $x_1(t), x_2(t), \ldots, x_n(t)$ such that

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(t, x_1, \ldots, x_n) \\ \frac{dx_2}{dt} &= f_2(t, x_1, \ldots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(t, x_1, \ldots, x_n) \end{aligned} \tag{0.2}$$

on some interval I.

For example $x_1(t) = t$ and $x_2(t) = t^2$ is a solution of the system

$$\begin{cases} \dfrac{dx_1}{dt} = 1 \\ \dfrac{dx_2}{dt} = x_1 + t \end{cases}$$

since

$$\frac{dx_1}{dt} = 1 \quad and \quad \frac{dx_2}{dt} = 2t = x_1(t) + t.$$

Let $x_1^0, x_2^0, \dots, x_n^0$ be given real numbers. The problem of finding a solution of the system

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, \dots, x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, \dots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, \dots, x_n)
\end{aligned}$$

satisying the initial condition

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \dots, \quad x_n(t_0) = x_n^0,$$

is called an **initial value problem**.

For example, $x_1(t) = e^t$ and $x_2(t) = 1 + e^{2t}/2$ is a solution of the initial value problem

$$\begin{cases} \dfrac{dx_1}{dt} = x_1, & x_1(0) = 1 \\ \dfrac{dx_2}{dt} = x_1^2, & x_2(0) = \dfrac{3}{2} \end{cases}$$

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since

$$\frac{dx_1}{dt} = e^t = x_1(t), \; \frac{dx_2}{dt} = e^{2t} = x_1^2(t),$$

$$x_1(0) = 1, \; and \; x_2(0) = \frac{3}{2}.$$

5.1.1 INTRODUCTION

Conversion of Higher Order Equations to First Order Systems

Every nth order differential equation for the single variable y

$$y^{(n)}(t) = f\left(t, y, y', ..., y^{(n-1)}\right)$$

can be converted into a system of n first-order equations for the variables

$$x_1(t) = y,\ x_2(t) = y'(t), ..., x_n(t) = y^{(n-1)}(t).$$

Example 1.1

Write the following 2^{nd} order differential equations as a system of first order, linear differential equations.

$$y'' - 5y' + y = 0, , \quad y(0) = 1, y'(0) = 2.$$

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Now notice that if we differentiate both sides of these functions, we get

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$$x_1'(t) := y'(t) = x_2(t), \quad x_2'(t) = y''(t) = 5y'(t) - y(t) = 5x_2(t) - x_1(t).$$

So we get the linear system of differential equations

$$\begin{cases} \frac{dx_1}{dt} &= x_2, & x_1(0) = 1 \\ \frac{dx_2}{dt} &= -x_1 + 5x_2 & x_2(0) = 2. \end{cases}$$

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Solution: Just as we did in the last example we will need to define some new functions. This time we will need 4 new functions

$$x_1(t) := y(t), \quad x_2(t) = y'(t), \quad x_3(t) := y''(t), \quad x_4(t) = y'''(t)$$

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$x_1'(t) := y'(t) = x_2(t),$

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$$x_4'(t) = y^{(4)}(t) = 3y'' + (\sin t)y' - 8y + t^2 = 3x_3(t) + (\sin t)x_2(t) - 8x_1(t) + t^2.$$

So we get the linear systems of differential equations:

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_3 \\ \frac{dx_3}{dt} = x_4 \\ \frac{dx_4}{dt} = -8x_1 + (\sin t)x_2 + 3x_3 + t^2 \end{cases}$$

5.1.1 INTRODUCTION

The most general system of n first order linear equations has the form

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t)\\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + f_2(t)\\ &\vdots\\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t)\end{aligned} \tag{0.3}$$

If each of the functions $f_j(t)$ is identically zero, then the system (0.3) is said to be **homogeneous**; otherwise it is **nonhomogeneous**. In this chapter, we only consider the case when the coefficients a_{ij} do not depend on t.

5.1.1 INTRODUCTION

If $x_1 = x_1(t), x_2 = x_2(t), ..., x_n = x_n(t)$, then

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is called a **vector valued function**. Its derivative is the vector valued function

$$\frac{dX}{dt} = X'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

Let

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \text{ and } F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

Then the left-hand side of (0.3) are the components of the vector $X'(t)$, while the right-hand side of (0.3) are the components of the vector $A(t)X(t) + F(t)$ and we can write Equation (0.3) in the concise form

$$X'(t) = A(t)X(t) + F(t).$$

5.1.1 INTRODUCTION

Moreover, if $x_1(t), x_2(t), \ldots, x_n(t)$ satisfy the initial conditions

$$x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0, \ldots,\ x_n(t_0) = x_n^0,$$

then $X(t)$ satisfies the *initial value problem*

$$X'(t) = A(t)X(t) + F(t),\ X(t_0) = X^0 \quad \text{where} \quad X^0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}.$$

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then $X(t)$ satisfies the *initial value problem*

$$X'(t) = A(t)X(t) + F(t),\ X(t_0) = X^0 \quad \text{where} \quad X^0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}.$$

For example, the initial value problem

$$\begin{aligned} x_1' &= x_1 - x_2 + x_3, & x_1(0) &= 1 \\ x_2' &= 5x_1 + 3x_2 - x_3, & x_2(0) &= 0 \\ x_3' &= x_1 + 7x_3, & x_3(0) &= -1 \end{aligned}$$

can be written in the concise form

$$X' = \begin{bmatrix} 1 & -1 & 1 \\ 5 & 3 & -1 \\ 1 & 0 & 7 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

5.1.2 GENERAL SOLUTION OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

Theorem 1.1 (**Existence and Uniqueness**)

Suppose that $A(t)$ and $F(t)$ are continuous on an open interval I that contains the point t_0. Then, for any choice of the initial vector $X^0 = [x_1^0, x_2^0, ..., x_n^0]^T$, there exists a unique solution $X(t)$ on the whole interval I to the initial value problem

$$X'(t) = A(t)X(t) + F(t), \quad X(t_0) = X^0. \tag{0.4}$$

If we rewrite system (0.4) as $X' - AX = F$ and define the operator $L[X] = X' - AX$, then we can express system (0.4) in the operator form $L[X] = F$. Moreover, L is a linear operator and so

Any linear combination of solutions of the homogeneous system $X' = AX$ is again a solution of $X' = AX$.

That is to say, if $X_1(t), X_2(t), ..., X_k(t)$ are solutions of $X' = AX$, then $c_1X_1(t) + c_2X_2(t) + \cdots + c_kX_k(t)$ is again a solution for any choice of constants $c_1, c_2, ...c_k$.

Linear Independence and the Wronskian

Definition 1.1

The m vector functions $X_1, X_2, ..., X_m$ are said to be **linearly dependent on an interval** I if there exist constants $c_1, c_2, ..., c_m$, not all zero, such that

$$c_1X_1(t) + c_2X_2(t) + \cdots + c_mX_m(t) = 0$$

for all t in I. If the vectors are not linearly dependent, they are said to be **linearly independent on** I.

5.1.2 GENERAL SOLUTION OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

Definition 1.2

The **Wronskian** of n vector functions

$$X_1(t) = [x_{11}(t), ..., x_{n1}(t)]^T, ..., X_n(t) = [x_{1n}(t), ..., x_{nn}(t)]^T$$

is defined to be the *real* valued function

$$W[X_1, ..., X_n](t) := \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}.$$

5.1.2 GENERAL SOLUTION OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

Some properties of the Wronskian:

(a) *The Wronskian of n solutions of $X'(t) = A(t)X(t)$ is either identically zero or never zero on I.*

(b) *A set of n solutions $X_1, X_2, ..., X_n$ of $X'(t) = A(t)X(t)$ on I is independent on I if and only if their Wronskian is never zero on I.*

5.1.2 GENERAL SOLUTION OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

Theorem 1.2 (**Representation of Solutions** (**Homogeneous Case**))

Let $X_1, X_2, ..., X_n$ be linearly independent solutions to the homogeneous system

$$X'(t) = A(t)X(t) \tag{0.5}$$

on the interval I, where $A(t)$ is an $n \times n$ matrix function continuous on I. Then every solution to (0.5) on I can be expressed in the form

$$c_1X_1(t) + c_2X_2(t) + \cdots + c_nX_n(t), \tag{0.6}$$

where $c_1, c_2, ..., c_n$ are constants.

The set of solutions $\{X_1, X_2, ..., X_n\}$ that are linearly independent on I is called a **fundamental set of solutions** for (0.5) on I. The linear combination in (0.6) is referred to as a **general solution** of (0.5).

5.1.2 GENERAL SOLUTION OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

Since the operator $L[X] := X' - AX$ is linear, we have the **Superposition Principle**:

If X_1 and X_2 are solutions, respectively, to the nonhomogeneous systems

$$L[X] = F_1 \quad \text{and} \quad L[X] = F_2,$$

then $c_1 X_1 + c_1 X_2$ is a solution to

$$L[X] = c_1 F_1 + c_2 F_2.$$

5.1.2 GENERAL SOLUTION OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

Theorem 1.3 (**Representation of Solutions (Nonhomogeneous Case)**)

Let X_p be a particular solution to the nonhomogeneous system

$$X'(t) = A(t)X(t) + F(t) \tag{0.7}$$

on the interval I, and let $\{X_1, X_2, .., X_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $X'(t) = A(t)X(t)$. Then every solution to (0.7) on I can be expressed in the form

$$c_1X_1(t) + c_2X_2(t) + \cdots + c_nX_n(t) + X_p(t), \tag{0.8}$$

where $c_1, c_2, ..., c_n$ are constants.

The linear combination of $X_1, X_2, .., X_n, X_p$ in (0.8) written with arbitrary constants $c_1, c_2, ..., c_n$ is called a **general solution** of (0.7).

5.1.2 GENERAL SOLUTION OF SYSTEMS OF FIRST ORDER LINEAR EQUATIONS

Consider a first-order linear homogeneous differential system with constant coefficients

$$X' = AX, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \qquad (0.9)$$

Our goal is to find n linearly independent solutions $X_1(t), ..., X_n(t)$.

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Our goal is to find n linearly independent solutions $X_1(t), ..., X_n(t)$.

We will try

$$X(t) = e^{rt} C,$$

where $C \neq 0$ is a constant vector, as a solution of (0.9).

5.2 HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

Observe that

$$\frac{d}{dt}e^{rt}C = re^{rt}C \quad \text{and} \quad A(e^{rt}C) = e^{rt}AC.$$

Hence $X(t) = e^{rt}C$ is a solution of $X' = AX$ if and only if $rC = AC$ or, equivalently,

$$(A - rI)C = O. \tag{0.10}$$

Since $C \neq \mathbf{0}$,

$$|A - rI| = 0. \tag{0.11}$$

Equation (0.11) is called the **characteristic equation** of the matrix A.

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Since $C \neq \mathbf{0}$,

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Equation (0.11) is called the **characteristic equation** of the matrix A.

The roots of the characteristic equation of A are called **eigenvalues** of the matrix A. A nonzero vector C, which is a solution of Equation (0.10), is called an **eigenvector** of the matrix A corresponding to the eigenvalue r.

5.2 HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

If A has *n linearly independent eigenvectors* $V_1, V_2, ..., V_n$ with eigenvalues $r_1, r_2, ..., r_n$ respectively ($r_1, r_2, ..., r_n$ need not be distinct), then

$$X_i(t) = e^{r_i t} V_i, \qquad i = 1, 2, ..., n$$

are n linearly independent solutions of $X' = AX$ and every solution $X(t)$ of $X' = AX$ is of the form

$$X(t) = c_1 e^{r_1 t} V_1 + c_2 e^{r_2 t} V_2 + \cdots + c_n e^{r_n t} V_n.$$

The situation is simplest when A has n distinct real eigenvalues $r_1, r_2, ..., r_n$ with eigenvectors $V_1, V_2, ..., V_n$ respectively, for in this case $V_1, V_2, ..., V_n$ are linearly independent.

Theorem 2.1

If $V_1, V_2, ..., V_n$ are n eigenvectors of A corresponding to n distinct eigenvalues $r_1, r_2, ..., r_n$ respectively, then the general solution of $X' = AX$ is

$$X(t) = c_1 e^{r_1 t} V_1 + c_2 e^{r_2 t} V_2 + \cdots + c_n e^{r_n t} V_n.$$

Example 2.1 Solve the linear system of differential equations

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

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Solution: The eigenvalues of the matrix $A := \begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix}$ are roots of the characteristic equation

$$\det(\lambda I_2 - A) = \det \begin{pmatrix} \lambda + 4 & 3 \\ -2 & \lambda - 3 \end{pmatrix} = 0 \Leftrightarrow (\lambda + 4)(\lambda - 3) + 6 = 0.$$

This gives $\lambda_1 = 2, \lambda_2 = -3.$

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This gives $\lambda_1 = 2, \lambda_2 = -3.$
We now find eigenvectors associated with $\lambda_1 = 2, \lambda_2 = -3.$
Eigenvectors associated with $\lambda_1 = 2$ are solutions of the linear system

$$\begin{pmatrix} 6 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Leftrightarrow 6a + 3b = 0.$$

So an eigenvector associated with $\lambda_1 = 2$ is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Eigenvectors associated with $\lambda_2 = -3$ are solutions of the linear system

$$\begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Leftrightarrow a + 3b = 0.$$

So an eigenvector associated with $\lambda_2 = -3$ is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Thus two linearly independent solutions of the given system are

$$e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad and \qquad e^{-3t} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

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$$e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \textit{and} \qquad e^{-3t} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Finally, the general solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

Example 2.2 Solve the initial value problem

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

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$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

SOLUTION: Let $X(t) := \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$. The given system can be rewritten as $X'(t) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} X(t)$.

The characteristic equation is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{vmatrix} = \lambda^2(2-\lambda) - 2 + \lambda = (\lambda^2 - 1)(2-\lambda)$$

Therefore the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$. For $\lambda_1 = -1$, the eigenvector equation is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_2 + v_3 \\ -2v_1 + v_2 + 3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The first two equations give $v_1 = -v_2$ and $v_3 = -v_2$. These two equations make the third equation redundant (the reader may check that). Choosing $v_2 = -1$ we get the eigenvector $[1, -1, 1]^T$, so that a solution becomes

$$X_1(t) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t}.$$

Likewise, for $\lambda_2 = 1$, we get the eigenvector conditions as $v_1 = v_2$ and $v_3 = v_2$ (the third equation being redundant), and setting $v_2 = 1$, we obtain another solution

$$X_2(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t}.$$

For $\lambda_3 = 2$, we get the eigenvector conditions as $v_2 = 2v_1$ and $v_3 = 2v_2 = 4v_1$. Setting $v_1 = 1$, we get another solution

$$X_3(t) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t}.$$

Thus the general solution is $X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t)$

or equivalently,
$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t}.$$

5.3 COMPLEX EIGENVALUES

If $r = \alpha + i\beta$ is a complex eigenvalue of A with complex eigenvector $V = V_1 + iV_2$, then $X(t) = e^{rt}V$ is a complex-valued solution of the differential equation

$$X' = AX.$$

This complex-valued solution gives two real-value solutions.

Theorem 3.1

Let $X(t) = Y(t) + iZ(t)$ be a complex-valued solution of $X' = AX$. Then both $Y(t)$ and $Z(t)$ are real-valued solutions of $X' = AX$.

5.3 COMPLEX EIGENVALUES

If $r = \alpha + i\beta$ is an eigenvalue of A with eigenvector $V = V_1 + iV_2$, then

$$Y(t) = e^{\alpha t}((\cos \beta t)V_1 - (\sin \beta t)V_2)$$

and

$$Z(t) = e^{\alpha t}((\sin \beta t)V_1 + (\cos \beta t)V_2)$$

are two real-valued solutions of $X' = AX$. Moreover, these two solutions must be linearly independent. Thus,

If the real matrix A has complex conjugate eigenvalues $r = \alpha \pm i\beta$ with corresponding eigenvectors $V = V_1 \pm iV_2$, then two linearly independent real vector solutions of $X' = AX$ are

$$e^{\alpha t}((\cos \beta t)V_1 - (\sin \beta t)V_2) \quad \text{and} \quad e^{\alpha t}((\sin \beta t)V_1 + (\cos \beta t)V_2)$$

Example 3.1 Solve

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Example 3.1 Solve

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution: The characteristic equation is

$$\det \begin{pmatrix} \lambda + \frac{1}{2} & -1 \\ 1 & \lambda + \frac{1}{2} \end{pmatrix} = 0,$$

therefore the eigenvalues are $\lambda_1 = -\frac{1}{2} + i, \lambda_2 = -\frac{1}{2} - i$. We find an eigenvector for $-\frac{1}{2} + i$ by solving the system:

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow ai - b = 0.$$

Choose $a = 1, b = i$. Then a complex solution of the given system is

$$e^{(-\frac{1}{2}+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t/2}(\cos t + i \sin t) = \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + i \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}.$$

Hence

$$\mathbf{u}(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \mathbf{v}(t) = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \tag{18}$$

is a set of real-valued solutions. To verify that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, we compute their Wronskian:

$$W(\mathbf{u}, \mathbf{v})(t) = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix}$$
$$= e^{-t}(\cos^2 t + \sin^2 t) = e^{-t}.$$

Therefore the general solution is:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) = c_1 \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2}\sin t \\ -e^{-t/2}\cos t \end{pmatrix}$$

Example 3.1 Solve

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Example 3.1 Solve

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution: Let $A := \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$. We find eigenvalues of A:

$$\det \begin{pmatrix} \lambda - 3 & 5 \\ -1 & \lambda + 1 \end{pmatrix} = 0.$$

This is equivalent to

$$\lambda^2 - 2\lambda + 2 = 0.$$

Thus

$$\lambda = 1 + i, 1 - i.$$

Eigenvectors:

$$\begin{pmatrix} -2+i & 5 \\ -1 & 2+i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -v_1 + (2+i)v_2 = 0.$$

We choose $v_1 = 2+i$, $v_2 = 1$.

$$\boxed{\begin{aligned} e^{\lambda t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} &= e^{(1+i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^{t}(\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^{t}\begin{pmatrix} (2+i)(\cos t + i\sin t) \\ \cos t + i\sin t \end{pmatrix} \\ &= e^{t}\begin{pmatrix} (2\cos t - \sin t) + i(\cos t + 2\sin t) \\ \cos t + i\sin t \end{pmatrix} \\ &= e^{t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^{t}\begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}. \end{aligned}}$$

The general solution:

$$\boxed{\begin{aligned}\mathbf{Y}(t) &= c_1 e^t \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \\ &= e^t \begin{pmatrix} c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ c_1 \cos t + c_2 \sin t \end{pmatrix}.\end{aligned}}$$

5.4 REPEATED EIGENVALUES

Example 4.1 Solve the system

$$X' = AX \qquad \text{for} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

When an $n \times n$ matrix A has a repeated eigenvalue r of multiplicity m, then *it is possible that* A do not have n linearly independent eigenvectors. However, we have the following

Remark: *If V is an eigenvector corresponding to the eigenvalue r of an $n \times n$ matrix A, then $X(t) = te^{rt}V + e^{rt}C$ is a solution of $X' = AX$ if and only if*

$$\boxed{(A - rI)C = V}$$

Example 4.2 Find the general solution of

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Example 4.2 Find the general solution of

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution: The characteristic equation is

$$\det \begin{pmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 3 \end{pmatrix} = 0,$$

therefore the eigenvalues are $\lambda_1 = 2, \lambda_2 = 2$. We find an eigenvector for 2 by solving the system:

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Leftrightarrow a + b = 0.$$

Thus, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvetor (or any non-zero multiple of this vector).

Then one solution of the given system is $\mathbf{u}(t) = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Based on the procedure used for second order linear equations, it may be natural to attempt to find a second solution of the system of the form

$$\mathbf{v}(t) = te^{2t} \begin{pmatrix} c \\ d \end{pmatrix} \qquad (*).$$

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$$\mathbf{v}(t) = te^{2t}\begin{pmatrix} c \\ d \end{pmatrix} \qquad (*).$$

Substituting $\mathbf{v}(t)$ into the given system gives

$$2te^{2t}\begin{pmatrix} c \\ d \end{pmatrix} + e^{2t}\begin{pmatrix} c \\ d \end{pmatrix} - te^{2t}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $c = d = 0$. Hence there is no nonzero solution of given system of the form $(*)$.

We seek the second solution of the given system of the form

$$\mathbf{v}(t) = t\mathbf{u}(t) + e^{2t}\begin{pmatrix} c \\ d \end{pmatrix}$$

We seek the second solution of the given system of the form

$$\mathbf{v}(t) = t\mathbf{u}(t) + e^{2t}\begin{pmatrix} c \\ d \end{pmatrix} = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} c \\ d \end{pmatrix}.$$

Here c and d satisfy:

$$\left(\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This gives $-c-d=1$. Choosing $c=0, d=-1$, we get the second solution of the given system:

$$\mathbf{v}(t) = te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Finally, the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right].$$

5.4 REPEATED EIGENVALUES

Example 4.3 Solve the system

$$\begin{aligned}
\frac{dx}{dt} &= -4x + 2y + 5z \\
\frac{dy}{dt} &= 6x - y - 6z \\
\frac{dz}{dt} &= -8x + 3y + 9z
\end{aligned}$$

Exercises and Assignments

Pages	Exercises	Assignments
398-401	12, 15	13, 17, 19, 20, 29
410-415	7, 10	3, 6, 8, 10, 18, 25
428-431	4, 5, 10	2, 6, 7, 12, 15