P5.1. (a) $\nabla \cdot \nabla \times \mathbf{A}$

$$= \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= 0$$

(b) $\nabla \times \nabla\Phi$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} & \frac{\partial \Phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{0}$$

P5.2. (a) $\nabla \bullet (xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z) = y + z + x \neq 0$

$\nabla \times (xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z)$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$= -y\mathbf{a}_x - z\mathbf{a}_y - x\mathbf{a}_z$

$\neq \mathbf{0}$

$\therefore$ cannot be expressed as the curl of another vector or the gradient of a scalar.

(b) $\nabla \bullet \frac{1}{r^2}(\cos\phi\ \mathbf{a}_r + \sin\phi\ \mathbf{a}_\phi)$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\cos\phi}{r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{\sin\phi}{r^2}\right)$$

$$= -\frac{1}{r^3}\cos\phi + \frac{1}{r^3}\cos\phi$$

$= 0$

$\nabla \times \frac{1}{r^2}(\cos\phi\ \mathbf{a}_r + \sin\phi\ \mathbf{a}_\phi)$

$$= \begin{vmatrix} \frac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \frac{\mathbf{a}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{\cos\phi}{r^2} & \frac{\sin\phi}{r} & 0 \end{vmatrix}$$

$$= \frac{\mathbf{a}_z}{r}\left(-\frac{\sin\phi}{r^2} + \frac{\sin\phi}{r^2}\right)$$

$= \mathbf{0}$

$\therefore$ can be expressed as the curl of another vector and also as the gradient of a scalar.

P5.2. (continued)

(c) $$\nabla \bullet \frac{\sin\theta}{r}\,\mathbf{a}_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\sin\theta}{r}\right) = 0$$

$$\nabla \times \frac{\sin\theta}{r}\mathbf{a}_\phi$$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2\sin\theta} & \dfrac{\mathbf{a}_\theta}{r\sin\theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ 0 & 0 & \sin^2\theta \end{vmatrix}$$

$$= \frac{2}{r^2}\cos\theta\,\mathbf{a}_r$$

$$\neq \mathbf{0}$$

$\therefore$ can be expressed as the curl of another vector but not as the gradient of a scalar.

P5.3. (a) $e^{-y}(\cos x\,\mathbf{a}_x - \sin x\,\mathbf{a}_y) \bullet (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z)$

$$= e^{-y}\cos x\,dx - e^{-y}\sin x\,dy$$

$$= d(e^{-y}\sin x)$$

$$\Phi(x, y, z) = e^{-y}\sin x$$

(b) $(\cos\phi\,\mathbf{a}_r - \sin\phi\,\mathbf{a}_\phi) \bullet (dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z)$

$$= \cos\phi\,dr - r\sin\phi\,d\phi$$

$$= d(r\cos\phi)$$

$$\Phi(r, \phi, z) = r\cos\phi$$

(c) $\frac{1}{r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta) \bullet (dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi)$

$$= \frac{2\cos\theta}{r^3}dr + \frac{\sin\theta}{r^2}d\theta$$

$$= d\left(-\frac{\cos\theta}{r^2}\right)$$

$$\Phi(r, \theta, \phi) = -\frac{\cos\theta}{r^2}$$

P5.4. The unit vector along the line of intersection of two planes is perpendicular to the normal vectors to the two planes. For the planes

$$a_1x + a_2\,y + a_3z = c_1$$

$$b_1x + b_2\,y + b_3z = c_2$$

the normal vectors are $(a_1\mathbf{a}_x + a_2\mathbf{a}_y + a_3\mathbf{a}_z)$ and $(b_1\mathbf{a}_x + b_2\mathbf{a}_y + b_3\mathbf{a}_z)$, respectively. Thus the required unit vector is

$$\pm\frac{(a_1\mathbf{a}_x + a_2\mathbf{a}_y + a_3\mathbf{a}_z)\mathbf{x}(b_1\mathbf{a}_x + b_2\mathbf{a}_y + b_3\mathbf{a}_z)}{\left|(a_1\mathbf{a}_x + a_2\mathbf{a}_y + a_3\mathbf{a}_z)\mathbf{x}(b_1\mathbf{a}_x + b_2\mathbf{a}_y + b_3\mathbf{a}_z)\right|}$$

$$=\pm\frac{(a_2b_3 - a_3b_2)\mathbf{a}_x + (a_3b_1 - a_1b_3)\mathbf{a}_y + (a_1b_2 - a_2b_1)\mathbf{a}_z}{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}$$

For the planes $x + y + z = 3$ and $y = x$,

$$a_1 = a_2 = a_3 = 1,\ b_1 = 1,\ b_2 = -1,\ b_3 = 0.$$

The unit vector along their intersection is

$$\pm\frac{\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z}{\sqrt{6}}$$

P5.5. Let the equation of the plane be

$$Ax + By + Cz = D$$

Then since the plane is normal to the vector $(a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z)$,

$$(A\mathbf{a}_x + B\mathbf{a}_y + C\mathbf{a}_z) \times (a\mathbf{a}_x + b\mathbf{a}_y + c\mathbf{a}_z) = 0$$

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = m$$

$\therefore$ The equation is

$$ax + by + cz = \frac{D}{m} = D'$$

Since the plane passes through the point (x_0, y_0, z_0),

$$ax_0 + by_0 + cz_0 = D'$$

$$\therefore\ ax + by + cz = ax_0 + by_0 + cz_0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

For the surface $xyz = 1$, a normal vector is

$$\nabla(xyz) = yz\mathbf{a}_x + zx\mathbf{a}_y + xy\mathbf{a}_z$$

At $\left(\frac{1}{2}, \frac{1}{4}, 8\right)$, this vector is $2\mathbf{a}_x + 4\mathbf{a}_y + \frac{1}{8}\mathbf{a}_z$.

Thus the required equation is

$$2\left(x - \frac{1}{2}\right) + 4\left(y - \frac{1}{4}\right) + \frac{1}{8}(z - 8) = 0$$

$$2x + 4y + \frac{1}{8}z = 3$$

$$16x + 32y + z = 24$$

P5.6. $\nabla^2\mathbf{A} = \nabla(\nabla \bullet \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$

$$[\nabla(\nabla \bullet \mathbf{A})]_r = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\right]$$

$$= \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r}\frac{\partial A_r}{\partial r} - \frac{A_r}{r^2} - \frac{1}{r^2}\frac{\partial A_\phi}{\partial \phi} + \frac{1}{r}\frac{\partial^2 A_\phi}{\partial r\,\partial \phi} + \frac{\partial^2 A_z}{\partial r\,\partial z}$$

$$[\nabla \times \nabla \times \mathbf{A}]_r = \frac{1}{r}\left\{\frac{\partial}{\partial \phi}[\nabla \times \mathbf{A}]_z - \frac{\partial}{\partial z} r[\nabla \times \mathbf{A}]_\phi\right\}$$

$$= \frac{1}{r}\frac{\partial}{\partial \phi}\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]\right\} - \frac{\partial}{\partial z}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)$$

$$= \frac{1}{r^2}\frac{\partial A_\phi}{\partial \phi} + \frac{1}{r}\frac{\partial^2 A_\phi}{\partial \phi\,\partial r} - \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \phi^2} - \frac{\partial^2 A_r}{\partial z^2} + \frac{\partial^2 A_z}{\partial z\,\partial r}$$

$$[\nabla^2\mathbf{A}]_r = [\nabla(\nabla \bullet \mathbf{A})]_r - [\nabla \times \nabla \times \mathbf{A}]_r$$

$$= \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r}\frac{\partial A_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \phi^2} + \frac{\partial^2 A_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2}$$

$$= \nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi}$$

$$[\nabla(\nabla \bullet \mathbf{A})]_\phi = \frac{1}{r}\frac{\partial}{\partial \phi}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\right]$$

$$= \frac{1}{r}\frac{\partial^2 A_r}{\partial r\,\partial \phi} + \frac{1}{r^2}\frac{\partial A_r}{\partial \phi} + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial \phi^2} + \frac{1}{r}\frac{\partial^2 A_z}{\partial \phi\,\partial z}$$

$$[\nabla \times \nabla \times \mathbf{A}]_\phi = \frac{\partial}{\partial z}[\nabla \times \mathbf{A}]_r - \frac{\partial}{\partial r}[\nabla \times \mathbf{A}]_z$$

$$= \frac{\partial}{\partial z}\left\{\frac{1}{r}\left[\frac{\partial A_z}{\partial \phi} - \frac{\partial}{\partial z}(rA_\phi)\right]\right\} - \frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]\right\}$$

$$= \frac{1}{r}\frac{\partial^2 A_z}{\partial \phi\,\partial z} - \frac{\partial^2 A_\phi}{\partial z^2} - \frac{\partial^2 A_\phi}{\partial r^2} + \frac{A_\phi}{r^2} - \frac{1}{r}\frac{\partial A_\phi}{\partial r} + \frac{1}{r}\frac{\partial^2 A_r}{\partial r\,\partial \phi} - \frac{1}{r^2}\frac{\partial A_r}{\partial \phi}$$

P5.6.. (continued)

$$[\nabla^2 \mathbf{A}]_\phi = [\nabla(\nabla \bullet \mathbf{A}]_\phi - [\nabla \times \nabla \times \mathbf{A}]_\phi$$

$$= \left(\frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial A_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial \phi^2} + \frac{\partial^2 A_\phi}{\partial z^2} \right) - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi}$$

$$= \nabla^2 A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi}$$

$$[\nabla(\nabla \bullet \mathbf{A}]_z = \frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \right]$$

$$= \frac{\partial^2 A_r}{\partial r\, \partial z} + \frac{1}{r}\frac{\partial A_r}{\partial z} + \frac{1}{r}\frac{\partial^2 A_\phi}{\partial \phi\, \partial z} + \frac{\partial^2 A_z}{\partial z^2}$$

$$[\nabla \times \nabla \times \mathbf{A}]_z = \frac{1}{r}\left\{ \frac{\partial}{\partial r} r[\nabla \times \mathbf{A}]_\phi - \frac{\partial}{\partial \phi}[\nabla \times \mathbf{A}]_r \right\}$$

$$= \frac{1}{r}\left\{ \frac{\partial}{\partial r}\left[r\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \right] - \frac{\partial}{\partial \phi}\frac{1}{r}\left[\frac{\partial A_z}{\partial \phi} - \frac{\partial}{\partial z}(rA_\phi) \right] \right\}$$

$$= \frac{\partial^2 A_r}{\partial r\, \partial z} - \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_r}{\partial z} - \frac{1}{r}\frac{\partial A_z}{\partial r} - \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{1}{r}\frac{\partial^2 A_\phi}{\partial \phi\, \partial z}$$

$$[\nabla^2 \mathbf{A}]_z = [\nabla(\nabla \bullet \mathbf{A}]_z - [\nabla \times \nabla \times \mathbf{A}]_z$$

$$= \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2}$$

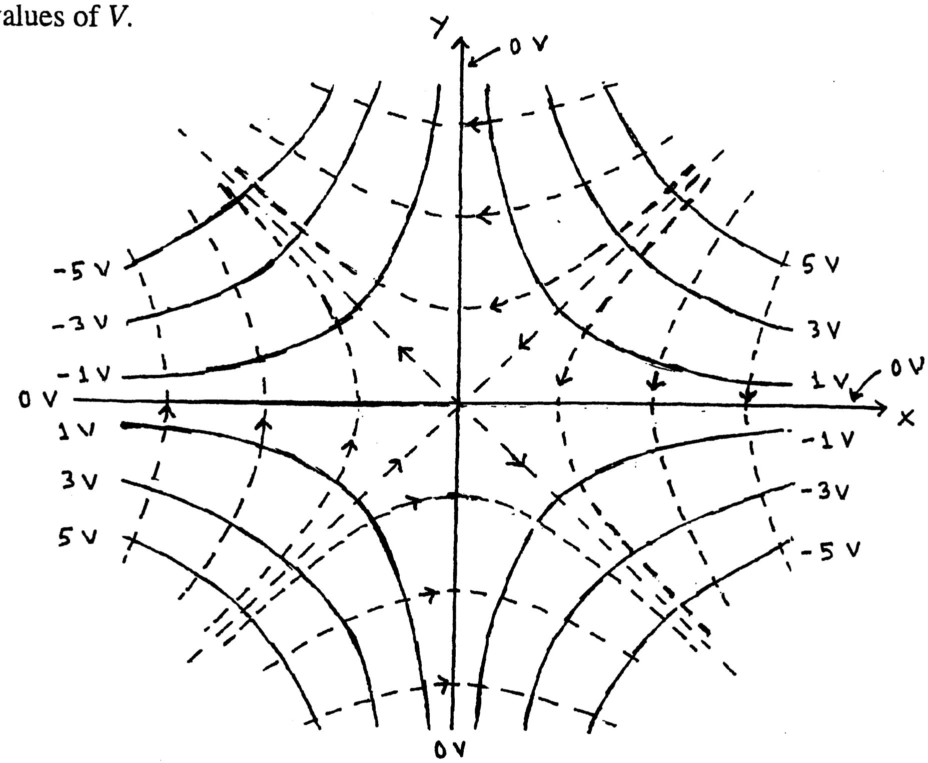
$$= \nabla^2 A_z$$

Thus

$$\nabla^2 \mathbf{A} = \left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} \right)\mathbf{a}_r + \left(\nabla^2 A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} \right)\mathbf{a}_\phi + (\nabla^2 A_z)\,\mathbf{a}_z$$

P5.7. $V(x, y) = xy$

The potential field is two-dimensional in x and y, and independent of z. Hence, it is sufficient to plot the cross sections of the equipotential surfaces in a constant–z plane. These are given by xy = constant. Thus the x and y axes correspond to zero potential. A few equipotential lines for nonzero values of V are shown by the solid curves in the figure. It can be seen that the first and third quadrants are characterized by positive values of V, whereas the second and fourth quadrants are characterized by negative values of V.



The direction lines of the electric field can now be sketched as shown by the dashed curves in the figure, by recognizing that they must be everywhere perpendicular to the equipotential lines. To do this formally, we find that

$$\mathbf{E} = -\nabla V = -\nabla(xy)$$

$$= -y\mathbf{a}_x - x\mathbf{a}_y$$

and the equation for the direction lines is given by

$$\frac{dx}{-y} = \frac{dy}{-x}$$

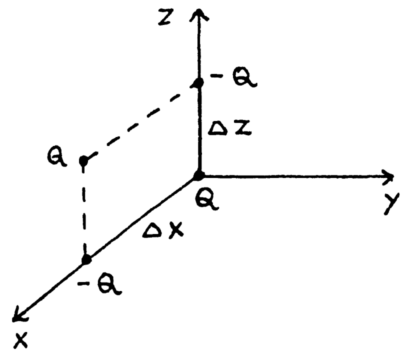
$$x\,dx - y\,dy = 0$$

$$x^2 - y^2 = \text{constant}$$

which corresponds to hyperbolas, as shown by the dashed curves.

P5.8.

$$V = \frac{Q}{4\pi\varepsilon_0\sqrt{x^2+y^2+z^2}}$$
$$-\frac{Q}{4\pi\varepsilon_0\sqrt{(x-\Delta x)^2+y^2+z^2}}$$
$$+\frac{Q}{4\pi\varepsilon_0\sqrt{(x-\Delta x)^2+y^2+(z-\Delta z)^2}}$$
$$-\frac{Q}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-\Delta z)^2}}$$



$$= \frac{Q}{4\pi\varepsilon_0 r}\left[1-\left\{1-\left[\frac{2\,\Delta x}{r}\sin\theta\cos\phi-\left(\frac{\Delta x}{r}\right)^2\right]\right\}^{-1/2}\right.$$

$$-\left\{1-\left[\frac{2\,\Delta z}{r}\cos\theta-\left(\frac{\Delta z}{r}\right)^2\right]\right\}^{-1/2}$$

$$\left.+\left\{1-\left[\frac{2\,\Delta x}{r}\sin\theta\cos\phi+\frac{2\,\Delta z}{r}\cos\theta-\left(\frac{\Delta x}{r}\right)^2-\left(\frac{\Delta z}{r}\right)^2\right]\right\}^{-1/2}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0 r}\left\{1-\left(1+\frac{1}{2}a+\frac{3}{8}a^2+\ldots\right)-\left(1+\frac{1}{2}b+\frac{3}{8}b^2+\ldots\right)\right.$$

$$\left.+\left[1+\frac{1}{2}(a+b)+\frac{3}{8}(a+b)^2+\ldots\right]\right\}$$

where $a = \left[\frac{2\,\Delta x}{r}\sin\theta\cos\phi-\left(\frac{\Delta x}{r}\right)^2\right]$ and $b = \left[\frac{2\,\Delta z}{r}\cos\theta-\left(\frac{\Delta z}{r}\right)^2\right]$.

Continuing, we have

$$V \approx \frac{Q}{4\pi\varepsilon_0 r}\frac{3}{4}ab$$

$$= \frac{3Q}{16\pi\varepsilon_0 r}\left[\frac{2\,\Delta x}{r}\sin\theta\cos\phi-\left(\frac{\Delta x}{r}\right)^2\right]\left[\frac{2\,\Delta z}{r}\cos\theta-\left(\frac{\Delta z}{r}\right)^2\right]$$

$$\approx \frac{3Q\,\Delta x\,\Delta z}{4\pi\varepsilon_0 r^3}\sin\theta\cos\theta\cos\phi$$

P5.8. (continued)

$$\mathbf{E} = -\nabla V$$

$$= -\left(\frac{\partial V}{\partial r}\,\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi\right)$$

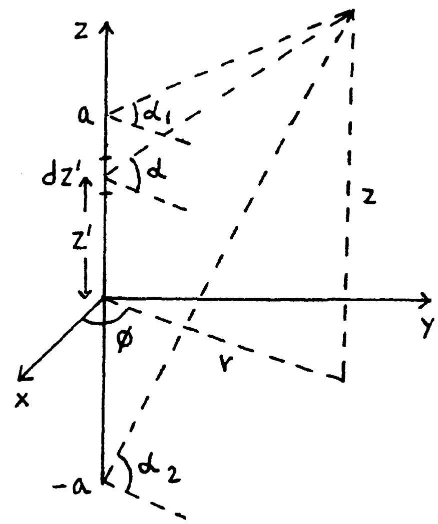
$$= -\frac{3Q\,\Delta x\,\Delta z}{4\pi\varepsilon_0}\left(-\frac{3}{r^4}\sin\theta\cos\theta\cos\phi\,\mathbf{a}_r\right.$$

$$+\frac{1}{r^4}\cos 2\theta\cos\phi\,\mathbf{a}_\theta$$

$$\left.+\frac{1}{r^4}\cos\theta\sin\phi\,\mathbf{a}_\phi\right)$$

$$= \frac{3Q\,\Delta x\,\Delta z}{4\pi\varepsilon_0 r^4}(3\sin\theta\cos\theta\cos\phi\,\mathbf{a}_r - \cos 2\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\sin\phi\,\mathbf{a}_\phi)$$

P5.9. Considering a differential length dz' at $(0, 0, z')$, we have



$$dV = \frac{\rho_{L0}\, dz'}{4\pi\varepsilon\sqrt{r^2 + (z - z')^2}}$$

Hence,

$$V = \int_{z'=-a}^{a} \frac{\rho_{L0}\, dz'}{4\pi\varepsilon\sqrt{r^2 + (z - z')^2}}$$

Making the change of variable $(z - z') = r \tan \alpha$,

$dz' = -r \sec^2 \alpha \, d\alpha$, we have

$$V = \int_{\alpha=\alpha_2}^{\alpha_1} \frac{-\rho_{L0}\, r \sec^2 \alpha \, d\alpha}{4\pi\varepsilon r \sec \alpha}$$

$$= -\frac{\rho_{L0}}{4\pi\varepsilon}\int_{\alpha=\alpha_2}^{\alpha_1} \sec \alpha \, d\alpha$$

$$= -\frac{\rho_{L0}}{4\pi\varepsilon}\left[\ln(\sec\alpha + \tan\alpha)\right]_{\alpha_2}^{\alpha_1}$$

$$= \frac{\rho_{L0}}{4\pi\varepsilon}\ln\frac{(\sec\alpha_2 + \tan\alpha_2)}{(\sec\alpha_1 + \tan\alpha_1)}$$

$$= \frac{\rho_{L0}}{4\pi\varepsilon}\ln\frac{\sqrt{r^2 + (z+a)^2} + (z+a)}{\sqrt{r^2 + (z-a)^2} + (z-a)}$$

Equipotential surfaces are given by

$$\frac{\sqrt{r^2 + (z+a)^2} + (z+a)}{\sqrt{r^2 + (z-a)^2} + (z-a)} = \text{constant, say } c$$

Then

$$\sqrt{r^2 + (z+a)^2} - c\sqrt{r^2 + (z-a)^2} = c(z-a) - (z+a)$$

Squaring both sides and simplifying, rearranging, and then squaring both sides and simplifying, we get

P5.9. (continued)

$$(c^2 - 1)^2 r^2 + 4c(c - 1)^2 z^2 - 4c(c + 1)^2 a^2 = 0$$

$$\frac{(c-1)^2}{4c}\left(\frac{r}{a}\right)^2 + \frac{(c-1)^2}{(c+1)^2}\left(\frac{z}{a}\right)^2 = 1$$

This is the equation of an ellipse in the ϕ = constant plane, having semimajor axis $\frac{c+1}{c-1}\,a$ along the z-direction and semiminor axis $\frac{2\sqrt{c}}{c-1}\,a$ along the r-direction. Distance from center to either focus is

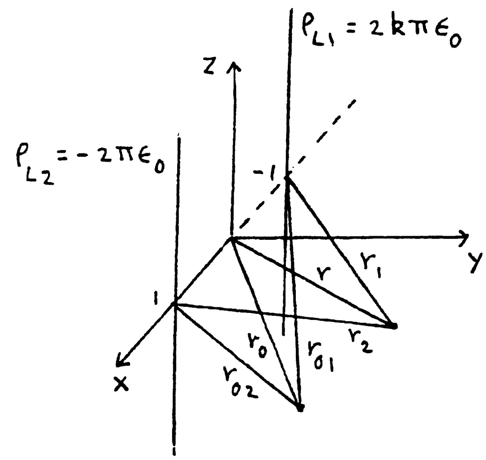
$$\sqrt{\left(\frac{c+1}{c-1}a\right)^2 - \left(\frac{2\sqrt{c}}{c-1}a\right)^2} = a$$

Thus the equipotential surfaces are ellipsoids with the ends of the line as their focii.

P5.10. From Example 5.4, for a single infinitely-long line charge of uniform density ρ_{L0} along the z-axis,

$$V = -\frac{\rho_{L0}}{2\pi\varepsilon_0} \ln \frac{r}{r_0}$$

For two line charges,



$$V = -\frac{2k\pi\varepsilon_0}{2\pi\varepsilon_0} \ln \frac{r_1}{r_{01}}$$

$$+\frac{2\pi\varepsilon_0}{2\pi\varepsilon_0} \ln \frac{r_2}{r_{02}}$$

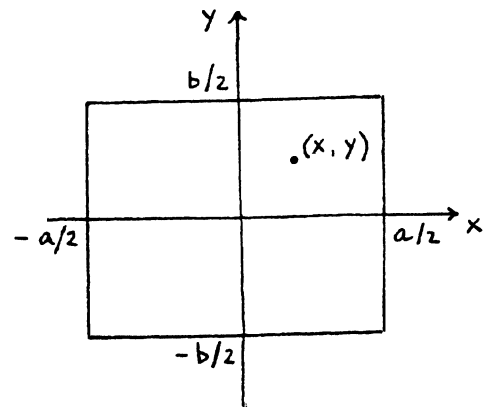
$$= \ln\left[\frac{r_2}{r_{02}}\left(\frac{r_{01}}{r_1}\right)^k\right]$$

$$= \ln\left[\frac{r_2}{r_1^k}\frac{r_{01}^k}{r_{02}}\right]$$

Choosing r_{01}^k / r_{02} to be unity, we have

$$V = \ln \frac{r_2}{r_1^k}$$

P5.11. With reference to the figure, V at center



$$= \frac{4\rho_{S0}}{4\pi\varepsilon_0}\int_{x=0}^{a/2}\int_{y=0}^{b/2}\frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \frac{\rho_{S0}}{\pi\varepsilon_0}\int_{x=0}^{a/2}\left[\ln\left(y+\sqrt{x^2+y^2}\right)\right]_0^{b/2}dx$$

$$= \frac{\rho_{S0}}{\pi\varepsilon_0}\int_{x=0}^{a/2}\ln\left(\frac{b/2+\sqrt{x^2+b^2/4}}{x}\right)dx$$

$$= \frac{\rho_{S0}}{\pi\varepsilon_0}\frac{b}{2}\int_0^{\tan^{-1}a/b}\ln\frac{\frac{b}{2}+\frac{b}{2}\sec\theta}{\frac{b}{2}\tan\theta}\sec^2\theta\,d\theta$$

$$= \frac{\rho_{S0}b}{2\pi\varepsilon_0}\int_0^{\tan^{-1}a/b}\ln\left(\cot\theta+\operatorname{cosec}\theta\right)\sec^2\theta\,d\theta$$

$$= \frac{\rho_{S0}b}{2\pi\varepsilon_0}\left[\tan\theta\ln\left(\cot\theta+\operatorname{cosec}\theta\right)+\ln\left(\sec\theta+\tan\theta\right)\right]_{\theta=0}^{\tan^{-1}a/b}$$

$$= \frac{\rho_{S0}b}{2\pi\varepsilon_0}\left[\frac{a}{b}\ln\left(\frac{b}{a}+\frac{\sqrt{a^2+b^2}}{a}\right)+\ln\left(\frac{\sqrt{a^2+b^2}}{b}+\frac{a}{b}\right)\right]$$

$$= \frac{\rho_{S0}}{2\pi\varepsilon_0}\left[a\ln\left(\frac{\sqrt{a^2+b^2}+b}{a}\right)+b\ln\left(\frac{\sqrt{a^2+b^2}+a}{b}\right)\right]$$

where we have used $x = \frac{b}{2}\tan\theta$, $dx = \frac{b}{2}\sec^2\theta\,d\theta$, and

$$\lim_{\theta\to 0}\tan\theta\ln\left(\cot\theta+\operatorname{cosec}\theta\right) = \lim_{\theta\to 0}\frac{\ln\left(\cot\theta+\operatorname{cosec}\theta\right)}{\cot\theta}$$

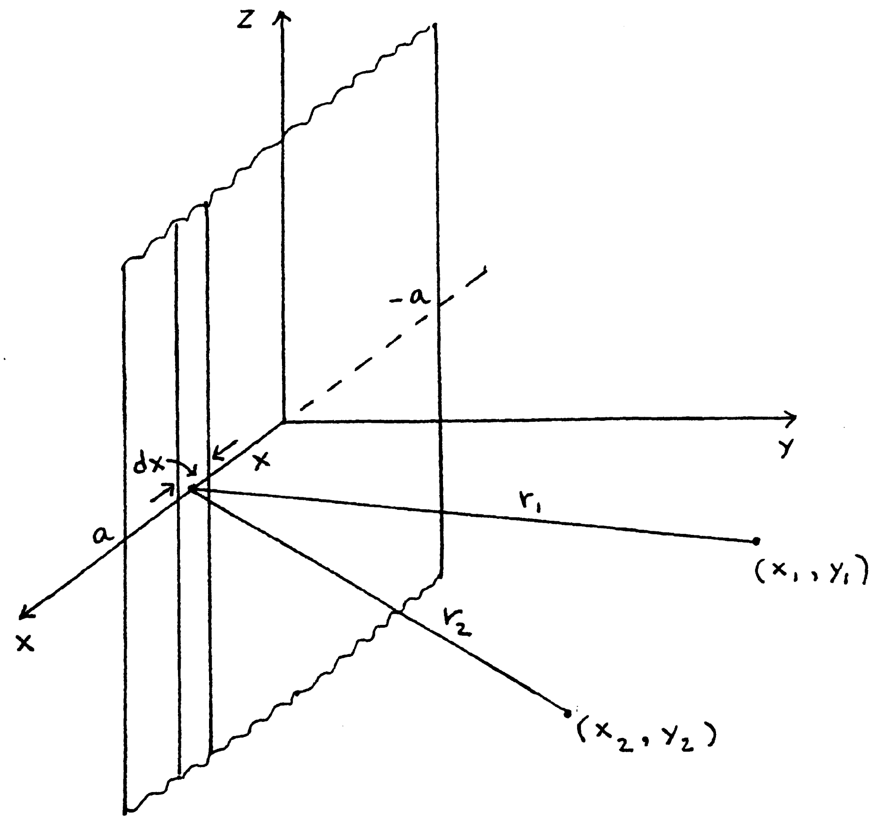
$$= \lim_{\theta\to 0}\frac{-\operatorname{cosec}^2\theta-\cot\theta\operatorname{cosec}\theta}{-\operatorname{cosec}^2\theta\left(\cot\theta+\operatorname{cosec}\theta\right)} = \lim_{\theta\to 0}\sin\theta = 0$$

For a square-shaped conductor of sides a, $b = a$, and V at center

$$= \frac{\rho_{S0}}{2\pi\varepsilon_0}\left[a\ln\left(\frac{\sqrt{a^2+a^2}+a}{a}\right)+a\ln\left(\frac{\sqrt{a^2+a^2}+a}{a}\right)\right]$$

$$= \frac{\rho_{S0}a}{\pi\varepsilon_0}\ln\left(1+\sqrt{2}\right)$$

P5.12.



$$
\begin{aligned}
V_1 - V_2 &= \int_{x=-a}^{a} -\frac{\rho_{S0}\,dx}{2\pi\varepsilon_0} \ln \frac{r_1}{r_2} \\
&= -\frac{\rho_{S0}}{2\pi\varepsilon_0} \int_{x=-a}^{a} \left[\ln \sqrt{(x_1 - x)^2 + y_1^2} - \ln \sqrt{(x_2 - x)^2 + y_2^2} \right] dx \\
&= -\frac{\rho_{S0}}{4\pi\varepsilon_0} \int_{x=-a}^{a} \left\{ \ln\left[(x - x_1)^2 + y_1^2\right] - \ln\left[(x - x_2)^2 + y_2^2\right] \right\} dx \\
&= -\frac{\rho_{S0}}{4\pi\varepsilon_0} \int_{x'=-a-x_1}^{a-x_1} \ln (x'^2 + y_1^2)\, dx' + \frac{\rho_{S0}}{4\pi\varepsilon_0} \int_{x'=-a-x_2}^{a-x_2} \ln (x'^2 + y_2^2)\, dx' \\
&= -\frac{\rho_{S0}}{4\pi\varepsilon_0} \left[x' \ln (x'^2 + y_1^2) - 2x' + 2y_1 \tan^{-1} \frac{x'}{y_1} \right]_{-a-x_1}^{a-x_1} \\
&\quad + \frac{\rho_{S0}}{4\pi\varepsilon_0} \left[x' \ln (x'^2 + y_2^2) - 2x' + 2y_2 \tan^{-1} \frac{x'}{y_2} \right]_{-a-x_2}^{a-x_2}
\end{aligned}
$$

P5.12. (continued)

$$= -\frac{\rho_{S0}}{4\pi\varepsilon_0}\left\{(a-x_1)\ln[(a-x_1)^2+y_1^2]\right.$$

$$+(a+x_1)\ln[(a+x_1)^2+y_1{}^2]-4a$$

$$\left.+2y_1\left[\tan^{-1}\left(\frac{a-x_1}{y_1}\right)+\tan^{-1}\left(\frac{a+x_1}{y_1}\right)\right]\right\}$$

$$+\frac{\rho_{S0}}{4\pi\varepsilon_0}\left\{(a-x_2)\ln[(a-x_2)^2+y_2^2]\right.$$

$$+(a+x_2)\ln[(a+x_2)^2+y_2{}^2]-4a$$

$$\left.+2y_2\left[\tan^{-1}\left(\frac{a-x_2}{y_2}\right)+\tan^{-1}\left(\frac{a+x_2}{y_2}\right)\right]\right\}$$

$$=\frac{\rho_{S0}}{4\pi\varepsilon_0}\left\{(a-x_2)\ln[(a-x_2)^2+y_2^2]\right.$$

$$+(a+x_2)\ln[(a+x_2)^2+y_2{}^2]$$

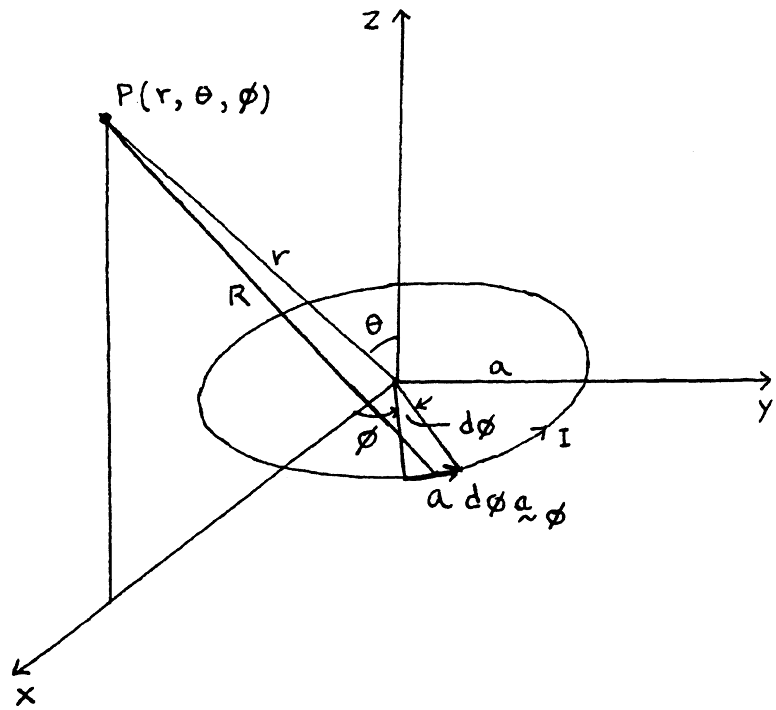
$$-(a-x_1)\ln[(a-x_1)^2+y_1{}^2]$$

$$-(a+x_1)\ln[(a+x_1)^2+y_1{}^2]$$

$$+2y_2\left[\tan^{-1}\left(\frac{a-x_2}{y_2}\right)+\tan^{-1}\left(\frac{a+x_2}{y_2}\right)\right]$$

$$\left.-2y_1\left[\tan^{-1}\left(\frac{a-x_1}{y_1}\right)+\tan^{-1}\left(\frac{a+x_1}{y_1}\right)\right]\right\}$$

P5.13.



From symmetry considerations, **A** is independent of ϕ. Hence it is sufficient to find **A** at $P(r, \theta, 0)$ and then generalize it. Thus

$$d\mathbf{A} = \frac{\mu I a\, d\phi\, \mathbf{a}_\phi}{4\pi R}$$

$$= \frac{\mu I a\, d\phi\, (-\sin\phi\, \mathbf{a}_x + \cos\phi\, \mathbf{a}_y)}{4\pi \sqrt{(r\sin\theta - a\cos\phi)^2 + (0 - a\sin\phi)^2 + (r\cos\theta - 0)^2}}$$

$$= \frac{\mu I a\, d\phi\, (-\sin\phi\, \mathbf{a}_x + \cos\phi\, \mathbf{a}_y)}{4\pi \sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi}}$$

$$= \frac{\mu I a\, d\phi\, (-\sin\phi\, \mathbf{a}_x + \cos\phi\, \mathbf{a}_y)}{4\pi r \left(1 + \frac{a^2}{r^2} - \frac{2a}{r}\sin\theta\cos\phi\right)^{1/2}}$$

$$\approx \frac{\mu I a\, d\phi}{4\pi r}\left(1 + \frac{a}{r}\sin\theta\cos\phi\right)(-\sin\phi\, \mathbf{a}_x + \cos\phi\, \mathbf{a}_y)$$

P5.13. (continued)

$$\mathbf{A} = \int_{\phi=0}^{2\pi} d\mathbf{A}$$

$$= \frac{\mu I a}{4\pi r}\left\{\left[\int_{\phi=0}^{2\pi}\left(-\sin\phi - \frac{a}{r}\sin\theta\sin\phi\cos\phi\right)d\phi\right]\mathbf{a}_x\right.$$

$$\left. + \left[\int_{\phi=0}^{2\pi}\left(\cos\phi + \frac{a}{r}\sin\theta\cos^2\phi\right)d\phi\right]\mathbf{a}_y\right\}$$

$$= \frac{\mu I a}{4\pi r}\frac{\pi a}{r}\sin\theta\,\mathbf{a}_y = \frac{\mu I \pi a^2}{4\pi r^2}\sin\theta\,\mathbf{a}_y$$

Generalizing, we have at an arbitrary point (r, θ, ϕ), where $r >> a$,

$$\mathbf{A} = \frac{\mu I \pi a^2}{4\pi r^2}\sin\theta\,\mathbf{a}_\phi = \frac{\mu \mathbf{m} \times \mathbf{a}_r}{4\pi r^2}$$

where $\mathbf{m} = I\pi a^2\mathbf{a}_z$ is the dipole moment of the loop. Proceeding further

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2\sin\theta} & \dfrac{\mathbf{a}_\theta}{r\sin\theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ 0 & 0 & \dfrac{\mu m\sin^2\theta}{4\pi r} \end{vmatrix}$$

$$= \frac{\mu m}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

P5.14.

$$\nabla \times (\Phi \mathbf{A}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \Phi A_x & \Phi A_y & \Phi A_z \end{vmatrix}$$

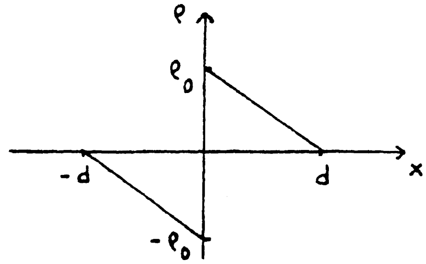
$$= \Phi \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} + \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial \Phi}{\partial x} & \dfrac{\partial \Phi}{\partial y} & \dfrac{\partial \Phi}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \Phi \nabla \times \mathbf{A} + \nabla \Phi \times \mathbf{A}$$

$$= \Phi \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \Phi$$

$$\mathbf{A} \times \nabla \Phi = \Phi \nabla \times \mathbf{A} - \nabla \times (\Phi \mathbf{A})$$

P5.15.



$$\frac{d^2V}{dx^2} = \begin{cases} 0 & \text{for } x < -d \\ \frac{\rho_0}{\varepsilon}\left(1 + \frac{x}{d}\right) & \text{for } -d < x < 0 \\ -\frac{\rho_0}{\varepsilon}\left(1 - \frac{x}{d}\right) & \text{for } 0 < x < d \\ 0 & \text{for } x > d \end{cases}$$

$$\frac{dV}{dx} = \begin{cases} C_1 & \text{for } x < -d \\ \frac{\rho_0}{\varepsilon}\left(x + \frac{x^2}{2d}\right) + C_2 & \text{for } -d < x < 0 \\ -\frac{\rho_0}{\varepsilon}\left(x - \frac{x^2}{2d}\right) + C_3 & \text{for } 0 < x < d \\ C_4 & \text{for } x > d \end{cases}$$

$$C_1 = -\frac{\rho_0 d}{2\varepsilon} + C_2 = -\frac{\rho_0 d}{2\varepsilon} + C_3 = C_4 = 0$$

$$C_2 = C_3 = \frac{\rho_0 d}{2\varepsilon}$$

$$\frac{dV}{dx} = \begin{cases} 0 & \text{for } x < -d \\ \frac{\rho_0}{\varepsilon}\left(x + \frac{x^2}{2d}\right) + \frac{\rho_0 d}{2\varepsilon} & \text{for } -d < x < 0 \\ -\frac{\rho_0}{\varepsilon}\left(x - \frac{x^2}{2d}\right) + \frac{\rho_0 d}{2\varepsilon} & \text{for } 0 < x < d \\ 0 & \text{for } x > d \end{cases}$$

P5.15. (continued)

$$V = \begin{cases} C_5 & \text{for } x < -d \\ \dfrac{\rho_0}{\varepsilon}\left(\dfrac{x^2}{2} + \dfrac{x^3}{6d} + \dfrac{dx}{2}\right) + C_6 & \text{for } -d < x < 0 \\ -\dfrac{\rho_0}{\varepsilon}\left(\dfrac{x^2}{2} - \dfrac{x^3}{6d} - \dfrac{dx}{2}\right) + C_7 & \text{for } 0 < x < d \\ C_8 & \text{for } x > d \end{cases}$$

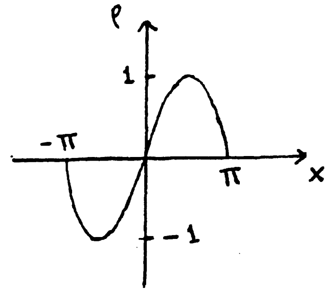
From $V = 0$ at $x = 0$, and continuity of V at $x = \pm d$, we obtain $C_6 = C_7 = 0$, $C_5 = -\dfrac{\rho_0 d^2}{6\varepsilon}$, $C_8 = \dfrac{\rho_0 d^2}{6\varepsilon}$. Thus

$$V = \begin{cases} -\dfrac{\rho_0 d^2}{6\varepsilon} & \text{for } x < -d \\ \dfrac{\rho_0}{2\varepsilon}\left(dx + x^2 + \dfrac{x^3}{3d}\right) & \text{for } -d < x < 0 \\ \dfrac{\rho_0}{2\varepsilon}\left(dx - x^2 + \dfrac{x^3}{3d}\right) & \text{for } 0 < x < d \\ \dfrac{\rho_0 d^2}{6\varepsilon} & \text{for } x > d \end{cases}$$

P5.16. Since $\rho = \rho(x)$, $V = V(x)$

$$\nabla^2 V = \frac{d^2V}{dx^2} = -\frac{\rho}{\varepsilon}$$

or
$$\frac{d^2V}{dx^2} = \begin{cases} 0 & \text{for } x < -\pi \\ -\dfrac{1}{\varepsilon}\sin x & \text{for } -\pi < x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$



$$\frac{dV}{dx} = \begin{cases} C_1 & \text{for } x < -\pi \\ \dfrac{1}{\varepsilon}\cos x + C_2 & \text{for } -\pi < x < \pi \\ C_3 & \text{for } x > \pi \end{cases}$$

Since electric field lines begin on positive charges and end on negative charges,

$$[E_x]_{x=\pm\pi} = \left[-\frac{dV}{dx}\right]_{x=\pm\pi} = 0$$

$$\therefore\ C_1 = C_3 = \frac{1}{\varepsilon}\cos(\pm\pi) + C_2 = 0$$

$$C_1 = C_3 = 0,\ C_2 = \frac{1}{\varepsilon}$$

$$\frac{dV}{dx} = \begin{cases} 0 & \text{for } x < -\pi \\ \dfrac{1}{\varepsilon}(\cos x + 1) & \text{for } -\pi < x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

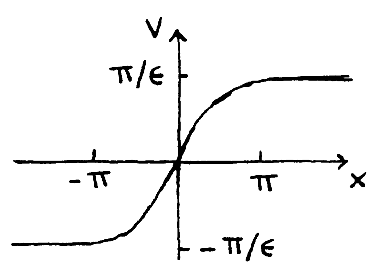
$$V = \begin{cases} C_4 & \text{for } x < -\pi \\ \dfrac{1}{\varepsilon}(\sin x + x) + C_5 & \text{for } -\pi < x < \pi \\ C_6 & \text{for } x > \pi \end{cases}$$

Setting $V = 0$ at $x = 0$ gives $C_5 = 0$. Then from the continuity of potentials,

$$C_4 = [V]_{x=-\pi} = -\frac{\pi}{\varepsilon} \text{ and } C_6 = [V]_{x=\pi} = \frac{\pi}{\varepsilon}$$

Thus

$$V = \begin{cases} -\pi/\varepsilon & \text{for } x < -\pi \\ \dfrac{1}{\varepsilon}(\sin x + x) & \text{for } -\pi < x < \pi \\ \pi/\varepsilon & \text{for } x > \pi \end{cases}$$



P5.17. Since $\rho = \rho(r)$, $V = V(r)$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = -\frac{\rho}{\varepsilon}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \begin{cases} -\dfrac{\rho_0}{\varepsilon} & \text{for } a < r < 2a \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \begin{cases} 0 & \text{for } r < a \\ -\dfrac{\rho_0 r^2}{\varepsilon} & \text{for } a < r < 2a \\ 0 & \text{for } r > 2a \end{cases}$$

$$r^2 \frac{\partial V}{\partial r} = \begin{cases} C_1 & \text{for } r < a \\ -\dfrac{\rho_0 r^3}{3\varepsilon} + C_2 & \text{for } a < r < 2a \\ C_3 & \text{for } r > 2a \end{cases}$$

$$\frac{\partial V}{\partial r} = \begin{cases} \dfrac{C_1}{r^2} & \text{for } r < a \\ -\dfrac{\rho_0 r}{3\varepsilon} + \dfrac{C_2}{r^2} & \text{for } a < r < 2a \\ \dfrac{C_3}{r^2} & \text{for } r > 2a \end{cases}$$

From $\mathbf{E} = -\nabla V = -\dfrac{\partial V}{\partial r}\mathbf{a}_r$, $\dfrac{\partial V}{\partial r} = -E_r$. From Gauss' law in integral form applied to a sphere of radius $r(< a)$ and centered at the origin, $4\pi r^2 \varepsilon E_r = 0$, so that $E_r = 0$ for $r < a$. Thus $C_1 = 0$. Then from continuity of D_r at $r = a$ and $r = 2a$,

$$-\frac{\rho_0 a}{3} + \frac{C_2 \varepsilon}{a^2} = 0 \qquad \text{or,} \qquad C_2 = \frac{\rho_0 a^3}{3\varepsilon}$$

$$\frac{C_3}{4a^2} = -\frac{2\rho_0 a}{3\varepsilon} + \frac{\rho_0 a}{12\varepsilon} \qquad \text{or,} \qquad C_3 = -\frac{7\rho_0 a^3}{3\varepsilon}$$

Thus

$$\frac{\partial V}{\partial r} = \begin{cases} 0 & \text{for } r < a \\ -\dfrac{\rho_0 r}{3\varepsilon} + \dfrac{\rho_0 a^3}{3\varepsilon r^2} & \text{for } a < r < 2a \\ -\dfrac{7\rho_0 a^3}{3\varepsilon r^2} & \text{for } r > 2a \end{cases}$$

P5.17. (continued)

$$V = \begin{cases} C_3 & \text{for } r < a \\ -\dfrac{\rho_0 r^2}{6\varepsilon} - \dfrac{\rho_0 a^3}{3\varepsilon r} + C_4 & \text{for } a < r < 2a \\ \dfrac{7\rho_0 a^3}{3\varepsilon r} + C_5 & \text{for } r > 2a \end{cases}$$

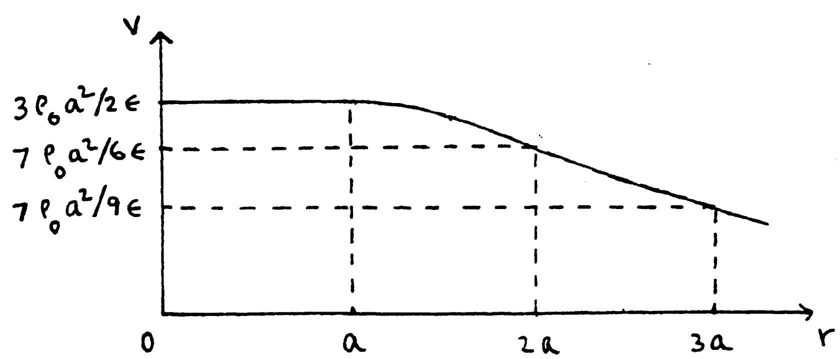
Setting $V = 0$ for $r = \infty$, we obtain $C_5 = 0$. Then from continuity of V at $r = 2a$ and $r = a$ (in the absence of impulse type discontinuities in E_r), we have

$$-\frac{\rho_0 (2a)^2}{6\varepsilon} - \frac{\rho_0 a^3}{3\varepsilon(2a)} + C_4 = \frac{7\rho_0 a^3}{3\varepsilon(2a)} \qquad \text{or,} \qquad C_4 = \frac{2\rho_0 a^2}{\varepsilon}$$

$$C_3 = -\frac{\rho_0 a^2}{6\varepsilon} - \frac{\rho_0 a^2}{3\varepsilon} + \frac{2\rho_0 a^2}{\varepsilon} \qquad \text{or,} \qquad C_3 = \frac{3\rho_0 a^2}{2\varepsilon}$$

Thus the final solution for V is given by

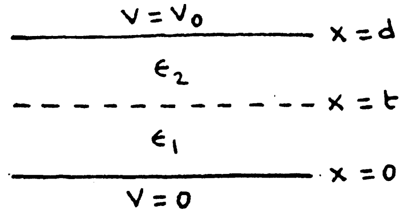
$$V = \begin{cases} \dfrac{3\rho_0 a^2}{2\varepsilon} & \text{for } r < a \\ \dfrac{\rho_0}{6\varepsilon}\left(12a^2 - r^2 - \dfrac{2a^3}{r}\right) & \text{for } a < r < 2a \\ \dfrac{7\rho_0 a^3}{3\varepsilon r} & \text{for } r > 2a \end{cases}$$



P5.18. (a) From Ex. 5.6, the general solutions for the potentials in the two regions are

$$V_1 = A_1 x + B_1, \; 0 < x < t$$

$$V_2 = A_2 x + B_2, \; t < x < d$$



The boundary conditions are

$V_1 = 0 \quad$ for $x = 0$

$V_2 = V_0 \quad$ for $x = d$

$V_1 = V_2 \quad$ for $x = t$

$D_{x1} = D_{x2}$, or, $\varepsilon_1 \dfrac{\partial V_1}{\partial x} = \varepsilon_2 \dfrac{\partial V_2}{\partial x}$ for $x = t$

Using these and solving for $A_1, B_1, A_2,$ and B_2, we obtain

$$V = \begin{cases} V_1 = \dfrac{\varepsilon_2 x}{\varepsilon_2 t + \varepsilon_1 (d - t)} V_0, & 0 < x < t \\ V_2 = \dfrac{\varepsilon_2 t + \varepsilon_1 (x - t)}{\varepsilon_2 t + \varepsilon_1 (d - t)} V_0, & t < x < d \end{cases}$$

(b) The magnitude of charge per unit area on either plate is

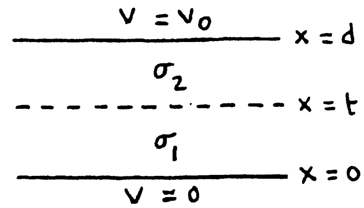
$$\left| -\varepsilon_1 \left[\frac{\partial V_1}{\partial x} \right]_{x=0} \right| \quad \text{or} \quad \varepsilon_2 \left[\frac{\partial V_2}{\partial x} \right]_{x=d}$$

$$= \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_2 t + \varepsilon_1 (d - t)} V_0$$

Capacitance per unit area of the plates

$$= \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_2 t + \varepsilon_1 (d - t)}$$

P5.19. **(a)** The boundary conditions at $x = t$ are



$$V_1 = V_2$$

$$J_{x1} = J_{x2}, \text{ or, } \sigma_1 \frac{dV_1}{dx} = \sigma_2 \frac{dV_2}{dx}$$

(b) $V_1 = A_1 x + B_1$ for $0 < x < t$

$V_2 = A_2 x + B_2$ for $t < x < d$

Using the boundary conditions $V_1 = 0$ for $x = 0$, and $V_2 = V_0$ for $x = d$, we obtain

$V_1 = A_1 x \qquad$ for $0 < x < t$

$V_2 = A_2(x - d) + V_0 \qquad$ for $t < x < d$

Then using the boundary conditions in **(a)**, we have

$$A_1 t = A_2(t - d) + V_0$$

$$\sigma_1 A_1 = \sigma_2 A_2$$

or,

$$A_1 = \frac{\sigma_2 V_0}{\sigma_2 t + \sigma_1 (d - t)}, \quad A_2 = \frac{\sigma_1 V_0}{\sigma_2 t + \sigma_1 (d - t)}$$

Thus,

$$V = \begin{cases} \dfrac{\sigma_2 x}{\sigma_2 t + \sigma_1 (d - t)} V_0 & \text{for } 0 < x < t \\[2ex] \dfrac{\sigma_2 t + \sigma_1 (x - t)}{\sigma_2 t + \sigma_1 (d - t)} V_0 & \text{for } t < x < d \end{cases}$$

(c) $$[V]_{x=t} = \frac{\sigma_2 t}{\sigma_2 t + \sigma_1 (d - t)} V_0$$

P5.20. In the region between the plates

$$\varepsilon\nabla^2 V + \nabla\varepsilon \bullet \nabla V = 0$$

For $\varepsilon = \varepsilon(x)$ and $V = V(x)$, this becomes

$$\varepsilon\frac{\partial^2 V}{\partial x^2} + \left(\frac{\partial \varepsilon}{\partial x}\right)\left(\frac{\partial V}{\partial x}\right) = 0$$

or $$\frac{d}{dx}\left(\varepsilon\frac{\partial V}{\partial x}\right) = 0$$

$$\varepsilon\frac{\partial V}{\partial x} = C_1, \text{ as constant}$$

$$\frac{dV}{dx} = \frac{C_1}{\varepsilon} = \frac{C_1}{\varepsilon_0}\left(1 - \frac{x}{2d}\right)$$

$$V = \frac{C_1}{\varepsilon_0}\left(x - \frac{x^2}{4d}\right) + C_2$$

Using the boundary conditions

$$V = 0 \text{ for } x = 0 \text{ and } V = V_0 \text{ for } x = d,$$

we obtain $C_2 = 0$ and $C_1 = \dfrac{4\varepsilon_0 V_0}{3d}$. Thus the solution for the potential between the plates is

$$V = \frac{4V_0}{3d}\left(x - \frac{x^2}{4d}\right)$$

To find the capacitance, we note that

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{a}_x = -\frac{4V_0}{3d}\left(1 - \frac{x}{2d}\right)\mathbf{a}_x$$

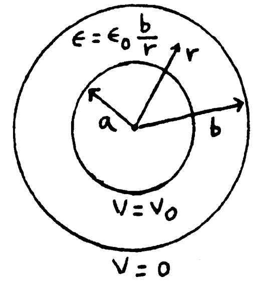
$$\mathbf{D} = \varepsilon\mathbf{E} = -\frac{4\varepsilon_0 V_0}{3d}\mathbf{a}_x$$

$$|\rho_S|_{x=0} = |\rho_S|_{x=d} = \frac{4\varepsilon_0 V_0}{3d}$$

$|Q|$, magnitude of charge on either plate $= \dfrac{4\varepsilon_0 V_0 A}{3d}$

Thus, $\dfrac{C}{A} = \dfrac{|Q|/A}{V_0} = \dfrac{4\varepsilon_0}{3d}$

P5.21. $\varepsilon\nabla^2 V + \nabla\varepsilon \bullet \nabla V = 0$



$$\frac{\varepsilon}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \left(\frac{\partial \varepsilon}{\partial r}\right)\left(\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\varepsilon_0 b}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) - \frac{\varepsilon_0 b}{r^2}\frac{\partial V}{\partial r} = 0$$

$$r\frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial r} - \frac{\partial V}{\partial r} = 0$$

$$\frac{\partial^2 V}{\partial r^2} = 0$$

$V = Ar + B$

$V = 0$ for $r = b \rightarrow Ab + B = 0,\ B = -Ab$

$V = V_0$ for $r = a \rightarrow Aa - Ab = V_0,\ A = \dfrac{V_0}{a-b}$

$$\therefore\ V = \frac{V_0}{a-b}r - \frac{V_0}{a-b}b$$

$$= \frac{V_0(r-b)}{a-b}$$

To find the capacitance per unit length, we obtain

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\mathbf{a}_r = \frac{V_0}{b-a}\mathbf{a}_r$$

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\frac{b}{r}\frac{V_0}{b-a}\mathbf{a}_r$$

$$[\rho_s]_{r=a} = \mathbf{a}_r \bullet [\mathbf{D}]_{r=a} = \varepsilon_0\frac{b}{a}\frac{V_0}{(b-a)}$$

$$[\rho_s]_{r=b} = -\mathbf{a}_r \bullet [\mathbf{D}]_{r=b} = -\varepsilon_0\frac{V_0}{b-a}$$

Magnitude of charge per unit length on either conductor

$$= \left|2\pi a\,[\rho_s]_{r=a}\right| = \left|2\pi b\,[\rho_s]_{r=b}\right| = 2\pi\varepsilon_0 b\frac{V_0}{b-a}$$

Capacitance per unit length $= \dfrac{2\pi\varepsilon_0 b}{b-a}$

P5.22. $C = \dfrac{\pi\varepsilon}{\ln\left[\left(d + \sqrt{d^2 - a^2}\right)/a\right]} = \dfrac{\pi\varepsilon}{\cosh^{-1}(d/a)}$

For $d >> a$, $C \approx \dfrac{\pi\varepsilon}{\ln[(d+d)/a]} = \dfrac{\pi\varepsilon}{\ln(2d/a)}$

Setting $\dfrac{\pi\varepsilon}{\cosh^{-1}\left(\dfrac{d}{a}\right)} = 1.05\dfrac{\pi\varepsilon}{\ln\left(\dfrac{2d}{a}\right)}$, we have

$$\cosh^{-1}\left(\frac{d}{a}\right) = \frac{1}{1.05}\ln\left(\frac{2d}{a}\right)$$

$$\cosh\left[\frac{1}{1.05}\ln\left(\frac{2d}{a}\right)\right] = \frac{d}{a}$$

$$e^{\frac{1}{1.05}\ln(2d/a)} + e^{-\frac{1}{1.05}\ln(2d/a)} = \frac{2d}{a}$$

Solving this equation for d/a, we obtain

$$\frac{d}{a} = \frac{4.0673}{2} = 2.03365$$

P5.23. The equipotential surfaces are given by

$$\frac{(x+b)^2+y^2}{(x-b)^2+y^2}=k^2$$

Differentiating both sides, we obtain

$$[(x-b)^2+y^2][2(x+b)\,dx+2y\,dy]$$

$$-\,[(x+b^2+y^2][2(x-b)\,dx+2y\,dy]=0$$

$$[(x^2-b^2)(-2b)+y^2(2b)]\,dx-4xby\,dy=0$$

$$[y^2-(x^2-b^2)]\,dx=2xy\,dy$$

Thus the slope of the equipotentials is given by

$$m=\frac{dy}{dx}=\frac{y^2-(x^2-b^2)}{2xy}$$

Since the electric field lines are orthogonal to the equipotential surfaces, their slope is given by

$$m'=-\frac{1}{m}=-\frac{2xy}{y^2-(x^2-b^2)}$$

Setting $\dfrac{dy}{dx}=-\dfrac{2xy}{y^2-(x^2-b^2)}$, we then have

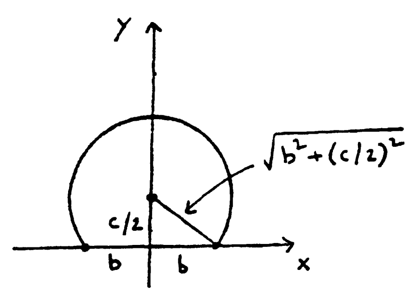
$$2xy\,dx+[y^2-(x^2-b^2)]\,dy=0$$

$$2xy\,dx+2y^2\,dy-[(x^2-b^2)+y^2]\,dy=0$$

$$y[2x\,dx+2y\,dy]-[(x^2-b^2)+y^2]\,dy=0$$

$$d\left[\frac{(x^2-b^2)+y^2}{y}\right]=0$$

$$\frac{(x^2-b^2)+y^2}{y}=c\text{, a constant}$$



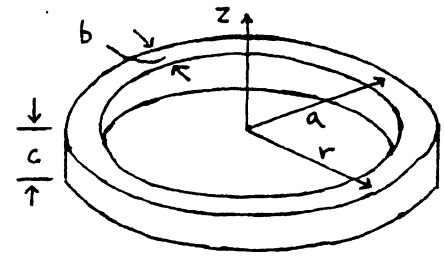
$$(x^2-b^2)+y^2=cy$$

$$x^2+(y-c/2)^2=b^2+(c/2)^2$$

This equation represents circles with centers at (0, $c/2$) and radii equal to $\sqrt{b^2+(c/2)^2}$. Thus the circles pass through the line charge locations $x=\pm b$, $y=0$.

P5.24. Choosing the z-axis to be the axis of the toroid and applying Ampere's circuital law to a closed path of radius r inside the toroid and centered on the z-axis, we obtain

$$2\pi r H_\phi = 2\pi a N I$$



$$H_\phi = \frac{NIa}{r}$$

$$\mathbf{B} = \frac{\mu N I a}{r}\mathbf{a}_\phi$$

The magnetic flux inside the toroid is then given by

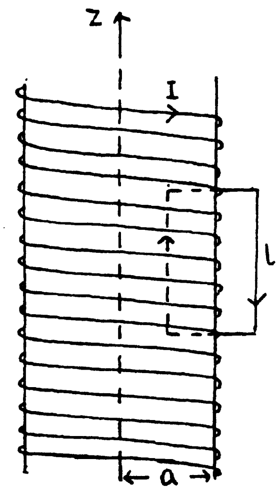
$$\begin{aligned}\psi &= \int_{r=a-b/2}^{a+b/2}\int_{z=0}^{c} \frac{\mu N I a}{r}\mathbf{a}_\phi \bullet dr\, dz\, \mathbf{a}_\phi \\ &= \mu N I a c\left[\ln r\right]_{r=a-b/2}^{a+b/2} \\ &= \mu N I a c \ln\frac{2a+b}{2a-b}\end{aligned}$$

The inductance is given by

$$L = 2\pi a N\frac{\psi}{I}$$

$$= 2\pi\mu N^2 a^2 c \ln\frac{2a+b}{2a-b}$$

P5.25. Choosing z-axis as the axis of the solenoid and noting that (a) no field exists outside the solenoid, and (b) the field inside the solenoid is z-directed and uniform, we write



$$\mathbf{H} = \begin{cases} H_0 \mathbf{a}_z & \text{for } r < a \\ \\ \mathbf{0} & \text{for } r > a \end{cases}$$

Then applying Ampere's circuital law in integral form around the rectangular closed path shown, we have

$$H_z l = NlI$$

$$H_z = NI$$

$$\mathbf{B} = \mu_0 NI \mathbf{a}_z$$

The magnetic flux inside the solenoid is then given by

$$\psi = \pi a^2 B_z = \pi a^2 \mu_0 NI$$

The inductance per unit length of the solenoid is

$$\mathcal{L} = \frac{N\psi}{I} = \pi a^2 \mu_0 N^2$$

P5.26. Noting from symmetry considerations that the magnetic field intensity is entirely in the ϕ direction and independent of ϕ, and applying Ampere's circuital law to a circular contour of radius r ($< a$), as in Ex. 5.8, we have

$$2\pi r H_\phi = \int_{r=0}^{r}\int_{\phi=0}^{2\pi} J_0 \frac{r^2}{a^2}\mathbf{a}_z \bullet r\,dr\,d\phi\,\mathbf{a}_z$$

$$= \frac{2\pi J_0}{a^2}\int_{r=0}^{r} r^3\,dr$$

$$= \frac{\pi J_0 r^4}{2a^2} \qquad \text{for } r \le a$$

$$H_\phi = \frac{J_0 r^3}{4a^2} \qquad \text{for } r \le a$$

$$\mathbf{H} = \frac{J_0 r^3}{4a^2}\,\mathbf{a}_\phi \qquad \text{for } r \le a$$

$$\mathbf{B} = \frac{\mu J_0 r^3}{4a^2}\,\mathbf{a}_\phi \qquad \text{for } r \le a$$

Considering a rectangle of infinitesimal width dr in the r-direction and length l in the z-direction at a distance r from the axis, as in Ex. 5.8, we obtain the magnetic flux $d\psi_i$ crossing the rectangular surface to be

$$d\psi_i = B_\phi(l\,dr)$$

$$= \frac{\mu J_0 r^3 l\,dr}{4a^2}$$

The fraction of the total current linked by this flux is

$$N = \frac{\pi J_0 r^4/2a^2}{\pi J_0 a^4/2a^2} = \frac{r^4}{a^4}$$

Thus

$$\psi_i = \int_{r=0}^{a} N\,d\psi_i = \int_{r=0}^{a}\frac{r^4}{a^4}\frac{\mu J_0 r^3 l}{4a^2}\,dr$$

$$= \frac{\mu J_0 l}{4a^6}\left[\frac{r^8}{8}\right]_0^a = \frac{\mu J_0 a^2 l}{32}$$

$$\mathcal{L}_i = \frac{\psi_i}{lI} = \frac{\mu J_0 a^2 l/32}{l(\pi J_0 a^2/2)}$$

$$= \frac{\mu}{16\pi}$$

P5.27. From Ex. 5.8, the magnetic field intensity internal to the current distribution is given by

$$\mathbf{H} = \frac{J_0 r}{2}\mathbf{a}_\phi, \quad r < a$$

The energy stored in the magnetic field internal to the current distribution per unit length of the arrangement is given by

$$W_m / l = \int_{r=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\left(\frac{1}{2}\mu H_\phi^2\right) dv$$

$$= \int_{r=0}^{a}\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\frac{1}{2}\mu\frac{J_0^2 r^2}{4} r\, dr\, d\phi\, dz$$

$$= \frac{\mu J_0^2 \pi}{4}\left[\frac{r^4}{4}\right]_{r=0}^{a}$$

$$= \frac{\mu J_0^2 \pi a^4}{16}$$

$$= \frac{1}{2}\left(\frac{\mu}{8\pi}\right)(J_0 \pi a^2)^2$$

$$= \frac{1}{2}\mathcal{L}_i\, I^2$$

P5.28. From the solution for P5.25, the magnetic flux density produced by the solenoid of radius b is $\mu_0 N_2 I_2$, where I_2 is the current in the winding of that solenoid. Thus the flux produced by the solenoid of radius b and linking one turn of the solenoid of radius a is given by

$$\psi_{12} = \mu_0 N_2 I_2 \, (\pi a^2)$$

The mutual inductance per unit length of the arrangement is

$$\begin{aligned} \mathcal{L}_{12} &= N_1 \frac{\psi_{12}}{I_2} \\ &= \mu_0 \pi a^2 N_1 N_2 \end{aligned}$$

Alternatively, the magnetic flux produced by the solenoid of radius a and linking one turn of the solenoid of radius b is given by

$$\begin{aligned} \psi_{21} &= \mu_0 N_1 I_1 \, (\pi a^2) + 0[\pi(b^2 - a^2)] \\ &= \mu_0 N_1 I_1 \pi a^2 \end{aligned}$$

The mutual inductance per unit length of the arrangement is

$$\begin{aligned} \mathcal{L}_{21} &= N_2 \frac{\psi_{21}}{I_1} \\ &= \mu_0 \pi a^2 N_1 N_2 \\ &= \mathcal{L}_{12} \end{aligned}$$

P5.29.

$$\frac{1}{2\pi\sqrt{\mu\varepsilon}\,l} = \frac{1}{2\pi\sqrt{4\pi\times10^{-7}\times\dfrac{10^{-9}}{36\pi}\times0.1}}$$

$$= 4.775\times10^{8}\text{ Hz}$$

(a) For $I(t) = 1\cos 10^6\pi t$,

$$f = \frac{10^6\pi}{2\pi} = 5\times10^5\text{ Hz} \ll \frac{1}{2\pi\sqrt{\mu\varepsilon}\,l}$$

The input behavior of the structure is essentially that of a single inductor of value

$$L = \frac{\mu_0\,dl}{w}$$

$$\bar{V} = j\omega L\bar{I} = j10^6\pi\frac{4\pi\times10^{-7}\times5\times10^{-3}\times0.1}{5\times10^{-2}}$$

$$= j0.0395$$

$$V(t) = 0.0395\cos(10^6\pi t + \pi/2)\text{ V}$$

$$= -0.0395\sin 10^6\pi t\text{ V}$$

(b) For $I(t) = 1\cos 10^9\pi t$

$$f = \frac{10^9\pi}{2\pi} = 5\times10^8\text{ Hz is not} \ll \frac{1}{2\pi\sqrt{\mu\varepsilon}\,l}$$

$$\left[\bar{E}_x\right]_{z=-l} = j\sqrt{\frac{\mu}{\varepsilon}}\frac{I_0}{w}\tan\omega\sqrt{\mu\varepsilon}l$$

$$= j\sqrt{\frac{4\pi\times10^{-7}}{10^{-9}/36\pi}}\frac{1}{5\times10^{-2}}\tan\frac{5\times10^8}{4.775\times10^8}$$

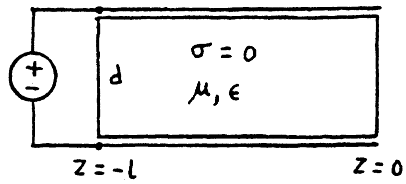
$$= j1.3057\times10^4$$

$$\left[\bar{V}\right]_{z=-l} = \left[\bar{E}_x\right]_{z=-l}d = j1.3057\times10^4\times5\times10^{-3}$$

$$= j65.285$$

$$V(t) = -65.285\sin 10^9\pi t\text{ V}$$

P5.30. From (5.112a) and (5.112b), for $\sigma = 0$,



$$\bar{E}_{x0} = \frac{V_0}{d},\ \bar{H}_{y1} = -j\omega\varepsilon\frac{V_0}{d}z$$

Then $\dfrac{\partial \bar{E}_x}{\partial z} = -j\omega\mu\bar{H}_{y1} = -\omega^2\mu\varepsilon\dfrac{V_0}{d}z$

$$\bar{E}_{x2} = -\omega^2\mu\varepsilon\frac{V_0}{d}\frac{z^2}{2} + \bar{C}''$$

Since $\left[\bar{E}_x\right]_{z=-l} d = V_0$ is satisfied by $\bar{E}_{x0}$ alone, $\left[\bar{E}_{x2}\right]_{z=-l} = 0$.

Hence $\bar{C}'' = \omega^2\mu\varepsilon\dfrac{V_0}{d}\dfrac{l^2}{2}$

$$\bar{E}_{x2} = -\omega^2\mu\varepsilon\frac{V_0}{d}(z^2 - l^2)$$

$$\frac{\partial \bar{H}_y}{\partial z} = j\omega\varepsilon\bar{E}_{x2} = -j\omega^3\mu\varepsilon^2\frac{V_0}{2d}(z^2 - l^2)$$

$$\bar{H}_{y3} = -j\omega^3\mu\varepsilon^2\frac{V_0}{2d}\left(\frac{z^3}{3} - l^2 z\right) + C'''$$

Since $\left[\bar{H}_y\right]_{z=0} = 0,\ \bar{C}''' = 0$

Thus $\bar{H}_{y3} = -j\omega^3\mu\varepsilon^2\dfrac{V_0}{2d}\left(\dfrac{z^3}{3} - l^2 z\right)$

$$\left[\bar{H}_y\right]_{z=-l} = \left[\bar{H}_{y1}\right]_{z=-l} + \left[\bar{H}_{y3}\right]_{z=-l}$$

$$= j\omega\varepsilon l\frac{V_0}{d} + j\frac{\omega^3\mu\varepsilon^2 l^3}{3}\frac{V_0}{d}$$

$$\left[\bar{I}\right]_{z=-l} = w\left[\bar{H}_y\right]_{z=-l} = j\omega\frac{\varepsilon wl}{d}\left(1 + \frac{\omega^2\mu\varepsilon l^2}{3}\right)\left[\bar{V}\right]_{z=-l}$$

$$\frac{\left[\bar{V}\right]_{z=-l}}{\left[\bar{I}\right]_{z=-l}} \approx \frac{1}{j\omega\dfrac{\varepsilon wl}{d}}\left(1 - \frac{1}{3}\omega^2\mu\varepsilon l^2\right)$$

$$= \frac{1}{j\omega\dfrac{\varepsilon wl}{d}} + j\frac{\omega\mu dl}{3w} = \frac{1}{j\omega C} + j\omega\frac{L}{3}$$

P5.31. (a) From (5.113), the condition is

$$\frac{\varepsilon wl}{d} = \frac{\mu\sigma^2 wl^3}{3d}$$

$$\sigma\sqrt{\frac{\mu}{\varepsilon}}\,l = \sqrt{3}$$

(b) From (5.113), the condition is

$$\frac{\varepsilon wl}{d} >> \frac{\mu\sigma^2 wl^3}{3d}$$

$$\sigma\sqrt{\frac{\mu}{\varepsilon}}\,l << 1$$

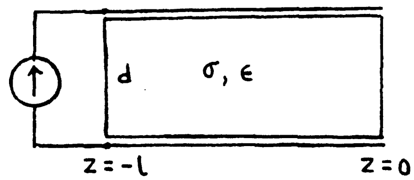
(c) From (5.114), the conditions are

$$\frac{\omega\mu\sigma l^2}{3} << 1 \text{ and } \frac{\sigma wl}{d} >> \omega\frac{\varepsilon wl}{d}$$

or,

$$\sqrt{\omega\mu\sigma}\,l << 1 \text{ and } \frac{\sigma}{\omega\varepsilon} >> 1$$

P5.32. (a) $\bar{I} = I_0, \; \bar{H}_{y0} = \frac{I_0}{w}$



$$\frac{\partial \bar{E}_{x1}}{\partial z} = -j\omega\mu \bar{H}_{y0} = -j\omega\mu\left(\frac{I_0}{w}\right)$$

$$\bar{E}_{x1} = -j\omega\mu\left(\frac{I_0}{w}\right)z + \bar{C}_1$$

The constant $\bar{C}_1$ is zero, since $\left[\bar{E}_{x1}\right]_{z=0} = 0$. Thus

$$\bar{E}_{x1} = -j\omega\mu z\left(\frac{I_0}{w}\right)$$

$$\bar{V}_1 = \left[\bar{E}_{x1}\right]_{z=-l} d = j\omega\mu\left(\frac{dl}{w}\right) I_0 = j\omega L I_0$$

$\therefore$ The input impedance is the same as if $\sigma = 0$.

(b) $$\frac{\partial \bar{H}_{y1}}{\partial z} = -\sigma \bar{E}_{x1} = j\omega\mu\sigma z\left(\frac{I_0}{w}\right)$$

$$\bar{H}_{y1} = j\omega\mu\sigma\frac{z^2}{2}\left(\frac{I_0}{w}\right) + \bar{C}_2$$

To obtain $\bar{C}_2$, we set $\left[\bar{H}_{y1}\right]_{z=-l} = 0$, since the boundary condition at $z = -l$ is satisfied by $\bar{H}_{y0}$. Thus

$$j\omega\mu\sigma\frac{l^2}{w}\left(\frac{I_0}{w}\right) + \bar{C}_2 = 0 \qquad \text{or} \qquad \bar{C}_2 = -j\omega\mu\sigma\frac{l^2}{2}$$

$$\therefore \bar{H}_{y1} = j\omega\mu\sigma\left(\frac{z^2 - l^2}{2}\right)\frac{I_0}{w}$$

$$\frac{\partial \bar{E}_{x2}}{\partial z} = -j\omega\mu \bar{H}_{y1} = \omega^2\mu\sigma\left(\frac{z^2 - l^2}{2}\right)\frac{I_0}{w}$$

$$\bar{E}_{x2} = \omega^2\mu\sigma\frac{I_0}{2w}\left(\frac{z^3}{3} - l^2 z\right) + \bar{C}_3$$

The constant $\bar{C}_3$ is zero since $\left[\bar{E}_{x2}\right]_{z=0} = 0$. Thus

P5.32. (continued)

$$\bar{V}_2 = \left[\bar{E}_{x2}\right]_{z=-l} d = \frac{\omega^2 \mu \sigma I_0}{2w}\left(-\frac{l^3}{3} + l^3\right) = \frac{\omega^2 \mu^2 \sigma I_0 l^3 d}{3w}$$

$$\bar{V} = \bar{V}_1 + \bar{V}_2 = \left(\frac{j\omega\mu dl}{w} + \frac{\omega^2 \mu^2 \sigma l^3 d}{3w}\right) I_0$$

$$= \frac{j\omega\mu dl}{w}\left(1 - \frac{j\omega\mu\sigma l^2}{3}\right)\bar{I}$$

The input impedance of the arrangement is

$$\bar{Z}_{\text{in}} = \frac{\bar{V}}{\bar{I}} = \frac{j\omega\mu dl}{w}\left(1 - \frac{j\omega\mu\sigma l^2}{3}\right)$$

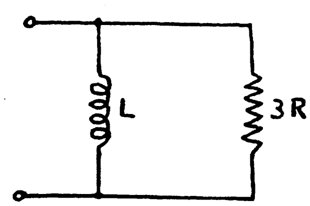
To determine the equivalent circuit, we consider the input admittance of the arrangement:

$$\bar{Y}_{\text{in}} = \frac{1}{\bar{Z}_{\text{in}}} = \frac{1}{j\omega\frac{\mu dl}{w}\left(1 - \frac{j\omega\mu\sigma l^2}{3}\right)}$$

$$\approx \frac{1}{j\omega\frac{\mu dl}{w}}\left(1 + j\frac{\omega\mu\sigma l^2}{3}\right)$$

$$= \frac{1}{j\omega\frac{\mu dl}{w}} + \frac{\sigma l w}{3d}$$

$$= \frac{1}{j\omega L} + \frac{1}{3R}$$



Thus the equivalent circuit consists of the parallel combination of $L = \frac{\mu dl}{w}$ and $3R$, where $R = \frac{d}{\sigma l w}$.

P5.33. From Eqs. (5.95), (5.102), and (5.103) of the text,

$$\left[\bar{E}_x\right]_{z=l} \approx j\omega\mu l \frac{I_0}{w} + j\frac{\omega^3\mu^2\varepsilon l^2}{3}\frac{I_0}{w} + j\frac{2\omega^5\mu^3\varepsilon^2 l^3}{15}\frac{I_0}{w}$$

$$\bar{V} = j\omega\frac{\mu dl}{w}I_0 + j\frac{\omega^3\mu^2\varepsilon dl^2}{3}\frac{I_0}{w} + j\frac{2\omega^5\mu^3\varepsilon^2 dl^3}{15}\frac{I_0}{w}$$

$$= j\omega L I_0\left(1+\frac{1}{3}\omega^2 LC + \frac{2}{15}\omega^4 L^2C^2\right)$$

$$I_0 = \frac{\bar{V}}{j\omega L}\left(1+\frac{1}{3}\omega^2 LC + \frac{2}{15}\omega^4 L^2C^2\right)^{-1}$$

$$= \frac{\bar{V}}{j\omega L}\left(1-\frac{1}{3}\omega^2 LC - \frac{1}{45}\omega^4 L^2C^2 + \text{higher order terms}\right)$$

$$\approx \frac{\bar{V}}{j\omega L}\left(1-\frac{1}{3}\omega^2 LC - \frac{1}{45}\omega^4 L^2C^2\right)$$

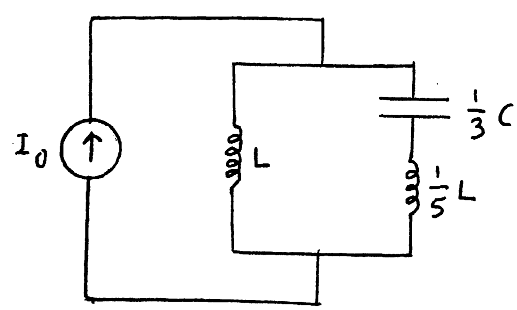
$$= \frac{\bar{V}}{j\omega L} + \bar{V}\left(j\frac{\omega C}{3} + j\frac{\omega^3 LC^2}{45}\right)$$

$$= \frac{\bar{V}}{j\omega L} + \frac{\bar{V}}{1\Big/\left[\left(\frac{j\omega C}{3}\right)\left(1+\frac{\omega^2 LC}{15}\right)\right]}$$

$$\approx \frac{\bar{V}}{j\omega L} + \frac{\bar{V}}{\frac{3}{j\omega C}\left(1-\frac{\omega^2 LC}{15}\right)}$$

$$= \frac{\bar{V}}{j\omega L} + \frac{\bar{V}}{\frac{3}{j\omega C} + \frac{j\omega L}{5}}$$

$$= \frac{\bar{V}}{j\omega L} + \frac{\bar{V}}{\frac{1}{j\omega(C/3)} + j\omega\left(\frac{L}{5}\right)}$$



The equivalent circuit is as shown.

P5.34. **(a)** From $NI = Hl$,

$$H = \frac{NI}{l} = \frac{200}{0.2} = 1000 \text{ A/m}$$

From $\psi = BA$,

$$B = \frac{\psi}{A} = \frac{8\times10^{-4}}{5\times10^{-4}} = 1.6 \text{ Wb/m}^2$$

$$\mu = \frac{B}{H} = \frac{1.6}{1000} = 1.6 \times 10^{-3} \text{ H/m}$$

(b) In the air gap, B is the same as in the core since fringing of flux is neglected.

$$\therefore\ H_{\text{gap}} = \frac{B_{\text{gap}}}{\mu_0} = \frac{1.6}{4\pi\times10^{-7}} \text{ A/m}$$

Thus the new value of NI

$$= H_{\text{core}}l_{\text{core}} + H_{\text{gap}}l_{\text{gap}}$$

$$= 100 \times 0.2 + \frac{1.6}{4\pi\times10^{-7}}\times10^{-4}$$

$$= 200 + 127.324$$

$$= 327.3 \text{ A-t}$$

P5.35. $H_1 l_1 = H_3 l_3$

Since $l_1 = l_3$, $H_1 = H_3$, and $B_1 = B_3$.

Then since $A_1 = A_3$, $\psi_1 = \psi_3 = \dfrac{\psi_2}{2} = 4.5 \times 10^{-4}$ Wb.

$$B_1 = B_3 = \frac{4.5\times10^{-4}}{3\times10^{-4}} = 1.5 \text{ Wb/m}^2$$

Also, $B_2 = \dfrac{9\times10^{-4}}{6\times10^{-4}} = 1.5$ Wb/m^2

From Fig. 5.26, $H_1 = H_2 = H_3 = 1000$ A/m

$\psi_g = 9 \times 10^{-4}$ Wb

$$\left(A_g\right)_{\text{eff}} = \left(\sqrt{6} + l_g\right)^2 = \left(\sqrt{6} + 0.02\right)^2 = 6.09838 \text{ cm}^2$$

$$B_g = \frac{\psi_g}{\left(A_g\right)_{\text{eff}}} = \frac{9\times10^{-4}}{6.09838\times10^{-4}} = 1.4758$$

$$H_g = \frac{B_g}{\mu_0} = \frac{1.4758}{4\pi\times10^{-7}} = 0.11744 \times 10^7$$

$$NI = H_g l_g + H_2 l_2 + H_1 l_1$$

$$= 0.11744 \times 10^7 \times 0.02 \times 10^{-2} + 1000 \times 0.1 + 1000 \times 0.2$$

$$= 534.881 \text{ A-t}$$

P5.36. From symmetry considerations

$$\psi_1 = \psi_3 = 4 \times 10^{-4} \text{ Wb}$$

$$\psi_2 = \psi_1 + \psi_3 = 8 \times 10^{-4} \text{ Wb}$$

$$B_2 = \frac{\psi_2}{A_2} = \frac{8 \times 10^{-4}}{6 \times 10^{-4}} = 1.333 \text{ Wb/m}^2$$

$$B_3 = \frac{\psi_3}{A_3} = \frac{4 \times 10^{-4}}{3 \times 10^{-4}} = 1.333 \text{ Wb/m}^2$$

From Fig. 5.26,

$$H_2 = H_3 = 475 \text{ A/m}$$

In the air gap,

$$H_g = \frac{B_g}{\mu_0} = \frac{\psi_g}{(A_g)_{\text{eff}}\mu_0}$$

$$= \frac{4 \times 10^{-4}}{(3.07 \times 10^{-4}) \times 4\pi \times 10^{-7}}$$

$$= 0.1037 \times 10^7 \text{ A/m}$$

$$NI = H_2 l_2 + H_3 l_3 + H_g l_g$$

$$= 475 \times 0.1 + 475 \times 0.2 + 0.1037 \times 10^7 \times 2 \times 10^{-4}$$

$$= 349.9 \text{ A-t}$$

P5.37. Since $l_1 = l_3$ and $A_1 = A_3$, $\psi_1 = \psi_3 = \dfrac{\psi_2}{2}$.

Then since $A_1 = A_3 = \dfrac{A_2}{2}$, $B_1 = B_3 = B_2$.

$\therefore H_1 = H_3 = H_2$

$NI = H_2 l_2 + H_1 l_1 = H_2 \times 0.1 + H_2 \times 0.2 = 0.3 H_2$

$H_2 = \dfrac{NI}{0.3} = \dfrac{180}{0.3} = 600$ A/m

From Fig. 5.26, $B_2 = 1.4$ Wb/m^2

$\therefore \ \psi_2 = 1.4 \times 6 \times 10^{-4} = 8.4 \times 10^{-4}$ Wb

P5.38. $A_1 = 5\ \text{cm}^2, A_2 = 6\ \text{cm}^2, A_3 = 3\ \text{cm}^2$

$l_1 = l_3 = 20$ cm, $l_2 = 10$ cm

Since $l_1 = l_3$, $H_1 = H_3$ and $B_1 = B_3$.

From $\psi_2 = \psi_1 + \psi_3$, we have

$$B_2A_2 = B_1A_1 + B_3A_3 = B_1(A_1 + A_3)$$

$$B_2 = B_1\frac{A_1 + A_3}{A_2} = \frac{4}{3}B_1 \quad \text{—(1)}$$

From $NI = H_2l_2 + H_1l_1$, we have

$$150 = 0.1H_2 + 0.2H_1$$

or, $2H_1 + H_2 = 1500$ —(2)

To find B_2, we solve (1) and (2), simultaneously.

Choosing $H_1 = 230$ A/m, we find from Fig. 5.26 that $B_1 = 1.13\ \text{Wb/m}^2$. Then From (1),

$B_2 = \frac{4}{3}B_1 = 1.51\ \text{Wb/m}^2$, and from Fig. 5.26, $H_2 = 1050$ A/m.

$$2H_1 + H_2 = 460 + 1050 = 1510 \approx 1500$$

$\therefore\ B_2 = 1.51\ \text{Wb/m}^2$

P5.39. $$W_e = \frac{1}{2}\varepsilon\left(\frac{V}{d}\right)^2 dwx + \frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2 dw(L-x)$$

$$= \frac{V^2 w}{2d}\left[\varepsilon x + \varepsilon_0(L-x)\right]$$

$$\frac{dW_e}{dx} = \frac{V^2 w}{2d}(\varepsilon - \varepsilon_0)$$

$$Q = \frac{\varepsilon V}{d}wx + \frac{\varepsilon_0 V}{d}w(L-x)$$

$$\frac{dQ}{dx} = \frac{Vw}{d}(\varepsilon - \varepsilon_0)$$

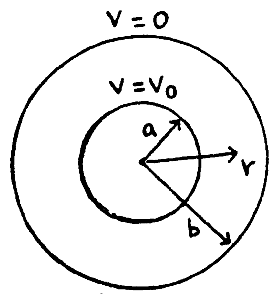
$$F_{ex} = -\frac{dW_e}{dx} + V\frac{dQ}{dx}$$

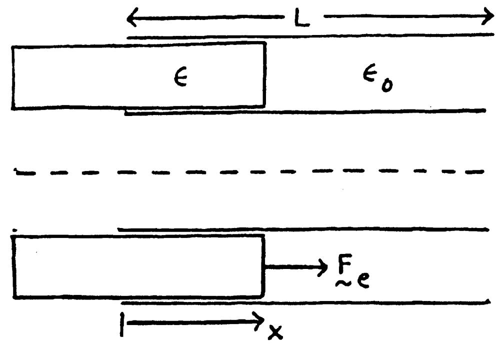
$$= -\frac{V^2 w}{2d}(\varepsilon - \varepsilon_0) + \frac{V^2 w}{d}(\varepsilon - \varepsilon_0)$$

$$= \frac{V^2 w}{2d}(\varepsilon - \varepsilon_0)$$

$$\mathbf{F}_e = \frac{V^2 w}{2d}(\varepsilon - \varepsilon_0)\mathbf{a}_x$$

P5.40.





From Table 5.1, the electric field intensity inside the capacitor is given by

$$\mathbf{E} = \frac{V_0}{r \ln (b/a)} \mathbf{a}_r \text{ for } a < r < b$$

Thus

$$W_e = x \int_{r=a}^{b} \int_{\phi=0}^{2\pi} \frac{1}{2} \varepsilon \left[\frac{V_0}{r \ln (b/a)} \right]^2 r \, dr \, d\phi$$

$$+ (L-x) \int_{r=a}^{b} \int_{\phi=0}^{2\pi} \frac{1}{2} \varepsilon_0 \left[\frac{V_0}{r \ln (b/a)} \right]^2 r \, dr \, d\phi$$

$$= \frac{\pi V_0^2}{\ln (b/a)} [\varepsilon x + \varepsilon_0 (L-x)]$$

$$\frac{dW_e}{dx} = \frac{\pi V_0^2}{\ln (b/a)} (\varepsilon - \varepsilon_0)$$

$$Q = x\varepsilon \frac{2\pi a V_0}{a \ln (b/a)} + (L-x)\varepsilon_0 \frac{2\pi a V_0}{a \ln (b/a)}$$

$$= \frac{2\pi V_0}{\ln (b/a)} [\varepsilon x + \varepsilon_0 (L-x)]$$

$$\frac{dQ}{dx} = \frac{2\pi V_0}{\ln (b/a)} (\varepsilon - \varepsilon_0)$$

$$F_{ex} = -\frac{dW_e}{dx} + V_0 \frac{dQ}{dx}$$

$$= -\frac{\pi V_0^2}{\ln (b/a)} (\varepsilon - \varepsilon_0) + \frac{2\pi V_0^2}{\ln (b/a)} (\varepsilon - \varepsilon_0)$$

$$= \frac{\pi V_0^2}{\ln (b/a)} (\varepsilon - \varepsilon_0)$$

$$\mathbf{F}_e = \frac{V_0^2 \pi (\varepsilon - \varepsilon_0)}{\ln (b/a)} \mathbf{a}_x$$

P5.41. $$W_{\substack{\text{mechanical}\\ \text{input}}} = -\oint_{ABCA} F_{ex}\,dx$$

$$= -\int_B^C F_{ex}\,dx - \int_C^A F_{ex}\,dx$$

$$F_{ex} = -\frac{1}{2}\frac{\varepsilon_0 A V^2}{x^2}$$

$$= \begin{cases} -\dfrac{\varepsilon_0 A V_0^2}{2x^2}\left(3-\dfrac{x}{d}\right)^2 & \text{from } B \text{ to } C \\ -\dfrac{\varepsilon_0 A V_0^2}{2x^2} & \text{from } C \text{ to } A \end{cases}$$

$$\therefore\ W_{\substack{\text{mechanical}\\ \text{input}}} = \int_{x=d}^{2d} \frac{\varepsilon_0 A V_0^2}{2}\left(\frac{9}{x^2} - \frac{6}{dx} + \frac{1}{d^2}\right)dx$$

$$+\int_{x=2d}^{d} \frac{\varepsilon_0 A V_0^2}{2x^2}\,dx$$

$$= \frac{\varepsilon_0 A V_0^2}{2}\left[-\frac{9}{x} - \frac{6}{d}\ln x + \frac{x}{d^2}\right]_{x=d}^{2d}$$

$$+\frac{\varepsilon_0 A V_0^2}{2}\left[-\frac{1}{x}\right]_{x=2d}^{d}$$

$$= \frac{\varepsilon_0 A V_0^2}{2d}\left[\frac{9}{2} - 6\ln 2 + 1 - \frac{1}{2}\right]$$

$$= 0.4206\,\frac{\varepsilon_0 A V_0^2}{d}$$

Thus, an amount of energy $0.4206\,\dfrac{\varepsilon_0 A V_0^2}{d}$ is converted from mechanical to electrical form.

P5.42. (a) $F_{ex}\,dx = dW_m$

$$F_{ex} = \frac{dW_m}{dx}$$

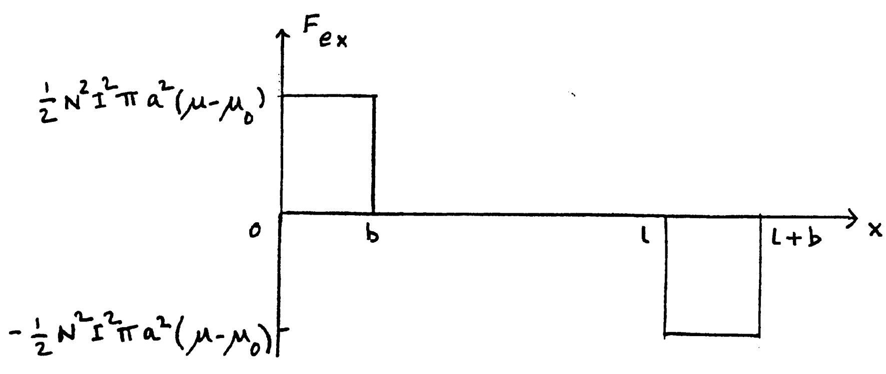
To find W_m, we note that inside the solenoid,

$$H = NI$$

$$\frac{1}{2}\mu H^2 = \begin{cases} \frac{1}{2}\mu N^2 I^2 & \text{in the core} \\ \frac{1}{2}\mu_0 N^2 I^2 & \text{outside the core} \end{cases}$$

$$W_m = \begin{cases} \frac{1}{2}\mu N^2 I^2 \pi a^2 x + \frac{1}{2}\mu_0 N^2 I^2 \pi a^2 (l - x) & \text{for } 0 < x < b \\ \frac{1}{2}\mu N^2 I^2 \pi a^2 b + \frac{1}{2}\mu_0 N^2 I^2 \pi a^2 (l - b) & \text{for } b < x < l \\ \frac{1}{2}\mu_0 N^2 I^2 \pi a^2 (x - b) + \frac{1}{2}\mu N^2 I^2 \pi a^2 (b - x - l) & \text{for } l < x < (l + b) \\ \frac{1}{2}\mu_0 N^2 I^2 \pi a^2 l & \text{for } x > (l + b) \end{cases}$$

$$\frac{dW_m}{dx} = \begin{cases} \frac{1}{2} N^2 I^2 \pi a^2 (\mu - \mu_0) & \text{for } 0 < x < b \\ 0 & \text{for } b < x < l \\ \frac{1}{2} N^2 I^2 \pi a^2 (\mu_0 - \mu) & \text{for } l < x < (l + b) \\ 0 & \text{for } x > (l + b) \end{cases}$$



P5.42. (continued)

(b) Note that $b < (b + l)/2$, since $b < l$.

$$\oint_{ABCDA} F_{ex}\, dx = \int_{b/2}^{b} \frac{1}{2} N^2 I_0^2 \pi a^2 (\mu - \mu_0)\, dx$$

$$+ \int_{b}^{b/2} \frac{1}{2} N^2 (2I_0)^2 \pi a^2 (\mu - \mu_0)\, dx$$

$$= \frac{1}{4} N^2 I_0^2 \pi a^2 b(\mu - \mu_0) - N^2 I_0^2 \pi a^2 b(\mu - \mu_0)$$

$$= -\frac{3}{4} N^2 I_0^2 \pi a^2 b(\mu - \mu_0)$$

$$\underset{\text{input}}{W_{\text{mech.}}} = -\oint F_{ex}\, dx$$

$$= \frac{3}{4} N^2 I_0^2 \pi a^2 b(\mu - \mu_0)$$

Thus an amount of energy $\frac{3}{4} N^2 I_0^2 \pi a^2 b(\mu - \mu_0)$ is converted from mechanical form to electrical form.

R5.1. Angle between the two planes

= Angle between the normal vectors to the planes

Unit vector normal to the plane $a_1x + a_2y + a_3z = C_1$ is

$$\frac{a_1\mathbf{a}_x + a_2\mathbf{a}_y + a_3\mathbf{a}_z}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

Unit vector normal to the plane $b_1x + b_2y + b_3z = C_2$ is

$$\frac{b_1\mathbf{a}_x + b_2\mathbf{a}_y + b_3\mathbf{a}_z}{\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$\therefore \cos\alpha = \frac{a_1\mathbf{a}_x + a_2\mathbf{a}_y + a_3\mathbf{a}_z}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \bullet \frac{b_1\mathbf{a}_x + b_2\mathbf{a}_y + b_3\mathbf{a}_z}{\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$\alpha = \cos^{-1} \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)}}$$

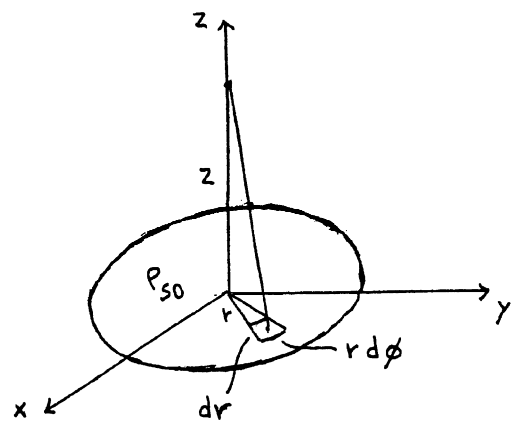
Angle between the planes $x + y + z = 1$ and $z = 0$ is

$$\cos^{-1} \frac{(1)(0) + (1)(0) + (1)(1)}{\sqrt{(1+1+1)(0+0+1)}}$$

$$= \cos^{-1} \frac{1}{\sqrt{3}}$$

$$= 54.736^\circ$$

R5.2.



$$dV = \frac{\rho_{S0}(dr)(r\,d\phi)}{4\pi\varepsilon_0\sqrt{r^2+z^2}}$$

$$V = \int_{r=0}^{a}\int_{\phi=0}^{2\pi} dV$$

$$= \int_{r=0}^{a}\int_{\phi=0}^{2\pi} \frac{\rho_{S0}\, r\,dr\,d\phi}{4\pi\varepsilon_0\sqrt{r^2+z^2}}$$

$$= \frac{\rho_{S0}}{2\varepsilon_0}\int_0^a \frac{r\,dr}{\sqrt{r^2+z^2}}$$

$$= \frac{\rho_{S0}}{2\varepsilon_0}\left[\sqrt{r^2+z^2}\,\right]_{r=0}^{a}$$

$$= \frac{\rho_{S0}}{2\varepsilon_0}\left(\sqrt{a^2+z^2} - |z|\right)$$

For $|z| >> a$,

$$\sqrt{a^2+z^2} = |z|\left(1+\frac{a^2}{z^2}\right)^{1/2}$$

$$\approx |z|\left(1+\frac{a^2}{2z^2}\right)$$

$$= |z| + \frac{a^2}{2|z|}$$

R5.2. (continued)

$$V \approx \frac{\rho_{S0}}{2\varepsilon_0}\left(|z| + \frac{a^2}{2|z|} - |z|\right)$$

$$= \frac{\rho_{S0}a^2}{4\varepsilon_0|z|} = \frac{\rho_{S0}(\pi a^2)}{4\pi\varepsilon_0|z|}$$

same as that for a point charge equal to the total charge on the disk located at the origin, as it should be since for $|z| >> a$ the disk appears to be almost like a point.

$$\mathbf{E} = -\nabla V = -\frac{\partial}{\partial z}\left[\frac{\rho_{S0}}{2\varepsilon_0}\left(\sqrt{a^2 + z^2} - |z|\right)\right]$$

$$= -\frac{\rho_{S0}}{2\varepsilon_0}\left(\frac{z}{\sqrt{a^2 + z^2}} \mp 1\right)\mathbf{a}_z \text{ for } z \gtrless 0$$

$$= \pm\frac{\rho_{S0}}{2\varepsilon_0}\left(1 \mp \frac{z}{\sqrt{a^2 + z^2}}\right)\mathbf{a}_z \text{ for } z \gtrless 0$$

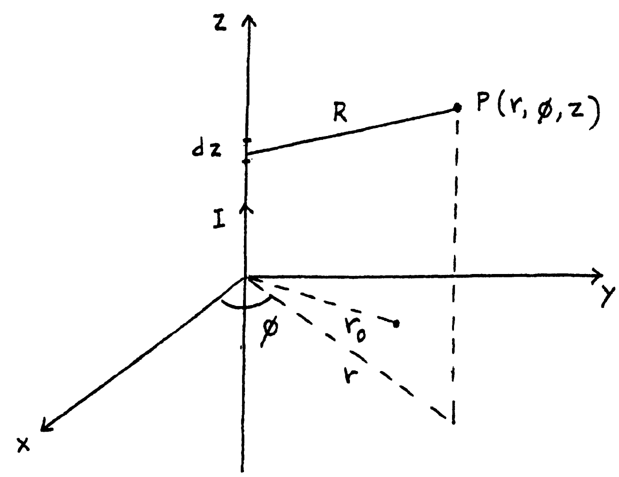
$$= \pm\frac{\rho_{S0}}{2\varepsilon_0}\left(1 - \frac{|z|}{\sqrt{a^2 + z^2}}\right)\mathbf{a}_z \text{ for } z \gtrless 0$$

For $|z| << a$, $\dfrac{|z|}{\sqrt{a^2 + z^2}} \approx \dfrac{|z|}{a} << 1$,

$$\mathbf{E} \approx \pm\frac{\rho_{S0}}{2\varepsilon_0}\mathbf{a}_z \text{ for } z \gtrless 0$$

same as that for an infinite plane sheet of surface charge density ρ_{S0}, as it should be, since for $|z| << a$ the disk appears to be very large in extent.

R5.3.



The magnetic vector potential due to an infinitesimal current element $I\,dz\,\mathbf{a}_z$ on the z-axis at a point $P(r, \phi, z)$ is given by

$$d\mathbf{A} = \frac{\mu_0 I\,dz\,\mathbf{a}_z}{4\pi R}$$

This is analogous to the electric potential at P due to an infinitesimal length of line charge $\rho_{L0}\,dz$ on the z-axis at the same location as the current element, which is given by

$$dV = \frac{\rho_{L0}\,dz}{4\pi\varepsilon_0 R}$$

Because of this, we can write the expression for $\mathbf{A}$ due to the entire line current from analogy with V due to an infinitely long line charge of uniform density ρ_{L0} on the z-axis. From Ex. 5.4 of the text, this is given by

$$V = -\frac{\rho_{L0}}{2\pi\varepsilon_0}\ln\frac{r}{r_0}$$

where r_0 is the reference value of r. Thus, the magnetic vector potential due to the infinitely long line current along the z-axis is given by

$$\mathbf{A} = -\left(\frac{\mu_0 I}{2\pi}\ln\frac{r}{r_0}\right)\mathbf{a}_z$$

R5.3. (continued)

The magnetic flux density is then given by

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix}$$

$$= -\frac{\partial A_z}{\partial r}\mathbf{a}_\phi$$

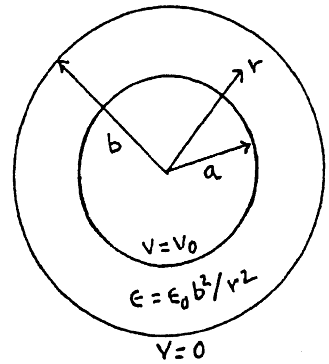
$$= \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi$$

as it should be.

R5.4. $\varepsilon\nabla^2 V + \nabla\varepsilon \bullet \nabla V = 0$

$$\frac{\varepsilon}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \left(\frac{\partial \varepsilon}{\partial r}\right)\left(\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\varepsilon_0 b^2}{r^4}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) - \frac{2\varepsilon_0 b^2}{r^3}\frac{\partial V}{\partial r} = 0$$



$$r^2\frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} - 2r\frac{\partial V}{\partial r} = 0$$

$$\frac{\partial^2 V}{\partial r^2} = 0$$

$$V = Ar + B$$

$V = 0$ for $r = b \rightarrow Ab + B = 0,\ B = -Ab$

$V = V_0$ for $r = a \rightarrow Aa - Ab = V_0,\ A = \dfrac{V_0}{(a-b)}$

$$\therefore\quad V = \frac{V_0}{(a-b)}r - \frac{V_0}{(a-b)}b = \frac{V_0(r-b)}{(a-b)}$$

To find the capacitance of the arrangement, we obtain

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\mathbf{a}_r = \frac{V_0}{(b-a)}\mathbf{a}_r$$

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\frac{b^2}{r^2}\frac{V_0}{(b-a)}\mathbf{a}_r$$

$$[\rho_S]_{r=a} = \mathbf{a}_r \bullet [\mathbf{D}]_{r=a} = \varepsilon_0\frac{b^2}{a^2}\frac{V_0}{(b-a)}$$

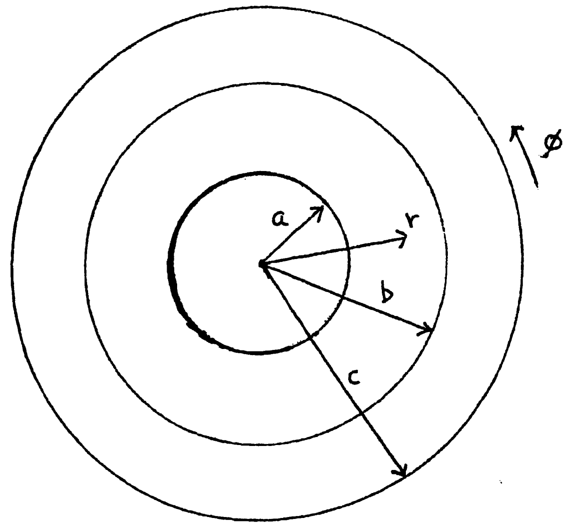
$$[\rho_S]_{r=b} = -\mathbf{a}_r \bullet [\mathbf{D}]_{r=b} = -\varepsilon_0\frac{V_0}{(b-a)}$$

Magnitude of charge on either conductor

$$= 4\pi a^2|\rho_S|_{r=a} = 4\pi b^2|\rho_S|_{r=b} = \frac{4\pi\varepsilon_0 b^2 V_0}{(b-a)}$$

$$\text{Capacitance} = \frac{4\pi\varepsilon_0 b^2}{(b-a)}$$

R5.5.



Using Ampere's circuital law in integral form, we have for $a < r < b$

$$2\pi r H_\phi = I\frac{\pi r^2 - \pi a^2}{\pi b^2 - \pi a^2}$$

$$H_\phi = \frac{I}{2\pi r}\left(\frac{r^2 - a^2}{b^2 - a^2}\right)$$

Considering a rectangle of infinitesimal width dr in the r-direction and length l in the z-direction at a distance r from the z-axis, where $a < r < b$, we obtain

$$d\psi_i = \frac{\mu I l}{2\pi r}\left(\frac{r^2 - a^2}{b^2 - a^2}\right) dr$$

$$N = \frac{r^2 - a^2}{b^2 - a^2}$$

$$N\, d\psi_i = \frac{\mu I l}{2\pi r}\left(\frac{r^2 - a^2}{b^2 - a^2}\right)^2 dr$$

R5.5. (continued)

$$\psi_i = \int N \, d\psi_i$$

$$= \int_a^b \frac{\mu I l}{2\pi r}\left(\frac{r^2 - a^2}{b^2 - a^2}\right)^2 dr$$

$$= \frac{\mu I l}{2\pi(b^2 - a^2)^2}\int_a^b \left(r^3 - 2a^2 r + \frac{a^4}{r}\right) dr$$

$$= \frac{\mu I l}{2\pi(b^2 - a^2)^2}\left[\frac{r^4}{4} - a^2 r^2 + a^4 \ln r\right]_a^b$$

$$= \frac{\mu I l}{2\pi(b^2 - a^2)^2}\left[\frac{b^4 - a^4}{4} - a^2(b^2 - a^2) + a^4 \ln \frac{b}{a}\right]$$

$$= \frac{\mu I l}{8\pi(b^2 - a^2)^2}\left(b^4 + 3a^4 - 4a^2 b^2 + 4a^4 \ln \frac{b}{a}\right)$$

The internal inductance per unit length is given by

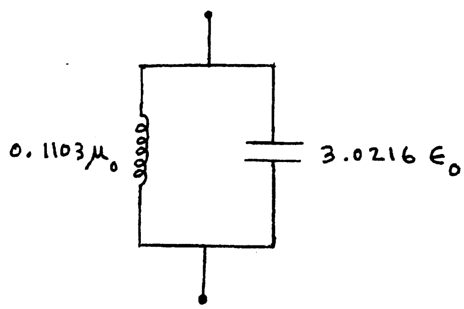
$$\mathcal{L}_i = \frac{\psi_i}{Il} = \frac{\mu_0}{8\pi(b^2 - a^2)^2}\left(b^4 + 3a^4 - 4a^2 b^2 + 4a^4 \ln \frac{b}{a}\right)$$

R5.6.

$$L = \frac{\mu_0 l}{2\pi} \ln \frac{b}{a} = \frac{\mu_0}{2\pi} \ln 2 = 0.1103\mu_0$$

$$C = \frac{2\pi\varepsilon_0}{\ln \frac{b}{a}} = \frac{2\pi\varepsilon_0}{\ln 2} = 9.0647\varepsilon_0$$

The equivalent circuit is given by



Resonant frequency is

$$f_0 = \frac{1}{2\pi\sqrt{(0.1103\mu_0)(3.0216\varepsilon_0)}}$$

$$= 82.7 \times 10^6 \text{ Hz} = 82.7 \text{ MHz}$$

For quasistatic approximations,

$$f << \frac{1}{2\pi\sqrt{\mu\varepsilon l}} = \frac{3 \times 10^8}{2\pi} = 47.77 \text{ MHz}$$

f_0 is much larger than the frequencies for which the quasistatic approximation is valid.

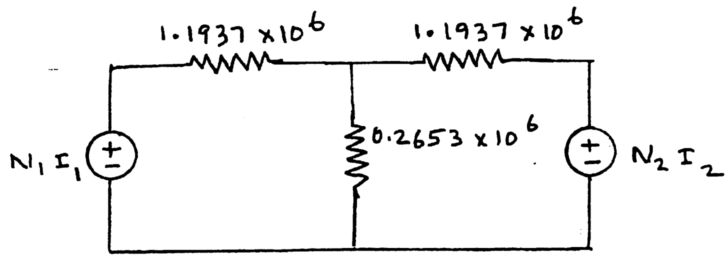
R5.7. **(a)** The reluctances of the three legs are given by

$$\mathcal{R}_1 = \frac{l_1}{\mu A_1} = \frac{0.3}{1000\mu_0 \times 2 \times 10^{-4}} = 1.1937 \times 10^6 \text{ A-t/Wb}$$

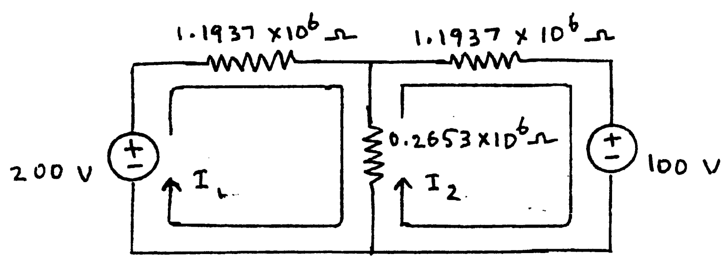
$$\mathcal{R}_2 = \frac{l_2}{\mu A_2} = \frac{0.1}{1000\mu_0 \times 3 \times 10^{-4}} = 0.2653 \times 10^6 \text{ A-t/Wb}$$

$$\mathcal{R}_3 = \frac{l_3}{\mu A_3} = \frac{0.3}{1000\mu_0 \times 2 \times 10^{-4}} = 1.1937 \times 10^6 \text{ A-t/Wb}$$

The equivalent electric circuit is given by



(b)



$$1.459 \times 10^6 I_1 - 0.2653 \times 10^6 I_2 = 200$$

$$-0.2653 \times 10^6 I_1 + 1.459 \times 10^6 I_2 = -100$$

Solving, we get

$$I_1 = \frac{265.27 \times 10^6}{2.0583 \times 10^{12}} = 1.2888 \times 10^{-4}$$

$$I_2 = \frac{-92.84 \times 10^6}{2.0583 \times 10^{12}} = -0.4511 \times 10^{-4}$$

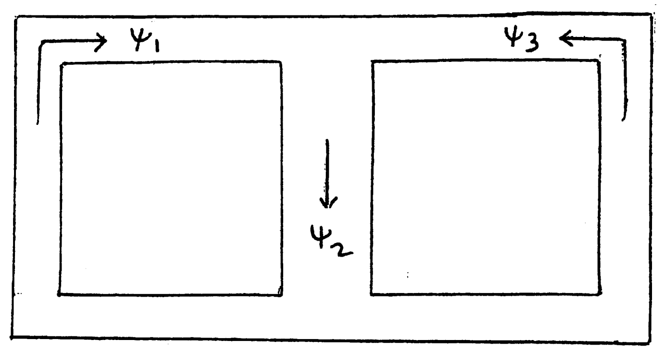
R5.7. (continued)

The magnetic fluxes are

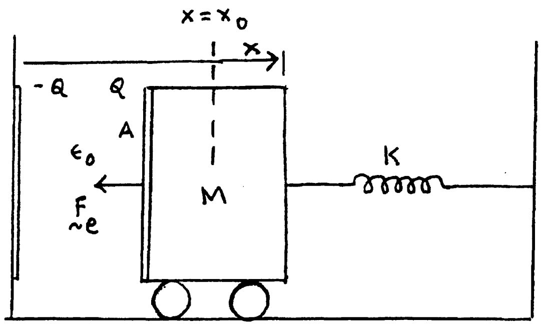
$\psi_1 = 1.2888 \times 10^{-4}$ Wb

$\psi_2 = (1.2888 + 0.4511) \times 10^{-4} = 1.7399 \times 10^{-4}$ Wb

$\psi_3 = 0.4511 \times 10^{-4}$ Wb



R5.8.



The differential equation of motion of M is

$$M\frac{d^2x}{dt^2} = -K(x - x_0) + F_{ex}$$

$$= -K(x - x_0) - \frac{Q^2}{2A\varepsilon_0}$$

$$M\frac{d^2x}{dt^2} + Kx = Kx_0 - \frac{Q^2}{2A\varepsilon_0}$$

The solution is given by

$$x = \left(x_0 - \frac{Q^2}{2A\varepsilon_0 K}\right) + C\cos\sqrt{\frac{K}{M}}\,t$$

The initial condition is $x = x_0$ at $t = 0$. Therefore

$$x_0 = x_0 - \frac{Q^2}{2A\varepsilon_0 K} + C$$

$$C = \frac{Q^2}{2A\varepsilon_0 K}$$

$$x = \left(x_0 - \frac{Q^2}{2A\varepsilon_0 K}\right) + \frac{Q^2}{2A\varepsilon_0 K}\cos\sqrt{\frac{K}{M}}\,t$$