Hence the difference of any two solutions of the nonhomogeneous equation (2) is a solution of the homogeneous equation (4). Since any solution of the homogeneous equation can be expressed as a linear combination of a fundamental set of solutions $y_1, \ldots, y_n$, it follows that any solution of Eq. (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \tag{9}$$

where Y is some particular solution of the nonhomogeneous equation (2). The linear combination (9) is called the general solution of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions $y_1, \ldots, y_n$ of the homogeneous equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

The method of reduction of order (Section 3.5) also applies to nth order linear equations. If y_1 is one solution of Eq. (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n-1$ for v' (see Problem 26 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

PROBLEMS

In each of Problems 1 through 6 determine intervals in which solutions are sure to exist.

1. $y^{(4)} + 4y''' + 3y = t$
2. $ty''' + (\sin t)y'' + 3y = \cos t$
3. $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$
4. $y''' + ty'' + t^2 y' + t^3 y = \ln t$
5. $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$
6. $(x^2 - 4)y^{(6)} + x^2 y''' + 9y = 0$

In each of Problems 7 through 10 determine whether the given set of functions is linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

7. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$
8. $f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t$
9. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$
10. $f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$

In each of Problems 11 through 16 verify that the given functions are solutions of the differential equation, and determine their Wronskian.

11. $y''' + y' = 0; \qquad 1, \quad \cos t, \quad \sin t$
12. $y^{(4)} + y'' = 0; \qquad 1, \quad t, \quad \cos t, \quad \sin t$
13. $y''' + 2y'' - y' - 2y = 0; \qquad e^t, \quad e^{-t}, \quad e^{-2t}$
14. $y^{(4)} + 2y''' + y'' = 0; \qquad 1, \quad t, \quad e^{-t}, \quad te^{-t}$
15. $xy''' - y'' = 0; \qquad 1, \quad x, \quad x^3$
16. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \qquad x, \quad x^2, \quad 1/x$
17. Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?
18. Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2],$$

where y_1 and y_2 are n times differentiable functions and c_1 and c_2 are arbitrary constants. Hence, show that if $y_1, y_2, \ldots, y_n$ are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \cdots + c_ny_n$ is also a solution of $L[y] = 0$.

19. Let the linear differential operator L be defined by

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny,$$

where $a_0, a_1, \ldots, a_n$ are real constants.

(a) Find $L[t^n]$.

(b) Find $L[e^{rt}]$.

(c) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Why?

20. In this problem we show how to generalize Theorem 3.3.2 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2, and y_3 be solutions of this equation on an interval I.

(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

(b) Substitute for y_1''', y_2''', and y_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c \exp\left[-\int p_1(t)\,dt\right].$$

It follows that W is either always zero or nowhere zero on I.

(d) Generalize this argument to the nth order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions $y_1, \ldots, y_n$. That is, establish Abel's formula,

$$W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\,dt\right],$$

for this case.

In each of Problems 21 through 24 use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

21. $y''' + 2y'' - y' - 3y = 0$
22. $y^{(4)} + y = 0$
23. $ty''' + 2y'' - y' + ty = 0$
24. $t^2y^{(4)} + ty''' + y'' - 4y = 0$

25. The purpose of this problem is to show that if $W(y_1, \dots, y_n)(t_0) \neq 0$ for some t_0 in an interval I, then $y_1, \dots, y_n$ are linearly independent on I, and if they are linearly independent and solutions of

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{i}$$

on I, then $W(y_1, \dots, y_n)$ is nowhere zero in I.

(a) Suppose that $W(y_1, \dots, y_n)(t_0) \neq 0$, and suppose that

$$c_1y_1(t) + \cdots + c_ny_n(t) = 0 \tag{ii}$$

for all t in I. By writing the equations corresponding to the first $n-1$ derivatives of Eq. (ii) at t_0, show that $c_1 = \cdots = c_n = 0$. Therefore, $y_1, \dots, y_n$ are linearly independent.

(b) Suppose that $y_1, \dots, y_n$ are linearly independent solutions of Eq. (i).
If $W(y_1, \dots, y_n)(t_0) = 0$ for some t_0, show that there is a nonzero solution of Eq. (i) satisfying the initial conditions

$$y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0.$$

Since $y = 0$ is a solution of this initial value problem, the uniqueness part of Theorem 4.1.1 yields a contradiction. Thus W is never zero.

26. Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second order equation for v':

$$y_1v''' + (3y_1' + p_1y_1)v'' + (3y_1'' + 2p_1y_1' + p_2y_1)v' = 0.$$

In each of Problems 27 and 28 use the method of reduction of order (Problem 26) to solve the given differential equation.

27. $(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2; \qquad y_1(t) = e^t$

28. $t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^3$

4.2 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0, \tag{1}$$

where $a_0, a_1, \dots, a_n$ are real constants. From our knowledge of second order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (1) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt}Z(r) \tag{2}$$

for all r, where

$$Z(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n. \tag{3}$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution of Eq. (1). The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (1).

The constants are determined by substituting into the individual differential equations; they are $A_0 = -\frac{1}{8}$, $A_1 = 0$, $B = 0$, $C = -\frac{3}{5}$, and $E = \frac{1}{8}$. Hence a particular solution of Eq. (8) is

$$Y(t) = -\tfrac{1}{8}t^2 - \tfrac{3}{5}\sin t + \tfrac{1}{8}te^{-2t}. \tag{9}$$

You should keep in mind that the amount of algebra required to calculate the coefficients may be quite substantial for higher order equations, especially if the nonhomogeneous term is even moderately complicated. A computer algebra system can be extremely helpful in executing these algebraic calculations.

The method of undetermined coefficients can be used whenever it is possible to guess the correct form for $Y(t)$. However, this is usually impossible for differential equations not having constant coefficients, or for nonhomogeneous terms other than the type described previously. For more complicated problems we can use the method of variation of parameters, which is discussed in the next section.

PROBLEMS

In each of Problems 1 through 8 determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$
2. $y^{(4)} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$
4. $y''' - y' = 2\sin t$
5. $y^{(4)} - 4y'' = t^2 + e^t$
6. $y^{(4)} + 2y'' + y = 3 + \cos 2t$
7. $y^{(6)} + y''' = t$
8. $y^{(4)} + y''' = \sin 2t$

In each of Problems 9 through 12 find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + 4y' = t$; $\quad y(0) = y'(0) = 0, \quad y''(0) = 1$
10. $y^{(4)} + 2y'' + y = 3t + 4$; $\quad y(0) = y'(0) = 0, \quad y''(0) = y'''(0) = 1$
11. $y''' - 3y'' + 2y' = t + e^t$; $\quad y(0) = 1, \quad y'(0) = -\frac{1}{4}, \quad y''(0) = -\frac{3}{2}$
12. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t}$; $\quad y(0) = 3, \quad y'(0) = 0,$ $y''(0) = -1, \quad y'''(0) = 2$

In each of Problems 13 through 18 determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

13. $y''' - 2y'' + y' = t^3 + 2e^t$
14. $y''' - y' = te^{-t} + 2\cos t$
15. $y^{(4)} - 2y'' + y = e^t + \sin t$
16. $y^{(4)} + 4y'' = \sin 2t + te^t + 4$
17. $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$
18. $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$
19. Consider the nonhomogeneous nth order linear differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = g(t), \tag{i}$$

where $a_0, \ldots, a_n$ are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t}(b_0t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t}u(t)$ reduces Eq. (i) to the form

$$k_0u^{(n)} + k_1u^{(n-1)} + \cdots + k_nu = b_0t^m + \cdots + b_m, \tag{ii}$$

where $k_0, \ldots, k_n$ are constants. Determine k_0 and k_n in terms of the a's and α. Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Method of Annihilators. In Problems 20 through 22 we consider another way of arriving at the proper form of $Y(t)$ for use in the method of undetermined coefficients. The procedure is based on the observation that exponential, polynomial, or sinusoidal terms (or sums and products of such terms) can be viewed as solutions of certain linear homogeneous differential equations with constant coefficients. It is convenient to use the symbol D for d/dt. Then, for example, e^{-t} is a solution of $(D+1)y = 0$; the differential operator $D+1$ is said to *annihilate*, or to be an *annihilator* of, e^{-t}. Similarly, D^2+4 is an annihilator of $\sin 2t$ or $\cos 2t$, $(D-3)^2 = D^2 - 6D + 9$ is an annihilator of e^{3t} or te^{3t}, and so forth.

20. Show that linear differential operators with constant coefficients obey the commutative law. That is, show that

$$(D-a)(D-b)f = (D-b)(D-a)f$$

for any twice differentiable function f and any constants a and b. The result extends at once to any finite number of factors.

21. Consider the problem of finding the form of a particular solution $Y(t)$ of

$$(D-2)^3(D+1)Y = 3e^{2t} - te^{-t}, \tag{i}$$

where the left side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.

(a) Show that $D-2$ and $(D+1)^2$, respectively, are annihilators of the terms on the right side of Eq. (i), and that the combined operator $(D-2)(D+1)^2$ annihilates both terms on the right side of Eq. (i) simultaneously.

(b) Apply the operator $(D-2)(D+1)^2$ to Eq. (i) and use the result of Problem 20 to obtain

$$(D-2)^4(D+1)^3Y = 0. \tag{ii}$$

Thus Y is a solution of the homogeneous equation (ii). By solving Eq. (ii), show that

$$Y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4t^3e^{2t} + c_5e^{-t} + c_6te^{-t} + c_7t^2e^{-t}, \tag{iii}$$

where $c_1, \ldots, c_7$ are constants, as yet undetermined.

(c) Observe that e^{2t}, te^{2t}, t^2e^{2t}, and e^{-t} are solutions of the homogeneous equation corresponding to Eq. (i); hence these terms are not useful in solving the nonhomogeneous equation. Therefore, choose c_1, c_2, c_3, and c_5 to be zero in Eq. (iii), so that

$$Y(t) = c_4t^3e^{2t} + c_6te^{-t} + c_7t^2e^{-t}. \tag{iv}$$

This is the form of the particular solution Y of Eq. (i). The values of the coefficients c_4, c_6, and c_7 can be found by substituting from Eq. (iv) in the differential equation (i).

Summary. Suppose that

$$L(D)y = g(t), \tag{v}$$

where $L(D)$ is a linear differential operator with constant coefficients, and $g(t)$ is a sum or product of exponential, polynomial, or sinusoidal terms. To find the form of a particular solution of Eq. (v), you can proceed as follows:

(a) Find a differential operator $H(D)$ with constant coefficients that annihilates $g(t)$, that is, an operator such that $H(D)g(t) = 0$.

(b) Apply $H(D)$ to Eq. (v), obtaining

$$H(D)L(D)y = 0, \tag{vi}$$

which is a homogeneous equation of higher order.

(c) Solve Eq. (vi).

(d) Eliminate from the solution found in step (c) the terms that also appear in the solution of $L(D)y = 0$. The remaining terms constitute the correct form of a particular solution of Eq. (v).

22. Use the method of annihilators to find the form of a particular solution $Y(t)$ for each of the equations in Problems 13 through 18. Do not evaluate the coefficients.

4.4 The Method of Variation of Parameters

The method of variation of parameters for determining a particular solution of the nonhomogeneous nth order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{1}$$

is a direct extension of the method for the second order differential equation (see Section 3.7). As before, to use the method of variation of parameters, it is first necessary to solve the corresponding homogeneous differential equation. In general, this may be difficult unless the coefficients are constants. However, the method of variation of parameters is still more general than the method of undetermined coefficients in that it leads to an expression for the particular solution for *any* continuous function g, whereas the method of undetermined coefficients is restricted in practice to a limited class of functions g.

Suppose then that we know a fundamental set of solutions $y_1, y_2, \ldots, y_n$ of the homogeneous equation. Then the general solution of the homogeneous equation is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t). \tag{2}$$

The method of variation of parameters for determining a particular solution of Eq. (1) rests on the possibility of determining n functions $u_1, u_2, \ldots, u_n$ such that $Y(t)$ is of the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t). \tag{3}$$

Since we have n functions to determine, we will have to specify n conditions. One of these is clearly that Y satisfy Eq. (1). The other $n-1$ conditions are chosen so as to make the calculations as simple as possible. Since we can hardly expect a simplification in determining Y if we must solve high order differential equations for $u_1, \ldots, u_n$, it is natural to impose conditions to suppress the terms that lead to higher derivatives of $u_1, \ldots, u_n$. From Eq. (3) we obtain

$$Y' = (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n), \tag{4}$$