Chapter 2: DIFFERENTIATION

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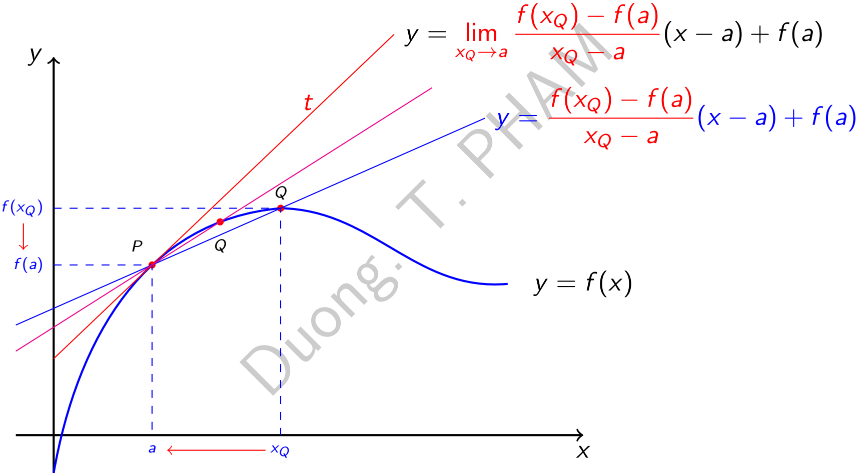
CALCULUS I

Outline

Tangent line



Tangent line

Definition.

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Ex: Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

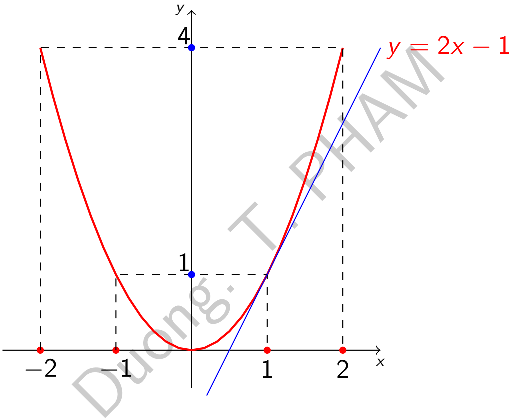
Ans: The slope of the tangent line at the point $P(1, 1)$ is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

The tangent line is

$$y - 1 = 2(x - 1) \quad (y = 2x - 1)$$

Tangent line



Graph of function $f(x) = x^2$

Figure: The CN Tower in Toronto

Example. Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Ans. Denote by $s(t)$: the distance fallen after t seconds. Galileo's law gives

$$s(t) = 4.9t^2.$$

We can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49\text{m/s}.\end{aligned}$$

The Velocity Problem

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The instantaneous velocity when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus the (instantaneous) velocity after 5 s is

$$v(5) = 49\,\text{m/s}.$$

Definition.

The derivative of a function f at a number $x = a$, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists

Remark: The limit in the above definition can be replaced by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Ex: Find the derivative of $f(x) = x^2 + 2x + 3$ at the number $x = a$

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{(a+h)^2 + 2(a+h) + 3 - (a^2 + 2a + 3)}{h} \\
&= \lim_{h\to 0} \frac{(a+h)^2 - a^2 + 2h}{h} = \lim_{h\to 0} \frac{(2a+h)h + 2h}{h} \\
&= \lim_{h\to 0} (2a + h + 2) \\
&= 2a + 2
\end{aligned}
$$

Corollary.

The tangent line to to the curve $y = f(x)$ at the point $(a, f(a))$ is given by

$$y - f(a) = f'(a)(x - a)$$

Ex: Find an equation of the tangent line to the parabola $y = x^2 + 2x + 3$ at the point $(0, 3)$

Ans: In the previous example, we have found that

$$f'(a) = 2a + 2.$$

Thus, $f'(0) = 2$. Applying the above corollary, the desired tangent line is

$$y - 3 = 2(x - 0) \quad \text{or} \quad y = 2x + 3$$

Rate of change

- Given a function $y = f(x)$, if the variable x change from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the correponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

- The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of** y **with respect to** x **over the interval** $[x_1, x_2]$

Rate of change

- The **instantaneous rate of change of y w.r.t.** x at $x = x_1$ is

$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- Note here that

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Definition.

Given $f : D \to \mathbb{R}$, denote $D^* := \{x \in D \text{ such that } f'(x) \text{ exists}\}$. The mapping

$$\begin{aligned} f' : \quad & D^* \to \mathbb{R} \\ & x \mapsto f'(x) \end{aligned}$$

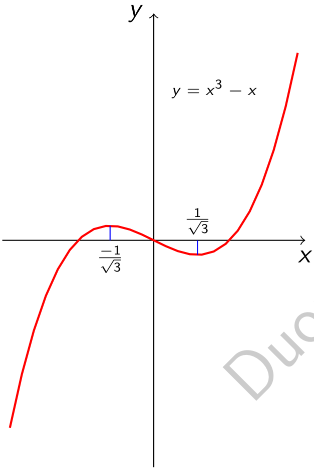
is a function of x and called **the derivative of** f

Ex: Given $f(x) = x^3 - x$. Find $f'(x)$.

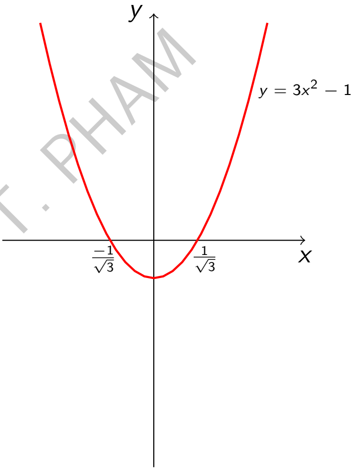
Ans: We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h} \\ &= \lim_{h\to 0} \frac{h[(x+h)^2 + (x+h)x + x^2] - h}{h} \\ &= \lim_{h\to 0} \left((x+h)^2 + (x+h)x + x^2 - 1\right) \\ &= 3x^2 - 1 \end{aligned}$$

Derivative



y
$y = x^3 - x$
$\frac{1}{\sqrt{3}}$
$\frac{-1}{\sqrt{3}}$
x



y
$y = 3x^2 - 1$
$\frac{-1}{\sqrt{3}}$
$\frac{1}{\sqrt{3}}$
x

Differentiable functions

Definition.

A function f is **differentiable at** $x = a$ if $f'(a)$ exists.
It is **differentiable on an interval** (a, b) $\big($or $(-\infty, a)$ or (a, ∞) or $(-\infty, \infty)$ $\big)$ if it is differentiable at every point in the interval

Remark: The following notations can be used to indicate the the derivative of a function $y = f(x)$ at the number x:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Ex: Determine when $f(x) = |x|$ is differentiable?

- $x > 0$: then $f(x) = |x| = x$ and for sufficiently small $|h|$, we have $x + h > 0$. Thus
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{x+h-x}{h}$$
$$= \lim_{h\to 0} 1 = 1$$

$\Longrightarrow f$ is differentiable on $(0, \infty)$

- $x < 0$: then $f(x) = |x| = -x$ and for sufficiently small $|h|$, we have $x + h < 0$. Thus
$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h\to 0} (-1) = -1$$

$\Longrightarrow f$ is differentiable on $(-\infty, 0)$

Ex: Determine when $f(x) = |x|$ is differentiable?

- $x = 0$: then

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

and

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1.$$

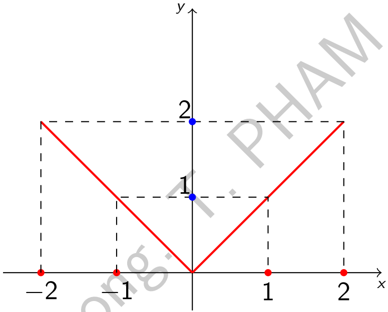
We note here that

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \neq \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$

$\Longrightarrow \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ does not exist

$\Longrightarrow f$ is NOT differentiable at $x = 0$

Conclusion: f is differentiable in $(-\infty, 0) \cup (0, \infty)$.



Graph of function $f(x) = |x|$

Theorem.

If f is differentiable at a then f is continuous at a

Proof: f is differentiable at a $\Longrightarrow \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists ($= L$)

- Then
$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right)$$
$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)$$
$$= L\cdot 0 = 0.$$

- Thus,
$$\lim_{x\to a} f(x) = \lim_{x\to a}[f(x)-f(a)+f(a)]$$
$$= \lim_{x\to a}[f(x)-f(a)] + \lim_{x\to a} f(a)$$
$$= 0 + f(a) = f(a).$$

- $\Longrightarrow$ f is continuous at a.

If f is a differentiable function, then f' is also function. If f' also has a derivative, we then denote $f'' = (f')'$, and f'' is called the **second derivative** of f. We can write

$$f'' = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}$$

Ex: Given $f(x) = x^3 - x$. Find $f''(x)$.

Ans: On slide 52, we have found that $f'(x) = 3x^2 - 1$. Thus

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2 - 1 - (3x^2 - 1)}{h} \\ &= \lim_{h\to 0} \frac{3h(2x+h)}{h} = \lim_{h\to 0}[3(2x+h)] \\ &= 6x \end{aligned}$$

Def: The third derivative f''' is defined to be the derivative of f'', i.e., $f''' = (f'')'$ and so on ...

Derivative of a constant function

Let c be a constant. Then

$$\frac{d}{dx}(c) = 0$$

Proof: We have $f(x) = c$. Then

$$\begin{aligned}\frac{d}{dx}(c) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

Derivatives of power functions

The power rule: If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof: If $f(x) = x^n$, then

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{h\big((x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}\big)}{h} \\
&= \lim_{h\to 0} \big((x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}\big) \\
&= nx^{n-1}
\end{aligned}$$

The power rule (General version): If α is any real number , then

$$\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$$

Ex: Find $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$ and $\dfrac{d}{dx}\left(\sqrt{x}\right)$

Ans:

- $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = \dfrac{d}{dx}\left(x^{-2}\right) = (-2)x^{-2-1} = -2x^{-3} = \dfrac{-2}{x^3}$
- $\dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{d}{dx}\left(x^{1/2}\right) = \dfrac{1}{2}x^{\frac{1}{2}-1} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

The constant multiple rule

The constant multiple rule: If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Proof: Let $g(x) = cf(x)$, then

$$\begin{aligned}\frac{d}{dx}g(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0}\left(c\frac{f(x+h)-f(x)}{h}\right) = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= c\frac{d}{dx}f(x)\end{aligned}$$

Ex: $\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3\cdot 4x^3 = 12x^3$

The sum rule

The sum rule: If f are g are both differentiable functions, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof: Let $k(x) = f(x) + g(x)$, then

$$\begin{aligned}
\frac{d}{dx}k(x) &= \lim_{h\to 0}\frac{k(x+h)-k(x)}{h} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)
\end{aligned}$$

Remark: The sum rule can be extended to sums of any number of functions. For example,

$$(f+g+k)' = f' + g' + k'$$

Derivative of exponential functions

- $(e^x)' = e^x$
- $(a^x)' = a^x \ln a$
- $(\ln x)' = \dfrac{1}{x}$
- $(\log_a x)' = \dfrac{1}{x \ln a}$

The product rule

The product rule: If f and g are differentiable function, then

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

Proof: Let $k(x) = f(x)g(x)$, then

$$\begin{aligned}
\frac{d}{dx}k(x) &= \lim_{h\to 0}\frac{k(x+h)-k(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{[f(x+h)-f(x)]g(x+h)}{h} + \frac{f(x)[g(x+h)-g(x)]}{h}\right) \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\lim_{h\to 0}g(x+h) + f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)
\end{aligned}$$

The quotient rule: If u and v are differentiable function, then

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

Proof: Exercise

Ex: Let $y = \dfrac{x^2 - x + 3}{x + 2}$. Find y'.

Ans:

$$\begin{aligned} y' &= \frac{(x^2 - x + 3)'(x+2) - (x^2 - x + 3)(x+2)'}{(x+2)^2} \\ &= \frac{(2x-1)(x+2) - (x^2 - x + 3)}{(x+2)^2} \\ &= \frac{x^2 + 5x - 5}{(x+2)^2} \end{aligned}$$

Differentiation formulae

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$(cf)' = cf' \qquad (f \pm g)' = f' \pm g'$$

$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Derivatives of trigonometric functions

Recall that $\sec x = \dfrac{1}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$. The following identities are true:

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{\cos^2 x} \qquad (\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\csc x)' = -\csc x \cot x \qquad (\sec x)' = \sec x \tan x$$

The chain rule: Let f be differentiable at a and let g be differentiable at $f(a) \Longrightarrow$ the composition $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof: We have

$$\begin{aligned}(g \circ f)'(a) &= \lim_{h \to 0} \frac{(g \circ f)(a+h) - (g \circ f)(a)}{h} = \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h} \\ &= \lim_{h \to 0} \left(\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h} \right) \\ &= \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= g'(f(a)) \cdot f'(a)\end{aligned}$$

Note that in the last argument we use the fact that f is continuous at a because it is differentiable at a, and thus $f(a+h) \to f(a)$ as h goes to 0.

Ex: Let $k(x) = \sqrt{x^2+1}$. Find $f'(x)$

Ans:

- Denote $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$. Then

$$k(x) = g(f(x)) = (g \circ f)(x).$$

- Applying the Chain Rule,

$$k'(x) = g'(f(x)) \cdot f'(x).$$

- $g(x) = \sqrt{x} \Longrightarrow g'(x) = \frac{1}{2\sqrt{x}} \Longrightarrow g'(f(x)) = \frac{1}{2\sqrt{f(x)}} = \frac{1}{2\sqrt{x^2+1}}$

  $f(x) = x^2 + 1 \Longrightarrow f'(x) = 2x$

  $\Longrightarrow k'(x) = \dfrac{2x}{2\sqrt{x^2+1}} = \dfrac{x}{\sqrt{x^2+1}}$

For any $\alpha \in \mathbb{R}$, we have

$$\frac{d}{dx}u^{\alpha} = \alpha u^{\alpha-1}\frac{du}{dx}$$

Ex: Differentiate $y = (x^3 - 1)^{100}$.

Ans: We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(x^3-1)^{100}] = 100(x^3-1)^{100-1}(x^3-1)' \\ &= 100(x^3-1)^{99}(3x^2) \\ &= 300(x^3-1)^{99}x^2\end{aligned}$$

Example. The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(i) Find the velocity at time t.

(ii) What is the velocity after 2 s? After 4 s?

(iii) When is the particle at rest?

(iv) When is the particle moving forward (that is, in the positive direction)?

(v) Draw a diagram to represent the motion of the particle.

(vi) Find the total distance traveled by the particle during the first five seconds.

(vii) Find the acceleration at time and after 4s.

(viii) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(ix) When is the particle speeding up? When is it slowing down?

Ans.

(i) The position function: $s = f(t) = t^3 - 6t^2 + 9t$. The velocity is

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9.$$

(ii) The velocity after 2s means the instantaneous velocity when $t = 2$, that is,

$$v(2) = \left.\frac{ds}{dt}\right|_{t=2} = 3 \times 2^2 - 12 \times 2 + 9 = -3\text{m/s}.$$

The velocity after 4 s is

$$v(4) = 3 \times 4^2 - 12 \times 4 + 9 = 9\text{m/s}.$$

(iii) When is the particle at rest? The particle is at rest when $v(t) = 0$, that is

$$3t^2 - 12t + 9 = 0 \iff 3(t-1)(t-3) = 0 \iff t = 1 \text{ or } t = 3.$$

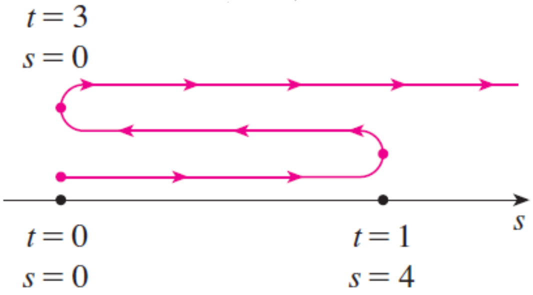
Thus the particle is at rest after 1 s and after 3 s.

(iv) When is the particle moving forward? The particle moves in the positive direction when $v(t) > 0$, that is,

$$v(t) > 0 \iff 3(t-1)(t-3) > 0 \iff t < 1 \text{ or } t > 3.$$

The particle is moving backward when $1 \leq t \leq 3$.

(v) Draw a diagram to represent the motion of the particle.



(vi) Total distance traveled after 5s? We need to calculate the distances traveled during the time intervals $[0, 1]$, $[1, 3]$, and $[3, 5]$ separately. The distance traveled in the first second is

$$|f(1) - f(0)| = |4 - 0| = 4.$$

From $t = 1$ to $t = 3$ the distance traveled is

$$|f(3) - f(1)| = |0 - 4| = 4.$$

From $t = 3$ to $t = 5$ the distance traveled is

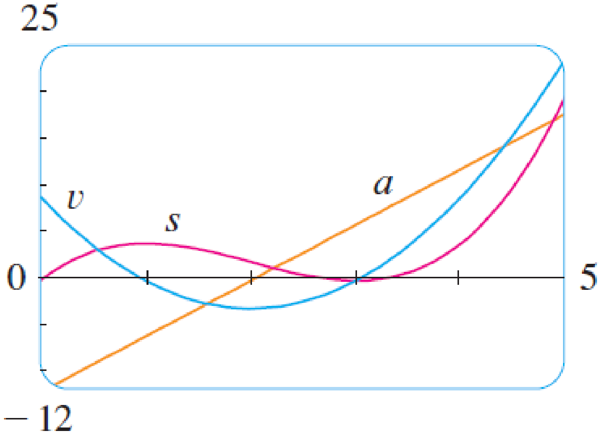
$$|f(5) - f(3)| = |20 - 0| = 20.$$

The total distance is $4 + 4 + 20 = 28$m.

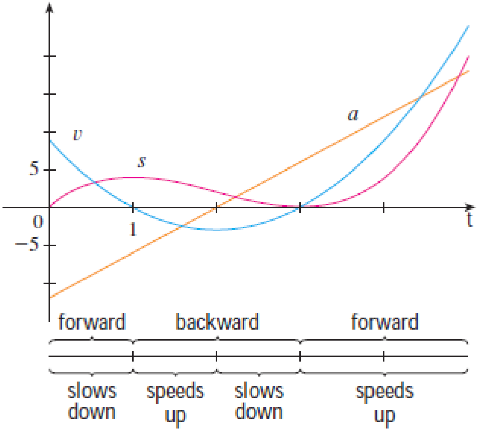
(vii) The acceleration is the derivative of the velocity function:

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = (3t^2 - 12t + 9)'_t = 6t - 12$$
$$a(4) = 6 \times 4 - 12 = 12m/s^2$$

(viii) the graphs of s, v and a:



(ix) When is the particle speeding up? When is it slowing down?



Example. If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/\ell$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point is $m = f(x)$

The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

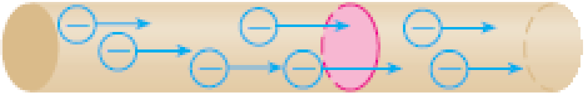
$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The linear density ρ at x_1 is the limit of these average densities as $\Delta x \to 0$, that is, the linear density is

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}.$$

Thus the linear density of the rod is the derivative of mass w.r.t. length.

Example. A current exists whenever electric charges move. The figure shows part of a wire and electrons moving through a shaded plane surface.



If ΔQ is the net charge that passes through this surface during a time period Δt, then the average current during this time interval is defined as

$$\text{average current} = \frac{\Delta Q}{\Delta t} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

If we take the limit of this average current over smaller and smaller time intervals, we get what is called the current I at a given time t_1:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

Thus the current is the rate at which charge flows through a surface. It is measured in units of charge per unit time (often coulombs per second, called amperes).

Remark. Velocity, density, and current are not the only rates of change that are important in physics. Others include power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

Example. $2H_2 + O_2 \longrightarrow 2H_2O$: two molecules of hydrogen and one molecule of oxygen form two molecules of water.
Consider the reaction: $A + B \longrightarrow C$, where A, B: reactants and C: product.

- The concentration of a reactant A is the number of moles (1 mole $= 6.022 \times 10^{23}$ molecules) per liter and is denoted by $[A]$.
- The concentration varies during a reaction, so $[A]$, $[B]$, and $[C]$ are all functions of t.
- The average rate of reaction of the product C over a time interval $t_1 \leq t \leq t_2$ is

$$\frac{\Delta[C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$$

- Instantaneous rate of reaction is

$$\text{rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[C]}{\Delta t} = \frac{d[C]}{dt}$$

Consider the reaction: $\mathrm{A} + \mathrm{B} \longrightarrow \mathrm{C}.$

- Since the concentration of the product increases as the reaction proceeds, the derivative $d[\mathrm{C}]/\mathrm{dt}$ will be positive, and so the rate of reaction of C is positive.
- The concentrations of the reactants, however, decrease during the reaction. Thus, $d[\mathrm{A}]/\mathrm{dt}$ and $d[\mathrm{B}]/\mathrm{dt}$ are negative.
- Since A and B each decrease at the same rate that C increases, we have

$$\text{rate of reaction} = \frac{d[\mathrm{C}]}{dt} = -\frac{d[\mathrm{A}]}{dt} = -\frac{d[\mathrm{B}]}{dt}.$$

Consider the reaction: $a\mathrm{A} + \mathrm{bB} \longrightarrow \mathrm{cC} + \mathrm{dD}.$ There holds

$$-\frac{1}{a}\frac{d[\mathrm{A}]}{dt} = -\frac{1}{b}\frac{d[\mathrm{B}]}{dt} = \frac{1}{c}\frac{d[\mathrm{C}]}{dt} = \frac{1}{d}\frac{d[\mathrm{D}]}{dt}$$

Example. Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period is

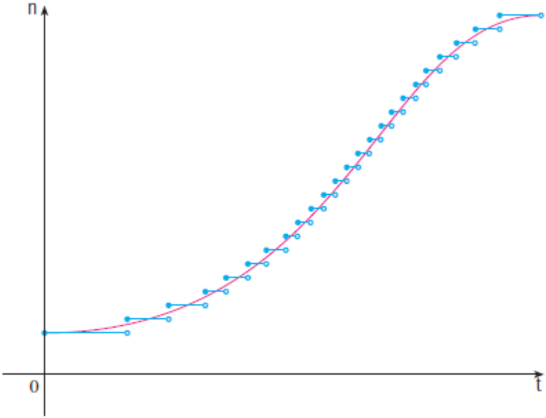
$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The instantaneous rate of growth is

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}.$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve as in the figure.

Biology



Example. Consider a population of bacteria in a homogeneous nutrient medium. Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$f(1) = 2f(0) = 2n_0, \; f(2) = 2f(1) = 2^2 n_0, \ldots, f(t) = 2^t n_0$$

The population function is $n = 2^t n_0$. So the rate of growth of the bacteria population at time t is

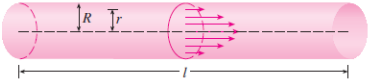
$$\frac{dn}{dt} = \frac{d}{dt}(2^t n_0) = n_0 2^t \ln 2.$$

For example, if $n_0 = 100$ bacteria. The rate of growth after 4 hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \times 2^4 \ln 2 \approx 1109.$$

This means that, after 4 hours, the bacteria population is growing at a rate of about 1109 bacteria per hour.

Example. When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l as illustrated in the figure



- Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall.
- The relationship between v and r is given by the law of laminar flow discovered by the French physician Jean-Louis-Marie Poiseuille in 1840.
- This law states that

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube

$$v = \frac{P}{4\eta l}(R^2 - r^2).$$

- If P and l are constant, then v is a function of r with domain $[0, R]$.
- The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

- Velocity gradient, i.e. the instantaneous rate of change of velocity v w.r.t. r is:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr} = -\frac{Pr}{2\eta l}$$

- For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$cm, $l = 2$cm, $P = 4000$ dynes$/\mathrm{m}^2$.

$$v = \frac{4000}{4 \times 0.027 \times 2}(0.008^2 - r^2)$$

- At $r = 0.002\text{cm}$, the blood is flowing at a speed of

$$v(0.002) = \frac{4000}{4 \times 0.027 \times 2}(0.008^2 - 0.002^2) \approx 1.11\text{cm/s}.$$

- The velocity gradient at that point is

$$\left.\frac{dv}{dt}\right|_{t=0.002} = -\frac{4000 \times 0.002}{2 \times 0.027 \times 2} \approx -74(\text{cm/s})/\text{cm}.$$

- $(1\text{cm} = 10,000\mu\text{m})$. Then the radius of the artery is 80μm. The velocity at the central axis is $11,850\mu\text{m/s}$, which decreases to $11,110\mu\text{m/s}$ at a distance of $r = 20\mu\text{m}$.
- The fact that $dv/dr = -74(\text{cm/s})/\text{cm}$ means that, when $r = 20\mu\text{m}$, the velocity is decreasing at a rate of about $74(\text{cm/s})$ for each micrometer that we proceed away from the center.

Example. Suppose that $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a cost function.

- If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$.
- The average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}.$$

- The instantaneous rate of change of cost w.r.t. the number of items produced, is called the marginal cost by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dt}.$$

Remark. Since x often takes on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as in the previous example.

Economics

- Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n+1) - C(n)$$

Thus the marginal cost of producing units is approximately equal to the cost of producing one more unit [the $n+1$st unit]

- It is often appropriate to represent a total cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the overhead cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to x, but labor costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

Example. For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10,000 + 5x + 0.01x^2.$$

- Then the marginal cost function is $C'(x) = 5 + 0.02x.$
- The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02 \times 500 = \$15/\text{item}.$$

- This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.
- The actual cost of producing the 501st item is

$$\begin{aligned} C(501) - C(500) &= [10^4 + 5{\cdot}501 + 0.01{\cdot}501^2] - [10^4 + 5{\cdot}500 + 0.01{\cdot}500^2] \\ &= \$15.01. \end{aligned}$$

- Notice that $C'(500) \approx C(501) - C(500)$.

Implicit differentiation

- If a function is given as an expression of the variable $y = f(x)$, then we can use definition and differentiation rules to compute y'.

  E.g., given $y = \sqrt{x+1}$. Then $y' = \frac{1}{2\sqrt{x+1}}$

- However, if the function y is given implicitly as a relation between x and y, then we need to use the method of **implicit differentiation**

  E.g., given $x^3 + y^3 = 6xy$. We need to find y'?

  E.g., Differentiating both sides, noting that y is a function of x,

$$
\begin{aligned}
(x^3 + y^3)'_x = (6xy)'_x &\Longleftrightarrow 3x^2 + 3y^2 y' = 6y + 6xy' \\
&\Longleftrightarrow (y^2 - 2x)y' = 2y - x^2 \\
&\Longleftrightarrow y' = \frac{2y - x^2}{y^2 - 2x}
\end{aligned}
$$

Implicit differentiation

Ex: Find y'' if $x^4 + y^4 = 16$

Ans: Differentiating both sides to obtain

$$(x^4 + y^4)'_x = (16)'_x \Longleftrightarrow 4x^3 + 4y^3 y' = 0 \Longrightarrow y' = -\frac{x^3}{y^3}$$

- Differentiating both sides of the blue equation

$$y'' = -\left(\frac{x^3}{y^3}\right)'_x = -\frac{(x^3)'_x y^3 - x^3 (y^3)'_x}{y^6} = -\frac{3x^2y^3 - 3x^3y^2y'}{y^6}$$

$$= -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} = -3x^2\frac{x^4 + y^4}{y^7}$$

$$= -\frac{48x^2}{y^7}$$

Derivative of inverse trigonometric functions

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\tan^{-1} x)' = \frac{1}{1+x^2}$ $\qquad$ $(\cot^{-1} x)' = -\frac{1}{1+x^2}$

$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$ $\qquad$ $(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$

- The inverse of f is denoted f^{-1}. Their two derivatives, assuming they exist, are reciprocal, as the Leibniz notation suggests; that is:

$$\frac{dx}{dy}\cdot\frac{dy}{dx}=1.$$

- Hence,

$$\left[f^{-1}\right]'(a)=\frac{1}{f'\left(f^{-1}(a)\right)}.$$

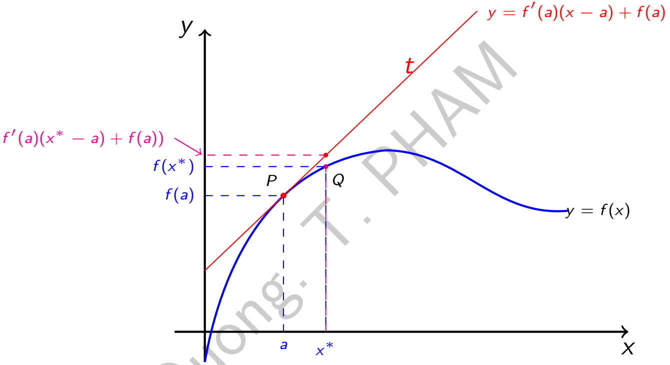
Example. $y=x^2$ (for positive x) has inverse $x=\sqrt{y}$. We have

$$\frac{dy}{dx}=2x,\quad \frac{dx}{dy}=\frac{1}{2\sqrt{y}}=\frac{1}{2x}$$

Here, there holds

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=2x\cdot\frac{1}{2x}=1.$$

Linear approximation



- The approximation $f(x^*) \approx f'(a)(x^* - a) + f(a)$ is called the **linear approximation** of f at a^*.
- The function $L(x) = f'(a)(x-a) + f(a)$ is the **linearization** of f

Ex: Write the linearization of $f(x) = \sqrt{x+3}$ at $x = 1$, then use it to approximate the values $\sqrt{3.95}$ and $\sqrt{4.05}$

Ans:

- The linearization at $x = 1$ is

$$L(x) = f'(1)(x-1) + f(1) = \frac{1}{2\sqrt{1+3}}(x-1) + \sqrt{1+3} = \frac{x}{4} + \frac{7}{4}$$

- The corresponding linear approximation is

$$\sqrt{x+3} \approx \frac{x}{4} + \frac{7}{4} \quad \text{(when } x \text{ is near 1)}$$

- In particular, $\sqrt{3.95} = \sqrt{0.95+3} \approx \frac{0.95}{4} + \frac{7}{4} = 1.9875$

  and $\sqrt{4.05} = \sqrt{1.05+3} \approx \frac{1.05}{4} + \frac{7}{4} = 2.0125$

- If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.
- In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).
- The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Ex. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

- Given information: the rate of increase of the volume of air is 100 cm^3/s
  Unknown: the rate of increase of the radius when the diameter is 50 cm?
- Denote $V(t)$: volume of the balloon at time t
  $r(t)$: radius of the balloon at time t.
- The rate of increase of the volume $V(t)$ w.r.t. time is dV/dt
  The rate of increase of the radius $r(t)$ w.r.t. time is dr/dt
  Hence, $\dfrac{dV}{dt} = 100\mathrm{cm}^3/\mathrm{s}$. **Question:** $\dfrac{dr}{dt} =?$ when $r = 25\mathrm{cm}$.

- We have $V = \frac{4}{3}\pi r^3$. Differentiate with respect to t

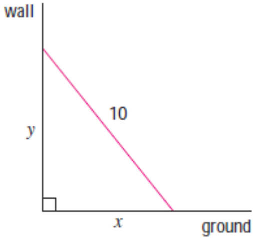
$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt} \Longrightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

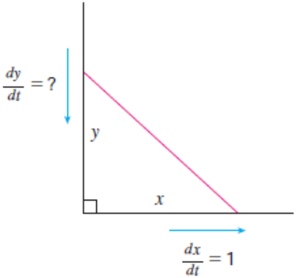
When $r = 25$ and $\frac{dV}{dt} = 100$, we have

$$\frac{dr}{dt} = \frac{1}{4\pi 25^2}100 = \frac{1}{25\pi}$$

- The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s

Ex. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?





- $x(t), y(t)$
- When $\dfrac{dx}{dt} = 1$ ft/s,

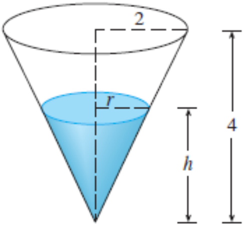
  $x = 6$ ft , $\dfrac{dy}{dt} = ?$

Pythagorean Theorem: $x^2 + y^2 = 10^2$. Differentiate both sides w.r.t. t,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longrightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$ when $x = 6, y = \sqrt{10^2 - 6^2} = 8$,

thus $\dfrac{dy}{dt} = -\dfrac{6}{8} \cdot 1 = -\dfrac{3}{4}$ ft/s

Ex. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2\mathrm{m}^3/\mathrm{min}$, find the rate at which the water level is rising when the water is 3 m deep.



- Denote $V(t)$: volume of water at time t
  $r(t)$: radius of the surface at time t
  $h(t)$: height of water at time t
- Given $\frac{dV}{dt} = 2\mathrm{m}^3/\mathrm{min}$. **Q:** $\frac{dh}{dt}$ when $h = 3$m?
- We have $V = \dfrac{1}{3}\pi r^2 h$

$\frac{r}{h} = \frac{2}{4} \Longrightarrow r = \frac{h}{2} \Longrightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \dfrac{\pi}{12}h^3$. Differentiate w.r.t. t,

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}.$$ Substitute $h = 3\mathrm{m}, \dfrac{\mathrm{dV}}{\mathrm{dt}} = 2\mathrm{m}^3/\mathrm{min}$

$\dfrac{dh}{dt} = \dfrac{4}{\pi 3^2}\cdot 2 = \dfrac{8}{9\pi}$. The water level is rising at a rate of $\frac{8}{9\pi} \approx 0.28\mathrm{m}/\mathrm{min}$

Strategy:

- Read the problem carefully.
- Draw a diagram if possible.
- Introduce notation. Assign symbols to all quantities that are functions of time.
- Express the given information and the required rate in terms of derivatives.
- Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution
- Use the Chain Rule to differentiate both sides of the equation with respect to t.
- Substitute the given information into the resulting equation and solve for the unknown rate.