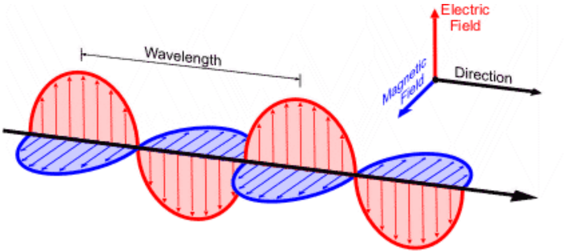
# CHAPTER 2: ANALYTIC GEOMETRY OF SPACE, VECTOR FUNCTIONS



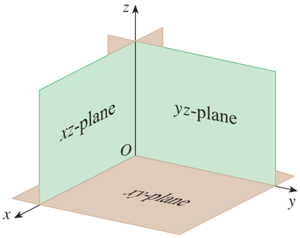
# CONTENTS

# Three-Dimensional (3D) Coordinate

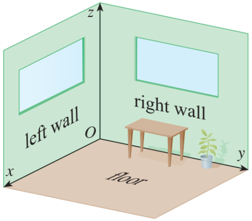
- We will introduce vectors and coordinate systems for 3D space. This will be the setting for our study of the calculus of functions of two variables

- We will see that vectors provide particularly simple descriptions of lines and planes in space.

- Reference for Chapter 2: Chapters 12-13 of the textbook by J. Stewart.




(a) Coordinate planes

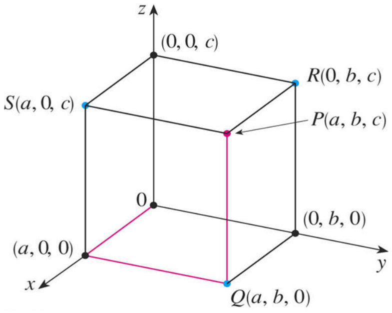



(b)

# Three-Dimensional (3D) Coordinate

- The Cartesian **coordinates** $(a, b, c)$ of a point $P(a, b, c)$ in space are the numbers at which the planes through $P$ perpendicular to the axes cut the axes. The value $a$ is the $x$-coordinate, $b$ is the $y$-coordinate, and $c$ is the $z$-coordinate.

- If we drop a perpendicular from $P(a, b, c)$ to the $xy$-plane, we get a point $Q(a, b, 0)$ called the **projection** of $P$ on the $xy$-plane.



# Three-Dimensional (3D) Coordinate

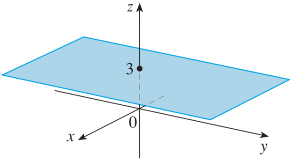
- The Cartesian product

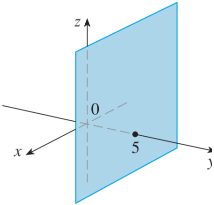
$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by $\mathbb{R}^3$. It is called a **three-dimensional rectangular coordinate system**.

- In three-dimensional analytic geometry, an equation in $x$, $y$, and $z$ represents a **surface** in $\mathbb{R}^3$.

- The equation $z = 3$ represents the set of all points in $\mathbb{R}^3$ whose $z$-coordinate is 3. The right figure is the plane $y = 5$.



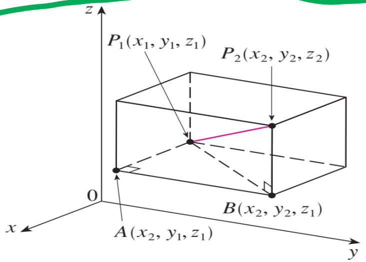



# Three-Dimensional (3D) Coordinate

## Distance between two points

*The distance $|P_1P_2|$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$




**Example** The distance between $P_1(2, -1, 7)$ and $P_2(1, -3, 5)$ is

$$|P_1P_2| = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2} = 3.$$

# Three-Dimensional (3D) Coordinate

**Example** An equation of a sphere with center $C(a, b, c)$ and radius $r$ is

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2}$$

In particular, if the center is the origin $O$, then an equation of the sphere is $\boxed{x^2 + y^2 + z^2 = r^2}$

**Example** Show that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

is the equation of a sphere, and find its center and radius.

**Solution** We have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4}$$

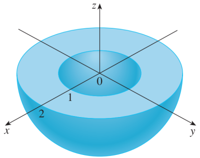
It is the equation of a sphere with center $(-3/2, 0, 2)$ and radius $\sqrt{21}/2$.

# Three-Dimensional (3D) Coordinate

**Example**

| Equations/inequalities | Description |
|---|---|
| a) $x^2 + y^2 + z^2 \leq 4$ | The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. |
| b) $x^2 + y^2 + z^2 = 4$, $z \leq 0$ | The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the $xy$-plane. |

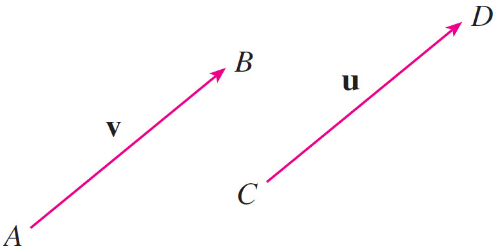
**Example** What region in $\mathbb{R}^3$ is represented by $1 \leq x^2 + y^2 + z^2 \leq 4$, $z \leq 0$?




Answer: Between (or on) the spheres and beneath (or on) the $xy-$plane.

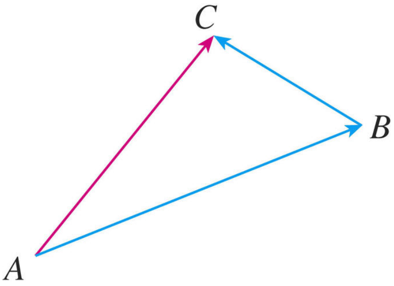
# Combining Vectors

- The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.
- A vector is often represented by an arrow or a directed line segment.
- For example, a particle moves along a line segment from point $A$ to point $B$. One can describe this moving by the **displacement vector** $\mathbf{v} = \overrightarrow{AB}$.



# Combining Vectors

- Two vectors are **equivalent** or **equal** if they have the same length and direction.
- The zero vector, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.
- Suppose a particle moves from $A$ to $B$, and changes direction and moves from $B$ to $C$. The resulting displacement vector $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

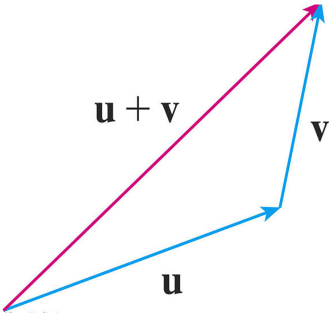


# Combining Vectors

## Definition

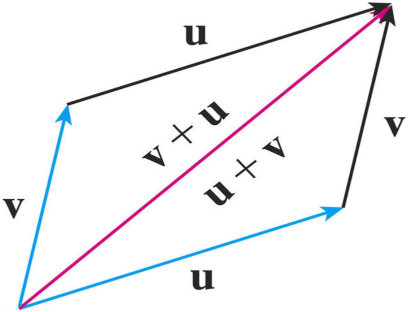
If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

This definition is sometimes called the **Triangle Law**.



# Combining Vectors

If we place **u** and **v** so they start at the same point, then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with **u** and **v** as sides. This is called the **Parallelogram Law**.

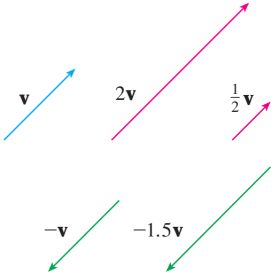



The Parallelogram Law

# Combining Vectors

## Definition

If $c$ is a scalar (a real number) and $\mathbf{v}$ is a vector, then scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.



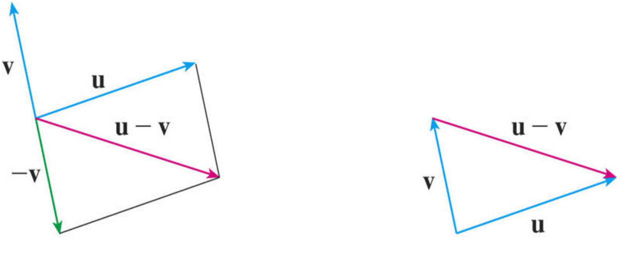

Note: Two nonzero vectors are **parallel** if they are scalar multiples of one another. Also, we call $-\mathbf{v}$ the **negative** of $\mathbf{v}$.

# Combining Vectors

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

So we can construct $\mathbf{u} - \mathbf{v}$ by first drawing the negative of $\mathbf{v}$, $-\mathbf{v}$, and then adding it to $\mathbf{u}$ by the Parallelogram Law (Fig. (a) below). Alternatively, since $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$ the vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$. So we could construct $\mathbf{u} - \mathbf{v}$ by means of the Triangle Law as in Fig. (b).




(a) (b)

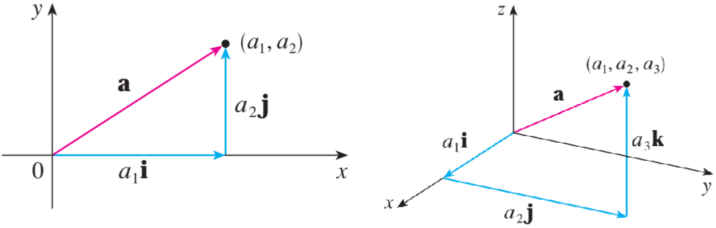
# Components of a vector

- If we place the initial point of a vector **a** at the origin, then the terminal point of **a** has coordinates of the form $(a_1, a_2) \in \mathbb{R}^2$ or $(a_1, a_2, a_3) \in \mathbb{R}^3$.

- These coordinates are called the **components** of **a** and we write

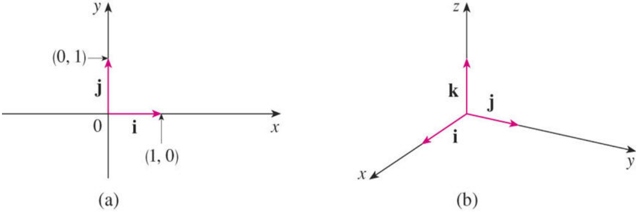
$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle.$$

- The vector $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are the **basic vectors**.

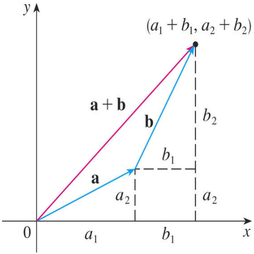



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ (b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

# Components of a vector



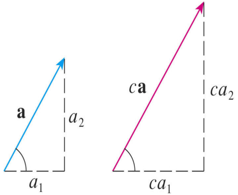

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive. So



# Components of a vector

Similarly, *to subtract vectors we subtract components*.

From the similar triangles, we see that the components of $c\mathbf{a}$ are $ca_1$ and $ca_2$. So *to multiply a vector by a scalar we multiply each component by that scalar*.




If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} \pm \mathbf{b} = \langle a_1 \pm b_1, a_2 \pm b_2 \rangle$$
$$c\mathbf{a} = \langle ca_1, ca_2 \rangle.$$

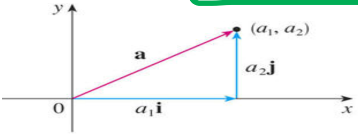
Since $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, we have $\boxed{\langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}}$

# Components of a vector

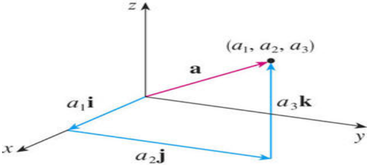
Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3\rangle \pm \langle b_1, b_2, b_3\rangle = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3\rangle$$
$$c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

Since $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, we have $\boxed{\langle a_1, a_2, a_3\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}}$




(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$




(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

# Components of a vector

- Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in $\mathbb{R}^3$, then

$$\boxed{\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle}$$

- Similarly, in two dimensions, the vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\boxed{\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle}$$

- The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}\,.$$

- The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\,.$$

**Example** The length of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is $\sqrt{14}$.

# Components of a vector

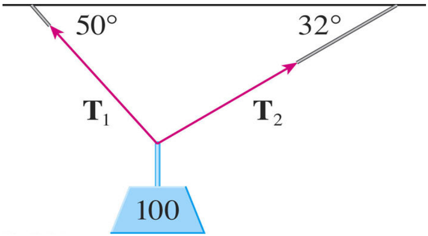
- Any vector whose length is 1 is a **unit vector**.
- For instance, the vector $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors.
- If $\mathbf{v} \neq \mathbf{0}$, $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector, called the **direction** of $\mathbf{v}$ or the **unit vector in the direction of** $\mathbf{v}$.
- Any nonzero vector can be expressed as a product of its length and direction:

$$\boxed{\mathbf{v} = |\mathbf{v}| \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (\text{length of } \mathbf{v}) \cdot (\text{direction of } \mathbf{v})}$$

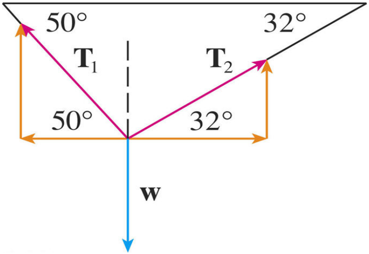
- A force is represented by a vector because it has both a magnitude and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

# Components of a vector

**Example** A 100-lb weight hangs from two wires as shown in the figure below. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and their magnitudes.



# Components of a vector




**Solution** We see that

$$T_1 = -|T_1|\cos 50^\circ \mathbf{i} + |T_1|\sin 50^\circ \mathbf{j}$$
$$T_2 = |T_2|\cos 32^\circ \mathbf{i} + |T_2|\sin 32^\circ \mathbf{j}$$

The resultant $T_1 + T_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have $T_1 + T_2 = -\mathbf{w} = 100\mathbf{j}$:

$$\big(-|T_1|\cos 50^\circ + |T_2|\cos 32^\circ\big)\mathbf{i} + \big(|T_1|\sin 50^\circ + |T_2|\sin 32^\circ\big)\mathbf{j} = 100\mathbf{j}.$$

# Components of a vector

Equating components, we get

$$\begin{aligned} -|T_1|\cos 50^\circ + |T_2|\cos 32^\circ &= 0 \\ |T_1|\sin 50^\circ + |T_2|\sin 32^\circ &= 100. \end{aligned}$$

Solving gives

$$|T_1|\sin 50^\circ + \frac{|T_1|\cos 50^\circ}{\cos 32^\circ}\sin 32^\circ = 100.$$

So

$$\begin{aligned} |T_1| &= \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb} \\ |T_2| &= \frac{|T_1|\cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb} \end{aligned}$$

Hence the tension vectors are

$$T_1 \approx -55.05\mathbf{i} + 65.60\mathbf{j} \quad \text{and} \quad T_2 \approx 55.05\mathbf{i} + 34.40\mathbf{j}.$$

# The dot product

## Definition

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Similarly, if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

The dot product is sometimes called the **scalar product** (or **inner product**).

# The dot product

## Theorem

*If* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors and* $\lambda$ *is a scalar, then*

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$;
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$;
4. $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b})$;
5. $\mathbf{0} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{0} = 0$.

## Theorem

*If* $\theta$ *is the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$, *then*

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cdot \cos\theta}$$

# The dot product

## Corollary

*If $\theta$ is the angle between the nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$, *then*

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}}$$

**Example** Find the angle between the vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. **Solution**

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= 1\times 6 + (-2)\times 3 + (-2)\times 2 = -4 \\
|\mathbf{a}| &= \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \\
|\mathbf{b}| &= \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7 \\
\theta &= \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \cos^{-1}\left(\frac{-4}{3\times 7}\right) \approx 1.76 \ \text{ rad}\,.
\end{aligned}$$

# The dot product

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Thus,

$$\boxed{\text{Two vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal} \iff \mathbf{a} \cdot \mathbf{b} = 0}$$

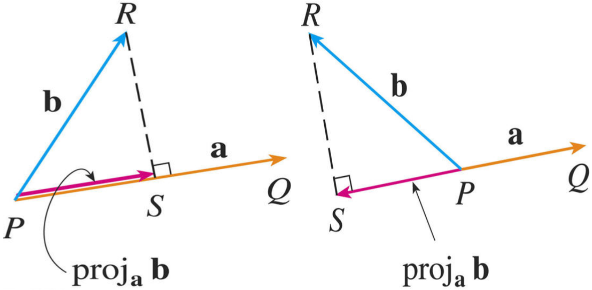
## Example

$\mathbf{a} = \langle 3, -2, 1 \rangle$ and $\mathbf{b} = \langle 0, 2, 4 \rangle$ are orthogonal because

$$\mathbf{a} \cdot \mathbf{b} = (3)(0) + (-2)(2) + (1)(4) = 0.$$

# The dot product

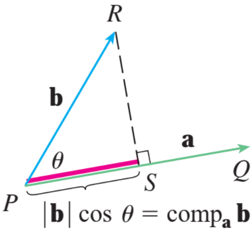
**Projection** Suppose that $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$. If $S$ is the foot of the perpendicular from $R$ to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of $\mathbf{b}$ onto $\mathbf{a}$ and is denoted by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$.




The vector projection of **b** onto **a**

# The dot product

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the **component of b along a**) is defined to be the number $|\mathbf{b}|\cos\theta$, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This is denoted by $\text{comp}_{\mathbf{a}}\,\mathbf{b}$.




$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

# The dot product

## Example

Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

**Solution** Since $\mathbf{a} = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$,

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2) \times 1 + 3 \times 1 + 1 \times 2}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

Thus

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

# The dot product

One use of projections occurs in physics in calculating work. If the force moves the object from $P$ to $Q$, then the **displacement vector** is $\overrightarrow{PQ}$.

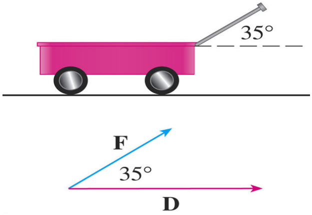
## Definition

The **work** done by a constant force $\boldsymbol{F}$ acting through a displacement $\overrightarrow{PQ}$ is

$$\text{Work} = \mathbf{F} \cdot \overrightarrow{PQ} = |\boldsymbol{F}||\overrightarrow{PQ}| \cos\theta.$$

# The dot product

**Example** A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.




**Solution** If **F** and **D** are the force and displacement vectors, then the work done is

$$\begin{aligned} W &= \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos 35^\circ \\ &= (70)(100)\cos 35^\circ \approx 5734\ \mathrm{N}\cdot\mathrm{m} = 5734\ \mathrm{J}. \end{aligned}$$

# The cross product

## Definition

$a = a_1, a_2, a_3$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Cross product is also called the **vector product**.

$\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are three-dimensional vectors.

# The cross product

A **determinant of order** 2 is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

A **determinant of order** 3 can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

# The cross product

Then the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \quad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

## Example

Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in $\mathbb{R}^3$.

**Solution** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\begin{aligned} \mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= (a_2a_3 - a_3a_2)\mathbf{i} + (a_3a_1 - a_1a_3)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} \quad = \mathbf{0}. \end{aligned}$$

# The cross product

**Example** Show that

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$
$$\mathbf{j}\times\mathbf{i}=-\mathbf{k} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j}.$$

**Theorem**

*The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.*

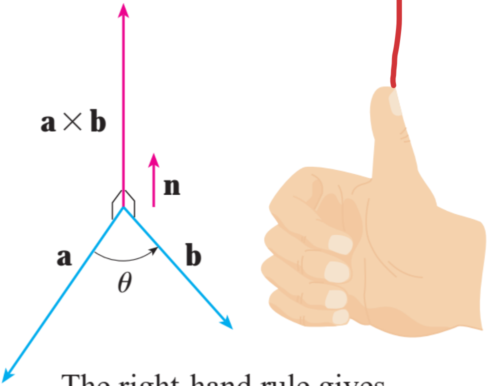
Proof Let $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$. Then

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= (a_2b_3-a_3b_2)a_1+(a_3b_1-a_1b_3)a_2 \\ &\quad +(a_1b_2-a_2b_1)a_3 \\ &= 0\end{aligned}$$

A similar computation shows that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$.

# The cross product

The direction of $\mathbf{a} \times \mathbf{b}$ is given by the right-hand rule: If the curled fingers of the right hand are rotated from the direction of $\mathbf{a}$ to the direction of $\mathbf{b}$, the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.




The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

# The cross product

**Example** Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

**Solution** The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\begin{aligned}
\overrightarrow{PQ} &= (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} &&= \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\
\overrightarrow{PR} &= (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} &&= -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\
&= 6\mathbf{i} + 6\mathbf{k}.
\end{aligned}$$

# The cross product

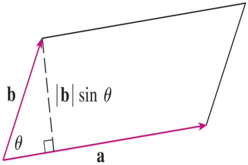
## Theorem

*If $\theta$ is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *(so $0 \leq \theta \leq \pi$), then*

$$\boxed{|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta}$$

Thus,

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.




$$|\mathbf{a} \times \mathbf{b}| = \text{area of parallelogram}$$

# The cross product

## Corollary

*Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if* $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

**Example** Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

**Solution** The area of the parallelogram determined by $P$, $Q$, and $R$ is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}| = 6\sqrt{2}.$$

The triangle's area is half of this, $3\sqrt{2}$.

# The cross product. Properties

## Theorem

*If* **a**, **b**, *and* **b** *are vectors and* $\lambda$ *is a scalar, then*

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{b}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{b} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{b}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{b})$ that occurs in Property 6 is called the **vector triple product** of **a**, **b**, and **b**.

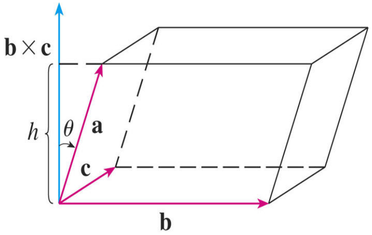
# The cross product. Triple Products

- The product that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. It can be shown that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

- The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$



# The cross product

## Example

Find the volume of the box (parallelepiped) determined by $\mathbf{a} = \langle 1, 2, -1 \rangle$, $\mathbf{b} = \langle -2, 0, 3 \rangle$, and $\mathbf{c} = \langle 0, 7, -4 \rangle$.

**Solution**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}$$
$$= -21 - 16 + 14 = -23.$$

The volume is $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 23$.

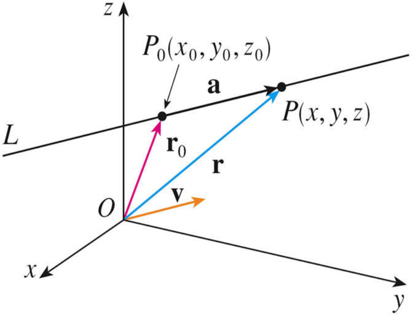
Note that if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, then the vectors must lie in the same plane; that is, they are **coplanar**.

# Equations of Lines and Planes

**Equations for Lines** Suppose $L$ is a line in three-dimensional space that passes a point $P_0(x_0, y_0, z_0)$. Let $\mathbf{v}$ be a vector parallel to $L$, $P(x, y, z)$ be an arbitrary point on $L$ and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$, respectively. Then

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

which is a **vector equation** of $L$.



# Equations of Lines

Suppose $\mathbf{v} = \langle a, b, c\rangle$, then we have the three scalar equations:

$$\boxed{x = x_0 + ta, \quad y = y_0 + tb \quad z = z_0 + tc, \quad t \in \mathbb{R}} \qquad (1)$$

These equations are called **parametric equations** of the line through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$.

**Note** The vector equation and parametric equations of a line are not unique.

# Equations of Lines

## Example

(a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(b) Find two other points on the line.

**Solution** (a) The vector equation is

$$\begin{aligned}\mathbf{r} &= (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &= (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}.\end{aligned}$$

Parametric equations are
$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t, \quad t \in \mathbb{R}.$
(b) Choosing the parameter value $t = 1$ gives $x = 6,\ y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$.

# Equations of Lines

If none of $a$, $b$, or $c$ is 0, we can solve each of Equations (1) for $t$, equate the results, and obtain

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These equations are called **symmetric equations** of $L$. If $a = 0$, we can write the equations of $L$ as

$$x = x_0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

**Equations for Linesegments** *The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation*

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1, \quad 0 \leq t \leq 1}$$

# Equations of Lines

**Example** Show that the lines $L_1$ and $L_2$ with parametric equations

$$\begin{array}{lll} x = 1 + t & y = -2 + 3t & z = 4 - t \\ x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

**Solution** The lines are not parallel because the corresponding vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. If $L_1$ and $L_2$ had a point of intersection, there would be values of $t$ and $s$ such that

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

These equations have no solution, so $L_1$ and $L_2$ do not intersect. Thus $L_1$ and $L_2$ are skew lines.

# Equations of Lines

## Example

Show that the midpoint of the line segment joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

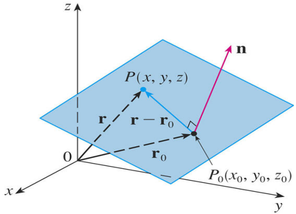
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

**Solution**

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \overrightarrow{P_1M} = \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} \\ &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

# Equations of Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector **n** that is orthogonal to the plane. This orthogonal vector **n** is called a **normal vector**. The plane consists of all points $P(x, y, z)$ for which $\overrightarrow{P_0P} = \langle x - x_0, x - y_0, x - z_0 \rangle$ is orthogonal to **n**.




We have **vector equation** of the plane:

$$\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0} \qquad (2)$$

which can be rewritten as

$$\boxed{\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0} \qquad (3)$$

# Equations of Planes

- Suppose $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation (2) becomes

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \tag{4}$$

Equation (4) is the **scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector** $\mathbf{n} = \langle a, b, c \rangle$.

- We can rewrite the equation of a plane as

$$\boxed{ax + by + cz + d = 0} \tag{5}$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation (5) is called a **linear equation** in $x$, $y$, and $z$.

# Equations of Planes

**Example** Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

**Solution** Since both $\overrightarrow{PQ} = \langle 2,-4,4\rangle$ and $\overrightarrow{PR} = \langle 4,-1,-2\rangle$ lie in the plane, $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is a normal vector of the plane. Thus

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}.$$

An equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

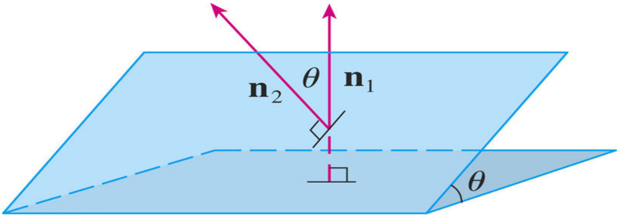
or

$$6x + 10y + 7z = 50.$$

# Equations of Planes

**Angles Between Planes** Two planes are parallel if their normal vectors are parallel.

If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.




The angle between planes

# Equations of Planes

**Example** Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

**Solution** The normal vectors of these planes are $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3 \rangle$ and so, if $\theta$ is the angle between the planes, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1 \cdot 1 + 1(-2) + 1 \cdot 3}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ.$$

# Equations of Planes

## Distance from a Point to a Plane

**Example** Find a formula for the distance from a point $P_1(x_1, y_1, z_1)$ to the plane

$$ax + by + cz + d = 0.$$

**Solution** Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b} = \overrightarrow{P_0P_1}$. Then $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$. The distance from $P_1$ to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$.

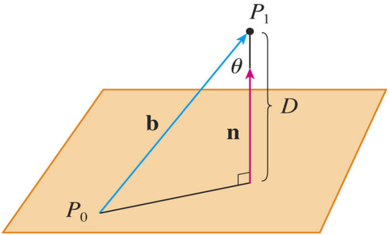
$$\begin{aligned} D = |\operatorname{comp}_{\mathbf{n}} \mathbf{b}| &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

# Equations of Planes

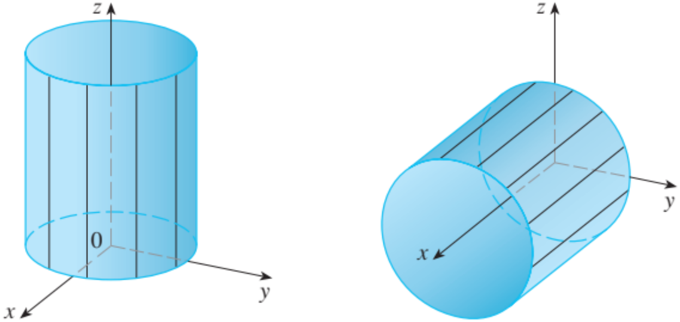
$$D = \frac{\left|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)\right|}{\sqrt{a^2 + b^2 + c^2}}$$

Since $P_0$ lies in the plane, $ax_0 + by_0 + cz_0 + d = 0$. Thus

$$\boxed{D = \frac{\left|ax_1 + by_1 + cz_1 + d\right|}{\sqrt{a^2 + b^2 + c^2}}}$$



# Equations of Cylinders

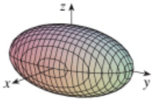
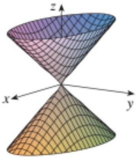



When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ (left) or $y^2 + z^2 = 1$ (right) represents a cylinder and not a circle.

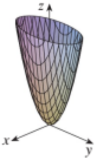
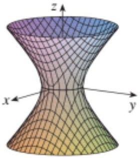
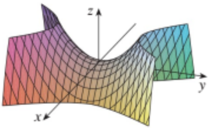
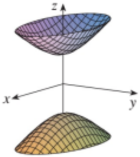
# Equations of Quadric Surfaces

## Quadric Surfaces

A quadric surface is the graph of a second-degree equation in three variables $x, y$, and $z$. The most general such equation is $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ where $A, B, C, \ldots, J$ are constants.

| Surface | Equation | Surface | Equation |
|---|---|---|---|
| Ellipsoid   | $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$<br>All traces are ellipses.<br>If $a = b = c$, the ellipsoid is a sphere. | Cone   | $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$<br>Horizontal traces are ellipses.<br>Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$. |

# Equations of Quadric Surfaces

<table>
<tr>
<td>Elliptic Paraboloid<br><br></td>
<td>$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$<br>Horizontal traces are ellipses.<br>Vertical traces are parabolas.<br>The variable raised to the first power indicates the axis of the paraboloid.</td>
<td>Hyperboloid of One Sheet<br><br></td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$<br>Horizontal traces are ellipses.<br>Vertical traces are hyperbolas.<br>The axis of symmetry corresponds to the variable whose coefficient is negative.</td>
</tr>
<tr>
<td>Hyperbolic Paraboloid<br><br></td>
<td>$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$<br>Horizontal traces are hyperbolas.<br>Vertical traces are parabolas.<br>The case where $c < 0$ is illustrated.</td>
<td>Hyperboloid of Two Sheets<br><br></td>
<td>$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$<br>Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$.<br>Vertical traces are hyperbolas.<br>The two minus signs indicate two sheets.</td>
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# Vector Functions and Space Curves

## Definitions

When a particle moves through space during a time interval $I$, we think of the particle's coordinates as functions defined on $I$:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I. \tag{6}$$

The points $(x, y, z) = \big(f(t), g(t), h(t)\big)$, $t \in I$, make up the **curve** in space that we call the particle's **path**. The equation and interval in (6) **parametrize** the curve.

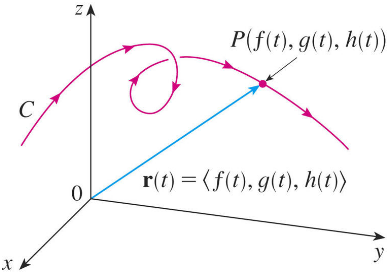
The vector $\mathbf{r}(t) = \overrightarrow{OP} = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ from the origin to the particle's **position** $P = \big(f(t), g(t), h(t)\big)$ at time $t$ is the particle's **position vector**. The functions $f$, $g$, and $h$ are the **components** or **coordinate functions** of the position vector.

# Vector Functions and Space Curves

## Definitions (cont'd)

More generally, a **vector-valued function** or **vector function** is a function whose range is a set of vectors. The vector function's domain to be the intersection of the domains of its component functions.

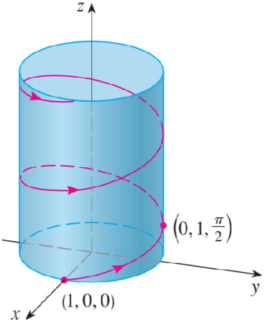
When we need to distinguish real-valued functions from vector functions, we refer to real-valued functions as **scalar functions**.



# Vector Functions and Space Curves

**Example: Space Curves** Sketch the curve whose vector equation is $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. **Solution** We have

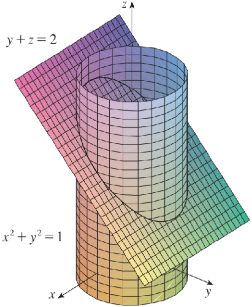
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Thus, the curve must lies on the circular cylinder $x^2 + y^2 = 1$. The curve spirals upward around the cylinder as $z = t$ increases. Each time $t$ increases by $2\pi$, the curve completes one turn around the cylinder. The curve is called a **helix**.

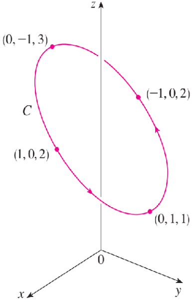



# Vector Functions and Space Curves

## Example

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$




Answer: $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2 - \sin t)\mathbf{k}$, $0 \leq t \leq 2\pi$.

# Vector Functions. Limits and Continuity

## Definition

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

**Example** If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned}\lim_{t\to\pi/4} \mathbf{r}(t) &= \left(\lim_{t\to\pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t\to\pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t\to\pi/4} t\right)\mathbf{k} \\ &= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.\end{aligned}$$

# Vector Functions. Continuity

## Definition

A vector function $\mathbf{r}(t)$ is **continuous** at $a$ if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

The function is **continuous** if it is continuous at every point in its domain.

A vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at $a$ if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous at $a$.

**Example** The function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ is continuous.

# Derivatives of vector functions

## Definition

The derivative of $\mathbf{r}(t)$ is the limit of the difference quotient

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}}$$

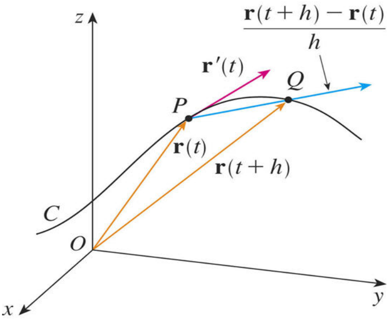
if this limit exists.

The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}(t)$ at the point $P$, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

The **tangent line** to $C$ at $P$ is defined to be the line through $P$ parallel to the tangent vector $\mathbf{r}'(t)$. The **unit tangent vector** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}(t)|}.$$

# Derivatives of vector functions




## Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

# Derivatives of vector functions

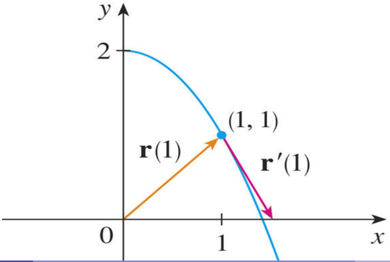
**Example** For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2-t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$. Find the corresponding unit tangent vector.

**Solution**

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}.$$

The unit tangent vector at the point where $t = 1$ is

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\frac{1}{2}\mathbf{i} - \mathbf{j}}{\sqrt{5}/2} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}.$$



# Derivatives of vector functions

## Definition

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is **differentiable** at $t = a$ if $f$, $g$, and $h$ are differentiable at $a$. Also, $\mathbf{r}$ is said to be **differentiable** if it is differentiable at every point of its domain. The curve traced by $\mathbf{r}$ is **smooth** if $d\mathbf{r}/dt$ is continuous and never equal to $\mathbf{0}$, i.e., if $f$, $g$, and $h$ have first derivatives that are not simultaneously 0.

- A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion so that the initial point of one curve is the terminal point of the immediately preceding one is called **piecewise smooth**.

- The second derivative of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

# Derivatives of vector functions

## Definition

If $\mathbf{r}(t)$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**. At any time $t$, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative

$$\mathbf{a} = d\mathbf{v}/dt,$$

when it exists, is the particle's **acceleration vector**.

**Note**

$$\boxed{\text{Velocity} = |\mathbf{v}| \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (\text{Speed}) \cdot (\text{Direction})}$$

# Derivatives of vector functions

**Example** The vector $\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$ gives the position of a moving body at time $t$. Find the body's speed and direction when $t = 2$. At what times, if any, are the body's velocity and acceleration orthogonal?

**Solution**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k},$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}.$$

At $t = 2$, the body's speed and direction are $|\mathbf{v}(2)| = 5$ and

$$\frac{\mathbf{v}(2)}{|\mathbf{v}(2)|} = \Big(-\frac{3}{5}\sin 2\Big)\mathbf{i} + \Big(\frac{3}{5}\cos 2\Big)\mathbf{j} + \frac{4}{5}\mathbf{k},$$

The body's velocity and acceleration are orthogonal when

$$\mathbf{v} \cdot \mathbf{a} = 9\sin t\cos t - 9\cos t\sin t + 4t = 4t = 0.$$

Therefore, $t = 0$.

# Derivatives of vector functions

## Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, $c$ is a scalar, and $f$ is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t).$
4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)).$

# Derivatives of vector functions

**Example** If $\mathbf{r}(t)$ is a differentiable vector function of constant length, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$:

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0.$$

**Solution** Since $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$ is constant,

$$0 = \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r}'\cdot\mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 2\mathbf{r} \cdot \mathbf{r}'.$$

Thus, $\mathbf{r} \cdot \mathbf{r}' = 0$.

# Integrals of vector functions

## Definition

If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over the interval $a \leq t \leq b$, then $\mathbf{r}$ is **integrable** over $[a, b]$ and the **definite integral** of $\mathbf{r}$ from $a$ to $b$ is

$$\int_a^b \mathbf{r}(t)dt = \Big(\int_a^b f(t)dt\Big)\mathbf{i} + \Big(\int_a^b g(t)dt\Big)\mathbf{j} + \Big(\int_a^b k(t)dt\Big)\mathbf{k}.$$

For example,

$$\int_0^\pi \langle 1, t, \sin t\rangle dt = \Big\langle \int_0^\pi 1dt, \int_0^\pi tdt, \int_0^\pi \sin tdt \Big\rangle \ = \Big\langle \pi, \frac{1}{2}\pi^2, 2\Big\rangle.$$

# Integrals of vector functions

- An **antiderivative** of $\mathbf{r}(t)$ on an interval $I$ is a vector function $\mathbf{R}(t)$ such that $\mathbf{R}'(t) = \mathbf{r}(t)$ at each point of $I$.

- If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on $I$, it can be shown that every antiderivative of $\mathbf{r}(t)$ on $I$ has the form $\mathbf{R}(t) + \mathbf{C}$ for some constant $\mathbf{C}$.

- The set of all antiderivatives of $\mathbf{r}$ on $I$ is the **indefinite integral** of $\mathbf{r}$ on $I$ and denoted by $\int \mathbf{r}(t)dt$.

- Thus, if $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$$\boxed{\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}}$$

# Integrals of vector functions

**Example** The velocity of a particle moving in the space is

$$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k}.$$

Find the particle's position as a function of $t$ if $\mathbf{r} = 2\mathbf{i} + \mathbf{k}$ when $t = 0$.

**Solution**

$$\mathbf{r}(t) = \int \mathbf{r}'(t)dt = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k} + \mathbf{C}.$$

To determine $\mathbf{C}$, we use the initial condition $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{k}$:

$$\begin{aligned}(\sin 0)\mathbf{i} + (\cos 0)\mathbf{j} + 0\mathbf{k} + \mathbf{C} &= 2\mathbf{i} + \mathbf{k}\\ \mathbf{C} &= 2\mathbf{i} - \mathbf{j} + \mathbf{k}.\end{aligned}$$

The particle's position as a function of $t$ is

$$\mathbf{r}(t) = (\sin t + 2)\mathbf{i} + (\cos t - 1)\mathbf{j} + (t + 1)\mathbf{k}.$$

# Length of space curves

**Arc Length** Suppose $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t), y = g(t), z = h(t)$, where $f'$, $g'$, and $h'$ are continuous.

If the curve is traversed exactly once as increases from $t = a$ to $t = b$, then it can be shown that its **length** is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

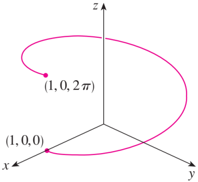
That is,

$$L = \int_a^b |\mathbf{r}'(\mathbf{t})| dt$$

# Length of space curves

## Example

Find the length of the arc of the helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.




Solution We have $\mathbf{r}'(\mathbf{t}) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 1\mathbf{k}$, so $|\mathbf{r}'(\mathbf{t})| = \sqrt{2}$. The arc length is

$$L = \int_0^{2\pi} |\mathbf{r}'(\mathbf{t})| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$

–END OF CHAPTER 2–