Sinusoidal Steady-State Analysis

We consider circuits energized by time-varying voltage or current sources.

Textbook:

Electric Circuits

James W. Nilsson & Susan A. Riedel

9th Edition.

Outline

- **Complex Numbers Tutorial**
- **Sinusoids**
- **Phasors**
- **Techniques of Circuit Analysis**
- **Phasor Diagrams**

Complex Numbers Tutorial

Notation :

Rectangular form:

$$n = a + jb$$

a : real component

b : imaginary component

$$j = \sqrt{-1}$$

Polar form:

$$n = ce^{j\theta} \quad \text{or} \quad n = c\angle\theta°$$

c : magnitude

θ : angle

e : natural logarithm

The conjugate of a complex number :

$$n^* = a - jb$$

$$n^* = c\angle-\theta°$$

Complex Numbers Tutorial

Transition between rectangular and polar forms :

<u>From polar form to rectangular form :</u>

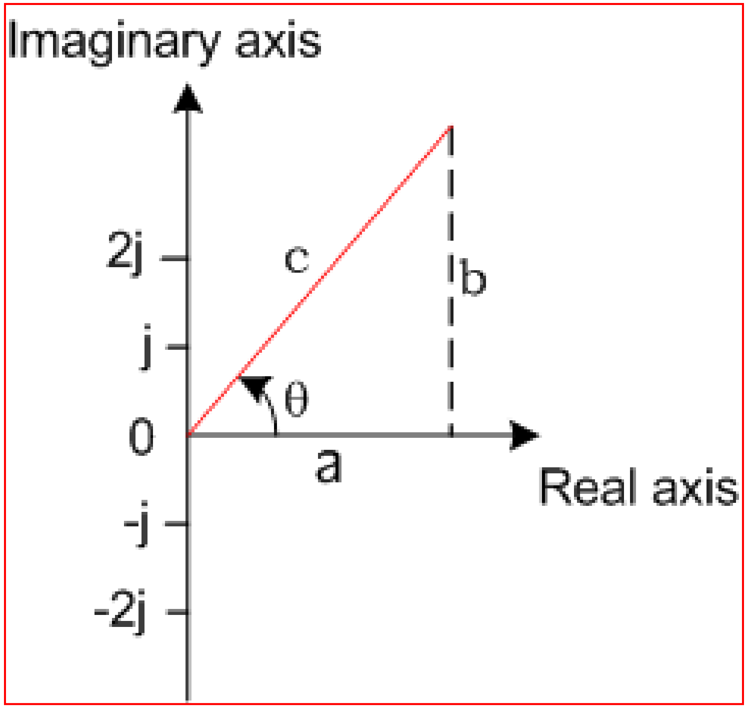
$$ce^{j\theta} = c(\cos\theta + j\sin\theta)$$
$$= c\cos\theta + jc\sin\theta$$
$$= a + jb$$

<u>From rectangular form to polar form :</u>

$$a + jb = ce^{j\theta}$$

where : $\begin{cases} c = \sqrt{a^2 + b^2} \\ \tan\theta = \dfrac{b}{a} \end{cases}$

Complex plane



$$\boldsymbol{n} = \boldsymbol{a} + \boldsymbol{jb} = \boldsymbol{c}\angle\theta = \boldsymbol{c}(\cos\theta + \boldsymbol{j}\sin\theta)$$

Complex Numbers Tutorial

Useful Identities :

$\mathrm{Re}(e^{j\theta})$ $\mathrm{Im}(e^{j\theta})$

Euler's identity: $$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\pm j^2 = \mp 1$$

$$(-j)(j) = 1$$

$$j = \frac{1}{-j}$$

$$e^{\pm j\pi} = -1$$

$$e^{\pm j\pi/2} = \pm j$$

$$radian = \frac{180^\circ}{\pi}$$

$$n + n^* = 2a$$

$$n - n^* = j2b$$

$$nn^* = a^2 + b^2 = c^2$$

$$\frac{n}{n^*} = 1\angle 2\theta^\circ$$

Complex Numbers Tutorial

Given complex numbers:

$$n_1 = a_1 + jb_1 = c_1\angle\theta_1$$
$$n_2 = a_2 + jb_2 = c_2\angle\theta_2$$

Addition : $$n_1 + n_2 = (a_1 + a_2) + j(b_1 + b_2)$$

Subtraction : $$n_1 - n_2 = (a_1 - a_2) + j(b_1 - b_2)$$

If the number to be added or subtracted are given in polar form, they are first converted to rectangular form.

Multiplication :

$$\begin{aligned} n_1 n_2 &= (a_1 + jb_1).(a_2 + jb_2) \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \end{aligned}$$

$$\begin{aligned} n_1 n_2 &= (c_1\angle\theta_1)(c_2\angle\theta_2) \\ &= c_1 c_2\angle(\theta_1 + \theta_2) \end{aligned}$$

Complex Numbers Tutorial

Division :

$$\frac{n_1}{n_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \times \frac{a_2 - jb_2}{a_2 - jb_2}$$

$$= \frac{(a_1 + jb_1)(a_2 - jb_2)}{a_2^2 + b_2^2}$$

$$\frac{n_1}{n_2} = \frac{(c_1 \angle \theta_1)}{(c_2 \angle \theta_2)} = \frac{c_1}{c_2} \angle (\theta_1 - \theta_2)$$

Example : Find $\frac{2\angle 90^\circ}{4\angle 75^\circ}$

Answer:

$$\frac{2\angle 90^\circ}{4\angle 75^\circ} = \frac{2}{4}\angle(90^\circ - 75^\circ) = \frac{1}{2}\angle 15^\circ$$

Example : Find $\frac{3\angle 20^\circ}{9\angle 60^\circ}$

Answer:

$$\frac{3\angle 20^\circ}{9\angle 60^\circ} = \frac{3}{9}\angle(20^\circ - 60^\circ) = \frac{1}{3}\angle -40^\circ$$

Complex Numbers Tutorial

Square root:

$$\sqrt{n} = \sqrt{c}(\theta/2)$$

Complex conjugate:

$$\boldsymbol{n}^* = \boldsymbol{a} - \boldsymbol{jb} = \boldsymbol{c}\angle(-\theta) = \boldsymbol{ce}^{-j\theta})$$

Complex Numbers Tutorial

Q: Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Sol

(a) $40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$

$20\angle -30^\circ = 20\big(\cos(-30^\circ) + j\sin(-30^\circ)\big) = 17.32 - j10$

$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$

$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = \underline{6.91\angle 12.81^\circ}$

Complex Numbers Tutorial

$$\text{(b)}\quad \frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - 4j)}{(2 + j4)(3 + j5)}$$

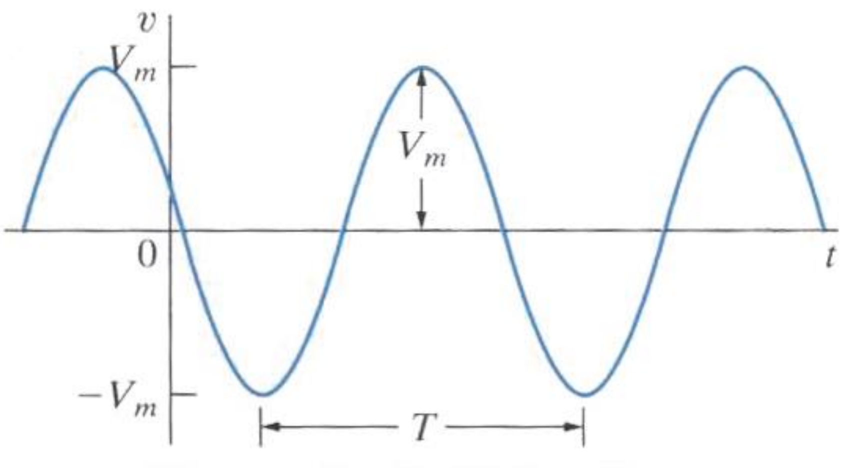
$$= \frac{11.33 + j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$

$$= \underline{0.565\angle -160.13^\circ}$$

Introduction to Sinusoid

- **Sinusoid**: a signal that has the form of the sine or cosine function.
- Why sinusoidal waveforms are useful to engineers?
  1. Appears everywhere: Vibration of a string, ripples of ocean surface, and natural response of underdamped second-order systems.
  2. Easily generated.
  3. Every practical periodic signal can be represented by a linear combination of sinusoidal signals.
  4. Easily analyzed.

The Sinusoidal Source



$$v = V_m \cos(\omega t + \phi)$$

A sinusoidal voltage/current source produces a voltage/current that varies sinusoidally with time.

Ωt + Φ : the *argument of the sinusoid*

T : period of the function (s)

f : frequency of the function (Hz)

$$f = \frac{1}{T}$$

ω : angular frequency (radians/second)

$$\omega = 2\pi f = 2\pi / T$$

Φ: phase angle (degree)

$$(\text{number of degrees}) = \frac{180}{\pi}(\text{number of radians})$$

V_m : maximum amplitude (V)

V_{rms}: root mean square value

$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)dt} = \frac{V_m}{\sqrt{2}}$$

The Sinusoidal Source

Phase Lead or Lag

Two sinusoids with the same frequency.

$$v_1(t) = V_{m1}\cos(\omega t + \phi_1),\ \ V_{m1} > 0$$

$$v_2(t) = V_{m2}\cos(\omega t + \phi_2), V_{m2} > 0$$

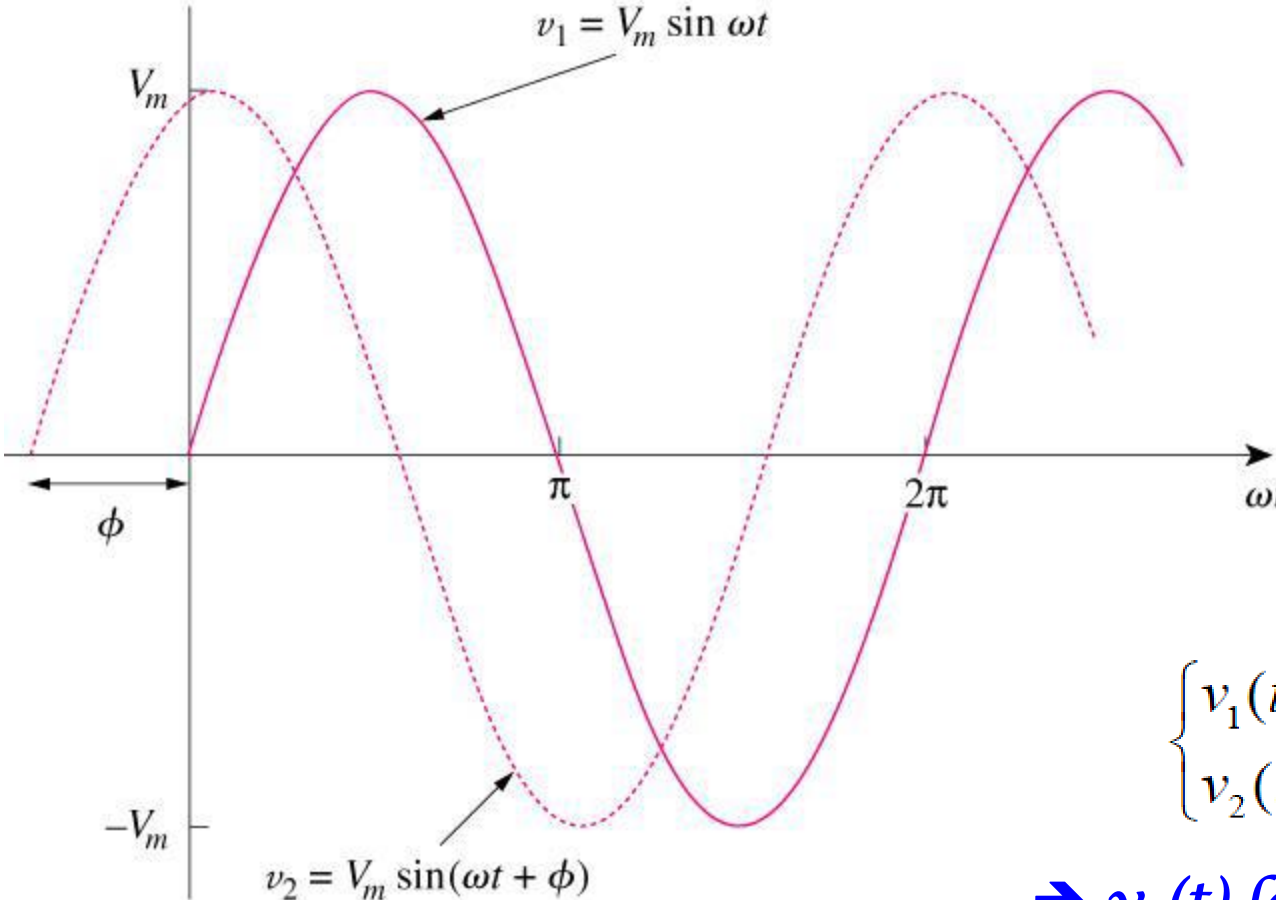
$\phi_1 - \phi_2 = 0:$ $v_1(t)$ and $v_2(t)$ are *in phase*.

$\phi_1 - \phi_2 \neq 0:$ $v_1(t)$ and $v_2(t)$ are *out of phase*.

$$\begin{cases} \phi_1 - \phi_2 > 0,\ v_1(t) \text{ leads } v_2(t) \text{ by } \phi_1 - \phi_2 \\ \phi_1 - \phi_2 < 0,\ v_1(t) \text{ lags } v_2(t) \text{ by } \phi_2 - \phi_1 \end{cases}$$

The Sinusoidal Source

Phase Lead or Lag



$$\begin{cases} v_1(t) = V_m \sin(\omega t) \\ v_2(t) = V_m \sin(\omega t + \phi) \end{cases}$$

➔ ***$v_2(t)$ leads $v_1(t)$ by ϕ***

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\Rightarrow \begin{cases} \sin(\omega t \pm 180^\circ) = -\sin \omega t \\ \cos(\omega t \pm 180^\circ) = -\cos \omega t \\ \sin(\omega t \pm 90^\circ) = \pm \cos \omega t \\ \cos(\omega t \pm 90^\circ) = \mp \sin \omega t \end{cases}$$

$$\Rightarrow A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta)$$

$$C = \sqrt{A^2 + B^2} \qquad \theta = tan^{-1} \frac{B}{A}$$

Ex. 1: Finding the Characteristics of a Sinusoidal Current

A sinusoidal current has a maximum amplitude of **20 A**. The current passes through one complete cycle in **1 ms**. The magnitude of the current at zero time is **10 A**.

a) What is the frequency of the current in hertz?

b) What is the frequency in radians per second?

c) Write the expression for *i(t) using the cosine* function. Express ϕ in degrees.

d) What is the rms value of the current?

Sol. of example 1:

a) From the statement of the problem, *T = 1 ms;*
Hence f = 1/T = 1000 Hz.

b) $\omega = 2\pi f = 2000\pi$ *rad/s.*

c) We have $i(t) = I_m \cos(\omega t + \phi) = 20 \cos(2000\pi t + \phi)$, *but* $i(0) = 10$ *A.*
Therefore $10 = 20\cos\phi \rightarrow \phi = 60^\circ$
Thus the expression for i(t) becomes $i(t) = 20\cos(2000\pi t + 60^\circ)$.

d) From the derivation of $V_{rms} = V_m/\sqrt{2}$, the rms value of a sinusoidal current is $20/\sqrt{2}$. Therefore the rms value is $20/\sqrt{2}$, or 14.14 A.

Ex. 2: Finding the Characteristics of a Sinusoidal Voltage

A sinusoidal voltage is given by the expression

$$v = 300 \cos(120\pi t + 30°).$$

a) What is the period of the voltage in milliseconds?

b) What is the frequency in hertz?

c) What is the magnitude of v at $t = 2.778$ ms?

d) What is the rms value of v?

Sol. of example 2:

a) From the expression for v, $\omega = 120\pi$ rad/s. Because $\omega = 2\pi/T$, $T = 2\pi/\omega = \frac{1}{60}$ s, or 16.667 ms.

b) The frequency is $1/T$, or 60 Hz.

c) From (a), $\omega = 2\pi/16.667$; thus, at $t = 2.778$ ms, ωt is nearly 1.047 rad, or 60°. Therefore, $v(2.778 \text{ ms}) = 300 \cos(60^\circ + 30^\circ) = 0$ V.

d) $V_{\text{rms}} = 300/\sqrt{2} = 212.13$ V.

Ex. 3: Translating a Sine Expression to a Cosine Expression

We can translate the sine function to the cosine function by subtracting 90° *(π/2 rad) from the argument* of the sine function.

a) Verify this translation by showing that

$$\sin(\omega t + \theta) = \cos(\omega t + \theta - 90°).$$

b) Use the result in (a) to express $\sin(\omega t + 30°)$ as a cosine function.

Sol. of example 3:

a) Verification involves direct application of the trigonometric identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

We let $\alpha = \omega t + \theta$ and $\beta = 90^\circ$. As $\cos 90^\circ = 0$ and $\sin 90^\circ = 1$, we have

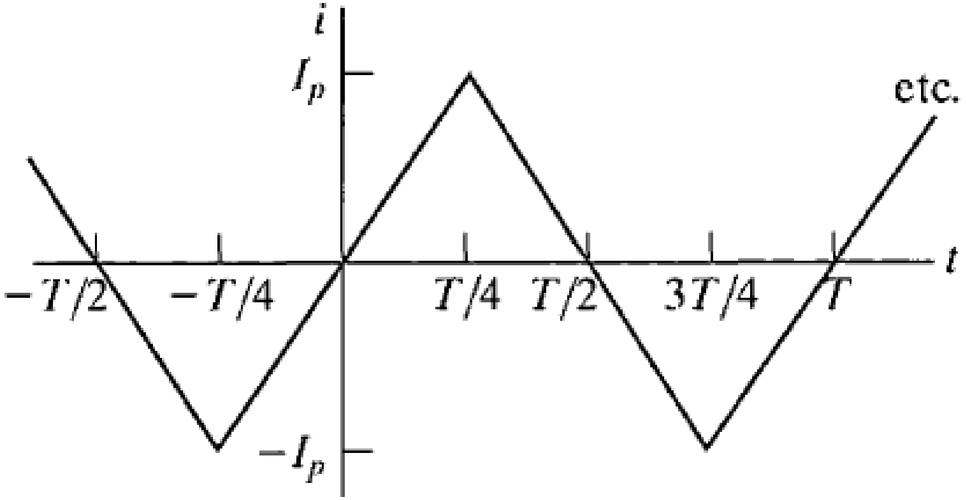
$$\cos(\alpha - \beta) = \sin\alpha = \sin(\omega t + \theta) = \cos(\omega t + \theta - 90^\circ).$$

b) From (a) we have

$$\sin(\omega t + 30^\circ) = \cos(\omega t + 30^\circ - 90^\circ) = \cos(\omega t - 60^\circ).$$

Ex. 4: Calculating the rms Value of a Triangular Waveform

Calculate the rms value of the periodic triangular current shown in Fig.. Express your answer in terms of the peak current I_p.

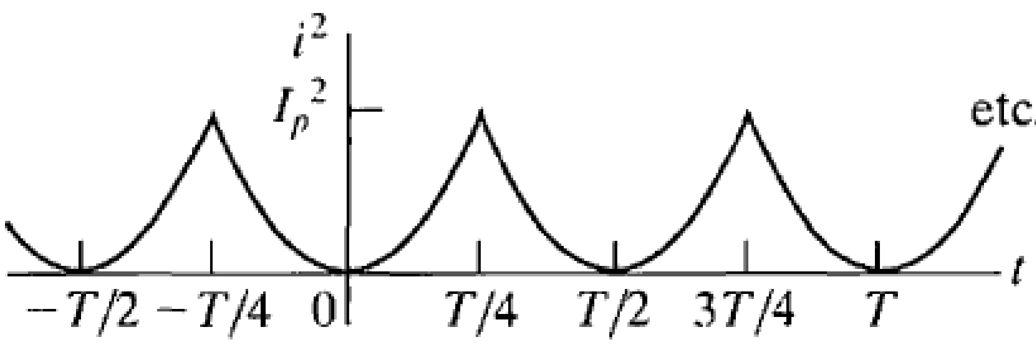


Sol. of example 4:

The rms value of i is
$$I_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2 dt}.$$

Interpreting the integral under the radical sign as the area under the squared function for an interval of one period is helpful in finding the rms value. The squared function with the area between 0 and *T shaded is shown in Fig., which also indicates* that for this particular function, the area under the squared current for an interval of one period is equal to four times the area under the squared current for the interval 0 to *T/4 seconds; that is,*

$$\int_{t_0}^{t_0+T} i^2 dt = 4\int_0^{T/4} i^2 dt.$$



Sol. of example 4 cont.:

The analytical expression for i in the interval 0 to $T/4$ is

$$i = \frac{4I_p}{T}t, \quad 0 < t < T/4.$$

The area under the squared function for one period is

$$\int_{t_0}^{t_0+T} i^2 dt = 4\int_0^{T/4} \frac{16I_p^2}{T^2}t^2 dt = \frac{I_p^2 T}{3}.$$

Sol. of example 4 cont.:

The mean, or average, value of the function is simply the area for one period divided by the period. Thus

$$i_{\text{mean}} = \frac{1}{T}\frac{I_p^2 T}{3} = \frac{1}{3}I_p^2.$$

The rms value of the current is the square root of this mean value. Hence

$$I_{\text{rms}} = \frac{I_p}{\sqrt{3}}.$$

Ex. 5

Calculus the phase angle between $v_1 = -10\cos(\omega t + 50°)$ and $v_2 = 12\sin(\omega t - 10°)$. *State which sinusoid is leading.*

Sol. of Ex. 5

v_1 = -10 cos (ωt + 50°) = 10 cos (ωt + 50° - 180°)

v_1 = 10 cos (ωt - 130°)

and

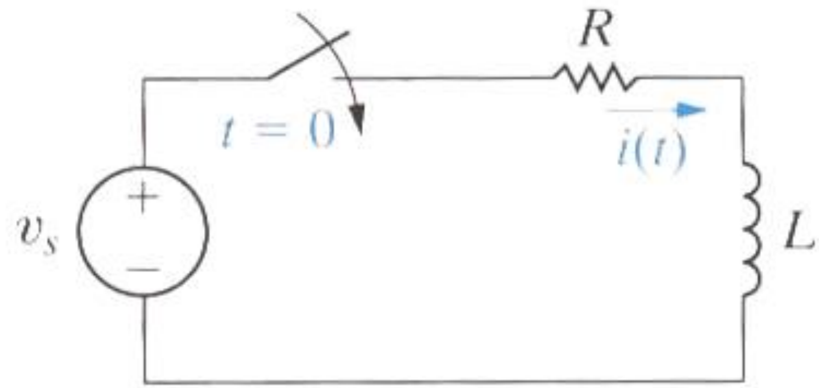
v_2 = 12 sin (ωt - 10°) = 12 cos (ωt - 10° - 90°)

v_2 = 12cos (ωt - 100°)

ϕ_1=-130°, ϕ_2=-100°; ϕ_2-ϕ_1=30°

➔ **v_2 leads v_1 by 30°**

The Sinusoidal Response



There, v_s *is a sinusoidal voltage, or*

$$v_s = V_m \cos(\omega t + \phi)$$

Apply KVL to the circuit

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

The formal solution

$$i = \underbrace{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}}_{\textbf{Transient}} + \underbrace{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}_{\textbf{Steady-state}}$$

The Sinusoidal Response

For the steady-state solution:

1. The steady-state solution is a sinusoidal function.
2. The frequency of the response signal is identical to the frequency of the source signal.
3. The maximum amplitude of the steady-state response differs from the maximum amplitude of the source.
4. The phase angle of the response signal differs from the phase angle of the source.

The Phasor

Euler's identity: $$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

For a given sinusoidal voltage function:

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t+\phi)})$$

$$= \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

Thus $v(t)$ can be written as

$$v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

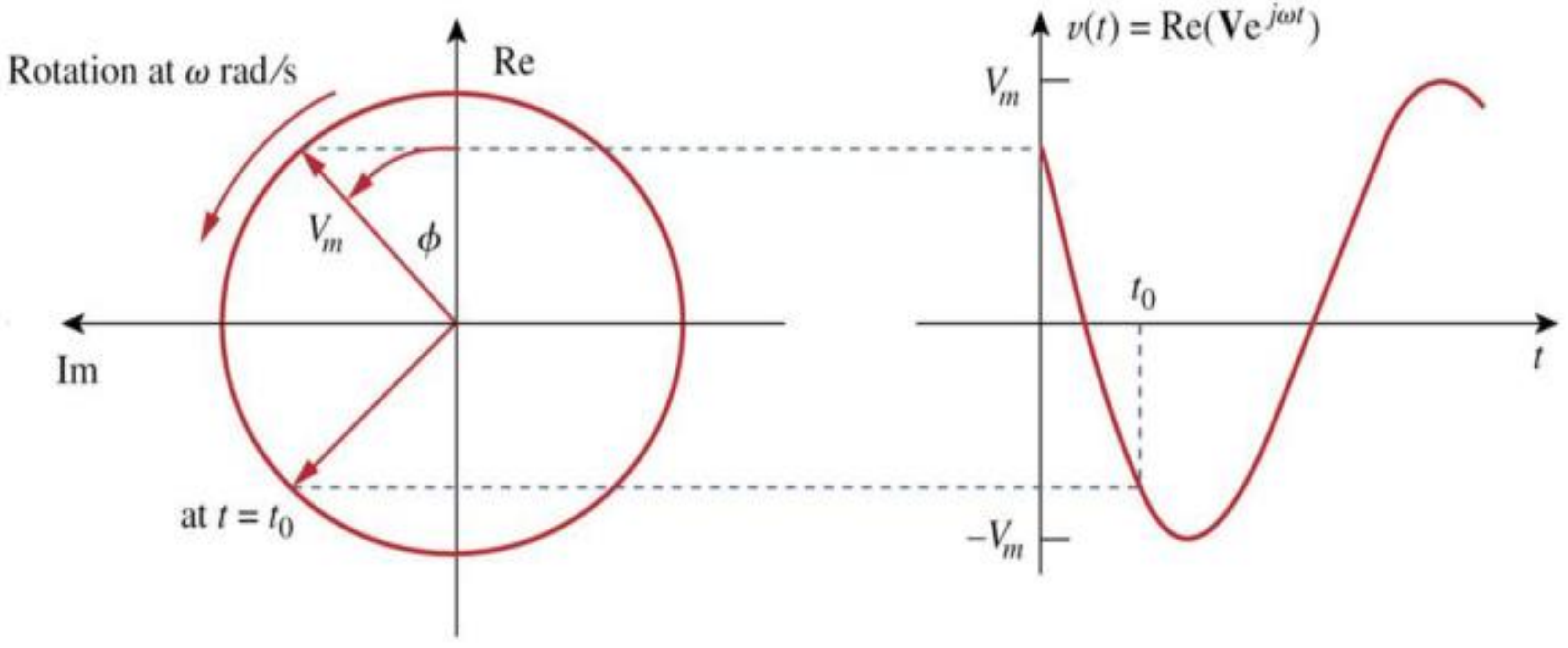
Here $\mathbf{V} = V_m e^{j\phi} = V_m \angle\phi$ is the *phasor representation* of $v(t)$.

The phasor is a complex number that carries the amplitude and phase angle information of a sinusoidal function.

The **phasor transform** of the given sinusoidal function is:

$$\mathbf{V} = V_m e^{j\phi} = \mathrm{P}\left\{V_m \cos\left(\omega t + \phi\right)\right\}$$

The Phasor



(a) (b)

The Phasor

The phasor transform transfers the sinusoidal function from the ***time domain*** to the complex-number domain, called the ***frequency domain***.

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

Given the frequency ω, the phasor can equivalently represent the signal v(t).

The polar form: $\mathbf{V} = V_m e^{j\phi}$

The rectangular form: $\mathbf{V} = V_m \cos\phi + jV_m \sin\phi$

The angle notation: $\boldsymbol{V_m \angle \phi^\circ = V_m e^{j\phi}}$

Inverse phasor transform is found by multiplying the phasor by $e^{j\omega t}$ and then extracting the real part of the product.

$$\mathrm{P}^{-1}\left\{V_m e^{j\phi}\right\} = \mathrm{R}\left\{V_m e^{j\phi} e^{j\omega t}\right\}$$

The Phasor

Sinusoid-PhasorTransformation

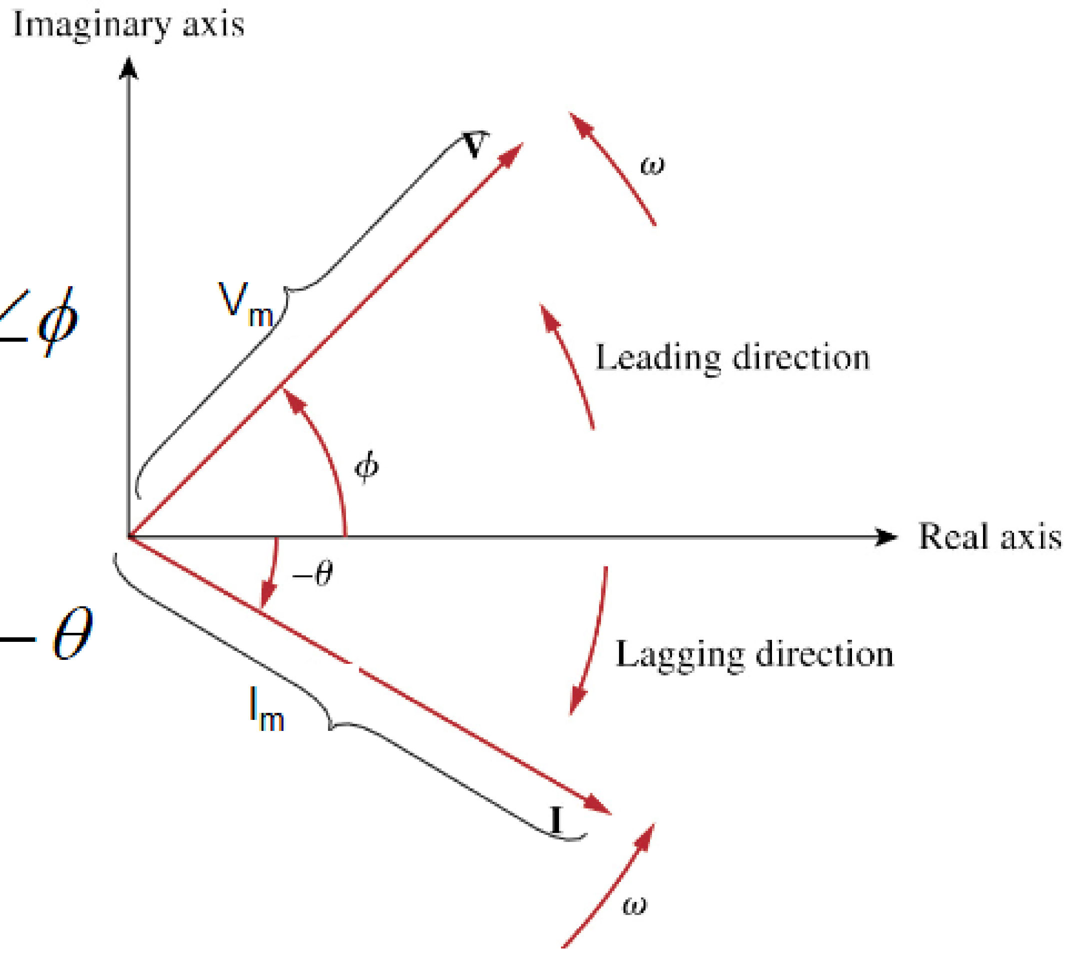
Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle (\phi - 90°)$
$I_m \cos(\omega t + \phi)$	$I_m \angle \phi$
$I_m \sin(\omega t + \phi)$	$I_m \angle (\phi - 90°)$

The Phasor

Phasor Diagram

$$\mathbf{V} = V_m \angle \phi$$

$$\mathbf{I} = I_m \angle -\theta$$



Ex. 6:

Transform these sinusoid to phasors:

(a) $i = 6\cos(50t - 40^\circ)$ A

(b) $v = -4\sin(30t + 50^\circ)$ V

Sol. of Ex. 6:

(a) $i = 6\cos(50t - 40°)$ has the phasor

$$\mathbf{I} = 6\angle -40° \text{ A}$$

(b) Since $-\sin A = \cos(A + 90°)$

$$v = -4\sin(30t + 50°) = 4\cos(30t + 50° + 90°)$$

$$= 4\cos(30t + 140°) \text{ V}$$

The phasor of v is $\mathbf{V} = 4\angle 140° \text{ V}$

Ex. 7: Find the sinusoid representation by these phasors:

(a) $\mathbf{I} = -3 + j4 \text{ A}$

(b) $\mathbf{V} = j8e^{-j20^\circ} \text{ V}$

Sol of Ex. 7:

(a) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$

$i(t) = 5\cos(\omega t + 126.87^\circ) \text{ A}$

(b) $j = 1\angle 90^\circ,$

$$\mathbf{V} = j8\angle -20^\circ = (1\angle 90^\circ) \times (8\angle -20^\circ)$$

$$= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ \text{ V}$$

$v(t) = 8\cos(\omega t + 70^\circ) \text{ V}$

Ex. 8: Adding Cosines Using Phasors

If $y_1 = 20\cos(\omega t - 30^\circ)$ and $y_2 = 40\cos(\omega t + 60^\circ)$, express $y = y_1 + y_2$ as a single sinusoidal function.

a) Solve by using trigonometric identities.

b) Solve by using the phasor concept.

Sol. of Ex. 8:

a) First we expand both y_1 and y_2, using the cosine of the sum of two angles, to get

$$y_1 = 20 \cos \omega t \cos 30^\circ + 20 \sin \omega t \sin 30^\circ;$$

$$y_2 = 40 \cos \omega t \cos 60^\circ - 40 \sin \omega t \sin 60^\circ.$$

Adding y_1 and y_2, we obtain

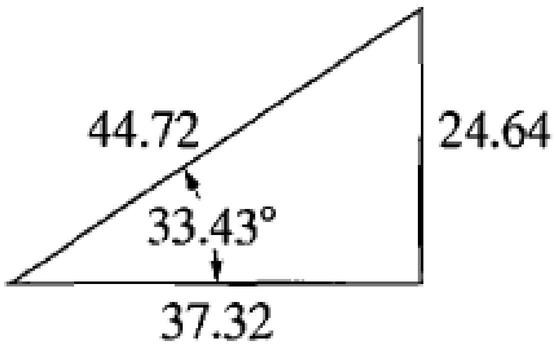
$$y = (20 \cos 30 + 40 \cos 60) \cos \omega t$$

$$+ (20 \sin 30 - 40 \sin 60) \sin \omega t$$

$$= 37.32 \cos \omega t - 24.64 \sin \omega t.$$

Sol. of Ex. 8 cont.:

To combine these two terms we treat the co-efficients of the cosine and sine as sides of a right triangle (Fig. and then multiply and divide the right-hand side by the hypotenuse. Our expression for y becomes



$$y = 44.72\left(\frac{37.32}{44.72}\cos\omega t - \frac{24.64}{44.72}\sin\omega t\right)$$

$$= 44.72(\cos 33.43^\circ \cos\omega t - \sin 33.43^\circ \sin\omega t).$$

Again, we invoke the identity involving the cosine of the sum of two angles and write

$$y = 44.72\cos(\omega t + 33.43^\circ).$$

Sol. of Ex. 8 cont.:

b) We can solve the problem by using phasors as follows: Because

$$y = y_1 + y_2,$$

then, from Eq. 9.24,

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$

$$= 20\angle{-30^\circ} + 40\angle{60^\circ}$$

$$= (17.32 - j10) + (20 + j34.64)$$

$$= 37.32 + j24.64$$

$$= 44.72\angle{33.43^\circ}.$$

Sol. of Ex. 8 cont.:

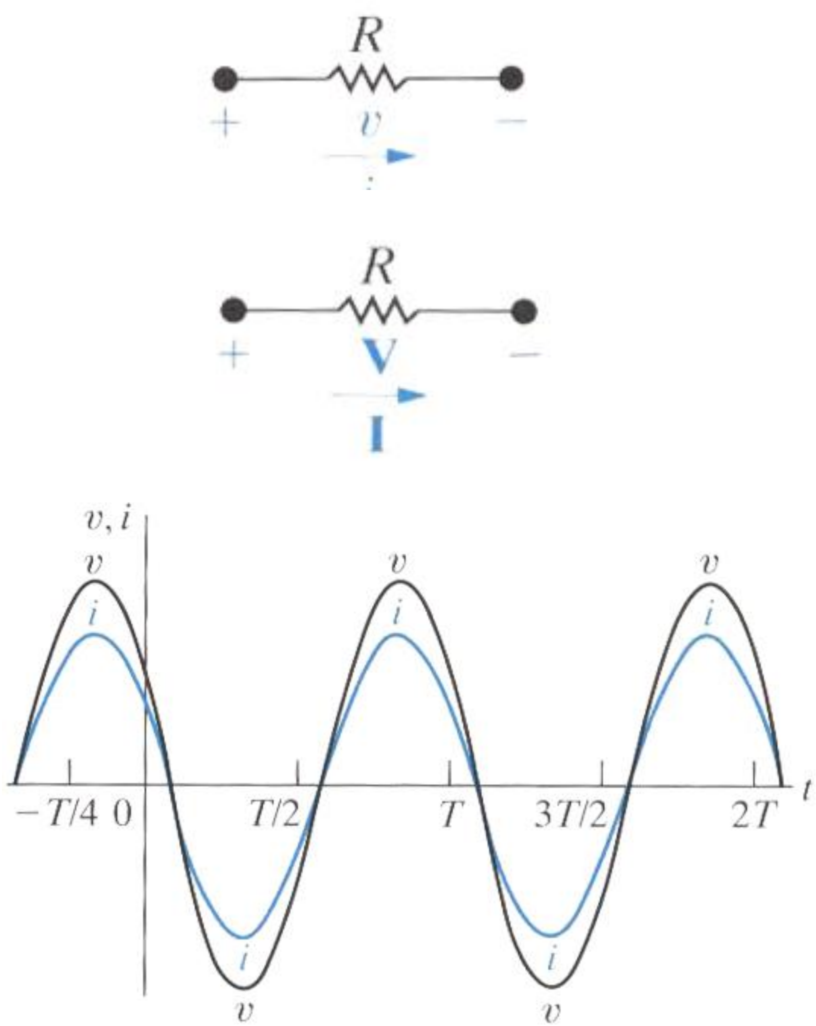
Once we know the phasor **Y**, we can write the corresponding trigonometric function for y by taking the inverse phasor transform:

$$y = \mathcal{P}^{-1}\{44.72e^{j33.43}\} = \Re\{44.72e^{j33.43}e^{j\omega t}\}$$

$$= 44.72\cos(\omega t + 33.43°).$$

The superiority of the phasor approach for adding sinusoidal functions should be apparent. Note that it requires the ability to move back and forth between the polar and rectangular forms of complex numbers.

The V-I Relationship for a Resistor



Given a current in a resistor:

$$i = I_m \cos(\omega t + \theta_i)$$

The voltage of the resistor is:

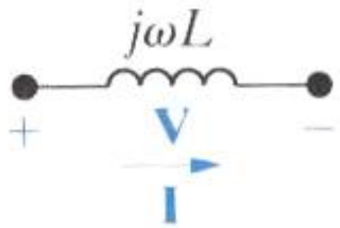
$$v = Ri = RI_m \cos(\omega t + \theta_i)$$

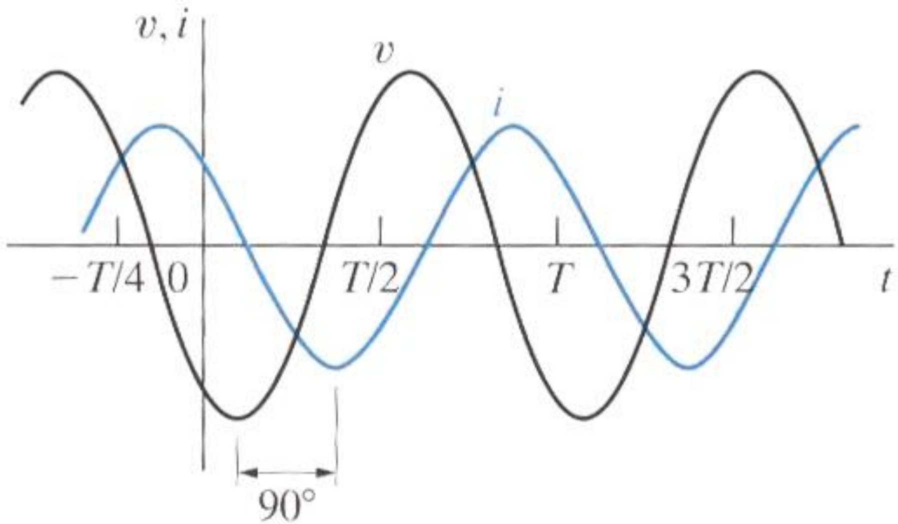
Phasor presentation:

$$\mathbf{V} = RI_m e^{j\theta_i} = R\mathbf{I}$$

Voltage and current of a resistor are ***in phase*** .

The V-I Relationship for an Inductor





Given a current in an inductor:

$$i = I_m \cos(\omega t + \theta_i)$$

The voltage is:

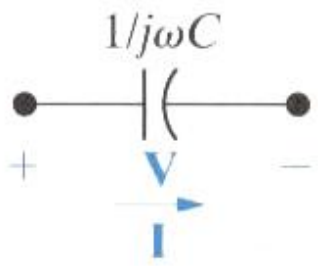
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i)$$
$$= -\omega L I_m \cos(\omega t + \theta_i - 90°)$$

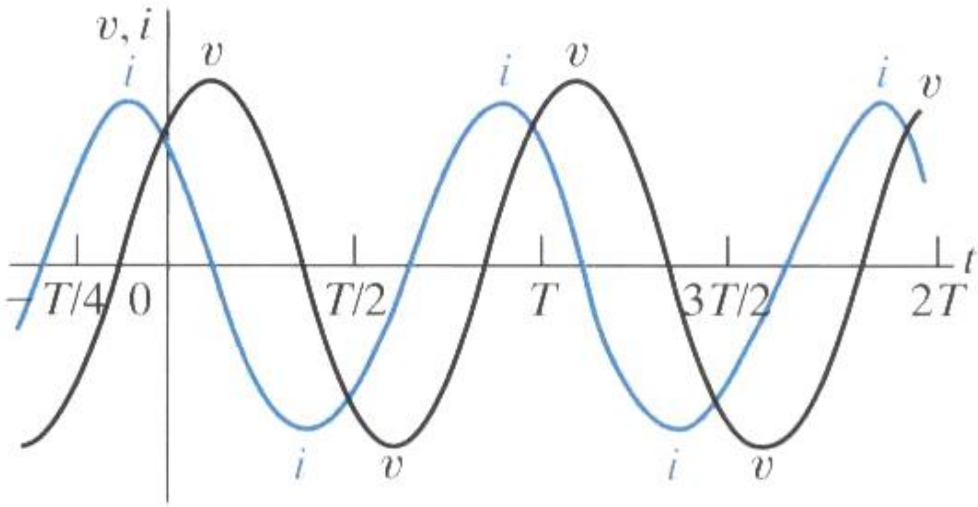
The phasor presentation:

$$\mathbf{V} = -\omega L I_m e^{j(\theta_1 - 90°)}$$
$$= -\omega L I_m e^{j\theta_i} e^{-j90°}$$
$$= j\omega L I_m e^{j\theta_i}$$
$$= j\omega L \mathbf{I}$$

In an inductor, the ***voltage leads*** the current by 90⁰ or the ***current lags*** behind the voltage by 90⁰.

The V-I Relationship for a Capacitor





Given a voltage in a capacitor:

$$v = V_m \cos(\omega t + \theta_v)$$

The current is:

$$i = C\frac{dv}{dt} = -\omega C V_m \cos(\omega t + \theta_v - 90°)$$

The phasor presentation:

$$\mathbf{I} = -\omega C V_m e^{j(\theta_v - 90°)} = j\omega C \mathbf{V}$$

or :

$$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$$

In a capacitor, the ***voltage lags*** behind the current by 90^0 or the ***current leads*** the voltage by 90^0 .

Summary of Voltage-Current Relationships

Element	Time domain	Frequency domain
R	$V = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Ex. 9

The voltage ***$v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.***

Sol. of Ex. 9

$\mathbf{V} = 12\angle 45^\circ \text{ V} = j\omega L\mathbf{I}$, where $\omega = 60\text{rad/s}$.

Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\angle 45^\circ}{j60 \times 0.1} = \frac{12\angle 45^\circ}{6\angle 90^\circ} = 2\angle -45^\circ \text{ A}$$

Converting,

$$i(t) = 2\cos(60t - 45^\circ) \text{ A}$$

Impedance and Reactance

Appy Ohm's law in frequency domain:

$$\mathbf{V} = Z\mathbf{I}$$

Z is the **impedance** of the circuit element, which is measured in ohms.

The imaginary part of the impedance is the **reactance** .

$$Y = \frac{1}{Z} = G + jB$$

Y is the **admittance** of the circuit element, which is measured in siemens.

Admittance is a complex number, whose real part, *G*, is called **conductance**, and whose impaginary part, *B*, is called **susceptance**.

Circuit Element	*Impedance (Z)*	*Reactance*	*Admittance (Y)*	*Susceptance*
Resistor	R	--	G	--
Inductor	$j\omega L$	ωL	$j(-1/\omega L)$	$-1/\omega L$
Capacitor	$j(-1/\omega C)$	$-1/\omega C$	$j\omega C$	ωC

General Passive Circuit In Phasor Domain

$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}| \angle \theta}$$

where

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R}$$

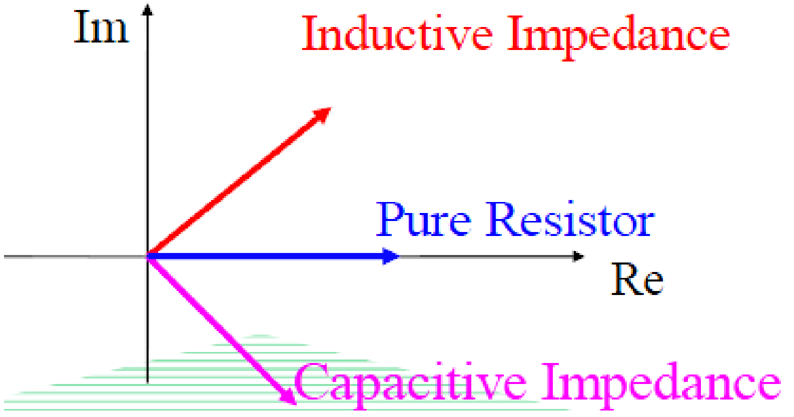
and

$$R = |\mathbf{Z}|\cos\theta = \mathrm{Re}(\mathbf{Z}): \text{ Resistance of } \mathbf{Z},$$

$$X = |\mathbf{Z}|\sin\theta = \mathrm{Im}(\mathbf{Z}): \text{ Reactance of } \mathbf{Z}$$

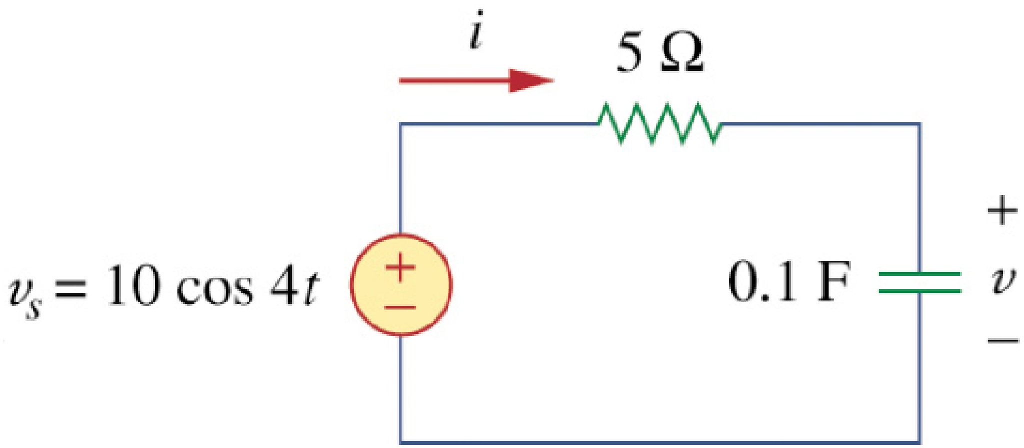
$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}| \angle \theta}$$

- $X > 0$: inductive impedance
- X=0: pure resistor
- X<0: capacitive impedance



Ex. 10

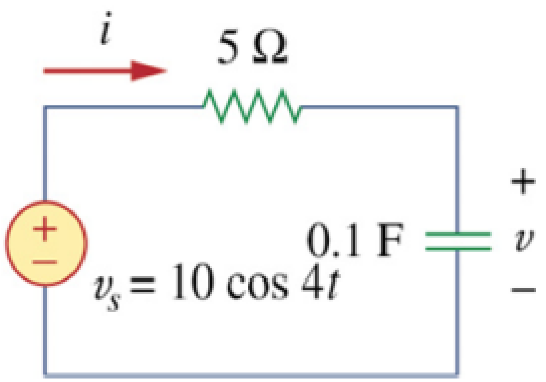
Find *v(t) and i(t) in the circuit*



Sol. of Ex. 10

- From the voltage source 10 cos 4t, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$



- The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1}$$
$$= 5 - j2.5\Omega$$

- Hence the current

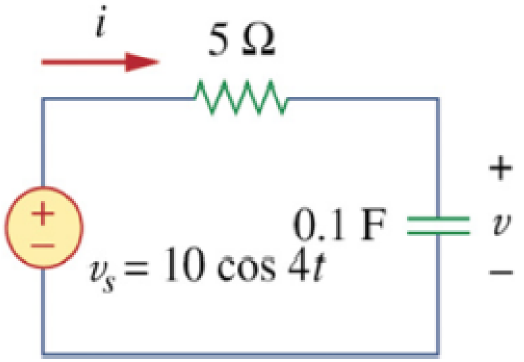
$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

Sol. of Ex. 10 Cont.

- The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z_c} = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4\times 0.1}$$

$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V}$$



- Converting **I** and **V**, we get

$$i(t) = 1.789\cos(4t + 26.57^\circ) \text{ A}$$

$$v(t) = 4.47\cos(4t - 63.43^\circ) \text{ V}$$

Kirchhoff's Laws

The KCL and KVL are still applicable in the frequency domain.

Time domain | Frequency domain

KVL $\quad v_1 + v_2 + \cdots + v_n = 0 \quad \Rightarrow \quad \mathbf{V}_1 + \mathbf{V}_2 + \cdots \mathbf{V}_n = 0$

KCL $\quad i_1 + i_2 + \cdots + i_n = 0 \quad \Rightarrow \quad \mathbf{I}_1 + \mathbf{I}_2 + \cdots \mathbf{I}_n = 0$

For KVL, let $v_1, v_2, \ldots, v_n$, be the voltages around a closed loop.

$$\Rightarrow v_1 + v_2 + \cdots + v_n = 0$$

In the sinusoidal steady state, each voltage may be written in cosine form.

$$\Rightarrow V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0$$

This can be written as

$$\mathrm{Re}(V_{m1}e^{j\theta_1}e^{j\omega t}) + \mathrm{Re}(V_{m2}e^{j\theta_2}e^{j\omega t}) + \cdots + \mathrm{Re}(V_{mn}e^{j\theta_n}e^{j\omega t}) = 0$$

$$\Rightarrow \mathrm{Re}\left[\left(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n}\right)e^{j\omega t}\right] = 0$$

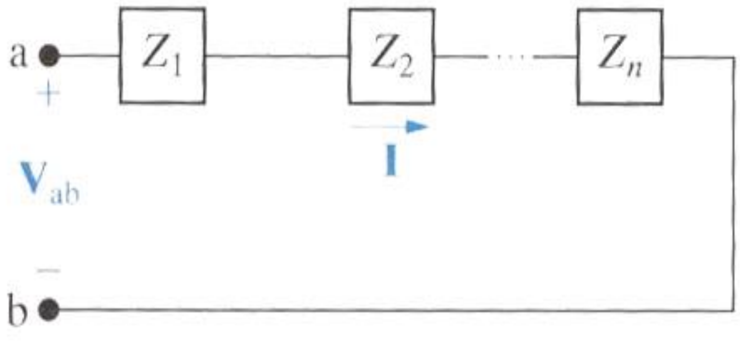
$$\Rightarrow \mathrm{Re}\left[\left(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n\right)e^{j\omega t}\right] = 0; \quad (\mathbf{V}_K = V_{mk}e^{j\theta_k})$$

$$\Rightarrow \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0; \quad (\because\ e^{j\omega t} \neq 0\ \forall t)$$

$\Rightarrow$ KVL holds for phasor! Similarily, KCL holds for phasor!

Series and Parallel Combination

Series Combination



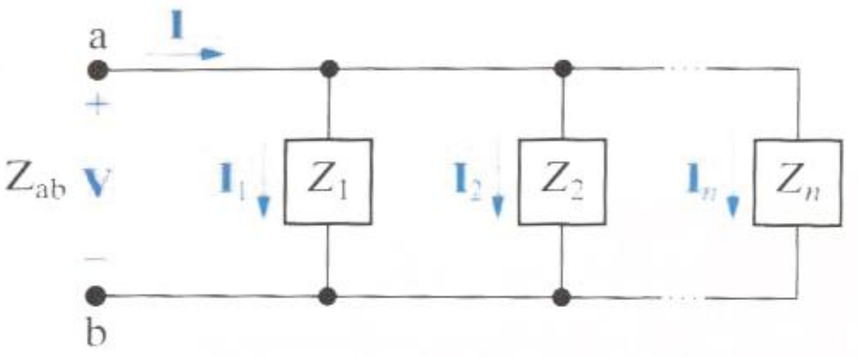
Voltage :

$$\mathbf{V}_{ab} = Z_1\mathbf{I} + Z_2\mathbf{I} + \ldots + Z_n\mathbf{I}$$
$$= (Z_1 + Z_2 + \ldots + Z_n)\mathbf{I}$$

Equivalent impedance :

$$Z_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{I}} = Z_1 + Z_2 + \ldots + Z_n$$

Parallel Combination



Current :

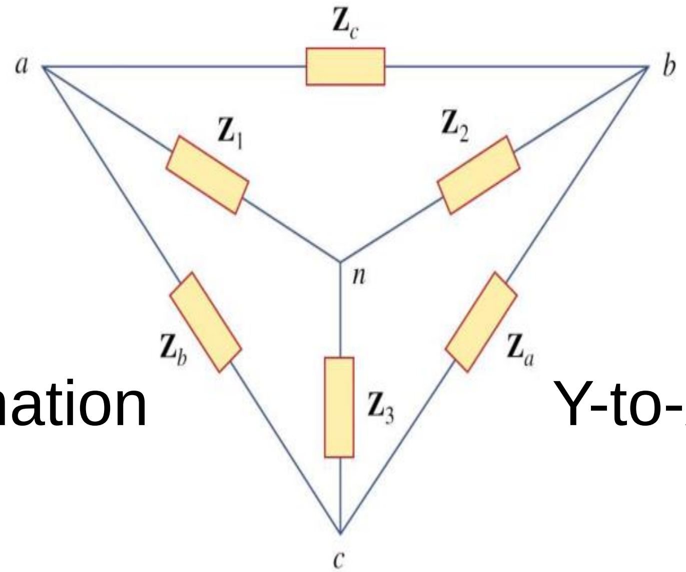
$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \ldots + \mathbf{I}_n = \frac{\mathbf{V}}{Z_{ab}}$$

Equivalent impedance :

$$\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_n}$$

$$Z_{ab} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Delta-to-Wye Simplifications



Δ-to-Y transformation

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

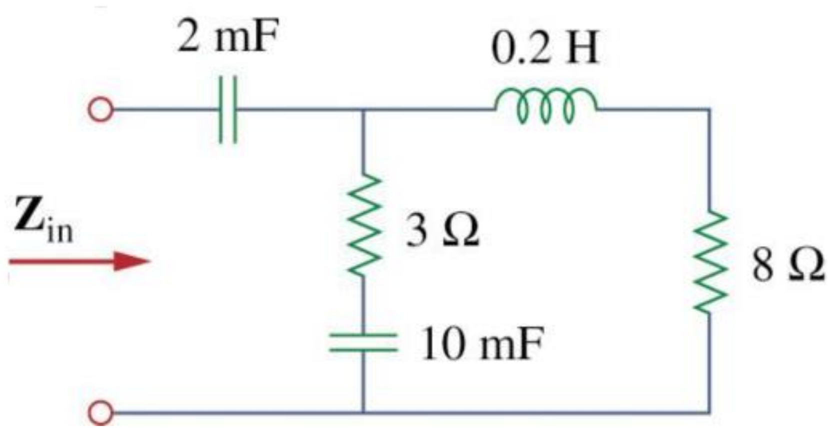
Y-to-Δ transformation

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

Ex. 11 **Find the input impedance of the circuit. Assume that the circuit operations at ω = 50 red/s.**

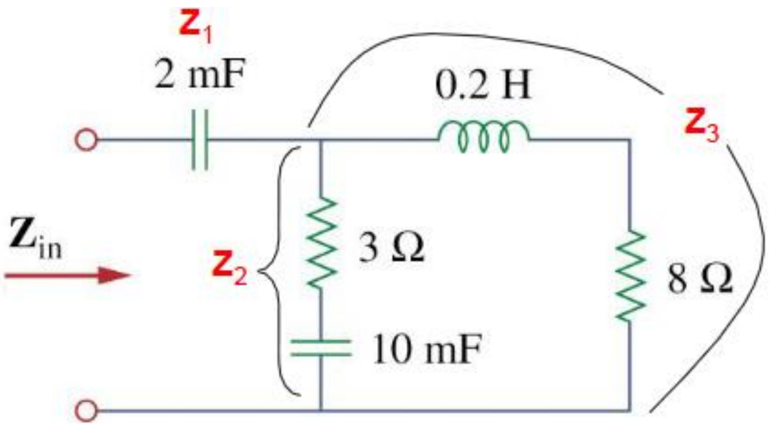


Sol. of Ex. 11

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50\times 2\times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50\times 10\times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50\times 0.2 = (8 + j10)\ \Omega$$



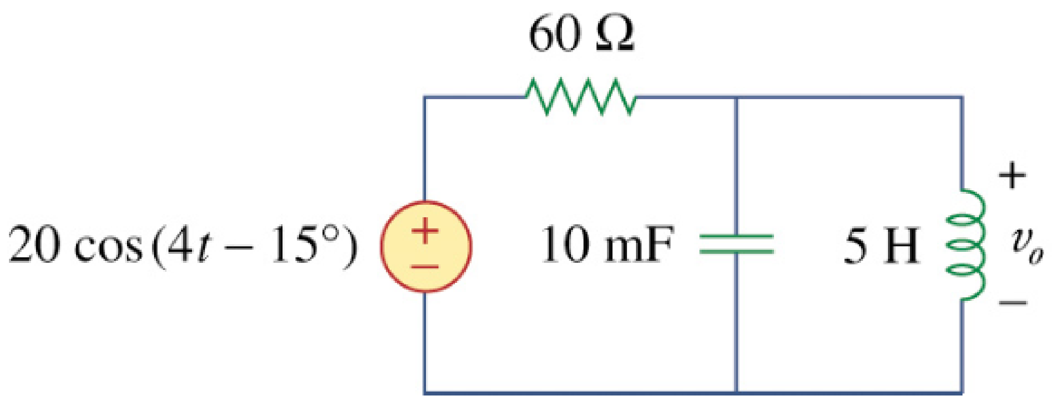
The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \| \mathbf{Z}_3 = -j10 + \frac{(3-j2)(8+j10)}{11+j8}$$

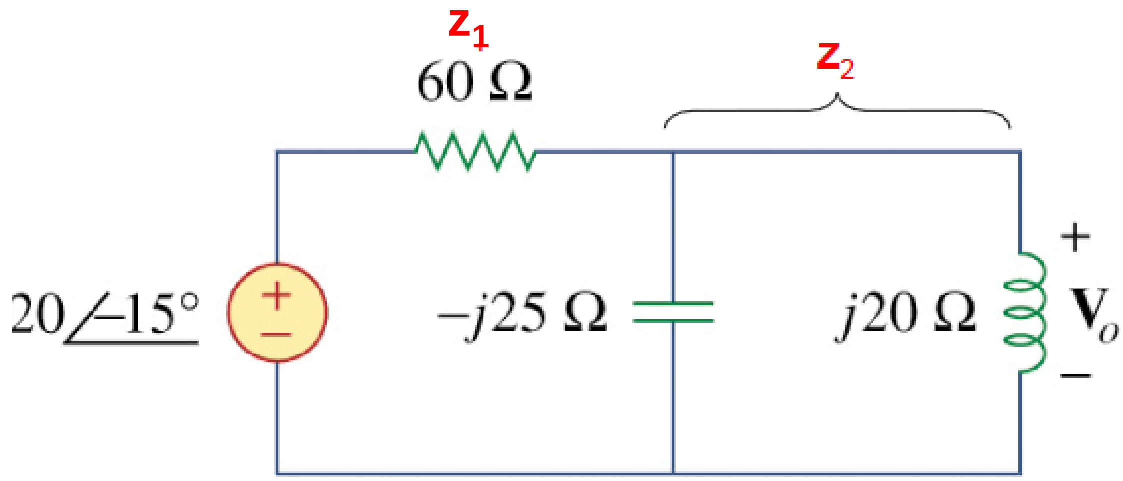
$$= -j10 + \frac{(44+j14)(11-j8)}{11^2+8^2} = -j10 + 3.22 - j1.07\ \Omega$$

$$= 3.22 - j11.07\ \Omega$$

Ex. 12 **Determine $v_0(t)$ in the circuit.**



Sol. of Ex. 12



$$v_s = 20\cos(4t - 15°) \quad \Rightarrow \quad \mathbf{V}_s = 20\angle -15° \text{ V}, \quad \omega = 4$$

$$10\text{mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4\times 10\times 10^{-3}} = -j25\Omega$$

$$5 \text{ H} \quad \Rightarrow \quad j\omega L = j4\times 5 = j20\ \Omega$$

Let

Z_1 = Impedance of the 60-Ω resistor; Z_2 = Impedance of the parallel combination of the 10 mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \| j20 = \frac{-j25\times j20}{-j25 + j20} = j100\ \Omega$$

Sol. of Ex. 12 Cont.

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100} \times (20\angle -15^\circ)$$
$$= (0.8575\angle 30.96^\circ) \times (20\angle -15^\circ) = 17.15\angle 15.96^\circ \text{ V}$$

Convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ) \text{ V}$$

Techniques of Circuit Analysis

- Techniques of circuit analysis introduced in Lecture 3 – 4 such as:
  - Source transformations
  - The node-voltage method
  - The mesh-current method
  - Superposition
  - Thevenin – Norton equivalent

  also can be applied to frequency-domain circuits.

- The validity of these techniques is followed the <u>same process</u> used in lectures 3-4 except that we <u>substitute impedance (Z) for resistance (R)</u>.

- Many examples are shown in textbook.

Techniques of Circuit Analysis

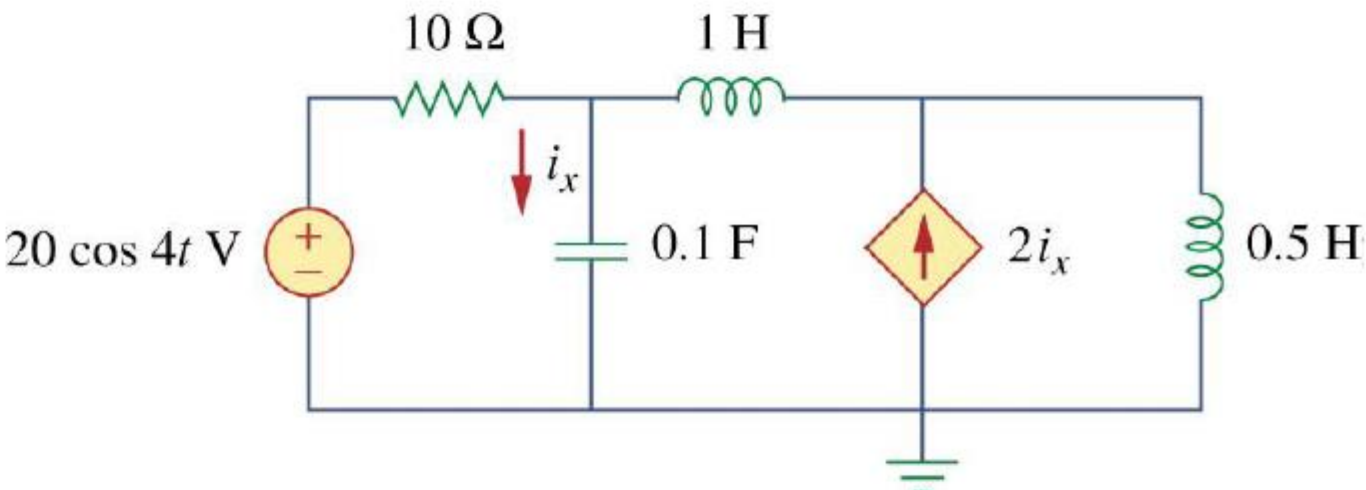
- In steady-state circuit response with sinusoidal excitation, the phasor method enables the R, L, C as an element of impedance whose function is the same as a resistor such that **generalized Ohm's law** can be applied.
- Hence, all circuit **analysis methods** (Nodal, Mesh), **theorems** (Superposition, Source transformation, Thevenin and Norton equivalent circuits) can be applied to analyze ac circuits.

Steps to Analyze AC Circuits

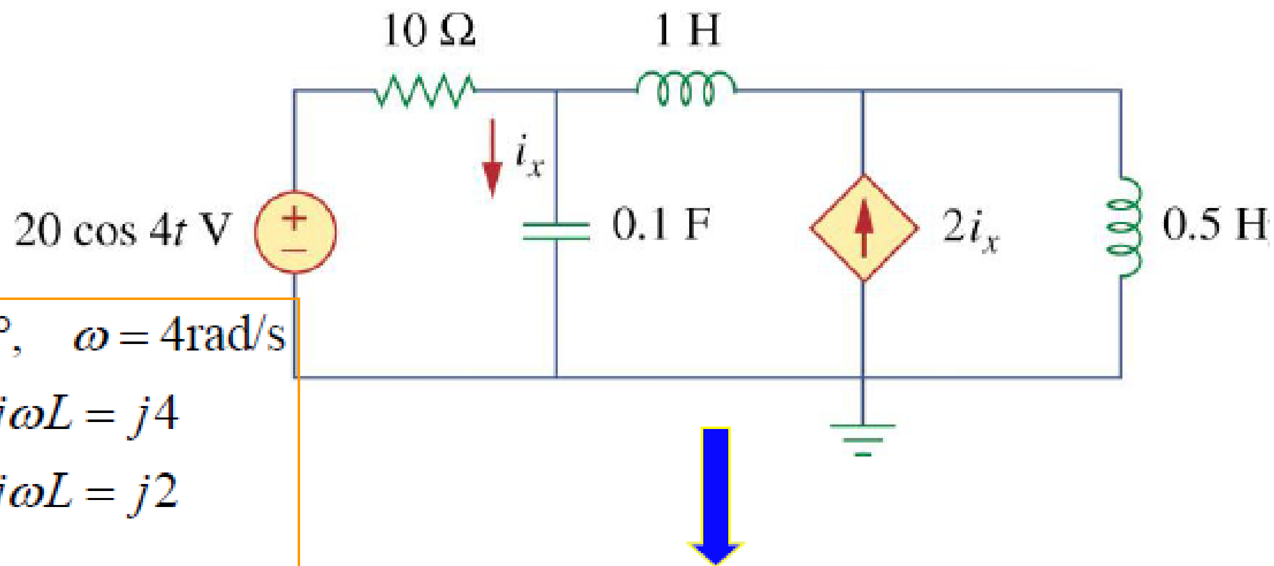
- Steps to Analyze AC Circuits:
  1. Transform the circuit to the phasor or frequency domain.
  2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
  3. Transform the resulting phasor to the time domain.

Ex. 13 - Nodal Analysis

Find i_x in the circuit using nodal analysis.



Sol. of Ex. 13:

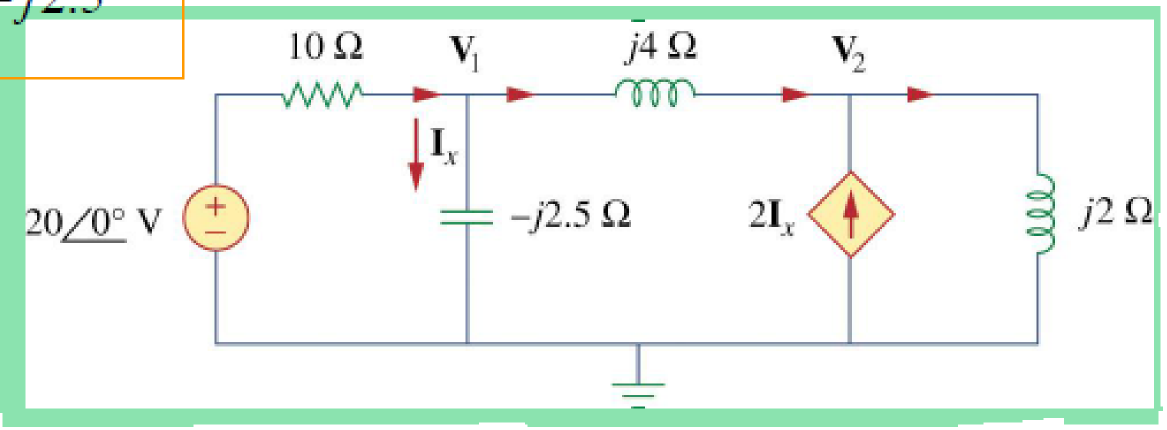


$$20\cos 4t \quad \Rightarrow \quad 20\angle 0^\circ, \quad \omega = 4\text{rad/s}$$

$$1 \text{ H} \quad \Rightarrow \quad j\omega L = j4$$

$$0.5 \text{ H} \quad \Rightarrow \quad j\omega L = j2$$

$$0.1 \text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j2.5$$

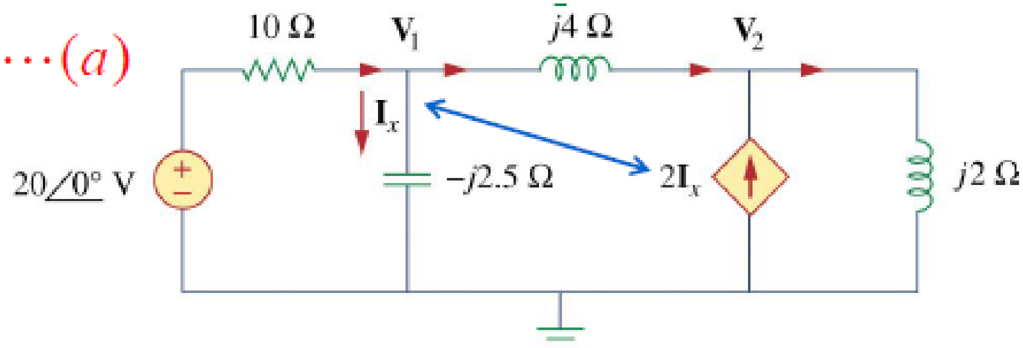


Sol. of Ex. 13:

- KCL at node 1

$$\frac{20-\mathbf{V}_1}{10}=\frac{\mathbf{V}_1}{-j2.5}+\frac{\mathbf{V}_1-\mathbf{V}_2}{j4}$$

$$\Rightarrow (1+j1.5)\mathbf{V}_1+j2.5\mathbf{V}_2=20\cdots(a)$$



- KCL at node 2

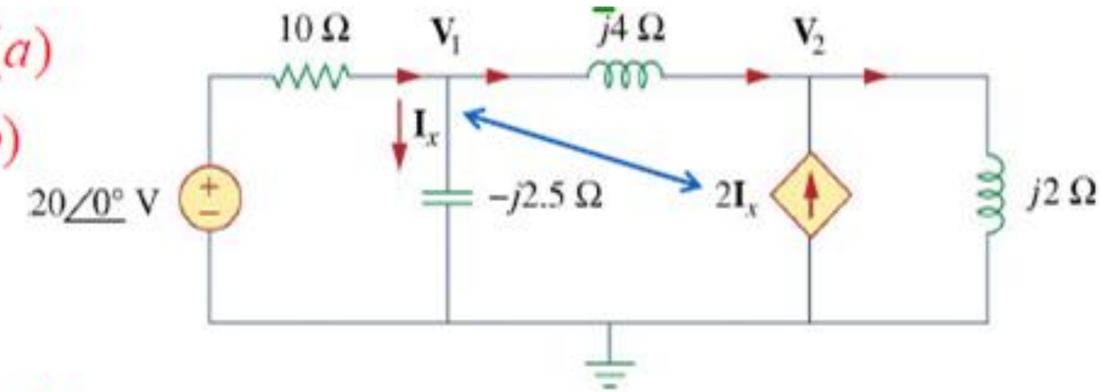
$$2\mathbf{I}_x+\frac{\mathbf{V}_1-\mathbf{V}_2}{j4}=\frac{\mathbf{V}_2}{j2}$$

$$\Rightarrow \frac{2\mathbf{V}_1}{-j2.5}+\frac{\mathbf{V}_1-\mathbf{V}_2}{j4}=\frac{\mathbf{V}_2}{j2};\ \left(\mathbf{I}_x=\mathbf{V}_1/-j2.5,\ \right)$$

$$\Rightarrow 11\mathbf{V}_1+15\mathbf{V}_2=0\cdots(b)$$

Sol. of Ex. 13:

$$\begin{cases} (1+j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20 \cdots (a) \\ 11\mathbf{V}_1 + 15\mathbf{V}_2 = 0 \cdots (b) \end{cases}$$



By (a) and (b) $\Rightarrow$

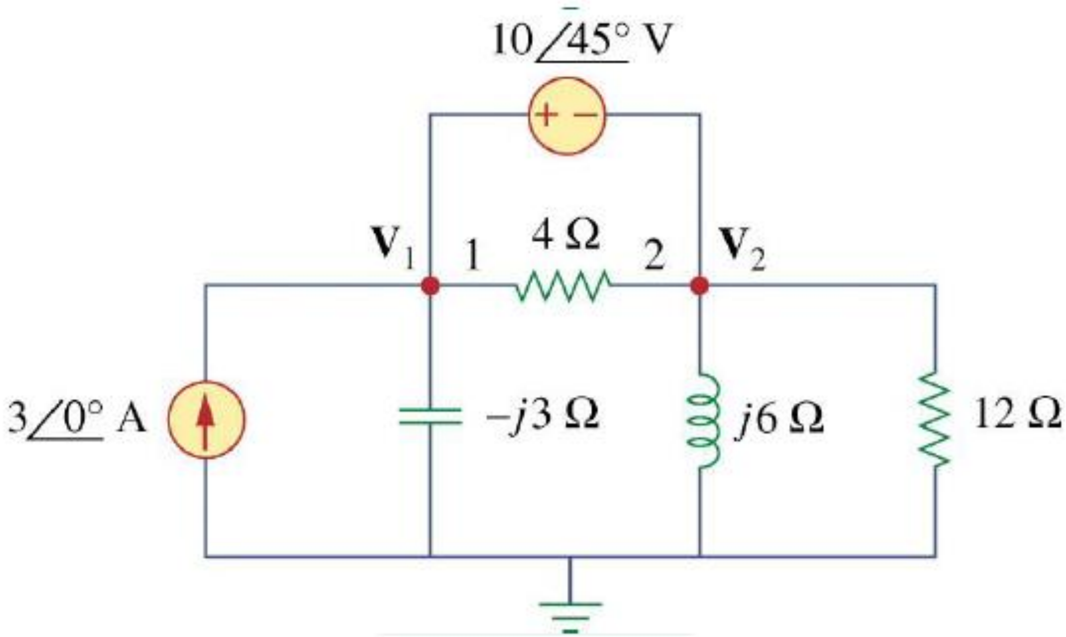
$$\mathbf{V}_1 = \frac{300}{15-j5} = 18.97\angle 18.43° \text{ V}$$

$$\mathbf{V}_2 = \frac{-220}{15-j5} = 13.91\angle 198.3° \text{ V}$$

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\angle 18.43°}{2.5\angle -90°} = 7.59\angle 108.4° \text{A}$$

$$\Rightarrow i_x(t) = 7.59\cos(4t + 108.4°) \text{ A}$$

Ex. 14: Compute V_1 & V_2 in the circuit

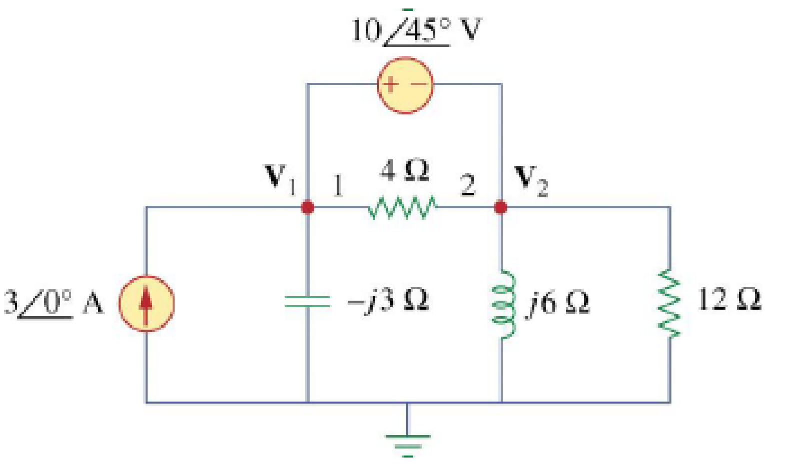


Sol. of Ex. 14:

KCL at supernode:

$$3 = \frac{\mathbf{V}_1}{-j3} + \frac{\mathbf{V}_2}{j6} + \frac{\mathbf{V}_2}{12}$$

$$\Rightarrow 36 = j4\mathbf{V}_1 + (1 - j2)\mathbf{V}_2 \cdots (a)$$



At supernode:

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ \cdots (b)$$

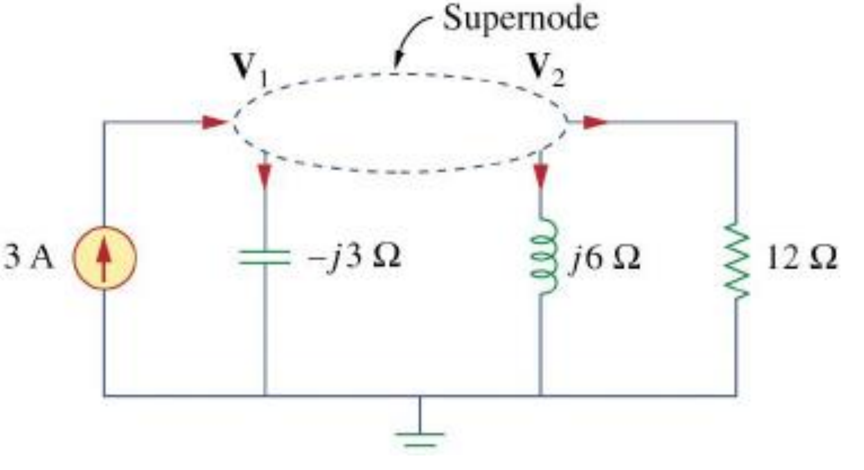
By (a) and (b) $\Rightarrow$

$\mathbf{V}_2 = 31.41\angle -87.18^\circ$ V

so,

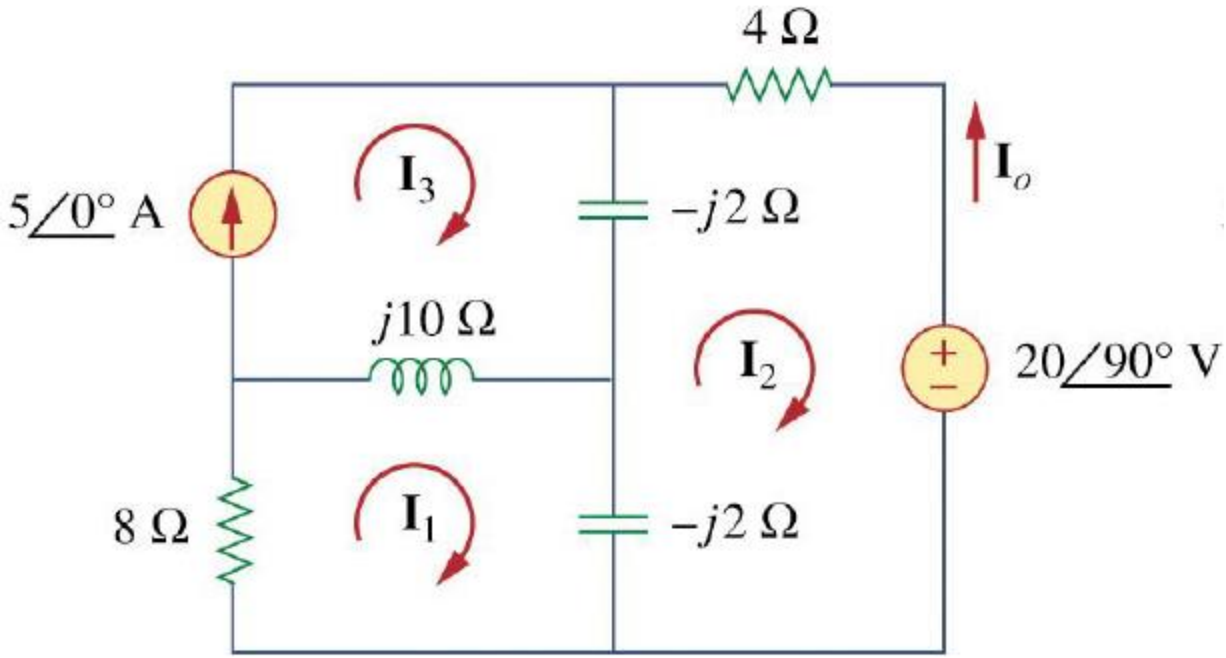
$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ$

$= 25.87\angle -70.48^\circ$ V



Ex. 15: Mesh Analysis

Determine current Io in the circuit using mesh analysis.



Sol. of Ex. 15:

KVL for mesh 1: $(8+j10-j2)\mathbf{I}_1-(-j2)\mathbf{I}_2-j10\mathbf{I}_3=0$

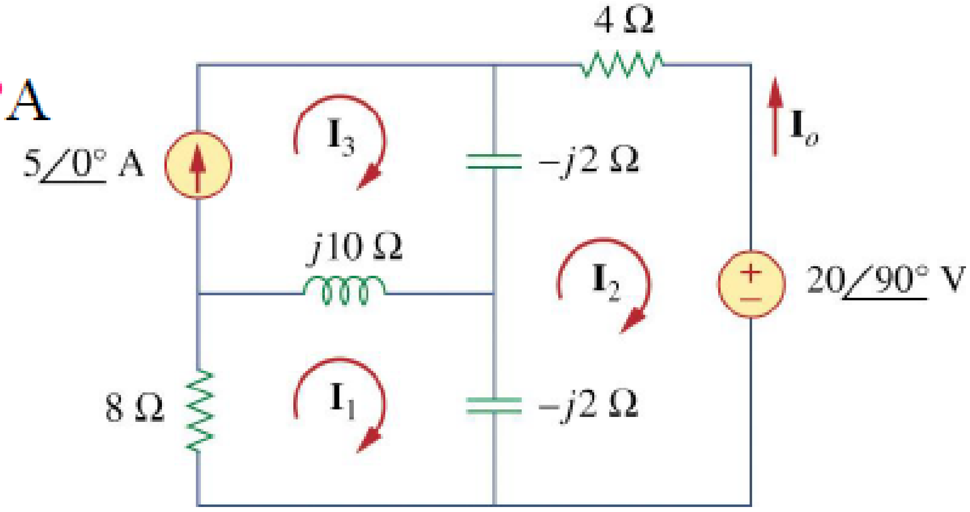
KVL for mesh 2: $(4-j2-j2)\mathbf{I}_2-(-j2)\mathbf{I}_1-(-j2)\mathbf{I}_3+20\angle 90°=0$

For mesh 3: $\mathbf{I}_3=5$

$$\Rightarrow \begin{cases} (8+j8)\mathbf{I}_1+j2\mathbf{I}_2=j50\cdots\cdots\cdots\cdots(a) \\ j2\mathbf{I}_1+(4-j4)\mathbf{I}_2=-j20-j10\cdots(b) \end{cases}$$

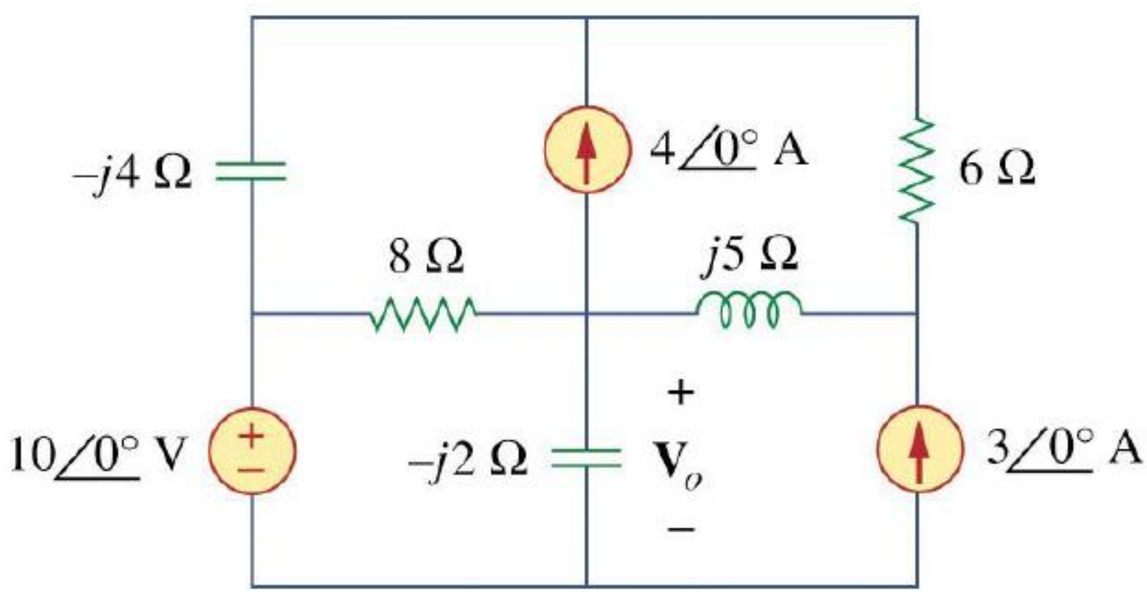
$$\Rightarrow \mathbf{I}_2=6.12\angle -35.22°\text{A}$$

$$\mathbf{I}_o=-\mathbf{I}_2=6.12\angle 144.78°\text{A}$$



Ex. 16: Mesh Analysis – super mesh

Solve V_o in the circuit using mesh analysis



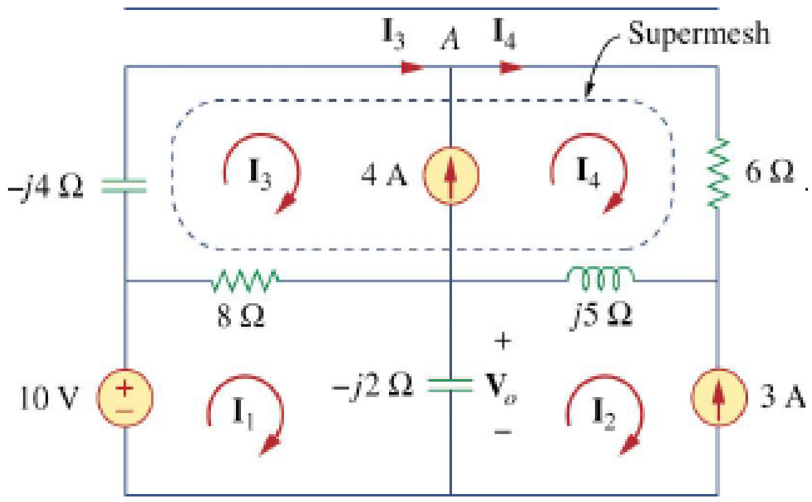
Sol. of Ex. 16:

$$\text{KVL for mesh 1: } -10+(8-j2)\mathbf{I}_1-(-j2)\mathbf{I}_2-8\mathbf{I}_3=0\cdots(a)$$

$$\text{KVL for supermesh : } (8-j4)\mathbf{I}_3-8\mathbf{I}_1+(6+j5)\mathbf{I}_4-j5\mathbf{I}_2=0\cdots(b)$$

$$\text{For mesh 2: } \mathbf{I}_2=-3\cdots(c)$$

Because of the current source between meshes 3 and 4, at node $A \Rightarrow \mathbf{I}_4=\mathbf{I}_3+4\cdots(d)$



By $(a)\sim(d)$

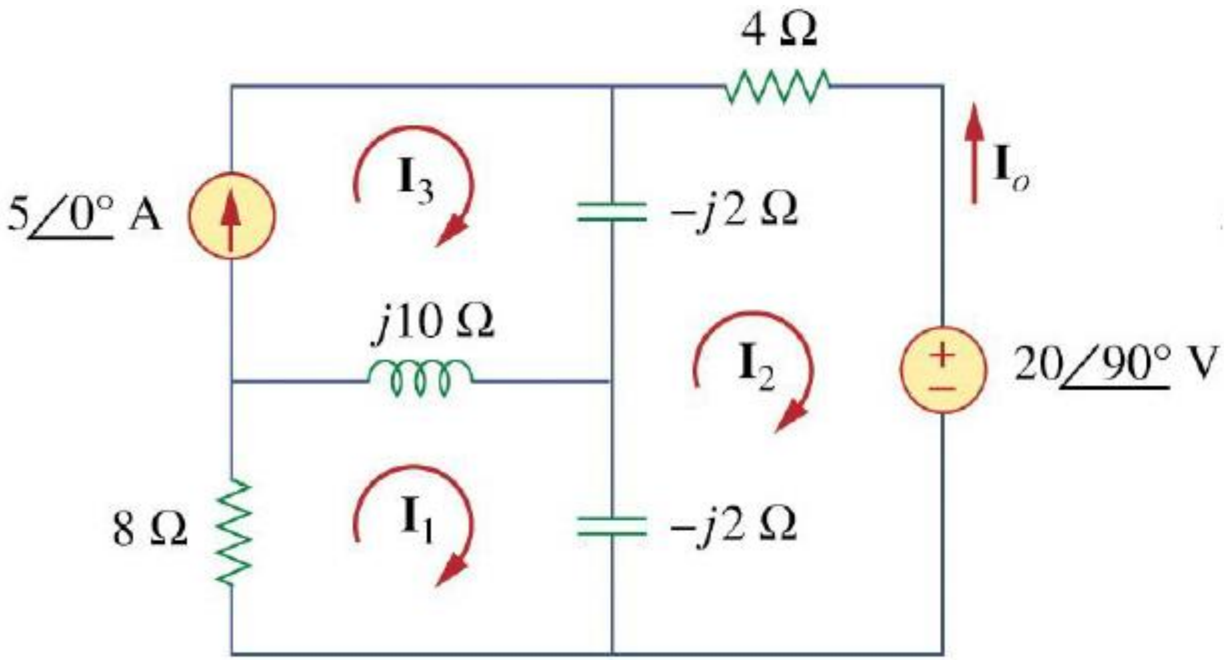
$$\Rightarrow \mathbf{I}_1=3.618\angle 274.5^\circ\ \text{A}$$

Hence,

$$\mathbf{V}_o=-j2(\mathbf{I}_1-\mathbf{I}_2)=-j2(3.618\angle 274.5^\circ+3)$$
$$=-7.2134-j6.568=9.756\angle -137.68^\circ\ \text{V}$$

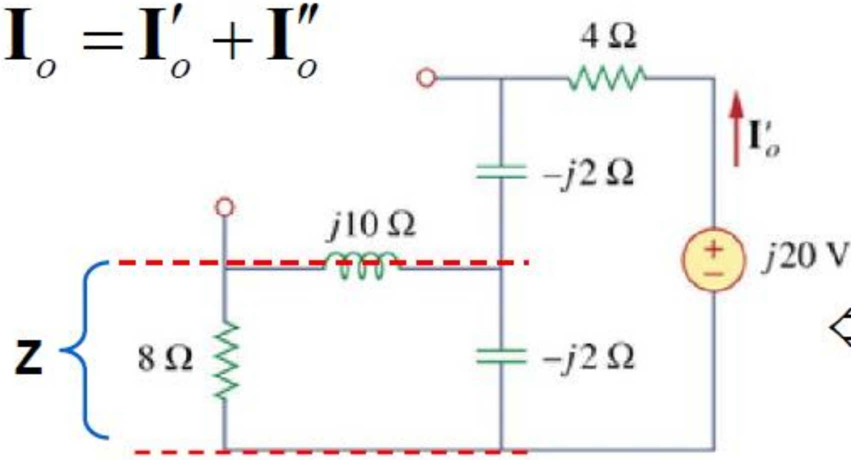
Ex. 17: Superposition Theorem

Use the superposition theorem to find I_o in the circuit.

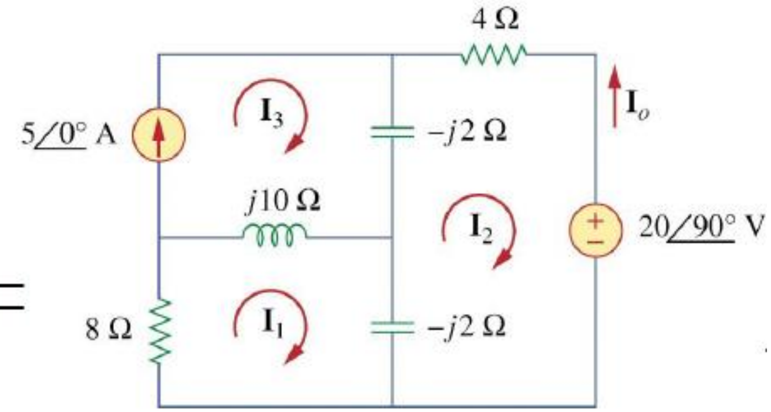


Sol. of Ex. 17: Superposition Theorem

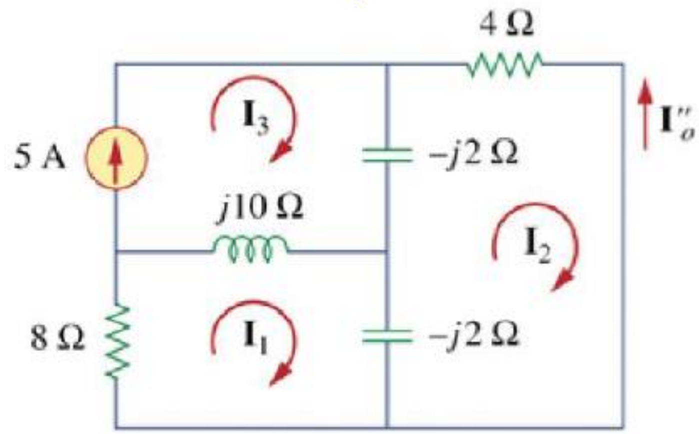
Let $\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o$



$\Leftarrow$



$\Downarrow$



For $\mathbf{I}'_o$

$$Z = \frac{-j2(8+j10)}{-2j+8+j10} = 0.25 - j2.25$$

$$\mathbf{I}'_o = \frac{j20}{4-j2+Z} = \frac{j20}{4.25-j4.25}$$

$$= -2.353 + j2.353$$

Sol. of Ex. 17: cont.

For $\mathbf{I}_o''$

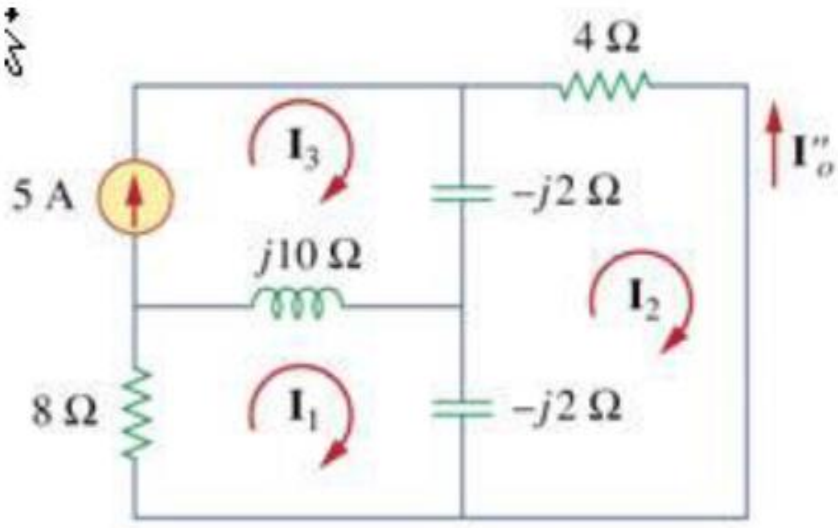
KVL fo rmesh 1: $(8+j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0 \cdots (a)$

KVL fo rmesh 2: $(4-j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0 \cdots (b)$

For mesh 3: $\mathbf{I}_3 = 5 \cdots (c)$

$$\Rightarrow \mathbf{I}_2 = \frac{90-j40}{34} = 2.647 - j1.176$$

$$\mathbf{I}_o'' = -\mathbf{I}_2$$



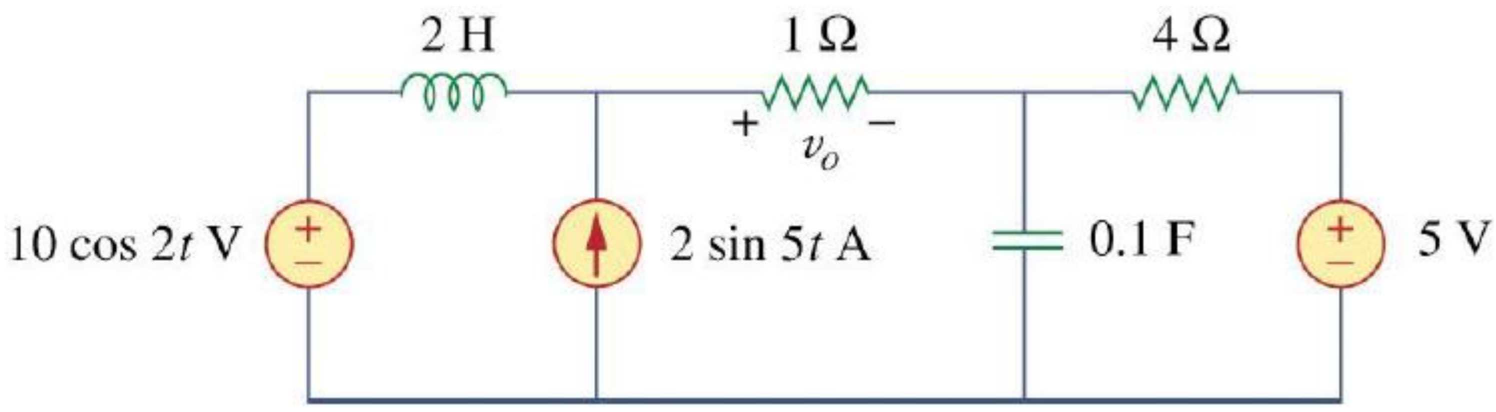
Hence,

$$\mathbf{I}_o = \mathbf{I}_o' + \mathbf{I}_o'' = -5 + j3.529$$

$$= 6.12\angle 144.78° \text{A}$$

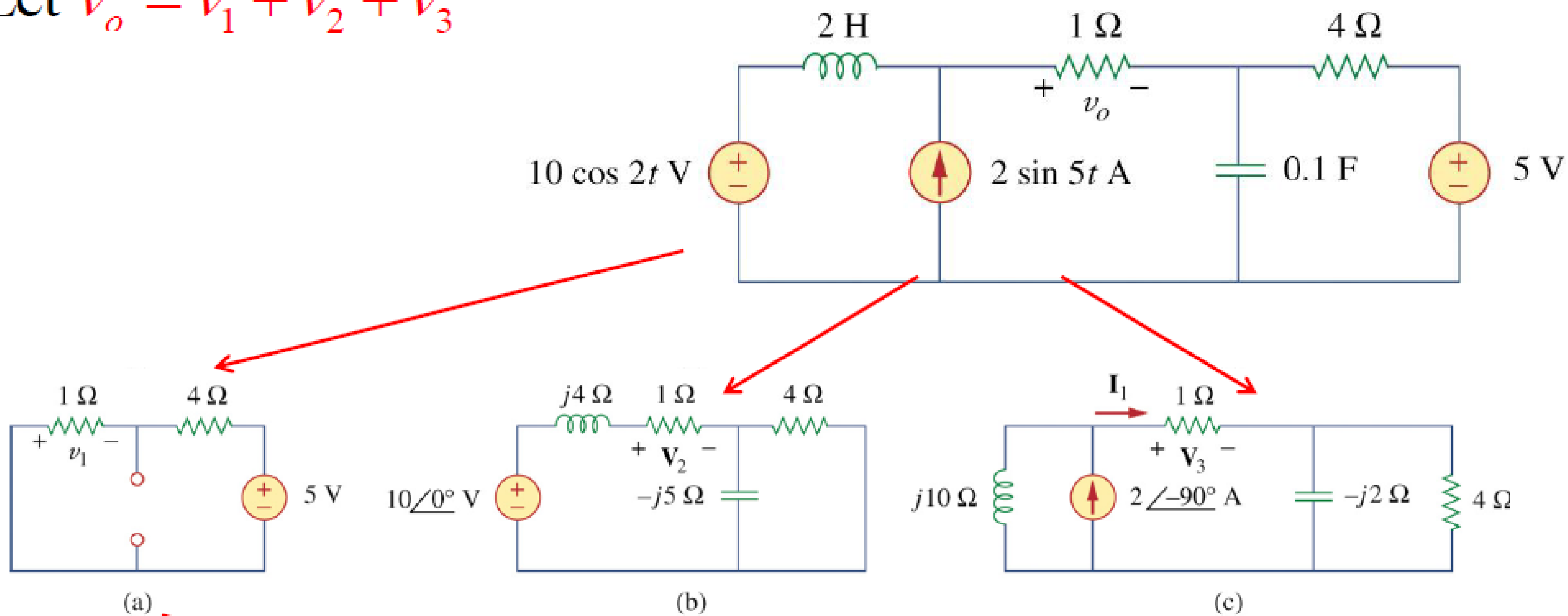
Ex. 18: Superposition Theorem

Find v_o of the circuit using the superposition theorem.



Sol. of Ex. 18:

Let $v_o = v_1 + v_2 + v_3$



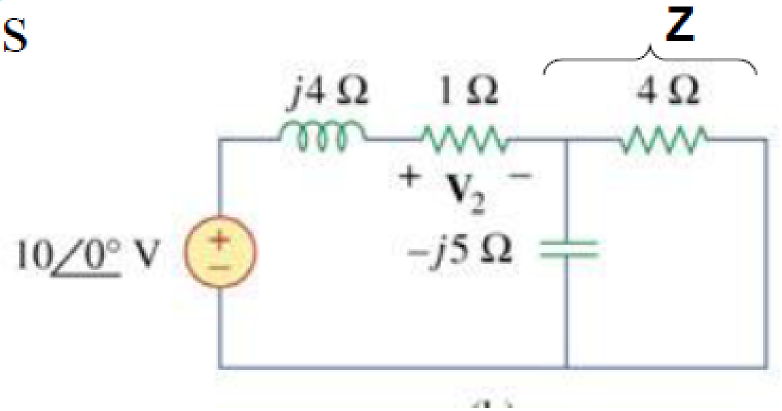
By voltage division $\Rightarrow -v_1 = \dfrac{1}{1+4}\times 5 = 1$ V

Sol. of Ex. 18: cont.

$$10\cos 2t \quad \Rightarrow \quad 10\angle 0°, \quad \omega = 2 \text{ red/s}$$

$$2\text{ H} \quad \Rightarrow \quad j\omega L = j4\,\Omega$$

$$0.1\text{ F} \quad \Rightarrow \quad \frac{1}{j\omega L} = -j5\,\Omega$$



- Let

$$\mathbf{Z} = -j5 \| 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

- By division,

$$\mathbf{V}_2 = \frac{1}{1 + j4 + \mathbf{Z}}(10\angle 0°) = \frac{10}{3.439 + j2.049} = 2.498\angle -30.79°$$

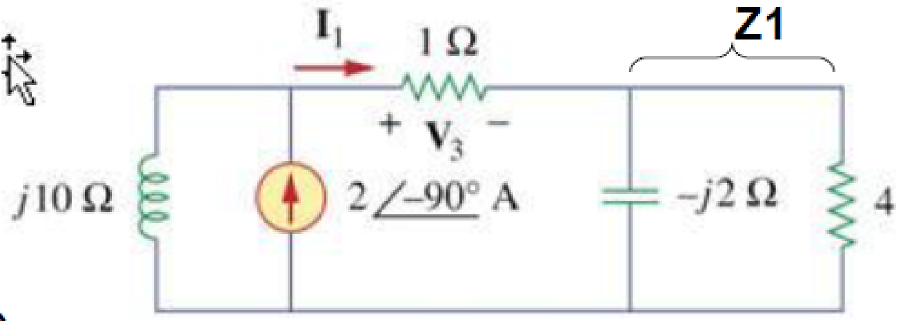
$$v_2 = 2.498\cos(2t - 30.79°)$$

Sol. of Ex. 18: cont.

$$2\sin 5t \quad \Rightarrow \quad 2\angle -90°, \quad \omega = 5 \text{ rad/s}$$

$$2 \text{ H} \quad \Rightarrow \quad j\omega L = j10\,\Omega$$

$$0.1 \text{ F} \quad \Rightarrow \quad \frac{1}{j\omega L} = -j2\,\Omega$$



$$\text{Let } \mathbf{Z}_1 = -j2 \| 4 = \frac{-j2\times 4}{4-j2} = 0.8 - j1.6\,\Omega$$

By current division $\mathbf{I}_1 = \dfrac{j10}{j10+1+\mathbf{Z}_1}(2\angle -90°)$ A

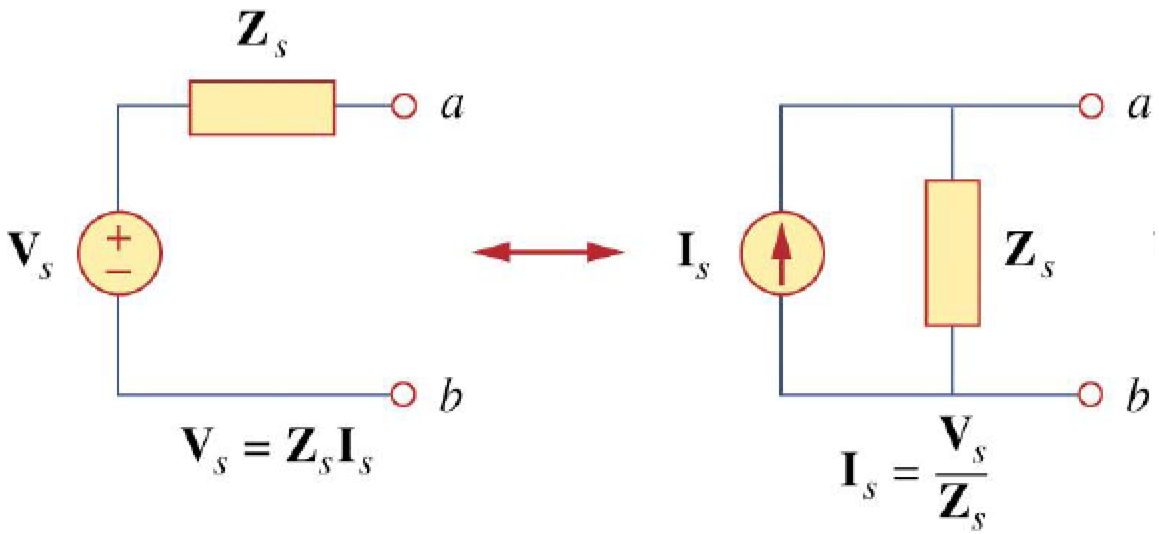
$$\Rightarrow \mathbf{V}_3 = \mathbf{I}_1 \times 1 = \frac{j10}{1.8+j8.4}(-j2) = 2.328\angle 80° \text{ V}$$

$$v_3 = 2.33\cos(5t - 80°) = 2.33\sin(5t+10°) \text{ V}$$

$$\boxed{v_0(t) = -1 + 2.498\cos(2t - 30.79°) + 2.33\sin(5t+10°)\text{V}}$$

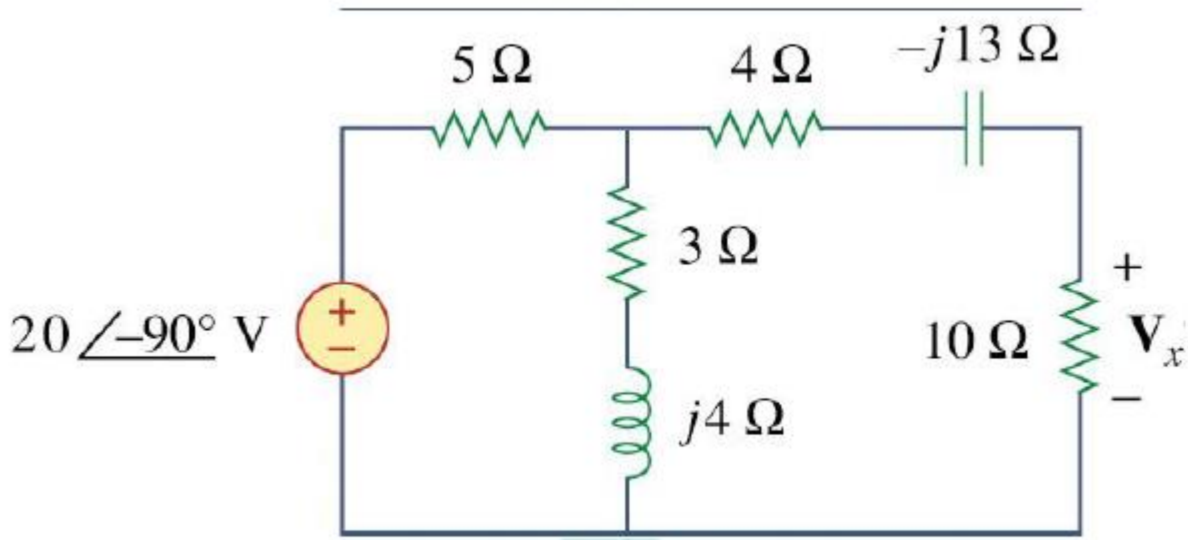
Ex. 19: Source Transformation

$$\boxed{\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}}$$



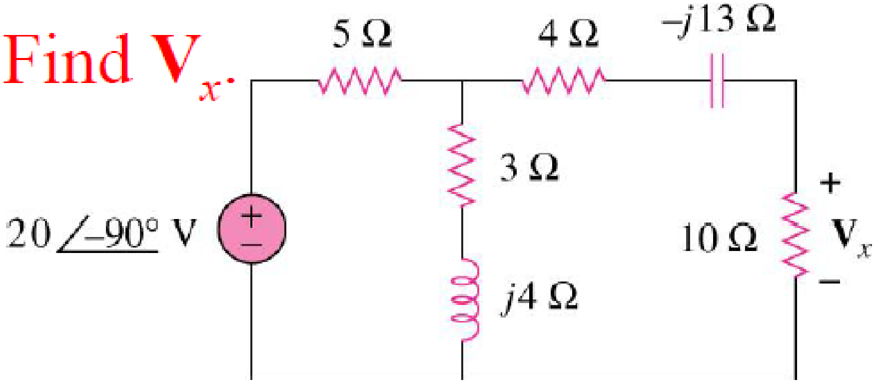
Ex. 19: Source Transformation

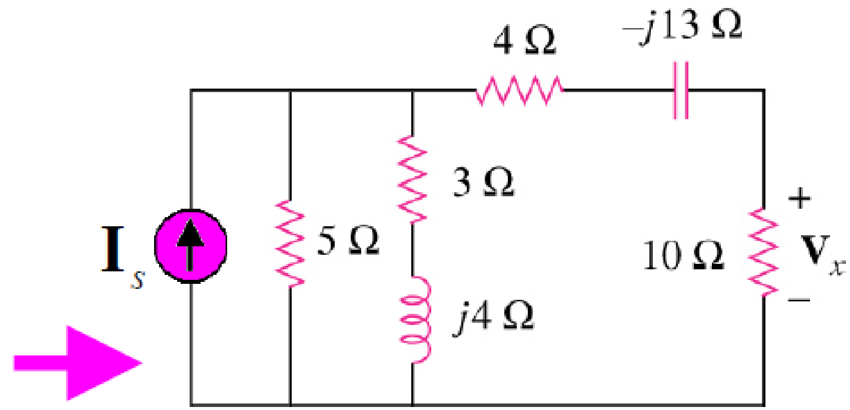
Calculus $\mathbf{V}_x$ in the circuit using the method of source transformation.



Sol. of Ex. 19:

Find $\mathbf{V}_x$.

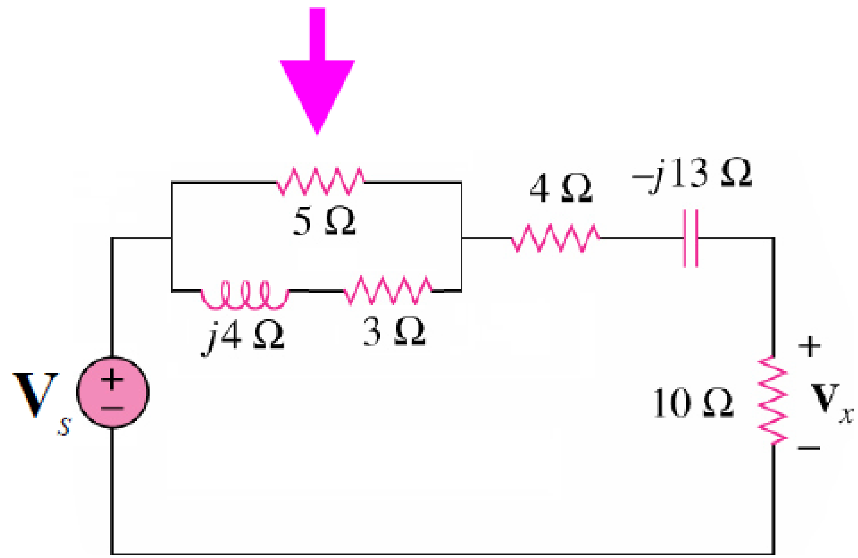




$$\mathbf{I}_s = \frac{20\angle-90^\circ}{5} = 4\angle-90^\circ = -j4$$

$$\mathbf{V}_s = \mathbf{I}_s \times \left(5 \,\|\, (3+j4)\right) = -j4\,\frac{5(3+j4)}{8+j4}$$

$$= -j4(2.5 + j1.25) = 5 - j10$$



By voltage division,

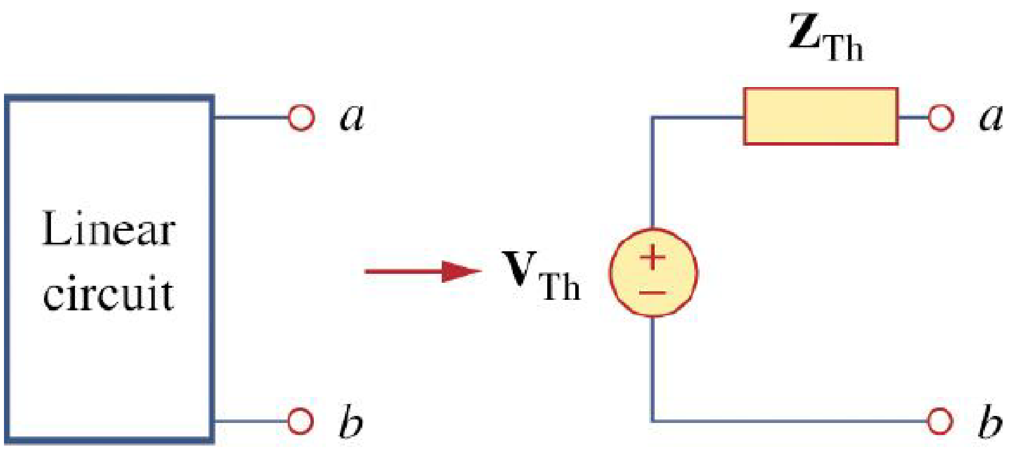
$$\mathbf{V}_x = \frac{10}{2.5 + j1.25 + 4 - j13 + 10}(5 - j10)$$

$$= 5.519\angle-28^\circ \text{ V}$$

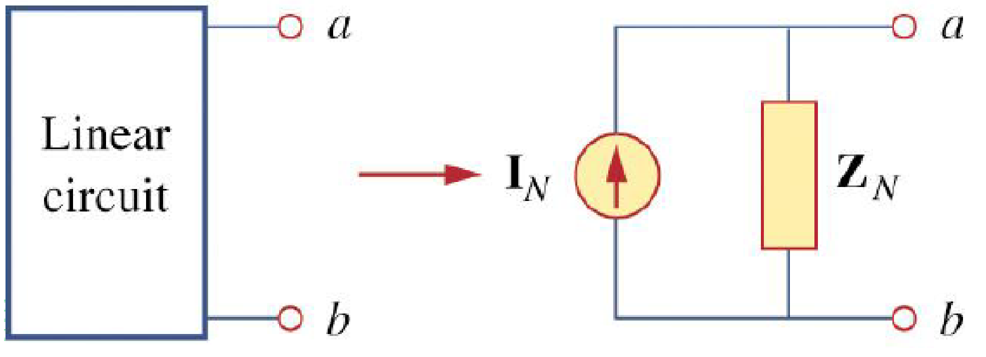
Ex. 20: Thevenin & Norton Equivalent Circuits

$$\mathbf{V}_{\mathrm{TH}} = \mathbf{Z}_N \mathbf{I}_N,$$

$$\mathbf{Z}_{\mathrm{TH}} = \mathbf{Z}_N$$

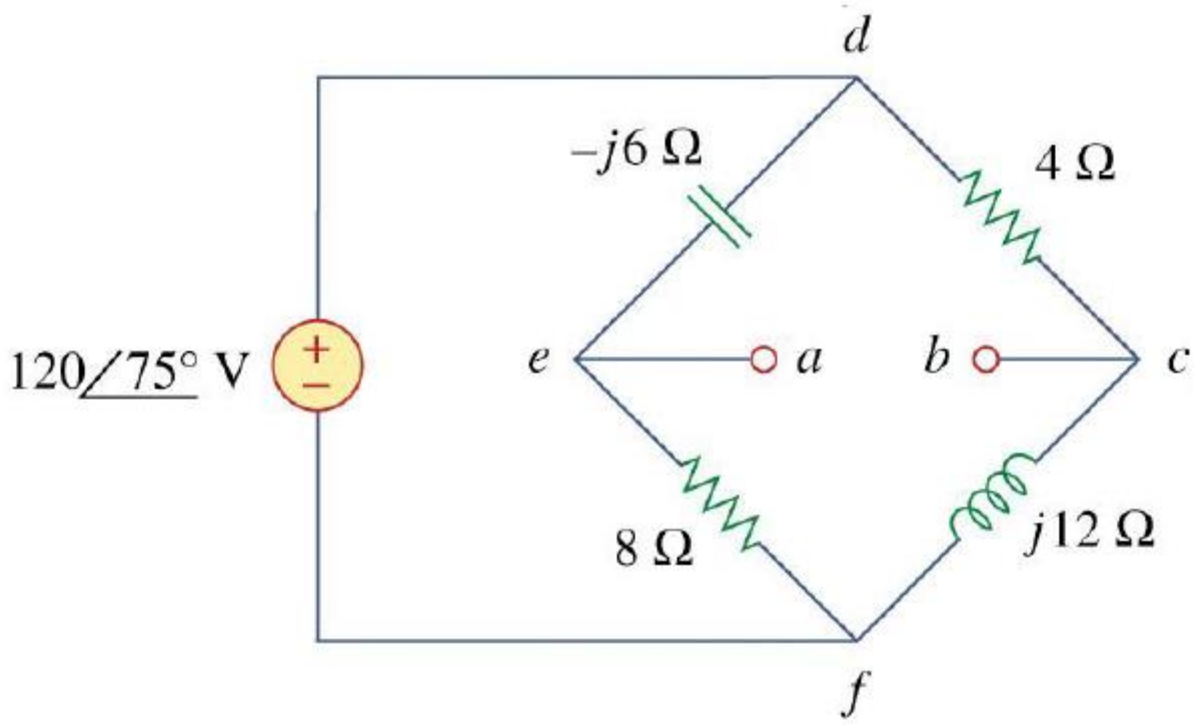


$$\boxed{\mathbf{Z}_{\mathrm{Th}} = \mathbf{Z}_{\mathrm{N}} = \frac{\mathbf{V}_{\mathrm{Th}}}{\mathbf{I}_{\mathrm{N}}}}$$

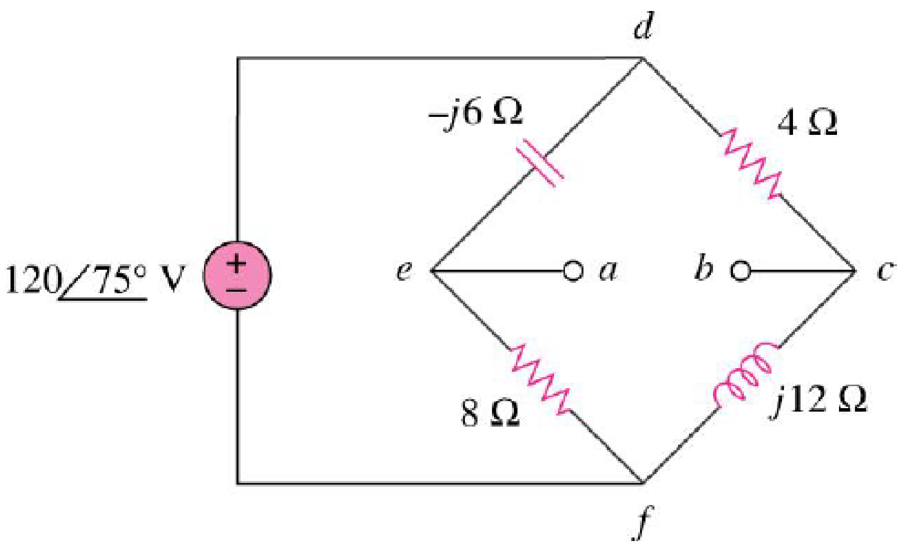


Ex. 20: Thevenin & Norton Equivalent Circuits

Obtain the Thevenin equivalent at terminals *a*-*b* of the circuit.

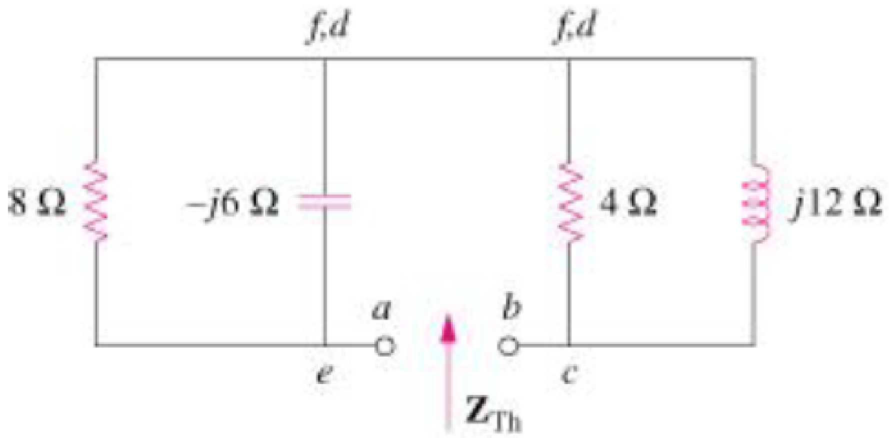


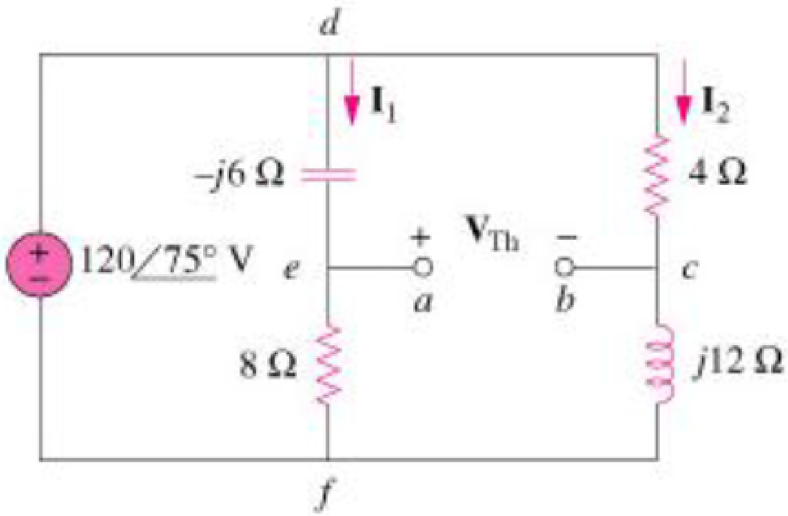
Sol. of Ex. 20:



$$\mathbf{Z}_{\text{Th}} = (8 \,||\, -j6) + (4 \,||\, j12)$$
$$= 6.48 - j2.64$$

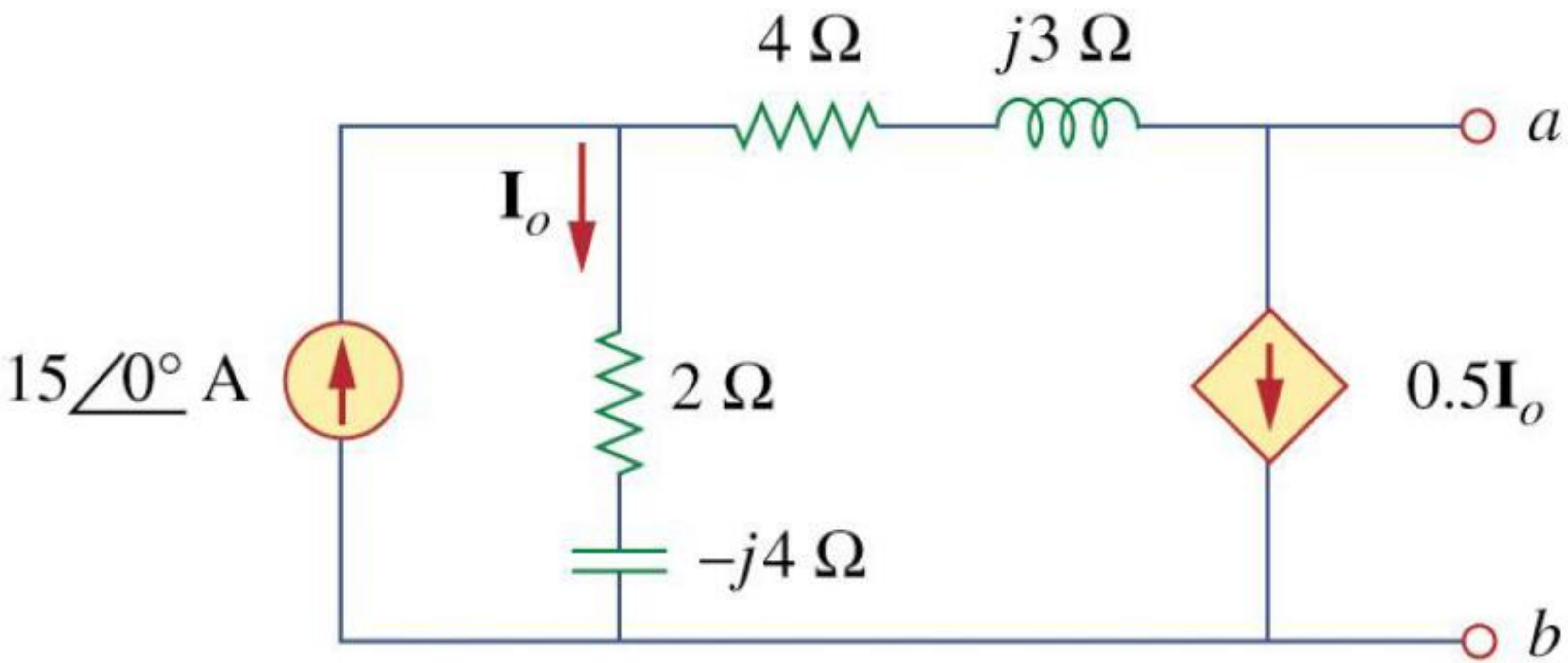
$$\mathbf{V}_{\text{Th}} = \left(\frac{8}{8-j6} - \frac{j12}{4+j12} \right) \times 120\angle 75^\circ$$
$$= 37.95\angle 220.31^\circ \text{ V}$$



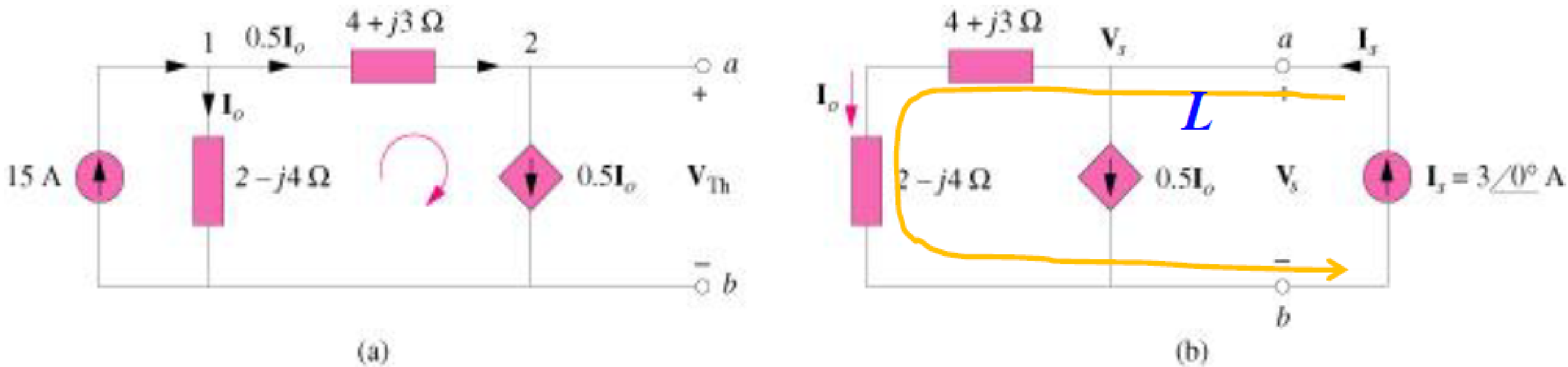


Ex. 21:

Find the Thevenin equivalent of the circuit as seen from terminals *a*-*b*.



Sol. of Ex. 21:



KCL at node 1:

$$15 = \mathbf{I}_0 + 0.5\mathbf{I}_0 \Rightarrow \mathbf{I}_0 = 10$$

KVL for loop:

$$-\mathbf{I}_0(2 - j4) + 0.5\mathbf{I}_0(4 + j3) + \mathbf{V}_{\text{Th}} = 0$$

$$\Rightarrow \mathbf{V}_{\text{Th}} = 10(2 - j4) - 5(4 + j3)$$

$$= -j55$$

$$= 55\angle -90^\circ \text{ V}$$

Set $\mathbf{I}_s = 3$ for simplicity,

KCL at node a:

$$\mathbf{I}_s = 3 = \mathbf{I}_0 + 0.5\mathbf{I}_0 \Rightarrow \mathbf{I}_0 = 2$$

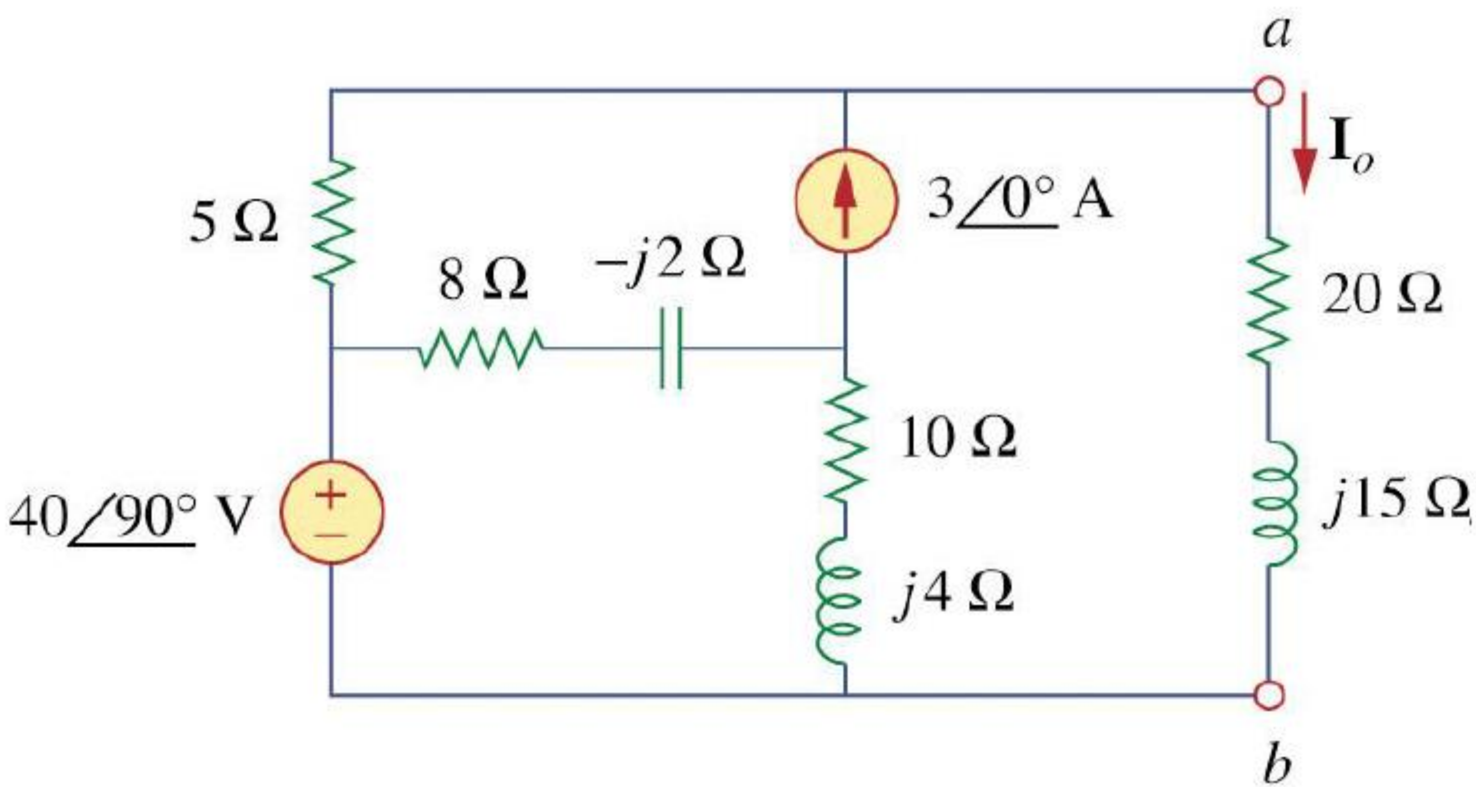
KVL for loop L:

$$\mathbf{V}_s = \mathbf{I}_0(4 + j3 + 2 - j4) = 2(6 - j)$$

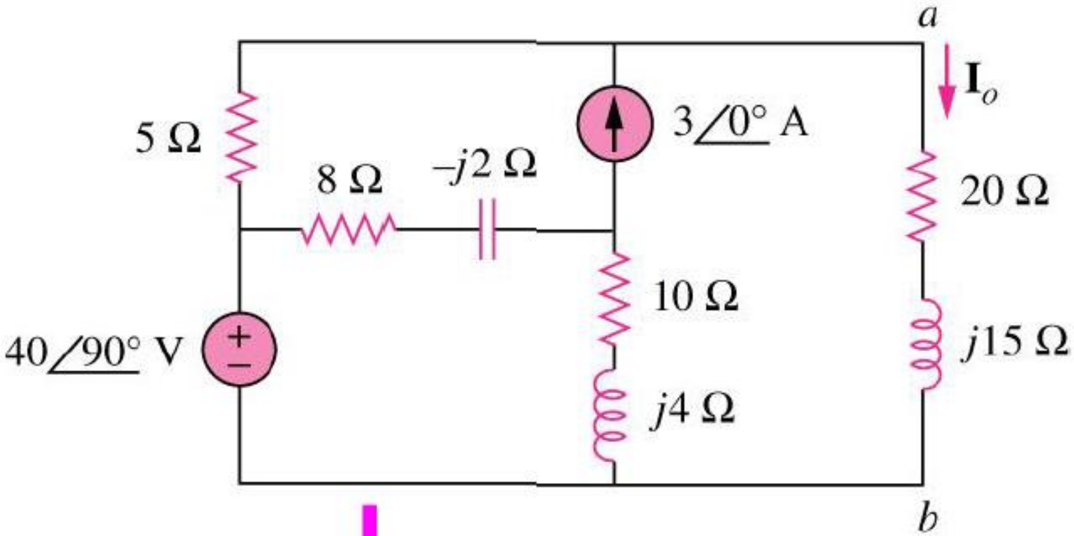
$$\Rightarrow \mathbf{Z}_{\text{Th}} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6 - j)}{3}$$

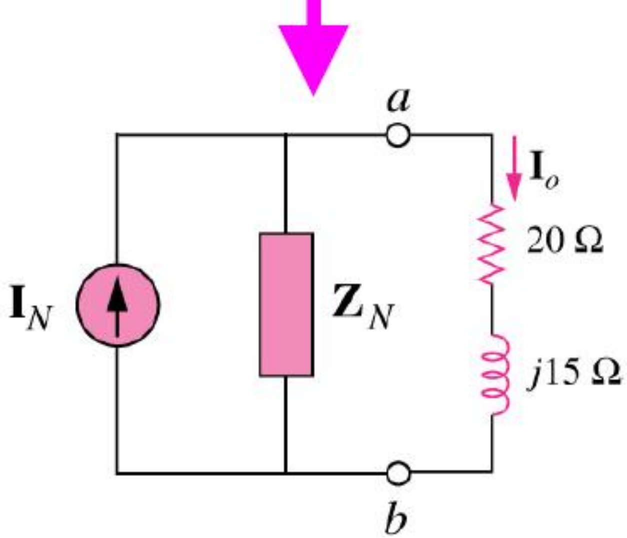
Ex. 22:

Obtain current I_o using Norton's theorem.



Sol. of Ex. 22:

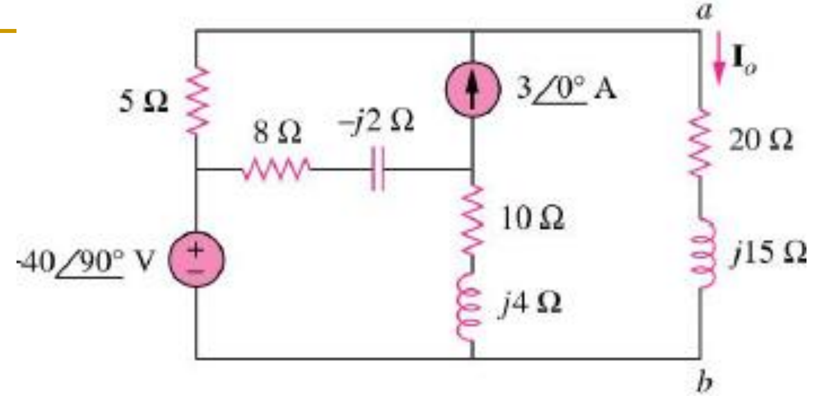




By current division,

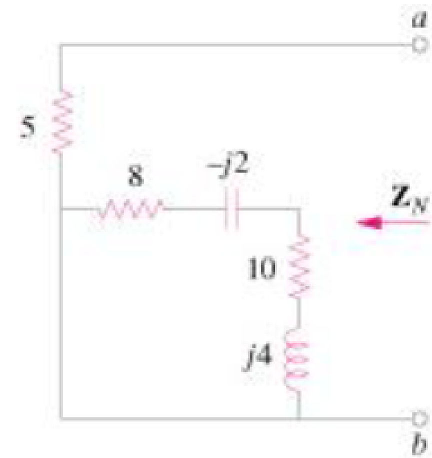
$$\mathbf{I}_0 = \frac{\mathbf{Z}_N}{\mathbf{Z}_N + (20 + j15)} \mathbf{I}_N$$

Sol. of Ex. 22:   cont.



(1) $\mathbf{Z}_N$ can be found easily, $\mathbf{Z}_N = 5$

(2) Apply mesh analysis to get $\mathbf{I}_N$.



KVL for mesh 1:

$$-j40 + (18 + j2)\mathbf{I}_1 - (8 - j2)\mathbf{I}_2 - (10 + j4)\mathbf{I}_3 = 0 \cdots (a)$$

KVL for the supermesh:

$$(13 - j2)\mathbf{I}_2 + (10 + j4)\mathbf{I}_3 - (18 + j2)\mathbf{I}_1 = 0 \cdots (b)$$

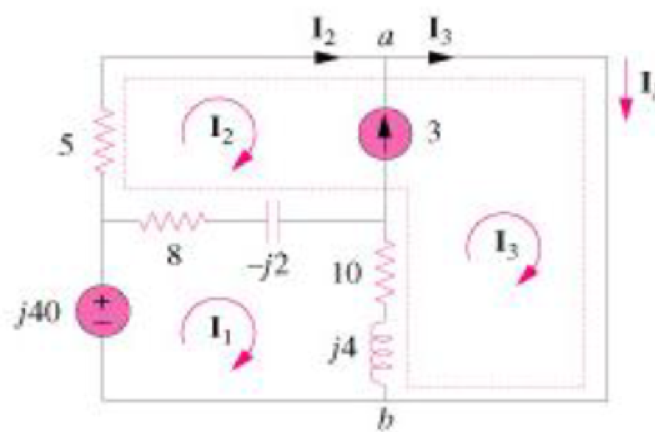
KCL at node a :

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 \cdots (C)$$

$(a) \sim (c)$ give

$$\mathbf{I}_N = \mathbf{I}_3 = 3 + j8$$

$$\Rightarrow \mathbf{I}_0 = \frac{5}{5 + 20 + j15}\,\mathbf{I}_N = 1.465\angle 38.48^\circ \text{ A}$$



Ex. 23

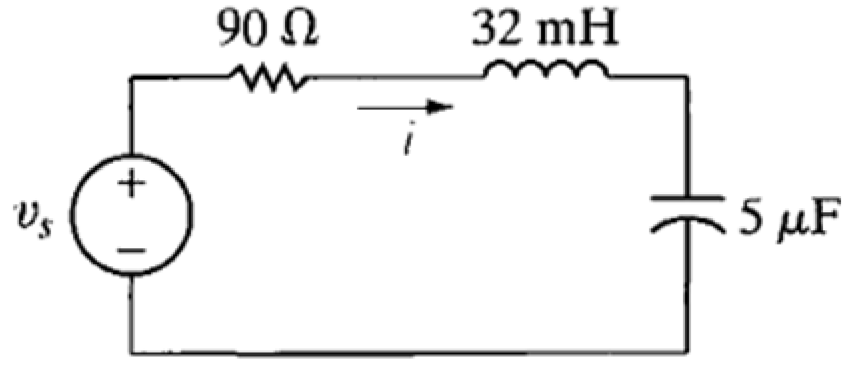
A 90 Ω resistor, a 32 mH inductor, and a 5 μF capacitor are connected in series across the terminals of a sinusoidal voltage source, as shown in the Fig.

The steady-state expression for the source voltage

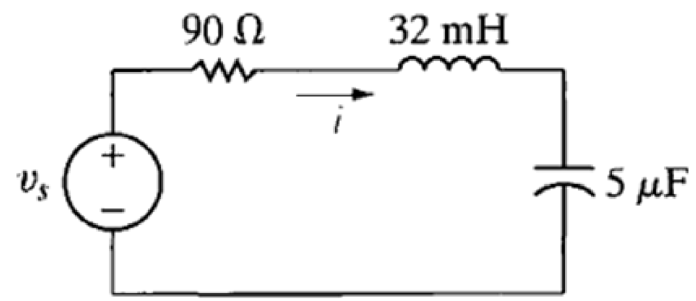
$$v_s = 750\cos(5000t + 30°)\ \text{V}.$$

a) Construct the frequency-domain equivalent circuit.

b) Calculate the steady-state current i by the phasor method.



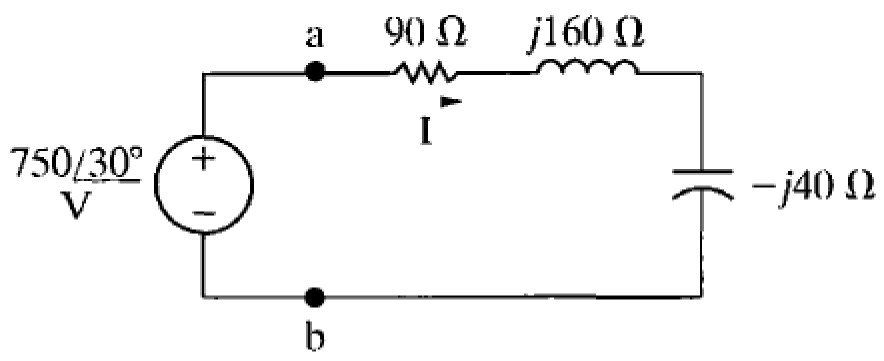
Ex. 23: Sol.



a) From the expression for v_s, we have ω = 5000 rad/s. Therefore the impedance of the 32 mH inductor is
$Z_L = j\omega L = j(5000)(32 \times 10^{-3}) = j160\ \Omega$,

and the impedance of the capacitor is $Z_C = j\dfrac{-1}{\omega C} = -j\dfrac{10^6}{(5000)(5)} = -j40\ \Omega.$

The phasor transform of v_s is $\mathbf{V}_s = 750\angle 30^\circ\ \text{V}.$



Ex. 23: Sol.

b) We compute the phasor current simply by dividing the voltage of the voltage source by the equivalent impedance between the terminals a, b.

$$\begin{aligned} Z_{ab} &= 90 + j160 - j40 \\ &= 90 + j120 = 150\underline{/53.13^\circ}\ \Omega. \end{aligned}$$

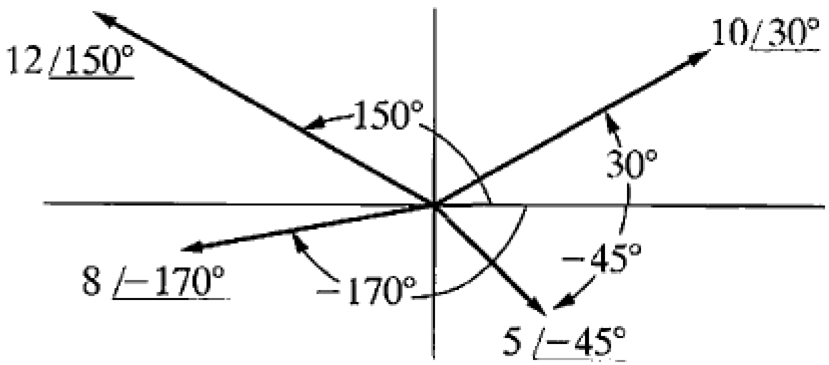
$$\mathbf{I} = \frac{750\underline{/30^\circ}}{150\underline{/53.13^\circ}} = 5\underline{/-23.13^\circ}\ \text{A}.$$

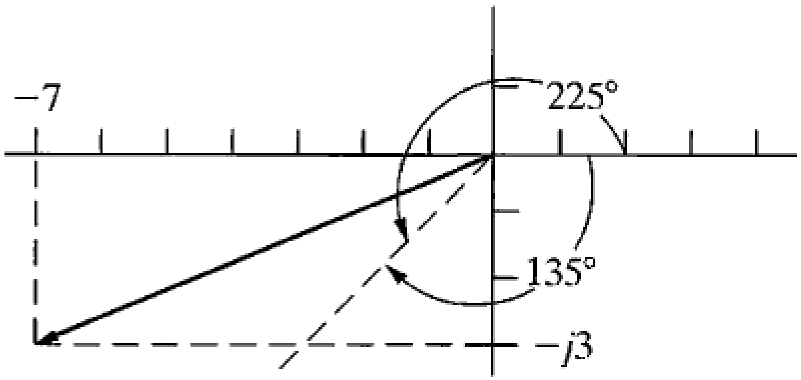
Thus, the steady-state expression for i directly:

i = 5cos(5000t - 23.13°) A.

Phasor Diagrams

- A phasor diagram shows the magnitude and phase angle of each phasor quantity in the complex-number plane.
- Phase angles are measured counterclockwise from the positive real axis.
- Magnitudes are measured from the origin of the axes.

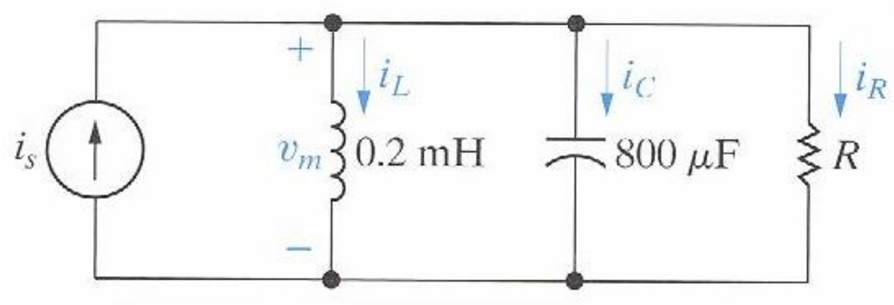




$$-7 - j3 = 7.62 \underline{/-156.80^\circ}.$$

Phasor diagram is useful for checking calculator calculations

Phasor Diagrams – An example



Use a phasor diagram to find the value of R that will cause the current through that resistor, i_R , to lag the source current, i_S , by 45º when ω = 5 krad/s

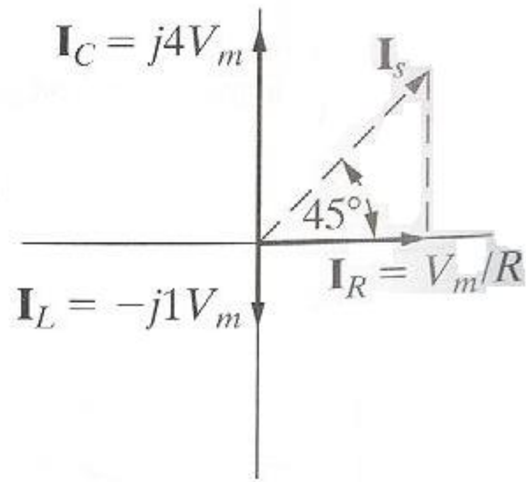
$$I_S = I_L + I_C + I_R$$

$$Assume \quad V_m = V_m\angle 0^\circ$$

$$I_L = \frac{V_m\angle 0^\circ}{j(5000)(0.2\times 10^{-3})} = V_m\angle -90^\circ$$

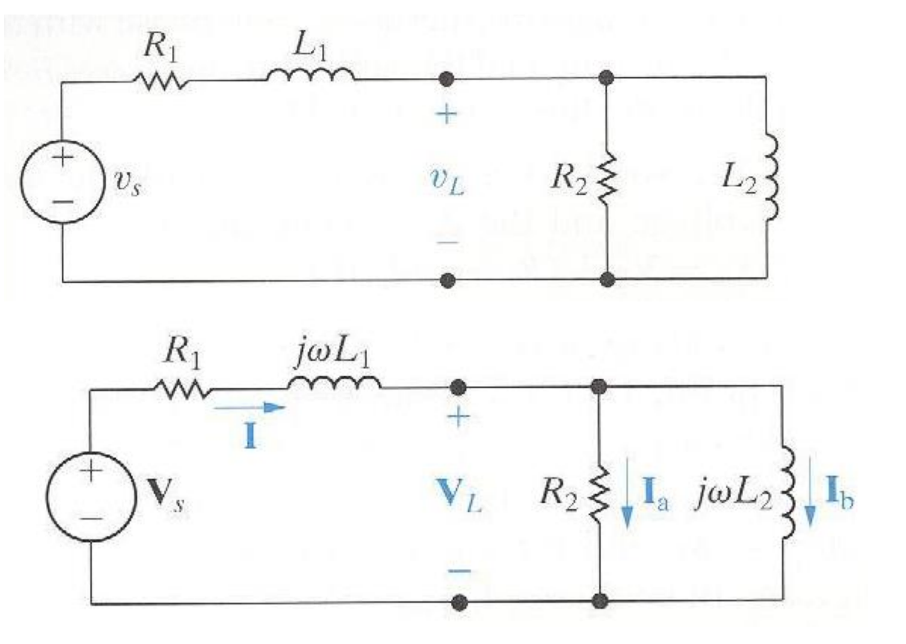
$$I_C = \frac{V_m\angle 0^\circ}{-j/(5000)(800\times 10^{-6})} = 4V_m\angle 90^\circ$$

$$I_R = \frac{V_m\angle 0^\circ}{R} = \frac{V_m}{R}\angle 0^\circ$$

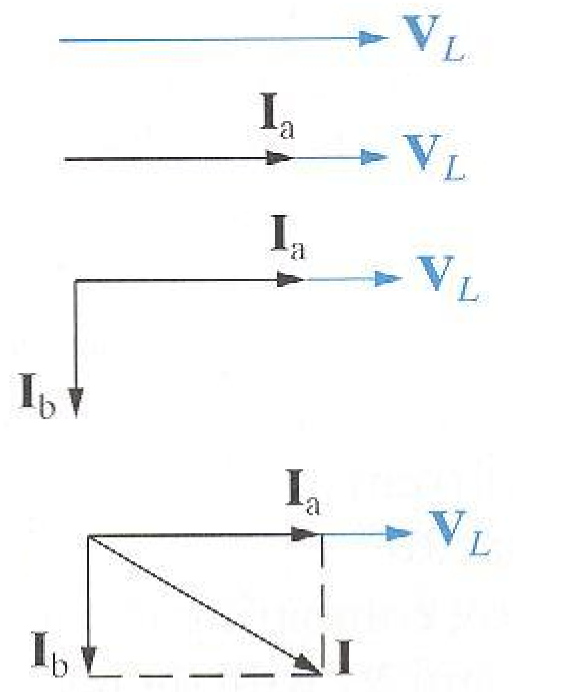


From phasor diagram, we have
R = 1/3 Ω

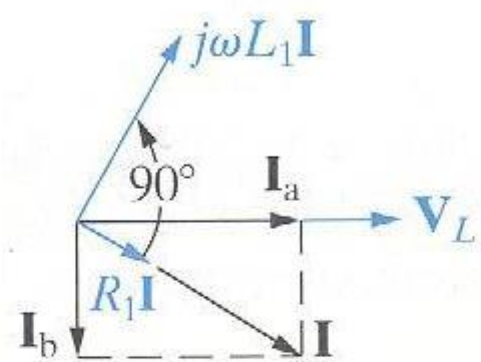
Example



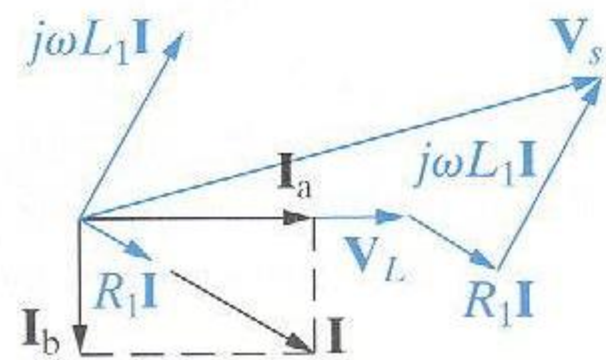
R_1
L_1
v_s
v_L
R_2
L_2
R_1
$j\omega L_1$
I
V_s
V_L
R_2
I_a
$j\omega L_2$
I_b



V_L
I_a
V_L
I_a
V_L
I_b
I_a
V_L
I_b
I



$j\omega L_1 I$
90°
I_a
V_L
$R_1 I$
I_b
I



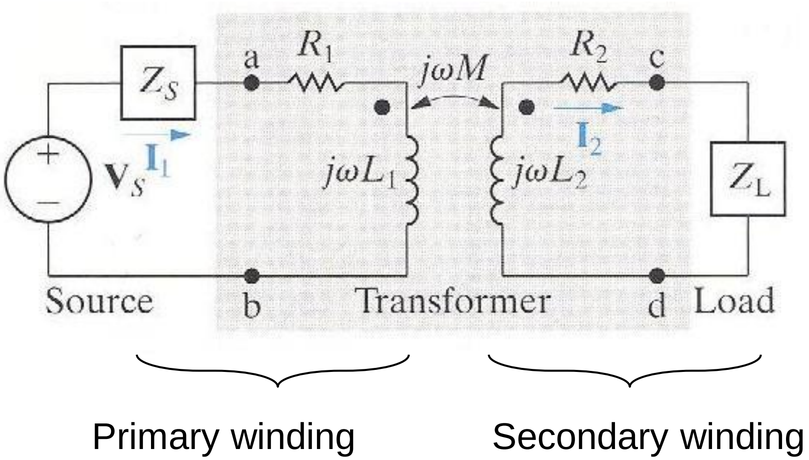
$j\omega L_1 I$
V_s
I_a
$j\omega L_1 I$
V_L
$R_1 I$
$R_1 I$
I_b
I

The transformer

- A transformer is a device that is based on magnetic coupling.
- Are used in both communication and power circuits.
- In communication circuits: transformer is used to matched impedance and eliminate dc signals from portions of the systems
- In power circuits: transformer is used to establish ac voltage levels that facilitate the transmission, distribution and consumption of electrical power.

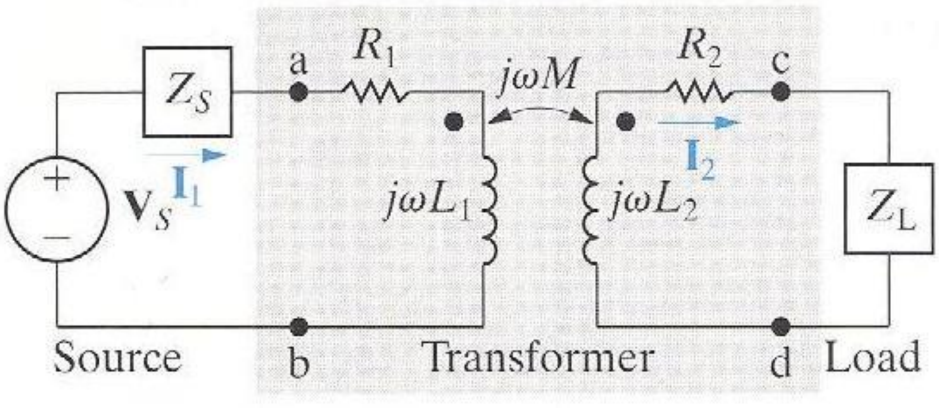
Linear transformer

- **Primarily used in communication circuits.**
- **Is formed when two coils are wound on a single core to ensure magnetic coupling.**
- **Frequency domain circuit model of a transformer**



R_1 = the resistance of the primary winding
R_2 = the resistance of the secondary winding
L_1 = the self-inductance of the primary winding
L_2 = the self-inductance of the secondary winding
M = the mutual inductance
V_s = sinusoidal source
Z_s = internal impedance of the source
Z_L = the load
I_1 = primary current
I_2 = secondary current

Transformer circuit analysis



Mesh-current equations:

$$\boldsymbol{V_s} = (\boldsymbol{Z_s} + \boldsymbol{R_1} + \boldsymbol{j\omega L_1})\boldsymbol{I_1} - \boldsymbol{j\omega M I_2}$$

$$0 = -\boldsymbol{j\omega M I_1} + (\boldsymbol{R_2} + \boldsymbol{j\omega L_2} + \boldsymbol{Z_2})\boldsymbol{I_2}$$

Call: $Z_{11} = Z_s + R_1 + j\omega L_1$ = total self-impedance of the primary winding

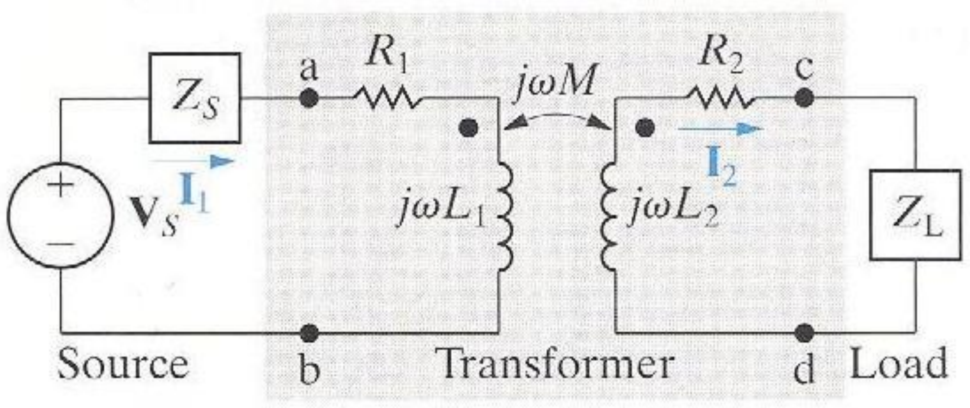
$Z_{22} = R_2 + j\omega L_2 + Z_2$ = total self-impedance of the secondary winding

Yield:

$$\boldsymbol{I_1} = \frac{\boldsymbol{Z_{22}}}{\boldsymbol{Z_{11}Z_{22}} + \omega^2 \boldsymbol{M}^2}\boldsymbol{V_s}$$

$$\boldsymbol{I_2} = \frac{\boldsymbol{j\omega M}}{\boldsymbol{Z_{11}Z_{22}} + \omega^2 \boldsymbol{M}^2}\boldsymbol{V_s} = \frac{\boldsymbol{j\omega M}}{\boldsymbol{Z_{22}}}\boldsymbol{I_1}$$

Transformer circuit analysis



Impedance at the terminal of the source:

$$\boldsymbol{Z_{ab}} = \boldsymbol{R}_1 + \boldsymbol{j}\omega\boldsymbol{L}_1 + \frac{\omega^2 \boldsymbol{M}^2}{\left(\boldsymbol{R}_2 + \boldsymbol{j}\omega\boldsymbol{L}_2 + \boldsymbol{Z}_L\right)}$$

Z_{ab} is independent of the magnetic polarity of the transformer.

Z_{ab} shows how the transformer affects the impedance of the load as seen from the source

Reflected impedance

$$\boldsymbol{Z_{ab}} = \boldsymbol{R_1} + \boldsymbol{j\omega L_1} + \underbrace{\frac{\omega^2 \boldsymbol{M}^2}{\left(\boldsymbol{R_2} + \boldsymbol{j\omega L_2} + \boldsymbol{Z_L}\right)}}$$

Z_r = reflected impedance

= the impedance of the secondary circuit as seen from the terminals of the primary circuit or vice versa.

<u>Notes:</u>

1) The reflected impedance is due solely to the existence of mutual inductance
2) The linear transformer reflects the conjugate of the self-impedance of the secondary circuit (Z^*_{22}) into the primary winding by a scalar multiplier

$$Z_r = \frac{\omega^2 M^2}{|Z_{22}|^2} \underbrace{\left[\left(R_2 + R_L\right) - j\left(\omega L_2 + X_L\right)\right]}_{\mathbf{Z^*_{22}}}$$

Example

The parameters of a certain linear transformer are R_1 = 200 Ω, R_2 = *100* Ω, L_1 = *9H*, L_2 = *4H, and k = 0.5. The transformer couples an impedance* consisting of an 800 Ω resistor in series with a 1 μF capacitor to a sinusoidal voltage source. The 300 V (rms) source has an internal impedance of 500 + j100 Ω and a frequency of 400 rad/s.

a) Construct a frequency-domain equivalent circuit of the system.

b) Calculate the self-impedance of the primary circuit.

c) Calculate the self-impedance of the secondary circuit.

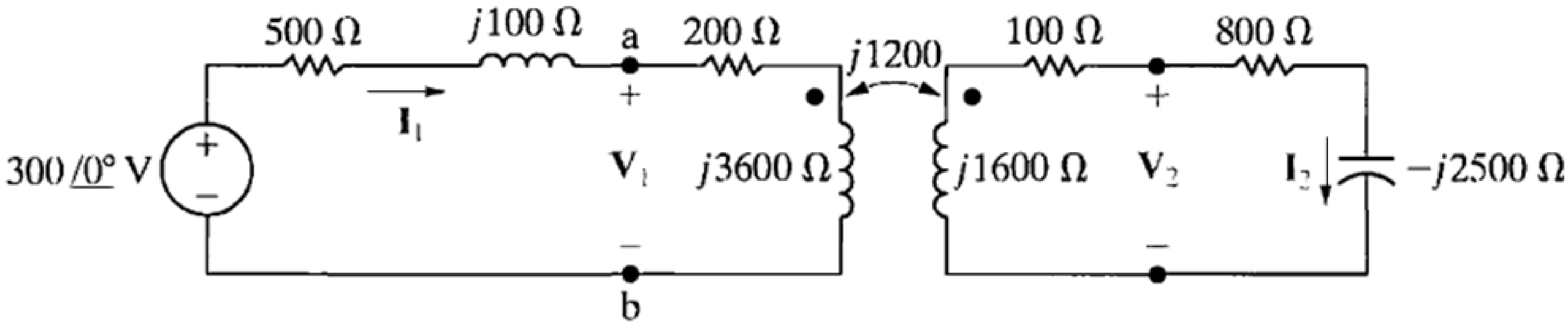
d) Calculate the impedance reflected into the primary winding.

e) Calculate the scaling factor for the reflected impedance.

f) Calculate the impedance seen looking into the primary terminals of the transformer.

g) Calculate the Thevenin equivalent with respect to the terminals c, d.

Example – Sol.



The figure shows the frequency-domain equivalent circuit. Note that the internal voltage of the source serves as the reference phasor, and that $\mathbf{V}_1$ and $\mathbf{V}_2$ represent the terminal voltages of the transformer. In the circuit of the figure, we made the following calculations:

$$j\omega L_1 = j(400)(9) = j3600\ \Omega,$$

$$j\omega L_2 = j(400)(4) = j1600\ \Omega,$$

$$M = 0.5\sqrt{(9)(4)} = 3\ \mathrm{H},$$

$$j\omega M = j(400)(3) = j1200\ \Omega,$$

$$\frac{1}{j\omega C} = \frac{10^6}{j400} = -j2500\ \Omega.$$

Example – Sol.

b) The self-impedance of the primary circuit is

$$Z_{11} = 500 + j100 + 200 + j3600 = 700 + j3700\ \Omega.$$

c) The self-impedance of the secondary circuit is

$$Z_{22} = 100 + j1600 + 800 - j2500 = 900 - j900\ \Omega.$$

d) The impedance reflected into the primary winding is

$$Z_r = \left(\frac{1200}{|900 - j900|}\right)^2 (900 + j900)$$

$$= \frac{8}{9}(900 + j900) = 800 + j800\ \Omega.$$

e) The scaling factor by which Z^*_{22} *is reflected is 8/9.*

f) The impedance seen looking into the primary terminals of the transformer is the impedance of the primary winding plus the reflected impedance; thus

$$Z_{ab} = 200 + j3600 + 800 + j800 = 1000 + j4400\ \Omega.$$

Example – Sol.

g) The Thevenin voltage will equal the open circuit value of $\mathbf{V}_{cd}$. The open circuit value of $\boldsymbol{V}_{cd}$ *will* equal j1200 times the open circuit value of $\mathbf{I}_1$. The open circuit value of $\mathbf{I}_1$ is

$$\mathbf{I}_1 = \frac{300\,\angle 0^\circ}{700 + j3700} = 79.67\,\angle -79.29^\circ \text{ mA.}$$

Therefore

$$\mathbf{V}_{\text{Th}} = j1200(79.67\,\angle -79.29^\circ) \times 10^{-3}$$

$$= 95.60\,\angle 10.71^\circ \text{ V.}$$

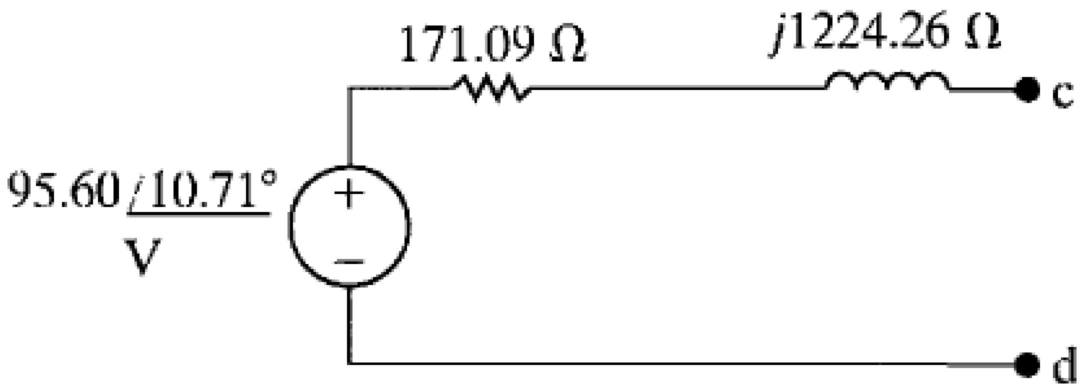
The Thevenin impedance will be equal to the impedance of the secondary winding plus the impedance reflected from the primary when the voltage source is replaced by a short-circuit. Thus

$$Z_{\text{Th}} = 100 + j1600 + \left(\frac{1200}{|700 + j3700|}\right)^2 (700 - j3700)$$

$$= 171.09 + j1224.26\ \Omega.$$

Example – Sol.

The Thevenin equivalent is shown in the figure bellow



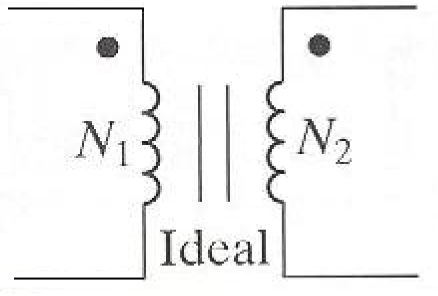
Ideal transformer

- Usually used to model the ferromagnetic transformer in power systems.
- An ideal transformer consists of two magnetically coupled coils having N1 and N2 turns, respectively, and exhibiting these three properties:

    1) The coefficient of coupling is unity ($k = 1$)

    2) The self-inductance of each coil is infinite ($L_1 = L_2 = \infty$)

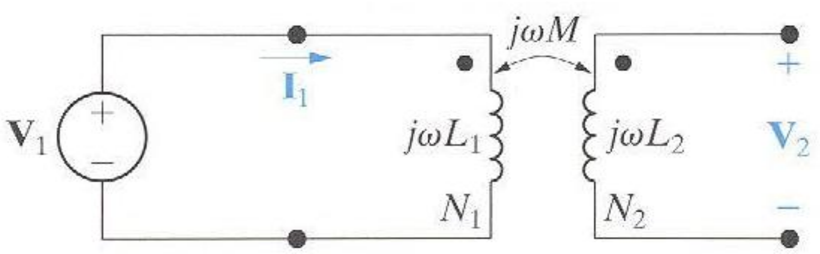
    3) The coil losses, due to parasitic resistance, are negligible ($R_1 = R_2 = 0$)



Ideal transformer

The circuit behavior is governed by the turns ratio a = N_2/N_1

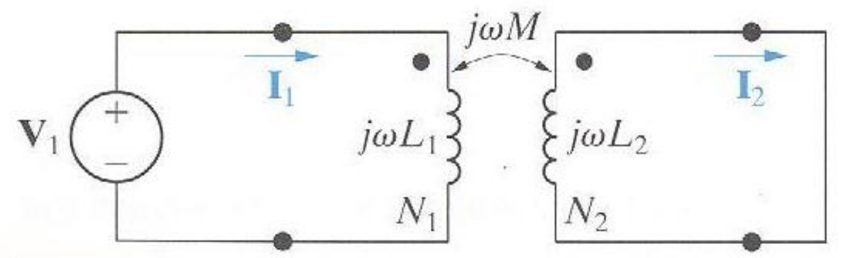
Volts per turns is the same for each winding



$$\left|\frac{V_1}{N_1}\right| = \left|\frac{V_2}{N_2}\right|$$

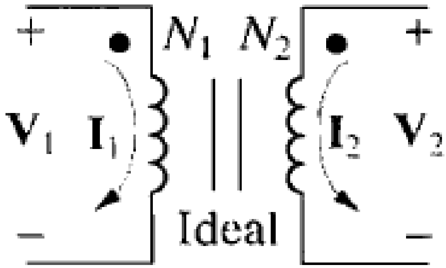
If the coil voltages V1 and V2 are both positive or negative at the dot-marked terminal, use a plus (+) sign. Otherwise, use a negative (-) sign.

Ampere turns are the same for each winding



$$|I_1 N_1| = |I_2 N_2|$$

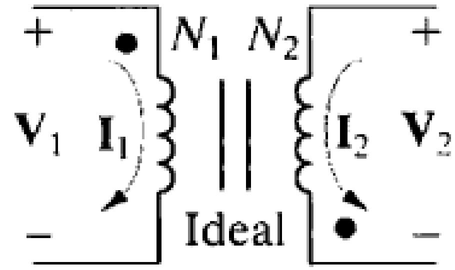
If the coil current I1 and I2 are both directed into or out of the dot-marked terminal, use a minus (-) sign. Otherwise, use a plus (+) sign.



$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2},$$

$$N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$$

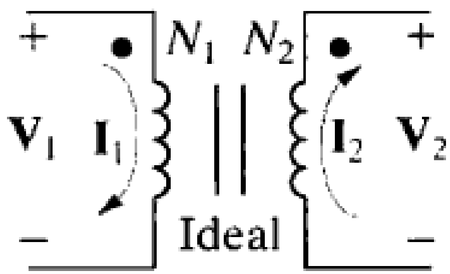
(a)



$$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2},$$

$$N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

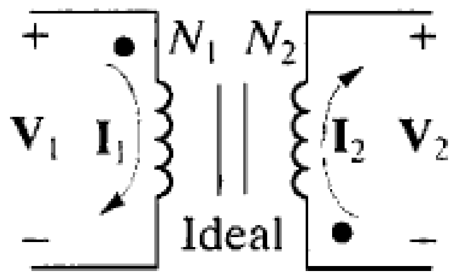
(b)



$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2},$$

$$N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

(c)



$$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2},$$

$$N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$$

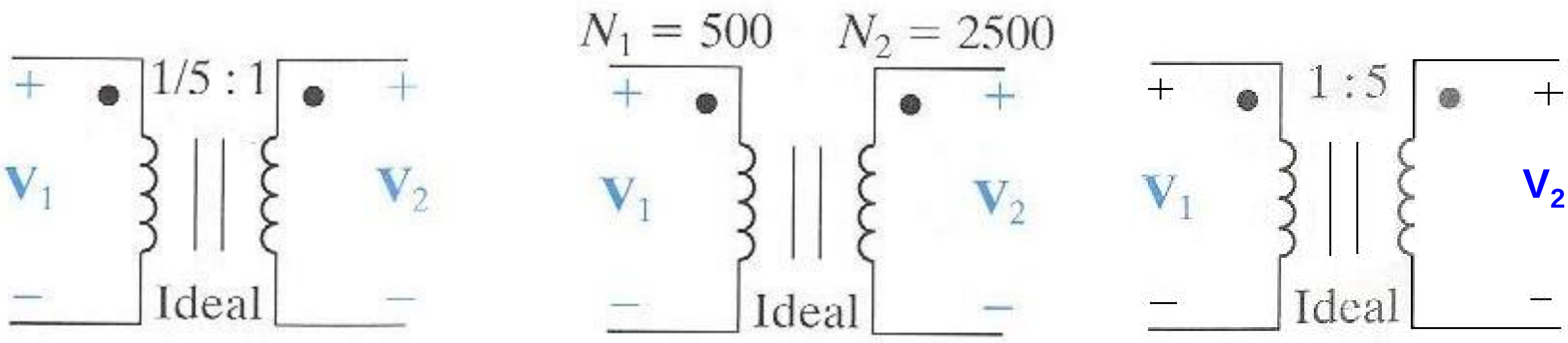
(d)

Ideal transformer

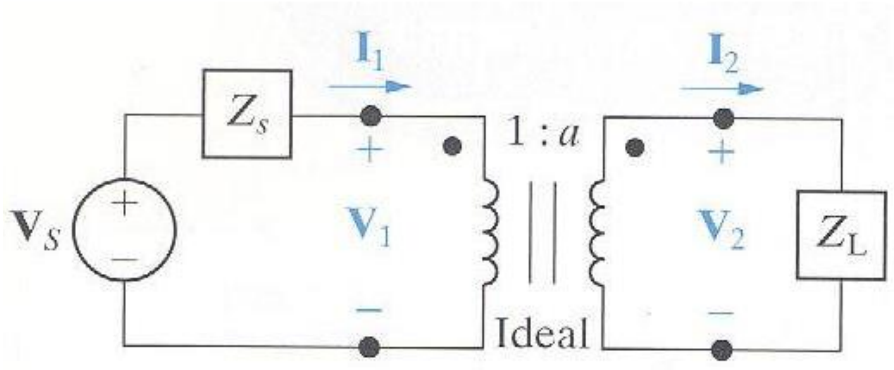
Turns ratio:

$$\boxed{a = \frac{N_2}{N_1}}$$

Three ways to show the turns ratio of an ideal transformer



Impedance matching by using ideal transformer



Relation of V_1 and I_1 by the transformer turns ratio:

$$V_1 = \frac{V_2}{a} \quad \text{and} \quad I_1 = aI_2$$

Impedance seen by the source and load respectively:

$$Z_{IN} = \frac{V_1}{I_1} \quad \text{and} \quad Z_L = \frac{V_2}{I_2}$$

Yield:

$$\boxed{Z_{IN} = \frac{1}{a^2} Z_L}$$

The ideal transformer's secondary coil reflects the load impedance back to the primary coil with the scaling factor $1/a^2$.