VIETNAM NATIONAL UNIVERSITY-HCMC
International University

Lecture Notes/Slides for

# APPLIED LINEAR ALGEBRA

## Chapter 3A. Vector Spaces



## What is $\mathbb{R}^n$?

### Notation and Terminology

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^2$ denotes the set of all column vectors with two entries.
- $\mathbb{R}^3$ denotes the set of all column vectors with three entries.
- In general, $\mathbb{R}^n$ denotes the set of all column vectors with $n$ entries.

## Scalar quantities versus vector quantities

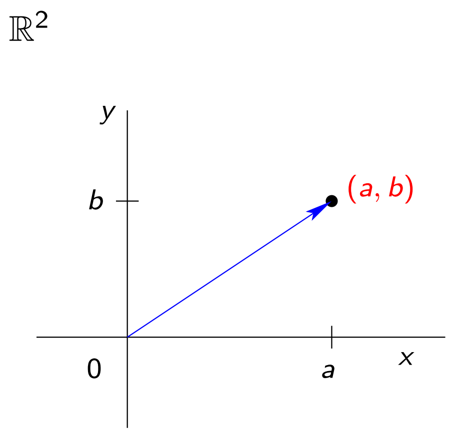
- A scalar quantity has only magnitude; e.g. time, temperature.
- A (non-zero) vector quantity has both magnitude and direction; e.g. displacement, force, wind velocity.

Whereas two scalar quantities are equal if they are represented by the same value, two vector quantities are equal if and only if they have the same magnitude and direction.

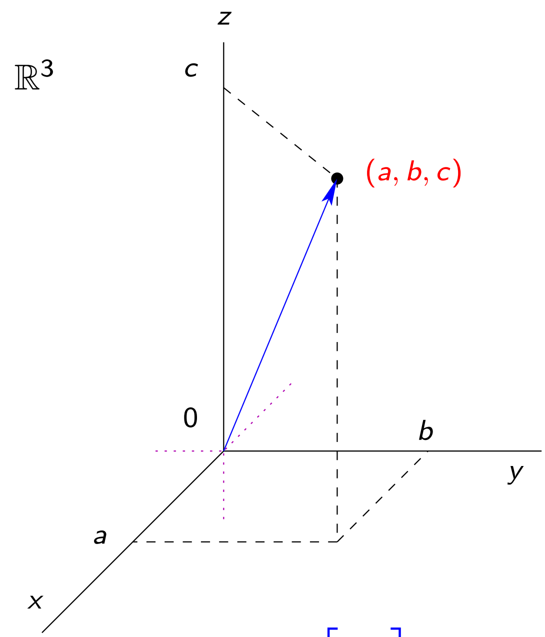


### $\mathbb{R}^2$ and $\mathbb{R}^3$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric representations as **position vectors** of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.




The vector $\begin{bmatrix} a \\ b \end{bmatrix}$.




The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.



## Notation

- If $P$ is a point in $\mathbb{R}^n$ with coordinates $(p_1, p_2, ..., p_n)$ we denote this by $P = (p_1, p_2, ..., p_n)$.
- If $P = (p_1, p_2, \ldots, p_n)$ is a point in $\mathbb{R}^n$, then

$$\overrightarrow{0P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

  is often used to denote the position vector of the point.
- Instead of using a capital letter to denote the vector (as we generally do with matrices), we emphasize the importance of the geometry and the direction with an arrow over the name of the vector.

## Notation and Terminology

- The notation $\overrightarrow{0P}$ emphasizes that this vector goes from the origin 0 to the point $P$. We can also use lower case letters for names of vectors. In this case, we write $\overrightarrow{0P} = \vec{p}$.
- Any vector

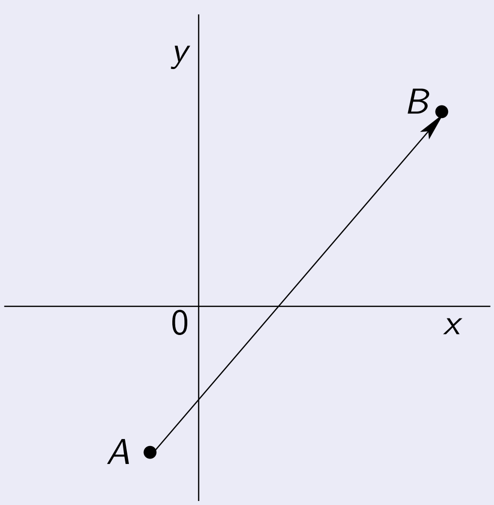
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n$$

  is associated with the point $(x_1, x_2, \ldots, x_n)$.
- Often, there is no distinction made between the vector $\vec{x}$ and the point $(x_1, x_2, \ldots, x_n)$, and we say that both $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.



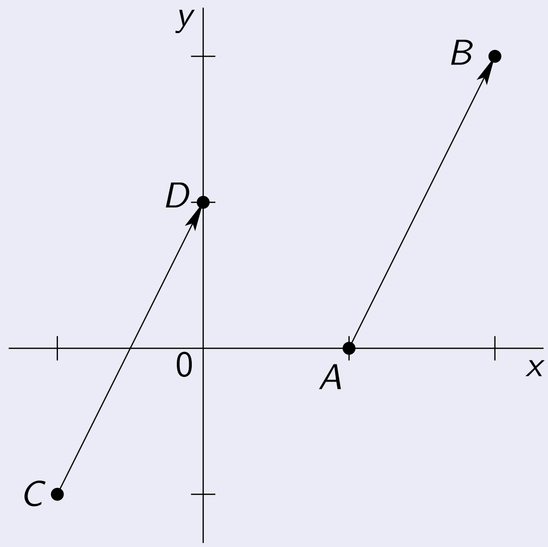
## Geometric Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let $A$ and $B$ be two points in $\mathbb{R}^2$ or $\mathbb{R}^3$.




- $\overrightarrow{AB}$ is the geometric vector from $A$ to $B$.
- $A$ is the tail of $\overrightarrow{AB}$.
- $B$ is the tip of $\overrightarrow{AB}$.
- the magnitude of $\overrightarrow{AB}$ is its length, and is denoted $||\overrightarrow{AB}||$.

## Equality of geometric vectors

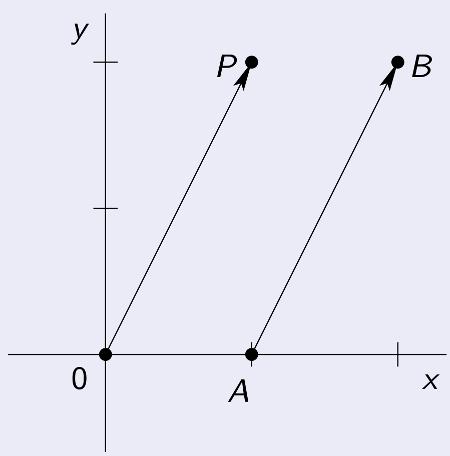



- $\overrightarrow{AB}$ is the vector from $A = (1, 0)$. to $B = (2, 2)$.
- $\overrightarrow{CD}$ is the vector from $C = (-1, -1)$ to $D = (0, 1)$.
- $\overrightarrow{AB} = \overrightarrow{CD}$ because the vectors have the same length and direction.

The fact that the points $A$ and $B$ are different from the points $C$ and $D$ is not important. For geometric vectors, the location of the vector in the plane (or in 3-dimensional space) is not important; the important properties are its length and direction.



## Coordinatizing Vectors – Part 1




$\overrightarrow{0P}$ is the position vector for $P = (1, 2)$,

and $\overrightarrow{0P} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Since $\overrightarrow{AB} = \overrightarrow{0P}$, it should be the case that $\overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. This can be seen by moving $\overrightarrow{AB}$ so that its tail is at the origin.

A geometric vector is coordinatized by putting it in standard position, meaning with its tail at the origin, and then identifying the vector with its tip.

# Algebra in $\mathbb{R}^n$

## Addition in $\mathbb{R}^n$

Since vectors in $\mathbb{R}^n$ are $n \times 1$ matrices, addition in $\mathbb{R}^n$ is precisely matrix addition using column matrices, i.e.,

- If $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^n$, then $\vec{u} + \vec{v}$ is obtained by adding together corresponding entries of the vectors.
- The zero vector in $\mathbb{R}^n$ is the $n \times 1$ zero matrix, and is denoted $\vec{0}$.

## Example

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. Then,

$$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$



# Properties of Vector Addition

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$. Then the following properties hold.

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (vector addition is commutative).
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (vector addition is associative).
3. $\vec{u} + \vec{0} = \vec{u}$ (existence of an additive identity).
4. $\vec{u} + (-\vec{u}) = \vec{0}$ (existence of an additive inverse).

## Scalar Multiplication

Since vectors in $\mathbb{R}^n$ are $n \times 1$ matrices, scalar multiplication in $\mathbb{R}^n$ is precisely matrix scalar multiplication using column matrices, i.e., If $\vec{u}$ is a vector in $\mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k\vec{u}$ is obtained by multiplying every entry of $\vec{u}$ by $k$.

### Example

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $k = 4$. Then,

$$k\vec{u} = 4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$



## Properties of Scalar Multiplication

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be vectors and $k, p \in \mathbb{R}$ be scalars. Then the following properties hold.

1. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (scalar multiplication distributes over vector addition).
2. $(k + p)\vec{u} = k\vec{u} + p\vec{u}$ (addition distributes over scalar multiplication).
3. $k(p\vec{u}) = (kp)\vec{u}$ (scalar multiplication is associative).
4. $1\vec{u} = \vec{u}$ (existence of a multiplicative identity).

## Some notation you may encounter

Often, in $\mathbb{R}^2$ the notation $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is used. Whereas in $\mathbb{R}^3$ the notation is $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\vec{i} + b\vec{j}$$

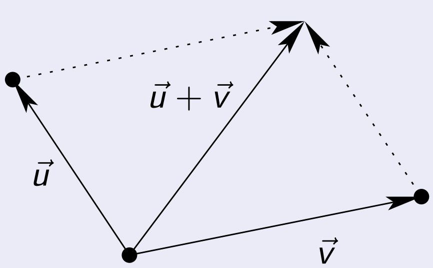
and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{i} + b\vec{j} + c\vec{k}$$



## The Geometry of Vector Addition

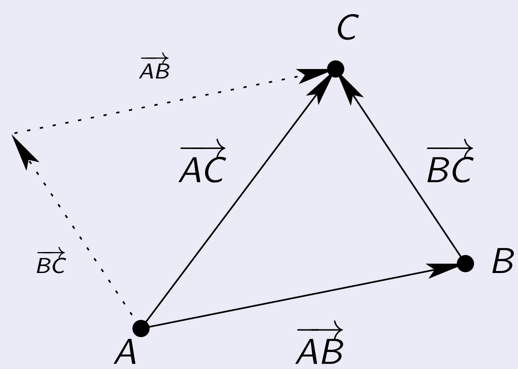
1. Vector Equality. The vectors have the same length and direction.
2. The zero vector, $\vec{0}$ has length zero and no direction.
3. Addition. Let $\vec{u}, \vec{v}$ be vectors. Then $\vec{u} + \vec{v}$ is the diagonal of the parallelogram defined by $\vec{u}$ and $\vec{v}$, and having the same tail as $\vec{u}$ and $\vec{v}$.



## Tip-to-Tail Method for Vector Addition

For points $A$, $B$ and $C$,

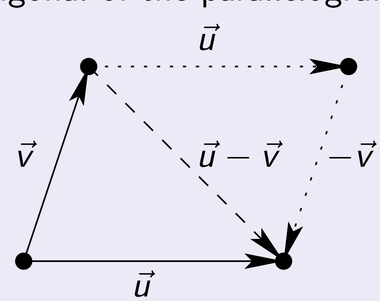
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$





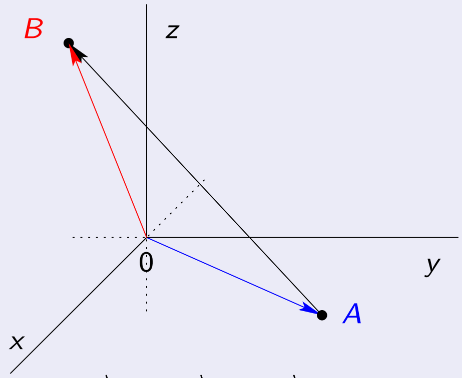

## The Geometry of Vector Subtraction

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ is obtained from the parallelogram defined by $\vec{u}$ and $\vec{v}$ by taking the vector from the tip of $\vec{v}$ to the tip of $\vec{u}$, i.e., the diagonal of the parallelogram, directed towards the tip of $\vec{u}$.



## Coordinatizing Vectors – Part 2

Let $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.



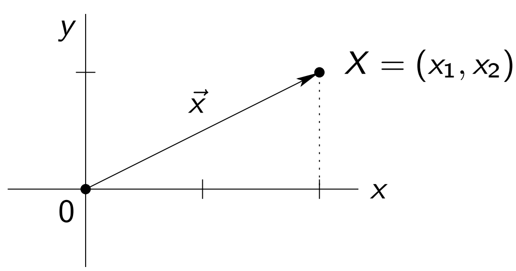

We see from the figure that $\overrightarrow{0A} + \overrightarrow{AB} = \overrightarrow{0B}$, and hence

$$\overrightarrow{AB} = \overrightarrow{0B} - \overrightarrow{0A} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$



## Length of a Vector, $\mathbb{R}^2$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,




then the length of the vector $\vec{x}$ is the distance from the origin 0 to the point $X = (x_1, x_2)$ given by $d(0, X)$.

The length of $\vec{x}$, denoted $\|\vec{x}\|$, is given by:

$$d(0, X) = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2}.$$

# Length of a Vector, $\mathbb{R}^3$

This extends clearly to $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

The length of $\vec{x}$ is the distance from the origin 0 to the point $X = (x_1, x_2, x_3)$ given by $d(0, X)$.

$$d(0, X) = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Suppose we want to find the distance between points other than the origin?



# Length of a Vector, $\mathbb{R}^3$

Consider two arbitrary points in $\mathbb{R}^3$, $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$. Then the distance between them is written $d(A, B)$ and is given by the distance formula.

## Distance Formula

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Now let $P = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$, and

$$\vec{p} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

Then the length of $\vec{p}$ is equal to the distance between the origin and $P$, which are both equal to the distance between points $A$ and $B$

$$\|\vec{p}\| = d(0, P) = d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

# Length of a Vector, $\mathbb{R}^n$

More generally, if $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$ are points in $\mathbb{R}^n$, then the distance between $P$ and $Q$ is the length of the vector $\overrightarrow{PQ}$, written $\|\overrightarrow{PQ}\|$.

$$d(P, Q) = \|\overrightarrow{PQ}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \cdots + (q_n - p_n)^2}.$$

The formula for calculating the length of a vector generalizes to $\mathbb{R}^n$: if

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n,$$

then

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

which represents the distance from the origin to the point $(x_1, x_2, \dots, x_n)$.



# Properties of Distance

Let $P$ and $Q$ be two points in $\mathbb{R}^n$, and $d(P, Q)$ the distance between them. Then the following properties hold.

1. The distance between $P$ and $Q$ is equal to the distance between $Q$ and $P$, i.e., $d(P, Q) = d(Q, P)$.
2. $d(P, Q) \geq 0$ with equality if and only if $P = Q$.

## Example

For $P = (1, -1, 3)$ and $Q = (3, 1, 0)$, the distance between $P$ and $Q$ is $d(P, Q) = \sqrt{2^2 + 2^2 + (-3)^2} = \sqrt{17}$.

## Example

Let $\vec{p} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $-2\vec{q} = (-2)\vec{q} = \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$.
The lengths of these vectors are

$$||\vec{p}|| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = 5,$$

$$||\vec{q}|| = \sqrt{(3)^2 + (-1)^2 + (-2)^2} = \sqrt{9 + 1 + 4} = \sqrt{14},$$

and

$$\begin{aligned} ||-2\vec{q}|| &= \sqrt{(-6)^2 + 2^2 + 4^2} \\ &= \sqrt{36 + 4 + 16} \\ &= \sqrt{56} = \sqrt{4 \times 14} \\ &= 2\sqrt{14} = 2||\vec{q}||. \end{aligned}$$



# The Geometry of Scalar Multiplication

- Scalar Multiplication. If $\vec{v} \neq \vec{0}$ and $a \in \mathbb{R}$, $a \neq 0$, then $a\vec{v}$ has length $||a\vec{v}|| = |a| \cdot ||\vec{v}||$, and
  - has the same direction as $\vec{v}$ if $a > 0$;
  - has direction opposite to $\vec{v}$ if $a < 0$.
- Parallel Vectors. Two nonzero vectors are called parallel if they have the same direction or opposite directions. It follows that nonzero vectors $\vec{v}$ and $\vec{w}$ are parallel if and only if one is a scalar multiple of the other.

## Problem

Let $P = (1, -2, 1), Q = (-3, 0, 5)$, $X = (2, -1, 5)$ and $Y = (4, -2, 3)$ be points in $\mathbb{R}^3$. Is $\overrightarrow{PQ}$ parallel to $\overrightarrow{XY}$? Is $\overrightarrow{PX}$ parallel to $\overrightarrow{QY}$?

## Solution

$\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$, $\overrightarrow{XY} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$, and these vectors are parallel if $\overrightarrow{PQ} = k\overrightarrow{XY}$ for some scalar $k$, i.e.,

$$\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \text{ or } \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ -2k \end{bmatrix}.$$

This gives a system of three equations in one variable, which is consistent, and has unique solution $k = -2$. Therefore, $\overrightarrow{PQ}$ is parallel to $\overrightarrow{XY}$.

$\overrightarrow{PX} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $\overrightarrow{QY} = \begin{bmatrix} 7 \\ -2 \\ -2 \end{bmatrix}$, and these vectors are parallel if $\overrightarrow{PX} = \ell\overrightarrow{QY}$ for some scalar $\ell$. You will find that no such $\ell$ exists, so $\overrightarrow{PX}$ is not parallel to $\overrightarrow{QY}$.



# Unit Vectors

## Definition

A unit vector is a vector of length one.

## Example

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, are examples of unit vectors.

## Example

If $\vec{v} \neq \vec{0}$, then

$$\frac{1}{\|\vec{v}\|}\vec{v}$$

is a unit vector in the same direction as $\vec{v}$.

## Example

$\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ is not a unit vector, since $\|\vec{v}\| = \sqrt{14}$. However,

$$\vec{u} = \frac{1}{\sqrt{14}}\vec{v} = \begin{bmatrix} \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}$$

is a unit vector in the same direction as $\vec{v}$, i.e.,

$$\|\vec{u}\| = \frac{1}{\sqrt{14}}\|\vec{v}\| = \frac{1}{\sqrt{14}}\sqrt{14} = 1.$$

## Example

If $\vec{v}$ and $\vec{w}$ are nonzero that have

- the same direction, then $\vec{v} = \frac{\|\vec{v}\|}{\|\vec{w}\|}\vec{w}$;
- opposite directions, then $\vec{v} = -\frac{\|\vec{v}\|}{\|\vec{w}\|}\vec{w}$.



# The Dot Product

## Definition

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \bullet \vec{v} = x_1x_2 + y_1y_2 + z_1z_2,$$

i.e., $\vec{u} \bullet \vec{v}$ is a scalar.

## Problem

Find $\vec{u} \bullet \vec{v}$ for $\vec{u} = \begin{bmatrix} 1 & 2 & 0 & -1 \end{bmatrix}^T, \vec{v} = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$.

## Solution

$$\begin{aligned} \vec{u} \bullet \vec{v} &= (1)(0) + (2)(1) + (0)(2) + (-1)(3) \\ &= 0 + 2 + 0 + -3 = -1 \end{aligned}$$

## Note

If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

are in $\mathbb{R}^n$, then another way to think about the dot product $\vec{u} \bullet \vec{v}$ is as the $1 \times 1$ matrix

$$\vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \end{bmatrix}$$

which is treated as a scalar given by $u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$



# Properties of the Dot Product

## Theorem

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$ and let $k \in \mathbb{R}$.

1. $\vec{u} \bullet \vec{v}$ is a real number
2. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
3. $\vec{u} \bullet \vec{0} = 0$
4. $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2$
5. $(k\vec{u}) \bullet \vec{v} = k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v})$
6. $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$
   $\vec{u} \bullet (\vec{v} - \vec{w}) = \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{w}$

Since, for $\vec{u} \in \mathbb{R}^n$, $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2$, we have an alternate (but equivalent) expression for the length of $\vec{u}$:

$$\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}}.$$

## Length of a Vector

We can use the properties of the dot product to find the length of a vector.

### Problem

Find the length of the vector $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \end{bmatrix}$.

### Solution

By the properties of the dot product, $\|\vec{u}\|^2 = \vec{u} \bullet \vec{u}$.

$$\begin{aligned} \vec{u} \bullet \vec{u} &= (1)(1) + (3)(3) + (5)(5) + (2)(2) \\ &= 1 + 9 + 25 + 4 \\ &= 39 \end{aligned}$$

Therefore, $\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}} = \sqrt{39}$



## Two Important Inequalities

### Theorem

The **Cauchy-Schwarz Inequality** is given as follows. For $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} \bullet \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Equality is obtained if one vector is a scalar multiple of the other.

### Theorem

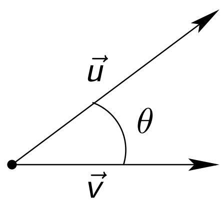
The **Triangle Inequality** is given as follows. For $\vec{u}, \vec{v} \in \mathbb{R}^n$,

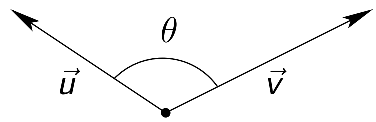
$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Equality is obtained if one vector is a non-negative scalar multiple of the other.

# The Included Angle

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^n$ $(n \geq 2)$, positioned so they have the same tail. Then there is a unique angle $\theta$ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$. This angle $\theta$ is called the included angle.




## Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let $\theta$ denote the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \ \|\vec{v}\| \cos\theta.$$



## Finding the included angle for nonzero vectors

As a consequence of the Theorem, if $\vec{u}$ and $\vec{v}$ are nonzero vectors with included angle $\theta$, then $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$,
and

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

1. If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$, implying that $\vec{u} \bullet \vec{v} > 0$. Conversely, if $\vec{u} \bullet \vec{v} > 0$, then $0 \leq \theta < \frac{\pi}{2}$.
2. If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$, implying that $\vec{u} \bullet \vec{v} = 0$. Conversely, if $\vec{u} \bullet \vec{v} = 0$, then $\theta = \frac{\pi}{2}$.
3. If $\frac{\pi}{2} < \theta \leq \pi$, then $\cos\theta < 0$, implying that $\vec{u} \bullet \vec{v} < 0$. Conversely, if $\vec{u} \bullet \vec{v} < 0$, then $\frac{\pi}{2} < \theta \leq \pi$.

# Included Angle

## Problem

Find the angle between $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

## Solution

$\vec{u} \bullet \vec{v} = 1$, $\|\vec{u}\| = \sqrt{2}$ and $\|\vec{v}\| = \sqrt{2}$.
Therefore,

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$, $\theta = \frac{\pi}{3}$.

Therefore, the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{3}$.



## Problem

Find the included angle for $\vec{u} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$.

## Solution

$$\vec{u} \bullet \vec{v} = -9, \ \|\vec{u}\| = \sqrt{54} = 3\sqrt{6}, \ \text{ and } \ \|\vec{v}\| = \sqrt{6}.$$

Let $\theta$ denote the included angle for $\vec{u}$ and $\vec{v}$. Then

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-9}{3\sqrt{6} \times \sqrt{6}} = \frac{-9}{18} = -\frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$, the included angle is $\theta = \frac{2\pi}{3}$.

### Problem

Find the included angle for $\vec{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

### Solution

Let $\theta$ denote included angle.

$$\vec{u} \bullet \vec{v} = 0.$$

Regardless of $\|\vec{u}\|$ and $\|\vec{v}\|$, $\cos\theta = 0$, and therefore the included angle is $\theta = \frac{\pi}{2}$.



## Orthogonal Vectors

### Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal, also called perpendicular, if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\theta = \frac{\pi}{2}$.

### Theorem

Nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \bullet \vec{v} = 0$.

### Proof

We have $\vec{u} \perp \vec{v}$ if and only if $\|\vec{u} - \vec{v}\| = \|\vec{u} + \vec{v}\|$ (see the picture).
This is equivalent to

$$(\vec{u} - \vec{v}) \bullet (\vec{u} - \vec{v}) = (\vec{u} + \vec{v}) \bullet (\vec{u} + \vec{v})$$

which gives $-2\vec{u} \bullet \vec{v} = 2\vec{u} \bullet \vec{v}$ and therefore $\vec{u} \bullet \vec{v} = 0$.

## Problem

Find all vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both $\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

## Solution

There are infinitely many such vectors.
Since $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{w}$,

$$\begin{aligned} \vec{v} \bullet \vec{u} &= -x - 3y + 2z = 0 \\ \vec{v} \bullet \vec{w} &= y + z = 0 \end{aligned}$$



## Solution (continued)

This is a homogeneous system of two linear equation in three variables.

$$\left[\begin{array}{rrr|r} -1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$\left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$ implies that $\vec{v} = \begin{bmatrix} 5t \\ -t \\ t \end{bmatrix}$ for $t \in \mathbb{R}$.

Therefore, $\vec{v} = t \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$ for all $t \in \mathbb{R}$.

## Problem

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

## Solution

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

- $\overrightarrow{AB} \bullet \overrightarrow{AC} = 6 + 187 + 75 \neq 0$.
- $\overrightarrow{BA} \bullet \overrightarrow{BC} = (-\overrightarrow{AB}) \bullet \overrightarrow{BC} = -2 - 66 - 50 \neq 0$.
- $\overrightarrow{CA} \bullet \overrightarrow{CB} = (-\overrightarrow{AC}) \bullet (-\overrightarrow{BC}) = \overrightarrow{AC} \bullet \overrightarrow{BC} = 3 + 102 + 150 \neq 0$.

None of the angles is $\frac{\pi}{2}$, and therefore the triangle is not a right angle triangle.



# Projections

## Theorem

Given nonzero vectors $\vec{v}$ and $\vec{u}$ in $\mathbb{R}^n$(for $n = 2, 3...$), there exist unique vectors $\vec{v}_{||}, \vec{v}_{\perp}$ such that $\vec{v}$ can be written as a sum

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$$

where $\vec{v}_{||}$ is parallel to $\vec{u}$ and $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$.


$\vec{v}_{||}$ is the projection of $\vec{v}$ onto $\vec{u}$, written $\vec{v}_{||} = \text{proj}_{\vec{u}}\vec{v}$ and $\vec{v}_{\perp} = \vec{v} - \vec{v}_{||}$.

## Projections

### A formula for $\text{proj}_{\vec{u}}\vec{v}$

The defining properties of $\vec{v}_{||}$ and $\vec{v}_{\perp}$ are

1. $\vec{v}_{||}$ is parallel to $\vec{u}$;
2. $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$;
3. $\vec{v}_{||} + \vec{v}_{\perp} = \vec{v}$.

Since $\vec{v}_{||}$ is parallel to $\vec{u}$, $\vec{v}_{||} = t\vec{u}$ for some $t \in \mathbb{R}$. Furthermore, $\vec{v}_{\perp} = \vec{v} - \vec{v}_{||}$ and $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$, so

$$0 = \vec{v}_{\perp} \bullet \vec{u} = (\vec{v} - \vec{v}_{||}) \bullet \vec{u} = (\vec{v} - t\vec{u}) \bullet \vec{u} = \vec{v} \bullet \vec{u} - t(\vec{u} \bullet \vec{u}).$$

Since $\vec{u} \neq \vec{0}$, it follows that $t = \frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}$. Therefore

$$\vec{v}_{||} = \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}, \quad \text{and} \quad \vec{v}_{\perp} = \vec{v} - \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}.$$



## Projections

### Theorem

Let $\vec{v}$ and $\vec{u}$ be vectors with $\vec{u} \neq \vec{0}$.

1. $\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}$
2. $\vec{v} - \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}$ is orthogonal to $\vec{u}$.

### Problem

Let $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Find vectors $\vec{v}_{||}$ and $\vec{v}_{\perp}$ so that $\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$, with $\vec{v}_{||}$ parallel to $\vec{u}$ and $\vec{v}_{\perp}$ orthogonal to $\vec{u}$.

### Solution

$$\vec{v}_{||} = \text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}\right)\vec{u} = \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 15/11 \\ 5/11 \\ -5/11 \end{bmatrix}.$$

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{||} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 \\ -16 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -16/11 \\ 5/11 \end{bmatrix}.$$



## Distance from a Point to a Line

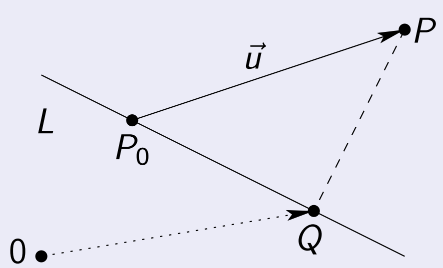
### Problem

Let $P = (3, 2, -1)$ be a point in $\mathbb{R}^3$ and $L$ a line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}.$$

Find the shortest distance from $P$ to $L$, and find the point $Q$ on $L$ that is closest to $P$.

### Solution




Let $P_0 = (2, 1, 3)$ be a point on $L$, and let $\vec{d} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}^T$. Then $\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P}$, $\overrightarrow{0Q} = \overrightarrow{0P_0} + \overrightarrow{P_0Q}$, and the shortest distance from $P$ to $L$ is the length of $\overrightarrow{QP}$, where $\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q}$.

## Solution (continued)

$\overrightarrow{P_0P} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

$$\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{d}}{\|\vec{d}\|^2}\right)\vec{d} = \frac{10}{14}\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix}.$$

Therefore,

$$\overrightarrow{OQ} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 29 \\ 2 \\ 11 \end{bmatrix},$$

so $Q = \left(\frac{29}{7}, \frac{2}{7}, \frac{11}{7}\right)$.



## Solution (continued)

Finally, the shortest distance from $P(3, 2, -1)$ to $L$ is the length of $\overrightarrow{QP}$, where

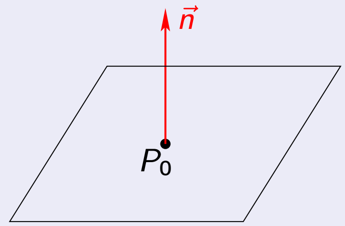
$$\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} -4 \\ 6 \\ -9 \end{bmatrix}.$$

Therefore the shortest distance from $P$ to $L$ is

$$\|\overrightarrow{QP}\| = \frac{2}{7}\sqrt{(-4)^2 + 6^2 + (-9)^2} = \frac{2}{7}\sqrt{133}.$$

## Equations of Planes

Given a point $P_0$ and a nonzero vector $\vec{n}$, there is a unique plane containing $P_0$ and orthogonal to $\vec{n}$.

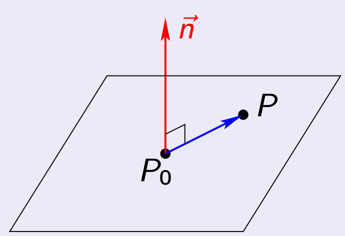



### Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \bullet \vec{v} = 0$ for every vector $\vec{v}$ in the plane, i.e., $\vec{n}$ is orthogonal to every vector in the plane.



Consider a plane containing a point $P_0$ and orthogonal to vector $\vec{n}$, and let $P$ be an arbitrary point on this plane. Then $\vec{n} \bullet \overrightarrow{P_0P} = 0$,




or, equivalently,

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0,$$

and is called a vector equation of the plane. The vector equation can also be written as

$$\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}.$$

Suppose a plane contains a fixed point $P_0 = (x_0, y_0, z_0)$ and has normal vector

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Let $P = (x, y, z)$ denote an arbitrary point on the plane. Since $\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}$,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}.$$

Thus

$$ax + by + cz = ax_0 + by_0 + cz_0,$$

where $d = ax_0 + by_0 + cz_0$ is simply a scalar.
A scalar equation of the plane has the form

$$ax + by + cz = d, \text{ where } a, b, c, d \in \mathbb{R}.$$



## Problem

Find an equation of the plane containing $P_0(1, -1, 0)$ and orthogonal to $\vec{n} = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}^T$.

## Solution

A vector equation of this plane is

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} x-1 \\ y+1 \\ z \end{bmatrix} = 0.$$

Thus, a scalar equation of this plane is

$$-3x + 5y + 2z = -3(1) + 5(-1) + 2(0) = -8,$$

i.e., the plane has scalar equation

$$-3x + 5y + 2z = -8.$$

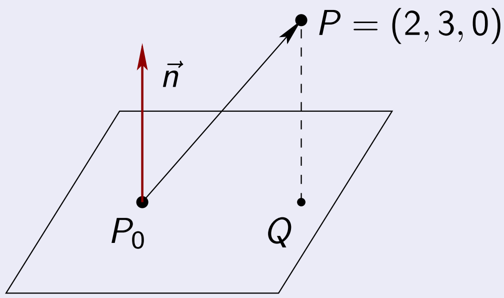
# Shortest distance from a point to a plane

## Problem

Find the shortest distance from the point $P = (2, 3, 0)$ to the plane with equation $5x + y + z = -1$, and find the point $Q$ on the plane that is closest to $P$.

(wb example)

## Solution



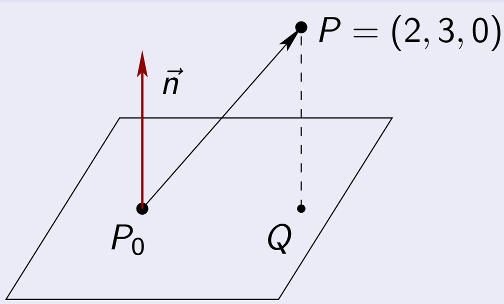

Pick an arbitrary point $P_0$ on the plane.

Then $\overrightarrow{QP} = \text{proj}_{\vec{n}} \overrightarrow{P_0P}$,
$\|\overrightarrow{QP}\|$ is the shortest distance,
and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

$\vec{n} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$. Choose $P_0 = (0, 0, -1)$. Then $\overrightarrow{P_0P} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.



## Solution (continued)




$$\overrightarrow{P_0P} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

$$\overrightarrow{QP} = \text{proj}_{\vec{n}} \overrightarrow{P_0P} = \left( \frac{\overrightarrow{P_0P} \bullet \vec{n}}{\|\vec{n}\|^2} \right) \vec{n} = \frac{14}{27} \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

Since $\|\overrightarrow{QP}\| = \frac{14}{27}\sqrt{27} = \frac{14\sqrt{3}}{9}$, the shortest distance from $P$ to the plane is $\frac{14\sqrt{3}}{9}$.

### Solution (continued)

To find $Q$, we have

$$\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - \frac{14}{27}\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -16 \\ 67 \\ -14 \end{bmatrix}.$$

Therefore $Q = \left(-\frac{16}{27}, \frac{67}{27}, -\frac{14}{27}\right)$.



## The Cross Product

### Definition

Let $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$. Then

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}.$$

**Note.** $\vec{u} \times \vec{v}$ is a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$.

A helpful way to remember (once we cover determinants):

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \text{ where } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

## Computing the Cross Product

### Problem

Find $\vec{u} \times \vec{v}$ for $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

### Solution

We will use the equation:

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Therefore,

$$\vec{u} \times \vec{v} = \begin{bmatrix} (-1)(1) - (2)(-2) \\ -((1)(1) - (2)(3)) \\ (1)(-2) - (-1)(3) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$



## Properties of the Cross Product

### Theorem

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be in $\mathbb{R}^3$.

1. $\vec{u} \times \vec{v}$ is a vector.
2. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
3. $\vec{u} \times \vec{0} = \vec{0}$ and $\vec{0} \times \vec{u} = \vec{0}$.
4. $\vec{u} \times \vec{u} = \vec{0}$.
5. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ for any scalar $k$.
7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
8. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.

## Problem

Find all vectors orthogonal to both $\vec{u} = \begin{bmatrix} -1 & -3 & 2 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$.

## Solution

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & -1 & 0 \\ \vec{j} & -3 & 1 \\ \vec{k} & 2 & 1 \end{vmatrix} = -5\vec{i} + \vec{j} - \vec{k} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}.$$

Any scalar multiple of $\vec{u} \times \vec{v}$ is also orthogonal to both $\vec{u}$ and $\vec{v}$, so

$$t \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, t \in \mathbb{R},$$

gives all vectors orthogonal to both $\vec{u}$ and $\vec{v}$.

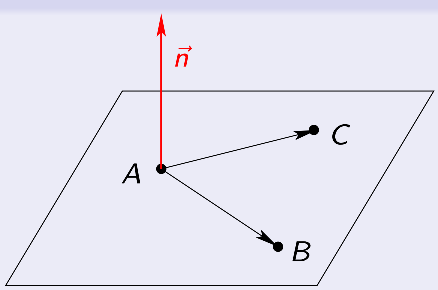


## Problem

Let $A = (1, -1, 2)$, $B = (2, 0, -1)$ and $C = (0, -2, 3)$ be points in $\mathbb{R}^3$. These points do no all lie on the same line (how can you tell?). Find an equation for the plane containing $A$, $B$, and $C$.

(wb example)

## Solution




$\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in the plane, so $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is a normal to the plane.

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \vec{n} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}.$$

$$\text{Therefore } -2x + 2y = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = -4$$

i.e. $-2x + 2y = -4$ is an equation of the plane.

# Area and Volume

## The Lagrange Identity

If $\vec{u}, \vec{v} \in \mathbb{R}^3$, then
$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2.$$

## Proof.

Write $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, and work out all the terms. □



## The length of the cross product

As a consequence of the Lagrange Identity and the fact that
$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \ \|\vec{v}\| \cos\theta,$$
we have
$$\begin{aligned} \|\vec{u} \times \vec{v}\|^2 &= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2 \\ &= \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta \\ &= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta) \\ &= \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta. \end{aligned}$$

Taking square roots on both sides yields,
$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta.$$

Note that since $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$.

If $\theta = 0$ or $\theta = \pi$, then $\sin\theta = 0$, and $\|\vec{u} \times \vec{v}\| = 0$. This is consistent with our earlier observation that if $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.

# Area of a Parallelogram

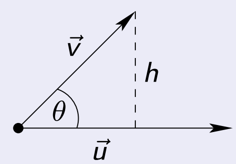
## Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$ with included angle $\theta$.

1. $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$, and is the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.
2. $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

## Proof of area of parallelogram.

The area of the parallelogram defined by $\vec{u}$ and $\vec{v}$ is $||\vec{u}||h$, where $h$ is the height of the parallelogram.




$\sin\theta = \frac{h}{||\vec{v}||}$, implying that $h = ||\vec{v}|| \sin\theta$. Therefore, the area is

$$||\vec{u}||||\vec{v}|| \sin\theta.$$

□



# Area of a Triangle

## Problem

Find the area of the triangle having vertices $A(3, -1, 2)$, $B(1, 1, 0)$ and $C(1, 2, -1)$.

## Solution

The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$. Therefore

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

so the area of the triangle is $\frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \sqrt{2}$.

## The Box Product

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Then

$$\begin{aligned} \vec{u} \bullet (\vec{v} \times \vec{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ -(v_1 w_3 - v_3 w_1) \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \\ &= u_1(v_2 w_3 - v_3 w_2) - u_2(v_1 w_3 - v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= u_1 \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}. \end{aligned}$$



## The Box Product

### Theorem

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Then the box product is

$$\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}.$$

**Shorthand:** $\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$.

### Theorem

The order of the box product is defined as follows:

$$(\vec{u} \times \vec{v}) \bullet \vec{w} = \vec{u} \bullet (\vec{v} \times \vec{w}).$$

# The Volume of a Parallelepiped

## Theorem

The volume of the parallelepiped determined by the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ is

$$|\vec{u} \bullet (\vec{v} \times \vec{w})|.$$

## Problem

Find the volume of the parallelepiped determined by the vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.



## Solution

The volume of the parallelepiped is $|\vec{u} \bullet (\vec{v} \times \vec{w})|$.
Since $\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$, and

$$\det \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} = -2,$$

the volume of the parallelepiped is $|-2| = 2$ cubic units.