HO CHI MINH CITY INTERNATIONAL UNIVERSITY

MIDTERM EXAMINATION

Academic year 2020-2021, Semester 2
Duration: 90 minutes

SUBJECT: **Differential Equations (MA024IU)**	
Head of Department of Mathematics	Lecturer:
Signature:	Signature:
Professor Pham Huu Anh Ngoc	Full name: Pham Huu Anh Ngoc

Instructions:

- *Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.*

Question 1. *(20 marks)* A model for the population $P(t)$ in a suburb of a large city is given by the initial value problem

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \quad P(0) = 5000,$$

where t is measured in months. What is the limiting value of the population? At what time will the population be equal to one-half of this limiting value?

Question 2. *(20 marks)* Prove that the differential equation

$$\big(\cos x \sin x - xy^2 + y\big)dx + \big(y(1-x^2) + x\big)dy = 0,$$

is exact. Solve the differential equation.

Question 3. *(20 marks)* Find the solution to the initial value problem

$$(x+1)y' + (x+2)y = 2xe^{-x}, \qquad y(0) = 0.$$

Question 4. *(20 marks)* Find a particular solution of the following differential equation

$$y'' - 6y' + 9y = 6x^2 + 2020 + 2021e^{3x}.$$

Question 5. *(20 marks)* Find the general solution of the following differential equation

$$x^2y'' - 5xy' + 9y = 0, \qquad x \in (0, \infty).$$

END.

SOLUTIONS:

Question 1. The limiting value of the population is $1,000,000$. The population will reach $500,000$ in 5.29 months.
Question 2. The given differential equation is rewritten as

$$(e^{2y}dx + xde^{2y}) - \cos(xy)(xdy + ydx) + dy^2 = 0.$$

Then, we get

$$d(e^{2y}x) - \cos(xy)d(xy) + dy^2 = d(e^{2y}x) + d(-\sin(xy)) + dy^2 = 0.$$

Therefore,

$$d(e^{2y}x - \sin(xy) + y^2) = 0.$$

Thus the general solution is given by

$$e^{2y}x - \sin(xy) + y^2 = C.$$

Question 3. Consider the differential equation

$$y' - (\sin x)y = 2\sin x.$$

The integrating factor is given by $I(x) = e^{\cos x}$. Thus, we get

$$e^{\cos x}y' - e^{\cos x}(\sin x)y = 2e^{\cos x}\sin x.$$

This gives

$$\frac{d}{dx}(e^{\cos x}y) = 2\int e^{\cos x}\sin x dx = -2e^{\cos x} + C.$$

Therefore, the general solution is

$$y(x) = -2 + \frac{C}{e^{\cos x}}.$$

Since $y(\frac{\pi}{2}) = 1$, the particular solution is $y(x) = -2 + \frac{3}{e^{\cos x}}$.
Question 4. a) The form of a particular solution of the differential equation

$$y'' - 4y' + 3y = e^{2x}(x^3 + 1) + e^x(x + 1)$$

is given by

$$y_p(x) = e^{2x}(Ax^3 + Bx^2 + Cx + D) + e^x(Ex^2 + Fx).$$

The general solution of the differential equation

$$y'' - 4y' + 3y = e^x(x + 1)$$

is given by

$$y(x) = c_1e^x + c_2e^{3x} - e^x(\frac{1}{4}x^2 + \frac{3}{4}x).$$

Question 5. a) $a = b = q, q \in \mathbb{R}$.
b) Note that $y_1(x) = x + 1$ is a particular solution of the differential equation

$$(1 - 2x - x^2)y'' + 2(x + 1)y' - 2y = 0.$$

By the Liouville formula, $y_2(x) = x^2 + x + 2$ is a solution of this equation such that y_1, y_2 are linearly independent. So, the general solution is given by

$$y(x) = c_1(x + 1) + c_2(x^2 + x + 2).$$