APPLIED LINEAR ALGEBRA

3. Vector Spaces

(Sections 7.4-7.5 of Kreyszig)

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2. Subspaces, Linear Combination and Span
3. Linear Independence
4. Basis and Dimensions
5. Row and Column Spaces - Rank of a Matrix
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Section 1

Vector Spaces

Introduction

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Introduction

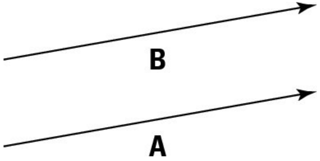
- Many physical quantities, such as area, length, mass, and temperature, are completely determined by its magnitude. Such quantities are called *scalars*.
- Other physical quantities can only be determined by both a magnitude and a direction, e.g. forces, velocities, electromagnetic fields,... These quantities are called *vectors*.
- The goal of this section is to study *vector spaces*, and its fundamental properties:

  1. Linear dependence/independence.
  2. Spanning sets.
  3. Bases and dimensions.

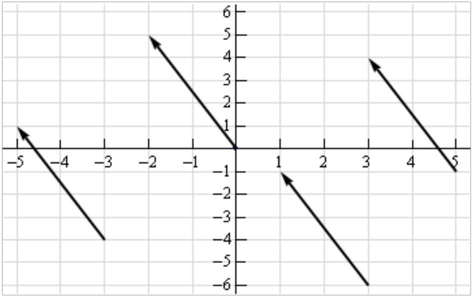
What is a Vector?

- A vector, represented by an arrow, has both a *direction and a magnitude*. Magnitude is shown as the length of a line segment. Direction is shown by the orientation of the line segment, and by an arrow at one end.

- Equal vectors have the same length and direction but may have different starting points.



What is a Vector?



Each of the directed line segments in the above figure represents *the same vector*. In each case the vector starts at a specific point then moves 2 units to the left and 5 units up.
Notation: $\vec{v} = \langle -2, 5 \rangle$ or $\vec{v} = (-2, 5)$.
Note: It is important to distinguish the vector $\vec{v} = (-2, 5)$ from the point $A(-2, 5)$.

Vectors

- Given the two points $A(a_1, a_2)$ and $B(b_1, b_2)$, the vector with the representation $\vec{AB}$ is $\vec{AB} = (b_1 - a_1, b_2 - a_2)$.
- The magnitude, or length, of the vector $\vec{v} = (a, b)$ is given by,
$$\|\vec{v}\| = \sqrt{a^2 + b^2}$$
- Example, if $\vec{v} = (-3, 5)$ then its magnitude
$$\|\vec{v}\| = \sqrt{9 + 16} = 5$$
- Any vector with magnitude of 1 is called a unit vector, e.g., $\vec{v_1} = (0, 1)$, or $\vec{v_2} = (1, 0)$ (standard basis vectors).
- The zero vector, $\vec{0} = (0, 0)$, is a vector that has magnitude zero, but no specific direction.

Vector Spaces

A vector space is a nonempty set V on which are defined two operations, called *addition and multiplication* by scalars (real numbers), satisfying the following nine axioms for all $u, v, w \in V$ and for all $c, d \in \mathbb{R}$:

1. The sum $u + v$ and product cu are in V.
2. $u + v = v + u$.
3. $(u + v) + w = u + (v + w)$.
4. There exists a zero vector $\mathbf{0}$ in V such that $u + \mathbf{0} = u$.
5. For each u in V, there is a vector $-u$ in V such that $u + (-u) = \mathbf{0}$.
6. $c(u + v) = cu + cv$.
7. $(c + d)u = cu + du$.
8. $c(du) = (cd)u$.
9. $1u = u$.

Vector Spaces

- Technically, V is a real vector space. All of the theory in this chapter also holds for a complex vector space in which the scalars are complex numbers. From now on, all scalars are assumed to be real.
- The zero vector in Axiom 4 is unique. The vector $-u$ called the negative vector of u.

Properties

For any u in V and scalar c,
1. $0u = \mathbf{0}$, where $\mathbf{0}$ is the zero vector of V.
2. $c\mathbf{0} = \mathbf{0}$.
3. $-u = (-1)u$.

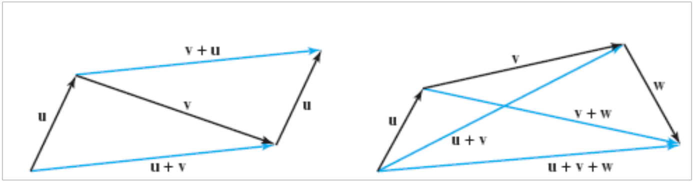
Vector Spaces

Example: Three-dimensional vector space

Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction.

Define addition by the parallelogram rule and for each v in V , define cv to be the arrow whose length is $|c|$ times the length of v, pointing in the same direction as v if $c > 0$ and otherwise pointing in the opposite direction.

Show that V is a vector space. This space is a common model in physical problems for various forces



Spaces of Matrices

The set of all $m \times n$ matrices with matrix addition and multiplication of a matrix by a real number (scalar multiplication), is a vector space (verify). We denote this vector space by M_{mn}.

Vector Spaces

Example: Vector space of Matrices with zero trace

Let V be the set of all 2×2 matrices with trace equal to zero, that is,

$$V = \left\{ A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] : Tr\,(A) = a + d = 0 \right\}$$

V is a vector space with the standard matrix addition, and the standard scalar multiplication of matrices.

Vector Spaces

Example: n-dimensional vector space

Let $\mathbb{R}^n$ be the set of all vector in the following form

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

This is the set of all matrices of size $n \times 1$, a specific case of the previous example. So $\mathbb{R}^n$ is a vector space.

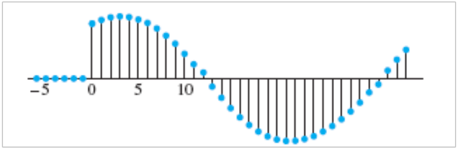
Vector Spaces

Example: discrete-time signals

Let S be the space of all doubly infinite sequences of numbers (usually written in a row rather than a column) with operations

$$\{y_k\} + \{z_k\} = \{y_k + z_k\}\,;\, c\,\{y_k\} = \{cy_k\}$$

Elements of S arise in engineering, for example, whenever a signal is measured (or sampled) at discrete times. A signal might be electrical, mechanical, optical, and so on. For convenience, we will call S the space of (discrete-time) signals.



Example: The vector spaces of polynomials of degree nth

For $n > 0$, the set P_n of polynomials of degree at most n consists of all polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

where the coefficients $a_0, a_1, \ldots, a_n$ and the variable x are real numbers. If all the coefficients are zero, p is called the *zero polynomial*.

If $q(x) = b_0 + b_1 x + \ldots + b_n x^n$, then we define

$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$$

$$(cp)(x) = ca_0 + (ca_1)x + (ca_2)x^2 + \ldots + (ca_n)x^n$$

Then P_n is a vector space.

Vector Spaces

Example: The vector space of all real-valued functions

Let V be the set of all real-valued functions defined on a set D. Addition of two functions and multiplication with scalar are defined pointwise

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x).$$

The zero vector in V is the function that is identically zero, i.e. $f(x) = 0$ for all x.
The axioms are easily shown to hold, hence V is a vector space.

Section 2

Subspaces, Linear Combination and Span

Subspaces

Definition

A *subspace* of a vector space V is a nonempty subset H of V that is

a. closed under vector addition, i.e. if $u, v \in H$ then $u + v \in H$.

b. and closed under multiplication by scalars, i.e. if $u \in H$ then $cu \in H$ for any $c \in \mathbb{R}$.

Subspaces

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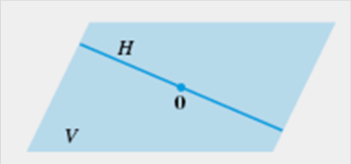
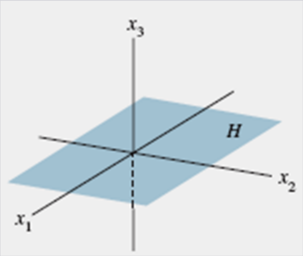
b. and closed under multiplication by scalars, i.e. if $u \in H$ then $cu \in H$ for any $c \in \mathbb{R}$.

Note: A subspace H of V must contain the zero vector.

Subspaces

Example





(a) A line H through 0 is a subspace of $V = \mathbb{R}^2$.
(b)The x_1x_2-plane

$$H = \{(x, y, 0) : x, y \in \mathbb{R}\}$$

is a subspace of $V = \mathbb{R}^3$.

Subspaces

Example

The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{\mathbf{0}\}$.

Example

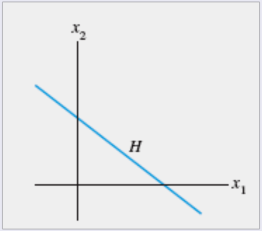
Let P be the set of all polynomials with real coefficients, with operations in P defined as for functions. Then P is a subspace of the space of all real-valued functions defined on $\mathbb{R}$. Also, for each $n > 0$, P_n is a subspace of P.

Subspaces

Example

A line in $\mathbb{R}^2$ not containing the origin is not a subspace of $\mathbb{R}^2$.

A plane in $\mathbb{R}^3$ not containing the origin is not a subspace of $\mathbb{R}^3$.



Subspaces

Example

Which of the given subsets of the vector space P_2 are subspace?
(a) $a_2t^2 + a_1t + a_0, \quad \text{where } a_1 = 0, a_0 = 0$
(b) $a_2t^2 + a_1t + a_0, \quad \text{where } a_1 = 2a_0$
(c) $a_2t^2 + a_1t + a_0, \quad \text{where } a_2 + a_1 + a_0 = 2$

Subspaces

Example

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(c) $a_2t^2 + a_1t + a_0, \quad \text{where } a_2 + a_1 + a_0 = 2$

Answers: (a) and (b).

Subspaces

Exercises

Let W be the set of all 3×3 matrices of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$$

Show that W is a subspace of M_{33}.

Subspace

Exercises

Which of the following subsets of the vector space M_{mn} are subspaces?

(a) The set of all $n \times n$ symmetric matrices.

(b) The set of all $n \times n$ diagonal matrices.

(c) The set of all $n \times n$ invertible matrices.

Subspace

Exercises

Which of the following subsets of the vector space M_{mn} are subspaces?
(a) The set of all $n \times n$ symmetric matrices.
(b) The set of all $n \times n$ diagonal matrices.
(c) The set of all $n \times n$ invertible matrices.

Answer: The subsets in (a) and (b) are subspaces.
The subset in (c) is not, because it does not contain the zero matrix.

Null Spaces

Example

if A is an $m \times n$ matrix, then the homogeneous system of m equations in n unknowns with coefficient matrix A can be written as

$$Ax = 0$$

where x is a vector in $\mathbb{R}^n$ and 0 is the zero vector. Show that set W of all solutions is a subspace of $\mathbb{R}^n$.

Null Spaces

Example

if A is an $m \times n$ matrix, then the homogeneous system of m equations in n unknowns with coefficient matrix A can be written as

$$Ax = 0$$

where x is a vector in $\mathbb{R}^n$ and 0 is the zero vector. Show that set W of all solutions is a subspace of $\mathbb{R}^n$.

W is called the solution space of the homogeneous system, or the *null space* of the matrix A.

Linear Combinations

Definition

Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V. A vector v in V is a linear combination of $v_1, v_2, \ldots, v_k$ if

$$v = a_1 v_1 + a_2 v_2 + + \ldots + a_k v_k = \sum_{j=1}^{k} a_j v_j$$

where $a_j \in \mathbb{R}$.

Linear Combinations

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where $a_j \in \mathbb{R}$.

Example

If $v_1 = [1\,0\,1]^T$ and $v_2 = [0\,1\,1]^T$ then every $w = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ is a linear combination of v_1, v_2 since $w = av_1 + bv_2$.

Subspace Spanned by a Set

The next theorem gives one of the most common ways to define a subspace.

Theorem

If $v_1, v_2, ..., v_k$ are vectors in a space V , then

$$span\{v_1, v_2, ..., v_k\} = \{a_1 v_1 + a_2 v_2 + + ... + a_k v_k : a_j \in \mathbb{R}\}$$

is a subspace of V.

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is a subspace of V.

We call $span\{v_1, v_2, ..., v_k\}$ the subspace spanned (or generated) by $\{v_1, v_2, ..., v_k\}$

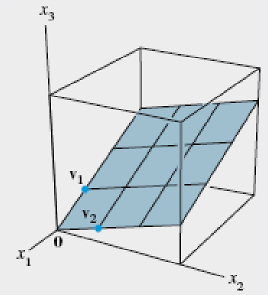
A Subspace Spanned by a Set

Example

Given v_1 and v_2 in $\mathbb{R}^3$,

$$H = span\{v_1, v_2\} = \{av_1 + bv_2 : a, b \in \mathbb{R}\}$$

is a plane, which is a subspace of $\mathbb{R}^3$.



A Subspace Spanned by a Set

Example

Consider the set S of 2×3 matrices given by

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Then span S is the set in M_{23} consisting of all vectors of the form

$$\begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & a_4 \end{bmatrix}$$

where $a_j \in \mathbb{R}$.

A Subspace Spanned by a Set

Example

Let $S = \{t^2, t, 1\}$, then we have span S be a subset of P_2. Then span S is the subspace of all polynomials of the form $a_2t^2 + a_1t + a_0$.

Example

Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then span S is the subspace of all all 2×2 diagonal matrices .

A Subspace Spanned by a Set

Example

Let
$$H = \left\{ (a - 3b, b - a, a, b)^T : a, b \in \mathbb{R} \right\}$$
Show that H is a subspace of $\mathbb{R}^4$.

A Subspace Spanned by a Set

Example

Let

$$H = \left\{ (a - 3b, b - a, a, b)^T : a, b \in \mathbb{R} \right\}$$

Show that H is a subspace of $\mathbb{R}^4$.

Proof:
An arbitrary vector in H has the form

$$\begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $H = span\left\{v_1, v_2\right\}$, where

$$v_1 = (1, -1, 1, 0)^T, \;\; v_2 = (-3, 1, 0, 1)^T.$$

Hence H is a subspace of $\mathbb{R}^4$.

A Subspace Spanned by a Set

Example

Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Is $v \in span\,\{v_1, v_2, v_3\}$?

A Subspace Spanned by a Set

Example

Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Is $v \in span \{v_1, v_2, v_3\}$?

Solution: Find a_1, a_2, a_3 such that

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Solve this linear system to obtain $a_1 = 1, a_2 = 2, a_3 = -1$.
Thus, $v = v_1 + 2v_2 - v_3$ so $v \in span \{v_1, v_2, v_3\}$

A Subspace Spanned by a Set

Example

For what value(s) of a will v be in the subspace of $\mathbb{R}^3$ spanned by v_1, v_2, v_3, if

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ and } v = \begin{bmatrix} -4 \\ 3 \\ a \end{bmatrix}$$

A Subspace Spanned by a Set

Example

For what value(s) of a will v be in the subspace of $\mathbb{R}^3$ spanned by v_1, v_2, v_3, if

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ and } v = \begin{bmatrix} -4 \\ 3 \\ a \end{bmatrix}$$

Answer: $a = 5$.

A Subspace Spanned by a Set

Example

In P_2 let

$$v_1 = 2t^2+t+2,\, v_2 = t^2-2t,\, v_3 = 5t^2-5t+2,\, v_4 = -t^2-3t-2$$

Determine whether the vector

$$v = t^2 + t + 2$$

belongs to span $\{v_1, v_2, v_3, v_4\}$.

A Subspace Spanned by a Set

Solution: Find scalars a_1, a_2, a_3, a_4 such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = v$$

$$(2a_1 + a_2 + 5a_3 - a_4)\, t^2 + (a_1 - 2a_2 - 5a_3 - 3a_4)\, t + (a_1 + 2a_3 - 2a_4) = t^2 + t + 2$$

Thus we get the linear system:

$$2a_1 + a_2 + 5a_3 - a_4 = 1$$

$$a_1 - 2a_2 - 5a_3 - 3a_4 = 1$$

$$a_1 + 2a_3 - 2a_4 = 2$$

A Subspace Spanned by a Set

To determine whether this system of linear equations is consistent. We form the augmented matrix and transform it to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

which indicates that the system is inconsistent; that is, it has no solution. Hence v does not belong to span $\{v_1, v_2, v_3, v_4\}$.

A Subspace Spanned by a Set

Example

Let V be the vector space R^3. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Show that $span\left\{v_1, v_2, v_3\right\} = \mathbb{R}^3$.

A Subspace Spanned by a Set

Example

Let V be the vector space R^3. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Show that $span\left\{v_1, v_2, v_3\right\} = \mathbb{R}^3$.

Solution:
Pick any

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in V$$

Solution (Cont.):
This leads to the linear system

$$a_1 + a_2 + a_3 = a$$

$$2a_1 + a_3 = b$$

$$a_1 + 2a_2 = c$$

A solution is (verify)

$$a_1 = \frac{-2a + 2b + c}{3}, a_2 = \frac{a - b + c}{3}, a_3 = \frac{4a - b - 2c}{3}$$

A Subspace Spanned by a Set

Example

Explain why the set S is not a spanning set for the vector space V.

(a) $S = \{t^3, t^2, t\}$, $V = P_3$

(b)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}, V = \mathbb{R}^2$$

(c)

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, V = M_{22}$$

Section 3

Linear Independence

Linear Independence

Definition

The vectors $v_1, v_2, v_3, ..., v_k$ in a vector space V are said to be linearly dependent if there exist constants $a_1, a_2, ..., a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j v_j = \mathbf{0}$$

Otherwise, $v_1, v_2, v_3, ..., v_k$ are called linearly independent.

That is, $v_1, v_2, v_3, ..., v_k$ are linearly independent if $\sum_{j=1}^{k} a_j v_j = \mathbf{0} \Leftrightarrow a_j = 0, \forall j = 1, .., k$.

Linear Independence

Example

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

are linearly dependent since $v_1 + v_2 - v_3 = \mathbf{0}$.

Example

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

are linearly *independent* since $av_1 + bv_2 = \mathbf{0}$ iff $a = b = 0$.

Linear Independence

Example

Determine whether the vectors

$$v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

are linearly independent.

Linear Independence

Example

Determine whether the vectors

$$v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

are linearly independent.

Solution

Forming equation:

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \mathbf{0}$$

$$a_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Independence

We obtain the homogeneous system

$$3a_1 + a_2 - a_3 = 0$$

$$2a_1 + 2a_2 + 2a_3 = 0$$

$$a_1 - a_3 = 0$$

Doing the row operations

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{Row\ \ operations} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The nontrivial solution is

$$k\,(1,-2,1)^T\,, k \neq 0$$

so the vectors are linearly dependent!

Linear Independence

Example

Determine whether the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are linearly independent.

Linear Independence

Example

Determine whether the vectors

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are linearly independent.

Solution

Forming equation:

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear Independence

Doing the row operations

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus the only solution is the trivial solution $a_1 = a_2 = a_3 = 0$, so the vectors are linearly independent.

Linear Independence

Example

Are the vectors

$$v_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

in M_{22} linearly independent?

Linear Independence

Example

Are the vectors

$$v_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

in M_{22} linearly independent?

Solution:

Setting up the equation:

$$a_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Linear Independence

$$\begin{bmatrix} 2a_1 + a_2 & a_1 + 2a_2 + a_3 \\ a_2 - 2a_3 & a_1 + a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving the linear system to find a_j:

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{Row\ \ operations} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The nontrivial solution is

$$k\,(-1, 2, 1)^T, k \neq 0$$

so the vectors are linearly dependent.

Linear Independence

Example

Are the vectors

$$v_1 = t^2 + t + 2, v_2 = 2t^2 + t, v_3 = 3t^2 + 2t + 2$$

in P_2 linearly independent?

Linear Independence

Example

Are the vectors

$$v_1 = t^2 + t + 2, v_2 = 2t^2 + t, v_3 = 3t^2 + 2t + 2$$

in P_2 linearly independent?

Answer: The given vectors are linearly dependent

Linear Independence

Theorem

Let $S = \{v_1, v_2, ..., v_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let A be the matrix whose columns (rows) are the elements of S. Then S is linearly independent if and only if $\det(A) \neq 0$.

Linear Independence

Theorem

Let $S = \{v_1, v_2, ..., v_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let A be the matrix whose columns (rows) are the elements of S. Then S is linearly independent if and only if $\det(A) \neq 0$.

Proof:

We will prove the result for column-vectors.

Suppose that S is linearly independent. Then it follows that the reduced row echelon form of A is I_n. Thus, A is row equivalent to I_n, and hence $\det(A) \neq 0$.

Conversely, if $\det(A) \neq 0$, then A is row equivalent to I_n. Hence, the rows of A are linearly independent.

Linear Independence

Example

Is $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$ a linearly independent set of vector in $\mathbb{R}^3$?

Linear Independence

Example

Is $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$ a linearly independent set of vector in $\mathbb{R}^3$?

Solution

We form the matrix A whose columns are the vectors in S:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

$$\det(A) = 2$$

So S is linearly independent.

Linear Independence

Theorem

Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2. Then the following statements are true:

(a) If S_1 is linearly dependent, so is S_2.

(b) If S_2 is linearly independent, so is S_1.

Linear Independence

Theorem

Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2. Then the following statements are true:

(a) If S_1 is linearly dependent, so is S_2.

(b) If S_2 is linearly independent, so is S_1.

Proof: Let

$$S_1 = \{v_1, v_2, ..., v_k\}, S_2 = \{v_1, v_2, ..., v_k, v_{k+1}, .., v_m\}$$

(a) Since S_1 is linearly dependent, there exist constants $a_1, a_2, ..., a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j v_j = \mathbf{0}$$

Linear Independence

Proof (Cont.) Therefore,

$$a_1v_1 + a_2v_2 + \ldots + a_kv_k + 0v_{k+1} + \ldots + 0v_m = \mathbf{0}$$

Since not all the coefficients in the equations above are zero, we conclude that S_2 is linearly dependent.

Statement (b) is the contrapositive of statement (a), so it is logically equivalent to statement (a).

Linear Independence

Remarks

- The set $S = \{\mathbf{0}\}$ consisting of only the vector $\mathbf{0}$ is linearly dependent.

  From this it follows that if S is any set of vectors that contains 0, then S must be linearly dependent.

- A set of vectors consisting of a *single nonzero* vector is linearly independent.

- If $v_1, v_2, \ldots, v_k$ are vectors in a vector space V and for some $i \neq j$, $v_i = v_j$, then $v_1, v_2, \ldots, v_k$ are linearly dependent.

Linear Independence

Theorem

The nonzero vectors $v_1, v_2, ..., v_k$ in a vector space V are linearly dependent if and only if one of the vectors $v_j (j \geq 2)$ is a linear combination of the preceding vectors $v_1, v_2, ..., v_{j-1}$.

Linear Independence

Theorem

The nonzero vectors $v_1, v_2, \ldots, v_k$ in a vector space V are linearly dependent if and only if one of the vectors $v_j (j \geq 2)$ is a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{j-1}$.

Example

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 + 0v_3 - v_4 = \mathbf{0}$$

so v_1, v_2, v_3, and v_4 are linearly dependent. We then have

$$v_4 = v_1 + v_2 + 0v_3.$$

Section 4

Basis and Dimensions

Basis

Definition

The vectors $v_1, v_2, ..., v_k$ in a vector space V are said to form a basis for V if
(a) $v_1, v_2, ..., v_k$ span V and
(b) $v_1, v_2, ..., v_k$ are linearly independent.

Basis

Definition

The vectors $v_1, v_2, \ldots, v_k$ in a vector space V are said to form a basis for V if
(a) $v_1, v_2, \ldots, v_k$ span V and
(b) $v_1, v_2, \ldots, v_k$ are linearly independent.

Example

Let $V = \mathbb{R}^3$. The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$, called the natural basis or standard basis for $\mathbb{R}^3$.

Basis

Example

Generally, the natural basis or standard basis for $\mathbb{R}^n$ is denoted by

$$\{e_1, e_2, \dots, e_n\}$$

where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Basis

Example

Show that the set

$$S = \left\{t^2 + 1, t - 1, 2t + 2\right\}$$

is a basis for the vector space P_2.

Basis

Example

Show that the set

$$S = \left\{t^2 + 1, t - 1, 2t + 2\right\}$$

is a basis for the vector space P_2.

Solution We must show that S spans V and is linearly independent.

To show that it spans V, we take any vector in V, that is a polynomial $at^2 + bt + c$ and find constants a_1, a_2 and a_3 such that

$$at^2 + bt + c = a_1\left(t^2 + 1\right) + a_2\left(t - 1\right) + a_3\left(2t + 2\right)$$

Basis

We find

$$a_1 = a, a_2 = \frac{a+b-c}{2}, a_3 = \frac{c+b-a}{4}.$$

Hence S spans V.
To show that S is linearly independent, we solve

$$a_1\left(t^2+1\right) + a_2\left(t-1\right) + a_3\left(2t+2\right) = \mathbf{0}$$

$$a_1 t^2 + \left(a_2 + 2a_3\right)t + \left(a_1 - a_2 + 2a_3\right) = \mathbf{0}$$

This can hold for all values of t only if

$$a_1 = a_2 + 2a_3 = a_1 - a_2 + 2a_3 = 0$$

Thus $a_1 = a_2 = a_3 = 0$.

Basis

We find

$$a_1 = a, a_2 = \frac{a+b-c}{2}, a_3 = \frac{c+b-a}{4}.$$

Hence S spans V.
To show that S is linearly independent, we solve

$$a_1\left(t^2+1\right) + a_2\left(t-1\right) + a_3\left(2t+2\right) = \mathbf{0}$$

$$a_1 t^2 + \left(a_2 + 2a_3\right)t + \left(a_1 - a_2 + 2a_3\right) = \mathbf{0}$$

This can hold for all values of t only if

$$a_1 = a_2 + 2a_3 = a_1 - a_2 + 2a_3 = 0$$

Thus $a_1 = a_2 = a_3 = 0$.
Remark: The set $S = \{t^n, t^{n-1}, .., t, 1\}$ forms a basis for the vector space P_n. S is called the *natural, or standard basis*, for P_n.

Basis

Example

Show that the set

$$S = \{v_1, v_2, v_3, v_4\}$$

where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is a basis for the vector space $\mathbb{R}^4$.

Hint

(a) To show that S spans $\mathbb{R}^4$, we let

$$v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

in $\mathbb{R}^4$ and find a_1, a_2, a_3 and a_4 such that
$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$.
(b) S is linearly independent since $\det(A) = 1$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

Basis

Example

Find a basis for the subspace V of P_2, consisting of all vectors of the form at $at^2 + bt + c$ where $c = a - b$.

Hint:

$$S = \left\{t^2 + 1, t - 1\right\}$$

Basis

Example

Find a basis for the subspace V of P_2, consisting of all vectors of the form at $at^2 + bt + c$ where $c = a - b$.

Hint:

$$S = \left\{t^2 + 1, t - 1\right\}$$

Remarks

A vector space V is called *finite-dimensional if there is a finite subset of V that is a basis for V*. If there is no such finite subset of V, then V is called infinite-dimensional.

Basis

Theorem

If $S = \{v_1, v_2, .., v_n\}$ is a basis for a vector space V, then every vector in V can be written in *one and only one* way as a linear combination of the vectors in S.

Basis

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Theorem

Let $S = \{v_1, v_2, .., v_n\}$ be a set of nonzero vectors in a vector space V and let $W = $span$(S)$. Then some subset of S is a basis for W.

Basis

Theorem

If $S = \{v_1, v_2, .., v_n\}$ is a basis for a vector space V, then every vector in V can be written in *one and only one* way as a linear combination of the vectors in S.

Theorem

Let $S = \{v_1, v_2, .., v_n\}$ be a set of nonzero vectors in a vector space V and let W =span(S). Then some subset of S is a basis for W.

Procedure for finding a basis

- Let A be the matrix with columns $v_1, \ldots, v_n$.

Basis

Theorem

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Procedure for finding a basis

- Let A be the matrix with columns $v_1, \ldots, v_n$.
- Use row transformations to bring A to row echelon form.

Basis

Theorem

If $S = \{v_1, v_2, .., v_n\}$ is a basis for a vector space V, then every vector in V can be written in *one and only one* way as a linear combination of the vectors in S.

Theorem

Let $S = \{v_1, v_2, .., v_n\}$ be a set of nonzero vectors in a vector space V and let W =span(S). Then some subset of S is a basis for W.

Procedure for finding a basis

- Let A be the matrix with columns $v_1, \ldots, v_n$.
- Use row transformations to bring A to row echelon form.
- The set of vectors v_j's corresponding to the pivot columns of the row echelon form is a basis for span(S).

Example

Find a basis for span(S) where $S = \{v_1, v_2, .., v_5\}$, where

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

Find a basis for span(S) where $S = \{v_1, v_2, .., v_5\}$, where

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: A is already in echelon form and its pivot columns have indices 1, 3, and 5. Thus,

$$B = \{v_1, v_3, v_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is a basis for span(S).

Example

Let

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}.$$

Find a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.

Example

Let

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}.$$

Find a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.
Solution: We have

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two column are the pivot columns. Hence $S = \{v_1, v_2\}$ is a basis for W.

Basis

Theorem

Let $S = \{v_1, v_2, .., v_n\}$ be a basis for a vector space V, and $T = \{w_1, w_2, .., w_m\}$.

- If T is a linearly independent then $m \leq n$.
- If T spans V then $m \geq n$.

Basis

Theorem

Let $S = \{v_1, v_2, .., v_n\}$ be a basis for a vector space V, and $T = \{w_1, w_2, .., w_m\}$.

- If T is a linearly independent then $m \leq n$.
- If T spans V then $m \geq n$.

Corollary

If $S = \{v_1, v_2, .., v_n\}$ and $T = \{w_1, w_2, .., w_m\}$ are bases for a vector V, then $n = m$.

Basis

Definition

Let S be a set of vectors in a vector space V. A subset T of S is called a maximal independent subset of S if T is a linearly independent set of vectors that is not properly contained in any other linearly independent subset of S.

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Example

Let
$$S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$
Then maximal independent subsets of S are $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_1, v_3\}$.

Basis

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Let S be a set of vectors in a vector space V. A subset T of S is called a maximal independent subset of S if T is a linearly independent set of vectors that is not properly contained in any other linearly independent subset of S.

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Then maximal independent subsets of S are $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_1, v_3\}$.

Theorem

Let S be a finite subset of the vector space V that spans V. A maximal independent subset T of S is a basis for V.

Dimensions

Definition

The dimension of a nonzero vector space V is the number of vectors in a basis for V, denoted by $\dim V$.

We also define the dimension of the trivial vector space $\{\mathbf{0}\}$ to be zero.

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The dimension of a nonzero vector space V is the number of vectors in a basis for V, denoted by $\dim V$.

We also define the dimension of the trivial vector space $\{\mathbf{0}\}$ to be zero.

Example

- If $S = \{t^2, t, 1\}$ is a basis for P_2, so $\dim\ P_2 = 3$.
- $\dim \mathbb{R}^n = n$.
- $\dim M_{m,n} = mn$.

For a set of exactly $\dim V$ vectors, only one of the two conditions for being a basis is needed. I.e., For a set of exactly $\dim V$ vectors, only one of the two conditions for being a basis is needed. I.e.,

Theorem

Let V be an n-dimensional vector space.

- If $S = \{v_1, v_2, ..., v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
- If $S = \{v_1, v_2, ..., v_n\}$ spans V, then S is a basis for V.

Section 5

Row and Column Spaces - Rank of a Matrix

Row and Column Spaces

Definition

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

be an $m \times n$ matrix.

The subspace of $\mathbb{R}^n$ spanned by the rows of A is called the row space of A, denoted by $Row\ A$.

The subspace of $\mathbb{R}^m$ spanned by the columns of A is called the column space of A, denoted by $Col\ A$.

Definition

The dimension of the row (column) space of A is called the row (column) rank of A.

Row and Column Spaces

Theorem

If A and B are two $m \times n$ row (column) equivalent matrices, then the row (column) spaces of A and B are equal.

Row and Column Spaces

Theorem

If A and B are two $m \times n$ row (column) equivalent matrices, then the row (column) spaces of A and B are equal.

Note: To find a basis for Row(A), use row operations to reduce it to B in echelon form. The non-zero rows of B form a basis for Row(A).

Row and Column Spaces

Theorem

If A and B are two $m \times n$ row (column) equivalent matrices, then the row (column) spaces of A and B are equal.

Note: To find a basis for Row(A), use row operations to reduce it to B in echelon form. The non-zero rows of B form a basis for Row(A).

Example

Find the row space, the null space, and the column space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

To find a basis for row/column space of a matrix A:

- Use row operations to reduce it to an echelon form B.

Note the difference: For the row space, we must use the rows of B. For the column space, we must use the columns of A.

To find a basis for row/column space of a matrix A:

- Use row operations to reduce it to an echelon form B.
- The non-zero rows of B form a basis for $\mathrm{Row}(A)$.

Note the difference: For the row space, we must use the rows of B. For the column space, we must use the columns of A.

To find a basis for row/column space of a matrix A:

- Use row operations to reduce it to an echelon form B.
- The non-zero rows of B form a basis for Row(A).
- Find the pivot columns of B (the columns that contain a pivot). The corresponding columns of A form a basis for Col(A)

Note the difference: For the row space, we must use the rows of B. For the column space, we must use the columns of A.

Solution

We have

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of B form a basis for for Row(A).
For Col(A), observe that the pivots of B are in columns $1, 2, 4$.
So a basis for Col(A) is

$$\left\{ [-2, 1, 3, 1]^T, [-5, 3, 11, 7]^T, [0, 1, 7, 5]^T \right\}$$

For the null space of A, we solve $Bx = 0$. Corresponding to nonpivot columns, $x_3 = s$ and $x_5 = t$ are arbitrary. By back substitution, we obtain

$$x_1 = -s - t, x_2 = 2s - 3t, x_4 = 5t.$$

A basis for Null(A) is $\{[-1, 2, 1, 0, 0]^T, [-1 - 3, 0, 5, 1]^T\}$.

Rank-Nullity

Theorem

Let A be an $m \times n$ matrix. Then

- The row rank and column rank of A are equal. They equal the number of pivot columns of the echelon form of A.
- The nullity of A equals the number of non-pivot columns of the echelon form of A.
- It follows that rank A + nullity $A = n$.

Rank-Nullity

Theorem

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- It follows that $\text{rank } A + \text{nullity } A = n$.

Definition

The nullity of A is the dimension of the null space of A, that is, the dimension of the solution space of $Ax = 0$.

Rank-Nullity

Theorem

Let A be an $m \times n$ matrix. Then

- The row rank and column rank of A are equal. They equal the number of pivot columns of the echelon form of A.
- The nullity of A equals the number of non-pivot columns of the echelon form of A.
- It follows that $\operatorname{rank} A + \operatorname{nullity} A = n$.

Definition

The nullity of A is the dimension of the null space of A, that is, the dimension of the solution space of $Ax = 0$.

In the previous example, $\operatorname{nullity}(A) = 2$, $\operatorname{rank}(A) = 3$.

Rank-Nullity

Example

Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix}$$

We have

$$A \sim B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, nullity A=1, rank A $= 2$ and nullity A $+$ rank A $= 3 =$ numbers of column of A.

Rank-Nullity

Example

Let

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

We have

$$A \sim B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, nullity A=2, rank A $= 3$ and nullity A $+$ rank A $= 5 =$ number of columns of A.

Rank and Invertibility

Suppose A is a square matrix of size n. Since rank A is the number of pivot columns of A, it follows that rank $A = n$ if and only if A is invertible. Thus,

Corollary

Let A be an $n \times n$ matrix. The following are equivalent

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Corollary

Let A be an $n \times n$ matrix. The following are equivalent

(a) A is invertible

(b) $\det(A) \neq 0$.

Rank and Invertibility

Suppose A is a square matrix of size n. Since rank A is the number of pivot columns of A, it follows that rank $A = n$ if and only if A is invertible. Thus,

Corollary

Let A be an $n \times n$ matrix. The following are equivalent

(a) A is invertible

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(e) rank $A = n$.

Section 6

Coordinates and Changes of Bases

Coordinates

Definition: Coordinates

Let V be an n-dimensional vector space, with a basis

$$S = \{v_1, v_2, ..., v_n\}.$$

Any vector $v \in V$ can be uniquely expressed in the form:

$$v = a_1 v_1 + a_2 v_2 + ... + a_n v_n.$$

We define

$$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

and call $[v]_S \in \mathbb{R}^n$ the coordinate vector of v with respect to the basis S.

Coordinates

Example

Consider the vector space $\mathbb{R}^3$ and let $S = \{v_1, v_2, v_3\}$ be the basis for $\mathbb{R}^3$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

If

$$v = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$$

compute $[v]_S$

Coordinates

Solution

To find $[v]_S$, we need to find the constants a_1, a_2, a_3 such that $a_1 v_1 + a_2 v_2 + a_3 v_3 = v$. Solve the linear system

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -5 \end{array}\right]$$

We get $a_1 = 3, a_2 = -1, a_3 = -2$. Thus,

$$[v]_S = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Isomorphism

Definition

Let V and W be vector spaces.

1. A map L from V to W is called a *linear map* if for any $u, v \in V$ and $c \in \mathbb{R}$
   (a) $L(u+v) = L(u) + L(v)$,
   (b) $L(cv) = cL(v)$.

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   (a) $L(u+v) = L(u) + L(v)$,
   (b) $L(cv) = cL(v)$.
2. A linear map L from V to W that is also a bijection (i.e. one-to-one and onto) is called an *isomorphism* between V and W.
3. If there is an isomorphism from V to W, we say that V is *isomorphic* to W.

Theorem

(a) Two finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal.

(b) If V is an $n-$dimensional real vector space, then V is isomorphic to $\mathbb{R}^n$.

Changes of bases - Transition matrices

Thus, let $S = \{v_1, v_2, ..., v_n\}$ and $T = \{w_1, w_2, ..., w_n\}$ be two ordered bases for the $n-$dimensional vector space V. Let v be a vector in V and let

$$[v]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Changes of bases - Transition matrices

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$$[v]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Question: Can we obtain $[v]_S$ from $[v]_T$?

We have

$$[v]_S = [c_1 w_1 + c_2 w_2 + ... + c_n w_n]_S$$

$$[v]_S = [c_1 w_1]_S + [c_2 w_2]_S + ... + [c_n w_n]_S$$

$$[v]_S = c_1 [w_1]_S + c_2 [w_2]_S + ... + c_n [w_n]_S$$

Changes of bases - Transition matrices

Let the coordinate vector of w_j with respect to S be denoted by

$$[w_j]_S = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

The $n \times n$ matrix whose jth column is $[w_j]_S$ is called the *transition matrix* (or the change-of-coordinates matrix) from the T-basis to the S-basis and is denoted by $P_{T \to S}$. That is,

$$P_{T \to S} = \left([w_1]_S \,, [w_2]_S \,, \ldots, [w_n]_S\right)$$

Therefore,

$$[v]_S = P_{T \to S}[v]_T$$

Changes of bases - Transition matrices

Example

Let $T = \{w_1, w_2\}$, $S = \{v_1, v_2\}$, where

$$w_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Find the transition matrix $P_{T \to S}$ from the T-basis to the S-basis.

Example

Let $T = \{w_1, w_2\}$, $S = \{v_1, v_2\}$, where

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Find the transition matrix $P_{T \to S}$ from the T-basis to the S-basis.

Solution: Let

$$[w_1]_S = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, [w_2]_S = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We need to solve the following linear systems

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = w_1, \quad \text{and} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = w_2$$

Example

We can solve both systems simultaneously.

$$\left[\begin{array}{cc|cc} v_1 & v_2 & w_1 & w_2 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

$$[w_1]_S = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, [w_2]_S = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Thus,

$$P_{T \to S} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Changes of bases - Transition matrices

Example

Let

$$T = \{w_1, w_2, w_3\}, S = \{v_1, v_2, v_3\},$$

where

$$w_1 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, w_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, w_3 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the transition matrix $P_{T \to S}$ from the T-basis to the S-basis.

To find $[w_j]_S$, $j = 1, 2, 3$, we can solve three systems simultaneously

$$\left[\begin{array}{ccc|ccc} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 6 & 4 & 5 \\ 0 & 2 & 1 & 3 & -1 & 5 \\ 1 & 0 & 1 & 3 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

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Changes of bases - Transition matrices

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$$P_{T \to S} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Q: Verify $[v]_S = P_{T \to S}[v]_T$?

Exercises

Section 7.4 (p. 287): 7-10, 12-21, 27-31.