Chapter 9
Z TRANSFORM

MOTIVATION BEHIND THE Z TRANSFORM

- Another important mathematical tool in the study of signals and systems is known as the z transform.
- The z transform can be viewed as a ***generalization of the (classical) Fourier transform***.
- Due to its more general nature, the z transform has a number of ***advantages*** over the (classical) Fourier transform.
- First, the z transform representation ***exists for some sequences that do not have a Fourier transform representation***. So, we can handle some sequences with the z transform that cannot be handled with the Fourier transform.
- Second, since the z transform is a more general tool, it can provide ***additional insights*** beyond those facilitated by the Fourier transform.

MOTIVATION BEHIND THE Z TRANSFORM

- Earlier, we saw that complex exponentials are eigensequences of LTI systems.
- In particular, for a LTI system $\mathcal{H}$ with impulse response h, we have that

$$\mathcal{H}\{z^n\}(n) = H(z)z^n \quad \text{where} \quad H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}.$$

- Previously, we referred to H as the system function.
- As it turns out, H is the z transform of h.
- Since the z transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.
- Furthermore, as we will see, the z transform has many additional uses.

BILATERAL Z TRANSFORM

- The (bilateral) **z transform** of the sequence x, denoted $\mathcal{Z}x$ or X, is defined as

$$\mathcal{Z}x(z) = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}.$$

- The **inverse z transform** of X, denoted $\mathcal{Z}^{-1}X$ or x, is then given by

$$\mathcal{Z}^{-1}X(n) = x(n) = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz,$$

  where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X.
- We refer to x and X as a **z transform pair** and denote this relationship as

$$x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z).$$

- In practice, we do not usually compute the inverse z transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

BILATERAL AND UNILATERAL Z TRANSFORM

- Two different versions of the z transform are commonly used:
  1. the ***bilateral*** (or ***two-sided***) z transform; and
  2. the ***unilateral*** (or ***one-sided***) z transform.
- The unilateral z transform is most frequently used to solve systems of linear difference equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral z transforms is in the ***lower limit of summation***.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0.
- For the most part, we will focus our attention primarily on the bilateral z transform.
- We will, however, briefly introduce the unilateral z transform as a tool for solving difference equations.
- Unless otherwise noted, all subsequent references to the z transform should be understood to mean ***bilateral*** z transform.

RELATIONSHIP BETWEEN Z AND FOURIER TRANSFORM

- Let X and X_{F} denote the z and (DT) Fourier transforms of x, respectively.
- The function $X(z)$ evaluated at $z = e^{j\Omega}$ (where Ω is real) yields $X_{\mathrm{F}}(\Omega)$. That is,

$$X(e^{j\Omega}) = X_{\mathrm{F}}(\Omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(e^{j\Omega})$.
- The function $X(z)$ evaluated at an arbitrary complex value $z = re^{j\Omega}$ (where $r = |z|$ and $\Omega = \arg z$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(re^{j\Omega}) = X'_{\mathrm{F}}(\Omega),$$

where X'_{F} is the (DT) Fourier transform of $x'(n) = r^{-n}x(n)$.
- So, in general, the z transform of x is the Fourier transform of an exponentially-weighted version of x.
- Due to this weighting, the z transform of a sequence may exist when the Fourier transform of the same sequence does not.

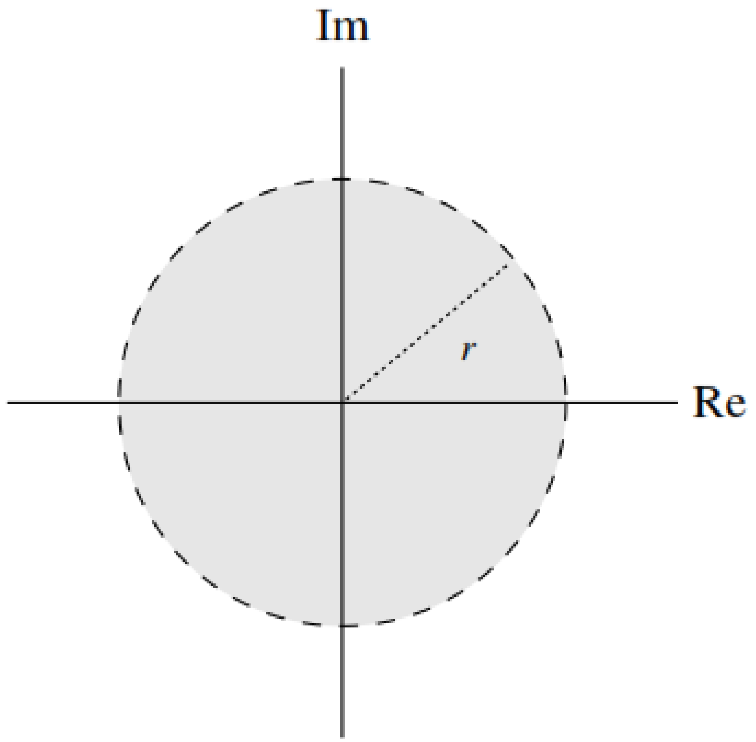
Region of Convergence (ROC)

DISK

- A **disk** with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| < r,$$

where r is a real constant and $r > 0$.

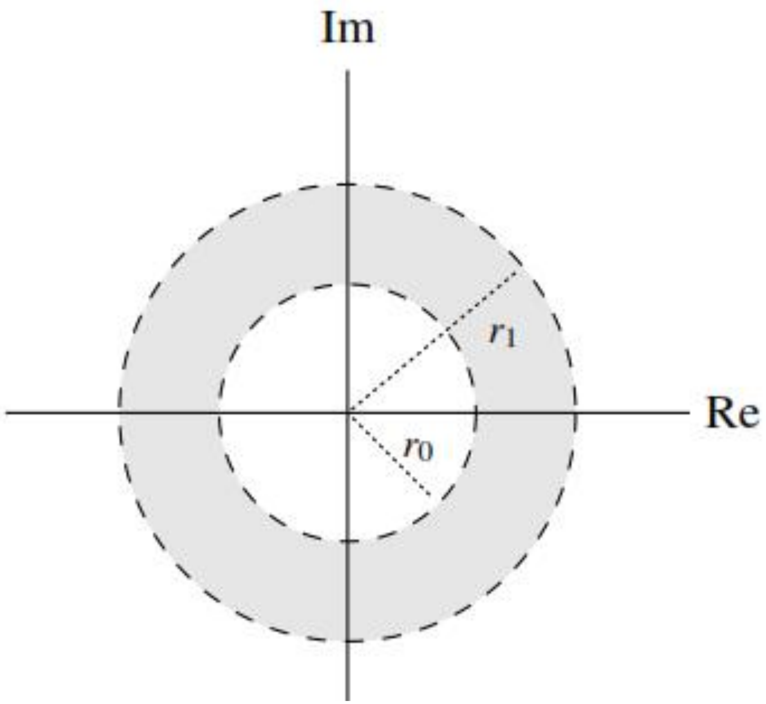


ANNULUS

- An **annulus** with center 0, inner radius r_0, and outer radius r_1 is the set of all complex numbers z satisfying

$$r_0 < |z| < r_1,$$

where r_0 and r_1 are real constants and $0 < r_0 < r_1$.

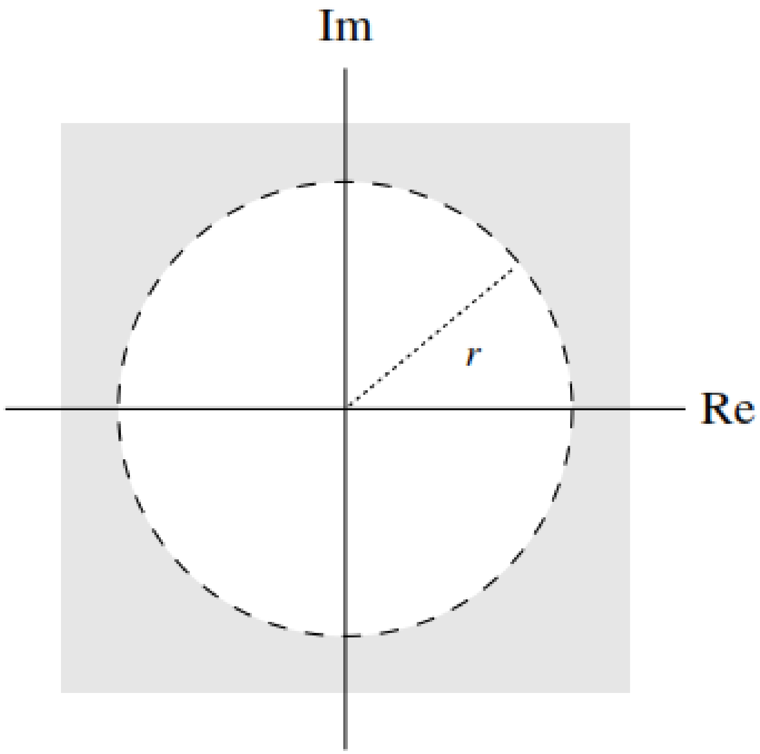


CIRCLE EXTERIOR

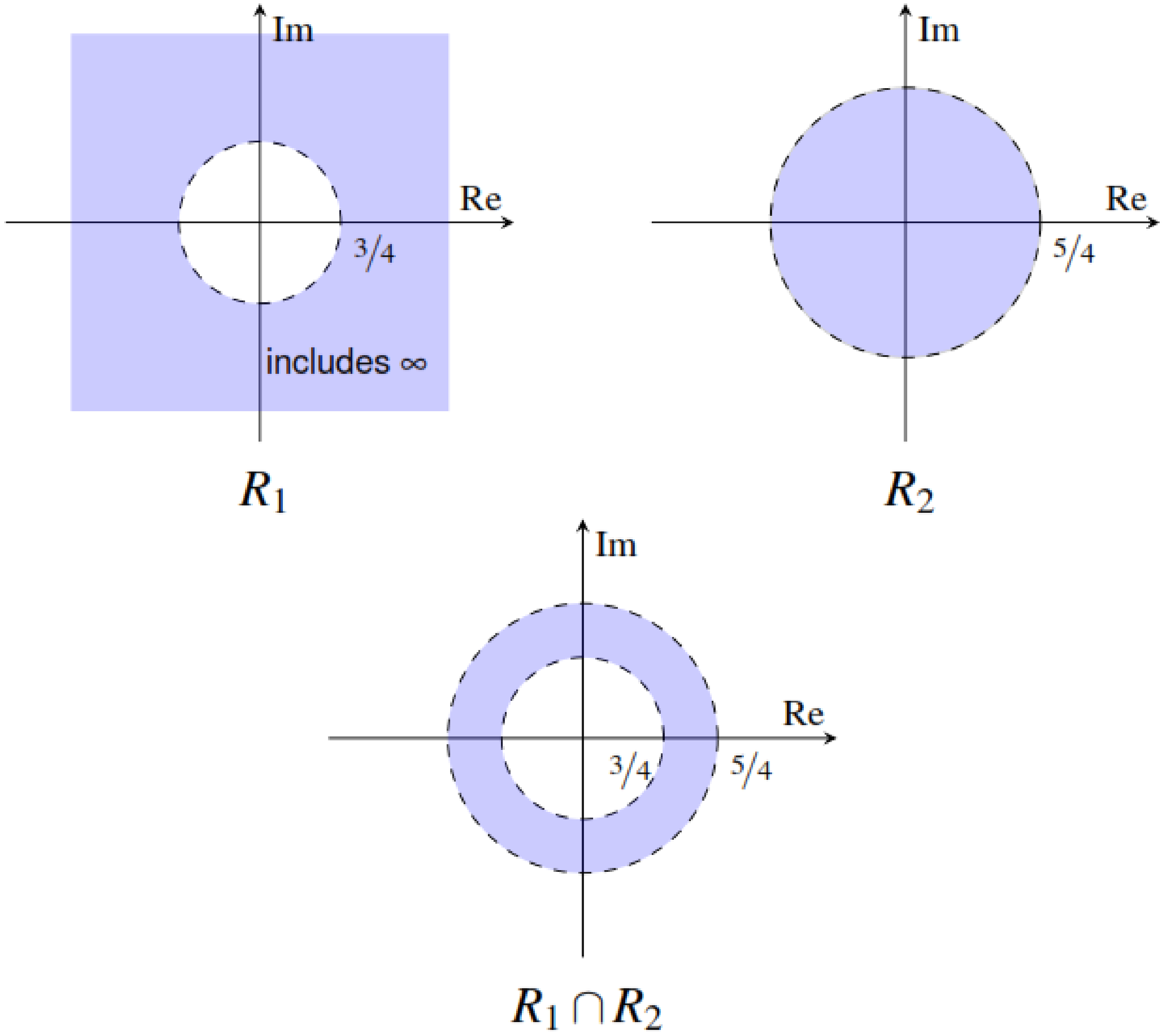
- The **exterior of a circle** with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| > r,$$

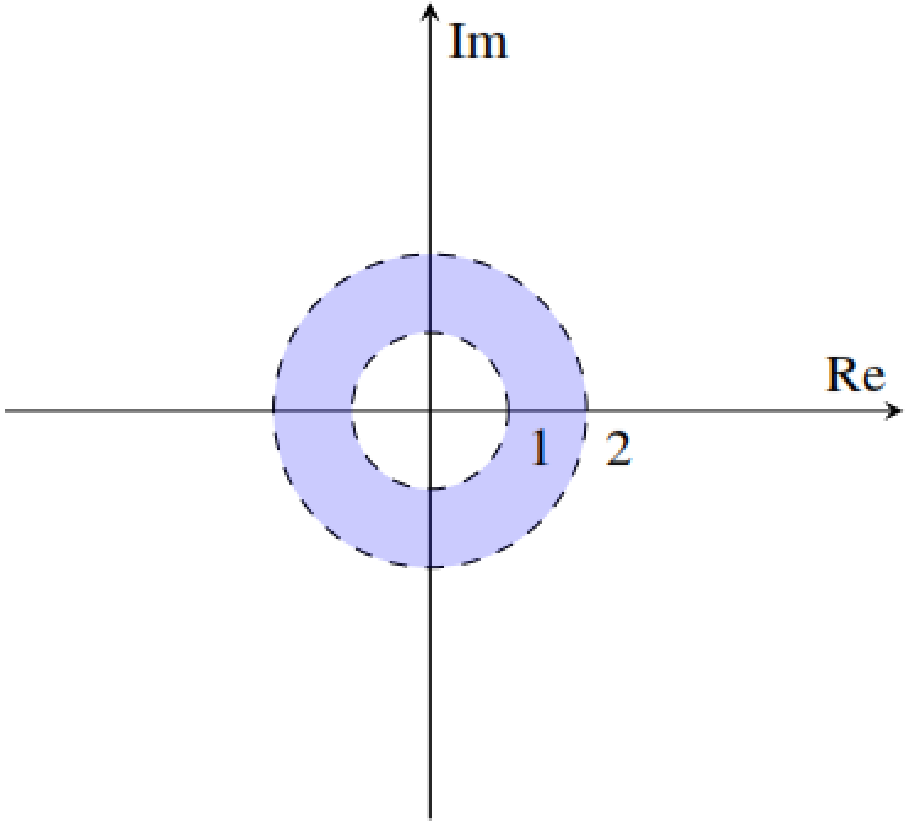
where r is a real constant and $r > 0$.



EXAMPLE: SET INTERSECTION



EXAMPLE: SCALAR MULTIPLE OF A SET



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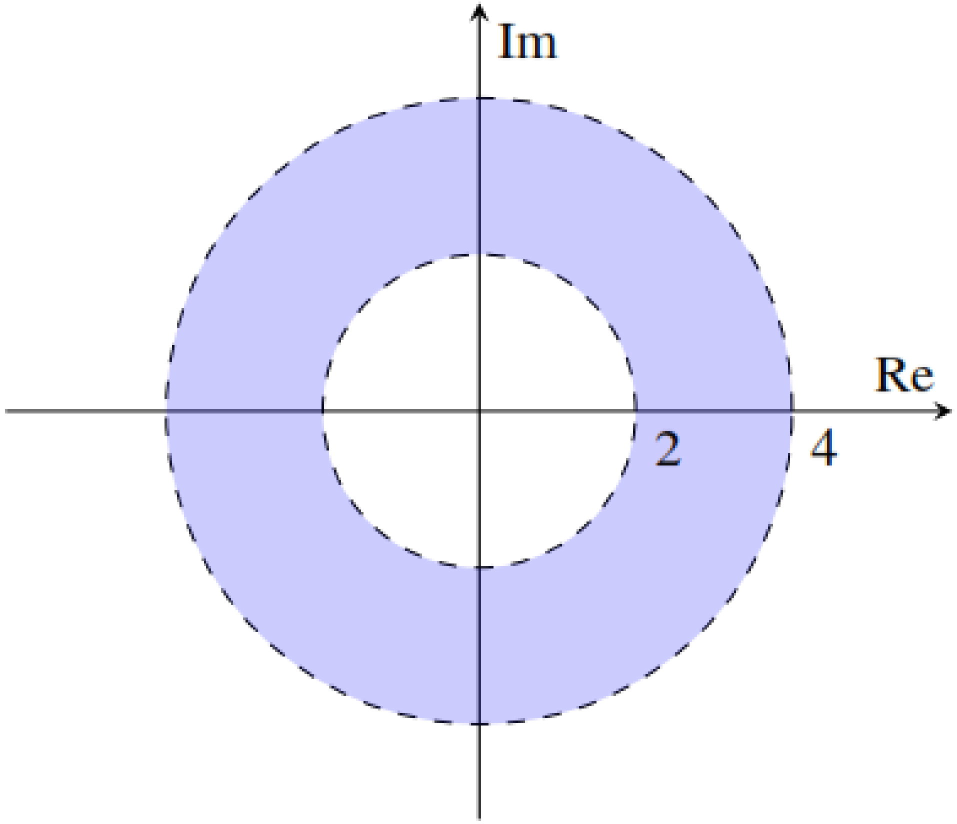
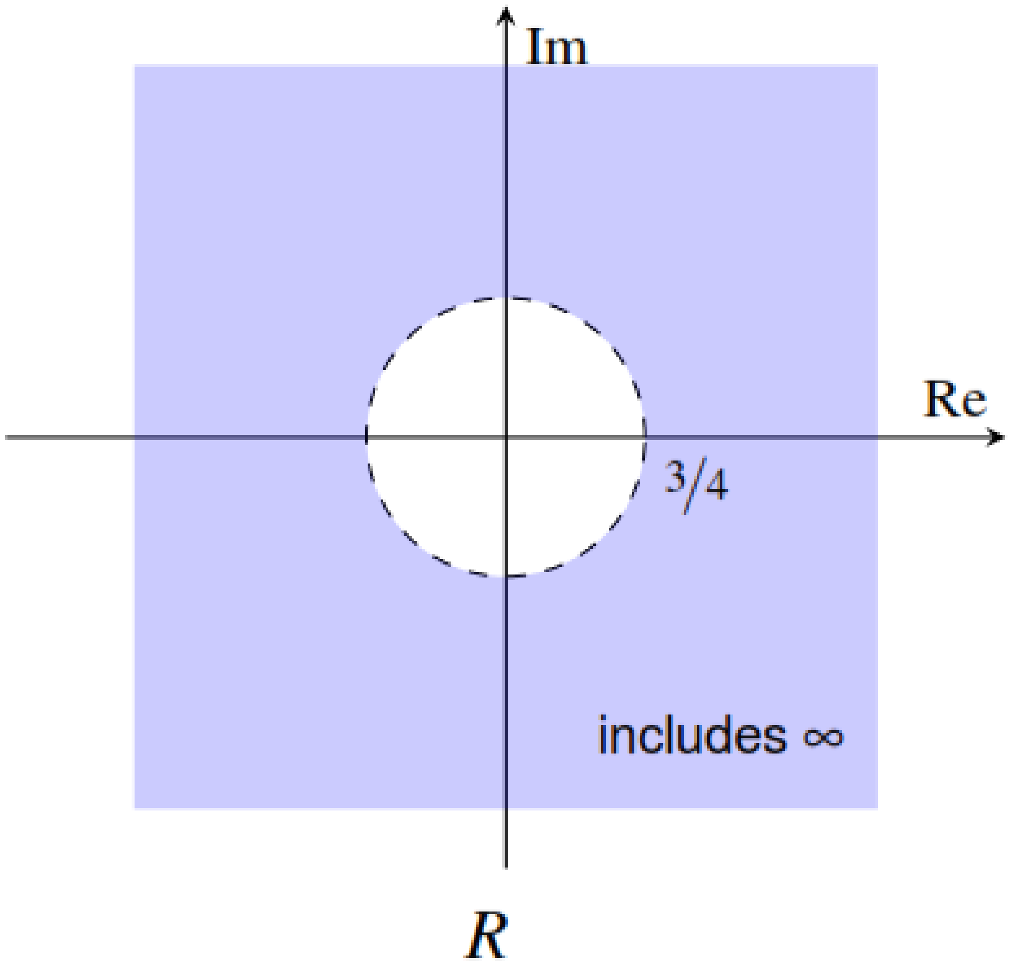
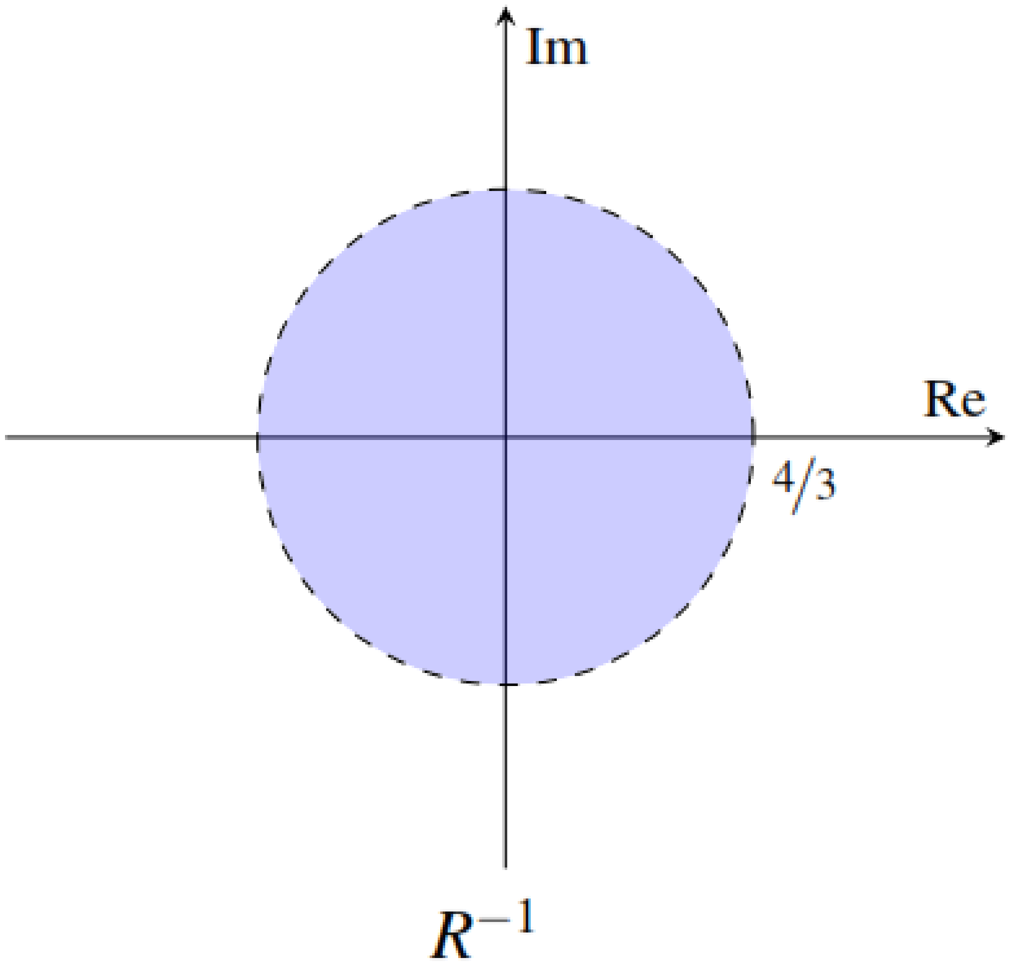


$2R$

EXAMPLE: RECIPROCAL OF A SET



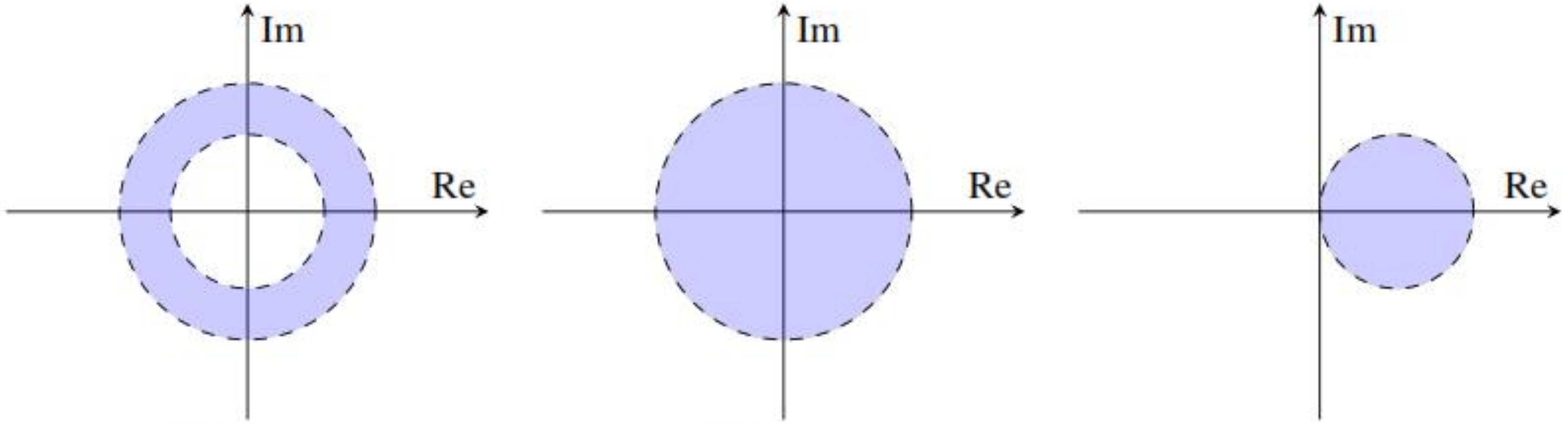


REGION OF CONVERGENCE (ROC)

- As we saw earlier, for a sequence x, the complete specification of its z transform X requires not only an algebraic expression for X, but also the ROC associated with X.
- Two very different sequences can have the same algebraic expressions for X.
- Now, we examine some of the constraints on the ROC (of the z transform) for various classes of sequences.

PROPERTY 1: GENERAL FORM

- The ROC of a z transform consists of *concentric circles centered at* 0 in the complex plane.
- That is, if a point z_0 is in the ROC, then the circle centered at 0 passing through z_0 (i.e., $|z| = |z_0|$) is also in the ROC.
- Some examples of sets that would be either valid or invalid as ROCs are shown below.



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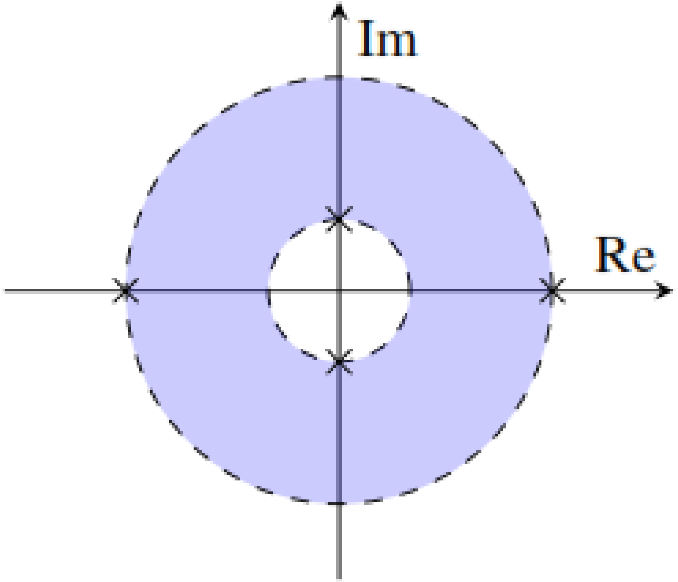
PROPERTY 2: RATIONAL Z TRANSFORMS

- If a z transform X is a ***rational*** function, then the ROC of X ***does not contain any poles*** and is ***bounded by poles or extends to infinity***.
- Some examples of sets that would be either valid or invalid as ROCs of rational z transforms are shown below.



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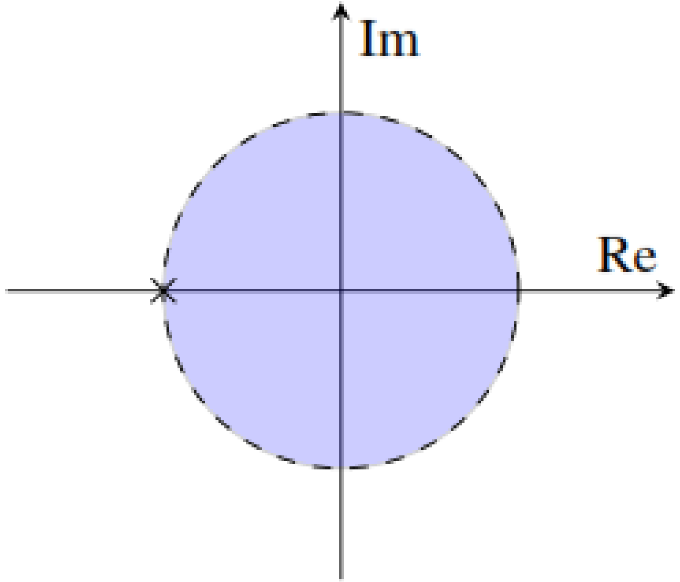

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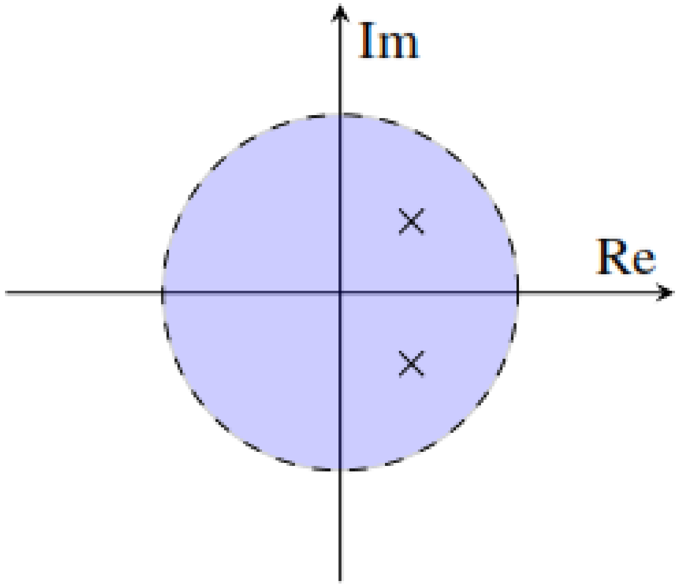

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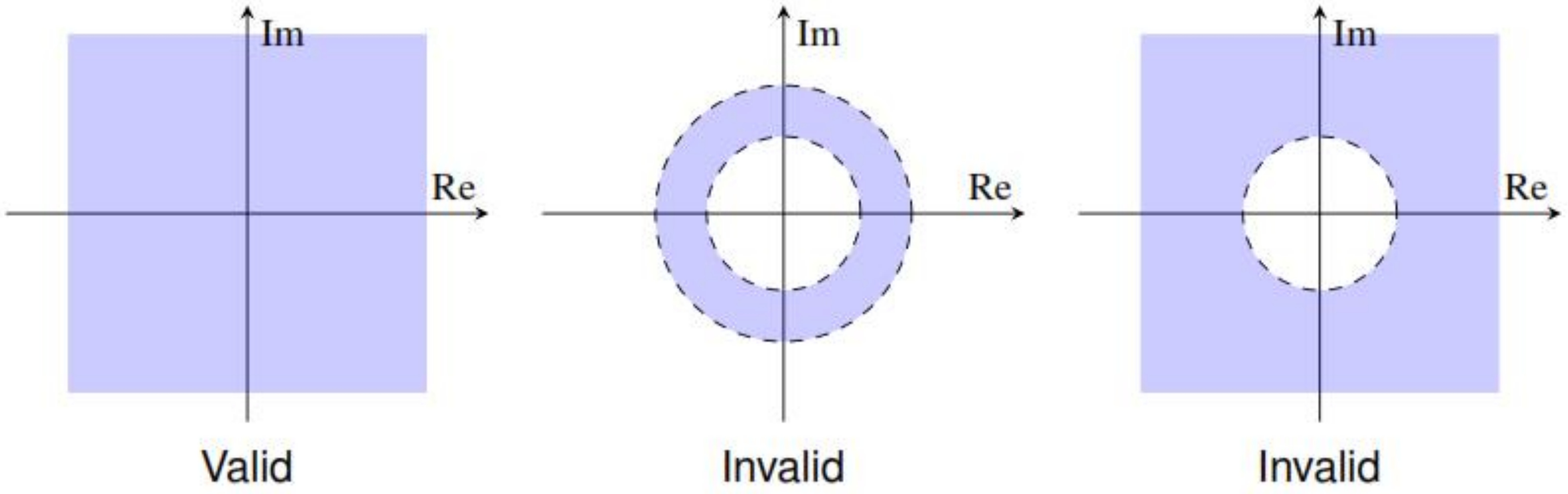
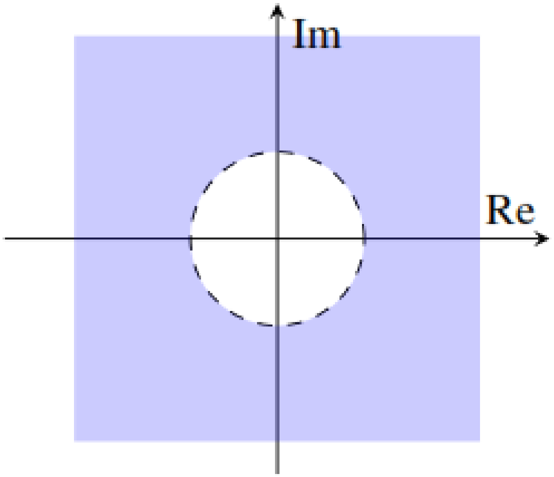
PROPERTY 3: FINITE DURATION SEQUENCES

- If a sequence x is ***finite duration*** and its z transform X converges for at least one point, then X converges for ***all points*** the complex plane, ***except possibly*** 0 ***and/or*** ∞.
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown below.



PROPERTY 4: RIGHT SIDED SEQUENCES

- If a sequence x is ***right sided*** and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then all (finite) values of z for which $|z| > r_0$ will also be in the ROC of X (i.e., the ROC contains the exterior of a circle centered at 0, possibly including ∞).
- Thus, if x is ***right sided but not left sided***, the ROC of X is the ***exterior of a circle centered at*** 0, possibly including ∞.
- Examples of sets that would be either valid or invalid as ROCs for X, if x is right sided but not left sided, are shown below.



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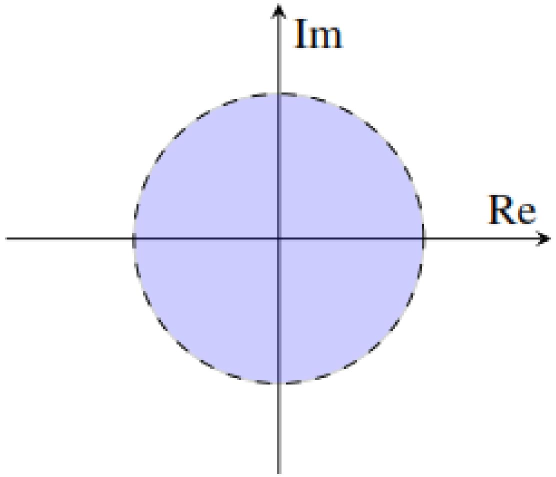

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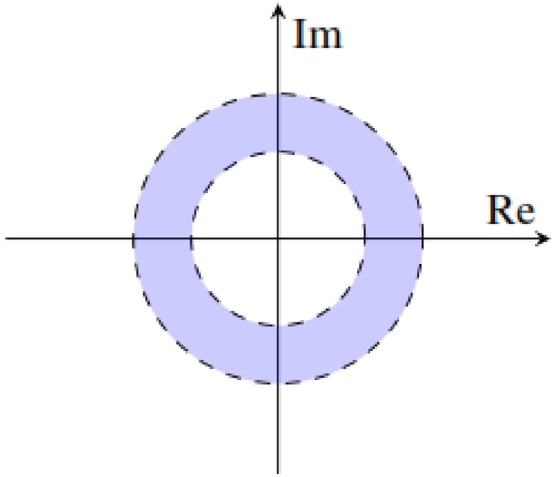

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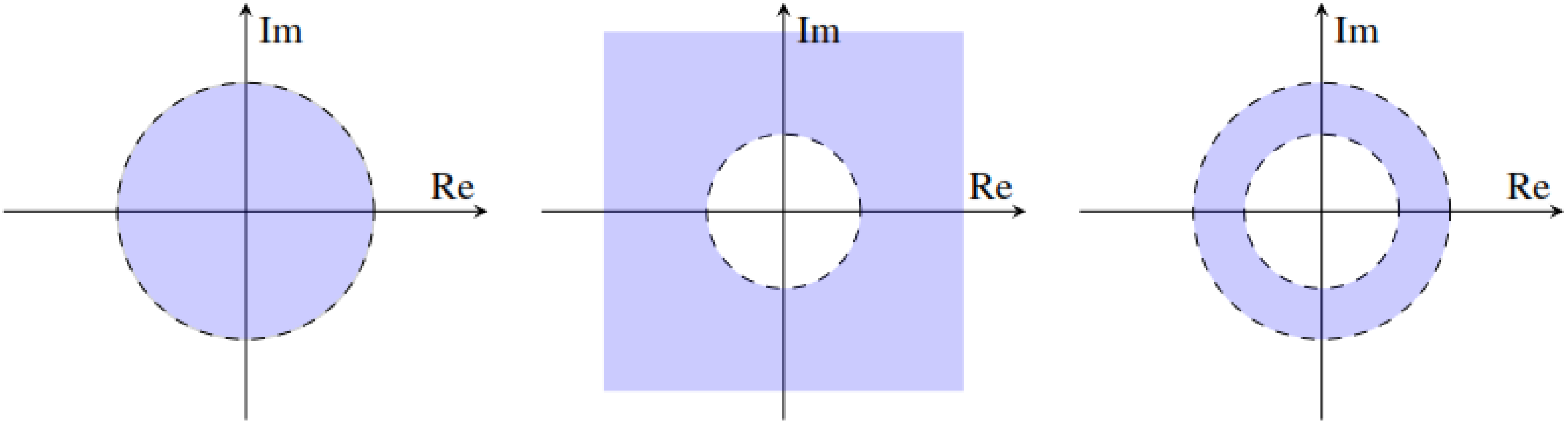
PROPERTY 5: LEFT SIDED SEQUENCES

- If a sequence x is ***left sided*** and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then all values of z for which $0 < |z| < r_0$ will also be in the ROC of X (i.e., the ROC contains a disk centered at 0, possibly excluding 0).
- Thus, if x is ***left sided but not right sided***, the ROC of X is a ***disk centered at*** 0, possibly excluding 0.
- Examples of sets that would be either valid or invalid as ROCs for X, if x is left sided but not right sided, are shown below.



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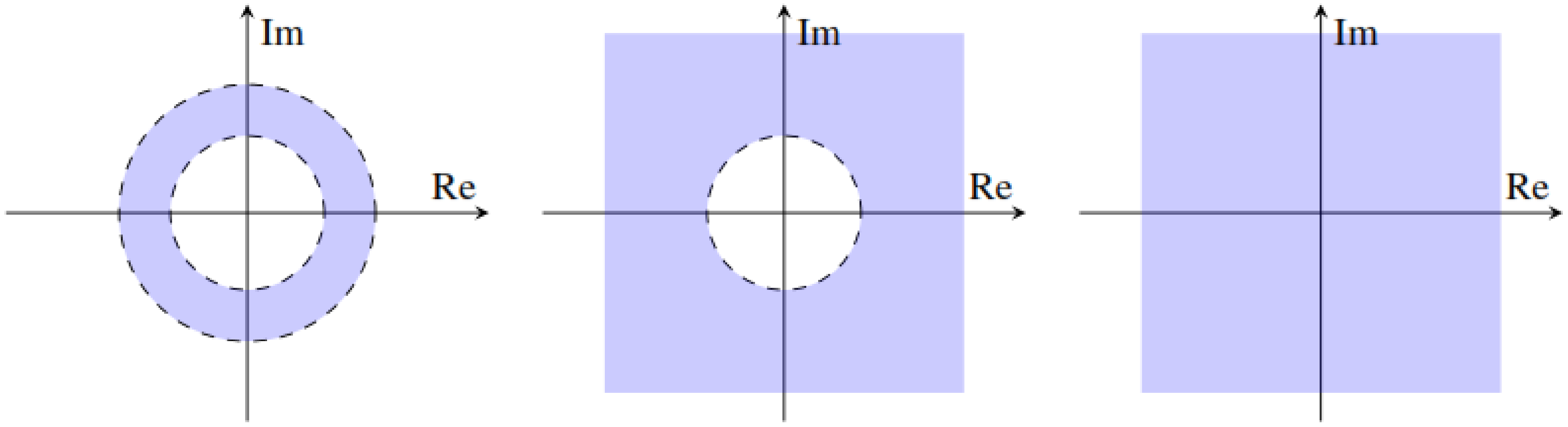
PROPERTY 6: TWO SIDED SEQUENCES

- If a sequence x is ***two sided*** and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then the ROC of X will consist of a ring that contains this circle (i.e., the ROC is an ***annulus centered at*** 0).
- Examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown below.



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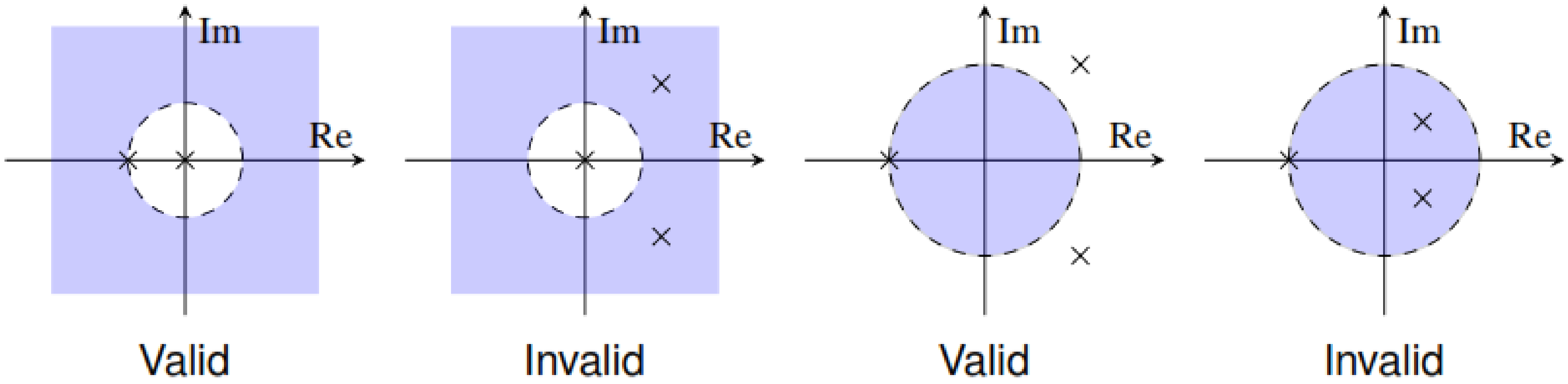
PROPERTY 7: MORE ON RATIONAL Z TRANSFORM

- If a sequence x has a ***rational*** z transform X (with at least one pole), then:
  1. If x is ***right sided***, then the ROC of X is the region outside the circle of radius equal to the largest magnitude of the poles of X (i.e., ***outside the outermost pole***), possibly including ∞.
  2. If x is ***left sided***, then the ROC of X is the region inside the circle of radius equal to the smallest magnitude of the nonzero poles of X and extending inward to, and possibly including, 0 (i.e., ***inside the innermost nonzero pole***).
- This property is implied by properties 1, 2, 4, and 5.
- Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are given below.

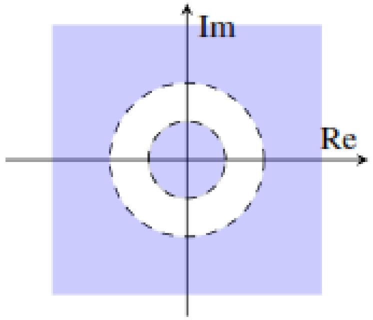


GENERAL FORM OF THE ROC

- To summarize the results of properties 3, 4, 5, and 6, if the z transform X of the sequence x exists, the ROC of X depends on the left- and right-sidedness of x as follows:

x		ROC of X
left sided	right sided	
yes	yes	everywhere, except possibly 0 and/or ∞
no	yes	exterior of circle centered at 0, possibly including ∞
yes	no	disk centered at 0, possibly excluding 0
no	no	annulus centered at 0

- Thus, we can infer that, if X exists, the ROC can only be of one of the forms listed above.
- For example, the sets shown below would not be valid as ROCs.



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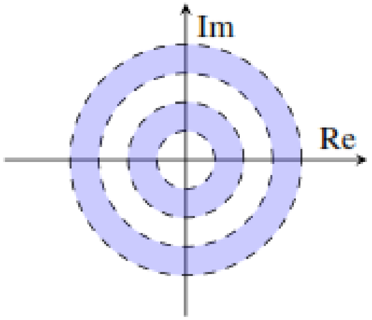

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Properties of the z Transform

PROPERTIES OF THE Z TRANSFORM

Property	Time Domain	Z Domain	ROC
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$	At least $R_1 \cap R_2$
Translation	$x(n-n_0)$	$z^{-n_0}X(z)$	R except possible addition/deletion of 0
Modulation	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert R$
Conjugation	$x^*(n)$	$X^*(z^*)$	R
Time Reversal	$x(-n)$	$X(1/z)$	R^{-1}
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$	$R^{1/M}$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right)$	R^M
Convolution	$x_1 * x_2(n)$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
Z-Domain Diff.	$nx(n)$	$-z\frac{d}{dz}X(z)$	R
Differencing	$x(n)-x(n-1)$	$(1-z^{-1})X(z)$	At least $R \cap \lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{z}{z-1}X(z)$	At least $R \cap \lvert z\rvert > 1$

Property	
Initial Value Theorem	$x(0) = \lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1)X(z)]$

Z TRANSFORM PAIR

Pair	$x(n)$	$X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$-u(-n-1)$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4	$nu(n)$	$\frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
5	$-nu(-n-1)$	$\frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert < 1$
6	$a^n u(n)$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
7	$-a^n u(-n-1)$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
8	$na^n u(n)$	$\frac{az}{(z-a)^2} = \frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$-na^n u(-n-1)$	$\frac{az}{(z-a)^2} = \frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
10	$\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(n)$	$\frac{z^m}{(z-a)^m} = \frac{1}{(1-az^{-1})^m}$	$\lvert z\rvert > \lvert a\rvert$
11	$-\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(-n-1)$	$\frac{z^m}{(z-a)^m} = \frac{1}{(1-az^{-1})^m}$	$\lvert z\rvert < \lvert a\rvert$

Z TRANSFORM PAIR

Pair	$x(n)$	$X(z)$	ROC
12	$\cos(\Omega_0 n)u(n)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1} = \frac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
13	$-\cos(\Omega_0 n)u(-n-1)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1} = \frac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert < 1$
14	$\sin(\Omega_0 n)u(n)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1} = \frac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
15	$-\sin(\Omega_0 n)u(-n-1)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1} = \frac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert < 1$
16	$a^n\cos(\Omega_0 n)u(n)$	$\frac{z(z-a\cos\Omega_0)}{z^2-2az\cos\Omega_0+a^2} = \frac{1-(a\cos\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
17	$a^n\sin(\Omega_0 n)u(n)$	$\frac{az\sin\Omega_0}{z^2-2az\cos\Omega_0+a^2} = \frac{(a\sin\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
18	$u(n)-u(n-M), M>0$	$\frac{z(1-z^{-M})}{z-1} = \frac{1-z^{-M}}{1-z^{-1}}$	$\lvert z\rvert > 0$
19	$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\frac{(a-a^{-1})z}{(z-a)(z-a^{-1})}$	$\lvert a\rvert < \lvert z\rvert < \lvert a^{-1}\rvert$

LINEARITY

- If $x_1(n) \xleftrightarrow{\text{ZT}} X_1(z)$ with ROC R_1 and $x_2(n) \xleftrightarrow{\text{ZT}} X_2(z)$ with ROC R_2, then

  $$a_1x_1(n) + a_2x_2(n) \xleftrightarrow{\text{ZT}} a_1X_1(z) + a_2X_2(z) \quad \text{with ROC } R \text{ containing } R_1 \cap R_2,$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the z transform.
- The ROC always contains the intersection but could be larger (in the case that pole-zero cancellation occurs).

TRANSLATING

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x(n-n_0) \overset{\text{ZT}}{\longleftrightarrow} z^{-n_0}X(z) \;\text{ with ROC } R',$$

  where n_0 is an integer constant and R' is the same as R except for the possible addition or deletion of zero or infinity.
- This is known as the **translation (or time-shifting) property** of the z transform.

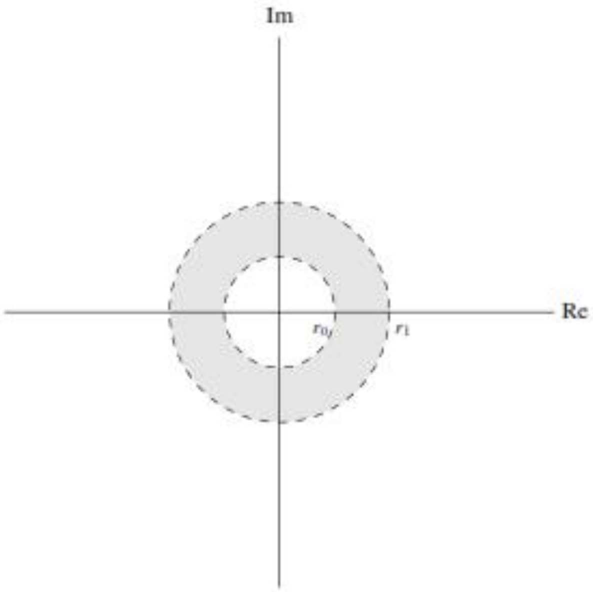
Z-DOMAIN SCALING

- If $x(n) \xleftrightarrow{\text{ZT}} X(z)$ with ROC R, then

$$a^n x(n) \xleftrightarrow{\text{ZT}} X(z/a) \quad \text{with ROC } |a|R,$$

  where a is a nonzero constant.

- This is known as the **z-domain scaling property** of the z transform.
- As illustrated below, the ROC R is *scaled* by $|a|$.



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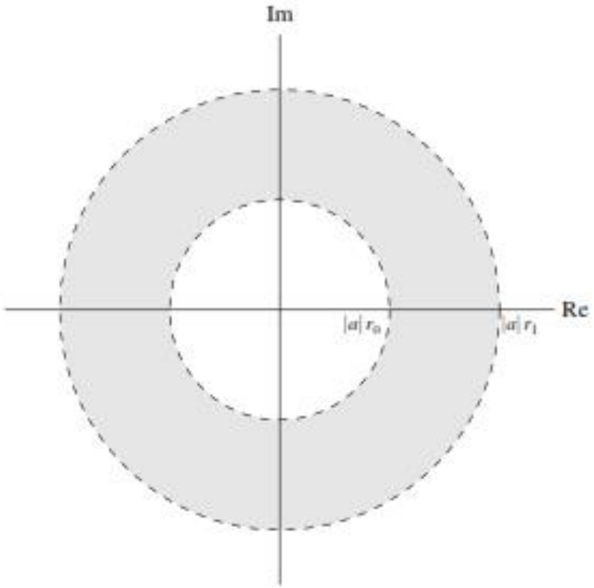


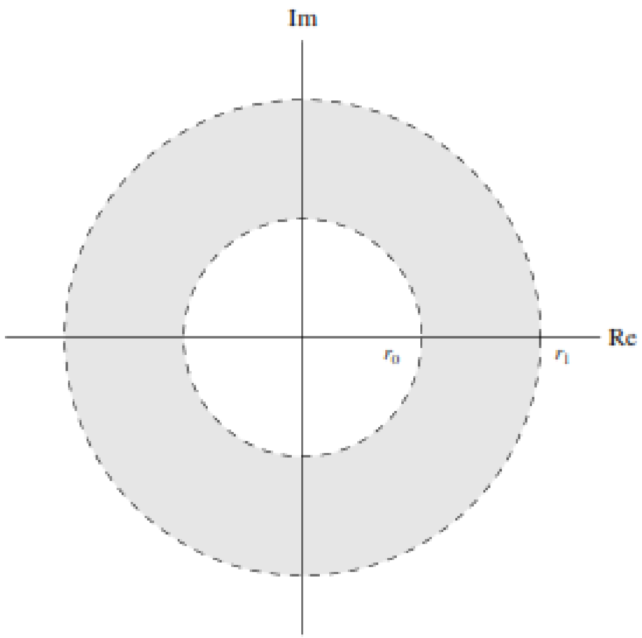
$|a|R$

TIME REVERSAL

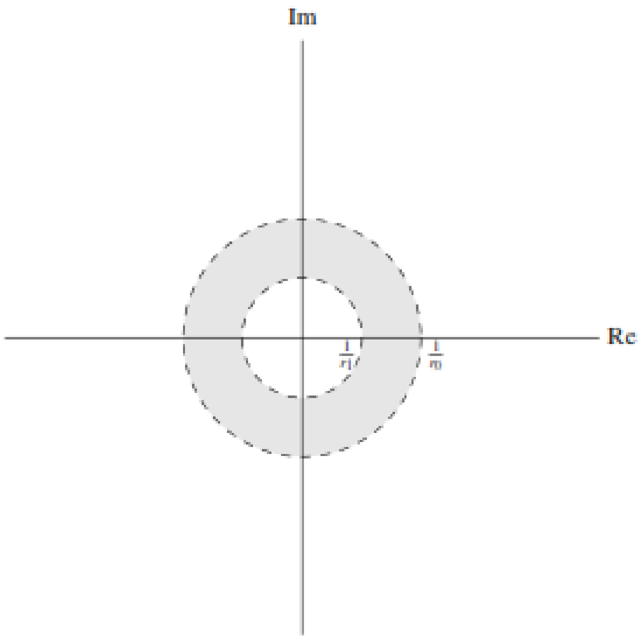
- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x(-n) \overset{\text{ZT}}{\longleftrightarrow} X(1/z) \quad \text{with ROC } 1/R.$$

- This is known as the **time-reversal property** of the z transform.
- As illustrated below, the ROC R is ***reciprocated***.



R



$1/R$

UPSAMPLING

- Define $(\uparrow M)x(n)$ as

$$(\uparrow M)x(n) = \begin{cases} x(n/M) & n/M \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

- If $x(n) \stackrel{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$(\uparrow M)x(n) \stackrel{\text{ZT}}{\longleftrightarrow} X(z^M) \quad \text{with ROC } R^{1/M}.$$

- This is known as the **upsampling (or time-expansion) property** of the z transform.

DOWNSAMPLING

- If $x(n) \stackrel{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$(\downarrow M)x(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M} z^{1/M}\right) \quad \text{with ROC } R^M.$$

- This is known as the **downsampling property** of the z transform.

CONJUGATION

- If $x(n) \xleftrightarrow{\text{ZT}} X(z)$ with ROC R, then

$$x^*(n) \xleftrightarrow{\text{ZT}} X^*(z^*) \quad \text{with ROC } R.$$

- This is known as the **conjugation property** of the z transform.

CONVOLUTION

- If $x_1(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_1(z)$ with ROC R_1 and $x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_2(z)$ with ROC R_2, then

$$x_1 * x_2(n) \stackrel{\text{ZT}}{\longleftrightarrow} X_1(z)X_2(z) \quad \text{with ROC containing } R_1 \cap R_2.$$

- This is known that the **convolution (or time-domain convolution) property** of the z transform.
- The ROC always contains the intersection but can be larger than the intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes ***multiplication*** in the z domain.
- This can make dealing with LTI systems much easier in the z domain than in the time domain.

Z-DOMAIN DIFFERENTIATION

- If $x(n) \xleftrightarrow{\text{ZT}} X(z)$ with ROC R, then

$$nx(n) \xleftrightarrow{\text{ZT}} -z\frac{d}{dz}X(z) \quad \text{with ROC } R.$$

- This is known as the **z-domain differentiation property** of the z transform.

DIFFERENCING

- If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then

$$x(n) - x(n-1) \overset{\text{ZT}}{\longleftrightarrow} (1 - z^{-1})X(z) \text{ for ROC containing } R \cap |z| > 0.$$

- This is known as the **differencing property** of the z transform.
- Differencing in the time domain becomes multiplication by $1 - z^{-1}$ in the z domain.
- This can make dealing with difference equations much easier in the z domain than in the time domain.

ACCUMULATION

- If $x(n) \xleftrightarrow{\text{ZT}} X(z)$ with ROC R, then

$$\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{\text{ZT}} \frac{z}{z-1}X(z) \text{ for ROC containing } R \cap |z| > 1.$$

- This is known as the **accumulation property** of the z transform.

INITIAL VALUE THEOREM

- For a sequence x with z transform X, if x is causal, then

$$x(0) = \lim_{z \to \infty} X(z).$$

- This result is known as the **initial-value theorem**.

FINAL VALUE THEOREM

- For a sequence x with z transform X, if x is causal and $\lim_{n\to\infty} x(n)$ exists, then

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1)X(z)].$$

- This result is known as the **final-value theorem**.

Determination of Inverse z Transform

FINDING THE INVERSE Z TRANSFORM

- Recall that the inverse z transform x of X is given by

$$x(n) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} dz,$$

  where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X.
- Unfortunately, the above contour integration can often be ***quite tedious*** to compute.
- Consequently, we do not usually compute the inverse z transform directly using the above equation.
- For rational functions, the inverse z transform can be more easily computed using ***partial fraction expansions***.
- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse z transforms can typically be found in tables.

z Transform and LTI Systems

SYSTEM FUNCTION OF LTI SYSTEMS

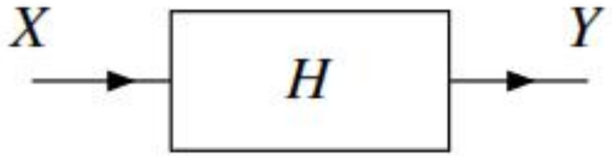
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the z transforms of x, y, and h, respectively.
- Since $y(n) = x * h(n)$, the system is characterized in the z domain by

$$Y(z) = X(z)H(z).$$

- As a matter of terminology, we refer to H as the **system function** (or **transfer function**) of the system (i.e., the system function is the z transform of the impulse response).
- When viewed in the z domain, a LTI system forms its output by multiplying its input with its system function.
- A LTI system is completely characterized by its system function H.
- If the ROC of H includes the unit circle $|z| = 1$, then $H(e^{j\Omega})$ is the ***frequency response*** of the LTI system.

BLOCK DIAGRAM REPRESENTATION OF LTI SYSTEMS

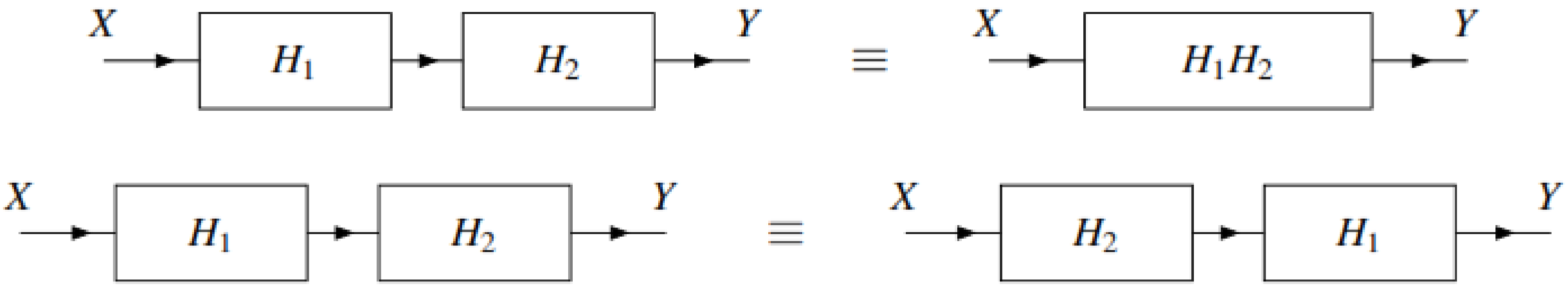
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the z transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the z domain as shown below.



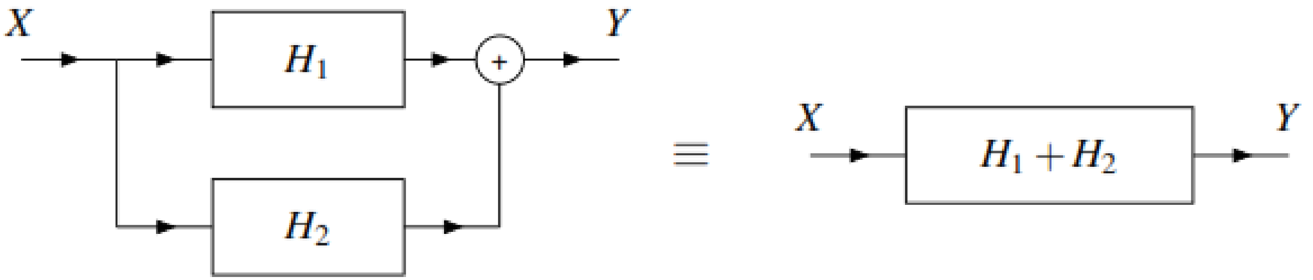
- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

INTERCONNECTION OF LTI SYSTEMS

- The *series* interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function $H = H_1 H_2$. That is, we have the equivalences shown below.



- The *parallel* interconnection of the LTI systems with impulse responses H_1 and H_2 is a LTI system with the system function $H = H_1 + H_2$. That is we have the equivalence shown below.



CAUSALITY

- If a LTI system is *causal*, its impulse response is causal, and therefore *right sided*. From this, we have the result below.
- **Theorem.** A LTI system is *causal* if and only if the ROC of the system function is:
  1. the *exterior of a circle, including* ∞; or
  2. the *entire complex plane, including* ∞ and possibly excluding 0.
- **Theorem.** A LTI system with a *rational* system function H is causal if and only if:
  1. the ROC of H is the exterior of a (possibly degenerate) circle *outside the outermost pole* of H or, if H has no poles, the entire complex plane; and
  2. H is *proper* (i.e., when $H(z)$ is expressed as a ratio of polynomials in z, the order of the numerator polynomial does not exceed the order of the denominator polynomial).

BIBO STABILITY

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is ***BIBO stable*** if and only if the ROC of its system function contains the ***unit circle*** (i.e., $|z| = 1$).
- **Theorem.** A ***causal*** LTI system with a ***rational*** system function H is BIBO stable if and only if all of the poles of H lie inside the unit circle (i.e., each of the poles has a ***magnitude less than one***).

INVERTIBILITY

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(z)H_{\text{inv}}(z) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(z) = \frac{1}{H(z)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is ***not necessarily unique***.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in ***one specific choice*** of inverse system (due to these additional constraints of stability and/or causality).

LTI SYSTEMS AND DIFFERENCE EQUATIONS

- Many LTI systems of practical interest can be represented using an *Nth-order linear difference equation with constant coefficients*.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k) \quad \text{where} \quad M \leq N.$$

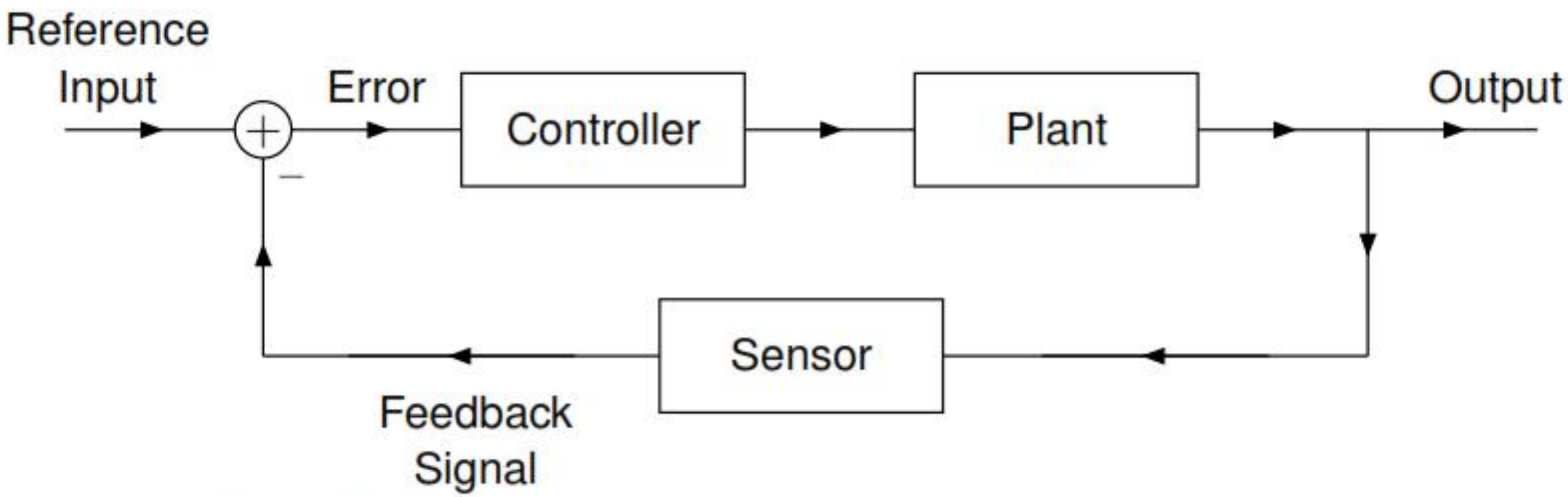
- Let h denote the impulse response of the system, and let X, Y, and H denote the z transforms of x, y, and h, respectively.
- One can show that $H(z)$ is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}.$$

- Observe that, for a system of the form considered above, the system function is always *rational*.

Application: Analysis of Control Systems

FEEDBACK CONTROL SYSTEMS



- **input**: *desired value* of the quantity to be controlled
- **output**: *actual value* of the quantity to be controlled
- **error**: *difference* between the desired and actual values
- **plant**: system to be controlled
- **sensor**: device used to measure the actual output
- **controller**: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

STABILITY ANALYSIS OF FEEDBACK CONTROL SYSTEMS

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the z domain than in the time domain.
- Therefore, the z domain is extremely useful for the stability analysis of systems.

Unilateral z Transform

UNILATERAL Z TRANSFORM

- The **unilateral z transform** of the sequence x, denoted $\mathcal{Z}_u x$ or X, is defined as

$$\mathcal{Z}_u x(z) = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

- The unilateral z transform is related to the bilateral z transform as follows:

$$\mathcal{Z}_u x(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) u(n) z^{-n} = \mathcal{Z}\{xu\}(z).$$

- In other words, the unilateral z transform of the sequence x is simply the bilateral z transform of the sequence xu.
- Since $\mathcal{Z}_u x = \mathcal{Z}\{xu\}$ and xu is always a ***right-sided*** sequence, the ROC associated with $\mathcal{Z}_u x$ is always the ***exterior of a circle***.
- For this reason, we often ***do not explicitly indicate the ROC*** when working with the unilateral z transform.

UNILATERAL Z TRANSFORM

- With the unilateral z transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral z transform is ***only invertible for causal sequences***. In particular, we have

$$
\begin{aligned}
\mathcal{Z}_u^{-1}\{\mathcal{Z}_u\{x\}\}(n) &= \mathcal{Z}_u^{-1}\{\mathcal{Z}\{xu\}\}(n) \\
&= \mathcal{Z}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\
&= x(n)u(n) \\
&= \begin{cases} x(n) & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

- For a noncausal sequence x, we can only recover $x(n)$ for $n \geq 0$.
- Due to the close relationship between the unilateral and bilateral z transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.

PROPERTIES OF THE UNILATERAL Z TRANSFORM

Property	Time Domain	Z Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$
Time Delay	$x(n-1)$	$z^{-1}X(z)+x(-1)$
Time Advance	$x(n+1)$	$zX(z)-zx(0)$
Modulation	$a^nx(n)$	$X(a^{-1}z)$
	$e^{j\Omega_0 n}x(n)$	$X(e^{-j\Omega_0}z)$
Conjugation	$x^*(n)$	$X^*(z^*)$
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1}X\left(e^{-j2\pi k/M}z^{1/M}\right)$
Convolution	$x_1 * x_2(n)$, x_1 and x_2 are causal	$X_1(z)X_2(z)$
Z-Domain Diff.	$nx(n)$	$-z\frac{d}{dz}X(z)$
Differencing	$x(n)-x(n-1)$	$(1-z^{-1})X(z)-x(-1)$
Accumulation	$\sum_{k=0}^{n}x(k)$	$\frac{1}{1-z^{-1}}X(z)$

Property	
Initial Value Theorem	$x(0)=\lim_{z\to\infty}X(z)$
Final Value Theorem	$\lim_{n\to\infty}x(n)=\lim_{z\to 1}[(z-1)X(z)]$

UNILATERAL Z TRANSFORM PAIR

Pair	$x(n),\ n \geq 0$	$X(z)$
1	$\delta(n)$	1
2	1	$\frac{z}{z-1}$
3	n	$\frac{z}{(z-1)^2}$
4	a^n	$\frac{z}{z-a}$
5	$a^n n$	$\frac{az}{(z-a)^2}$
6	$\cos(\Omega_0 n)$	$\frac{z(z-\cos\Omega_0)}{z^2-2(\cos\Omega_0)z+1}$
7	$\sin(\Omega_0 n)$	$\frac{z\sin\Omega_0}{z^2-2(\cos\Omega_0)z+1}$
8	$\|a\|^n\cos(\Omega_0 n)$	$\frac{z(z-\|a\|\cos\Omega_0)}{z^2-2\|a\|(\cos\Omega_0)z+\|a\|^2}$
9	$\|a\|^n\sin(\Omega_0 n)$	$\frac{z\|a\|\sin\Omega_0}{z^2-2\|a\|(\cos\Omega_0)z+\|a\|^2}$

SOLVING DIFFERENCE EQUATIONS USING THE UNILATERAL Z-T

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear difference equations.
- One common use of the unilateral z transform is in solving constant-coefficient linear difference equations with nonzero initial conditions.

EXAMPLE