DE AY1516 S2

Q1.

Old Solution:

Due to Newton's Cooling Law:

$$\frac{dT}{dt} = -k(T - R)\ (*)$$

Where:

T: Temperature of a body at time t.

k: Positive constant characteristic of the system.

R: Environment temperature.

$$(*) \to \frac{dT}{T - R} = -kdt$$
$$\to \ln(T - R) = -kt + C\ (1)$$

Converting temperature: $70°F\ = 21.1111°C$; $50°F = 10°C$; $15°F = -9.4444°C$; $10°F = -12.2222°C$

With the condition given in the prolem:

$$\begin{cases} T(0) = 21.1111 \\ T(0.5) = 10 \end{cases} \to \begin{cases} \ln(21.1111 + 12.22222) = -k.0 + C \\ \ln(10 + 12.22222) = -k \times 0.5 + C \end{cases} \leftrightarrow \begin{cases} C = 3.5066 \\ k = 0.8110 \end{cases}$$

From (1), Solve for $T(t)$, we obtain:

$$T(t) = e^{-kt+C} + R$$

If $T(t) = -9.4444$, Solve for t, we get $t = 3.06$ (minutes)

Therefore, it took 3.06 minutes to reads $15°F$

New Solution:

Due to Newton's Cooling Law:

$$\frac{dT}{dt} = -k(T - R)\ (*)$$

Where:

T: Temperature of a body at time t.

k: Positive constant characteristic of the system.

R: Environment temperature.

$$(*) \to \frac{dT}{T - R} = -kdt$$
$$\to \ln(T - R) = -kt + C\ (1)$$

With the condition given in the prolem:

$$\begin{cases} T(0) = 70 \\ T(0.5) = 50 \end{cases} \to \begin{cases} \ln(70 - 10) = -k.0 + C \\ \ln(50 - 10) = -k \times 0.5 + C \end{cases} \leftrightarrow \begin{cases} C = \ln(60) \approx 4.0943 \\ k = 2\ln(1.5) \approx 0.8110 \end{cases}$$

From (1), Solve for $T(t)$, we obtain:

$$T(t) = e^{-kt+C} + R$$

If $T(t) = 15$, Solve for t, we get $t = 3.06$ (minutes)

Therefore, it took 3.06 minutes to reads $15°F$

Q2.

Given that: $$(3x^2y + e^y)dx + (x^3 + xe^y - 2016y)dy = 0\ (*)$$
$$\leftrightarrow M(x,y)dx + N(x,y)dy = 0$$

Where: $\begin{cases} M(x,y) = 3x^2y + e^y \\ N(x,y) = x^3 + xe^y - 2016y \end{cases}$

And: $\begin{cases} \dfrac{\partial M}{\partial y} = 3x^2 + e^y \\ \dfrac{\partial N}{\partial x} = 3x^2 + e^y \end{cases}$

$$\rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore the given differential equation is exact.

Solve the given differential equation:

$$(*) \leftrightarrow 3x^2ydx + e^ydx + x^3dy + xe^ydy - 2016ydy = 0$$
$$\leftrightarrow yd(x^3) + e^ydx + x^3dy + xd(e^y) - 1008d(y^2) = 0$$
$$\leftrightarrow yd(x^3) + x^3dy + e^ydx + xd(e^y) - 1008d(y^2) = 0$$
$$\leftrightarrow d(x^3y) + d(xe^y) - 1008d(y^2) = 0$$
$$\leftrightarrow d(x^3y + xe^y - 1008y^2) = 0$$

Integrating both sides we obtain the final result:

$$\leftrightarrow x^3y + xe^y - 1008y^2 + C = 0$$

Q3.

Given that: $$xy' + (3x+1)y = e^{-3x}\ (*), \qquad y(1) = 2$$
$$(*) \leftrightarrow xe^{3x}y' + (3x+1)e^{3x}y = 1$$
$$\leftrightarrow xe^{3x}\frac{dy}{dx} + \frac{d(xe^{3x})}{dx}y = 1$$
$$\leftrightarrow \frac{d(xe^{3x}y)}{dx} = 1$$
$$\leftrightarrow d(xe^{3x}y) = dx$$
$$\leftrightarrow xe^{3x}y = x + C$$

With the initial condition: $y(1) = 1$, it leads to:

$$1.e^3.2 = 1 + C \leftrightarrow C = 2e^3 - 1$$

Hence, the solution of the equation is:

$$xe^{3x}y = x + 2e^3 - 1$$

Or:

$$y = e^{-3x} + \frac{(2e^3 - 1)e^{-3x}}{x}$$

Q4.

a) Given that: $$y'' - 2y' + y = e^{2x}(x^3 + 1) + e^x(x+1)$$
$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y'' - 2y' + y \\ g_1(x) = e^{2x}(x^3 + 1) \\ g_2(x) = e^x(x+1) \end{cases}$

Characteristic equation of the given ODE: $r^2 - 2r + 1 = 0$

$$\rightarrow r_1 = r_2 = 1$$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two term: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore y_{p1} from: $\mathrm{L}[y_{p1}] = g_1(x) \leftrightarrow y_{p1}'' - 2y_{p1}' + y_{p1} = e^{2x}(x^3 + 1)$ $(\alpha = 2)$

Since, $\alpha = 2$ is not a root of characteristic equation.

Hence: $y_{p1} = e^{2x}(Ax^3 + Bx^2 + Cx + D)$

Solve fore y_{p2} from: $\mathrm{L}[y_{p2}] = g_2(x) \leftrightarrow y_{p2}'' - 2y_{p2}' + y_{p2} = e^{x}(x + 1)$ $(\alpha = 1)$

Since, $\alpha = 1$ is double root of characteristic equation.

Hence: $y_{p2} = x^2 e^x (Ex + F)$

So: $y_p = y_{p1} + y_{p2}$

$$= e^{2x}(Ax^3 + Bx^2 + Cx + D) + x^2 e^x (Ex + F)$$

b) Given that: $$y'' - 4y' + 5y = e^x(x + 1) + 2015$$

$$\leftrightarrow \mathrm{L}[y] = g_1(x) + g_2(x)$$

Where: $\begin{cases} \mathrm{L}[y] = y'' - 4y' + 5y \\ g_1(x) = e^x(x+1) \\ g_2(x) = 2015 \end{cases}$

Characteristic equation of the given ODE: $r^2 - 4r + 5 = 0$

$$\rightarrow r_1 = 2 - i;\ r_2 = 2 + i$$

So, the complement solution is: $y_c = C_1 e^{2x} \sin x + C_2 e^{2x} \cos x$

Since the right hand side of the given equation has two terms $g_1(x)$ and $g_2(x)$, therefore the particular solution also has two term: $y_p = y_{p1} + y_{p2}$, respectively.

Solve fore y_{p1} from: $\mathrm{L}[y_{p1}] = g_1(x) \leftrightarrow y_{p1}'' - 4y_{p1}' + 5y_{p1} = e^{x}(x + 1)$ $(\alpha = 1)$

Since, $\alpha = 1$ is not a root of characteristic equation.

So, y_{p1} has the following form: $y_{p1} = (Ax + B)e^x$

$$\rightarrow y_{p1}' = (Ax + B + A)e^x$$

$$\rightarrow y_{p1}'' = (Ax + B + 2A)e^x$$

Substituting into the equation we obtain:

$$e^x(2Ax + 2B - 2A) = e^x(x + 1)$$

$$\rightarrow \begin{cases} 2A = 1 \\ 2B - 2A = 1 \end{cases} \leftrightarrow \begin{cases} A = \dfrac{1}{2} \\ B = 1 \end{cases}$$

Therefore: $y_{p1} = \left(\frac{1}{2}x + 1\right) e^x$

Solve fore y_{p2} from: $\mathrm{L}[y_{p2}] = g_2(x) \leftrightarrow y_{p2}'' - 4y_{p2}' + 5y_{p2} = 2015$ $(\alpha = 0)$

Since, $\alpha = 0$ is not a root of characteristic equation.

So, y_{p2} has the following form: $y_{p2} = A$

$$\rightarrow y_{p2}' = 0$$

$$\rightarrow y_{p2}'' = 0$$

Substituting into the equation we obtain:

$$5A = 2015$$

$$\leftrightarrow A = 403$$

Therefore: $y_{p2} = 403$

So: $y_p = y_{p1} + y_{p2}$

$$= \left(\frac{1}{2}x + 1\right)e^x + 403$$

Thus, the general solution of the given differential equation is:

$$y_G = y_c + y_p$$

$$= C_1 e^{2x}\sin x + C_2 e^{2x}\cos x + \left(\frac{1}{2}x + 1\right)e^x + 403$$

Q5.

a) Given that: $4x^2y'' + 8xy' + y = 0\ (*)$

We have: $y_1 = x^\alpha; \to y_1' = \alpha x^{\alpha-1} \to y_1'' = \alpha(\alpha-1)x^{\alpha-2}$.

We know that y_1 is a solution of $(*)$, therefore substituting y_1 into $(*)$, we get:

$$4x^2\alpha(\alpha-1)x^{\alpha-2} + 8x\alpha x^{\alpha-1} + x^\alpha = 0$$

$$\leftrightarrow 4\alpha(\alpha-1) + 8\alpha + 1 = 0$$

$$\leftrightarrow \alpha = -\frac{1}{2}$$

Thus, with $\alpha = -\frac{1}{2}$, $y_1 = x^\alpha = \frac{1}{\sqrt{x}}$ is a solution of $(*)$

b) To find the general solution of $(*)$, we rewire $(*)$ in the following form:

$$y'' + \frac{2}{x}y' + \frac{1}{4x^2}\,y = 0$$

$$(y'' + p(x)y' + q(x) = 0)$$

The Wronskian determinant for the equation is:

$$\mathrm{W}[y_1, y_2] = C_1 e^{-\int p(x)dx} = C_1 e^{-\int \frac{2}{x}dx}$$

$$\to \mathrm{W}[y_1, y_2] = C_1 x^{-2}$$

Hence:

$$y_2 = y_1\left[\int \frac{\mathrm{W}[y_1, y_2]}{y_1^2}dx + C_2\right]$$

$$\to y_2 = x^{-\frac{1}{2}}\left[\int \frac{C_1 x^{-2}}{\left(x^{-\frac{1}{2}}\right)^2}dx + C_2\right]$$

$$\to y_2 = x^{-\frac{1}{2}}[C_1 \ln x + C_2]$$

$$\to y_2 = C_1 x^{-\frac{1}{2}}\ln x + C_2(x+1)$$

Choose $C_1 = 1, C_2 = 0 \to y_2 = x^{-\frac{1}{2}}\ln x$

Since, the Wronskian determinant different from 0 for some x, therefore y_1 and y_2 are linearly independence solution of the equation.

Thus, the general solution of the equation is:

$$y_G = C_1 y_1 + C_2 y_2 = C_1 x^{-\frac{1}{2}} + C_2 x^{-\frac{1}{2}}\ln x$$