Lecture notes: Differential Equations for ISE (MA029IU)

Week 4 *

February 22, 2022

1 Exact equations

Another type of equation that comes up quite often in physics and engineering is an *exact equation*. Suppose $F(x, y)$ is a function of two variables, which we call the *potential function*. The naming should suggest potential energy, or electric potential. Exact equations and potential functions appear when there is a conservation law at play, such as conservation of energy. Let us make up a simple example. Let

$$F(x, y) = x^2 + y^2.$$

We are interested in the lines of constant energy, that is lines where the energy is conserved; we want curves where $F(x, y) = C$, for some constant C. In our example, the curves $x^2 + y^2 = C$ are circles. See Figure 1.

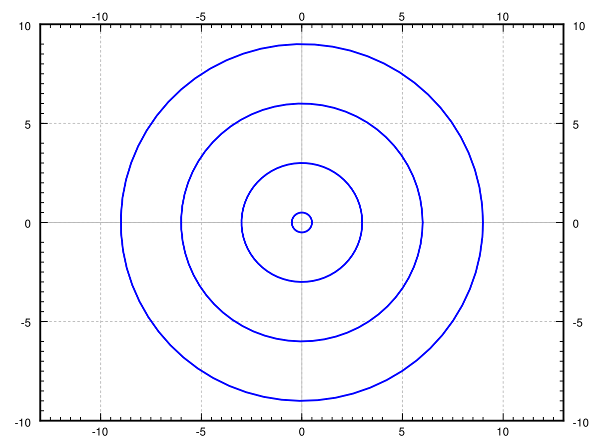

Figure 1: Solutions to $F(x, y) = x^2 + y^2 = C$ for various C.

We take the *total derivative* of F:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

For convenience, we will make use of the notation of $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. In our example,

$$dF = 2x\, dx + 2y\, dy.$$

We apply the total derivative to $F(x, y) = C$, to find the differential equation $dF = 0$. The differential equation we obtain in such a way has the form

$$M\, dx + N\, dy = 0, \qquad \text{or} \qquad M + N \frac{dy}{dx} = 0.$$

An equation of this form is called *exact* if it was obtained as $dF = 0$ for some potential function F. In our simple example, we obtain the equation

$$2x\, dx + 2y\, dy = 0, \qquad \text{or} \qquad 2x + 2y \frac{dy}{dx} = 0.$$

Since we obtained this equation by differentiating $x^2 + y^2 = C$, the equation is exact. We often wish to solve for y in terms of x. In our example,

$$y = \pm\sqrt{C^2 - x^2}.$$

*This note is taken from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl

An interpretation of the setup is that at each point $\vec{v} = (M, N)$ is a vector in the plane, that is, a direction and a magnitude. As M and N are functions of (x, y), we have a *vector field*. The particular vector field $\vec{v}$ that comes from an exact equation is a so-called *conservative vector field*, that is, a vector field that comes with a potential function $F(x, y)$, such that

$$\vec{v} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right).$$

Let γ be a path in the plane starting at (x_1, y_1) and ending at (x_2, y_2). If we think of $\vec{v}$ as force, then the work required to move along γ is

$$\int_\gamma \vec{v}(\vec{r}) \cdot d\vec{r} = \int_\gamma M\,dx + N\,dy = F(x_2, y_2) - F(x_1, y_1).$$

That is, the work done only depends on endpoints, that is where we start and where we end. For example, suppose F is gravitational potential. The derivative of F given by $\vec{v}$ is the gravitational force. What we are saying is that the work required to move a heavy box from the ground floor to the roof, only depends on the change in potential energy. That is, the work done is the same no matter what path we took; if we took the stairs or the elevator. Although if we took the elevator, the elevator is doing the work for us. The curves $F(x, y) = C$ are those where no work need be done, such as the heavy box sliding along without accelerating or breaking on a perfectly flat roof, on a cart with incredibly well oiled wheels.

An exact equation is a conservative vector field, and the implicit solution of this equation is the potential function.

1.1 Solving exact equations

Now you, the reader, should ask: Where did we solve a differential equation? Well, in applications we generally know M and N, but we do not know F. That is, we may have just started with $2x + 2y\frac{dy}{dx} = 0$, or perhaps even

$$x + y\frac{dy}{dx} = 0.$$

It is up to us to find some potential F that works. Many different F will work; adding a constant to F does not change the equation. Once we have a potential function F, the equation $F\big(x, y(x)\big) = C$ gives an implicit solution of the ODE.

Example 1.1: Let us find the general solution to $2x + 2y\frac{dy}{dx} = 0$. Forget we knew what F was.

If we know that this is an exact equation, we start looking for a potential function F. We have $M = 2x$ and $N = 2y$. If F exists, it must be such that $F_x(x, y) = 2x$. Integrate in the x variable to find

$$F(x, y) = x^2 + A(y), \tag{1}$$

for some function $A(y)$. The function A is the "constant of integration", though it is only constant as far as x is concerned, and may still depend on y. Now differentiate (1) in y and set it equal to N, which is what F_y is supposed to be:

$$2y = F_y(x, y) = A'(y).$$

Integrating, we find $A(y) = y^2$. We could add a constant of integration if we wanted to, but there is no need. We found $F(x, y) = x^2 + y^2$. Next for a constant C, we solve

$$F\big(x, y(x)\big) = C.$$

for y in terms of x. In this case, we obtain $y = \pm\sqrt{C^2 - x^2}$ as we did before.

***Exercise* 1.1:** *Why did we not need to add a constant of integration when integrating $A'(y) = 2y$? Add a constant of integration, say 3, and see what F you get. What is the difference from what we got above, and why does it not matter?*

The procedure, once we know that the equation is exact, is:

(i) Integrate $F_x = M$ in x resulting in $F(x, y) = \text{something} + A(y)$.

(ii) Differentiate this F in y, and set that equal to N, so that we may find $A(y)$ by integration.

The procedure can also be done by first integrating in y and then differentiating in x. Pretty easy huh? Let's try this again.

Example 1.2: Consider now $2x + y + xy\frac{dy}{dx} = 0$.

OK, so $M = 2x + y$ and $N = xy$. We try to proceed as before. Suppose F exists. Then $F_x(x, y) = 2x + y$. We integrate:

$$F(x, y) = x^2 + xy + A(y)$$

for some function $A(y)$. Differentiate in y and set equal to N:

$$N = xy = F_y(x, y) = x + A'(y).$$

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y. The equation is not exact; no potential function F exists.

Is there an easier way to check for the existence of F, other than failing in trying to find it? Turns out there is. Suppose $M = F_x$ and $N = F_y$. Then as long as the second derivatives are continuous,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Let us state it as a theorem. Usually this is called the Poincaré Lemma*.

Theorem 1.1 (Poincaré)**.** *If M and N are continuously differentiable functions of (x, y), and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x, y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.*

The theorem doesn't give us a global F defined everywhere. In general, we can only find the potential locally, near some initial point. By this time, we have come to expect this from differential equations.

Let us return to Example 1.2 where $M = 2x + y$ and $N = xy$. Notice $M_y = 1$ and $N_x = y$, which are clearly not equal. The equation is not exact.

Example 1.3: Solve

$$\frac{dy}{dx} = \frac{-2x - y}{x - 1}, \qquad y(0) = 1.$$

We write the equation as

$$(2x + y) + (x - 1)\frac{dy}{dx} = 0,$$

so $M = 2x + y$ and $N = x - 1$. Then

$$M_y = 1 = N_x.$$

The equation is exact. Integrating M in x, we find

$$F(x, y) = x^2 + xy + A(y).$$

Differentiating in y and setting to N, we find

$$x - 1 = x + A'(y).$$

*Named for the French polymath Jules Henri Poincaré (1854–1912).

So $A'(y) = -1$, and $A(y) = -y$ will work. Take $F(x, y) = x^2 + xy - y$. We wish to solve $x^2 + xy - y = C$. First let us find C. As $y(0) = 1$ then $F(0, 1) = C$. Therefore $0^2 + 0 \times 1 - 1 = C$, so $C = -1$. Now we solve $x^2 + xy - y = -1$ for y to get

$$y = \frac{-x^2 - 1}{x - 1}.$$

Example 1.4: Solve

$$-\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = 0, \qquad y(1) = 2.$$

We leave to the reader to check that $M_y = N_x$.

This vector field (M, N) is not conservative if considered as a vector field of the entire plane minus the origin. The problem is that if the curve γ is a circle around the origin, say starting at $(1, 0)$ and ending at $(1, 0)$ going counterclockwise, then if F existed we would expect

$$0 = F(1, 0) - F(1, 0) = \int_\gamma F_x \, dx + F_y \, dy = \int_\gamma \frac{-y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy = 2\pi.$$

That is nonsense! We leave the computation of the path integral to the interested reader, or you can consult your multivariable calculus textbook. So there is no potential function F defined everywhere outside the origin $(0, 0)$.

If we think back to the theorem, it does not guarantee such a function anyway. It only guarantees a potential function locally, that is only in some region near the initial point. As $y(1) = 2$ we start at the point $(1, 2)$. Considering $x > 0$ and integrating M in x or N in y, we find

$$F(x, y) = \arctan\left(y/x\right).$$

The implicit solution is $\arctan\left(y/x\right) = C$. Solving, $y = \tan(C)x$. That is, the solution is a straight line. Solving $y(1) = 2$ gives us that $\tan(C) = 2$, and so $y = 2x$ is the desired solution. See Figure 2, and note that the solution only exists for $x > 0$.

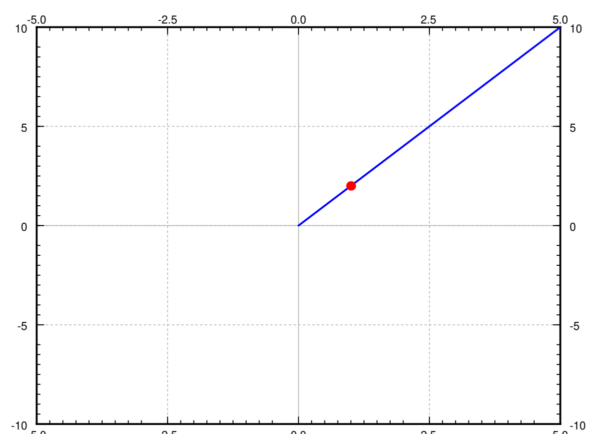

Figure 2: Solution to $-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy = 0$, $y(1) = 2$, *with initial point marked.*

Example 1.5: Solve

$$x^2 + y^2 + 2y(x + 1)\frac{dy}{dx} = 0.$$

The reader should check that this equation is exact. Let $M = x^2 + y^2$ and $N = 2y(x+1)$. We follow the procedure for exact equations

$$F(x,y) = \frac{1}{3}x^3 + xy^2 + A(y),$$

and

$$2y(x+1) = 2xy + A'(y).$$

Therefore $A'(y) = 2y$ or $A(y) = y^2$ and $F(x,y) = \frac{1}{3}x^3 + xy^2 + y^2$. We try to solve $F(x,y) = C$. We easily solve for y^2 and then just take the square root:

$$y^2 = \frac{C - (1/3)x^3}{x+1}, \qquad \text{so} \qquad y = \pm\sqrt{\frac{C - (1/3)x^3}{x+1}}.$$

When $x = -1$, the term in front of $\frac{dy}{dx}$ vanishes. You can also see that our solution is not valid in that case. However, one could in that case try to solve for x in terms of y starting from the implicit solution $\frac{1}{3}x^3 + xy^2 + y^2 = C$. The solution is somewhat messy and we leave it as implicit.

1.2 Integrating factors

Sometimes an equation $M\,dx + N\,dy = 0$ is not exact, but it can be made exact by multiplying with a function $u(x,y)$. That is, perhaps for some nonzero function $u(x,y)$,

$$u(x,y)M(x,y)\,dx + u(x,y)N(x,y)\,dy = 0$$

is exact. Any solution to this new equation is also a solution to $M\,dx + N\,dy = 0$.

In fact, a linear equation

$$\frac{dy}{dx} + p(x)y = f(x), \qquad \text{or} \qquad \bigl(p(x)y - f(x)\bigr)\,dx + dy = 0$$

is always such an equation. Let $r(x) = e^{\int p(x)\,dx}$ be the integrating factor for a linear equation. Multiply the equation by $r(x)$ and write it in the form of $M + N\frac{dy}{dx} = 0$.

$$r(x)p(x)y - r(x)f(x) + r(x)\frac{dy}{dx} = 0.$$

Then $M = r(x)p(x)y - r(x)f(x)$, so $M_y = r(x)p(x)$, while $N = r(x)$, so $N_x = r'(x) = r(x)p(x)$. In other words, we have an exact equation. Integrating factors for linear functions are just a special case of integrating factors for exact equations.

But how do we find the integrating factor u? Well, given an equation

$$M\,dx + N\,dy = 0,$$

u should be a function such that

$$\frac{\partial}{\partial y}\bigl[uM\bigr] = u_yM + uM_y = \frac{\partial}{\partial x}\bigl[uN\bigr] = u_xN + uN_x.$$

Therefore,

$$(M_y - N_x)u = u_xN - u_yM.$$

At first it may seem we replaced one differential equation by another. True, but all hope is not lost.

A strategy that often works is to look for a u that is a function of x alone, or a function of y alone. If u is a function of x alone, that is $u(x)$, then we write $u'(x)$ instead of u_x, and u_y is just zero. Then

$$\frac{M_y - N_x}{N}u = u'.$$

In particular, $\frac{M_y - N_x}{N}$ ought to be a function of x alone (not depend on y). If so, then we have a linear equation

$$u' - \frac{M_y - N_x}{N} u = 0.$$

Letting $P(x) = \frac{M_y - N_x}{N}$, we solve using the standard integrating factor method, to find $u(x) = Ce^{\int P(x)\,dx}$. The constant in the solution is not relevant, we need any nonzero solution, so we take $C = 1$. Then $u(x) = e^{\int P(x)\,dx}$ is the integrating factor.

Similarly we could try a function of the form $u(y)$. Then

$$\frac{M_y - N_x}{M} u = -u'.$$

In particular, $\frac{M_y - N_x}{M}$ ought to be a function of y alone. If so, then we have a linear equation

$$u' + \frac{M_y - N_x}{M} u = 0.$$

Letting $Q(y) = \frac{M_y - N_x}{M}$, we find $u(y) = Ce^{-\int Q(y)\,dy}$. We take $C = 1$. So $u(y) = e^{-\int Q(y)\,dy}$ is the integrating factor.

Example 1.6: Solve

$$\frac{x^2 + y^2}{x + 1} + 2y\frac{dy}{dx} = 0.$$

Let $M = \frac{x^2+y^2}{x+1}$ and $N = 2y$. Compute

$$M_y - N_x = \frac{2y}{x+1} - 0 = \frac{2y}{x+1}.$$

As this is not zero, the equation is not exact. We notice

$$P(x) = \frac{M_y - N_x}{N} = \frac{2y}{x+1}\frac{1}{2y} = \frac{1}{x+1}$$

is a function of x alone. We compute the integrating factor

$$e^{\int P(x)\,dx} = e^{\ln(x+1)} = x + 1.$$

We multiply our given equation by $(x + 1)$ to obtain

$$x^2 + y^2 + 2y(x+1)\frac{dy}{dx} = 0,$$

which is an exact equation that we solved in Example 1.5. The solution was

$$y = \pm\sqrt{\frac{C - (1/3)x^3}{x + 1}}.$$

Example 1.7: Solve

$$y^2 + (xy + 1)\frac{dy}{dx} = 0.$$

First compute

$$M_y - N_x = 2y - y = y.$$

As this is not zero, the equation is not exact. We observe

$$Q(y) = \frac{M_y - N_x}{M} = \frac{y}{y^2} = \frac{1}{y}$$

is a function of y alone. We compute the integrating factor

$$e^{-\int Q(y)\,dy} = e^{-\ln y} = \frac{1}{y}.$$

Therefore we look at the exact equation

$$y + \frac{xy+1}{y}\frac{dy}{dx} = 0.$$

The reader should double check that this equation is exact. We follow the procedure for exact equations

$$F(x,y) = xy + A(y),$$

and

$$\frac{xy+1}{y} = x + \frac{1}{y} = x + A'(y). \tag{2}$$

Consequently $A'(y) = \frac{1}{y}$ or $A(y) = \ln y$. Thus $F(x,y) = xy + \ln y$. It is not possible to solve $F(x,y) = C$ for y in terms of elementary functions, so let us be content with the implicit solution:

$$xy + \ln y = C.$$

We are looking for the general solution and we divided by y above. We should check what happens when $y = 0$, as the equation itself makes perfect sense in that case. We plug in $y = 0$ to find the equation is satisfied. So $y = 0$ is also a solution.

1.3 Exercises

***Exercise* 1.2:** *Solve the following exact equations, implicit general solutions will suffice*:

a) $(2xy + x^2)\,dx + (x^2 + y^2 + 1)\,dy = 0$

b) $x^5 + y^5\frac{dy}{dx} = 0$

c) $e^x + y^3 + 3xy^2\frac{dy}{dx} = 0$

d) $(x+y)\cos(x) + \sin(x) + \sin(x)y' = 0$

***Exercise* 1.3:** *Find the integrating factor for the following equations making them into exact equations*:

a) $e^{xy}\,dx + \frac{y}{x}e^{xy}\,dy = 0$

b) $\frac{e^x + y^3}{y^2}\,dx + 3x\,dy = 0$

c) $4(y^2 + x)\,dx + \frac{2x+2y^2}{y}\,dy = 0$

d) $2\sin(y)\,dx + x\cos(y)\,dy = 0$

***Exercise* 1.4:** *Suppose you have an equation of the form*: $f(x) + g(y)\frac{dy}{dx} = 0$.

a) *Show it is exact.*

b) *Find the form of the potential function in terms of f and g.*

***Exercise* 1.5:** *Suppose that we have the equation $f(x)\,dx - dy = 0$.*

a) *Is this equation exact?*

b) *Find the general solution using a definite integral.*

Exercise* 1.6*: *Find the potential function $F(x,y)$ of the exact equation $\frac{1+xy}{x}\,dx + \left(1/y + x\right)dy = 0$ in two different ways.*

a) *Integrate M in terms of x and then differentiate in y and set to N.*

b) *Integrate N in terms of y and then differentiate in x and set to M.*

Exercise* 1.7*: *A function $u(x,y)$ is said to be a* harmonic function *if $u_{xx} + u_{yy} = 0$.*

a) *Show if u is harmonic, $-u_y\,dx + u_x\,dy = 0$ is an exact equation. So there exists (at least locally) the so-called* harmonic conjugate *function $v(x,y)$ such that $v_x = -u_y$ and $v_y = u_x$.*

Verify that the following u are harmonic and find the corresponding harmonic conjugates v:

b) $u = 2xy$

c) $u = e^x \cos y$

d) $u = x^3 - 3xy^2$

Exercise* 1.101*: *Solve the following exact equations, implicit general solutions will suffice:*

a) $\cos(x) + ye^{xy} + xe^{xy}y' = 0$

b) $(2x+y)\,dx + (x-4y)\,dy = 0$

c) $e^x + e^y \frac{dy}{dx} = 0$

d) $(3x^2+3y)\,dx + (3y^2+3x)\,dy = 0$

Exercise* 1.102*: *Find the integrating factor for the following equations making them into exact equations:*

a) $\frac{1}{y}\,dx + 3y\,dy = 0$

b) $dx - e^{-x-y}\,dy = 0$

c) $\left(\frac{\cos(x)}{y^2} + \frac{1}{y}\right)dx + \frac{x}{y^2}\,dy = 0$

d) $\left(2y + \frac{y^2}{x}\right)dx + (2y+x)\,dy = 0$

Exercise* 1.103*:

a) *Show that every separable equation $y' = f(x)g(y)$ can be written as an exact equation, and verify that it is indeed exact.*

b) *Using this rewrite $y' = xy$ as an exact equation, solve it and verify that the solution is the same as it was in Example* **??**.

2 First order linear PDE

Note: 1 lecture, can safely be skipped

We only considered ODE so far, so let us solve a linear first order PDE. Consider the equation

$$a(x,t)\,u_x + b(x,t)\,u_t + c(x,t)\,u = g(x,t), \qquad u(x,0) = f(x), \qquad -\infty < x < \infty, \quad t > 0,$$

where $u(x,t)$ is a function of x and t. The *initial condition* $u(x,0) = f(x)$ is now a function of x rather than just a number. In these problems, it is useful to think of x as position and t as time. The equation describes the evolution of a function of x as time goes on. Below, the coefficients a, b, c, and the function g are mostly going to be constant or zero. The method we describe works with nonconstant coefficients, although the computations may get difficult quickly.

This method we use is the *method of characteristics*. The idea is that we find lines along which the equation is an ODE that we solve. We will see this technique again for second order PDE when we encounter the wave equation in § **??**.

Example 2.1: Consider the equation

$$u_t + \alpha u_x = 0, \qquad u(x,0) = f(x).$$

This particular equation, $u_t + \alpha u_x = 0$, is called the *transport equation*.

The data will propagate along curves called characteristics. The idea is to change to the so-called *characteristic coordinates*. If we change to these coordinates, the equation simplifies. The change of variables for this equation is

$$\xi = x - \alpha t, \qquad s = t.$$

Let's see what the equation becomes. Remember the chain rule in several variables.

$$\begin{aligned} u_t &= u_\xi \xi_t + u_s s_t = -\alpha u_\xi + u_s, \\ u_x &= u_\xi \xi_x + u_s s_x = u_\xi. \end{aligned}$$

The equation in the coordinates ξ and s becomes

$$\underbrace{(-\alpha u_\xi + u_s)}_{u_t} + \alpha \underbrace{(u_\xi)}_{u_x} = 0,$$

or in other words

$$u_s = 0.$$

That is trivial to solve. Treating ξ as simply a parameter, we have obtained the ODE $\frac{du}{ds} = 0$.

The solution is a function that does not depend on s (but it does depend on ξ). That is, there is some function A such that

$$u = A(\xi) = A(x - \alpha t).$$

The initial condition says that:

$$f(x) = u(x,0) = A(x - \alpha 0) = A(x),$$

so $A = f$. In other words,

$$u(x,t) = f(x - \alpha t).$$

Everything is simply moving right at speed α as t increases. The curve given by the equation

$$\xi = \text{constant}$$

is called the characteristic. See Figure 3 on the following page. In this case, the solution does not change along the characteristic.

In the (x, t) coordinates, the characteristic curves satisfy $t = \frac{1}{\alpha}(x - \xi)$, and are in fact lines. The slope of characteristic lines is $\frac{1}{\alpha}$, and for each different ξ we get a different characteristic line.

We see why $u_t + \alpha u_x = 0$ is called the transport equation: everything travels at some constant speed. Sometimes this is called *convection*. An example application is material being moved by a river where the material does not diffuse and is simply carried along. In this setup, x is the position along the river, t is the time, and $u(x, t)$ the concentration the material at position x and time t. See Figure 4 for an example.

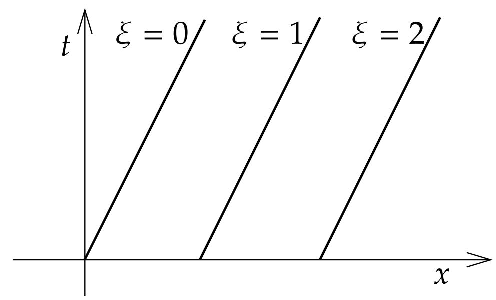

Figure 3: Characteristic curves.

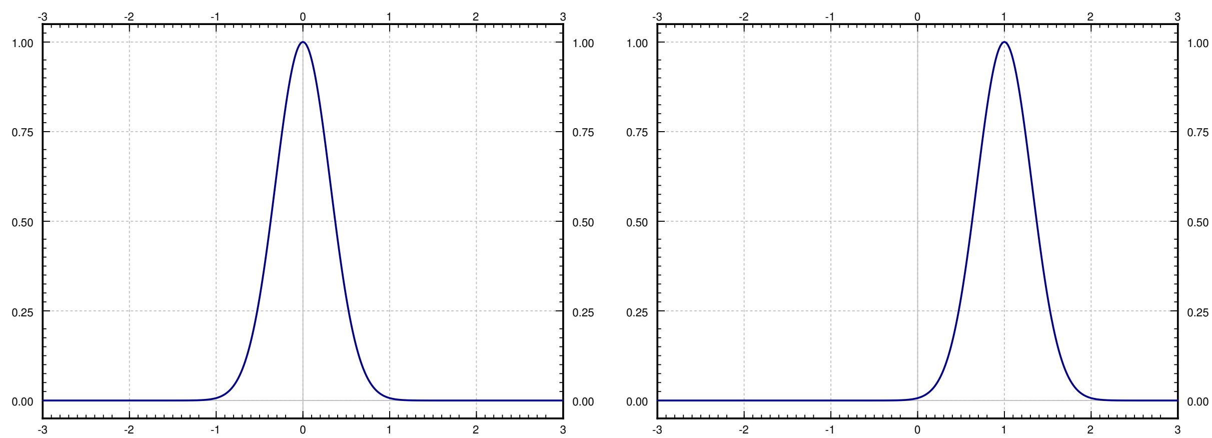

Figure 4: Example of "transport" in $u_t - u_x = 0$ (that is, $\alpha = 1$) where the initial condition $f(x)$ is a peak at the origin. On the left is a graph of the initial condition $u(x, 0)$. On the right is a graph of the function $u(x, 1)$, that is at time $t = 1$. Notice it is the same graph shifted one unit to the right.

We use similar idea in the more general case:

$$a u_x + b u_t + c u = g, \qquad u(x, 0) = f(x).$$

We change coordinates to the characteristic coordinates. Let us call these coordinates (ξ, s). These are coordinates where $a u_x + b u_t$ becomes differentiation in the s variable.

Along the characteristic curves (where ξ is constant), we get a new ODE in the s variable. In the transport equation, we got the simple $\frac{du}{ds} = 0$. In general, we get the linear equation

$$\frac{du}{ds} + cu = g. \tag{3}$$

We think of everything as a function of ξ and s, although we are thinking of ξ as a parameter rather than an independent variable. So the equation is an ODE. It is a linear ODE that we can solve using the integrating factor.

To find the characteristics, think of a curve given parametrically $\big(x(s), t(s)\big)$. We try to have the curve satisfy

$$\frac{dx}{ds} = a, \qquad \frac{dt}{ds} = b.$$

Why? Because when we think of x and t as functions of s we find, using the chain rule,

$$\frac{du}{ds} + cu = \underbrace{\left(u_x \frac{dx}{ds} + u_t \frac{dt}{ds}\right)}_{\frac{du}{ds}} + cu = au_x + bu_t + cu = g.$$

So we get the ODE (3), which then describes the value of the solution u of the PDE along this characteristic curve. It is also convenient to make sure that $s = 0$ corresponds to $t = 0$, that is $t(0) = 0$. It will be convenient also for $x(0) = \xi$. See Figure 5.

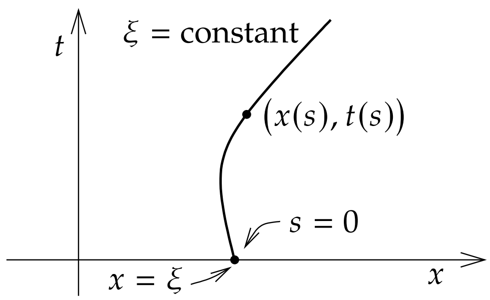

Figure 5: General characteristic curve.

Example 2.2: Consider

$$u_x + u_t + u = x, \qquad u(x,0) = e^{-x^2}.$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = 1, \qquad \frac{dt}{ds} = 1.$$

So

$$x = s + c_1, \qquad t = s + c_2,$$

for some c_1 and c_2. At $s = 0$ we want $t = 0$, and x should be ξ. So we let $c_1 = \xi$ and $c_2 = 0$:

$$x = s + \xi, \qquad t = s.$$

The ODE is $\frac{du}{ds} + u = x$, and $x = s + \xi$. So, the ODE to solve along the characteristic is

$$\frac{du}{ds} + u = s + \xi.$$

The general solution of this equation, treating ξ as a parameter, is $u = Ce^{-s} + s + \xi - 1$, for some constant C. At $s = 0$, our initial condition is that u is $e^{-\xi^2}$, since at $s = 0$ we have $x = \xi$. Given this initial condition, we find $C = e^{-\xi^2} - \xi + 1$. So,

$$\begin{aligned} u &= \left(e^{-\xi^2} - \xi + 1\right)e^{-s} + s + \xi - 1 \\ &= e^{-\xi^2 - s} + (1 - \xi)e^{-s} + s + \xi - 1. \end{aligned}$$

Substitute $\xi = x - t$ and $s = t$ to find u in terms of x and t:

$$\begin{aligned} u &= e^{-\xi^2 - s} + (1 - \xi)e^{-s} + s + \xi - 1 \\ &= e^{-(x-t)^2 - t} + (1 - x + t)e^{-t} + x - 1. \end{aligned}$$

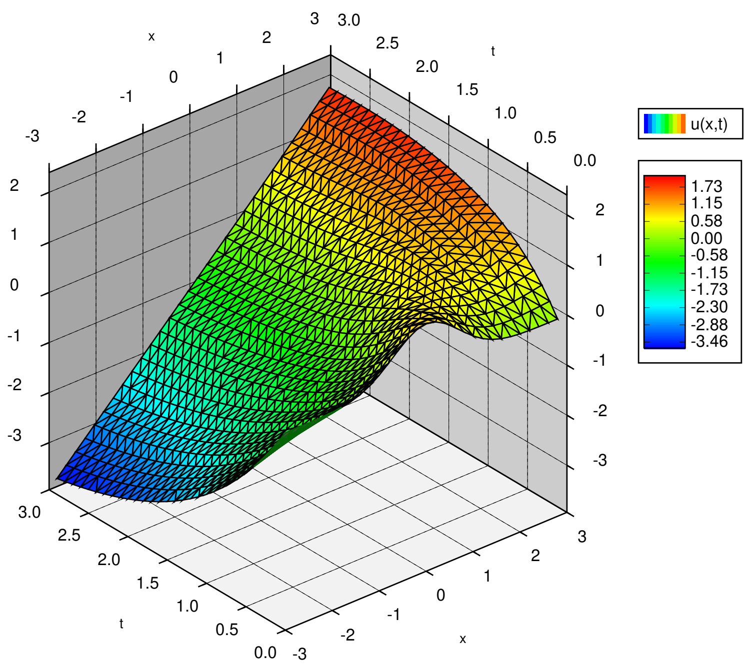

Figure 6: Plot of the solution $u(x,t)$ to $u_x + u_t + u = x$, $u(x,0) = e^{-x^2}$.

See Figure 6 for a plot of $u(x,t)$ as a function of two variables.

When the coefficients are not constants, the characteristic curves are not going to be straight lines anymore.

Example 2.3: Consider the following variable coefficient equation:

$$xu_x + u_t + 2u = 0, \qquad u(x,0) = \cos(x).$$

We find the characteristics, that is, the curves given by

$$\frac{dx}{ds} = x, \qquad \frac{dt}{ds} = 1.$$

So

$$x = c_1 e^s, \qquad t = s + c_2.$$

At $s = 0$, we wish to get the line $t = 0$, and x should be ξ. So

$$x = \xi e^s, \qquad t = s.$$

OK, the ODE we need to solve is

$$\frac{du}{ds} + 2u = 0.$$

This is for a fixed ξ. At $s = 0$, we should get that u is $\cos(\xi)$, so that is our initial condition. Consequently,

$$u = e^{-2s}\cos(\xi) = e^{-2t}\cos(xe^{-t}).$$

We make a few closing remarks. One thing to keep in mind is that we would get into trouble if the coefficient in front of u_t, that is the b, is ever zero. Let us consider a quick example of what can go wrong:

$$u_x + u = 0, \qquad u(x,0) = \sin(x).$$

This problem has no solution. If we had a solution, it would imply that $u_x(x,0) = \cos(x)$, but $u_x(x,0)+u(x,0) = \cos(x) + \sin(x) \neq 0$. The problem is that the characteristic curve is now the line $t = 0$, and the solution is already provided on that line!

As long as b is nonzero, it is convenient to ensure that b is positive by multiplying by -1 if necessary, so that positive s means positive t.

Another remark is that if a or b in the equation are variable, the computations can quickly get out of hand, as the expressions for the characteristic coordinates become messy and then solving the ODE becomes even messier. In the examples above, b was always 1, meaning we got $s = t$ in the characteristic coordinates. If b is not constant, your expression for s will be more complicated.

Finding the characteristic coordinates is really a system of ODE in general if a depends on t or if b depends on x. In that case, we would need techniques of systems of ODE to solve, see chapter **??** or chapter **??**. In general, if a and b are not linear functions or constants, finding closed form expressions for the characteristic coordinates may be impossible.

Finally, the method of characteristics applies to nonlinear first order PDE as well. In the nonlinear case, the characteristics depend not only on the differential equation, but also on the initial data. This leads to not only more difficult computations, but also the formation of singularities where the solution breaks down at a certain point in time. An example application where first order nonlinear PDE come up is traffic flow theory, and you have probably experienced the formation of singularities: traffic jams. But we digress.

2.1 Exercises

Exercise* 2.1*: *Solve*

a) $u_t + 9u_x = 0, \quad u(x,0) = \sin(x),$

b) $u_t - 8u_x = 0, \quad u(x,0) = \sin(x),$

c) $u_t + \pi u_x = 0, \quad u(x,0) = \sin(x),$

d) $u_t + \pi u_x + u = 0, \quad u(x,0) = \sin(x).$

Exercise* 2.2*: *Solve* $u_t + 3u_x = 1$, $u(x,0) = x^2$.

Exercise* 2.3*: *Solve* $u_t + 3u_x = x$, $u(x,0) = e^x$.

Exercise* 2.4*: *Solve* $u_x + u_t + xu = 0$, $u(x,0) = \cos(x)$.

Exercise* 2.5*:

a) *Find the characteristic coordinates for the following equations*:

1) $u_x + u_t + u = 1, \ u(x,0) = \cos(x),$ 2) $2u_x + 2u_t + 2u = 2, \ u(x,0) = \cos(x).$

b) *Solve the two equations using the coordinates.*

c) *Explain why you got the same solution, although the characteristic coordinates you found were different.*

Exercise* 2.6*: *Solve* $(1 + x^2)u_t + x^2u_x + e^xu = 0$, $u(x,0) = 0$. *Hint*: *Think a little out of the box.*

Exercise* 2.101*: *Solve*

a) $u_t - 5u_x = 0$, $u(x,0) = \frac{1}{1+x^2}$,

b) $u_t + 2u_x = 0$, $u(x,0) = \cos(x)$.

Exercise* 2.102*: *Solve* $u_x + u_t + tu = 0$, $u(x,0) = \cos(x)$.

Exercise* 2.103*: *Solve* $u_x + u_t = 5$, $u(x,0) = x$.