Counting techniques

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- To calculate $P(E)$, we need to count the number of elements in E and Ω.
- Need to learn counting technique.

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Notation

- Denote number of elements in A by $n(A)$

Example

If $A = \{1, 3, 9\}$ then $n(A) = 3$

- if $A = \emptyset$ then $n(A) = 0$

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Inclusion–Exclusion Principle

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

- $n(A \cup B)$: the number of elements in either A or B
- In $n(A) + n(B)$, an element lies in both A and B, it is counted twice
- To make up for this double counting, we must subtract the number of elements counted twice, namely, $n(A \cap B)$
- So $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example

In the year 2012, Executive magazine surveyed the presidents of the 500 largest corporations in the United States. Of these 500 people, 310 had degrees (of any sort) in business, 238 had undergraduate degrees in business, and 184 had graduate degrees in business. How many presidents had both undergraduate and graduate degrees in business

- A = {presidents with an undergraduate degree in business}
- B ={presidents with a graduate degree in business}.

then $A \cup B$ = {presidents with at least one degree in business} and
$A \cap B$ = {presidents had both undergraduate and graduate degrees in business}

We have

$$n(A) = 238, n(B) = 184, n(A \cup B) = 310$$

By inclusion - exclusion principle, we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

So

$$n(A \cap B) = n(A) + n(B) - n(A \cup B) = 238+$$

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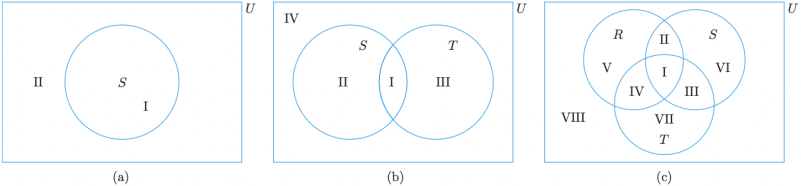
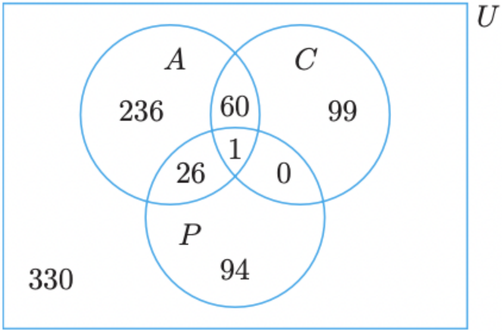
Each Venn diagram divides the sample space U into a certain number of regions.



Example

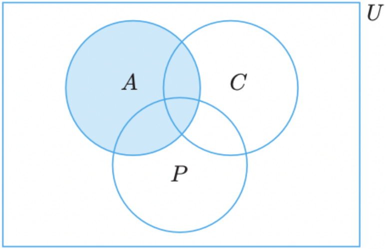
Let

- U = {Nobel laureates during the period 1901–2011}
- A = American Nobel laureates during the period 1901–2011
- C = Chemistry Nobel laureates during the period 1901–2011
- P = Nobel Peace Prize laureates during the period 1901–2011

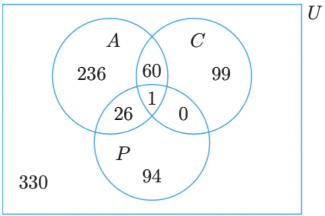


How many Americans received a Nobel Prize during the period 1901–2011?

Solution



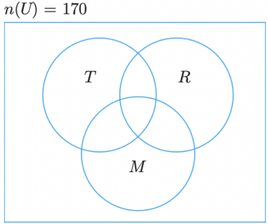
The number of Americans who received a Nobel Prize is the total contained in the circle A , which is 236+26+1+60 = 323.



1. How many Americans received Nobel Prizes in fields other than chemistry and peace during this period?
2. How many Americans received the Nobel Peace Prize during this period?

Practice

Among 170 clients using media, there are 115 use television (T), 100 use radio (R), 130 use magazines (M), 75 use television and radio, 95 use radio and magazines, 85 use television and magazines, 70 use all three



Complete the Venn diagram to display the clients' use of mass media.

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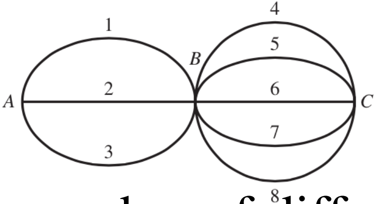
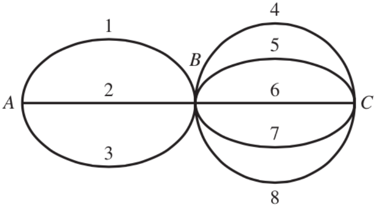
Suppose that there are three different routes from city A to city B and five different routes from city B to city C.



Count the number of different routes from A to C that pass through B



All possible routes is

$$\left\{\begin{matrix} (1,4) & (1,5) & (1,6) & (1,7) & (1,8) \\ (2,4) & (2,5) & (2,6) & (2,7) & (2,8) \\ (3,4) & (3,5) & (3,6) & (3,7) & (3,8) \end{matrix}\right\}$$

The number of routes is $3 \times 5 = 15$

Example - Experiment in Two Parts

Consider an experiment that has the following two characteristics:

1. The experiment is performed in two parts.
2. The first part of the experiment has m possible outcomes $x_1, \ldots, x_m$, and, regardless of which one of these outcomes x_i occurs, the second part of the experiment has n possible outcomes $y_1, \ldots, y_n$.

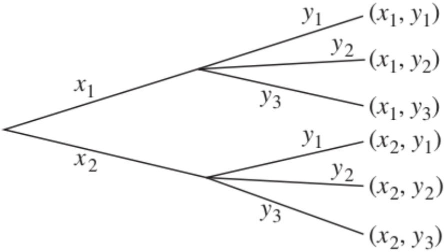
Each outcome in the sample space Ω of such an experiment will therefore be a pair having the form (x_i, y_j)

$$\Omega = \left\{ \begin{array}{llll} (x_1, y_1) & (x_1, y_2) & \dots & (x_1, y_n) \\ (x_2, y_1) & (x_2, y_2) & \dots & (x_2, y_n) \\ \quad \dots & & & \\ (x_m, y_1) & (x_m, y_2) & \dots & (x_m, y_n) \end{array} \right\}$$

Total number of outcomes in Ω is

$$n(\Omega) = mn$$

Tree diagram in which end-nodes represent outcomes



Multiplication rule

- Suppose there is a job that has 2 steps
- There are m ways to do step 1
- There are n ways to do step 2
- There are $m \times n$ ways to do the job.

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General multiplication rule

Suppose that a task consists of k steps performed consecutively. Suppose that step 1 can be performed in m_1 ways; for each of these, step 2 in m_2 ways; for each of these, step 3 in m_3 ways; and so forth. Then the task can be performed in

$$m = m_1 m_2 \ldots m_k$$

ways

Example

A corporation has a board of directors consisting of 10 members. The board must select from among its members a chairperson, vice chairperson, and secretary. In how many ways can this be done?

Solution

- The task Select the three officers can be divided into three consecutive steps
  Select chairperson $\rightarrow$ Select vice chairperson $\rightarrow$ Select secretary
- Step 1: there are 10 ways to select chairperson
- After select chairperson, there are 9 directors left
- Step 2: three are 9 ways to select vice chairperson

- After this step, there are 8 directors who are possible candidates for secretary
- Step 3: there are 8 ways to select secretary
- the number of possible ways to perform the sequence of three choices equals 10 .9 . 8, or 720
- So the officers of the board can be selected in 720 ways.

Sequential selection process

Number of ways is $10 \times 9 \times 8 = 720$

Example

There are 6 balls of different colors. How many ways to arrange them on a straight line?

Solution

- Think of a line with 6 positions
- 6 ways to choose ball for 1st position
- 5 ways for 2nd position, 4 for 3rd ...
- Total $6 \cdot 5 \cdot 4 \ldots 2 \cdot 1 = 6! = 720$ ways

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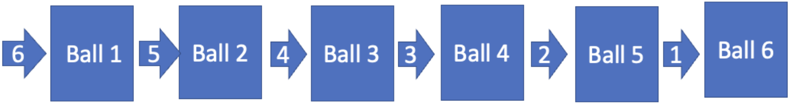

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Problem A

How many words (by which we mean strings of letters) of two distinct letters can be formed from the letters a, b, c?

Problem B

Problem B A construction crew has three members. A team of two must be chosen for a particular job. In

Solution

Enumerate all possibilities

Problem A

There are six possible words

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Problem B

Designate the three crew members by a, b, and c. Then there are three possible two-person teams

Both problems are concerned with counting the numbers of arrangements of the elements of the set $\{a, b, c\}$, taken two at a time, without allowing repetition. (For example, aa was not allowed.) However, in Problem A, the **order of the arrangement** *mattered*, whereas in Problem B it did *not*. Arrangements of the sort considered in Problem A are called *permutations*, whereas those in Problem B are called *combinations*.

Permutation

Given a set of n distinguishable objects. Then **a permutation of n objects taken r at a time** is an arrangement of r of the n objects in a specific order.

Combination

A combination of n objects taken r at a time is a selection of r objects from among the n, with order disregarded.

Permutation formula

The number of permutations of n objects taken r at a time is

$$P(n, r) = n(n-1)\ldots(n-r+1)$$

Combination formula

The number of combinations of n objects taken r at a time is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n \dots (n-r+1)}{r(r-1)\dots 1}$$

Another convenient formula

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Alternative notation of $C(n,r)$ is $\binom{n}{r}$ read "n choose r"

Example

The board of directors of a corporation has 10 members. In how many ways can they choose a committee of three board members to negotiate a merger?

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Solution

Since the committee of three involves no ordering of its members, we are concerned here with combinations. The number of combinations of 10 people taken 3 at a time is $C(10, 3) = 120$

Example

Eight horses are entered in a race in which a first, second, and third prize will be awarded. Assuming no ties, how many different outcomes are possible?

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Eight horses are entered in a race in which a first, second, and third prize will be awarded. Assuming no ties, how many different outcomes are possible?

Solution

we are considering ordered arrangements of three horses, so we are dealing with permutations. The number of permutations of eight horses taken three at a time is

$$P(8,3) = 8.7.6 = 336$$

Practice

A club has 10 members. In how many ways can they choose a slate of four officers, consisting of a president, vice president, secretary, and treasurer?

Practice

Suppose that an experiment consists of tossing a coin 10 times and observing the sequence of heads and tails.

1. How many different outcomes are possible?
2. How many different outcomes have exactly four heads?

Practice

An urn contains 25 numbered balls, of which 15 are red and 10 are white. A sample of 5 balls is to be selected.

1. How many different samples are possible?
2. How many samples contain all red balls?
3. How many samples contain 3 red balls and 2 white balls?
4. How many samples contain at least 4 red balls?