Lecture notes: Differential Equations for ISE (MA029IU)

Week 6 *

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*This note is taken from "Notes on Diffy Qs: Differential Equations for Engineers" by Jiri Lebl

1 Second order linear ODEs

*Note: 1 lecture, reduction of order optional, first part of §3.1 in [**?**], parts of §3.1 and §3.2 in [**?**]*

Let us consider the general *second order linear differential equation*

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by $A(x)$ to get

$$y'' + p(x)y' + q(x)y = f(x), \tag{1}$$

where $p(x) = {}^{B(x)}/_{A(x)}$, $q(x) = {}^{C(x)}/_{A(x)}$, and $f(x) = {}^{F(x)}/_{A(x)}$. The word *linear* means that the equation contains no powers nor functions of y, y', and y''.

In the special case when $f(x) = 0$, we have a so-called *homogeneous* equation

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

We have already seen some second order linear homogeneous equations.

$$y'' + k^2 y = 0 \;\text{ Two solutions are:} \quad y_1 = \cos(kx), \quad y_2 = \sin(kx).$$
$$y'' - k^2 y = 0 \;\text{ Two solutions are:} \quad y_1 = e^{kx}, \quad y_2 = e^{-kx}.$$

If we know two solutions of a linear homogeneous equation, we know many more of them.

Theorem 1.1 (Superposition)**.** *Suppose y_1 and y_2 are two solutions of the homogeneous equation* (2)*. Then*

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

also solves (2) *for arbitrary constants C_1 and C_2.*

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2. Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work.

Proof: Let $y = C_1 y_1 + C_2 y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1 y_1 + C_2 y_2)'' + p(C_1 y_1 + C_2 y_2)' + q(C_1 y_1 + C_2 y_2) \\ &= C_1 y_1'' + C_2 y_2'' + C_1 p y_1' + C_2 p y_2' + C_1 q y_1 + C_2 q y_2 \\ &= C_1(y_1'' + py_1' + qy_1) + C_2(y_2'' + py_2' + qy_2) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0. \quad \square \end{aligned}$$

The proof becomes even simpler to state if we use the operator notation. An *operator* is an object that eats functions and spits out functions (kind of like what a function is, but a function eats numbers and spits out numbers). Define the operator L by

$$Ly = y'' + py' + qy.$$

The differential equation now becomes $Ly = 0$. The operator (and the equation) L being *linear* means that $L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2$. The proof above becomes

$$Ly = L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 = C_1 \cdot 0 + C_2 \cdot 0 = 0.$$

Two different solutions to the second equation $y'' - k^2 y = 0$ are $y_1 = \cosh(kx)$ and $y_2 = \sinh(kx)$. Let us remind ourselves of the definition, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Therefore, these are solutions by superposition as they are linear combinations of the two exponential solutions.

The functions sinh and cosh are sometimes more convenient to use than the exponential. Let us review some of their properties:

$$\begin{aligned} &\cosh 0 = 1, && \sinh 0 = 0, \\ &\frac{d}{dx}\Big[\cosh x\Big] = \sinh x, && \frac{d}{dx}\Big[\sinh x\Big] = \cosh x, \\ &\cosh^2 x - \sinh^2 x = 1. \end{aligned}$$

***Exercise* 1.1:** *Derive these properties using the definitions of* sinh *and* cosh *in terms of exponentials.*

Linear equations have nice and simple answers to the existence and uniqueness question.

Theorem 1.2 (Existence and uniqueness). *Suppose p, q, f are continuous functions on some interval I, a is a number in I, and a, b_0, b_1 are constants. The equation*

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

For example, the equation $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

The equation $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx).$$

Using cosh and sinh in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Question: Suppose we find two different solutions y_1 and y_2 to the homogeneous equation (2). Can every solution be written (using superposition) in the form $y = C_1 y_1 + C_2 y_2$?

Answer is affirmative! Provided that y_1 and y_2 are different enough in the following sense. We say y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem 1.3. *Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation* (2)*. Then every other solution is of the form*

$$y = C_1 y_1 + C_2 y_2.$$

That is, $y = C_1 y_1 + C_2 y_2$ is the general solution.

For example, we found the solutions $y_1 = \sin x$ and $y_2 = \cos x$ for the equation $y'' + y = 0$. It is not hard to see that sine and cosine are not constant multiples of each other. If $\sin x = A \cos x$ for some constant A, we let $x = 0$ and this would imply $A = 0$. But then $\sin x = 0$ for all x, which is preposterous. So y_1 and y_2 are linearly independent. Hence,

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

For two functions, checking linear independence is rather simple. Let us see another example. Consider $y'' - 2x^{-2}y = 0$. Then $y_1 = x^2$ and $y_2 = {}^1/_x$ are solutions. To see that they are linearly indepedent, suppose one is a multple of the other: $y_1 = Ay_2$, we just have to find out that A cannot be a constant. In this case we have $A = {}^{y_1}/_{y_2} = x^3$, this most decidedly not a constant. So $y = C_1x^2 + C_2{}^1/_x$ is the general solution.

If you have one solution to a second order linear homogeneous equation, then you can find another one. This is the *reduction of order method*. The idea is that if we somehow found y_1 as a solution of $y'' + p(x)y' + q(x)y = 0$ we try a second solution of the form $y_2(x) =$

$y_1(x)v(x)$. We just need to find v. We plug y_2 into the equation:

$$
\begin{aligned}
0 = y_2'' + p(x)y_2' + q(x)y_2 &= y_1''v + 2y_1'v' + y_1v'' + p(x)(y_1'v + y_1v') + q(z)y_1v \\
&= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\big(y_1'' + p(x)y_1' + q(x)y_1\big)}v.
\end{aligned}
$$

In other words, $y_1v'' + (2y_1' + p(x)y_1)v' = 0$. Using $w = v'$ we have the first order linear equation $y_1w' + (2y_1' + p(x)y_1)w = 0$. After solving this equation for w (integrating factor), we find v by antidifferentiating w. We then form y_2 by computing y_1v. For example, suppose we somehow know $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$. The equation for w is then $xw' + 3w = 0$. We find a solution, $w = Cx^{-3}$, and we find an antiderivative $v = \frac{-C}{2x^2}$. Hence $y_2 = y_1v = \frac{-C}{2x}$. Any C works and so $C = -2$ makes $y_2 = {}^1/_x$. Thus, the general solution is $y = C_1x + C_2{}^1/_x$.

Since we have a formula for the solution to the first order linear equation, we can write a formula for y_2:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{\big(y_1(x)\big)^2}\,dx$$

However, it is much easier to remember that we just need to try $y_2(x) = y_1(x)v(x)$ and find $v(x)$ as we did above. Also, the technique works for higher order equations too: you get to reduce the order for each solution you find. So it is better to remember how to do it rather than a specific formula.

We will study the solution of nonhomogeneous equations in § **??**. We will first focus on finding general solutions to homogeneous equations.

1.1 Exercises

***Exercise* 1.2:** *Show that $y = e^x$ and $y = e^{2x}$ are linearly independent.*

***Exercise* 1.3:** *Take $y'' + 5y = 10x + 5$. Find (guess!) a solution.*

***Exercise* 1.4:** *Prove the superposition principle for nonhomogeneous equations. Suppose that y_1 is a solution to $Ly_1 = f(x)$ and y_2 is a solution to $Ly_2 = g(x)$ (same linear operator L). Show that $y = y_1 + y_2$ solves $Ly = f(x) + g(x)$.*

***Exercise* 1.5:** *For the equation $x^2y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution. Hint: Try $y = x^r$.*

Equations of the form $ax^2y'' + bxy' + cy = 0$ are called *Euler's equations* or *Cauchy–Euler equations*. They are solved by trying $y = x^r$ and solving for r (assume that $x \geq 0$ for simplicity).

***Exercise* 1.6:** *Suppose that $(b - a)^2 - 4ac > 0$.*

a) *Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Try $y = x^r$ and find a formula for r.*

b) *What happens when $(b - a)^2 - 4ac = 0$ or $(b - a)^2 - 4ac < 0$?*

We will revisit the case when $(b - a)^2 - 4ac < 0$ later.

***Exercise* 1.7:** *Same equation as in Exercise 1.6. Suppose $(b-a)^2 - 4ac = 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Try $y = x^r \ln x$ for the second solution.*

***Exercise* 1.8** (reduction of order)**:** *Suppose y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$. By directly plugging into the equation, show that*

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{\big(y_1(x)\big)^2}\,dx$$

is also a solution.

***Exercise* 1.9** (Chebyshev's equation of order 1)**:** *Take $(1-x^2)y'' - xy' + y = 0$.*

a) Show that $y = x$ is a solution.

b) Use reduction of order to find a second linearly independent solution.

c) Write down the general solution.

***Exercise* 1.10** (Hermite's equation of order 2)**:** *Take* $y'' - 2xy' + 4y = 0$.

a) *Show that* $y = 1 - 2x^2$ *is a solution.*

b) *Use reduction of order to find a second linearly independent solution.* (*It's OK to leave a definite integral in the formula.*)

c) *Write down the general solution.*

***Exercise* 1.101:** *Are* $\sin(x)$ *and* e^x *linearly independent? Justify.*

***Exercise* 1.102:** *Are* e^x *and* e^{x+2} *linearly independent? Justify.*

***Exercise* 1.103:** *Guess a solution to* $y'' + y' + y = 5$.

***Exercise* 1.104:** *Find the general solution to* $xy'' + y' = 0$. *Hint: It is a first order ODE in* y'.

***Exercise* 1.105:** *Write down an equation* (*guess*) *for which we have the solutions* e^x *and* e^{2x}. *Hint: Try an equation of the form* $y'' + Ay' + By = 0$ *for constants* A *and* B, *plug in both* e^x *and* e^{2x} *and solve for* A *and* B.

2 Constant coefficient second order linear ODEs

2.1 Solving constant coefficient equations

Consider the problem

$$y'' - 6y' + 8y = 0, \qquad y(0) = -2, \qquad y'(0) = 6.$$

This is a second order linear homogeneous equation with constant coefficients. *Constant coefficients* means that the functions in front of y'', y', and y are constants, they do not depend on x.

To guess a solution, think of a function that stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero. Yes, we are talking about the exponential.

Let us try* a solution of the form $y = e^{rx}$. Then $y' = re^{rx}$ and

*Making an educated guess with some parameters to solve for is such a central technique in differential

$y'' = r^2e^{rx}$. Plug in to get

$$
\begin{aligned}
y'' - 6y' + 8y &= 0, \\
\underbrace{r^2e^{rx}}_{y''} - 6\underbrace{re^{rx}}_{y'} + 8\underbrace{e^{rx}}_{y} &= 0, \\
r^2 - 6r + 8 &= 0 \qquad \text{(divide through by } e^{rx}\text{)}, \\
(r-2)(r-4) &= 0.
\end{aligned}
$$

Hence, if $r = 2$ or $r = 4$, then e^{rx} is a solution. So let $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

***Exercise* 2.1:** *Check that y_1 and y_2 are solutions.*

The functions e^{2x} and e^{4x} are linearly independent. If they were not linearly independent, we could write $e^{4x} = Ce^{2x}$ for some constant C,

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equations, that people sometimes use a fancy name for such a guess: *ansatz*, German for "initial placement of a tool at a work piece." Yes, the Germans have a word for that.

implying that $e^{2x} = C$ for all x, which is clearly not possible. Hence, we can write the general solution as

$$y = C_1 e^{2x} + C_2 e^{4x}.$$

We need to solve for C_1 and C_2. To apply the initial conditions, we first find $y' = 2C_1 e^{2x} + 4C_2 e^{4x}$. We plug $x = 0$ into y and y' and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2, \\ 6 &= y'(0) = 2C_1 + 4C_2. \end{aligned}$$

Either apply some matrix algebra, or just solve these by high school math. For example, divide the second equation by 2 to obtain $3 = C_1 + 2C_2$, and subtract the two equations to get $5 = C_2$. Then $C_1 = -7$ as $-2 = C_1 + 5$. Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}.$$

Let us generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{3}$$

where a, b, c are constants. Try the solution $y = e^{rx}$ to obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Divide by e^{rx} to obtain the so-called *characteristic equation* of the ODE:

$$ar^2 + br + c = 0.$$

Solve for the r by using the quadratic formula.

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So e^{r_1x} and e^{r_2x} are solutions. There is still a difficulty if $r_1 = r_2$, but it is not hard to overcome.

Theorem 2.1. *Suppose that r_1 and r_2 are the roots of the characteristic equation.*

(*i*) *If* r_1 *and* r_2 *are distinct and real* (*when* $b^2 - 4ac > 0$)*, then* (3) *has the general solution*

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

(*ii*) *If* $r_1 = r_2$ (*happens when* $b^2 - 4ac = 0$)*, then* (3) *has the general solution*

$$y = (C_1 + C_2 x)\, e^{r_1 x}.$$

Example 2.1: Solve

$$y'' - k^2 y = 0.$$

The characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions, and the general solution is

$$y = C_1 e^{kx} + C_2 e^{-kx}.$$

Since $\cosh s = \frac{e^s + e^{-s}}{2}$ and $\sinh s = \frac{e^s - e^{-s}}{2}$, we can also write the general solution as

$$y = D_1 \cosh(kx) + D_2 \sinh(kx).$$

Example 2.2: Find the general solution of

$$y'' - 8y' + 16y = 0.$$

The characteristic equation is $r^2 - 8r + 16 = (r-4)^2 = 0$. The equation has a double root $r_1 = r_2 = 4$. The general solution is, therefore,

$$y = (C_1 + C_2 x)\, e^{4x} = C_1 e^{4x} + C_2 x e^{4x}.$$

***Exercise* 2.2:** *Check that e^{4x} and xe^{4x} are linearly independent.*

That e^{4x} solves the equation is clear. If xe^{4x} solves the equation, then we know we are done. Let us compute $y' = e^{4x} + 4xe^{4x}$ and $y'' = 8e^{4x} + 16xe^{4x}$. Plug in

$$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8(e^{4x} + 4xe^{4x}) + 16xe^{4x} = 0.$$

In some sense, a doubled root rarely happens. If coefficients are picked randomly, a doubled root is unlikely. There are, however, some natural phenomena (such as resonance as we will see) where a doubled root does happen, so we cannot just dismiss this case.

Let us give a short argument for why the solution xe^{rx} works when the root is doubled. This case is really a limiting case of when the two roots are distinct and very close. Note that $\frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1}$ is a solution when the roots are distinct. When we take the limit as r_1 goes to r_2, we are really taking the derivative of e^{rx} using r as the variable. Therefore, the limit is xe^{rx}, and hence this is a solution in the doubled root case.

2.2 Complex numbers and Euler's formula

A polynomial may have complex roots. The equation $r^2 + 1 = 0$ has no real roots, but it does have two complex roots. Here we review some properties of complex numbers.

Complex numbers may seem a strange concept, especially because of the terminology. There is nothing imaginary or really complicated about complex numbers. A complex number is simply a pair of real numbers,

(a, b). Think of a complex number as a point in the plane. We add complex numbers in the straightforward way: $(a, b) + (c, d) = (a + c, b + d)$. We define multiplication by

$$(a, b) \times (c, d) \stackrel{\text{def}}{=} (ac - bd, ad + bc).$$

It turns out that with this multiplication rule, all the standard properties of arithmetic hold. Further, and most importantly $(0, 1) \times (0, 1) = (-1, 0)$.

Generally we write (a, b) as $a + ib$, and we treat i as if it were an unknown. When b is zero, then $(a, 0)$ is just the number a. We do arithmetic with complex numbers just as we would with polynomials. The property we just mentioned becomes $i^2 = -1$. So whenever we see i^2, we replace it by -1. For example,

$$(2 + 3i)(4i) - 5i = (2 \times 4)i + (3 \times 4)i^2 - 5i = 8i + 12(-1) - 5i = -12 + 3i.$$

The numbers i and $-i$ are the two roots of $r^2 + 1 = 0$. Engineers often use the letter j instead of i for the square root of -1. We use the mathematicians' convention and use i.

Exercise 2.3: *Make sure you understand (that you can justify) the following identities:*

a) $i^2 = -1$, $i^3 = -i$, $i^4 = 1$,

b) $\dfrac{1}{i} = -i$,

c) $(3-7i)(-2-9i) = \cdots = -69 - 13i$,

d) $(3-2i)(3+2i) = 3^2 - (2i)^2 = 3^2 + 2^2 = 13$,

e) $\frac{1}{3-2i} = \frac{1}{3-2i}\frac{3+2i}{3+2i} = \frac{3+2i}{13} = \frac{3}{13} + \frac{2}{13}i$.

We also define the exponential e^{a+ib} of a complex number. We do this by writing down the Taylor series and plugging in the complex number. Because most properties of the exponential can be proved by looking at the Taylor series, these properties still hold for the complex exponential. For example the very important property: $e^{x+y} = e^x e^y$. This means that $e^{a+ib} = e^a e^{ib}$. Hence if we can compute e^{ib}, we can compute e^{a+ib}. For e^{ib} we use the so-called *Euler's formula*.

Theorem 2.2 (Euler's formula)**.**

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta \qquad \textit{and} \qquad e^{-i\theta} = \cos\theta - i\sin\theta.}$$

In other words, $e^{a+ib} = e^a\big(\cos(b) + i\sin(b)\big) = e^a\cos(b) + ie^a\sin(b)$.

***Exercise* 2.4:** *Using Euler's formula, check the identities*:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \textit{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

***Exercise* 2.5:** *Double angle identities*: *Start with* $e^{i(2\theta)} = \big(e^{i\theta}\big)^2$. *Use Euler on each side and deduce*:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \qquad \textit{and} \qquad \sin(2\theta) = 2\sin\theta\cos\theta.$$

For a complex number $a + ib$ we call a the *real part* and b the *imaginary part* of the number. Often the following notation is used,

$$\operatorname{Re}(a+ib) = a \qquad \text{and} \qquad \operatorname{Im}(a+ib) = b.$$

2.3 Complex roots

Suppose the equation $ay'' + by' + cy = 0$ has the characteristic equation $ar^2 + br + c = 0$ that has complex roots. By the quadratic formula, the roots are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$. These roots are complex if $b^2 - 4ac < 0$. In this case the roots are

$$r_1, r_2 = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}.$$

As you can see, we always get a pair of roots of the form $\alpha \pm i\beta$. In this case we can still write the solution as

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}.$$

However, the exponential is now complex-valued. We need to allow C_1 and C_2 to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

Here we can use Euler's formula. Let

$$y_1 = e^{(\alpha+i\beta)x} \qquad \text{and} \qquad y_2 = e^{(\alpha-i\beta)x}.$$

Then

$$\begin{aligned} y_1 &= e^{\alpha x}\cos(\beta x) + i e^{\alpha x}\sin(\beta x), \\ y_2 &= e^{\alpha x}\cos(\beta x) - i e^{\alpha x}\sin(\beta x). \end{aligned}$$

Linear combinations of solutions are also solutions. Hence,

$$\begin{aligned} y_3 &= \frac{y_1 + y_2}{2} = e^{\alpha x}\cos(\beta x), \\ y_4 &= \frac{y_1 - y_2}{2i} = e^{\alpha x}\sin(\beta x), \end{aligned}$$

are also solutions. Furthermore, they are real-valued. It is not hard to see that they are linearly independent (not multiples of each other). Therefore, we have the following theorem.

Theorem 2.3. *Take the equation*

$$ay'' + by' + cy = 0.$$

If the characteristic equation has the roots $\alpha \pm i\beta$ *(when* $b^2 - 4ac < 0$*), then the general solution is*

$$y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x).$$

Example 2.3: Find the general solution of $y'' + k^2 y = 0$, for a constant $k > 0$.

The characteristic equation is $r^2 + k^2 = 0$. Therefore, the roots are $r = \pm ik$, and by the theorem, we have the general solution

$$y = C_1 \cos(kx) + C_2 \sin(kx).$$

Example 2.4: Find the solution of $y'' - 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 10$.

The characteristic equation is $r^2 - 6r + 13 = 0$. By completing the square we get $(r-3)^2 + 2^2 = 0$ and hence the roots are $r = 3 \pm 2i$. By the theorem we have the general solution

$$y = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x).$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0 = y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1.$$

Hence, $C_1 = 0$ and $y = C_2 e^{3x} \sin(2x)$. We differentiate,

$$y' = 3C_2 e^{3x} \sin(2x) + 2C_2 e^{3x} \cos(2x).$$

We again plug in the initial condition and obtain $10 = y'(0) = 2C_2$, or $C_2 = 5$. The solution we are seeking is

$$y = 5e^{3x} \sin(2x).$$

2.4 Exercises

***Exercise* 2.6:** *Find the general solution of* $2y'' + 2y' - 4y = 0$.

***Exercise* 2.7:** *Find the general solution of* $y'' + 9y' - 10y = 0$.

***Exercise* 2.8:** *Solve* $y'' - 8y' + 16y = 0$ *for* $y(0) = 2$, $y'(0) = 0$.

***Exercise* 2.9:** *Solve* $y'' + 9y' = 0$ *for* $y(0) = 1$, $y'(0) = 1$.

***Exercise* 2.10:** *Find the general solution of* $2y'' + 50y = 0$.

***Exercise* 2.11:** *Find the general solution of* $y'' + 6y' + 13y = 0$.

***Exercise* 2.12:** *Find the general solution of* $y'' = 0$ *using the methods of this section.*

***Exercise* 2.13:** *The method of this section applies to equations of other orders than two. We will see higher orders later. Try to solve the first order equation* $2y' + 3y = 0$ *using the methods of this section.*

***Exercise* 2.14:** *Let us revisit the Cauchy–Euler equations of Exercise 1.6 on page 11. Suppose now that* $(b-a)^2 - 4ac < 0$. *Find a formula for the general solution of* $ax^2y'' + bxy' + cy = 0$. *Hint*: *Note that* $x^r = e^{r \ln x}$.

Exercise* 2.15*: *Find the solution to* $y'' - (2\alpha)y' + \alpha^2 y = 0$, $y(0) = a$, $y'(0) = b$, *where* α, a, *and* b *are real numbers.*

Exercise* 2.16*: *Construct an equation such that* $y = C_1 e^{-2x}\cos(3x) + C_2 e^{-2x}\sin(3x)$ *is the general solution.*

Exercise* 2.101*: *Find the general solution to* $y'' + 4y' + 2y = 0$.

Exercise* 2.102*: *Find the general solution to* $y'' - 6y' + 9y = 0$.

Exercise* 2.103*: *Find the solution to* $2y'' + y' + y = 0$, $y(0) = 1$, $y'(0) = -2$.

Exercise* 2.104*: *Find the solution to* $2y'' + y' - 3y = 0$, $y(0) = a$, $y'(0) = b$.

Exercise* 2.105*: *Find the solution to* $z''(t) = -2z'(t) - 2z(t)$, $z(0) = 2$, $z'(0) = -2$.

Exercise* 2.106*: *Find the solution to* $y'' - (\alpha + \beta)y' + \alpha\beta y = 0$, $y(0) = a$, $y'(0) = b$, *where* α, β, a, *and* b *are real numbers, and* $\alpha \neq \beta$.

Exercise* 2.107*: *Construct an equation such that* $y = C_1 e^{3x} + C_2 e^{-2x}$ *is the general solution.*