Applied Linear Algebra
Spring 2024

Homework

Chapter 1

Week 1

Recall that *A system of linear equations is said to be* **consistent** *if it has either one solution or infinitely many solutions; a system is* **inconsistent** *if it has no solution.*

1. Determine if the following system is consistent:

$$\begin{cases} x_1 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{cases}$$

2. Determine which matrices are in reduced echelon form and which others are only in echelon form.

$$\text{a. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{b. } \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{d. } \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

3. Reduced the matrices to echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

4. Find the general solutions of the systems whose augmented matrices

a) $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$

b) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$

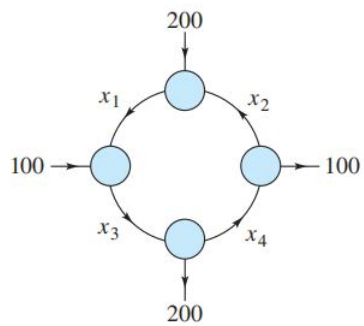
5 Solve the system

a) $$\begin{cases} 4x + 2y + z = 18 \\ 4x - 2y - 2z = 28 \\ 2x - 3y + 2z = -8 \end{cases}$$

b) $$\begin{cases} 2x_1 + x_2 + x_3 + 2x_4 = -1 \\ 5x_1 - 2x_2 + x_3 - 3x_4 = 0 \\ -x_1 + 3x_2 + 2x_3 + 2x_4 = 1 \\ 3x_1 + 2x_2 + 3x_3 - 5x_4 = 12 \end{cases}$$

Applications

6. *(Network Analysis)* The figure shows the flow of traffic (in vehicles per hour) through a network of streets.



a) Solve this system for $x_i, i = 1, 2, 3, 4$

b) Find the traffic flow when $x_4 = 0$.

c) Find the traffic flow when $x_4 = 100$.

Week 2

1. Answer the following questions

a) If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

b) How many rows does B have if BC is a 3×4 matrix?

2. Let $A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 1 & -2 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 & -3 \\ -1 & 2 & 1 \\ -3 & -1 & 0 \end{pmatrix}$.
Find the following if possible.

a) $A + 20B$, $B - 5A^T$, and BA

b) $A + 4C^T$, AC and CA

3. Let $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -5 \\ 3 & c \end{pmatrix}$. What is value of c such that $AB = BA$?

4. Let $A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$. Find matrix B such that $AB = 0$

5. Consider the following system of equation

$$\begin{cases} 3x_1 + x_2 + x_3 = 3 \\ x_1 - x_2 - x_3 = 1 \\ x_1 + 2x_2 + 2x_3 = 1 \end{cases}$$

Denote $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ the vector solution of the equation. Express your solution in the form $x = tv + su$, where v and u are column vector in three dimensions, $t, s \in \mathbb{R}$.

Week 3

Inverse matrices

1. Suppose A, B, and X are $n \times n$matrices with A, X, and $A - AX$ is invertible and and suppose

$$(A - AX)^{-1} = X^{-1}B \qquad (*)$$

a) Explain why B is invertible

b) Solve $(*)$ for X. If you need to invert a matrix, explain why that matrix is invertible.

2. Find the inverses of the matrices, if they exist

a)

$$\begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{pmatrix}$$

c)

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

3. If $A, B,$ and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A+X)B^{-1} = I_n$ have a solution, X? If so, find it.

4. Use an inverse matrix to solve system of linear equations.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= -1 \\ x_1 - 2x_2 + x_3 &= 2 \\ x_2 - x_2 - x_3 &= 0 \end{aligned}$$

5 Prove that if $A^2 = A$ then $I - 2A = (I - 2A)^{-1}$.

Chapter 2

Week 4

Determinants

1. Find the determinants in the following problems by row reduction to echelon form.

a)

$$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$$

b)

$$\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix}$$

2. We know that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Find the determinant of the following matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix}$$

3. Compute $det(B^4)$ where

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

4. Using cofactors, find inverse of the following matrices

a)

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 6 & -2 & 5 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

5. Use Gaussian elimination and Cramer's Rule to solve the systems.

a)

$$\begin{cases} 7x_1 + x_2 - 4x_3 = 3 \\ -6x_1 - 4x_2 + x_3 = 0 \\ 4x_1 - x_2 - 2x_3 = 6 \end{cases}$$

b)

$$\begin{cases} 2x_1 + 3x_2 - 5x_3 = 2 \\ 3x_1 - x_2 + 2x_3 = 1 \\ 5x_1 + 4x_2 - 6x_3 = 3 \end{cases}$$

Chapter 3. Vector Spaces

Week 5

1. Determine whether the set, together with the standard operations, is a vector space?

a) The set $S = \{(x, y) : x \geq 0, y \in \mathbb{R}\}$

b) The set $S = \{(x, x/2) : x \in \mathbb{R}\}$

2. Determine whether the set $\mathbb{R}^2$ with the operations
$(x_1, y_1) + (x_2, y_2) = (x_1y_1, x_2y_2)$
and $c(x_1, y_1) = (cx_1, cy_1)$ where $c \in \mathbb{R}$,
is a vector space. If it is, verify each vector space axiom; if it is not, state all vector space axioms that fail.

3. Determine whether the set W is a subspace of $\mathbb{R}^3$ with the standard operations. Justify your answer.

a) $W = \{(0, x_2, x_3) : x_2; x_3 \quad \text{are real numbers}\}$

b) $W = \{(x_1, x_2, 4) : x_1 \quad \text{and} \quad x_2 \quad \text{are real numbers}\}$

4. Write each vector as a linear combination of the vectors in S (if possible).

$$S = \{(2, 0, 7), (2, 4, 5), (2, -12, 13)\}$$

a) $u = (-1, 5, -6)$

b) $v = (-3, 15, 18)$

5. Determine whether the set S spans $\mathbb{R}^3$.

a) $S = \{(4,7,3),(-1,2,6),(2,-3,5)\}$

b) $S = \{(5,6,5),(2,1,-5),(0,-4,1)\}$

6. Determine whether the set S is linearly independent or linearly dependent.

a) $S = \{(-2,1,3),(2,9,-3),(2,3,-3)\}$

b) $S = \{(-4,-3,4),(1,-2,3),(6,0,0)\}$

7. For which values of t is each set linearly independent?

$$S = \{(t,1,1),(1,t,1),(1,1,t)\}$$

Week 6

1. Find rank, nullity (dim Null) of the following matrix

a)

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ -2 & -6 & 4 & -8 \end{pmatrix}$$

2. Finding a Basis for a Row Space and Rank of the following matrix

a)

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} -2 & -4 & 4 & -5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{pmatrix}$$

3. Determining whether a Set is a Basis. If it is, write $u = (8, 3, 8)$ as a linear combination of the vectors in S

a) $S = \{(4, 3, 2), (0, 3, 2), (0, 0, 2)\}$

b) $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

c) $S = \{(0, 0, 0), (1, 3, 4), (6, 1, -2)\}$

4. Find the dimension and basic of the subspace

$$H = \left\{ \begin{pmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{pmatrix}, a, b, c \in \mathbb{R} \right\}$$