P3.1. (a) $\nabla \times (zx\mathbf{a}_x + xy\mathbf{a}_y + yz\mathbf{a}_z)$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ zx & xy & yz \end{vmatrix} = z\mathbf{a}_x + x\mathbf{a}_y + y\mathbf{a}_z$$

(b) $\nabla \times (\cos y\, \mathbf{a}_x - x \sin y\, \mathbf{a}_y)$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y & -x \sin y & 0 \end{vmatrix} = (-\sin y + \sin y)\, \mathbf{a}_z = \mathbf{0}$$

(c) $\nabla \times \dfrac{e^{-r^2}}{r}\, \mathbf{a}_\phi$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & e^{-r^2} & 0 \end{vmatrix} = \frac{\mathbf{a}_z}{r}(-2re^{-r^2}) = -2e^{-r^2}\, \mathbf{a}_z$$

(d) $\nabla \times (2r \cos \theta\, \mathbf{a}_r + r\, \mathbf{a}_\phi)$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2 \sin \theta} & \dfrac{\mathbf{a}_\theta}{r \sin \theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ 2r \cos \theta & r^2 & 0 \end{vmatrix} = \frac{\mathbf{a}_\phi}{r}\, (2r + 2r \sin \theta)$$

$$= 2\,(1 + \sin \theta)\, \mathbf{a}_\phi$$

P3.2. **(a)** $\mathbf{E} = E_0 \cos 3\pi z \cos 9\pi \times 10^8 t\, \mathbf{a}_x$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\frac{\partial E_x}{\partial z}\mathbf{a}_y$$

$$= 3\pi E_0 \sin 3\pi z \cos 9\pi \times 10^8 t\, \mathbf{a}_y$$

$$\mathbf{B} = \frac{3\pi E_0}{9\pi \times 10^8} \sin 3\pi z \sin 9\pi \times 10^8 t\, \mathbf{a}_y$$

$$= \frac{E_0}{3 \times 10^8} \sin 3\pi z \sin 9\pi \times 10^8 t\, \mathbf{a}_y$$

(b) $\mathbf{E} = E_0 \mathbf{a}_y \cos [3\pi \times 10^8 t + 0.2\pi(4x + 3z)]$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = \frac{\partial E_y}{\partial z}\mathbf{a}_x - \frac{\partial E_y}{\partial x}\mathbf{a}_z$$

$$= E_0(-0.6\pi \mathbf{a}_x + 0.8\pi \mathbf{a}_z) \sin [3\pi \times 10^8 t + 0.2\pi(4x + 3z)]$$

$$\mathbf{B} = -\frac{E_0(-0.6\pi \mathbf{a}_x + 0.8\pi \mathbf{a}_z)}{3\pi \times 10^8} \cos [3\pi \times 10^8 t + 0.2\pi(4x + 3z)]$$

$$= \frac{E_0(0.6\mathbf{a}_x - 0.8\mathbf{a}_z)}{3 \times 10^8} \cos [3\pi \times 10^8 t + 0.2\pi(4x + 3z)]$$

P3.3. **(a)** $\mathbf{H} = H_x(z, t)\,\mathbf{a}_x$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\frac{\partial H_x}{\partial z} = J_y + \frac{\partial D_y}{\partial t}$$

(b) $\mathbf{E} = E_\phi(r, t)\,\mathbf{a}_\phi$

$$\begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & 0 & 0 \\ 0 & rE_\phi & 0 \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) = -\frac{\partial B_z}{\partial t}$$

P3.4. $\mathbf{E} = E_0 e^{-\alpha z} \cos \omega t \, \mathbf{a}_x$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$$= \frac{\partial E_x}{\partial z} \mathbf{a}_y = -\alpha E_0 e^{-\alpha z} \cos \omega t \, \mathbf{a}_y$$

$$\mathbf{B} = \frac{\alpha E_0}{\omega} e^{-\alpha z} \sin \omega t \, \mathbf{a}_y$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\alpha E_0}{\omega \mu_0} e^{-\alpha z} \sin \omega t \, \mathbf{a}_y$$

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix}$$

$$= -\frac{\partial H_y}{\partial z} \mathbf{a}_x = \frac{\alpha^2 E_0}{\omega \mu_0} e^{-\alpha z} \sin \omega t \, \mathbf{a}_x$$

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon_0 E_0 e^{-\alpha z} \cos \omega t \, \mathbf{a}_x\right)$$

$$= -\omega \varepsilon_0 E_0 e^{-\alpha z} \sin \omega t \, \mathbf{a}_x$$

For $\nabla \times \mathbf{H}$ to be equal to $\frac{\partial \mathbf{D}}{\partial t}$, $\frac{\alpha^2 E_0}{\omega \mu_0}$ must be equal to $-\omega \varepsilon_0 E_0$, or α^2 must be equal to $-\omega^2 \mu_0 \varepsilon_0$. Since this is not possible for real values of α, the pair of **E** and **B** do not satisfy Ampere's circuital law in differential form.

P3.5. $\mathbf{E} = E_0 \cos(\omega t - \alpha y - \beta z)\,\mathbf{a}_x$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$$= \frac{\partial E_x}{\partial z}\mathbf{a}_y - \frac{\partial E_x}{\partial y}\mathbf{a}_z$$

$$= \beta E_0 \sin(\omega t - \alpha y - \beta z)\,\mathbf{a}_y - \alpha E_0 \sin(\omega t - \alpha y - \beta z)\,\mathbf{a}_z$$

$$\mathbf{B} = (\beta \mathbf{a}_y - \alpha \mathbf{a}_z)\,\frac{E_0}{\omega}\cos(\omega t - \alpha y - \beta z)$$

$$\mathbf{H} = (\beta \mathbf{a}_y - \alpha \mathbf{a}_z)\,\frac{E_0}{\mu_0 \omega}\cos(\omega t - \alpha y - \beta z)$$

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & H_z \end{vmatrix}$$

$$= \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x$$

$$= (-\alpha^2 - \beta^2)\,\frac{E_0}{\mu_0 \omega}\sin(\omega t - \alpha y - \beta z)\,\mathbf{a}_x$$

$$\mathbf{D} = (\alpha^2 + \beta^2)\,\frac{E_0}{\mu_0 \omega^2}\cos(\omega t - \alpha y - \beta z)\,\mathbf{a}_x$$

$$\mathbf{E} = (\alpha^2 + \beta^2)\,\frac{E_0}{\omega^2 \mu_0 \varepsilon_0}\cos(\omega t - \alpha y - \beta z)\,\mathbf{a}_x$$

$$\therefore\ \alpha^2 + \beta^2 = \omega^2 \mu_0 \varepsilon_0$$

P3.6. $\mathbf{E} = E_0 e^{-kx} \cos(2 \times 10^8 t - y)\, \mathbf{a}_z$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix}$$

$$= -\frac{\partial E_z}{\partial y}\mathbf{a}_x + \frac{\partial E_z}{\partial x}\mathbf{a}_y$$

$$= -E_0 e^{-kx} \sin(2 \times 10^8 t - y)\, \mathbf{a}_x - kE_0 e^{-kx} \cos(2 \times 10^8 t - y)\, \mathbf{a}_y$$

$$\mathbf{B} = \frac{E_0}{2 \times 10^8}\left[e^{-kx} \cos(2 \times 10^8 t - y)\, \mathbf{a}_x - ke^{-kx} \sin(2 \times 10^8 t - y)\, \mathbf{a}_y\right]$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\mathbf{B}}{4\pi \times 10^{-7}}$$

$$= \frac{E_0}{80\pi}\left[e^{-kx} \cos(2 \times 10^8 t - y)\, \mathbf{a}_x - ke^{-kx} \sin(2 \times 10^8 t - y)\, \mathbf{a}_y\right]$$

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & 0 \end{vmatrix}$$

$$= -\frac{\partial H_y}{\partial z}\mathbf{a}_x + \frac{\partial H_x}{\partial z}\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

$$= \frac{E_0}{80\pi}\left[k^2 e^{-kx} \sin(2 \times 10^8 t - y) - e^{-kx} \sin(2 \times 10^8 t - y)\right] \mathbf{a}_z$$

$$= \frac{E_0}{80\pi}(k^2 - 1)\, e^{-kx} \sin(2 \times 10^8 t - y)\, \mathbf{a}_z$$

$$\mathbf{D} = -\frac{E_0(k^2 - 1)}{80\pi \times 2 \times 10^8}\, e^{-kx} \cos(2 \times 10^8 t - y)\, \mathbf{a}_z$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \frac{\mathbf{D}}{10^{-9}/36\pi}$$

$$= -\frac{E_0(k^2 - 1)36\pi}{160\pi \times 10^{-1}}\, e^{-kx} \cos(2 \times 10^8 t - y)\, \mathbf{a}_z$$

$$= \frac{9}{4}E_0(1 - k^2)\, e^{-kx} \cos(2 \times 10^8 t - y)\, \mathbf{a}_z$$

P3.6. (continued)

For this field to be equal to the original **E**,

$$\frac{9}{4}E_0(1-k^2)=E_0$$

$$1-k^2=\frac{4}{9}$$

$$k^2=\frac{5}{9}$$

$$k=\pm\frac{\sqrt{5}}{3}$$

P3.7. (a)

$$\mathbf{J} = \begin{cases} J_0 \dfrac{z}{a}\mathbf{a}_x & \text{for } -a < z < a \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Since $\mathbf{J}$ has only an x-component, independent of x and y, $\nabla \times \mathbf{H} = \mathbf{J}$ reduces to

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = J_x \mathbf{a}_x$$

$$\frac{\partial H_y}{\partial z} = -J_x$$

$$H_y = -\int J_x(z)\,dz + C$$

$$= \begin{cases} -\int_{-\infty}^{z} 0\,dz + C & \text{for } z < -a \\ -\int_{-\infty}^{-a} 0\,dz - \int_{-a}^{z} J_0 \frac{z}{a}\,dz + C & \text{for } -a < z < a \\ -\int_{-\infty}^{-a} 0\,dz - \int_{-a}^{a} J_0 \frac{z}{a}\,dz - \int_{a}^{z} 0\,dz + C & \text{for } z > a \end{cases}$$

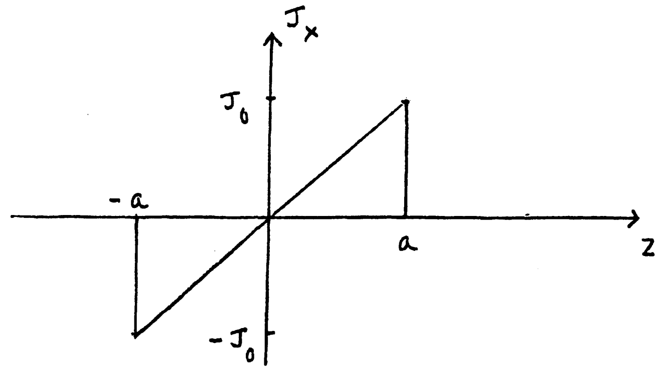
$$= \begin{cases} C & \text{for } z < -a \\ \dfrac{J_0}{2a}(a^2 - z^2) + C & \text{for } -a < z < a \\ C & \text{for } z > a \end{cases}$$

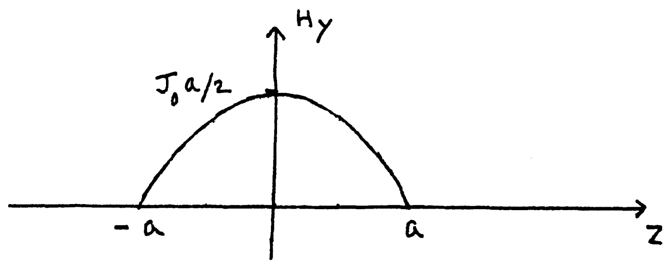
Note that the function $\frac{J_0}{2a}(a^2 - z^2)$ is symmetric about $z = 0$ and a nonzero constant simply raises or lowers the function above or below the $H_y = 0$ axis, thereby always maintaining the symmetry about $z = 0$. But since the entire field has to go to zero for $J_0 \to 0$, C has to be zero. Thus

$$\mathbf{H} = \begin{cases} \mathbf{0} & \text{for } z < -a \\ \dfrac{J_0}{2a}(a^2 - z^2)\,\mathbf{a}_y & \text{for } -a < z < a \\ \mathbf{0} & \text{for } z > a \end{cases}$$

The plots of J_x and H_y versus z are shown in the figure.

P3.7. (continued)





(b)

$$\mathbf{J} = \begin{cases} J_0\mathbf{a}_\phi & \text{for } a < r < 2a \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Since $\mathbf{J}$ has only a ϕ-component, independent of ϕ and z, $\nabla \times \mathbf{H} = \mathbf{J}$ reduces to

$$\begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & 0 & 0 \\ H_r & rH_\phi & H_z \end{vmatrix} = J_\phi \mathbf{a}_\phi$$

$$\frac{\partial H_z}{\partial r} = -J_\phi$$

$$H_z = -\int J_\phi(r)\, dr + C$$

$$= \begin{cases} -\int_0^r 0\, dr + C & \text{for } 0 < r < a \\ -\int_0^a 0\, dr - \int_a^r J_0\, dr + C & \text{for } a < r < 2a \\ -\int_0^a 0\, dr - \int_a^{2a} J_0\, dr + \int_{2a}^r 0\, dr + C & \text{for } r > 2a \end{cases}$$

$$= \begin{cases} C & \text{for } 0 < r < a \\ J_0(a-r)+C & \text{for } a < r < 2a \\ -J_0 a + C & \text{for } r > 2a \end{cases}$$

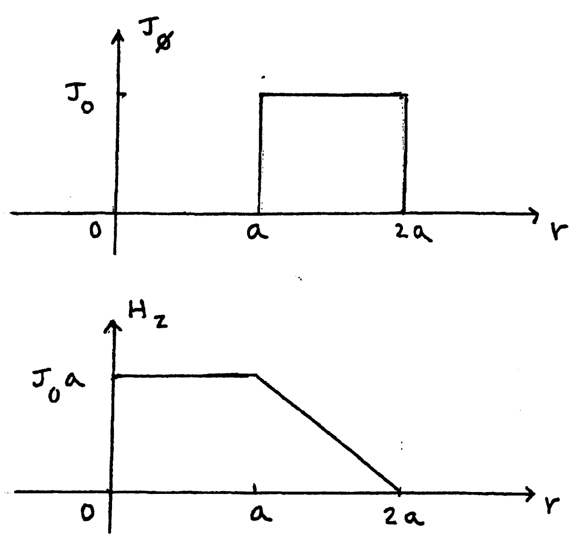
To determine the value of C, we note that a nonzero value of H_y for $r > 2a$ results in an infinite magnetic flux crossing a constant-z plane. Thus

$$-J_0 a + C = 0$$

$$C = J_0 a$$

$$\mathbf{H} = \begin{cases} J_0 a \mathbf{a}_z & \text{for } 0 < r < a \\ J_0(2a - r)\mathbf{a}_z & \text{for } a < r < 2a \\ \mathbf{0} & \text{for } r > 2a \end{cases}$$

The plots of J_ϕ and H_z are shown in the figure.



P3.8. **(a)** $\nabla \bullet (zx\mathbf{a}_x + xy\mathbf{a}_y + yz\mathbf{a}_z)$

$$= \frac{\partial}{\partial x}(zx) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(yz)$$

$$= z + x + y$$

$$= x + y + z$$

(b) $\nabla \bullet [3\mathbf{a}_x + (y-3)\mathbf{a}_y + (2+z)\mathbf{a}_z]$

$$= \frac{\partial}{\partial x}(3) + \frac{\partial}{\partial y}(y-3) + \frac{\partial}{\partial z}(2+z)$$

$$= 0 + 1 + 1$$

$$= 2$$

(c) $\nabla \bullet r \sin\phi\, \mathbf{a}_\phi$

$$= \frac{1}{r}\frac{\partial}{\partial r}(0) + \frac{1}{r}\frac{\partial}{\partial \phi}(r\sin\phi) + \frac{\partial}{\partial z}(0)$$

$$= \cos\phi$$

(d) $\nabla \bullet r\cos\theta\,(\cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta)$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\cos^2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(-r\cos\theta\sin^2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0)$$

$$= 3\cos^2\theta + \frac{1}{\sin\theta}(\sin^3\theta - 2\sin\theta\cos^2\theta)$$

$$= 3\cos^2\theta + \sin^2\theta - 2\cos^2\theta$$

$$= 1$$

P3.9. (a)

$$\rho = \begin{cases} \rho_0 x/a & \text{for } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

From $\nabla \bullet \mathbf{D} = \rho$, $\dfrac{\partial D_x}{\partial x} = \rho$

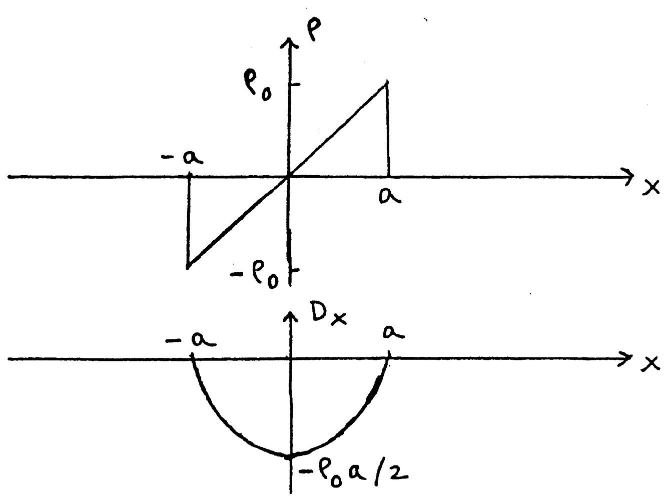
$$D_x = \int_{-\infty}^{x} \rho\, dx + C$$

$$= \begin{cases} \int_{-\infty}^{x} 0\, dx + C & \text{for } x < -a \\ \int_{-\infty}^{-a} 0\, dx + \int_{-a}^{x} \frac{\rho_0 x}{a}\, dx + C & \text{for } -a < x < a \\ \int_{-\infty}^{-a} 0\, dx + \int_{-a}^{a} \frac{\rho_0 x}{a}\, dx + C & \text{for } x > a \end{cases}$$

$$= \begin{cases} C & \text{for } x < -a \\ \frac{\rho_0}{2a}(x^2 - a^2) + C & \text{for } -a < x < a \\ 0 + C & \text{for } x > a \end{cases}$$

Since for $\rho_0 \to 0$, $\rho \to 0$, D_x must $\to 0$. Therefore $C = 0$.

$$\mathbf{D} = \begin{cases} \frac{\rho_0}{2a}(x^2 - a^2)\mathbf{a}_x & \text{for } -a < x < a \\ \mathbf{0} & \text{otherwise} \end{cases}$$



P3.9. (continued)

(b)

$$\rho = \begin{cases} \rho_0 r/a & \text{for } 0 < r < a \\ 0 & \text{otherwise} \end{cases}$$

From $\nabla \bullet \mathbf{D} = \rho$, $\dfrac{1}{r}\dfrac{\partial}{\partial r}(rD_r) = \rho$

$$rD_r = \int_0^r r\rho\, dr + C$$

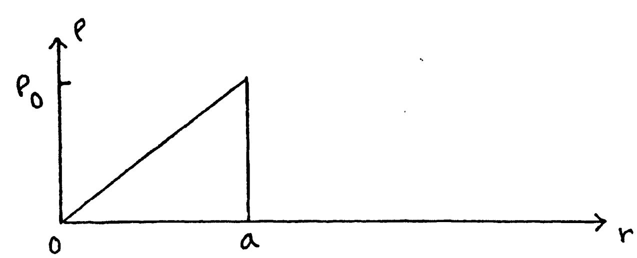
$$D_r = \frac{1}{r}\int_0^r r\rho\, dr + \frac{C}{r}$$

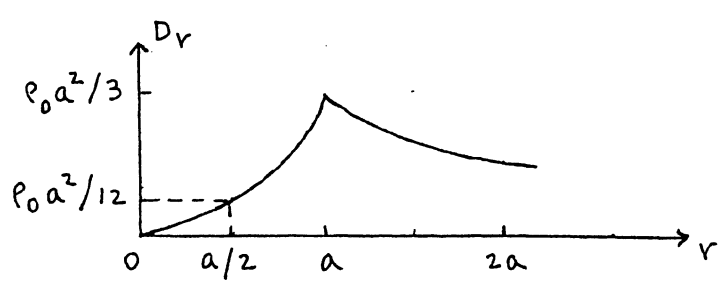
In the absence of a line charge at $r = 0$, $C = 0$. Thus

$$D_r = \frac{1}{r}\int_0^r r\rho\, dr$$

$$= \begin{cases} \dfrac{1}{r}\displaystyle\int_0^r \frac{\rho_0 r^2}{a}\, dr & \text{for } 0 < r < a \\ \dfrac{1}{r}\displaystyle\int_0^a \frac{\rho_0 r^2}{a}\, dr + \int_a^r r(0) dr & \text{for } r > a \end{cases}$$

$$\mathbf{D} = \begin{cases} (\rho_0 r^2/3a)\mathbf{a}_r & \text{for } 0 < r < a \\ (\rho_0 a^2/3r)\mathbf{a}_r & \text{for } r > a \end{cases}$$





P3.10. (a) $\nabla \bullet \frac{1}{y^k}(2x\mathbf{a}_x + y\mathbf{a}_y)$

$$= \frac{\partial}{\partial x}\left(\frac{2x}{y^k}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y^{k-1}}\right)$$

$$= \frac{2}{y^k} + \frac{1-k}{y^k}$$

$$= \frac{3-k}{y^k} = 0$$

$$\therefore\ k = 3$$

(b) $\nabla \bullet [r(\sin k\phi\, \mathbf{a}_r + \cos k\phi\, \mathbf{a}_\phi)]$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r^2 \sin k\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(r \cos k\phi)$$

$$= 2 \sin k\phi - k \sin k\phi = 0$$

$$\therefore\ k = 2$$

(c) $\nabla \bullet \left[\left(1 + \frac{2}{r^3}\right)\cos\theta\, \mathbf{a}_r + k\left(1 - \frac{1}{r^3}\right)\sin\theta\, \mathbf{a}_\theta\right]$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(r^2 + \frac{2}{r}\right)\cos\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[k\left(1 - \frac{1}{r^3}\right)\sin^2\theta\right]$$

$$= \frac{1}{r}\left(2 - \frac{2}{r^3}\right)\cos\theta + \frac{2k}{r}\left(1 - \frac{1}{r^3}\right)\cos\theta = 0$$

$$\therefore\ k = -1$$

P3.11. For a field to be realized both as an electric field in a charge-free region and a magnetic field in a current-free region, its divergence has to be zero, as well as its curl must be a null vector.

(a) $$\nabla \bullet (y\mathbf{a}_x + x\mathbf{a}_y) = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$$

$$\nabla \times (y\mathbf{a}_x + x\mathbf{a}_y)$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = \mathbf{0}$$

$\therefore$ Yes.

(b) $$\nabla \bullet \left[\left(1+\frac{1}{r^2}\right)\cos\phi\,\mathbf{a}_r - \left(1-\frac{1}{r^2}\right)\sin\phi\,\mathbf{a}_r\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[\left(r+\frac{1}{r}\right)\cos\phi\right] + \frac{1}{r}\frac{\partial}{\partial \phi}\left[-\left(1-\frac{1}{r^2}\right)\sin\phi\right]$$

$$= \frac{1}{r}\left(1-\frac{1}{r^2}\right)\cos\phi - \frac{1}{r}\left(1-\frac{1}{r^2}\right)\cos\phi = 0$$

$$\nabla \times \left[\left(1+\frac{1}{r^2}\right)\cos\phi\,\mathbf{a}_r - \left(1-\frac{1}{r^2}\right)\sin\phi\,\mathbf{a}_r\right]$$

$$= \begin{vmatrix} \frac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \frac{\mathbf{a}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \left(1+\frac{1}{r^2}\right)\cos\phi & -r\left(1-\frac{1}{r^2}\right)\sin\phi & 0 \end{vmatrix}$$

$$= \frac{\mathbf{a}_z}{r}\left[-\left(1+\frac{1}{r^2}\right)\sin\phi + \left(1+\frac{1}{r^2}\right)\sin\phi\right] = \mathbf{0}$$

$\therefore$ Yes

P3.11. (continued)

(c) $\nabla \bullet r \sin\theta\, \mathbf{a}_\phi$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta) = 0$$

$\nabla \mathbf{x}\, r\sin\theta\, \mathbf{a}_\phi$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r^2\sin\theta} & \dfrac{\mathbf{a}_\theta}{r\sin\theta} & \dfrac{\mathbf{a}_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix}$$

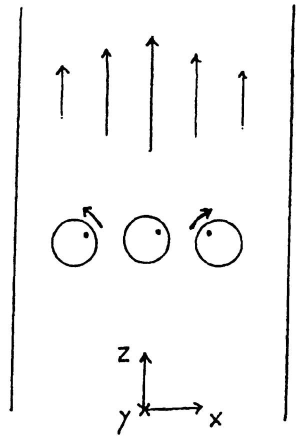
$$= \frac{1}{r^2\sin\theta}(2r^2\sin\theta\cos\theta)\,\mathbf{a}_r + \frac{\mathbf{a}_\theta}{r\sin\theta}(-2r\sin^2\theta)\,\mathbf{a}_\theta$$

$$= 2(\cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta) \neq \mathbf{0}$$

$\therefore$ No.

P3.12. $\mathbf{v} = \dfrac{4v_0}{a^2}(ax - x^2)\mathbf{a}_z$

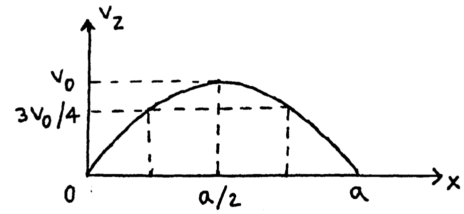
Curl meter when placed with axis along the y-axis rotates ccw to the left of the center, cw to the right of the center, and does not rotate at the center. Therefore, $[\nabla \times \mathbf{v}]_y$ is negative to the left, positive to the right, and zero at the center. Also on either side its magnitude is not constant because the velocity differential is not constant.

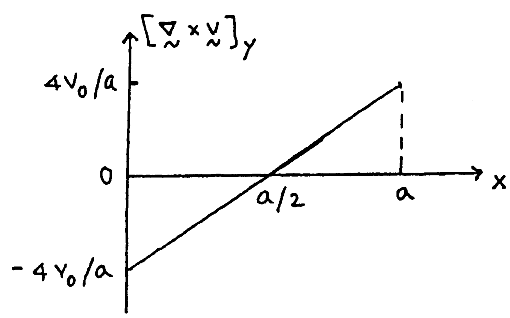


$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v_z \end{vmatrix}$$

$$= -\frac{\partial v_z}{\partial x}\mathbf{a}_y$$

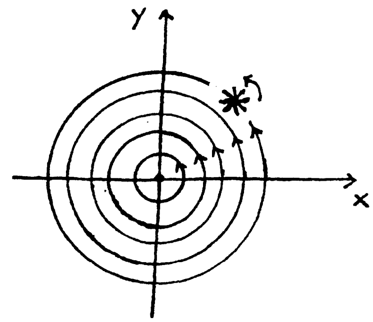
$$= -\frac{4v_0}{a^2}(a - 2x)\mathbf{a}_y$$





P3.13. $\mathbf{v} = \omega r \mathbf{a}_\phi$

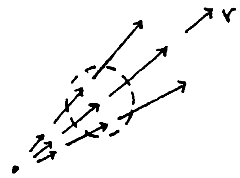
Curl meter when placed with its axis along the z-axis rotates cw everywhere. Therefore, $[\nabla \times \mathbf{v}]_z$ is positive everywhere.



$$\nabla \times \mathbf{v} = \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & \omega r^2 & 0 \end{vmatrix} = 2\omega \mathbf{a}_z$$

P3.14. **(a)** $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z = r\mathbf{a}_r$

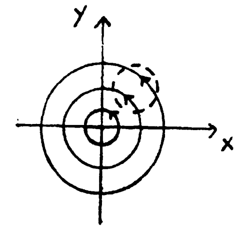
Since the field increases in magnitude along its own direction, the divergence meter expands.



$\therefore$ Divergence is positive.

$$\nabla \bullet \mathbf{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3) = \frac{3r^2}{r^2} = 3 > 0$$

(b) $\mathbf{v} = \omega r\mathbf{a}_\phi = \omega(-y\mathbf{a}_x + x\mathbf{a}_y)$

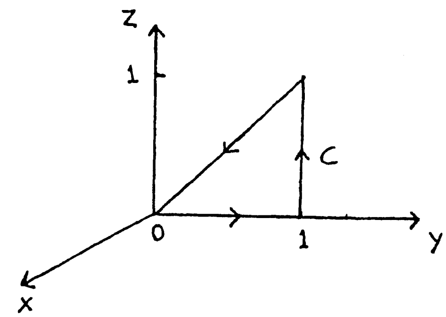


Since the field does not vary in magnitude in its own direction, the divergence meter is unaffected.

$\therefore$ Divergence is zero.

$$\nabla \bullet \omega(-y\mathbf{a}_x + x\mathbf{a}_y) = \omega\left[\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)\right] = 0$$

P3.15. **(a)** $\mathbf{A} = zx\mathbf{a}_x + xy\mathbf{a}_y + yz\mathbf{a}_z$



From (0, 0, 0) to (0, 1, 0),

$x = 0, z = 0; dx = dz = 0$

$d\mathbf{l} = dy\,\mathbf{a}_y$

$\mathbf{A} = \mathbf{0}$

$$\therefore \int_{(0,0,0)}^{(0,1,0)} \mathbf{A} \bullet d\mathbf{l} = 0$$

From (0, 1, 0) to (0, 1, 1),

$x = 0, y = 1; dx = dy = 0, d\mathbf{l} = dz\,\mathbf{a}_z$

$\mathbf{A} = z\mathbf{a}_z, \mathbf{A} \bullet d\mathbf{l} = z\,dz$

$$\int_{(0,1,0)}^{(0,1,1)} \mathbf{A} \bullet d\mathbf{l} = \int_0^1 z\,dz = \frac{1}{2}$$

From (0, 1, 1) to (0, 0, 0),

$x = 0, y = z; dx = 0, dy = dz, d\mathbf{l} = dz\,\mathbf{a}_y + dz\,\mathbf{a}_z$

$\mathbf{A} = z^2\mathbf{a}_z, \mathbf{A} \bullet d\mathbf{l} = z^2\,dz$

$$\int_{(0,1,1)}^{(0,0,0)} \mathbf{A} \bullet d\mathbf{l} = \int_1^0 z^2\,dz = -\frac{1}{3}$$

$$\therefore \oint_C \mathbf{A} \bullet d\mathbf{l} = 0 + \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx & xy & yz \end{vmatrix}$$

$$= z\mathbf{a}_x + x\mathbf{a}_y + y\mathbf{a}_z$$

$d\mathbf{S} = dy\,dz\,\mathbf{a}_x$

$$\int_S (\nabla \times \mathbf{A}) \bullet dS = \int_{y=0}^{1} \int_{z=0}^{y} z\,dy\,dz = \int_{y=0}^{1} \frac{y^2}{2}\,dy = \frac{1}{6}$$

Thus Stokes' theorem is verified.

P3.15. (continued)

(b) $\mathbf{A} = \cos y\, \mathbf{a}_x - x \sin y\, \mathbf{a}_y$

$$\oint_C \mathbf{A} \bullet d\mathbf{l} = \oint_C (\cos y\, \mathbf{a}_x - x \sin y\, \mathbf{a}_y) \bullet (dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z)$$

$$= \oint_C \cos y\, dx - x \sin y\, dy$$

$$= \oint_C d(x \cos y)$$

$$= \left[x \cos y\right]_{x_1,\, y_1,\, z_1}^{x_1,\, y_1,\, z_1}$$

$$= 0 \text{ for any } C$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y & -x \sin y & 0 \end{vmatrix}$$

$$= (-\sin y + \sin y)\, \mathbf{a}_z$$

$$= \mathbf{0}$$

$$\therefore \int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S} = 0 \text{ for any } S$$

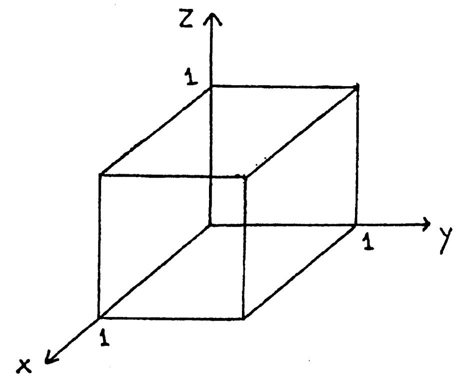
Thus Stokes' theorem is verified, without choosing any particular path.

P3.16. (a) $\oint_S (xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z) \bullet d\mathbf{S}$

$$= \int_{y=0}^{1}\int_{z=0}^{1} y\,dy\,dz + \int_{x=0}^{1}\int_{z=0}^{1} z\,dx\,dz$$

$$+\int_{x=0}^{1}\int_{y=0}^{1} x\,dx\,dy$$

$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$



$$\int_V \nabla \bullet (xy\mathbf{a}_x + yz\mathbf{a}_y + zx\mathbf{a}_z)\,dv$$

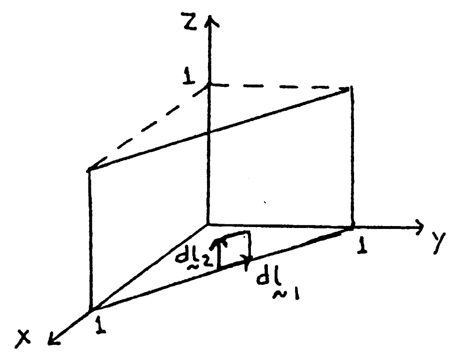
$$= \int_V (y+z+x)\,dv$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (x+y+z)\,dx\,dy\,dz$$

$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$

Thus divergence theorem is verified.

(b) For the surface $x + y = 1$



$d\mathbf{S} = d\mathbf{l}_1 \times d\mathbf{l}_2$

$= (dy\,\mathbf{a}_y - dy\,\mathbf{a}_x) \times dz\,\mathbf{a}_z$

$= dy\,dz\,(\mathbf{a}_x + \mathbf{a}_y)$

$$\oint_S (y^2\mathbf{a}_y - 2yz\mathbf{a}_z) \bullet d\mathbf{S}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} (-2y)\,dx\,dy + \int_{y=0}^{1}\int_{z=0}^{1} (y^2\mathbf{a}_y - 2y\mathbf{a}_z) \bullet (\mathbf{a}_x + \mathbf{a}_y)\,dy\,dz$$

$$= \int_{x=0}^{1} -(1-x)^2\,dx + \int_{y=0}^{1}\int_{z=0}^{1} y^2\,dy\,dz$$

$$= \left[\frac{1-x^3}{3}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1 = -\frac{1}{3}+\frac{1}{3} = 0$$

$$\int_V \nabla \bullet (y^2\mathbf{a}_y - 2yz\mathbf{a}_z)\,dv = \int_V 0\,dv = 0$$

Thus divergence theorem is verified.

P3.17. $\mathbf{J} = J_z(y, t)\,\mathbf{a}_z$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & \frac{\partial}{\partial y} & 0 \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & \frac{\partial}{\partial y} & 0 \\ H_x & H_y & H_z \end{vmatrix} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\frac{\partial E_z}{\partial y} = -\frac{\partial B_x}{\partial t} \qquad \frac{\partial H_z}{\partial y} = \frac{\partial D_x}{\partial t}$$

$$0 = -\frac{\partial B_y}{\partial t} \qquad 0 = \frac{\partial D_y}{\partial t}$$

$$-\frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \qquad -\frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t}$$

The pertinent differential equations are

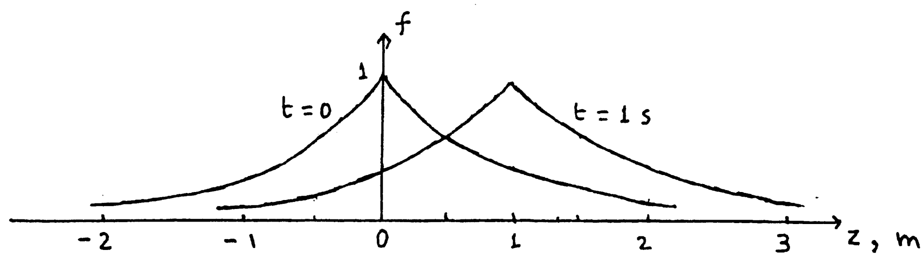
$$\frac{\partial E_z}{\partial y} = -\frac{\partial B_x}{\partial t}$$

$$\frac{\partial H_x}{\partial y} = -J_z - \frac{\partial D_z}{\partial t}$$

P3.18. (a) $f = e^{-|t-z|}$

$$t = 0:\ f = e^{-|-z|} = \begin{cases} e^{-z} & \text{for } z > 0 \\ e^{z} & \text{for } z < 0 \end{cases}$$

$$t = 1\text{ s}:\ f = e^{-|1-z|} = \begin{cases} e^{-(z-1)} & \text{for } z > 1 \\ e^{-(1-z)} & \text{for } z < 1 \end{cases}$$

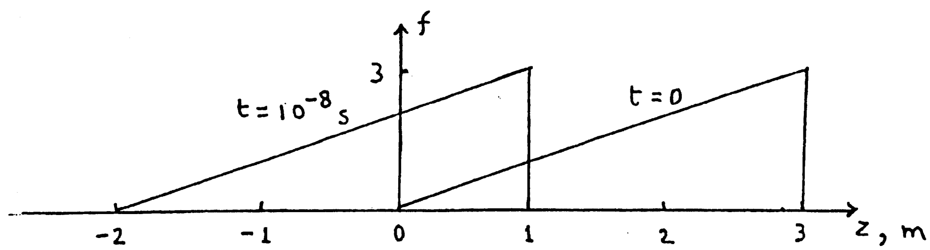


f represents a traveling wave propagating in the $+z$ direction with velocity $\frac{1}{1}$ or 1 m/s.

(b) $f = (2 \times 10^8 t + z)\,[u(2 \times 10^8 t + z) - u(2 \times 10^8 t + z - 3)]$

$t = 0$: $f = z[u(z) - u(z - 3)]$

$t = 10^{-8}s$: $f = (2 + z)[u(2 + z) - u(z - 1)]$



f represents a traveling wave propagating in the $-z$ direction with velocity $\frac{2}{10^{-8}}$ or 2×10^8 m/s.

P3.19. (a) $f(x, t) = f\left(t + \frac{x}{0.5}\right) = f(t + 2x)$

$f(0, t) = 10u(t)$

$\therefore f(x, t) = 10u(t + 2x)$

(b) $f(y, t) = f\left(t - \frac{y}{4}\right)$

$f(0, t) = t \sin 20t$

$\therefore f(y, t) = \left(t - \frac{y}{4}\right) \sin 20\left(t - \frac{y}{4}\right)$

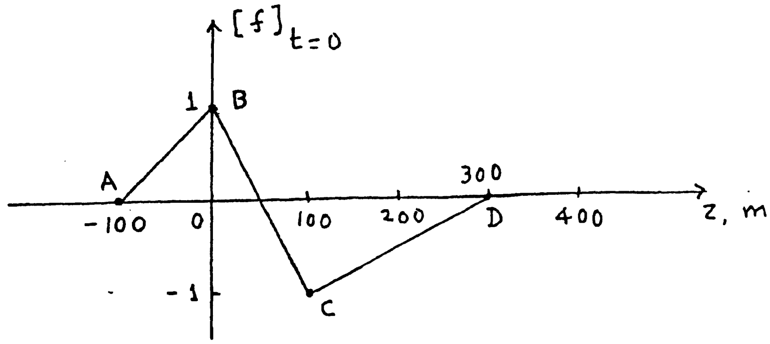
$= \left(t - \frac{y}{4}\right) \sin (20t - 5y)$

(c) $f(z, t) = f\left(t + \frac{z}{2}\right) = f(z + 2t)$

$f(z, 0) = 5z^3 e^{-z^2}$

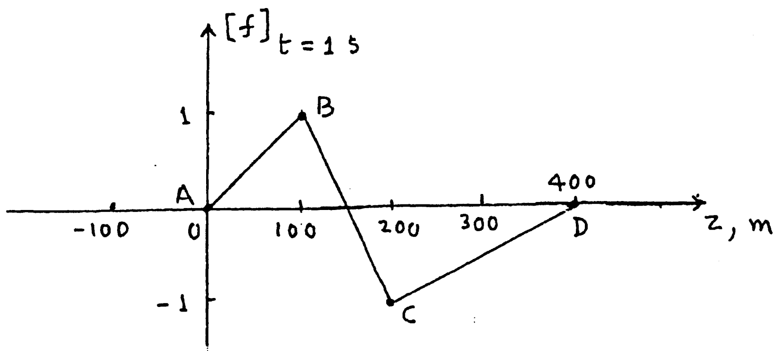
$f(z, t) = 5(z + 2t)^3 e^{-(z + 2t)^2}$

P3.20.

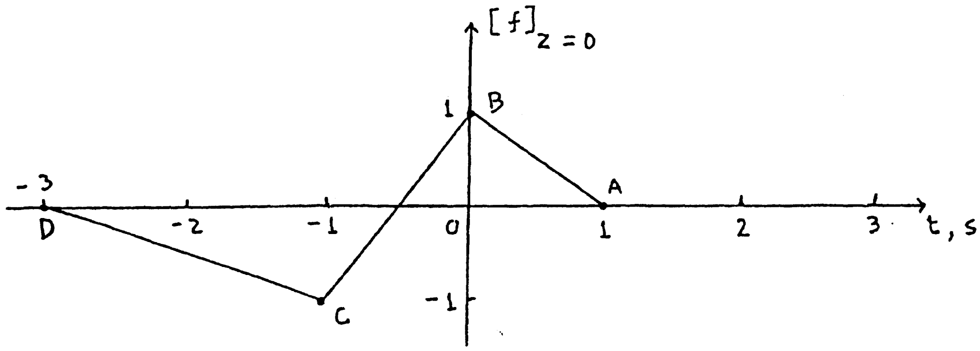


v_p = 100 m/s in the +z direction.

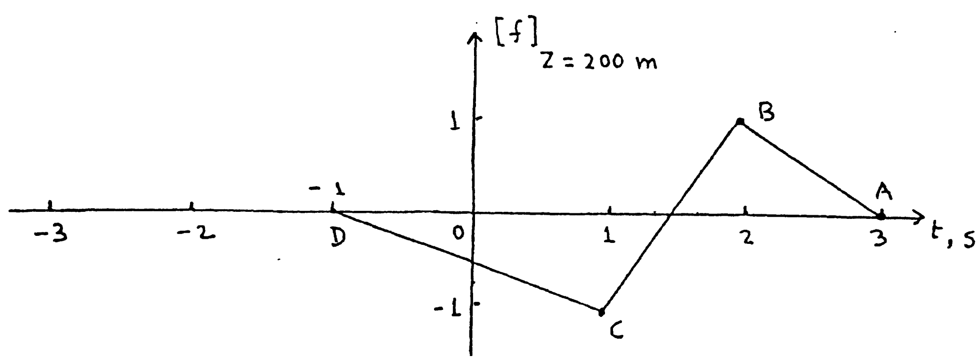
(a)



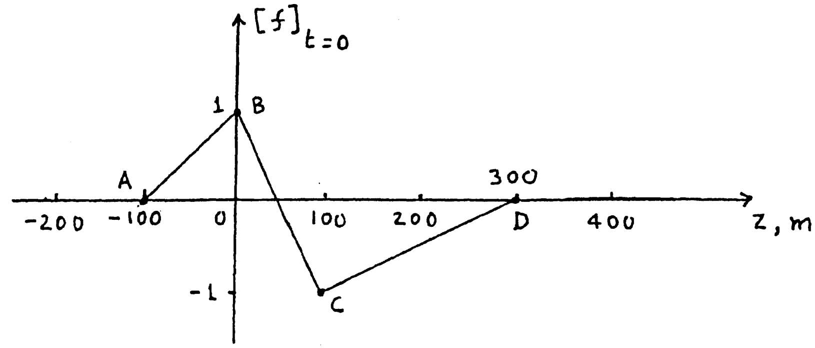
(b)



(c)

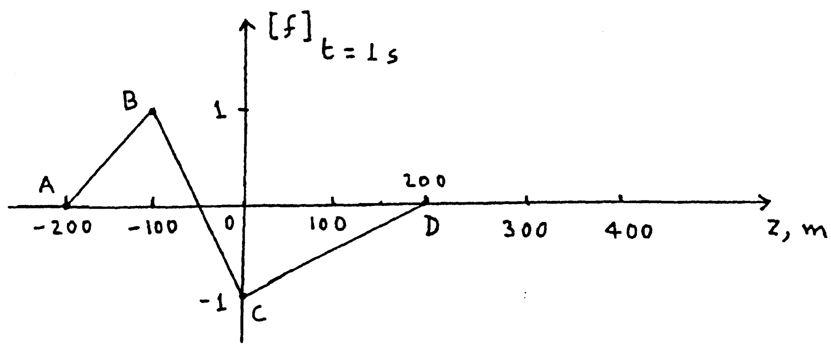


P3.21.

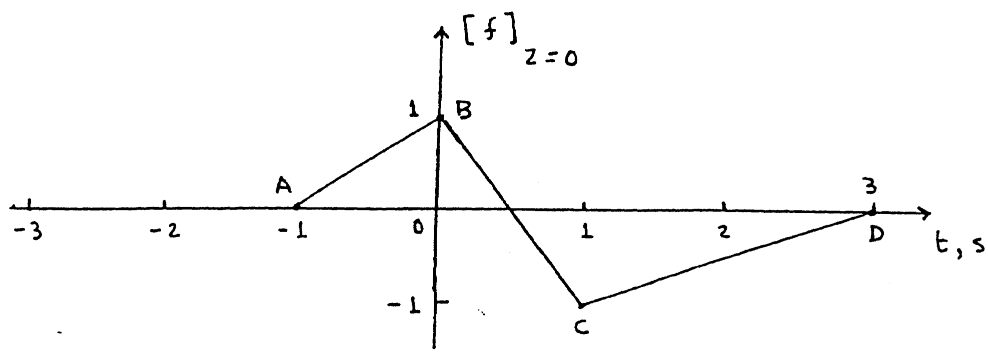


$v_p = 100$ m/s in the $-z$ direction.

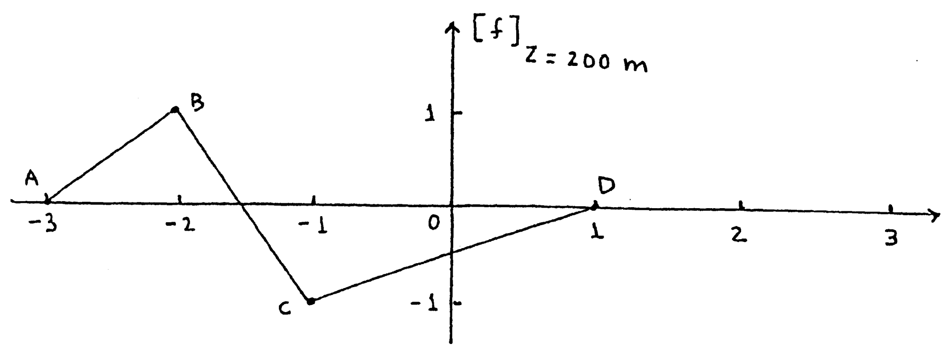
(a)



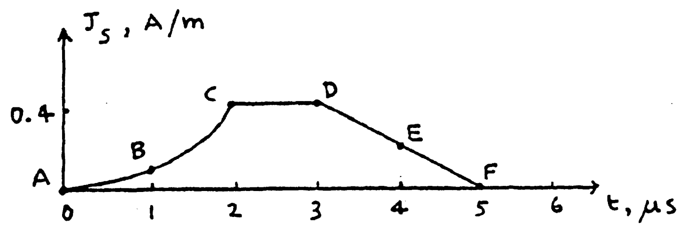
(b)



(c)

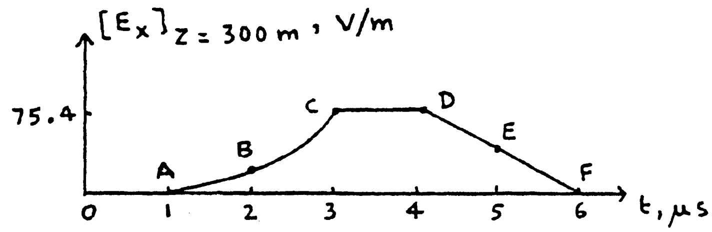


P3.22.



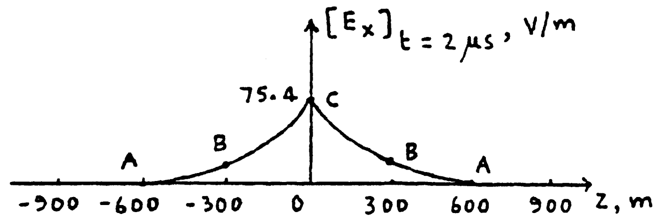
J_S, A/m
0.4
A
B
C
D
E
F
0
1
2
3
4
5
6
t, μs

(a)



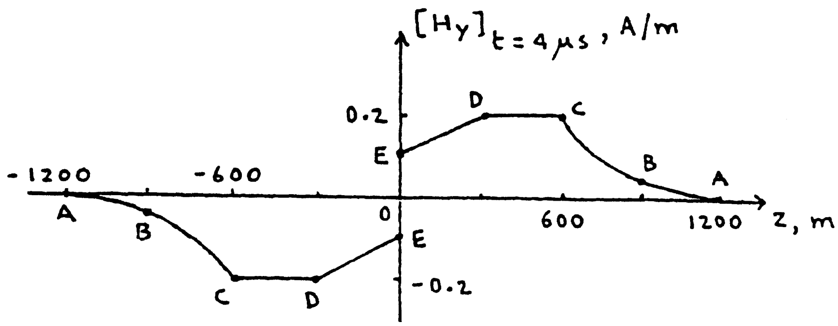
$[E_x]_{z=300\,m}$, V/m
75.4
A
B
C
D
E
F
0
1
2
3
4
5
6
t, μs

(b)



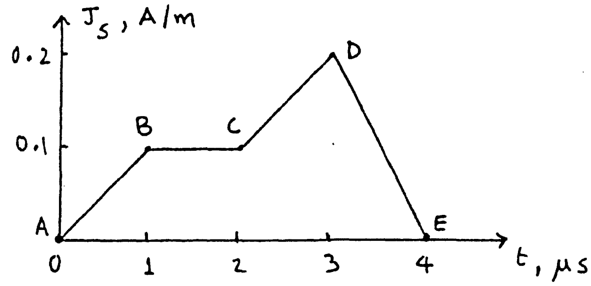
$[E_x]_{t=2\,\mu s}$, V/m
75.4
C
B
A
-900
-600
-300
0
300
600
900
z, m

(c)

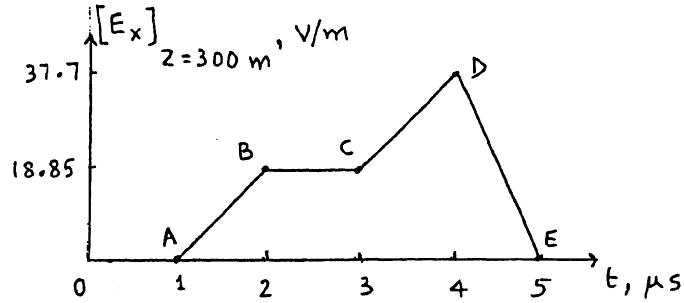


$[H_y]_{t=4\,\mu s}$, A/m
0.2
-0.2
A
B
C
D
E
-1200
-600
0
600
1200
z, m

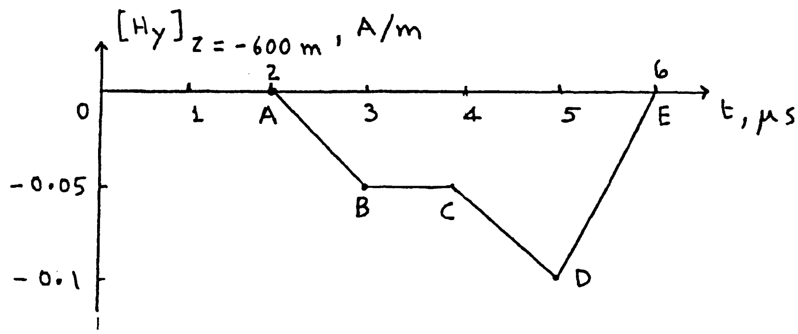
P3.23.


J_S, A/m
0.2
0.1
A
B
C
D
E
0
1
2
3
4
t, μs

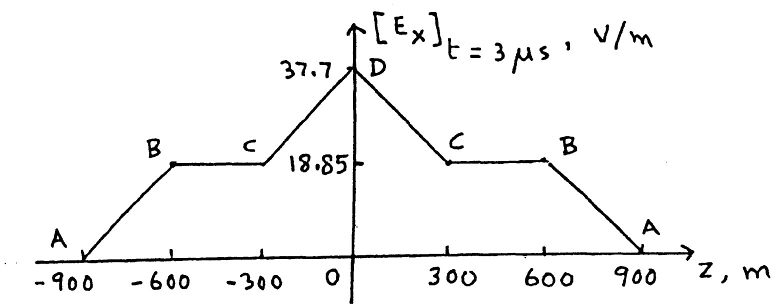
(a)


$[E_x]_{z=300\text{ m}}$, V/m
37.7
18.85
A
B
C
D
E
0
1
2
3
4
5
t, μs

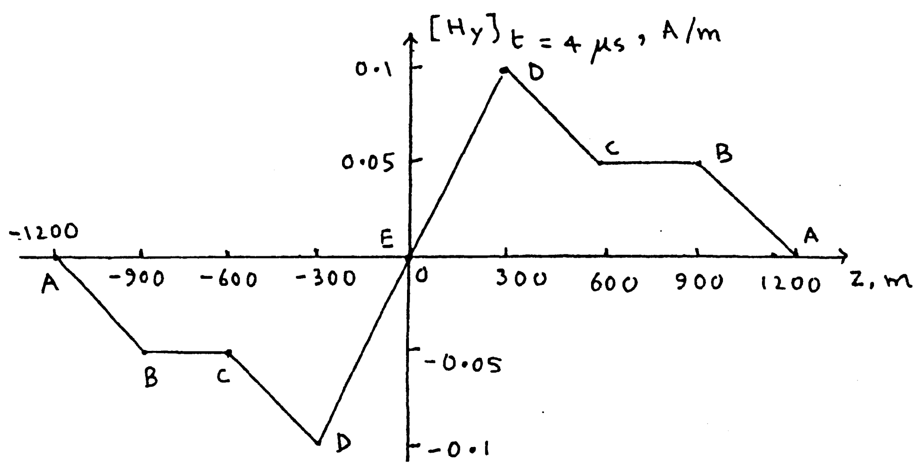
(b)


$[H_y]_{z=-600\text{ m}}$, A/m
0
-0.05
-0.1
1
2
3
4
5
6
A
B
C
D
E
t, μs

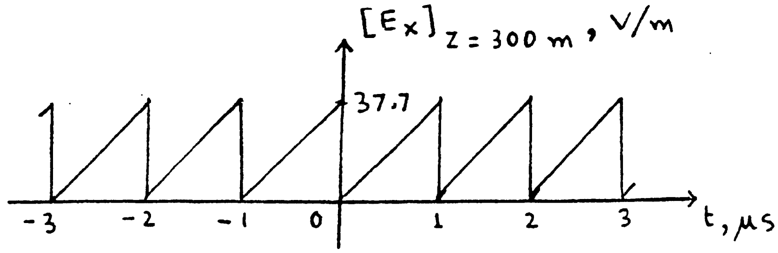
(c)


$[E_x]_{t=3\,\mu s}$, V/m
37.7
18.85
A
B
C
D
-900
-600
-300
0
300
600
900
z, m

(d)

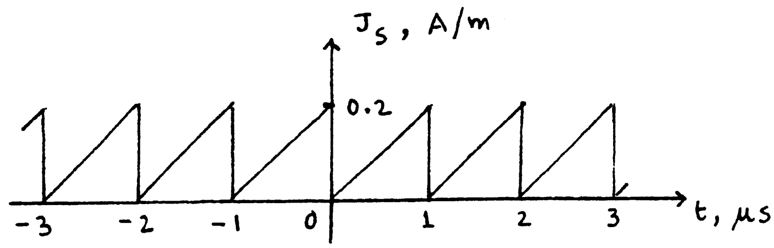

$[H_y]_{t=4\,\mu s}$, A/m
0.1
0.05
-0.05
-0.1
-1200
-900
-600
-300
0
300
600
900
1200
z, m
A
B
C
D
E

P3.24.



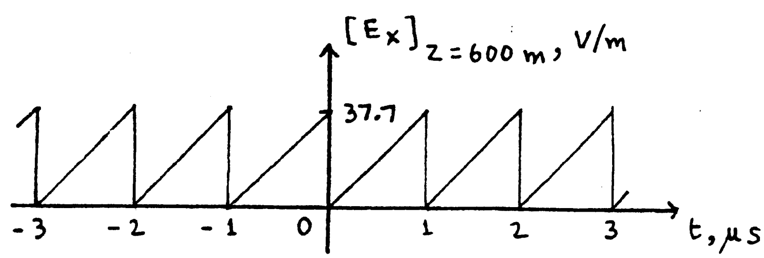
[Ex] z = 300 m , V/m
37.7
-3
-2
-1
0
1
2
3
t, μs

(a)



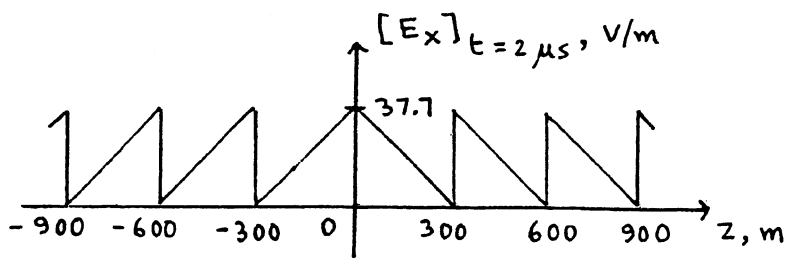
Js , A/m
0.2
-3
-2
-1
0
1
2
3
t, μs

(b)



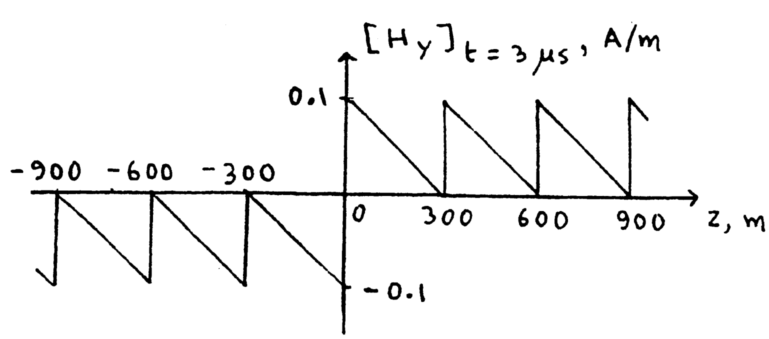
[Ex] z = 600 m , V/m
37.7
-3
-2
-1
0
1
2
3
t, μs

(c)



[Ex] t = 2 μs , V/m
37.7
-900
-600
-300
0
300
600
900
z, m

(d)



[Hy] t = 3 μs , A/m
0.1
-900
-600
-300
0
300
600
900
z, m
-0.1

P3.25. $\mathbf{E} = 37.7 \cos (9\pi \times 10^7 t + 0.3\pi y)\, \mathbf{a}_x$ V/m

(a) $\omega = 9\pi \times 10^7$

$$f = \frac{\omega}{2\pi} = 4.5 \times 10^7 \text{ Hz} = 45 \text{ MHz}$$

(b) $\beta = 0.3\pi$

$$\lambda = \frac{2\pi}{\beta} = \frac{2}{0.3} = 6\frac{2}{3}\text{m}$$

(c) Direction of propagation is the $-y$ direction in view of the argument $(9\pi \times 10^7 t + 0.3\pi y)$ for the cosine function.

(d) Amplitude of $\mathbf{H} = \dfrac{37.7}{\eta_0} = 0.1$

At $y = 0$, $t = 0$, direction of **E** is along $\mathbf{a}_x$. For $\mathbf{E} \times \mathbf{H}$ to be along $-\mathbf{a}_y$, direction of **H** at $x = 0$, $t = 0$ must be along $\mathbf{a}_z$. Thus

$$\mathbf{H} = 0.1 \cos (9\pi \times 10^7 t + 0.3\pi y)\, \mathbf{a}_z \text{ A/m}$$

P3.26. $\mathbf{J}_S = 0.2(\sqrt{3}\mathbf{a}_x + \mathbf{a}_y) \cos 6\pi \times 10^9 t$ A/m in the $z = 0$ plane

<u>Step 1:</u>

$$[\mathbf{H}]_{z=0\pm} = \frac{1}{2}\mathbf{J}_S \times \mathbf{a}_n = \frac{1}{2}\mathbf{J}_S \times (\pm\mathbf{a}_z)$$

$$= 0.1(\sqrt{3}\mathbf{a}_x + \mathbf{a}_y) \cos 6\pi \times 10^9 t \times (\pm\mathbf{a}_z)$$

$$= \pm 0.1(\mathbf{a}_x - \sqrt{3}\mathbf{a}_y) \cos 6\pi \times 10^9 t \text{ A/m}$$

<u>Step 2:</u>

For $\omega = 6\pi \times 10^9$, $\beta = \dfrac{\omega}{v_p} = \dfrac{6\pi \times 10^9}{3 \times 10^8} = 20\pi$

$$[\mathbf{H}]_{z \gtrless 0} = \pm 0.1(\mathbf{a}_x - \sqrt{3}\mathbf{a}_y) \cos (6\pi \times 10^9 t \mp 20\pi z) \text{ A/m}$$

<u>Step 3:</u>

$$[\mathbf{E}]_{z \gtrless 0} = \eta_0 \, [\mathbf{H}]_{z \gtrless 0} \times \mathbf{a}_n$$

$$= \pm\eta_0 \, [0.1(\mathbf{a}_x - \sqrt{3}\mathbf{a}_y) \cos (6\pi \times 10^9 t \mp 20\pi z)] \times (\pm\mathbf{a}_z)$$

$$= -37.7(\sqrt{3}\mathbf{a}_x + \mathbf{a}_y) \cos (6\pi \times 10^9 t \mp 20\pi z) \text{ V/m}$$

P3.27. $\mathbf{J}_S = 0.2 \sin 15\pi \times 10^7 t\, \mathbf{a}_y$ A/m in the $x = 0$ plane.

Step 1:

$$[\mathbf{H}]_{x=0\pm} = \frac{1}{2}\mathbf{J}_S \times \mathbf{a}_n = \frac{1}{2}\mathbf{J}_S \times (\pm\mathbf{a}_x)$$

$$= \frac{1}{2}(0.2 \sin 15\pi \times 10^7 t\ \mathbf{a}_y) \times (\pm\mathbf{a}_x)$$

$$= \mp 0.1 \sin 15\pi \times 10^7 t\, \mathbf{a}_z \text{ A/m}$$

Step 2:

For $\omega = 15\pi \times 10^7$, $\beta = \dfrac{\omega}{v_p} = \dfrac{15\pi \times 10^7}{3 \times 10^8} = 0.5\pi$

$$[\mathbf{H}]_{x \gtrless 0} = \mp 0.1 \sin (15\pi \times 10^7 t \mp 0.5\pi x)\, \mathbf{a}_z \text{ A/m}$$

Step 3:

$$[\mathbf{E}]_{x \gtrless 0} = \eta_0\, [\mathbf{H}]_{x \gtrless 0} \times \mathbf{a}_n$$

$$= \eta_0\, [\mp 0.1 \sin (15\pi \times 10^7 t \mp 0.5\pi x)\mathbf{a}_z] \times (\pm\mathbf{a}_x)$$

$$= -37.7 \sin (15\pi \times 10^7 t \mp 0.5\pi x)\, \mathbf{a}_y \text{ V/m}$$

P3.28. $\mathbf{J}_{S1} = -J_{S0} \cos \omega t \, \mathbf{a}_x, \; z = 0$

$$\mathbf{J}_{S2} = -kJ_{S0} \cos \omega t \, \mathbf{a}_x, \; z = \frac{\lambda}{2}$$

$$\mathbf{E}_1 = \eta_0 \frac{J_{S0}}{2} \cos (\omega t \mp \beta z) \, \mathbf{a}_x \text{ for } z \gtrless 0$$

$$\mathbf{E}_2 = \frac{k\eta_0 J_{S0}}{2} \cos \left[\omega t \mp \beta \left(z - \frac{\lambda}{2} \right) \right] \mathbf{a}_x \text{ for } z \gtrless \frac{\lambda}{2}$$

$$= \frac{k\eta_0 J_{S0}}{2} \cos (\omega t \mp \beta z \mp \pi) \, \mathbf{a}_x \text{ for } z \gtrless \frac{\lambda}{2}$$

$$= -\frac{k\eta_0 J_{S0}}{2} \cos (\omega t \mp \beta z) \, \mathbf{a}_x \text{ for } z \gtrless \frac{\lambda}{2}$$

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

(a) $$\mathbf{E} = \frac{(1-k)\eta_0 J_{S0}}{2} \cos (\omega t + \beta z) \, \mathbf{a}_x \text{ for } z < 0$$

(b) $$\mathbf{E} = \frac{\eta_0 J_{S0}}{2} [\cos (\omega t - \beta z) - k \cos (\omega t + \beta z)] \, \mathbf{a}_x \text{ for } 0 < z < \frac{\lambda}{2}$$

(c) $$\mathbf{E} = \frac{(1-k)\eta_0 J_{S0}}{2} \cos (\omega t - \beta z) \, \mathbf{a}_x \text{ for } z > \frac{\lambda}{2}$$

P3.29. $\mathbf{J}_{S1} = -J_{S0} \cos \omega t \, \mathbf{a}_x, \; z = 0$

$$\mathbf{J}_{S2} = -kJ_{S0} \sin \omega t \, \mathbf{a}_x, \; z = \frac{\lambda}{4}$$

$$\mathbf{J}_{S3} = -2kJ_{S0} \cos \omega t \, \mathbf{a}_x, \; z = \frac{\lambda}{2}$$

For $z < 0$,

$$\mathbf{E} = \left\{ \frac{\eta_0 J_{S0}}{2} \cos(\omega t + \beta z) + \frac{k\eta_0 J_{S0}}{2} \sin\left[\omega t + \beta\left(z - \frac{\lambda}{4}\right)\right] \right.$$
$$\left. + \frac{2k\eta_0 J_{S0}}{2} \cos\left[\omega t + \beta\left(z - \frac{\lambda}{2}\right)\right] \right\} \mathbf{a}_x$$
$$= \left[\frac{\eta_0 J_{S0}}{2} \cos(\omega t + \beta z) - \frac{k\eta_0 J_{S0}}{2} \cos(\omega t + \beta z) \right.$$
$$\left. - \frac{2k\eta_0 J_{S0}}{2} \cos(\omega t + \beta z) \right] \mathbf{a}_x$$
$$= \frac{\eta_0 J_{S0}}{2} (1 - 3k) \cos(\omega t + \beta z) \, \mathbf{a}_x$$

For $z > \lambda/2$,

$$\mathbf{E} = \left\{ \frac{\eta_0 J_{S0}}{2} \cos(\omega t - \beta z) + \frac{k\eta_0 J_{S0}}{2} \sin\left[\omega t + \beta\left(z - \frac{\lambda}{4}\right)\right] \right.$$
$$\left. + \frac{2k\eta_0 J_{S0}}{2} \cos\left[\omega t - \beta\left(z - \frac{\lambda}{2}\right)\right] \right\} \mathbf{a}_x$$
$$= \left[\frac{\eta_0 J_{S0}}{2} \cos(\omega t - \beta z) + \frac{k\eta_0 J_{S0}}{2} \cos(\omega t - \beta z) \right.$$
$$\left. - \frac{2k\eta_0 J_{S0}}{2} \cos(\omega t - \beta z) \right] \mathbf{a}_x$$
$$= \frac{\eta_0 J_{S0}}{2} (1 - k) \cos(\omega t - \beta z) \, \mathbf{a}_x$$

$$\therefore \frac{\text{Amplitude of } [\mathbf{E}]_{z>\lambda/2}}{\text{Amplitude of } [\mathbf{E}]_{z<0}} = \frac{|1-k|}{|1-3k|}$$

(a) $k = -1$, ratio $= \left| \dfrac{2}{4} \right| = \dfrac{1}{2}$

(b) $k = \dfrac{1}{2}$, ratio $= \left| \dfrac{1/2}{-1/2} \right| = 1$

(c) $k = 1$, ratio $= \left| \dfrac{0}{2} \right| = 0$

P3.29. (continued)

Let r be the ratio. Then

$$r = \frac{|1-k|}{|1-3k|}$$

$$(1-3k)^2 r^2 = (1-k)^2$$

$$(1-6k+9k^2)r^2 = 1-2k+k^2$$

$$(9r^2-1)k^2 - (6r^2-2)k + (r^2-1) = 0$$

(a) For $r = \frac{1}{3}$,

$$-\left(\frac{2}{3}-2\right)k + \left(\frac{1}{9}-1\right) = 0$$

$$k = \frac{8}{9} \times \frac{3}{4} = \frac{2}{3}$$

(b) For $r = 3$,

$$80k^2 - 52k + 8 = 0$$

$$20k^2 - 13k + 2 = 0$$

$$k = \frac{13 \pm \sqrt{169-160}}{40}$$

$$= \frac{13 \pm 3}{40}$$

$$= \frac{2}{5} \text{ or } \frac{1}{4}$$

P3.30. $\mathbf{F}_1 = \sqrt{3}\,\mathbf{a}_x \cos(2\pi \times 10^6 t + 30°)$

$\mathbf{F}_2 = \mathbf{a}_z \cos(2\pi \times 10^6 t + 30°)$

$$\mathbf{F}_3 = \left(\frac{1}{2}\,\mathbf{a}_x + \sqrt{3}\mathbf{a}_y + \frac{\sqrt{3}}{2}\,\mathbf{a}_z\right)\cos(2\pi \times 10^6 t - 60°)$$

(a) The vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are in phase.

$\therefore\ \mathbf{F}_1 + \mathbf{F}_2 = (\sqrt{3}\,\mathbf{a}_x + \mathbf{a}_z)\cos(2\pi \times 10^6 t + 30°)$
is *linearly* polarized.

(b) Considering $(\mathbf{F}_1 + \mathbf{F}_2)$ and $\mathbf{F}_3$, we note that although these have equal amplitudes and differ in phase by 90°, they do not differ in direction by 90° since $(\mathbf{F}_1 + \mathbf{F}_2) \bullet \mathbf{F}_3 \neq 0$.

$\therefore\ (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3)$ is *elliptically* polarized.

(c) Noting that $\mathbf{F}_1 - \mathbf{F}_2 = (\sqrt{3}\,\mathbf{a}_x - \mathbf{a}_z)\cos(2\pi \times 10^6 t + 30°)$ and considering $(\mathbf{F}_1 - \mathbf{F}_2)$ and $\mathbf{F}_3$, we find that these have equal amplitudes, differ in phase by 90°, and differ in direction by 90° since $(\mathbf{F}_1 - \mathbf{F}_2) \bullet \mathbf{F}_3 = 0$.

$\therefore\ (\mathbf{F}_1 - \mathbf{F}_2 + \mathbf{F}_3)$ is *circularly* polarized.

P3.31. $\mathbf{F}_1 = (C\mathbf{a}_x + C\mathbf{a}_y + \mathbf{a}_z)\cos 2\pi \times 10^6 t$

$\mathbf{F}_2 = (C\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)\sin 2\pi \times 10^6 t$

(a) For $C = 2$,

$\mathbf{F}_1 = (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)\cos 2\pi \times 10^6 t$

$\mathbf{F}_2 = (2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)\sin 2\pi \times 10^6 t$

The polarization of $\mathbf{F}_1 + \mathbf{F}_2$ is elliptical, since although $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in amplitude and out of phase by 90°, they are not perpendicular.

(b) $(C\mathbf{a}_x + C\mathbf{a}_y + \mathbf{a}_z) \bullet (C\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = 0$

$C^2 + C - 2 = 0$

$(C + 2)(C - 1) = 0$

$C = -2$ or 1

But amplitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal for

$C^2 + C^2 + 1 = C^2 + 1 + 4$

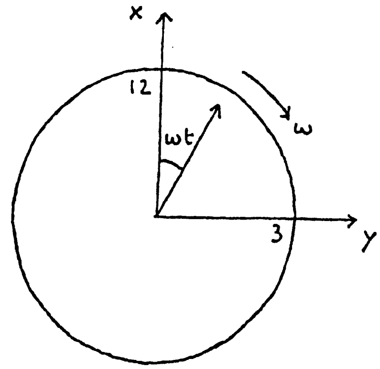
or $C = \pm 2$

$\therefore\ C = -2$

P3.32. (a) $\mathbf{a}_h = \cos \omega t \, \mathbf{a}_x + \sin \omega t \, \mathbf{a}_y$

$$\omega = \frac{2\pi}{12 \times 60 \times 60} = \frac{\pi}{21600}$$

$$\therefore \; \mathbf{a}_h = \cos \frac{\pi t}{21600} \, \mathbf{a}_x + \sin \frac{\pi t}{21600} \, \mathbf{a}_y$$



(b) $$\omega = \frac{2\pi}{60 \times 60} = \frac{\pi}{1800}$$

$$\therefore \; \mathbf{a}_m = \cos \frac{\pi t}{1800} \, \mathbf{a}_x + \sin \frac{\pi t}{1800} \, \mathbf{a}_y$$

(c) Let t be 5 hours and s seconds, where $1500 < s < 1800$. Then

$$\mathbf{a}_h = \cos \frac{\pi(18000 + s)}{21600} \, \mathbf{a}_x + \sin \frac{\pi(18000 + s)}{21600} \, \mathbf{a}_y$$

$$\mathbf{a}_m = \cos \frac{\pi s}{1800} \, \mathbf{a}_x + \sin \frac{\pi s}{1800} \, \mathbf{a}_y$$

For $\mathbf{a}_h = \mathbf{a}_m$,

$$\frac{\pi(18000 + s)}{21600} = \frac{\pi s}{1800}$$

$$s = \frac{18000}{11} = 1636 \frac{4}{11}$$

$$\mathbf{a}_h = \mathbf{a}_m = \cos \frac{10\pi}{11} \, \mathbf{a}_x + \sin \frac{10\pi}{11} \, \mathbf{a}_y$$

$$= -0.9595 \, \mathbf{a}_x + 0.2817 \, \mathbf{a}_y$$

P3.33. $f = 100$ MHz, $\omega = 2\pi \times 10^8$

$$\beta = \frac{\omega}{v_p} = \frac{2\pi \times 10^8}{3 \times 10^8} = \frac{2\pi}{3}$$

Let the electric field be

$$\mathbf{E} = E_1 \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + \phi\right)\mathbf{a}_x + E_1 \sin\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + \phi\right)\mathbf{a}_y$$

Note that for $\phi = 0$ and $z = 0$, $\mathbf{E} = E_1\mathbf{a}_x$ for $\omega t = 0$, $E_1\mathbf{a}_y$ for $\omega t = \frac{\pi}{2}$, and hence $\mathbf{E}$ is right circularly polarized. Now from (c),

$E_1 \cos\phi = E_0$, $E_1 \sin\phi = 0.75E_0$

$\therefore\ E_1 = \sqrt{E_0^2 + (0.75E_0)^2} = 1.25E_0$

$\cos\phi = 0.8$, $\sin\phi = 0.6$

$\phi = 36.87^\circ$ or 0.2048π

Thus

$$\begin{aligned}
\mathbf{E} &= 1.25E_0 \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_x \\
&\quad + 1.25E_0 \sin\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_y \\
&= 1.25E_0\left[\cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_x\right. \\
&\quad \left. + \sin\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_y\right]
\end{aligned}$$

$$\begin{aligned}
\mathbf{H} &= \frac{1.25E_0}{120\pi}\left[-\sin\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_x\right. \\
&\quad \left. + \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_y\right] \\
&= \frac{E_0}{96\pi}\left[-\sin\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_x\right. \\
&\quad \left. + \cos\left(2\pi \times 10^8 t - \frac{2\pi}{3}z + 0.2048\pi\right)\mathbf{a}_y\right]
\end{aligned}$$

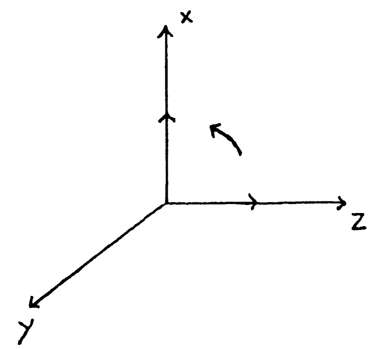
P3.34. (a) $E_0 \cos(\omega t - \beta y)\,\mathbf{a}_z + E_0 \sin(\omega t - \beta y)\,\mathbf{a}_x$

For $y = 0$, $\omega t = 0$, the field is $E_0\mathbf{a}_z$.

For $y = 0$, $\omega t = \frac{\pi}{2}$, the field is $E_0\mathbf{a}_x$.

Direction of propagation is $+y$.

$\therefore$ Polarization is right circular.



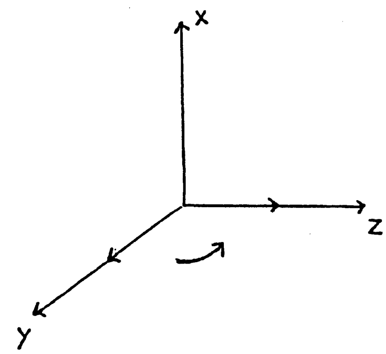
(b) $E_0 \cos(\omega t + \beta x)\,\mathbf{a}_y + E_0 \sin(\omega t + \beta x)\,\mathbf{a}_z$

For $x = 0$, $\omega t = 0$, the field is $E_0\mathbf{a}_y$.

For $x = 0$, $\omega t = \frac{\pi}{2}$, the field is $E_0\mathbf{a}_z$.

Direction of propagation is $-x$.

$\therefore$ Polarization is left circular.



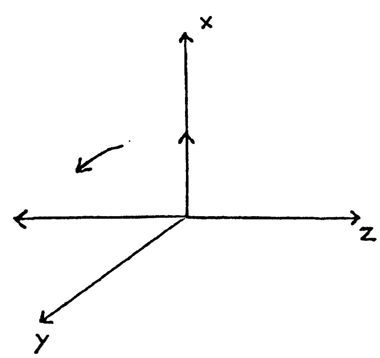
(c) $E_0 \cos(\omega t + \beta y)\,\mathbf{a}_x - 2E_0 \sin(\omega t + \beta y)\,\mathbf{a}_z$

For $y = 0$, $\omega t = 0$, the field is $E_0\mathbf{a}_x$.

For $y = 0$, $\omega t = \frac{\pi}{2}$, the field is $-2E_0\mathbf{a}_z$.

Direction of propagation is $-y$.

$\therefore$ Polarization is left elliptical.



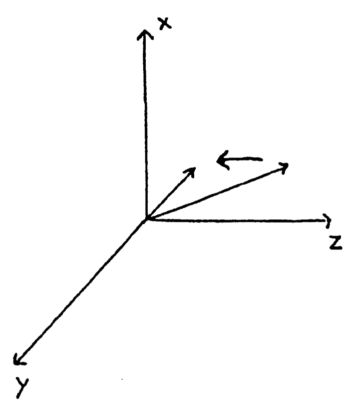
(d) $E_0 \cos(\omega t - \beta x)\,\mathbf{a}_z - E_0 \sin(\omega t - \beta x + \pi/4)\,\mathbf{a}_y$

For $x = 0$, $\omega t = 0$, the field is $E_0\left(\mathbf{a}_z - \frac{1}{\sqrt{2}}\mathbf{a}_y\right)$

For $x = 0$, $\omega t = \frac{\pi}{2}$, the field is $-\frac{1}{\sqrt{2}}E_0\mathbf{a}_y$.

The direction of propagation is $+x$.

$\therefore$ Polarization is right elliptical.



P3.35. (a) Let $E_0\mathbf{a}_x \cos(\omega t + \beta z)$

$$= A[\cos(\omega t + \beta z)\,\mathbf{a}_x - \sin(\omega t + \beta z)\,\mathbf{a}_y]$$

$$+ B[\cos(\omega t + \beta z)\,\mathbf{a}_x + \sin(\omega t + \beta z)\,\mathbf{a}_y]$$

Then

$$A + B = E_0$$

$$-A + B = 0$$

or

$$A = B = \frac{E_0}{2}$$

Thus

$$E_0\mathbf{a}_x \cos(\omega t + \beta z)$$

$$= \frac{E_0}{2}[\cos(\omega t + \beta z)\,\mathbf{a}_x - \sin(\omega t + \beta z)\,\mathbf{a}_y]$$

$$+ \frac{E_0}{2}[\cos(\omega t + \beta z)\,\mathbf{a}_x + \sin(\omega t + \beta z)\,\mathbf{a}_y]$$

(b) $E_0\mathbf{a}_x \cos(\omega t - \beta z + \pi/3) - E_0\mathbf{a}_y \cos(\omega t - \beta z + \pi/6)$

$$= E_0\mathbf{a}_x\left[\frac{1}{2}\cos(\omega t - \beta z) - \frac{\sqrt{3}}{2}\sin(\omega t - \beta z)\right]$$

$$-E_0\mathbf{a}_y\left[\frac{\sqrt{3}}{2}\cos(\omega t - \beta z) - \frac{1}{2}\sin(\omega t - \beta z)\right]$$

$$= \frac{E_0}{2}[\cos(\omega t - \beta z)\,\mathbf{a}_x + \sin(\omega t - \beta z)\,\mathbf{a}_y]$$

$$- \frac{\sqrt{3}}{2}E_0[\sin(\omega t - \beta z)\,\mathbf{a}_x + \cos(\omega t - \beta z)\,\mathbf{a}_y]$$

P3.36. (a) $\mathbf{E} = E_0 \cos(\omega t - \beta z)\,\mathbf{a}_x + 2E_0 \cos(\omega t - \beta z)\,\mathbf{a}_y$

$$\mathbf{H} = \frac{E_0}{\eta_0}\,[-2\cos(\omega t - \beta z)\,\mathbf{a}_x + \cos(\omega t - \beta z)\,\mathbf{a}_y]$$

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = (E_x H_y - E_y H_x)\,\mathbf{a}_z$$

$$= \frac{5E_0^2}{\eta_0}\cos^2(\omega t - \beta z)\,\mathbf{a}_z$$

$$\langle \mathbf{P} \rangle = \frac{2.5E_0^2}{\eta_0}\,\mathbf{a}_z$$

(b) $\mathbf{E} = E_0 \cos(\omega t - \beta z)\,\mathbf{a}_x - E_0 \sin(\omega t - \beta z)\,\mathbf{a}_y$

$$\mathbf{H} = \frac{E_0}{\eta_0}\,[\sin(\omega t - \beta z)\,\mathbf{a}_x + \cos(\omega t - \beta z)\,\mathbf{a}_y]$$

$$\mathbf{P} = \frac{E_0^2}{\eta_0}\,[\cos^2(\omega t - \beta z) + \sin^2(\omega t - \beta z)]\,\mathbf{a}_z$$

$$= \frac{E_0^2}{\eta_0}\,\mathbf{a}_z$$

$$\langle \mathbf{P} \rangle = \frac{E_0^2}{\eta_0}\,\mathbf{a}_z$$

(c) $\mathbf{E} = E_0 \cos(\omega t - \beta z)\,\mathbf{a}_x + 2E_0 \sin(\omega t - \beta z)\,\mathbf{a}_y$

$$\mathbf{H} = \frac{E_0}{\eta_0}\,[-2E_0\sin(\omega t - \beta z)\,\mathbf{a}_x + E_0\cos(\omega t - \beta z)\,\mathbf{a}_y]$$

$$\mathbf{P} = \frac{E_0^2}{\eta_0}\,[\cos^2(\omega t - \beta z) + 4\sin^2(\omega t - \beta z)]\,\mathbf{a}_z$$

$$= \frac{E_0^2}{\eta_0}\,[1 + 3\sin^2(\omega t - \beta z)]\,\mathbf{a}_z$$

$$\langle \mathbf{P} \rangle = \frac{E_0^2}{\eta_0}\left(1 + \frac{3}{2}\right)\mathbf{a}_z$$

$$= \frac{2.5E_0^2}{\eta_0}\,\mathbf{a}_z$$

P3.37. (a) $\mathbf{P} = \mathbf{E} \times \mathbf{H}$

$$= \frac{V_0 I_0}{2\pi r^2 \ln \frac{b}{a}} \cos^2 \omega(t - \sqrt{\mu_0 \varepsilon_0}\, z)\, \mathbf{a}_z$$

$$\langle \mathbf{P} \rangle = \frac{V_0 I_0}{2\pi r^2 \ln \frac{b}{a}} \langle \cos^2 \omega(t - \sqrt{\mu_0 \varepsilon_0}\, z) \rangle\, \mathbf{a}_z$$

$$= \frac{V_0 I_0}{4\pi r^2 \ln \frac{b}{a}}\, \mathbf{a}_z$$

(b) $$\langle \mathbf{P} \rangle = \int_{r=a}^{b} \int_{\phi=0}^{2\pi} \langle \mathbf{P} \rangle \bullet r\, dr\, d\phi\, \mathbf{a}_z$$

$$= \int_{r=a}^{b} \int_{\phi=0}^{2\pi} \frac{V_0 I_0}{4\pi r \ln \frac{b}{a}}\, dr\, d\phi$$

$$= \frac{1}{2} V_0 I_0$$

P3.38. (a) $\mathbf{P} = \mathbf{E} \times \mathbf{H} = E_\theta H_\phi \mathbf{a}_r$

$$= \frac{E_0^2}{\sqrt{\mu_0/\varepsilon_0}} \frac{\sin^2\theta}{r^2} \cos^2 \omega\left(t - r\sqrt{\mu_0\varepsilon_0}\right) \mathbf{a}_r$$

(b) Instantaneous power radiated

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \mathbf{P} \bullet r^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{E_0^2}{\sqrt{\mu_0/\varepsilon_0}} \cos^2 \omega\left(t - r\sqrt{\mu_0\varepsilon_0}\right) \sin^3\theta \, d\theta \, d\phi$$

$$= \frac{8\pi E_0^2}{3\sqrt{\mu_0/\varepsilon_0}} \cos^2 \omega\left(t - r\sqrt{\mu_0\varepsilon_0}\right)$$

(c) Average power radiated

$$= \left< \frac{8\pi E_0^2}{3\sqrt{\mu_0/\varepsilon_0}} \cos^2 \omega\left(t - r\sqrt{\mu_0\varepsilon_0}\right) \right>$$

$$= \frac{8\pi E_0^2}{3\sqrt{\mu_0/\varepsilon_0}} \left< \cos^2 \omega\left(t - r\sqrt{\mu_0\varepsilon_0}\right) \right>$$

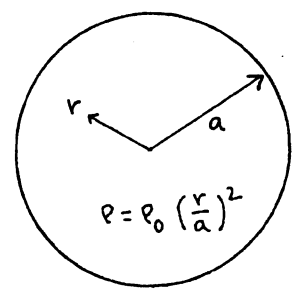
$$= \frac{4\pi E_0^2}{3\sqrt{\mu_0/\varepsilon_0}}$$

P3.39. (a) For

$$\rho = \begin{cases} \rho_0 \left(\dfrac{r}{a}\right)^2 & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

$$4\pi r^2 D_r = \begin{cases} \displaystyle\int_{r=0}^{r}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \rho_0 \left(\frac{r}{a}\right)^2 r^2 \sin\theta\, dr\, d\theta\, d\phi & \text{for } r < a \\ \displaystyle\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \rho_0 \left(\frac{r}{a}\right)^2 r^2 \sin\theta\, dr\, d\theta\, d\phi & \text{for } r > a \end{cases}$$

$$= \begin{cases} \dfrac{4\pi\rho_0 r^5}{5a^2} & \text{for } r < a \\ \dfrac{4\pi\rho_0 a^3}{5} & \text{for } r > a \end{cases}$$



$$E_r = \begin{cases} \dfrac{\rho_0 r^3}{5\varepsilon_0 a^2} & \text{for } r < a \\ \dfrac{\rho_0 a^3}{5\varepsilon_0 r^2} & \text{for } r > a \end{cases}$$

$$W_e = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\rho_0^2 r^6}{50\varepsilon_0 a^4} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$+ \int_{r=a}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\rho_0^2 a^6}{50\varepsilon_0 r^4} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4\pi\rho_0^2 a^5}{450\varepsilon_0} + \frac{4\pi\rho_0^2 a^5}{50\varepsilon_0} = \frac{4\pi\rho_0^2 a^5}{45\varepsilon_0}$$

$$= 0.2793\, \frac{\rho_0^2 a^5}{\varepsilon_0}$$

Total charge inside the sphere $= \dfrac{4\pi\rho_0 a^3}{5}$

If the charge is redistributed with uniform density within the region $r < a$, the new charge density is

$$\rho' = \frac{4\pi\rho_0 a^3}{5} \Big/ \frac{4}{3}\pi a^3 = 0.6\rho_0$$

P3.39. (continued)

$$W_e = \frac{4\pi}{15\varepsilon_0}(0.6\rho_0)^2a^5$$

$$= 0.3016\,\frac{\rho_0^2a^5}{\varepsilon_0}$$

$\therefore$ Work required to rearrange the charge distribution

$$= 0.3016\,\frac{\rho_0^2a^5}{\varepsilon_0} - 0.2793\,\frac{\rho_0^2a^5}{\varepsilon_0}$$

$$= 0.0223\,\frac{\rho_0^2a^5}{\varepsilon_0}$$

(b) Let the radius of the sphere be k_1a and the charge density by $k_2\rho_0$. Then

$$\frac{4}{3}\pi(k_1a)^3(k_2\rho_0) = \frac{4\pi\rho_0a^3}{5} \text{ or } k_1^3\,k_2 = \frac{3}{5} \quad \text{— (1)}$$

$$\frac{4\pi}{15\varepsilon_0}(k_2\rho_0)^2(k_1a)^5 = \frac{4\pi\rho_0^2a^5}{45\varepsilon_0} \text{ or } k_1^5\,k_2^2 = \frac{1}{3} \quad \text{— (2)}$$

Dividing (2) by (1), we obtain

$$k_2 = \frac{5}{9k_1^2}$$

Then from (1), $k_1 = \frac{27}{25} = 1.08$

$$k_2 = \frac{5\times25^2}{9\times27^2} = \frac{3125}{6561} = 0.4763$$

Thus, the required radius = $1.08a$

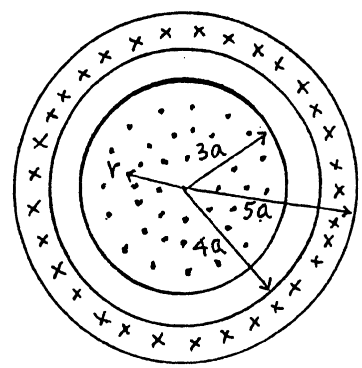
The corresponding charge density = $0.4763\rho_0$

P3.40. For

$$\mathbf{J} = \begin{cases} J_0 \mathbf{a}_z & \text{for } r < 3a \\ -J_0 \mathbf{a}_z & \text{for } 4a < r < 5a \end{cases}$$

$$2\pi r H_\phi = \begin{cases} J_0 \pi r^2 & \text{for } r < 3a \\ J_0 \pi (3a)^2 & \text{for } 3a < r < 4a \\ J_0 \pi (3a)^2 - J_0[\pi r^2 - \pi (4a)^2] & \text{for } 4a < r < 5a \\ J_0 \pi (3a)^2 - J_0[\pi (5a)^2 - \pi (4a)^2] & \text{for } r > 5a \end{cases}$$

$$= \begin{cases} J_0 \pi r^2 & \text{for } r < 3a \\ J_0 (9\pi a^2) & \text{for } 3a < r < 4a \\ J_0 \pi (25a^2 - r^2) & \text{for } 4a < r < 5a \\ 0 & \text{for } r > 5a \end{cases}$$



$$H_\phi = \begin{cases} J_0 r/2 & \text{for } r < 3a \\ 9 J_0 a^2/2r & \text{for } 3a < r < 4a \\ J_0 (25a^2 - r^2)/2r & \text{for } 4a < r < 5a \\ 0 & \text{for } r > 5a \end{cases}$$

$$W_m/l = \int_{r=0}^{3a} \int_{\phi=0}^{2\pi} \frac{\mu_0 J_0^2 r^2}{8} r\, dr\, d\phi + \int_{r=3a}^{4a} \int_{\phi=0}^{2\pi} \frac{81 \mu_0 J_0^2 a^4}{8r^2} r\, dr\, d\phi$$

$$+ \int_{r=4a}^{5a} \int_{\phi=0}^{2\pi} \frac{\mu_0 J_0^2 (25a^2 - r^2)^2}{8r^2} r\, dr\, d\phi$$

$$= \frac{\mu_0 J_0^2 \pi}{4} \left\{ \left[\frac{r^4}{4}\right]_0^{3a} + 81a^4 [\ln r]_{3a}^{4a} + 625a^4 [\ln r]_{4a}^{5a} - 50a^2 \left[\frac{r^2}{2}\right]_{4a}^{5a} + \left[\frac{r^4}{4}\right]_{4a}^{5a} \right\}$$

$$= \frac{\mu_0 J_0^2 \pi}{4} \left[\frac{81a^4}{4} + 81a^4 \ln\frac{4}{3} + 625a^4 \ln\frac{5}{4} - 225a^4 + \frac{369a^4}{4} \right]$$

$$= 39.48 \mu_0 J_0^2 \text{ J/m}$$

R3.1. $\mathbf{E} = E_0 \sin 6x \sin (3\times 10^9 t - kz)\,\mathbf{a}_y$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$= -\frac{\partial E_y}{\partial z}\,\mathbf{a}_x + \frac{\partial E_y}{\partial x}\,\mathbf{a}_z$$

$$= kE_0 \sin 6x \cos (3\times 10^9 t - kz)\,\mathbf{a}_x$$

$$+ 6E_0 \cos 6x \sin (3\times 10^9 t - kz)\,\mathbf{a}_z$$

$$\mathbf{B} = \frac{E_0}{3\times 10^9}\,[-k \sin 6x \sin (3\times 10^9 t - kz)\,\mathbf{a}_x$$

$$+ 6 \cos 6x \cos (3\times 10^9 t - kz)\,\mathbf{a}_z]$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\mathbf{B}}{4\pi\times 10^{-7}}$$

$$= \frac{E_0}{1200\pi}\,[-k \sin 6x \sin (3\times 10^9 t - kz)\,\mathbf{a}_x$$

$$+ 6 \cos 6x \cos (3\times 10^9 t - kz)\,\mathbf{a}_z]$$

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ H_x & 0 & H_z \end{vmatrix}$$

$$= \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y$$

R3.1. (continued)

$$= \frac{E_0}{1200\pi} [k^2 \sin 6x \cos (3 \times 10^9 t - kz)$$

$$+ 36 \sin 6x \cos (3 \times 10^9 t - kz)] \mathbf{a}_y$$

$$= \frac{E_0(k^2 + 36)}{1200\pi} [\sin 6x \cos (3 \times 10^9 t - kz)] \mathbf{a}_y$$

$$\mathbf{D} = \frac{E_0(k^2 + 36)}{3600 \times 10^9 \pi} [\sin 6x \sin (3 \times 10^9 t - kz)] \mathbf{a}_y$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \frac{\mathbf{D}}{10^{-9}/36\pi}$$

$$= \frac{E_0(k^2 + 36)}{100} \sin 6x \sin (3 \times 10^9 t - kz) \mathbf{a}_y$$

Equating this to the original **E**, we have

$$\frac{k^2 + 36}{100} = 1$$

$$k^2 = 64$$

$$k = \pm 8$$

R3.2. $\mathbf{E} = E_0 \sin\frac{\pi x}{a} \sin\frac{\pi z}{d} \cos\omega t\, \mathbf{a}_y$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$= -\frac{\partial E_y}{\partial z}\mathbf{a}_x + \frac{\partial E_y}{\partial x}\mathbf{a}_z$$

$$= -\frac{\pi E_0}{d} \sin\frac{\pi x}{a} \cos\frac{\pi z}{d} \cos\omega t\, \mathbf{a}_x + \frac{\pi E_0}{a} \cos\frac{\pi x}{a} \sin\frac{\pi z}{d} \cos\omega t\, \mathbf{a}_z$$

$$\mathbf{B} = \frac{\pi E_0}{d\omega} \sin\frac{\pi x}{a} \cos\frac{\pi z}{d} \sin\omega t\, \mathbf{a}_x - \frac{\pi E_0}{a\omega} \cos\frac{\pi x}{a} \sin\frac{\pi z}{d} \sin\omega t\, \mathbf{a}_z$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0}$$

$$= \frac{\pi E_0}{\mu_0 d\omega} \sin\frac{\pi x}{a} \cos\frac{\pi z}{d} \sin\omega t\, \mathbf{a}_x - \frac{\pi E_0}{\mu_0 a\omega} \cos\frac{\pi x}{a} \sin\frac{\pi z}{d} \sin\omega t\, \mathbf{a}_z$$

which is of the form of the given **H**, and which simply means, by comparison

$$H_{01} = \frac{\pi E_0}{\mu_0 d\omega} \text{ and } H_{02} = \frac{\pi E_0}{\mu_0 a\omega}$$

Proceeding further,

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H}$$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & H_z \end{vmatrix}$$

$$= \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y$$

$$= -\frac{\pi^2 E_0}{\mu_0 \omega}\left(\frac{1}{a^2} + \frac{1}{d^2}\right) \sin\frac{\pi x}{a} \sin\frac{\pi z}{d} \sin\omega t\, \mathbf{a}_y$$

R3.2. (continued)

$$\mathbf{D} = \frac{\pi^2 E_0}{\mu_0 \omega^2}\left(\frac{1}{a^2}+\frac{1}{d^2}\right) \sin\frac{\pi x}{a} \sin\frac{\pi z}{d} \cos\omega t \, \mathbf{a}_y$$

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0}$$

$$= \frac{\pi^2 E_0}{\mu_0 \varepsilon_0 \omega^2}\left(\frac{1}{a^2}+\frac{1}{d^2}\right) \sin\frac{\pi x}{a} \sin\frac{\pi z}{d} \cos\omega t \, \mathbf{a}_y$$

Comparing with the given **E**, we have

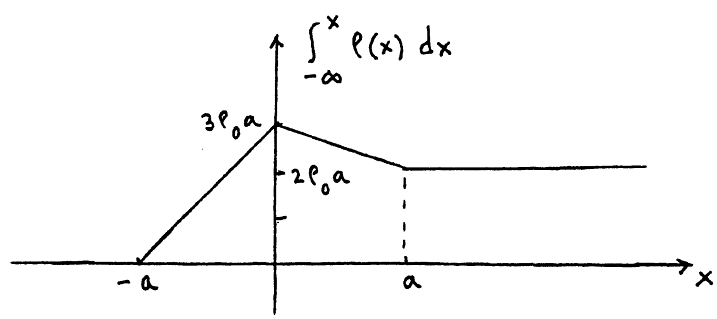
$$\frac{\pi^2 E_0}{\mu_0 \varepsilon_0 \omega^2}\left(\frac{1}{a^2}+\frac{1}{d^2}\right) = E_0$$

$$\omega = \frac{\pi}{\sqrt{\mu_0 \varepsilon_0}}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{d}\right)^2}$$

R3.3. Since $\rho = \rho(x)$, $\mathbf{D} = \mathbf{D}(x)$, and $\nabla \bullet \mathbf{D}$ reduces to $\dfrac{\partial D_x}{\partial x} = \rho$. Then

$$D_x(x) = \int_{-\infty}^{x} \rho(x)\, dx + C$$

Evaluating the integral graphically, we obtain the function shown in the figure.

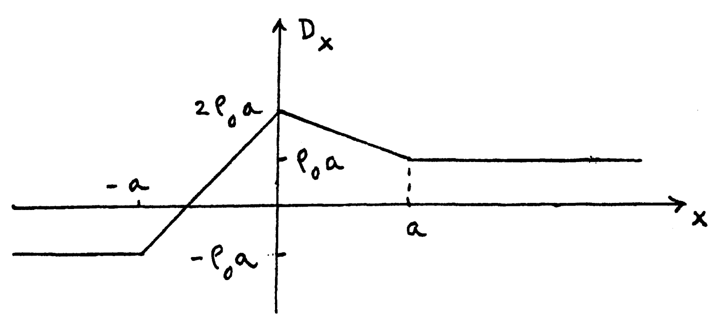


To determine the constant C, we can consider the given charge distribution as the superposition of two charge distributions, one within the interval $-a < x < 0$ and the other within the interval $0 < x < a$. From symmetry consideration, for each of these distributions, the field must be equal in magnitude on either side of the interval. Therefore for the given charge distribution, the field must be equal in magnitude on either side of the interval $-a < x < a$, so that the constant is

$$C = -\rho_0 a$$

Thus D_x is given by the plot shown in the figure, and

$$\mathbf{D} = \begin{cases} -\rho_0 a \mathbf{a}_x & \text{for } x < -a \\ \rho_0(3x + 2a)\mathbf{a}_x & \text{for } -a < x < 0 \\ \rho_0(2a - x)\mathbf{a}_x & \text{for } 0 < x < a \\ \rho_0 a \mathbf{a}_x & \text{for } x > a \end{cases}$$



R3.4. For group (i), curl is zero and divergence is nonzero.
For group (ii), divergence is zero and curl is nonzero.
For group (iii), both curl and divergence are zero.
For group (iv), both curl and divergence are nonzero.

(a) $\nabla \times (x\mathbf{a}_x + y\mathbf{a}_y)$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \mathbf{0}$$

$\nabla \bullet (x\mathbf{a}_x + y\mathbf{a}_y)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$$

$\therefore$ The field belongs to group (i).

(b) $\nabla \times [(x^2 - y^2)\,\mathbf{a}_x - 2xy\mathbf{a}_y + 4\mathbf{a}_z]$

$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & -2xy & 4 \end{vmatrix}$$

$$= (-2y + 2y)\mathbf{a}_x = \mathbf{0}$$

$\nabla \bullet [(x^2 - y^2)\mathbf{a}_x - 2xy\mathbf{a}_y + 4\mathbf{a}_z]$

$$= \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(4)$$

$$= 2x - 2x = 0$$

$\therefore$ The field belongs to group (iii).

R3.4. (continued)

(c) $\nabla \times \dfrac{e^{-r}}{r}\mathbf{a}_\phi$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & e^{-r} & 0 \end{vmatrix}$$

$$= -\frac{e^{-r}}{r}\mathbf{a}_z$$

$\nabla \bullet \dfrac{e^{-r}}{r}\mathbf{a}_\phi$

$$= \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{e^{-r}}{r}\right) = 0$$

$\therefore$ The field belongs to group (ii).

(d) $\nabla \times \dfrac{1}{r}(\cos\phi\, \mathbf{a}_r + \sin\phi\, \mathbf{a}_\phi)$

$$= \begin{vmatrix} \dfrac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \dfrac{\mathbf{a}_z}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \dfrac{\cos\phi}{r} & \sin\phi & 0 \end{vmatrix}$$

$$= \frac{\sin\phi}{r^2}\mathbf{a}_z$$

$\nabla \bullet \dfrac{1}{r}(\cos\phi\, \mathbf{a}_r + \sin\phi\, \mathbf{a}_\phi)$

$$= \frac{1}{r}\frac{\partial}{\partial r}(\cos\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{\sin\phi}{r}\right)$$

$$= \frac{\cos\phi}{r^2}$$

$\therefore$ The field belongs to group (iv).

R3.5. $$f(z, t) = f\left(t - \frac{z}{100}\right) = f(z - 100t)$$

$$g(z, t) = g\left(t + \frac{z}{100}\right) = g(z + 100t)$$

Let

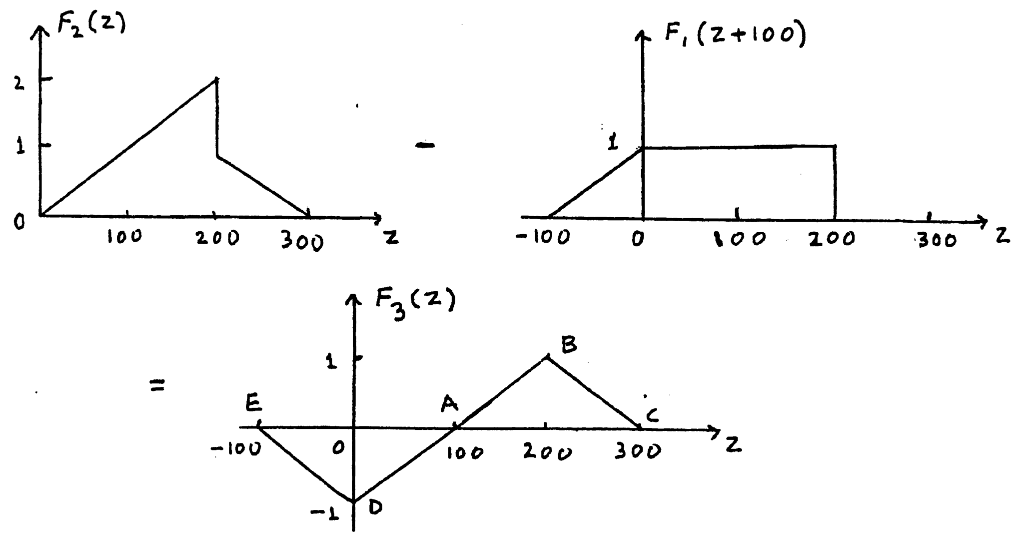
$$F_1(z) = f(z, 0) + g(z, 0) = f(z) + g(z)$$

$$F_2(z) = f(z, 1) + g(z, 1) = f(z - 100) + g(z + 100)$$

Then

$$F_1(z + 100) = f(z + 100) + g(z + 100)$$

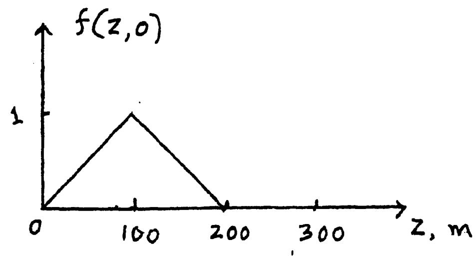
$$F_3(z) = F_2(z) - F_1(z + 100) = f(z - 100) - f(z + 100)$$



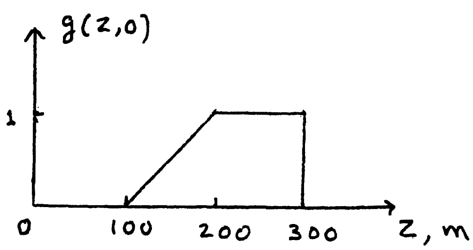
Now, since f and g both travel with velocity 100 m/s and their durations do not exceed 3 s, their widths along the z-direction do not exceed 300 m.

From $F_3(z) = f(z - 100) - f(z + 100)$, we can therefore sketch $f(z, 0)$ as follows.

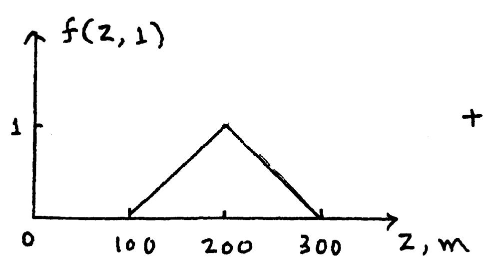
R3.5. (continued)



Subtracting $f(z, 0)$ from $F_1(z)$, we then obtain $g(z, 0)$ as follows.



Note that for verification,



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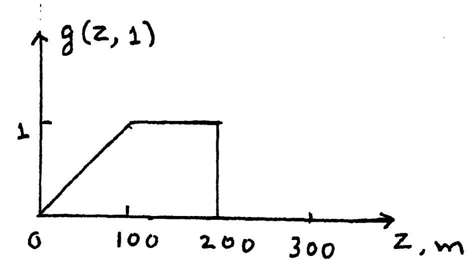

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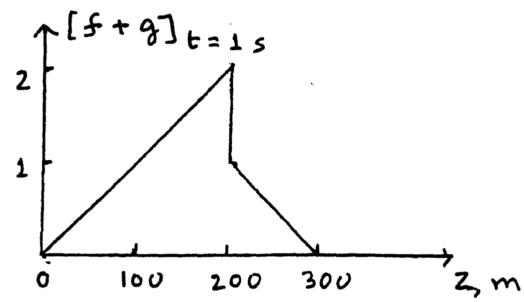
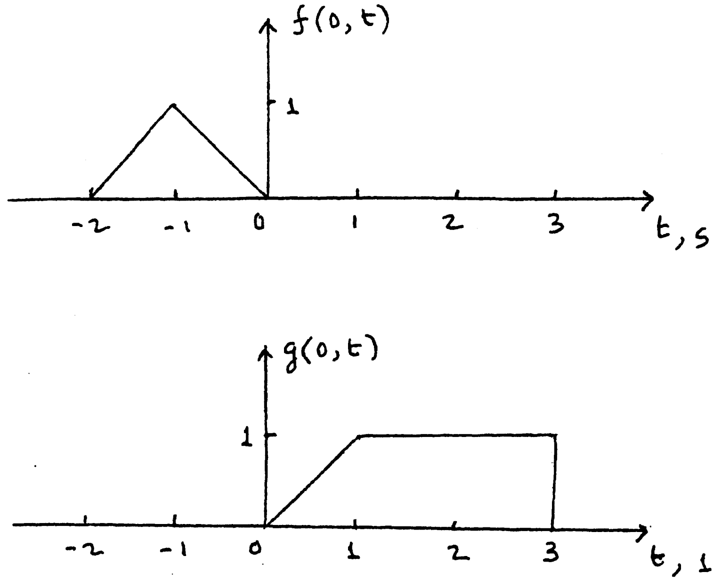


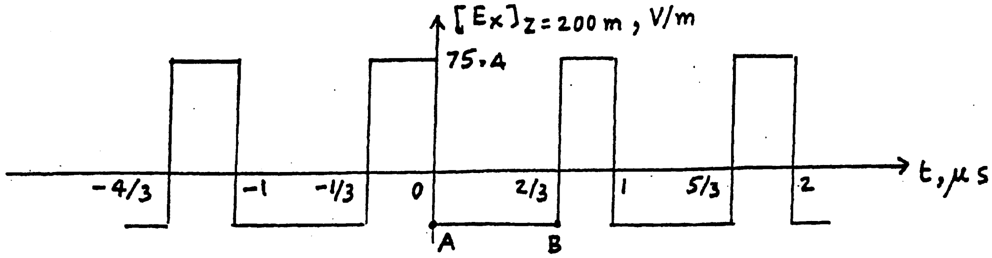
is indeed the same as the given plot for $[f + g]_{t = 1\text{ s}}$.

R3.5. (continued)

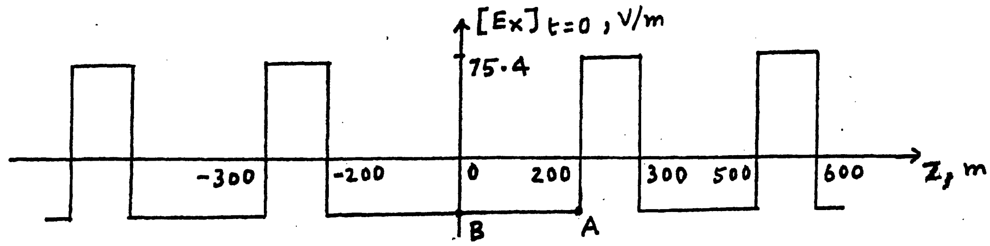
Finally, from the sketches of $f(z, 0)$ and $g(z, 0)$, we obtain $f(0, t)$ and $g(0, t)$ as follows.



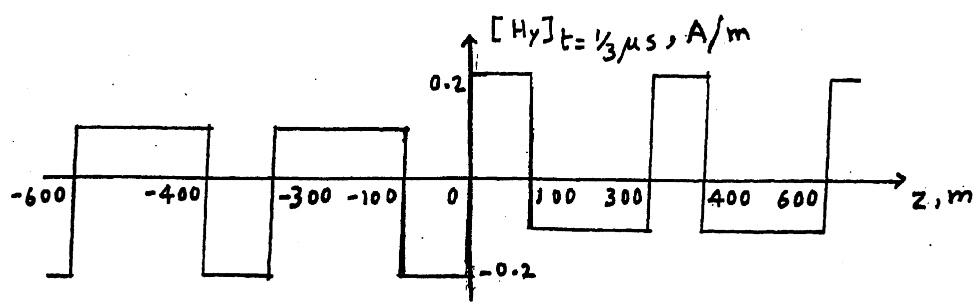
R3.6. **(a)** $[E_x(t)]_{z=200\text{ m}}$ is the same as $[E_x(t)]_{z=600\text{ m}}$ displaced to the left by $400/(3 \times 10^8)$ second, or 4/3 μs.



(b) $[E_x(z)]_{t=0}$ is an even function of z with points A and B in (a) corresponding to points A and B in (b).



(c) $[H_y(z)]_{t=\frac{1}{3}\mu s}$ is an odd function of z, with the right half obtained by shifting the right half of (b) by 100 m to the right and dividing by η_0.



R3.7. For $\mathbf{J}_{S1} = -J_{S0} \cos \omega t \, \mathbf{a}_x$ in the $z = 0$ plane

$$\mathbf{E}_1 = \begin{cases} \frac{\eta_0 J_{S0}}{2} \cos(\omega t - \beta z)\, \mathbf{a}_x & \text{for } z > 0 \\ \frac{\eta_0 J_{S0}}{2} \cos(\omega t + \beta z)\, \mathbf{a}_x & \text{for } z < 0 \end{cases}$$

For $\mathbf{J}_{S2} = -J_{S0} \sin(\omega t + \alpha)\, \mathbf{a}_x$ in the $z = \lambda/4$ lane

$$\mathbf{E}_2 = \begin{cases} \frac{\eta_0 J_{S0}}{2} \sin[\omega t + \alpha - \beta(z - \lambda/4)]\, \mathbf{a}_x & \text{for } z > \lambda/4 \\ \frac{\eta_0 J_{S0}}{2} \sin[\omega t + \alpha + \beta(z - \lambda/4)]\, \mathbf{a}_x & \text{for } z < \lambda/4 \end{cases}$$

$$= \begin{cases} \frac{\eta_0 J_{S0}}{2} \sin(\omega t - \beta z + \alpha + \pi/2)\, \mathbf{a}_x & \text{for } z > \lambda/4 \\ \frac{\eta_0 J_{S0}}{2} \sin(\omega t + \beta z + \alpha - \pi/2)\, \mathbf{a}_x & \text{for } z < \lambda/4 \end{cases}$$

$$= \begin{cases} \frac{\eta_0 J_{S0}}{2} \cos(\omega t - \beta z + \alpha)\, \mathbf{a}_x & \text{for } z > \lambda/4 \\ -\frac{\eta_0 J_{S0}}{2} \cos(\omega t + \beta z + \alpha)\, \mathbf{a}_x & \text{for } z < \lambda/4 \end{cases}$$

By superposition,

$$[\mathbf{E}]_{z>\lambda/4} = \frac{\eta_0 J_{S0}}{2} [\cos(\omega t - \beta z) + \cos(\omega t - \beta z + \alpha)]\, \mathbf{a}_x$$

$$[\mathbf{E}]_{z<0} = \frac{\eta_0 J_{S0}}{2} [\cos(\omega t + \beta z) - \cos(\omega t + \beta z + \alpha)]\, \mathbf{a}_x$$

Using phasor technique, the amplitudes of **E** for the two regions are given by

for $z > \lambda/4 : \frac{\eta_0 J_{S0}}{2} |1 + 1\underline{/\alpha}|$

for $z < 0 : \frac{\eta_0 J_{S0}}{2} |1 - 1\underline{/\alpha}|$

The required ratio is

$$\frac{\text{Amplitude of } \mathbf{E} \text{ for } z > \lambda/4}{\text{Amplitude of } \mathbf{E} \text{ for } z < 0} = \frac{|1 + 1\underline{/\alpha}|}{|1 - 1\underline{/\alpha}|}$$

$$= \frac{|1 + \cos\alpha + j\sin\alpha|}{|1 - \cos\alpha - j\sin\alpha|}$$

$$= \frac{\sqrt{2 + 2\cos\alpha}}{\sqrt{2 - 2\cos\alpha}}$$

$$= \sqrt{\frac{1 + \cos\alpha}{1 - \cos\alpha}} = \frac{1 + \cos\alpha}{\sin\alpha} = \text{cosec } \alpha + \cot\alpha$$

R3.7. (continued)

(a) $\alpha = \pi/4$

Ratio $= \sqrt{2} + 1$

$= 2.4142$

(b) Ratio $= 2$

$$\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = 2$$

$1 + \cos\alpha = 4\,(1 - \cos\alpha)$

$\cos\alpha = 0.6$

$\alpha = 0.2952\pi$

R3.8. For $\mathbf{J}_{S1} = 0.2 \cos 6\pi \times 10^8 t\, \mathbf{a}_x$ A/m in the $y = 0$ plane,

$$\mathbf{H}_1 = \frac{1}{2} \mathbf{J}_{S1} \times (\pm \mathbf{a}_y) \text{ for } y = 0\pm$$

$$= \pm 0.1 \cos 6\pi \times 10^8 t\, \mathbf{a}_z \text{ for } y = 0\pm$$

$$\mathbf{H}_1 = \pm 0.1 \cos (6\pi \times 10^8 t \mp 2\pi y)\, \mathbf{a}_z \text{ for } y \gtrless 0$$

$$\mathbf{E}_1 = \eta_0 \mathbf{H}_1 \times (\pm \mathbf{a}_y) \text{ for } y \gtrless 0$$

$$= -12\pi \cos (6\pi \times 10^8 t \mp 2\pi y)\, \mathbf{a}_x \text{ for } y \gtrless 0$$

For $\mathbf{J}_{S2} = 0.2 \cos 6\pi \times 10^8 t\, \mathbf{a}_z$ A/m in the $y = 0.25$ m plane,

$$\mathbf{H}_2 = \frac{1}{2} \mathbf{J}_{S2} \times (\pm \mathbf{a}_y) \text{ for } y = 0.25\pm$$

$$= \mp 0.1 \cos 6\pi \times 10^8 t\, \mathbf{a}_x \text{ for } y = 0.25\pm$$

$$\mathbf{H}_2 = \mp 0.1 \cos[6\pi \times 10^8 t \mp 2\pi(y - 0.25)]\, \mathbf{a}_x$$

$$= \mp 0.1 \cos (6\pi \times 10^8 t \mp 2\pi y \pm 0.5\pi)\, \mathbf{a}_x$$

$$= 0.1 \sin (6\pi \times 10^8 t \mp 2\pi y)\, \mathbf{a}_x \text{ for } y \gtrless 0.25$$

$$\mathbf{E}_2 = \eta_0 \mathbf{H}_2 \times (\pm \mathbf{a}_y) \text{ for } y \gtrless 0.25$$

$$= \pm 12\pi \sin (6\pi \times 10^8 t \mp 2\pi y)\, \mathbf{a}_z \text{ for } y \gtrless 0.25$$

By superposition, we then have

$$[\mathbf{E}]_{y<0} = -12\pi \cos (6\pi \times 10^8 t + 2\pi y)\, \mathbf{a}_x - 12\pi \sin (6\pi \times 10^8 t + 2\pi y)\, \mathbf{a}_z$$

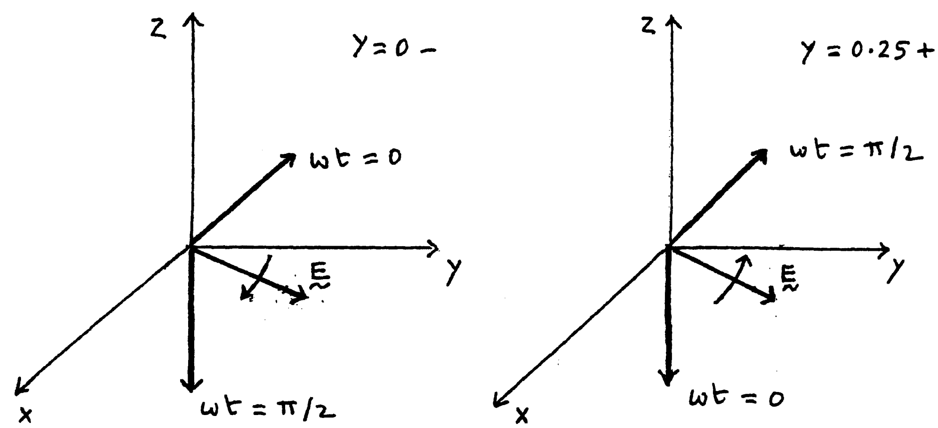
$$[\mathbf{H}]_{y<0} = -0.1 \cos (6\pi \times 10^8 t + 2\pi y)\, \mathbf{a}_z + 0.1 \sin (6\pi \times 10^8 t + 2\pi y)\, \mathbf{a}_z$$

$$[\mathbf{E}]_{y>0.25} = -12\pi \cos (6\pi \times 10^8 t - 2\pi y)\, \mathbf{a}_x + 12\pi \sin (6\pi \times 10^8 t - 2\pi y)\, \mathbf{a}_z$$

$$[\mathbf{H}]_{y>0.25} = 0.1 \cos (6\pi \times 10^8 t - 2\pi y)\, \mathbf{a}_z + 0.1 \sin (6\pi \times 10^8 t - 2\pi y)\, \mathbf{a}_z$$

R3.8. (continued)

To discuss the polarizations in the two regions, we note that in both regions, the fields are the superposition of two components, which are equal in amplitude, perpendicular to each other, and differ in phase by 90°. Therefore the polarizations are circular. To determine the senses of rotation, we look at the field components of **E** at a fixed value of y in the two regions and at two instants of time, as shown in Figs. (a) and (b) for $y = 0$ and $y = 0.25+$, respectively.



Since the directions of propagation of the waves are $-y$ and y for $y = 0-$ and $y = 0.25+$, respectively, the sense of rotation of **E** in both cases is clockwise. Therefore, the polarization is cw circular for both $y < 0$ and $y > 0.25$.

R3.9. Although F_x and F_y are equal in amplitude and perpendicular, they differ in phase by 60°. Also, the vector lies in the xy-plane, since $F_z = 0$. Thus the field is elliptically polarized in the xy-plane.

To find the equation for the ellipse, we set

$$x = 1 \cos \omega t \tag{1}$$

$$\begin{aligned} y &= 1 \cos(\omega t + 60^\circ) \\ &= 1 \cos \omega t \cos 60^\circ - 1 \sin \omega t \sin 60^\circ \\ &= \frac{1}{2}\cos \omega t - \frac{\sqrt{3}}{2} \sin \omega t \end{aligned} \tag{2}$$

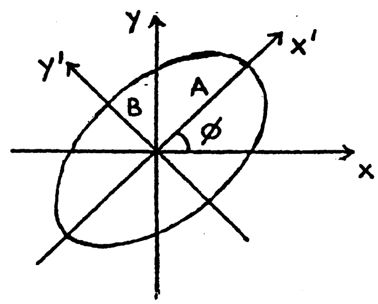
Substituting (1) into (2), we get

$$y = \frac{1}{2}x - \frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$$x - 2y = \sqrt{3(1-x^2)}$$

$$x^2 - xy + y^2 = \frac{3}{4} \tag{3}$$

This is the equation of an ellipse whose major and minor axes do not coincide with the x- and y-axes. Let the major and minor axes be along x'- and y'-axes, making angle ϕ with the x- and y-axes, respectively. Then



$$x' = x \cos \phi + y \sin \phi$$

$$y' = -x \sin \phi + y \cos \phi$$

and the equation for the ellipse is

$$\left(\frac{x \cos \phi + y \sin \phi}{A}\right)^2 + \left(\frac{-x \sin \phi + y \cos \phi}{B}\right)^2 = 1 \tag{4}$$

where A and B are the semi-major and semi-minor axes, respectively.

Now, expressing (3) in the form of (4), we obtain

$$\left(\frac{x+y}{\sqrt{3}}\right)^2 + \left(\frac{-x+y}{1}\right)^2 = 1$$

so that

R3.9. (continued)

$$\frac{\cos\phi}{A} = \frac{1}{\sqrt{3}}, \frac{\sin\phi}{A} = \frac{1}{\sqrt{3}}$$

$$\frac{\sin\phi}{B} = 1, \frac{\cos\phi}{B} = 1$$

$$\phi = 45^\circ, A = \sqrt{1.5}, B = \sqrt{0.5}$$

Thus,

$$\text{axial ratio of the ellipse} = \frac{\sqrt{1.5}}{\sqrt{0.5}} = \sqrt{3}$$

$$\text{tilt angle of the ellipse} = 45^\circ$$

R3.10. The electric field for the region $r > a$ is the same for both the charge distributions. The electric field for $r < a$ is zero for the surface charge case, whereas it is nonzero for the volume charge case. Therefore, the amount of work required is simply the energy stored in the electric field in the region $r < a$ for the volume charge. From Gauss' law for the electric field in integral form and assuming the center of the charge distribution to be at the origin, we have for $r < a$ for the volume charge,

$$4\pi r^2 D_r = \frac{Q}{\left(\frac{4}{3}\pi a^3\right)}\left(\frac{4}{3}\pi r^3\right)$$

$$= Q\,\frac{r^3}{a^3}$$

$$D_r = \frac{Qr^3}{4\pi r^2 a^3} = \frac{Qr}{4\pi a^3}$$

$$E_r = \frac{Qr}{4\pi\varepsilon_0 a^3}$$

$$W_e = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{2}\varepsilon_0\left(\frac{Qr}{4\pi\varepsilon_0 a^3}\right)^2 r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{Q^2}{8\pi\varepsilon_0 a^6}\int_0^a r^4\, dr$$

$$= \frac{Q^2}{40\pi\varepsilon_0 a}$$

$\therefore$ Work required is $Q^2/40\pi\varepsilon_0 a$.