Sinusoidal Steady-State Power Calculations

Reference:

Electric Circuits

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Outline

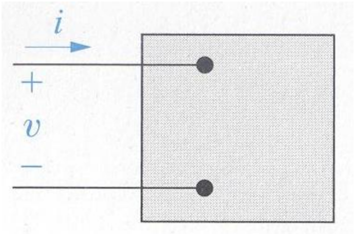
- Instantaneous power
- Average (real) power
- Reactive power
- Power in purely resistive, inductive and capacitive circuits
- Power factor and reactive factor
- Complex power and apparent power
- Maximum power transfer

Instantaneous Power

- Instantaneous power is the product of the instantaneous terminal voltage and current.

$$\boxed{p(t) = \pm v(t)i(t)}$$

- The positive sign is used when the reference direction for the current is from the positive to the negative reference polarity of the voltage.

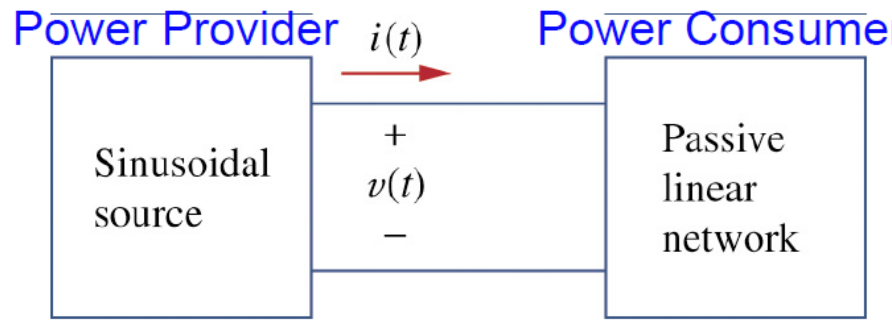


- Instantaneous power is measured in watts when the voltage is in volts and current is in amperes.

Instantaneous Power

Given: $v(t) = V_m \cos(\omega t + \theta_v)$

$$i(t) = I_m \cos(\omega t + \theta_i)$$



Choose convenient reference for zero time:

$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i)$$

$$i(t) = I_m \cos(\omega t)$$

Instantaneous power is calculated by:

$$p(t) = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos \omega t$$

Expanded formula:

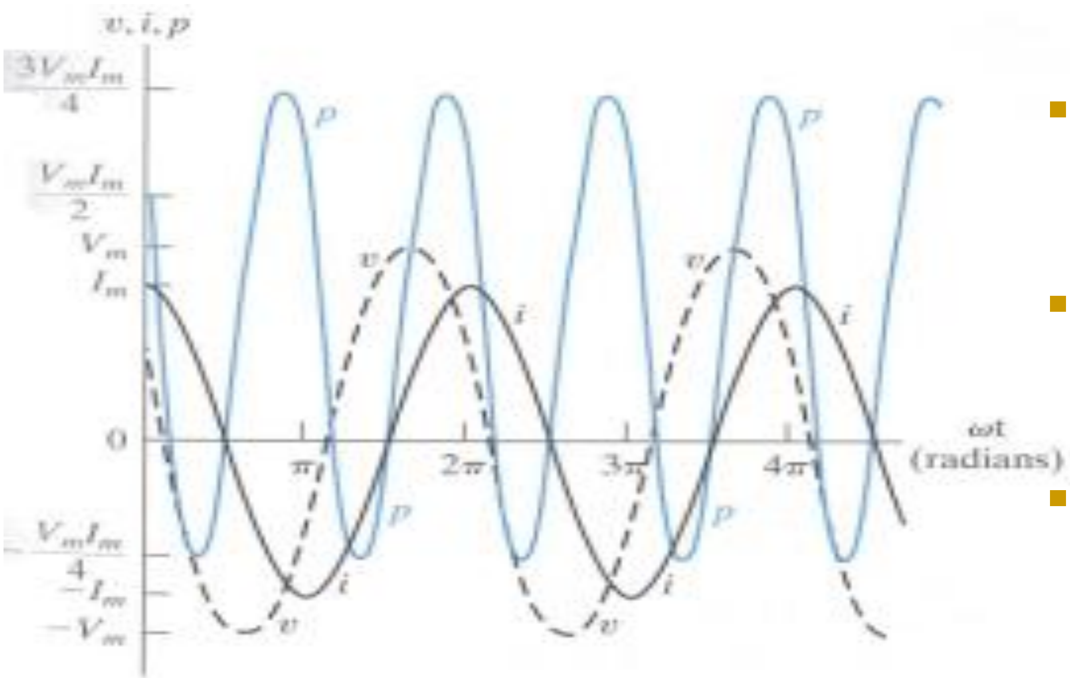
$$\boxed{\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)}$$

$$\boxed{\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta}$$

$$p(t) = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \cos 2\omega t - \frac{V_m I_m}{2} \sin(\theta_v - \theta_i) \sin 2\omega t$$

Instantaneous Power

$$p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t$$



Instantaneous power, voltage and current versus ωt for steady-state sinusoidal operation (θ_v = 60° and θ_i = 0°)

- The frequency of the instantaneous power is **twice** the frequency of the voltage (or current).
- The instantaneous power goes through two complete cycles for every cycle of either the voltage or the current.
- Instantaneous power may be negative for a portion of each cycle

Average and Reactive Power

Instantaneous power:

$$p(t) = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t$$

Instantaneous power is rewrite as follow:

$$p = P + P\cos 2\omega t - Q\sin 2\omega t$$

Average power:

$$\boxed{P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)}$$

Reactive power:

$$\boxed{Q = \frac{1}{2}V_m I_m \sin(\theta_v - \theta_i)}$$

Average (Real) Power

- Average power is the average value of the instantaneous power over one period.

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt$$

- Average power is expressed as:

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = V_{eff} I_{eff} \cos(\theta_v - \theta_i)$$

- Average power is also called **real power** because it describes the power in a circuit that is transformed from electric to nonelectric energy.

$$\begin{aligned} P &= \frac{1}{T}\int_0^T p(t)\,dt \\ &= \frac{1}{T}\int_0^T V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)\,dt \\ &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \\ &= \frac{1}{2}\mathrm{Re}\left[\mathbf{VI}^*\right]; \qquad \left(\because \mathbf{VI}^* = V_m I_m \angle(\theta_v - \theta_i)\right) \end{aligned}$$

$$\begin{cases} v(t) = V_m \cos(\omega t + \theta_v) \\ i(t) = I_m \cos(\omega t + \theta_i) \end{cases}$$

Average (Real) Power

Two Special Cases of Average Power

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

Purely resistive circuit: $\theta_v = \theta_i$

$$\Rightarrow P = \frac{1}{2} |\mathbf{I}|^2 R_{eq}$$

Purely reactive circuit: $\theta_v - \theta_i = \pm 90°$

$$\Rightarrow P = \frac{1}{2} V_m I_m \cos 90° = 0$$

The resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

Average (Real) Power

Two Special Cases of Average Power

$$\mathbf{Z} = R \quad : \quad P = \frac{1}{2} V_m I_m \angle 0^\circ = \frac{1}{2} |\mathbf{I}|^2 R = \frac{|\mathbf{V}|^2}{2R}$$

$$\mathbf{Z} = j\omega L \quad : \quad P = \frac{1}{2} V_m I_m \angle 90^\circ = 0$$

$$\mathbf{Z} = \frac{1}{j\omega C} \quad : \quad P = \frac{1}{2} V_m I_m \angle -90^\circ = 0$$

Reactive Power

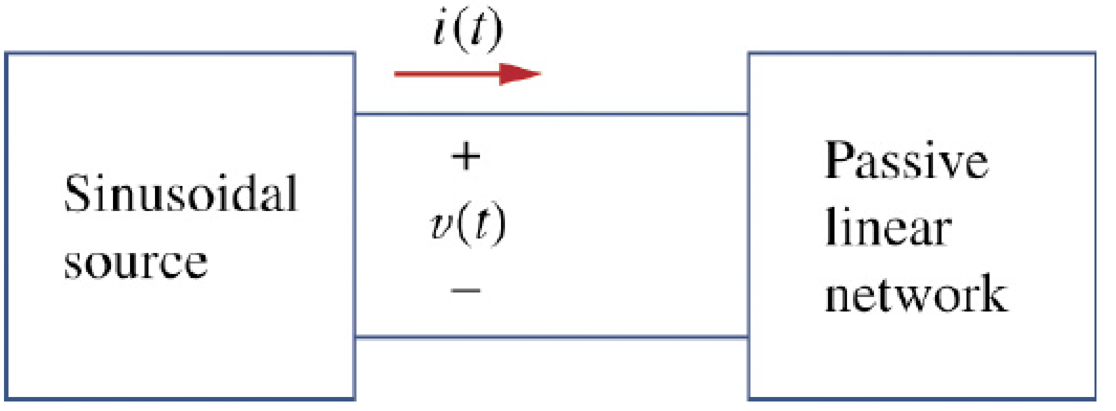
- Reactive power is the electric power exchanged between the magnetic field of an inductor and the source that drives it or between the electric field of a capacitor and the source that drives it.
- Reactive power is never converted to non-electric power.
- Reactive power is expressed as:

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{eff} I_{eff} \sin(\theta_v - \theta_i)$$

- Both average power and reactive power can be expressed in terms of either peak (V_m,I_m) or effective (V_{eff} , I_{eff}) current and voltage.
- Effective values are widely used in both household and industrial application.
- Effective value and rms value are interchangeable for the same value.

Example 1.

Given that $v(t)$=120cos(377t+45°)V, $i(t)$= 10cos(377t-10°)A, find the instantaneous power and the average power absorbed by the passive linear network.



$$p = iv = 1200\cos(377t + 45°)\cos(377t - 10°)$$

$$= 600\left(\cos(754t + 35°) + \cos 55°\right)$$

$$= \underline{344.2 + 600\cos(754t + 35°)\ \text{W}}$$

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}1200\cos\left(45° - (-10°)\right)$$

$$= 600\cos 55° = 344.2\ \text{W}$$

Example 2.

Calculus the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0^\circ$ is applied across it.

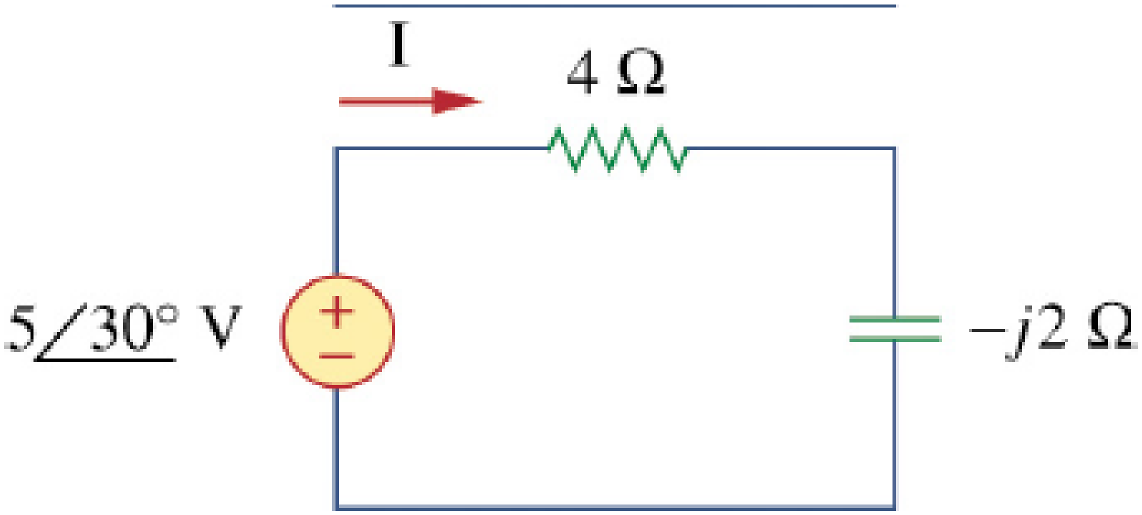
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\angle 0^\circ}{76.16\angle -66.8^\circ} = 1.576\angle 66.8^\circ \text{ A}$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$= \frac{1}{2}(120)(1.576)\cos 0 - 66.8^\circ) = \underline{37.24} \text{ W}$$

Example 3.

For the circuit, find the average power supplied by the source and the average power absorbed by the resistor.



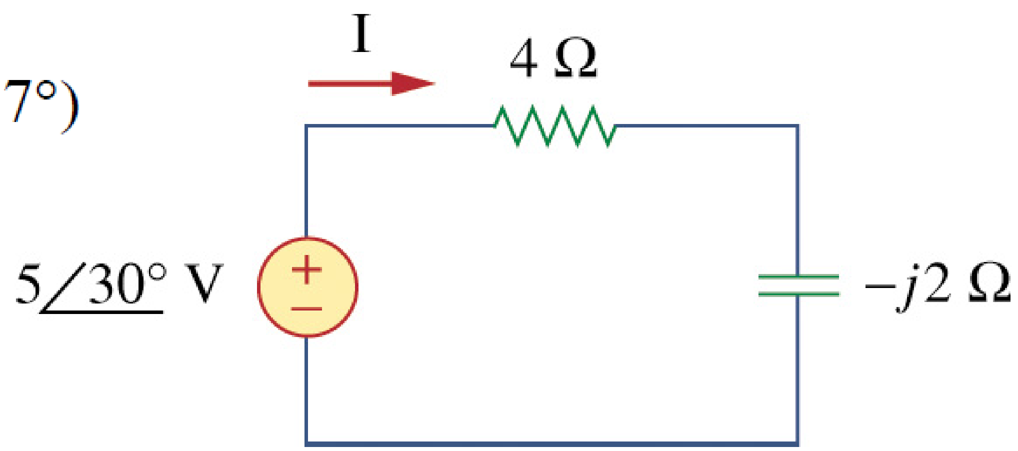
Example 3 - Solution

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

$$P = \frac{1}{2}\text{Re}\left[\mathbf{VI}^*\right]$$

$$= \frac{1}{2} \times 5 \times 1.118 \cos(30^\circ - 56.57^\circ)$$

$$= \underline{2.5} \text{ W}$$



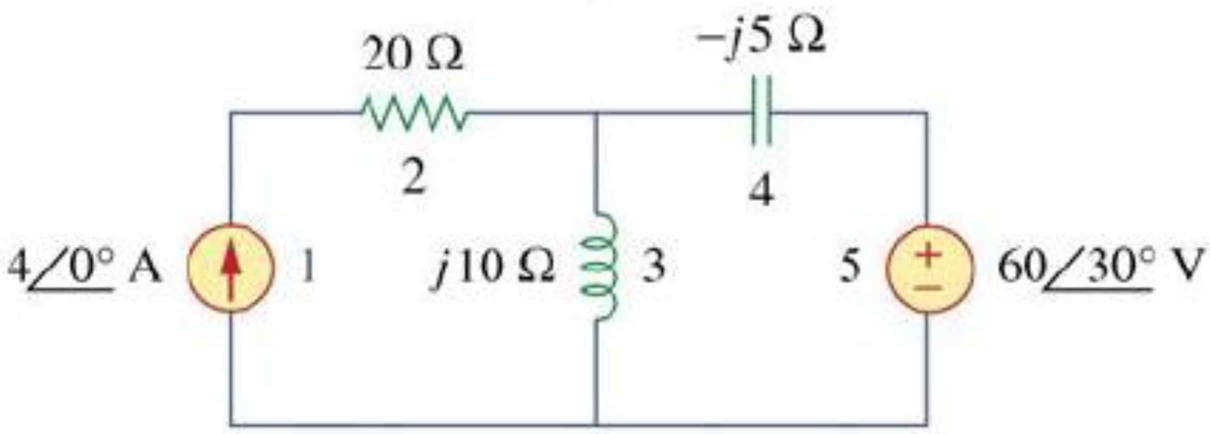
$$\mathbf{I}_R = \mathbf{I} = 1.118\angle 56.57^\circ \text{ A}$$

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ \text{ V}$$

$$P_R = \frac{1}{2} \times 4.472 \times 1.118 = \underline{2.5} \text{ W}$$

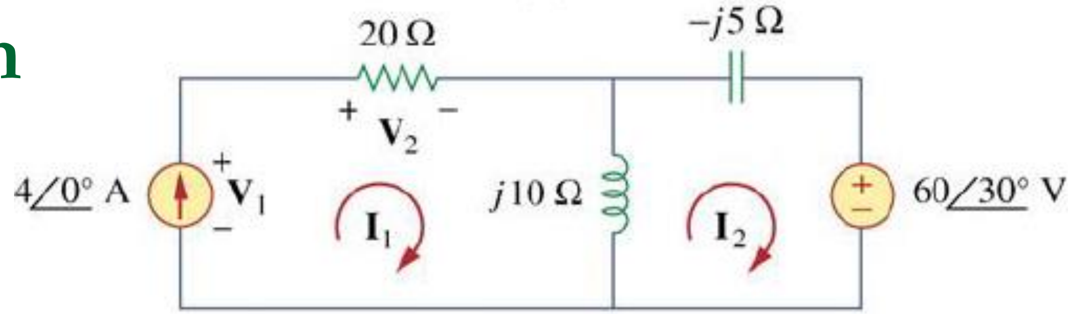
Example 4

Determine the average power generated by each source and the average power absorbed by each passive element in circuit (a).



(a)

Example 4 - Solution



(b)

For mesh 1: $\mathbf{I}_1 = 4 \text{ A} \cdots (a)$

KCL for mesh 2: $(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\angle 30° = 0 \cdots (b)$

By (a) and $(b) \Rightarrow \quad \mathbf{I}_2 = -12\angle -60° + 8 = \underline{10.58\angle 79.1°} \text{ A}$

For the voltage source:

$$P_V = \frac{1}{2}\text{Re}\left[\mathbf{V}\mathbf{I}_2^*\right] = \frac{1}{2} \times 60 \times 10.58 \times \cos(30° - 79.1°) = \underline{207.8} \text{ W}$$

$P_V > 0$. Hence, this average power is absorbed by the source.

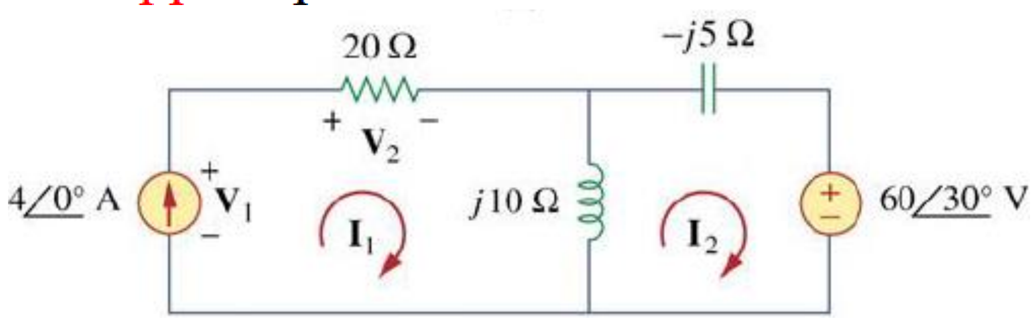
Example 4 - Solution

For the current source:

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 184.984\angle 6.21°\ V$$

$$P_I = -\frac{1}{2}\mathrm{Re}\left[\mathbf{V}_1\mathbf{I}_1^*\right] = -\frac{1}{2}\times 184.984\times 4\cos(6.21° - 0)$$
$$= \underline{-367.8}\ \mathrm{W}$$

$P_I < 0$. Hence, this current source supplies power to the circuit.



(b)

For the resistor:

$$\mathbf{I}_1 = 4\angle 79.1°\ \mathrm{A}$$

$$\mathbf{V}_2 = 20\mathbf{I}_1 = 80\angle 79.1°\ \mathrm{V}$$

$$\Rightarrow P_R = \frac{1}{2}\mathrm{Re}\left[\mathbf{V}_2\mathbf{I}_1^*\right] = \frac{1}{2}\times 80\times 4 = 160\ \mathrm{W}$$

Example 4 - Solution

For the capacitor:

$$\mathbf{I}_2 = 10.58\angle 79.1°,$$

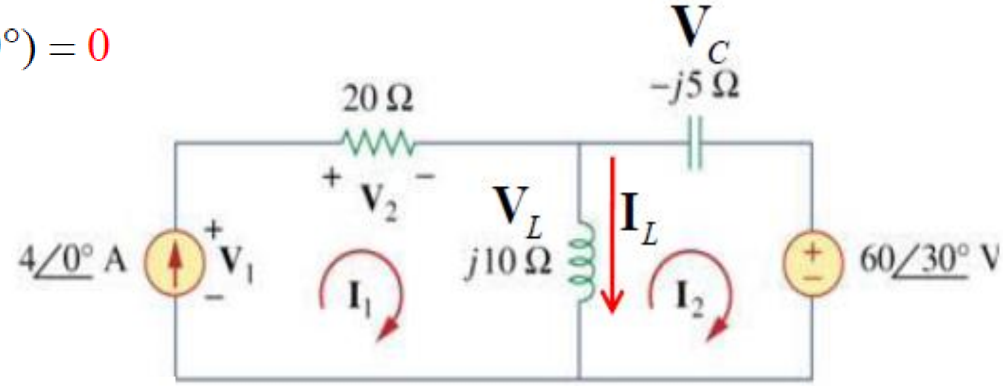
$$\mathbf{V}_C = -j5\mathbf{I}_2 = (5\angle -90°)(10.58\angle 79.1°) = 52.9\angle(79.1° - 90°)$$

$$\Rightarrow P_C = \frac{1}{2}\mathrm{Re}\left[\mathbf{V}_C\mathbf{I}_2^*\right] = \frac{1}{2}(52.9)(10.58)\cos(-90°) = 0$$

For the inductor:

$$\mathbf{I}_L = \mathbf{I}_1 - \mathbf{I}_2 = 2 - j10.39 = 10.58\angle -79.1°$$

$$\mathbf{V}_L = j10(\mathbf{I}_1 - \mathbf{I}_2) = 10.58\angle(-79.1° + 90°)$$



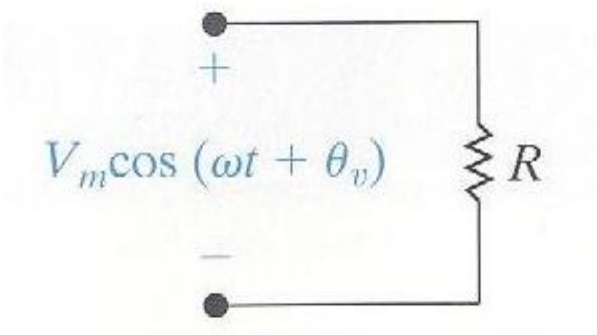
$$\Rightarrow P_L = \frac{1}{2}\mathrm{Re}\left[\mathbf{V}_L\mathbf{I}_L^*\right] = \frac{1}{2}(105.8)(10.58)\cos 90° = 0$$

Finally,

$$P_V + P_I + P_R + P_C + P_L$$

$$= -367.8 + 160 + 0 + 0 + 207.8 = 0$$

The rms value and power calculation



Definition of V_{rms} : $$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t+\theta_v)dt}$$

Average power delivered to the resistor:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_m^2 \cos^2(\omega t+\theta_v)}{R}dt = \frac{1}{R}\left[\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t+\theta_v)dt\right]$$

It can be seen that the average power delivered to R is simply the rms value of the voltage squared divided by R.

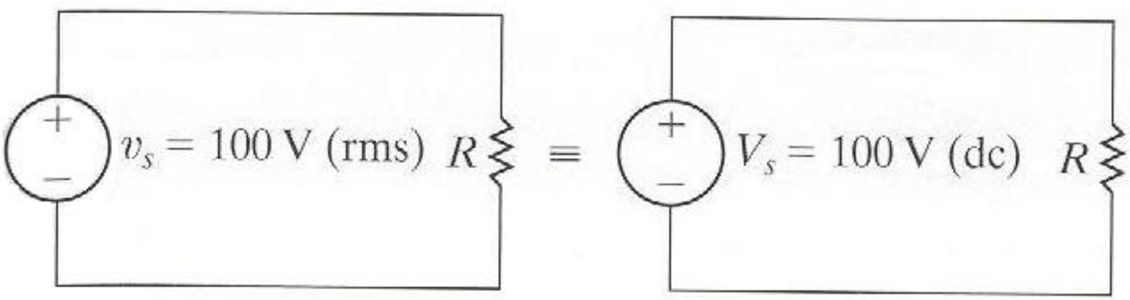
$$P = \frac{V_{rms}^2}{R}$$

The rms value and power calculation

If the resistor is carrying a sinusoidal current, then average power delivered to the resistor is:

$$P = I_{rms}^{2} R$$

1) The rms value is also referred to as the effective value of the sinusoidal voltage (or current).
2) The rms value of a sinusoidal source delivers the same energy to R as does a dc source of the same value. Energywise, the effect of the two sources is identical. Therefore, the term *effective value* being used interchangeable with *rms value.*

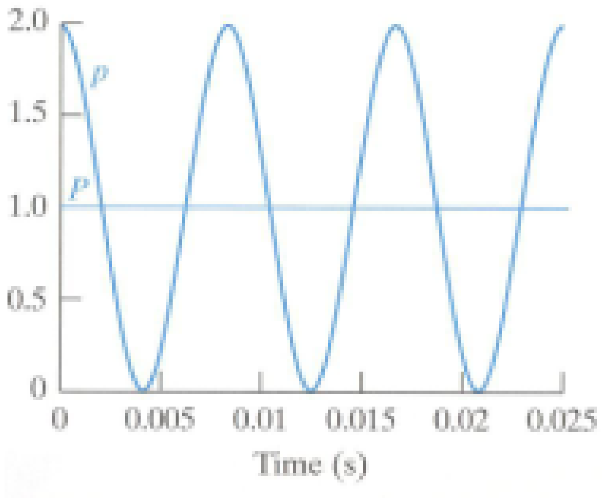


Power for Purely Resistive Circuits

In purely resistive circuits, the voltage and current are in phase $\theta_v - \theta_i = 0$

The instantaneous becomes:

$$p = P + P\cos 2\omega t$$



Instantaneous and average powers (W)

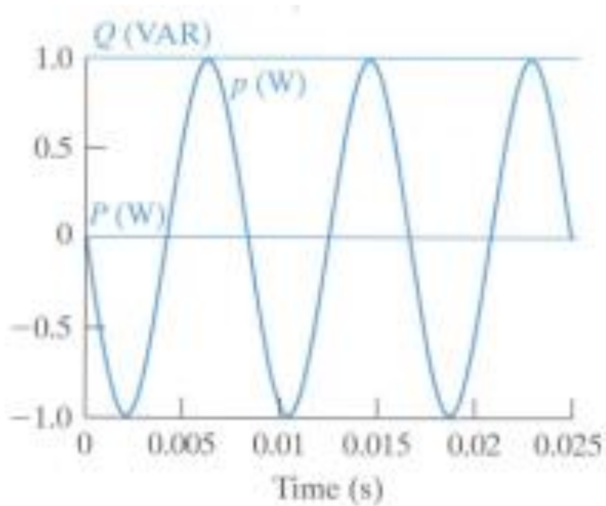
The instantaneous power in purely resistive circuits is always positive.

Or, power cannot be extracted form a purely resistive network.

Power for Purely Inductive Circuits

In purely inductive circuits, the voltage leads the current by 90º : $\theta_v - \theta_i = +90^\circ$

The instantaneous becomes: $p = -Q \sin 2\omega t$



Instantaneous, average and reactive powers for a purely inductive circuit

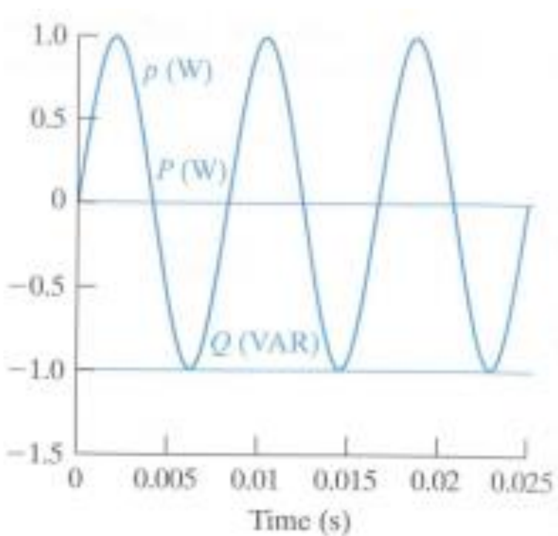
In a purely inductive circuit:

1) Average power is zero. Therefore, no transformation of energy form electric to nonelectric form takes place.
2) The instantaneous power is continually exchanged between the circuit and the source driving the circuit at 2ω frequency. When p>0, energy is being stored in the magnetic fields associated with the inductive elements. When p <0, energy is being extracted from the magnetic fields.
3) Reactive power is named from the characterization of an inductor as a reactive element.
4) The unit of reactive power is **var**

Power for Purely Capacitive Circuits

In purely capacitive circuits, the voltage lags the current by 90°: $\theta_v - \theta_i = -90^\circ$

The instantaneous becomes: $p = Q \sin 2\omega t$



Instantaneous, average and reactive powers for a purely capacitive circuit

In a purely capacitive circuit:

1) Average power is zero. Therefore, no transformation of energy form electric to nonelectric form takes place.
2) The instantaneous power is continually exchanged between the electric field associated with the capacitive elements and the source driving the circuit at 2ω frequency.
3) Depend on the algebraic sign of Q, it can be said that <u>inductors absorb</u> magnetizing vars, and <u>capacitors deliver</u> magnetizing vars.

Power Factor and Reactive Factor

- The power factor is the cosine of the phase angle between the voltage and the current

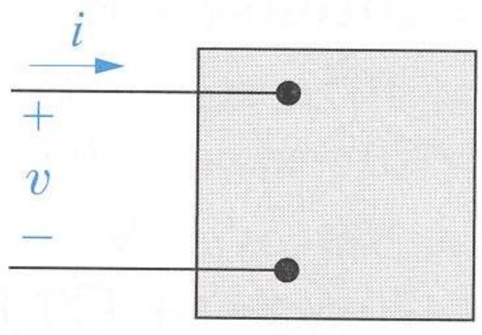
$$pf = \cos(\theta_v - \theta_i)$$

- The reactive factor is the sine of the phase angle between the voltage and the current.

$$rf = \sin(\theta_v - \theta_i)$$

- Lagging power factor implies that the current lags voltage – hence an inductive load.
- Leading power factor implies that the current leads the voltage – hence a capacitive load.

Example 5



Given:

$$v = 100\cos(\omega t + 15^\circ) \quad \text{V}$$

$$i = 4\sin(\omega t - 15^\circ) \quad \text{A}$$

1) Calculate the average power and the reactive power at the terminals of the network.
2) State whether the network inside the box is absorbing or delivering average power.
3) State whether the network inside the box is absorbing or supplying magnetizing vars.

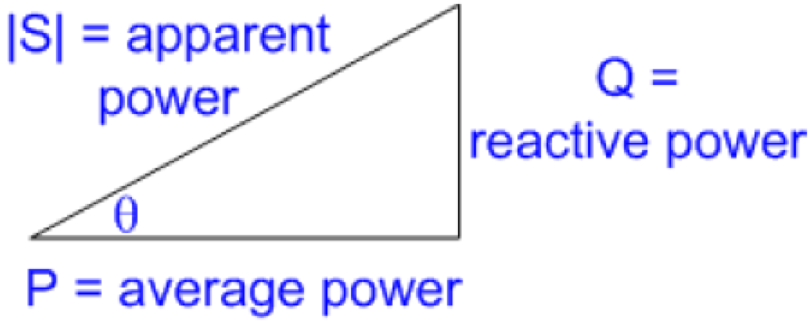
Complex Power

- Complex power is the complex sum of the real and reactive powers

$$S = P + jQ$$

- The **watt** is used as the unit for both instantaneous and real power.
- The **var** (volt ampere reactive, or VAR) is used as the unit for reactive power.
- The **volt-amp** (VA) is used as the unit for complex and apparent power.

Apparent Power



Power triangle

The geometric relations mean that four power triangle dimensions (the three sides and the power factor angle) can be determined if any two of the four are known.

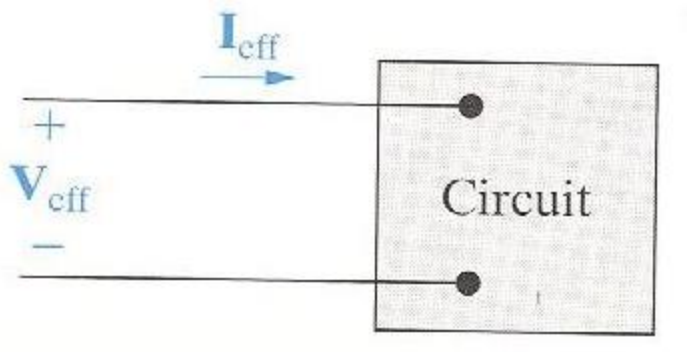
Apparent power is the magnitude of the complex power:

$$|S| = \sqrt{P^2 + Q^2}$$

The angle θ in the power triangle is the power factor angle $\theta_v - \theta_i$:

$$\tan\theta = \frac{Q}{P}$$

Power Calculation

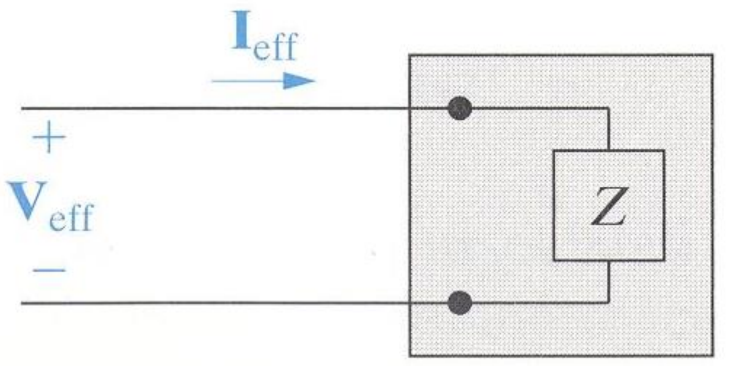


Complex power:

$$S = P + jQ = \frac{1}{2}VI^{*} = V_{eff} I_{eff}^{*}$$

Can you prove it?

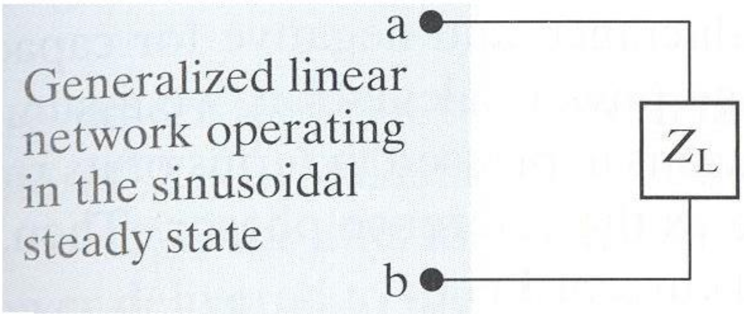
Alternate form for complex power

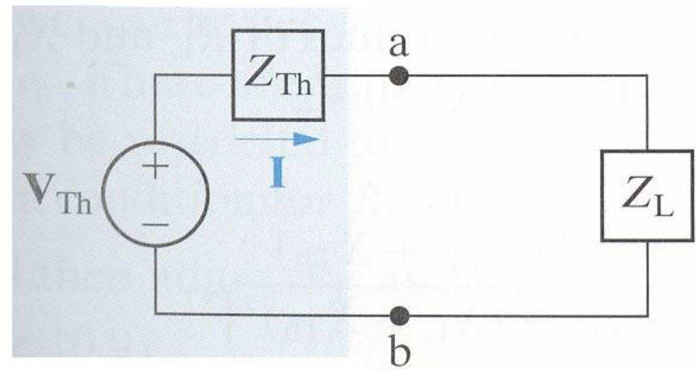


$$S = P + jQ = \frac{1}{2}VI^* = V_{eff}I_{eff}^* = I_{eff}^2 Z = \frac{V_{eff}^2}{Z^*}$$

Can you prove it?

Maximum Power Transfer



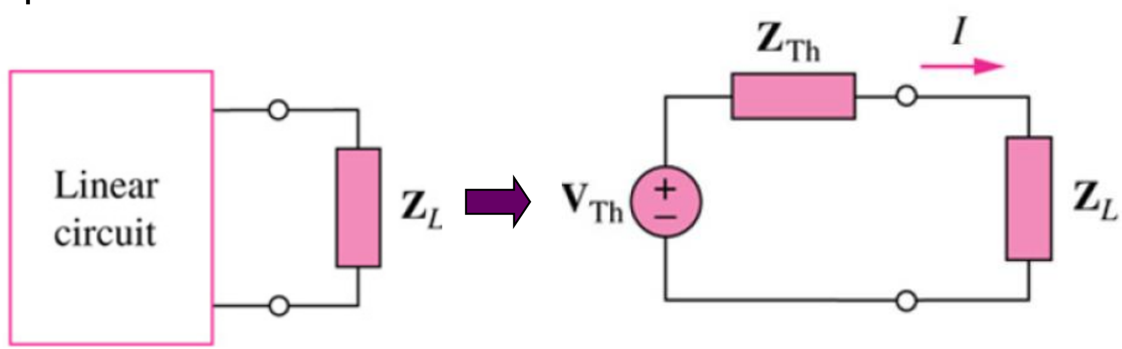


Maximum power transfer occurs in circuits operating in the sinusoidal steady state when the load impedance is the conjugate on the Thevenin impedance as viewed from the terminals of the load impedance.

$$\boxed{Z_L = Z_{Th}^*}$$

Maximum average power absorbed:

$$\boxed{P_{\max} = \frac{1}{4}\frac{|V_{Th}|^2}{R_L} = \frac{1}{8}\frac{V_m^2}{R_L}}$$



Maximum Power Transfer

Derivation of Maximum Average Power Transfer

$$\mathbf{Z}_{\mathrm{Th}} = R_{\mathrm{Th}} + jX_{\mathrm{Th}};\ \mathbf{Z}_L = R_L + jX_L$$

$$\mathbf{I} = \frac{\mathbf{V}_{\mathrm{Th}}}{\mathbf{Z}_{\mathrm{Th}} + \mathbf{Z}_L} = \frac{\mathbf{V}_{\mathrm{Th}}}{(R_{\mathrm{Th}} + R_L) + j(X_{\mathrm{Th}} + X_L)}$$

$$P = \frac{1}{2}\mathrm{Re}\left[\mathbf{V}_L\mathbf{I}^*\right] = \frac{1}{2}\mathrm{Re}\left[\mathbf{Z}_L\mathbf{I}\mathbf{I}^*\right] = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}\frac{|\mathbf{V}_{\mathrm{Th}}|^2 R_L}{(R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2}$$

To find the condition with maximum power,

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\mathrm{Th}}|^2 R_L (X_{\mathrm{Th}} + X_L)}{\left((R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2\right)^2} = 0$$

$$\frac{\partial P}{\partial R_L} = -\frac{|\mathbf{V}_{\mathrm{Th}}|^2 \left((R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2 - 2R_L(R_{\mathrm{Th}} + R_L)\right)}{2\left((R_{\mathrm{Th}} + R_L)^2 + (X_{\mathrm{Th}} + X_L)^2\right)^2} = 0$$

Maximum Power Transfer

Derivation of Maximum Average Power Transfer

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{\text{Th}}|^2 R_L (X_{\text{Th}} + X_L)}{\left((R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2\right)^2} = 0$$

$$\Rightarrow \underline{X_L = -X_{\text{Th}}}$$

$$\frac{\partial P}{\partial R_L} = -\frac{|\mathbf{V}_{\text{Th}}|^2 \left((R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2 - 2R_L(R_{\text{Th}} + R_L)\right)}{2\left((R_{\text{Th}} + R_L)^2 + (X_{\text{Th}} + X_L)^2\right)^2} = 0$$

$$\Rightarrow R_L = \sqrt{R_{th}^2 + (X_{th} + X_L)^2}$$

$$= R_{\text{Th}};\ (\because X_{\text{Th}} = -X_L)$$

$$\Rightarrow \underline{\mathbf{Z}_L = R_{\text{Th}} - jX_{\text{Th}} = \mathbf{Z}_{\text{Th}}^*}$$

$$\Rightarrow P_{\max} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}}$$