Differential Equation in a Nutshell

Chapter 1: First order Differential Equations

1. General Form

Given a first order DE[1] in general form

$$y' + p(x)y = q(x) \tag{Eq 1.1}$$

2. Solving Steps

1. Calculating outside integrating factor

$$u(x) = e^{\int p(x)dx}$$

Since, we have $u'(x) = p(x)e^{\int p(x)dx} = p(x)u(x)$.

2. Multiply both sides of (Eq 1.1) by $u(x)$

$$\begin{aligned}\text{(Eq 1.1)} &\leftrightarrow y'u(x) + p(x)u(x)y = q(x)u(x)\\ &\leftrightarrow y'u(x) + u'(x)y = q(x)u(x)\\ &\leftrightarrow (u(x)y)' = q(x)u(x)\end{aligned}$$

3. Integrating both sides, we obtain

$$u(x)y = \int q(x)u(x)dx + C$$

4. Divide both sides by integrating factor to obtain the final result

$$y = \frac{1}{u(x)}\left(\int q(x)u(x)dx + C\right)$$

[1] DE: Differential equation.

Chapter 2: Exact Equations

1. General Form

Given a DE in general form

$$M(x,y)dx + N(x,y)dy = 0 \qquad \text{(Eq 2.1)}$$

1. The given equation is called exact equation if and only if:

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \qquad \text{(Eq 2.2)}$$

2. If the equation is exact, there is exists a function $F(x,y)$, so that:

$$\begin{cases} F_x(x,y) = M(x,y) \ \ (1) \\ F_y(x,y) = N(x,y) \ \ (2) \end{cases} \qquad \text{(Eq 2.3)}$$

2. Solving Steps

Integrating both sides of (1) of (Eq 2.3), with respect to x, we get:

$$F(x,y) = \int M(x,y)dx = f(x,y) + \varphi(y)$$

Differentiating the result above with respect to y

$$\rightarrow F_y(x,y) = f_y(x,y) + \varphi'(y) \ (3)$$

Compare (3) and (2):

$$\varphi'(y) = N(x,y) - f_y(x,y)$$

$$\rightarrow \varphi(y) = \int \left[N(x,y) - f_y(x,y)\right]dy + C = g(y) + C$$

The solution becomes

$$f(x,y) + g(y) + C = 0$$

Chapter 3: Second order Differential Equations

1. General Solution

Given a DE in general form (a, b, c are constant coefficients)

$$ay'' + by' + cy = g(x) \tag{Eq 3.1}$$

The ***General solution*** of the DE is the ***sum*** of ***Complement solution*** and ***Particular solution***

$$y_G = y_c + y_p \tag{Eq 3.2}$$

2. Complement Solution

Complement solution is the solution of ***Homogeneous equation*** or $g(x) = 0$. In fact, complement solution is also a general solution of homogeneous equation.

2. 1. Homogeneous with Constant Coefficients

Characteristic equation (CE) is given by

$$\begin{gathered} ar^2 + br + c = 0 \\ \leftrightarrow (r - r_1)(r - r_2) = 0 \end{gathered} \tag{Eq 3.3}$$

Complement solution of second order DE depends on the root of CE as follows:

1. Two distinct real roots r_1, r_2:

$$y_c = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

2. Double root $r_1 = r_2 = r$:

$$y_c = C_1 e^{rx} + C_2 x e^{rx}$$

3. Complex root $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$:

$$y_c = C_1 e^{\alpha x} \sin \beta x + C_2 e^{\alpha x} \cos \beta x$$

2. 2. Homogeneous with Non-constant Coefficients

In the case of constants a, b, c become variable parameters:

$$y'' + p(x)y' + q(x)y = 0 \tag{Eq 3.4}$$

If y_1, y_2 are solution of the above differential equation and satisfy Wronskian determinant different from zero for interval I, we call that y_1, y_2 belong to ***Fundamental solution set*** of the equation. It leads to the complement solution is:

$$y_c = C_1 y_1 + C_2 y_2 \tag{Eq 3.5}$$

Wronskian determinant:

$$\mathrm{W}[y_1, y_2] = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2 = C_1 e^{-\int p(x)dx} \tag{Eq 3.6}$$

In some cases, $y_1 = ax + b$, $y_1 = x^\alpha$, or $y_1 = x^\alpha \ln x$ maybe helpful for us to check whether or not it is a solution of the Homogeneous equation.

When y_1 already is a solution of homogeneous equation, we have to find another solution y_2. This solution is given by the formula

$$y_2 = y_1 \left[\int \frac{\mathrm{W}[y_1, y_2]}{y_1^2} dx + C_2 \right] \qquad \text{(Eq 3.7)}$$

To simplifier, choosing $C_2 = 0$ and C_1 is a particular constant to obtain the simplifies result.

Combine it to get the complement solution due to (Eq 3.5).

3. Particular Solution

3. 1. Constant Coefficients

Particular solution is a nontrivial solution of the given DE (included RHS[2]).

The particular solution depends on the RHS; almost of cases, it copies the form of the RHS

Case 1:

$$g(x) = P_n(x)e^{\alpha x} = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)e^{\alpha x}$$

(Product of polynomial and exponential)

- $\alpha \neq r_1, r_2$: α is not a root of CE:

$$y_p = Q_n(x)e^{\alpha x}$$

- $\alpha \equiv r_1$: α is one of single roots of CE:

$$y_p = xQ_n(x)e^{\alpha x}$$

- $\alpha \equiv r_1 \equiv r_2$: α is a double root of CE:

$$y_p = x^2 Q_n(x)e^{\alpha x}$$

Case 2:

$$g(x) = P_n(x)e^{\alpha x} \times \begin{cases} \cos\beta x \\ \sin\beta x \end{cases}$$

(Product of polynomial, exponential, and trigonometric)

- $\alpha + i\beta \neq r_1, r_2$: $\alpha + i\beta$ is not a root of CE:

$$y_p = [Q_n(x)\cos\beta x + R_n(x)\sin\beta x]e^{\alpha x}$$

- $\alpha + i\beta \equiv r_1$: $\alpha + i\beta$ is a root of CE:

$$y_p = x[Q_n(x)\cos\beta x + R_n(x)\sin\beta x]e^{\alpha x}$$

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[2] RHS: Right hand side, $g(x)$ of (Eq 3.1).

3. 2. Non-constant Coefficients

The particular solution in this case is given by

$$y_p = u_1(x)y_1 + u_2(x)y_2 \qquad \text{(Eq 3.8)}$$

Where u_1, u_2 are unknown functions we have to find and y_1, y_2 are fundamental solution set which show at process of finding complement solution.

Cramer's rule immediately gives us:

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(x) \end{cases} \qquad \text{(Eq 3.9)}$$

Or,

$$u_1' = -\frac{g(x)y_2}{\mathrm{W}[y_1, y_2]}, \qquad u_2' = \frac{g(x)y_1}{\mathrm{W}[y_1, y_2]} \qquad \text{(Eq 3.10)}$$

Then find out u_1, u_2 by integrating the above result and completing the particular solution due to (Eq 3.8).

Chapter 4: Higher order Differential Equations

1. General Solution

Given a n-order DE in general form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = g(x) \qquad \text{(Eq 4.1)}$$

The General solution of the DE is the ***sum*** of ***Complement solution*** and ***Particular solution***

$$y_G = y_c + y_p \qquad \text{(Eq 4.2)}$$

Complement solution depends on the order of DE, see at **section 4.2**.

Particular solution depends on the right hand side (RHS) of DE. If $g(x) = 0$, it leads to $y_p = 0$. If the RHS is different from 0, see at **section 4.3**.

2. Complement Solution

Complement solution is the solution of ***Homogeneous equation*** or $g(x) = 0$. In fact, complement solution is also a general solution of homogeneous equation.

2. 1. Homogeneous with Constant Coefficients

In the case of all coefficients of (Eq 4.1), $a_{n-1}, \ldots, a_2, a_1, a_0$, are constant. We obtain the ***Characteristic equation (CE)*** as follows

$$\begin{gathered} r^n + a_{n-1}r^{n-1} + \cdots + a_2r^2 + a_1r + a_0 = 0 \\ \leftrightarrow (r - r_1)(r - r_2) \ldots (r - r_{n-1})(r - r_n) = 0 \end{gathered} \qquad \text{(Eq 4.3)}$$

The CE with n-order has n roots, includes its multiplicities and complex roots. The compliment solution is classified as follows:

1. For distinct real roots $r_1, r_2, \ldots, r_n$:

$$y_c = C_1e^{r_1x} + C_2e^{r_2x} + \cdots + C_ne^{r_nx}$$

2. For a pair of complex roots $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$:

$$y_c = C_1e^{\alpha x}\sin\beta x + C_2e^{\alpha x}\cos\beta x$$

3. For multiplicity real roots $r_1 = r_2 = \cdots = r_n = r_0$:

$$y_c = C_1e^{r_0x} + C_2xe^{r_0x} + \cdots C_nx^ne^{r_0x}$$

(For each time of the root repeated, the power of x increases by 1)

4. For multiplicity complex roots $r_1 = r_3 = \alpha + i\beta, r_2 = r_4 = \alpha - i\beta$:

$$y_c = C_1e^{\alpha x}\sin\beta x + C_2e^{\alpha x}\cos\beta x + x(C_1e^{\alpha x}\sin\beta x + C_2e^{\alpha x}\cos\beta x)$$

(If multiplicity greater than 1, we continue the process same as multiplicity real root)

In practice, $r_1, r_2, \ldots, r_n$ are not always all single real roots or complex roots. It may combine some cases from 1 to 4. We just sum up the combination of these cases.

2. 2. Homogeneous with Non-constant Coefficients

In the case of some coefficients of (Eq 4.1), $a_{n-1}, \dots, a_2, a_1, a_0$, are variable parameters

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad \text{(Eq 4.4)}$$

If we have $y_1, y_2, \dots, y_n$ belong to Fundamental solution set of (Eq 4.3), the complement solution of (Eq 4.3) is given by:

$$y_c = C_1y_1 + C_2y_2 + \cdots + C_ny_n \qquad \text{(Eq 4.5)}$$

If $y_1, y_2, \dots, y_n$ are solutions of (Eq 4.3) and satisfy Wronskian determinant different from zero for interval I, we call that $y_1, y_2, \dots, y_n$ belong to Fundamental solution set of (Eq 4.3).

Wronskian determinant:

$$\mathrm{W}[y_1, y_2, \dots, y_n] = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{bmatrix} \qquad \text{(Eq 4.6)}$$

3. Particular Solution

3. 1. Constant Coefficients

The particular solution depends on the RHS, almost of cases; it copies the form of the RHS

Case 1:

$$g(x) = P_n(x)e^{\alpha x} = (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)e^{\alpha x}$$

(Product of polynomial and exponential)

- $\alpha \neq r_1, r_2, \dots, r_n$: α is not a root of CE:

$$y_p = Q_n(x)e^{\alpha x}$$

- $\alpha \equiv r_1$: α is one of all single roots of CE:

$$y_p = xQ_n(x)e^{\alpha x}$$

- $\alpha \equiv r_1 \equiv r_2$: α is a double root of CE:

$$y_p = x^2Q_n(x)e^{\alpha x}$$

- α is a root of CE with multiplicity of n:

$$y_p = x^nQ_n(x)e^{\alpha x}$$

Case 2:

$$g(x) = P_n(x)e^{\alpha x} \times \begin{cases} \cos\beta x \\ \sin\beta x \end{cases}$$

(Product of polynomial, exponential, and trigonometric)

- $\alpha + i\beta \neq r_1, r_2, \dots, r_n$: $\alpha + i\beta$ is not a root of CE.

$$y_p = [Q_n(x)\cos\beta x + R_n(x)\sin\beta x]e^{\alpha x}$$

- $\alpha + i\beta \equiv r_1$: $\alpha + i\beta$ is a root of CE.

$$y_p = x[Q_n(x)\cos\beta x + R_n(x)\sin\beta x]e^{\alpha x}$$

- $\alpha + i\beta$ is a root of CE with multiplicity of n:

$$y_p = x^n[Q_n(x)\cos\beta x + R_n(x)\sin\beta x]e^{\alpha x}$$

3. 2. Non-constant Coefficients

The particular solution in this case is given by

$$y_p = u_1(x)y_1 + u_2(x)y_2 + \cdots + u_n(x)y_n \qquad \text{(Eq 4.7)}$$

Where $u_1, u_2, \dots, u_n$ are unknown functions that we have to find and $y_1, y_2, \dots, y_n$ are fundamental solution set which show at process of finding complement solution.

To obtain $u_1, u_2, \dots, u_n$, we have to solve the following system:

$$\begin{cases} y_1 u_1'(x) + y_2 u_2'(x) + \cdots + y_n u_n'(x) = 0 \\ y_1' u_1'(x) + y_2' u_2'(x) + \cdots + y_n' u_n'(x) = 0 \\ y_1'' u_1'(x) + y_2'' u_2'(x) + \cdots + y_n'' u_n'(x) = 0 \\ \vdots \\ y_1^{(n-1)} u_1'(x) + y_2^{(n-1)} u_2'(x) + \cdots + y_n^{(n-1)} u_n'(x) = g(x) \end{cases} \qquad \text{(Eq 4.8)}$$

Then find out $u_1, u_2, \dots, u_n$ by integrating the above result and completing the particular solution due to (Eq 4.5).

Chapter 5: System of Linear First order Differential Equations

1. Solution of SLFDE[3]

Given a SLFDE in matrix form

$$\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t) \quad \text{(Eq 5.1)}$$

The ***General solution*** of the SLFDE is the ***sum*** of ***Complement solution*** and ***Particular solution***

$$\boldsymbol{x_G} = \boldsymbol{x_c} + \boldsymbol{x_p} \quad \text{(Eq 5.2)}$$

1. Fundamental matrix:

$$\boldsymbol{\Phi}(t) = e^{\int \boldsymbol{A}(t)dt} \quad \text{(Eq 5.3)}$$

2. Complement solution

$$\boldsymbol{x_c} = \boldsymbol{\Phi}(t)\boldsymbol{C} \quad \text{(Eq 5.4)}$$

is corresponding to the solution of $\boldsymbol{x}'(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$, where $\boldsymbol{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$; C_1, C_2 are arbitrary constants.

3. Particular solution $\boldsymbol{x_p}$ corresponding to the solution of $\boldsymbol{x}'_{\boldsymbol{p}}(t) = \boldsymbol{A}(t)\boldsymbol{x_p}(t) + \boldsymbol{B}(t)$ can be found as follow

$$\boldsymbol{x_p} = \boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(\tau)\boldsymbol{B}(\tau)d\tau \quad \text{(Eq 5.5)}$$

4. If we have the initial value $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, then the solution becomes

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(t_0)\boldsymbol{x}_0 + \boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(\tau)\boldsymbol{B}(\tau)d\tau \quad \text{(Eq 5.6)}$$

2. Homogeneous SLFDE with Constant Coefficients

Given homogeneous SLFDE in the following form

$$\begin{cases} \dfrac{dx}{dt} = a_{11}x + a_{12}y \\ \dfrac{dy}{dt} = a_{21}x + a_{22}y \end{cases} \leftrightarrow \boldsymbol{x}'(t) = A\boldsymbol{x}(t) \quad \text{(Eq 5.7)}$$

where

$$\boldsymbol{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix}; \quad \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The solution of SLFDE is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{C}, \qquad \boldsymbol{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad \text{(Eq 5.8)}$$

[3] SLFDE: System of linear first order differential equations.

If the given system has initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, the solution becomes

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(t_0)\boldsymbol{x}_0 \qquad \text{(Eq 5.9)}$$

2. 1. Eigenvalue

The CE of SLFDE is given by

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = 0 \qquad \text{(Eq 5.10)}$$

Then the characteristic polynomial $p(\lambda)$ of $\boldsymbol{A}$ is

$$p(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \cdots = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

Solve for $p(\lambda) = 0$ to obtain the eigenvalue which leads to result of fundamental matrix.

2. 2. Fundamental Matrix and Solution

The fundamental matrix and solution are consequence of value of eigenvalues.

1. Two distinct real roots λ_1, λ_2:

Let $\boldsymbol{v_1} = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$, and $\boldsymbol{v_2} = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$, which are associated eigenvectors satisfy the following equation

$$(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{v_1} = 0; (\boldsymbol{A} - \lambda_2\boldsymbol{I})\boldsymbol{v_2} = 0$$

Then the fundamental matrix and the solution is given by

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{bmatrix}$$

And

$$\boldsymbol{x}(t) = C_1\boldsymbol{v_1}e^{\lambda_1 t} + C_2\boldsymbol{v_2}e^{\lambda_2 t}$$

2. Double root λ_0:

- If $\boldsymbol{A} - \lambda_0\boldsymbol{I} = \boldsymbol{0}$ Let $\boldsymbol{v_1} = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$, which are associated linear independent eigenvectors.

Then the fundamental matrix and the solution is given by

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} e^{\lambda_0 t}$$

And

$$\boldsymbol{x}(t) = (C_1\boldsymbol{v_1} + C_2\boldsymbol{v_2})e^{\lambda_0 t}$$

- If $\boldsymbol{A} - \lambda_0\boldsymbol{I} \neq \boldsymbol{0}$ Let $\boldsymbol{v_1} = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ is the only associated eigenvector and $\boldsymbol{v_2} = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$ are solution of

$$(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{v_1} = 0; (\boldsymbol{A} - \lambda_2\boldsymbol{I})\boldsymbol{v_2} = \boldsymbol{v_1}$$

Then the fundamental matrix and the solution is given by

$$\mathbf{\Phi}(t) = \begin{bmatrix} v_{11} & (v_{11}t + v_{12}) \\ v_{21} & (v_{21}t + v_{22}) \end{bmatrix} e^{\lambda_0 t}$$

And

$$\boldsymbol{x}(t) = \left(C_1\boldsymbol{v_1} + C_2(\boldsymbol{v_1}t + \boldsymbol{v_2})\right)e^{\lambda_0 t}$$

3. Two complex conjugate roots $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$:

Let $\boldsymbol{v}_{[2\times1]}$ is the only associated eigenvector which satisfy the following equation

$$(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{v} = 0$$

From $\boldsymbol{v} = \begin{bmatrix} v_{11} + iv_{12} \\ v_{21} + iv_{22} \end{bmatrix}$, we obtain $\boldsymbol{v_1} = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$, and $\boldsymbol{v_2} = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$

Then the fundamental matrix and the solution is given by

$$\mathbf{\Phi}(t) = \begin{bmatrix} v_{11}\cos(\beta t) - v_{12}\sin(\beta t) & v_{12}\cos(\beta t) + v_{11}\sin(\beta t) \\ v_{21}\cos(\beta t) - v_{22}\sin(\beta t) & v_{22}\cos(\beta t) + v_{21}\sin(\beta t) \end{bmatrix} e^{\alpha t}$$

And

$$\boldsymbol{x}(t) = [C_1(\boldsymbol{v_1}\cos(\beta t) - \boldsymbol{v_2}\sin(\beta t)) + C_2(\boldsymbol{v_2}\cos(\beta t) + \boldsymbol{v_1}\sin(\beta t))]e^{\alpha t}$$

2. 3. Particular Solution for Initial Value Problem (IVP)

If the given system has initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, the solution becomes

$$\boldsymbol{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\boldsymbol{x}_0 \qquad \text{(Eq 5.11)}$$

Where $\mathbf{\Phi}(t)$ already mentioned in the previous section.

3. Substitution Method to Find Solution of SLFDE

Given a linear first order differential equation in the standard form

$$\begin{cases} \dfrac{dx}{dt} = ax + by \ (1) \\ \dfrac{dy}{dt} = cx + dy \ (2) \end{cases} \qquad \text{(Eq 5.12)}$$

To solve this system of equation, we follow the below steps:

1. Differentiating both sides with respect to t of equation (1), we get

$$x'' = ax' + by' \ \ (3)$$

2. Taking $b \times (2) - d \times (1)$, we obtain

$$by' - dx' = (bc - ad)x \leftrightarrow by' = dx' + \ (bc - ad)x \ (4)$$

3. Substituting (4) into (3), it leads to

$$x'' = ax' + dx' + \ (bc - ad)x$$

$$\leftrightarrow x'' - (a + d)x' + (ad - bc)x = 0$$

Characteristic equation

$$r^2 - (a + d)r + (ad - bc) = 0 \to r_1, r_2$$

Based on **Chapter 3 section 2** to derive the expression of $x(t)$, then differentiate to obtain $x'(t)$

3. From (1)

$$y(t) = \frac{1}{b}\left(x'(t) - ax(t)\right)$$

4. Thus, the solution of the given system of differential equations is

$$\begin{cases} x(t) = \cdots \\ y(t) = \cdots \end{cases}$$

If the given system is not in standard form, but it is in the following form

$$\begin{cases} \dfrac{dx}{dt} = ax + by + g(x) \\ \dfrac{dy}{dt} = cx + dy + h(x) \end{cases}$$

We also apply the same method as the standard form to solve this system.