Cal 3 2017/08

Q1.

a)

$$\mathcal{L}\{\delta(t-1)+u(t-4)\}=e^{-s}+\frac{1}{s}e^{-4s}$$

b)

Given that:

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}-4y=u(t-1)+u(t-2)\ \ (*),\qquad y(0)=0,\quad y'(0)=1$$

Let $Y(s)=\mathcal{L}\{y(t)\}$, it holds that:

$$\mathcal{L}\{y'(t)\}=sY(s)-y(0)=sY(s)$$
$$\mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-1$$

Taking Laplace transform both sides of $(*)$, we obtain:

$$[s^2Y(s)-1]-3[sY(s)]-4[Y(s)]=\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}$$
$$\leftrightarrow Y(s)(s^2-3s-4)=1+\frac{e^{-s}+e^{-2s}}{s}$$
$$\leftrightarrow Y(s)=\frac{1}{s^2-3s-4}+\frac{e^{-s}+e^{-2s}}{s(s^2-3s-4)}$$
$$\leftrightarrow Y(s)=\frac{1}{5}\left(\frac{1}{s-4}-\frac{1}{s+1}\right)+\frac{1}{20}\left(\frac{4}{s+1}+\frac{1}{s-4}-\frac{5}{s}\right)(e^{-s}+e^{-2s})$$

$$\to y(t)=\mathcal{L}^{-1}\{Y(s)\}$$
$$=\frac{1}{5}(e^{4t}-e^{-t})u(t)+\frac{1}{20}\left(4e^{-(t-1)}+e^{4(t-1)}-5\right)u(t-1)+\frac{1}{20}\left(4e^{-(t-2)}+e^{4(t-2)}-5\right)u(t-2)$$

Thus, the solution of the given differential equation is:

$$y(t)=\frac{1}{5}(e^{4t}-e^{-t})u(t)+\frac{1}{20}\left(4e^{-(t-1)}+e^{4(t-1)}-5\right)u(t-1)$$
$$+\frac{1}{20}\left(4e^{-(t-2)}+e^{4(t-2)}-5\right)u(t-2)$$

Q2.

Given that:

$$10y_{n+2}-11y_{n+1}+3y_n=10\ \ (*),\qquad y_0=0,\quad y_1=0$$

Let $Y(z)=\mathcal{Z}\{y_n\}$, it holds that:

$$\mathcal{Z}\{y_{n+1}\}=zY(z)-zy_0=zY(z)$$
$$\mathcal{Z}\{y_{n+2}\}=z^2Y(z)-z^2y_0-zy_1=z^2Y(z)$$

Taking $\mathcal{Z}$-transform both side of $(*)$, we obtain:

$$10z^2Y(z)-11zY(z)+10Y(z)=\frac{10z}{z-1}$$
$$\leftrightarrow Y(z)(10z^2-11z+10)=\frac{10z}{z-1}$$
$$\to\frac{Y(z)}{z}=\frac{10}{(10z^2-11z+10)(z-1)}$$
$$\leftrightarrow\frac{Y(z)}{z}=\frac{5}{z-1}+\frac{20}{z-1/2}-\frac{25}{z-3/5}$$
$$\to Y(z)=\frac{5z}{z-1}+\frac{20z}{z-1/2}-\frac{25z}{z-3/5}$$

$$\to y_n=\mathcal{Z}^{-1}\{Y(z)\}=5+20\left(\frac{1}{2}\right)^n-25\left(\frac{3}{5}\right)^n$$

Thus, the solution of the given system difference equations is:

$$y_n = 5 + 20\left(\frac{1}{2}\right)^n - 25\left(\frac{3}{5}\right)^n$$

Q3.

a)

Let: $f(t) = \cos t \to F(s) = \mathcal{L}\{f(t)\} = \dfrac{s}{s^2+1}$

Let: $g(t) = (f * f)(t)$, it leads to:

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{(f*f)(t)\} = F(s).F(s)$$

$$\to G(s) = \frac{s^2}{(s^2+1)^2} = \frac{1}{2}\frac{s^2+1+s^2-1}{(s^2+1)^2} = \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{s^2-1}{(s^2+1)^2}\right)$$

$$\to g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{2}\sin t + \frac{1}{2}t\cos t$$

Thus,

$$\cos t * \cos t = \frac{1}{2}\sin t + \frac{1}{2}t\cos t$$

b)

Given that: $f(x) = 1, \quad 0 < x < 5, \quad L = 5$

The half range sine series is given by:

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{5}\int_0^5 1\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{2}{5}\left(-\frac{5}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{5}\right)\right]\Big|_0^5 = -\frac{2}{n\pi}((-1)^n - 1)$$

$$= \frac{2(1-(-1)^n)}{n\pi}$$

Thus,

$$f(x) = \sum_{n=1}^{+\infty} \frac{2(1-(-1)^n)}{n\pi}\sin\left(\frac{n\pi x}{5}\right)$$

Q4.

Given that: $f(x) = 2|x| - 1, \quad -1 \le x \le 1 \qquad T = 2 \to \omega = \dfrac{2\pi}{T} = \pi$

Since, we have $f(x)$ is an even function on $(-1,1)$ which leads to $b_n = 0$

- $a_0 = \dfrac{4}{T}\displaystyle\int_0^{\frac{T}{2}} f(x)dx = \dfrac{4}{2}\int_0^1 (2|x|-1)dx$

  $= 2\displaystyle\int_0^1 (2x-1)dx = 0$

- $a_n = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(x)\cos(n\omega x)\,dx = \dfrac{4}{2}\int_0^1 (2|x|-1)\cos(n\pi x)\,dx$

$$= 2\int_0^1 (2x-1)\cos(n\pi x)\,dx$$
$$= 2\left[\frac{2x-1}{n\pi}\sin(n\pi x) + \frac{2}{n^2\pi^2}\cos(n\pi x)\right]\Bigg|_0^\pi$$
$$= \frac{4((-1)^n - 1)}{\pi^2 n^2}$$

The Fourier series is given by:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\omega x) + \sum_{n=1}^{+\infty} b_n \sin(n\omega x)$$
$$= \sum_{n=1}^{+\infty} \frac{4((-1)^n - 1)}{\pi^2 n^2}\cos(nx)$$

Since we have: $f(x) = 2|x| - 1, \quad -1 \le x \le 1 \quad \to f(0) = -1$

Therefore,

$$f(0) = \sum_{n=1}^{+\infty} \frac{4((-1)^n - 1)}{\pi^2 n^2} = -1$$
$$\to \sum_{n=1}^{+\infty} \frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$
$$\leftrightarrow \sum_{n=1}^{+\infty} \frac{1-(-1)^n}{2n^2} = \frac{\pi^2}{8}$$
$$\to \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^3} + \cdots = \frac{\pi^2}{8}$$