MTH 252 - Fall 2023

REVIEW PROBLEMS - ANSWERS - EXAM II

The following problems should help you prepare for our Exam 2. Some of the problems here are more difficult than the problems that you are going to find in the exam, we will discuss these problems in class on Wednesday, November 1, 2023. Our Exam 2 is schedule for November 3, 2023.

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(1) Find and sketch the domain of the each of the following functions:

(a) $f(x,y) = \sqrt{4-x^2-y^2} + \sqrt{1-x^2}$: note that the function f is defined only when $4-x^2-y^2 \geq 0 \iff x^2+y^2 \leq 4$, and $1-x^2 \geq 0 \iff x^2 \leq 1$, so the domain of f is $D_f = \{(x,y)\,|\,x^2+y^2 \leq 4, -1 \leq x \leq 1\}$.

(b) $f(x,y) = \ln(x+y+1)$: f is defined only when $x+y+1 > 0 \iff y > -x-1$, so the domain of f is $D_f = \{(x,y)\,|\,y > -x-1\}$, these are all points above the line $y = -x-1$.

(2) Evaluate the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \cos(\ln(1+x^2+y^2)) = \cos(\ln(1)) = 1.$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = 2.$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ **DNE**.

- along the x-axis with $x \neq 0$: $f(x,0) = 0$.
- along the line $y = x$ with $x \neq 0$: $f(x,x) = 1$.

(d) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2} = 2/3$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$ **DNE**.

- along the x-axis with $x \neq 0$: $f(x,0) = 0$.
- along the line $y = x$ with $x \neq 0$: $f(x,x) = 2/3$.

(3) Find the first partial derivatives of each of the following functions:

(a) $f(x, y) = (5y^3 + 2x^2y)^8$

(b) $s(x, y) = \dfrac{x + 2y}{x^2 + y^2}$

(c) $g(x, y) = x^2 \ln(x^2 + y^2)$

(d) $h(x, y, z) = e^{x/y} \sin(y/z)$

(4) Find the second partial derivatives of each of the following functions:

(a) $f(x, y) = 4x^3 - xy^2$

(b) $g(x, y) = xe^{-2y}$

(c) $h(x, y, z) = x \cos(y + 2z)$

(5) Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicit as a function of x and y by the equation

$$x^2 + y^2 + z^2 = 5xyz$$

Let $F(x, yz) = x^2 + y^2 + z^2 - 5xyz = 0$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$. Therefore

$$\frac{\partial z}{\partial x} = \frac{5yz - 2x}{2z - 5xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{5xz - 2y}{2z - 5xy}$$

(6) If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Here we have to show that $z = f(x, y)$ is a solution to the given differential equation. If $z = f(x, y) \Longrightarrow \dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos(\theta) + \dfrac{\partial z}{\partial y}\sin(\theta)$, and $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin(\theta)) + \dfrac{\partial z}{\partial y}(r\cos(\theta))$. Now show that $\left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2 = (\sin^2\theta + \cos^2\theta)\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right] = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$.

(7) Find equations for (a) tangent plane and (b) the normal line to the given surface at the given point:

(a) $z = 3x^2 - y^2 + 2x$, $(1, -2, 1)$

(a) $z_x = 6x + 2 \Rightarrow z_x(1, -2) = 8$ and $z_y = -2y \Rightarrow z_y(1, -2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x - 1) + 4(y + 2)$.

(b) A normal vector to the tangent plane (and the surface) at $(1, -2, 1)$ is $\langle 8, 4, -1\rangle$. Then the parametric equations of the normal line are: $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$, for al $t \in \mathbb{R}$.

(b) $z = e^x \cos(y)$, $(0, 0, 1)$

(a) $z_x = e^x \cos(y) \Rightarrow z_x(0, 0) = 1$ and $z_y = -e^x \sin(y) \Rightarrow z_y(0, 0) = 0$, so an equation of the tangent plane is $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1\rangle$. Then the parametric equations of the normal line are: $x = t$, $y = 0$, $z = 1 - t$, for al $t \in \mathbb{R}$.

(c) $xy + yz + zx = 3$, $(1, 1, 1)$

(a) Let $F(x, y, z) = xy + yz + zx$, $F_x = y + z \Rightarrow F_x(1, 1, 1) = 2$, $F_y = x + z \Rightarrow F_y(1, 1, 1) = 2$, $F_z = x + y \Rightarrow F_z(1, 1, 1) = 2$ and so an equation of the tangent plane is $x + y + z = 3$.

(b) A normal vector to the tangent plane (and the surface) at $(1, 1, 1)$ is $\langle 2, 2, 2\rangle$. Then the symmetric equations for the normal line are: $\dfrac{x-1}{2} = \dfrac{y-1}{2} = \dfrac{z-1}{2}$.

(8) (Solved in class) Find the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$, where the tangent plane is parallel to the plane $2x + 2y + z = 5$.

(9) Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point $(2, 3, 4)$ and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

$f_x(x, y, z) = 3x^2\sqrt{y^2 + z^2}$, $f_y(x, y, z) = \dfrac{yx^3}{\sqrt{y^2 + z^2}}$, and $f_z(x, y, z) = \dfrac{zx^3}{\sqrt{y^2 + z^2}}$, so $f(2, 3, 4) = 40$, $f_x(2, 3, 4) = 60$, $f_y(2, 3, 4) = \frac{24}{5}$, $f_z(2, 3, 4) = \frac{32}{5}$. The the linear approximation of function f at $(2, 3, 4)$ is $f(x, y, z) \approx f(2, 3, 4) + f_x(2, 3, 4)(x - 2) + f_y(2, 3, 4)(y - 3) + f_z(2, 3, 4)(z - 4) = 40 + 60(x - 2) + \frac{24}{5}(y - 3) + \frac{32}{5}(z - 4)$.

(10) If $\cos(xyz) = 1 + x^2y^2 + z^2$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Let $F(x, y, z) = 1+x^2y^2+z^2-\cos(xyz) = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2xy^2 + yz\sin(xyz)}{2z + xy\sin(xyz)}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2x^2y + xz\sin(xyz)}{2z + xy\sin(xyz)}$.

(11) Find an equation of the tangent plane to the surface $2ze^{xz} - y^2 = 2$ at the point $P(0, -2, 3)$.

Note that this is equivalent to find an equation of the tangent plane $\mathcal{T}$ to the level surface $F(x, y, z) = 2ze^{xz} - y^2 = 2$ at the point $P(0, -2, 3)$. Here $\nabla F(0, -2, 3) = \langle 18, 4, 2\rangle$, and an equation for $\mathcal{T}$ is: $18x + 4(y - 2) + 2(z - 3) = .0$

(12) Find all points (x, y) where the graph of $f(x, y) = e^y(x^2+xy)$ has a horizontal tangent plane.

The tangent plane is horizontal if and only if $f_x = e^y(2x + y) = 0$ and $f_y = e^y(x + x^2 + xy) = 0 \iff 2x + y = 0$ and $x + x^2 + xy = 0$, and the solution s to this equations are points are: $(0, 0)$ and $(1, -2)$.

(13) Find the parametric equations for the normal line to the level surface $g(x, y, z) = x^2 + 2y^2 - 3z^2 = 5$ at the point $(2, -1, 1)$.

The normal line $\mathcal{L}$ has direction vector $\mathbf{v} = \nabla g(2, -1, 1) = \langle 4, -4, -6\rangle$. The parametric equations of $\mathcal{L}$ are: $x = 2 + 4t$, $y = -1 - 4t$ and $z = 1 - 6t$, $t \in \mathbb{R}$.

(14) Show that every plane that is tangent to the cone $z^2 = x^2 + y^2$ passes through the origin.

Consider the level surface $F(x, y, z) = x^2 + y^2 - z^2 = 0$. Then $\nabla F = \langle 2x, 2y, -2z\rangle$. Given any point $P(a, b, c)$ in the level surface $F(x, y, z) = 0$. It follows that the equation of the tangent plane to $F(x, y, z) = 0$ is: $2ax + 2by - 2cz = 0$. Note that all these planes pass through the origin.

(15) Find the gradient of the function $f(x, y, z) = x^2e^{yz^2}$.

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2}\rangle.$$

(16) Let $f(x, y) = 2xy - y^2 - 3x + 5$.

(a) Find the directional derivative of f at the point $(1, 3)$ in the direction of the vector $\mathbf{v} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$.
Let $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\langle 2, 1\rangle$ be a unit vector in the direction of $\mathbf{v}$. Then $D_{\hat{\mathbf{v}}}f(1, 3) = \nabla f(1, 3) \cdot \hat{v} = \frac{1}{\sqrt{5}}\langle 3, -4\rangle \cdot \langle 2, 1\rangle = \dfrac{2}{\sqrt{5}}$.

(b) Find the **unit vector** in the direction of maximum increase of f at the point $(1, 3)$.
Let $\hat{\mathbf{u}} = \dfrac{\nabla f(1, 3)}{\|\nabla f(1, 3)\|} = \frac{1}{5}\langle 3, -4\rangle$

(c) What is the value of the directional derivative at the point $(1, 3)$ in the direction of maximum increase of f $\|\nabla f(1, 3)\| = 5$

(d) Find a **unit vector u** giving $D_{\mathbf{u}}f(1, 3) = 0$. Let $\mathbf{u} = \langle a, b\rangle$. Now $D_{\mathbf{u}}f(1, 3) = \nabla f(1, 3) \cdot \mathbf{u} = 0 \iff \langle 3, -4\rangle \cdot \langle a, b\rangle = 0 \iff 3a - 4b = 0$. Take $\mathbf{u} = \frac{1}{5}\langle 4, 3\rangle$.

(17) Find the directional derivative of $f(x, y) = x^2 e^{-y}$ at the point $(2, -3)$ in the direction of the vector $\mathbf{u} = \langle 2, -3\rangle$.

$\nabla f = \langle 2xe^{-y}, -x^2 e^{-y}\rangle$, $\nabla f(2, -3) = \langle -4e^3, -4e^3\rangle$. The unit vector in the direction of $\langle 2, -3\rangle$ is $\mathbf{u} = \frac{1}{\sqrt{13}}\langle 2, -3\rangle$. The $D_{\mathbf{u}}f(2, -3) = \frac{1}{\sqrt{13}}\langle -4e^3, -4e^3\rangle \cdot \langle 2, -3\rangle$.

(18) Find the maximum rate of change of $f(x, y) = x^2 y + \sqrt{y}$ at the point $(2, 1)$. In what direction does it occur?

$\nabla f = \langle 2xy, x^2 + 1/(2\sqrt{y})\rangle$, $||\nabla f(2, 1)|| = ||\langle 4, \frac{9}{2}\rangle||$. Thus the maximum rate of change of f at $(2, 1)$ is $\frac{\sqrt{145}}{2}$.

(19) Find the local maximum and minimum values and the saddle points of each of the following functions:

(a) $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10 \Rightarrow f_x = 2x - y + 9, f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}, f_{xy} = -1$. Then$f_x = 0$ and $f_y = 0$ imply $y = 1, x = -4$. Thus the only critical point is $(-4, 1)$ and $f_{xx}(-4, 1) > 0, D(-4, 1) = 3 > 0$, so $f(-4, 1) = -11$ is a local minimum.

(b) $f(x, y) = (x^2 + y)e^{y/2} \Rightarrow f_x = 2xe^{y/2}, f_y = e^{y/2}(2 + x^2 + y)/2, f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2}(4+x^2+y)/4, f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies

$y = -2$. But $f_{xx}(0,-2) > 0, D(0,-2) = e^{-2} - 0 > 0$, so $f(0,-2) = -2/e$ is a local minimum.

(20) Find the absolute maximum and minimum values of f on the set D.

(a) $f(x,y) = 4xy^2 - x^2y^2 - xy^3$; D is the closed triangular region in the xy-plane with vertices $(0,0)$, $(0,6)$, and $(6,0)$.

First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3 = y^2(4-2x-y)$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4-2x$. Since $y = 0$ isn't inside D. Substituting $y = 4-2x$ iinto $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$. But $x = 0$ is not inside D, and when $x = 2$, $y = 0$ but $(2,0)$ is not inside D. Thus the only critical point inside D is $(1,2)$ and $f(1,2) = 4$.

Next, we consider the boundary of D: Let L_1 be the segment joining the points$(0,0)$ and $(6,0)$, let L_2 be the segment joining the points $(6,0)$ and $(0,6)$, and let L_3 be the segment joining the points$(0,0)$ and $(0,6)$. On L_1: $f(x,0) = 0$ so $f = 0$ on L_1. On $L_2 : x = -y+6$ and $f(-y+6,y) = y^2(6-y)(-2) = -2(6y^2 - y^3)$ which has a critical point at $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3: $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

(b) $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$; D is the disk $x^2 + y^2 \leq 4$.

Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, then $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ given the critical points $(0,0), (0,\pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ given the critical points $(\pm 1, 0)$. Now $f(0,0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0,\pm 1) = 2e^{-1}$. Next, we consider the boundary of D: $x^2 + y^2 = 4$, so $f(x,y) = e^{-4}(4+y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. Since $f(\pm 2, 0) = 4e^{-4}$, $f(0,\pm 2) = 8e^{-4}$. Then on D the absolute maximum of f is $f(0,\pm 1) = 2e^{-1}$ and the absolute minimum is $f(0,0) = 0$.

(21) Calculate the value of each of the following multiple integrals:

(a) $$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} dydx = \frac{1}{4}(e-1)$$

(b) $\displaystyle\iint_R ye^{xy} dA$, where $R = \{(x,y)\,|\,0 \leq x \leq 2,\ 0 \leq y \leq 3\}$.

$$\int_0^3 \int_0^2 ye^{xy}\, dxdy = \frac{1}{2}(e^6 - 7).$$

(c) $\displaystyle\iint_R xy dA$, where $R = \{(x,y)\,|\,0 \leq y \leq 1,\ y^2 \leq x \leq y+2\}$.

$$\int_0^1 \int_{y^2}^{y+2} xy\, dxdy = \frac{41}{24}.$$

(d) $\displaystyle\iint_D \frac{y}{1+x^2} dA$, where D is bounded by $y = \sqrt{x}$, $y = 0$ and $x = 1$.

$$\int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dydx = \frac{1}{4}\ln(2).$$

(e) $\displaystyle\iint_D \frac{1}{1+x^2} dA$, where D is the triangular region with vertices $(0,0)$, $(1,1)$, $(0,1)$.

$$\int_0^1 \int_x^1 \frac{1}{1+x^2}\, dydx = \frac{\pi}{4} - \frac{1}{2}\ln(2).$$