Modelling Beam Uncertainties in Global 21cm Experiments through Bayesian Data Analysis

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URSI GASS 2023
August 19 (Sat) – 26 (Sat), 2023 SAPPORO, JAPAN

Beam Chromaticity and Uncertainties

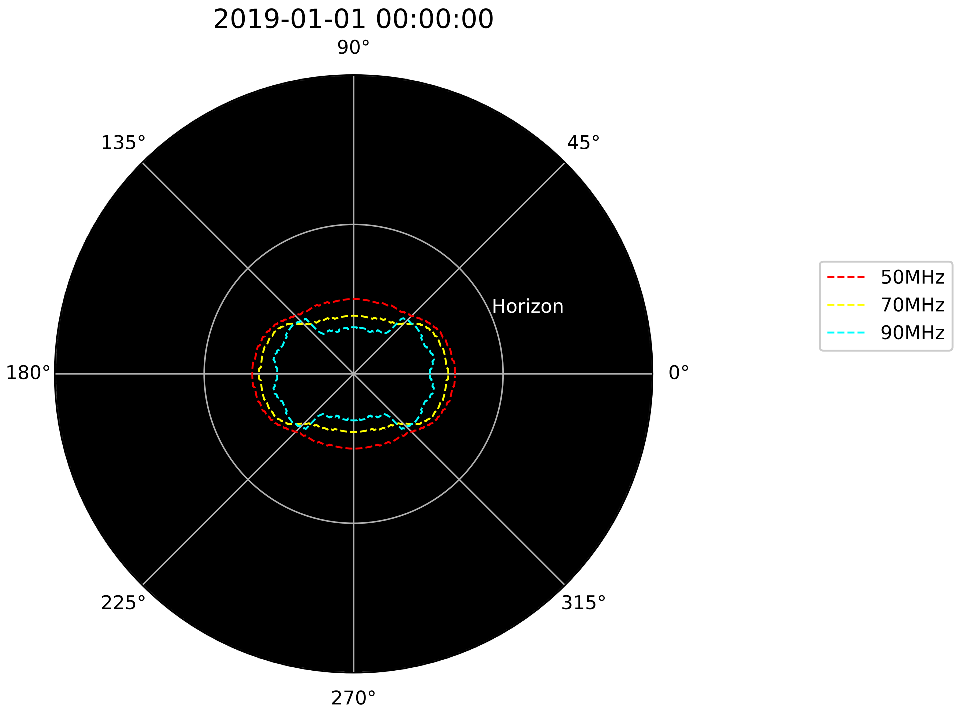
$$T_{\mathrm{A}}\left(\nu,t\right)=\frac{1}{4\pi}\int_{0}^{4\pi}D\left(\Omega,\nu\right)T_{\mathrm{F}}\left(\Omega,\nu,t\right)d\Omega+T_{21}\left(\nu\right)$$

Effects that can introduce beam errors:

- Errors in EM simulation

- Imperfections in antenna construction

- Unknown physical factors not included in the simulation
    - soil permittivity
    - weather effects
    - material on the instrument



Beam Errors and Uncertainties

- Produced 2 test beams in EM simulations, A and B

- B differs from A by realistic construction errors:
    - blade vertices randomly shifted by order ~ >1mm
    - height above ground plane offset by ~few mm
    - size of ground plane altered by ~few cm

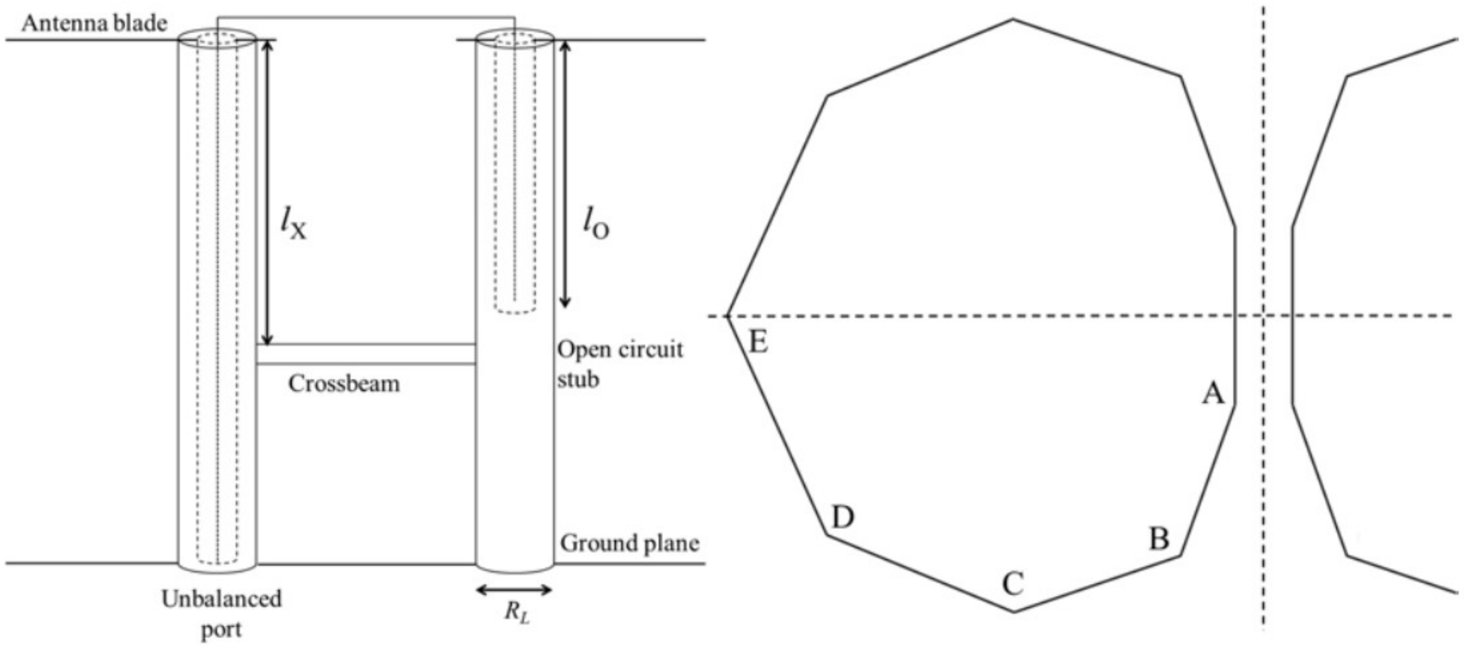


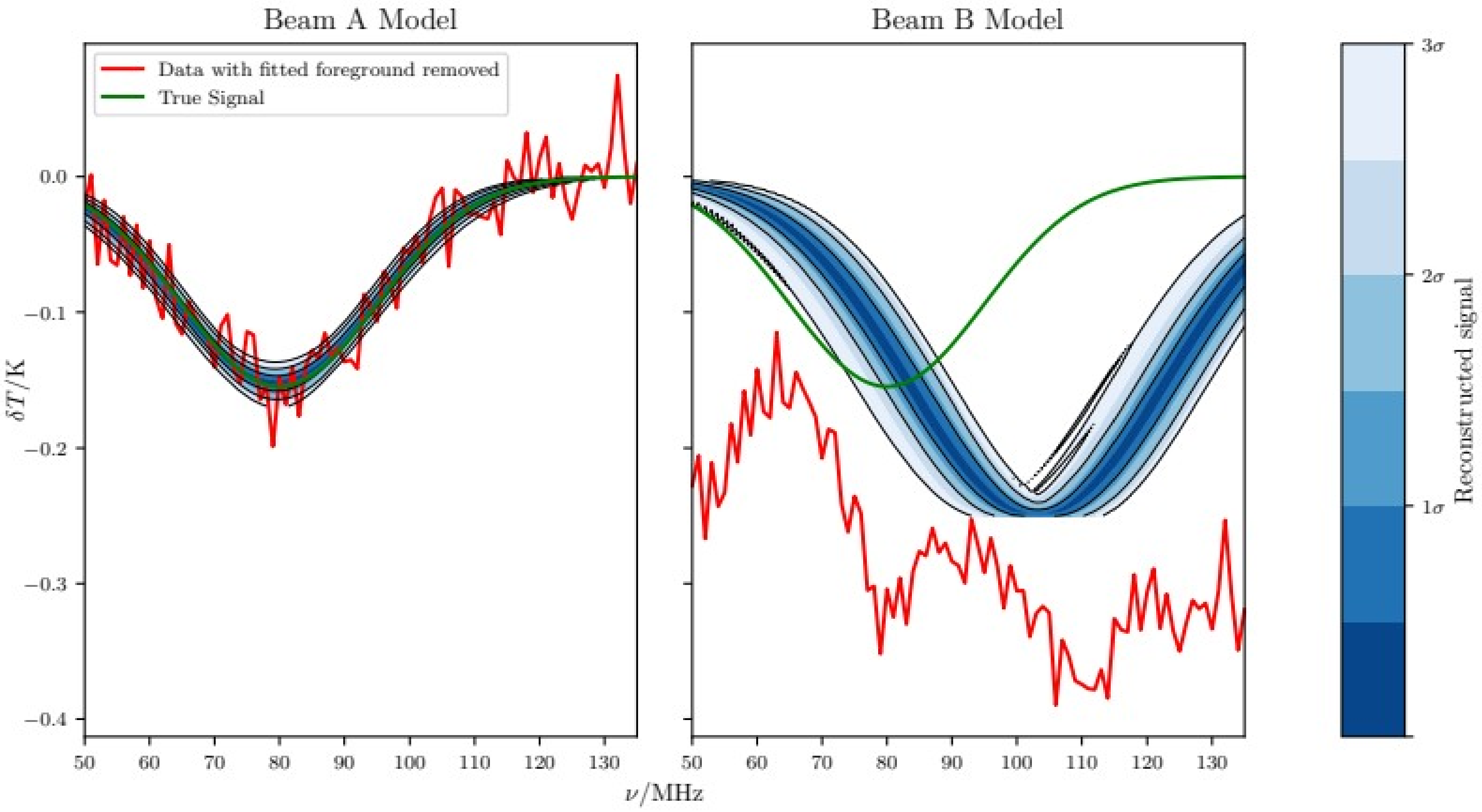
Figure 24 from Cumner et al. 2022

Position/ mm	Ax	Bx	By	Cx	Cy	Dx	Dy	Ex	gap	h	groundplane_size	R_L	l_x	l_o
Beam A	17.956	78.147	565.57	271.74	454.42	724.74	415.64	894.35	11.208	694.9	14196	29.694	485.65	433.16
Beam B	18	79	566	271	454	725	415	895	11	700	14250	30	486	433
Difference	0.044	0.853	0.43	0.74	0.42	0.26	0.64	0.65	0.208	5.1	54	0.306	0.35	0.16

Impact of Errors

Foreground known exactly

Beam directivity variation <=0.5% (~-23dB)



Beam Parametrisation

Beam need to be parametrised and fit for:

$$D\left(\Omega,\nu\right)=\sum_{k}^{N_{\mathrm{basis}}}\Gamma_{k}\left(\nu,\theta\right)Y_{k}\left(\Omega\right)$$

Parametrised coefficient functions

Basis functions

$$T_{\mathrm{A}}\left(\nu,t\right)=\frac{1}{4\pi}\int_{0}^{4\pi}\sum_{k}^{N_{\mathrm{basis}}}\Gamma_{k}\left(\nu,\theta\right)Y_{k}\left(\Omega\right)T_{\mathrm{F}}\left(\Omega,\nu,t\right)d\Omega+T_{21}\left(\nu\right)$$

Coefficient Numbers

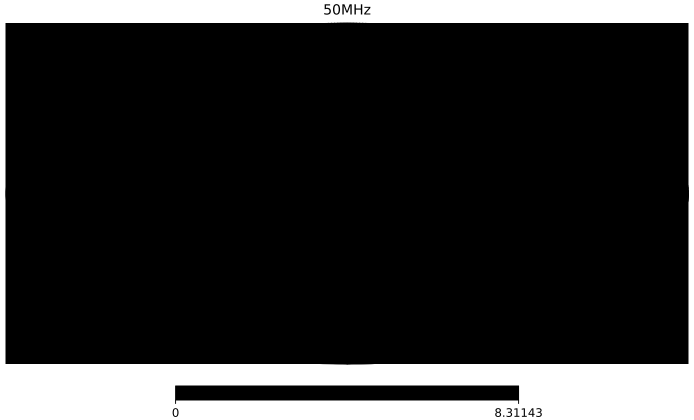
- Spherical Harmonics:
  > 1000s of basis functions

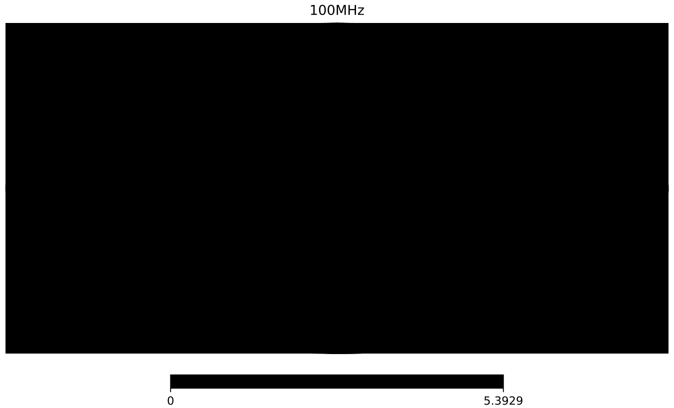
- More sophisticated basis functions
  ~ 10-30

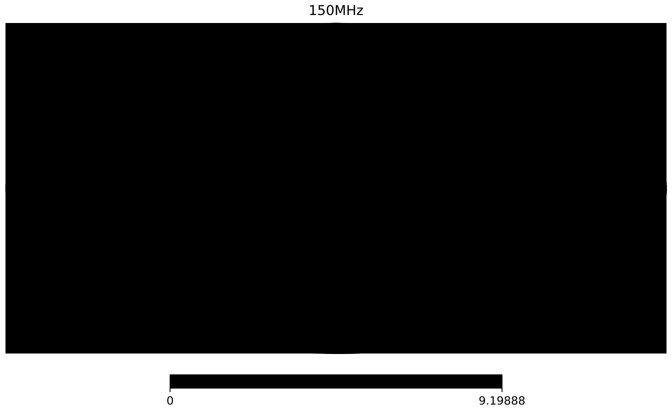
Each basis functions requires a parameterised coefficient function.
Order 5-10 parameters each.

Current best case requires of order ~50-100 parameters.

Two key Issues:
- Very slow to fit directly
- High freedom in the model







Nuisance Parameters

$$\mathcal{P}\left(\theta_{\mathrm{S}}, \theta_{\mathrm{N}}\right) = \frac{\mathcal{L}\left(\theta_{\mathrm{S}}, \theta_{\mathrm{N}}\right) \Pi\left(\theta_{\mathrm{S}}\right) \Pi\left(\theta_{\mathrm{N}}\right)}{\mathcal{Z}}$$

$$\mathcal{P}_{\mathrm{eff}}\left(\theta_{\mathrm{S}}\right) = \int \mathcal{P}\left(\theta_{\mathrm{S}}, \theta_{\mathrm{N}}\right) \Pi\left(\theta_{\mathrm{N}}\right) d\theta_{\mathrm{N}}$$

$$\mathcal{P}_{\mathrm{eff}}\left(\theta_{\mathrm{S}}\right) = \frac{\Pi\left(\theta_{\mathrm{S}}\right) \int \mathcal{L}\left(\theta_{\mathrm{S}}, \theta_{\mathrm{N}}\right) \Pi\left(\theta_{\mathrm{N}}\right) d\theta_{\mathrm{N}}}{\mathcal{Z}} = \frac{\mathcal{L}_{\mathrm{eff}}\left(\theta_{\mathrm{S}}\right) \Pi\left(\theta_{\mathrm{S}}\right)}{\mathcal{Z}}$$

$$\mathcal{L}_{\mathrm{eff}}\left(\theta_{\mathrm{S}}\right) = \int \mathcal{L}\left(\theta_{\mathrm{S}}, \theta_{\mathrm{N}}\right) \Pi\left(\theta_{\mathrm{N}}\right) d\theta_{\mathrm{N}}$$

Analytic Marginalisation of Linear Parameters

Tauscher et al. 2021 - arXiv:2105.01672,
Murray et al. 2022 - arXiv:2209.03459

$$\log \mathcal{L}\left(\theta\right) = -\frac{1}{2}\log\left|2\pi\underline{\underline{C}}\right| - \frac{1}{2}\left(\underline{y} - \underline{M}\left(\theta\right)\right)^{\intercal}\underline{\underline{C}}^{-1}\left(\underline{y} - \underline{M}\left(\theta\right)\right)$$

If the model is linear in some parameters:

$$\underline{M}\left(\theta\right) = \underline{\underline{A}}\left(\theta_{\mathrm{non-linear}}\right)\underline{\theta}_{\mathrm{linear}} = \underline{\underline{A}}\,\underline{\theta}$$

$$\log \mathcal{L} = -\frac{1}{2}\log\left|2\pi\underline{\underline{C}}\right| - \frac{1}{2}\left(\underline{y} - \underline{\underline{A}}\,\underline{\theta}\right)^{\intercal}\underline{\underline{C}}^{-1}\left(\underline{y} - \underline{\underline{A}}\,\underline{\theta}\right)$$

Analytic Marginalisation of Linear Parameters

For uniform priors

$$\log \mathcal{L} = -\frac{1}{2}\log|2\pi\underline{\underline{C}}| - \frac{1}{2}\left(\underline{y} - \underline{\underline{A}}\,\underline{\theta}\right)^{\intercal}\underline{\underline{C}}^{-1}\left(\underline{y} - \underline{\underline{A}}\,\underline{\theta}\right)$$

Make the following substitutions:

$$\underline{\underline{\Sigma}}^{-1} = \underline{\underline{A}}^{\intercal}\,\underline{\underline{C}}^{-1}\underline{\underline{A}} \qquad\qquad \underline{\mu} = \underline{\underline{\Sigma}}\,\underline{\underline{A}}^{\intercal}\underline{\underline{C}}^{-1}\underline{y}$$

$$\log \mathcal{L} = -\frac{1}{2}\log|2\pi\underline{\underline{C}}| - \frac{1}{2}\underline{y}^{\intercal}\underline{\underline{C}}^{-1}\underline{y} + \frac{1}{2}\underline{\mu}^{\intercal}\underline{\underline{\Sigma}}^{-1}\underline{\mu} - \frac{1}{2}\left(\underline{\mu} - \underline{\theta}\right)^{\intercal}\underline{\underline{\Sigma}}^{-1}\left(\underline{\mu} - \underline{\theta}\right)$$

$$\log \mathcal{L}_{\text{eff}} = -\frac{1}{2}\log|2\pi\underline{\underline{C}}| - \frac{1}{2}\underline{y}^{\intercal}\underline{\underline{C}}^{-1}\underline{y} + \frac{1}{2}\underline{\mu}^{\intercal}\underline{\underline{\Sigma}}^{-1}\underline{\mu} + \frac{1}{2}\log|2\pi\underline{\underline{\Sigma}}|$$

Implementation for Beam Fitting

Require the coefficient parameters to be linear
- e.g. polynomials

$$T_{\mathrm{A}}\left(\nu, t\right) = \frac{1}{4\pi}\int_0^{4\pi}\sum_k^{N_{\mathrm{basis}}} \Gamma_k\left(\nu, \theta\right) Y_k\left(\Omega\right) T_{\mathrm{F}}\left(\Omega, \nu, t\right) d\Omega + T_{21}\left(\nu\right)$$

Fitting a beam entirely introduces significant freedom to the model

Require basis functions specific to the antenna used

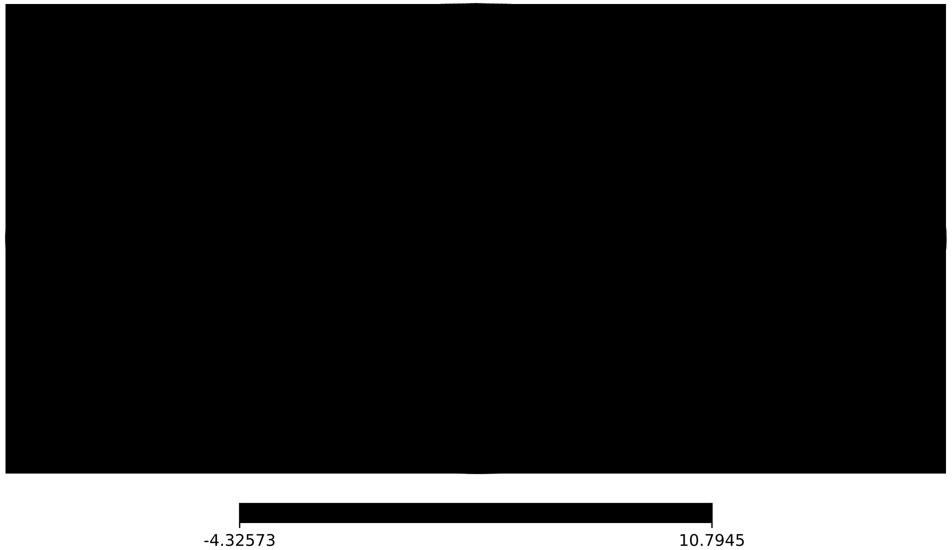
Require very tight priors
- fit variations to the coefficients

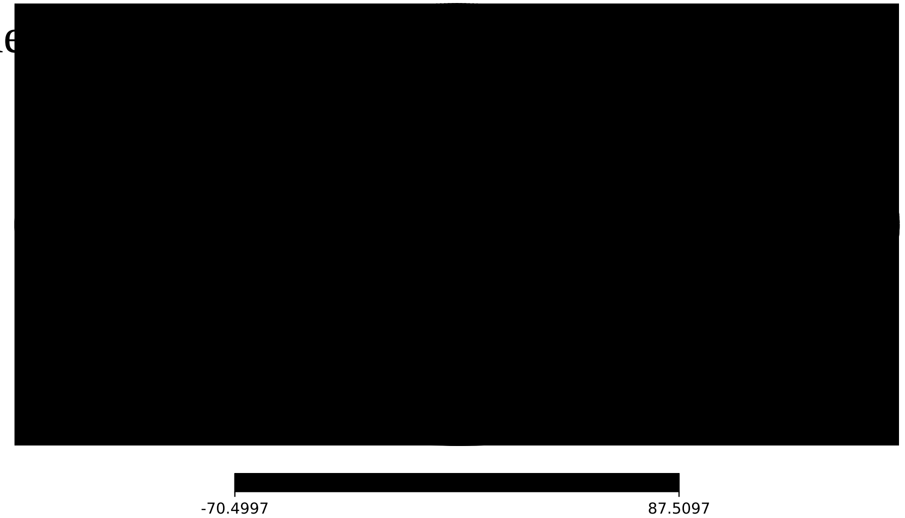
$$\Gamma_k\left(\nu, \theta\right) = C_k\left(\nu\right) + V_k\left(\nu, \theta\right)$$

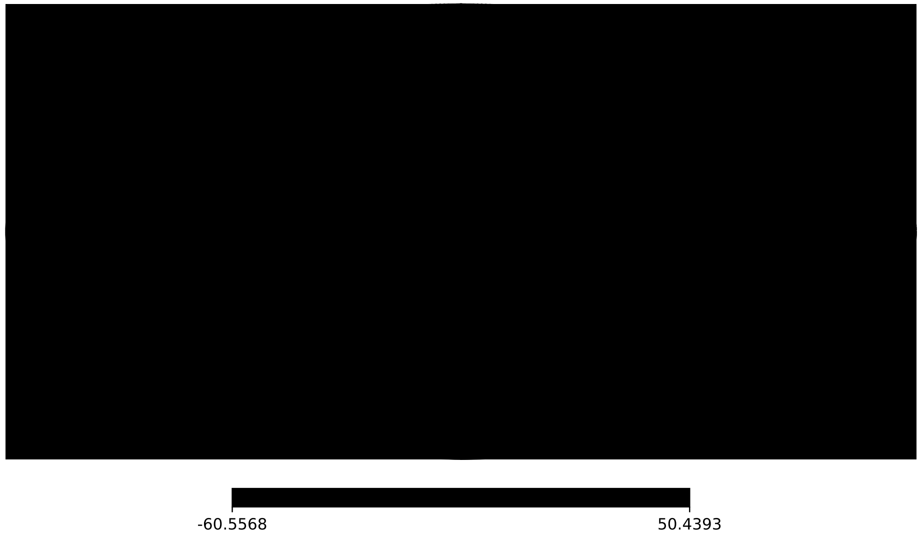
Trial Model

4 basis functions

Derived from SVD of a dipole over a serrated square ground plane

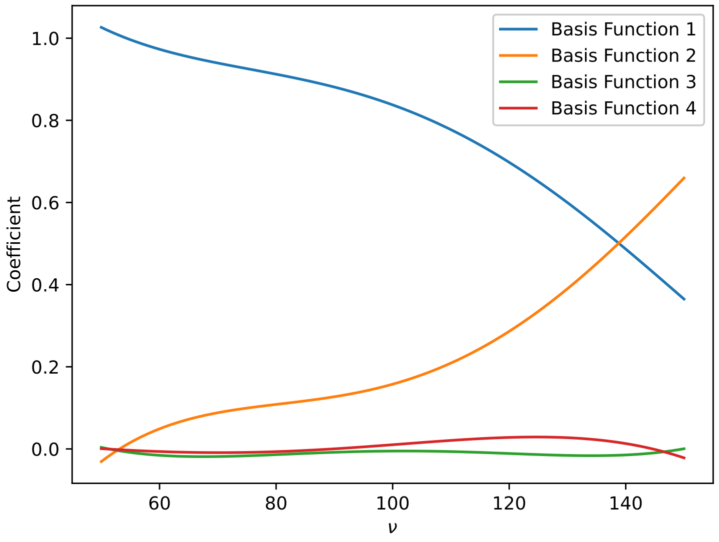




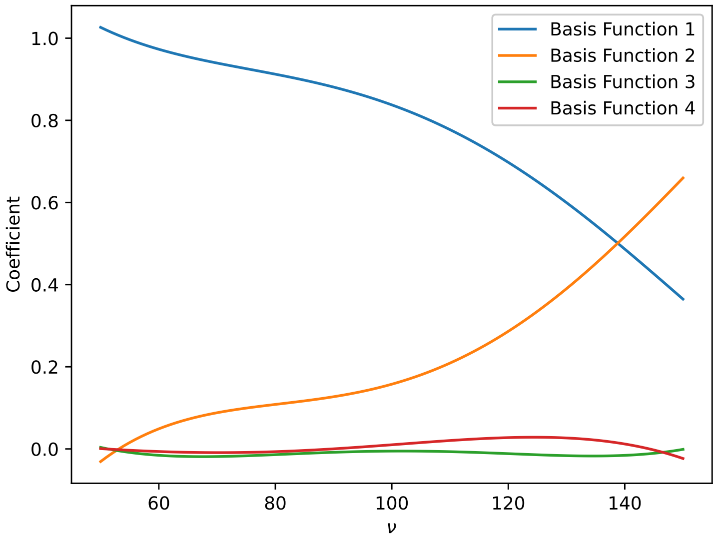


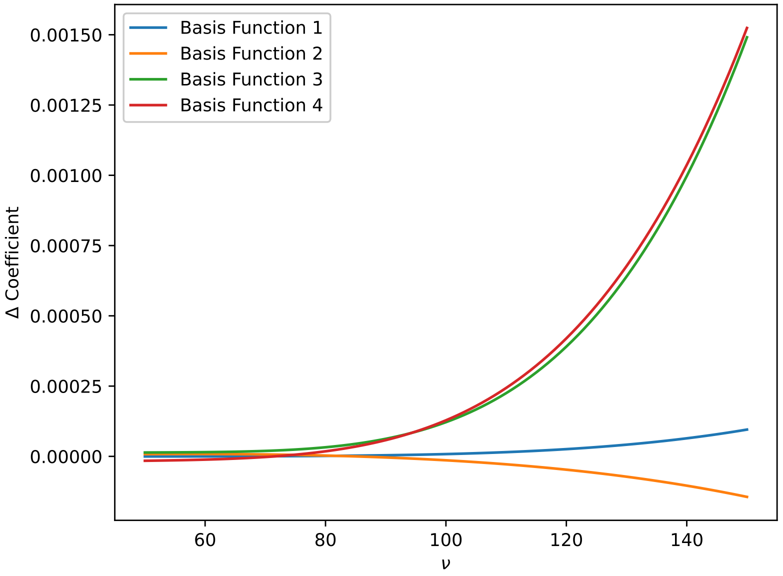
Trial Model

Beam 1



Beam 2

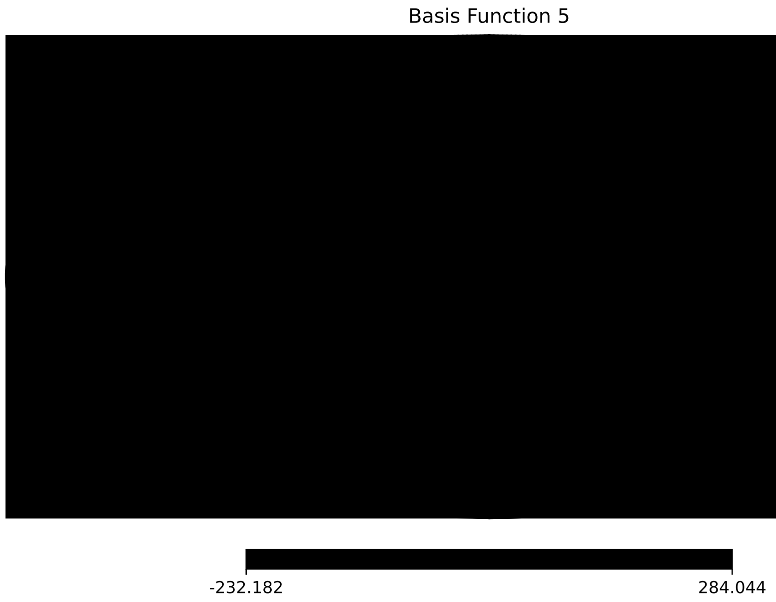


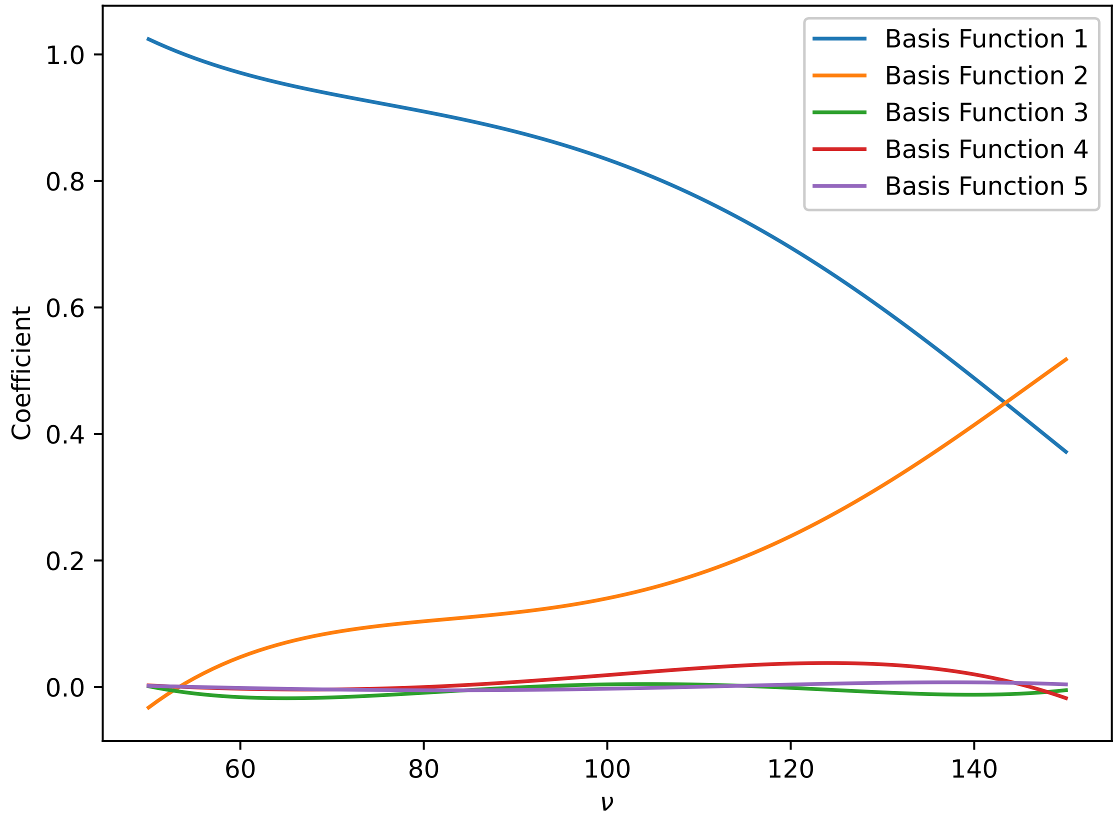


Changes of order 10^{-3}

Trial Model

Beam 3



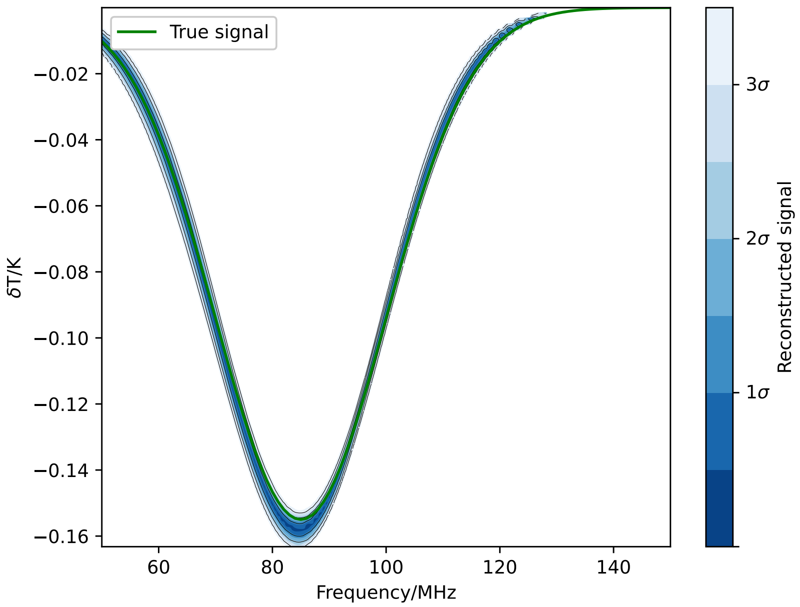


Effect of Beam Errors Without Modelling

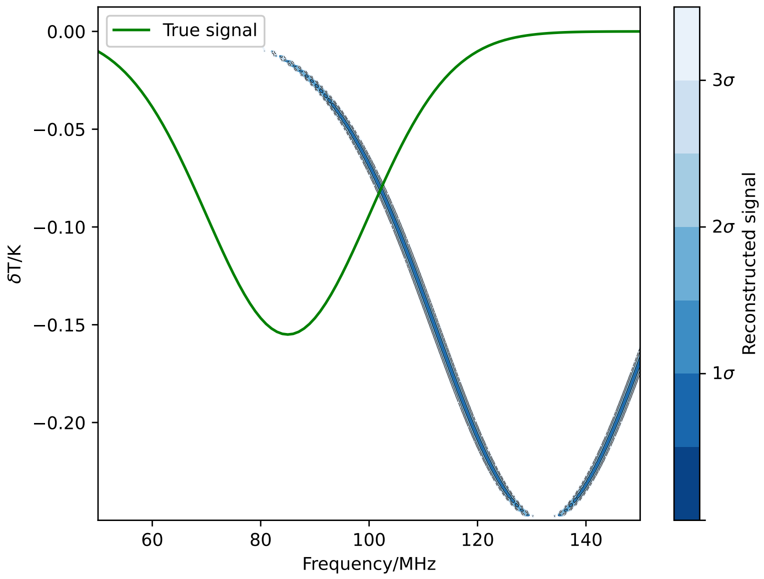
Simulated data generated using specified test beam and a uniform power law foreground

Single spectral index foreground parameter fitted with chromaticity modelling assuming beam 1

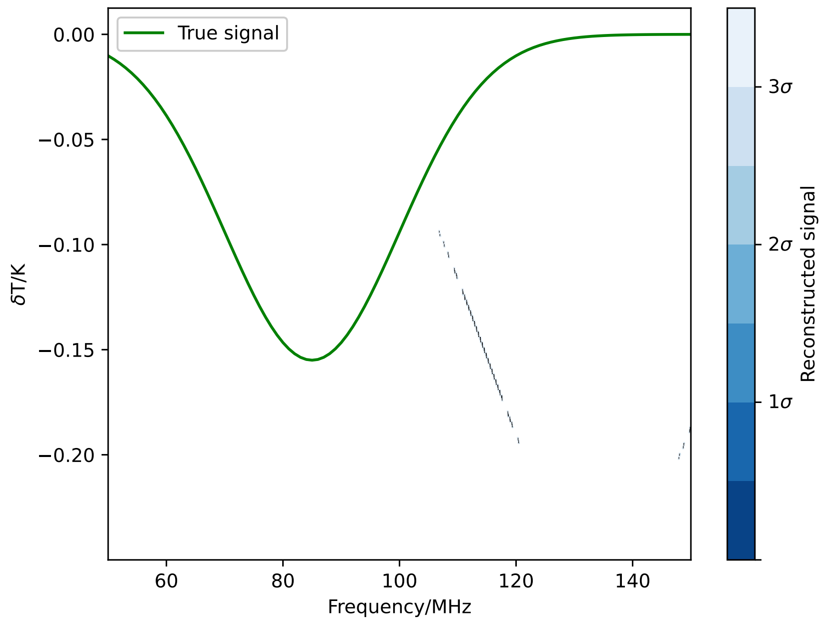
Beam 1



Beam 2



Beam 3

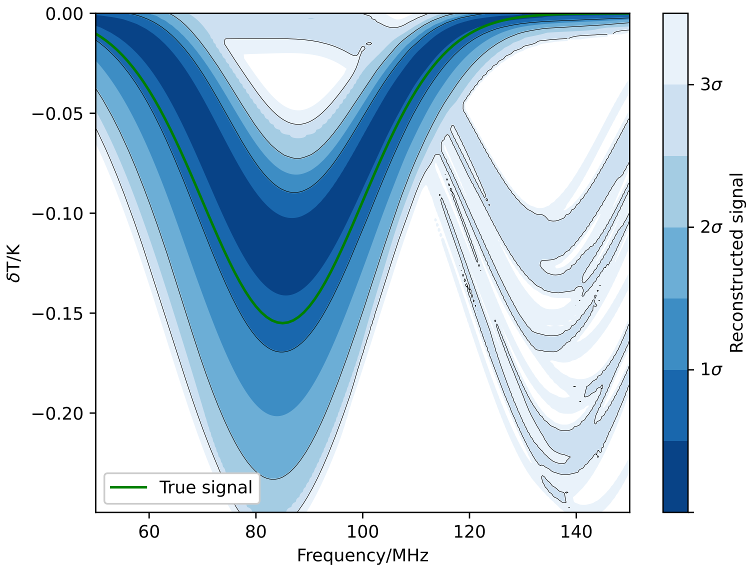


Results with AMLP Beam Modelling

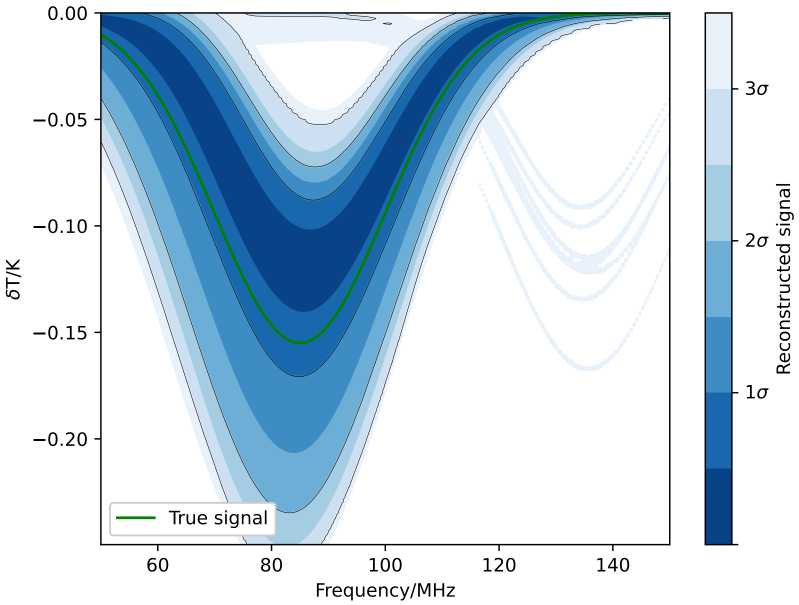
Simulated data generated using specified test beam and a spatially dependant power law foreground

Full foreground with 15 spectral index parameters fitted with AMLP of 5th order polynomial variations on beam 1

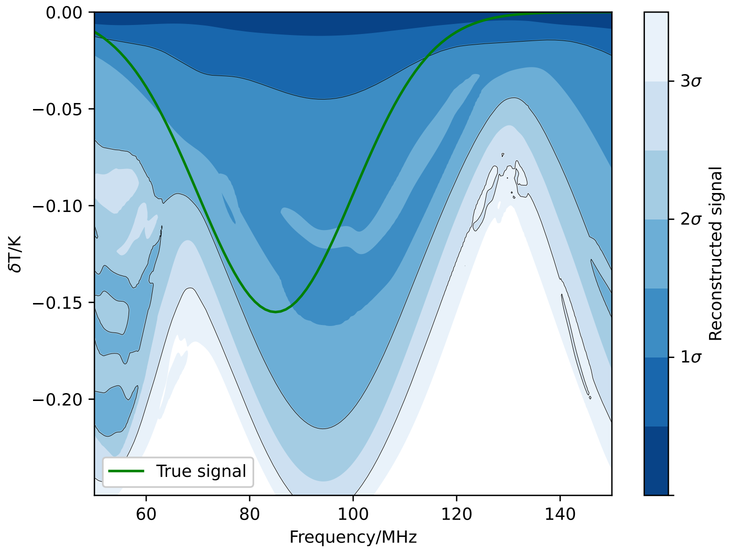
Beam 1



Beam 2



Beam 3



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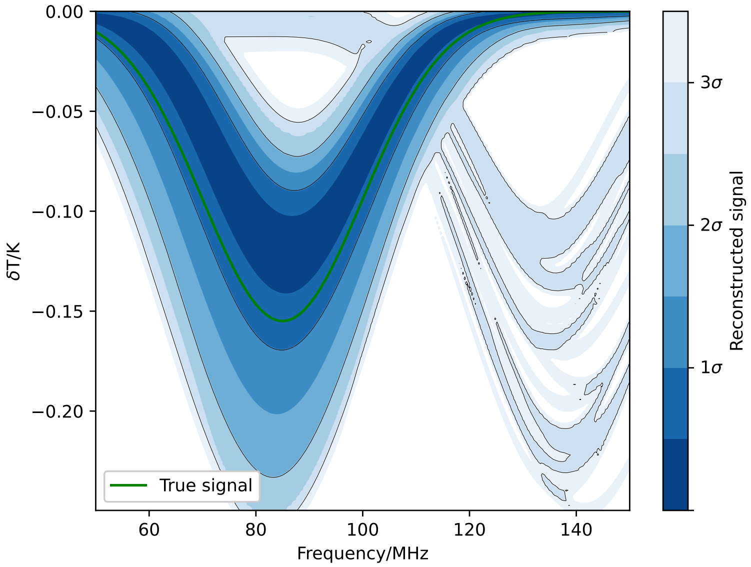
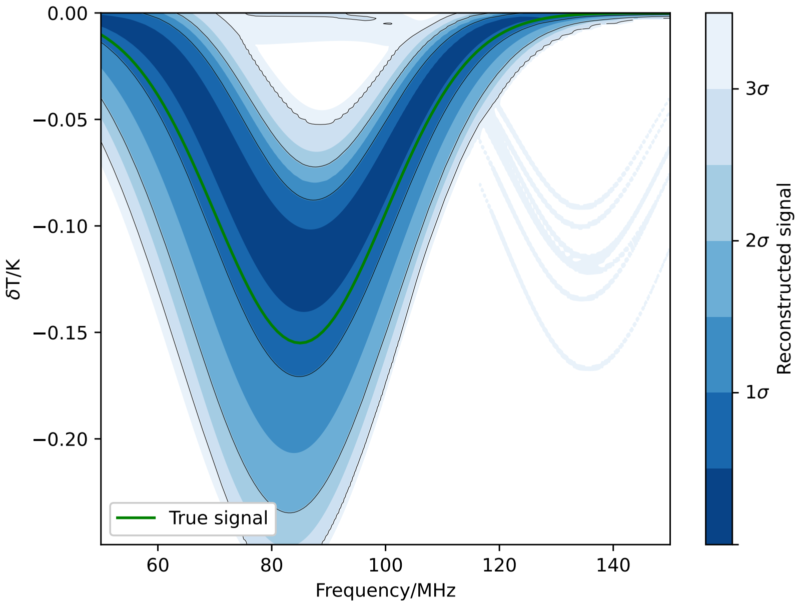
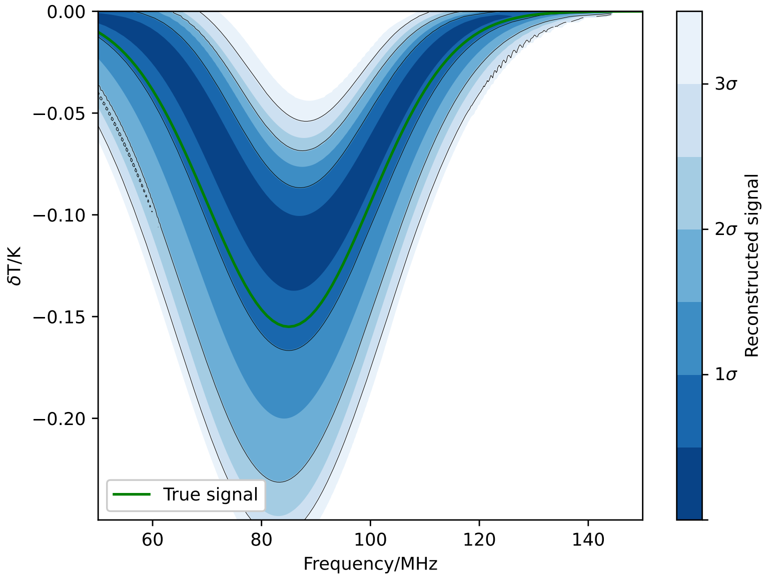


Beam 2



Beam 3



5 basis functions

Correcting for Construction Errors

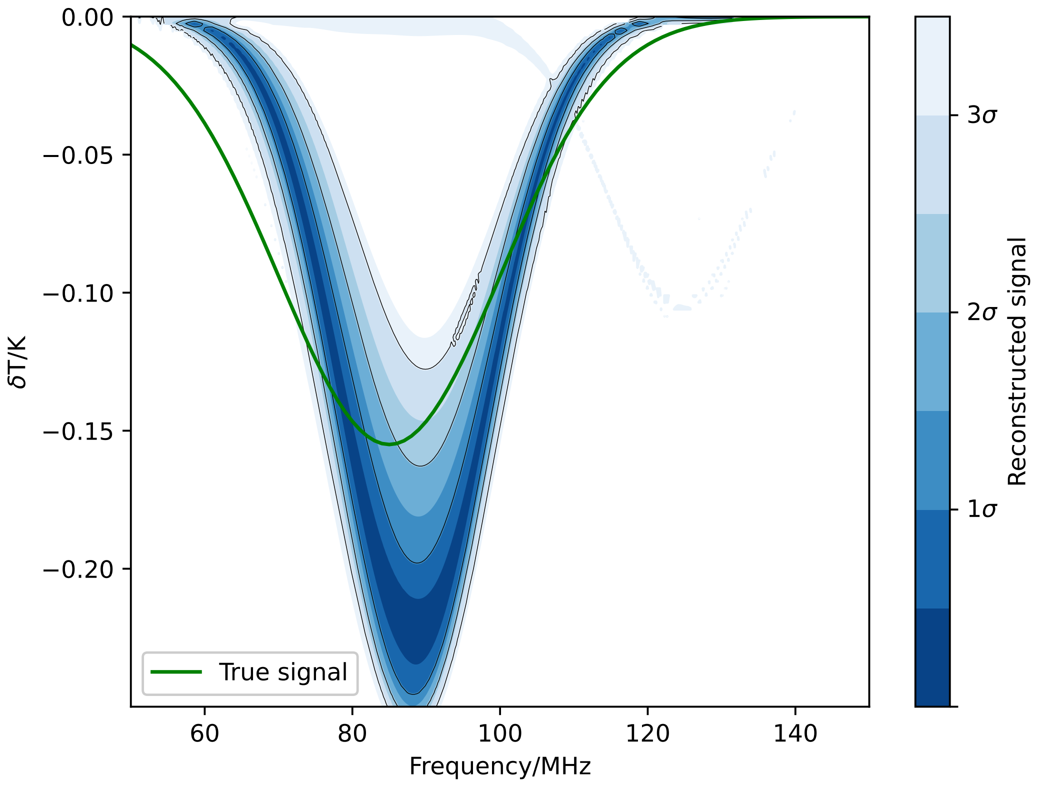
Simulated data generated using beam B and a spatially dependant power law foreground

Full foreground with 15 spectral index parameters fitted with AMLP of 6^{h} order polynomial variations of coefficients on beam A, for 11 basis functions derived by decomposition of beam A.

Correcting for Construction Errors

Simulated data generated using beam B and a spatially dependant power law foreground

Full foreground with 15 spectral index parameters fitted with AMLP of 6h order polynomial variations of coefficients on beam A, for 11 basis functions derived by decomposition of beam A.

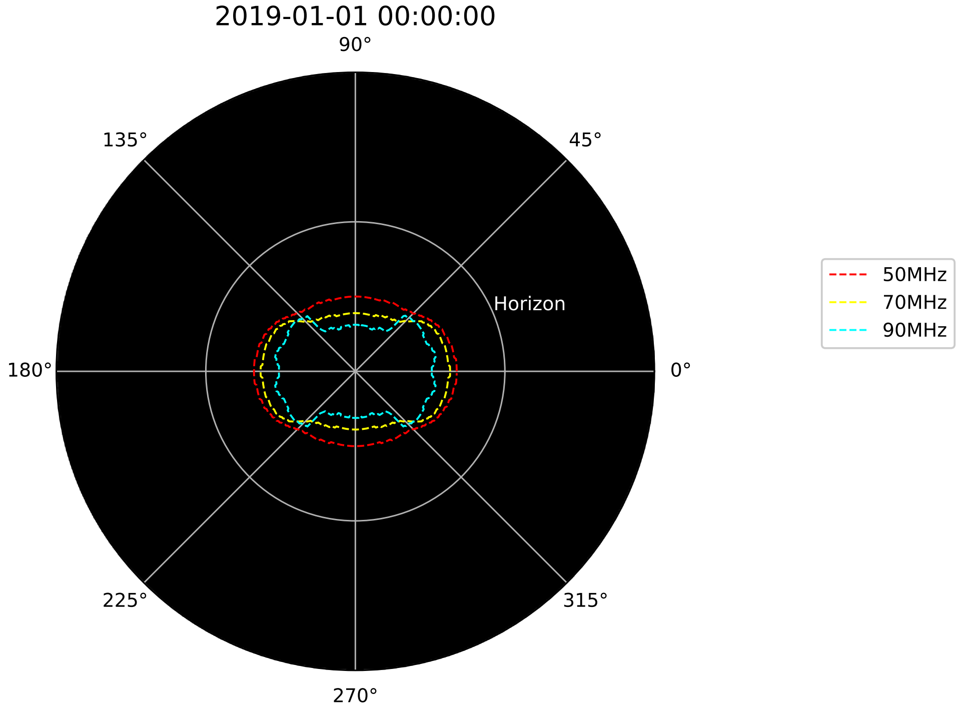


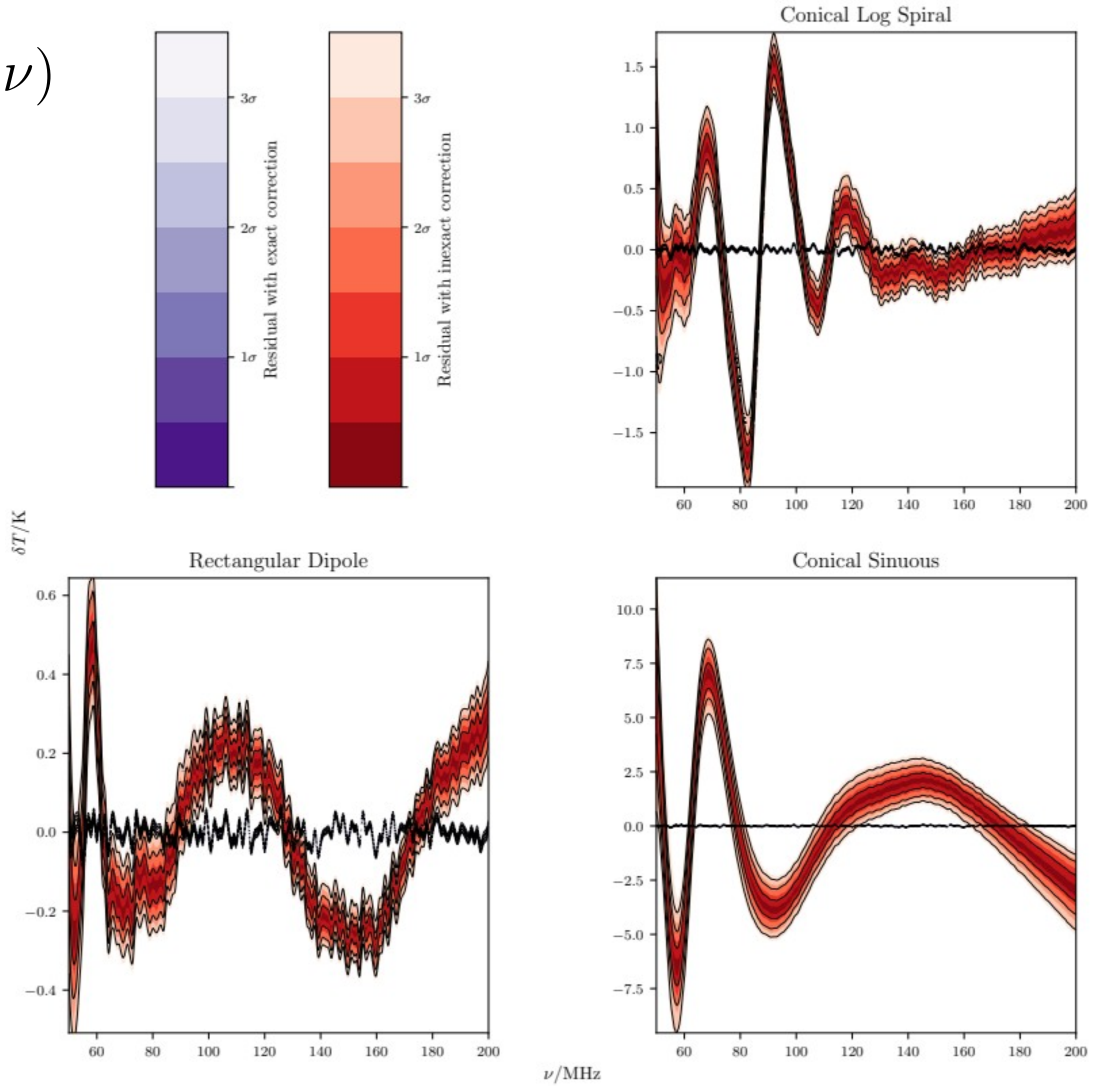
Conclusions

- Inaccuracies in a beam model in 21cm experiments lead to uncorrected chromatic distortions that can mask the signal
- Many realistic effects can produce sufficient beam errors to cause this issue
- Parametrised beam modelling is required
- Typical antenna directivities are complex enough to require more basis functions and parameters than can be practically fitted for
- AMLP enables beams to be fitted for efficiently
- Tight priors aid in avoiding fitting away the signal
- Implementing AMLP enables beam uncertainties to be accounted for sufficiently for the 21cm signal to be recovered

Beam Chromaticity

$$T_{\mathrm{A}}(\nu, t) = \frac{1}{4\pi}\int_{0}^{4\pi} D(\Omega, \nu)\, T_{\mathrm{F}}(\Omega, \nu, t)\, d\Omega + T_{21}(\nu)$$





Slight errors in the predicted beam or foreground can produce residual chromaticity large enough to mask the 21cm signal

Analytic Marginalisation of Linear Parameters

For Gaussian priors

$$\Pi\left(\underline{\theta}\right) = \frac{1}{\left|2\pi\underline{\underline{\Lambda}}\right|^{\frac{1}{2}}} e^{-\frac{1}{2}\left(\underline{\theta}-\underline{\xi}\right)^{\intercal}\underline{\underline{\Lambda}}^{-1}\left(\underline{\theta}-\underline{\xi}\right)}$$

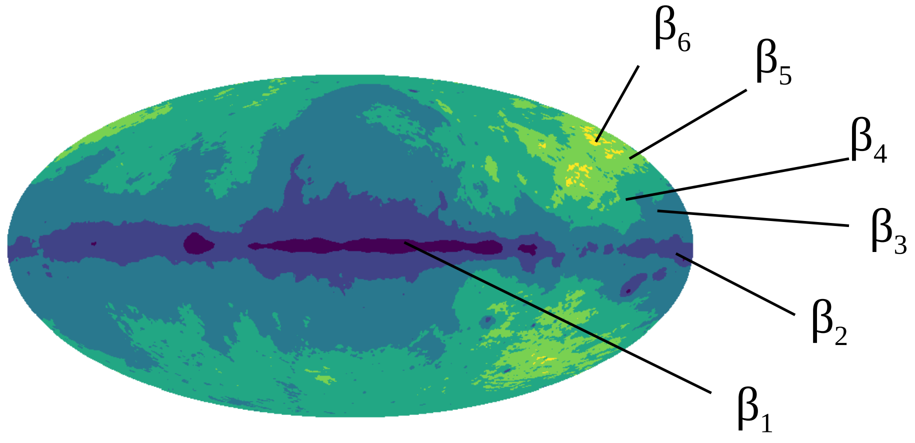
Further substitutions:

$$\underline{\underline{\Omega}}^{-1} = \underline{\underline{\Sigma}}^{-1} + \underline{\underline{\Lambda}}^{-1}$$

$$\underline{\underline{\Omega}}^{-1}\underline{\omega} = \underline{\underline{\Sigma}}^{-1}\underline{\mu} + \underline{\underline{\Lambda}}^{-1}\underline{\xi}$$

$$\log\mathcal{L}_{\mathrm{eff}} = -\frac{1}{2}\log\left|2\pi\underline{\underline{C}}\right| - \frac{1}{2}\underline{y}^{\intercal}\underline{\underline{C}}^{-1}\underline{y} + \frac{1}{2}\underline{\omega}^{\intercal}\underline{\underline{\Omega}}^{-1}\underline{\omega} - \frac{1}{2}\underline{\xi}^{\intercal}\underline{\underline{\Lambda}}^{-1}\underline{\xi} - \frac{1}{2}\log\left|2\pi\underline{\underline{\Lambda}}\right| + \frac{1}{2}\log\left|2\pi\underline{\underline{\Omega}}\right|$$

Implementation for Beam Fitting



$$\sum_{i}^{N_{\mathrm{reg}}} \left(\frac{\nu}{\nu_{\mathrm{base}}}\right)^{-\beta_i} \frac{1}{4\pi} \int_0^{4\pi} Y_j\left(\Omega\right) M_i\left(\Omega\right) \left(T_{\mathrm{base}}\left(\Omega, t\right) - T_{\mathrm{CMB}}\right) d\Omega$$

x coefficient factor

$\left(\mathrm{e.g.}\ \nu, \nu^2, \nu^3 \ldots.\right)$

$$\underline{\underline{A}} = \begin{array}{cc} N_\nu & \begin{pmatrix} & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \end{pmatrix} \\ & N_{\mathrm{basis}} \times N_{\mathrm{coeff}} \end{array}$$

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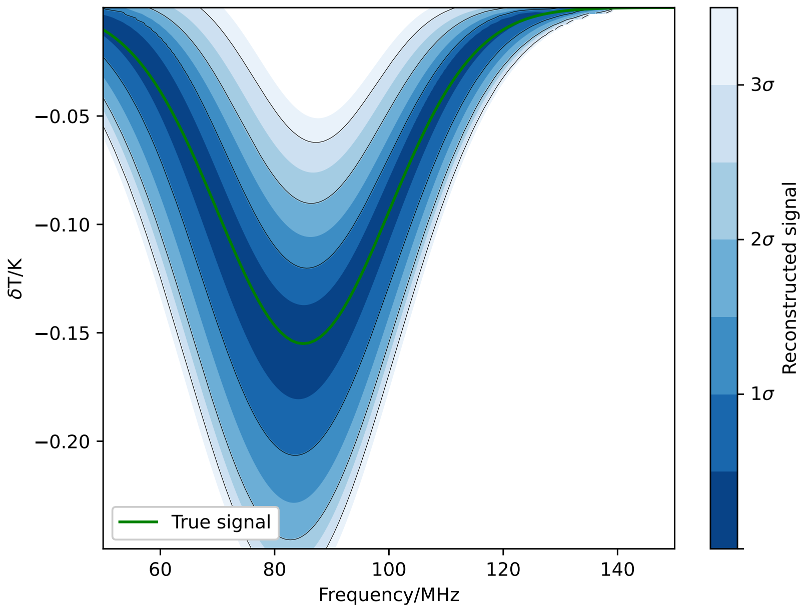
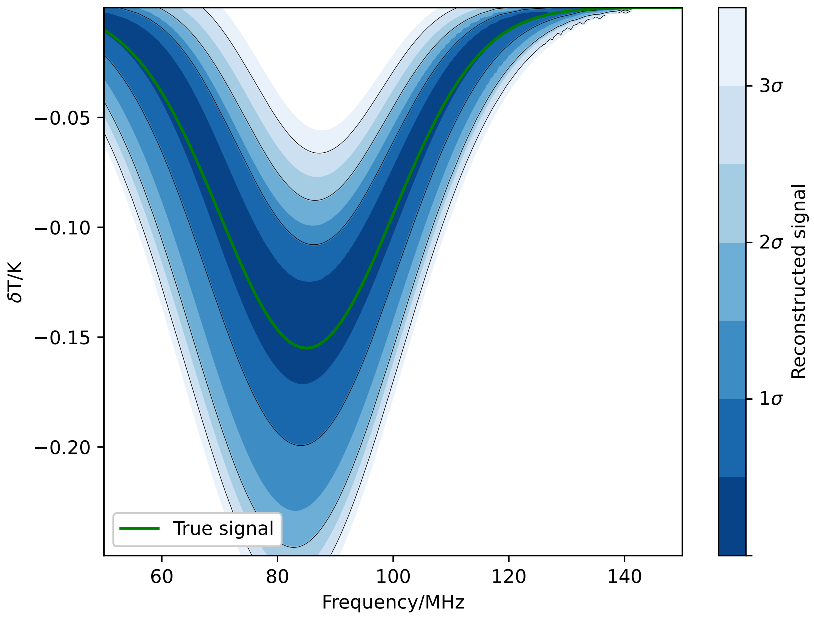
Single spectral index foreground parameter fitted with AMLP of 5^{th} order polynomial variations on beam 1

Beam 1



Beam 2



Beam 3