# On the Geometry of $\varphi$ -Static Perfect Fluid Space-Times

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In loving memory of Rebo (2015-2024)

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Abstract. In this paper we study the geometry of  $\varphi$ -static perfect fluid spacetimes ( $\varphi$ -SPFST, for short). In the context of Einstein's General Relativity, they arise from a space-time whose matter content is described by a perfect fluid in addition to a nonlinear field expressed by a smooth map  $\varphi$  with values in a Riemannian manifold. Considering the Lorentzian manifold  $\widehat{M}$  in the form of a static warped product, we derive the fundamental equations via reduction of Einstein's Field Equations to the factors of the product. To set the stage for our main results, we discuss the validity of the classical Energy Conditions in the present setting and we introduce the formalism of  $\varphi$ -curvatures, which is a fundamental tool to merge the geometry of the manifold with that of the smooth map  $\varphi$ . We then present several mathematical settings in which similar structures arise. After computing two integrability conditions, we apply them to prove a number of rigidity results, both for manifolds with or without boundary. In each of the aforementioned results, the main assumption is given by the vanishing of some  $\varphi$ -curvature tensors and the conclusion is a local splitting of the metric into a warped product. Inspired by the classical Cosmic No Hair Conjecture of Boucher, Gibbons and Horowitz, we find sharp sufficient conditions on a compact  $\varphi$ -SPFST with boundary to be isometric to the standard hemisphere. We then describe the geometry of relatively compact domains in M subject to an upper bound on the mean curvature of their boundaries. Finally, we study non-existence results for  $\varphi$ -SPFSTs, both viathe existence of zeroes of the solutions of an appropriate ODE and with the aid of a suitable integral formula generalizing in a precise sense the well-known Kazdan-Warner obstruction.

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#### CHAPTER 1

# Introduction

#### 1.1. Mathematical Settings and Main Results

Over the last decades, there has been an ever-growing interest of the mathematical and physical communities towards the study of Riemannian manifolds endowed with special metrics, arising, for instance, as critical metrics of curvature functionals, or with special structures coming, for example, as soliton solutions of particular geometric flows or as special solutions of some (physically and geometrically) relevant equations. Four important examples are, respectively, Einstein metrics, which are critical points of the Einstein-Hilbert functional with constrained volume (see e.g. the classical [6]), Ricci solitons, that is, self-similar solutions of the celebrated Ricci flow (see e.g. [19] and references therein), static perfect fluid space-times (SPFSTs, for short; see e.g. [41] and references therein), characterized by the system

(1.1) 
$$\begin{cases} \operatorname{Hess}(u) - u \left( \operatorname{Ric} - \frac{1}{m-1} \left( \frac{S}{2} - p \right) g \right) = 0, \\ \Delta u = \frac{u}{m-1} \left( mp + \frac{m-2}{2} S \right), \\ \mu = \frac{1}{2} S, \\ (\mu + p) \nabla u = -u \nabla p, \end{cases}$$

where  $u \in C^{\infty}(M)$ , Ric and S denote the Ricci tensor and the scalar curvature of (M, g), respectively,  $\mu$  represents the *energy density* and p the pressure of a perfect fluid, and, finally, vacuum static spaces (also known as static triples) (M, g, u) (see e.g. [61], [5], [22], [44] and the very recent [21]), which consist of a Riemannian manifold (M, g) endowed with a positive, smooth solution u on M of the equation

(1.2) 
$$\operatorname{Hess}(u) - u\left(\operatorname{Ric} - \frac{S}{m-1}g\right) = 0.$$

In the last two cases (and this will be explicitated for an extension of (1.1)), the set of equations comes from a reduction of the Einstein equations, governing the corresponding phenomenon in General Relativity, on a Lorentzian warped product  $\overline{M} = \mathbbmss{R}_{e^{-f}} \times M$  to the Riemannian "space" component (M,g) and to the "time" component  $\mathbbmss{R}$  of  $\overline{M}$ .

In this paper we consider a further generalization of SPFSTs, that is, we study the geometry of an m-dimensional, connected, Riemannian manifold (M, g),

possibly with boundary, carrying a structure that we shall call a  $\varphi$ -static perfect fluid space-time (from now on,  $\varphi$ -SPFST). This means that for some smooth map  $\varphi:(M,g)\to(N,h)$  and smooth functions  $U:N\to\mathbb{R}$  and  $\mu,p:M\to\mathbb{R}$ , (M,g) admits a non-negative smooth solution u of the  $\varphi$ -SPFST system

(1.3) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \left\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) g \right\} = 0, \\ ii) \Delta u = \frac{u}{m-1} \left[ mp - mU(\varphi) + \frac{m-2}{2} S^{\varphi} \right], \\ iii) u\tau(\varphi) = -d\varphi(\nabla u) + \frac{u}{\alpha} (\nabla U)(\varphi), \\ iv) \mu + U(\varphi) = \frac{1}{2} S^{\varphi}, \\ v) (\mu + p) \nabla u = -u \nabla p, \end{cases}$$

where we are using the shorthand notation

$$\operatorname{Ric}^{\varphi} := \operatorname{Ric} - \alpha \varphi^* h,$$

 $\alpha \in \mathbb{R} \setminus \{0\}$ , for the  $\varphi$ -Ricci tensor (introduced in [53] and studied, for instance, in [68], [3] and [20]),  $S^{\varphi}$  for the  $\varphi$ -scalar curvature, i.e. the trace of  $\operatorname{Ric}^{\varphi}$  with respect to g, and  $\tau(\varphi)$  for the tension field of the map  $\varphi$ . We recall that  $\tau(\varphi)$  is the trace, with respect to g, of the generalized second fundamental tensor  $\nabla d\varphi$  of the map  $\varphi$ : it plays a central role in the geometry of smooth, harmonic maps, extending the Laplace operator to the general case where  $\varphi$  takes values in a curved space, and it is responsible of the non-linearity of the theory (see e.g. the classical [26], [25] and [27]).

We also require, motivated by physical reasons,

$$(1.4) u > 0 on int(M)$$

and

$$\partial M = u^{-1}(\{0\})$$

in case  $\partial M \neq \emptyset$ . We refer to Chapter 2 for the derivation of the system *via* the reduction of the Einstein equations and for the interpretation of the various terms (see also the next subsection for some remarks on the physical motivations and, again, Chapter 2 for more details on  $\varphi$ -curvatures).

Note that, when  $\varphi$  is constant and  $U \equiv 0$ , system (1.3) reduces to system (1.1), that is the classical SPFST system.

We observe that, for a vacuum static space, in 1984 Boucher, Gibbons and Horowitz (see [11] and [10]) conjectured that the only m-dimensional, simply connected, static triple (M, g, u) with a single horizon (that is, a connected boundary  $\partial M \neq \emptyset$ ) and positive constant scalar curvature S is isometric to a Euclidean hemisphere  $\mathbb{S}_{+}^{m}(r)$ , for some appropriate radius r > 0; this conjecture, which is known as the Cosmic No-Hair Conjecture, suggests that, under certain assumptions, the evolution of the universe leads to the dominance of a highly symmetric geometry.

getting rid of more complicated, "hairy" solutions. The conjecture has been confirmed under different further hypotheses, but disproved for  $\dim(M) \geq 4$  (for more information, see, for instance, [33], [35], [21]).

A great deal of the present work is devoted to the study, under various assumptions, of this type of rigidity results. In fact, in doing so, we go through a number of intermediate steps, in particular a local description of the metric g of M as a (local) warped product; this is achieved by a careful study of the geometry of the regular level sets of u. To achieve our goal we use a number of new or relatively new concepts; for instance, the next definition will be needed in a shortwhile to state some of our results:

**Definition** 1.1. Let (M,g) and (N,h) be two Riemannian manifolds. For a smooth map  $\varphi:(M,g)\to (N,h)$  and for a constant  $\alpha\in\mathbb{R},\ \alpha\neq 0$ , we say that (M,g) is harmonic-Einstein if, for some  $\Lambda\in\mathbb{R}$ ,

(1.6) 
$$\begin{cases} \operatorname{Ric}^{\varphi} = \operatorname{Ric} - \alpha \varphi^* h = \Lambda g, \\ \tau(\varphi) = 0. \end{cases}$$

**Remark** 1.2. Note that, instead of (M, g) to be harmonic-Einstein, one should refer to  $(M, g, \varphi, \alpha, \Lambda)$  to be harmonic-Einstein; however, for the sake of readability we shall avoid this, and possibly other, cumbersome notation, the correct meaning being clear from the context.

Observe that, for  $m \geq 3$ , an analogous of Schur's lemma for Einstein metrics holds: indeed, if  $\Lambda$  is a function on M, then (1.6) automatically implies that  $\Lambda$  is constant (see e.g. [3]). Harmonic-Einstein manifolds are strictly related to harmonic-Einstein solitons, which are special solutions of the Ricci-harmonic flow introduced by B. List in [47].

The present paper can be roughly divided into three parts: in the first we describe the setting and we analyze some special cases and motivations, in the second part we concentrate on rigidity results, while in the third we provide some sufficient conditions for the non-existence of positive solutions u on M for a large class of systems containing (1.3). We now describe some of our main results. The first, which is local in nature, is motivated by the work [43] of Kobayashi and Obata (in Chapter 4 we shall go into details to justify the genesis of our result and the reason why it is an extension of [43]). We need one more definition:

**Definition** 1.3. Let  $\varphi:(M,g)\to (N,h)$  be a smooth map and  $G:N\to\mathbb{R}$  a smooth function. Then  $\varphi$  is said to be G-harmonic if

(1.7) 
$$\tau(\varphi) = (\nabla G)(\varphi).$$

Note that, for G constant, we obtain the case of harmonic maps. The previous definition goes back to the work of Fardoun, Ratto and Regbaoui ([28], [29], [62]); it comes from a variational setting that has been vastly analyzed by Lemaire in his Ph.D. Thesis, ([46]). Indeed, (1.7) is the Euler-Lagrange equation of the functional

$$E(\varphi) = \frac{1}{2} \int_{\Omega} \left\{ |d\varphi|^2 + 2G(\varphi) \right\} dV_g,$$

where  $\Omega$  is a relatively compact domain in M (see [46] for the details).

The stress-energy tensor associated to  $\varphi$  and G is

$$S^{G} = \left\{ \frac{\left| d\varphi \right|^{2}}{2} + G(\varphi) \right\} g - \varphi^{*} h;$$

a simple computation shows that

$$\operatorname{div} S^G = h(d\varphi, (\nabla G)(\varphi) - \tau(\varphi)),$$

so that a G-harmonic map  $\varphi$  is in particular G-conservative, that is

$$\operatorname{div} S^G = 0.$$

Thus, for G constant, a map  $\varphi$  is conservative if and only if

$$h(\tau(\varphi), d\varphi) = 0.$$

If, for some  $\psi \in C^{\infty}(M)$ ,  $\psi > 0$ , we perform a conformal change of metric  $\widetilde{g} = \psi^2 g$ , we will set  $\widetilde{\varphi}$  to indicate the map  $\varphi$  but now considered as a map from  $(M,\widetilde{g})$  to (N,h). In general, "tilded" quantities will refer to the metric  $\widetilde{g}$  (thus, for example,  $\widetilde{S}^{\widetilde{\varphi}}$  is the  $\widetilde{\varphi}$ -scalar curvature of  $(M,\widetilde{g})$ ).

In many instances, setting  $f = -\log u$  when u > 0, we will transform system (1.3) into a particular case of a system of the general type

(1.8) 
$$\begin{cases} \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi) \end{cases}$$

for some  $\alpha, \eta \in \mathbb{R}, \alpha \neq 0, \lambda \in C^{\infty}(M)$ .

In the following, we will often consider a regular level set  $\Sigma$  of f endowed with the metric  $g_{\Sigma} = g_{|_{\Sigma}}$ ; we will set  $\operatorname{Ric}^{\varphi_{|_{\Sigma}}}$  and  $S^{\varphi_{|_{\Sigma}}}$  to denote the  $\varphi$ -Ricci tensor and the  $\varphi$ -Scalar curvature of  $(\Sigma, g_{\Sigma})$ , respectively.

We are now ready to state our first

**Theorem** 1.4. Let (M,g) be a manifold of dimension  $m \geq 3$ , with  $\partial M = \emptyset$ . Let  $\varphi : (M,g) \to (N,h)$  be a smooth map,  $U : N \to \mathbb{R}$  be a smooth function,  $\alpha \in \mathbb{R} \setminus \{0\}$ ,  $\lambda \in C^{\infty}(M)$  and let  $f \in C^{\infty}(M)$  be a solution of system (1.8) on M, with  $\eta \neq -\frac{1}{m-2}$ . Assume that  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic and suppose that, for the conformal change of metric

$$\widetilde{g} = e^{-\frac{2}{m-2}f}g,$$

we have

$$(1.10) 2(m-1)\widetilde{C}^{\widetilde{\varphi}} = -\operatorname{div}_1(U(\varphi)g \otimes g),$$

where  $\widetilde{C}^{\widetilde{\varphi}}$  is the  $\widetilde{\varphi}$ -Cotton tensor of the metric  $(M, \widetilde{g})$ . Then, for each  $p \in \Sigma$ , where  $\Sigma$  is a regular level set of f, there exists an open set  $A \subset M$  with  $p \in A$  such that  $g_{|_A}$  is a warped product metric. Moreover,  $U(\varphi)$  and  $S^{\varphi_{|_{\Sigma}}}$  are constant on  $\Sigma$  and we have

(1.11) 
$$\begin{cases} \operatorname{Ric}^{\varphi_{|_{\Sigma}}} = \frac{S^{\varphi_{|_{\Sigma}}}}{m-1} g_{\Sigma}, \\ h(\tau(\varphi_{|_{\Sigma}}), d\varphi_{|_{\Sigma}}) = 0 \end{cases}$$

on  $(\Sigma, g_{\Sigma})$ .

A few remarks are in order to explain the statement of the Theorem. Assume that  $(A,g_{|A})$  can be identified with the warped product  $I\times_{\rho}(\Sigma\cap A)$ , with metric  $g_{|A}=dr^2+\rho^2g_{|\Sigma}$ . Then one can prove that, under this identification, the signed distance function  $r:A\to\mathbb{R}$  from  $\Sigma\cap A$  coincides with the projection  $p:I\times(\Sigma\cap A)\to I$ . If, for some  $t\in I$ , we consider the hypersurface  $\{t\}\times(\Sigma\cap A)$ , then its mean curvature with respect to the inner unit normal is constant; we will denote it by H(t). With these notation in mind, we can express the warping factor  $\rho$  by means of the relation  $\rho(r)=e^{-\int_0^r H(t)dt}$ . This, and an explicit description of  $S^{\varphi_{|\Sigma}}$ , is the content of Theorem 4.14 below.

The tensor  $\widetilde{C}^{\widetilde{\varphi}}$  is defined in Chapter 2, Section 2.3, while the notation  $\operatorname{div}_1 T$ , for some tensor T (for instance, a 4-times covariant tensor) means, in a local orthonormal coframe,

$$(\operatorname{div}_1 T)_{ikt} = T_{ijkt,i}.$$

Note that it is necessary to make this type of distinctions because T might not have special symmetries, and thus  $\operatorname{div}_1 T$  could be different from  $\operatorname{div}_2 T$ ; as a matter of fact, for instance,

$$\operatorname{div}_1(U(\varphi)g \otimes g) = -\operatorname{div}_2(U(\varphi)g \otimes g)$$

(here  $\bigcirc$  denotes the Kulkarni-Nomizu product).

Note also that, if we suppose that  $\varphi_{|_{\Sigma}}$  is a harmonic submersion defined on an open set  $B \subset \Sigma$ , then  $\tau(\varphi_{|_{\Sigma}}) = 0$  on B and, by the unique continuation property for harmonic maps,  $\tau(\varphi) = 0$ , see [66]. In this case,  $\Sigma$  becomes a harmonic-Einstein manifold; indeed, system (1.11) reduces to

(1.12) 
$$\begin{cases} \operatorname{Ric}^{\varphi_{|_{\Sigma}}} = \frac{S^{\varphi_{|_{\Sigma}}}}{m-1} g_{\Sigma}, \\ \tau(\varphi_{|_{\Sigma}}) = 0. \end{cases}$$

Observe that, in case  $m \geq 4$ , equation (1.12) forces  $S^{\varphi_{|_{\Sigma}}}$  to be constant (see [3] for a proof).

Replacing hypothesis (1.10) with a different set of assumptions yields the following theorem, which asserts a stronger conclusion than Theorem 1.4:

**Theorem** 1.5. Let (M,g) be a complete manifold of dimension  $m \geq 3$  and with  $\partial M = \emptyset$ . Let  $\varphi : (M,g) \to (N,h)$  be a smooth map,  $U : N \to \mathbb{R}$  be a smooth function,  $\alpha \in \mathbb{R} \setminus \{0\}$ ,  $\lambda \in C^{\infty}(M)$  and let  $f \in C^{\infty}(M)$  be a solution on M of system (1.8). Let  $\Sigma$  be a regular level set of f. Assume the validity of the following assumptions:

- f is proper;
- either  $\alpha > 0$  and  $\eta > -\frac{1}{m-2}$  or  $\alpha < 0$  and  $\eta < -\frac{1}{m-2}$ ;
- $B^{\varphi}(\nabla f, \nabla f) = 0$ , where  $B^{\varphi}$  is the  $\varphi$ -Bach tensor;
- for each regular  $p \in M$ ,  $\nabla f_p$  is an eigenvector of  $\operatorname{Ric}_p^{\varphi}$ ;
- $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic.

Then, for each  $p \in \Sigma$ , there exists an open set A in M, with  $p \in A$ , such that  $g_{|A}$  is a warped product metric. Moreover,  $U(\varphi)$  and  $S^{\varphi_{|\Sigma}}$  are constant on  $\Sigma$  and  $(\Sigma, g_{\Sigma})$  is harmonic-Einstein, that is, it satisfies

(1.13) 
$$\begin{cases} \operatorname{Ric}^{\varphi_{|\Sigma}} = \frac{S^{\varphi_{|\Sigma}}}{m-1} g_{\Sigma}, \\ \tau(\varphi_{|\Sigma}) = 0. \end{cases}$$

The definition of  $B^{\varphi}$  will be given in formula (2.55).

The request that  $\nabla f_p$  is an eigenvector of  $\operatorname{Ric}_p$  at any regular point p of f is necessary, as we will see in Corollary 4.7.

We mention here a further result, where we prove that the - geometrically significant - tensor  $\overline{D}^{\varphi}$ , defined in (4.5), is identically null. As a consequence of Theorem 4.14, this fact will give a local splitting of the metric as in the previous Theorems.

**Theorem** 1.6. Let (M,g) be a complete manifold of dimension  $m \geq 3$  and with  $\partial M = \emptyset$ . Let  $\varphi : (M,g) \to (N,h)$  be a smooth map,  $\lambda \in C^{\infty}(M)$  and let  $f \in C^{\infty}(M)$  be a solution on M of system

(1.14) 
$$\operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g$$

for some  $\alpha, \eta \in \mathbb{R}, \alpha \neq 0$  and  $\lambda \in C^{\infty}(M)$ . Suppose the validity of the following assumptions:

- $\bullet \ \eta \neq -\frac{1}{m-2};$
- if M is non-compact,  $f(x) \to +\infty$  as  $x \to +\infty$  in M;
- $\varphi$  is conservative, that is

$$h(\tau(\varphi), d\varphi) = 0,$$

•  $\operatorname{div}^3 C^{\varphi} \equiv 0$ .

Then there are two possibilities:

i) if  $\eta \neq 0$  and we further assume

(1.15) 
$$W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) \equiv 0,$$

then we have

$$C^{\varphi} \equiv 0.$$

Moreover, if, for some smooth  $U: N \to \mathbb{R}$  we also assume the validity of

(1.16) 
$$\tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi),$$

then we obtain

$$\overline{D}^{\varphi} \equiv 0.$$

ii) If  $\eta = 0$ , then we have

$$C^{\varphi} \equiv 0$$
.

Moreover, if we also assume both (1.15) and (1.16), then we have

$$\overline{D}^{\varphi} \equiv 0.$$

 $C^{\varphi}$  and  $W^{\varphi}$ , the  $\varphi$ -Cotton tensor and the  $\varphi$ -Weyl tensor, are defined respectively in (2.48) and (2.50), while the symbol div<sup>3</sup>  $C^{\varphi}$  stands for the "total divergence"

$$\operatorname{div}^3 C^{\varphi} = C^{\varphi}_{ijk,kji}$$

(pay attention to the order of indices).

Note that the conservativity of  $\varphi$  and div<sup>3</sup>  $C^{\varphi} \equiv 0$  are both necessary conditions for the validity of  $C^{\varphi} \equiv 0$ . It is also worth to observe that condition (1.15), which for  $\varphi$  constant takes the form

$$W(\nabla f, \cdot, \cdot, \cdot) = 0,$$

in the recent literature is called the zero radial Weyl curvature assumption, and has in fact a precise geometric meaning. Indeed, if we pointwise conformally deform g to  $\tilde{g}$  as in (1.9), we obtain

(1.17) 
$$\widetilde{C}^{\widetilde{\varphi}} = C^{\varphi} + W^{\varphi}(\nabla f, \cdot, \cdot, \cdot);$$

thus,  $W^{\varphi}(\nabla f, \cdot, \cdot, \cdot)$  is the obstruction to  $C^{\varphi}$  being invariant under the conformal deformation (1.9). This also shows that the case  $\eta = -\frac{1}{m-2}$ , for  $m \geq 3$ , is special. Indeed, as we shall see in Chapter 2, the system

(1.18) 
$$\begin{cases} \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) + \frac{1}{m-2} df \otimes df = \lambda g, \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases}$$

is obtained via the conformal deformation (1.9) of the metric of a harmonic-Einstein manifold (see Proposition 2.19 below). It is immediate to verify that for a harmonic-Einstein manifold the  $\varphi$ -Cotton tensor is identically null; thus, formula (1.17) shows that, when  $\eta = -\frac{1}{m-2}$ ,

$$C^{\varphi} = -W^{\varphi}(\nabla f, \cdot, \cdot, \cdot)$$

and therefore

$$C^{\varphi} \equiv 0$$
 if and only if  $W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) \equiv 0$ .

Interpreting Theorem 1.6 in case of a  $Ricci-harmonic \ soliton$ , that is, for a smooth solution f of the system

(1.19) 
$$\begin{cases} \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) = \lambda g, \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases}$$

with  $\alpha, \lambda \in \mathbb{R}, \alpha \neq 0$ , we deduce

Corollary 1.7. Let (M,g) be a complete Ricci-harmonic soliton of dimension  $m \geq 3$  such that  $\varphi$  is conservative and  $\operatorname{div}^3 C^{\varphi} \equiv 0$ . Then  $C^{\varphi} \equiv 0$ .

Similarly, in the case of a Ricci almost soliton (M, g) (see [59]), we have a solution  $f \in C^{\infty}(M)$  of the system

$$Ric + Hess(f) = \lambda g$$
,

with  $\lambda \in C^{\infty}(M)$ , and we deduce the

**Corollary** 1.8. Let  $(M, g, f, \lambda)$  be a complete almost Ricci soliton of dimension  $m \geq 4$  such that

$$\operatorname{div}^{3}\left(\operatorname{div}_{1}W\right) \equiv 0.$$

Then (M, g) has harmonic Weyl tensor.

**Remark** 1.9. According to the notation we just introduced, expression (1.20) reads, in a local orthonormal coframe,

$$W_{tijk,tkji} \equiv 0.$$

As for the proof of Corollary 1.8, we observe that, in this case, (1.20) is equivalent to  $\operatorname{div}^3 C \equiv 0$ , and the conclusion  $C \equiv 0$  implies  $W_{tijk,t} \equiv 0$ , (this follows by equation (2.51) when  $\varphi$  is constant), that is, the Weyl tensor is harmonic. Thus, if we assume that  $\lambda$  is a positive constant, or, in other words, that  $(M, g, f, \lambda)$  is a complete, shrinking Ricci soliton, then (M, g) is either Einstein or it is isometric to a finite quotient of  $P^{m-k} \times \mathbb{R}^k$ ,  $k \in \mathbb{N}$ , k > 0, the product of an Einstein manifold P with the Gaussian shrinking soliton  $(\mathbb{R}^k, g_E, \frac{\lambda}{2}|x|^2, \lambda)$ . Note that Corollary 1.8, with  $\lambda$  constant, is due to Catino, Mastrolia, Monticelli [16], while the classification of Weyl-harmonic complete, shrinking Ricci solitons is due to Fernandez-Lopez,

Garcia-Rio [30] and Munteanu, Sesum [55].

The proof of Theorem 1.6 is based on a basic formula related to (1.14) and proved in Proposition 5.5 of Chapter 5; however, rearranging the formula under different assumptions, we obtain the following slightly different version:

**Theorem** 1.10. Let (M,g) be a complete Riemannian manifold of dimension  $m \geq 3$ , with  $\partial M = \emptyset$ , supporting system (1.8), that is,

$$\begin{cases} \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi) \end{cases}$$

for some  $\alpha, \eta \in \mathbb{R}$ ,  $\alpha \neq 0$ , and  $\lambda \in C^{\infty}(M)$ . Assume that

- $\bullet \ \eta \neq -\frac{1}{m-2};$
- if M is non compact,  $f(x) \to +\infty$  as  $x \to \infty$  on M;
- $\operatorname{div}^3 C^{\varphi} = 0$ :
- $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic;
- $\nabla f_p$  is an eigenvector of  $\operatorname{Ric}_p^{\varphi}$ , for each regular point p of f.

Then we have two possibilities:

i) if  $\eta \neq 0$  and we further assume

$$(1.21) W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0,$$

then we have that

(1.22) 
$$C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_{1}\left(U(\varphi)g \bigotimes g\right)$$

and

$$\overline{D}^{\varphi}=0.$$

ii) If  $\eta = 0$ , (1.22) holds and, if we further assume (1.21), we also infer  $\overline{D}^{\varphi} = 0.$ 

**Remark** 1.11. We note that, since  $\overline{D}^{\varphi} \equiv 0$  and  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic, we can apply the local description of the metric given in Theorem 1.4.

In Chapter 5 we will give a version of Theorem 1.10 for (M,g) compact with non empty-boundary (see Theorem 5.8); it will be necessary to identify  $\partial M$  with  $u^{-1}(\{0\})$ , and the Theorem will therefore be expressed in terms of u (and not of f). Furthermore, the system we consider is the more restrictive

(1.23) 
$$\begin{cases} \eta u \operatorname{Ric}^{\varphi} - \operatorname{Hess}(u) = \frac{1}{m} (\eta u S^{\varphi} - \Delta u) g, \\ \eta u \tau(\varphi) = -d\varphi(\nabla u) + \frac{\eta}{\alpha} u(\nabla U)(\varphi), \end{cases}$$

which can be obtained from (1.8) via the change of variable  $u = e^{-\eta f}$ .

In Chapter 6, one of the main results goes in the direction of the *Cosmic No-Hair Conjecture* in the case of SPFST's, which is a case more general than static

triples; this result is strictly related to the Null Energy Condition and the Strong Energy Condition (see Section 2.2 in Chapter 2). Indeed, in this setting, the *Null Energy Condition* is implied by

while the conditions

(1.25) 
$$p + \mu \ge 0, (m-2)\mu + mp \ge 2U(\varphi)$$

imply the validity of the Strong Energy Condition.

**Remark** 1.12. The second requirement in (1.25) is also necessary; moreover, we underline that, when  $\varphi$  is constant and  $U \equiv 0$ , (1.24) and (1.25) are both necessary, respectively, for the Null Energy Condition and the Strong Energy Condition.

The following simple proposition will make assumption (1.27) in Theorem 1.14 below meaningful. Note that (1.27) in the literature is called a *Boundary Gravity Condition* or a *Shear Stress Condition*: we shall discuss its sharpness after the proof of Theorem 6.3.

**Proposition** 1.13. Let (M, g) be a compact  $\varphi$ -SPFST such that  $\partial M \neq \emptyset$ . Then there exists  $p \in \text{int}(M)$  such that

$$(m-2)\mu + mp < 2U(\varphi)$$
 at  $p$ .

PROOF. Inserting (1.3) iv) into (1.3) ii) we obtain

$$\Delta u = \frac{1}{m-1}[(m-2)\mu + mp - 2U(\varphi)]u;$$

thus, if by contradiction we assume

$$(m-2)\mu + mp > 2U(\varphi),$$

then u > 0 imply

$$\Delta u \geq 0$$
 on  $M$ .

Since u=0 on  $\partial M$  and u>0 on  $\mathrm{int}M$ , by the maximum principle we infer  $u\equiv 0$ , which is a contradiction.

**Theorem** 1.14. Let (M,g) be an m-dimensional compact  $\varphi$ -SPFST with connected, non-empty boundary and  $\alpha > 0$ . Assume that

$$(m-2)\mu + mp$$

is constant, that

$$(1.26) p + \mu \ge 0$$

and that

$$(1.27) m(m-1)|\nabla u|_{|_{\partial M}}^2 \le \max_{M} \{ [2U(\varphi) - ((m-2)\mu + mp)] \},$$

with U weakly convex. Then  $\varphi, \mu, p$  and S are constant on M, with  $\mu$  and S positive and  $\mu = -p$ ; moreover, (M, g) is isometric to the hemisphere

$$(1.28) S_+^m \left( \frac{S}{m(m-1)} \right) \subset \mathbb{R}^{m+1}.$$

**Remark** 1.15. The weak convexity assumption means that the Hessian of the function U is positive semi-definite (see Definition 3.7 in Chapter 3).

**Remark** 1.16. The constancy assumption in Theorem 1.14 and in Corollary 1.18 below can be relaxed to an appropriate subharmonicity of the function itself (see Theorem 6.3).

**Remark** 1.17. In the "classical" case  $\varphi$  constant and  $U \equiv 0$ , a request as in (1.27) is a *surface gravity* condition (see e.g. [9]).

Finally, we exploit the weak convexity of U, in order to establish the inequality

$$(\operatorname{Hess}(u))(d\varphi(e_i), d\varphi(e_i)) \ge 0,$$

where  $\{e_i\}$  denotes an orthonormal basis for  $T_pM$ . In case we deal with system (1.1) we obtain the following

**Corollary** 1.18. Let (M,g) be an m-dimensional,  $m \geq 3$ , compact, SPFST with connected, non-empty boundary. Assume that

$$(m-2)\mu + mp$$

is constant, that

$$p + \mu \ge 0$$

and that

$$m(m-1)|\nabla u|_{|_{\partial M}}^2 \leq \max_{M} \left\{ -((m-2)\mu + mp)u^2 \right\}.$$

Then  $\mu$ , p and S are constant with  $\mu$  and S positive and  $\mu = -p$ ; moreover, (M, g) is isometric to

$$S_+^m \left( \frac{S}{m(m-1)} \right) \subset \mathbb{R}^{m+1}.$$

**Remark** 1.19. Having chosen S = m(m-1),  $\mu = \frac{1}{2}m(m-1)$  and

$$p = -\mu$$

system (1.1) yields

$$\begin{cases} u \operatorname{Ric} - \operatorname{Hess}(u) = mug, \\ \Delta u = -mu, \end{cases}$$

while assumption (1.27) becomes

$$|\nabla u|_{|_{\partial M}}^2 \le \max_M u^2.$$

Thus, Corollary 1.18 recovers Theorem 4.2 of Borghini and Mazzieri ([9]).

**Remark** 1.20. We underline the basic fact that M compact,  $\partial M \neq \emptyset$  and  $\mu$  constant always imply, by Proposition 3.5,

$$p + \mu = 0.$$

The next result, motivated by the work of Fogagnolo and Pinamonti in a different setting ([32]), describes the geometry of relatively compact domains  $\Omega$  in  $\mathrm{int}(M)$  with smooth boundary  $\partial\Omega$  subject to an upper bound  $-\overline{H}$  on the mean curvature H of  $\partial\Omega$  in the direction of the inward unit normal. Precisely, we have the following

**Theorem** 1.21. Let (M,g) be a  $\varphi$ -SPFST of dimension  $m \geq 2$  and let  $\Omega \subset \operatorname{int}(M)$  with smooth boundary. Let

$$\overline{H} = \frac{1}{m} \frac{\int_{\partial \Omega} u}{\int_{\Omega u}}$$

and assume

$$(1.30) H \le -\overline{H},$$

where H is the mean curvature of  $\partial\Omega$  in the direction of the inward unit normal. Furthermore, suppose

$$\mu + p \ge 0$$
 on  $M$ .

Then

$$i:\partial\Omega\hookrightarrow M$$

is totally umbilical and  $\mu$  and p are constant on  $\Omega$ , with  $\mu = -p$ .

**Remark** 1.22. If we compute H with respect to the outward unit normal, (1.30) becomes

$$H > \overline{H}$$
.

As we said before, the third part of the paper is devoted to non-existence results. The idea is simple: the existence of a positive solution of the  $\varphi$ -SPFST system (1.3) implies the existence of u solving

(1.31) 
$$\begin{cases} \Delta u + \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)u = 0, \\ u > 0 \text{ on } \text{int}(M), \end{cases}$$

thus we only need to give sufficient conditions on the geometry of M and on the coefficient

$$(1.32) 2U(\varphi) - mp - (m-2)\mu$$

to ensure that (1.31) has no positive solutions. Since, for M complete with empty boundary, the existence of a positive solution implies that the differential operator L defined by

$$Lv = \Delta v + \frac{1}{m-1}(2U(\varphi) - mp - (m-2)\mu)v$$

has non-negative spectral radius (see e.g. [31], [52]), to prove non-existence we look, under appropriate assumptions, for a contradiction to this property. Towards this aim, we study conditions for the existence of a first zero or an oscillatory behaviour of a solution of a Cauchy problem of the type

(1.33) 
$$\begin{cases} (v(t)z')' + A(t)v(t)z = 0 \text{ on } \mathbb{R}^+, \\ z(0^+) = z_0 > 0, \\ (vz')(0^+) = 0, \end{cases}$$

where we set

$$(1.34) v(t) = \operatorname{Vol}(\partial B_t),$$

with  $B_t$  the geodesic ball of radius t with a fixed origin  $o \in M$  and A(t) is given by

$$A(t) = \frac{1}{v(t)} \int_{\partial B_t} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)$$

(see e.g. [8] for more details on (1.33)). We then compare a solution of (1.33) having a first zero with u to obtain the desired contradiction. We refer the reader to Section 7 for the details; let us however state one of our results to have some flavor on the subject.

**Theorem** 1.23. Let (M,g) be a complete manifold of dimension  $m \geq 2$  with  $\partial M = \emptyset$ . Assume

$$\frac{1}{\operatorname{Vol}(\partial B_t)} \notin L^1(+\infty)$$

and, for some  $t_0 > 0$ ,

(1.36) 
$$\lim_{t \to +\infty} \int_{B_t \setminus B_{t_0}} \left[ 2U(\varphi) - mp - (m-2)\mu \right] = +\infty.$$

Then there is no positive solution u of the  $\varphi$ -SPFST system (1.3).

We should point out that condition (1.35) is not so restrictive as it may appear. Indeed, one can construct examples of complete manifolds with, for instance, exponential growth of the volume of geodesic balls but still satisfying (1.35). As for assumption (1.36), we observe that it does not imply that  $2U(\varphi) > (m-2)\mu + mp$  on the entire manifold, but only "at infinity", so that (M,g) could in fact even satisfy the Strong Energy Condition in a region  $\Omega$  of M provided that, on  $\Omega$ , the negative contribution of the integrand in (1.36) does not affect the infinite limit.

The paper ends with an intrinsic (that is, independent of the existence of a conformal vector field on M) Kazdan-Warner-type obstruction on the compact manifold M, related to the elementary symmetric functions of the tensor

$$A^{\varphi} - \frac{U(\varphi)}{m-1}g$$

provided the latter is Codazzi, that is, equivalently,

$$2(m-1)C^{\varphi} = -\operatorname{div}_1(U(\varphi)g \otimes g).$$

#### 1.2. Some Remarks on the Physical Motivations

At the beginning of the last century, Einstein was led to postulate that the gravitational forces should be expressed by the curvature of a Lorentzian metric  $\bar{g}$  on a 4-dimensional manifold, on the basis of four (by now famous) principles, that is, the principles of Relativity, Equivalence, General Covariance and Causality. He also postulated, maybe inspired by Riemann and Herbart, that space-time is curved in itself, and that its curvature is locally determined by the distribution of the sources encoded in what is called the *stress-energy tensor*  $\bar{T}$ . As a first attempt to describe the dynamic of gravitational forces, he and Grossman proposed the equation

$$(1.37) \overline{Ric} = k\overline{T},$$

where k is a positive coupling constant, but Einstein soon realized that equation (1.37) was not satisfactory, both from the physical and the mathematical point of view. Indeed, if  $\overline{T}$  describes any reasonable kind of matter, for instance a perfect fluid for which

$$\overline{T} = (p + \mu)du \otimes du - p\overline{q},$$

where p is the pressure,  $\mu$  the density of the fluid and  $du \otimes du$  a comoving observer, then the continuity equation for matter requires

$$\operatorname{div} \overline{T} = 0,$$

which is compatible with (1.37) only in case  $(\overline{M}, \overline{g})$  has identically zero scalar curvature  $\overline{S}$ . Moreover, from the mathematical point of view, equation (1.37), as pointed out by Hilbert (but note that this is not completely correct), has no variational origin. To overcome this latter difficulty, Einstein and Hilbert, independently, concluded that (1.37) has to be replaced by

$$(1.39) \overline{Ric} - \frac{1}{2} \bar{S}\bar{g} = k\overline{T},$$

that, because of the Bianchi identities, is compatible with (1.38) and has a left-hand side member coinciding with the left-hand side of the Euler-Lagrange equations for the Einstein-Hilbert functional

$$(1.40) \qquad \int_{\Omega} \bar{S} dV_{\bar{g}}, \quad \Omega \subset \subset \overline{M},$$

while the right-hand side comes from the action of a matter Lagrangian with "variational derivative"  $\overline{T}$  (see for instance Hawking and Ellis, [41]).

Equation (1.39) is the "simplest choice", both from the physical and mathematical point of view; however, this is by no means the only possible choice, and recent astrophysical observations and current cosmological hypothesis indicate that equation (1.39) is inadequate to describe the gravitational forces at the level of extra-galactic and cosmic scale, unless we are keen to admit the existence, in the universe, of some kind of unknown matter and energy, that is, the dark matter and the dark energy. A second approach to solve the above problem could rest on modifying the left-hand side of the field equations by considering an extended action functional, more complex than (1.40) (these are usually called extended f(R) theories of gravitation, see e.g. [13]). However, we remark that taking powers of  $\overline{S}$  in the action, which seems to be the first simplest choice to test, might not be the right one, indeed for  $\bar{\varphi}: \overline{M} \to (N,h)$  the geometry associated to  $\int_{\Omega} \left(\bar{S} - \frac{1}{2} |d\bar{\varphi}|^2\right)^2 dV_{\bar{g}}$  is quite rigid (see the Riemannian case treated in [65]). On the other hand, Cotton gravity, introduced by Harada [38] and based on a different action functional, has been able to describe the rotational motion of eightyfour galaxies using the gravitational potential obtained from a solution of the field equations, well fitting with the experimental data, without having to use dark matter (see [39]). As explained before, in this paper the setting we shall consider falls in the realm of dark matter.

#### CHAPTER 2

## **Preliminaries**

The aim of this chapter is to present a number of facts and results that, at the same time, justify the analysis of system (1.3) and will reveal to be important investigation tools in the rest of the paper. In particular, we will be dealing with the following subjects:

- given the extended Einstein field equations in a warped product Lorentzian context, we derive system (1.3) by splitting a suitable stress-energy tensor with respect to the "temporal" and "spatial" components.
- We recast the "Energy Conditions" relating to the energy-momentum tensor in the setting of  $\varphi$ -SPFST, extending the recent work ([48]), based on the analysis of Hawking and Ellis in [41]. Some of them will play the role of assumptions in part of the results, while, in some other situations, their violation will be necessarily needed to gain rigidity of the structure.
- We introduce the notion of  $\varphi$ -curvatures, extending the standard curvature tensors; these new tensors will be an invaluable tool to merge the geometry of (M,g) with that of the smooth map  $\varphi:(M,g)\to(N,h)$ . We shall also prove some relevant formulas.
- We consider a generalized version of system (1.3) and we show, in some special cases, that it can be deduced from well-known mathematical settings; indeed, beside the variational derivation of the Euler-Lagrange equations from an action functional involving a non-linear field, we also highlight how the system can be considered as a "soliton" structure coming from the coupled "Ricci-harmonic" flow, introduced by List in [47]. A further derivation is obtained from a warped product manifold carrying a harmonic-Einstein structure with potential; finally, we obtain the system for  $U \equiv 0$ , by a conformal deformation of a harmonic-Einstein manifold.

#### 2.1. Deduction of the System

A Lorentzian manifold  $(\widehat{M}, \widehat{g})$  is a smooth (m+1)-dimensional manifold with a non-degenerate (0,2)-symmetric tensor  $\widehat{g}$  with signature (-,+,...,+). We fix the indexes range  $0 \le \alpha, \beta, \ldots \le m$ : then we can find a local coframe  $\{\omega^{\alpha}\}_{\alpha=0}^{m}$  on an open set  $U \subseteq \widehat{M}$  with the property that

(2.1) 
$$\widehat{g} = -\omega^0 \otimes \omega^0 + \omega^1 \otimes \omega^1 + \dots + \omega^m \otimes \omega^m;$$

we can write (2.1) as

$$\widehat{q} = q_{\alpha\beta}\omega^{\alpha} \otimes \omega^{\beta}$$

where  $g_{\alpha\beta}$  are the entries of the diagonal matrix with  $g_{00} = -1$ ,  $g_{ii} = 1$  for i = 1, ..., m (no sum over the index i); we denote by  $(g^{-1})^{\alpha\beta} = g^{\alpha\beta} (= g_{\alpha\beta})$  the entries of the inverse matrix of  $g_{\alpha\beta}$  and by  $\{e_{\alpha}\}_{\alpha=0}^{m}$  the frame dual to  $\{\omega^{\alpha}\}_{\alpha=0}^{m}$ . The Levi-Civita connection is defined via the formula

$$\nabla e_{\alpha} = \omega_{\alpha}^{\beta} \otimes e_{\beta},$$

where the Levi-Civita connection forms  $\left\{\omega_{\beta}^{\alpha}\right\}$  are uniquely defined by the requirements

(2.2) 
$$\begin{cases} d\omega^{\alpha} = -\omega^{\alpha}_{\beta} \wedge \omega^{\beta} & \text{(first structure equations),} \\ \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0, \end{cases}$$

with  $\omega_{\alpha\beta} = g_{\alpha\gamma}\omega_{\beta}^{\gamma}$ .

The second structure equations read as

$$(2.3) d\omega_{\beta}^{\alpha} = -\omega_{\gamma}^{\alpha} \wedge \omega_{\beta}^{\gamma} + \Omega_{\beta}^{\alpha},$$

where the 2-forms  $\left\{\Omega_{\beta}^{\alpha}\right\}$  are the *curvature forms* associated to the orthonormal coframe  $\left\{\omega^{\alpha}\right\}_{\alpha=0}^{m}$ . Note that, having defined

$$\Omega_{\alpha\beta} = g_{\alpha\gamma}\Omega_{\beta}^{\gamma}$$

(and thus  $\Omega_{\beta}^{\alpha} = g^{\alpha\gamma}\Omega_{\gamma_{\beta}}$ ), we have

$$\Omega_{\alpha\beta} + \Omega_{\beta\alpha} = 0$$
,

and the components  $\widehat{R}_{\alpha\beta\gamma\delta}$  of the (0,4)-version of the Riemann curvature tensor are given by

(2.4) 
$$\Omega_{\alpha\beta} = \frac{1}{2} \widehat{R}_{\alpha\beta\gamma\delta} \omega^{\gamma} \wedge \omega^{\delta},$$

while the components  $\widehat{R}^{\alpha}_{\beta\gamma\delta}$  of the (1, 3)-version are given by

$$\widehat{R}^{\alpha}_{\beta\gamma\delta} = g^{\alpha\eta} \widehat{R}_{\eta\beta\gamma\delta}.$$

Observe that  $\hat{R}_{\alpha\beta\gamma\delta}$  satisfy the usual symmetries

$$\widehat{R}_{\alpha\beta\gamma\delta} = \widehat{R}_{\gamma\delta\alpha\beta} = -\widehat{R}_{\delta\gamma\alpha\beta} = \widehat{R}_{\delta\gamma\beta\alpha}$$

and so on.

Suppose now that  $(\widehat{M}, \widehat{g}) = (\mathbb{R} \times M, \widehat{g})$ , where  $\widehat{g}$  is the warped product metric

$$\widehat{g} = -e^{-2f}dt^2 + g,$$

where (M,g) is an m-dimensional Riemannian manifold,  $f \in C^{\infty}(M)$  and t is the standard coordinate on  $\mathbb{R}$ . To deduce system (1.3), we explicitly calculate the relation between  $\widehat{\text{Ric}}$  and  $\widehat{\text{Ric}}$ ,  $\tau(\widehat{\varphi})$  and  $\tau(\varphi)$ . We fix the further index range  $1 \leq i, j, \ldots, m$  and we let  $\left\{\theta^i\right\}_{i=1}^m$  be a local orthonormal coframe for g with Levi-Civita connection forms  $\left\{\theta^i_j\right\}$  and curvature forms  $\left\{\Theta^i_j\right\}$ , i, j = 1, ..., m. Note that

(2.6) 
$$\Theta_j^i = \frac{1}{2} R_{jkl}^i \theta^k \wedge \theta^l,$$

where  $R^i_{jkl}$  are the components of the curvature tensor of (M,g); therefore,

(2.7) 
$$\Omega_{\beta}^{\alpha} = \frac{1}{2} \widehat{R}_{\beta\gamma\delta}^{\alpha} \omega^{\gamma} \wedge \omega^{\delta}.$$

We set

(2.8) 
$$\omega^{i} = \theta^{i}, \quad i = 1, ..., m, \quad \omega^{0} = e^{-f}dt;$$

in this way, the forms  $\{\omega^{\alpha}\}_{\alpha=0}^{m}$  so defined give a local orthonormal coframe for the Lorentzian metric  $\hat{g}$  in (2.5). The validity of the first structure equation for g, (2.8) together with  $df = f_k \theta^k$  yield

$$d\omega^{i} = d\theta^{i} = -\theta^{i}_{j} \wedge \theta^{j} = -\omega^{i}_{j} \wedge \omega^{j},$$
  
$$d\omega^{0} = -e^{-f} df \wedge dt = -e^{-f} f_{k} \theta^{k} \wedge dt = -f_{k} \omega^{k} \wedge \omega^{0}.$$

Thus, defining

(2.9) 
$$\omega_i^0 = -\omega_0^i = -f_i \omega^0 = -f_i e^{-f} dt,$$

$$\omega_j^i = \theta_j^i,$$

$$\omega_0^0 = 0,$$

we have that (2.2) are satisfied. It follows that

$$\Omega_j^i = \Theta_j^i,$$
  

$$\Omega_j^0 = (f_{jk} - f_j f_k) \omega^0 \wedge \omega^k;$$

as a consequence, we have

$$\frac{1}{2} \widehat{R}^i_{j\sigma\mu} \omega^\sigma \wedge \omega^\mu = \frac{1}{2} R^i_{jkl} \theta^k \wedge \theta^l$$

and

$$\frac{1}{2}\widehat{R}_{j\sigma\mu}^{0}\omega^{\sigma}\wedge\omega^{\mu}=-(f_{jk}-f_{j}f_{k})\omega^{k}\wedge\omega^{0}.$$

We let

$$\widetilde{a}_{j\sigma} = \begin{cases} 0, & \text{if } \sigma = 0; \\ -(f_{jk} - f_j f_k), & \text{if } \sigma = k. \end{cases}$$

Then

$$\begin{split} \frac{1}{2}\widehat{R}^0_{j\sigma\mu}\omega^\sigma\wedge\omega^\mu &= \widetilde{a}_{j\sigma}\omega^\sigma\wedge\omega^0\\ &= \widetilde{a}_{j\sigma}\delta_{0\mu}\omega^\sigma\wedge\omega^\mu\\ &= \frac{1}{2}(\widetilde{a}_{j\sigma}\delta_{0\mu}-\widetilde{a}_{j\mu}\delta_{0\sigma})\omega^\sigma\wedge\omega^\mu, \end{split}$$

so that

(2.10) 
$$\widehat{R}_{i\sigma\mu}^{0} = \widetilde{a}_{j\sigma}\delta_{0\mu} - \widetilde{a}_{j\mu}\delta_{0\sigma}.$$

In particular, we have

$$\widehat{R}_{j0\mu}^{0} = \widetilde{a}_{j0}\delta_{0\mu} - \widetilde{a}_{j\mu} = -\widetilde{a}_{j\mu}$$

and

(2.11) 
$$\widehat{R}_{j0k}^{0} = -\widetilde{a}_{jk} = f_{jk} - f_{j}f_{k}.$$

Similarly, let

$$\widetilde{R}^{i}_{j\sigma l} = \begin{cases} 0, & \text{if } \sigma = 0, \\ R^{i}_{jkl}, & \text{if } \sigma = k; \end{cases}$$

then

$$\begin{split} \frac{1}{2} \widehat{R}^i_{j\sigma\mu} \omega^\sigma \wedge \omega^\mu &= \frac{1}{2} \widehat{R}^i_{j\sigma l} \omega^\sigma \wedge \omega^l \\ &= \frac{1}{2} \widehat{R}^i_{j\sigma l} \delta_{l\mu} \omega^\sigma \wedge \omega^\mu \\ &= \frac{1}{4} \Big( \widehat{R}^i_{j\sigma l} \delta_{l\mu} - \widehat{R}^i_{j\mu l} \delta_{l\sigma} \Big) \omega^\sigma \wedge \omega^\mu, \end{split}$$

so that

(2.12) 
$$\widehat{R}_{j\sigma\mu}^{i} = \frac{1}{2} \left( \widehat{R}_{j\sigma l}^{i} \delta_{l\mu} - \widehat{R}_{j\mu l}^{i} \delta_{l\sigma} \right)$$

and, in particular,

(2.13) 
$$\widehat{R}^{i}_{jkl} = R^{i}_{jkl}, \qquad \widehat{R}^{i}_{j0l} = 0 = R^{i}_{jl0}.$$

As far as the Ricci tensor is concerned, from equations (2.11) and (2.13) we have, for  $\hat{R}_{\mu\nu} = \hat{R}^{\alpha}_{\mu\alpha\nu}$ ,

$$\widehat{R}_{ij} = f_{ij} - f_i f_j + R_{ij} = \widehat{R}_{ji},$$

$$\widehat{R}_{00} = -\Delta f + |\nabla f|^2,$$

$$\widehat{R}_{i0} = 0 = \widehat{R}_{0i}$$

(here  $|\nabla f|^2 = |\nabla f|_q^2$ ). Since

$$\widehat{\mathrm{Ric}} = \widehat{R}_{00}\omega^0 \otimes \omega^0 + \widehat{R}_{ij}\omega^i \otimes \omega^j,$$

we deduce

(2.14) 
$$\widehat{\mathrm{Ric}} = \mathrm{Ric} + \mathrm{Hess}(f) - df \otimes df - \left(\Delta f - |\nabla f|^2\right) e^{-2f} dt \otimes dt.$$

Let us now consider a Riemannian manifold (N,h) of dimension n, and let  $\varphi:(M,g)\to (N,h)$  be a smooth map; let  $\widehat{\varphi}:=\varphi\circ\pi_M:(\widehat{M},\widehat{g})\to (N,h)$ , where  $\pi_M$  is the projection of  $\widehat{M}$  on M. Fix a local orthonormal coframe  $\{\eta^a\}_{a=1}^n$  on N, with dual frame  $\{E_a\}$ , and let  $\{\eta^a_b\}$ , a,b=1,...,n, be the corresponding Levi-Civita connection forms. We have

$$d\varphi = \varphi_* = \varphi_i^a \theta^i \otimes E_a$$
 and  $d\widehat{\varphi} = \widehat{\varphi}_* = \widehat{\varphi}_{\nu}^a \omega^{\nu} \otimes E_a$ ,

but since

$$d\widehat{\varphi} = \varphi_* \circ (\pi_M)_* = d\varphi \circ (\pi_M)_*,$$

a simple computation shows that

(2.15) 
$$\widehat{\varphi}_0^a = 0, \quad \widehat{\varphi}_i^a = \varphi_i^a,$$

that is

$$d\widehat{\varphi} = d\varphi$$
.

We recall that, by definition (and omitting the pullback notation for simplicity),

(2.16) 
$$\widehat{\varphi}^a_{\mu\nu}\omega^{\nu} = d\widehat{\varphi}^a_{\mu} - \widehat{\varphi}^a_{\nu}\omega^{\nu}_{\mu} + \widehat{\varphi}^b_{\mu}\eta^a_b;$$

therefore, using (2.15) and simplifying we obtain

$$\widehat{\varphi}_{i\nu}^a \omega^\nu = \varphi_{ij}^a \omega^j = \varphi_{ij}^a \theta^j,$$

which implies

(2.17) 
$$\widehat{\varphi}_{ij}^a = \varphi_{ij}^a \quad \text{and} \quad \widehat{\varphi}_{i0}^a = 0.$$

To compute the coefficients  $\widehat{\varphi}_{00}^a$ , note that, again from (2.16), we have

$$\widehat{\varphi}_{0\nu}^a\omega^\nu=d\widehat{\varphi}_0^a-\widehat{\varphi}_\nu^a\omega_0^\nu+\widehat{\varphi}_0^b\eta_b^a=-\varphi_k^a\omega_0^k=\varphi_k^af_k\omega^0,$$

and thus

$$\widehat{\varphi}_{00}^a = f_k \varphi_k^a,$$

which immediately implies

(2.19) 
$$\tau(\widehat{\varphi}) = \tau(\varphi) - d\varphi(\nabla f).$$

Note that, when  $f = -\log u$  for some  $u \in C^{\infty}(M)$ , u > 0, equations (2.14) and (2.19) become

(2.20) 
$$\widehat{\mathrm{Ric}} = \mathrm{Ric} - \frac{\mathrm{Hess}(u)}{u} + u\Delta u \, dt \otimes dt$$

and

(2.21) 
$$\tau(\widehat{\varphi}) = \tau(\varphi) + \frac{1}{u} d\varphi(\nabla u).$$

Moreover, contracting (2.20), we get

$$\widehat{S} = S - 2\frac{\Delta u}{u}.$$

A  $\varphi$ -static perfect fluid space-time,  $\varphi$ -SPFST for short, is characterized as a solution of the Einstein equations

(2.23) 
$$\widehat{\mathrm{Ric}} - \frac{\widehat{S}}{2}\widehat{g} = \widehat{T},$$

where  $\widehat{T}$  is the stress-energy tensor defined as the sum of the stress-energy tensor of a static perfect fluid  $\widehat{T}_F$  with energy density  $\mu \in C^{\infty}(M)$  and pressure  $p \in C^{\infty}(M)$  with the stress-energy tensor  $\widehat{T}_{\widehat{\varphi}}$  relative to the static "non-linear field"  $\widehat{\varphi}: (\widehat{M}, \widehat{g}) \to (N, h)$ , interacting with the potential  $U \in C^{\infty}(N)$ . To define  $\widehat{T}_F$ , we let  $v = \frac{1}{u} \frac{\partial}{\partial t}$  and set  $v^{\flat}$  for its dual form; note that v is a unit, future directed timelike vector field. Then,  $\widehat{T}_F$  is phenomenologically defined by

(2.24) 
$$\widehat{T}_F = [(\mu + p) \circ \pi_M] v^{\flat} \otimes v^{\flat} + [p \circ \pi_M] \widehat{g}$$

(see [21],[22],[41],[43],[48]), while  $\widehat{T}_{\widehat{\varphi}}$  is given by the prescription

(2.25) 
$$\widehat{T}_{\widehat{\varphi}} = \alpha \widehat{\varphi}^* h - \left( U(\widehat{\varphi}) + \alpha \frac{|d\widehat{\varphi}|^2}{2} \right) \widehat{g}$$

(see [48]).

Hence, combining the two expressions, observing that  $\widehat{\varphi}^*h = \varphi^*h$  and that  $|d\widehat{\varphi}|^2 = |d\varphi|^2$  and omitting the compositions with  $\pi_M$  to simplify the writing, we obtain

$$(2.26) \qquad \widehat{T} = \left(\mu + \alpha \frac{\left|d\varphi\right|^2}{2} + U(\varphi)\right) v^{\flat} \otimes v^{\flat} + \left(p - \alpha \frac{\left|d\varphi\right|^2}{2} - U(\varphi)\right) g + \alpha \varphi^* h.$$

Remark 2.1. The stress-energy tensor of a SPFST and that of a non-linear field can be both obtained variationally. However, the approach used in the two cases is quite different: indeed, as remarked in [41], in the first case one has to take the variation of an appropriate functional depending on  $\mu$  with respect to the flow lines of the tangent vector field v, while in the second case it is sufficient to compute the variation of a functional depending on  $\widehat{\varphi}$  and the metric  $\widehat{g}$ , with respect to  $\widehat{\varphi}$  and the components of the metric (see [48] for more details).

We are now ready to show how (1.3) is deduced; for the convenience of the reader, we recall here the system:

(2.27) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \left\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) g \right\} = 0, \\ ii) \Delta u = \frac{u}{m-1} \left[ mp - mU(\varphi) + \frac{m-2}{2} S^{\varphi} \right], \\ iii) u\tau(\varphi) = -d\varphi(\nabla u) + \frac{u}{\alpha} (\nabla U)(\varphi), \\ iv) \mu + U(\varphi) = \frac{1}{2} S^{\varphi}, \\ v) (\mu + p) \nabla u = -u \nabla p. \end{cases}$$

First, note that the splitting of  $\widehat{T}$  in (2.26) is with respect to the time and the space components. Inserting (2.20), (2.22) and  $\widehat{g} = g - v^{\flat} \otimes v^{\flat}$  into the definition of  $\widehat{T}$ , we have

$$\widehat{T} = \widehat{\text{Ric}} - \frac{\widehat{S}}{2} \widehat{g}$$

$$= \text{Ric} - \frac{\text{Hess}(u)}{u} - \left(\frac{S}{2} - \frac{\Delta u}{u}\right) g + \frac{S}{2} v^{\flat} \otimes v^{\flat}$$

$$(2.28) \quad = \text{Ric} - \frac{\text{Hess}(u)}{u} - \left(\frac{S^{\varphi}}{2} + \alpha \frac{|d\varphi|^{2}}{2} - \frac{\Delta u}{u}\right) g + \left(\frac{S^{\varphi}}{2} + \alpha \frac{|d\varphi|^{2}}{2}\right) v^{\flat} \otimes v^{\flat}.$$

Comparing space and time components of (2.28) and (2.26), we infer equation (1.3) iv), that is,

$$\mu + U(\varphi) = \frac{S^{\varphi}}{2}$$

and

(2.29) 
$$\operatorname{Ric} - \alpha \varphi^* h - \frac{\operatorname{Hess}(u)}{u} - \frac{S^{\varphi}}{2} g + \frac{\Delta u}{u} g = (p - U(\varphi)) g.$$

Taking the trace of (2.29) yields

(2.30) 
$$m(p - U(\varphi)) = (2 - m)\frac{S^{\varphi}}{2} + (m - 1)\frac{\Delta u}{u},$$

that immediately gives equation (1.3) ii). Then, replacing (2.30) into (2.29), we obtain

(2.31) 
$$\operatorname{Ric} - \alpha \varphi^* h - \frac{\operatorname{Hess}(u)}{u} = \frac{1}{m} \left( S^{\varphi} - \frac{\Delta u}{u} \right) g.$$

Substituting the expression for  $\Delta u$ , obtained in (1.3) ii), into (2.31) we easily get equation i) of (1.3). Equation iii) and v) of system (1.3) are a consequence of the energy-momentum conservation,  $\operatorname{div}_{\widehat{g}}\widehat{T} = 0$ : indeed, a computation using the fact that

$$\frac{\partial \mu}{\partial t} = \frac{\partial p}{\partial t} = 0$$

shows the validity of the equations of motion

(2.32) 
$$(\mu + p) \frac{\nabla u}{u} + \nabla [p - U(\varphi)] + \alpha [h(\tau(\widehat{\varphi}), d\widehat{\varphi})]^{\sharp} = 0.$$

Since  $\tau(\widehat{\varphi})$  is related to the tension field  $\tau(\varphi)$  by (2.21), from (2.32) we have

(2.33) 
$$\alpha[h(u\tau(\varphi) + d\varphi(\nabla u), d\varphi)]^{\sharp} + (\mu + p)\nabla u = u\nabla[U(\varphi) - p].$$

**Remark** 2.2. Note that equation (2.33) is weaker than the pair of equations 1.3 iii) and 1.3 v), that is,

(2.34) 
$$u\tau(\varphi) = -d\varphi(\nabla u) + \frac{u}{\alpha}(\nabla U)(\varphi), \qquad (\mu + p)\nabla u = -u\nabla p.$$

We can now give the following

**Definition** 2.3. We say that  $(\widehat{M}, \widehat{g})$  is a  $\varphi$ -static perfect fluid space-time, or  $\varphi$ -SPFST, if (M, g) is an m-dimensional Riemannian manifold with a smooth solution u of the system (1.3), that is

(2.35) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \Big\{ \operatorname{Ric} - \alpha \varphi^* h - \frac{1}{m-1} \Big( \frac{S - |d\varphi|^2}{2} - p + U(\varphi) \Big) g \Big\} = 0, \\ ii) \Delta u = \frac{u}{m-1} \Big[ -mU(\varphi) + mp + \frac{m-2}{2} (S - \alpha |d\varphi|^2) \Big], \\ iii) u\tau(\varphi) = -d\varphi(\nabla u) + \frac{u}{\alpha} (\nabla U)(\varphi), \\ iv) \mu + U(\varphi) = \frac{S - \alpha |d\varphi|^2}{2}, \\ v) (\mu + p) \nabla u = -u \nabla p. \end{cases}$$

We also require u > 0 on int(M) and  $\partial M = u^{-1}(\{0\})$ , in case  $\partial M \neq \emptyset$ .

Throughout this paper, we will denote the spatial factor of a  $\varphi$ -SPFST simply as  $\varphi$ -SPFST for simplicity, although this is a slight abuse of terminology. Observe also that when  $\varphi$  is constant and  $U(\varphi) \equiv 0$ , system (1.3) reduces to (1.1), that is the one characterizing a static perfect fluid space time. Moreover, as we shall see in Lemma 3.3, equation (2.35) v) does not add further information to the other equations of the system.

### 2.2. Energy Conditions

As we pointed out in the introduction, some of our assumptions are expressed in terms of energy conditions. Some of the latter are discussed in detail in [41], for the case  $\dim \widehat{M} = 4$ , and have been recently extended to any dimension in [48]. However, the General Relativity model cases they consider are different from our, so that we feel obliged to give a short treatment to justify our assumptions.

It is a general fact that in many theories of gravity the distribution of the sources of the gravitational field is encoded in the stress-energy tensor  $\widehat{T}$ ; natural assumptions on these sources that are expected to be satisfied by any reasonable physical model of space-time are related to the general principle of "positivity of the energy" and they translate into requests on  $\widehat{T}$  called *energy conditions*. We first recall that, in Lorentzian Geometry, with respect to a metric  $\widehat{g}$  a vector w is said to be:

• a null vector, if  $\widehat{q}(w, w) = 0$ ;

- a time-like vector, if  $\widehat{g}(w,w) < 0$ ;
- a space-like vector, if  $\widehat{g}(w, w) > 0$ .

If  $(\widehat{M}, \widehat{g})$  is an (m+1)-dimensional Lorentzian manifold, the following is a list of the most standard conditions in the classical literature:

1. the Null Energy Condition (NEC) is satisfied if, for any null vector w, we have

$$\widehat{T}(w,w) \ge 0.$$

2. The Weak Energy Condition (WEC) is satisfied if, for any time-like vector w, we have

$$\widehat{T}(w, w) \ge 0.$$

3. The Strong Energy Condition (SEC) is satisfied if, for any time-like vector w, we have

$$\widehat{T}(w,w) \ge \frac{1}{m-1} (\operatorname{tr}_{\widehat{g}}\widehat{T})\widehat{g}(w,w),$$

where

$$\operatorname{tr}_{\widehat{g}}\widehat{T} = g^{\alpha\beta}\widehat{T}_{\alpha\beta}.$$

4. The Flux Energy Condition (**FEC**) is satisfied if, for any time-like vector w, the flux vector

$$J_w := -\widehat{T}(w, \cdot)^{\#}$$

satisfies

$$\widehat{g}(J_w, J_w) \leq 0.$$

5. The *Dominant Energy Condition* (**DEC**) is satisfied if, for any time-like vector w, we have

$$\widehat{T}(w,w) \ge 0$$
 and  $\widehat{g}(J_w,J_w) \le 0$ .

For some physical and geometrical interpretations of the above energy conditions see, for instance, [23]. Here we only observe that:

- i) The WEC is saying that any time-like observer measures a non-negative energy density.
- ii) The NEC states the same fact for null-observers.
- iii) The SEC is more enlightening if we assume the validity of Einstein field equations

$$\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{S}\widehat{g} = \widehat{T};$$

indeed, in this case the SEC is equivalent to

$$\widehat{\mathrm{Ric}}(w,w) \ge 0$$

for each time-like vector w. This latter fact is interpreted as gravity being "essentially" an attractive force.

iv) The FEC states that every flux vector does not propagate faster than the speed of light.

In the next Proposition we analyze the above energy conditions for the stress-energy tensor  $\hat{T}$  of a  $\varphi$ -SPFST, given, as we showed before, by

$$(2.36) \qquad \widehat{T} = (\mu + p)u^2 dt \otimes dt + p\widehat{g} + \alpha \widehat{\varphi}^* h - \frac{1}{2} (\alpha |d\widehat{\varphi}|^2 + 2U(\widehat{\varphi}))\widehat{g},$$

on the Lorentzian warped product  $\widehat{M} = \mathbb{R}_n \times M$  with metric

$$\widehat{g} = -u^2 dt \otimes dt + g,$$

u > 0 on M.

**Proposition** 2.4. For the stress-energy tensor  $\widehat{T}$  as in (2.36), with  $\alpha > 0$ , we have the following properties:

1. the condition

$$p + \mu \ge 0$$

implies the validity of the NEC;

2. the conditions

$$p + \mu \ge 0$$
 and  $\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi) \ge 0$ 

imply the validity of the WEC;

3. the conditions

$$p + \mu \ge 0$$
 and  $(m-2)\mu + mp \ge 2U(\varphi)$ 

imply the validity of the SEC;

4. the conditions

$$U(\varphi) \ge p$$
 and  $\left(\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi)\right)^2 \ge \left(p - \frac{\alpha}{2}|d\varphi|^2 - U(\varphi)\right)^2$ 

imply the validity of the FEC;

5. the conditions

$$U(\varphi) \geq p \; , \quad \mu + p \geq 0 \quad and \quad \mu + \frac{\alpha}{2} |d\varphi|^2 + U(\varphi) \geq 0$$

imply the validity of the DEC.

Remark 2.5. In Proposition 2.4 we claim that certain hypotheses are sufficient to guarantee the validity of the corresponding energy conditions. For the case of SPFSTs, which is obtained taking  $\varphi$  constant and  $U \equiv 0$ , the hypotheses we give reduce to the classical ones and in this case we know that not only they are sufficient conditions, but also necessary. In our setting, we can prove that

$$\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi) \ge 0$$

is necessary for the validity of the DEC and the WEC, while

$$(m-2)\mu + mp \ge 2U(\varphi)$$

is necessary for the validity of the SEC.

The proof of Proposition 2.4 will be divided in several lemmas. To set the stage, recall that, using the expression for  $\hat{g}$  given in (2.37) (and simplifying again the notation), we can rewrite  $\hat{T}$  as

(2.38) 
$$\widehat{T} = \left(\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi)\right)u^2dt \otimes dt + \left(p - \frac{\alpha}{2}|d\varphi|^2 - U(\varphi)\right)g + \alpha\varphi^*h.$$

Let  $\{e_i\}_{i=1}^m$  be a local orthonormal frame for g and let  $e_0 = \frac{1}{u} \frac{\partial}{\partial t}$ . For a time-like vector w we have that, up to a multiplication for a scalar, we can write w as

$$(2.39) w = e_0 + w^i e_i$$

(sum over i, from 1 to m), for some coefficients  $w^{i}$ 's satisfying

$$\sum_{i=1}^{m} (w^i)^2 < 1.$$

In the next lemma we collect some easy but useful calculations.

Lemma 2.6. With the notations above, we have

(2.40) 
$$\widehat{T}(w,w) = \left(1 - \sum_{i=1}^{m} (w^{i})^{2}\right) \left[\mu + \frac{\alpha}{2} |d\varphi|^{2} + U(\varphi)\right] + \left(\sum_{i=1}^{m} (w^{i})^{2}\right) (\mu + p) + \alpha(\varphi^{*}h)(w,w),$$

and

$$\left(\widehat{T} - \frac{\operatorname{tr}_{\widehat{g}}\widehat{T}}{m-1}\widehat{g}\right)(w,w) = \left(1 - \sum_{i=1}^{m} (w^{i})^{2}\right) \left[\left(\frac{m-2}{m-1}\right)\mu + \frac{m}{m-1}p - \frac{2}{m-1}U(\varphi)\right] + \left(\sum_{i=1}^{m} (w^{i})^{2}\right)(\mu+p) + \alpha(\varphi^{*}h)(w,w).$$

PROOF. Equation (2.40) is a direct consequence of (2.38) and (2.39); indeed,

$$\widehat{T}(w,w) = \mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi) + \left(\sum_{i=1}^m (w^i)^2\right) \left[p - \frac{\alpha}{2}|d\varphi|^2 - U(\varphi)\right]$$

$$+ \alpha(\varphi^*h)(w,w)$$

$$= \left(1 - \sum_{i=1}^m (w^i)^2\right) \left[\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi)\right] + \left(\sum_{i=1}^m (w^i)^2\right) (p+\mu)$$

$$+ \alpha(\varphi^*h)(w,w).$$

To prove (2.41), we first show the validity of the relation

(2.42) 
$$\widehat{T} - \frac{\operatorname{tr}_{\widehat{g}}\widehat{T}}{m-1}\widehat{g} = \frac{u^2}{m-1} \left[ (m-2)\mu + mp - 2U(\varphi) \right] dt \otimes dt + \frac{1}{m-1} \left[ -p + \mu + 2U(\varphi) \right] g + \alpha \varphi^* h.$$

To prove it, observe that from (2.36) it follows

$$\operatorname{tr}_{\widehat{g}}\widehat{T} = -\mu + mp - \alpha \frac{m-1}{2} |d\varphi|^2 - (m+1)U(\varphi),$$

and thus

$$\begin{split} \widehat{T} - \frac{\operatorname{tr}_{\widehat{g}} \widehat{T}}{m-1} \widehat{g} &= \frac{u^2}{m-1} \left[ (m-1)\mu + \alpha \frac{m-1}{2} |d\varphi|^2 + (m-1)U(\varphi) - \mu + mp \right. \\ &- \alpha \frac{m-1}{2} |d\varphi|^2 - (m+1)U(\varphi) \right] dt \otimes dt \\ &+ \frac{1}{m-1} \left[ (m-1)p - \alpha \frac{m-1}{2} |d\varphi|^2 - (m-1)U(\varphi) + \mu - mp \right. \\ &+ \alpha \frac{m-1}{2} |d\varphi|^2 + (m+1)U(\varphi) \right] g + \alpha \varphi^* h \\ &= \frac{u^2}{m-1} [(m-2)\mu + mp - 2U(\varphi)] dt \otimes dt + \frac{1}{m-1} [-p + \mu + 2U(\varphi)] g \\ &+ \alpha \varphi^* h, \end{split}$$

which gives (2.42). Equation (2.41) is now an immediate consequence of (2.42).  $\square$ 

For a time-like vector w, a direct computation shows that the flux vector  $J_w$  satisfies

$$J_w = -\widehat{T}(w,\cdot)^{\#} = \left(\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi)\right)e_0$$
$$-\left(p - \frac{\alpha}{2}|d\varphi|^2 - U(\varphi)\right)\left(\sum_{i=1}^m w^i e_i\right) - \alpha(\varphi^*h)(w,\cdot)^{\#}.$$

Moreover, since  $\widehat{\varphi}^* h = \varphi^* h$ ,

$$(\varphi^*h)(w,\cdot)^\# = \sum_{i,j=1}^m w^j \varphi_j^a \varphi_i^a e_i,$$

thus

(2.43) 
$$J_w = \left(\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi)\right)e_0$$
$$-\left(p - \frac{\alpha}{2}|d\varphi|^2 - U(\varphi)\right)\sum_{i=1}^m w^i e_i - \alpha \sum_{i,j=1}^m w^j \varphi_j^a \varphi_i^a e_i.$$

**Lemma** 2.7. With the previous notations we have (2.44)

$$\widehat{g}(J_w, J_w) = -\left(1 - \sum_{i=1}^m (w^i)^2\right) \left(\mu + \frac{\alpha}{2} |d\varphi|^2 + U(\varphi)\right)^2$$

$$+ \left(\sum_{i=1}^m (w^i)^2\right) \left[\left(p - \frac{\alpha}{2} |d\varphi|^2 - U(\varphi)\right)^2 - \left(\mu + \frac{\alpha}{2} |d\varphi|^2 + U(\varphi)\right)^2\right]$$

$$+ \alpha^2 \sum_{i,j,k=1}^m w^i \varphi_i^a \varphi_j^a \varphi_j^b \varphi_k^b a_k$$

$$+ 2\alpha \left(p - \frac{\alpha}{2} |d\varphi|^2 - U(\varphi)\right) \sum_{i,j=1}^m w^i \varphi_i^a \varphi_j^a a_j.$$

PROOF. The proof is a direct consequence of equation (2.43).

We will also need the following elementary lemma.

**Lemma** 2.8. Let A be a  $n \times n$  real, symmetric, positive semi-definite matrix. Then

$$\langle Au, u \rangle \le (\operatorname{tr} A) \langle u, u \rangle$$
,

where  $u \in \mathbb{R}^n$  and  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product.

PROOF. Since A is positive semi-definite we have

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 \ge \sum_{i=1}^{n} \lambda_i^2$$

where the  $\lambda_i$ 's are the eigenvalues of A. If we let ||A|| denote the operator norm of A, we deduce

$$\operatorname{tr} A \ge ||A|| = \sqrt{\sum_{i=1}^{n} \lambda_i^2}.$$

Since

$$\langle Au, u \rangle \leq ||A|| \langle u, u \rangle$$
,

the conclusion follows.

We are now ready for the proof of Proposition 2.4.

PROOF (OF PROPOSITION 2.4). The WEC follows from equation (2.40) once we assume, as we are doing, that  $\alpha > 0$ . Indeed,  $\varphi^*h$  is positive semi-definite, so that from (2.40) we deduce

$$\widehat{T}(w,w) \ge \left(1 - \sum_{i=1}^m a_i^2\right) \left[\mu + \frac{\alpha}{2} |d\varphi|^2 + U(\varphi)\right] + \left(\sum_{i=1}^m a_i^2\right) (p+\mu).$$

If in this expression we allow  $\sum_{i=1}^{m} (w^{i})^{2} = 1$ , then w becomes a null vector, and we also obtain the validity of the NEC. Reasoning as above, the SEC is implied by equation (2.41). To deduce the FEC from Lemma 2.7, we apply Lemma 2.8 to the matrix A of entries

$$A_{ab} = \sum_{i=1}^{m} \varphi_i^a \varphi_i^b$$

and the vector u of components

$$u^a = \sum_{i=1}^m w^i \varphi_i^a$$

to conclude, since  $trA = |d\varphi|^2$ ,

$$\sum_{i,j,k=1}^m w^i \varphi_i^a \varphi_j^a \varphi_j^b \varphi_k^b a_k \le |d\varphi|^2 \sum_{i,j=1} w^i \varphi_i^a \varphi_j^a a_j.$$

Inserting this information into (2.44) we obtain

$$\widehat{g}(J_w, J_w) \le -\left(1 - \sum_{i=1}^m (w^i)^2\right) \left(\mu + \frac{\alpha}{2} |d\varphi|^2 + U(\varphi)\right)^2$$

$$+ \left(\sum_{i=1}^m (w^i)^2\right) \left[\left(p - \frac{\alpha}{2} |d\varphi|^2 - U(\varphi)\right)^2 - \left(\mu + \frac{\alpha}{2} |d\varphi|^2 + U(\varphi)\right)^2\right]$$

$$+ 2\alpha(p - U(\varphi)) \sum_{i,j=1}^m w^i \varphi_i^a \varphi_j^a a_j.$$

which gives the FEC. Now note that DEC is satisfied exactly when the FEC and the WEC are satisfied simultaneously; from what we previously proved we deduce that DEC is implied by the validity of the following four conditions:

- I)  $\mu + p \ge 0$ ;
- II)  $\mu + \frac{\alpha}{2} |d\varphi|^2 + U(\varphi) \ge 0;$
- III)  $U(\varphi) \ge p$ ;

IV) 
$$\left(\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi)\right)^2 \ge \left(p - \frac{\alpha}{2}|d\varphi|^2 - U(\varphi)\right)^2$$
.

Then, to conclude, it is sufficient to prove that condition IV) is redundant. Indeed, computing the squares and simplifying we get

$$\mu^2 + \alpha \mu |d\varphi|^2 + 2\mu U(\varphi) \ge p^2 - \alpha p |d\varphi|^2 - 2pU(\varphi)$$

which is equivalent to

$$\begin{split} 0 & \leq \mu^2 - p^2 + (\mu + p) \left[ \alpha |d\varphi|^2 + 2U(\varphi) \right] \\ & = (\mu + p)(\mu - p) + (\mu + p) \left[ \alpha |d\varphi|^2 + 2U(\varphi) \right] \\ & = (\mu + p) \left[ \mu - p + \alpha |d\varphi|^2 + 2U(\varphi) \right]. \end{split}$$

Therefore, from I), we need to show that

$$\mu - p + \alpha |d\varphi|^2 + 2U(\varphi) \ge 0;$$

since  $U(\varphi) \ge p$  from III), we only need  $\mu + \alpha |d\varphi|^2 + U(p) \ge 0$ , which is implied by II).

**Remark** 2.9. As observed in Remark 2.5, we prove here the necessity of

$$\mu + \frac{\alpha}{2}|d\varphi|^2 + U(\varphi) \ge 0$$

and

$$(m-2)\mu + mp \ge 2U(\varphi)$$

for the respective energy conditions. Indeed, taking  $w^i = 0$ ,  $\forall i = 1, ..., m$  in the expression (2.39) of w, we deduce

$$\varphi^*h(w,w) = \varphi_0^a \varphi_0^a = 0,$$

so that, choosing  $w^i = 0$ ,  $\forall i = 1,...,m$  in (2.40) and (2.41), we conclude.

#### 2.3. $\varphi$ -Curvatures

As we have seen in Section 2.1 of this chapter, Einstein field equations transform, in our setting, into system (2.35). As we shall see, it is worth to encode some information on the non-linear field  $\varphi$  into the curvature tensors of (M, g), in order to see at the same time both the combined action of  $\varphi$  and that of the Riemannian metric g of M. With these motivations (others can be found in [3], [20], [50]) we introduce modified curvature tensors depending on the map  $\varphi: (M, g) \to (N, h)$ .

The first step in this direction, that is the definition of the  $\varphi$ -Ricci tensor, is due to B. List, that merged the Ricci flow with the harmonic map flow (for more details and background see [47]).

For some fixed coupling constant  $\alpha \neq 0$ , we set

(2.45) 
$$\operatorname{Ric}^{\varphi} = \operatorname{Ric} - \alpha \varphi^* h$$

for the  $\varphi$ -Ricci tensor, and the  $\varphi$ -scalar curvature will be its contraction with the metric g, that is

$$(2.46) S^{\varphi} = S - \alpha |d\varphi|^2.$$

In components we have

$$R_{ij}^{\varphi} = R_{ij} - \alpha \varphi_i^a \varphi_j^a$$

(note that List uses the notation  $S_{ij}$  instead of  $R_{ij}^{\varphi}$ ). The next formula, the  $\varphi$ Schur's identity will be repeatedly used in the sequel:

$$R_{ij,i}^{\varphi} = \frac{1}{2} S_j^{\varphi} - \alpha \varphi_{tt}^a \varphi_j^a$$

(see [3] for a simple proof). Observe that it is immediate from here to show that, for  $m \geq 3$ ,  $\Lambda$  as in (1.6) of Definition 1.1 is indeed constant. In fact, for

$$Ric^{\varphi} = \Lambda g$$
,

with the usual procedure we obtain

$$(m-2)\nabla\Lambda = 2\alpha h(\tau(\varphi), d\varphi)$$

so that, for  $m \geq 3$ ,  $\Lambda$  is constant any time  $\varphi$  is conservative.

By analogy with the classical case, we define the  $\varphi$ -Schouten tensor by setting

(2.47) 
$$A^{\varphi} = \operatorname{Ric}^{\varphi} - \frac{S^{\varphi}}{2(m-1)}g,$$

while the  $\varphi$ -Cotton tensor  $C^{\varphi}$  is just the obstruction to  $A^{\varphi}$  being Codazzi (see equation (2.50) below). The aforementioned tensors have been used in the statements of the results reported in the Introduction; note that, with the above definitions, system (2.35) can be written in the form (1.3). We are now going to introduce some more  $\varphi$ -curvature tensors that will be used later on and whose formal properties will speed up computations.

The  $\varphi$ -Weyl tensor  $W^{\varphi}$  is defined to formally respect the usual decomposition of the Riemann curvature tensor, that is,

$$W^{\varphi} := \operatorname{Riem} - \frac{1}{m-2} A^{\varphi} \otimes g$$

$$= \operatorname{Riem} - \frac{1}{m-2} \operatorname{Ric}^{\varphi} \otimes g + \frac{S^{\varphi}}{2(m-1)(m-2)} g \otimes g.$$
(2.48)

Although  $W^{\varphi}$  has the same algebraic symmetries of Riem, it is not totally trace-free. Indeed, a computation in local orthonormal coframes (on M and on N) shows that

$$(2.49) W_{kikj}^{\varphi} = \alpha(\varphi^*h)_{ij} = \alpha\varphi_i^a\varphi_j^a = W_{ikjk}^{\varphi},$$

(here, and in the rest of the paper, we fix the indices ranges  $1 \leq i, j, \ldots \leq m = \dim M$  and  $1 \leq a, b, \ldots \leq n = \dim N$ ), while the remaining traces are identically zero.

Note that, in terms of the classical counterparts and of  $\varphi$ , we have

$$A^{\varphi} = A - \alpha A(\varphi^* h),$$
 
$$W^{\varphi} = W + \frac{\alpha}{m-2} A(\varphi^* h) \otimes g,$$

where

$$A(\varphi^*h) = \varphi^*h - \frac{|d\varphi|^2}{2(m-1)}g$$

is the "Schouten tensor" of the symmetric 2-covariant tensor  $\varphi^*h$ . For the  $\varphi$ -Cotton tensor  $C^{\varphi}$ , since

$$(2.50) C_{ijk}^{\varphi} = A_{ij,k}^{\varphi} - A_{ik,j}^{\varphi},$$

we deduce

$$C_{ijk}^{\varphi} = C_{ijk} - \alpha \left[ \varphi_{ik}^a \varphi_j^a - \varphi_{ij}^a \varphi_k^a - \frac{\varphi_t^a}{m-1} \left( \varphi_{tk}^a \delta_{ij} - \varphi_{tj}^a \delta_{ik} \right) \right].$$

 $C^{\varphi}$  has the same symmetries of C: indeed,

$$C_{ijk}^{\varphi} = -C_{ikj}^{\varphi}$$
 and therefore  $C_{ijj}^{\varphi} = 0$ ;

however, it is not totally trace-free, since

$$C_{kki}^{\varphi} = \alpha \varphi_{kk}^{a} \varphi_{i}^{a}.$$

Note that  $C^{\varphi}$  also satisfies the "Bianchi identity"

$$C_{ijk}^{\varphi} + C_{kij}^{\varphi} + C_{jki}^{\varphi} = 0,$$

as one immediately verifies.

The following alternative definition of the  $\varphi$ -Cotton tensor, for  $m \geq 4$ , points out at deep differences between the classical and the  $\varphi$ -curvatures (see Proposition 2.64 of [3]):

(2.51)

$$-\bigg(\frac{m-3}{m-2}\bigg)C^{\varphi}_{jkt} = W^{\varphi}_{sjkt,s} - \alpha \big(\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k\big) - \frac{\alpha}{m-2}\varphi^a_{ss}(\varphi^a_k\delta_{jt} - \varphi^a_t\delta_{jk}).$$

Observe that the above formula reduces to the classical one, that is,

$$W_{tijk,t} = -\left(\frac{m-3}{m-2}\right)C_{ijk}$$

in case  $\varphi$  is constant and to

$$W_{tijk,t}^{\varphi} = -\left(\frac{m-3}{m-2}\right)C_{ijk}^{\varphi}$$

for  $\varphi$  conservative (see the discussion after Definition 1.3) and  $\varphi^*h$  Codazzi. In both cases, it follows immediately the validity of

$$W_{tijk,tkji}^{\varphi} = -\left(\frac{m-3}{m-2}\right)C_{ijk,kji}^{\varphi}.$$

In the general case we have the following

**Proposition** 2.10. Let (M,g) be a Riemannian manifold of dimension  $m \ge 2$  and  $\varphi: (M,g) \to (N,h)$  be a smooth map. Then

(2.52)

$$W_{tijk,tkji}^{\varphi} = -\left(\frac{m-3}{m-2}\right)C_{ijk,kji}^{\varphi} + \alpha \left\{R_{tijk}\varphi_j^a\varphi_{tk}^a\right\}_i + \frac{1}{m-2} \left\{\left(R_{si}^{\varphi} + \alpha\varphi_s^b\varphi_i^b\right)C_{tts}^{\varphi}\right\}_i.$$

PROOF. We start from equation (2.51). We set

$$E_{ijk} = \alpha \varphi_{ij}^a \varphi_k^a - \alpha \varphi_{ik}^a \varphi_j^a + \frac{\alpha}{m-2} \varphi_{tt}^a (\varphi_j^a \delta_{ik} - \varphi_k^a \delta_{ij});$$

note that  $E_{ijk} = -E_{ikj}$ .

Then, equation (2.51) becomes

$$W_{tijk,t}^{\varphi} = -\frac{m-3}{m-2}C_{ijk}^{\varphi} + E_{ijk}.$$

Taking the divergence of the previous equation with respect to k and then with respect to j we get

$$(2.53) W_{tijk,tkj}^{\varphi} = -\left(\frac{m-3}{m-2}\right)C_{ijk,kj}^{\varphi} + E_{ijk,kj}.$$

Using the skew symmetry of  $E_{ijk}$  and the Ricci commutation relations we deduce

$$\begin{split} E_{ijk,kj} &= \frac{1}{2} (E_{ijk,kj} - E_{ikj,kj}) = \frac{1}{2} (E_{ijk,kj} - E_{ijk,jk}) \\ &= \frac{1}{2} R_{pikj} E_{pjk} + \frac{1}{2} R_{pjkj} E_{ipk} + \frac{1}{2} R_{pkkj} E_{ijp} \\ &= \frac{1}{2} R_{pikj} E_{pjk} + \frac{1}{2} R_{pk} E_{ipk} - \frac{1}{2} R_{pj} E_{ijp} \\ &= \frac{1}{2} R_{pikj} E_{pjk}; \end{split}$$

from the definition of E and the symmetries of the Riemann curvature tensor, a simple computation gives

$$E_{ijk,kj} = \alpha R_{pikj} \varphi_{pj}^a \varphi_k^a + \frac{\alpha}{m-2} \varphi_{tt}^a R_{ij} \varphi_j^a.$$

Using the definition of  $Ric^{\varphi}$  and formula

$$(2.54) C_{ttk}^{\varphi} = \alpha \varphi_{tt}^{a} \varphi_{k}^{a}$$

we obtain

$$E_{ijk,kj} = \alpha R_{pikj} \varphi_{pj}^a \varphi_k^a + \frac{\alpha}{m-2} \varphi_{tt}^a \left( R_{ij}^{\varphi} \varphi_j^a + \alpha \varphi_i^b \varphi_j^b \varphi_j^a \right)$$
$$= \alpha R_{pikj} \varphi_{pj}^a \varphi_k^a + \frac{C_{ppj}^{\varphi}}{m-2} \varphi_{tt}^a \left( R_{ij}^{\varphi} + \alpha \varphi_i^a \varphi_j^a \right).$$

Substituting into (2.53) we get

$$W_{tijk,tkj}^{\varphi} = -\left(\frac{m-3}{m-2}\right)C_{ijk,kj}^{\varphi} + \alpha R_{tijk}\varphi_j^a\varphi_{tk}^a + \frac{1}{m-2}\left(R_{si}^{\varphi} + \alpha\varphi_s^b\varphi_i^b\right)C_{tts}^{\varphi}.$$

Computing the divergence with respect to the index i we obtain (2.52).

Other  $\varphi$ -curvatures, for instance the  $\varphi$ -Bach tensor, are not defined in analogy with the classical ones; indeed, for  $m \geq 3$ , we have

$$(m-2)B_{ij}^{\varphi} = C_{ijk,k}^{\varphi} + R_{tk}^{\varphi} W_{tikj}^{\varphi} - \alpha R_{tk}^{\varphi} \varphi_{t}^{a} \varphi_{i}^{a} \delta_{jk}$$

$$+ \alpha \left( \varphi_{ij}^{a} \varphi_{kk}^{a} - \varphi_{kkj}^{a} \varphi_{i}^{a} - \frac{|\tau(\varphi)|^{2}}{m-2} \delta_{ij} \right).$$

Note that  $B^{\varphi}$  is a symmetric tensor, see [3], but it is not trace free; indeed,

(2.56) 
$$B_{ii}^{\varphi} = \alpha \frac{(m-4)}{(m-2)^2} |\tau(\varphi)|^2.$$

We refer to [3] for a more detailed discussion. The next result will be needed in Chapter 5 for the proof of Theorem 5.8.

**Lemma** 2.11. Let (M,g) be a manifold of dimension  $m \geq 3$  and  $\varphi : (M,g) \rightarrow (N,h)$  a smooth map. Then

$$(2.57) \qquad (m-2)B_{ik,k}^{\varphi} = \left(\frac{m-4}{m-2}\right) \left[R_{jk}^{\varphi}C_{jki}^{\varphi} + \alpha\varphi_{tt}^{a}\left(\varphi_{kki}^{a} + R_{ij}^{\varphi}\varphi_{j}^{a}\right)\right]$$

$$+ \alpha\varphi_{i}^{a} \left[\frac{m}{(m-1)(m-2)}\varphi_{tt}^{a}S^{\varphi} + 2\alpha\varphi_{tt}^{b}\varphi_{j}^{b}\varphi_{j}^{a}\right]$$

$$- \frac{\alpha}{2}\left(\frac{m-2}{m-1}\right)\varphi_{i}^{a}\varphi_{j}^{a}S_{j}^{\varphi} - 2\alpha\varphi_{jk}^{a}R_{jk}^{\varphi}\varphi_{i}^{a}$$

$$- \alpha\varphi_{i}^{a}\tau_{2}^{a}(\varphi),$$

where

$$\tau_2^a(\varphi) = \varphi_{ttss}^a - {}^{N}R_{bcd}^a \varphi_s^b \varphi_s^c \varphi_{tt}^d$$

are the components of the bi-tension field of the map  $\varphi$  and  ${}^{N}R^{a}_{bcd}$  are the components of the curvature tensor of N.

#### Remark 2.12. We recall that

$$\tau_2(\varphi) = 0$$

is the Euler-Lagrange equation of the bi-energy functional  $E_{\tau}^{\varphi}$  given on a relatively compact domain  $\Omega$  in M by the prescription

$$E_{\tau}^{\varphi}(\Omega) = \frac{1}{2} \int_{\Omega} |\tau(\varphi)|^2.$$

Remark 2.13. Lemma 2.11 was first proved in [2], but note that the two formulas are slightly different due to some minor typos that are present in [2]. The proof that we give here for completeness is essentially the same as the one presented there, with some slight modifications.

PROOF. Define the tensors of components

$$\mathcal{L}_{ij} = \varphi_{tt}^a \varphi_{ij}^a - \frac{1}{m-2} |\tau(\varphi)|^2 \delta_{ij},$$

$$\mathcal{M}_{ij} = R_{tk}^{\varphi} W_{tikj}^{\varphi},$$

$$\mathcal{N}_{ij} = C_{ijk,k}^{\varphi} - \alpha \varphi_i^a \Big( \varphi_{kkj}^a + R_{jk}^{\varphi} \varphi_k^a \Big).$$

Then, using (2.55), it is immediate to see that

$$(m-2)B_{ij}^{\varphi} = \alpha \mathcal{L}_{ij} + \mathcal{M}_{ij} + \mathcal{N}_{ij}.$$

We compute separately the divergences of these three tensors. First, recall the validity of the following commutation formula (see Section 1.7 of [1] for a proof):

$$\varphi_{ijk}^a = \varphi_{ikj}^a + \varphi_t^a R_{ijk}^t - {}^{N}R_{bcd}^a \varphi_i^b \varphi_j^c \varphi_k^d.$$

We have

$$\mathcal{L}_{ij,j} = \varphi_{ttj}^a \varphi_{ij}^a + \varphi_{tt}^a \varphi_{ij}^a - \frac{2}{m-2} \varphi_{tti}^a \varphi_{pp}^a$$

$$= \varphi_{ttj}^a \varphi_{ij}^a + \left(\frac{m-4}{m-2}\right) \varphi_{tt}^a \varphi_{jji}^a + \varphi_{pp}^a \varphi_t^a R_{ti} - {}^N R_{bcd}^a \varphi_j^b \varphi_i^c \varphi_j^d \varphi_{tt}^a$$

Using the definition of  $Ric^{\varphi}$  we deduce

$$(2.58) \qquad \mathcal{L}_{ij,j} = \varphi_{ttj}^a \varphi_{ij}^a + \left(\frac{m-4}{m-2}\right) \varphi_{tt}^a \varphi_{jji}^a + \varphi_{pp}^a \varphi_t^a R_{ti}^\varphi + \alpha \varphi_{pp}^a \varphi_t^a \varphi_t^b \varphi_i^b$$
$$- {}^{N}R_{bcd}^a \varphi_j^b \varphi_i^c \varphi_j^d \varphi_{tt}^a.$$

Next, for the tensor  $\mathcal{M}$ , we use the symmetries of  $W^{\varphi}$  and formula (2.51) to get

$$\begin{split} \mathcal{M}_{ij,j} = & R_{tk,j}^{\varphi} W_{tikj}^{\varphi} + R_{tk}^{\varphi} W_{tikj,j}^{\varphi} \\ = & \frac{1}{2} \bigg( R_{tk,j}^{\varphi} - R_{tj,k}^{\varphi} \bigg) W_{tikj}^{\varphi} + \bigg( \frac{m-3}{m-2} \bigg) R_{tk}^{\varphi} C_{tki}^{\varphi} + \alpha R_{tk}^{\varphi} \varphi_{ki}^{a} \varphi_{t}^{a} \\ & - \alpha R_{tk}^{\varphi} \varphi_{tk}^{a} \varphi_{i}^{a} + \frac{\alpha}{m-2} \varphi_{tt}^{a} \big( S^{\varphi} \varphi_{i}^{a} - R_{pi}^{\varphi} \varphi_{p}^{a} \big). \end{split}$$

Using the definition of  $C^{\varphi}$  and equation (2.49) we obtain

$$(2.59) M_{ij,j} = \frac{1}{2} C_{tkj}^{\varphi} W_{tikj}^{\varphi} + \frac{\alpha}{2(m-1)} S_j^{\varphi} \varphi_j^a \varphi_i^a + \left(\frac{m-3}{m-2}\right) R_{tk}^{\varphi} C_{tki}^{\varphi}$$
$$+ \alpha R_{tk}^{\varphi} \varphi_{ki}^a \varphi_t^a - \alpha R_{tk}^{\varphi} \varphi_{tk}^a \varphi_i^a + \frac{\alpha}{m-2} \varphi_{tt}^a \left(S^{\varphi} \varphi_i^a - R_{pi}^{\varphi} \varphi_p^a\right).$$

For the tensor  $\mathcal{N}$ , we use the  $\varphi$ -Schur identity to compute

(2.60) 
$$\mathcal{N}_{ij,j} = C^{\varphi}_{ijk,kj} - \alpha \varphi^{a}_{ij} \left( \varphi^{a}_{ttj} + R^{\varphi}_{jk} \varphi^{a}_{k} \right) - \alpha \varphi^{a}_{i} \left( \varphi^{a}_{ttjj} + \frac{1}{2} S^{\varphi}_{t} \varphi^{a}_{t} - \alpha \varphi^{b}_{tt} \varphi^{b}_{j} \varphi^{a}_{j} + R^{\varphi}_{jk} \varphi^{a}_{jk} \right).$$

From the Ricci commutation relations and the symmetries of  $C^{\varphi}$  we get

$$C_{ijk,kj}^{\varphi} = \frac{1}{2} \left( C_{ijk,kj}^{\varphi} - C_{ikj,kj}^{\varphi} \right) = \frac{1}{2} \left( C_{ijk,kj}^{\varphi} - C_{ijk,jk}^{\varphi} \right)$$

$$= \frac{1}{2} \left( R_{pikj} C_{pjk}^{\varphi} + R_{pjkj} C_{ipk}^{\varphi} + R_{pkkj} C_{ijp}^{\varphi} \right)$$

$$= \frac{1}{2} \left( R_{pikj} C_{pjk}^{\varphi} + R_{pk} C_{ipk}^{\varphi} - R_{pj} C_{ijp}^{\varphi} \right)$$

$$= \frac{1}{2} R_{pikj} C_{pjk}^{\varphi}.$$

Using the definition of  $W^{\varphi}$  we deduce

$$\begin{split} C_{ijk,kj}^{\varphi} = & \frac{1}{2} W_{pikj}^{\varphi} C_{pjk}^{\varphi} + \frac{1}{m-2} R_{pk}^{\varphi} C_{pik}^{\varphi} - \frac{\alpha}{m-2} R_{ij}^{\varphi} \varphi_{tt}^{a} \varphi_{j}^{a} \\ & + \frac{\alpha S^{\varphi}}{(m-1)(m-2)} \varphi_{tt}^{a} \varphi_{i}^{a}. \end{split}$$

Inserting this information into (2.60) we get

$$(2.61) \qquad \mathcal{N}_{ij,j} = \frac{1}{2} W^{\varphi}_{pikj} C^{\varphi}_{pjk} + \frac{1}{m-2} R^{\varphi}_{pk} C^{\varphi}_{pik} - \frac{\alpha}{m-2} R^{\varphi}_{ij} \varphi^{a}_{tt} \varphi^{a}_{j}$$
$$+ \frac{\alpha S^{\varphi}}{(m-1)(m-2)} \varphi^{a}_{tt} \varphi^{a}_{i} - \alpha \varphi^{a}_{ij} \left( \varphi^{a}_{ttj} + R^{\varphi}_{jk} \varphi^{a}_{k} \right)$$

$$-\alpha\varphi_i^a\bigg(\varphi_{ttjj}^a + \frac{1}{2}S_t^\varphi\varphi_t^a - \alpha\varphi_{tt}^b\varphi_j^b\varphi_j^a + R_{jk}^\varphi\varphi_{jk}^a\bigg).$$

Putting together equations (2.58), (2.59) and (2.61) we obtain equation (2.57).

# 2.4. Perspectives on a more general System

Through this section, we are going to study system (2.35) in a more general version that, in some special cases, can be derived from different mathematical settings; in particular, we focus on a generalization of the system

(2.62) 
$$\begin{cases} i) u \operatorname{Ric}^{\varphi} - \operatorname{Hess}(u) = \frac{1}{m} (u S^{\varphi} - \Delta u) g, \\ ii) u \tau(\varphi) = -d\varphi(\nabla u) + \frac{1}{\alpha} (\nabla U)(\varphi) u; \end{cases}$$

note that the first equation of (2.62) is (2.31), while the second is the third equation of (2.35); we consider system (2.62) on a Riemannian manifold (M, g) with empty boundary, so that u > 0 on M. Setting

$$f = -\log u$$
,

the system reduces to

(2.63) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha} (\nabla U)(\varphi), \end{cases}$$

where  $m\lambda = S^{\varphi} + \Delta f - |\nabla f|^2$ . As we have already seen in Section 2.1, these equations can be obtained decomposing the stress-energy tensor  $\widehat{T}$  into its spatial and temporal parts, in addition to the energy-momentum conservation. Introducing a parameter  $\eta \in \mathbb{R}$ , we consider

(2.64) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha} (\nabla U)(\varphi), \end{cases}$$

where  $\lambda \in C^{\infty}(M)$ . In the following subsections, we will focus on the different contexts in which some special versions of this generalized system emerge.

**2.4.1.** An Euler-Lagrange equation. In this subsection we show how a particular case of system (2.64) (i.e., the one with  $\eta=1$ ) appears as Euler-Lagrange equations of the functional

(2.65) 
$$\mathcal{F} = \mathcal{F}(\widehat{g}, \widehat{\varphi}) = \int_{\Omega} \left( \widehat{S}^{\widehat{\varphi}} - 2U(\widehat{\varphi}) \right) dV_{\widehat{g}},$$

where  $\Omega \subset \widehat{M}$  is a relatively compact domain and  $\widehat{S}^{\widehat{\varphi}}$  is the  $\widehat{\varphi}$ -scalar curvature with respect to  $\widehat{g}$ . We now compute the variation of  $\mathcal{F}$ , first with respect to the metric  $\widehat{g}$  and then with respect to  $\widehat{\varphi}$ . For a symmetric 2-covariant tensor  $\psi$  on  $\widehat{M}$  and  $t \in (-\varepsilon, \varepsilon)$ , with  $\varepsilon > 0$  sufficiently small, define

$$\widehat{g}_t = \widehat{g} + t\psi.$$

Then we have

$$\left. \frac{d}{dt} \mathcal{F}(\widehat{g}_t, \widehat{\varphi}) \right|_{t=0} = \int_{\Omega} \frac{d}{dt} \left( \widehat{S}_t^{\widehat{\varphi}} - 2U(\widehat{\varphi}) \right) \Big|_{t=0} dV_{\widehat{g}} + \int_{\Omega} \left[ \left( \widehat{S}_t^{\widehat{\varphi}} - 2U(\widehat{\varphi}) \right) \frac{d}{dt} (dV_{\widehat{g}_t}) \Big|_{t=0} \right],$$

where  $\widehat{S}_t^{\widehat{\varphi}}$  is the  $\widehat{\varphi}$ -Scalar curvature with respect to  $\widehat{g}_t$ . Recall that

$$\widehat{S}^{\widehat{\varphi}} = \widehat{S} - \alpha |d\widehat{\varphi}|_{\widehat{g}}^2,$$

where

$$|d\widehat{\varphi}|_{\widehat{g}}^2 = g^{\alpha\beta}\widehat{\varphi}_{\alpha}^a\widehat{\varphi}_{\beta}^a$$

so that

$$\left. \frac{d}{dt} \left( |d\widehat{\varphi}|_{\widehat{g}_{t}}^{2} \right) \right|_{t=0} = -\psi_{\alpha\beta} \widehat{\varphi}_{\mu}^{a} \widehat{\varphi}_{\nu}^{a} g^{\alpha\mu} g^{\beta\nu}$$

(for a proof see e.g. [15], Chapter 2, and [18]); moreover, we have the validity of the following formulas:

$$\left. \frac{d}{dt} (dV_{\widehat{g}_t}) \right|_{t=0} = \frac{1}{2} \Psi dV_{\widehat{g}},$$

where

$$\Psi = g^{\alpha\beta} k_{\alpha\beta}$$

and

$$\frac{d}{dt}(\widehat{S}_t)\bigg|_{t=0} = -k\psi_{\mu\beta,\nu\gamma}g^{\mu\beta}g^{\nu\gamma} + \psi_{\mu\nu,\beta\gamma}g^{\mu\beta}g^{\nu\gamma} - \widehat{R}_{\mu\eta}g^{\mu\gamma}g^{\eta\beta}\psi_{\gamma\beta}.$$

Therefore we have

$$\frac{d}{dt}\mathcal{F}(\widehat{g}_{t},\widehat{\varphi})\bigg|_{t=0} = \int_{\Omega} \left( -\psi_{\mu\beta,\nu\gamma}g^{\mu\beta}g^{\nu\gamma} + \psi_{\mu\nu,\beta\gamma}g^{\mu\beta}g^{\nu\gamma} - \widehat{R}_{\mu\eta}g^{\mu\gamma}g^{\eta\beta}\psi_{\gamma\beta} \right. \\
\left. + \alpha\psi_{\mu\nu}\widehat{\varphi}_{\beta}^{a}\widehat{\varphi}_{\gamma}^{a}g^{\mu\beta}g^{\nu\gamma}\right)dV_{\widehat{g}} \\
+ \int_{\Omega} \left( \frac{1}{2}\widehat{S}\Psi - U(\widehat{\varphi})\Psi - \frac{1}{2}\alpha|d\widehat{\varphi}|_{\widehat{g}}^{2}\Psi \right)dV_{\widehat{g}},$$

and using the divergence theorem we get

$$\left. \frac{d}{dt} \mathcal{F}(\widehat{g}_t, \widehat{\varphi}) \right|_{t=0} = \int_{\Omega} \left( -\widehat{R}_{\mu\eta}^{\widehat{\varphi}} + \frac{1}{2} \widehat{S}^{\widehat{\varphi}} g_{\mu\eta} - U(\widehat{\varphi}) g_{\mu\eta} \right) g^{\mu\gamma} g^{\eta\beta} \psi_{\gamma\beta} dV_{\widehat{g}}.$$

Hence, the first Euler-Lagrange equation of the functional  $\mathcal{F}$  is

$$\widehat{\mathrm{Ric}}^{\widehat{\varphi}} - \frac{1}{2} \widehat{S}^{\widehat{\varphi}} \widehat{g} = -U(\widehat{\varphi}) \widehat{g}.$$

Now we take the variation of  $\mathcal{F}$  with respect to  $\widehat{\varphi}$ . Following [27], let V be a vector field along  $\widehat{\varphi}$ , that is  $\pi(V(q)) = \widehat{\varphi}(q)$  for all  $q \in \widehat{M}$  (where  $\pi : TN \to N$  is the canonical projection), and define  $\widehat{\varphi}_t(x) = \exp_{\widehat{\varphi}(x)}(tV)$  for  $t \in (-\varepsilon, \varepsilon)$ , with  $\varepsilon > 0$  sufficiently small. Then the variation of  $\varphi$  with respect to V is given by

$$\left. \frac{d}{dt} \mathcal{F}(\widehat{g}, \widehat{\varphi}_t) \right|_{t=0} = \int_{\Omega} \frac{d}{dt} \left( -\alpha |d\widehat{\varphi}_t|_{\widehat{g}}^2 - 2U(\widehat{\varphi}_t) \right) \Big|_{t=0} dV_{\widehat{g}}.$$

By Lemma B of section 2 of [27], which holds also in Lorentzian signature, as one can easily verify, we get

$$\int_{\Omega} -\alpha \frac{d}{dt} (|d\widehat{\varphi}_t|_{\widehat{g}}^2) \bigg|_{t=0} dV_{\widehat{g}} = 2\alpha \int_{\Omega} \tau(\widehat{\varphi})^a V^a dV_{\widehat{g}},$$

while a simple computation shows that

$$\frac{d}{dt}U(\widehat{\varphi}_t)_{|_{t=0}} = (U^a \circ \widehat{\varphi})V^a.$$

Therefore we get

$$\left. \frac{d}{dt} \mathcal{F}(\widehat{g}, \widehat{\varphi}_t) \right|_{t=0} = 2 \int_{\Omega} \left( \alpha \tau(\widehat{\varphi})^a - (U^a \circ \widehat{\varphi}) \right) V^a dV_{\widehat{g}}.$$

Hence, the second Euler-Lagrange equation of the functional  $\mathcal{F}$  is given by

$$\tau(\widehat{\varphi}) = \frac{1}{\alpha} \nabla U(\widehat{\varphi}),$$

and therefore the critical points of  $\mathcal F$  satisfy the system

(2.66) 
$$\begin{cases} -\widehat{\text{Ric}} + \frac{1}{2}\widehat{S}\widehat{g} - U(\widehat{\varphi})\widehat{g} + \alpha\widehat{\varphi}^*h - \frac{\alpha}{2}|d\widehat{\varphi}|_{\widehat{g}}\widehat{g} = 0, \\ \alpha\tau(\widehat{\varphi}) = \nabla U(\widehat{\varphi}). \end{cases}$$

Let us now consider manifolds  $\widehat{M}$  of the form  $\widehat{M}=M\times_{e^{-2f}}\mathbb{R}$ , with a warped product metric

$$\widehat{g} = g - e^{-2f} dt \otimes dt,$$

where g is the lifting on  $\widehat{M}$  of a Riemannian metric on M,  $f \in C^{\infty}(M)$  and  $t: \widehat{M} \to \mathbb{R}$  denotes the projection. To perform computations, we will consider, at a fixed point  $p \in \widehat{M}$ , a local orthonormal frame  $\{e_i\}_{i=0,...m}$  such that

$$e_0 = e^f \frac{\partial}{\partial t},$$

while  $e_1, ..., e_m$  span the tangent space of M. Moreover, we will assume that  $\widehat{\varphi}$  is the lift on  $\widehat{M}$  of a map  $\varphi: M \to N$ , so that, as we have seen in previous computations,

$$\widehat{\varphi}_0^a = 0;$$

note that this also implies, as we have noted before, that

$$|d\widehat{\varphi}|_{\widehat{g}}^2 = |d\varphi|_g^2.$$

If we let Ric and S be the lifts on  $\widehat{M}$  of the Ricci tensor and the scalar curvature of (M, g), respectively, then we have the following expressions:

(2.67) 
$$\widehat{\text{Ric}} = \text{Ric} + \text{Hess}(f) - df \otimes df - (\Delta f - |\nabla f|^2)e^{-2f}dt \otimes dt$$

and

$$\widehat{S} = S + 2\Delta f - 2|\nabla f|^2.$$

Moreover,

$$\widehat{\varphi}_{00}^a = \varphi_i^a f_i, \quad \widehat{\varphi}_{0i}^a = 0, \quad \widehat{\varphi}_{ij}^a = \varphi_{ij}^a.$$

The second equation of (2.66) becomes

$$0 = \alpha \tau(\widehat{\varphi}) - \nabla U(\widehat{\varphi}) = \alpha \tau(\varphi) - \alpha d\varphi(\nabla f) - \nabla U(\varphi),$$

and therefore

$$\tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}\nabla U(\varphi).$$

Furthermore, contracting the first equation of (2.66) with  $\hat{g}$  we get

$$-\widehat{S} + \frac{1}{2}(m+1)\widehat{S} - (m+1)U(\widehat{\varphi}) + \alpha|d\widehat{\varphi}|^2 - \frac{\alpha}{2}(m+1)|d\widehat{\varphi}|^2 = 0,$$

so that

$$\frac{1}{2}(m-1)\widehat{S} = (m+1)U(\widehat{\varphi}) + \frac{\alpha}{2}(m-1)|d\widehat{\varphi}|^2.$$

Inserting this information into the first equation of (2.66) we have

$$(2.68) -\widehat{\text{Ric}} + \frac{m+1}{m-1}U(\widehat{\varphi})\widehat{g} - U(\widehat{\varphi})\widehat{g} + \frac{\alpha}{2}|d\widehat{\varphi}|^2\widehat{g} + \alpha\varphi^*h - \frac{\alpha}{2}|d\widehat{\varphi}|^2\widehat{g} = 0;$$

rearranging terms and simplifying, we get

$$\widehat{\mathrm{Ric}}^{\varphi} = \frac{2}{m-1} U(\widehat{\varphi}) \widehat{g}.$$

Using equation (2.67) into (2.68), we deduce

$$-\operatorname{Ric}^{\varphi} - \operatorname{Hess}(f) + df \otimes df + (\Delta f - |\nabla f|^{2})e^{-2f}dt \otimes dt + \frac{2}{m-1}U(\widehat{\varphi})g$$
$$-\frac{2}{m-1}U(\widehat{\varphi})e^{-2f}dt \otimes dt = 0$$

Splitting the spatial and temporal part we deduce

$$\begin{cases} \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - df \otimes df = \frac{2}{m-1}U(\varphi)g, \\ \Delta f - |\nabla f|^2 = \frac{2}{m-1}U(\varphi). \end{cases}$$

In conclusion, we have derived the system

$$\begin{cases} \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - df \otimes df = \frac{2}{m-1}U(\varphi)g, \\ \Delta f - |\nabla f|^2 = \frac{2}{m-1}U(\varphi), \\ \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi). \end{cases}$$

**2.4.2.** On some solutions of Ricci-Harmonic flow. We now recall the definition of the *Ricci-Harmonic flow*, introduced by List in [47]. This is given by a solution  $(g(t), \varphi(t))_{t \in [0,\varepsilon)}$  of the evolution system

(2.69) 
$$\begin{cases} \frac{d}{dt}g(t) = -2\operatorname{Ric}_{g(t)} + 2\alpha\varphi(t)^*h, \\ \frac{d}{dt}\varphi(t) = \tau_{g(t)}(\varphi(t)), \end{cases}$$

for a family of Riemannian metrics  $\{g(t)\}$  on M, smooth maps  $\varphi(t):(M,g)\to (N,h)$  and a constant  $\alpha$ . The self-similar solutions of this flow are solutions whose metric evolves by diffeomorphisms and rescaling, while the map  $\varphi$  evolves by diffeomorphisms. In other terms, there exists a one-parameter family  $\{F_t\}_{t\in[0,\varepsilon)}$  of diffeomorphisms of M such that  $F_0=\mathrm{id}_M$  and a positive function  $c:[0,\varepsilon)\to\mathbb{R}$  such that c(0)=1 for which we have

(2.70) 
$$\begin{cases} g(t) = c(t)F_t^*g(0), \\ \varphi(t) = F_t^*\varphi(0). \end{cases}$$

In the next Proposition we will show how solutions of (2.64) with  $\eta = 0$ ,  $\lambda$  constant and non-constant U can also be characterized in terms of special solutions of (2.69).

**Proposition** 2.14. For a smooth function  $U \in C^{\infty}(N)$ , let  $\{G_t\}_{t \in [0,\varepsilon)}$  be a one-parameter family of diffeomorphisms of N such that  $G_0 = \operatorname{id}_N$  and  $\frac{d}{dt}G_t|_{t=0} = \frac{1}{\alpha}\nabla U$ . Let  $\{F_t\}_{t \in [0,\varepsilon)}$  be a one-parameter family of diffeomorphisms of M such that  $F_0 = \operatorname{id}_M$  and  $\frac{d}{dt}F_t|_{t=0} = \nabla f$ . Consider a solution of (2.69) of the form

(2.71) 
$$\begin{cases} g(t) = c(t)F_t^*g(0), \\ \varphi(t) = G_t(F_t^*\varphi(0)), \end{cases}$$

where  $c:[0,\varepsilon)\to\mathbb{R}$  is a positive function satisfying c(0)=1. Then,  $(M,g(0),\varphi(0))$  is a solution of (2.64) with  $\eta=0$  and  $\lambda$  constant.

PROOF. Given a solution of (2.69) satisfying (2.71) we deduce

$$-2\operatorname{Ric}_{g(0)} + 2\alpha\varphi(0)^* h = \frac{d}{dt}g(t) \Big|_{t=0}$$

$$= \frac{d}{dt}(c(t)F_t^*g(0)) \Big|_{t=0} = \dot{c}(0)g(0) + c(0)\mathcal{L}_{\nabla f}g(0)$$

$$= \dot{c}(0)g(0) + 2c(0)\operatorname{Hess}(f).$$

Setting  $\dot{c}(0) = -2\lambda$  we obtain the first equation of (2.64) with  $\eta = 0$ . For the second equation, we compute

$$\tau_{g(0)}(\varphi(0)) = \frac{d}{dt}\varphi(t)_{|_{t=0}}$$
$$= \frac{d}{dt}(G_t(F_t^*\varphi(0)))\Big|_{t=0}$$

$$= \left(\frac{d}{dt}G_t\right)_{|_{t=0}} (F_0^*\varphi(0)) + G_0\left(\frac{d}{dt}(F_t^*\varphi(0))\Big|_{t=0}\right)$$
$$= \frac{(\nabla U)}{\alpha}(\varphi(0)) + d\varphi(0)(\nabla f).$$

**Remark** 2.15. If in (2.71) we let c(t) be a function c(t, x) of both time and space, we obtain that at time t = 0 the corresponding solution of (2.64) does not need to have  $\lambda$  constant. This has been first observed in the case of Ricci solitons by Gomes, Wang and Xia in [36].

**Remark** 2.16. Unfortunately, to the best of our knowledge, it is not known if the converse statement of the above Proposition is true: in other words, a solution of (2.64) with  $\eta = 0$  and  $\lambda$  constant might not be enough to construct a solution of (2.71).

**2.4.3.** Harmonic-Einstein warped products. Under suitable assumptions, Riemannian manifolds satisfying system (2.64), that is

$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha} (\nabla U)(\varphi), \end{cases}$$

arise as warping factor of a Riemaniann manifold that is  $\frac{1}{\alpha}U$ -harmonic Einstein, where  $\alpha \in \mathbb{R} \setminus \{0\}$ .

Let (M,g) and  $(F,g_F)$  be two Riemannian manifolds of dimensions m and d, respectively, and let  $f \in C^{\infty}(M)$ ; we denote by  $\overline{M} = M \times_{e^{-f/d}} F$  the warped product manifold endowed with the metric

$$\overline{g} = g + e^{-2f/d}g_F.$$

Then, we have the validity of the following

**Proposition** 2.17. [2, Corollary 4.13] In the notation above, let  $\overline{\text{Ric}}$  and  $\overline{S}$  be the Ricci tensor and the scalar curvature of  $(\overline{M}, \overline{g})$ , respectively; let Ric, <sup>F</sup>Ric and S, <sup>F</sup>S denote the lifts to  $\overline{M}$  of the Ricci tensors and the scalar curvatures of (M, g) and  $(F, g_F)$ , respectively. Then, the non-vanishing components of  $\overline{\text{Ric}}$  are given by

$$(2.72) \quad \overline{R}_{ij} = R_{ij} + f_{ij} - \frac{1}{d} f_i f_j, \quad \overline{R}_{(A+m)(B+m)} = \frac{\Delta_f f}{d} (g_F)_{AB} + e^{2f/d F} R_{AB},$$

where i, j = 1, ..., m, A, B = 1, ..., d and  $\Delta_f f$  is the f-laplacian of f in the metric g, that is,

$$\Delta_f f = \Delta f - |\nabla f|^2.$$

Let  $\varphi:(M,g)\to (N,h)$  be a smooth map form M to a second Riemannian manifold of dimension n. We denote  $\overline{\varphi}:(\overline{M},\overline{g})\to (N,h)$  the smooth map defined

$$\overline{\varphi} := \varphi \circ \pi_M,$$

where  $\pi_M : \overline{M} \to M$  is the projection on the first factor. Then (see [2, Proposition 4.17] for a proof) we have

(2.73) 
$$\overline{\varphi}_i^a = \varphi_i^a, \quad \overline{\varphi}_{A+m}^a = 0, \quad \tau(\overline{\varphi}) = \tau(\varphi) - d\varphi(\nabla f),$$

where  $\tau(\overline{\varphi})$  is the tension field of  $\overline{\varphi}$ .

**Theorem** 2.18. Let (M,g) and  $(F,g_F)$  be Riemannian manifolds of dimension m and d respectively, with  $d \geq 3$ . Let  $f \in C^{\infty}(M)$ ,  $\varphi : (M,g) \to (N,h)$  and  $U : (N,h) \to \mathbb{R}$  be smooth maps. Consider the warped product manifold  $(\overline{M},\overline{g}) = (M \times_{e^{-f/d}} F, g + e^{-2f/d}g_F)$  and let  $\overline{\varphi} : (\overline{M},\overline{g}) \to (N,h)$  be as above. Then  $(\overline{M},\overline{g})$  is  $\frac{1}{\alpha}U$ -harmonic-Einstein,  $\alpha \in \mathbb{R} \setminus \{0\}$ , i.e. there exists  $\lambda \in \mathbb{R}$  such that

(2.74) 
$$\begin{cases} \overline{\mathrm{Ric}}^{\overline{\varphi}} = \lambda \overline{g}, \\ \tau(\overline{\varphi}) = \frac{1}{\alpha} (\nabla U)(\varphi), \end{cases}$$

if and only if (M,g) satisfies (2.64) with  $\eta = \frac{1}{d}$  and  $(F,g_F)$  is Einstein, with Einstein constant  $\Lambda$  satisfying

(2.75) 
$$\Lambda - e^{-2f/d}\lambda + e^{-2f/d}\frac{\Delta_f f}{d} = 0.$$

PROOF. Assume that  $(\overline{M}, \overline{g})$  satisfies (2.74): then, by (2.72) and (2.73) we get

$$\lambda(g + e^{-2f/d}g_F) = \overline{\mathrm{Ric}}^{\varphi}$$

$$= \operatorname{Ric} + \operatorname{Hess}(f) - \frac{1}{d} df \otimes df + \frac{\Delta_f f}{d} e^{-2f/d} g_F + {}^F \operatorname{Ric} - \alpha \varphi^* h,$$

from which we deduce

$$\operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \frac{1}{d} df \otimes df = \lambda g,$$

$${}^{F}\operatorname{Ric} = e^{-2f/d} \left(\lambda - \frac{\Delta_{f} f}{d}\right) g_{F} = \Lambda g_{F}$$

and by (2.73) we have

$$\frac{1}{\alpha}(\nabla U)(\varphi) = \tau(\overline{\varphi}) = \tau(\varphi) - d\varphi(\nabla f).$$

Conversely, let us assume that (M, g) satisfies (2.64) and that  $(F, g_F)$  is Einstein, with Einstein constant  $\Lambda$  satisfying (2.75). Then, by (2.72) we deduce

(2.76) 
$$\overline{\text{Ric}} = \text{Ric} + \text{Hess}(f) - \frac{1}{d} df \otimes df + \frac{\Delta_f f}{d} g_F + {}^F \text{Ric}.$$

Since (2.64) holds, we have

(2.77) 
$$\operatorname{Ric} = \alpha \varphi^* h - \operatorname{Hess}(f) + \frac{1}{d} df \otimes df + \lambda g;$$

moreover, by the definition of  $\Lambda$ , we get

(2.78) 
$${}^{F}\text{Ric} = \Lambda g_{F} = e^{-2f/d} \left( \frac{\Delta_{f} f}{d} + \lambda \right) g_{F}.$$

Inserting (2.77) and (2.78) into (2.76), we obtain

$$\overline{\text{Ric}} = \alpha \varphi^* h + \lambda (g + e^{-2f/d} g_F);$$

hence, (2.74) follows by (2.73).

**2.4.4.** Conformally harmonic-Einsten. When  $\eta = -\frac{1}{m-2}$  and  $U \equiv 0$ , the structure (2.64) can be obtained *via* a conformal deformation of a harmonic-Einstein structure; we recall that, as we have seen in the Introduction, a Riemannian manifold (M, g) is said to be harmonic-Einstein if it carries a solution of the system

$$\begin{cases} \operatorname{Ric}^{\varphi} = \Lambda g, \\ \tau(\varphi) = 0, \end{cases}$$

where  $\Lambda \in C^{\infty}(M)$ . Note that, since  $\tau(\varphi) = 0$ , the  $\varphi$ -Schur identity rewrites as the usual one, that is

$$R_{ij,i}^{\varphi} = \frac{S_j^{\varphi}}{2} - \varphi_{tt}^a \varphi_j^a = \frac{S_j^{\varphi}}{2},$$

and therefore  $\Lambda$  is constant.

We say that a Riemannian manifold (M,g) of dimension  $m \geq 3$  is conformally harmonic-Einstein if there exists a smooth positive function  $\psi \in C^{\infty}(M)$  such that, given the conformal change

$$\widetilde{g} = \psi^2 g$$
,

the manifold  $(M, \widetilde{g})$  is harmonic-Einstein.

Note that, when (M, g) carries a solution of system (2.64) that is constant and  $U \equiv 0$ , then (M, g) is harmonic-Einstein. More in general, when f is not constant and  $U \equiv 0$ , we have the validity of the following

**Proposition** 2.19. Let (M,g) be a Riemannian manifold of dimension  $m \geq 3$ ,  $\varphi: (M,g) \to (N,h)$  be a smooth map and let  $\alpha \in \mathbb{R} \setminus \{0\}$ . Then (M,g) carries a solution of the system

(2.79) 
$$\begin{cases} \operatorname{Ric} + \operatorname{Hess}(f) + \frac{1}{m-2} df \otimes df = \lambda g \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases}$$

if and only if, given the conformal change of metric

$$\widetilde{g} = e^{-\frac{2f}{m-2}}g,$$

there exists  $\Lambda \in C^{\infty}(M)$  such that

(2.80) 
$$\begin{cases} \widetilde{\mathrm{Ric}}^{\widetilde{\varphi}} = \widetilde{\mathrm{Ric}} - \alpha \widetilde{\varphi}^* h = \Lambda \widetilde{g}, \\ \tau(\widetilde{\varphi}) = 0, \end{cases}$$

where  $\widetilde{\mathrm{Ric}}$  and  $\widetilde{\mathrm{Ric}}^{\widetilde{\varphi}}$  are the Ricci and the  $\varphi$ -Ricci tensor of  $(M, \widetilde{g})$ , respectively and  $\widetilde{\varphi}$  denotes the map  $\varphi$  from  $(M, \widetilde{g})$  to (N, h). Moreover,  $\Lambda$  satisfies

$$\Lambda = \frac{1}{m-2} e^{\frac{2f}{m-2}} (\Delta f - |\nabla f| + (m-2)\lambda).$$

**Remark** 2.20. Note that  $\Lambda$  is constant by the  $\varphi$ -Schur identity.

PROOF. To prove Proposition 2.19 we recall the transformation laws of the Ricci tensor Ric (see e.g. [15]) and of the tension field  $\tau(\varphi)$  (see e.g. [24]), that are, respectively,

(2.81) 
$$\widetilde{\operatorname{Ric}} = \operatorname{Ric} + \operatorname{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{1}{m-2} \left( \Delta f - |\nabla f|^2 \right) g,$$

(2.82) 
$$\tau(\widetilde{\varphi}) = e^{-\frac{2f}{m-2}} (\tau(\varphi) - d\varphi(\nabla f)).$$

First note that, by (2.82), we deduce

$$\tau(\widetilde{\varphi}) = 0$$

if and only if

$$\tau(\varphi) = d\varphi(\nabla f).$$

Suppose that (2.79) holds for some  $f, \lambda \in C^{\infty}(M)$ : by (2.81) and the first equation of (2.79), we obtain

$$\begin{aligned} \widetilde{\mathrm{Ric}}^{\widetilde{\varphi}} &= \widetilde{\mathrm{Ric}} - \alpha \widetilde{\varphi}^* h \\ &= \mathrm{Ric} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{1}{m-2} \Big( \Delta f - |\nabla f|^2 \Big) g - \alpha \varphi^* h \\ &= \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{1}{m-2} \Big( \Delta f - |\nabla f|^2 \Big) g \\ &= \lambda g + \frac{1}{m-2} \Big( \Delta f - |\nabla f|^2 \Big) g \\ &= e^{\frac{2f}{m-2}} \frac{1}{m-2} \Big( \Delta f - |\nabla f|^2 + (m-2)\lambda \Big) \widetilde{g} \\ &= \Lambda \widetilde{g} \end{aligned}$$

Conversely, assume that (2.80) holds for some  $\Lambda \in \mathbb{R}$ . By the transformation law (2.81), we deduce

$$\begin{split} & \Lambda \widetilde{g} = \widetilde{\operatorname{Ric}}^{\widetilde{\varphi}} \\ & = \widetilde{\operatorname{Ric}} - \alpha \widetilde{\varphi}^* h \\ & = \operatorname{Ric} - \alpha \varphi^* h + \operatorname{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{1}{m-2} \left( \Delta f - \left| \nabla f \right|^2 \right) g, \end{split}$$

that is

$$\operatorname{Ric} - \alpha \varphi^* h + \operatorname{Hess}(f) + \frac{1}{m-2} df \otimes df = \left[ e^{-\frac{2f}{m-2}} \Lambda + \frac{1}{m-2} \left( \Delta f - |\nabla f|^2 \right) \right] g.$$

Remark 2.21. When we perform the conformal change

$$\widetilde{g} = e^{-\frac{2f}{m-2}}g,$$

we deduce, taking the trace of (2.81),

$$(2.83) e^{-\frac{2f}{m-2}}\widetilde{S}^{\widetilde{\varphi}} = S^{\varphi} + \frac{m-1}{m-2} \Big(2\Delta f - |\nabla f|^2\Big).$$

Moreover, by the definition of the  $\varphi$ -Schouten tensor (2.47) and the transformation laws (2.81) and (2.83) we have

$$\widetilde{A}^{\widetilde{\varphi}} = A^{\varphi} + \operatorname{Hess}(f) + \frac{1}{m-2} \left( df \otimes df - \frac{|\nabla f|^2}{2} g \right).$$

As a consequence, using the definition of the covariant derivative of the Levi-Civita connection associated to the metric  $\tilde{g}$  and the definition of  $\tilde{C}^{\tilde{\varphi}}$ , a simple but tedious computation shows that the conformal change for the  $\varphi$ -Cotton tensor is given by

(2.84) 
$$\widetilde{C}^{\widetilde{\varphi}} = C^{\varphi} + W^{\varphi}(\nabla f, \cdot, \cdot, \cdot).$$

#### CHAPTER 3

# Elementary Considerations on System (1.3)

#### 3.1. Some Observations

The aim of this chapter is to analyze the structure of system (1.3), that we recall here for the sake of readability, and some consequences which can be deduced by only considering some parts of it:

$$\begin{cases} i) \operatorname{Hess}(u) - u \Big\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \Big( \frac{S^{\varphi}}{2} - p + U(\varphi) \Big) g \Big\} = 0, \\ ii) \Delta u = \frac{u}{m-1} \Big[ mp - mU(\varphi) + \frac{m-2}{2} S^{\varphi} \Big], \\ iii) u\tau(\varphi) = -d\varphi(\nabla u) + \frac{u}{\alpha} (\nabla U)(\varphi), \\ iv) \mu + U(\varphi) = \frac{1}{2} S^{\varphi}, \\ v) (\mu + p) \nabla u = -u \nabla p. \end{cases}$$

The next observations will be also useful in the proofs of some of the main theorems: for instance, in Theorem 1.14 we rely on Proposition 3.1 to deduce that  $i:\partial M\to M$  is totally geodesic, which allows us to apply a result of Reilly ([63]) and conclude the validity of the statement. In what follows, we shall always assume, unless otherwise stated,

$$u > 0$$
 on  $int(M)$  and  $u^{-1}(\{0\}) = \partial M$  lif  $\partial M \neq \emptyset$ .

In case  $\partial M \neq \emptyset$ , the boundary will also be assumed to be **connected**.

**Proposition** 3.1. Let (M,g) be a smooth manifold of dimension  $m \geq 2$ , with smooth boundary  $\partial M \neq \emptyset$ . Let u be a solution on M of

(3.1) 
$$\operatorname{Hess}(u) - u\{\operatorname{Ric}^{\varphi} - \Lambda(x)g\} = 0,$$

for some  $\Lambda \in C^2(M)$ . Then  $|\nabla u|$  is a positive constant on  $\partial M$  and  $i: \partial M \hookrightarrow M$  is totally geodesic.

**Remark** 3.2. Obviously, (3.1) can be replaced by (1.3) i).

PROOF. Let  $e_1, e_2, \dots e_{m-1}, e_m$  be a Darboux frame along  $i : \partial M \hookrightarrow M$ , with  $e_m = \nu$ , the inward unit normal to  $\partial M$ . From (3.1) we obtain

$$|\nabla u|_j^2 = 2u_{jk}u_k = 2u\left(R_{jk}^{\varphi} - \Lambda(x)\delta_{jk}\right)u_k,$$

and therefore, since  $u \equiv 0$  on  $\partial M$ , it follows that  $|\nabla u|$  is constant on  $\partial M$ . To show that  $|\nabla u|^2$  is not zero we reason by contradiction: fix  $p \in \partial M$  and, for some  $\varepsilon > 0$  sufficiently small, let  $\gamma : [0, \varepsilon) \to M$  be the geodesic such that  $\gamma(0) = p$ ,  $\dot{\gamma}(0) = \nu$ . We define

$$v(t) := (u \circ \gamma)(t);$$

then, since  $\gamma$  is a geodesic, we have

$$\begin{cases} v''(t) = \operatorname{Hess}(u)(\dot{\gamma}, \dot{\gamma})(t) = v(t) \left[ \operatorname{Ric}^{\varphi}(\dot{\gamma}, \dot{\gamma}) - \Lambda |\dot{\gamma}|^{2} \right], \\ v'(0) = g(\nabla u(p), \dot{\gamma}(0)), \\ v(0) = u(p) = 0. \end{cases}$$

Therefore, if  $\nabla u(p) = 0$ , then v'(0) = v(0) = 0 and  $v \equiv 0$  on  $[0, \varepsilon')$  for some  $0 < \varepsilon' \le \varepsilon$ . This is a contradiction, since u > 0 and  $\gamma((0, \varepsilon')) \subseteq \text{int} M$ . It follows that

$$\nu = \frac{\nabla u}{|\nabla u|} \quad \text{on } \partial M$$

and the second fundamental form II in the direction of  $\nu$  is

$$II = -\frac{\operatorname{Hess}(u)|_{\mathfrak{X}(\partial M) \times \mathfrak{X}(\partial M)}}{|\nabla u|}.$$

From (3.1), since u = 0 on  $\partial M$ , we deduce II = 0, that is,  $i : \partial M \hookrightarrow M$  is totally geodesic.

We next show another relevant fact, that is, equation (1.3) v), which is physically motivated by the preservation of the energy momentum, can be deduced from the other equations of the system. To do this, note that (1.3) v) is equivalent to

(3.2) 
$$u\nabla\mu = \nabla[u(\mu+p)] \quad \text{on } M,$$

and this latter can be obtained as follows:

**Proposition** 3.3. Let (M,g) be a manifold of dimension  $m \geq 2$  and u a solution of

$$\begin{cases} i)\operatorname{Hess}(u) - u\Big\{\operatorname{Ric}^{\varphi} - \frac{1}{m-1}\big(\frac{S^{\varphi}}{2} - p + U(\varphi)\big)g\Big\} = 0, \\ ii)\Delta u = \frac{u}{m-1}\big[mp - mU(\varphi) + \frac{m-2}{2}S^{\varphi}\big], \\ iii')h(\alpha u \tau(\varphi) + \alpha d\varphi(\nabla u) - u(\nabla U)(\varphi), d\varphi) = 0, \\ iv)\mu + U(\varphi) = \frac{1}{2}S^{\varphi} \end{cases}$$

on M. Then (3.2) holds.

PROOF. We insert (3.3) iv) into (3.3) ii) and we take covariant derivative to obtain

(3.4) 
$$u_{iit} = \frac{u_t}{m-1} [mp - 2U(\varphi) + (m-2)\mu] + \frac{u}{m-1} (mp_t - 2U^a \varphi_t^a + (m-2)\mu_t);$$

next we compute the covariant derivative of (3.3) i):

$$0 = u_{itj} - u_j R_{it}^{\varphi} + \frac{u_j}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) \delta_{it}$$
  
+ 
$$\frac{u}{m-1} \left( \frac{S_j^{\varphi}}{2} - p_j + U^a \varphi_j^a \right) \delta_{it} - u R_{it,j}^{\varphi};$$

contracting with respect to i and j, using the  $\varphi$ -Schur's identity, Ricci commutation relations, (3.3) iii'), (3.3) ii) and (3.4) we obtain

$$\begin{split} 0 = & u_{iit} + \alpha u_s \varphi_s^a \varphi_t^a - u \left( \frac{S_t^{\varphi}}{2} - \alpha \varphi_{ii}^a \varphi_t^a \right) + \frac{u}{m-1} \left( \frac{S_t^{\varphi}}{2} - p_t + U^a \varphi_t^a \right) \\ & + \frac{u_t}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) \\ = & \frac{u_t}{m-1} (mp - 2U(\varphi) + (m-2)\mu) + \frac{u}{m-1} (mp_t - 2U^a \varphi_t^a + (m-2)\mu_t) \\ & - \alpha u \varphi_{ii}^a \varphi_t^a + u U^a \varphi_t^a - u \frac{S_t^{\varphi}}{2} + \alpha u \varphi_{ii}^a \varphi_t^a + \frac{u}{m-1} \left( \frac{S_t^{\varphi}}{2} - p_t + U^a \varphi_t^a \right) \\ & + \frac{u_t}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) \\ = & \frac{u}{m-1} \left[ (m-1)p_t - U^a \varphi_t^a + (m-1)U^a \varphi_t^a + (m-2)\mu_t + \frac{S_t^{\varphi}}{2} - (m-1) \frac{S_t^{\varphi}}{2} \right] \\ & + \frac{u_t}{m-1} \left[ mp - 2U(\varphi) + (m-2)\mu + \frac{S^{\varphi}}{2} - p + U(\varphi) \right] \\ = & - u \frac{S_t^{\varphi}}{2} + u U^a \varphi_t^a + \frac{1}{m-1} \left[ u \left( (m-1)p - U(\varphi) + (m-2)\mu + \frac{S^{\varphi}}{2} \right) \right]_t, \end{split}$$

that is,

$$(3.5) \quad u\nabla\left(\frac{S^{\varphi}}{2} - U(\varphi)\right) = \frac{1}{m-1}\nabla\left[u\left((m-1)p - U(\varphi) + (m-2)\mu + \frac{S^{\varphi}}{2}\right)\right].$$
 Using (3.3) iv) we infer (3.2).

**Remark** 3.4. Note that, if in (3.3) iii')  $d\varphi$  is a submersion at each point of M, it gives the validity of (1.3) iii); however, in general the latter is stronger.

**Proposition** 3.5. In the assumptions of Proposition 3.3, suppose that  $\partial M \neq \emptyset$ . Then  $\mu$  is constant if and only if  $p = -\mu$ .

PROOF. By Proposition 3.3, if  $\mu$  is constant on M, we have

(3.6) 
$$\nabla[u(\mu+p)] = u\nabla\mu = 0,$$

that is,  $u(\mu + p)$  is constant on M. Since u > 0 on int(M) and u = 0 on  $\partial M$ , we deduce

$$u(\mu + p) = 0$$
 on  $M$ ,

and thus

$$\mu + p = 0$$
 on  $M$ .

Conversely, if we assume  $p + \mu = 0$ , Proposition 3.3 implies

$$u\nabla\mu=0;$$

since u > 0 on int(M), we deduce that  $\mu$  is constant on int(M) and therefore on M.

If we assume  $p = -\mu$ , using (1.3) i), ii) and iv) we get

$$\operatorname{Hess}(u) - u \left( \operatorname{Ric}^{\varphi} - \frac{S^{\varphi}}{m-1} g \right) = 0;$$

adding the assumption U constant, from (1.3) iii) we infer

(3.7) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \left( \operatorname{Ric}^{\varphi} - \frac{S^{\varphi}}{m-1} g \right) = 0, \\ ii) \Delta u = -\frac{S^{\varphi}}{m-1} u, \\ iii) u\tau(\varphi) + d\varphi(\nabla u) = 0, \end{cases}$$

where the second equation is the trace of the first. In other words, the  $\varphi$ -static space system (3.7) can be deduced from (1.3) i), ii), iii) and iv) in the assumptions  $p = -\mu$  and U constant (i.e., assuming lowest level of the Null Energy Condition and constant scalar potential).

It is well-known, at least for  $\varphi$  constant, that (3.7) implies  $S^{\varphi}$  constant; as a matter of fact, we have

**Proposition** 3.6. Let (M,g) be a manifold of dimension  $m \geq 2$  and  $u \geq 0$ ,  $u \not\equiv 0$  a solution on M of the system

(3.8) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \left( \operatorname{Ric}^{\varphi} - \frac{S^{\varphi}}{m-1} g \right) = 0, \\ ii) h(\alpha u \tau(\varphi) + \alpha d \varphi(\nabla u) - u(\nabla U)(\varphi), d\varphi) = 0. \end{cases}$$

Then

$$\frac{1}{2}u\nabla S^{\varphi} = u\nabla(U(\varphi)).$$

In particular, if  $U(\varphi)$  is constant, then  $S^{\varphi}$  is also constant.

Proof. Tracing (3.8) i) we obtain

(3.9) 
$$\Delta u = -\frac{S^{\varphi}}{m-1}u,$$

which enables us to rewrite (3.8) i) in the form

(3.10) 
$$\operatorname{Hess}(u) - u\operatorname{Ric}^{\varphi} - \Delta ug = 0.$$

From (3.8) ii), that is,

(3.11) 
$$\alpha u \varphi_{tt}^a \varphi_j^a = -\alpha \varphi_i^a u_i \varphi_j^a + u U^a \varphi_j^a$$

and from the  $\varphi$ -Schur's identity,

$$\frac{1}{2}uS_{j}^{\varphi}=\left(uR_{ij}^{\varphi}\right)_{i}-u_{i}R_{ij}^{\varphi}+\alpha u\varphi_{tt}^{a}\varphi_{j}^{a};$$

we now use (3.9), (3.10) and (3.11) into the above to obtain

$$\begin{split} \frac{1}{2}uS_j^{\varphi} &= \left(uR_{ij}^{\varphi}\right)_i - u_iR_{ij}^{\varphi} - \alpha u_i\varphi_i^a\varphi_j^a + uU^a\varphi_j^a \\ &= \left(u_{ij} - \Delta u\delta_{ij}\right)_i - u_i\left(R_{ij}^{\varphi} + \alpha\varphi_i^a\varphi_j^a\right) + uU^a\varphi_j^a \\ &= u_{iji} - u_{ttj} - u_iR_{ij} + uU^a\varphi_i^a = uU^a\varphi_i^a, \end{split}$$

where in the last equality we have used the Ricci commutation relations. In other words

$$\frac{1}{2}u\nabla S^{\varphi} = uU^{a}\varphi_{j}^{a},$$

which gives the first part of the statement. For the second part, we set

$$\Sigma_0 := \{ x \in M : u(x) = 0 \};$$

then  $S^{\varphi}$  is constant on each component of  $M \setminus \Sigma_0$ . If  $\operatorname{int}(\Sigma_0) = \emptyset$ , then, by continuity,  $S^{\varphi}$  is constant on M. If  $\operatorname{int}(\Sigma_0) \neq \emptyset$ , then  $u \equiv 0$  on an open set of M, and it satisfies (3.9); hence, by the unique continuation property (see Appendix A of [60] and also [42]), we have  $u \equiv 0$  on M, which is a contradiction.

## 3.2. On the Constancy of the Map $\varphi$

As mentioned above, it is worth to try to determine sufficient conditions for the constancy of the map  $\varphi$ ; towards this aim, we recall the classical Bochner-Weitzenböck formula for the energy density of a smooth map  $\varphi:(M,g)\to(N,h)$  (for more details, see for instance [26] and Proposition 1.5 of [1]). We have

$$(3.13) \qquad \frac{1}{2}\Delta |d\varphi|^2 = |\nabla d\varphi|^2 + \varphi_i^a \varphi_{kki}^a + {}^N R_{bcd}^a \varphi_i^a \varphi_k^b \varphi_k^c \varphi_i^d + R_{ti} \varphi_t^a \varphi_i^a,$$

where  ${}^{N}R^{a}_{bcd}$  denote the components of the Riemann curvature tensor of (N, h) with respect to a local orthonormal coframe on N.

We consider the system

(3.14) 
$$\left\{ \begin{aligned} i)\operatorname{Hess}(u) - u \Big\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} (S^{\varphi} - (p+\mu))g \Big\} &= 0, \\ ii) \, u\tau(\varphi) &= -d\varphi(\nabla u) + \frac{u}{\alpha} (\nabla U)(\varphi), \end{aligned} \right.$$

with u > 0 on int(M). Note that (3.14) is obtained from (1.3) in the following way: with the aid of (1.3) ii), we rewrite (1.3) i) in the form

(3.15) 
$$\operatorname{Hess}(u) - u \left\{ \operatorname{Ric}^{\varphi} - \frac{1}{m} \left( S^{\varphi} - \frac{\Delta u}{u} \right) \right\} = 0.$$

Next, using (1.3) iv) into (1.3) ii) we obtain

(3.16) 
$$\frac{\Delta u}{u} = \frac{1}{m-1} [m(\mu+p) - S^{\varphi}];$$

inserting (3.16) into (3.15) yields (3.14) i). The second equation, that is (3.14) ii), is simply (1.3) iii).

We proceed by introducing a definition that will be useful to prove the next result; we refer to it also in the statement of Theorem 1.14.

**Definition** 3.7. Let (M,g) be a Riemannian manifold of dimension m and let  $F: M \to \mathbb{R}$  be a smooth function; then, we say that F is weakly convex if the Hessian of F, Hess(F), is positive semi-definite.

Now let

$$(3.17) f := -\log u on int(M);$$

we rewrite system (3.14) in the form

(3.18) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - df \otimes df = \frac{1}{m-1} (S^{\varphi} - (p+\mu))g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha} (\nabla U)(\varphi). \end{cases}$$

We contract (3.18) i) by  $\varphi_i^a \varphi_j^a$  and we use (3.18) ii) to obtain

$$(3.19) \quad R_{ij}^{\varphi}\varphi_{i}^{a}\varphi_{j}^{a} = \frac{1}{m-1}(S^{\varphi} - (p+\mu))|d\varphi|^{2} - f_{ij}\varphi_{i}^{a}\varphi_{j}^{a} + |\tau(\varphi)|^{2} + \frac{1}{\alpha^{2}}|\nabla U(\varphi)|^{2} - \frac{2}{\alpha}h(\tau(\varphi), \nabla U(\varphi)).$$

Inserting (3.19) into (3.13) and using (3.18) ii) we get

$$\begin{aligned} &(3.20) \\ &\frac{1}{2}\Delta|d\varphi|^2 = |\nabla d\varphi|^2 + \varphi_i^a \varphi_{kki}^a + {}^N R_{bcd}^a \varphi_i^a \varphi_k^b \varphi_k^c \varphi_i^d + \alpha \varphi_i^a \varphi_t^a \varphi_i^b \varphi_k^b \\ &\quad + \frac{1}{m-1}(S^\varphi - (p+\mu))|d\varphi|^2 - f_{ij}\varphi_i^a \varphi_j^a + |\tau(\varphi)|^2 + \frac{1}{\alpha^2}|\nabla U(\varphi)|^2 \\ &\quad - \frac{2}{\alpha}h(\tau(\varphi), \nabla U(\varphi)) \\ &= |\nabla d\varphi|^2 + \varphi_{si}^a \varphi_i^a f_s + \varphi_i^a \varphi_s^a f_{si} + \frac{1}{\alpha}U^{ab}\varphi_i^a \varphi_i^b + {}^N R_{bcd}^a \varphi_i^a \varphi_k^b \varphi_k^c \varphi_i^d \\ &\quad + \alpha \varphi_i^a \varphi_i^a \varphi_i^b \varphi_t^b + \frac{1}{m-1}(S^\varphi - (p+\mu))|d\varphi|^2 - f_{ij}\varphi_i^a \varphi_j^a + |\tau(\varphi)|^2 \\ &\quad + \frac{1}{\alpha^2}|\nabla U(\varphi)|^2 - \frac{2}{\alpha}h(\tau(\varphi), \nabla U(\varphi)) \\ &= |\nabla d\varphi|^2 + \frac{1}{2}g\Big(\nabla|d\varphi|^2, \nabla f\Big) + \frac{1}{\alpha}U^{ab}\varphi_i^a \varphi_i^b + \alpha \varphi_i^a \varphi_i^a \varphi_i^b \varphi_t^b + |\tau(\varphi)|^2 \\ &\quad + \frac{1}{m-1}(S^\varphi - (p+\mu))|d\varphi|^2 + \frac{1}{\alpha^2}|\nabla U(\varphi)|^2 + {}^N R_{bcd}^a \varphi_i^a \varphi_k^b \varphi_k^c \varphi_i^d \\ &\quad - \frac{2}{\alpha}h(\tau(\varphi), \nabla U(\varphi)). \end{aligned}$$

We are thus ready to prove

**Lemma** 3.8. Let (M, g) be a manifold of dimension  $m \ge 2$  satisfying (3.18) with  $\alpha > 0$ . Assume

where Sect(N) is the sectional curvature of N and A is a real constant. Then, on int(M),

$$(3.22) \qquad \frac{1}{2}\Delta_{f}|d\varphi|^{2} \geq |\nabla d\varphi|^{2} + \left(\frac{\alpha}{m} - A\right)|d\varphi|^{4} + |d\varphi(\nabla f)|^{2} + \frac{1}{m-1}(S^{\varphi} - (p+\mu))|d\varphi|^{2} + \frac{1}{\alpha}\operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)),$$

where  $\Delta_f = \Delta - g(\nabla f, \cdot)$  is the f-Laplacian.

PROOF. Observe that, using (3.18) ii),

$$\frac{1}{\alpha^2} |\nabla U(\varphi)|^2 - \frac{2}{\alpha} h(\tau(\varphi), \nabla U(\varphi)) + |\tau(\varphi)|^2 = \left| \frac{1}{\alpha} \nabla U - \tau(\varphi) \right|^2 = |d\varphi(\nabla f)|^2.$$

Furthermore, since  $\alpha > 0$ ,

$$\alpha \varphi_i^a \varphi_t^a \varphi_i^b \varphi_t^b \ge \alpha \frac{|d\varphi|^4}{m},$$

and (3.21) implies

$${}^{N}R^{a}_{bcd}\varphi^{a}_{i}\varphi^{b}_{k}\varphi^{c}_{k}\varphi^{d}_{i} \geq -A|d\varphi|^{4}.$$

Inserting this information into (3.20) immediately yields the validity of (3.22).  $\Box$ 

Now observe that, for  $v = |d\varphi|^2$ ,  $\frac{\alpha}{m} > A$ , U weakly convex and supposing  $\alpha > 0$  and  $\sigma := \inf_M (S^{\varphi} - (p + \mu)) > -\infty$ , (3.22) gives

(3.23) 
$$\frac{1}{2}\Delta_f v \ge \left(\frac{\alpha}{m} - A\right)v^2 + \frac{\sigma}{m-1}v.$$

We recall that  $\partial M = u^{-1}(\{0\})$  if not empty and therefore, since  $f = -\log u$ , the operator  $\Delta_f$  is not well defined on  $\partial M$ . However, when  $\partial M \neq \emptyset$  we require  $d\varphi \equiv 0$  on  $\partial M$ ; it follows that, when  $v \not\equiv 0$ , if  $\gamma > 0$  and near to  $v^* = \sup_M v > 0$  (possibly infinite), equation (3.23) is well defined on the super-level set

$$\Omega_{\gamma} = \{ x \in M : v(x) > \gamma \},\$$

since the latter has empty intersection with  $\partial M$ . This enables us to apply Theorems 4.2 and 4.1 of [1] to  $\Omega_{\gamma}$  and conclude via (3.23) that  $v^* < +\infty$  and

$$(3.24) v^* \le -\frac{\sigma}{(m-1)(\frac{\alpha}{m}-A)},$$

provided the growth condition

$$\liminf_{r \to +\infty} \frac{1}{r^2} \log \int_{B_r} e^{-f} < +\infty,$$

where  $B_r$  is a geodesic ball of radius r centered at some origin  $o \in M$ . Thus, assume that (M, g), with  $\partial M \neq \emptyset$ , is complete in the sense of Cauchy; in order to guarantee

(3.25), we suppose that  $\partial M$  is compact.

To continue, we introduce the *topological double* of M, defined gluing two copies of M along their boundaries:

$$\mathcal{D}(M) = \frac{M_0 \cup M_1}{2} = \frac{M \times \{0\} \cup M \times \{1\}}{2}$$

where, for each  $x \in \partial M$ ,  $\sim$  identifies the points (x,0) and (x,1) (see [54] and [60] for more details). To define a differentiable structure on  $\mathcal{D}(M)$ , we consider two open sets  $W_i$ , i = 0, 1, such that  $\partial M \subset W_i \subset M_i$  for each copy of M, and two diffeomorphisms

$$h_0: W_0 \to \partial M \times [0,1), \qquad h_1: W_1 \to \partial M \times (-1,0]$$

such that

$$h_0(x) = (x, 0) = h_1(x) \quad \forall x \in \partial M.$$

Let  $W = W_0 \cup W_1$  be the union of  $W_0$  and  $W_1$  in  $\mathcal{D}(M)$ ; then there is a homeomorphism

$$h: W \to \partial M \times (-1,1)$$

such that  $h|_{W_i} = h_i$ , i = 0, 1; moreover, consider the inclusions

$$i_0: \partial M \times \{0\} \hookrightarrow \mathcal{D}(M),$$

$$i_1: \partial M \times \{1\} \hookrightarrow \mathcal{D}(M).$$

A differentiable structure on  $\mathcal{D}(M)$  is obtained imposing that h is a diffeomorphism and  $i_0, i_1$  are smooth embeddings.

The Riemannian metric g on M induces Riemannian metrics  $g_0, g_1$  on  $M_0$  and  $M_1$ , respectively. Moreover, let

$$h: W \to \partial M \times (-1,1)$$

be a diffeomorphism such that  $h|_{W_i} = h_i$ , i = 0, 1; then we define

$$g_W := h^* (g_{|_{\partial M}} + dt \otimes dt).$$

Given  $V \subset W$  open and relatively compact such that  $\partial M \subset V$ , we can define a Riemannian metric on  $\mathcal{D}(M)$ ,

$$g^{\mathcal{D}} = \psi_0 g_0 + \psi_1 g_1 + \psi_V g_W$$

where  $\{\psi_0, \psi_1 \psi_V\}$  is a partition of unity relative to the open cover

$$\{ int(M_0), int(M_1), V \}.$$

Note that Cauchy sequences for the metric  $g^{\mathcal{D}}$  are convergent in  $\mathcal{D}(M)$  and, since  $\partial \mathcal{D}(M) = \emptyset$ , we have that  $\mathcal{D}(M)$  is geodesically complete by the Hopf-Rinow Theorem. Moreover, by the definition of  $g^{\mathcal{D}}$  and since  $\partial M$  is compact, the volume of

 $B_r(o) \subset M$  is bounded above by the volume of a ball in  $\mathcal{D}(M)$  centered at o and of radius possibly dilated by a positive constant a. Thus, to control

$$\int_{B_r(o)} e^{-f}$$

from above, it is enough to bound

$$\int_{B_{ar}(0)\backslash U} e^{-f}$$

for U open, relatively compact such that  $\partial M \subset U$  and with f extended on  $\mathcal{D}(M)$  outside of U.

Now we know, in the case we are interested in, that f satisfies (3.18) i), so that, for  $\alpha > 0$  and

(3.26) 
$$\sigma = \inf_{M} \left( S^{\varphi} - (p + \mu) \right),$$

assuming  $\sigma > -\infty$ , we have

(3.27) 
$$\operatorname{Ric} + \operatorname{Hess}(f) = \alpha \varphi^* h + df \otimes df + \frac{1}{m-1} (S^{\varphi} - (p+\mu)) g \ge \sigma g.$$

Thus, inequality (8.115) of [1] gives

(3.28) 
$$\int_{B_{ar}(o)\setminus U} e^{-f} \le D + \int_0^{ar} e^{Ct - \frac{\sigma}{2(m-1)}t^2} dt$$

for some sufficiently large constants C, D > 0. It follows that

(3.29) 
$$\int_{B_r(o)} u \quad \text{grows at most like} \quad \begin{cases} \frac{e^{\frac{|\sigma|}{2}a^2r^2}}{ar} & \text{if } \sigma \neq 0, \\ \frac{e^{Car}}{2} & \text{if } \sigma = 0. \end{cases}$$

when  $r \to +\infty$ . In particular, (3.25) is satisfied.

We are now ready to state the next

**Theorem** 3.9. Let (M,g) and (N,h) be two Riemannian manifolds, such that (M,g) is complete with compact boundary  $\partial M$  and dimension  $m \geq 2$ . Let u be a solution of (3.14), where  $\varphi : (M,g) \to (N,h)$  is a smooth map,  $U: N \to \mathbb{R}$ ,  $p, \mu : M \to \mathbb{R}$  are smooth functions, U is weakly convex,  $\alpha > 0$ , and assume

$$\begin{split} d\varphi &\equiv 0 &\quad on \; \partial M, \\ \sigma &= \inf_M \left\{ S^\varphi - (p+\mu) \right\} > -\infty, \\ \mathrm{Sect}(N) &\leq A, \qquad A < \frac{\alpha}{m}. \end{split}$$

Then, either  $\varphi$  is constant or

$$(3.30) |d\varphi|^2 \le -\frac{\sigma}{(m-1)(\frac{\alpha}{m}-A)} on M.$$

In particular, if  $S^{\varphi} - (p + \mu) \ge 0$ , then  $\varphi$  is constant.

**Remark** 3.10. If we assume the validity of (3.14) up to the boundary, that is, also on  $\partial M$  if not empty, then it is not hard to see that for  $\alpha > 0$  and  $\operatorname{Sect}(N) \leq A$  we obtain the validity of the following formula: (3.31)

$$\frac{1}{2}\operatorname{div}\left(u\nabla|d\varphi|^{2}\right) \geq u|\nabla d\varphi|^{2} + \left(\frac{\alpha}{m} - A\right)|d\varphi|^{4}u + \left|\frac{1}{\alpha}\nabla U(\varphi) - \tau(\varphi)\right|^{2}u + \frac{1}{m-1}(S^{\varphi} - (p+\mu))|d\varphi|^{2}u + \frac{u}{\alpha}\operatorname{tr}\left(\operatorname{Hess}(U)\right)(d\varphi, d\varphi).$$

Thus, recalling that u > 0 on int(M) and u = 0 on  $\partial M$ , we have the following

**Proposition** 3.11. Let (M,g) be a compact manifold of dimension  $m \geq 2$ , with  $\partial M \neq \emptyset$ . Suppose that u is a solution of (3.14) up to the boundary with  $\alpha > 0$  and let U be weakly convex,  $\operatorname{Sect}(N) \leq A$ ,  $A < \frac{\alpha}{m}$ . Assume

$$S^{\varphi} - (p + \mu) \ge 0$$
 on  $M$ .

Then  $\varphi$  is constant.

PROOF. Simply integrate (3.31) on M, use  $u \equiv 0$  on  $\partial M$  and the remaining assumptions of the proposition, to deduce that

$$u\left(\frac{\alpha}{m} - A\right) |d\varphi|^2 \equiv 0$$
 on  $int(M)$ ,

and thus  $|d\varphi|^2 \equiv 0$  on M.

#### CHAPTER 4

# A Kobayashi-Obata type Theorem

In 1980 Kobayashi and Obata (see [43]) proved that a conformally flat manifold (M, g) that admits a non-constant positive solution u of the equation

(4.1) 
$$\operatorname{Hess}(u) - u \left( \operatorname{Ric} - \frac{1}{m} (uS - \Delta u) g \right) = 0$$

is such that, for any regular value c of u, the hypersurface  $\Sigma = u^{-1}(\{c\})$  has constant curvature and the metric g splits locally as a warped product  $g = dt^2 + \rho(t)^2 g_{\Sigma}$ , where

$$dt = \frac{du}{|du|}.$$

Needless to say, a prominent role in the proof of the result is played by the assumption  $W \equiv 0$ .

Note that, if we make the change of variable

$$f = -\log u$$
,

(4.1) transforms into the system

(4.2) 
$$\operatorname{Ric} + \operatorname{Hess}(f) - df \otimes df = \lambda g,$$

with

$$\lambda = \frac{1}{m} \left( S + \Delta f - |\nabla f|^2 \right) \in C^{\infty}(M).$$

We want to interpret the assumption  $W \equiv 0$  in an alternative way. Towards this aim, we consider the pointwise conformal change of metric g of M given by

$$\widetilde{g} = e^{-\frac{2}{m-2}f}g,$$

where  $m = \dim M \geq 3$ . It is well known (see for instance [6]) that the Cotton tensor C of g changes accordingly to the formula

$$\widetilde{C} = C + W(\nabla f, \cdot, \cdot, \cdot),$$

where  $\widetilde{C}$  is the Cotton tensor with respect to  $\widetilde{g}$ . Thus, the condition  $W \equiv 0$  implies that the conformal change of metric  $\widetilde{g}$  gives rise to a Cotton-flat metric on M. This suggests two distinct aims: the first is that an appropriate formulation of  $\widetilde{C}$  is related to the geometry of the level set hypersurfaces of f; the second is that we

can probably weaken the assumption  $W \equiv 0$  to get the result.

In our setting, we are considering a more general system than (4.2), precisely

(4.4) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi), \end{cases}$$

 $\lambda \in C^{\infty}(M)$ . Let  $\widetilde{\varphi} = \varphi : (M, \widetilde{g}) \to (N, h)$ , then the conformal change (4.3) gives the transformation (2.84), that we recall here,

$$\widetilde{C}^{\widetilde{\varphi}} = C^{\varphi} + W^{\varphi}(\nabla f, \cdot, \cdot, \cdot),$$

but information on U and  $\nabla U$  is missing and cannot depend on (4.3). The latter fact is, somehow, an indication that we have to consider an "intrinsic" point of view. The right idea seems to be suggested by the introduction of a modification of the well-known tensor D, first introduced by Cao and Chen ([12]) to study gradient Ricci solitons. In particular, they derive the so-called integrability condition for a Ricci soliton,

$$D = C + W(\nabla f, \cdot, \cdot, \cdot).$$

Therefore, we introduce a suitable modification  $\overline{D}^{\varphi}$  of D and then we determine the first (and the second) integrability conditions for system (4.4); then, we study the geometry of a regular level set hypersurface of f under the assumption

$$\overline{D}^{\varphi} \equiv 0.$$

## 4.1. First and Second Integrability Conditions

As promised in the Introduction, we study system (4.4), putting a special emphasis on the role of the tensor

$$\overline{D}_{ijk}^{\varphi} = \frac{1}{m-2} \left[ R_{ij}^{\varphi} f_k - R_{ik}^{\varphi} f_j + \frac{1}{m-1} f_t \left( R_{tk}^{\varphi} \delta_{ij} - R_{tj}^{\varphi} \delta_{ik} \right) - \frac{S^{\varphi}}{m-1} (f_k \delta_{ij} - f_j \delta_{ik}) \right],$$

which is well defined for  $m \geq 3$ . Note that we have the validity of the symmetries

$$\overline{D}_{ijk}^{\varphi} = -\overline{D}_{ikj}^{\varphi}, \qquad \overline{D}_{iik}^{\varphi} = 0$$

and

$$\overline{D}_{ijk}^{\varphi} + \overline{D}_{kij}^{\varphi} + \overline{D}_{jki}^{\varphi} = 0;$$

moreover, it is an easy matter to check that when (4.4) is satisfied, we can express  $\overline{D}^{\varphi}$  in the form

$$\overline{D}_{ijk}^{\varphi} = \frac{1}{m-2} \left[ f_{ik} f_j - f_{ij} f_k + \frac{1}{m-1} f_t (f_{tj} \delta_{ik} - f_{tk} \delta_{ij}) - \frac{\Delta f}{m-1} (f_j \delta_{ik} - f_k \delta_{ij}) \right].$$

Note that  $\lambda$  does not appear in (4.6).

**Proposition** 4.1 (First and Second integrability conditions). Let (M,g) be a manifold of dimension  $m \geq 3$ . Let  $\varphi : (M,g) \to (N,h)$ ,  $U : (N,h) \to \mathbb{R}$ ,  $\lambda : (M,g) \to \mathbb{R}$  be smooth maps,  $\alpha \in \mathbb{R} \setminus \{0\}$  and let  $f \in C^{\infty}(M)$  be a solution of (4.4) on  $\operatorname{int}(M)$ . We then have

$$(4.7) [1 + \eta(m-2)]\overline{D}_{ijk}^{\varphi} = C_{ijk}^{\varphi} + f_t W_{tijk}^{\varphi} + \frac{U^a}{m-1} (\varphi_j^a \delta_{ik} - \varphi_k^a \delta_{ij})$$

and

(4.8)

$$\begin{split} (m-2)B_{ij}^{\varphi} = & [1+\eta(m-2)] \bigg\{ \overline{D}_{ijk,k}^{\varphi} - \frac{\alpha}{m-2} \varphi_{ss}^a \varphi_i^a f_j \bigg\} + \frac{m-3}{m-2} f_k C_{jik}^{\varphi} - \eta W_{tijk}^{\varphi} f_t f_k \\ & + \frac{U^a}{m-1} \bigg[ (m-2) \varphi_{ij}^a - \frac{1}{m-2} \varphi_{ss}^a \delta_{ij} \bigg] \\ & + \frac{U^{ab}}{m-1} \big[ \varphi_k^a \varphi_k^b \delta_{ij} - m \varphi_i^a \varphi_j^b \big] \\ & + \eta f_i U^a \varphi_i^a. \end{split}$$

**Remark** 4.2. When U is constant, equations (4.7) and (4.8) reduce to the first and second integrability conditions of a Riemannian manifold endowed with a non-trivial Einstein-type structure (see [3]).

**Remark** 4.3. Formula (4.7) is strictly related to the conformal change of metric given, for  $m \geq 3$ , by

$$\widetilde{g} = e^{-\frac{2}{m-2}f}g.$$

In view of the conformal transformation law (2.84), that is

$$\widetilde{C}^{\widetilde{\varphi}} = C^{\varphi} + W^{\varphi}(\nabla f, \cdot, \cdot, \cdot),$$

equation (4.7) takes the form

$$[1 + \eta(m-2)]\overline{D}^{\varphi} = \widetilde{C}^{\widetilde{\varphi}} + \frac{1}{2(m-1)}\operatorname{div}_{1}(U(\varphi)g \otimes g).$$

Note that, when

$$2(m-1)\widetilde{C}^{\widetilde{\varphi}} = -\operatorname{div}_1(U(\varphi)g \bigotimes g),$$

then  $\overline{D}^{\varphi} = 0$ .

**Remark** 4.4. Note that, since  $B_{ij}^{\varphi}$  (see [3, Proposition 2.38]) and the terms on the second and third lines of (4.8) are symmetric, then

$$[1+\eta(m-2)]\left\{\overline{D}_{ijk,k}^{\varphi}-\frac{\alpha}{m-2}\varphi_{ss}^{a}\varphi_{i}^{a}f_{j}\right\}+\frac{m-3}{m-2}f_{k}C_{ijk}^{\varphi}-\eta W_{tijk}^{\varphi}f_{t}f_{k}+\eta f_{j}U^{a}\varphi_{i}^{a}$$

has to be symmetric with respect to the indices i and j.

PROOF (OF PROPOSITION 4.1). To obtain equation (4.7), we need to find a "good" expression for  $C^{\varphi}$ . We use the first equation of system (4.4) and the Ricci commutation relations to deduce

$$\begin{split} C_{ijk}^{\varphi} = & R_{ij,k}^{\varphi} - R_{ik,j}^{\varphi} - \frac{1}{2(m-1)} \left( S_k^{\varphi} \delta_{ij} - S_j^{\varphi} \delta_{ik} \right) \\ = & - f_{ijk} + f_{ikj} + \eta (f_{ik} f_j + f_i f_{jk} - f_{ij} f_k - f_i f_{jk}) \\ & + \lambda_k \delta_{ij} - \lambda_j \delta_{ik} - \frac{1}{2(m-1)} \left( S_k^{\varphi} \delta_{ij} - S_j^{\varphi} \delta_{ik} \right) \\ = & - f_p R_{pijk} + \eta (f_{ik} f_j - f_{ij} f_k) + \left( \lambda_k - \frac{1}{2(m-1)} S_k^{\varphi} \right) \delta_{ij} \\ & - \left( \lambda_j - \frac{1}{2(m-1)} S_j^{\varphi} \right) \delta_{ik}, \end{split}$$

that is,

$$(4.9) C_{ijk}^{\varphi} = -f_p R_{pijk} + \eta (f_{ik} f_j - f_{ij} f_k) + \left(\lambda_k - \frac{1}{2(m-1)} S_k^{\varphi}\right) \delta_{ij}$$
$$-\left(\lambda_j - \frac{1}{2(m-1)} S_j^{\varphi}\right) \delta_{ik}.$$

Now we contract (4.9) with respect to i and j and recall that  $C_{ttk}^{\varphi} = \alpha \varphi_{tt}^{a} \varphi_{k}^{a}$ , to obtain

$$\alpha \varphi_{tt}^a \varphi_k^a = f_t R_{tk} + \eta (f_{tk} f_t - \Delta f f_k) + (m-1)\lambda_k - \frac{1}{2} S_k^{\varphi}.$$

Using the definition of  $\mathrm{Ric}^{\varphi}$  and rearranging the terms we get

$$\frac{1}{2}S_k^{\varphi} - (m-1)\lambda_k = f_t R_{tk}^{\varphi} + \eta (f_{tk}f_t - \Delta f f_k) - \alpha \varphi_k^a (\varphi_{tt}^a - \varphi_t^a f_t).$$

Inserting the second equation of (4.4) into the above formula we deduce

(4.10) 
$$\frac{1}{2}S_k^{\varphi} - (m-1)\lambda_k = f_t R_{tk}^{\varphi} + \eta (f_{tk}f_t - \Delta f f_k) - U^a \varphi_k^a,$$

while using (4.10) into (4.9) we obtain

(4.11) 
$$C_{ijk}^{\varphi} = -f_{p}R_{pijk} + \eta(f_{ik}f_{j} - f_{ij}f_{k}) - \frac{\delta_{ij}}{m-1}[f_{t}R_{tk}^{\varphi} + \eta(f_{tk}f_{t} - \Delta f f_{k}) - U^{a}\varphi_{k}^{a}] + \frac{\delta_{ik}}{m-1}[f_{t}R_{tj}^{\varphi} + \eta(f_{tj}f_{t} - \Delta f f_{j}) - U^{a}\varphi_{j}^{a}].$$

We now multiply the first equation of system (4.4) by  $\eta f_k$ , that is

$$\eta f_{ij} f_k = -\eta R_{ij}^{\varphi} f_k + \eta f_i f_j f_k + \eta \lambda f_k \delta_{ij},$$

and skew-symmetrize the latter equation, in order to deduce

(4.12) 
$$\eta(f_{ik}f_j - f_{ij}f_k) = \eta(R_{ij}^{\varphi}f_k - R_{ik}^{\varphi}f_j + \lambda f_j \delta_{ik} - \lambda f_k \delta_{ij});$$

contracting (4.12) with respect to i and j we also have

(4.13) 
$$\eta(f_{tk}f_t - \Delta f f_k) = \eta(S^{\varphi}f_k - f_t R_{tk}^{\varphi} - (m-1)\lambda f_k).$$

Using (4.12) and (4.13) into (4.11) we obtain

$$\begin{split} C_{ijk}^{\varphi} &= -f_p R_{pijk} + \eta (R_{ij}^{\varphi} f_k - R_{ik}^{\varphi} f_j + \lambda f_j \delta_{ik} - \lambda f_k \delta_{ij}) \\ &- \frac{\delta_{ij}}{m-1} [(1-\eta) f_t R_{tk}^{\varphi} + \eta S^{\varphi} f_k - (m-1) \eta \lambda f_k - U^a \varphi_k^a] \\ &+ \frac{\delta_{ik}}{m-1} [(1-\eta) f_t R_{tj}^{\varphi} + \eta S^{\varphi} f_j - (m-1) \eta \lambda f_j - U^a \varphi_j^a], \end{split}$$

which can be written as

(4.14) 
$$C_{ijk}^{\varphi} = -f_{p}R_{pijk} + \eta(R_{ij}^{\varphi}f_{k} - R_{ik}^{\varphi}f_{j})$$
$$-\frac{\delta_{ij}}{m-1}[(1-\eta)f_{t}R_{tk}^{\varphi} + \eta S^{\varphi}f_{k} - U^{a}\varphi_{k}^{a}]$$
$$+\frac{\delta_{ik}}{m-1}[(1-\eta)f_{t}R_{tj}^{\varphi} + \eta S^{\varphi}f_{j} - U^{a}\varphi_{j}^{a}].$$

Now we use the definition of  $W^{\varphi}$  to get

$$f_{p}R_{pijk} = f_{p}W_{pijk}^{\varphi} + \frac{f_{j}}{m-2}R_{ik}^{\varphi} + \frac{f_{p}R_{pj}^{\varphi}}{m-2}\delta_{ik} - \frac{f_{k}}{m-2}R_{ij}^{\varphi}$$
$$-\frac{f_{p}R_{pk}^{\varphi}}{m-2}\delta_{ij} - \frac{S^{\varphi}}{(m-1)(m-2)}(f_{j}\delta_{ik} - f_{k}\delta_{ij}).$$

Inserting the previous relation into (4.14) we obtain, after some simplifications,

$$\begin{split} C_{ijk}^{\varphi} = & \frac{1 + \eta(m-2)}{m-2} \left[ R_{ij}^{\varphi} f_k - R_{ik}^{\varphi} f_j - \frac{f_t}{m-1} \left( R_{tj}^{\varphi} \delta_{ik} - R_{tk}^{\varphi} \delta_{ij} \right) \right] \\ & + \frac{1 + \eta(m-2)}{(m-1)(m-2)} S^{\varphi} (f_j \delta_{ik} - f_k \delta_{ij}) - f_p W_{pijk}^{\varphi} + \frac{U^a}{m-1} \varphi_k^a \delta_{ij} \\ & - \frac{U^a}{m-1} \varphi_j^a \delta_{ik}. \end{split}$$

Using the definition (4.5) of  $\overline{D}^{\varphi}$  we get

$$C_{ijk}^{\varphi} = [1 + \eta(m-2)]D_{ijk}^{\varphi} - f_p W_{pijk}^{\varphi} + \frac{U^a}{m-1} \varphi_k^a \delta_{ij} - \frac{U^a}{m-1} \varphi_j^a \delta_{ik},$$

and therefore (4.7).

To obtain (4.8), we compute div<sub>3</sub> of (4.7):

(4.15)

$$[1 + \eta(m-2)]\overline{D}_{ijk,k}^{\varphi} = C_{ijk,k}^{\varphi} + f_{tk}W_{tijk}^{\varphi} + f_{t}W_{tijk,k}^{\varphi} + \frac{U^{a}}{m-1}(\varphi_{ij}^{a} - \varphi_{kk}^{a}\delta_{ij}) + \frac{U^{ab}}{m-1}\varphi_{k}^{b}(\varphi_{j}^{a}\delta_{ik} - \varphi_{k}^{a}\delta_{ij}).$$

From (2.51), we have

$$W_{tijk,k}^{\varphi} = \frac{m-3}{m-2} C_{jti}^{\varphi} + \alpha (\varphi_{ij}^{a} \varphi_{t}^{a} - \varphi_{jt}^{a} \varphi_{i}^{a}) + \frac{\alpha}{m-2} \varphi_{ss}^{a} (\varphi_{i}^{a} \delta_{jt} - \varphi_{t}^{a} \delta_{ij}).$$

Inserting the previous relation into (4.15) and using (4.4) we get

$$[1+\eta(m-2)]\overline{D}_{ijk,k}^{\varphi} = C_{ijk,k}^{\varphi} + W_{tijk}^{\varphi}(-R_{tk}^{\varphi} + \eta f_t f_k + \lambda \delta_{tk})$$

$$\begin{split} &+f_t\bigg[\bigg(\frac{m-3}{m-2}\bigg)C^{\varphi}_{jti} + \alpha \big(\varphi^a_{ij}\varphi^a_t - \varphi^a_{jt}\varphi^a_i\big) + \frac{\alpha}{m-2}\varphi^a_{ss}\big(\varphi^a_i\delta_{jt} - \varphi^a_t\delta_{ij}\big)\bigg] \\ &+ \frac{U^a}{m-1}\big(\varphi^a_{ij} - \varphi^a_{kk}\delta_{ij}\big) + \frac{U^{ab}}{m-1}\varphi^b_k\big(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}\big) \\ &= C^{\varphi}_{ijk,k} - W^{\varphi}_{tijk}R^{\varphi}_{tk} + \eta W^{\varphi}_{tijk}f_tf_k + \lambda W^{\varphi}_{tijk}\delta_{tk} + \bigg(\frac{m-3}{m-2}\bigg)f_kC^{\varphi}_{jki} \\ &+ \alpha \big(\varphi^a_{ij}f_t\varphi^a_t - \varphi^a_{jk}f_k\varphi^a_i\big) + \frac{\alpha}{m-2}\varphi^a_{ss}\big(\varphi^a_if_j - \varphi^a_kf_k\delta_{ij}\big) \\ &+ \frac{U^a}{m-1}\big(\varphi^a_{ij} - \varphi^a_{kk}\delta_{ij}\big) + \frac{U^{ab}}{m-1}\varphi^b_k\big(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}\big). \end{split}$$

Using the definition of the  $\varphi$ -Bach tensor (see (2.55)) we deduce

$$(4.16) \quad [1+\eta(m-2)]\overline{D}_{ijk,k}^{\varphi} = (m-2)B_{ij}^{\varphi} + \alpha R_{tj}^{\varphi}\varphi_{i}^{a}\varphi_{i}^{a} - \alpha \left(\varphi_{ij}^{a}\varphi_{kk}^{a} - \varphi_{kkj}^{a}\varphi_{i}^{a}\right)$$

$$+ \frac{\alpha}{m-2}|\tau(\varphi)|^{2}\delta_{ij} + \eta W_{tijk}^{\varphi}f_{t}f_{k} + \lambda W_{tijk}^{\varphi}\delta_{tk} + \frac{m-3}{m-2}f_{k}C_{jki}^{\varphi}$$

$$+ \alpha \left(\varphi_{ij}^{a}f_{t}\varphi_{t}^{a} - \varphi_{jk}^{a}f_{k}\varphi_{i}^{a}\right) + \frac{\alpha}{m-2}\varphi_{ss}^{a}(\varphi_{i}^{a}f_{j} - \varphi_{k}^{a}f_{k}\delta_{ij})$$

$$+ \frac{U^{a}}{m-1}\left(\varphi_{ij}^{a} - \varphi_{kk}^{a}\delta_{ij}\right) + \frac{U^{ab}}{m-1}\left(\varphi_{i}^{a}\varphi_{j}^{b} - \varphi_{k}^{a}\varphi_{k}^{b}\delta_{ij}\right).$$

Observing that, by (4.4),

$$R_{tj}^{\varphi}\varphi_t^a = -f_{tj}\varphi_t^a + \eta\varphi_{ss}^a f_j - \frac{\eta}{\alpha}U^a f_j + \lambda\varphi_j^a,$$

and using (4.4) ii) and equation (2.49), (4.16) yields

$$\begin{split} [1+\eta(m-2)] \overline{D}_{ijk,k}^{\varphi} &= (m-2)B_{ij}^{\varphi} - \alpha f_{tj} \varphi_{t}^{a} \varphi_{i}^{a} + \alpha \eta f_{j} \varphi_{ss}^{a} \varphi_{i}^{a} - \eta f_{j} U^{a} \varphi_{i}^{a} \\ &+ \alpha \lambda \varphi_{i}^{a} \varphi_{j}^{a} - \alpha \left( \varphi_{ij}^{a} \varphi_{kk}^{a} - \varphi_{kkj}^{a} \varphi_{i}^{a} - \frac{|\tau(\varphi)|^{2}}{m-2} \delta_{ij} \right) + \eta f_{t} f_{k} W_{tijk}^{\varphi} \\ &- \alpha \lambda \varphi_{i}^{a} \varphi_{j}^{a} + f_{t} \left( \frac{m-3}{m-2} \right) C_{jti}^{\varphi} + \alpha \left( \varphi_{ij}^{a} \varphi_{ss}^{a} - \frac{1}{\alpha} \varphi_{ij}^{a} U^{a} - f_{t} \varphi_{jt}^{a} \varphi_{i}^{a} \right) \\ &+ \frac{\alpha}{m-2} \varphi_{ss}^{a} \left( f_{j} \varphi_{i}^{a} - \varphi_{tt}^{a} \delta_{ij} + \frac{1}{\alpha} U^{a} \delta_{ij} \right) + \frac{U^{a}}{m-1} \left( \varphi_{ij}^{a} - \varphi_{kk}^{a} \delta_{ij} \right) \\ &+ \frac{U^{ab}}{m-1} \left( \varphi_{j}^{a} \varphi_{i}^{b} - \varphi_{k}^{a} \varphi_{k}^{b} \delta_{ij} \right) \\ &= (m-2)B_{ij}^{\varphi} - \alpha f_{tj} \varphi_{t}^{a} \varphi_{i}^{a} + \alpha \frac{1+\eta(m-2)}{m-2} f_{j} \varphi_{ss}^{a} \varphi_{i}^{a} \\ &- \eta f_{j} U^{a} \varphi_{i}^{a} + \alpha \varphi_{kkj}^{a} \varphi_{i}^{a} + \eta f_{t} f_{k} W_{tijk}^{\varphi} + f_{t} \left( \frac{m-3}{m-2} \right) C_{jti}^{\varphi} \\ &- \alpha f_{t} \varphi_{jt}^{a} \varphi_{i}^{a} + \frac{1}{(m-1)(m-2)} U^{a} \varphi_{kk}^{a} \delta_{ij} \\ &- \left( \frac{m-2}{m-1} \right) U^{a} \varphi_{ij}^{a} + \frac{U^{ab}}{m-1} \varphi_{j}^{a} \varphi_{i}^{b} - \frac{U^{ab}}{m-1} \varphi_{k}^{a} \varphi_{k}^{b} \delta_{ij}, \end{split}$$

that is,

$$(4.17) \quad (m-2)B_{ij}^{\varphi} = [1+\eta(m-2)]\overline{D}_{ijk,k}^{\varphi} + \alpha f_{tj}\varphi_t^a\varphi_i^a - \alpha \frac{1+\eta(m-2)}{m-2}f_j\varphi_{ss}^a\varphi_i^a$$

$$+ \eta f_j U^a\varphi_i^a - \alpha \varphi_{kkj}^a\varphi_i^a - \eta f_t f_k W_{tijk}^{\varphi} - f_t \left(\frac{m-3}{m-2}\right)C_{jti}^{\varphi} + \alpha f_t\varphi_{jt}^a\varphi_i^a$$

$$- \frac{1}{(m-1)(m-2)}U^a\varphi_{kk}^a\delta_{ij} + \left(\frac{m-2}{m-1}\right)U^a\varphi_{ij}^a - \frac{U^{ab}}{m-1}\varphi_j^a\varphi_i^b$$

$$+ \frac{U^{ab}}{m-1}\varphi_k^a\varphi_k^b\delta_{ij}.$$

Taking the covariant derivative of (4.4) ii), we obtain

$$\alpha \varphi_i^a \varphi_{kkj}^a = \varphi_j^b \varphi_i^a U^{ab} + \alpha \varphi_i^a \varphi_t^a f_{tj} + \alpha \varphi_i^a \varphi_{tj}^a f_t.$$

Using the equation above we rewrite (4.17) as

$$\begin{split} (m-2)B_{ij}^{\varphi} = & [1+\eta(m-2)]\overline{D}_{ijk,k}^{\varphi} + \alpha f_{tj}\varphi_{t}^{a}\varphi_{i}^{a} - \alpha \frac{1+\eta(m-2)}{m-2}f_{j}\varphi_{ss}^{a}\varphi_{i}^{a} \\ & + \eta f_{j}U^{a}\varphi_{i}^{a} - \varphi_{j}^{a}\varphi_{i}^{b}U^{ab} - \alpha \varphi_{i}^{a}\varphi_{t}^{a}f_{jt} - \alpha \varphi_{i}^{a}\varphi_{tj}^{a}f_{t} - \eta f_{t}f_{k}W_{tijk}^{\varphi} \\ & - f_{t}\left(\frac{m-3}{m-2}\right)C_{jti}^{\varphi} + \alpha f_{t}\varphi_{jt}^{a}\varphi_{i}^{a} - \frac{1}{(m-1)(m-2)}U^{a}\varphi_{kk}^{a}\delta_{ij} \\ & + \frac{m-2}{m-1}U^{a}\varphi_{ij}^{a} - \frac{U^{ab}}{m-1}\varphi_{j}^{a}\varphi_{i}^{a} + \frac{U^{ab}}{m-1}\varphi_{k}^{a}\varphi_{k}^{b}\delta_{ij} \\ = & [1+\eta(m-2)]\overline{D}_{ijk,k}^{\varphi} - \alpha \frac{1+\eta(m-2)}{m-2}f_{j}\varphi_{ss}^{a}\varphi_{i}^{a} + \eta f_{j}U^{a}\varphi_{i}^{a} \\ & - \frac{m}{m-1}\varphi_{j}^{a}\varphi_{i}^{b}U^{ab} - \eta f_{k}f_{t}W_{tijk}^{\varphi} - f_{t}\left(\frac{m-3}{m-2}\right)C_{jti}^{\varphi} \\ & - \frac{1}{(m-1)(m-2)}U^{a}\varphi_{kk}^{a}\delta_{ij} + \frac{m-2}{m-1}U^{a}\varphi_{ij}^{a} + \frac{U^{ab}}{m-1}\varphi_{k}^{a}\varphi_{k}^{b}\delta_{ij}. \end{split}$$

**Remark** 4.5. It is interesting to compare (4.7) to an equation appearing in the context of *Cotton Gravity*, a gravitational theory that, as we already discussed in the Introduction, has been recently introduced by Harada ([38]). One can prove, after some computations, that, on a static space-time  $\widehat{M} = M \times_u I$  of metric

$$\widehat{q} = -u^2 dt \otimes dt + q$$

for some Riemannian metric g on M and  $u \in C^{\infty}(M)$ , the field equations of Cotton Gravity for the stress-energy tensor (2.26) of the Introduction with conservation laws

$$\begin{cases} (\mu + p)\nabla u = -u\nabla p, \\ \alpha h(\tau(\widehat{\varphi}), d\widehat{\varphi}) = h(\nabla U, d\widehat{\varphi}) \end{cases}$$

reduce to

$$\begin{cases}
C_{ijk}^{\varphi} + f_t W_{tijk}^{\varphi} - \frac{U^a \varphi_k^a}{m-1} \delta_{ij} + \frac{U^a \varphi_j^a}{m-1} \delta_{ik} - D_{ijk}^A - (m-2) D_{ijk}^B = 0, \\
-\frac{m-1}{m} f_{tti} + \frac{m-2}{m} f_{ti} f_t + \Delta f f_i + \frac{1}{2m} S_i^{\varphi} - R_{it}^{\varphi} f_t + \frac{U^a \varphi_i^a}{m} - \frac{m-1}{m} \mu_i = 0,
\end{cases}$$

where  $f = -\log(u)$ ,  $D^A$  is given by the right hand side of (4.5) and  $D^B$  is given by the right hand side of (4.6). Note that, for a  $\varphi$ -SPFST, equation (4.4) i) with  $\eta = 1$  is satisfied so that, as we already discussed,  $D^A = D^B = \overline{D}^{\varphi}$  and (4.18) i) reduces to (4.7).

It is also interesting to observe that, for the stress energy tensor

$$\widehat{T} = \alpha \widehat{\varphi}^* h - \left( \alpha \frac{|d\widehat{\varphi}|^2}{2} + U(\widehat{\varphi}) \right) \widehat{g},$$

the field equations of Cotton Gravity become, for a general m-dimensional Lorentzian manifold  $(\widehat{M}, \widehat{g})$ ,

$$\widehat{C}^{\widehat{\varphi}} = -\frac{1}{2(m-1)}\operatorname{div}_1(U(\widehat{\varphi})\widehat{g} \otimes \widehat{g}).$$

The above equation, in Riemannian signature, will appear several times in the rest of this paper, starting from Proposition 4.13.

## **4.2.** The Geometry of the Level Sets of *f*

Let (M,g) and (N,h) be two Riemannian manifolds, let  $\varphi:(M,g)\to (N,h)$  be a smooth map,  $f,\lambda\in C^\infty(M)$  and  $\eta\in\mathbb{R}$  such that

(4.19) 
$$\operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda q.$$

The aim of this subsection is to understand how the vanishing of the tensor  $\overline{D}^{\varphi}$  affects the geometry of the level sets of f. From now on, indices in lower case letters  $i, j, k, \ldots$  will run between 1 and m, while indices in upper case letters  $A, B, \ldots$  will run between 1 and m-1.

**Proposition** 4.6. Let (M,g) be a Riemannian manifold,  $f \in C^{\infty}(M)$  satisfying (4.19), c be a regular value of f and  $\Sigma$  the corresponding level set. For each  $p \in \Sigma$ , let  $\{e_i\}_{i=1}^m$  be a local frame such that  $e_1, ..., e_{m-1}$  are tangent to  $\Sigma$  and

$$e_m = \frac{\nabla f}{|\nabla f|}$$

denotes a unit normal with respect to  $\Sigma$ . Then, at  $p \in \Sigma$ , we have the validity of (4.20)

$$\frac{(m-2)^2}{2|\nabla f|^2} \left| \overline{D}^{\varphi} \right|^2 = \left| \mathring{\text{II}} \right|^2 |\nabla f|^2 + \left( \frac{m-2}{m-1} \right) \frac{1}{|\nabla f|^2} g \left( \text{Ric}^{\varphi} (\nabla f, \cdot)^{\sharp}, \text{Ric}^{\varphi} (\nabla f, \cdot)^{\sharp} \right) - \left( \frac{m-2}{m-1} \right) \frac{1}{|\nabla f|^4} \left( \text{Ric}^{\varphi} (\nabla f, \nabla f) \right)^2,$$

where  $\check{\Pi}$  is the traceless part of the second fundamental form of  $\Sigma$  with respect to the inward unit normal.

PROOF. For the sake of simplicity, we divide the proof in two steps. **Step 1**. We start by proving the validity of

(4.21) 
$$\frac{(m-2)^2}{2|\nabla f|^2} \left| \overline{D}^{\varphi} \right|^2 = \left| \operatorname{Ric}^{\varphi} \right|^2 - \frac{m}{m-1} R_{Am}^{\varphi} R_{Am}^{\varphi} - \frac{m}{m-1} (R_{mm}^{\varphi})^2 + \frac{2}{m-1} S^{\varphi} R_{mm}^{\varphi} - \frac{1}{m-1} (S^{\varphi})^2.$$

Using (4.5) together with the fact that  $\overline{D}^{\varphi}$  is totally trace free and skew symmetric in the last two indices, we immediately get

$$\begin{split} \left| \overline{D}^{\varphi} \right|^2 = & \overline{D}^{\varphi}_{ijk} \overline{D}^{\varphi}_{ijk} \\ = & \frac{2}{m-2} \overline{D}^{\varphi}_{ijk} R^{\varphi}_{ij} f_k, \end{split}$$

that is,

Then, we insert (4.5) into (4.22) to get

$$\begin{split} \frac{m-2}{2} \left| \overline{D}^{\varphi} \right|^{2} &= \overline{D}_{ijk}^{\varphi} R_{ij}^{\varphi} f_{k} \\ &= \frac{R_{ij}^{\varphi} f_{k}}{m-2} \left[ R_{ij}^{\varphi} f_{k} - R_{ik}^{\varphi} f_{j} + \frac{1}{m-1} f_{t} \left( R_{tk}^{\varphi} \delta_{ij} - R_{tj}^{\varphi} \delta_{ik} \right) \right. \\ &\quad \left. - \frac{S^{\varphi}}{m-1} (f_{k} \delta_{ij} - f_{j} \delta_{ik}) \right] \\ &= \frac{1}{m-2} \left[ \left| \operatorname{Ric}^{\varphi} \right|^{2} \left| \nabla f \right|^{2} - \left| \nabla f \right|^{2} R_{im}^{\varphi} R_{im}^{\varphi} + \frac{S^{\varphi}}{m-1} R_{mm}^{\varphi} \left| \nabla f \right|^{2} \right. \\ &\quad \left. - \frac{1}{m-1} \left| \nabla f \right|^{2} R_{im}^{\varphi} R_{im}^{\varphi} - \frac{(S^{\varphi})^{2}}{m-1} \left| \nabla f \right|^{2} + \frac{S^{\varphi}}{m-1} \left| \nabla f \right|^{2} R_{mm}^{\varphi} \right] \\ &= \frac{\left| \nabla f \right|^{2}}{m-2} \left[ \left| \operatorname{Ric}^{\varphi} \right|^{2} - \frac{m}{m-1} R_{im}^{\varphi} R_{im}^{\varphi} + \frac{2}{m-1} S^{\varphi} R_{mm}^{\varphi} - \frac{(S^{\varphi})^{2}}{m-1} \right] \\ &= \frac{\left| \nabla f \right|^{2}}{m-2} \left[ \left| \operatorname{Ric}^{\varphi} \right|^{2} - \frac{m}{m-1} R_{Am}^{\varphi} R_{Am}^{\varphi} - \frac{m}{m-1} (R_{mm}^{\varphi})^{2} \right. \\ &\quad \left. + \frac{2}{m-1} S^{\varphi} R_{mm}^{\varphi} - \frac{(S^{\varphi})^{2}}{m-1} \right], \end{split}$$

that is, (4.21).

**Step 2** With respect to the given frame  $\{e_i\}_{i=1}^m$  we have

$$f_A = 0, f_m = |\nabla f|,$$

where A = 1, ..., m - 1. Let  $II_{AB}$  be the second fundamental form of  $\Sigma$ ; then, by [17, Proposition 8.1], we have

$$II_{AB} = -\theta_A^m(e_B) = -\frac{f_{AB}}{|\nabla f|}.$$

Moreover, by (4.19), we get

(4.23) 
$$II_{AB} = \frac{1}{|\nabla f|} (R_{AB}^{\varphi} - \eta f_A f_B - \lambda \delta_{AB})$$
$$= \frac{1}{|\nabla f|} (R_{AB}^{\varphi} - \lambda \delta_{AB});$$

furthermore, the (normalized) mean curvature of  $\Sigma$  is

$$H = \frac{II_{AA}}{m-1},$$

so that, by (4.23), we deduce

(4.24) 
$$H = \frac{1}{|\nabla f|} \left( \frac{S^{\varphi} - R_{mm}^{\varphi}}{m - 1} - \lambda \right)$$

and

(4.25) 
$$\mathring{\Pi}_{AB} = \Pi_{AB} - H\delta_{AB}$$

$$= \frac{1}{|\nabla f|} \left( R_{AB}^{\varphi} - \frac{S^{\varphi} - R_{mm}^{\varphi}}{m - 1} \delta_{AB} \right).$$

Computing the norm we have

(4.26)

$$\begin{split} |\nabla f|^2 \Big| \mathring{\Pi} \Big|^2 &= R_{AB}^{\varphi} R_{AB}^{\varphi} + \frac{1}{m-1} (S^{\varphi} - R_{mm}^{\varphi})^2 - \frac{2}{m-1} R_{AA}^{\varphi} (S^{\varphi} - R_{mm}^{\varphi}) \\ &= |\mathrm{Ric}^{\varphi}|^2 - 2 R_{Am}^{\varphi} R_{Am}^{\varphi} - R_{mm}^{\varphi} R_{mm}^{\varphi} + \frac{1}{m-1} (S^{\varphi} - R_{mm}^{\varphi})^2 \\ &- \frac{2}{m-1} (S^{\varphi} - R_{mm}^{\varphi})^2 \\ &= |\mathrm{Ric}^{\varphi}|^2 - 2 R_{Am}^{\varphi} R_{Am}^{\varphi} - \frac{m}{m-1} R_{mm}^{\varphi} R_{mm}^{\varphi} - \frac{(S^{\varphi})^2}{m-1} + \frac{2}{m-1} S^{\varphi} R_{mm}^{\varphi}. \end{split}$$

Hence, inserting (4.26) into (4.21) we obtain (4.20).

**Corollary** 4.7. Let (M,g) be a Riemannian manifold and let f be a solution of (4.19). Then the tensor  $\overline{D}^{\varphi}$  vanishes identically on M if and only if the following two conditions are simultaneously satisfied:

- i) any regular level set of f is totally umbilical;
- ii) for any regular point p of f,  $\nabla f_p$  is an eigenvector of  $\operatorname{Ric}_p^{\varphi}$ .

PROOF. From (4.20), we only need to prove that  $R_{Am}^{\varphi} = 0$ , A = 1, ..., m - 1, if and only if  $\nabla f$  is an eigenvector of  $\mathrm{Ric}^{\varphi}$ ; this fact follows by the choice of the frame: indeed,

$$\operatorname{Ric}^{\varphi}(\nabla f, \cdot)^{\sharp} = |\nabla f| R_{im}^{\varphi} e_i.$$

Therefore,  $R_{Am}^{\varphi} = 0$  for each A if and only if  $\operatorname{Ric}^{\varphi}(\nabla f, \cdot)^{\sharp}$  is proportional to  $e_m = \frac{\nabla f}{|\nabla f|}$ , that is, if and only if  $\nabla f$  is an eigenvector of  $\operatorname{Ric}^{\varphi}$ . Moreover, (4.20) reads, in components,

$$\frac{(m-2)^2}{2|\nabla f|^2} \left| \overline{D}^{\varphi} \right|^2 = \left| \mathring{\Pi} \right|^2 |\nabla f|^2 + \frac{m-2}{m-1} R_{Am}^{\varphi} R_{Am}^{\varphi}$$

so that  $\overline{D}^\varphi$  vanishes if and only if  $\mathring{\Pi}$  and  $R^\varphi_{Am}$  are zero.

**Remark** 4.8. From Corollary 4.7 we deduce that condition ii) is necessary to the validity of  $\overline{D}^{\varphi} \equiv 0$ . This condition, that will be assumed in Theorem 4.17, has some interesting geometric consequences that we explore in the next

**Lemma** 4.9. Let (M, g) be a Riemannian manifold and f a solution of (4.4). Then the following conditions are equivalent:

- i) at any regular point p of f, we have that  $\nabla f_p$  is an eigenvector of  $\operatorname{Ric}_p^{\varphi}$ ;
- ii)  $|\nabla f|$  is constant on any regular level set of f;
- iii) for any fixed regular level set  $\Sigma$  and  $p \in \Sigma$ , there exists an open neighbourhood U of p in M and a distance function  $r: U \to \mathbb{R}$  such that f depends only on r;
- iv) we have the validity of

$$\overline{D}^{\varphi}(\nabla f,\cdot,\cdot)=0;$$

v) we have the validity of

$$C^{\varphi}(\nabla f, \cdot, \cdot) + \frac{1}{2(m-1)} \operatorname{div}_{1}(U(\varphi)g \otimes g)(\nabla f, \cdot, \cdot) = 0.$$

**Remark** 4.10. Petersen and Wylie, in [57], studied Ricci solitons with a potential function satisfying condition iii). There, such a function is said to be *rectifiable* on U.

PROOF (OF LEMMA 4.9). To prove the equivalence of conditions i) and ii) it is sufficient to compute, in the same frame introduced in Proposition 4.6,

$$\frac{1}{2}|\nabla f|_A^2 = f_{Ai}f_i$$

$$= |\nabla f|f_{Am}$$

$$= |\nabla f|(-R_{Am}^{\varphi} + \eta f_A f_m + \lambda \delta_{Am})$$

$$= -|\nabla f|R_{Am}^{\varphi},$$

where we have used the first equation of (4.4). It follows that  $|\nabla f|$  is constant on  $\Sigma$  if and only if  $R_{Am}^{\varphi} = 0$ ,  $\forall A \in \{1, ..., m-1\}$ .

It is clear that iii) implies ii); to see the converse, consider an open neighbourhood U of p in M such that every point of U is regular for f. We want to show that every

integral curve  $\gamma$  of  $\frac{\nabla f}{|\nabla f|}$  is a unit speed geodesic. Consider a local orthonormal frame  $\{e_i\}_{i=1}^m$  on U such that  $e_m = \frac{\nabla f}{|\nabla f|}$  and its dual coframe  $\{\theta^i\}_{i=1}^m$ . By assumptions,

$$e_A(f) = 0$$
 and  $e_A(|\nabla f|) = 0$ ,  $\forall A \in \{1, ..., m-1\}$ .

The first structure equations give

$$d\left(\frac{df}{|\nabla f|}\right) = -\theta_j^m \wedge \theta^j,$$

so that

$$d\left(\frac{1}{|\nabla f|}\right) \wedge df = -\theta_j^m \wedge \theta^j.$$

Contracting the above with the local tensor field  $e_A \otimes e_m$  gives

$$e_A\bigg(\frac{1}{|\nabla f|}\bigg)e_m(f) - e_m\bigg(\frac{1}{|\nabla f|}\bigg)e_A(f) = -\theta_m^m(e_A) + \theta_A^m(e_m).$$

Since  $e_A(|\nabla f|) = 0 = e_A(f)$  and  $\theta_m^m = 0$  we get

$$\theta_A^m(e_m) = 0.$$

By definition of covariant derivative, this implies

$$\nabla_{e_m} e_m = 0,$$

so that  $\gamma$  is a geodesic. Therefore, choosing  $r:U\to\mathbb{R}$  to be the signed distance function from  $U\cap\Sigma$ , oriented accordingly to the chosen unit normal, we obtain

$$\nabla r = \frac{\nabla f}{|\nabla f|}$$

at every point of U, so that f only depends on r.

Next we prove the equivalence of i) and iv). From (4.5) we get

$$\begin{split} (m-2)f_{i}\overline{D}_{ijk}^{\varphi} &= f_{i} \left[ R_{ij}^{\varphi} f_{k} - R_{ik}^{\varphi} f_{j} - \frac{f_{t}}{m-1} \left( R_{jt}^{\varphi} \delta_{ik} - R_{kt}^{\varphi} \delta_{ij} \right) + \frac{S^{\varphi}}{m-1} (f_{j} \delta_{ik} - f_{k} \delta_{ij}) \right] \\ &= f_{i} R_{ij}^{\varphi} f_{k} - f_{i} R_{ik}^{\varphi} f_{j} - \frac{f_{t}}{m-1} \left( R_{jt}^{\varphi} f_{k} - R_{kt}^{\varphi} f_{j} \right) + \frac{S^{\varphi}}{m-1} (f_{j} f_{k} - f_{k} f_{j}) \\ &= \frac{m-2}{m-1} \left( f_{i} R_{ij}^{\varphi} f_{k} - f_{i} R_{ik}^{\varphi} f_{j} \right). \end{split}$$

Therefore, at any regular point of f,

$$(4.27) (m-1)\overline{D}_{mjk}^{\varphi} = R_{mj}^{\varphi} f_k - R_{mk}^{\varphi} f_j.$$

Since we have

$$f_m = |\nabla f|, \quad f_A = 0, \ \forall A \in \{1, ..., m-1\}.$$

we deduce that the right hand side, and hence also the left hand side, of (4.27) vanishes if j = m = k or if  $j, k \in \{1, ..., m-1\}$ , so that we are only left with

$$(m-1)\overline{D}_{mAm}^{\varphi} = |\nabla f| R_{mA}^{\varphi}.$$

Therefore,  $f_i \overline{D}_{ijk}^{\varphi}$  vanishes identically on M if and only if  $R_{Am}^{\varphi} = 0$  for any regular point.

To conclude, we prove the equivalence of iv) and v). This is a simple application of the first integrability condition of system (4.4). Indeed, contracting (4.7) with  $f_i$  we deduce

$$[1 + \eta(m-2)]f_i\overline{D}_{ijk}^{\varphi} = f_i f_t W_{tijk}^{\varphi} + f_i \left[ C_{ijk}^{\varphi} - \frac{U^a \varphi_k^a}{m-1} \delta_{ij} + \frac{U^a \varphi_j^a}{m-1} \delta_{ik} \right]$$
$$= f_i \left[ C_{ijk}^{\varphi} - \frac{U^a \varphi_k^a}{m-1} \delta_{ij} + \frac{U^a \varphi_j^a}{m-1} \delta_{ik} \right],$$

and we conclude.

**Proposition** 4.11. Let (M,g) and (N,h) be two Riemannian manifolds and let  $\varphi: (M,g) \to (N,h)$  be a smooth map; assume that  $\overline{D}^{\varphi} \equiv 0$  and let f be a solution of (4.4), where  $U \in C^{\infty}(N)$ ,  $\lambda \in C^{\infty}(M)$  and  $\alpha \in \mathbb{R} \setminus \{0\}$ . Assume that  $\varphi$  is  $\frac{U}{\alpha}$ -harmonic, i.e.

(4.28) 
$$\tau(\varphi) = \frac{1}{\alpha} (\nabla U)(\varphi).$$

Then, for any regular value c of f,  $|\nabla f|$  and H are constant on every connected component of  $\Sigma = f^{-1}(c)$  and  $\varphi|_{\Sigma}$  is  $\frac{U}{\alpha}$ -harmonic. Moreover,  $S^{\varphi} - (m-1)\lambda$  and  $S^{\varphi|_{\Sigma}} - (m-1)\lambda|_{\Sigma}$  are also constant on every connected component of  $\Sigma$ .

**Remark** 4.12. Note that the constancy of  $S^{\varphi} - (m-1)\lambda$  and  $S^{\varphi|_{\Sigma}} - (m-1)\lambda|_{\Sigma}$  is not needed in the proof of the Kobayashi-Obata type theorem; however, we give a proof for the sake of completeness.

PROOF. By Lemma 4.9 we have that  $|\nabla f|$  is constant on  $\Sigma$ . Using Codazzi equations,  $R_{Am}^{\varphi}=0$  and

$$II_{AB} = H\delta_{AB}$$
,

we obtain

$$(m-1)H_B = II_{AA,B} = II_{AB,A} - R_{mAAB} = II_{AB,A} + R_{mB} = H_B + R_{mB}^{\varphi} + \alpha \varphi_m^a \varphi_B^a$$
$$= H_B + \alpha \varphi_m^a \varphi_B^a;$$

therefore,

$$(m-2)H_B = \alpha \varphi_m^a \varphi_B^a.$$

Since

$$0 = \varphi_{tt}^a - \frac{1}{\alpha}U^a = \varphi_i^a f_i = |\nabla f| \varphi_m^a,$$

we have that, on a regular level set,  $\varphi_m^a = 0$  and hence

$$H_B = 0.$$

Since  $i:\Sigma\hookrightarrow M$  is an isometric immersion, a computation shows that

$$\tau^{a}(\varphi|_{\Sigma}) = \tau^{a}(\varphi) - \varphi_{mm}^{a} + (m-1)H\varphi_{m}^{a},$$

that is, since  $\varphi_m^a = 0$ ,

(4.29) 
$$\tau^{a}(\varphi|_{\Sigma}) = \tau^{a}(\varphi) - \varphi^{a}_{mm}.$$

Taking the covariant derivative of

$$0 = \varphi_i^a f_i$$

we obtain

$$0 = \varphi_{ij}^a f_i + \varphi_i^a f_{ij} = \varphi_{mi}^a |\nabla f| + \varphi_i^a f_{ij}.$$

When j = m we have

by (4.4), and using the vanishing of  $\varphi_m^a, f_A$  and  $R_{Am}^{\varphi}$  we get

$$\varphi_{mm}^{a}|\nabla f| = -\varphi_{i}^{a}f_{im} = \varphi_{i}^{a}R_{im}^{\varphi} - f_{i}f_{m}\varphi_{i}^{a} + \lambda\varphi_{m}^{a} = 0.$$

Since c is a regular value,  $|\nabla f| \neq 0$ , thus  $\varphi_{mm}^a = 0$  and (4.29) implies

$$\tau(\varphi_{|_{\Sigma}}) = \frac{1}{\alpha} (\nabla U)(\varphi_{|_{\Sigma}}).$$

To prove the constancy of  $S^{\varphi} - (m-1)\lambda$ , we first show the validity of

(4.31) 
$$\frac{1}{2}S_k^{\varphi} = R_{ik}^{\varphi}f_i + \eta(f_{ik}f_i - \Delta f f_k) + (m-1)\lambda_k - U^a \varphi_k^a.$$

Indeed, taking the covariant derivative of (4.4) i), we get

$$R_{ij,k}^{\varphi} + f_{ijk} - \eta f_{ik} f_j - \eta f_i f_{jk} = \lambda_k \delta_{ij};$$

exchanging the role of i and j in the above equation

$$R_{ik,j}^{\varphi} + f_{ikj} - \eta f_{ij} f_k - \eta f_i f_{kj} = \lambda_j \delta_{ik}$$

and subtracting the second equation to the first one, we deduce

$$R_{ij,k}^{\varphi} - R_{ik,j}^{\varphi} = f_{ikj} - f_{ijk} - \eta(f_{ij}f_k - f_{ik}f_j) + \lambda_k \delta_{ij} - \lambda_j \delta_{ik}.$$

Using the Ricci commutation relations, the previous equation reduces to

$$R_{ij,k}^{\varphi} - R_{ik,j}^{\varphi} = f_t R_{tikj} - \eta (f_{ij} f_k - f_{ik} f_j) + \lambda_k \delta_{ij} - \lambda_j \delta_{ik}.$$

Then, taking the trace with respect to i and j and using the  $\varphi$ -Schur's identity, we get

$$\frac{1}{2}S_k^{\varphi} + \alpha \varphi_{tt}^a \varphi_k^a = f_t R_{tk} + \eta (f_{tk} f_t - \Delta f f_k) + (m-1)\lambda_k;$$

now we use the definition of  $Ric^{\varphi}$  and (4.4) ii) to obtain

$$\frac{1}{2}S_k^{\varphi} = f_t R_{tk}^{\varphi} + \eta (f_{tk}f_t - \Delta f f_k) + (m-1)\lambda_k + \alpha (f_t \varphi_t^a \varphi_k^a - \varphi_{tt}^a \varphi_k^a)$$
$$= f_t R_{tk}^{\varphi} + \eta (f_{tk}f_t - \Delta f f_k) + (m-1)\lambda_k - U^a \varphi_k^a,$$

that is, (4.31).

Taking k = B in (4.31) we have

$$\frac{1}{2}S_B^{\varphi} = R_{Bj}^{\varphi} f_j + \eta (f_{Bj}f_j - \Delta f f_B) + (m-1)\lambda_B - U^a \varphi_B^a$$

$$= (m-1)\lambda_B - U^a \varphi_B^a,$$
(4.32)

where, in the last equality, we have used the fact that  $R_{Bm}^{\varphi} = 0$ ,  $f_B = 0$  and

$$f_{mB} = -R_{mB}^{\varphi} + \eta f_B f_m + \lambda \delta_{mB} = 0,$$

which is a consequence of the first equation of (4.4). Therefore, by (4.32), we have that  $S^{\varphi} - (m-1)\lambda + U(\varphi)$  is constant on every connected component of  $\Sigma$ . By (4.24) we have

$$\begin{split} (m-1)|\nabla f|^2 H &= S^{\varphi} - (m-1)\lambda - R_{mm}^{\varphi} \\ &= \left(\frac{S^{\varphi}}{2} - (m-1)\lambda + U(\varphi)\right) + \frac{S^{\varphi}}{2} - R_{mm}^{\varphi} - U(\varphi). \end{split}$$

Thus,  $\frac{S^{\varphi}}{2} - R_{mm}^{\varphi} - U(\varphi)$  is constant on every connected component of  $\Sigma$ . Since  $\overline{D}^{\varphi} \equiv 0$ , we have that  $\mathring{\Pi} \equiv 0$ ; then, by (4.25), we have

$$(4.33) R_{AB}^{\varphi} = \frac{1}{m-1} (S^{\varphi} - R_{mm}^{\varphi}) \delta_{AB}$$

that, together with (4.4) ii), gives

$$f_{AA} = -R_{AA}^{\varphi} + \eta f_A f_A + (m-1)\lambda = -S^{\varphi} + R_{mm}^{\varphi} + (m-1)\lambda = -(m-1)|\nabla f|H.$$

Therefore, using (4.31) with k = m,  $\varphi_m^a = 0$ ,  $R_{Am}^{\varphi} = 0$  and

$$II_{AB} = -\frac{f_{AB}}{|\nabla f|},$$

we deduce

$$\frac{1}{2}S_m^{\varphi} = R_{mm}^{\varphi} + \eta(f_{mm}|\nabla f| - \Delta f|\nabla f|) + (m-1)\lambda_m - U^a \varphi_m^a$$

$$= R_{mm}^{\varphi}|\nabla f| - \eta|\nabla f|f_{AA} + (m-1)\lambda_m$$

$$= R_{mm}^{\varphi}|\nabla f| + (m-1)H|\nabla f|^2 + (m-1)\lambda_m.$$

Since we have already proved the constancy of  $|\nabla f|$  and H, we conclude that, since  $(U(\varphi))_m = 0$ ,

$$\frac{1}{2}S_m^{\varphi} - R_{mm}^{\varphi}|\nabla f| - (m-1)\lambda_m + (U(\varphi))_m$$

is constant on the connected components of  $\Sigma$ . Hence,

$$0 = \left\{ \frac{1}{2} S_m^{\varphi} - R_{mm}^{\varphi} |\nabla f| - (m-1)\lambda_m + (U(\varphi))_m \right\}_A$$

$$= \frac{1}{2} S_{mA}^{\varphi} - R_{mm,A}^{\varphi} |\nabla f| - R_{mm}^{\varphi} |\nabla f|_A - (m-1)\lambda_{mA} + (U(\varphi))_{mA}$$

$$= \left\{ \left[ \frac{1}{2} S^{\varphi} - (m-1)\lambda + U(\varphi) \right]_A \right\}_m - R_{mm,A}^{\varphi} |\nabla f|$$

$$=-R_{mm,A}^{\varphi}|\nabla f|,$$

since  $|\nabla f|$  and  $\frac{1}{2}S^{\varphi} - (m-1)\lambda + U(\varphi)$  are constant on the connected components of  $\Sigma$ . It follows that, since  $|\nabla f| \neq 0$ ,  $R^{\varphi}_{mm,A} = 0$  and  $R^{\varphi}_{mm}$  is constant on the connected components of  $\Sigma$ . By the constancy of  $\frac{1}{2}S^{\varphi} + R^{\varphi}_{mm} - U(\varphi)$  and  $R^{\varphi}_{mm}$  we have that

$$\frac{1}{2}S^{\varphi} - U(\varphi)$$

is constant on every connected component of  $\Sigma$ , then

$$(4.34) S^{\varphi} - (m-1)\lambda = \frac{1}{2}S^{\varphi} - (m-1)\lambda + U(\varphi) + \frac{1}{2}S^{\varphi} - U(\varphi)$$

is also constant.

Since  $i:\Sigma\hookrightarrow M$  is totally umbilical, Gauss equation yields

$$(4.35) ^{\Sigma}S = S - 2R_{mm} + (m-1)(m-2)H^{2},$$

where  ${}^{\Sigma}S$  is the scalar curvature of  $\Sigma$ . By the definition of  $S^{\varphi_{|_{\Sigma}}}$  and the fact that  $\varphi_m^a = 0$ , we get

$$\begin{split} S^{\varphi_{|\Sigma}} &= {}^{\Sigma}S - \alpha \varphi_A^a \varphi_A^a \\ &= S - 2R_{mm} + (m-1)(m-2)H^2 - \alpha |d\varphi|^2 + \alpha \varphi_m^a \varphi_m^a \\ &= S^{\varphi} - 2R_{mm}^{\varphi} + (m-1)(m-2)H^2 \\ &= S^{\varphi} - 2R_{mm}^{\varphi} + (m-1)(m-2)H^2 \end{split}$$

By the constancy of (4.34) we conclude that

$$S^{\varphi|_{\Sigma}} - (m-1)\lambda|_{\Sigma}$$

is constant on the connected components of  $\Sigma$ .

**Proposition** 4.13. Let (M,g) be a manifold of dimension  $m \geq 3$ . Assume  $\overline{D}^{\varphi} \equiv 0$  on M and that (4.4) is satisfied by  $f \in C^{\infty}(M)$ . Then, at any regular point p of f, we have

$$(4.36) \hspace{1cm} C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_{1}\left(U(\varphi)g \bigotimes g\right).$$

PROOF. By (4.7) and  $D^{\varphi} \equiv 0$ , we have

(4.37) 
$$C_{ijk}^{\varphi} = -f_t W_{tijk}^{\varphi} - \frac{U^a}{m-1} \left( \varphi_j^a \delta_{ik} - \varphi_k^a \delta_{ij} \right).$$

Note that (4.36) is automatically satisfied when f is constant. When f is not constant, we need to show the validity of (4.36) on  $\{x \in M : \nabla f \neq 0\}$ , hence, if c is a regular value of f it is sufficient to show (4.36) on the corresponding level set  $\Sigma$ .

Fix  $p \in \Sigma$ , let us choose a Darboux frame as in Proposition 4.11. By (4.37) and the symmetries of the  $\varphi$ -Weyl tensor, we have

$$0 = f_i f_t W_{tijk}^{\varphi} = \frac{U^a}{m-1} \left( f_k \varphi_j^a - f_j \varphi_k^a \right) + f_i C_{ijk}^{\varphi}$$
$$= \frac{U^a}{m-1} \left( f_k \varphi_j^a - f_j \varphi_k^a \right) + |\nabla f| C_{mjk}^{\varphi};$$

therefore,

$$C_{mjk}^{\varphi} = -\frac{1}{|\nabla f|} \frac{U^a}{m-1} \left( f_k \varphi_j^a - f_j \varphi_k^a \right)$$
 on  $\Sigma$ .

Thus, (4.36) holds for  $C_{mjk}^{\varphi}$ . Since  $i:\Sigma\hookrightarrow M$  is totally umbilical with constant mean curvature H, we have

$$II_{AB,C} = 0$$
,

where  $II_{AB}$ , A, B = 1, ..., m - 1, is the second fundamental form of i. Thus, by Codazzi equations (see e.g. [1]), we deduce

$$R_{mABC} = 0.$$

Therefore, by the definition of  $W^{\varphi}$  and using  $R_{Am}^{\varphi} = 0$ , we obtain

$$0 = R_{mABC} = W_{mABC}^{\varphi} + \frac{1}{m-2} (R_{mB}^{\varphi} \delta_{AC} + R_{AC}^{\varphi} \delta_{mB} - R_{mC}^{\varphi} \delta_{AB} - R_{AB}^{\varphi} \delta_{mC})$$

$$- \frac{S^{\varphi}}{(m-1)(m-2)} (\delta_{mB} \delta_{AC} - \delta_{mc} \delta_{AB})$$

$$= W_{mABC}^{\varphi} + \frac{1}{m-2} (R_{mB}^{\varphi} \delta_{AC} - R_{mC}^{\varphi} \delta_{AB})$$

$$= W_{mABC}^{\varphi}.$$

$$(4.38)$$

Hence, by (4.37) and (4.38)

$$C_{ABC}^{\varphi} = -f_t W_{tABC}^{\varphi} - \frac{U^a}{m-1} (\varphi_B^a \delta_{AC} - \varphi_C^a \delta_{AB})$$

$$= -f_m W_{mABC}^{\varphi} - \frac{U^a}{m-1} (\varphi_B^a \delta_{AC} - \varphi_C^a \delta_{AB})$$

$$= -\frac{U^a}{m-1} (\varphi_B^a \delta_{AC} - \varphi_C^a \delta_{AB}).$$

Note that, when j, k = m, we have  $C_{imm}^{\varphi} = 0$  and

$$\frac{U^a}{m-1}(\varphi_m^a \delta_{mi} - \varphi_m^a \delta_{mi}) = 0$$

therefore, (4.36) holds. Hence we only need to prove the validity of (4.36) for  $C_{mmB}^{\varphi}$  and  $C_{AmB}^{\varphi}$ . By (4.37)

$$C_{mmB}^{\varphi} = -\frac{U^a}{m-1} (\varphi_m^a \delta_{mB} - \varphi_B^a) - f_t W_{tmmB}^{\varphi}$$
$$= \frac{U^a}{m-1} \varphi_B^a - f_A W_{AmmB}^{\varphi} - f_m W_{mmmB}^{\varphi}$$

$$=\frac{U^a}{m-1}\varphi_B^a,$$

so that (4.36) is verified in this case too. Let us show the last case. Since  $R_{Am}^{\varphi} = 0$ , we deduce

$$\begin{split} 0 &= dR_{Am}^{\varphi} = &R_{Am,k}^{\varphi}\theta^k + R_{km}^{\varphi}\theta_A^k + R_{Ak}^{\varphi}\theta_m^k \\ &= &R_{Am,k}^{\varphi}\theta^k + R_{Bm}^{\varphi}\theta_A^B + R_{mm}^{\varphi}\theta_A^m + R_{AB}^{\varphi}\theta_m^B + R_{Am}\theta_m^m \\ &= &R_{Am,k}^{\varphi}\theta^k + R_{mm}^{\varphi}\theta_A^m + R_{AB}^{\varphi}\theta_m^B \end{split}$$

(note that we are not taking the sum over m). Therefore, using (4.33) in the above equation, we obtain

By the definition of the  $\varphi$ -Cotton tensor and (4.33), we have

$$C_{ABm}^{\varphi} = R_{AB,m}^{\varphi} - R_{Am,B}^{\varphi} - \frac{1}{2(m-1)} (S_{m}^{\varphi} \delta_{AB} - S_{B}^{\varphi} \delta_{Am})$$

$$= R_{AB,m}^{\varphi} - R_{Am,B}^{\varphi} - \frac{S_{m}^{\varphi}}{2(m-1)} \delta_{AB}$$

$$= \frac{1}{m-1} [(S^{\varphi} - R_{mm}^{\varphi}) \delta_{AB}]_{m} - R_{Am,B}^{\varphi} - \frac{S_{m}^{\varphi}}{2(m-1)} \delta_{AB}.$$
(4.40)

Then, from (4.39) we obtain, on  $\Sigma$ ,

$$\begin{split} R_{Am,B}^{\varphi} &= \frac{1}{m-1} (S^{\varphi} - m R_{mm}^{\varphi}) \theta_A^m(e_B) \\ &= \frac{1}{m-1} (S^{\varphi} - m R_{mm}^{\varphi}) \Pi_{AB} \\ &= -\frac{1}{m-1} (S^{\varphi} - m R_{mm}^{\varphi}) \frac{f_{AB}}{|\nabla f|}; \end{split}$$

therefore, on  $\Sigma$  we have

(4.41) 
$$R_{Am,B}^{\varphi} = -\frac{1}{m-1} (S^{\varphi} - mR_{mm}^{\varphi}) \frac{f_{AB}}{|\nabla f|}.$$

Inserting (4.41) into (4.40) we get

$$C_{ABm}^{\varphi} = \frac{1}{m-1} \left[ \left( S_{m}^{\varphi} - R_{mm,m}^{\varphi} \right) \delta_{AB} \right] + \frac{1}{m-1} \left( S^{\varphi} - m R_{mm}^{\varphi} \right) \frac{f_{AB}}{|\nabla f|} - \frac{S_{m}^{\varphi}}{2(m-1)} \delta_{AB}$$

$$(4.42)$$

$$= \frac{S_{m}^{\varphi}}{2(m-1)} \delta_{AB} - \frac{1}{m-1} R_{mm,m}^{\varphi} \delta_{AB} + \frac{1}{m-1} \left( S^{\varphi} - m R_{mm}^{\varphi} \right) \frac{f_{AB}}{|\nabla f|}.$$

By the  $\varphi$ -Schur's identity

$$\begin{split} \frac{1}{2}S_{m}^{\varphi} = &\alpha\varphi_{tt}^{a}\varphi_{m}^{a} + R_{im,i}^{\varphi} \\ = &\alpha\varphi_{tt}^{a}\varphi_{m}^{a} + R_{Am,A}^{\varphi} + R_{mm,m}^{\varphi}. \end{split}$$

Using that  $\varphi_{tt}^a = \frac{1}{\alpha}U^a$ , we obtain

$$\frac{S_m^{\varphi}}{2(m-1)} = \frac{U^a}{m-1} \varphi_m^a + \frac{1}{m-1} R_{Am,A}^{\varphi} + \frac{1}{m-1} R_{mm,m}^{\varphi}.$$

Hence,

$$(4.43) \quad C_{ABm}^{\varphi} = \frac{U^a}{m-1} \varphi_m^a \delta_{AB} + \frac{1}{m-1} R_{Cm,C}^{\varphi} \delta_{AB} + \frac{1}{m-1} (S^{\varphi} - m R_{mm}^{\varphi}) \frac{f_{AB}}{|\nabla f|}.$$

Taking the trace of (4.41), we have

(4.44) 
$$R_{Bm,B}^{\varphi} = -\frac{1}{m-1} (S^{\varphi} - mR_{mm}^{\varphi}) \frac{f_{BB}}{|\nabla f|}.$$

Moreover, by (4.19).

$$f_{AB} = -R_{AB}^{\varphi} + \eta f_A f_B + \lambda \delta_{AB}$$

$$= -\frac{1}{m-1} (S^{\varphi} - R_{mm}^{\varphi}) \delta_{AB} + \lambda \delta_{AB}$$

$$= -\frac{1}{m-1} (S^{\varphi} - R_{mm}^{\varphi} - (m-1)\lambda) \delta_{AB};$$

$$(4.45)$$

thus, taking the trace, we get

$$f_{BB} = -S^{\varphi} + R_{mm}^{\varphi} + (m-1)\lambda.$$

From (4.44) we deduce

$$(4.46) R_{Cm,C}^{\varphi} = \frac{1}{|\nabla f|} \frac{1}{m-1} (S^{\varphi} - R_{mm}^{\varphi} - (m-1)\lambda)(S^{\varphi} - mR_{mm}^{\varphi});$$

on the other hand, from (4.45), we have

(4.47)

$$\frac{1}{|\nabla f|} \frac{(S^{\varphi} - mR_{mm}^{\varphi})}{m - 1} f_{AB} = -\frac{1}{|\nabla f|} \frac{(S^{\varphi} - mR_{mm}^{\varphi})}{(m - 1)^2} (S^{\varphi} - R_{mm}^{\varphi} - (m - 1)\lambda) \delta_{AB}.$$

Inserting (4.46) and (4.47) into (4.43), we get

$$\begin{split} C_{ABm}^{\varphi} &= \frac{1}{m-1} U^{a} \varphi_{m}^{a} \delta_{AB} \\ &+ \frac{1}{(m-1)^{2}} \frac{1}{|\nabla f|} (S^{\varphi} - R_{mm}^{\varphi} - (m-1)\lambda) (S^{\varphi} - m R_{mm}^{\varphi}) \delta_{AB} \\ &- \frac{1}{(m-1)^{2}} \frac{1}{|\nabla f|} (S^{\varphi} - R_{mm}^{\varphi} - (m-1)\lambda) (S^{\varphi} - m R_{mm}^{\varphi}) \delta_{AB} \\ &= \frac{1}{m-1} U^{a} \varphi_{m}^{a} \delta_{AB}, \end{split}$$

that is (4.36).

In the next Theorem we will prove, among other things, that, under the assumptions of Proposition 4.11, for any regular level set  $\Sigma$  of f and for any point  $p \in \Sigma$  there exists an open neighbourhood A of p such that  $f_{|A}$  only depends on the

signed distance function  $r: A \to \mathbb{R}$  from  $A \cap \Sigma$ . With a slight abuse of notation, we will write

$$f = f(r),$$

identifying f with a function  $f: \mathbb{R} \to \mathbb{R}$ .

**Theorem** 4.14. Let (M,g), (N,h) be two Riemannian manifolds of dimension  $m \geq 3$  and n, respectively. Let  $\varphi : (M,g) \to (N,h)$  be a smooth map and f a solution of (4.4), where  $U \in C^{\infty}(N)$ ,  $\alpha \in \mathbb{R} \setminus \{0\}$  and

$$\lambda = \frac{1}{m} \Big( S^{\varphi} + \Delta f - \eta |\nabla f|^2 \Big).$$

Assume that  $\overline{D}^{\varphi} \equiv 0$  and that  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic; let c be a regular value of f and  $\Sigma$  the corresponding level set. Then, for each  $p \in \Sigma$ , there exists an open set A, with  $p \in A \subseteq M$ , such that  $g|_A$  is a warped product  $(I \times_{\rho} (\Sigma \cap A), dr^2 + \rho^2 g_{\Sigma})$ , with I an open interval. When M is compact we can choose A such that  $\Sigma \subset A$ . Let r denote the signed distance function from  $\Sigma$ : then

- 1.  $f_{|A|}$  only depends on r.
- 2.  $S^{\varphi_{|\Sigma}}$  is constant and it holds

$$(4.48) 0 = \frac{S^{\varphi_{|\Sigma}}}{\rho^{2}(r)} + (m-1)(m-2) \left[ \frac{\rho''(r)}{\rho(r)} - \left( \frac{\rho'(r)}{\rho(r)} \right)^{2} \right] - (m-1)f''(r) + \eta(m-1)(f'(r))^{2} + (m-1)f'(r) \frac{\rho'(r)}{\rho(r)}.$$

3.  $U(\varphi)$  is constant on the connected components of  $\Sigma$  and  $(\Sigma, g_{\Sigma})$  satisfies

(4.49) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi_{|\Sigma}} = \frac{S^{\varphi_{|\Sigma}}}{m-1} g_{\Sigma}, \\ ii) h(d\varphi_{|\Sigma}, \tau(\varphi_{|\Sigma})) = 0. \end{cases}$$

PROOF. We divide the proof into three steps.

**Step 1**. We start by showing that the metric g locally splits as a warped product around p; to do this we follow the proof in [14].

Let r be the signed distance function from  $\Sigma = \Sigma_c = f^{-1}(c)$ , chosen so that its gradient is

$$\nabla r = \frac{\nabla f}{|\nabla f|}.$$

On an open neighborhood  $A \subset M$  of p, there exist adapted coordinates, namely Fermi coordinates,  $\left\{x^i\right\}_{i=1}^m$  (see [45, Corollary 6.42] for more details), such that  $x_m = r$  and the tangent space of  $\Sigma \cap A$  is spanned by

$$\bigg\{\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^{m-1}}\bigg\}.$$

By restricting A if necessary, we can assume that all the points of A are regular for f. For the ease of notations, let

$$\underline{x} = (x^1, ..., x^{m-1}).$$

In these coordinates the metric g takes the form

$$(4.50) g = dr \otimes dr + g_{AB}(\underline{x}, r)dx^A \otimes dx^B,$$

where  $1 \leq A, B \leq m-1$  and

$$(4.51) g_{AB}(\underline{x},0)dx^A \otimes dx^B = g_{\Sigma}.$$

Furthermore, the Christoffel symbols of the Levi-Civita connection satisfy

(4.52) 
$$\begin{cases} \Gamma_{mm}^{m} = \Gamma_{mm}^{A} = \Gamma_{Am}^{m} = 0, \\ \Gamma_{AB}^{m} = -\frac{1}{2}\partial_{r}g_{AB}, \\ \Gamma_{Bm}^{A} = \frac{1}{2}g^{AC}\partial_{r}g_{BC}. \end{cases}$$

Since, by Proposition 4.11,  $|\nabla f|$  is a positive constant on any regular level set of f, it follows that f only depends on r, if we remain sufficiently close to  $\Sigma$ . In particular, near  $\Sigma$ ,

$$(4.53) df = f'dr,$$

(4.54) 
$$\nabla f = f' \frac{\partial}{\partial r},$$

$$\operatorname{Hess}(f) = f''dr \otimes dr + f'\operatorname{Hess}(r)$$

$$= f'' dr \otimes dr + \frac{f'}{2} \partial_r g_{AB} dx^A \otimes dx^B.$$

From (4.54) and our choice of r we get

$$\nabla f = \frac{f'}{|\nabla f|} \nabla f$$

so that f' > 0. Therefore,  $f : \mathbb{R} \to \mathbb{R}$  is monotone increasing and hence injective, so that there exists a one to one correspondence between the regular level sets of r and those of f that meets A. Equation (4.55) implies

$$(4.56) f_{AB} = \frac{f'}{2} \partial_r g_{AB}.$$

Since, by Corollary 4.7,  $i: \Sigma_{\overline{c}} \hookrightarrow M$  is totally umbilical, with  $\overline{c}$  a regular value sufficiently close to c,

(4.57) 
$$\frac{f_{AB}}{f'} = -II_{AB} = -Hg_{AB}.$$

Note that c = f(0) and  $\bar{c} = f(\bar{r})$  for some  $\bar{r}$  close to 0; moreover, H is constant on the connected components of  $\Sigma_{\bar{c}}$ , as we have proved in Proposition 4.11. Comparing (4.56) and (4.57), we deduce

$$-f'Hg_{AB} = \frac{f'}{2}\partial_r g_{AB};$$

since  $f' \neq 0$  for  $0 \leq r \ll 1$ , we have

$$-Hg_{AB} = \frac{\partial_r g_{AB}}{2}$$

and integrating the above expression with respect to r, we get

$$g_{AB}(\underline{x},r) = e^{-2\int_0^r H(t)dt} g_{AB}(\underline{x},0).$$

Inserting the above into (4.50), we obtain that g is a warped product metric.

**Step 2.** In this step we prove the validity of (4.48), i.e.,

$$0 = \frac{S^{\varphi_{|\Sigma}}}{\rho^{2}(r)} + (m-1)(m-2) \left[ \frac{\rho''(r)}{\rho(r)} - \left( \frac{\rho'(r)}{\rho(r)} \right)^{2} \right] - (m-1)f''(r) + \eta(m-1)(f'(r))^{2} + (m-1)f'(r)\frac{\rho'(r)}{\rho(r)}.$$

Assume that g locally splits as in Step 1 in a neighborhood of p, that is, there is an open neighborhood A of p such that

$$A \simeq (-\varepsilon, \varepsilon) \times (\Sigma \cap A),$$

where we identify  $\Sigma \cap A$  with  $\{0\} \times (\Sigma \cap A)$  and

$$g = dr \otimes dr + \rho^2 g_{\Sigma};$$

moreover, by (4.51),  $\rho$  satisfies  $\rho(0) = 1$ ,  $\rho > 0$ . Consider a local orthonormal coframe  $\{\theta^i\}_{i=1}^m$  such that  $\theta^m = dr$  and  $\{\theta^i\}_{i=1}^{m-1}$  is a local orthonormal coframe for  $g_{\Sigma}$ ; then we have

$$(4.58) f_{mm} = f'',$$

(4.59) 
$$\Delta f = f'' + (m-1)f'\frac{\rho'}{\rho},$$

(4.60) 
$$S = \frac{\Sigma S}{\rho^2} - (m-1)(m-2) \left(\frac{\rho'}{\rho}\right)^2 - 2(m-1)\frac{\rho''}{\rho},$$

(4.61) 
$$R_{mm} = -(m-1)\frac{\rho''}{\rho},$$

where, for the sake of simplicity, we have omitted the dependence on r of f = f(r) and  $\rho$  (see e.g. [43], and page 50 of [1] for more details on the previous formulas). Observe that, since  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic by assumption and (4.4) ii) holds, we have

$$\varphi_m^a = 0;$$

thus

$$\left|d\varphi\right|_g^2 = \left|d\varphi\right|_{\rho^2 g_{\Sigma}}^2 = \frac{1}{\rho^2} \left|d\varphi\right|_{g_{\Sigma}}^2,$$

so that

$$S^{\varphi} = S - \alpha |d\varphi|_q^2$$

$$\begin{split} &= \frac{\Sigma S}{\rho^2} - \alpha |d\varphi|_g^2 - (m-1)(m-2) \left(\frac{\rho'}{\rho}\right)^2 - 2(m-1)\frac{\rho''}{\rho} \\ &= \frac{S^{\varphi_{|\Sigma}}}{\rho^2} - (m-1)(m-2) \left(\frac{\rho'}{\rho}\right)^2 - 2(m-1)\frac{\rho''}{\rho}. \end{split}$$
(4.62)

From (4.4) i) and the definition of  $\lambda$ , we get

$$(4.63) R_{mm}^{\varphi} + f_{mm} - \eta |\nabla f|^2 = \frac{1}{m} \left( S^{\varphi} + \Delta f - \eta |\nabla f|^2 \right).$$

Inserting (4.58), (4.59) and (4.61) into (4.63), we obtain

$$S^{\varphi} + m(m-1)\frac{\rho''}{\rho} - (m-1)f'' + \eta(m-1)(f')^{2} + (m-1)f'\frac{\rho'}{\rho} = 0;$$

then, using (4.62), we conclude the validity of (4.48), i.e.

$$0 = \frac{S^{\varphi_{|\Sigma}}}{\rho^{2}(r)} + (m-1)(m-2) \left[ \frac{\rho''(r)}{\rho(r)} - \left( \frac{\rho'(r)}{\rho(r)} \right)^{2} \right] - (m-1)f''(r) + \eta(m-1)(f'(r))^{2} + (m-1)f'(r) \frac{\rho'(r)}{\rho(r)},$$

where we have expressed the dependence on r in the above formula; from this we deduce that  $S^{\varphi_{|\Sigma}}$  is constant on  $\Sigma$ .

Step 3. In this step we prove the final part of the statement.

By the assumption  $\overline{D}^{\varphi} \equiv 0$ , the first integrability condition (4.7) rewrites as

$$0 = C_{AmB}^{\varphi} + |\nabla f| W_{mAmB}^{\varphi} - \frac{U^a}{m-1} \varphi_B^a \delta_{Am} + \frac{U^a}{m-1} \varphi_m^a \delta_{AB};$$

by Proposition 4.13

$$C_{AmB}^{\varphi} = -C_{ABm}^{\varphi} = -\frac{U^a}{m-1}\varphi_m^a\delta_{AB},$$

hence

$$W_{mAmB}^{\varphi} = 0$$

on  $\Sigma$ . By the definition of the  $\varphi$ -Weyl tensor we deduce

$$(4.64) R_{mAmB} = \frac{1}{m-2} \left( R_{mm}^{\varphi} \delta_{AB} + R_{AB}^{\varphi} - \frac{S^{\varphi}}{m-1} \delta_{AB} \right)$$

on  $\Sigma$ . Using (4.33) of Proposition 4.11, i.e.

$$R_{AB}^{\varphi} = \frac{1}{m-1} (S^{\varphi} - R_{mm}^{\varphi}) \delta_{AB},$$

into (4.64), we obtain

$$R_{mAmB} = \frac{1}{m-2} \left( \frac{S^{\varphi}}{m-1} \delta_{AB} - \frac{1}{m-1} R^{\varphi}_{mm} \delta_{AB} + R^{\varphi}_{mm} \delta_{AB} - \frac{S^{\varphi}}{m-1} \delta_{AB} \right)$$

$$(4.65) \qquad = \frac{1}{m-1} R^{\varphi}_{mm} \delta_{AB}.$$

From the Gauss equation for the Ricci tensor of  $i:\Sigma\hookrightarrow M$ ,  ${}^\Sigma\mathrm{Ric}$ , and since i is totally umbilical we have

$${}^{\Sigma}R_{AC} = R_{AC} - R_{AmCm} + (m-2)H^2\delta_{AC};$$

from the definition of  $\mathrm{Ric}^{\varphi_{|_{\Sigma}}}$  we deduce

(4.66) 
$$R_{AC}^{\varphi_{|\Sigma}} = {}^{\Sigma}R_{AC} - \alpha\varphi_A^a\varphi_C^a = R_{AC}^{\varphi} - R_{AmCm} + (m-2)H^2\delta_{AC}.$$

Using (4.33) and (4.65) into (4.66), we obtain

$$R_{AC}^{\varphi_{|\Sigma}} = \frac{S^{\varphi} - R_{mm}^{\varphi}}{m - 1} \delta_{AC} - \frac{1}{m - 1} R_{mm}^{\varphi} \delta_{AC} + (m - 2)H^{2} \delta_{AC}$$
$$= \left[ \frac{1}{m - 1} (S^{\varphi} - 2R_{mm}^{\varphi}) + (m - 2)H^{2} \right] \delta_{AC}.$$

Contracting with respect to the indexes A and C, we get

$$S^{\varphi|_{\Sigma}} = S^{\varphi} - 2R_{mm} + (m-1)(m-2)H^2.$$

Therefore, the above can be rewritten as

$$R_{AC}^{\varphi_{\mid_{\Sigma}}} = \frac{S^{\varphi_{\mid_{\Sigma}}}}{m-1} \delta_{AC},$$

that is, we have the validity of (4.49) i). Taking the divergence of (4.49) i) and using the  $\varphi$ -Schur's identity we deduce

$$R_{BA,B}^{\varphi|_{\Sigma}} = \frac{1}{2} S_A^{\varphi|_{\Sigma}} - \alpha \tau^a(\varphi|_{\Sigma}) \varphi_A^a = \frac{1}{m-1} S_A^{\varphi|_{\Sigma}};$$

by the constancy of  $S^{\varphi|_{\Sigma}}$  we have

$$\tau^a(\varphi|_{\Sigma})\varphi_A^a = 0,$$

from which we deduce

$$0 = \tau^a(\varphi|_{\Sigma})\varphi_A^a = \frac{1}{\alpha}U^a\varphi_A^a = \frac{1}{\alpha}(U(\varphi))_A,$$

where we have used that, according to Proposition 4.11,  $\varphi_{|_{\Sigma}}$  is  $\frac{U}{\alpha}$ -harmonic. This proves that  $U(\varphi)$  is constant on the connected components of  $\Sigma$ .

### 4.3. Main Results

In this subsection we are finally ready to give a proof of Theorem 1.4, that we recall here for the ease of readability.

**Theorem** 4.15. Let (M,g) be a Riemannian manifold of dimension  $m \geq 3$  with  $\partial M = \emptyset$ . Let  $\varphi : (M,g) \to (N,h)$ ,  $U : (N,h) \to \mathbb{R}$  be smooth maps,  $\alpha \in \mathbb{R} \setminus \{0\}$  and let  $f \in C^2(M)$  be a solution on M of the system

(4.67) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi) \end{cases}$$

where  $\lambda = \frac{1}{m} \left( S^{\varphi} + \Delta f - \eta |\nabla f|^2 \right)$  and  $\eta \neq -\frac{1}{m-2}$ . Consider the conformal change of metric

$$\widetilde{g} = e^{-\frac{2}{m-2}f}g;$$

suppose that  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic and that

$$2(m-1)\widetilde{C}^{\widetilde{\varphi}} = -\operatorname{div}_1(U(\varphi)g \otimes g).$$

Then, for each  $\Sigma$  regular level set of f and  $p \in \Sigma$ , there exists  $A \subseteq M$  open such that  $p \in A$  and  $g|_A$  is a warped product metric. Moreover  $U(\varphi)$  is constant on M,  $S^{\varphi|_{\Sigma}}$  is the constant in (4.48) and  $(\Sigma, g_{\Sigma})$  satisfies

(4.68) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi_{\mid_{\Sigma}}} = \frac{S^{\varphi_{\mid_{\Sigma}}}}{m-1} g_{\Sigma}, \\ ii) h(\tau(\varphi_{\mid_{\Sigma}}), d\varphi_{\mid_{\Sigma}}) = 0. \end{cases}$$

PROOF. Since

$$2(m-1)\widetilde{C}^{\widetilde{\varphi}} = -\operatorname{div}_1(U(\varphi)g \otimes g),$$

by (4.7) and Remark 4.3, we have  $\overline{D}^{\varphi} \equiv 0$ . Therefore the claim follows by Theorem 4.14.

Assume now, again, the validity of system (4.4), that is

$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi), \end{cases}$$

where  $\lambda \in C^{\infty}(M)$ , and define on int(M) the vector field Y of (local) components

$$(4.69) Y_k = [1 + \eta(m-2)]\overline{D}_{ijk}^{\varphi} f_i f_j - \frac{U^a}{m-1} \Big(\varphi_j^a f_j f_k - \varphi_k^a |\nabla f|^2\Big).$$

Note that

$$(4.70) g(Y, \nabla f) = Y_k f_k \equiv 0,$$

so that on the level set of a regular value of f

$$(4.71) g(Y,\nu) = g\left(Y,\frac{\nabla f}{|\nabla f|}\right) = 0,$$

where  $\nu = \frac{\nabla f}{|\nabla f|}$  is a unit normal vector field.

**Lemma** 4.16. Let (M,g) be a Riemannian manifold of dimension  $m \geq 3$ , let  $\varphi: (M,g) \to (N,h)$  be a smooth map to a second Riemannian manifold,  $U: (N,h) \to \mathbb{R}$  be a smooth function and let Y be as in (4.69). Under the validity of (4.4) we have

(4.72)

$$\operatorname{div} Y = \frac{1}{2} [1 + \eta(m-2)](m-2) \left| \overline{D}^{\varphi} \right|^{2} + (m-2) B^{\varphi}(\nabla f, \nabla f) - h(\nabla U, \nabla d\varphi(\nabla f, \nabla f))$$

$$+ \operatorname{Hess}(U)(d\varphi(\nabla f), d\varphi(\nabla f)) + \frac{1}{2(m-1)} h\Big(\nabla U, d\varphi\Big(\nabla |\nabla f|^2\Big)\Big) \\ - \frac{\Delta f}{m-1} h(\nabla U, d\varphi(\nabla f)) + \frac{\alpha}{m-2} |\nabla f|^2 |\tau(\varphi)|^2 + \alpha \eta |\nabla f|^2 \Big|\tau(\varphi) - \frac{1}{\alpha} \nabla U\Big|^2.$$

PROOF. Recall the validity of equation (4.22), that is,

$$(4.73) \frac{2}{m-2}\overline{D}_{ijk}^{\varphi}R_{ij}^{\varphi}f_k = \left|\overline{D}^{\varphi}\right|^2;$$

from the first equation of (4.4) and the symmetries of  $\overline{D}^{\varphi}$  we get

$$(4.74) -\frac{2}{m-2}\overline{D}_{ijk}^{\varphi}f_{ij}f_k = \left|\overline{D}^{\varphi}\right|^2.$$

This identity will be used later on. We compute the divergence of Y

$$\begin{aligned} \operatorname{div} Y = & [1 + \eta(m-2)] \left( f_{ik} f_j \overline{D}_{ijk}^{\varphi} + f_i f_j \overline{D}_{ijk,k}^{\varphi} \right) \\ & + \frac{|\nabla f|^2}{m-1} U^{ab} \varphi_t^a \varphi_t^b + \frac{|\nabla f|^2}{m-1} U^a \varphi_{tt}^a + \frac{2}{m-1} f_{tk} f_t U^a \varphi_k^a \\ & - \frac{1}{m-1} U^{ab} \varphi_t^a f_t \varphi_k^b f_k - \frac{1}{m-1} U^a \varphi_{tk}^a f_t f_k - \frac{1}{m-1} U^a \varphi_t^a f_{tk} f_k \\ & - \frac{\Delta f}{m-1} U^a \varphi_j^a f_j. \end{aligned}$$

Using the second integrability condition (4.8) and equation (4.74) we get

$$\operatorname{div} Y = \frac{1}{2} [1 + \eta(m-2)](m-2) \left| \overline{D}^{\varphi} \right|^{2} + (m-2) B_{ij}^{\varphi} f_{i} f_{j} - \frac{m-2}{m-1} U^{a} \varphi_{ij}^{a} f_{i} f_{j}$$

$$+ \alpha \frac{1 + \eta(m-2)}{m-2} |\nabla f|^{2} \varphi_{tt}^{a} \varphi_{k}^{a} f_{k} + \frac{|\nabla f|^{2}}{(m-1)(m-2)} U^{a} \varphi_{tt}^{a}$$

$$- \frac{|\nabla f|^{2}}{m-1} U^{ab} \varphi_{t}^{a} \varphi_{t}^{a} + \frac{m}{m-1} U^{ab} \varphi_{i}^{a} f_{i} \varphi_{j}^{a} f_{j} - \eta |\nabla f|^{2} U^{a} \varphi_{t}^{a} f_{t}$$

$$+ \frac{|\nabla f|^{2}}{m-1} U^{ab} \varphi_{t}^{a} \varphi_{t}^{b} + \frac{|\nabla f|^{2}}{m-1} U^{a} \varphi_{tt}^{a} + \frac{2}{m-1} U^{a} \varphi_{k}^{a} f_{tk} f_{t}$$

$$- \frac{1}{m-1} U^{ab} \varphi_{k}^{a} f_{k} \varphi_{t}^{b} f_{t} - \frac{1}{m-1} U^{a} \varphi_{tk}^{a} f_{t} f_{t} - \frac{1}{m-1} U^{a} \varphi_{t}^{a} f_{tk} f_{k}$$

$$- \frac{\Delta f}{m-1} U^{a} \varphi_{t}^{a} f_{t}.$$

After some simplifications, the previous relation becomes

$$(4.75) \quad \operatorname{div} Y = \frac{1}{2} [1 + \eta(m-2)](m-2) \left| \overline{D}^{\varphi} \right|^{2} + (m-2) B_{ij}^{\varphi} f_{i} f_{j} - U^{a} \varphi_{ij}^{a} f_{i} f_{j}$$

$$+ \alpha \frac{1 + \eta(m-2)}{m-2} |\nabla f|^{2} \varphi_{tt}^{a} \varphi_{k}^{a} f_{k} + \frac{|\nabla f|^{2}}{m-2} U^{a} \varphi_{tt}^{a} + U^{ab} \varphi_{t}^{a} f_{t} \varphi_{k}^{b} f_{k}$$

$$- \eta |\nabla f|^{2} U^{a} \varphi_{t}^{a} f_{t} + \frac{1}{m-1} U^{a} \varphi_{t}^{a} f_{tk} f_{k} - \frac{\Delta f}{m-1} U^{a} \varphi_{k}^{a} f_{k}.$$

Using the second equation of (4.4), that is

$$\varphi_t^a f_t = \varphi_{tt}^a - \frac{1}{\alpha} U^a,$$

we compute

$$\begin{split} &\alpha\frac{1+\eta(m-2)}{m-2}|\nabla f|^2\varphi_{tt}^a\varphi_k^af_k + \frac{|\nabla f|^2}{m-2}U^a\varphi_{tt}^a - \eta|\nabla f|^2U^a\varphi_t^af_t\\ &= \alpha\frac{1+\eta(m-2)}{m-2}|\nabla f|^2|\tau(\varphi)|^2 - \frac{1+\eta(m-2)}{m-2}|\nabla f|^2U^a\varphi_{tt}^a + \frac{|\nabla f|^2}{m-2}U^a\varphi_{tt}^a\\ &+ \frac{\eta}{\alpha}|\nabla f|^2|\nabla U|^2 - \eta|\nabla f|^2U^a\varphi_{tt}^a\\ &= \frac{\alpha}{m-2}|\nabla f|^2|\tau(\varphi)|^2 + \alpha\eta|\nabla f|^2|\tau(\varphi)|^2 + \frac{\eta}{\alpha}|\nabla U|^2 - 2\eta|\nabla f|^2U^a\varphi_{tt}^a\\ &= \frac{\alpha}{m-2}|\nabla f|^2|\tau(\varphi)|^2 + \alpha\eta|\nabla f|^2\Big|\tau(\varphi) - \frac{1}{\alpha}\nabla U\Big|^2, \end{split}$$

that is

$$\alpha \frac{1 + \eta(m-2)}{m-2} |\nabla f|^2 \varphi_{tt}^a \varphi_k^a f_k + \frac{|\nabla f|^2}{m-2} U^a \varphi_{tt}^a - \eta |\nabla f|^2 U^a \varphi_t^a f_t$$
$$= \frac{\alpha}{m-2} |\nabla f|^2 |\tau(\varphi)|^2 + \alpha \eta |\nabla f|^2 |\tau(\varphi) - \frac{1}{\alpha} \nabla U|^2.$$

Inserting the above formula into (4.75), we obtain

$$\operatorname{div} Y = \frac{1}{2} [1 + \eta(m-2)](m-2) \left| \overline{D}^{\varphi} \right|^{2} + (m-2) B_{ij}^{\varphi} f_{i} f_{j} - U^{a} \varphi_{ij}^{a} f_{i} f_{j}$$

$$+ U^{ab} \varphi_{t}^{a} f_{t} \varphi_{k}^{b} f_{k} + \frac{1}{m-1} U^{a} \varphi_{t}^{a} f_{tk} f_{k} - \frac{\Delta f}{m-1} U^{a} \varphi_{k}^{a} f_{k}$$

$$+ \frac{\alpha}{m-2} |\nabla f|^{2} |\tau(\varphi)|^{2} + \alpha \eta |\nabla f|^{2} \left| \tau(\varphi) - \frac{1}{\alpha} \nabla U \right|^{2},$$

that is, (4.72).

Note that, when  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic, that is,

(4.76) 
$$0 = \tau(\varphi) - \frac{1}{\alpha} (\nabla U)(\varphi) = d\varphi(\nabla f),$$

then the divergence of Y rewrites as

(4.77) 
$$\operatorname{div} Y = \frac{1}{2} [1 + \eta(m-2)](m-2) \left| \overline{D}^{\varphi} \right|^{2} + (m-2) B_{ij}^{\varphi} f_{i} f_{j} - U^{a} \varphi_{ij}^{a} f_{i} f_{j} + \frac{1}{m-1} U^{a} \varphi_{t}^{a} f_{tk} f_{k} + \frac{\alpha}{m-2} |\nabla f|^{2} |\tau(\varphi)|^{2}.$$

Moreover, by (4.76), we have

$$(4.78) 0 = (\varphi_i^a f_i)_j = \varphi_{ij}^a f_i + \varphi_j^a f_{ij},$$

thus, (4.77) rewrites as

$$\operatorname{div} Y = \frac{1}{2} [1 + \eta(m-2)](m-2) \left| \overline{D}^{\varphi} \right|^{2} + (m-2) B_{ij}^{\varphi} f_{i} f_{j}$$
$$+ \frac{m}{m-1} U^{a} \varphi_{t}^{a} f_{tk} f_{k} + \frac{\alpha}{m-2} |\nabla f|^{2} |\tau(\varphi)|^{2}$$

and, since  $\varphi_k^a f_k = 0$ , by the first equation of system (4.4) we deduce

$$f_{tk}f_k\varphi_t^a = -R_{tk}^{\varphi}f_k\varphi_t^a + \eta|\nabla f|^2 f_t\varphi_t^a + \lambda\varphi_t^a f_t$$
$$= -R_{tk}^{\varphi}f_k\varphi_t^a.$$

Therefore,

$$\operatorname{div} Y = (m-2)B^{\varphi}(\nabla f, \nabla f) - \frac{m}{m-1}U^{a}\varphi_{j}^{a}f_{k}R_{jk}^{\varphi} + \frac{\alpha}{m-2}|\tau(\varphi)|^{2}|\nabla f|^{2} + \frac{1}{2}(1+\eta(m-2))(m-2)|\overline{D}^{\varphi}|^{2}.$$
(4.79)

Note that (4.77) simplifies under the assumption  $B^{\varphi}(\nabla f, \nabla f) = 0$ : this observation, together with the divergence theorem, is exploited in the proof of the following theorem.

**Theorem** 4.17. Let (M,g) be a complete manifold of dimension  $m \geq 3$  and with  $\partial M = \emptyset$ . Let  $\varphi : (M,g) \to (N,h)$ ,  $U:N \to \mathbb{R}$  be smooth maps,  $\lambda \in C^{\infty}(M)$  and let  $f \in C^{\infty}(M)$  be a solution of (4.4). Let  $\Sigma$  be a regular level set of f. Assume that:

- 1. f is proper;
- 2. either  $\alpha > 0$  and  $\eta > -\frac{1}{m-2}$  or  $\alpha < 0$  and  $\eta < -\frac{1}{m-2}$ ;
- 3. we have the validity of

$$(4.80) B^{\varphi}(\nabla f, \nabla f) = 0;$$

- 4. for each regular  $p \in M$ ,  $\nabla f_p$  is an eigenvector of  $\mathrm{Ric}_p^{\varphi}$ ;
- 5.  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic

Then, for each  $p \in \Sigma$ , there exists  $p \in A \subset M$ , A open in M such that  $g_{|A}$  is a warped product metric. Moreover,  $U(\varphi)$  is constant on  $\Sigma$  and  $(\Sigma, g_{\Sigma})$  satisfies

$$\begin{cases} \operatorname{Ric}^{\varphi_{\mid_{\Sigma}}} = \frac{S^{\varphi_{\mid_{\Sigma}}}}{m-1} g_{\Sigma} \\ \tau(\varphi_{\mid_{\Sigma}}) = 0 \end{cases}$$

and  $S^{\varphi_{|\Sigma}}$  is constant.

**Remark** 4.18. Note that, by Corollary 4.7, the request of  $\nabla f$  being an eigenvector of  $\operatorname{Ric}^{\varphi}$  is necessary for the validity of  $\overline{D}^{\varphi} \equiv 0$ , which is a fundamental tool in the proof of the theorem.

PROOF. By hypothesis we have that  $\nabla f$  is an eigenvector of  $\mathrm{Ric}^{\varphi}$  at any regular point of f, that is, at such a point,

$$(4.81) R_{ik}^{\varphi} f_k = \Lambda f_i$$

for some  $\Lambda \in \mathbb{R}$ . Therefore, since  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic

$$\frac{m}{m-1}U^a\varphi_j^a f_k R_{jk}^{\varphi} = \Lambda \frac{m}{m-1}U^a\varphi_j^a f_j = 0$$

and by assumption (4.80), (4.77) rewrites as

(4.83) 
$$\operatorname{div} Y = \frac{\alpha}{m-2} |\tau(\varphi)|^2 |\nabla f|^2 + \frac{1}{2} (1 + \eta(m-2))(m-2) |\overline{D}^{\varphi}|^2.$$

For two regular values  $\delta, \sigma$  of f, with  $\sigma < \delta$  , consider the set

$$\Omega_{\delta,\sigma} = \{ x \in M : \sigma \le f(x) \le \delta \};$$

since the map f is proper we can integrate (4.83) over  $\Omega_{\delta,\sigma}$  and we can apply the divergence theorem to obtain

$$\frac{1}{2}(1 + \eta(m-2))(m-2) \int_{\Omega_{\delta,\sigma}} \left| \overline{D}^{\varphi} \right|^{2} + \frac{\alpha}{m-2} \int_{\Omega_{\delta,\sigma}} \left| \tau(\varphi) \right|^{2} \left| \nabla f \right|^{2} \\
= \int_{\Omega_{\delta,\sigma}} \operatorname{div} Y = \int_{\partial\Omega_{\delta,\sigma}} -g(Y,\nu) = 0,$$

where the last equality is implied by (4.71). Therefore,

$$\frac{1}{2}(1+\eta(m-2))(m-2)\int_{\Omega_{\delta,\sigma}}\left|\overline{D}^{\varphi}\right|^{2}+\frac{\alpha}{m-2}\int_{\Omega_{\delta,\sigma}}\left|\tau(\varphi)\right|^{2}\left|\nabla f\right|^{2}=0$$

and letting  $\delta \to +\infty, \sigma \to -\infty$ , we have

$$\frac{1}{2}(1 + \eta(m-2))(m-2) \int_{M} \left| \overline{D}^{\varphi} \right|^{2} + \frac{\alpha}{m-2} \int_{M} |\tau(\varphi)|^{2} |\nabla f|^{2} = 0.$$

If  $\alpha>0$  and  $\eta>-\frac{1}{m-2}$  the left hand side of the above expression is non-negative, while if  $\alpha<0$  and  $\eta<-\frac{1}{m-2}$  it is non-positive. Either way, we get  $\overline{D}^{\varphi}=0=\tau(\varphi)$  and the claim follows by Theorem 4.14.

#### CHAPTER 5

# Other rigidity and related results

# 5.1. Other Rigidity Results

In this Chapter we prove further rigidity results for a Riemannian manifold (M, g) supporting a structure of the type

(5.1) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi). \end{cases}$$

As in Chapter 4, our main goal is to prove the vanishing of the tensor  $\overline{D}^{\varphi}$  defined in (4.5). In Theorem 4.15, this was established through the study of the first integrability condition

$$[1 + \eta(m-2)]\overline{D}_{ijk}^{\varphi} = C_{ijk}^{\varphi} + f_t W_{tijk}^{\varphi} - \frac{U^a}{m-1} (\varphi_k^a \delta_{ij} - \varphi_j^a \delta_{ik})$$

of system (5.1). Using it, one can prove that condition

(5.3) 
$$2(m-1)\widetilde{C}^{\widetilde{\varphi}} + \operatorname{div}_1(U(\varphi)g \otimes g) = 0,$$

where, as before, tilded quantities are relative to the conformal metric  $\tilde{g} = e^{-\frac{2}{m-2}f}g$ , is equivalent to  $\overline{D}^{\varphi} \equiv 0$ . Our aim is now to relax assumption (5.3): in particular, we study the consequences of the vanishing of the total divergence  $\operatorname{div}^3 C^{\varphi}$  of the  $\varphi$ -Cotton tensor. In components, this is defined by

$$\operatorname{div}^3 C^{\varphi} = C^{\varphi}_{ijk,kji}.$$

In this Chapter we provide a proof of Theorem 1.6 and Theorem 1.10 of the Introduction, starting with the latter, whose assumptions are closer to those of Theorem 4.17.

**Theorem** 5.1. Let (M, g) be a complete Riemannian manifold satisfying system (5.1), where f is a proper function and  $\eta \neq -\frac{1}{m-2}$ . If M is non-compact, we also require

$$f(x) \to +\infty$$
 as  $x \to +\infty$ .

Assume that

(5.4) 
$$\operatorname{div}^3 C^{\varphi} = 0,$$

(5.5) 
$$\varphi$$
 is  $\frac{1}{\alpha}U$ -harmonic,

(5.6)  $\nabla f_p$  is an eigenvector of  $\operatorname{Ric}_p^{\varphi}$  for every regular point p of f.

Then we have two possibilities:

i) if  $\eta \neq 0$  and we further assume

(5.7) 
$$W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0,$$

then we have

(5.8) 
$$C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_{1}\left(U(\varphi)g \bigotimes g\right)$$

and

$$(5.9) \overline{D}^{\varphi} = 0;$$

ii) if  $\eta = 0$  we have

$$C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_{1}\left(U(\varphi)g \bigotimes g\right)$$

and, if we further assume (5.7), also

$$\overline{D}^{\varphi} = 0.$$

**Remark** 5.2. Condition (5.6) is necessary for (5.8), as observed in Lemma 4.9.

**Theorem** 5.3. Let (M,g) be a complete Riemannian manifold satisfying system (5.1), where f is a proper function and  $\eta \neq -\frac{1}{m-2}$ . If M is non-compact, we will also require

$$f(x) \to +\infty$$
 as  $x \to +\infty$ .

Assume that

(5.11) 
$$h(\tau(\varphi), d\varphi) = 0.$$

Then we have two possibilities:

i) if  $\eta \neq 0$  and we further assume

$$(5.12) W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0,$$

then we have

(5.13) 
$$C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_{1}\left(U(\varphi)g \bigotimes g\right)$$

and

$$(5.14) \overline{D}^{\varphi} = 0;$$

ii) if  $\eta = 0$  we have

$$C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_{1}\left(U(\varphi)g \bigotimes g\right)$$

and, if we further assume (5.12), also

$$\overline{D}^{\varphi} = 0.$$

Although the assumptions of the two Theorems that we have presented are different, the techniques of their proofs are similar and we get the same conclusion. To prove Theorem 5.1 and Theorem 5.3 we will need some preliminary results; we begin with a very general lemma.

**Lemma** 5.4. If  $\omega \in \Lambda^2(M)$  is a 2-form, its total divergence  $\operatorname{div}^2 \omega$  vanishes. In components,

$$\omega_{tk,kt} = 0.$$

Proof. The proof is a simple application of the Ricci commutation formulas. Indeed

$$\begin{aligned} \omega_{tk,kt} &= \frac{1}{2} (\omega_{tk,kt} - \omega_{kt,kt}) \\ &= \frac{1}{2} (\omega_{tk,kt} - \omega_{tk,tk}) \\ &= \frac{1}{2} (R_{ptkt} \omega_{pk} + R_{pkkt} \omega_{tp}) \\ &= R_{pk} \omega_{pk} = 0. \end{aligned}$$

The core of the proof of Theorem 5.1 lies in the next Proposition.

**Proposition** 5.5. Let (M,g) be a Riemannian manifold admitting a solution  $f \in C^{\infty}(M)$  of

(5.15) 
$$\operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g,$$

for  $\eta \neq -\frac{1}{m-2}$ . Then we have the validity of the following formula:

(5.16) 
$$\frac{1}{2[(m-2)\eta+1]} |C^{\varphi}|^{2} = f_{t} C^{\varphi}_{tjk,jk} - \frac{\eta(m-2)}{2(m-2)\eta+2} f_{t} W^{\varphi}_{ptjk} C^{\varphi}_{pjk} + \frac{1}{(m-2)\eta+1} \left[ \lambda_{k} - \frac{1}{2(m-1)} S^{\varphi}_{k} - \eta \lambda f_{k} \right] C^{\varphi}_{ttk} - \frac{\eta}{(m-2)\eta+1} \left[ f_{t} R^{\varphi}_{tk} - \frac{S^{\varphi}}{(m-1)} f_{k} \right] C^{\varphi}_{ttk}.$$

PROOF. We apply Lemma 5.4 to the 2-form of components

$$f_t C_{tjk}^{\varphi}$$

to obtain

$$0 = \left( f_t C_{tjk}^{\varphi} \right)_{jk} = f_{tjk} C_{tjk}^{\varphi} + f_{tj} C_{tjk,k}^{\varphi} + f_{tk} C_{tjk,j}^{\varphi} + f_t C_{tjk,jk}^{\varphi}$$
$$= f_{tjk} C_{tjk}^{\varphi} + f_t C_{tjk,jk}^{\varphi},$$

that is,

$$(5.17) 0 = f_{tjk}C_{tjk}^{\varphi} + f_tC_{tjk,jk}^{\varphi}.$$

We claim the validity of the equation

(5.18) 
$$0 = f_t C_{tjk,jk}^{\varphi} - \frac{1}{2} |C^{\varphi}|^2 + \eta f_k R_{tj}^{\varphi} C_{tjk}^{\varphi} + C_{ttk}^{\varphi} \left( \lambda_k - \frac{1}{2(m-1)} S_k^{\varphi} - \eta \lambda f_k \right).$$

To prove it, take covariant derivative of equation (5.15) and skew symmetrize with respect to the last two indexes to deduce

$$f_{tjk} - f_{tkj} = -R_{tj,k}^{\varphi} + R_{tk,j}^{\varphi} + \eta f_{tk} f_j - \eta f_{tj} f_k + \lambda_k \delta_{tj} - \lambda_j \delta_{tk}.$$

Using the definition of  $C^{\varphi}$  we get

$$(5.19) f_{tjk} - f_{tkj} = -C_{tjk}^{\varphi} - \frac{1}{2(m-1)} \left( S_k^{\varphi} \delta_{tj} - S_j^{\varphi} \delta_{tk} \right) + \eta f_{tk} f_j - \eta f_{tj} f_k$$
$$+ \lambda_k \delta_{tj} - \lambda_j \delta_{tk}.$$

From the skew-symmetry of the last two indexes of  $C^{\varphi}$  and equation (5.17) we obtain

$$0 = f_t C_{tjk,jk}^{\varphi} + \frac{1}{2} (f_{tjk} - f_{tkj}) C_{tjk}^{\varphi}.$$

Inserting (5.19) into the above formula we have

$$0 = f_t C_{tjk,jk}^{\varphi} - \frac{1}{2} |C^{\varphi}|^2 - \frac{1}{2(m-1)} S_k^{\varphi} C_{ttk}^{\varphi} + \lambda_k C_{ttk}^{\varphi} + \eta f_{tk} f_j C_{tjk}^{\varphi}.$$

Using again equation (5.15), which we rewrite as

$$f_{tk} = -R_{tk}^{\varphi} + \eta f_t f_k + \lambda \delta_{tk},$$

into the latter, we obtain (5.18).

The last formula that we will need to conclude is

$$(5.20) \frac{(m-2)\eta + 1}{(m-2)} f_k R_{pj}^{\varphi} C_{pjk}^{\varphi} = \frac{1}{2} |C^{\varphi}|^2 - \frac{1}{2} f_t W_{ptjk}^{\varphi} C_{pjk}^{\varphi} - C_{ttk}^{\varphi} \left( \lambda_k - \frac{1}{2(m-1)} S_k^{\varphi} - \eta \lambda f_k + \frac{1}{m-2} f_t R_{tk}^{\varphi} - \frac{S^{\varphi}}{(m-1)(m-2)} f_k \right).$$

To deduce it, we apply the Ricci commutation rules to obtain

$$f_t C_{tjk,jk}^{\varphi} = \frac{1}{2} f_t \left( C_{tjk,jk}^{\varphi} - C_{tjk,kj}^{\varphi} \right)$$

$$\begin{split} &=\frac{1}{2}f_tR_{ptjk}C_{pjk}^{\varphi}+\frac{1}{2}f_tR_{pjjk}C_{tpk}^{\varphi}+\frac{1}{2}f_tR_{pkjk}C_{tjp}^{\varphi}\\ &=\frac{1}{2}f_tR_{ptjk}C_{pjk}^{\varphi}+f_tR_{pk}^{\varphi}C_{tpk}^{\varphi}\\ &=\frac{1}{2}f_tR_{ptjk}C_{pjk}^{\varphi}. \end{split}$$

Using the definition of  $W^{\varphi}$  into the above equation we get

$$\begin{split} f_t C^{\varphi}_{tjk,jk} &= \frac{1}{2} f_t W^{\varphi}_{ptjk} C^{\varphi}_{pjk} + \frac{1}{2(m-2)} \Big( f_t R^{\varphi}_{tk} C^{\varphi}_{ppk} - f_t R^{\varphi}_{tj} C^{\varphi}_{pjp} \Big) \\ &+ \frac{1}{2(m-2)} \Big( f_k R^{\varphi}_{pj} C^{\varphi}_{pjk} - f_j R^{\varphi}_{pk} C^{\varphi}_{pjk} \Big) \\ &- \frac{S^{\varphi}}{2(m-1)(m-2)} \Big( f_k C^{\varphi}_{ttk} - f_j C^{\varphi}_{tjt} \Big) \\ &= \frac{1}{2} f_t W^{\varphi}_{ptjk} C^{\varphi}_{pjk} + \frac{1}{(m-2)} f_t R^{\varphi}_{tk} C^{\varphi}_{ppk} \\ &+ \frac{1}{(m-2)} f_k R^{\varphi}_{pj} C^{\varphi}_{pjk} - \frac{S^{\varphi}}{(m-1)(m-2)} f_k C^{\varphi}_{ttk}, \end{split}$$

that is,

(5.21) 
$$f_t C_{tjk,jk}^{\varphi} = \frac{1}{2} f_t W_{ptjk}^{\varphi} C_{pjk}^{\varphi} + \frac{1}{(m-2)} f_t R_{tk}^{\varphi} C_{ppk}^{\varphi} + \frac{1}{(m-2)} f_k R_{pj}^{\varphi} C_{pjk}^{\varphi} - \frac{S^{\varphi}}{(m-1)(m-2)} f_k C_{ttk}^{\varphi}.$$

Inserting (5.21) into (5.18) and rearranging we get (5.20). Using (5.20) into (5.18) we obtain

$$\begin{split} 0 = & f_t C_{tjk,jk}^{\varphi} - \frac{1}{2} |C^{\varphi}|^2 + C_{ttk}^{\varphi} \left( \lambda_k - \frac{1}{2(m-1)} S_k^{\varphi} - \eta \lambda f_k \right) \\ & + \frac{\eta(m-2)}{(m-2)\eta + 1} \left( \frac{1}{2} |C^{\varphi}|^2 - \frac{1}{2} f_t W_{ptjk}^{\varphi} C_{pjk}^{\varphi} \right) \\ & - \frac{\eta(m-2)}{(m-2)\eta + 1} C_{ttk}^{\varphi} \left( \lambda_k - \frac{1}{2(m-1)} S_k^{\varphi} \right. \\ & \left. - \eta \lambda f_k + \frac{1}{m-2} f_t R_{tk}^{\varphi} - \frac{S^{\varphi}}{(m-1)(m-2)} f_k \right), \end{split}$$

which, after some simplifications, becomes (5.16).

**Proposition** 5.6. Let (M, g) be a Riemannian manifold admitting a solution of

$$\operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g$$

for  $\eta \neq -\frac{1}{m-2}$ . Let  $z: \mathbb{R} \to \mathbb{R}$  be a smooth function such that z(f) is compactly supported on M. Then we have the validity of the following integral formula:

$$(5.22) \quad \frac{1}{2[(m-2)\eta+1]} \int_{M} \left| C^{\varphi} \right|^{2} [z(f) - z'(f)] e^{-f} = -\int_{M} C^{\varphi}_{tjk,kjt} z(f) e^{-f}$$

$$-\frac{\eta(m-2)}{2(m-2)\eta+2} \int_{M} f_{t}W_{ptjk}^{\varphi} C_{pjk}^{\varphi}[z(f)-z'(f)]e^{-f}$$

$$+\frac{1}{(m-2)\eta+1} \int_{M} \left(\lambda_{k} - \frac{1}{2(m-1)} S_{k}^{\varphi} - \eta \lambda f_{k}\right) C_{ttk}^{\varphi}[z(f)-z'(f)]e^{-f}$$

$$-\frac{1}{(m-2)\eta+1} \left(f_{t}R_{tk}^{\varphi} - \frac{S^{\varphi}}{(m-1)} f_{k}\right) C_{ttk}^{\varphi}[z(f)-z'(f)]e^{-f}.$$

PROOF. Integrating by parts, we have

$$\int_{M} f_{t} C_{tik,ki}^{\varphi} z(f) e^{-f} = \int_{M} C_{tik,kit}^{\varphi} z(f) e^{-f} + \int_{M} f_{t} C_{tik,ki}^{\varphi} z'(f) e^{-f},$$

which implies

(5.23) 
$$\int_{M} f_{t} C_{tik,ki}^{\varphi}[z(f) - z'(f)]e^{-f} = \int_{M} \operatorname{div}^{3} C^{\varphi} z(f)e^{-f}.$$

Multiplying (5.16) by  $[z(f) - z'(f)]e^{-f}$ , integrating on M and using (5.23) we immediately obtain (5.22).

As a computational short-hand, we define

(5.24) 
$$F_{ijk} = C^{\varphi}_{ijk} - \frac{1}{m-1} U^a \varphi^a_k \delta_{ij} + \frac{1}{m-1} U^a \varphi^a_j \delta_{ik}.$$

Then, if we assume the validity of (5.5), we have

$$\begin{split} \left| F \right|^2 &= \left| C^\varphi \right|^2 + \frac{2m}{(m-1)^2} \left| \nabla (U(\varphi)) \right|^2 - \frac{2}{(m-1)^2} \left| \nabla (U(\varphi)) \right|^2 - \frac{4}{m-1} C^\varphi_{ijk} (U^a \varphi^a_k \delta_{ij}) \\ &= \left| C^\varphi \right|^2 + \frac{2}{m-1} \left| \nabla (U(\varphi)) \right|^2 - \frac{4\alpha}{m-1} \varphi^a_{tt} \varphi^a_k U^b \varphi^b_k \\ &= \left| C^\varphi \right|^2 - \frac{2}{m-1} \left| \nabla (U(\varphi)) \right|^2 \end{split}$$

where we have used

$$C_{ttk}^{\varphi} = \alpha \varphi_{tt}^{a} \varphi_{k}^{a}$$

and (5.5). Thus we have

(5.25) 
$$|F|^2 = |C^{\varphi}|^2 - \frac{2}{m-1} |\nabla(U(\varphi))|^2.$$

Corollary 5.7. Let (M,g) satisfy system (5.1) with  $\eta \neq -\frac{1}{m-2}$ . Assume that

(5.26) 
$$\varphi$$
 is  $\frac{1}{\alpha}U$ -harmonic,

(5.27) 
$$\nabla f_p$$
 is an eigenvector of  $\operatorname{Ric}_p^{\varphi}$ , for every regular point  $p$  of  $f$ .

Moreover, when  $\eta \neq 0$ , also assume

$$W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0.$$

Let  $z : \mathbb{R} \to \mathbb{R}$  be a smooth function such that z(f) is compactly supported on M. Then

(5.28) 
$$\frac{1}{2[1+(m-2)\eta]} \int_{M} |F|^{2} [z(f)-z'(f)] e^{-f} = -\int_{M} \operatorname{div}^{3} C^{\varphi} z(f) e^{-f},$$

where the components of F have been defined in (5.24).

PROOF. From the second equation of (5.1) and since  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic we get

$$\varphi_t^a f_t = 0.$$

Since

$$C_{ttk}^{\varphi} = \alpha \varphi_{tt}^{a} \varphi_{k}^{a},$$

we obtain that, from assumptions (5.26) and (5.27), equation (5.22) reduces to

$$\begin{split} &\frac{1}{2[(m-2)\eta+1]} \int_{M} \left| C^{\varphi} \right|^{2} [z(f)-z'(f)] e^{-f} = -\int_{M} \operatorname{div}^{3} C^{\varphi} z(f) e^{-f} \\ &+ \int_{M} \alpha \varphi_{tt}^{a} \varphi_{k}^{a} \left[ \frac{1}{(m-2)\eta+1} \left( \lambda_{k} - \frac{1}{2(m-1)} S_{k}^{\varphi} \right) \right] [z(f)-z'(f)] e^{-f}. \end{split}$$

To deal with the last term, recall the validity of (4.31), that is,

$$(5.29) \qquad \frac{1}{2}S_k^{\varphi} = R_{ik}^{\varphi}f_i + \eta(f_{ik}f_i - \Delta f f_k) + (m-1)\lambda_k - \alpha\varphi_k^a(\varphi_{tt}^a - \varphi_t^a f_t).$$

This, together with assumptions (5.26) and (5.27), implies

$$\alpha \varphi_{tt}^{a} \varphi_{k}^{a} \left[ \frac{1}{(m-2)\eta + 1} \left( \lambda_{k} - \frac{1}{2(m-1)} S_{k}^{\varphi} \right) \right]$$

$$= \frac{\alpha^{2}}{(m-1)[(m-2)\eta + 1]} \varphi_{tt}^{a} \varphi_{k}^{a} \varphi_{pp}^{b} \varphi_{k}^{b}$$

$$= \frac{1}{(m-1)[(m-2)\eta + 1]} |\nabla U(\varphi)|^{2},$$

and therefore

$$\begin{split} &\frac{1}{2[(m-2)\eta+1]} \int_{M} \left| C^{\varphi} \right|^{2} [z(f)-z'(f)] e^{-f} = - \int_{M} \operatorname{div}^{3} C^{\varphi} z(f) e^{-f} \\ &+ \frac{1}{(m-1)[(m-2)\eta+1]} \int_{M} \left| \nabla U(\varphi) \right|^{2} [z(f)-z'(f)] e^{-f}. \end{split}$$

Rearranging and using equation (5.25) we conclude.

We are ready for the proof of Theorem 5.1. We will use equation (5.28) together with a trick taken from [16].

PROOF (OF THEOREM 5.1). From equation (5.28) and assumption (5.4) we get

(5.30) 
$$\frac{1}{2[1+(m-2)\eta]} \int_{M} |F|^{2} e^{-f} (z(f) - z'(f)) = 0.$$

If M is compact, set  $z(f) \equiv 1$  to obtain  $F \equiv 0$  and therefore (5.8). If M is non-compact, let

$$z_k(t) = \begin{cases} 1 & t \in [-k, k], \\ \frac{2k+t}{k} & t \in [-2k, -k], \\ \frac{2k-t}{k} & t \in [k, 2k], \\ 0 & t \in (-\infty, -2k] \cup [2k, +\infty) \end{cases}$$

for a positive natural number k. Since  $f(x) \to +\infty$  as  $x \to +\infty$  by assumptions, there exists  $A \in \mathbb{R}$  such that

$$f(x) > -2A$$
 on  $M$ .

Let  $k \geq 2A$ . Then, for  $f(x) \in [-2k, -k]$ , we have

$$z_k(f(x)) - z'_k(f(x)) = 2 + \frac{f(x) - 1}{k} \ge 2 - \frac{2A + 1}{k} \ge 1 - \frac{1}{k}$$

and, for  $f(x) \in [k, 2k]$ , we have

$$z_k(f(x)) - z'_k(f(x)) = 2 - \frac{f(x) - 1}{k} \ge \frac{1}{k}.$$

Therefore, from (5.30) we deduce

$$0 = \int_{\overline{\Omega}_{2k}} |F|^2 e^{-f} (z_k(f) - z'_k(f)) \ge \frac{1}{k} \int_{\overline{\Omega}_{2k}} |F|^2 e^{-f}$$

where

$$\overline{\Omega}_{2k} = \overline{\{x \in M : f(x) \in [-2k, 2k]\}}.$$

Letting  $k \to +\infty$  we get  $F \equiv 0$  and therefore equation (5.8). To conclude, if  $F \equiv 0$  and  $f_t W_{tijk}^{\varphi} = 0$ , the first integrability condition (5.2) of system (5.1) implies  $\overline{D}^{\varphi} \equiv 0$ .

PROOF (OF THEOREM 5.3). Assumption (5.11) reads, in components,

$$\varphi_{tt}^a \varphi_k^a = 0, \quad \forall i = 1, ..., m.$$

Since  $C_{ttk}^{\varphi} = \alpha \varphi_{tt}^{a} \varphi_{k}^{a}$  we deduce

$$C_{ttk}^{\varphi} = 0.$$

Therefore, when either  $\eta=0$  or the zero radial Weyl curvature condition (5.12) holds, using (5.11) and (5.22), we deduce that, for any smooth, compactly supported function  $z: \mathbb{R} \to \mathbb{R}$ , we have

$$\frac{1}{2[(m-2)\eta+1]} \int_{M} \left| C^{\varphi} \right|^{2} [z(f)-z'(f)] e^{-f} = - \int_{M} C^{\varphi}_{tjk,kjt} z(f) e^{-f}.$$

The proof now continues as that of Theorem 5.1, replacing F with  $C^{\varphi}$ , to give

$$C^{\varphi} \equiv 0.$$

From the first integrability condition (5.2), we deduce

$$[1 + \eta(m-2)]\overline{D}_{ijk}^{\varphi} = f_t W_{tijk}^{\varphi} - \frac{U^a \varphi_k^a}{m-1} \delta_{ij} + \frac{U^a \varphi_j^a}{m-1} \delta_{ik}.$$

If we are assuming (5.12), then we have

$$[1+\eta(m-2)]\overline{D}_{ijk}^{\varphi} = -\frac{U^a \varphi_k^a}{m-1} \delta_{ij} + \frac{U^a \varphi_j^a}{m-1} \delta_{ik}.$$

Since  $\overline{D}^{\varphi}$  is totally trace-free, tracing the above equation with respect to i and j gives

$$0 = [1 + \eta(m-2)]\overline{D}_{ttk}^{\varphi} = -\frac{m}{m-1}U^{a}\varphi_{k}^{a} + \frac{1}{m-1}U^{a}\varphi_{k}^{a} = -U^{a}\varphi_{k}^{a}.$$

From equation (5.31) and assumption  $\eta \neq -\frac{1}{m-2}$  we deduce  $\overline{D}^{\varphi} \equiv 0$ , and this concludes the proof.

# 5.2. The Boundary Case

In this subsection we prove a rigidity result for Riemannian manifolds with non-empty boundary.

Before focusing on manifolds with boundary, we highlight some facts concerning system (5.1), that is,

$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi), \end{cases}$$

where

$$m\lambda = S^{\varphi} + \Delta f - \eta |\nabla f|^2.$$

If we let

$$(5.32) u = e^{-\eta f},$$

then system (5.1) transforms into

(5.33) 
$$\begin{cases} i) u\eta \operatorname{Ric}^{\varphi} - \operatorname{Hess}(u) = \frac{1}{m} (\eta u S^{\varphi} - \Delta u) g \\ ii) \eta u\tau(\varphi) = -d\varphi(\nabla u) + \frac{\eta u}{\alpha} (\nabla U)(\varphi). \end{cases}$$

Note that looking for solutions of system (5.1) is equivalent to finding positive solutions of system (5.33); however, the latter system can be considered an extension of (5.1) if one looks for possibly changing-sign solutions u. In our setting, system (5.33) will play a fundamental role in the study of manifolds with boundary,  $\partial M = u^{-1}(\{0\})$ .

From now on, let (M,g) be a connected Riemannian manifold of dimension  $m \geq 3$  with non-empty boundary; let  $\varphi : (M,g) \to (N,h)$ , where (N,h) is a Riemannian manifold of dimension n, be a smooth map, let  $U : (N,h) \to \mathbb{R}$  be a smooth

function and let  $u \in C^{\infty}(M)$  be a solution of (5.33) such that u > 0 on int(M) and  $\partial M = u^{-1}(\{0\})$ . In this setting, we have the following

**Theorem** 5.8. Let (M,g) be a connected Riemannian manifold of dimension  $m \geq 3$  and let  $u \in C^{\infty}(M)$  be a solution of system (5.33), with  $\alpha \in \mathbb{R} \setminus \{0\}$  and  $\eta \neq -\frac{1}{m-2}$ , 0. Assume that

(5.35) 
$$\varphi \text{ is } \frac{1}{\alpha}U\text{-harmonic};$$

(5.36) 
$$\nabla f_p$$
 is an eigenvector of  $\operatorname{Ric}_p^{\varphi}$ , for every regular point  $p$  of  $f$ ;

(5.37) 
$$W^{\varphi}(\nabla f, \cdot, \cdot, \cdot) = 0.$$

Then

(5.38) 
$$C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_{1}\left(U(\varphi)g \bigotimes g\right)$$

and

$$\overline{D}^{\varphi} \equiv 0.$$

For the proof of Theorem 5.8 we need some preliminary results. First we introduce the (0,3)-tensor  $D^{\varphi}$ , whose components are

(5.39)

$$(m-2)D_{ijk}^{\varphi} := u_{ik}u_j - u_{ij}u_k + \frac{u_t}{m-1}(u_{tj}\delta_{ik} - u_{tk}\delta_{ij}) + \frac{\Delta u}{m-1}(u_k\delta_{ij} - u_j\delta_{ik}).$$

**Lemma** 5.9. Let (M,g) be a connected Riemannian manifold of dimension  $m \geq 3$ , let  $u \in C^{\infty}(M)$  be a solution of system (5.33), with  $\alpha \in \mathbb{R} \setminus \{0\}$  and  $\eta \neq -\frac{1}{m-2}, 0$  and let  $D^{\varphi}$  be the tensor field on M defined in (5.39). Then,

$$\frac{[1+\eta(m-2)]}{\eta u}D^{\varphi}=\eta uC^{\varphi}-W^{\varphi}(\nabla u,\cdot,\cdot,\cdot)+\eta u\frac{1}{2(m-1)}\operatorname{div}_{1}(U(\varphi)g\bigotimes g).$$
 on  $\operatorname{int}(M)$ .

PROOF. Note that, when we consider the change of variable (5.32), the first integrability condition (5.40) of system (5.33) can be easily obtained by (5.2). Indeed, by (4.6), we have

$$\overline{D}_{ijk}^{\varphi} = (u\eta)^{-2} D_{ijk}^{\varphi};$$

hence, by (5.2), we deduce (5.40).

**Lemma** 5.10. Let (M,g) be a connected Riemannian manifold of dimension  $m \geq 3$  and let  $u \in C^{\infty}(M)$  be a solution of system (5.33), with  $\alpha \in \mathbb{R} \setminus \{0\}$  and  $\eta \neq -\frac{1}{m-2}$ , 0. Define the vector field Z on M of components

$$Z_i = u \left\{ R_{tk}^{\varphi} W_{tikj}^{\varphi} \right\}_j - \left( \frac{m-4}{m-2} \right) u R_{tk}^{\varphi} C_{tki}^{\varphi} + 2\alpha u \varphi_i^a \varphi_{jk}^a R_{jk}^{\varphi} - \alpha \left\{ u R_{tk}^{\varphi} (\varphi_i^a \varphi_i^a)_k \right\}$$

$$-\alpha \left\{ u \left( \frac{1}{2} S_t^{\varphi} \varphi_t^a \varphi_i^a - \alpha \varphi_{ss}^b \varphi_t^b \varphi_t^a \varphi_i^a \right) \right\}$$

$$(5.41) \quad + \left[ u \left( \varphi_{ik}^a U^a - U^{ab} \varphi_k^b \varphi_i^a - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \right)_k \right],$$

and assume that  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic and  $\nabla u$  is an eigenvector of  $\mathrm{Ric}^{\varphi}$ , for every regular point of u (that are assumptions (5.35) and (5.36), respectively). Then, on  $\mathrm{int}(M)$ , we have

(5.42)

$$\operatorname{div} Z = -u_t W_{tijk}^{\varphi} \left( \frac{1}{2(\eta u)^2} D_{ijk}^{\varphi} + R_{ij,k}^{\varphi} \right) - \frac{[1 + \eta(m-2)]}{(\eta u)^3} |D^{\varphi}|^2$$

$$- u C_{ikj,jki}^{\varphi} + \left( \frac{m-4}{m-2} \right) u \left[ U^a \left( \frac{1}{\alpha} U^{ab} \varphi_i^b + R_{ij}^{\varphi} \varphi_j^a \right) \right]_i$$

$$+ \left\{ \varphi_i^a \left( \frac{m}{(m-1)(m-2)} U^a S^{\varphi} + 2\alpha U^b \varphi_j^b \varphi_i^a \right) - \frac{\alpha}{2} \left( \frac{m-2}{m-1} \right) \varphi_i^a \varphi_j^a S_j^{\varphi} \right\}$$

$$- 2\alpha \varphi_{jk}^a R_{jk}^{\varphi} \varphi_i^a - \alpha \varphi_i^a \tau_2^a(\varphi) \right\}_i u.$$

If we further assume

$$u_t W_{tijk}^{\varphi} = 0,$$

equation (5.42) becomes

$$\begin{aligned} \operatorname{div} Z &= -\frac{\left[1 + \eta(m-2)\right]}{(\eta u)^3} \left|D^{\varphi}\right|^2 - u C^{\varphi}_{ikj,jki} + \left(\frac{m-4}{m-2}\right) u \left[U^a \left(U^{ab} \varphi^b_i + R^{\varphi}_{ij} \varphi^a_j\right)\right]_i \\ &+ \left\{\varphi^a_i \left(\frac{m}{(m-1)(m-2)} U^a S^{\varphi} + 2\alpha U^b \varphi^b_j \varphi^a_i\right) - \frac{\alpha}{2} \frac{m-2}{m-1} \varphi^a_i \varphi^a_j S^{\varphi}_j \right. \\ &\left. - 2\alpha \varphi^a_{jk} R^{\varphi}_{jk} \varphi^a_i - \alpha \varphi^a_i \tau^a_2(\varphi)\right\}_i u. \end{aligned}$$

PROOF. Let X be the vector field of components

$$X_{i} = u \left\{ R_{tk}^{\varphi} W_{tikj}^{\varphi} \right\}_{k} - \left( \frac{m-4}{m-2} \right) u R_{tk}^{\varphi} C_{tki}^{\varphi} + 2\alpha u \varphi_{i}^{a} \varphi_{jk}^{a} R_{jk}^{\varphi};$$

computing its divergence and using  $0 = \tau(\varphi) - \frac{1}{\alpha}(\nabla U)(\varphi)$  and hence  $d\varphi(\nabla u) = 0$ , we obtain

(5.44)

$$\begin{split} X_{ii} = & \left\{ u \left[ R_{tk}^{\varphi} W_{tikj}^{\varphi} \right]_k \right\}_i - \left( \frac{m-4}{m-2} \right) u_i R_{tk}^{\varphi} C_{tki}^{\varphi} - \left( \frac{m-4}{m-2} \right) u R_{tk,i}^{\varphi} C_{tki}^{\varphi} - \left( \frac{m-4}{m-2} \right) u R_{tk}^{\varphi} C_{tki,i}^{\varphi} \\ & + 2 \alpha u_i \varphi_i^a \varphi_{jk}^a R_{jk}^{\varphi} + 2 \alpha u \varphi_{ii}^a \varphi_{jk}^a R_{jk}^{\varphi} + 2 \alpha u \varphi_i^a \varphi_{jki}^a R_{jk}^{\varphi} + 2 \alpha u \varphi_i^a \varphi_{jk}^a R_{jk,i}^{\varphi} \\ = & \left\{ u \left[ R_{tk}^{\varphi} W_{tikj}^{\varphi} \right]_k \right\}_i - \left( \frac{m-4}{m-2} \right) u_i R_{tk}^{\varphi} C_{tki}^{\varphi} - \left( \frac{m-4}{m-2} \right) u R_{tk,i}^{\varphi} C_{tki}^{\varphi} - \left( \frac{m-4}{m-2} \right) u R_{tk}^{\varphi} C_{tki,i}^{\varphi} \\ & + 2 \alpha u \varphi_{ii}^a \varphi_{jk}^a R_{jk}^{\varphi} + 2 \alpha u \varphi_i^a \varphi_{jki}^a R_{jk}^{\varphi} + 2 \alpha u \varphi_i^a \varphi_{jk}^a R_{jk,i}^{\varphi} \end{split}$$

$$= \Bigl\{u\Bigl[R^{\varphi}_{tk}W^{\varphi}_{tikj}\Bigr]_k\Bigr\}_i - \Bigl(\frac{m-4}{m-2}\Bigr)u_iR^{\varphi}_{tk}C^{\varphi}_{tki} - \Bigl(\frac{m-4}{m-2}\Bigr)u\{R^{\varphi}_{tk}C^{\varphi}_{tki}\}_i + 2\alpha u\Bigl\{\varphi^a_i\varphi^a_{jk}R^{\varphi}_{jk}\Bigr\}_i.$$

Since  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic, equation (2.57) rewrites as

$$(5.45) \qquad (m-2)B_{ik,k}^{\varphi} = \frac{m-4}{m-2} \left[ R_{jk}^{\varphi} C_{jki}^{\varphi} + U^{a} \left( \frac{1}{\alpha} U^{ab} \varphi_{i}^{b} + R_{ij}^{\varphi} \varphi_{j}^{a} \right) \right]$$

$$+ \varphi_{i}^{a} \left( \frac{m}{(m-1)(m-2)} U^{a} S^{\varphi} + 2\alpha U^{b} \varphi_{j}^{b} \varphi_{i}^{a} \right)$$

$$- \frac{\alpha}{2} \frac{m-2}{m-1} \varphi_{i}^{a} \varphi_{j}^{a} S_{j}^{\varphi} - 2\alpha \varphi_{jk}^{a} R_{jk}^{\varphi} \varphi_{i}^{a} - \alpha \varphi_{i}^{a} \tau_{2}^{a} (\varphi);$$

therefore, taking its divergence, we obtain

(5.46)

$$(m-2)B_{ik,ki}^{\varphi} = \left(\frac{m-4}{m-2}\right) \left(R_{jk}^{\varphi}C_{jki}^{\varphi}\right)_{i} + \left(\frac{m-4}{m-2}\right) \left(U^{a}\left(\frac{1}{\alpha}U^{ab}\varphi_{i}^{b} + R_{ij}^{\varphi}\varphi_{j}^{a}\right)\right)_{i} + \left[\varphi_{i}^{a}\left(\frac{m}{(m-1)(m-2)}U^{a}S^{\varphi} + 2\alpha U^{b}\varphi_{j}^{b}\varphi_{i}^{a}\right) - \frac{\alpha}{2}\frac{m-2}{m-1}\varphi_{i}^{a}\varphi_{j}^{a}S_{j}^{\varphi} - 2\alpha\varphi_{jk}^{a}R_{jk}^{\varphi}\varphi_{i}^{a} - \alpha\varphi_{i}^{a}\tau_{2}^{a}(\varphi)\right]_{i}.$$

To simplify the writing we set

$$\mathcal{A} = -\left[\varphi_i^a \left(\frac{m}{(m-1)(m-2)} U^a S^{\varphi} + 2\alpha U^b \varphi_j^b \varphi_i^a\right) - \frac{\alpha}{2} \frac{m-2}{m-1} \varphi_i^a \varphi_j^a S_j^{\varphi} - \alpha \varphi_i^a \tau_2^a(\varphi)\right].$$

As a consequence, using (5.46), we have

$$(5.47) u \frac{m-4}{m-2} (R^{\varphi}_{jk} C^{\varphi}_{jki})_i - 2\alpha u (\varphi^a_{jk} R^{\varphi}_{jk} \varphi^a_i)_i$$

$$= u(m-2) B^{\varphi}_{ik,ki} - \left(\frac{m-4}{m-2}\right) u \left(\frac{1}{\alpha} U^a (U^{ab} \varphi^b_i + R^{\varphi}_{ij} \varphi^a_j)\right)_i + \mathcal{A}u.$$

Using the first equation of (5.33) into (5.39), we get

(5.48)

$$\begin{split} (m-2)D_{ijk}^{\varphi} = & \eta u R_{ik}^{\varphi} u_j - \frac{1}{m} (\eta u S^{\varphi} - \Delta u) \delta_{ik} u_j - \eta u R_{ij}^{\varphi} u_k + \frac{1}{m} (\eta u S^{\varphi} - \Delta u) \delta_{ij} u_k \\ & + \frac{u_t}{m-1} \left[ R_{tj}^{\varphi} \delta_{ik} \eta u - \frac{1}{m} (\eta u S^{\varphi} - \Delta u) \delta_{ik} \delta_{tj} - R_{tk}^{\varphi} \delta_{ij} \eta u \right. \\ & + \frac{1}{m} (\eta u S^{\varphi} - \Delta u) \delta_{ij} \delta_{tk} \right] + \frac{1}{m-1} \Delta u (u_k \delta_{ij} - u_j \delta_{ik}) \\ = & \eta u \left\{ R_{ik}^{\varphi} u_j - R_{ij}^{\varphi} u_k + \frac{u_t}{m-1} \left( R_{tj}^{\varphi} \delta_{ik} - R_{tk}^{\varphi} \delta_{ij} \right) \right. \end{split}$$

$$+\frac{S^{\varphi}}{m-1}(u_k\delta_{ij}-u_j\delta_{ik})$$
.

Hence, by (5.35), (5.36) and (5.48), we deduce

$$(5.49) (m-2)D_{ijk}^{\varphi}C_{ijk}^{\varphi} = -2\eta u u_k R_{ij}^{\varphi}C_{ijk}^{\varphi}.$$

Thus, inserting (5.49) and (5.47) into (5.44), we get the validity of

$$X_{ii} = \left\{ u \left[ R_{tk}^{\varphi} W_{tikj}^{\varphi} \right]_{j} \right\}_{i} + \frac{m-4}{2\eta u} D_{ijk}^{\varphi} C_{ijk}^{\varphi} - u(m-2) B_{ik,ki}^{\varphi}$$

$$+ \left( \frac{m-4}{m-2} \right) u \left[ U^{a} \left( \frac{1}{\alpha} U^{ab} \varphi_{i}^{b} + R_{ij}^{\varphi} \varphi_{j}^{a} \right) \right]_{i} - \mathcal{A}u$$

$$(5.50)$$

on int(M).

Next, using the definition of  $B^{\varphi}$  and that  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic, we obtain

$$\begin{split} u(m-2)B_{ik,ki}^{\varphi} = & uC_{ikj,jki}^{\varphi} + u(R_{st}^{\varphi}W_{sitk}^{\varphi})_{ki} - \alpha u(R_{tk}^{\varphi}\varphi_{t}^{a}\varphi_{i}^{a})_{ki} \\ & + u\bigg(\varphi_{ij}^{a}U^{a} - U^{ab}\varphi_{k}^{b}\varphi_{i}^{a} - \frac{\alpha}{m-2}|\tau(\varphi)|^{2}\delta_{ik}\bigg)_{ki} \\ = & uC_{ikj,jki}^{\varphi} + \left[u(R_{st}^{\varphi}W_{sitk}^{\varphi})_{k}\right]_{i} - u_{i}(R_{st}^{\varphi}W_{sitk}^{\varphi})_{k} \\ & - \alpha[u(R_{tk}^{\varphi}\varphi_{t}^{a}\varphi_{i}^{a})_{k}]_{i} + \alpha u_{i}(R_{tk}^{\varphi}\varphi_{t}^{a}\varphi_{i}^{a})_{k} \\ & + \left[u\bigg(\varphi_{ik}^{a}U^{a} - U^{ab}\varphi_{k}^{b}\varphi_{i}^{a} - \frac{\alpha}{m-2}|\tau(\varphi)|^{2}\delta_{ik}\bigg)_{k}\right]_{i} \\ & - u_{i}\bigg(\varphi_{ik}^{a}U^{a} - U^{ab}\varphi_{k}^{b}\varphi_{i}^{a} - \frac{\alpha}{m-2}|\tau(\varphi)|^{2}\delta_{ik}\bigg)_{k}. \end{split}$$

Inserting the latter into (5.50), we obtain, on int(M),

$$(5.51) \quad X_{ii} = \frac{m-4}{2\eta u} D^{\varphi}_{ijk} C^{\varphi}_{ijk} - u C^{\varphi}_{ikj,jki} + u_i \{R^{\varphi}_{st} W^{\varphi}_{sitk}\}_k + \alpha [u (R^{\varphi}_{tk} \varphi^a_t \varphi^a_i)_k]_i$$

$$- \alpha u_i (R^{\varphi}_{tk} \varphi^a_t \varphi^a_i)_k - \left[ u \left( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \right)_k \right]_i$$

$$+ u_i \left( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \right)_k$$

$$+ \left( \frac{m-4}{m-2} \right) u \left[ U^a \left( \frac{1}{\alpha} U^{ab} \varphi^b_i + R^{\varphi}_{ij} \varphi^a_j \right) \right]_i - \mathcal{A} u.$$

Observe that by (2.51), since  $\varphi$  is  $\frac{1}{\alpha}U$ -harmonic and  $\nabla u$  is an eigenvector of  $\mathrm{Ric}^{\varphi}$  for every regular point of u, we get

$$\begin{split} u_i \big\{ R_{st}^\varphi W_{sitk}^\varphi \big\}_k &= u_i R_{st,k}^\varphi W_{sitk}^\varphi + u_i R_{st}^\varphi W_{sitk,k}^\varphi \\ &= u_i R_{st,k}^\varphi W_{sitk}^\varphi + \left(\frac{m-3}{m-2}\right) u_i R_{st}^\varphi C_{tsi}^\varphi + \alpha u_i R_{ts}^\varphi \varphi_{ti}^a \varphi_s^a \end{split}$$

which by (5.49), on int(M), can be rewritten as

$$u_i \{R_{st}^{\varphi} W_{sitk}^{\varphi}\}_k = u_i R_{st,k}^{\varphi} W_{sitk}^{\varphi} + \left(\frac{m-3}{m-2}\right) u_i R_{st}^{\varphi} C_{tsi}^{\varphi} + \alpha u_i R_{ts}^{\varphi} (\varphi_i^a \varphi_s^a)_t$$

$$= u_i R_{st,k}^{\varphi} W_{sitk}^{\varphi} - \frac{m-3}{2\eta u} D_{tsi}^{\varphi} C_{tsi}^{\varphi} + \alpha u_i R_{ts}^{\varphi} (\varphi_i^a \varphi_s^a)_t.$$

Therefore, on int(M), we have

$$\begin{split} X_{ii} &= -\frac{1}{2\eta u} D^{\varphi}_{ijk} C^{\varphi}_{ijk} - u C^{\varphi}_{ikj,jki} + u_i R^{\varphi}_{st,k} W^{\varphi}_{sitk} + \alpha u_i R^{\varphi}_{ts} (\varphi^a_i \varphi^a_s)_t \\ &+ \alpha [u (R^{\varphi}_{tk} \varphi^a_t \varphi^a_i)_k]_i - \alpha u_i (R^{\varphi}_{tk} \varphi^a_t \varphi^a_i)_k \\ &- \left[ u \left( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \right)_k \right]_i \\ &+ u_i \left( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \right)_k \\ &+ \left( \frac{m-4}{m-2} \right) u \left[ U^a \left( \frac{1}{\alpha} U^{ab} \varphi^b_i + R^{\varphi}_{ij} \varphi^a_j \right) \right]_i - \mathcal{A}u; \end{split}$$

using the  $\varphi$ -Schur's identity, we have

$$\begin{split} X_{ii} &= -\frac{1}{2\eta u} D^{\varphi}_{ijk} C^{\varphi}_{ijk} - u C^{\varphi}_{ikj,jki} + u_i R^{\varphi}_{st,k} W^{\varphi}_{sitk} + \alpha u_i R^{\varphi}_{ts} (\varphi^a_i \varphi^a_s)_t \\ &+ \alpha \bigg\{ u \bigg( \frac{1}{2} S^{\varphi}_t \varphi^a_i \varphi^a_i - \alpha \varphi^b_{ss} \varphi^b_t \varphi^a_i \varphi^a_i \bigg) \bigg\}_i + \alpha \{ u R^{\varphi}_{tk} (\varphi^a_t \varphi^a_i)_k \}_i \\ &- \alpha u_i \bigg( \frac{1}{2} S^{\varphi}_t \varphi^a_i \varphi^a_i - \alpha \varphi^b_{ss} \varphi^b_t \varphi^a_i \varphi^a_i + R^{\varphi}_{tk} (\varphi^a_t \varphi^a_i)_k \bigg) \\ &- \bigg[ u \bigg( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \bigg)_k \bigg]_i \\ &+ u_i \bigg( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \bigg)_k \\ &+ \bigg( \frac{m-4}{m-2} \bigg) u \bigg[ U^a \bigg( \frac{1}{\alpha} U^{ab} \varphi^b_i + R^{\varphi}_{ij} \varphi^a_j \bigg) \bigg] - \mathcal{A}u; \end{split}$$

since  $\varphi_i^a u_i = 0$  we deduce

$$\begin{split} X_{ii} &= -\frac{1}{2\eta u} D^{\varphi}_{ijk} C^{\varphi}_{ijk} - u C^{\varphi}_{ikj,jki} + u_i R^{\varphi}_{st,k} W^{\varphi}_{sitk} + \alpha \bigg\{ u \bigg( \frac{1}{2} S^{\varphi}_t \varphi^a_i \varphi^a_i - \alpha \varphi^b_{ss} \varphi^b_t \varphi^a_i \varphi^a_i \bigg) \bigg\}_i \\ &+ \alpha \{ u R^{\varphi}_{tk} (\varphi^a_t \varphi^a_i)_k \}_i - \bigg[ u \bigg( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \bigg)_k \bigg]_i \\ &+ u_i \bigg( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \bigg)_k \\ &+ \frac{m-4}{m-2} u \bigg[ U^a \bigg( \frac{1}{\alpha} U^{ab} \varphi^b_i + R^{\varphi}_{ij} \varphi^a_j \bigg) \bigg]_i - \mathcal{A} u \end{split}$$

on int(M) and

$$u_i \left( \varphi_{ik}^a U^a - U^{ab} \varphi_k^b \varphi_i^a - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \right)_k$$

$$\begin{split} &=u_i\Bigg(\varphi^a_{ikk}U^a+\varphi^a_{ik}U^{ab}\varphi^b_k-U^{abc}\varphi^c_k\varphi^b_k\varphi^a_i-U^{ab}\varphi^b_{kk}\varphi^a_i\\ &-U^{ab}\varphi^b_k\varphi^a_{ik}-\alpha\frac{2}{m-2}\varphi^a_{tti}\varphi^a_{ss}\Bigg)\\ &=u_i\Bigg(\varphi^a_{ikk}U^a-\alpha\frac{2}{m-2}\varphi^a_{tti}\varphi^a_{ss}\Bigg). \end{split}$$

Using the commutation rule (see Section 1.7 of [1])

$$\varphi_{ijk}^a = \varphi_{ikj}^a + R_{tijk}\varphi_t^a - {}^{N}R_{bcd}^a\varphi_i^b\varphi_i^c\varphi_k^d,$$

we get

$$u_{i}\left(\varphi_{ik}^{a}U^{a}-U^{ab}\varphi_{k}^{b}\varphi_{i}^{a}-\frac{\alpha}{m-2}|\tau(\varphi)|^{2}\delta_{ik}\right)_{k}$$

$$=u_{i}\left[\left(\frac{m-4}{m-2}\right)\varphi_{kki}^{a}U^{a}+U^{a}R_{si}\varphi_{s}^{a}-{}^{N}R_{bcd}^{a}\varphi_{k}^{b}\varphi_{i}^{c}\varphi_{k}^{d}U^{a}\right]$$

$$=u_{i}\left(\frac{m-4}{\alpha(m-2)}U^{ab}\varphi_{i}^{b}U^{a}+U^{a}R_{si}^{\varphi}\varphi_{s}^{a}-{}^{N}R_{bcd}^{a}\varphi_{k}^{b}\varphi_{i}^{c}\varphi_{k}^{d}U^{a}\right)$$

$$=u_{i}U^{a}R_{si}^{\varphi}\varphi_{s}^{a}=0,$$

where the last equality follows by (5.36). As a consequence, (5.52) becomes

$$\begin{split} X_{ii} &= -\frac{1}{2\eta u} D^{\varphi}_{ijk} C^{\varphi}_{ijk} - u C^{\varphi}_{ikj,jki} + u_i R^{\varphi}_{st,k} W^{\varphi}_{sitk} \\ &+ \alpha \bigg\{ u \bigg( \frac{1}{2} S^{\varphi}_t \varphi^a_i \varphi^a_i - \alpha \varphi^b_{ss} \varphi^b_t \varphi^a_i \varphi^a_i \bigg) \bigg\}_i + \alpha \{ u R^{\varphi}_{tk} (\varphi^a_t \varphi^a_i)_k \}_i \\ &- \bigg[ u \bigg( \varphi^a_{ik} U^a - U^{ab} \varphi^b_k \varphi^a_i - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \bigg)_k \bigg]_i \\ &+ \frac{m-4}{m-2} u \big[ U^a \big( U^{ab} \varphi^b_i + R^{\varphi}_{ij} \varphi^a_j \big) \big]_i - \mathcal{A} u. \end{split}$$

On int(M), since u > 0, by (5.40), we obtain

(5.53)

$$\begin{split} C^{\varphi}_{ijk}D^{\varphi}_{ijk} &= \frac{1}{\eta u}D^{\varphi}_{ijk}\bigg\{u_tW^{\varphi}_{tijk} - \eta u\frac{U^a}{m-1}\big(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ik}\big) + \frac{[1+\eta(m-2)]}{\eta u}D^{\varphi}_{ijk}\bigg\} \\ &= \frac{1}{\eta u}u_tW^{\varphi}_{tijk}D^{\varphi}_{ijk} + \frac{[1+\eta(m-2)]}{(\eta u)^2}|D^{\varphi}|^2, \end{split}$$

where the last equality is a consequence of the fact that  $D^{\varphi}$  is totally trace-free (since  $D^{\varphi} = (u\eta)^2 \overline{D}^{\varphi}$ ). It follows that, on int(M), we have the validity of

$$X_{ii} = -\frac{1}{2(\eta u)^2} u_t W_{tijk}^{\varphi} D_{ijk}^{\varphi} - \frac{\left[1 + \eta(m-2)\right]}{(\eta u)^3} |D^{\varphi}|^2 - u C_{ikj,jki}^{\varphi} + u_i R_{st,k}^{\varphi} W_{sitk}^{\varphi}$$
$$+ \alpha \left\{ u \left(\frac{1}{2} S_t^{\varphi} \varphi_t^a \varphi_i^a - \alpha \varphi_{ss}^b \varphi_t^b \varphi_t^a \varphi_i^a\right) \right\}_i + \alpha \left\{ u R_{tk}^{\varphi} (\varphi_t^a \varphi_i^a)_k \right\}_i$$

$$-\left[u\left(\varphi_{ik}^{a}U^{a}-U^{ab}\varphi_{k}^{b}\varphi_{i}^{a}-\frac{\alpha}{m-2}|\tau(\varphi)|^{2}\delta_{ik}\right)_{k}\right]_{i}$$
$$+\frac{m-4}{m-2}u\left[U^{a}\left(\frac{1}{\alpha}U^{ab}\varphi_{i}^{b}+R_{ij}^{\varphi}\varphi_{j}^{a}\right)\right]_{i}-\mathcal{A}u.$$

Therefore, we have

$$\begin{split} X_{ii} - \alpha \bigg\{ u \bigg( \frac{1}{2} S_t^{\varphi} \varphi_t^a \varphi_i^a - \alpha \varphi_{ss}^b \varphi_t^b \varphi_i^a \varphi_i^a \bigg) \bigg\}_i - \alpha \big\{ u R_{tk}^{\varphi} (\varphi_t^a \varphi_i^a)_k \big\}_i \\ + \bigg[ u \bigg( \varphi_{ik}^a U^a - U^{ab} \varphi_k^b \varphi_i^a - \frac{\alpha}{m-2} |\tau(\varphi)|^2 \delta_{ik} \bigg)_k \bigg]_i &= -\frac{1}{2(\eta u)^2} u_t W_{tijk}^{\varphi} D_{ijk}^{\varphi} \\ - \frac{[1 + \eta(m-2)]}{(\eta u)^3} |D^{\varphi}|^2 - u C_{ikj,jki}^{\varphi} + u_i R_{st,k}^{\varphi} W_{sitk}^{\varphi} \\ + \frac{m-4}{m-2} u \bigg[ U^a \bigg( \frac{1}{\alpha} U^{ab} \varphi_i^b + R_{ij}^{\varphi} \varphi_j^a \bigg) \bigg]_i - \mathcal{A}u, \end{split}$$

and substituting the expression of  $\mathcal{A}$  into the latter, we get (5.42).

PROOF (OF THEOREM 5.8). Let Z be the vector field defined in (5.41); from Lemma 5.10 and assumptions (5.34) and (5.37) we get

$$\begin{split} \operatorname{div} Z &= -\frac{\left[1 + \eta(m-2)\right]}{(\eta u)^3} |D^{\varphi}|^2 + \left(\frac{m-4}{m-2}\right) \left[U^a \left(\frac{1}{\alpha} U^{ab} \varphi_i^b - R_{ij}^{\varphi} \varphi_j^a\right)\right]_i u \\ &+ \left\{ \varphi_i^a \left(\frac{m}{(m-1)(m-2)} U^a S^{\varphi} + 2\alpha U^b \varphi_j^b \varphi_i^a\right) - \frac{\alpha}{2} \left(\frac{m-2}{m-1}\right) \varphi_i^a \varphi_j^a S_j^{\varphi} \right. \\ &\left. - 2\alpha \varphi_{jk}^a R_{jk}^{\varphi} \varphi_i^a - \alpha \varphi_i^a \tau_2^a(\varphi) \right\}_i u. \end{split}$$

Let

$$M_{\varepsilon} := \{ x \in M : u(x) \ge \varepsilon \}, \quad \partial M_{\varepsilon} := \{ x \in M : u(x) = \varepsilon \}.$$

Then, using the divergence theorem, we deduce

$$\int_{\partial M_{\varepsilon}} -g(Z, \nu) = \int_{M_{\varepsilon}} \operatorname{div} Z,$$

where

$$\nu = \frac{\nabla u}{|\nabla u|}$$

is the inward unit normal. For the boundary part, we have, from the definition (5.41) of the vector field Z, that

$$\begin{split} \int_{\partial_{M_{\varepsilon}}} g(Z,\nu) = & \varepsilon \int_{\partial_{M_{\varepsilon}}} \left\{ \left( R_{tk}^{\varphi} W_{tikj}^{\varphi} \right)_{j} - \frac{m-4}{m-2} R_{tk}^{\varphi} C_{tki}^{\varphi} + 2\alpha \varphi_{i}^{a} \varphi_{jk}^{a} R_{jk}^{\varphi} \right\} \nu_{i} \\ & - \varepsilon \int_{\partial_{M_{\varepsilon}}} \left\{ R_{tk}^{\varphi} (\varphi_{i}^{a} \varphi_{t}^{a})_{k} + \alpha \left( \frac{1}{2} S_{t}^{\varphi} \varphi_{t}^{a} \varphi_{i}^{a} - \alpha \varphi_{tt}^{b} \varphi_{s}^{b} \varphi_{s}^{a} \varphi_{i}^{a} \right) \right\} \nu_{i} \\ & + \varepsilon \int_{\partial_{M_{\varepsilon}}} \left\{ \left( \varphi_{ik}^{a} U^{a} - U^{ab} \varphi_{k}^{b} \varphi_{i}^{a} - \frac{\alpha}{m-2} |\tau(\varphi)|^{2} \delta_{ik} \right)_{k} \right\} \nu_{i} \end{split}$$

and this term vanishes as  $\varepsilon \to 0^+$ . For the left hand side, we integrate by parts equation (5.54) to obtain

$$\begin{split} \int_{M_{\varepsilon}} \operatorname{div} Z &= -\left[1 + \eta(m-2)\right] \int_{M_{\varepsilon}} \frac{1}{(\eta u)^3} |D^{\varphi}|^2 \\ &- \frac{m-4}{m-2} \int_{M_{\varepsilon}} u_i \left[ U^a \left( \frac{1}{\alpha} U^{ab} \varphi_i^b + R_{ij}^{\varphi} \varphi_j^a \right) \right] \\ &- \frac{m-4}{m-2} \varepsilon \int_{\partial M_{\varepsilon}} \left[ U^a \left( \frac{1}{\alpha} U^{ab} \varphi_i^b + R_{ij}^{\varphi} \varphi_j^a \right) \right] \nu_i \\ &- \int_{M_{\varepsilon}} u_i \left( \frac{m}{(m-1)(m-2)} S^{\varphi} U^a + 2\alpha U^b \varphi_j^b \varphi_j^a \right) \varphi_i^a \\ &- \varepsilon \int_{\partial M_{\varepsilon}} \left( \frac{m}{(m-1)(m-2)} S^{\varphi} U^a + 2\alpha U^b \varphi_j^b \varphi_j^a \right) \varphi_i^a \nu_i \\ &+ \int_{M_{\varepsilon}} u_i \left( \frac{\alpha}{2} \frac{m-2}{m-1} S_j^{\varphi} \varphi_j^a \varphi_i^a + 2\alpha \varphi_{jk}^a R_{jk}^{\varphi} \varphi_i^a \right) \\ &+ \varepsilon \int_{\partial M_{\varepsilon}} \left( \frac{\alpha}{2} \frac{m-2}{m-1} S_j^{\varphi} \varphi_j^a \varphi_i^a + 2\alpha \varphi_{jk}^a R_{jk}^{\varphi} \varphi_i^a \right) \nu_i \\ &+ \int_{M_{\varepsilon}} \alpha \varphi_i^a \tau_2^a (\varphi) u_i + \varepsilon \int_{\partial M_{\varepsilon}} \alpha \varphi_i^a \tau_2^a (\varphi) \nu_i. \end{split}$$

Using (5.36) and (5.35) we deduce

$$\int_{M_{\varepsilon}} \operatorname{div} Z = -[1 + \eta(m-2)] \int_{M_{\varepsilon}} \frac{1}{(\eta u)^3} |D^{\varphi}|^2$$

and therefore, letting  $\varepsilon$  tend to zero, we conclude

$$D^{\varphi} \equiv 0.$$

## 5.3. Some Conditions on the Cotton Tensor

For the ease of readability, we recall here system (5.1):

(5.55) 
$$\begin{cases} i) \operatorname{Ric}^{\varphi} + \operatorname{Hess}(f) - \eta df \otimes df = \lambda g, \\ ii) \tau(\varphi) = d\varphi(\nabla f) + \frac{1}{\alpha}(\nabla U)(\varphi). \end{cases}$$

As we have seen in Theorems 5.1 and 5.8 and Proposition 4.13, a condition that is often met for a system of the type (5.55) is

(5.56) 
$$C_{ijk}^{\varphi} = \frac{U^a \varphi_k^a}{m-1} \delta_{ij} - \frac{U^a \varphi_j^a}{m-1} \delta_{ik}.$$

It is immediate to see that condition (5.56) is satisfied if and only if the modification of the  $\varphi$ -Schouten tensor given by

$$(5.57) A^{\varphi} - \frac{U(\varphi)}{m-1}g$$

is a Codazzi tensor. We will prove the following

**Theorem** 5.11. Let (M,g) be compact with empty boundary and satisfy system (5.55), for some non-constant function f and  $\alpha > 0$ . Assume that (5.56) holds and that the (normalized) k-th elementary symmetric polynomial  $\sigma_k$  in the eigenvalues of  $A^{\varphi} - \frac{U(\varphi)}{m-1}g$  is constant. If  $k \geq 2$ , assume, moreover, that  $\sigma_k$  is positive and that there exists a point of M at which all the eigenvalues of  $A^{\varphi} - \frac{U(\varphi)}{m-1}g$  are positive. Then  $\varphi$  is constant and (M,g) is isometric to a Euclidean sphere.

Before proceeding, we recall some results on the elementary symmetric polynomials, then we will prove an Obata-type result which will be instrumental in the proof of Theorem 5.11.

**5.3.1. On the elementary symmetric polynomials.** Here we collect some well-known facts about elementary symmetric polynomials and we fix the notation. Given m real numbers  $\lambda_1, \lambda_2, ..., \lambda_m$ , we set

$$S_k := \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdot \dots \cdot \lambda_{i_k}$$

for the k-th elementary symmetric polynomial and

$$\sigma_k := \binom{m}{k}^{-1} S_k$$

for its normalization. Then we have the validity of Newton's inequality,

$$\sigma_{k-1}\sigma_{k+1} \le \sigma_k^2$$

where, provided  $\sigma_{k-1} \neq 0$ , the equality is obtained if and only if all the  $\lambda_i$ 's coincide (see [40] for a proof). Moreover, we have Gårding's inequalities ([34])

$$\sigma_1 \ge \sigma_2^{\frac{1}{2}} \ge \dots \ge \sigma_k^{\frac{1}{k}},$$

which hold when all of the  $\sigma_j$ 's, j = 1, ..., k, are positive. Again, equality is obtained if and only if all the  $\lambda_i$ 's are equal. Combining them, one can prove the validity of the next lemma, which is Lemma 5.33 of [3].

**Lemma** 5.12. Let A be a 2-covariant symmetric tensor on (M,g) with  $m = \dim M \geq 3$ . Let  $\sigma_k$  be the k-th normalized symmetric function in the eigenvalues of A; if  $k \geq 2$ , assume that it is positive on M and assume that there exists a point  $p \in M$  at which all the eigenvalues of A are positive. Then

$$(5.58) \sigma_1 \sigma_k - \sigma_{k+1} \ge 0,$$

with equality holding at a point  $x \in M$  if and only if the eigenvalues of A at x coincide.

PROOF OF LEMMA 5.12. For k=1, (5.58) coincides with the Newton's inequality so that the conclusion immediately follows.

For  $k \geq 2$ , note that, by assumptions, the set

$$\{(\lambda_1(x), \lambda_2(x), ..., \lambda_m(x)) \in \mathbb{R}^m : x \in M\}$$

is contained in the connected component of  $\{x \in \mathbb{R}^m : \sigma_k(x) > 0\}$  that contains the positive orthant

$$\{(x_1, x_2, ..., x_m) \in \mathbb{R}^m : x_i > 0, \forall i = 1, ..., m\}.$$

As it is well-known, this connected component coincides with the set

$$\{x \in \mathbb{R}^m : \sigma_i(x) > 0, \forall i = 1, ..., k\}$$

so that Gårding's inequalities can be applied. Moreover, since  $\sigma_{k-1} > 0$ , we obtain from Newton's inequalities that, for  $k \geq 2$ ,

$$\sigma_{k+1} = \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_{k-1}} \le \frac{\sigma_k^2}{\sigma_{k-1}} = \sigma_k \frac{\sigma_k}{\sigma_{k-1}}.$$

We claim

$$\frac{\sigma_k}{\sigma_{k-1}} \le \sigma_1$$

which, since  $\sigma_k > 0$ , would imply (5.58). To prove the claim we use Gårding's inequalities and the positivity of  $\sigma_1, \sigma_k$  and  $\sigma_{k-1}$  to deduce

$$\sigma_k = \sigma_k^{1/k} \sigma_k^{(k-1)/k} \le \sigma_1 \sigma_k^{(k-1)/k} \le \sigma_1 \sigma_{k-1}.$$

This proves the claim. Moreover, equality at a point  $x \in M$  of (5.58) would imply equality in the Newton's inequalities and in Gårding's inequalities and this happens if and only if all of the eigenvalues of A at x coincide.

In the following, we will fix a symmetric 2-covariant tensor A on M. The  $\sigma_i$ 's will be intended as functions of the eigenvalues of A. The Newton endomorphisms

$$P_k = P_k(A) : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

associated to A, where  $\mathfrak{X}(M)$  is the  $C^{\infty}(M)$ -algebra of smooth vector fields on M, are inductively defined by

(5.59) 
$$P_0 = id, P_k = S_k id - A \circ P_{k-1}, 1 \le k \le m.$$

Here A is identified with the corresponding endomorphism  $A: \mathfrak{X}(M) \to \mathfrak{X}(M)$ . Note that, by Cayley-Hamilton's Theorem, we have  $P_m \equiv 0$ . It is well known that the  $P_k$ 's satisfy the following formulas (see [4]):

$$(5.60) tr P_k = (m-k)S_k,$$

(5.61) 
$$\operatorname{tr}(A \circ P_k) = (k+1)S_{k+1}.$$

Setting

$$(5.62) c_k = (m-k) \binom{m}{k},$$

a simple computation shows that equations (5.60) and (5.61) become

$$(5.63) tr P_k = c_k \sigma_k,$$

$$(5.64) tr(A \circ P_k) = c_k \sigma_{k+1}.$$

As a last well-known fact, we recall that if A is a Codazzi tensor, then all of the  $P_i$ 's are divergence-free; this is a consequence of the following

**Lemma** 5.13. Let A be a 2-covariant symmetric tensor and let  $P_k$  denote the k-th symmetric Newton operator given by (5.59). Then the following formula holds:

(5.65) 
$$\operatorname{div}(P_k)_i = -A_{ij}\operatorname{div}(P_{k-1})_j - C(A)_{iit}(P_{k-1})_{it},$$

where

$$C(A)_{jit} = A_{ji,t} - A_{jt,i}$$

is the obstruction to A being a Codazzi tensor.

PROOF. From (5.59) we deduce

$$(P_k)_{ij,j} = (S_k)_i - A_{ip,j}(P_{k-1})_{pj} - A_{ip}(P_{k-1})_{pj,j}.$$

To obtain (5.65) from it, we only need to prove the validity of

$$(5.66) (S_k)_i = A_{pj,i}(P_{k-1})_{pj}.$$

To deduce it, we fix  $p \in M$  such that each eigenvalue of A is smooth at p. As it is well-known (see Paragraph 16.10 of [6]), the set of points at which all the eigenvalues of A are smooth is dense in M, so that it is enough to prove the validity of (5.66) at p.

We choose a local reference frame in a neighbourhood of p that diagonalizes A: from the definition of  $S_k$ , we immediately obtain

$$(S_k)_i = (\lambda_1)_i S_{k-1}|_{\lambda_1 = 0} + (\lambda_2)_i S_{k-1}|_{\lambda_2 = 0} + \dots + (\lambda_m)_i S_{k-1}|_{\lambda_m = 0}$$

where  $S_{k-1}|_{\lambda_j=0}$  denotes the sum of those summands of  $S_{k-1}$  in which  $\lambda_j$  does not appear. With this notations in mind, a simple mathematical induction that uses only the definition (5.59) of  $P_k$  gives that, in the chosen frame,  $P_{k-1}$  takes the form

$$P_{k-1} = \operatorname{diag}(S_{k-1}|_{\lambda_1=0}, S_{k-1}|_{\lambda_2=0}, ..., S_{k-1}|_{\lambda_m=0}).$$

Therefore we have

$$A_{pj,i}(P_{k-1})_{pj} = (\lambda_1)_i S_{k-1}|_{\lambda_1=0} + (\lambda_2)_i S_{k-1}|_{\lambda_2=0} + \dots + (\lambda_m)_i S_{k-1}|_{\lambda_m=0}$$
 and (5.66) follows.  $\Box$ 

Corollary 5.14. Let A be a 2-covariant symmetric Codazzi tensor. Then all of the Newton operators in the eigenvalues of A are divergence-free. Moreover, we also have

$$[(P_{k-1}) \circ A]_{ij,i} = \frac{m-k}{m} (S_k)_j$$
$$= \frac{c_k}{m} (\sigma_k)_j$$

for k = 1, ..., m - 1, where

$$[(P_{k-1}) \circ A] = (P_{k-1}) \circ A - \frac{1}{m} tr[(P_{k-1}) \circ A]g.$$

PROOF. Since A is Codazzi, we have

$$div(P_1)_i = (S_1)_i - A_{ij,j}$$
  
=  $A_{jj,i} - A_{ij,j} = 0$ ,

and therefore the first part of the statement follows by induction on k, using (5.65). To prove (5.67), we use (5.61) to deduce

$$[(P_{k-1}) \circ A] = (P_{k-1}) \circ A - \frac{k}{m} S_k g.$$

Taking the divergence of the above expression we get

$$[(P_{k-1}) \circ A]_{ij,i} = (P_{k-1})_{it,i} A_{tj} + (P_{k-1})_{it} A_{tj,i} - \frac{k}{m} (S_k)_j.$$

Using div  $(P_k) = 0$ , the fact that A is Codazzi and (5.66) we deduce

$$[(P_{k-1})_{it}^{\circ} A_{tj}]_i = \frac{m-k}{k} (S_k)_j,$$

that is, (5.67).

**Remark** 5.15. Symmetrizing the last term of (5.65) we get

$$C(A)_{jit}(P_{k-1})_{jt} = \frac{1}{2}(P_{k-1})_{jt}[C(A)_{jit} + C(A)_{tij}].$$

Reasoning as in the first part of the proof of Corollary 5.14, we deduce that  $P_k$  is divergence-free assuming only

$$C(A)_{jit} + C(A)_{tij} = 0.$$

The Newton endomorphisms give rise to a family of second order differential operators,  $L_k = L_k(A)$ , defined as follows: let  $u \in C^2(M)$  and set  $\mathrm{Hess}(u)$  to denote both the 2-covariant tensor and the corresponding endomorphism; we set

$$(5.68) L_k u = \operatorname{tr}(P_k \circ \operatorname{Hess}(u)),$$

that we also rewrite in the useful form

$$L_k u = \sum_{i=0}^k (-1)^i S_{k-i} \operatorname{tr} (A^i \circ \operatorname{Hess}(u)),$$

where  $A^0 = \text{id}$  and, for  $i \ge 1$ ,  $A^i$  is the *i*-th iterated composition of A with itself. A computation shows that  $L_k u$  can also be expressed in the form

$$L_k u = \operatorname{div} (P_k(\nabla u)) - g(\operatorname{div} P_k, \nabla u).$$

From the above formula we see that  $L_k$  is semielliptic whenever  $P_k$  is positive semidefinite and it is in divergence form when div  $P_k = 0$ . From Corollary 5.14 we see that the second condition is satisfied when A is a Codazzi tensor.

Suppose now that, for some

$$p(x), q(x), l(x) \in C^{\infty}(M),$$

we can express  $\operatorname{Hess}(u)$  in the form

(5.69) 
$$\operatorname{Hess}(u) = p(x)g + q(x)du \otimes du - l(x)A.$$

Then from equations (5.64), (5.63) and (5.68) we obtain, after some algebraic manipulations,

(5.70) 
$$L_k u = c_k[p(x)\sigma_k - l(x)\sigma_{k+1}] + q(x)g(P_k(\nabla u), \nabla u)$$

for  $1 \le k \le m - 1$ .

**5.3.2.** An Obata-type result. Recall that a vector field X on (M,g) is said to be *conformal* if

(5.71) 
$$\mathcal{L}_X g = \frac{2 \operatorname{div} X}{m} g,$$

where  $\mathcal{L}_X g$  is the Lie derivative of the metric in the direction of X, while X is *Killing* when it is conformal and div  $X \equiv 0$ . Moreover, X is *closed* when, in an orthonormal coframe,

$$X_{ij} = X_{ji}$$
.

We now prove the next result, which is inspired by a classical theorem of Obata ([56]).

**Proposition** 5.16. Let (M,g) be a closed, connected Riemannian manifold supporting the structure

(5.72) 
$$\begin{cases} \operatorname{Ric}^{\varphi} = \frac{S^{\varphi}}{m}g, \\ \tau(\varphi) = \frac{1}{\alpha}\nabla U(\varphi). \end{cases}$$

Assume that there exists a closed, conformal, non-Killing vector field X on (M,g) and assume

$$(5.73) d\varphi(X) = 0.$$

Then  $\varphi$  is constant and (M,g) is isometric to a Euclidean sphere.

When U is constant, the above result has been proved, without assuming that X is closed, in Lemma 5.2 of [3]. We first prove the following

**Lemma** 5.17. Let (M,g) be a Riemannian manifold of dimension  $m \geq 3$  and X a conformal vector field on (M,g). Let  $\varphi:(M,g)\to (N,h)$  be a smooth map and  $\alpha\in\mathbb{R}\setminus\{0\}$ . Set

$$\gamma = \operatorname{div} X$$
.

Then

$$\operatorname{Hess}(\gamma) = \frac{S^{\varphi}}{(m-1)(m-2)} \gamma g + \frac{m}{2(m-1)(m-2)} g(\nabla S^{\varphi}, X) g - \frac{m}{m-2} \mathcal{L}_X \operatorname{Ric}^{\varphi}$$

$$(5.74) \qquad -\frac{m}{m-2} \alpha \left[ \mathcal{L}_X(\varphi^* h) - \frac{1}{2(m-1)} \operatorname{tr}(\mathcal{L}_X(\varphi^* h)) g \right].$$

In particular,

$$(5.75) \qquad \Delta \gamma = -\frac{S^{\varphi}}{m-1} \gamma - \frac{m}{2(m-1)} g(\nabla S^{\varphi}, X) - \frac{m}{2(m-1)} \alpha \operatorname{tr}(\mathcal{L}_X(\varphi^*h)).$$

PROOF. We first prove

(5.76)

$$\operatorname{Hess}(\gamma) = \frac{S}{(m-1)(m-2)} \gamma g + \frac{m}{2(m-1)(m-2)} g(\nabla S, X) g - \frac{m}{m-2} \mathcal{L}_X \operatorname{Ric}.$$

We apply Ricci commutations rules twice to deduce

$$\gamma_{ij} = X_{ttij} = X_{titj} + X_{pj}R_{ptti} + X_{p}R_{ptti,j} 
= X_{tijt} + X_{pi}R_{pttj} + X_{tp}R_{pitj} - X_{pj}R_{pi} - X_{p}R_{pi,j} 
= X_{tijt} - X_{pi}R_{pj} + X_{tp}R_{pitj} - X_{pj}R_{pi} - X_{p}R_{pi,j}.$$

Since X is conformal, we have

$$(5.77) X_{ij} + X_{ji} = \frac{2}{m} \gamma \delta_{ij}.$$

From (5.77) and the Ricci commutation rules we deduce

$$\begin{split} \gamma_{ij} &= -X_{itjt} + \frac{2}{m} X_{ttji} - X_{pi} R_{pj} + X_{tp} R_{pitj} - X_{pj} R_{pi} - X_{p} R_{pi,j} \\ &= -X_{ijtt} - X_{pt} R_{pitj} - X_{p} R_{pitj,t} + \frac{2}{m} X_{ttji} - X_{pi} R_{pj} + X_{tp} R_{pitj} \\ &- X_{pj} R_{pi} - X_{p} R_{pi,j}. \end{split}$$

Using the first and second Bianchi identities we obtain

$$\gamma_{ij} = -X_{ijtt} + \frac{2}{m} X_{ttji} - X_{pt} R_{ptij} - X_{p} R_{ij,p} + X_{p} R_{pj,i} - X_{p} R_{pi,j} - X_{pi} R_{pj} - X_{pj} R_{pi}.$$

Recalling that

$$(\mathcal{L}_X \operatorname{Ric})_{ij} = X_t R_{ij,t} + X_{ti} R_{tj} + X_{tj} R_{it}$$

we obtain, rearranging some terms,

$$\gamma_{ij} = -X_{ijtt} + \frac{2}{m}X_{ttji} - X_{pt}R_{ptij} + X_t(R_{tj,i} - R_{ti,j}) - (\mathcal{L}_X \operatorname{Ric})_{ij}.$$

Symmetrizing the above expression and using (5.77) we deduce

$$\gamma_{ij} = -\frac{1}{2}(X_{ij} + X_{ji})_{tt} + \frac{2}{m}\gamma_{ij} - (\mathcal{L}_X \operatorname{Ric})_{ij}$$
$$= -\frac{1}{m}\gamma_{tt}\delta_{ij} + \frac{2}{m}\gamma_{ij} - (\mathcal{L}_X \operatorname{Ric})_{ij},$$

which, after some simplifications, becomes, in global notation,

(5.78) 
$$\operatorname{Hess}(\gamma) = -\frac{\Delta \gamma}{m-2} g - \frac{m}{m-2} \mathcal{L}_X \operatorname{Ric}.$$

Tracing (5.78) and simplifying we get

$$\Delta \gamma = -\frac{S}{m-1} \gamma - \frac{m}{2(m-1)} g(\nabla S, X).$$

Substituting it into (5.78) we obtain (5.76). From the definitions of  $\operatorname{Ric}^{\varphi}$  and  $S^{\varphi}$  we immediately get

(5.79) 
$$\mathcal{L}_X \operatorname{Ric} = \mathcal{L}_X \operatorname{Ric}^{\varphi} + \alpha \mathcal{L}_X (\varphi^* h)$$

and

(5.80) 
$$S\gamma g + \frac{m}{2}g(\nabla S, X) = S^{\varphi}\gamma g + \frac{m}{2}g(\nabla S^{\varphi}, X)g + \alpha \frac{m}{2} tr(\mathcal{L}_X(\varphi^*h))g.$$

Using (5.79) and (5.80) into (5.76) we obtain (5.74), while tracing (5.74) we obtain (5.75).

**Remark** 5.18. Assume now that  $X \in \text{Ker}(d\varphi)$ , that is,

$$\varphi_i^a X_i = 0.$$

Taking the covariant derivative of the above equation we get

$$\varphi_{ij}^a X_i + \varphi_i^a X_{ij} = 0,$$

which implies  $\mathcal{L}_X(\varphi^*h) = 0$ . Therefore, (5.74) and (5.75) reduce, respectively, to (5.81)

$$\operatorname{Hess}(\gamma) = \frac{S^{\varphi}}{(m-1)(m-2)} \gamma g + \frac{m}{2(m-1)(m-2)} g(\nabla S^{\varphi}, X) g - \frac{m}{m-2} \mathcal{L}_X \operatorname{Ric}^{\varphi}$$

and

(5.82) 
$$\Delta \gamma = -\frac{S^{\varphi}}{m-1} - \frac{m}{2(m-1)}g(X, \nabla S^{\varphi}).$$

Suppose now

$$\begin{cases} \operatorname{Ric}^{\varphi} = \frac{S^{\varphi}}{m}g \\ X \in \operatorname{Ker}(d\varphi). \end{cases}$$

Then, taking the divergence of the first of the above equations and using the  $\varphi$ -Schur's identity, we get

$$\frac{1}{2}S_k^{\varphi} - \alpha \varphi_{tt}^a \varphi_k^a = R_{tk,t}^{\varphi} = \frac{S_k^{\varphi}}{m}$$

and

$$\frac{m-2}{2m}S_k^{\varphi} = \varphi_{tt}^a \varphi_k^a.$$

Since  $X \in \text{Ker}(d\varphi)$ , it follows that

$$\frac{m-2}{2m}S_k^{\varphi}X_k = \varphi_{tt}^a\varphi_k^aX_k = 0,$$

that is, since  $m \geq 3$ ,

$$(5.83) S_k^{\varphi} X_k = 0.$$

Furthermore, using  $\operatorname{Ric}^{\varphi} = \frac{S^{\varphi}}{m}g$  and (5.83),

$$(\mathcal{L}_X \operatorname{Ric}^{\varphi})_{ij} = X_t R_{ij,k}^{\varphi} + X_{ti} R_{tj}^{\varphi} + X_{tj} R_{ti}^{\varphi}$$

$$= \frac{1}{m} X_t S_t^{\varphi} \delta_{ij} + \frac{S^{\varphi}}{m} X_{ti} \delta_{tj} + \frac{S^{\varphi}}{m} X_{tj} \delta_{ti}$$

$$= \frac{1}{m} (S_t^{\varphi} X_t \delta_{ij} + S^{\varphi} X_{ji} + S^{\varphi} X_{ij})$$

$$= \frac{S^{\varphi}}{m} (\mathcal{L}_X g)_{ij} = \frac{2}{m^2} S^{\varphi} \gamma \delta_{ij},$$

that is,

(5.84) 
$$(\mathcal{L}_X \operatorname{Ric}^{\varphi})_{ij} = \frac{2}{m^2} S^{\varphi} \gamma \delta_{ij}.$$

Hence, (5.81) and (5.82) become, respectively,

(5.85) 
$$\operatorname{Hess}(\gamma) = -\frac{1}{m(m-1)} S^{\varphi} \gamma g$$

and

$$\Delta \gamma = -\frac{1}{m-1} S^{\varphi} \gamma.$$

**Remark** 5.19. From equation (5.86) we deduce (see Appendix A of [60] or [42]) that  $\gamma$  satisfies the *unique continuation property*. In particular, it vanishes on some open subset of M if and only if it vanishes on the connected components of M containing it.

**Remark** 5.20. Since we do not know weather  $S^{\varphi}$  is constant or not, it seems hard to prove Proposition 5.16 from equation (5.85) using the original ideas of Obata, or those used in [3]. Instead, to prove Proposition 5.16, we will prove the constancy of U under the further assumption that X is closed and then apply Lemma 5.2 of [3] to conclude. In this direction, the next Lemma will prove to be useful.

**Lemma** 5.21. Let (M,g) be a Riemannian manifold, let  $\varphi:(M,g) \to (N,h)$  be a smooth map with target another Riemannian manifold (N,h) and let X be a smooth, closed, conformal vector field on M. Assume that, for some  $\eta \in \mathbb{R}$ , (M,g) supports the structure

(5.87) 
$$\begin{cases} \operatorname{Ric}^{\varphi} + \frac{1}{2}\mathcal{L}_{X}g - \eta X^{\flat} \otimes X^{\flat} = \lambda g, \\ \tau(\varphi) = \frac{1}{\alpha}\nabla U(\varphi), \\ d\varphi(X) = 0, \end{cases}$$

where  $\lambda = \frac{1}{m} \Big( S^{\varphi} + \operatorname{div} X - \eta |X|^2 \Big)$ . Then  $\tau(\varphi)$  vanishes identically on the set  $\{ x \in M : (\operatorname{div} X)(x) \neq 0 \}$ .

PROOF. The third equation of (5.87) reads, in components,

$$\varphi_i^a X_i = 0.$$

Setting

$$\gamma = \operatorname{div} X$$

since X is conformal and closed we have

$$(5.89) X_{ij} = \frac{\gamma}{m} \delta_{ij}.$$

Take the covariant derivative of (5.88) and use (5.89) to deduce

$$0 = \varphi_{ij}^a X_i + \varphi_i^a X_{ij} = \varphi_{ij}^a X_i + \frac{\gamma}{m} \varphi_j^a.$$

Compute the divergence of the expression above and use (5.89) to get

$$0 = \varphi_{ijj}^a X_i + \varphi_{ij}^a X_{ij} + \frac{\gamma}{m} \varphi_{jj}^a + \frac{1}{m} \varphi_j^a \gamma_j$$
$$= \varphi_{ijj}^a X_i + 2 \frac{\gamma}{m} \varphi_{jj}^a + \frac{1}{m} \varphi_j^a \gamma_j,$$

that is,

$$(5.90) 0 = \varphi_{ijj}^a X_i + 2 \frac{\gamma}{m} \varphi_{jj}^a + \frac{1}{m} \varphi_j^a \gamma_j.$$

Since X is conformal and  $\lambda = \frac{1}{m} (S^{\varphi} + \operatorname{div} X - \eta |X|^2)$ , the first equation of (5.87) becomes

(5.91) 
$$\operatorname{Ric}^{\varphi} - \eta X^{\flat} \otimes X^{\flat} = \frac{1}{m} \left( S^{\varphi} - \eta |X|^{2} \right) g.$$

Taking the divergence of (5.89) we have

$$X_{jij} = \frac{\gamma_i}{m}.$$

Applying the Ricci commutation relations and using  $d\varphi(X) = 0$  we deduce

$$\frac{\gamma_i}{m} = X_t R_{ti} + X_{jji} = X_t R_{ti}^{\varphi} + \gamma_i.$$

Rearranging and using (5.91) we get

(5.92) 
$$\frac{m-1}{m}\gamma_i = -R_{ti}^{\varphi}X_t = -\frac{S^{\varphi}}{m}X_i - \eta \frac{m-1}{m}|X|^2X_i.$$

From equations (5.92) and (5.88) we deduce

$$\varphi_i^a \gamma_j = 0.$$

We want to prove

$$\varphi_{ijj}^a X_i = 0.$$

We exploit the commutation relation (see Section 1.7 of [1])

$$\varphi_{ijk}^a = \varphi_{ikj}^a + \varphi_t^a R_{tijk} - {}^{N}R_{bcd}^a \varphi_i^b \varphi_j^c \varphi_k^d,$$

which implies

$$\varphi_{ijj}^a = \varphi_{jij}^a = \varphi_{jji}^a + \varphi_t^a R_{ti} - {}^{N}R_{bcd}^a \varphi_j^b \varphi_i^c \varphi_j^d.$$

Contracting the above with  $X_i$  we get

(5.95) 
$$\varphi_{iji}^a X_i = \varphi_{iji}^a X_i + \varphi_t^a R_{ti} X_i - {}^N R_{bcd}^a \varphi_i^b \varphi_i^c \varphi_i^d X_i.$$

From (5.88) we deduce

$${}^{N}R^{a}_{bcd}\varphi^{b}_{j}\varphi^{c}_{i}\varphi^{d}_{j}X_{i} = 0,$$

and using (5.88) and (5.91) we obtain

$$\varphi_t^a R_{ti} X_i = \varphi_t^a R_{ti}^{\varphi} X_i = \frac{S^{\varphi}}{m} \varphi_t^a X_t + \eta \frac{m-1}{m} |X|^2 \varphi_t^a X_t = 0.$$

On the other hand, differentiating the second equation of (5.87) we have

$$\varphi_{tti}^a = \frac{1}{\alpha} U^{ab} \varphi_i^b,$$

and contracting with  $X_i$  and using (5.88) we deduce

$$\varphi_{tti}^a X_i = \frac{1}{\alpha} U^{ab} \varphi_i^b X_i = 0.$$

Inserting in (5.95) we obtain

$$\varphi_{ijj}^a X_i = 0,$$

that is, equation (5.94). Using (5.94) and (5.93) into (5.90) we deduce

$$(5.96) 0 = 2 \frac{\gamma}{m} \varphi_{tt}^a.$$

Therefore, on the set where  $\gamma = \operatorname{div} X$  does not vanish, we have that  $\tau(\varphi)$  vanishes.

PROOF (OF PROPOSITION 5.16). Since X is conformal and closed, we are in the assumptions of Lemma 5.21, that is, we are in the case  $\eta=0$  of that Lemma. Applying it, we deduce that  $\tau(\varphi)$  vanishes on the set

$$A := \{ x \in M : \gamma(x) \neq 0 \}.$$

From Remark 5.19, we deduce that  $\gamma$  satisfies the unique continuation property; therefore, it vanishes on an open subset of M if and only if it vanishes identically. Since X is non-Killing by assumption, this cannot happen so that the zero set of  $\gamma$  does not contain any open subset of M. As a consequence, A is dense in M. Since  $\tau(\varphi)$  vanishes there, we deduce that, by continuity,  $\tau(\varphi) = 0$  on all of M. Therefore (M,g) is a harmonic-Einstein manifold and we are in the conditions to apply Lemma 5.2 of [3] to conclude.

### **5.3.3.** Proof of the Main Theorem. From our assumptions, we know that

$$A^{\varphi} - \frac{U(\varphi)}{m-1}g$$

is a Codazzi tensor; in the following,  $P_k$  will be the k-th Newton operator "in its eigenvalues", which is divergence free by Corollary 5.14. Therefore

$$[(P_k)_{ij}f_i]_j = (P_k)_{ij}f_{ij},$$

that is

(5.97) 
$$\operatorname{div}(P_k(\nabla f)) = \operatorname{tr}(P_k \circ \operatorname{Hess}(f)).$$

We first prove the validity of the following identity:

(5.98) 
$$\operatorname{tr}\left\{\left(P_{k} - \frac{c_{k}}{m}\sigma_{k}\operatorname{id}\right) \circ \operatorname{Hess}(u)\right\} - g\left(\nabla \log u^{1-\frac{\eta}{\beta}}, \left(P_{k} - \frac{c_{k}}{m}\sigma_{k}\operatorname{id}\right)(\nabla u)\right) \\ = c_{k}\beta u(\sigma_{k+1} - \sigma_{1}\sigma_{k}),$$

where  $u = e^{-\beta f}$ , for some  $\beta \in \mathbb{R}$ ,  $\beta \neq 0$ . Note that, to prove it, we will only use the constancy of  $\sigma_k$ , equation (5.56) and the first equation of (5.55). Rewriting (5.55) i) we get

$$(5.99) f_{ij} = \eta f_i f_j - A_{ij}^{\varphi} + \frac{U(\varphi)}{m-1} \delta_{ij} - \frac{1}{2(m-1)} S^{\varphi} \delta_{ij} - \frac{U(\varphi)}{m-1} \delta_{ij} + \lambda \delta_{ij}.$$

We perform the substitution  $u = e^{-\beta f}$  to obtain

(5.100) 
$$\operatorname{Hess}(u) = \beta u \left( \frac{S^{\varphi}}{2(m-1)} - \lambda(x) + \frac{U(\varphi)}{m-1} \right) g + \left( 1 - \frac{\eta}{\beta} \right) \frac{1}{u} du \otimes du + \beta u \left( A^{\varphi} - \frac{U(\varphi)}{m-1} g \right),$$

which is a structure of the form (5.69) for the choices

$$p(x) = \beta u \left( \frac{S^{\varphi}}{2(m-1)} - \lambda + \frac{U(\varphi)}{m-1} \right) (x), \quad q(x) = \left( 1 - \frac{\eta}{\beta} \right) \frac{1}{u} (x),$$
$$l(x) = -\beta u(x)$$

and  $A = A^{\varphi} - \frac{U(\varphi)}{m-1}g$ . From equation (5.70) we deduce

$$\begin{split} L_k u = & c_k \beta u \bigg[ \bigg( \frac{S^{\varphi}}{2(m-1)} - \lambda(x) + \frac{U(\varphi)}{m-1} \bigg) \sigma_k + \sigma_{k+1} \bigg] \\ & + \bigg( 1 - \frac{\eta}{\beta} \bigg) \frac{1}{u} g(P_k(\nabla u), \nabla u). \end{split}$$

A simple computation gives the validity of

$$\frac{U(\varphi)}{m-1} = \frac{m-2}{2m(m-1)}S^{\varphi} - \sigma_1$$

and, therefore.

$$L_k u = c_k \beta u \left(-\sigma_1 \sigma_k + \sigma_{k+1}\right) + g\left(P_k(\nabla u), \nabla \log u^{1-\frac{\eta}{\beta}}\right) + c_k \beta u \sigma_k \frac{S^{\varphi}}{m} - c_k \beta u \sigma_k \lambda.$$

Tracing equation (5.100) we obtain

$$\Delta u = \frac{m}{2(m-1)} \beta u S^{\varphi} + \frac{m}{m-1} \beta u U(\varphi) - m \beta u \lambda(x) + \left(1 - \frac{\eta}{\beta}\right) \frac{1}{u} |\nabla u|^{2}$$
$$+ \frac{m-2}{2(m-1)} \beta u S^{\varphi} - \frac{m}{m-1} \beta u U(\varphi)$$
$$= S^{\varphi} \beta u - m \beta u \lambda(x) + \left(1 - \frac{\eta}{\beta}\right) \frac{1}{u} |\nabla u|^{2},$$

so that

$$L_k u - \frac{c_k}{m} \sigma_k \Delta u - g\left(P_k(\nabla u), \nabla \log u^{1-\frac{\eta}{\beta}}\right) + \frac{c_k}{m} \sigma_k g\left(\nabla u, \nabla \log u^{1-\frac{\eta}{\beta}}\right)$$
$$= c_k \beta u(\sigma_{k+1} - \sigma_1 \sigma_k);$$

since  $\sigma_k$  is constant, the above equation can be rewritten as

$$\operatorname{tr}\left\{\left(P_{k} - \frac{c_{k}}{m}\sigma_{k}\operatorname{id}\right) \circ \operatorname{Hess}(u)\right\} - g\left(\nabla \log u^{1-\frac{\eta}{\beta}}, \left(P_{k} - \frac{c_{k}}{m}\sigma_{k}\operatorname{id}\right)(\nabla u)\right)$$
$$= c_{k}\beta u(\sigma_{k+1} - \sigma_{1}\sigma_{k}),$$

which is (5.98). The importance of (5.98) stems from the fact that, when  $P_k$  is divergence-free, its left-hand side becomes a weighted divergence; indeed we have

$$\operatorname{tr}\left\{\left(P_{k} - \frac{c_{k}}{m}\sigma_{k}\operatorname{id}\right) \circ \operatorname{Hess}(u)\right\} - g\left(\nabla \log u^{1 - \frac{\eta}{\beta}}, \left(P_{k} - \frac{c_{k}}{m}\sigma_{k}\operatorname{id}\right)(\nabla u)\right)$$
$$= u^{1 - \frac{\eta}{\beta}}\operatorname{div}\left(u^{\frac{\eta}{\beta} - 1}\left(P_{k} - \frac{c_{k}}{m}\sigma_{k}\operatorname{id}\right)(\nabla u)\right).$$

Therefore, multiplying (5.98) by  $u^{\frac{\eta}{\beta}-1}$ , integrating the result over M, using the divergence theorem and  $\beta \neq 0$  we obtain

$$\int_{M} u^{\frac{\eta}{\beta}} (\sigma_{k+1} - \sigma_1 \sigma_k) = 0.$$

From u > 0 and (5.58) we get

$$\sigma_{k+1} - \sigma_1 \sigma_k = 0$$

so that the equality case in (5.58) is satisfied. Therefore, all the eigenvalues of  $A^{\varphi} - \frac{U(\varphi)}{m-1}g$  coincide so that  $A^{\varphi}$  is proportional to the metric, and hence  $\mathrm{Ric}^{\varphi}$  is too. We then have that

$$\operatorname{Ric}^{\varphi} = \gamma g$$

holds, for some  $\gamma \in C^{\infty}(M)$ . Tracing the equation above, we deduce

$$m\gamma = S^{\varphi}$$

and

$$\mathrm{Ric}^{\varphi} = \frac{S^{\varphi}}{m}g$$

so that the first equation of a harmonic-Einstein manifold is satisfied. From equation

$$C_{iik}^{\varphi} = \alpha \varphi_k^a \varphi_{ii}^a$$

and (5.56) we deduce

$$\alpha \varphi_k^a \varphi_{ii}^a = \frac{1}{\alpha} U^a \varphi_k^a.$$

Contracting the above equation with  $f_k$  and using the second equation of (5.55) we deduce

$$0 = \alpha \varphi_k^a f_k \left( \varphi_{tt}^a - \frac{1}{\alpha} U^a \right) = \alpha \left| \tau(\varphi) - \frac{1}{\alpha} (\nabla U)(\varphi) \right|^2.$$

In particular, we have that  $\varphi$  is  $\frac{U}{\alpha}$ -harmonic and, again from the second equation of (5.55),  $\nabla f \in \text{Ker}(d\varphi)$ . In other words, we have proved the validity of

(5.101) 
$$\begin{cases} \operatorname{Ric}^{\varphi} = \frac{S^{\varphi}}{m} g \\ \tau(\varphi) = \frac{1}{\alpha} \nabla U(\varphi) \end{cases}$$

and

$$d\varphi(\nabla f) = 0 = d\varphi(\nabla u).$$

Moreover, when  $\eta=0$ , since  $A^{\varphi}$  is proportional to the metric, (5.99) tells us that also  $\operatorname{Hess}(f)$  is, so that  $\nabla f$  is a closed conformal vector field. When  $\eta\neq 0$ , choosing  $\beta=\eta$  in (5.100) shows us that  $\nabla u$  is a closed conformal vector field. In both cases, we are in the assumptions of Proposition 5.16, and we can use it to conclude.

#### CHAPTER 6

# The Boucher-Gibbons-Horowitz conjecture and related results

To prove Theorem 1.14 we make a tricky use of the maximum principle, together with a *Shen-type* identity for a suitable vector field. Throughout the proof we are going to use the equations of  $\varphi$ -SPFST many times; thus, for the sake of simplicity, we recall system (2.35) here:

(6.1) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \Big\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) g \Big\} = 0, \\ ii) \Delta u = \frac{u}{m-1} \Big[ mp - mU(\varphi) + \frac{m-2}{2} S^{\varphi} \Big], \\ iii) u\tau(\varphi) = -d\varphi(\nabla u) + \frac{u}{\alpha} (\nabla U)(\varphi), \\ iv) \mu + U(\varphi) = \frac{1}{2} S^{\varphi}, \\ v) (\mu + p) \nabla u = -u \nabla p. \end{cases}$$

We now focus on the following result that, besides being interesting in its own, is instrumental for the proof of the theorem.

**Proposition** 6.1. Let (M,g) be a  $\varphi$ -SPFST of dimension  $m \geq 2$  with  $\alpha > 0$ . Consider the vector field

$$(6.2)$$

$$Z := \frac{1}{u} \nabla \left\{ |\nabla u|^2 - \frac{u^2}{m(m-1)} [((m-2)\mu + mp) - 2U(\varphi)] \right\} = \frac{1}{u} \nabla \left( |\nabla u|^2 - \frac{u}{m} \Delta u \right),$$
on int(M). Then

(6.3)

$$\operatorname{div} Z = \frac{2}{u} \left\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right\} + \frac{(2m-3)}{m(m-1)} g(\nabla((m-2)\mu + mp), \nabla u)$$

$$+ \frac{2u}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)) + \frac{2u}{\alpha m^2(m-1)} |\nabla U|^2(\varphi)$$

$$- \frac{u}{m(m-1)} \Delta((m-2)\mu + mp) + u \left| \sqrt{\frac{2}{\alpha}} \frac{1}{m} (\nabla U)(\varphi) - \sqrt{2\alpha} d\varphi \left( \frac{\nabla u}{u} \right) \right|^2$$

$$+ \frac{2}{u} (\mu + p) |\nabla u|^2.$$

**Remark** 6.2. Note that Z is a modification of an *Obata-type vector field*, as defined in [51].

PROOF. Inserting equation (6.1) iv) into (6.1) ii) we have

(6.4) 
$$\Delta u = \frac{u}{m-1} \{ [mp + (m-2)\mu] - 2U(\varphi) \};$$

next, we observe that the components of Z are given by

$$Z_{i} = \frac{1}{u} \left[ 2u_{ki}u_{k} - \frac{1}{m(m-1)} ((m-2)\mu + mp)_{i}u^{2} - \frac{2u}{m(m-1)} ((m-2)\mu + mp)u_{i} + \frac{2}{m(m-1)} U^{a}\varphi_{i}^{a}u^{2} + \frac{4U(\varphi)}{m(m-1)} uu_{i} \right];$$

computing the divergence of Z we get

$$Z_{ii} = \frac{1}{u} \left( 2|\text{Hess}(u)|^2 + 2u_{iki}u_k - \frac{2u}{m(m-1)}u_i((m-2)\mu + mp)_i - \frac{u^2}{m(m-1)}((m-2)\mu + mp)_{ii} - \frac{2}{m(m-1)}((m-2)\mu + mp)|\nabla u|^2 - \frac{2u}{m(m-1)}((m-2)\mu + mp)_iu_i - \frac{2u}{m(m-1)}((m-2)\mu + mp)\Delta u + \frac{2}{m(m-1)}U^{ab}\varphi_i^a\varphi_i^bu^2 + \frac{4u}{m(m-1)}U^a\varphi_i^au_i + \frac{2}{m(m-1)}U^a\varphi_{ii}^au^2 + \frac{4}{m(m-1)}U^a\varphi_i^au_iu + \frac{4U(\varphi)}{m(m-1)}|\nabla u|^2 + \frac{4U(\varphi)}{m(m-1)}uu_{ii} \right) - \frac{u_i}{u^2} \left[ 2u_{ki}u_k - \frac{1}{m(m-1)}((m-2)\mu + mp)_iu^2 - \frac{2u}{m(m-1)}((m-2)\mu + mp)u_i + \frac{2}{m(m-1)}U^a\varphi_i^au^2 + \frac{4U(\varphi)}{m(m-1)}uu_i \right].$$

Next, using the commutation rules for the covariant derivative of a smooth function we obtain

$$\operatorname{div} Z = \frac{1}{u} \left[ 2|\operatorname{Hess}(u)|^2 + 2g(\nabla \Delta u, \nabla u) + 2\operatorname{Ric}(\nabla u, \nabla u) \right.$$

$$- \frac{4u}{m(m-1)} g(\nabla u, \nabla((m-2)\mu + mp)) - \frac{u^2}{m(m-1)} \Delta((m-2)\mu + mp)$$

$$- \frac{2}{m(m-1)} ((m-2)\mu + mp) |\nabla u|^2 - \frac{2u}{m(m-1)} ((m-2)\mu + mp) \Delta u$$

$$+ \frac{2}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)) u^2 + \frac{8u}{m(m-1)} g(\nabla(U(\varphi)), \nabla u)$$

$$+ \frac{2}{m(m-1)} h(\nabla U, \tau(\varphi)) u^2 + \frac{4U(\varphi)}{m(m-1)} |\nabla u|^2 + \frac{4U(\varphi)u}{m(m-1)} \Delta u \right]$$

$$- \frac{2}{u^2} \operatorname{Hess}(u)(\nabla u, \nabla u) + \frac{1}{m(m-1)} g(\nabla u, \nabla((m-2)\mu + mp))$$

$$+ \frac{2}{u} \frac{((m-2)\mu + mp)}{m(m-1)} |\nabla u|^2 - \frac{2}{m(m-1)} g(\nabla(U(\varphi), \nabla u)) - \frac{4}{u} \frac{U(\varphi)}{m(m-1)} |\nabla u|^2.$$

Using equation (6.4) to express  $\Delta u$  we infer

$$\operatorname{div} Z = \frac{2}{u} |\operatorname{Hess}(u)|^2 + \frac{2}{u(m-1)} g(\nabla[((m-2)\mu + mp - 2U(\varphi))u], \nabla u) + \frac{2}{u} \operatorname{Ric}(\nabla u, \nabla u)$$

$$- \frac{3}{m(m-1)} g(\nabla u, \nabla((m-2)\mu + mp)) - \frac{u}{m(m-1)} \Delta((m-2)\mu + mp)$$

$$- \frac{2((m-2)\mu + mp)}{um(m-1)} |\nabla u|^2 - \frac{2}{mu} (\Delta u)^2 + \frac{2}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi))u$$

$$+ \frac{6}{m(m-1)} g(\nabla u, \nabla(U(\varphi))) + \frac{2u}{m(m-1)} h(\nabla U, \tau(\varphi))$$

$$+ \frac{4U(\varphi)}{um(m-1)} |\nabla u|^2 - \frac{2}{u^2} \operatorname{Hess}(u)(\nabla u, \nabla u)$$

$$+ \frac{2((m-2)\mu + mp)}{um(m-1)} |\nabla u|^2 - \frac{4U(\varphi)}{um(m-1)} |\nabla u|^2.$$

Writing

$$Ric = Ric^{\varphi} + \varphi^* h$$

and using the first equation of (6.1) we rewrite Hess(u) as:

(6.5) 
$$\operatorname{Hess}(u) = u \left\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) g \right\}.$$

Inserting (6.5) into the expression of div Z we obtain

$$\begin{split} \operatorname{div} Z &= \frac{2}{u} |\operatorname{Hess}(u)|^2 + \frac{2}{u(m-1)} g(\nabla[m((m-2)\mu + mp - 2U(\varphi))u], \nabla u) \\ &+ \frac{2}{u} \operatorname{Ric}(\nabla u, \nabla u) - \frac{3}{m(m-1)} g(\nabla u, \nabla((m-2)\mu + mp)) \\ &- \frac{u}{m(m-1)} \Delta((m-2)\mu + mp) - \frac{2}{mu} (\Delta u)^2 + \frac{2}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi))u \\ &+ \frac{6}{m(m-1)} g(\nabla u, \nabla(U(\varphi))) + \frac{2u}{m(m-1)} h(\nabla U, \tau(\varphi)) \\ &- \frac{2}{u} \bigg\{ \operatorname{Ric}(\nabla u, \nabla u) - \alpha |d\varphi(\nabla u)|^2 - \frac{1}{m-1} \bigg( \frac{S^{\varphi}}{2} - p + U(\varphi) \bigg) |\nabla u|^2 \bigg\}. \end{split}$$

Then, using (6.1) iii) we deduce

$$\operatorname{div} Z = \frac{2}{u} |\operatorname{Hess}(u)|^2 + \frac{2}{m-1} g(\nabla((m-2)\mu + mp), \nabla u)$$

$$- \frac{4}{m-1} g(\nabla(U(\varphi)), \nabla u) + \frac{2}{u(m-1)} ((m-2)\mu + mp - 2U(\varphi)) |\nabla u|^2$$

$$- \frac{3}{m(m-1)} g(\nabla u, \nabla((m-2)\mu + mp)) + \frac{6}{m(m-1)} g(\nabla u, \nabla(U(\varphi)))$$

$$+ \frac{2u}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)) + \frac{2}{m(m-1)} \left(\frac{1}{\alpha} |\nabla U|^2 (\varphi) u - g(\nabla(U(\varphi)), \nabla u)\right)$$

$$- \frac{2}{um} (\Delta u)^2 + \frac{2\alpha}{u} |d\varphi(\nabla u)|^2 + \frac{2}{2(m-1)} \frac{2}{u} \left(\frac{S^{\varphi}}{2} - p + U(\varphi)\right) |\nabla u|^2$$

$$-\frac{u}{m(m-1)}\Delta((m-2)\mu+mp),$$

that is

$$\begin{split} \operatorname{div} Z &= \frac{2}{u} \bigg\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \bigg\} + \frac{(2m-3)}{m(m-1)} g(\nabla((m-2)\mu + mp), \nabla u) \\ &- \frac{4}{m} g(\nabla(U(\varphi)), \nabla u) + \frac{2}{u} \frac{(m-2)}{(m-1)} \mu |\nabla u|^2 + \frac{2m}{u(m-1)} p |\nabla u|^2 - \frac{4U(\varphi)}{u(m-1)} |\nabla u|^2 \\ &+ \frac{2}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)) u + \frac{2|\nabla U|^2(\varphi)u}{m(m-1)\alpha} + \frac{2\alpha}{u} |d\varphi(\nabla u)|^2 + \frac{2|\nabla u|^2}{u(m-1)} \frac{S^{\varphi}}{2} \\ &- \frac{2}{u(m-1)} p |\nabla u|^2 + \frac{2U(\varphi)}{u(m-1)} |\nabla u|^2 - \frac{u}{m(m-1)} \Delta((m-2)\mu + mp) \\ &= \frac{2}{u} \bigg\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \bigg\} + \frac{(2m-3)}{m(m-1)} g(\nabla((m-2)\mu + mp), \nabla u) \\ &- \frac{4}{m} g(\nabla(U(\varphi)), \nabla u) + \frac{2}{u(m-1)} \bigg( \frac{S^{\varphi}}{2} + (m-2)\mu + (m-1)p - U(\varphi) \bigg) |\nabla u|^2 \\ &+ \frac{2}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)) u + \frac{2}{m(m-1)\alpha} |\nabla U|^2(\varphi) \\ &+ \frac{2\alpha}{u} |d\varphi(\nabla u)|^2 - \frac{u}{m(m-1)} \Delta((m-2)\mu + mp). \end{split}$$

Rewriting

(6.6) 
$$\frac{2}{m(m-1)\alpha}u|\nabla U|^2(\varphi) = \frac{2}{m^2\alpha}u|\nabla U|^2(\varphi) + \frac{2}{m^2(m-1)\alpha}u|\nabla U|^2(\varphi)$$

we obtain

$$\operatorname{div} Z = \frac{2}{u} \left\{ \left| \operatorname{Hess}(u) \right|^2 - \frac{(\Delta u)^2}{m} \right\} + \frac{(2m-3)}{m(m-1)} g(\nabla((m-2)\mu + mp), \nabla u)$$

$$+ \frac{2u}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)) + \frac{2u}{m^2(m-1)\alpha} |\nabla U|^2(\varphi)$$

$$- \frac{u}{m(m-1)} \Delta((m-2)\mu + mp) + u \left| \frac{\sqrt{2}}{m\sqrt{\alpha}} (\nabla U)(\varphi) - \sqrt{2\alpha} d\varphi \left( \frac{\nabla u}{u} \right) \right|^2$$

$$+ \frac{2}{u(m-1)} \left( \frac{S^{\varphi}}{2} + (m-2)\mu + (m-1)p - U(\varphi) \right) |\nabla u|^2.$$

Now, using (6.1) iv), that is  $\frac{S^{\varphi}}{2} = \mu + U(\varphi)$ , we obtain

$$\operatorname{div} Z = \frac{2}{u} \left\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right\} + \frac{(2m-3)}{m(m-1)} g(\nabla((m-2)\mu + mp), \nabla u)$$

$$+ \frac{2u}{m(m-1)} \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi)) + \frac{2u}{m^2(m-1)\alpha} |\nabla U|^2(\varphi)$$

$$- \frac{u}{m(m-1)} \Delta((m-2)\mu + mp) + u \left| \frac{\sqrt{2}}{m\sqrt{\alpha}} (\nabla U)(\varphi) - \sqrt{2\alpha} d\varphi \left(\frac{\nabla u}{u}\right) \right|^2$$

$$+ \frac{2}{u} (\mu + p) |\nabla u|^2.$$

that is, (6.3).

We now prove a generalization of Theorem 1.14.

**Theorem** 6.3. Let (M,g) be an m-dimensional,  $m \geq 3$ , compact  $\varphi$ -SPFST with connected, non-empty boundary and  $\alpha > 0$ . Assume that:

- U is weakly convex,
- $\Delta_{\log u^{2m-3}}((m-2)\mu + mp) \le 0;$
- $p + \mu \ge 0$

and that

(6.7) 
$$m(m-1)|\nabla u|_{|_{\partial M}}^2 \le \max_{M} \{ [2U(\varphi) - ((m-2)\mu + mp)]u^2 \}.$$

Then  $\varphi, \mu, p$  and S are constant on M, with  $\mu$  and S positive and  $\mu = -p$ ; finally, (M, g) is isometric to the hemisphere

(6.8) 
$$S_{+}^{m}\left(\frac{S}{m(m-1)}\right) \subset \mathbb{R}^{m+1}.$$

PROOF. We start by showing that the function

(6.9) 
$$v := |\nabla u|^2 - \frac{1}{m(m-1)}((m-2)\mu + mp - 2U(\varphi))u^2$$

is constant on M. Since  $|\nabla u| = c^2 > 0$  on  $\partial M$  by Proposition 3.1, we can fix  $\delta > 0$  sufficiently small such that the level set

$$\partial M_{\varepsilon} := \{ x \in M : u(x) = \varepsilon \}$$

is a smooth hypersurface for each  $0 < \varepsilon \le \delta$ , where

(6.11) 
$$M_{\varepsilon} := \{ x \in M : u(x) \ge \varepsilon \}.$$

Thus,  $\frac{1}{u}$  is positive and bounded on  $M_{\varepsilon}$ . Let Z be as in (6.2); by Proposition 6.1 we have

$$\operatorname{div} Z = \frac{2}{u} \left\{ \left| \operatorname{Hess}(u) \right|^2 - \frac{(\Delta u)^2}{m} \right\} + \frac{2}{m(m-1)} u \operatorname{tr}(\operatorname{Hess}(U)(d\varphi, d\varphi))$$

$$+ \frac{2}{m^2(m-1)\alpha} \left| \nabla U \right|^2(\varphi) + u \left| \frac{2}{\sqrt{2\alpha}m} (\nabla U)(\varphi) - \sqrt{2\alpha} d\varphi \left( \frac{\nabla u}{u} \right) \right|^2$$

$$+ \frac{2}{u} (\mu + p) |\nabla u|^2 - \frac{u}{m(m-1)} \Delta_{\log u^{2m-3}}((m-2)\mu + mp).$$

Hence, since U is weakly convex,  $p + \mu \ge 0$  and  $\Delta_{\log u^{2m-3}}((m-2)\mu + mp) \le 0$ , the above gives

$$\operatorname{div} Z \geq 0$$
 on  $\operatorname{int}(M)$ 

and by definition (6.2) of Z, that is,  $Z = \frac{1}{u}\nabla v$ , we deduce

(6.12) 
$$\Delta v - \frac{1}{u}g(\nabla u, \nabla v) \ge 0.$$

Thus, by the maximum principle

(6.13) 
$$\max_{M_{\varepsilon}} v = \max_{\partial M_{\varepsilon}} v \quad \text{for } 0 < \varepsilon \le \delta.$$

Since  $M_{\delta} \subseteq M_{\varepsilon}$  for  $0 < \varepsilon \le \delta$ , by (6.13) we obtain

(6.14) 
$$\max_{\partial M_{\delta}} v \le \max_{\partial M_{\varepsilon}} v.$$

Letting  $\varepsilon \to 0^+$ , we get

(6.15) 
$$\lim_{\varepsilon \to 0^+} \max_{\partial M_{\varepsilon}} v = |\nabla u|^2|_{\partial M}$$

and therefore, using (6.14), we have

(6.16) 
$$\max_{\partial M_s} v \le |\nabla u|^2|_{\partial M}.$$

We use the latter, the fact that u > 0 on int(M) and assumption (6.7) to infer

(6.17) 
$$\max_{\partial M_{\delta}} v \le \max_{M} \left\{ \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^{2} \right\}.$$

On the other hand, by (6.13) with  $\varepsilon = \delta$  and (6.17), we deduce

(6.18) 
$$\max_{M_{\delta}} v \le \max_{M} \left\{ \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^{2} \right\}.$$

Let K be the set of points of M at which the function  $(2U(\varphi) - (m-2)\mu - mp)u^2$  realizes its absolute maximum; K is closed, so that, since M is compact, it is compact and for each  $x \in K$ , u(x) > 0. It follows that

$$\min_K u = 2\eta$$

for some  $\eta > 0$ . Choosing  $\delta \leq \eta$ , we have

$$K \subseteq M_{\delta} \setminus \partial M_{\delta}$$

and using (6.18), for each  $p \in K$ , it follows

$$\max_{M} \left\{ \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^{2} \right\} 
= \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^{2}(p) 
\leq v(p) \leq \max_{M_{\delta}} v 
\leq \max_{M} \left\{ \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^{2} \right\}.$$

Hence, for each  $p \in K \subseteq M_{\delta} \setminus \partial M_{\delta}$ , we get

$$v(p) = \max_{M_{\delta}} v = \max_{M} \left\{ \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^{2} \right\},$$

i.e., v assumes its absolute maximum at an interior point of  $M_{\delta}$ . Using (6.12) and again the maximum principle we deduce that

(6.19) 
$$v = |\nabla u|^2 + \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^2$$

$$\equiv \max_{M} \left\{ \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u^2 \right\} \quad \text{on } M_{\delta}.$$

Letting  $\delta \to 0^+$ , we deduce that (6.19) holds on M and v is constant. From (6.3) and the assumptions of the theorem, we obtain, on intM,

(6.20) 
$$\operatorname{Hess}(u) = \frac{\Delta u}{m} g;$$

(6.21) 
$$\frac{2}{\sqrt{2\alpha}m}(\nabla U)(\varphi) - \sqrt{2\alpha}d\varphi\left(\frac{\nabla u}{u}\right) \equiv 0;$$

(6.22) 
$$(\nabla U)(\varphi) \equiv 0;$$

(6.23) 
$$\frac{2}{u}(\mu+p)|\nabla u|^2 \equiv 0.$$

We want to show that there is no open subset of M such that  $|\nabla u|$  vanishes identically on it: suppose by contradiction the existence of an open subset  $A \subset M$  where u is constant. Using (6.1) ii) and iv) we get

$$\Delta u = -\frac{1}{m(m-1)} [2U(\varphi) - (m-2)\mu - mp]u,$$

so that,

$$0 \equiv \frac{1}{m(m-1)} (2U(\varphi) - (m-2)\mu - mp)u \quad \text{on } A.$$

Therefore,

$$v := |\nabla u|^2 - \frac{1}{m(m-1)}((m-2)\mu + mp - 2U(\varphi))u^2$$

vanishes identically on A. Since v is constant, as we just proved, we deduce  $v \equiv 0$  on M. By Proposition 1.13, there exists  $p \in \text{int} M$  such that

$$2U(\varphi)-(m-2)\mu-mp>0\quad \text{ at } p$$

so that v(p) > 0, a contradiction. It follows by (6.23) and u > 0 on int(M) that

$$\mu + p = 0$$

on a dense open subset of M and therefore on M. From Proposition 3.5, we get that  $\mu$  and p are therefore constant.

Moreover, (6.22) implies

$$(6.24) U^a \varphi_i^a \equiv 0 on M.$$

It follows that  $U(\varphi)$  is constant on M and therefore

$$2U(\varphi) - (m-2)\mu - mp$$

is constant on M. Since  $\mu = -p$ , (6.1) iv) implies

(6.25) 
$$S^{\varphi} = 2U(\varphi) - (m-2)\mu - mp,$$

so that  $S^{\varphi}$  is a positive constant because of Proposition 1.13. From (6.1) ii), iv), (6.20) and (6.25) we deduce

(6.26) 
$$\operatorname{Hess}(u) = -\frac{u}{m(m-1)} S^{\varphi} g.$$

Since u is non-constant and  $i:\partial M\hookrightarrow M$  is totally geodesic we can apply Lemma 3 of [63] to conclude that (M,g) is isometric to

(6.27) 
$$\mathbb{S}_{+}^{m} \left( \frac{S^{\varphi}}{m(m-1)} \right).$$

Next, by (6.21) and (6.22), we have

$$d\varphi(\nabla u) \equiv 0$$
 and  $(\nabla U)(\varphi) \equiv 0$ ;

hence, by (6.1) iii),

$$\tau(\varphi) \equiv 0,$$

that is  $\varphi$  is harmonic. Moreover, using (6.26) and (6.1) i), iv) we deduce

(6.29) 
$$\operatorname{Ric}^{\varphi} = \frac{S^{\varphi}}{m}g.$$

From (6.28) and (6.29) we get

$$\begin{cases} \operatorname{Ric}^{\varphi} = \Lambda g \\ \tau(\varphi) = 0 \end{cases}$$

for some constant  $\Lambda$ , that is,  $(M, g, \varphi)$  is harmonic-Einstein; moreover, since (M, g) is isometric to a Euclidean hemisphere, it is also Einstein, i.e.

$$Ric = \zeta g$$
,

where  $\zeta \in \mathbb{R}$ . As a consequence  $\varphi^*h$  is proportional to the metric g,

$$\varphi^* h = \rho q$$

for some constant  $\rho \in \mathbb{R}$ , and by (6.21) we infer

$$\varphi_i^a u_i = 0,$$

which implies

$$0 = \varphi^* h(\nabla u, \nabla u) = \rho |\nabla u|^2.$$

Thus, since u is non-constant, we deduce  $\rho = 0$  and therefore the constancy of  $\varphi$ . As a consequence,  $S^{\varphi} = S$  and (M, g) is isometric to

$$\mathbb{S}_{+}^{m} \left( \frac{S}{m(m-1)} \right).$$

Remark 6.4. The validity of

$$Hess(u) = \frac{\Delta u}{m}g$$

gives rise to a structure on M which is usually called a *conformal gradient soliton*. They were studied in the late '60 (see for instance [67]) and recently reconsidered by Petersen and Wylie, [58], where they sketched a classification which has been fully proved in Catino, Mantegazza and Mazzieri, [14].

**Remark** 6.5. We discuss here the sharpness of assumption (6.7) in Theorem 6.3. To do so, we present an example of  $\varphi$ -vacuum static space, which is based on an example of Costa, Diogenes, Pinheiro and Ribeiro ([21]) and reduces to it when  $\alpha = 0$ . Let  $M = \mathbb{S}^{n+1}_+ \times \mathbb{S}^q$ , with q > 1, and

$$g = dr^2 + \sin^2(r)g_{\mathbb{S}^n} + \rho g_{\mathbb{S}^q}$$

where r is the height function of  $\mathbb{S}^{n+1}_+$  and  $\rho$  is a positive constant. In particular,  $r \equiv \frac{\pi}{2}$  on  $\partial \mathbb{S}^{n+1}_+$ . Denoting with m the dimension of M, we have m = n + 1 + q. Note that g is just the product metric

$$g = g_{\mathbb{S}^{n+1}} + \rho g_{\mathbb{S}^q}.$$

Choosing  $u(r)=\cos(r)\in C^\infty(M), \ \varphi:(M,g)\to (\mathbb{S}^q,\rho g_{\mathbb{S}^q})$  the projection,  $U\equiv 0$  and  $\alpha=\frac{q-1}{\rho}-(n+1)$ , we have that (M,g) becomes a  $\varphi$ -SPFST, as we are going to see. Note that the (0,2)-version of the Ricci tensor is invariant by a homothetic rescaling of the metric, so that

$$\operatorname{Ric}_{\rho g_{\mathbb{S}^q}} = \operatorname{Ric}_{g_{\mathbb{S}^q}} = (q-1)g_{\mathbb{S}^q}.$$

Since g is a product metric, its Ricci tensor, Ric, satisfies

(6.30) 
$$\begin{aligned} \operatorname{Ric} &= \operatorname{Ric}_{g_{\mathbb{S}^{n+1}}} + \operatorname{Ric}_{\rho g_{\mathbb{S}^q}} \\ &= n g_{\mathbb{S}^{n+1}} + (q-1) g_{\mathbb{S}^q}. \end{aligned}$$

Taking the covariant derivative of u we get

$$(6.31) \nabla u = -\sin(r)\nabla r$$

and

$$\left|\nabla u\right|^2 = \sin^2(r).$$

Therefore

Since g is a product metric and u only depends on the first factor, we have

(6.34) 
$$\operatorname{Hess}(u) = -\cos(r)dr^{2} - \cos(r)\sin^{2}(r)g_{\mathbb{S}^{2}}$$
$$= -ug_{\mathbb{S}^{n+1}_{+}}$$

and

$$(6.35) \Delta u = -(n+1)u.$$

From our choice of  $\varphi$ , we have

$$\varphi^* h = \rho g_{\mathbb{S}^q}$$

so that

(6.36) 
$$\operatorname{Ric}^{\varphi} = ng_{\mathbb{S}^{n+1}_{+}} + [q - 1 - \alpha \rho]g_{\mathbb{S}^{q}}.$$

Tracing (6.36) with respect to g we get, from our choice of  $\alpha$ ,

$$S^{\varphi} = n(n+1) + \frac{1}{\rho}[q-1-\alpha\rho]q$$
  
=  $n(n+1) + q(n+1) = (n+1)(n+q)$   
=  $(m-1)(n+1)$ ,

that is,

(6.37) 
$$S^{\varphi} = (m-1)(n+1).$$

Using (6.34), (6.35) and (6.36) we compute

$$\begin{split} -\Delta u g + \mathrm{Hess}(u) - u \mathrm{Ric}^{\varphi} \\ &= (n+1) u g - u g_{\mathbb{S}^{n+1}_+} - n u g_{\mathbb{S}^{n+1}_+} - u (q-1-\alpha \rho) g_{\mathbb{S}^q} \\ &= [(n+1) u - u - n u] g_{\mathbb{S}^{n+1}_+} + \{(n+1) \rho - [(q-1) - \alpha \rho]\} u g_{\mathbb{S}^q} \\ &- 0 \end{split}$$

where the last equality is due to our choice of  $\alpha$ . Therefore, we have

(6.38) 
$$-\Delta u g + \operatorname{Hess}(u) - u \operatorname{Ric}^{\varphi} = 0$$

and tracing (6.38) we also have

$$\Delta u = -\frac{S^{\varphi}}{m-1}u.$$

Since  $\varphi$  is the projection map, from (6.31) we get  $d\varphi(\nabla u) = 0$ . Setting  $U \equiv 0$ , since  $\varphi$  is clearly harmonic, we get

(6.40) 
$$u\tau(\varphi) = -d\varphi(\nabla u) + \frac{u}{\alpha}(\nabla U)(\varphi).$$

Setting  $\mu = \frac{1}{2}S^{\varphi}$  and  $p = -\mu$  we deduce, from (6.38), (6.39) and (6.40), that system (1.3) is satisfied so that (M, g) is a  $\varphi$ -SPFST. Note that, when  $\rho = \frac{q-1}{n+1}$ , we get  $\alpha = 0$  and our example reduces to Example 2 of [21].

We show that inequality (6.7), that is,

(6.41) 
$$m(m-1)|\nabla u|^2|_{\partial M} \le \max_{M} \{ [2U(\varphi) - ((m-2)\mu + mp)]u^2 \},$$

does not hold on (M, g).

From (6.33), the left hand side of (6.41) is

(6.42) 
$$m(m-1)|\nabla u|^2|_{\partial M} = m(m-1).$$

Since  $\mu = -p$  and  $U \equiv 0$ , the right hand side of (6.41) becomes

(6.43) 
$$\max_{M} \left\{ [2U(\varphi) - ((m-2)\mu + mp)]u^{2} \right\} = 2\max_{M} \left\{ \mu u^{2} \right\}.$$

Using (6.1) iv) in (6.43) we have

$$\max_{M} \left\{ [2U(\varphi) - ((m-2)\mu + mp)]u^2 \right\} = \max_{M} \left\{ S^{\varphi}u^2 \right\}.$$

From (6.33) and (6.37) we get

(6.44) 
$$\max_{M} \left\{ [2U(\varphi) - ((m-2)\mu + mp)]u^{2} \right\} = (m-1)(n+1) \max_{M} u^{2}$$
$$= (m-1)(n+1).$$

Thus, since m = n + 1 + q, (6.41) in this case becomes

$$1 \le \frac{n+1}{n+1+q},$$

which is false. However, note that for fixed q and n >> 1 we approach 1 with any desired precision. This shows that (6.7) is sharp at least for  $\varphi$ -vacuum static spaces.

In what follows, it will be useful to introduce the 2-covariant, symmetric tensor Q, depending on  $u \in C^{\infty}(M)$  and  $\varphi : (M, g) \to (N, h)$ , defined as

(6.45) 
$$Q := \overset{\circ}{\operatorname{Hess}}(u) - u \overset{\circ}{\operatorname{Ric}}^{\varphi},$$

where Hess and  $\text{Ric}^{\varphi}$  denote the traceless Hessian and the traceless  $\varphi$ -Ricci tensor, respectively. With a slight abuse of notation, as we did before, we shall indicate with Hess(u) also the associated endomorphism. Note that, if (M,g) is a  $\varphi$ -SPFST, then  $Q \equiv 0$ .

For the proof of Theorem 1.21 we shall also need some technical results.

**Lemma** 6.6. Let (M,g) be a Riemannian manifold of dimension m,  $\alpha \in \mathbb{R}$ ,  $\varphi : (M,g) \to (N,h)$  a smooth map and  $u,w \in C^2(M)$ , with u > 0 on M. Define the vector field

$$X := u \operatorname{Hess}(w)(\nabla w, \cdot)^{\#} + \frac{w^{2}}{u} \operatorname{Hess}(u)(\nabla u, \cdot)^{\#} - w \operatorname{Hess}(u)(\nabla w, \cdot)^{\#}$$

$$- w \operatorname{Hess}(w)(\nabla u, \cdot)^{\#} + \frac{1}{m}(u \nabla w - w \nabla u).$$

Let Q be defined as in (6.45). Then,

$$\operatorname{div} X = u \Big| \operatorname{Hess}(w) - \frac{w}{u} \operatorname{Hess}(u) \Big|^2 + \frac{u}{m} \Big( \Delta w - \frac{w}{u} \Delta u \Big) \Big( \Delta w - \frac{w}{u} \Delta u + 1 \Big) \\ - Q \Big( \nabla w - \frac{w}{u} \nabla u, \nabla w - \frac{w}{u} \nabla u \Big) - \frac{1}{m} \Big| \nabla w - \frac{w}{u} \nabla u \Big|^2 (\Delta u - S^{\varphi} u)$$

$$(6.47) + ug\left(\nabla w - \frac{w}{u}\nabla u, \nabla \Delta w - \frac{w}{u}\nabla \Delta u\right) + \alpha u \left|d\varphi\left(\nabla w - \frac{w}{u}\nabla u\right)\right|^{2}.$$

**Remark** 6.7. Note that X can also be written in the elegant form

$$X = u \left[ \text{Hess}(w) - \frac{w}{u} \text{Hess}(u) + \frac{1}{m} g \right] \left( \nabla w - \frac{w}{u} \nabla u, \cdot \right)^{\sharp}.$$

PROOF OF LEMMA 6.6. As a first step, we prove the validity of

(6.48)

$$\operatorname{div} X = u \left\{ |\operatorname{Hess}(w)|^2 + \frac{w^2}{u^2} |\operatorname{Hess}(u)|^2 - 2\frac{w}{u} \operatorname{tr}(\operatorname{Hess}(w) \circ \operatorname{Hess}(u)) \right\}$$

$$- \left\{ \operatorname{Hess}(u)(\nabla w, \nabla w) - 2\frac{w}{u} \operatorname{Hess}(u)(\nabla w, \nabla u) + \frac{w^2}{u^2} \operatorname{Hess}(u)(\nabla u, \nabla u) \right\}$$

$$+ u \left\{ \operatorname{Ric}^{\varphi}(\nabla w, \nabla w) - 2\frac{w}{u} \operatorname{Ric}^{\varphi}(\nabla w, \nabla u) + \frac{w^2}{u^2} \operatorname{Ric}^{\varphi}(\nabla u, \nabla u) \right\}$$

$$+ ug \left( \nabla w - \frac{w}{u} \nabla u, \nabla \Delta w - \frac{w}{u} \nabla \Delta u \right) + \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^2$$

$$+ \frac{u}{m} \left( \Delta w - \frac{w}{u} \Delta u \right).$$

In a local orthonormal coframe, the components of X are given by

$$X_{k} = uw_{ik}w_{i} + \frac{w^{2}}{u}u_{ik}u_{i} - wu_{ik}w_{i} - ww_{ik}u_{i} + \frac{1}{m}uw_{k} - \frac{1}{m}wu_{k}.$$

Then,

$$\begin{aligned} &\operatorname{div} X = X_{kk} \\ &= u_k w_{ik} w_i + u w_{ikk} w_i + u w_{ik} w_{ik} + 2 \frac{w}{u} w_k u_{ik} u_i + \frac{w^2}{u} u_{ikk} u_i + \frac{w^2}{u} u_{ik} u_{ik} \\ &- \frac{w^2}{u^2} u_k u_{ik} u_i - w_k u_{ik} w_i - w u_{ikk} w_i - w u_{ik} w_{ik} - w_k w_{ik} u_i - w w_{ikk} u_i \\ &- w w_{ik} u_{ik} + \frac{1}{m} u w_{kk} + \frac{1}{m} u_k w_k - \frac{1}{m} w u_{kk} - \frac{1}{m} w_k u_k \\ &= u w_i (w_{kki} + R_{it} w_t) + |\operatorname{Hess}(w)|^2 u + 2 \frac{w}{u} w_k u_{ik} u_i + \frac{w^2}{u} u_i (u_{kki} + R_{it} u_t) \\ &+ \frac{w^2}{u} |\operatorname{Hess}(u)|^2 - \frac{w^2}{u^2} u_k u_{ik} u_i - w_k u_{ik} w_i - w w_i (u_{kki} + R_{it} u_t) \\ &- w u_i (w_{kki} + R_{it} w_t) - 2 w w_{ik} u_{ik} + \frac{1}{m} u \Delta w - \frac{1}{m} w \Delta u \\ &= u \bigg\{ |\operatorname{Hess}(w)|^2 + \frac{w^2}{u^2} |\operatorname{Hess}(u)|^2 - 2 \frac{w}{u} w_{ik} u_{ik} + \frac{1}{m} \Delta w - \frac{1}{m} \frac{w}{u} \Delta u \bigg\} \\ &- \operatorname{Hess}(u) (\nabla w, \nabla w) + 2 \frac{w}{u} \operatorname{Hess}(u) (\nabla w, \nabla u) - \frac{w^2}{u^2} \operatorname{Hess}(u) (\nabla u, \nabla u) \\ &+ u g(\nabla \Delta w, \nabla w) + u \operatorname{Ric}(\nabla w, \nabla w) + \frac{w^2}{u} g(\nabla \Delta u, \nabla u) + \frac{w^2}{u} \operatorname{Ric}(\nabla u, \nabla w) \\ &- w g(\nabla \Delta u, \nabla w) - \operatorname{Ric}(\nabla w, \nabla w) - w g(\nabla \Delta w, \nabla w) - w \operatorname{Ric}(\nabla u, \nabla w) \end{aligned}$$

$$= u \bigg\{ |\operatorname{Hess}|(w) + \frac{w^2}{u^2} |\operatorname{Hess}(u)|^2 - 2\frac{w}{u} \operatorname{tr}(\operatorname{Hess}(u) \circ \operatorname{Hess}(w)) \bigg\}$$

$$+ \frac{u}{m} \Big( \Delta w - \frac{w}{u} \Delta u \Big)$$

$$- \bigg\{ \operatorname{Hess}(u) (\nabla w, \nabla w) - 2\frac{w}{u} \operatorname{Hess}(u) (\nabla u, \nabla w) + \frac{w^2}{u^2} \operatorname{Hess}(u) (\nabla u, \nabla u) \bigg\}$$

$$+ u \bigg\{ \operatorname{Ric}^{\varphi}(\nabla w, \nabla w) - 2\frac{w}{u} \operatorname{Ric}^{\varphi}(\nabla u, \nabla w) + \frac{w^2}{u^2} \operatorname{Ric}^{\varphi}(\nabla u, \nabla u) \bigg\}$$

$$+ u \alpha \bigg\{ \varphi^* h(\nabla w, \nabla w) - 2\frac{w}{u} \varphi^+ h(\nabla u, \nabla w) + \frac{w^2}{u^2} \varphi^* h(\nabla u, \nabla u) \bigg\}$$

$$+ u g \Big( \nabla w - \frac{w}{u} \nabla u, \nabla \Delta w - \frac{w}{u} \nabla \Delta u \Big),$$

which gives (6.48). To simplify the writing, set

(6.49)

$$\mathcal{A} = \left| \overset{\circ}{\operatorname{Hess}}(w) - \frac{w}{u} \overset{\circ}{\operatorname{Hess}}(u) \right|^{2}$$

$$= \left| \operatorname{Hess}(w) \right|^{2} + \frac{w^{2}}{u^{2}} \left| \operatorname{Hess}(u) \right|^{2} - 2 \frac{w}{u} \operatorname{tr} \left( \operatorname{Hess}(w) \circ \operatorname{Hess}(u) \right) - \frac{1}{m} \left( \Delta w - \frac{w}{u} \Delta u \right)^{2}$$

and observe that

$$Q\left(\nabla w - \frac{w}{u}\nabla u, \nabla w - \frac{w}{u}\nabla u\right) = \operatorname{Hess}(u)(\nabla w, \nabla w) - 2\frac{w}{u}\operatorname{Hess}(u)(\nabla w, \nabla u) + \frac{w^2}{u^2}\operatorname{Hess}(u)(\nabla u, \nabla u) - u\left\{\operatorname{Ric}^{\varphi}(\nabla w, \nabla w) - 2\frac{w}{u}\operatorname{Ric}^{\varphi}(\nabla w, \nabla u) + \frac{w^2}{u^2}\operatorname{Ric}^{\varphi}(\nabla u, \nabla u)\right\} - \frac{\Delta u}{m}\left|\nabla w - \frac{w}{u}\nabla u\right|^2 + u\frac{S^{\varphi}}{m}\left|\nabla w - \frac{w}{u}\nabla u\right|^2.$$

$$(6.50)$$

We check (6.50): by the definition of Q, we have

$$\begin{split} Q\Big(\nabla w - \frac{w}{u}\nabla u, \nabla w - \frac{w}{u}\nabla u\Big) &= \\ &= \left(u_{ij} - u\mathring{R}_{ij}^{\varphi} - \frac{1}{m}u_{kk}\delta_{ij}\right) \left(w_{i}w_{j} - \frac{w}{u}w_{i}u_{j} + \frac{w^{2}}{u^{2}}u_{i}u_{j} - \frac{w}{u}w_{j}u_{i}\right) \\ &= \operatorname{Hess}(u)(\nabla w, \nabla w) - \frac{1}{m}\Delta u|\nabla w|^{2} - u\mathring{\operatorname{Ric}}^{\varphi}(\nabla w, \nabla w) \\ &- \frac{w}{u}\operatorname{Hess}(u)(\nabla u, \nabla w) + \frac{1}{m}\frac{w}{u}\Delta ug(\nabla u, \nabla w) + w\mathring{\operatorname{Ric}}^{\varphi}(\nabla u, \nabla w) \\ &- \frac{w}{u}\operatorname{Hess}(u)(\nabla u, \nabla w)\frac{1}{m}\frac{w}{u}\Delta ug(\nabla u, \nabla w) + w\mathring{\operatorname{Ric}}^{\varphi}(\nabla u, \nabla w) \\ &+ \frac{w^{2}}{u^{2}}\operatorname{Hess}(u)(\nabla u, \nabla u) - \frac{1}{m}\frac{w^{2}}{u^{2}}\Delta u|\nabla u|^{2} - \frac{w^{2}}{u^{2}}u\mathring{\operatorname{Ric}}^{\varphi}(\nabla u, \nabla u) \\ &= \operatorname{Hess}(u)(\nabla w, \nabla w) - 2\frac{w}{u}\operatorname{Hess}(u)(\nabla w, \nabla u) + \frac{w^{2}}{u^{2}}\operatorname{Hess}(u)(\nabla u, \nabla u) \end{split}$$

$$-u\left(\operatorname{Ric}^{\varphi}(\nabla w, \nabla w) - 2\frac{w}{u}\operatorname{Ric}^{\varphi}(\nabla w, \nabla u) + \frac{w^{2}}{u^{2}}\operatorname{Ric}^{\varphi}(\nabla u, \nabla u)\right) + u\left(\frac{S^{\varphi}}{m}|\nabla w|^{2} - 2\frac{S^{\varphi}}{m}g(\nabla w, \nabla u) + \frac{w^{2}}{u^{2}}\frac{S^{\varphi}}{m}|\nabla u|^{2}\right) - \frac{\Delta u}{m}\left(|\nabla w|^{2} - 2\frac{w}{u}g(\nabla w, \nabla u) + \frac{w^{2}}{u^{2}}|\nabla u|^{2}\right),$$

from which we immediately deduce (6.50). Using (6.49) and (6.50) into (6.48), we obtain

$$\operatorname{div} X = u \left\{ |\operatorname{Hess}(w)|^2 + \frac{w^2}{u^2} |\operatorname{Hess}(u)|^2 - 2\frac{w}{u} \operatorname{tr} \left(\operatorname{Hess}(w) \circ \operatorname{Hess}(u)\right) \right\}$$

$$+ \frac{u}{m} \left( \Delta w - \frac{w}{u} \Delta u \right) - Q \left( \nabla w - \frac{w}{u} \nabla u, \nabla w - \frac{w}{u} \nabla u \right)$$

$$- \frac{1}{m} (\Delta u - u S^{\varphi}) \left| \nabla w - \frac{w}{u} \nabla u \right|^2 + u g \left( \nabla w - \frac{w}{u} \nabla u, \nabla \Delta w - \frac{w}{u} \nabla \Delta u \right)$$

$$+ \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^2$$

$$= u \mathcal{A} + \frac{u}{m} \left( \Delta w - \frac{w}{u} \Delta u \right) \left( \Delta w - \frac{w}{u} \Delta u + 1 \right) - Q \left( \nabla w - \frac{w}{u} \nabla u, \nabla w - \frac{w}{u} \nabla u \right)$$

$$- \frac{1}{m} (\Delta u - u S^{\varphi}) \left| \nabla w - \frac{w}{u} \nabla u \right|^2$$

$$+ u g \left( \nabla w - \frac{w}{u} \nabla u, \nabla \Delta w - \frac{w}{u} \nabla \Delta u \right) + \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^2.$$

from which we get (6.47).

**Lemma** 6.8. Assume that there exists  $u \in C^{\infty}(M)$ , u > 0, satisfying

(6.51) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \left\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) g \right\} = 0, \\ ii) \Delta u = \frac{u}{m-1} \left( mp - mU(\varphi) + \frac{m-2}{2} S^{\varphi} \right), \\ iii) \mu + U(\varphi) = \frac{1}{2} S^{\varphi} \end{cases}$$

and that, for some relatively compact  $\Omega \subset\subset \operatorname{int}(M)$ ,  $w\in C^\infty(\Omega)$  satisfies the differential equation

$$(6.52) \Delta w - \frac{w}{u} \Delta u = -1, \quad on \ \Omega.$$

Let X be the vector field on  $\Omega$  defined in (6.46). Then

(6.53) 
$$\operatorname{div} X = u \left| \operatorname{Hess}(w) - \frac{w}{u} \operatorname{Hess}(u) \right|^2 + \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^2 + (\mu + p) u \left| \nabla w - \frac{w}{u} \nabla u \right|^2 \quad on \ \Omega.$$

PROOF. Since w is a solution of (6.52), we have

(6.54) 
$$\frac{u}{m} \left( \Delta w - \frac{w}{u} \Delta u \right) \left( \Delta w - \frac{w}{u} \Delta u + 1 \right) \equiv 0;$$

moreover, by (6.51) i), ii), Q defined in (6.45) is identically null; indeed,

$$\begin{split} Q &= \overset{\circ}{\operatorname{Hess}}(u) - u \overset{\circ}{\operatorname{Ric}}^{\varphi} \\ &= \operatorname{Hess}(u) - \frac{\Delta u}{m} g - u \overset{\circ}{\operatorname{Ric}}^{\varphi} + u \frac{S^{\varphi}}{m} g \\ &= \operatorname{Hess}(u) - u \overset{\circ}{\operatorname{Ric}}^{\varphi} - \frac{u}{m(m-1)} \bigg( mp - mU(\varphi) + \frac{m-2}{2} S^{\varphi} \bigg) g + u \frac{S^{\varphi}}{m} g \\ &= \operatorname{Hess}(u) - u \bigg\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \bigg( \frac{S^{\varphi}}{2} - p + U(\varphi) \bigg) g \bigg\} = 0. \end{split}$$

As a consequence, (6.47) rewrites as

(6.55) 
$$\operatorname{div} X = u \left| \operatorname{Hess}(w) - \frac{w}{u} \operatorname{Hess}(u) \right|^{2} - \frac{1}{m} (\Delta u - u S^{\varphi}) \left| \nabla w - \frac{w}{u} \nabla u \right|^{2} + u g \left( \nabla w - \frac{w}{u} \nabla u, \nabla \Delta w - \frac{w}{u} \nabla \Delta u \right) + \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^{2}.$$

Equation (6.52) yields

(6.56)

$$\begin{split} ug\Big(\nabla w - \frac{w}{u}\nabla u, \nabla \Delta w - \frac{w}{u}\nabla \Delta u\Big) &= -ug\Big(\nabla w - \frac{w}{u}\nabla u, \Delta u \frac{\nabla w}{u} - \Delta u \frac{w}{u^2}\nabla u\Big) \\ &= \Delta u\Big|\nabla w - \frac{w}{u}\nabla u\Big|^2. \end{split}$$

Inserting (6.56) into (6.55) and using (6.51) ii), we have

$$\operatorname{div} X = u \left| \operatorname{Hess}(w) - \frac{w}{u} \operatorname{Hess}(u) \right|^{2} + \frac{u}{m} ((m-1)\Delta u + S^{\varphi}u) \left| \nabla w - \frac{w}{u} \nabla u \right|^{2}$$

$$+ \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^{2}$$

$$= u \left| \operatorname{Hess}(w) - \frac{w}{u} \operatorname{Hess}(u) \right|^{2} + \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^{2}$$

$$+ \frac{u}{m} \left| \nabla w - \frac{w}{u} \nabla u \right|^{2} \left( S^{\varphi} - mU(\varphi) + mp + \frac{m-2}{2} S^{\varphi} \right)$$

$$= u \left| \operatorname{Hess}(w) - \frac{w}{u} \operatorname{Hess}(u) \right|^{2} + \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^{2}$$

$$+ u \left| \nabla w - \frac{w}{u} \nabla u \right|^{2} (\mu + p),$$

where in the last equality we have used (6.51) iii). Thus, we have the validity of (6.53).

In order to ensure that (6.52) holds, we recall the following

**Lemma** 6.9. Let (M,g) be a manifold and  $\Omega \subset \operatorname{int}(M)$  open,  $u \in C^{\infty}(M)$ , such that  $\frac{\Delta u}{u} \in C^{\infty}(\overline{\Omega})$ . Then, there exists a solution  $w \in C^{\infty}(\Omega) \cap C^{2,\beta}(\overline{\Omega})$ ,  $\beta \in (0,1)$ , of the problem

(6.57) 
$$\begin{cases} \Delta w = \frac{\Delta u}{u} w - 1 & \text{on } \overline{\Omega} \\ w \equiv 0 & \text{on } \partial \Omega. \end{cases}$$

Moreover, w > 0 on  $\Omega$  and  $\partial \Omega = \{x \in M : w(x) = 0\}$  is a regular level set of w. For a proof see [32].

We are now ready to prove Theorem 1.21, that we recall for the ease of the reader.

**Theorem** 6.10. Let (M,g) be a  $\varphi$ -SPFST of dimension  $m \geq 2$  and let  $\Omega \subset \subset$  intM with smooth boundary. Let

$$\overline{H} = \frac{1}{m} \frac{\int_{\partial \Omega} u}{\int_{\Omega} u}$$

and assume

$$H < -\overline{H}$$
,

where H is the mean curvature of  $\partial\Omega$  in the direction of the inward unit normal. Furthermore, suppose

$$\mu + p \ge 0$$
 on  $M$ .

Then

$$i:\partial\Omega\hookrightarrow M$$

is totally umbilical and  $\mu, p$  are constant on  $\Omega$ , with

$$\mu = -p$$
.

PROOF (OF THEOREM 1.21). Let  $\Omega \subset\subset \operatorname{int}(M)$  be a domain with smooth boundary. Let w be the solution of (6.57) and recall that

$$\partial\Omega = w^{-1}(\{0\}),$$

which we know to be smooth with inward unit normal

$$\nu = \frac{\nabla w}{|\nabla w|}.$$

For the vector field X defined in (6.46), we have

(6.58) 
$$g(X,\nu) = \frac{u}{|\nabla w|} \text{Hess}(w)(\nabla w, \nabla w) + \frac{1}{m} u |\nabla w|$$

on  $\partial\Omega$ .

A simple computation shows that the mean curvature in the direction of  $\nu$  of the level set  $\partial\Omega$  of w is given by

$$(m-1)H = \frac{1}{|\nabla w|} \Delta w + g\left(\nabla\left(\frac{1}{|\nabla w|}\right), \nabla w\right) \text{ on } \partial\Omega$$

and therefore

(6.59) 
$$(m-1)H = \frac{1}{|\nabla w|} \Delta w - \frac{1}{|\nabla w|^3} \operatorname{Hess}(w)(\nabla w, \nabla w).$$

Using the latter and (6.57), we infer

(6.60) 
$$-\frac{1}{|\nabla w|} \operatorname{Hess}(w)(\nabla w, \nabla w) = (m-1)H|\nabla w|^2 + |\nabla w| \text{ on } \partial\Omega.$$

Integrating div X on  $\Omega$  and using (6.58) and (6.60), we obtain

(6.61) 
$$\int_{\Omega} \operatorname{div} X = \int_{\partial \Omega} -g(X, \nu) = \int_{\partial \Omega} \left( (m-1)Hu|\nabla w|^2 + \frac{m-1}{m}u|\nabla w| \right).$$

We set

(6.62) 
$$\Lambda = \frac{\int_{\Omega} u}{\int_{\partial \Omega} u}, \quad \overline{H} = \frac{1}{m} \frac{1}{\Lambda} = \frac{1}{m} \frac{\int_{\partial \Omega} u}{\int_{\Omega} u}.$$

Integrating  $\operatorname{div}(w\nabla u)$  on  $\Omega$  we obtain

(6.63) 
$$\int_{\Omega} w \Delta u = -\int_{\Omega} g(\nabla u, \nabla w).$$

Similarly, integrating  $\operatorname{div}(u\nabla w)$  on  $\Omega$  and using (6.57) and (6.63), we infer

$$\int_{\partial\Omega} -u|\nabla w| = \int_{\Omega} u\Delta w + \int_{\Omega} g(\nabla u, \nabla w)$$
$$= \int_{\Omega} (w\Delta u - u) - \int_{\Omega} w\Delta u$$
$$= -\int_{\Omega} u,$$

that is

(6.64) 
$$\int_{\partial\Omega} u |\nabla w| = \int_{\Omega} u.$$

Using the latter and (6.62), we have

$$\begin{split} \frac{1}{\Lambda} \int_{\partial \Omega} u (\Lambda - |\nabla w|)^2 &= \frac{1}{\Lambda} \int_{\partial \Omega} u |\nabla w|^2 - \int_{\partial \Omega} 2u |\nabla w| + \int_{\partial \Omega} \Lambda u \\ &= \frac{1}{\Lambda} \int_{\partial \Omega} u |\nabla w|^2 - 2 \int_{\partial \Omega} u |\nabla w| + \int_{\Omega} u \\ &= \frac{1}{\Lambda} \int_{\partial \Omega} u |\nabla w|^2 - \int_{\partial \Omega} u |\nabla w|. \end{split}$$

Now we use (6.53) and the previous equality into (6.61) to obtain

$$\begin{split} (m-1) \int_{\partial\Omega} \left\{ Hu |\nabla w|^2 - \frac{u}{m} |\nabla w| \right\} &= \int_{\Omega} u \Big| \mathring{\mathrm{Hess}}(w) - \frac{w}{u} \mathring{\mathrm{Hess}}(u) \Big|^2 \\ &+ \int_{\Omega} \alpha u \Big| d\varphi \Big( \nabla w - \frac{w}{u} \nabla u \Big) \Big|^2 \\ &+ \int_{\Omega} \Big| \nabla w - \frac{w}{u} \nabla u \Big|^2 u (\mu + p), \end{split}$$

that is

$$(m-1)\int_{\partial\Omega} \left\{ Hu|\nabla w|^2 + \frac{1}{m}\frac{1}{\Lambda}u|\nabla w|^2 \right\} = \int_{\Omega} u \left| \mathring{\operatorname{Hess}}(w) - \frac{w}{u}\mathring{\operatorname{Hess}}(u) \right|^2 + \int_{\Omega} \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u}\nabla u \right) \right|^2$$

$$+ \int_{\Omega} \left| \nabla w - \frac{w}{u} \nabla u \right|^{2} u(\mu + p)$$

$$+ \frac{m-1}{m} \frac{1}{\Lambda} \int_{\partial \Omega} u(\Lambda - |\nabla w|)^{2}.$$

On the other hand, by (6.63), we get

$$(6.65) \qquad \int_{\partial\Omega} (m-1) \left\{ Hu + \frac{1}{m} \frac{1}{\Lambda} u \right\} |\nabla w|^2 = (m-1) \int_{\partial\Omega} u |\nabla w|^2 \left( \overline{H} + H \right).$$

Hence we have

$$(m-1) \int_{\partial\Omega} u |\nabla w|^2 (\overline{H} + H) = \int_{\Omega} u \left| \mathring{\text{Hess}}(w) - \frac{w}{u} \mathring{\text{Hess}}(u) \right|^2$$

$$+ \int_{\Omega} \alpha u \left| d\varphi \left( \nabla w - \frac{w}{u} \nabla u \right) \right|^2$$

$$+ \int_{\Omega} \left| \nabla w - \frac{w}{u} \nabla u \right|^2 u (\mu + p)$$

$$+ \frac{m-1}{m} \frac{1}{\Lambda} \int_{\partial\Omega} u (\Lambda - |\nabla w|)^2.$$
(6.66)

Thus, for  $\alpha > 0$ ,  $\mu + p \ge 0$  and  $\overline{H} + H \le 0$  we deduce

$$\left| \overset{\circ}{\text{Hess}}(w) - \frac{w}{u} \overset{\circ}{\text{Hess}}(u) \right|^2 \equiv 0 \quad \text{on } \overline{\Omega};$$

moreover, since  $w \in C^{2,\beta}(\overline{\Omega})$  and  $w \equiv 0$  on  $\partial \Omega$ , we get

(6.67) 
$$\operatorname{Hess}(w) = \frac{\Delta w}{m} g \quad \text{on } \partial\Omega.$$

Let  $\Pi_{ab}$ ,  $1 \leq a, b, \dots \leq m-1$ , be the coefficients of the second fundamental form of  $i: \partial\Omega \hookrightarrow M$  in the direction of the inward unit normal  $\nu = \frac{\nabla w}{|\nabla w|}$  of  $\partial\Omega$ ; since

$$II_{ab} = -\frac{w_{ab}}{|\nabla w|}$$

from (6.67), we deduce that  $i:\partial\Omega\hookrightarrow M$  is totally umbilical. Furthermore, by (6.66), we infer

(6.68) 
$$(\mu + p) \left| \nabla w - \frac{w}{u} \nabla u \right|^2 \equiv 0 \quad \text{on } \Omega.$$

and we deduce that  $\mu$  and p are constant on  $\Omega$ , with  $\mu = -p$ ; indeed, suppose

$$A := \operatorname{int}\left(\left\{\left(\nabla w - \frac{w}{u}\nabla u\right)(x) = 0\right\}\right) \neq \emptyset.$$

Let  $\widehat{A}$  be a connected component of A; since

$$\nabla \left(\frac{w}{u}\right) = \frac{1}{u}\nabla w - \frac{w}{u^2}\nabla u = \frac{1}{u}\left(\nabla w - \frac{w}{u}\nabla u\right)$$

on  $\widehat{A}$ , then there exists  $c \in \mathbb{R}$  such that w = cu, but w solves

$$\Delta w - \frac{\Delta u}{u}w = -1,$$

thus

$$0 = c\Delta u - \frac{\Delta u}{u}cu = -1,$$

which is a contradiction.

Note that  $|\nabla w - \frac{w}{u}\nabla u|$  can be zero only on a set with empty interior of  $\Omega$ . On the other hand, (6.66) gives  $d\varphi(\nabla w - \frac{w}{u}\nabla u) \equiv 0$  on  $\Omega$ , so that necessarily  $\operatorname{Ker}(d\varphi)$  is not trivial, at least on the complement in  $\Omega$  of a set with empty interior.

#### CHAPTER 7

## Non-Existence results

In this Chapter we provide non-existence results for  $\varphi$ -SPFSTs; in the first section, the method we use is based on the introduction of the elliptic operator

$$Lu := \Delta u + \frac{u}{m-1}(2U(\varphi) - mp - (m-2)\mu)$$

and on finding sufficient conditions under which each solution of a related Cauchy problem admits a first zero. We observe that Lu = 0 is obtained by

(7.1) 
$$\begin{cases} i) \operatorname{Hess}(u) - u \Big\{ \operatorname{Ric}^{\varphi} - \frac{1}{m-1} \left( \frac{S^{\varphi}}{2} - p + U(\varphi) \right) g \Big\} = 0, \\ ii) \Delta u = \frac{u}{m-1} \Big[ mp - mU(\varphi) + \frac{m-2}{2} S^{\varphi} \Big], \\ iii) u\tau(\varphi) + d\varphi(\nabla u) = \frac{u}{\alpha} (\nabla U)(\varphi), \\ iv) \mu + U(\varphi) = \frac{1}{2} S^{\varphi}, \\ v) (\mu + p) \nabla u = -u \nabla p, \end{cases}$$

combining the second and the fourth equations.

Using a different technique, the second part of this chapter is devoted to finding an obstruction to the existence on a closed Riemannian manifold (M, g), of a structure satisfying

(7.2) 
$$\begin{cases} u \operatorname{Ric}^{\varphi} - \operatorname{Hess}(u) + \Delta u g = \lambda g, \\ u > 0, \end{cases}$$

where  $u, \lambda \in C^{\infty}(M)$ ; note that system (7.2) holds for every  $\varphi$ -SPFST with empty boundary, as it can be seen combining equations i), ii) and iv) of system (7.1).

#### 7.1. A non-existence result via oscillation theorey

As before, and for the rest of this chapter,  $\varphi:(M,g)\to (N,h)$  is a smooth map,  $U\in C^\infty(N), \mu, p\in C^\infty(M)$  and we use the notation of the previous chapters.

**Proposition** 7.1. Let (M,g) be a complete Riemannian manifold of dimension m,  $\partial M = \emptyset$ . For  $r \in \mathbb{R}^+$ , let

$$v(r) := vol(\partial B_r), \quad A(r) := \frac{1}{v(r)} \int_{\partial B_r} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x),$$

where  $B_r$  is the geodesic ball of radius r centered at a fixed origin  $o \in M$ . Let  $z \in \text{Lip}_{loc}(\mathbb{R}_0^+)$  be a solution of

(7.3) 
$$\begin{cases} (vz')' + Avz = 0 & on \mathbb{R}^+, \\ z(0^+) = z_0 > 0, \\ (vz')(0^+) = 0. \end{cases}$$

Suppose z admits a first zero  $R_0 \in \mathbb{R}^+$ ; then there exists no positive  $u \in C^{\infty}(M)$  satisfying

(7.4) 
$$Lu := \Delta u + \frac{u}{m-1} (2U(\varphi) - mp - (m-2)\mu) = 0.$$

PROOF. By contradiction, assume that (7.4) admits a positive solution on M; then by Theorem 1 of [31] and [52], the first eigenvalue of L on M,  $\lambda_L^1(M)$ , is non-negative; moreover, by Proposition 4.6 of [8] and the definition of A(t) we have that  $z \in \text{Lip}_{loc}(R_0^+)$  and its zeros are isolated (if any).

Let r be the distance function from the fixed origin  $o \in M$  and define

$$(7.5) \psi = z \circ r.$$

To obtain the desired contradiction, we consider the Rayleigh quotient of  $\psi$  on the geodesic ball  $B_{R_0}$  centered at  $o \in M$  and of radius  $R_0$ ,

(7.6)

$$Q(\psi) = \left(\int_{B_{R_0}} \psi^2\right)^{-1} \int_{B_{R_0}} \left( |\nabla \psi|^2 - \frac{1}{m-1} [(2U(\varphi) - mp - (m-2)\mu)(x)] \psi^2 \right).$$

Gauss lemma yields

$$|\nabla \psi|^2 = (z')^2;$$

applying the co-area formula (see e.g. [51])  $Q(\psi)$  rewrites as

$$Q(\psi) = \left( \int_0^{R_0} dt \int_{\partial B_t} z^2 \right)^{-1} \int_0^{R_0} dt \int_{\partial B_t} \left( (z')^2 - \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x) z^2 \right)$$
$$= \left( \int_0^{R_0} v z^2 dt \right)^{-1} \int_0^{R_0} \left( v (z')^2 - Av z^2 \right) dt,$$

where in the last equality we have used the definition of A. Integrating by parts, using (7.3) and the fact that  $z(R_0) = 0$ , we deduce

$$\int_0^{R_0} (z')^2 v dt = (zz'v)(R_0) - (zz'v)(0) - \int_0^{R_0} z(z'v)' dt$$
$$= \int_0^{R_0} Avz^2 dt.$$

It follows that  $Q(\psi) = 0$ . Hence  $\lambda_L^1(B_{R_0}) \leq 0$  and by monotonicity of the eigenvalues of L, we have  $\lambda_L^1(M) < 0$ , the desired contradiction.

The existence of a first zero for z has been a fundamental tool in the proof of Proposition 7.1; as a consequence, it is natural to study sufficient conditions under which a solution of (7.3) (that always exists by Proposition 3.2 of [8]) admits a first zero.

Towards this aim, let h be a function satisfying

- 1)  $h \in L^{\infty}_{loc}(\mathbb{R}^+_0);$
- $2) \ \frac{1}{h} \in L^{\infty}_{loc}(\mathbb{R}_0^+);$
- 3)  $0 \le v \le h$  on  $\mathbb{R}_0^+$ ,

with corresponding critical curve

(7.7) 
$$\chi_h(r) := \left\{ 2h(r) \int_r^{+\infty} \frac{1}{h(s)} ds \right\}^{-2}.$$

By Corollary 6.2 of [8], if

- (A1)  $A \in L^{\infty}_{loc}(R_0^+);$
- (V1)  $0 \le v(r) \in L^{\infty}_{loc}(R_0^+), \quad \frac{1}{v(r)} \in L^{\infty}_{loc}(R_0^+), \quad \lim_{r \to 0^+} v(r) = 0$

are satisfied,  $A \ge 0$  on  $\mathbb{R}^+$ ,  $A \not\equiv 0$  and, for some h satisfying the assumptions above, either

- i)  $\frac{1}{h} \notin L^1(+\infty)$ ;
- ii) or otherwise  $\frac{1}{h} \in L^1(+\infty)$  and there exists r > R > 0 such that  $A \not\equiv 0$  on [0,R] and

$$(7.8) \int_{R}^{r} \left(\sqrt{A(s)} - \sqrt{\chi_h(s)}\right) ds > -\frac{1}{2} \left(\log \int_{0}^{R} A(s)v(s)ds + \log \int_{R}^{+\infty} \frac{1}{h(s)} ds\right),$$

then the solution z of (7.3) admits a first zero.

A similar result is given for oscillation in Theorem 6.6 of [8].

**Remark** 7.2. Note that conditions (V1) in our case is satisfied since

$$v(r) = \operatorname{vol}(\partial B_r)$$

(see, for instance, Proposition 2.6 of [8]), while (A1) is satisfied by

$$A(r) = \frac{1}{v(r)} \int_{\partial B_r} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x),$$

because  $\frac{1}{v(r)} \in L^{\infty}_{loc}(R_0^+)$  and  $\frac{1}{m-1}(2U(\varphi) - mp - (m-2)\mu)(x) \in C^{\infty}(M)$ .

By Corollary 2.9 of [49] and the co-area formula, if  $v, 1/v \in L^{\infty}_{loc}(\mathbb{R}^+)$ , v > 0,  $1/v \notin L^1(+\infty)$  and, for some  $r_0 \in \mathbb{R}^+$ ,

$$\lim_{r \to +\infty} \int_{B_r \setminus B_{r_0}} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x) = +\infty,$$

then a solution z of (7.3) is oscillatory, so that it certainly admits a first zero.

As a consequence, we deduce the validity of the following

**Proposition** 7.3. Let (M,g) be a complete Riemannian manifold of dimension m. For  $r \in \mathbb{R}^+$ , let

$$v(r) = \operatorname{vol}(\partial B_r)$$

and assume that

$$\frac{1}{v(r)} \notin L^1(+\infty)$$

and

$$\lim_{r \to +\infty} \int_{B_r \setminus B_{r_0}} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x) = +\infty.$$

Then, there is no positive solution u to (7.4).

**Remark** 7.4. Note that it is possible to construct examples of manifolds satisfying

$$\frac{1}{v(r)} \notin L^1(+\infty)$$

and such that the volume grows exponentially (see e.g. [64]). Moreover, the second hypothesis does not require assumptions on the sign of

$$(2U(\varphi) - mp - (m-2)\mu)(x)$$

and hence, assumptions on the violation of the Strong Energy Condition (SEC) when we consider a solution of (7.1) with  $\alpha > 0$ .

When v(r) admits an upper bound it is possible to show that under suitable assumptions either (7.3) admits a first zero or it is oscillatory. Note that, in the first case hypotheses on the sign of  $(2U(\varphi) - mp - (m-2)\mu)(x)$  will be needed; however, to prove that (7.3) is oscillatory we rely on Theorem 6.6 [8] and Proposition 6.9 of [8], whose hypotheses do not force us to make an assumption on the sign of  $(2U(\varphi) - mp - (m-2)\mu)(x)$ .

**Proposition** 7.5. Let (M,g) be a complete Riemannian manifold of dimension m, let v(r), A(r),  $\frac{1}{m-1}(2U(\varphi)-mp-(m-2)\mu)(x)$  be as in Proposition 7.1. Assume that

$$(7.9) v(r) \le Cr^{\theta},$$

where  $C, \theta \in \mathbb{R}, C > 0, \theta > 1$ .

1) If 
$$A \geq 0$$
 on  $\mathbb{R}^+$ ,

(7.10) 
$$2U(\varphi) - mp - (m-2)\mu \ge 0 \text{ on } M$$
 and for some  $R, D \in \mathbb{R}^+, D > \frac{\theta-1}{2}$ , we have

(7.11) 
$$\int_{\partial B_R} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x) \ge \frac{D^2}{r^2} v(r), \quad r \ge R,$$
then (7.3) admits a first zero.

2) If 
$$A \ge 0$$
 in  $[r_0, +\infty)$ ,  $\mathbb{R} \ni r_0 > 0$ ,

(7.12) 
$$\int_{\partial B_n} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x) \notin L^1(+\infty)$$

and, for some  $R, D \in \mathbb{R}$ ,  $R > r_0$ ,  $D > \frac{\theta - 1}{2}$ , we have

(7.13) 
$$\int_{R}^{r} \sqrt{A(s)} ds \ge D \log (r/R),$$

then (7.3) is oscillatory.

**Remark** 7.6. Note that, when (7.10) is satisfied, half of the Strong Energy Condition (SEC) is violated.

PROOF. By (7.9), we set

$$h(r) = Cr^{\theta}$$
.

When  $\theta > 1$ ,  $h^{-1} \in L^1(+\infty)$  and by definition of h, we have

(7.14) 
$$\chi_h(r) = \left\{ 2h(r) \int_r^{+\infty} \frac{1}{Cs^{\theta}} ds \right\}^{-2}$$

$$= \left\{ 2Cr^{\theta} \frac{r^{1-\theta}}{C(\theta-1)} \right\}^{-2}$$

$$= \left( \frac{\theta-1}{2r} \right)^2.$$

We now prove the first part of the theorem. We want to show the validity of (7.8); by (7.14), we rewrite (7.8) as

$$\begin{split} \int_R^r \Big( \sqrt{A(s)} - \sqrt{\chi_h(s)} \Big) ds &= \int_R^r \left[ \sqrt{A(s)} - \left( \frac{\theta - 1}{2s} \right) \right] ds \\ &= \int_R^r \sqrt{A(s)} \, ds - \frac{\theta - 1}{2} (\log(r) - \log(R)) \\ &> -\frac{1}{2} \bigg[ \log \int_{B_R} \frac{1}{m - 1} (2U(\varphi) - mp - (m - 2)\mu)(x) \\ &+ \log \bigg( \frac{R^{1 - \theta}}{C(\theta - 1)} \bigg) \bigg], \end{split}$$

that is,

$$\int_{R}^{r} \sqrt{A(s)} \, ds - \frac{\theta - 1}{2} \log(r) > \frac{1}{2} \log \left[ C(\theta - 1) \right] - \frac{1}{2} \log \int_{B_{R}} \frac{1}{m - 1} (2U(\varphi) - mp - (m - 2)\mu)(x).$$

By definition of A(r) and (7.11), it follows

$$\sqrt{A(r)} \ge \frac{D}{r}$$
 for  $r \ge R$ .

Therefore, to conclude, it is sufficient to show

$$D \int_{R}^{r} \frac{ds}{s} - \frac{\theta - 1}{2} \log(r) > \frac{1}{2} \log \left[ C(\theta - 1) \right] - \frac{1}{2} \log \int_{R_{R}} \frac{1}{m - 1} (2U(\varphi) - mp - (m - 2)\mu)(x),$$

that is,

$$(7.15) \quad \left(D - \frac{\theta - 1}{2}\right) \log(r) > \log\left[R^D \sqrt{C(\theta - 1)}\right]$$
$$- \frac{1}{2} \log \int_{B_R} \frac{1}{m - 1} (2U(\varphi) - mp - (m - 2)\mu)(x).$$

Since  $D > \frac{\theta-1}{2}$ , (7.15) holds for a sufficient large r; moreover,  $A(r) \neq 0$  on [0, R] since equation (7.11) implies

$$\frac{1}{m-1}(2U(\varphi) - mp - (m-2)\mu) \not\equiv 0,$$

which concludes the first part of the statement.

We are now ready prove the second part of the Theorem. By the validity of (7.12), it is sufficient to show that condition ii) of Proposition 6.9 of [8] is satisfied to prove that (7.3) is oscillatory. It follows by (7.13) and our choice of D that

(7.16) 
$$\limsup_{r \to +\infty} \frac{\int_{R}^{r} \sqrt{A(s)} ds}{\int_{R}^{r} \sqrt{\chi_{h}(s)} ds} = \limsup_{r \to +\infty} \frac{\int_{R}^{r} \sqrt{A(s)} ds}{\left(\frac{\theta - 1}{2}\right) \log(r/R)}$$
$$\geq \limsup_{r \to +\infty} \frac{D \log(r/R)}{\left(\frac{\theta - 1}{2}\right) \log(r/R)} > 1.$$

So far we have considered a polynomial growth of v(r), see assumption (7.9). Our aim is now to allow a faster growth, even superexponential, as in assumption (7.17) below. In this way, we cover the reasonable ranges of the growth of the volume of geodesic spheres in a complete manifold.

**Proposition** 7.7. Let (M,g) be a complete Riemannian manifold of dimension m, and let v(r), A(r),  $\frac{1}{m-1}(2U(\varphi)-mp-(m-2)\mu)(x)$  be as in Proposition 7.1. Assume the validity of (7.10) on M and

(7.17) 
$$v(r) \le \Lambda \exp\left\{ar^{\gamma} \log^{\beta}(r)\right\},\,$$

for some constants  $\Lambda$ , a > 0 and either  $\gamma > 0$ ,  $\beta \ge 0$  or  $\gamma \ge 0$ ,  $\beta > 0$ .

1) If  $A \ge 0$  on  $\mathbb{R}^+$ , for some  $r > r_0 > 1$ , and for some  $b \in \mathbb{R}$ , b > 1 we have

(7.18) 
$$A(r) = \frac{1}{v(r)} \int_{\partial B_r} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x)$$
$$\geq b \frac{a^2}{4} (\gamma \log(r) + \beta)^2 (r)^{2(\gamma-1)} \log^{2(\beta-1)}(r),$$

then (7.3) admits a first zero.

2) If  $A \ge 0$  in  $[r_0, +\infty)$ , for some  $r_0 > 0$ ,

(7.19) 
$$\int_{\partial B} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x) \notin L^{1}(+\infty)$$

and for some  $R, C \in \mathbb{R}$  such that  $R > r_0, C > 1$  we have the validity of

(7.20) 
$$\int_{R}^{r} \sqrt{A(s)} ds \ge Car^{\gamma} \log^{\beta}(r),$$

then (7.3) is oscillatory.

PROOF. By (7.17), we let

$$h(r) = \Lambda \exp\left\{ar^{\gamma}\log^{\beta}(r)\right\};$$

note that  $h^{-1} \in L^1(+\infty)$ .

1) To prove the validity of the first part of the statement of Proposition 7.7, it is then sufficient to show that there exists  $r > R > r_0$  such that  $A \not\equiv 0$  on [0, R] and

$$\int_{R}^{r} \left( \sqrt{A(s)} - \sqrt{\chi_{h}(s)} \right) ds >$$

$$> -\frac{1}{2} \left( \log \int_{B_{R}} \frac{1}{m-1} (2U(\varphi) - mp - (m-2)\mu)(x) + \log \int_{R}^{+\infty} \frac{1}{h(s)} ds \right).$$
Let
$$\widetilde{\chi_{h}}(s) := \left( \frac{h'(s)}{2h(s)} \right)^{2};$$

then

$$\lim_{t \to +\infty} \frac{\sqrt{\widetilde{\chi_h}(s)}}{\sqrt{\chi_h(s)}} = 1,$$

and we refer the reader to [7] for a proof; therefore, note that when R is sufficiently large, for every  $t \ge R$ 

$$\sqrt{\chi_h(s)} < c\sqrt{\widetilde{\chi_h}(s)},$$

where  $c \in \mathbb{R}$ , c > 1. Hence,

(7.21) 
$$\int_{R}^{r} \sqrt{\chi_{h}(s)} dt < \int_{R}^{r} c\sqrt{\widetilde{\chi_{h}}(s)} ds$$

$$= \int_{R}^{r} c \frac{h'(s)}{2h(s)} ds$$

$$= \frac{c}{2} \log(h(r)) - \frac{c}{2} \log(h(R))$$

$$= \frac{c}{2} ar^{\gamma} \log^{\beta}(r) - \frac{c}{2} aR^{\gamma} \log^{\beta}(R).$$

Moreover, by assumption (7.18), we have

$$(7.22) \qquad \sqrt{A(t)} \ge \sqrt{b} \frac{a}{2} (\gamma \log(t) + \beta) t^{(\gamma - 1)} \log^{(\beta - 1)}(t) = \sqrt{b} \frac{a}{2} \left( t^{\gamma} \log^{\beta}(t) \right)'.$$

It follows by (7.21) and (7.22) that

(7.23) 
$$\int_{R}^{r} \left(\sqrt{A(t)} - \sqrt{\chi_{h}(t)}\right) dt >$$

$$> \int_{R}^{r} \sqrt{b} \frac{a}{2} \left(t^{\gamma} \log^{\beta}(t)\right)' dt - \frac{c}{2} a r^{\gamma} \log^{\beta}(r) + \frac{c}{2} a R^{\gamma} \log^{\beta}(R)$$

$$= \frac{\left(\sqrt{b} - c\right)}{2} a r^{\gamma} \log^{\beta}(r) - \frac{\left(\sqrt{b} - c\right)}{2} a R^{\gamma} \log^{\beta}(R).$$

Therefore, choosing  $c < \sqrt{b}$  we can conclude as in Proposition 7.5.

2) Since we have the validity of (7.19), it is sufficient to show that one of the equivalent hypotheses of Proposition 6.9 of [8] is satisfied to show that (7.3) is oscillatory. As in the previous proposition, we show that

$$\limsup_{r \to +\infty} \frac{\int_{R}^{r} \sqrt{A(s)} ds}{\int_{R}^{r} \sqrt{\chi_{h}(s)} ds} > 1$$

holds. By (7.21) we deduce

$$\begin{split} \limsup_{r \to +\infty} \frac{\int_R^r \sqrt{A(s)} ds}{\int_R^r \sqrt{\chi_h(s)} ds} &\geq \lim_{r \to +\infty} \frac{\int_R^r \sqrt{A(s)} ds}{\int_R^r \sqrt{\chi_h(s)} ds} \\ &\geq \lim_{r \to +\infty} \frac{\int_R^r \sqrt{A(s)} ds}{\int_R^r c \sqrt{\widetilde{\chi_h}(s)} ds} \\ &\geq \lim_{r \to +\infty} \frac{Car^\gamma \log^\beta(r)}{\frac{c}{2} ar^\gamma \log^\beta(r) - \frac{c}{2} aR^\gamma \log^\beta(R)} > 1. \end{split}$$

Hence, choosing  $C \ge \frac{c}{2} + \varepsilon$ ,  $\varepsilon > 0$ , we conclude that (7.3) is oscillatory by Proposition 6.9 of [8].

## 7.2. A Kazdan-Warner type obstruction

We now look for obstructions, on a compact manifold with empty boundary, to the existence of a positive solution u of the equation

(7.25) 
$$u\operatorname{Ric}^{\varphi} - \operatorname{Hess}(u) + \Delta uq = uE,$$

for a 2-covariant, symmetric tensor field E on M. Note that the first equation in (7.1) can be written in the form (7.25), with

(7.26) 
$$E = (\mu + p)q.$$

From equation (5.67), we obtain the validity of the next

**Lemma** 7.8. Let A be a 2-covariant, symmetric Codazzi tensor on M and X a vector field. Define the vector field Z of components

(7.27) 
$$Z_j = [P_{k-1} \circ A]_{ij} X_i,$$

where

$$[P_{k-1} \circ A]_{ij} = (P_{k-1})_{it} A_{tj} - \frac{1}{m} \operatorname{tr}(P_{k-1} \circ A) \delta_{ij}$$
$$= (P_{k-1})_{it} A_{tj} - \frac{k}{m} S_k \delta_{ij}.$$

Then,

(7.28) 
$$\operatorname{div} Z = \frac{m-k}{m} (S_k)_j X_j + \frac{1}{2} (\mathcal{L}_X g)_{ij} [P_{k-1} \circ A]_{ij}.$$

PROOF. Since A commutes with the Newton operator,

$$[P_{k-1} \circ A]_{ij} = [P_{k-1} \circ A]_{ji}.$$

Using (5.67), the fact that A is Codazzi and that, under this assumption,  $P_k$  is divergence free, we obtain

$$\begin{aligned} \operatorname{div} Z = & X_j \Big\{ [P_{k-1} \circ A] \Big\}_{ji,i} + X_{ji} [P_{k-1} \circ A]_{ij} \\ = & \frac{m-k}{m} (S_k)_j X_j + \frac{1}{2} (X_{ji} + X_{ij}) [P_{k-1} \circ A]_{ij} \\ = & \frac{m-k}{m} (S_k)_j X_j + \frac{1}{2} (\mathcal{L}_X g)_{ij} [P_{k-1} \circ A]_{ij}. \end{aligned}$$

**Remark** 7.9. From here we already obtain an interesting conclusion; indeed, suppose that X is a conformal vector field; then (7.28) reduces to

(7.29) 
$$\operatorname{div} Z = \frac{m-k}{m} g(X, \nabla S_k) = \frac{c_k}{m} g(X, \nabla \sigma_k),$$

which reminds us of the Kazdan-Warner-Pohozaev formula. Indeed, the above identity has been proved by Han, [37], for A the classical Schouten tensor, to generalize the Kazdan-Warner-Pohozaev formula to higher order symmetric functions.

We have

**Proposition** 7.10. Let (M,g) be a compact Riemannian manifold with possibly empty boundary. Let A be a 2-covariant, symmetric, Codazzi tensor on M and X a conformal vector field. Then, for  $1 \le k \le m-1$ ,

(7.30) 
$$\frac{m-k}{m} \int_{M} X(S_k) = -\int_{\partial M} \left[ P_{k-1} \circ A \right]_{ij} X_i \nu_j,$$

where  $\nu$  is the inward unit normal to  $\partial M$ .

Observe that for k=1 and Ric Codazzi, that is (M,g) a harmonic manifold, equation (7.30) yields

(7.31) 
$$\frac{m-1}{m} \int_{M} X(S) = -\int_{\partial M} \mathring{\operatorname{Ric}}(X, \nu).$$

The latter does not coincide with the classic Kazdan-Warner formula obtained without restrictions on Ric and that reads

(7.32) 
$$\frac{m-2}{2m} \int_{M} X(S) = -\int_{\partial M} \mathring{\text{Ric}}(X,\nu);$$

however, with this choice of A (A = Ric) and for k = 1 it is possible to avoid to suppose that Ricci is Codazzi: in fact, due to Schur's identity we have

$$\mathring{R}_{ij,i} = R_{ij,i} - \frac{S_j}{m} = \frac{m-2}{2m} S_j,$$

that gives the validity of (7.32) just proceeding as in the proof of Lemma 7.8. Let now  $A^{\varphi}$  be the  $\varphi$ -Schouten tensor and  $u \in C^{\infty}(M)$  a positive solution of

(7.33) 
$$u\operatorname{Ric}^{\varphi} - \operatorname{Hess}(u) + \Delta ug = uE,$$

for some 2-covariant, symmetric tensor E. We choose the vector field

$$X = \nabla u$$
,

so that

$$\mathcal{L}_X g = 2 \text{Hess}(u).$$

We set, for some  $V \in C^{\infty}(M)$ ,

$$A = A^{\varphi} - \frac{V}{m-1}g.$$

From (7.28) and (7.33) we obtain

(7.34) 
$$\operatorname{div} Z = \frac{m-k}{m} g(\nabla S_k, \nabla u) + u R_{ij}^{\varphi} [P_{k-1} \circ A]_{ij} - u E_{ij} [P_{k-1} \circ A]_{ij}.$$

By the definitions of  $A^{\varphi}$  and A we get

$$R_{ij}^{\varphi} = A_{ij}^{\varphi} + \frac{S^{\varphi}}{2(m-1)} \delta_{ij}$$
$$= A_{ij} + \frac{1}{m-1} \left[ V + \frac{S^{\varphi}}{2} \right] \delta_{ij},$$

thus, using (5.59) and (5.61), we obtain

$$\begin{split} uR_{ij}^{\varphi}[P_{k-1} \circ A]_{ij} &= uA_{ij}^{\varphi}[(P_{k-1}) \circ A]_{ij} \\ &= u\bigg\{A_{ij}(P_{k-1})_{it}A_{tj} - A_{ij}\frac{k}{m}S_k\delta_{ij}\bigg\} \\ &= u\bigg\{S_k\delta_{tj}A_{tj} - (P_k)_{tj}A_{tj} - \frac{k}{m}S_kS_1\bigg\} \\ &= u\bigg\{\frac{m-k}{m}S_1S_k - (k+1)S_{k+1}\bigg\}. \end{split}$$

Assuming

$$E = \lambda g$$

then (7.34) becomes

(7.35) 
$$\operatorname{div} Z = \frac{m-k}{m} \nabla u(S_k) + \left\{ \frac{m-k}{m} S_1 S_k - (k+1) S_{k+1} \right\} u$$
$$= \frac{(m-1)!}{k!(m-k-1)!} \{ \nabla u(\sigma_k) + m(\sigma_1 \sigma_k - \sigma_{k+1}) u \}.$$

Therefore, we have the following

**Theorem** 7.11. Let (M,g) be a compact Riemannian manifold of dimension  $m \geq 2$  with empty boundary and let  $u \in C^{\infty}(M)$  be a solution of

(7.36) 
$$\begin{cases} u \operatorname{Ric}^{\varphi} - \operatorname{Hess}(u) + \Delta u g = \lambda g, \\ u > 0, \end{cases}$$

where  $\lambda \in C^{\infty}(M)$ . Let  $V \in C^{\infty}(M)$  and let  $\sigma_k$  be the k-th normalized symmetric function in the eigenvalues of the 2-covariant symmetric tensor

$$A = A^{\varphi} - \frac{V}{m-1}g,$$

for some  $1 \le k \le m-1$ , and assume that A is Codazzi. Then

$$\int_{M} \nabla(u)(\sigma_{k}) = -\int_{M} m(\sigma_{1}\sigma_{k} - \sigma_{k+1})u.$$

Furthermore, for k = 1 or  $k \geq 2$ ,  $\sigma_k$  positive and A with positive eigenvalues at some point  $p \in M$ , then

$$\int_{M} \nabla u(\sigma_k) \le 0.$$

Here the last inequality is due to Lemma 5.58 and u > 0. In case of a  $\varphi$ -SPFST we can choose

$$\lambda = \mu + p, \qquad V = U(\varphi)$$

and thus A is Codazzi if and only if (5.56) is satisfied, that is,

$$C_{ijk}^{\varphi} = \frac{U^a}{m-1} (\varphi_k^a \delta_{ij} - \varphi_j^a \delta_{ik}).$$

Corollary 7.12. Let (M,g) be a compact  $\varphi$ -SPFST of dimension  $m \geq 2$  with empty boundary. Let

$$A = A^{\varphi} - \frac{U(\varphi)}{m-1}g$$

and suppose that

$$C^{\varphi} = -\frac{1}{2(m-1)}\operatorname{div}_1(U(\varphi)g \otimes g).$$

Let  $\sigma_k$  be the k-th normalized symmetric function in the eigenvalues of A, for some  $1 \le k \le m-1$ . Then

$$\int_{M} \nabla(u)(\sigma_{k}) = -\int_{M} m(\sigma_{1}\sigma_{k} - \sigma_{k+1})u.$$

Furthermore, for k = 1 or  $k \geq 2$ ,  $\sigma_k$  positive and A with positive eigenvalues at some point  $p \in M$ , then

$$\int_{M} \nabla u(\sigma_k) \le 0,$$

where equality holds if and only if  $\varphi$  is constant and (M,g) is isometric to a Euclidean sphere.

PROOF. We only have to prove the last part of the statement. In particular, we have

$$\int_{M} \nabla u(\sigma_{k}) = -\int_{M} m(\sigma_{1}\sigma_{k} - \sigma_{k+1})u = 0.$$

Then, Lemma 5.58 implies

$$\sigma_1 \sigma_k - \sigma_{k+1} = 0$$

and the eigenvalues of A coincide. The proof now continues as the one of Theorem 5.11 to give the desired conclusion.

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