The Laplace Equation

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9/15/22

Abstract

This lab tested the strength of using the Laplace Equation to predict the voltage across a conductor where the boundary constraints are known. This method of voltage calculation was compared to a geometric approximation called the Finite Difference Method. While both were successful in predicting the general contour and magnitude of the voltage drops across the conductive sheet, it was found that in this case, the finite difference method more successfully described the measured data.

1 Introduction

The Laplace Equation is a useful tool in describing the behavior of static electric fields. When the current is constant, the gradient, applied twice to the potential necessarily equals zero:

$$\nabla\varphi^2 = 0$$

Given a boundary condition that is known, this second-degree partial differential equation can often be solved (geometry permitting!). This technique of solving for the potential of an electric field is tested against measured data, and the success of the descriptiveness is compared to an alternative geometric solution called the Finite Difference Method that can be simply applied in excel when exploiting the symmetries of these particular boundary conditions.

2 Procedures

In this lab, the electric potential across a conductive sheet is measured when the electric potential at the 'boundaries' of the sheet are known. A conductive sheet is surrounded by two different conductors with known electric potential:

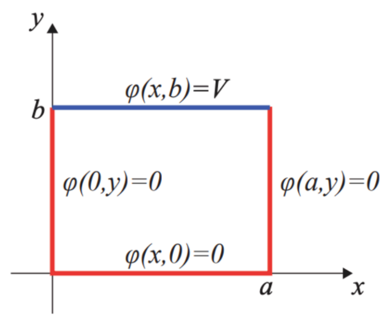

Figure 1: A visualization of the conductive sheet used in the lab.

The parameters, a (the width of the setup) = 28cm, and b (the height) = 20 cm. The voltage maintained at the top conductor is 19.62. A regular 2D grid of 19x27 is created with a conductive sheet and various measurements are taken spaced at 1cm apart.

Using these boundary conditions and applying the Laplace Equation, the following solution is found:

$$\varphi(x, y) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi y}{a} \sin\frac{n\pi x}{a}$$

Where:

$$A_n = \frac{4V}{\pi(2n-1)\sin\left(\frac{2n-1}{a}\pi b\right)}$$

3 Data & Analysis

The electric potential data were collected at every point and are represented on the plot in Figure 2.

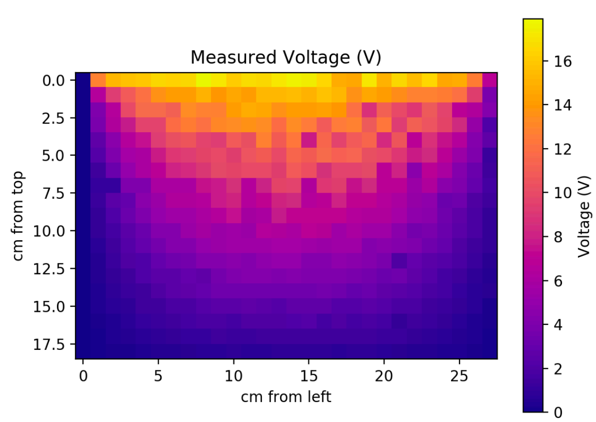

Figure 2: Electric potential data at each point.

Computationally, the analytic solution was solved. To verify that there were a sufficient number of terms to estimate the solution, the average estimate at 3 points (a/3, b/3), (a/2, b/2), and (2a/3, 2b/3) were found using different n number of terms, and the solutions were plotted.

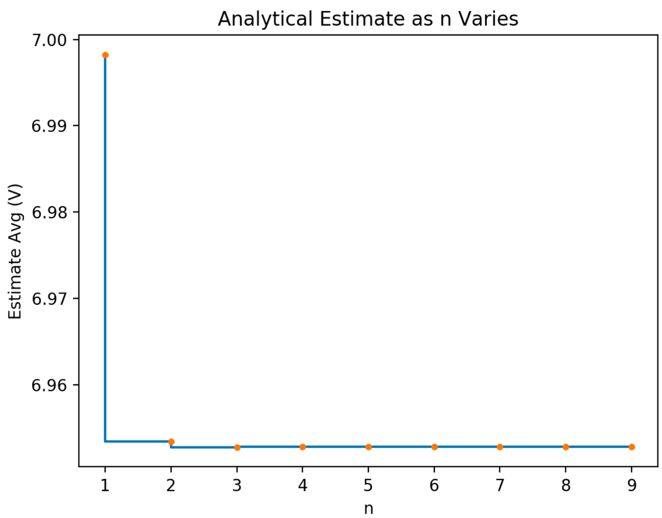

Figure 3: A plot showing the average estimate for three different points using n terms to calculate the estimate.

For n > 3, the benefit of adding more terms to the model is marginal.

Note: 10 terms were chosen as the maximum test number instead of simply 3 as using 10 did not change the computing time and should, in theory, be a better estimate of the analytic solution. This method of choosing n an estimate, not a proof, but it provides sufficient evidence that n>3 is sufficient for the purposes of this lab.

The analytic solution is plotted:

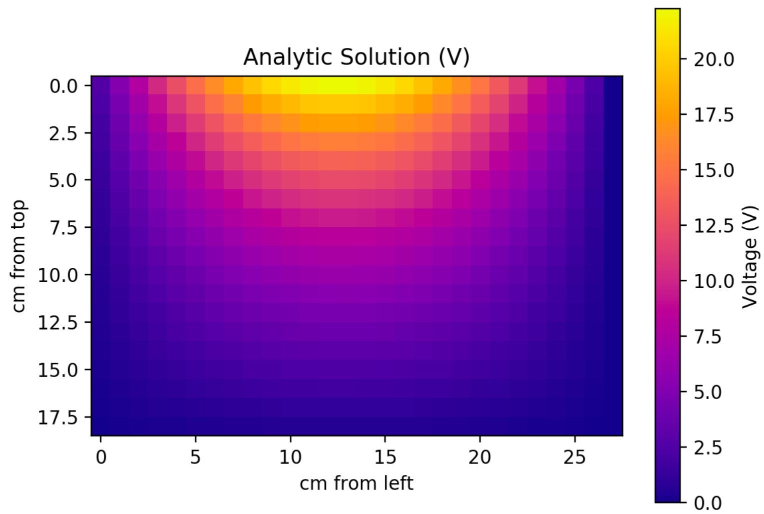

Figure 4: Analytic solution for the voltage over position for the set up provided in Figure 1.

It is somewhat self-evident from this plot alone that there exists a systematic issue in the analytical solution, perhaps due to some incorrect scaling of the position vectors, although investigation into the code did not reveal that as an issue, so the possible issue being a lab-collection error is more likely. And from the excel averaging technique, the solution of the method of Finite Difference is plotted below:

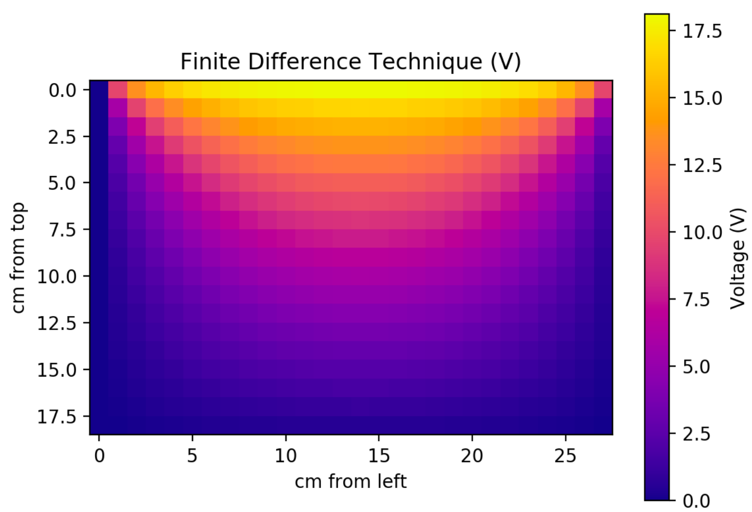

Figure 5: Finite Difference Technique solution for the voltage over position for the set up provided in Figure 1.

Then to get a visual of the success of each of these models at describing our collected data, the difference with the measured data is taken for each and plotted again.

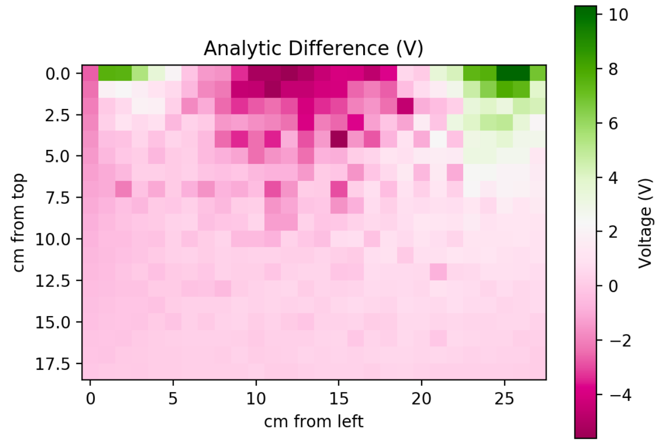

Figure 6: Residuals between analytic solution and raw data.

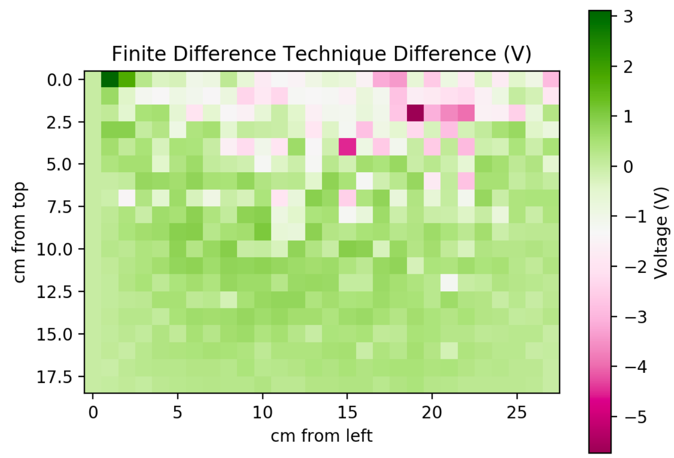

Figure 7: Residuals between Finite Difference Technique solution and raw data.

The scale of the voltage as seen on the color bar is significantly reduced which suggests that these models at least are reasonably descriptive of the behavior of the electric potential. The analytic model seems to have some significant problems as the position approaches the top of the plot. While at 10cm from the top and below are very close to zero, the data above that begins to deviate significantly, and there does seem to be a discernable pattern in the error which suggests that there exists some systematic reason the data do not match the model. For some points, the error even approaches 10V.

Were the model a better fit, the error would be both minimized and also apparently random across space closer to what is seen in the differences of the Finite Difference Technique plot. Since there is less of a pattern in these errors, the Finite Difference Technique seems to be a better solution for the current data. It is more likely that these errors come from little deviations in where the voltage was measured relative to the cross or waiting inconsistent times to get measurements across data points.

Histograms of the error are created in each case:

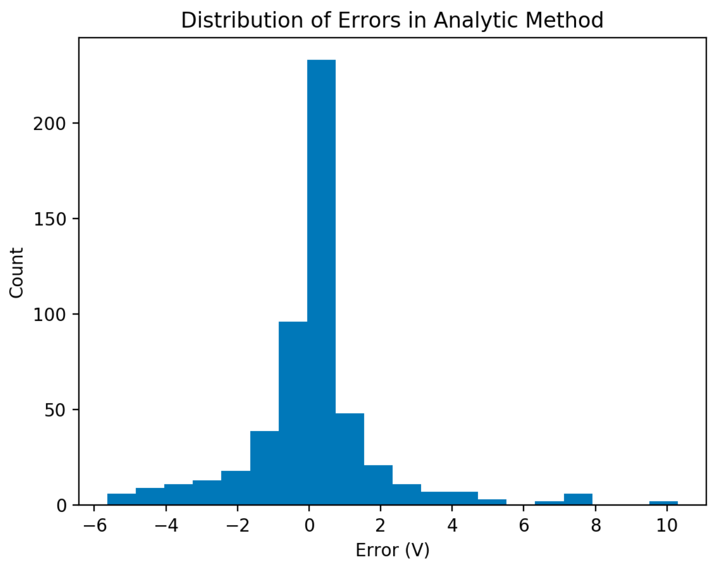

Figure 8: Histogram of residuals between the analytical solution and raw data.

The residual error in the analytic method does seem to look symmetric about zero, and despite a few outliers, seems to be contained mostly in error less than magnitude 5V. This might artificially suggest that the error is more statistical than systematic, but the pattern seen in the error of this method in Figure 6 undercuts this argument. It seems possible that the top of the analytic solution should more closely align with the boundary condition. The reason it does not is not obvious. Perhaps, it has to do with these solutions based in sines do not do as well at the edges (I am prompted to think of the "edges" of a Fourier Series approximation of a square wave). Whatever the reason, the apparent symmetry of the distribution of errors seems to be hidden when looking at Figure 8 out of context. The magnitude of the height towards the center does seem to go significantly higher than a normal distribution might suggest, so while the error might for some systematic reason be symmetric, this plot is not necessarily highly correlated to a normal distribution.

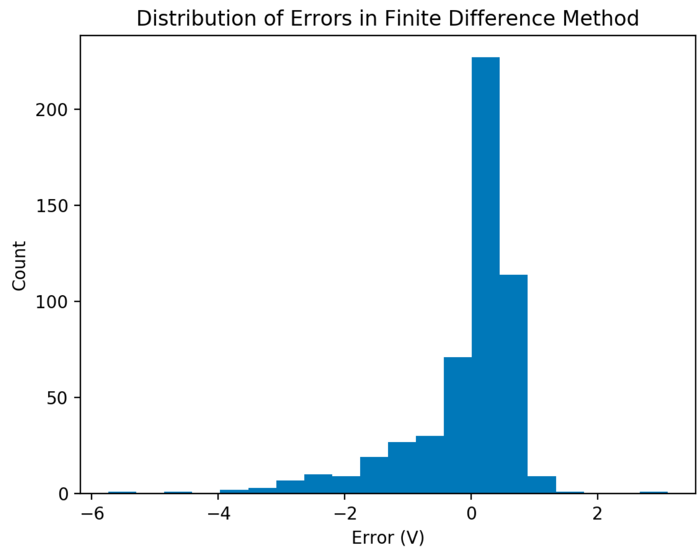

Figure 9: Histogram of residuals between the Finite Difference Solution and raw data.

Figure 9 certainly reveals a non-normal, left skewed distribution, which suggests some non-statistical error. This could have to do with some subset of measurements never fully reaching the total voltage that it could due to either impatience in the measuring process or damage to the conducting sheet. The skew might also come from the fact that this averaging technique is simply an approximation and might just be a skewed estimate of the true behavior in itself. It's worth noting that the range of errors in this plot is significantly smaller and closer to zero than the analytical solution.

Assuming the error to be constant 0.1V, χ^2 is calculated for both sets of errors.

The $\chi^2 \approx 19297$ for the Finite Difference Technique, and the $\chi^2 \approx 4514$ for the analytical solution. These numbers are not helpful until the degrees of freedom are taken into account and the reduced χ^2 or χ^2_v.

Dividing by the degrees of freedom, one finds that for the Finite Difference Technique $\chi^2_v = 36.3$, and for the analytical technique $\chi^2_v = 8.5$. Both of these numbers are large enough to suggest that the error deviates significantly from the expectation, likely implying that systematic error is present.

It is also worth noting that while this experiment was approximating electrostatic conditions, current was still flowing. This was necessary because it is impossible to make measurements of electric potential without any flow of current. Future work could include updating the theory to account for the flow of current and not relying on electrostatic assumptions.

4 Conclusion

The results seem to suggest that while the Laplace Equation solution is appropriate in approximating the electric potential in space (at the least this solution looked visually/graphically appealing), if a simple symmetry in the problem can be exploited for a geometric solution, this might be a better (and simpler!) approach to describing the behavior of electric potential.

References

[1] Griffiths, D.J., *Introduction to Electrodynamics, Third Edition*, Prentice Hall, Upper Saddle River, 1999.

[2] Jackson, J.D., *Classical Electrodynamics, Second Edition Wiley*, New York, 1975.