Chapter 2: Sets - The Lego Bricks of Mathematics

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"No one shall expel us from the paradise that Cantor created."
– David Hilbert

1. Introduction

This is the amazing thing about mathematics: The whole discipline is based on only **two** undefined notions, and these are:

- sets, and
- set membership $\in$.

Compare this to physics which comes with the notions of *time*, *distance*, *mass*, *electric charge*, and others, which are regarded as fundamental and are not defined in other terms. Lawyers are faced with an even more complicated situation, since basic notions such as *guilt*, *cause*, or *judgement*[1] are left undefined and are the source of difficult philosophical problems.

Set theory as a basis of the whole of mathematics was created[2] by **Georg Cantor** (1845-1918). The concept of a *set* is really intuitive for us as we implicitly use it in daily life whenever we speak of *teams*, or categorise *collections* of things into subcollections (for instance all living beings belong to one of a few *domains* such as the domain of animals, plants, fungi, or bacteriae).

Cantor himself said of the concept of a set:

> A *set* is a well-defined collection C of distinct objects, called the *elements* of C, and we consider C as an object in its own right.

2. Notation

In order to write down a set, we use the *bracket notation* $\{\ldots\}$. For instance the set F of the members of the Federal Council in Switzerland [3] can be written as

$$F = \{\text{Amherd, Sommaruga, Keller-Sutter, Cassis, Parmelin, Maurer, Berset}\}.$$

[1] a necessary condition for someone to be able to be found guilty.
[2] Some mathematicians, the "Platonists", would replace "created" by "discovered".
[3] as of July 2020

The order in which the elements are listed is of no importance; the set stays the same. Even if we listed a federal councillor twice, this changes nothing in the set - it stays the same.

Using the *set membership relation* $\in$ we can write for example

$$\text{Sommaruga} \in F.$$

(Despite all efforts we still have van der Zypen $\notin F$).

2.1. Special sets.

- $\varnothing$: the *empty set* contains no elements; alternative notation: $\{\}$.
- $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of *natural numbers*.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of the *integers*.
- $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ und } b > 0\}$, the set of *rational numbers* or *fractions*.
- $\mathbb{R}$, the set of *real numbers*.

2.2. **Describing sets.** We can write down the set F of all 7 federal councillors, as we did above, but this is impossible to do for the set E of all even natural numbers. We could write informally $E = \{2, 4, 6, \ldots\}$, but the "…"-part gives us an unsatisfied feeling. But there is a better way:

$$E = \{n \in \mathbb{N} : \exists k \in \mathbb{N}(n = 2k)\}.$$

We read the colon (:) above as "such that". So the descriptive form above reads as: E equals the set of all natural numbers n such that there exists a natural number k such that $n = 2k$ – which exactly captures the idea of an even number.

Exercise 2.1. Write the set U of odd (= non-even) numbers in 2 different ways using the descriptive form introduced above.

Exercise 2.2. Give a set description of the set of all prime numbers.

3. Subset relation $\subseteq$

Sometimes the elements of one set are contained in another, bigger set. For instance, the set H of all humans is contained in the set M of all mammals. In this case we say that the sets A, B are in a subset relationship. More precisely: If A, B are sets, we write

$$A \subseteq B$$

if every element of A is an element of B. In that case we also say that A *is contained* in B.

The sets A and B are *equal* if they contain exactly the same elements. We can formulate this using the subset relation:

$$A = B \iff (A \subseteq B) \land (B \subseteq A).$$

A is not a subset of B if we can find an element that is in A, but not in B. For instance the set F of the seven Federal Councillors is not a subset of the set S of the members of the Socalist Party ("SP") as Mrs Keller-Sutter is not a socialist.

Note that the empty set $\varnothing = \{\}$ is a subset of *every* set A: it is impossible to find an element in $\varnothing$ that is not contained in A, because there are no elements in $\varnothing$!

Example 3.1. Consider the set of the female members of the federal council:

$$F' = \{\text{Amherd, Sommaruga, Keller-Sutter}\}$$

and let's list all the subsets of F':

$$\begin{array}{l} \varnothing, \\ \{\text{Amherd}\}, \{\text{Sommaruga}\}, \{\text{Keller-Sutter}\}, \\ \{\text{Amherd, Sommaruga}\}, \{\text{Amherd, Keller-Sutter}\}, \{\text{Sommaruga, Keller-Sutter}\}, \\ F'. \end{array}$$

Exercise 3.2. For what sets A and B does at least one of the subset relationships $A \subseteq B$ or $B \subseteq A$ hold?

(1) $A =$ set of all humans, $B =$ set of all vertebrates;
(2) $A =$ set of employees of the Swiss Armed Forces, $B =$ set of people who have served at least 3 days in the Armed Forces;
(3) $A =$ set of the odd numbers, $B =$ set of the prime numbers;
(4) $A =$ set of the odd numbers ≥ 3, $B =$ set of the prime numbers ≥ 3;

Exercise 3.3. For every natural number $n \in \mathbb{N}$ let V_n be the set of all multiples of n, that is,

$$V_n = \{a \in \mathbb{N} : \exists k \in \mathbb{N}(k \cdot n = a)\}.$$

Show that for all $m, n \in \mathbb{N}$:

$$V_n \subseteq V_m \text{ if and only if } m|n \text{ (that is, } m \text{ divides } n).$$

Exercise 3.4. Consider the set F of all 7 members of the Federal Council: How many subsets does F have?

4. Sets of sets of sets of ...

Every employee of the company "FantasyCorp" belongs to a team. Each team is the set of all employees who work in that team.

Now we can build a *set of sets*: the set $\mathcal{T}$ of all teams in FantasyCorp. Note that no element of $\mathcal{T}$ is a person, the elements of $\mathcal{T}$ are teams, which are abstract constructs as opposed to people. So $\mathcal{T}$ has less elements than the set E of all employees of FantasyCorp employees. Again, $\mathcal{T}$ is made up of teams, and E is made up of people, so the sets $\mathcal{T}$ and E are completely different!

This line of thought will be important later.

5. Set operations

Given sets A, B we can form new sets from them in the following ways:

- *Intersection*: $A \cap B = \{x : x \in A \wedge x \in B\}$;
- Die *Union*: $A \cup B = \{x : x \in A \vee x \in B\}$ – Note: the logical "oder" means that x is an element of A or B, *or both*!
- Die *Set difference*: $A \setminus B = \{x \in A : x \notin B\}$.

We want to get familiar with these set operations by doing some exercises:

Exercise 5.1. Let $A = \{\text{Lisa, Mark, Tom, Sara}\}$ and $B = \{\text{Anna, Tom, Lisa}\}$. What do the following sets look like?

(1) $A \cap B$, and compare to $B \cap A$;
(2) $A \setminus B$, and compare to $B \setminus A$;
(3) $A \cup B$, and compare to $B \cap A$;
(4) $\{\text{Tom, Mark}\} \setminus A$?

Exercise 5.2. Consider the following subsets of $\mathbb{Q}$:

$A = \{x \in \mathbb{Q} : x < \frac{1}{4}\}$ and
$B = \{x \in \mathbb{Q} : x > \frac{1}{5}\}$.

Name an element of each of the following sets:

(1) $A \setminus B$,
(2) $B \setminus A$,
(3) $A \cap B$.

Can you find an element in $\mathbb{Q} \setminus (A \cup B)$?

Exercise 5.3. Let $n \in \mathbb{N}$, and let V_n be the set of multiples of n.

(1) What numbers belong to $V_2 \cap \{1, 2, \ldots, 11\}$?
(2) Compare $V_2 \cap V_3$ to V_6.
(3) Does the set $\mathbb{N} \setminus (V_2 \cap V_3)$ contain infinitely many numbers?

6. Operations on sets of sets

Let $\mathcal{A}$ be a set of sets, that is, each element of $\mathcal{A}$ is a set as well. We define:

(1) $\bigcup \mathcal{A} = \{x : \exists A \in \mathcal{A}(x \in A)\}$, und
(2) $\bigcap \mathcal{A} = \{x : \forall A \in \mathcal{A}(x \in A)\}$.

We call $\bigcup \mathcal{A}$ the *union* of all sets that are members of $\mathcal{A}$, and $\bigcap \mathcal{A}$ the *intersection* of all sets that are members of $\mathcal{A}$.

Exercise 6.1. Let P denote the set of members of the Swiss Parliament. Some members of P work in certain *committees*. Each committee K is a subset of P that is devoted to a special subject such as public health, defense, and so on, so $K \subseteq P$. Let $\mathcal{K}$ be the set of all committees. Note that $\mathcal{K}$ does not consist of members of Parliament, but of *committees*, which are not persons, but *sets of* persons! Who is contained in the following sets?

(1) $\bigcap \mathcal{K}$;
(2) $\bigcup \mathcal{K}$;
(3) $P \setminus (\bigcup \mathcal{K})$?

Exercise 6.2. For $n \in \mathbb{N}$ we define the set of *proper multiples* of n by

$$W_n = \{x \in \mathbb{N} : \exists k \in \mathbb{N}(k > 1 \wedge k \cdot n = x)\}.$$

Let $\mathcal{W} = \{W_n : n \in \mathbb{N} \wedge n \geq 2\}$. Moreover, let P be the set of all primes. Compare P to

$$\mathbb{N} \setminus \left(\bigcup \mathcal{W}\right).$$

Is one set contained in the other? Or are they even equal?

7. Additional homework

1. Let $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}$. What elements are in $(A \setminus B) \cup (B \setminus A)$?

2. Let A, B be sets (finite or infinite). How many elements does $(A \setminus B) \cap (B \setminus A)$ contain?

3. A school with 20 students offers 2 optional subjects, A and B. Subject A is taken by 10 students, and subject B by 8 students. Moreover, 6 students chose both A and B. How many students do not take any of the optional subjects A or B?

4. Again, the same school with 20 students offers the subjects A and B, Subject A is taken by 10 students, and subject B by 8 students. But this time, we do not know how many (if any) students attend *both* A and B. Without this information, how many students *at least* take one of the subjects A or B? And how many *at most*?

5. Consider the following sets:

$$A = \{1, \{2\}, \{1, 2\}, 3\}$$
$$B = \{\{\}, \{1\}, 2, 3\}$$
$$C = \{1, 2, 3\}$$
$$D = \{\{1, 2\}, 3, 4, 5\}.$$

Which of the following statements are true? Please give a short argument for your answer.

- $C \subseteq B$.
- A has 4 elements.
- $A \cap B = C \cap D$.
- $C \cup D$ has 32 subsets.

6. Consider the following sets:

$$
\begin{aligned}
&E = \{\{\}, 1, \{1\}\}\\
&F = \{1, 2, \{1, 2\}\}\\
&G = \{\{1\}, \{1, 2\}\}\\
&H = \{\{\}\}.
\end{aligned}
$$

(1) How many elements do the sets E, F, G, H have?
(2) Answer the following questions:

$$
\begin{aligned}
&E \cap F = ?\\
&E \cup (F \cap G) = ?\\
&(H \cup E) \setminus (G \setminus F) = ?\\
&H \cup (E \setminus (G \setminus F)) = ?
\end{aligned}
$$

7. Let A, B be sets. For each of the following statements, prove it or disprove it with a counterexample:

(1) $A \cup B = B \cup A$.
(2) $A \subseteq A \setminus B$.
(3) $B \subseteq A \setminus B$.
(4) $A \setminus B \subseteq A$.

8. Let A, B be sets. For each of the following statements, prove it or disprove it with a counterexample:

(1) $A \cup (B \setminus A) = A \cup B$.
(2) $A \cap (B \setminus A) = A \cap B$.

9. Let A, B be sets. Prove the following *distributive law*:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

10. *DeMorgan's Law.* Let X be a se, and let $\mathcal{A}$ be a set of subsets of X, which means tha for all $A \in \mathcal{A}$ we have: $A \subseteq X$. Then show that:

(1) $X \setminus \left(\bigcup \mathcal{A}\right) = \bigcap \left\{X \setminus A : A \in \mathcal{A}\right\}$.
(2) $X \setminus \left(\bigcap \mathcal{A}\right) = \bigcup \left\{X \setminus A : A \in \mathcal{A}\right\}$.