Chapter 6: Introduction to Combinatorics

Dominic van der Zypen
dominic.zypen@gmail.com

"When angry, count to 10 before you speak. If very angry, count to 100."
– Thomas Jefferson

1. Preliminary remarks

Counting is a basic activity that we all are familiar with since early childhood. But counting can get tricky as the following example shows:

> A hacker wants to crack a PIN. Via social engineering, the perpetrator gained the knowledge that all the digits are strictly ascending.[1] How many numerical PINs are there with this property? And how many PINs does the perpetrator have to try if he knows the first digit is `3`?

The mathematical branch called *combinatorics* is exactly about giving correct answers to questions such as the above. Moreover, in computer science you want to do estimations of memory or time needed to perform a certain program, and this is another important field for combinatorial methods.

2. Addition principle

Let's start with an introductory example.

Example 2.1. Let's assume that a restaurant offers 12 vegetarian meals and 9 non-veggie meals. If we can pick exactly one meal, in how many ways can we do that?

Solution. It is easy to see that we have to do the calculation $12 + 9 = 21$ to reach the correct solution.

> The **addition principle** says that if an event A can occur in m ways and event B can occur in n ways, and A and B are *disjoint* (that is, they cannot occur at the same time), then then event "A or B" can occur in $m + n$ ways.

[1] An example of such a PIN is `134689` or `024589`, but not `146589` because of `'65'`, or `235779` because of number repetition.

Example 2.2. We consider 2-letter combinations like `xy` or `am`. How many of these start with the letters `'a'` or `'b'`?

Solution. There are 26 combinations starting with `a`, that is, the 2-letter-strings `aa`, `ab`, ... `az`. With the same argument we get another 26 strings starting with `b`. Now use the addition principle because the *disjointness condition* is trivially met: No 2-letter string starts with `a` *and* with `b`! So we get the solution

$$26 + 26 = 52.$$

Exercise 2.3. How many 4-digit PIN codes (the "lowest" being `0000` and te "highest" being `9999`) start with one of the digits `3,4` or `5`?

We give another formulation of the addition principle in terms of set theory:

Addition principle for sets: Let A, B be disjoint sets, that is $A \cap B = \varnothing$. Then
$$|A \cup B| = |A| + |B|.$$

To emphasise the notion of disjointness for sets, here it is again:

Definition 2.4. Two sets A, B are said to be *disjoint*, if

$$A \cap B = \varnothing,$$

that is, A and B have no common elements.

3. Multiplication principle

Consider the following example.

Example 3.1. My summer wardrobe is relatively small: I have 5 shorts and 13 T-shirts. In how many ways can I combine these two kinds of garment?

Solution. If I pick my favourite pair of shorts, the orange one, I can pick one of 13 T-shirts to go with it. I can do the same thing with shorts number 2, the blue one, that I also love (but not as much as the orange one). Again, I can pick 13 T-shirts to go with the blue shorts. I can do the same for all my 5 pairs of shorts, so I get in total

$$5 \cdot 13 = 65$$

ways of combining shorts and T-shirts.

The **multiplication principle** says that if event A can occur in m ways and for *every* occurence of A, event B can occur in n ways, then the event "A and B" can occur in $m \cdot n$ ways.

Exercise 3.2. Some time ago, the Scotland's license plates consisted of 3 letters followed by 3 digits. How many such plates are possible?

We can also formulate the multiplication principle in the language of set theory using Cartesian products (see Course 3):

> **Multiplication principle for sets.**
> If A, B are finite sets, then
> $$|A \times B| = |A| \cdot |B|.$$

4. Inclusions-exclusion principle

Again we start with an introductory example.

Example 4.1. Remember the sports club from course 4 about `SQL`? Well, all these people love sports, and at the most recent club event, people were asked to choose their preferred sports:

- Tennis (A)
- Football[2] (B)
- Land hockey (C)

At the annual gathering of the club, the members are asked to mark the sports from A, B, C that they like. (Evey member can mark several sports, and they are allowed to mark none of A, B, C). This is the table of choices:

Sport liked	A	B	C	AB	AC	BC	ABC
Number of members	20	13	26	9	15	7	5

How many people like at least one of the sports A, B, C? The answer is **not** 95, the sum of the numbers above. Please try to find the solution for yourself and then watch the video with the solution to this example.

4.1. Only 2 sets.

If we know that a set A contains 10 elements, and B contains 8 elements, we cannot make a precise statement about

$$|A \cup B|,$$

the number of elements that are in at least one of A and B.

- It is conceivable that A, B are disjoint and therefore have no common elements. So we can use the *addition principle* and conclude that
$$|A \cup B| = |A| + |B| = 18.$$
- On the other hand it could also be the case that $B \subseteq A$ (that is, B is a *subset* of A, so all 8 elements of B already lie in A (which has 10 elements), which would imply that
$$|A \cup B| = |A| = 10.$$

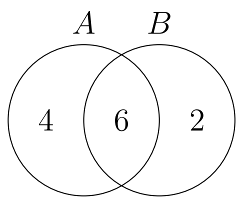
[2] The "real" one, not American football

Without additional information we cannot answer the question for $|A \cup B|$, the number of elements of $A \cup B$.

But if we know that A and B have 6 common elements (i.e. $|A \cap B| = 6$) then we can form disjoint sets and use the *addition principle*:

> If $A \cap B$ contains 6 elements, then $A \setminus B$ (the set of elements lying in A but not B) has 4 elements, because A contains 10 elements in total. The same argument leads us to $|B \setminus A| = 2$. So
>
> $$|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A| = 4 + 6 + 2 = 12,$$
>
> because all the sets involved are disjoint - and so the addition principle applies.

The most intuitive way to solve this problem is a pictorial one with a so-called **Venn diagram**:



But there is another way to reach the correct answer of 12:

> If we add the number of elements $|A| = 10$ und $|B| = 8$, we get 18, but then we counted the common elements of A and B, that is the elements of $A \cap B$, **twice**. So let's *subtract* them again and get $18 - 6 = 12$.

Exercise 4.2. Give an argument (or just convince yourself) that these two methods of counting always lead to the same result.

This second method can be expressed in a compact rule:

> **Inclusion-exclusion principle for 2 sets.**
> If A, B are finite sets, then
> $$|A \cup B| = |A| + |B| - |A \cap B|.$$

The name *inclusion-exclusion principle* tries to capture the notion that first we count the common element of A and B twice (inclusion), and then "kick them out again" (exclusion).

The more sets are involved, the more complicated this inclusion-exclusion process becomes.

Exercise 4.3. (*) Prove the inclusion-exclusion principle for 3 finite sets A, B, C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

5. Binomial coefficients

Let's start with an example again:

Example 5.1. There are 12 kinds of salad at a salad buffet. The standard menu includes choosing exactly 3 kinds of salad. How many choices of 3 kinds of salad out of 12 are there?

We will give a solution to this in a moment.

First we are going to introduce a number called "binomial coefficient", but more in a "semantic" and not mathematical way. Then we deduce how to calculate binomial coefficients and apply it to the above problem.

> **Binomial coefficient.**
> For ever non-negative integer $n \geq 0$ and every integer k mit $0 \leq k \leq n$ we define a natural number
> $$\binom{n}{k},$$
> read as "n choose k", which denotes the number of ways to choose k objects out of n objects.

Remark. We defined the number $\binom{n}{k}$ only *by its meaning (semantically)*, but not explicitly.

We do this step-by-step via a method called *recursion*. This is used often in programming and computer science, and the subject of recursion is worth an extra paragraph.

5.1. **Recursion.**

The process of *recursion*[3] means reducing a problem to a smaller problem[4]. So if we want to bake two cakes for our cousin's birthday (a lot of people with different tastes are expected!) we have to know how to bake cakes in the first place, how the oven works, and so on.

Here's how we define *factorials* via recursion - we will need them later on in the chapter on permutations.

Example 5.2. For every number $n \in \mathbb{N}$ we define the *factorial* $n!$ (read: "n factorial") as follows:

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1.$$

Note that we can calculate $5!$ as $5! = 5 \cdot 4!$. We can generalise this and write it down as a *recursion*:

- $n! = n \cdot (n-1)!$ for $n \geq 2$. That is the reduction of the case of n to the case of $n-1$ and it is also called the *recurrence relation*.

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[3] lat *re-currere*, "to run back, to run several times"

[4] "Inside every problem, there is a smaller problem trying to get out." – Arthur Cohn, *Murphy's Law Complete*

- $1! = 1$. You can't go down from n to $n-1$ for every - at some point this has to stop. So we have to give a concrete value for $n = 1$, and this is said to be the *initial condition.*

5.2. **Recurrence relaton for $\binom{n}{k}$.** So far we have defined $\binom{n}{k}$ only semantically. Recall that $\binom{n}{k}$ is the number of ways to choose exactly k objects out of n objects.

We are ready for a nifty trick!

Fix a "favourite object" amongst the n objects and call it x^*.

Now we can k objects out of n objects in 2 mutually exclusive ways:

- Case 1: x^* belongs to the k objects that we pick. So from the remaining $n-1$ elements we have to choose $k-1$ to complete our selection of k elements i total.
- Case 2: x^* does not belong to the k objects that we pick. So we pick k elements from the remaining $n-1$ elements.

Note that

(1) The number of ways to make a selection of k elements according to case 1 is $\binom{n-1}{k-1}$
(2) The number of ways to make a selection of k elements according to case 2 is $\binom{n-1}{k}$.

Since cases 1 and 2 are mutually exclusive, we can use the *addition principle* and we get

$$\underbrace{\binom{n}{k}}_{\text{Total number of selectios}} = \underbrace{\binom{n-1}{k-1}}_{\text{Case 1: } x^* \text{ in selection}} + \underbrace{\binom{n-1}{k}}_{\text{Case 2: } x^* \text{ not in selection}}$$

So we know how to reduce $n, k > 0$ to $n-1$ and $k-1$, that is, the recurrence relation. But we still have to formulate the *initial condition* for the simple reason we cannot go down from n, k to $n-1, k-1$ forever!

In how many ways can we choose $k = 0$ objects out of n objects? In exactly 1 way: do nothing! So we get

$$\binom{n}{0} = 1 \text{ for all } n \geq 0,$$

and this is our initial step. Taking stock we get:

Recursion for $\binom{n}{k}$.
For every natural number $n \geq 1$ and every natural number k with $1 \leq k \leq n$ we have:

- (Recurrence relation:) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $n \geq k \geq 1$;
- (Initial condition:) $\binom{n}{0} = 1$ for all $n \geq 0$.

Exercise 5.3. Using the above recurrence relation and the initial condition, determine:

(1) $\binom{1}{0}$,

(2) $\binom{2}{0}$,
(3) $\binom{2}{1}$,
(4) $\binom{n}{1}$ for arbitrary $n > 2$,
(5) $\binom{3}{2}$,
(6) $\binom{4}{2}$,
(7) $\binom{5}{2}$,
(8) $\binom{5}{3}$.

We will derive an explicit formula for $\binom{n}{k}$ later on.

6. Permutations

This section answers the following question:

> In how many ways can we arrange n in a row?

The key is in the *multiplication principle* introduced before: To occupy slot number 1 in our row, we have n ways to pick the first object. For every choice of object number 1, we have $(n-1)$ ways to pick object number 2 to go into slot 2. So there are $n(n-1)$ ways to occupy the first 2 slots in the row. It's easy to see how to continue: There are

$$n(n-1)(n-2) \cdot \ldots \cdot 1 = n!$$

ways to arrange all n objects in a row. Arranging objects in a row such that order matters is also referred to as *permutations*. For permutations we also use the notation $P(n)$, so $P(n) = n!$.

7. Variations with repetition

There are 26 letters and 10 digits that are used to form a valid British license plate. How many strings of length 7 (= format of the British number plates) can be made with these 36 characters? Answer:

$$36^7 \simeq 7.8 \cdot 10^{10},$$

which corresponds to 78 Billion (that's about 10 number plates for every human on Earth). Note that again we used the multiplication principle.

When order matters, we talk about "*variations*".

In general: We can form n^k variations with length k, using n symbols.

8. Variations without repetition

This is a generalisation of the concept of permutation. Given n objects and a fixed natural number $k \leq n$, in how many ways can we pick k objects out of the n objects and arrange them in a row (so that order matters)? Answer:

$$\underbrace{n(n-1)(n-2)\cdot\ldots\cdot(n-k+1)}_{k \text{ factors in total}}$$

Using the factorial we can write ths in a more compact way:

$$n(n-1)(n-2)\cdot\ldots\cdot(n-k+1) = \frac{n(n-1)(n-2)\cdot\ldots\cdot(n-k+1)\cdot\overbrace{(n-k)(n-k-1)\ldots\cdot 2\cdot 1}^{=(n-k)!}}{(n-k)(n-k-1)\ldots\cdot 2\cdot 1}$$
$$= \frac{n!}{(n-k)!}$$

This number of arrangements is called *variations*, and we use the notation $V(n,k)$, also $V(n,k) = \frac{n!}{(n-k)!}$.

9. Combinations (without ordering): Explicit formula for $\binom{n}{k}$

Returning to binomial coefficients, we were interested in the number of ways to choose k objects out of n objects *without caring about the ordering.* (This is often the case when we count the number of teams of a given size that we can form; we usually don't care about the order, you just want to count the number of selections.)

Exercise 9.1. Show every selection of k objects out of n objects is counted precisely $k!$ times if we use $V(n,k)$.

The statement of the exercise is the kay for an explicit formula for counting the selections of k objects out of n objects: divide $V(n,k)$ by $k!$ because we counted every selection $k!$ times.

So we get

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{1\cdot 2\cdot\ldots\cdot k}.$$

The number of ways to choose k objects out of n objects is called *combinations* and we use the notation $C(n,k)$, also $C(n,k) = \binom{n}{k}$.

10. Exercises

1. You want to spend a movie night with a friend who insists on watching some of his DVD collection. He has 12 romance movies, 4 dramas und 9 crime movies. You want to watch 3 movies, one each out of every genre (romance, drama, crime). In how many ways can you select the movies?

2. We have two sets A, B and we know that $|A| = 10$ and $|B| = 14$.

   (1) What is the greatest possible value of $|A \cap B|$?
   (2) What is the smallest possible value of $|A \cap B|$?

(3) What is the range of possible values for $|A \cup B|$?

3. We consider a company and their employees. Let P the set of employees speaking *Portuguese*, R is he set of employees speakig *Romanian*, and S is the set of employees speaking *Spanish*. Moreover, we know:

(1) $|P| = 50, |R| = 45, |S| = 40$,
(2) $|P \cap R| = 20, |P \cap S| = 15, |R \cap S| = 23$,
(3) $|P \cap R \cap S| = 12$.

Problems:

(1) Describe $(P \cup S) \setminus R$ in your own words: what set of employees does this set contain?
(2) Determine $|(P \cup S) \setminus R|$.

4. The prime factorisation of $735,000$ is

$$735,000 = 2^3 \cdot 3 \cdot 5^4 \cdot 7^2.$$

Determine the number of divisors of $735'000$.

5. My "formal" wardrobe consists of 5 shirts, 4 trousers and 12 ties. How many outfits can I create with these garments (assuming I always wear 1 of each kind of garment)?

6. A *binary string* is a sequence of 0's and 1's such that order matters. For example `0100111` is a binary string of length 7.

(1) How many binary strings of length 7 are there?
(2) How many of these strings do not contain the same digit twice in a row? (In other words, they do not contain `00` or `11` anywhere.)
(3) How many binary strings contain `00` or `11` at least once?
(4) How many binary strings of length 7 end in `1100`?
(5) How many binary strings of length 7 end in `1100` or `0011`?

7. How many natural number ≤ 1000 are a multiple of $3, 5$ or 7? Hint: Inklusion-exclusion principle.

8. In the Subway restaurant[5] you can assemble sandwiches by picking at most 3 ingredients (like onions, pickles, ...) out of 11 possible ingredients.

(1) In how many ways can you put your sandwich together if you pick exactly 3 ingredients?
(2) In how many ways can you put your sandwich together if you pick *at least* 1 ingredient (but at most 3)?

[5] I don't want to advertise for Subway's, but I do love it very much!

(3) In how many ways can you put your sandwich together if you want 3 ingredients, but please no onions because you have a meeting afterwards?
(4) In how many ways can you put your sandwich together if you want 3 ingredients, and this time one of the ingredients *has* to be onions because you want to make sure people keep the distance at the meeting?

9. For each of the following properties below, determine how many binary strings there are having length 8.

(1) The number of `1`'s they contain is 5.
(2) The number of `1`'s they contain is 5 and they end with `101`.
(3) They start with `11` or end in `101`, or both.
(4) They start with `11` or end in `101`, or both; and the number of `1`'s they contain is 5 .

10. Out of 11 employees we want to form a special team containing *at least* 3 employees. In how many ways can we do this?

11. (*) During a boring business meeting I empty my wallet and find 6 CHF-1 coins and 4 CHF-2 coins. In how many ways can I arrange these 10 coins in a row?

12. Answer the question at the top of page 1 of this course.

On the next page you find a cheat sheet, a summary of all the definitions and notations introduced in this combinatorics course.

11. Cheat Sheet

Formula	**Explanation**	**Example**
$A \cap B = \varnothing \implies$ $\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert$	Addition principle	I have 4 polo shirts and 5 other T-shirts. How many shirts in total?
$\lvert A \times B\rvert = \lvert A\rvert \cdot \lvert B\rvert$	Multiplication principle	I have 7 T-shirts and 5 jeans. In how many ways can I combine these garments?
$\lvert A \cup B\rvert =$ $\lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$	Inclusions-exklusion principle	If A has 4 elements and B has 6 el. and they have 3 el. in common. $\lvert A \cup B\rvert = ?$
$P(n) = n!$	Number of *permutations* of n objects such that order doe matter	How many 3-digit numbers without repeating a digit can be formed with $3, 4, 5$?
n^k	Variations w/o repetition	Length 7 strings ($k = 7$) out of 8 ($n = 8$) symbols.
$V(n,k) = \frac{n!}{(n-k)!}$	Variations with repetition	8 runners of a sprint are given 3 medals Gold, Silver, Bronze. How many possibilities?
$C(n,k) = \binom{n}{k}$ $= \frac{n(n-1)\ldots(n-k+1)}{1\cdot 2\cdot\ldots\cdot k}$	Combinations / Choices of k objects out of n objects *without order*	How many 4-member committees out of 200 members of parliament are possible?

Remark. Some calculators and smartphone apps have keys for $C(n,k)$ und $V(n,k)$ - so you don't need to enter the complicated formulas.