Chapter 4: Basics of Graph Theory

Dominic van der Zypen
dominic.zypen@gmail.com

"The origins of graph theory are humble, even frivolous."
– R J. Wilson et al, *Graph Theory: 1736-1936*

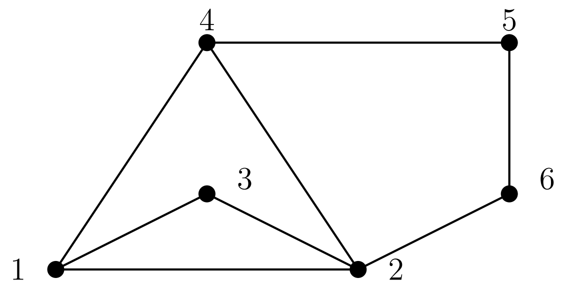
1. Introduction

A graph is a mathematical object that is particularly accessible to intuition. It consists of a set of *points* or *vertices* (we will use these terms interchangeably), and a set of *connections*, also called *edges*; every edge links up two points. Some pairs of points are connected by an edge, others aren't.

Graphs arise in many situations of daily life:

(1) Electricity distribution centers as vertices, and power lines connecting them
(2) Cities and direct train connections (without changing trains) between cities
(3) Computer networks
(4) Acquaintances between people
(5) Connections in social networks

This is how we depict a graph:



If we interpret these vertices as cites, for instance, and the edges as direct train connections, we see that there is a direct connection between cities 2 and 6. But if you want to get from city 3 to city 6 then you have to change trains at least once (if you pass by city 2), and you have to change at least twice if you want to geet from 3 to 5.

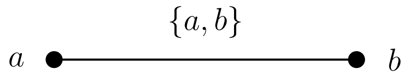
Graph theory was created by the great Swiss mathematician *Leonhard Euler* (1707–1783). When he spent some time in the city of Königsberg, he wanted to take a walk leading him exactly once over each of Königsberg's seven bridges[1]. Euler found an elegant argument proving that no such walk was possible - and graph theory was born.

2. Set of unordered pairs of a set

If X is a set, we denote by $[X]^2$ the set of all subsets of X that contain exactly 2 elements. We also call this the set of all unordered pairs. Formally, we write

$$[X]^2 = \big\{\{a, b\} : (a, b \in X) \wedge (a \neq b)\big\}.$$

So what do we use this for? The simple answer is: in graph theory we write a connection (edge) between two vertices a, b as the unordered pair $\{a, b\}$:



Example 2.1. Let $X = \{i, j, k\}$. Here is $[X]^2$ in explicit form:

$$[X]^2 = \big\{\{i, j\}, \{i, k\}, \{j, k\}\big\}.$$

We could have chosen another order:

$$[X]^2 = \big\{\{j, i\}, \{k, j\}, \{k, i\}\big\},$$

No matter what order you pick: order doesn't matter in sets!

Here's a question for the above example $X = \{i, j, k\}$: What about the set $\{i, i\}$, do we have

$$\{i, i\} \in [X]^2?$$

No, because $\{i, i\}$ only contains i as its only element, so

$$\{i, i\} = \{i\}.$$

But only sets with 2 elements are "admitted to the club" $[X]^2$, so

$$\{i\} = \{i, i\} \notin [X]^2.$$

Exercise 2.2. For $X = \{1, 2, 3, 4\}$ write down $[X]^2$.

Example 2.3. Let $X = \{1, 2, 3\}$. An example of a 2-element subset of X is $\{1, 3\}$. From course 2 (Set Theory) we know that order doesn't matter: the information stored in a set is only which elements are contained in a set - but order is irrelevant. So

$$\{1, 3\} = \{3, 1\}.$$

Exercise 2.4. If $X = \{1, \ldots, n\}$, how many elements does $[X]^2$ have?

Please try to solve the exercise yourself before looking up the solution below.

[1]for viewing the setup of Königsberg's brigdes at that time, google "seven bridges of Königsberg"

2.1. **Solution to exercise.** Let $X = \{1, \ldots, n\}$. Every element of $[X]^2$ has the form $\{a, b\}$, where a, b are members of $X = \{1, \ldots, n\}$. First, let's pick only one element: ther are n possible ways to pick a number from $\{1, \ldots, n\}$.

And how many ways are there to pick the 2nd member of the 2-element set? $n - 1$ ways because we pick a different number from the one already chosen.

So for every way of choosing the first element (n ways), there are $n-1$ ways of picking the 2nd element, so we end up with the

$$n(n-1)$$

ways of choosing both elements.

But wait...! In that way we count every 2-element set *twice*. we count $\{1, 3\}$ once, and $\{3, 1\}$ once - but they are the same!

So we have to divide the number $n(n-1)$ by two to compensate for this and end up with

$$\frac{n(n-1)}{2}$$

as the number of elements of $[X]^2$.

One last remark: Do we always get an integer if we divide $n(n-1)$ by 2?

Answer: Yes - because exactly one of the numbers n and $n-1$ is even (that is, divisible by 2), so $n(n-1)$ is even and we can divide by two without worry.

3. Formally correct definition

A graph G is an ordered pair[2] $G = (V, E)$, consisting of two sets:

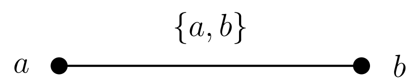
1. (1) V, the set of *vertices*
2. (2) E, the set of *edges*.

So this is the definition:

Definition 3.1. A *graph* G is an ordered pair $G = (V, E)$, where $V \neq \varnothing$ is a (non-empty) set, and $E \subseteq [V]^2$.

That's it!

So we see that the concept of a *connection* between vertices $a, b \in V$ is represented by the 2-element set $\{a, b\} \in E$:

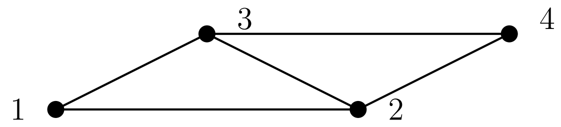


Exercise 3.2. Draw the following graph $G = (V, E)$:

[2] refer to Course 3 (Functions and Relations), section 1 (Ordered pairs)

(1) $V = \{1, 2, 3, 4, 5\}$ und $E = \big\{\{1, 2\}, \{2, 3\}, \{2, 4\}\big\}$;
(2) $V = \{a, b, c, d, e\}$, $E = \big\{\{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}\big\}$;

Exercise 3.3. Write down the sets V and E for the graph depicted below:



If $G = (V, E)$ is a graph on the vertex set V, and we are given $a, b \in V$ such that

$$\{a, b\} \in E,$$

that is, a, b form an *edge*, then we also say that a, b are *neighbours*.

Definition 3.4. If $G = (V, E)$ is a graph and $v \in V$ is a vertex, then we define the *neighbourhood* of v in G by

$$N(v) = \{w \in V : \{v, w\} \in E\}.$$

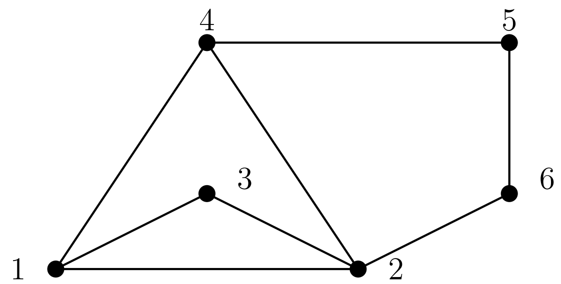
The *degree* of v is the number of neighbours:

$$\deg(v) = |N(x)|.$$

(Recall: if X is a finite set, then $|X|$ is the number of elements of X.)

Note that v is **not** a neighbour of itself, since $\{v, v\} = \{v\}$, and E only contains 2-element sets, so $\{v, v\} = \{v\} \notin E$!

Example 3.5. Consider the following graph:



We have

(1) $N(1) = \{2, 4\}$ and therefore $\deg(1) = 2$,
(2) $N(2) = \{1, 3, 4, 6\}$ and therefore $\deg(2) = 4$,
(3) $N(6) = \{2, 5\}$ and therefore $\deg(6) = 2$.

Exercise 3.6.

(1) For the graph of exercise 3.2 write down the neighbourhood $N(1)$ and determine $\deg(1)$.

(2) For the graph of exercise 3.3 write down the neighbourhood $N(4)$ and determine $\deg(4)$.

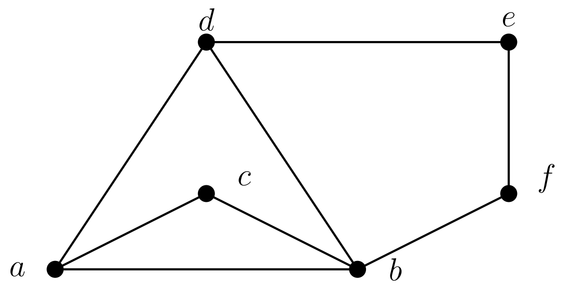
Next we define the important concept of a *sub*-graph. Given a graph $G = (V, E)$ and a subset $V' \subseteq V$ then n the smaller vertex set V' we let $a, b \in V'$ be connected if these points are already connected in the "parent" graph G. More formally:

Definition 3.7. If $G = (V, E)$ is a graph and $V' \subseteq V$ then

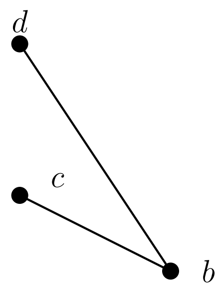
$$(V', E \cap [V']^2)$$

is the *(induced) subgraph on the set* V' of G.

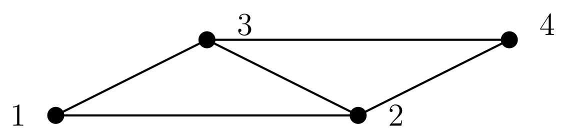
Example 3.8. Consider the following graph on the vertex set $V = \{a, b, c, d, e, f\}$:



Let $V' = \{b, c, d, e\}$. Then the induced subgraph on V' looks like this:



Exercise 3.9. Consider the following graph on the vertex set $V = \{1, 2, 3, 4\}$:



Let $V' = \{1, 3, 4\}$. Draw the subgraph induced by V'.

Exercise 3.10. Let $G = (V, E)$ be given by:

$$V = \{a, b, c, d, e\},\ E = \big\{\{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}\big\}.$$

Let $V' = V \setminus \{d\}$ and let $G' = (V', E')$ be the subgraph induced by V'. Write down the set $E' = E \cap [V']^2$ explicitly.

Definition 3.11. A graph $G = (V, E)$ is said to be *complete*, if $E = [V]^2$, that is if every point is connected to every other point. The complete graph on the set $\{1, \dots, n\}$ is denoted by K_n.

Exercise 3.12. Let $n > 1$ be an integer.

(1) How many edges does K_n have?
(2) Let $k \in \{1, \dots n\}$. Determine $N(k)$ and $\deg(k)$.
(3) Does an induced subgraph of a complete graph have to be complete again?

Exercise 3.13. (*) An infinite graph! We say that two distinct members $m, n \in \mathbb{N} \setminus \{1\}$ form an edge if their only common divisor is 1. Formally we let [3]:

$$E = \big\{\{m, n\} : m, n \in V \wedge (\forall k \in \mathbb{N} \setminus \{1\}) : \neg((k \mid m) \wedge (k \mid n))\big\},$$

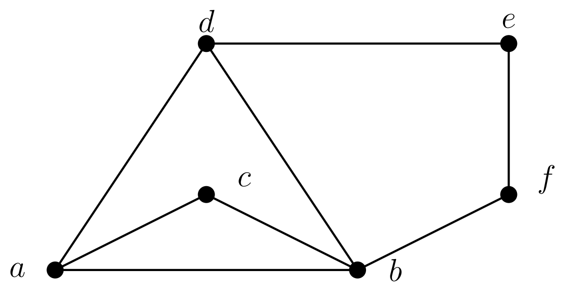
where $k \mid n$ means that k divides n[4]. So a 2-element set $\{m, n\} \in [\mathbb{N}]^2$ forms an edge if and only if there is no integer $k > 1$ that divides **both** m and n.

Let P be the set of primes $\{2, 3, 5, 7, 11, \dots\}$. Let $G = (\mathbb{N} \setminus \{1\}, E)$. Is the induced subgraph on $P \subseteq \mathbb{N} \setminus \{1\}$ complete?

4. Paths and connectivity

A *path* in a graph is conceptually exactly what you would imagine: a walk from vertex to vertex going along an edge for ever step.

Consider this graph:



We can walk from c to e via the following paths of different length:

(1) $c \to b \to f \to e$,
(2) $c \to a \to d \to e$,
(3) $c \to b \to c \to b \to a \to d \to e$.

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[3]please make sure you understand this formalism

[4]The expression $k \mid n$ can be expanded to $(\exists a \in \mathbb{N}) : ak = n$

There are infinitely many more paths. We see that even the shortest path from c to e needs 3 edges, and so we say that the *distance* between c and e is 3. We use the notation

$$d(c, e) = 3.$$

This is how we define the notion of a path formally:

Definition 4.1. Let $G = (V, E)$ be a graph, and let $v \neq w \in V$ and $n \in \mathbb{N}$. Then a map

$$p : \{1, \ldots n\} \to V$$

is said to be a *path from v to w*, if

(1) $p(1) = v$ and $p(n) = w$, that is, the beginning and end points of p are v and w, respectively, and
(2) $\{p(k), p(k+1)\} \in E$ for all $k \in \{1, \ldots, n-1\}$, that is we move along edges.

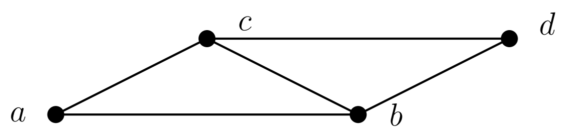
We call the number $n - 1$ the *length* of p by $n - 1$, because we move along $n - 1$ edges.

Example 4.2. In the graph used just before, one possible path from c to e is given by

$$n = 4,\ p : \{1, 2, 3, 4\} \to V \text{ wth } p(1) = c, p(2) = b, p(3) = f, p(4) = e.$$

The length of this path is $n - 1 = 3$.

Exercise 4.3. In the following graph, find a path of length 2 from a to d, and another path having length 3.

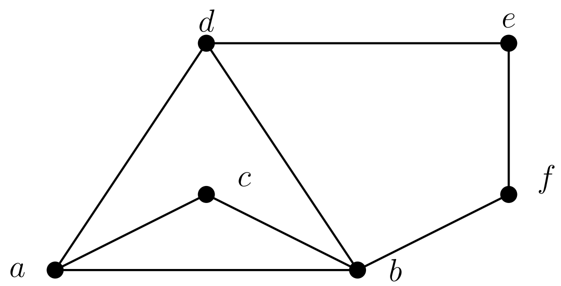


Using the concept of a path, we can easily define the notion of a connected graph:

Definition 4.4. A graph $G = (V, E)$ is said to be *connected*, if for every choice of two distinct vertices $v \neq w \in V$ there is a path from v to w.

If a graph is not connected then it splits up into several maximally connected bits which we call *connected components* - another very intuitive concept.

Exercise 4.5. Does the following graph always stay connected if we remove one connection (edge)?



How about removing 2 connections, or 3 connections, can we always keep the resulting graph connected?

If no, into how many components does the resulting graph split?