Chapter 3: Relations and Functions

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"Confusion is the most productive state that you can be in; the struggle out of that state is the primary drive for progress."

– Dror Bar-Natan

1. Ordered pairs

We recall from the last course: in a set such as {Andrea, Bernd}, the order does not matter. It is the same set as {Bernd, Andrea} because it contains the same members.

In some contexts however, we would like to have "sets with order": If we want to express that Andrea is Bernd's supervisor, then we cannot say this using the set {Andrea, Bernd}, because: how would we know from this set alone about who is the supervisor of whom?

To address this problem, there is the construction of the *ordered pair*. We use the notation

$$(x, y).$$

There, the order matters: if $x \neq y$, then we have $(x, y) \neq (y, x)$.

If you want to know exactly how the ordered pair is defined in the language of set theory: Given any objects x, y (numbers, persons, ...) we define the ordered pair (x, y) by

$$(x, y) = \big\{\{x\}, \{x, y\}\big\}.$$

Exercise 1.1. Prove that $(x, y) = (a, b)$ if and only if $x = a$ and $y = b$.

In particular, we have (Andrea, Bernd) $\neq$ (Bernd, Andrea), and $(0, 1) \neq (1, 0)$ and so on, so *the order does matter.*

2. Cartesian products

If A, B are sets, then the *Cartesian product* von A and B is the set of all ordered pairs such that the first entry is a member of A, and the second entry is a member of B, and we write

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

Example 2.1. If $X =$ {Andrea, Bernd} and $Y =$ {brown, black}, then $X \times Y =$ (Andrea, brown), (Andrea, black), (Bernd, brown), (Bernd, black).

Exercise 2.2. Let $P = \{\text{Alice, Bob, Charlie}\}$ be a small set of people sharing a house. If you live together, there are invariably regular tasks to be done. Let's list a small subset of these tasks: $T = \{\text{wash dishes, vacuum-clean, bring out the garbage, clean the toilet}\}$. Write down the set $P \times T$ explicitly. How many elements does $P \times T$ contain?

Exercise 2.3. If X is a set, what is $X \times \varnothing$?

Exercise 2.4. Let $A = \{1, 2, 3\}$. Write down $A \times A$ explicitly.

3. Relations

Let A, B be sets. We now know what $A \times B$ looks like.

Definition 3.1. A **Relation** R between two sets A, B a subset of $A \times B$, so

$$R \subseteq A \times B.$$

Example 3.2. We go back to the people sharing a house: The set of the people living there is $P = \{\text{Alice, Bob, Charlie}\}$, and the set of the regular tasks consists of washing dishes, vacuum-cleaning, bringing out the garbage, and cleaning the toilet, as defined in exercise 2.2. A typical member of the Cartesian product $P \times T$ has the form (p, t) haben, where p is a household member, and t a task. For example we have

$$(\text{Charlie, bring out the garbage}) \in P \times T.$$

(Again, take note of the order: the pair (bring out the garbage, Charlie) is not in $P \times T$!)

We define the "responsibility" relation $R \subseteq P \times T$ by:

$$R = \big\{(\text{Bob, clean toilet}), (\text{Alice, vacuum-clean}), (\text{Bob, vacuum-clean})\big\}.$$

Each member (p, t) of the relation R means that person p is responsible for getting task t done on a regular basis

Let's have a closer look at R. First we check that every element of R belongs to $P \times T$, so that indeed $R \subseteq P \times T$. Then we note that diligent Bob is responsible for two tasks, while lazy Charlie gets nothing done. Moreover, the task of vacuum-cleaning seems to be such a heavy burden that Alice and Bob share responsibility for it. And still, not all the tasks get done

Exercise 3.3. In the above example, which task(s) are being left out?

Exercise 3.4. Let $A = \{1, 2, 3\}$ and $B = \{9, 10, 11\}$. Consider the following relation $R \subseteq A \times B$:

$$R = \{(a, b) \in A \times B : a + b \in B\}.$$

Tasks:

(1) Enumerate all ordered pairs (members) of the relation R.
(2) Is there $(m, n) \in R$, such that its "mirror pair" (n, m) is an element of R as well?

Exercise 3.5. In the last excercise we looked at a relation on sets A, B such that $A \neq B$. Now we consider a relation of a set A with itself. Let $A = \{1, 2, 3, \ldots, 10\}$ and let $R \subseteq A \times A$ be defined by

$$R = \{(a, b) \in A \times A : \exists k \in \mathbb{N}(k \cdot a = b)\}.$$

Tasks:

(1) Describe R in your own words.
(2) Explicitly list all of the elements of R.
(3) Is the following true?
If $(a, b), (b, c) \in R$, then $(a, c) \in R$.

Exercise 3.6. We consider the set $\mathbb{N}$ of natural numbers and define a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ by

$$R = \big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \exists k \in \mathbb{N}(m - n = 6 \cdot k)\big\}.$$

Tasks:

(1) Describe in your own words what kind of ordered pairs are members of R.
(2) Give an example of natural numbers x, y such that $(x, y) \in R$.
(3) Is there $n \in \mathbb{N}$ such that $(1, n) \in R$?
(4) What is the smallest $n \in \mathbb{N}$ such that $(1, n) \in R$?
(5) What is the smallest $n \in \mathbb{N}$ such that $(1, n) \notin R$?
(6) Is there a largest natural number m such that $(1, m) \in R$?
(7) Is there a largest natural number m such that $(1, m) \notin R$?

3.1. **Composition of relations.** Relations can be composed or “applied after each other”, in the following manner.

Let A, B, C be sets and let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations. Then we define a relation $T \subseteq A \times C$ by:

$$T = \big\{(a, c) \in A \times C : \exists b \in B\big((a, b) \in R \wedge (b, c) \in S\big)\big\}.$$

Intuitively speaking you “get” from A to B via R, and then from B to C via S. The relation T introduced above is written

$$S \circ R.$$

Please note the counterintuitive order...! It will be justified when we learn about the composition of functions, which is really a special case of the composition of relations.

Exercise 3.7. Let $A = \{p, q\}, B = \{1, 2, 3\}, C = \{11, 12\}$. Calculate the composition $S \circ R$ for the relations $R \subseteq A \times B$ and $S \subseteq B \times C$ as given below:

(1) $R = \{(p, 1), (p, 2)\}$, $S = \{(2, 11), (3, 12)\}$.
(2) $R = \{(p, 1), (q, 1)\}$, $S = \{(2, 11), (3, 12)\}$.
(3) $R = \{(q, 2), (p, 1), (p, 3)\}$, $S = \{(1, 13), (3, 12), (2, 11), (3, 11)\}$.

Exercise 3.8. Let A, B, C be sets, and suppose that $R \subseteq A \times B$ and $S \subseteq B \times C$ are relations such that $S \circ R = A \times C$. Does this automatically imply that $R = A \times B$ or $S = B \times C$?

The remainder of this section about relations is optional.

If X is any set, a relation $R \subseteq X \times X$ is said to be

- *reflexive*, if $(x, x) \in R$ for all $x \in X$,
- *symmetric*, if for all $x, y \in X$ we have: if $(x, y) \in R$, then $(y, x) \in R$, and
- *transitive*, if for all $x, y, z \in X$ we hav: if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Exercise 3.9. Let V be the set of all employees of the Swiss Armed Forces. We say that an ordered pair of employees $(x, y) \in V \times V$ is in the relation $R \subseteq V \times V$, if x and y attended the same meeting at least once in the past. Which of the properties above (reflexivity, symmetry, transitivity) does R have?

Exercise 3.10. Consider the *divisibility relation* on $\mathbb{N}$:

$$R = \{(a, b) \in \mathbb{N} \times \mathbb{N} : (\exists k \in \mathbb{N}) k \cdot a = b\}.$$

Which of the properties above (reflexivity, symmetry, transitivity) does R have?

4. Functions

Definition 4.1. If X, Y are sets, then a relation $f \subseteq X \times Y$ is said to be a *function*, in symbols $f : X \to Y$, if the following two conditions are met:

(i) For every $x \in X$ there is $y \in Y$ such that $(x, y) \in f$. ("Left existence")
(ii) For every $x \in X$ we have: if $y \in Y$ and $(x, y) \in f$ and $z \in Y \setminus \{y\}$, then $(x, z) \notin f$. ("Right uniqueness")

Note that we can summarize the above conditions by:

for each $x \in X$ there is *exactly one* $y \in Y$ such that $(x, y) \in f$.

Instead of $(x, y) \in f$ we often write

$$f(x) = y,$$

since y is uniquely determined by x (right uniqueness).

Exercise 4.2. Let $X = \{a, b, c\}$ and $Y = \{0, 1\}$. Which of the following relations $f \subseteq X \times Y$ are functions?

(1) $f = \{(a, 0), (b, 0), (c, 1), (b, 1)\}$,
(2) $f = \{(a, 0), (b, 0), (c, 1)\}$,
(3) $f = \{(c, 0), (a, 1), (b, 0)\}$,
(4) $f = \{(a, 0), (b, 0), (c, 1), (b, 0)\}$,
(5) $f = \{(a, 0), (b, 1)\}$,
(6) $f = \{(c, 1), (b, 1), (a, 1)\}$.

Exercise 4.3. Let's go back to the house sharing example of 3.2.

(1) Argue whether the relation given there has the properties:
  (a) Left existence, and/or
  (b) right uniqueness.
(2) Is there a superset $f \supseteq R$, such that f is a function?

Exercise 4.4. Draw the functions $f : \mathbb{R} \to \mathbb{R}$ given below:

(1) $f(x) = x$ for all $x \in \mathbb{R}$,
(2) $f(x) = x^2$ for all $x \in \mathbb{R}$,
(3) $f(x) = e^2$ for all $x \in \mathbb{R}$,
(4) $f(x) = \log(x)$ for all $x \in \mathbb{R}$,
(5) $f(x) = \sin(x)$ for alle $x \in \mathbb{R}$.

4.1. **Composition of functions.** In subsection 3.1 we saw how to compose relations. Now, since functions are just a special case of relations, we can compose them in exactly the same manner. We noted in that subsection that the **order** in which we write the relations may seem counterintuitive. But the reason for this ordering will be seen when doing the following exercise:

Exercise 4.5. Let X, Y, Z be sets and suppose that $f : X \to Y, g : Y \to Z$ are functions. Show that $g \circ f \subseteq X \times Z$ is a *function* again and that it can be computed by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X.$$

Exercise 4.6. Let $A = \{1, 2, 3\}, B = \{c, d\}, C = \{99, 100, 101\}$. Let $f : A \to B$ be given by $f(1) = f(2) = c, f(3) = d$, and $g : B \to C$ by $g(c) = 99, g(d) = 101$. Compute $g \circ f : A \to C$.

Exercise 4.7. Consider $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = n + 1 \text{ für alle } n \in \mathbb{N}$$

and $g : \mathbb{N} \to \mathbb{N}$ defined by

$$g(1) = g(2) = 1, g(n) = g(n-1) \text{ for all } n \in \mathbb{N} \setminus \{1\}.$$

Describe and compare the functions $g \circ f : \mathbb{N} \to \mathbb{N}$ and $f \circ g : \mathbb{N} \to \mathbb{N}$.

Exercise 4.8. For the following functions $f, g : \mathbb{R} \to \mathbb{R}$ write down the functional equations for $g \circ f, f \circ g, f \circ f$ and $g \circ g$:

(1) $f(x) = \sin(x)$ for all $x \in \mathbb{R}$,
(2) $g(x) = x^2$ for all $x \in \mathbb{R}$.

Exercise 4.9. (*) Find a function $f : \mathbb{N} \to \mathbb{N}$ with the following properties:

(1) $f(n) \neq n$ for all $n \in \mathbb{N}$,
(2) $(f \circ f)(n) = n$ for all $n \in \mathbb{N}$ (i.e., $f \circ f$ is what is called the *identity function* on $\mathbb{N}$).

5. Images and preimages of functions (optional)

Definition 5.1. Let A, B be sets and $f : A \to B$ be a function. Let $S \subseteq A$ and $T \subseteq B$ be subsets of A and B, respectively.

- The *image of* $S \subseteq A$ *under* f is defined by

$$f(S) = \{f(s) : s \in S\} = \{b \in B : (\exists s \in S) f(s) = b\}.$$

- The *preimage of* $T \subseteq B$ *under* f is defined by

$$f^{-1}(T) = \{a \in A : f(a) \in T\}.$$

Exercise 5.2. Let $X = \{a, b, c\}, Y = \{1, 2, 3, 4\}$ and let $f : X \to Y$ be given by

$$f = \{(a, 1), (b, 1), (c, 4)\}.$$

(Using the alternative notiation: $f(a) = f(b) = 1, f(c) = 4$.) Determine the following sets:

(1) $f(\{b, c\})$;
(2) $f(X)$;
(3) $f^{-1}(\{2\})$;
(4) $f^{-1}(\{2, 3\})$;
(5) $f^{-1}(\{1\})$.

Exercise 5.3. Consider the sine function $\sin : \mathbb{R} \to \mathbb{R}$.

(1) $\sin(\mathbb{R}) = ?$
(2) $\sin^{-1}(\{0\}) = ?$
(3) $\sin^{-1}(\{2\}) = ?$

Exercise 5.4. (*) Similar question for the logarithm function $\log : \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+$ denotes the set of positive real numbers (this will be relevant for the course on logarithms).

(1) $\log([1, e]) = ?$ (e is the *Euler number* 2.71.., and $[1, e] = x \in \mathbb{R} : 1 \leq x \wedge x \leq e\}$.)
(2) $\log(\mathbb{R}_+) = ?$
(3) $\log^{-1}([0, 1]) = ?$
(4) $\sin^{-1}(\{x \in \mathbb{R} : x \leq 0\}) = ?$

Exercise 5.5. Let $f : \mathbb{N} \to \mathbb{N}$ be given by

$$f(n) = 2n + 1 \text{ for all } n \in \mathbb{N}.$$

(1) $f(\{7, 8, 9, 10\}) = ?$
(2) $f^{-1}(\{7, 8, 9\}) = ?$
(3) $f^{-1}(f(\{1, 2, 3\})) = ?$

6. Additional exercises

1. Let A, B be sets and $f : A \to B$ be a function. Show:

(1) If $S \subseteq A$, then $S \subseteq f^{-1}(f(S))$.
(2) If $T \subseteq B$, then $f(f^{-1}(T)) \subseteq T$.

Give examples of sets A, B and a function f that show that we cannot replace $\subseteq$ by $=$ in the above problem statements.

2. Let X, Y, Z be sets and $f : X \to Y, g : Y \to Z$ be functions. Let $A \subseteq X$, $C \subseteq Z$. Show:

(1) $(g \circ f)(A) = g\big(f(A)\big)$;
(2) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(x^2)$ for all $x \in \mathbb{R}$. Which real numbers belong to the set $f^{-1}([0, 1])$?

Definition 6.1. If A, B are sets, then a function $f : A \to B$ is said to be

- *injective*, if for all $b \in B$ the preimage $f^{-1}(\{b\})$ consists of at most 1 element,
- *surjective*, if $f(A) = B$, and
- *bijective*, if f is injective as well as surjective.

4. Let $X = \{a, b, c\}, Y = \{1, 2, 34\}$ and let $f : X \to Y$ be defined by

$$f = \{(a, 1), (b, 1), (c, 4)\}.$$

(In alternative notation: $f(a) = f(b) = 1, f(c) = 4$.) Check whether f is injective, surjective, or both.

5. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

(1) Find a function $f : X \to Y$ that is neither injective nor surjective.
(2) Find a function $f : X \to Y$ that is both injective and surjective (= bijective).

6. Show that a function $f : A \to B$ is surjective if and only if for all $b \in B$ we have $f^{-1}(\{b\}) \neq \varnothing$.

7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by:

- $f(x) = x^2$ for all $x \in \mathbb{R}$,
- $g(x) = x^3$ for all $x \in \mathbb{R}$.

Check which of these functions are injective, surjective, or bijective.

8. Let $f : \mathbb{N} \to \mathbb{N}$ be given by

$$f(n) = n + 1 \text{ for all } n \in \mathbb{N}$$

and $g : \mathbb{N} \to \mathbb{N}$ by

$$g(1) = g(2) = 1, g(n) = g(n-1) \text{ for all } n \in \mathbb{N} \setminus \{1\}.$$

Check which of these functions are injective, surjective, or bijective.

9. (*) Let X, Y, Z be sets and $f : X \to Y, g : Y \to Z$ be functions. Show:

(1) If $g \circ f : X \to Z$ is injective, then so is f.
(2) If $g \circ f : X \to Z$ is surjective, then so is g.

Extra task: What about the converse of these implications?

10. (*) Find $f, g : \mathbb{N} \to \mathbb{N}$ such that $g \circ f : \mathbb{N} \to \mathbb{N}$ is bijective, but neither f nor g are bijective.