Chapter 5: Digraphs and hypergraphs

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"Lost in the right direction."
– Album title of the band Etherwood

1. Introduction

So far, given any 2 vertices in a graph, either there was a edge (connection), or there was no edge. These graphs are the simplest kind of graphs, and that is why they are called *simple, undirected graphs.*

1.1. **Preliminary remarks concerning directed graphs.** Sometimes you would like to put more information into an edge between two points than "these two points are connected". Imagine being at a party. We would like to draw an edge from a to b present at the party if a knows b.

But wait. What does "to know" mean? Let's say that *a knows b* if a can identify b by name. But then we run into the "prominence problem": If I happen to be at the same party as Donald Trump, I can identify him – but not vice versa. So the image

does not represent the situation in a true manner, because it insinuates that "Donald Trump and me know each other". So the following picture would represent the asymmetric nature of the situation better:

This is meant to represent the notion that I can identify Donald Trump, but *not* the other way round.

And this is what *directed graphs* are all about.

1.2. **Preliminary remarks concerning hypergraphs.** Another restriction of graphs sometimes can stop us from modelling a situation correctly. Take the Swiss Parliament for example. Some members of parliament (MPs) sit in one or several *committees*,[1] all of which consist of more than 2 MPs. But the edges that we use in conventional graph theory only can link up exactly 2 points. With hypergraphs, we do not have this restriction; an edge (sometimes also called *hyperedge*) in a hypergraph can link up an arbitrary number of points.

In the following two sections we first treat directed graphs and then hypergraphs. There are even more "cousins" of simple graphs, but these three kinds of graphs are the ones most often encountered in data science.

2. Directed graphs (digraphs)

Going back to "Dominic knows Donald Trump, but Trump does not know Dominic" we realized that in some settings *order* matters. So it is of no surprise that the *ordered pairs* that we met in Course 3 (Functions) come into play.

We model a (directed) edge between points a, b by an ordered pair

$$(a, b).$$

Moreover we exclude "self-edges" (a, a), pictorially speaking arrows that point from a point a back to a, because they often model trivial situations like "I know my own name" and don't add valuable information to a given setting.

Definition 2.1. A *directed graph* or short *digraph* is a pair $G = (V, E)$ where V is a set, called the set of vertices or points, and

$$E \subseteq \{(a, b) \in V \times V : a \neq b\}$$

is a subset of the Cartesian product $V \times V$, referred to as the set of (directed) edges.

2.1. **In-degree and out-degree.** Recall that with simple undirected graphs $G = (V, E)$ the degree of a vertex is defined as

$$\deg(v) = |N(v)| = |\{w \in V : \{v, w\} \in E\}|.$$

(If A is a set, then $|A|$ denotes the number of elements of that set.)

Now in the context of directed graphs we have a *direction* in the edges. The *in-degree* $\deg^-(v)$ of a vertex $v \in V$ is informally speaking the number of arrows ending in v and the *out-degree* $\deg^+(v)$ is the number of arrows going away from v.

Exercise 2.2. This is an exercise in formalisation / set notation: Give a set proper definition of $\deg^-(v)$ and $\deg^+(v)$. The solution is written down below, but please try it for yourself before looking up the solution.

[1] A committee is devoted to a special political subject, such as public health ("Gesundheitskommission"), security ("Sicherheitskommisson") and so on.

Solution:

$$\deg^-(v) = |\{a \in V : (a, v) \in E\}|, \text{ and } \deg^+(v) = |\{z \in V : (v, z) \in E\}|.$$

Exercise 2.3. Consider the following directed graph $G = (V, E)$:

- $V = \{2, \ldots, 30\}$,
- $E = \{(a, b) \in V \times V : (a < b) \wedge (\exists k \in \mathbb{N} : ka = b)\}$.

Problems:

(1) Draw this digraph (points and arrows).
(2) Determine $\deg^-(12)$ und $\deg^+(12)$.
(3) Find a number $n \in V$ such that $\deg^+(v) = 0$ and $\deg^-(v) > 0$.
(4) What numbers $v \in V$ have $\deg^+(v) > 0$ and $\deg^-(v) = 0$?

Exercise 2.4. (*) Show that in a directed graph $G = (V, E)$ the sum of all in-degrees taken over all vertices equals the sum of all out-degrees taken over all vertices. In a formula:

$$\sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v).$$

(*Hint.* Every arrow (directed edge) contributes exactly 1 in-degree, and 1 out-degree.)

3. Hypergraphs

Sometimes, linking up only 2 points as we can do it in simple undirected graphs, does not adequately model a situation. Whenever we deal with committees of members of parliament (MPs), with teams at work, and so on, graphs are inappropriate.

For the definition of a hypergraph we need the concept of the *power sets* $\mathcal{P}(X)$ of a set X. This is the *set of all subsets of* X.

Example 3.1. For $X = \{1, 2\}$ we get

$$\mathcal{P}(X) = \big\{\varnothing, \{1\}, \{2\}, \{1, 2\}\big\},$$

so $\mathcal{P}(X)$ has exactly 4 elements.

Exercise 3.2. Write down all elements of $\mathcal{P}(\{a, b, c\})$.

Exercise 3.3. Let $n \in \mathbb{N}$. Show that $|\mathcal{P}(\{1, \ldots, n\})| = 2^n$.

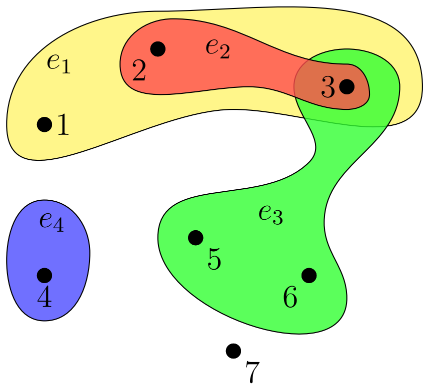
Definition 3.4. A *hypergraph* $H = (V, E)$ is an ordered pair consisting of a set V and a set E of *non-empty* subsets of V, that is,

$$E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

The elements of E are called again *edges*; another name to emphasise that we are dealing with hypergraphs is *hyperedges*. Note that a hypergraph with the property that all hyperedges consist of exactly 2 elements is a *graph*.

Example 3.5. This is how we can draw hypergraphs. Let $H = (V, E)$ with $V = \{1, 2, \ldots, 7\}$ and $E = \{e_1, \ldots, e_4\}$ where

- $e_1 = \{1, 2, 3\}$,
- $e_2 = \{2, 3\}$ (so we have $e_2 \subseteq e_1$ - this is allowed!),
- $e_3 = \{3, 5, 6\}$ and
- $e_4 = \{4\}$.



Note that poor vertex 7 is not contained in any edge.

We can define the notion of *degree* of a vertex similarly to how we did it for graphs.

Definition 3.6. If $H = (V, E)$ is a hypergraph, then the *degree* of $v \in V$ is the number of edges containing v, or formally

$$\deg(v) = |\{e \in E : v \in e\}|.$$

Exercise 3.7. Show that if a hypergraph $H = (V, E)$ is a graph (that is, all the edges have 2 elements), then this definition agrees with the definition of degree in simple, undirected graphs.

Exercise 3.8. Determine the degrees of the vertices $1, 2, \ldots, 7$ in the hypergraph of example 3.5.

Exercise 3.9. Consider the hypergraph $H = (V, E)$ defined by

- $V = \{1, 2, \ldots, 9\}$
- E is the collection of subsets T of V such that the sum of the elements of T equals 10 ist; so for instance we have $\{1, 4, 6\} \in E$.

Problems:

(1) Draw this hypergraph.
(2) What is the size of the largest edge?
(3) Determine $\deg(4)$.
(4) Is there $v \in V$ with $\deg(v) = 0$?

Exercise 3.10. (*) In a class of $N \geq 3$ students, every now and then 3 students get tasked to take care of the rooms after each lesson, cleaning the blackboard etc. Now the teacher wants to arrange these groups of 3 in such a way that every 2-set of students is together in exactly one of the care teams.

For what values of N can such an arrangement be found?