# Chapter 1: Logic in Maths and Everyday Life

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*"The whole of science is nothing more than a refinement of everyday thinking."* – Albert Einstein

## 1. Introduction

This series of courses on mathematical subjects has one main goal: your ability to handle mathematical concepts with confidence and to come up with sound mathematical arguments of your own.

A mathematical argument (also called a *proof*) consists of well-defined statements and a small set of rules of inference[1] that allow you to form new statements. We will find out that mathematical logic is firmly grounded in the kind of thinking we perform everyday – just as in Einstein's quote above.

## 2. Statements

A *statement* formulates a thought about the real world[2], abstract objects[3], or other objects accessible to our mind such that it is possible to say whether the statement is **true** or **false**. There is no third possibility besides true or false in classical logic. A statement is something that is fundamentally different from a *question* or an *order*, because these are not inherently true or false. Also, dadaistic non-sense such as "A rose and an apple lead to the Presidency of the United States" is not a statement because it is unintelligible rubbish.

Here are some example of true statements:

(1) Paris is the capital of France.
(2) 9 is divisible by 3.
(3) Every human is a vertebrate.

Here are some example of false statements:

(1) Switzerland has 40 cantons.
(2) 15 is a prime[4] number.
(3) Fungi are animals.

[1]synonymous for: conclusion
[2]trees, houses, peole,...
[3]legal terms, numbers, ...
[4]only divisible by 1 and itself

And here we list some non-statements:

(1) Every prime number is green and has little white dots. (Non-sensical)
(2) Sit down! (Command)
(3) Are red and blue different colors? (Question)

**Exercise 2.1.** Which of the following are statements? For each statement, indicate whether it is true or false.

(1) Everyone should buy an eletric car.
(2) At least one Swiss person has an electric car.
(3) If I add 5 to 3, I get 6.
(4) Get your homework done!
(5) If $p$ is a prime number, then the only divisors of $p$ are 1 and $p$ itself.
(6) Is 9 a prime number?

## 3. Operators on statements

This section is not about *concrete* statements, but about building new statements out of given statements $P, Q, \ldots$. To do this, we use **operators** like **and, or, not**. These are already familiar from their everyday use. The "mechanism" of each of the operators is given in a *truth table*. The truth values **true** and **false** will be denoted by **1** and **0** respectively.[5]

3.1. **Negation ("not") $\neg$.** If $P$ is a statement, then so is $\neg P$ and the effect of this operator is given by the following truth table:

| $P$ | $\neg P$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

This is how we read the truth table above: If a statement $P$ is assigned the truth value 0 ("false"), then $\neg P$ gets the truth value 1 ("true").[6]

3.2. **Conjunction ("and") $\wedge$.** If $P, Q$ are statements, then so is $P \wedge Q$ and the effect of this operator is given by the following truth table:

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

As in everyday life, we see that $P \wedge Q$ is true if and only if both $P$ and $Q$ are true.

[5] This 1 and 0 convention is widely used in programming languages such as C or Java.

[6] So mathematical logic is different from the famous quote by the physicist and Nobel prize laureate Niels Bohr: "The opposite of a deep truth can again be a deep truth."

3.3. **Disjunction ("or") $\vee$.** If $P, Q$ are statements, then so is $P \vee Q$ and the effect of this operator is given by the following truth table:

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

*Note.* In mathematical logic, $P \vee Q$ is true if *at least one* of $P, Q$ is true. In everyday life sometimes we use the "exclusive or": We say that "$P$ or $Q$" is true if exactly one, but not both, is true. Take note of this small but important difference.

3.4. **Implication ("implies") $\Rightarrow$.** If $P, Q$ are statements, then so is $P \vee Q$ and the effect of this operator is given by the following truth table:

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

We can arbitrarily combine operators and build truth tables, such as in the following example:

| $P$ | $Q$ | $S$ | $(\neg P) \wedge (Q \vee S)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

**Exercise 3.1.** Make a truth table $((\neg P) \wedge Q) \vee S$ auf. Note: brackets () matter...!

**Exercise 3.2.** Make truth tables for the following expressions:

(1) $P \wedge (\neg Q)$
(2) $\neg((\neg P) \wedge Q)$
(3) $(\neg Q) \Rightarrow P$
(4) $P \wedge (\neg P)$
(5) $P \vee (\neg P)$. This expression is *always true*, and we call such expressions *tautologies*.

We say that two expressions are *equivalent* if their truth table coincide (= are the same).

**Exercise 3.3.** Prove the following equivalences by comparing truth tables:

- $\neg(P \wedge Q)$ is equivalent to $(\neg P) \vee (\neg Q)$.
- $\neg(P \vee Q)$ is equivalent to $(\neg P) \wedge (\neg Q)$.
- $P \Rightarrow Q$ is equivalent to
  (1) $\neg(P \wedge (\neg Q))$ and
  (2) $(\neg P) \vee Q$.

**Exercise 3.4.** Formulate the negation of the following statement: "Sophia is Greek, 30 years old, and she has brown hair."

## 4. Rules of inference

In order to formulate mathematical proofs you need to be able to draw conclusions from statements you already know to be true. These conclusions are made with the help of a small number of *rules of inference* that we present below.

### 4.1. **Modus Ponens.**

| Schema | Example |
|---|---|
| $P \Rightarrow Q$ | If Jumbo is an elephant, then Jumbo is a mammal. |
| $P$ | Jumbo is an elephant. |
| $Q$ | Jumbo is a mammal. |

### 4.2. **Modus Tollens / Contrapositive.**

| Schema | Example |
|---|---|
| $P \Rightarrow Q$ | If Jumbo is an elephant, then Jumbo is a mammal. |
| $\neg Q$ | Jumbo is not a mammal. |
| $\neg P$ | Jumbo is not an elephant. |

### 4.3. **Composition.**

| Schema | Beispiel |
|---|---|
| $P \Rightarrow Q$ | If Sarah lives in Bern, then Sarah lives in Switzerland. |
| $Q \Rightarrow R$ | If Saray lives in Switzerland, then Sarah lives in Europe. |
| $P \Rightarrow R$ | If Sarah lives in Bern, then Sarah lives in Europe. |

### 4.4. **Case distinction.**

| Schema | Beispiel |
|---|---|
| $P \Rightarrow T$ | (Example see below) |
| $Q \Rightarrow T$ | ... |
| $(P \vee Q) \Rightarrow T$ | ... |

**Example 4.1.** There are two irrational numbers $a, b$ such that $a^b$ is rational.[7]

[7]Rational numbers are fractions like $\frac{2}{7}$, and irrational numbers such as $\sqrt{2}$ cannot be written as a fraction.

*Beweis.* We consider the real number $\sqrt{2}^{\sqrt{2}}$. This number is either rational or irrationa. So we have a case distinction:

Case 1: If $\sqrt{2}^{\sqrt{2}}$ is rational, we choose $a = b = \sqrt{2}$ and we are finished.

Case 2: If $\sqrt{2}^{\sqrt{2}}$ is irrational, we set $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ and we get

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\sqrt{2})} = \sqrt{2}^2 = 2,$$

which is a rational number again.

So, in either case, we were able to find irrational numbers $a, b$ such that $a^b$ is rational and the proof is finished. □

**Remark 4.2.** It is *not known* which case of the cases 1 and 2 above is true – but we know that exactly one of the cases much be true so that we have proved 4.1!

## 5. Quantors

As an abbrevation for the expressions

"there is (at least) ...", "for all ..."

we use the **existential quantor** $\exists$ and the **universal quantor** $\forall$. These quantors are only used in the context of *statements with variables* $P(x)$. This may sound intimidating but you will see it is quite natural when we work through the exercises after introducing the notation:

| Notation | Meaning |
|---|---|
| $\exists x \in X\ (P(x))$ | there is (at least) one $x$ in the set $X$ such that $P(x)$ is true. |
| $\forall x \in X\ (P(x))$ | for all $x$ in the set $X$ the statement $P(x)$ is true. |

**Exercise 5.1.** Let $S$ be the set of all Swiss people. Let $P(x)$ be the following the statement:

"$x$ belongs to the Federal Council."

Translate the following statements into everyday language and check whether they are true:

(1) $\forall x \in S\ (P(x))$.
(2) $\exists x \in S\ (P(x))$.

An important subject is the **negation of quantors**: The negation of $\forall x \in A\ (P(x))$ is

$$\exists x \in A\ (\neg P(x)).$$

It is **not** given by $\forall x \in A\ (\neg P(x)))$! This is easy to see with the example given in the last exercise. The statement $\forall x \in A\ (P(x))$ translates to: "Every Swiss is a Federal Councillor", which is rubbish. The negation of this false statement is: "There is at least one Swiss who is not Federal Councillor." That is the correct negation! Remember - if we negate a false statement, we need to get a true statement. - If we had put "Every Swiss is a Non-Federal-Councillor ." then we would get another false statement, because this would mean that there are *no* Federal Councillors in Switzerland, which we know to be false.

Similarly, the negation of $\exists x \in A\ (P(x))$ is the statement $\forall x \in A\ (\neg P(x))$.

It is time for some hands-on exercises.

**Exercise 5.2.** Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers. For every statement below say whether it is true or false, and write down the negation of the statement according to the rules above.

(1) $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ (x < y)$
(2) $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ (x \leq y)$
(3) $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ (y > x)$
(4) $\exists y \in \mathbb{N}\ \forall x \in \mathbb{N}\ (y > x)$. (Note: we get this statement from the one above it just by switching quantors - and it makes a difference!)
(5) (*) $\exists x \in \mathbb{N}\ (\exists y \in \mathbb{N}\ (x = y^2) \wedge \exists z \in \mathbb{N}\ (x = 4z + 3))$.

(The asterisk (*) denotes a challenging problem.)

**Exercise 5.3.** Let $M$ denote the set of all persons employed by the Swiss Armed Forces at this moment. We define a statement $S(x, y)$ depending on $x, y \in M$ as follows:

$S(x, y) =$ "$x$ and $y$ have sat in at least one common meeting in the past."

Are the following statements true?

(1) $\forall x \in M\ \forall y \in M\ (S(x, y) \Rightarrow S(y, x))$
(2) $\forall x \in M\ \forall y \in M\ \forall z \in M\ \big((S(x, y) \wedge S(y, z)) \Rightarrow S(x, z)\big)$

## 6. Fun with logic riddles

Try to solve at least 5 of the following problems. Difficult / challenging problems are marked with (*).

1. Jack is looking at Anne. Anne is looking at George. Jack is married, George is unmarried, and we do not know about Anne. Show that there is an married person looking at an unmarried person. (Hint: case distinction.)

2. Five friends ran a race. After celebrating hard at night, they can only remember some details:

(1) No two people reached the finish line at the same time.
(2) Tim reached the finish line before Lucas.
(3) Janina reached the finish line before Tim, Francis, or Lucas.
(4) Anna reached the finish line before Janina or Francis.
(5) At least one of Tim and Lucas reached the finish line before Anna.

(6) Francis reached the finish line before Tim.

In what order did the five friends finish the race?

3. Once upon a time there were four friends who couldn't agree on who is right.

Tom : "Exactly two of us are right."
Anna : "At least two of us are wrong."
Mark : "Anna's claim is false."
Lisa : "At most one of us is right."

Who is right, and who is wrong?

4. It's quiz time! You can win CHF 1 Million hiding behind exactly one of two doors. (Behind the other door you find nothing.) Attached to each door you find a note:

Door 1: "Behind this door you find CHF 1 Million, and behind the other door there is nothing."
Tür 2: "Behind one of the doors you find CHF 1 Million, and behind the other door there is nothing."

The quizmaster tells you that exactly 1 door is labelled with a true statement. Which door do you open to secure your million?

5. On Crazy Island there are 3 types of people: People of type 1 make only true statements, people of type 2 make only false statements, and people of type 3 lie and tell the truth at random. You get into a conversion with two Crazy Islanders $A$ and $B$:

$A$: At least one of us always tells the truth.
$B$: At least one of us always lies.
$A$: You are right.

Determine the type of $A$ and $B$.

6. You listen in on a conversation of 3 Crazy Islanders, and this time you know that one is type 1, one is type 2, one is type 3.

$X$: Our friend $Y$ always tells the truth.
$Y$: I'll tell you what: $Z$ sometimes lies and sometimes tells the truth.
$Z$: $X$ always lies!

Which person is which type?

7. "I barbieri da Berna." - A Bernese barber $X$ is said to be *normal* if there is a Bernese barber $Y$ (not necessarily distinct from $X$) such that $X$ shaves $Y$ at least once a month, and vice versa. Bernese barbers who are not normal are said to be *exclusive.*

The exclusive barbers have formed a club called “The Exclusive Club” One day a Bernese barber named Barrabas barges into the Barbers’ Pub and boasts that he shaves every member of The Exclusive Club at least once a month and that he shaves nobody else.

Can Barrabas’ story be true?

8. Count Eutin has been murdered. The culprit is one of 4 persons: Ms Ming, Professor Bloom, Mrs Weiss and Colonel von Gatow. Everyone makes a statement. Further investigations show that only the murderer lied and the others told the truth:

> Ms Ming: “When the murder happened, I was with Professor Bloom.”
> Professor Bloom: “Colonel von Gatow was in the saloon when the murder happened.”
> Mrs Weiss: “Ms Ming, Colonel von Gatow, and me were not in the saloon when the murder happened.”
> Oberst von Gatow: “I am innocent. The murder happened in the saloon.”

Who is the murderer?

9. (*) Professor Knusi is on the Bru Islands. Each inhabitant either always tells the truth or always lies. Interestingly, Prof Knusi found out that this depends on the age of the person: When a person reaches a certain age threshold, he or she starts lying. Before that they always tell the truth. Professor Knusi collects the following statements:

Aru: “Beru is older than 15.” Beru: “Ceru is older than 13.” Ceru: “Deru is younger than 17.” Deru: “Eru is not 12.” Eru: “Aru is older than 16.” Aru: “Deru is older than 11.” Beru: “Eru is younger than 15.” Ceru: “Aru is 14.” Deru: “Beru it 15.” Eru: “Ceru is younger than 13.”

What is the age threshold? And what other conclusions can Professor Knusi draw?

10. (*) School starts! “The requests of the first graders get stranger and stranger”, says the secretary. The 8 first graders shall be split into two (small) classes of 4 firs graders each. This is what the first graders ask for:

Ah: “If Yi and Mo are in the same class, I would like to be in that class too.”

Ba: “If Bo is in the same class as me, I would like for Ah to be in our class.”

El: “I want Ul and Ah to be in the same class as me.”

Ul: “I want to be in the same class as Ba, or Lu and Ba shall not be together in the same class.”

Yi: “If Mo is in class $A$, I don’t want to be in the same class as Bo.”

Lu: “Yi and Mo shall be in the same class, or Ah shall be in my class.”

Bo: "I don't want to be in Lu's class, except if Ul and El are not in the same class."

Mo: "I want to be in the same class as Ah, or in class $B$."

The schoolmaster sighs and says: "I cannot grant every request – but every request *except one*, this can be done."

Who is in class $A$, and who is in class $B$?