HADWIGER'S CONJECTURE FOR HYPERGRAPHS

D.Z., APRIL 27, 2021

1. Basic notions

1.1. **Hypergraphs.** A *hypergraph* $H = (V, E)$ consists of a set V and $E \subseteq \mathcal{P}(V)$, that is, E consists of subsets of V of arbitrary size.

If $H = (V, E)$ is a hypergraph and $S \subseteq V$, we define

$$E|_S = \{e \cap S : (e \in E) \wedge (e \cap S \neq \emptyset)\}$$

and call $(S, E|_S)$ the *induced sub-hypergraph* of H.

1.2. **Connectedness.** A hypergraph $H = (V, E)$ is *connected* if for all $X \subseteq V$ with $\emptyset \neq X \neq V$ there is $e \in E$ such that

$$e \cap X \neq \emptyset \neq e \cap (V \setminus X).$$

1.3. **Colouring.** Let $H = (V, E)$ be a hypergraph and $\kappa \neq \emptyset$ be a cardinal. Then a map $c : V \to \kappa$ is said to be a *colouring* if for every $e \in E$ with $|e| \geq 2$ we have that the restriction $c \restriction_e$ is non-constant. The *chromatic number* $\chi(H)$ of H is the smallest cardinal κ such there is a colouring $c : V \to \kappa$.

1.4. **Connected to each other.** . If $H = (V, E)$ is a hypergraph and $S_1, S_2 \subseteq V$ are disjoint, we say they are *connected to each other* if there is $e \in E$ such that

$$e \cap S_1 \neq \emptyset \neq e \cap S_2.$$

2. A form of Hadwiger's conjecture for hypergraphs

Assume that $H = (V, E)$ is a hypergraph and let's assume $V \neq \emptyset \neq E$ to avoid pathologies. Let κ be a cardinal such that there is *no* colouring $c : V \to \kappa$. Then there is a collection $\mathcal{S}$ of mutually disjoint subsets with $|\mathcal{S}| = \kappa$ such that

(1) $(S, E|_S)$ is a connected hypergraph for each $S \in \mathcal{S}$, and
(2) whenever $S \neq T \in \mathcal{S}$ then S, T are connected to each other.

(In the graph context, this amounts to saying that there is a complete minor of cardinality κ.)