HAMMING GRAPH REPRESENTATION

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Abstract. Graph embeddings deal with injective maps from a given simple, undirected graph $G = (V, E)$ into a metric space, such as $\mathbb{R}^n$ with the Euclidean metric. This concept is widely studied in computer science, see [1], but also offers attractive research in pure graph theory [2]. In this note we show that any graph can be embedded into a particularly simple metric space: $\{0, 1\}^n$ with the Hamming distance, for large enough n.

1. The Hamming Graph $H(n, k)$

We construct graph on the vertex set $\{0, 1\}^n$ where n is a positive integer. For $x, y \in \{0, 1\}^n$ the *Hamming distance* of x, y is the cardinality of the set

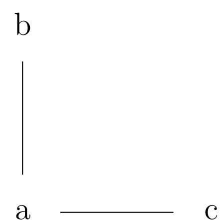
$$\big\{i \in \{0, ..., n-1\} : x(i) \neq y(i)\big\}.$$

That is, we count the positions on which x and y do not agree.

Fix a positive integer $k \leq n$. Two distinct elements of $\{0, 1\}^n$ form an edge if their Hamming distance is at most k (so they are in some sense "close" to each other). We denote the resulting graph on $\{0, 1\}^n$ by $H(n, k)$.

We say that a finite graph $G = (V, E)$ is *Hamming-representable* if there are positive integers $k \leq n$ such that G is isomorphic to an induced subgraph of $H(n, k)$.

As an easy example, we show that the following 3-point graph can be embedded into $H(2, 1)$:



The solution is best shown in the following picture, where it is easily seen that points connected with an edge have Hamming distance 1 and points not connected have Hamming distance 2:

$$\begin{tikzcd} 01 \arrow[d, no head] & \\ 00 \arrow[r, no head] & 10 \end{tikzcd}$$

As a further example, note that $H(n,n)$ is isomorphic to K_{2^n}, the complete graph on 2^n vertices.

Some notation: By $\mathrm{Mat}(\{0,1\}, n \times m)$ we denote the set of $n \times m$-matrices with entries in $\{0,1\}$. We identify $\mathrm{Mat}(\{0,1\}, n \times m)$ with $\{0,1\}^{nm}$ via the canonical bijection.

2. The Result

Proposition 2.1. *Every finite graph $G = (V,E)$ is Hamming-representable.*

Proof. We embed G into $H(|E| \cdot (|V|-1),\ 2|E|-2)$. To each vertex v of G, we will associate an $|E| \times (|V|-1)$ matrix M_v with rows indexed by the edges of G. There will be a single 1 in each row, with all other entries in that row equal to 0.

If $v \in e$, then the 1 in row e of M_v will be in the first column. If not, we will place a 1 in one of the other $|V|-2$ columns, so that each of the non-endpoints of e gets a 1 in a different position of row e.

If v and w are not joined by an edge, the Hamming distance between M_v and M_w is $2|E|$ because they have no 1's in common; if they are joined, then the Hamming distance is $2|E|-2$. $\square$.

3. Possible use cases

Representing graphs as subgraphs of some $H(n,k)$ can be useful in applications in computer science: the Hamming distance is computed by bitwise XOR, the fastest operation a CPU can do. So given two vertices represented by n-bit strings, it can be very quickly determined whether they form an edge (i.e. whether their Hamming distance is smaller than the limit given in k).

Moreover, for some graphs $G = (V,E)$ with $|V| = n$ we can represent the graph using bit strings of length $\mathcal{O}(\log n)$, making this technique potentially interesting for memory management.

4. Open questions

We define the *Hamming dimension* of a graph $G = (V, E)$ to be the minimum positive integer n such that there is $k \leq n$ such that G can be embedded into some induced subgraph of $H(n, k)$, and denote this by $\dim(G)$ Questions:

(1) If $G = (V, E)$ is a graph with $n = |V|$, do we necessarily have $\dim(G) \leq n$? If not, can we at least acheive for $\dim(G)$ to be $\mathcal{O}(|E| \log |V|)$?
(2) Given graphs G, H what is $\dim(G \times H)$ in terms of $\dim(G), \dim(H)$, where $G \times H$ denotes the categorical product?
(3) How (if at all) does $\dim(G)$ relate to the chromatic number $\chi(G)$?

5. Acknowledgement

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References

[1] Palash Goyal, Emilio Ferrara, *Graph Embedding Techniques, Applications, and Performance: A Survey*, `https://arxiv.org/abs/1705.02801`
[2] Hongyun Cai, Vincent W. Zheng, Kevin Chen-Chuan Chang, *A Comprehensive Survey of Graph Embedding: Problems, Techniques and Applications*, `https://arxiv.org/abs/1709.07604`

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