SOME NOTES ON THE HADWIGER CONJECTURE

D.Z., APRIL 21, 2021

1. Introductory remarks

Hugo Hadwiger formulated his conjecture in 1943. Some historical background on this conjecture can be found on Wikipedia using the search terms

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wiki hadwiger conjecture graph theory
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Erdős considered this conjecture to be one of the top three problems he would like to see solved in his lifetime.

2. Basic notions

2.1. **Graphs.** For any set X we let

$$[X]^2 = \big\{\{x, y\} : (x, y \in X) \wedge (x \neq y)\big\}.$$

A *graph* is a tuple $G = (V, E)$ where $E \subseteq [V]^2$.

If $G = (V, E)$ is a graph and $v \in V$, then the *neighborhood* of v is defined by

$$N(v) = \{w \in V : \{v, w\} \in E\}.$$

Note that we always have $v \notin N(v)$. The cardinal $\mathrm{card}(N(v)) = |N(v)|$ is said to be the *degree* of v, denoted by $\deg(v)$.

2.2. **(Induced) subgraphs.** For $G = (V, E)$ and $S \subseteq V$ we define the induced subgraph of S by

$$(S, E \cap [S]^2).$$

For $v \in V$ we denote the induced subgraph on $V \setminus \{v\}$ by $G \setminus \{v\}$.

2.3. **Hypergraphs.** A *hypergraph* $H = (V, E)$ consists of a set V and $E \subseteq \mathcal{P}(V)$, that is, E consists of subsets of V of arbitrary size. Obviously, a graph is a special kind of hypergraph.

2.4. **Colouring.** We define a notion of colouring for *hypergraphs* that agrees with the notion of graph coloring whenever the hypergraph at hand is a graph. Let $H = (V, E)$ be a hypergraph and $\kappa \neq \emptyset$ be a cardinal. Then a map $c : V \to \kappa$ is said to be a *colouring* if for every $e \in E$ with $|e| \geq 2$ we have that the restriction $c \restriction_e$ is non-constant. The *chromatic number* $\chi(H)$ of H is the smallest cardinal κ such there is a colouring $c : V \to \kappa$.

A few (non-trivial[1]) things on hypergraph colouring:

(1) If κ is an infinite cardinal and E is a collection of subsets of cardinality κ, and also we have $|E| = \kappa$, then $\chi(\kappa, E) = 2$.
(2) Let ω be the first infinite cardinal. Given $n \in \omega \cup \{\omega\}$, is there $E \subseteq \mathcal{P}(\omega)$ such that $\chi(\omega, E) = n$?
(3) Determine $\chi(\mathbb{N}, E)$ where
    (a) $E = \big\{\{a, b, a + b\} : a, b \in \mathbb{N}\big\}$,
    (b) E = collection of subsets of $\mathbb{N}$ such that its members are pairwise relatively prime.
(4) What is $\chi(\mathbb{R}, E)$ where E is the collection of infinite Lebesgue-measurable sets? (We have $|E| > 2^{\aleph_0} = |\mathbb{R}|$ if I am not mistaken.)

2.5. **Point contraction.** If $v \neq w \in V$ we can "contract" v and w in the following way. Consider $V' = V \setminus \{w\}$ and set

$$E' = (E \cap [V']^2) \cup \big\{\{v, x\} : x \in N(w) \setminus \{v\}\big\}.$$

Then we denote the graph (V', E') by $G/\{v, w\}$.

3. Connectedness

Usually, a graph $G = (V, E)$ is said to be *connected* if there is a path between any two vertices (points) $v, w \in V$. The following notion of connectedness seems more general, and it can be used in the context of hypergraphs too:

Definition 3.1. *$G = (V, E)$ is* connected *if for all $X \subseteq V$ with $\emptyset \neq X \neq V$ there is $e \in E$ with*

$$X \cap e \neq \emptyset \neq (V \setminus X) \cap e.$$

I was convinced that for infinite graphs the above definition was *weaker*, but it turns out there is a short inductive argument showing that this is equivalent to the path definition we are used to.

[1]Disclosure: I find them hard and couldn't do all, but some or all of these points could be trivial

4. Complete minors (no age requirement), and the Hadwiger number

Definition 4.1. *If S, T are disjoint subsets of V where $G = (V, E)$ is a graph, we say S, T are* connected to each other *if there is $s \in S, t \in T$ such that $\{s, t\} \in E$.*

A *complete minor* of $G = (V, E)$ is a collection $\mathcal{S}$ of *connected*, non-empty, subsets of V such that

(1) whenever $S \neq T \in \mathcal{S}$ then $S \cap T = \emptyset$, and
(2) whenever $S \neq T \in \mathcal{S}$ then S, T are connected to each other.

The *Hadwiger number* $\eta(G)$ of a graph $G = (V, E)$ is the supremum of the cardinalities that a complete minor can have, i.e.

$$\eta(G) = \sup\{|\mathcal{S}| : \mathcal{S} \subseteq \mathcal{P}(V) \text{ is a complete minor}\}.$$

5. Hadwiger's conjecture

Hadiwger proposes the following easy connection between the chromatic number $\chi(G)$ of a graph, and its Hadwiger number:

Hadwiger's conjecture. $\chi(G) \leq \eta(G)$ for all graphs G.

Some remarks:

(1) As soon as $E \neq \emptyset$ for a graph $G = (V, E)$, we have $\eta(G) \geq 2$.
(2) Note that even for $\chi(G) = 2$, the value of $\eta(G)$ can be arbitrarily large: Let $\kappa \geq 2$ be a cardinal (finite or infinite). Then $\eta(K_{\kappa,\kappa}) = \kappa$ where $K_{\kappa,\kappa}$ is the complete bipartite graph on $2 \cdot \kappa$ points. Obviously $\chi(K_{\kappa,\kappa}) = 2$.

6. Hadwiger's conjecture in the infinite

The situation for graphs with infinite chromatic number is much clearer than for finite graphs[2].

Proposition 6.1. *If $G = (V, E)$ is such that $\chi(G) \geq \aleph_0$, then $\chi(G) \leq \eta(G)$.*

This is a corollary of the main result of `https://arxiv.org/pdf/1312.2829.pdf`.

However, we can reformulate Hadwiger's conjecture as follows:

[2]this also covers infinite graphs with *finite chromatic number* because of the De Bruijn - Erdős theorem stating every such graph has a *finite* subgraph with the same chromatic number

Hadwiger Version 2. Any graph G has a complete minor of cardinality $\chi(G)$.

Version 2 is equivalent to the original statement for FINITE graphs, but is is *false* for graphs with infinite chromatic number:

Let $G = \bigcup_{n\in\mathbb{N}} K_n$ be the disjoint union of copies of the complete graph K_n. Then $\chi(G)$ is infinite, but G has only finite complete minors (but of arbitrarily large finite size).

7. Properties of a minimal hypothetical counterexample

7.1. **Criticality.** A graph $G = (V, E)$ is said to be *critical* if $\chi(G \setminus \{v\}) < \chi(G)$ for all $v \in V$. For the remainder of this section let n_0 be the *minimum cardinality* $|V|$ of any graph $G_0 = (V, E)$ that is a counterexample to Hadwiger, i.e. has

$$\eta(G_0) < \chi(G_0).$$

Proposition 7.1. *G_0 is critical.*

Proof. Otherwise, take $v \in V$ with $\chi(G_0 \setminus \{v\} = \chi(G_0)$. Then $\eta(G_0 \setminus \{v\}) \leq \eta(G_0) < \chi(G_0) = \chi(G_0 \setminus \{v\}$ so that $G_0 \setminus \{v\}$ is another counterexample to Hadwiger with $|V \setminus \{v\}| = n_0 - 1 < n_0$, contradicting minimality of n_0. □

Note that every critical graph is connected.

Now we revisit contractions (see "Basic notions").

Let $v \neq w \in V$ such that $\{v, w\} \notin E$.

Proposition 7.2. *If G is any graph and $v \neq w \in V$ with $\{v, w\} \notin E$, then*

$$\chi(G/\{v, w\}) \geq \chi(G).$$

Let's go back to our minimal Hadwiger counterexample G_0.

Proposition 7.3. *Whenever $v \neq w \in V(G_0)$ and $\{v, w\} \notin E$, then $\eta(G_0/\{v, w\}) > \eta(G_0)$.*

Proof. Suppose $\eta(G_0/\{v, w\}) \leq \eta(G_0)$. Then since $\chi(G_0/\{v, w\} \geq \chi(G_0)$ we get $\eta(G_0/\{v, w\}) < \chi(G_0/\{v, w\})$, so $G_0/\{v, w\}$ is a Hadwiger counterexample with a smaller number of vertices than G_0, contradicting minimality. □.

Proposition 7.4. *If $G = (V, E)$ is any graph and $v \neq w \in V$ with $\{v, w\} \notin E$, then*

$$\eta(G/\{v, w\}) \leq \eta(G \cup \{v, w\}),$$

where $G \cup \{v, w\}$ denotes the graph $(V, E \cup \{v, w\})$.

Together, propositions 7.3 and 7.4 imply that G_0 must have the following property:

> (E): Whenever a new edge is added in G_0, then the Hadwiger number increases.

So G_0 has property (E), and it is critical. I asked Paul Seymour[3] to come up with *any* finite critical graph G_1 having property (E), and he wrote after a while that he wasn't able to.

If some such G_1 could be found, this wouldn't refute Hadwiger but could give insight as to what kind of examples could be interesting regarding Hadwiger.

Any proof that shows no critical graph with property (E) can exist would prove the Hadwiger conjecture.

8. Extending to hypergraphs?

We can define connected hypergraphs, see "Basic notions", and we can extend the concept of colouring to hypergraphs. Maybe we can define a complete minor in a hypergraph.

Question. Is there a cool way to formulate Hadwiger's conjecture in terms of hypergraphs?

[3] Quite a famous combinatorialist, we exchange e-mail like once or twice a year