A TOPOLOGICAL SPACE WITH $\binom{\Omega}{T} \neq \binom{\Omega}{\Gamma}$

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Abstract. We show that the selection principles $\binom{\Omega}{T}$ and $\binom{\Omega}{\Gamma}$ are not equal by providing a space (X, τ) in which $\binom{\Omega}{T}$ holds, but not $\binom{\Omega}{\Gamma}$. This answers a question from the year 2003 in [2].

1. Introduction and definitions

Throughout this note, let (X, τ) be a topological space.

Definition 1.1. *We say that* $\mathcal{U} \subseteq \tau$ *is an* open cover*, or* cover *for short, if*

(1) $X \notin \mathcal{U}$,
(2) $\bigcup \mathcal{U} = X$.

For $x \in \mathcal{U}$ we set $\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}$ and call $\mathcal{U}_x$ the *star* of x (with respect to $\mathcal{U}$).

1.1. Thick covers.

We call a cover $\mathcal{U} \subseteq \tau$

(1) *large* if $\mathcal{U}_x$ is infinite for every $x \in X$,
(2) an ω-cover if every finite subset of X is contained in some member of $\mathcal{U}$,
(3) a t-cover if $\mathcal{U}$ is large, and for all $x, y \in V$ at least one of the sets $\mathcal{U}_x \setminus \mathcal{U}_y$ and $\mathcal{U}_y \setminus \mathcal{U}_x$ is finite, and
(4) a γ-cover if $\mathcal{U}$ is infinite, and for every $x \in V$ the set $\mathcal{U} \setminus \mathcal{U}_x$ is finite.

Let $\Lambda, \Omega, T, \Gamma$ denote the collections of (open) large covers, ω-covers, t-covers, and γ-covers, respectively. An easy argument shows that every γ-cover is large and therefore a t-cover, so $\Gamma \subseteq T$.

1.2. The selection principle.

If $\mathfrak{U}, \mathfrak{V}$ are families of covers of X, then we define the property $\binom{\mathfrak{U}}{\mathfrak{V}}$, read "$\mathfrak{U}$ choose $\mathfrak{V}$", as follows:

$$\binom{\mathfrak{U}}{\mathfrak{V}} \text{ : For each } \mathcal{U} \in \mathfrak{U} \text{ there is } \mathcal{V} \subseteq \mathcal{U} \text{ such that } \mathcal{V} \in \mathfrak{V}.$$

2. Construction of the example

We consider the space $(\aleph_1, \tau)$ where $\aleph_1$ is the smallest uncountable cardinal and τ is the collection of down-sets in the cardinal $\aleph_1$, that is $\tau = \aleph_1 \cup \{\aleph_1\}$.

Proposition 2.1. *If $\mathcal{U} \subseteq (\tau \setminus \{\aleph_1\})$ is a cover, then:*

(1) *$\mathcal{U}$ is a large cover,*
(2) *$\mathcal{U}$ is an ω-cover,*
(3) *$\mathcal{U}$ is a t-cover, but*
(4) *$\mathcal{U}$ is* never *a γ-cover.*

Proof.

(1) Let $\alpha \in \aleph_1$. Suppose α is only covered by finitely many members $U_1, \ldots U_n \in \mathcal{U}$. But then, $\beta := \bigcup \mathcal{U} \in \aleph_1$ is not covered by any member of $\{U_1, \ldots, U_n\}$. We have $\beta > \alpha$, and since $\mathcal{U}$ is a covering of $\aleph_1$, there is $U^* \in \mathcal{U}$ covering β and therefore α. Clearly

$$U^* \notin \{U_1, \ldots, U_n\},$$

contradicting the assumption that the only members of $\mathcal{U}$ covering α are $U_1, \ldots, U_n$.

(2) Let $S \subseteq \aleph_1$ be finite, and consider

$$\beta = \bigcup S \in \aleph_1.$$

Then β is contained in some $U \in \mathcal{U}$, so $S \subseteq U$.

(3) Let $\alpha, \beta \in \aleph_1$. We may assume that $\alpha < \beta$. Then $\mathcal{U}_\alpha \setminus \mathcal{U}_\beta = \emptyset$, which is finite, so $\mathcal{U}$ is a t-cover.

(4) Suppose that $\mathcal{U} \subseteq \tau \setminus \{\aleph_1\}$ is a γ-cover. So every $\alpha \in \aleph_1$ is contained in all but finitely many members of $\mathcal{U}$. Note that due to the special nature of cardinals, we have $\mathcal{U} \subseteq \aleph_1$. So $\mathcal{U}$ is a well-ordered set such that all members only finitely many predecessors. This implies that $\mathcal{U}$ is either a finite or countable collection of members of $\aleph_1$ such that $\bigcup \mathcal{U} = \aleph_1$. This contradicts the fact that $\aleph_1$ is a *regular* cardinal. □

By proposition 2.1 (2) and (3), every ω-cover in the space $(\aleph_1, \tau)$ with $\tau = \aleph_1 \cup \{\aleph_1\}$ is a t-cover, so the property $\binom{\Omega}{T}$ is trivially true. On the other hand, by proposition 2.1(4), no cover is a γ-cover, therefore property $\binom{\Omega}{\Gamma}$ is false.

So for the space $(\aleph_1, \aleph_1 \cup \{\aleph_1\})$ we have $\binom{\Omega}{T} \neq \binom{\Omega}{\Gamma}$.

As a final remark, the property $\binom{\Omega}{\Gamma}$ is the celebrated Gerlits-Nagy γ-property [1]. Since we have seen that $\Gamma \subseteq T$, property $\binom{\Omega}{\Gamma}$ always implies $\binom{\Omega}{T}$. But the converse is not true, as Proposition 2.1 shows that there is a space where $\binom{\Omega}{T}$ is true but $\binom{\Omega}{\Gamma}$ is false.

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References

[1] J. Gerlits and Zs. Nagy, *Some properties of* $C(X)$*, I*, Topology and its Applications **14** (1982), 151161.

[2] Boaz Tsaban, *SPM (Selection Principles in Mathematics) bulletin No. 1*, 2003, `https://arxiv.org/pdf/math/0301011`

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