$\binom{\Omega}{T} \neq \binom{\Omega}{\Gamma}$

DOMINIC VAN DER ZYPEN

Abstract. We show that the selection principles $\binom{\Omega}{T}$ and $\binom{\Omega}{\Gamma}$ are not equal constructing a topological space (X, τ) that satisfies $\binom{\Omega}{T}$, but not $\binom{\Omega}{\Gamma}$. This answers a question from the year 2003 in [2].

1. Introduction and definitions

Throughout this note, let (X, τ) be a topological space. [1]

Definition 1.1. *We say that $\mathcal{U} \subseteq \tau$ is an* open cover*, or* cover *for short, if*

(1) $X \notin \mathcal{U}$,
(2) $\bigcup \mathcal{U} = X$.

For $x \in X$ we set $\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}$ and call $\mathcal{U}_x$ the *star* of x (with respect to $\mathcal{U}$).

1.1. Thick covers.

We call a cover $\mathcal{U} \subseteq \tau$

(1) *large* if $\mathcal{U}_x$ is infinite for every $x \in X$,
(2) an ω-cover if every finite subset of X is contained in some member of $\mathcal{U}$,
(3) a t-cover if $\mathcal{U}$ is large, and for all $x, y \in X$ at least one of the sets $\mathcal{U}_x \setminus \mathcal{U}_y$ and $\mathcal{U}_y \setminus \mathcal{U}_x$ is finite, and
(4) a γ-cover if $\mathcal{U}$ is infinite, and for every $x \in X$ the set $\mathcal{U} \setminus \mathcal{U}_x$ is finite.

Let $\Lambda, \Omega, T, \Gamma$ denote the collections of (open) large covers, ω-covers, t-covers, and γ-covers, respectively. An easy argument shows that every γ-cover is large and therefore a t-cover, so $\Gamma \subseteq T$.

[1] The definitions in this note also make sense in the broader context of hypergraphs $H = (V, E)$, where V is any set and $E \subseteq \mathcal{P}(V)$.

1.2. **The selection principle.** If $\mathfrak{U}, \mathfrak{V}$ are families of covers of X, then we define the property $\binom{\mathfrak{U}}{\mathfrak{V}}$, read "$\mathfrak{U}$ choose $\mathfrak{V}$", as follows:

$$\binom{\mathfrak{U}}{\mathfrak{V}} \ : \text{For each } \mathcal{U} \in \mathfrak{U} \text{ there is } \mathcal{V} \subseteq \mathcal{U} \text{ such that } \mathcal{V} \in \mathfrak{V}.$$

2. Construction of the example

We consider the space $(\aleph_1, \tau)$ where $\aleph_1$ is the smallest uncountable cardinal and τ is the collection of down-sets in the cardinal $\aleph_1$, that is $\tau = \aleph_1 \cup \{\aleph_1\}$.

Proposition 2.1. *If* $\mathcal{U} \subsetneq (\tau \setminus \{\aleph_1\})$ *is a cover, then:*

(1) $\mathcal{U}$ *is a large cover,*
(2) $\mathcal{U}$ *is an* ω*-cover,*
(3) $\mathcal{U}$ *is a* t*-cover, but*
(4) $\mathcal{U}$ *is* never *a* γ*-cover.*

Proof.

(1) Let $\alpha \in \aleph_1$. Suppose α is only covered by finitely many members $U_1, \dots U_n \in \mathcal{U}$. But then, $\beta := \bigcup \mathcal{U} \in \aleph_1$ is not covered by any member of $\{U_1, \dots, U_n\}$. We have $\beta > \alpha$, and since $\mathcal{U}$ is a covering of $\aleph_1$, there is $U^* \in \mathcal{U}$ covering β and therefore α. Clearly

$$U^* \notin \{U_1, \dots, U_n\},$$

contradicting the assumption that the only members of $\mathcal{U}$ covering α are $U_1, \dots, U_n$.

(2) Let $S \subseteq \aleph_1$ be finite, and consider

$$\beta = \bigcup S \in \aleph_1.$$

Then β is contained in some $U \in \mathcal{U}$, so $S \subseteq U$.

(3) Let $\alpha, \beta \in \aleph_1$. We may assume that $\alpha < \beta$. Then $\mathcal{U}_\alpha \setminus \mathcal{U}_\beta = \emptyset$, which is finite, so $\mathcal{U}$ is a t-cover.

(4) Suppose that $\mathcal{U} \subseteq \tau \setminus \{\aleph_1\}$ is a γ-cover. So every $\alpha \in \aleph_1$ is contained in all but finitely many members of $\mathcal{U}$. Note that due to the special nature of cardinals, we have $\mathcal{U} \subseteq \aleph_1$. So $\mathcal{U}$ is a well-ordered set such that all members only finitely many predecessors. This implies that $\mathcal{U}$ is either a finite or countable collection of members of $\aleph_1$ such that $\bigcup \mathcal{U} = \aleph_1$. This contradicts the fact that $\aleph_1$ is a *regular* cardinal. $\square$

By proposition 2.1 (2) and (3), every ω-cover in the space $(\aleph_1, \tau)$ with $\tau = \aleph_1 \cup \{\aleph_1\}$ is a t-cover, so the property $\binom{\Omega}{T}$ is trivially true. On the other hand, by proposition 2.1(4), no cover is a γ-cover, therefore property $\binom{\Omega}{\Gamma}$ is false.

So for the space $(\aleph_1, \aleph_1 \cup \{\aleph_1\})$ we have $\binom{\Omega}{T} \neq \binom{\Omega}{\Gamma}$.

As a final remark, the property $\binom{\Omega}{\Gamma}$ is the celebrated Gerlits-Nagy γ-property [1]. Since we have seen that $\Gamma \subseteq T$, property $\binom{\Omega}{\Gamma}$ always implies $\binom{\Omega}{T}$. But the converse is not true, as Proposition 2.1 shows that there is a space where $\binom{\Omega}{T}$ is true but $\binom{\Omega}{\Gamma}$ is false.

Acknowledgement. I am grateful to Boaz Tsaban for a fruitful discussion via e-mail on the subject of this note.

References

[1] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$, I*, Topology and its Applications **14** (1982), 151161.

[2] Boaz Tsaban, *SPM (Selection Principles in Mathematics) bulletin No. 1*, 2003, https://arxiv.org/pdf/math/0301011

Federal Office of Borders and Customs Security, Taubenstrasse 16, CH-3003 Bern, Switzerland

E-mail address: dominic.zypen@gmail.com