# SOME NOTES ON $\square(\omega+1)^\omega$

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## 1. Topology on $\omega+1$

We define the topology on $\omega+1$ to be

$$\tau = \mathcal{P}(\omega) \cup \{A \subseteq (\omega+1) : (\omega+1) \setminus A \text{ is finite}\}.$$

It is not hard to verify that this is a (compact) topology on $\omega+1$. The space $(\omega+1, \tau)$ is (homeomorphic to) the 1-point compactification of the discrete topology on $\omega$.

## 2. The box product on a family of spaces $\{X_\lambda : \lambda \in \Lambda\}$

Let $\{(X_\lambda, \tau_\lambda : \lambda \in \Lambda\}$ be a family of topological spaces. Then the *box product topology* is the topological space on the base set $\prod_{\lambda\in\Lambda} X_\lambda$ generated by the basis

$$\{\prod_{\lambda\in\Lambda} U_\lambda : U_\lambda \in \tau_\lambda \text{ for all } \lambda \in \Lambda\}.$$

(It is a standard exercise to prove that the above set is indeed a basis.)

If all the spaces $X_\lambda$ are equal to space $X$, we denote the box product by

$$\square X^\Lambda.$$

## 3. Para- and metacompactness, and normalcy

Let $(X, \tau)$ be a topological space. We call a collection $\mathcal{U} \subseteq \tau$ a *cover* if $\bigcup \mathcal{U} = X$. We call $\mathcal{U}$ *locally finite* if for every $x \in X$ there is an open neighborhood $U_0$ of $x$ such that $U_0$ intersects only finitely many members of $\mathcal{U}$. This is a way of saying that $\mathcal{U}$ is "thinly spread" over $X$, speaking with a lot of hand-waving.

Moreover, $\mathcal{U}$ is *point-finite* if every $x \in X$ is only contained in finitely many members of $\mathcal{U}$. This is another, weaker notion of "thinness" than local finiteness, that is local finiteness implies point-finiteness.

A cover $\mathcal{V}$ of $X$ is said to be a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$.

Finally, $(X, \tau)$ is said to be **paracompact** if every open cover has a locally finite refinement. For the weaker notion of **metacompactness**, replace "locally finite" by "point-finite".

A space is said to be **normal** if disjoint closed sets can be separated by disjoint open sets. In $T_2$-spaces, normalcy and paracompactness is equivalent.

## 4. Is $\square(\omega+1)^\omega$ paracompact?

This is the big open question we are trying to think about, and I find it more accessible in some way than the equivalent question about normalcy, despite remembering that we treated some cases of disjoint closed sets being able to be separated by disjoint open sets.

(Is it known whether $\square(\omega+1)^\omega$ is metacompact?)

## 5. Introductory thoughts

As an entry point, let's put an observation without proof (dangerous!! Will write one later on, because "it's obvious" is always dangerous).

Let $X = \square(\omega+1)^\omega$, and let $\bar{\omega}$ be the constant $\omega$-sequence in $X$. For $n \in \omega$, let $\uparrow n = \{(\omega+1) \setminus n\}$.

**Lemma 5.1.** *If $U \subseteq X$ is an open neighborhood of $\bar{\omega}$, then there is a* monotone map *$f : \omega \to \omega$ such that*

$$\bar{\omega} \in \prod_{n\in\omega}(\uparrow f(n)) \subseteq U.$$

**Exercise I would like to solve as a first step.** Given monotonically increasing $f : \omega \to \omega$ with $f(0) > 0$, consider the following cover of $X$. Let $\mathcal{U}$ consist of

- $\prod_{n\in\omega}(\uparrow f(n))$, covering $\bar{\omega}$,
- the singletons $\{a\}$ for $a \in \omega \to \omega$, and finally
- $\prod U_x$ covering $x \in X \setminus \big(\{\bar{\omega}\} \cup \omega^\omega\big)$ where $U_x = \{x(n)\}$ if $x(n) \in \omega$, and $U_x = \uparrow f(n)$ if $x(n) = \omega$.

Questions: Is $\mathcal{U}$ locally finite? Point-finite? If no, does it have such a refinement?

(Maybe this is a very easy exercise, but variations on the cover sets in the third bullet point could make it interesting.)