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$\int f g' = fg - \int f' g$

$\cos t = \frac{e^{it} + e^{-it}}{2}$

$\sin t = \frac{e^{it} - e^{-it}}{2}$

$\cos(n\pi) = (-1)^n$

$\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$e^{-ixw} = \cos(wx) - i\sin(wx)$

$e^{ixw} = \cos(wx) + i\sin(wx)$

$\mathcal{L}(f''') = s^2 \mathcal{L}(f) - s^2 f(0) - s f'(0) - f''(0)$

Explicit Runge-Kutta

$$\begin{array}{c|cccc} 0 & & & \\ c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \vdots & \vdots & \vdots & \cdots \\ \hline & b_1 & b_2 & \cdots \end{array}$$

$K_1 = f(t_n, y_n)$

$K_2 = f(t_n + c_2 h, y_n + (a_{21} K_1)h)$

$K_3 = f(t_n + c_3 h, y_n + (a_{31} K_1 + a_{32} K_2)h)$

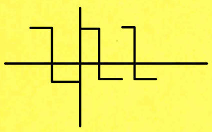
$K_s = f(t_n + c_s h, y_n + (a_{s1} K_1 + a_{s2} K_2 + \dots)h)$

$y_{n+1} = y_n + h \sum b_i \cdot K_i$

Fixed point

$g(x) = -1.93\ln(x) + 15.9$
$x \in [e, e^3]$
Converges if
1) $g'(x) \in C^0[e, e^3]$
2) $g([e, e^3]) \subset [e, e^3]$
3) $|g'(x)| < 1$ for all x

Fourier Series



Odd hvis speilvendt om y-aksen
Da kan a_n fjernes hvis odd
Even hvis ikke

D'Alemberts

$u_{tt}(x,t) = c^2 u_{xx}(x,t)$

$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$

$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(t)\, dt$

Crank-Nicholson

$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h}$

Eulers Eksplisitt

$+ \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{2h^2}$

Eulers Implisitt

Husk å fjerne 2-ern

Newtons Method

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Laplace Transform

$2y'' + y' - y = 3u(t-2)$

$\mathcal{L}(y'')/\mathcal{L}(y')/\mathcal{L}(y) = \dots$

$\mathcal{L}(3) = 3 \cdot \frac{1}{s}$

$\mathcal{L}(u(t-2)) = e^{-2s} \cdot F(s)$

$2s^2 Y + sY - Y = 3\frac{e^{-2s}}{2}$

$Y = \frac{3e^{-2s}}{s(2s^2+s-1)} = e^{-2s}F(s)$

$F(s) = \frac{3}{2(2s-1)(s+1)} = \frac{A}{s} + \frac{B}{2s-1} + \frac{C}{s+1}$

$F(s) = \frac{-3}{s} + \frac{2}{s-\frac{1}{2}} + \frac{1}{s+1}$

$= -3 + 2e^{t/2} + e^{-t}$

Apply $y(t) = u(t-2)f(t-2)$

$= u(t-2)(-3 + 2e^{-1+t/2} + e^{2-t})$

Runge-Kutta

0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\frac{3}{4}$	0	$\frac{3}{4}$	0
	0	1	0
	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

$y' = -y^2 \quad y(0) = 1$

a) $h_0 = 0.2$ preform 1 step

b) Compute local error $\hat{\varepsilon}_1 = |y_1 - \hat{y}_1|$

c) $tol = 10^{-3}$ $P = 0.8$ Compute h_{new}

a) $K_1 = f(t_0, y_0) = f(t_0, 1) = -1$

$K_2 = f(t_n + \frac{1}{2}h, 1 + (\frac{1}{2} \cdot -1) \cdot 0.2)$

$= f(\cdots, 0.9) = -0.81$

$K_3 = f(\cdots, 1 + (0 \cdot K_1 + \frac{3}{4} \cdot -0.81) \cdot 0.2)$

$= f(\cdots, 0.8785) = -0.7718$

$y_1 = 1 + \frac{2}{9} \cdot -1 + \frac{1}{3} \cdot -0.81 + \frac{4}{9} \cdot -0.7718$

$= 0.83295\overline{111}$

b)

$\hat{\varepsilon}_{n+1} = h|(b_1 - \hat{b}_1)K_1 + (b_2 - \hat{b}_2)K_2 + (b_3 - \hat{b}_3)K_3$

$\hat{\varepsilon}_1 = 0.2 \cdot (\frac{2}{9} \cdot -1 - \frac{2}{3} \cdot -0.81 + \frac{4}{9} \cdot -0.77)$

$= 5.04\overline{88} \cdot 10^{-3}$

c) $h_{new} = P\left(\frac{Tol}{\hat{\varepsilon}_1}\right)^{\frac{1}{P+1}} \cdot h$

$= P\sqrt[3]{\frac{Tol}{\hat{\varepsilon}_1}} \cdot h = 0.09324$

$u_t(x,t) = 4u_{xx}(x,t)$

$u(-1,t) = 0 \quad u(1,t) = 0 \quad t > 0$

$u(x,t) = F(x)G(t)$

$F(x)G'(t) = 4F''(x)G(t)$

$\frac{G'(t)}{4G(t)} = \frac{F''(x)}{F(x)} = k$

$F'' = kF$

$G' = 4kG$

$u(x,t) = F(x)G(t)$

$F(x)G''(t) = c^2 F'(x)G(t)$

$\frac{G''}{c^2 G} = \frac{F''}{F}$

Now we consider possible solutions to the equations:

$k > 0$: Denote $p = \sqrt{k} > 0$

$F(x) = Ae^{px} + Be^{-px}$

$F(-1) = Ae^{-p} + Be^{p} = 0$

$F(1) = Ae^{p} + Be^{-p} = 0$

We get $A = -Be^{2p}$

$0 = -Be^{3p} + Be^{-p} = -B(e^{3p} - e^{-p})$

Only solution is $B = 0$ from $A = -Be^{2p}$

Not good? Trivial

$k = 0$:

$F(x) = A + Bx$

$F(-1) = A - B = 0$

$F(1) = A + B = 0$

We get $A = 0$ and $B = 0$

Not good? Trivial

$k < 0$: $F(x) = A\cos(px) + B\sin(px)$

$F(-1) = A\cos(-p) + B\sin(-p) = 0$

$F(1) = A\cos(p) + B\sin(p) = 0$

We got $A \cdot \cos(p) = 0$ so $A = 0$ or $\cos(p) = 0$ for $p = \left(\frac{2n+1}{2}\right)\pi$

Also got $B \cdot \sin(p) = 0$ so $B = 0$ or $\sin(p) = 0$ for $p = n\pi$

① $F(x) = \cos\left(\left(\frac{2n+1}{2}\right)\pi x\right)$, $G(t) = Ce^{-4kt} = Ce^{-4\left(\frac{2n+1}{2}\right)^2\pi^2 t}$ → We get the non-trivial solutions $u(x,t) = Ae^{-4\left(\frac{2n+1}{2}\right)^2\pi^2 t} \cdot \cos\left(\left(\frac{2n+1}{2}\right)\pi x\right)$

② $F(x) = \sin(n\pi x)$, $G(t) = Ce^{-4kt} = Ce^{-4n^2\pi^2 t}$ → $u(x,t) = Ae^{-4n^2\pi^2 t} \cdot \sin(n\pi x)$

Convolution w/ Laplace

$y'(t) - 5\int_0^t y(t-\tau)\cos\tau \, d\tau = 8\sin t$

$y'(t) - 5y(t) * \cos t = 8\sin t$

Then regular $\mathcal{L}$(Laplace) and ABC

Fixed Point

$x - \sqrt{\sin x} = 0$, $x_{k+1} = \sqrt{\sin x_k}$, $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

a) Show there is a solution r

Conditions:

1) must exist positive constant $L < 1$ so that $|g'(x)| \leq L$ for all $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

$|g'(x)| = \frac{\cos x}{2\sqrt{\sin x}}$ is decreasing.

So we only check $x = \frac{\pi}{6}$

$g'(\frac{\pi}{6}) = \frac{\sqrt{6}}{4}$

2) $g(x)$ stays within interval $g(x) \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ for all $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

b) $x_0 = \pi/2$, find upper bound error $|x_{k+1} - r|$ after $k = 60$.

$\vdots$

$|x_{k+1} - r| = \frac{L^{k+1}}{1-L}|g(x_0) - x_0|$

L is the answer in condition 1 above

$L = \frac{\sqrt{6}}{4}$, og regne ut resten

$|x_{61} - r| \leq \frac{(\sqrt{6}/4)^{61}}{1 - (\sqrt{6}/4)}\left|\sqrt{\sin\frac{\pi}{2}} - \frac{\pi}{2}\right|$

$\approx 1.5 \cdot 10^{-13}$

La $f(x) = \sum c_n e^{inx}$ på $[-\pi, \pi]$ interval

Vis at $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}$

Ser først at

$\int_{-\pi}^{\pi} e^{i(n-m)x} dx = \begin{cases} 2\pi & n = m \\ 0 & n \neq m \end{cases}$

Dersom vi ganger $f(x) = \sum c_n e^{inx}$ my e^{-imx} på begge sider og integrerer

$\int_{-\pi}^{\pi} f(x)e^{-imx}dx = \int_{-\pi}^{\pi} c_n e^{inx}e^{-imx}dx$

$= \sum c_n \int_{-\pi}^{\pi} e^{i(n-m)x}dx = 2\pi c_m$

eller $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$

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Newton Iteration Non-linear

$g(x) = -1.93\ln(x) + 15.9$

Vi må derved ta $f(x) = x - g(x)$

også iterere som vanlig

La $f(x) = \cos x$

Bruk diff formel til å tilnærme $f'(1) \sim -0.841470\dots$ med feil $< 10^{-9}$

Bruke $\frac{f(x+h) - f(x-h)}{2h}$

med $h = 0.00001$ gjør jobben.

Differansen $\frac{f(x+h) - f(x)}{h}$

klarer ikke å oppnå høyere presisjon enn 10^{-9}

Fourier Transform

Let $L > 0$ and $f(x) = \begin{cases} \cos(\lambda x) & \text{for } |x| \leq L \\ 0 & \text{else} \end{cases}$

Compute the Fourier Transform

$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-L}^{L}\cos(\lambda x)e^{-iwx}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-L}^{L}\frac{1}{2}(e^{i\lambda x} + e^{-i\lambda x}) \cdot e^{-iwx}dx$

$= \frac{1}{2\sqrt{2\pi}}\int_{-L}^{L} e^{ix(\lambda-w)} + e^{-ix(\lambda+w)}dx$

$\hat{f}(w) = \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{i(\lambda-w)}e^{ix(\lambda-w)} - \frac{1}{i(\lambda+w)}e^{-ix(\lambda+w)}\Big|_{-L}^{L}\right)$

$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2i(\lambda-w)}(e^{iL(\lambda-w)} - e^{-iL(\lambda-w)})\right.$

$\left.+ \frac{1}{2i(\lambda-w)}(e^{iL(\lambda+w)} - e^{-iL(\lambda+w)})\right)$

$= \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(L(\lambda-w))}{\lambda-w} + \frac{\sin(L(\lambda+w))}{\lambda+w}\right)$

$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}(L \cdot \text{sinc}(L(\lambda-w)) + \text{sinc}(L(\lambda+w)))$

q.e.d.