Approximation

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Approximation

Basis functions: $\{\phi_0(t), \ldots, \phi_n(t)\}$

$$p(x) = \sum_{j=0}^{n} c_j \phi_j(x) \tag{1}$$

$x = [x_0, \ldots, x_m]$ $\; m > n$ (often $m \gg n$)

$$p(x_j) \approx f(x_j) \tag{2}$$

Approximation error E:

	Discrete	Continuous
max	$\min_{p \in \mathbb{P}_n} \max_{0 \le j \le m} \lvert f(x_j) - p(x_j) \rvert$	$\min_{p \in \mathbb{P}_n} \max_{x \in [a,b]} \lvert f(x) - p(x) \rvert$
L_2^2	$\min_{p \in \mathbb{P}_n} \sum_{j=0}^{m} \lvert f(x_j) - p(x_j) \rvert^2$	$\min_{p \in \mathbb{P}_n} \int_a^b \lvert f(x) - p(x) \rvert^2$

Discrete least squares approximation

$$p(x) = \sum_{j=0}^{n} c_j x^j \tag{3}$$

$$S_k = \sum_{i=0}^{n} x_i^k, \ k = 0, 1, \ldots, 2m \tag{4}$$

$$T_k = \sum_{i=0}^{n} x_i^k y_i, \ k = 0, 1, \ldots m \tag{5}$$

$$\begin{aligned}
&c_0 \cdot S_0 + c_1 \cdot S_1 \quad + \cdots + c_m \cdot S_m \quad = T_0 \\
&c_0 \cdot S_1 + c_1 \cdot S_2 \quad + \cdots + c_m \cdot S_{m+1} = T_1 \\
&c_0 \cdot S_2 + c_1 \cdot S_3 \quad + \cdots + c_m \cdot S_{m+2} = T_2 \\
&\ldots \\
&c_0 \cdot S_k + c_1 \cdot S_{k+1} + \cdots + c_m \cdot S_{m+k} = T_k \\
&\ldots \\
&c_0 \cdot S_m + c_1 \cdot S_{m+1} + \cdots + c_m \cdot S_{2m} \quad = T_m
\end{aligned} \tag{6}$$

Discrete least squares approximation

$$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix}, \; c = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix}, \; y = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \dots \\ f(x_m) \end{bmatrix} \tag{7}$$

Normal equation:

$$A^T A c = A^T y \tag{8}$$

$$c = (A^T A)^{-1} A^T y = A^+ y \tag{9}$$

Continuous least squares approximation

$p_\star(t)$ to be optimal requires approximation error $E = E(c_0, \ldots, c_n)$ to satisfy

$$\frac{\partial E}{\partial c_0} = \ldots \frac{\partial E}{\partial c_n} = 0 \tag{10}$$

Continuous least squares approximation

$$\langle f, g \rangle = \int_{-1}^{1} w(x) f(x) g(x) dx, \tag{11}$$

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \dots & \langle \phi_0, \phi_n \rangle \\ \langle \phi_1, \phi_0 \rangle & \dots & \langle \phi_1, \phi_n \rangle \\ \dots & \dots & \dots \\ \langle \phi_n, \phi_0 \rangle & \dots & \langle \phi_n, \phi_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \dots \\ \langle f, \phi_n \rangle \end{bmatrix} \tag{12}$$

Continuous least squares approximation

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & 0 & \dots & 0 \\ 0 & \langle \phi_1, \phi_1 \rangle & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \langle \phi_n, \phi_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \dots \\ \langle f, \phi_n \rangle \end{bmatrix} \tag{13}$$

$$c_k = \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \tag{14}$$

$$p_\star = \sum_{k=0}^{n} c_k \phi_k = \sum_{k=0}^{n} \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k \tag{15}$$

Orthogonal polynomials

Name	Symbol	Interval	Weight function
Legendre	P_n	$[-1, 1]$	1
Chebyshev, 1st kind	T_n	$[-1, 1]$	$(1 - t^2)^{-1/2}$
Chebyshev, 2st kind	U_n	$[-1, 1]$	$(1 - t^2)^{1/2}$
Jacobi	J_n	$[-1, 1]$	$(1 - t)^{\alpha}(1 - t)^{\beta}$
Lagurerre	L_n	$[0, \infty)$	e^{-t}
Hermite	H_n	$(-\infty, \infty)$	e^{-t^2}

Orthogonal trigonometric functions

Functions	Interval	Weight
$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx), n = 1, 2 \ldots$	$[-\pi, \pi]$	1
$\frac{2}{\sqrt{\pi}} \sin(nx), n = 1, 2 \ldots$	$[0, \pi]$	1
$\frac{1}{\sqrt{\pi}}, \frac{2}{\sqrt{\pi}} \cos(nx), n = 1, 2 \ldots$	$[0, \pi]$	1

References

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