Ordinary differential equations

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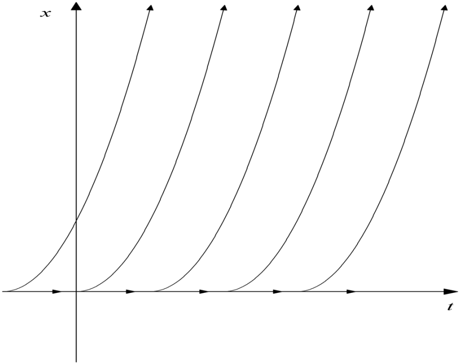
Existence and Uniqueness

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^{\beta},\ 0 < \beta < 1,\ x \geq 0$$



$$x(t) = (1-\beta)^{\frac{1}{1-\beta}}(t-C)^{\frac{1}{1-\beta}} \tag{1}$$

$$x(t) \equiv 0 \tag{2}$$

Existence and Uniqueness

Function $f(t, y)\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is *Lipschitz continuous* if there is constant L such, that for any $t \in [a, b]$ and any $y, \hat{y} \in \Omega$

$$||f(t, \hat{y}) - f(t, y)|| \leq L||\hat{y} - y||$$

If f is differentiable:

$$L = \max_{(t,y)\in D} ||J_f(t, y)||$$

Stability of a problem

$$y' = f(y,t)$$

$$y(t_0) = y_0$$
$$\hat{y}(t_0) = \hat{y}_0$$

Solution to $y' = f(y,t)$ is *stable* if for every ε there exists δ such that if

$$||\hat{y}(t_0) - y(t_0)|| < \delta$$

then

$$||\hat{y}(t) - y(t)|| < \varepsilon \text{ for } t \geq t_0.$$

Stable solution to $y' = f(y,t)$ is *asymptotically stable* if

$$||\hat{y}(t) - y(t)|| \to 0 \text{ for } t \to \infty.$$

Explicit Euler method

$$y_{k+1} = y_k + h_k f(t_k, y_k) \tag{3}$$

Stability:

$$|1 + h\lambda| < 1 \tag{4}$$

Implicit Euler method

$$y_{k+1} = y_k + h_k f(t_{k+1}, y_{k+1}) \tag{5}$$

Stability:

$$\left| \frac{1}{1 - h\lambda} \right| < 1 \tag{6}$$

Amplification factor

$$\varepsilon_{n+1} = Q(\lambda h)\varepsilon_n \quad (7)$$

Method	Amplification factor
Explicit Euler	$1 + \lambda h$
Implicit Euler	$\frac{1}{1-\lambda h}$
Trapezoidal	$\frac{1+\frac{1}{2}\lambda h}{1-\frac{1}{2}\lambda h}$
Modified Euler	$1 + \lambda h + \frac{1}{2}(\lambda h)^2$
RK4	$1 + \lambda h + \frac{1}{2}(\lambda h)^2 + \frac{1}{6}(\lambda h)^3 + \frac{1}{24}(\lambda h)^4$

Local and global error

Local error

$$\ell_k = \frac{y_k - u_{k-1}(t_k)}{h_k} \tag{8}$$

Global error

$$e_k = y_k - y(t_k) \tag{9}$$

Local error

Method	Local error
Explicit Euler	$O(h)$
Implicit Euler	$O(h)$
Trapezoidal	$O(h^2)$
Modified Euler	$O(h^2)$
RK4	$O(h^4)$

Methods

Fixed step	Adaptive step
Euler method	Runge-Kutta 1(2), Adaptive Heun
Midpoint method	Runge-Kutta 2(3), Bogacki-Shampine
Runge-Kutta 4, 3/8 rule	Runge-Kutta 4(5), Dormand-Prince
Explicit Adams-Bashforth	Runge-Kutta 7(8), Dormand-Prince-Shampine
Implicit Adams-Bashforth-Moulton	

References

1. http://heath.cs.illinois.edu/scicomp/notes/cs450_chapt09.pdf