$$V(X_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{\beta^t c_t^{1-\sigma}}{1-\sigma}$$

s.t. $c_t + X_{t+1} \le X_t$

$$V(x_0) = \max_{\Gamma(x_0)} \sum \frac{\beta^t}{1-\sigma}(x_t - x_{t+1})^{1-\sigma}$$

$$V(x_0) = \max_{\Gamma(x_0)} \left\{ \frac{1}{1-\sigma}(x_0 - x_1)^{1-\sigma} + \beta \sum_{t=1}^{\infty} \frac{\beta^{t-1}}{t=1} (x_t - x_{t+1})^{1-\sigma} \right\}$$

$$\Gamma(x_0) = \{0 \le x_{t+1} \le x_t\}$$

$$V(x) = \max_{0 \le y \le x} \left\{ \frac{1}{1-\sigma}(x-y)^{1-\sigma} + \beta v(y) \right\}$$

Gess:

$$v(x) = \frac{L x^{1-\sigma}}{1-\sigma}, \quad g(x) = \theta x$$

$$v'(x) = L x^{-\sigma}, \quad L, \theta = ?$$

$$\frac{\partial v}{\partial y} = -(x-y)^{-\sigma} + \beta v'(y) = 0$$

$$\beta v'(y) = (x-y)^{-\sigma}$$

$$\beta L y^{-\sigma} = (x-y)^{-\sigma}$$

⋮

$$y = \frac{\sqrt[\sigma]{L\beta}}{\sqrt[\sigma]{L\beta}+1} x \quad \Big| \quad \gamma = \sqrt[\sigma]{L\beta} = L^{1/\sigma}\beta^{1/\sigma}$$

$$\theta = \frac{\gamma}{\gamma+1} \leftarrow y = \frac{\gamma}{\gamma+1} x \quad \text{(optimal value)}$$

$$\gamma = L^{1/\sigma} \cdot \beta^{1/\sigma} \quad \uparrow 1-\sigma$$

$$\gamma^{1-\sigma} = L^{\frac{1-\sigma}{\sigma}} \beta^{\frac{1-\sigma}{\sigma}}$$

(substitute guess in Bellman)

$$\frac{L x^{1-\sigma}}{1-\sigma} = \frac{1}{1-\sigma}\left(x - \frac{\gamma}{1+\gamma}x\right)^{1-\sigma} + \frac{L\beta\left(\frac{\gamma}{1+\gamma}x\right)^{1-\sigma}}{1-\sigma}$$

$$= \frac{1}{1-\sigma}\left\{ \left(\frac{x}{1+\gamma}\right)^{1-\sigma} + \frac{L\beta(\gamma x)^{1-\sigma}}{(1+\gamma)^{1-\sigma}} \right\}$$

$$= \frac{1}{1-\sigma}\left\{ \frac{x^{1-\sigma} + L\beta(\gamma x)^{1-\sigma}}{(1+\gamma)^{1-\sigma}} \right\}$$

$$= \frac{x^{1-\sigma}}{1-\sigma}\left[\frac{1 + L\beta\gamma^{1-\sigma}}{(1+\gamma)^{1-\sigma}} \right] \text{— } L$$

$$L = \frac{1 + L\beta\gamma^{1-\sigma}}{(1+\gamma)^{1-\sigma}}$$

$$L(1+\gamma)^{1-\sigma} = 1 + L\beta\gamma^{1-\sigma}$$

$$L(1+\gamma)^{1-\sigma} = 1 + L^{1/\sigma} \cdot \beta^{1/\sigma}$$

$$(1+\gamma)\left[L \cdot (1+\gamma)^{-\sigma} - 1\right] = \emptyset$$

$$(1 + L^{1/\sigma}\beta^{1/\sigma})\left[L(1 + L^{1/\sigma}\beta^{1/\sigma})^{-\sigma} - 1\right] = 0$$

I.

$$1 + L^{1/\sigma}\beta^{1/\sigma} = \emptyset$$

$$L^{1/\sigma} = \frac{-1}{\beta^{1/\sigma}} \quad \uparrow \sigma$$

$$L = \frac{(-1)^{\sigma}}{\beta} \rightarrow \gamma = -1$$

$$\theta = \frac{-1}{0}$$ ⊗ infeasible!

II.

$$L(1 + L^{1/\sigma}\beta^{1/\sigma})^{-\sigma} - 1 = \emptyset$$

$$L = (1 + L^{1/\sigma}\beta^{1/\sigma})^{\sigma} \quad \uparrow 1/\sigma$$

$$L^{1/\sigma} = 1 + L^{1/\sigma}\beta^{1/\sigma}$$

$$L^{1/\sigma}\left[1 - \beta^{1/\sigma}\right] = 1 \quad \uparrow \sigma$$

$$\boxed{L = \frac{1}{(1-\beta^{1/\sigma})^{\sigma}}}$$

SOLUTION!

TASK B

(b) $\sum_{t=0}^{\infty} \beta^t \frac{(c_t - \gamma c_{t-1})^{1-\sigma}}{1-\sigma}$

$\boxed{\begin{array}{l} 0 \le X_{t+1} \le X_t \\ 0 \le X_t \le X_{t-1} \end{array}}$

$C_t = X_t - X_{t+1}$; $C_{t-1} = X_{t-1} - X_t$

- $\sum_{t=0}^{\infty} \frac{\beta^t}{1-\sigma} (X_t - X_{t+1} - \gamma X_{t-1} + \gamma X_t)^{1-\sigma}$

- $\sum_{t=0}^{\infty} \frac{\beta^t}{1-\sigma} \left[X_t (1+\gamma) - X_{t+1} - \gamma X_{t-1}\right]^{1-\sigma}$

- $\underbrace{\frac{1}{\beta} \frac{1}{1-\sigma} \left[-X_0\right]^{1-\sigma}}_{t=-1} + \underbrace{\frac{1}{1-\sigma} \left[X_0 (1+\gamma) - X_1\right]^{1-\sigma}}_{t=0} + \underbrace{\frac{\beta}{1-\sigma} \left[X_1 (1+\gamma) - X_2 + \gamma X_0\right]^{1-\sigma}}_{t=1} +$

$\beta \sum_{t=2} \frac{\beta^{t-1}}{1-\sigma} \left[X_t (1+\gamma) - X_{t+1} + \gamma X_{t-1}\right]^{1-\sigma}$

$X_{t-1} \rightarrow X$
$X_t \rightarrow Y$
$X_{t+1} \rightarrow Z$

- $V(X) = \max_{Y,Z} \left\{ \frac{1}{\beta} \cdot \frac{1}{1-\sigma} \left[-X\right]^{1-\sigma} + \frac{1}{1-\sigma} \left[X(1+\gamma) - Y\right]^{1-\sigma} + \frac{\beta}{1-\sigma} \left[Y(1+\gamma) - Z + \gamma X\right]^{1-\sigma} + \beta V(Y,Z) \right\}$

s.t. $0 \le Z \le Y$
$0 \le Y \le X$

TASK C

(c) $\sum_{t=1}^{\infty} \beta^t \frac{(c_t - \gamma c_{t-2})^{1-\sigma}}{1-\sigma} = \sum_{t=1}^{\infty} \frac{\beta^t}{1-\sigma} \left[X_t - X_{t+1} - \gamma X_{t-2} + \gamma X_{t-1}\right]^{1-\sigma}$

$\underbrace{\frac{1}{\beta^2} \frac{1}{1-\sigma} \left[-X_0\right]^{1-\sigma}}_{t=-1} + \underbrace{\frac{1}{1-\sigma} \left[X_0 - X_1\right]^{1-\sigma}}_{t=0} + \underbrace{\frac{\beta}{1-\sigma} \left[X_1 - X_2 + \gamma X_0\right]^{1-\sigma}}_{t=1} + \underbrace{\frac{\beta^2}{1-\sigma} \left[X_2 - X_3 - \gamma X_1 + \gamma X_0\right]^{1-\sigma}}_{t=2}$

$+ \beta \sum_{t=3} \frac{\beta^{t-1}}{1-\sigma} \left[X_t - X_{t+1} - \gamma X_{t-2} + \gamma X_{t-1}\right]^{1-\sigma}$

$\boxed{\begin{array}{l} C_t = X_t - X_{t+1} \\ C_{t-2} = X_{t-2} - X_{t-1} \end{array}}$

$V(X) = \max_{Y,Z,W} \left\{ \frac{1}{\beta} \frac{1}{1-\sigma} (-X)^{1-\sigma} + \frac{\beta}{1-\sigma} \left[X - Y\right]^{1-\sigma} + \frac{\beta^2}{1-\sigma} \left[Y - Z + \gamma X\right] + \frac{\beta^3}{1-\sigma} \left[Z - W - \gamma Y + \gamma X\right] + \beta V(Y,Z,W) \right\}$

$\boxed{\begin{array}{l} 0 \le W \le Z \\ 0 \le Z \le Y \\ 0 \le Y \le X \end{array}}$