Macro Ia
Fall 2025

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Problem Set 2

1 Eating a cake

Consider the problem of a consumer who has a cake x_t in period t and wants to decide how much of the cake to eat each period. Preferences over sequences of consumption are

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma},$$

where $0 < \beta < 1$ and $\sigma > 0$. The resource constraint is

$$c_t + x_{t+1} \leq x_t,$$

where $x_{t+1} \geq 0$ is the amount of cake left at the end of the current period. Consumption cannot be negative, $c_t \geq 0$.

a. Provide a recursive formulation for the consumer's decision problem. Guess that the value function v and optimal policy function $x' = g(x)$ are of the form:

$$v(x) = \alpha \frac{x^{1-\sigma}}{1-\sigma}, \quad g(x) = \theta x.$$

Find v and g.

b. Now assume that the consumer exhibits *habit persistence*, and that the utility function is given by

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t - \gamma c_{t-1})^{1-\sigma}}{1-\sigma},$$

where $0 < \gamma < 1$. What is the state variable for the consumer's decision problem? Formulate the Bellman equation corresponding to this problem (you do not need to solve it).

c. Now, assume that the consumer's habit operates with an additional lag of one period and the utility function is given by

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t - \gamma c_{t-2})^{1-\sigma}}{1-\sigma}.$$

Formulate the Bellman equation for this case (again, no need to solve it).

2 Computing the optimal growth model with dynamic programming

Dynamic programs can be solved using paper and pencil methods only in very special cases. This problem is intended to get you started in using the computer to solve more general problems. You will use the iteration on the value function method to compute the value function and the associated policy function for the optimal growth model with arbitrary depreciation.

The production technology in the economy is given by a production function $F(k) = k^\alpha$, $\alpha \in (0,1)$, where k is the capital stock used to produced output y. Given the initial capital stock k_0, the agent's problem is

$$\max_{\{c_t, k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \tag{1}$$
$$\text{s.t.} \qquad k_{t+1} = k_t^\alpha + (1-\delta)k_t - c_t, \qquad t \geq 0$$

where $0 < \beta < 1$ is the discount factor. We know that the maximized value of the objective function in (1) depends only on k_0. Call it $v^*(k_0)$. We know that this value function satisfies the Bellman Equation

$$v^*(k) = \max_{c,k'} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \beta v^*(k') \right\} \tag{2}$$
$$\text{s.t.} \qquad k' = k^\alpha + (1-\delta)k - c.$$

Unfortunately, just by looking at (2) it is hard to say much about v^*. But the functional equation can also be used to define an algorithm for computing v^* numerically. In particular for any function w define another function Tw by

$$Tw(k) = \max_{c,k'} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \beta w(k') \right\} \tag{3}$$
$$\text{s.t.} \qquad k' = k^\alpha + (1-\delta)k - c.$$

Then for any initial function v_0, we can define a sequence of functions $\{v_n\}_{n\geq 0}$ recursively by the formula $v_{n+1} = Tv_n$.

The exercise will illustrate that $\lim_{n\to\infty} v_n = v^*$ for any initial function v_0. It will also illustrate that a thoughtful choice of v_0 makes the procedure converge more quickly. (This will be shown by counting the number of iterations. The system is so simple that the computation time is very short, even if a poor guess is used.) For all the computations, use the parameter values

$$\alpha = 0.4,\ \delta = 0.04,\ \sigma = 1,\ \beta = 0.96,\ k = [5, 20].$$

Use a grid of 1000 equally spaced points and a convergence criterion of $\varepsilon = 0.0001$.

a. Let $v_0(k) = 0$ all k and compute the sequence $\{v_n\}$ until the convergence criterion

$$|v_N(k) - v_{N-1}(k)| < \varepsilon, \qquad \text{all } k$$

is satisfied. (i.e., the maximal difference between successive approximations is smaller than ε. How many iterations are required? Plot the calculated value and policy functions, $v_N(k)$, $c_N(k)$ and $k'_N(k)$.

b. What are the steady state levels of k^{ss}, c^{ss} for the problem in part (a)? Let $\eta = \frac{c^{ss}}{F(k^{ss})}$ be the steady state fraction of output that is consumed. Suppose that the agent follows a feasible strategy of simply consuming a constant fraction of output η each period, so

$$\hat{c}(k) = \eta k^{\alpha},\ \text{all } k.$$

What is the discounted lifetime utility, call it $\hat{v}_0(k)$, of an agent who follows this rule? Now use $\hat{v}_0(k)$ as a starting point for the iterative procedure. Use the same convergence criterion as before to stop. How many iterations does it take to converge? Plot the new calculated value function. Is it the same as in part (a)? Compare the value function v_N with the guess $\hat{v}_0$ to see how good it was as a guess.

c. Suppose that the initial capital stock k_0 is 25 percent below the steady state level. Use the optimal policy function $k'_N(k)$ to compute the optimal transition to the steady state. Plot the transition path.

Solve the problem in Matlab. Send Shuai a PDF copy of your solution, plus a working Matlab code.