Law of Large Numbers

Suppose we have $x_i : \Omega \to \mathbb{R}^d \quad i = 1, ..., \infty$ variables.

Definition (*Identically distributed*)
(a) The *law* of x_i is the measure

$$\mu_i := law(x_i) \ defined \ on \ \mathbb{R}^d \ \ by \mu_i[B] = \mathbb{P}\{x_i \ \epsilon \ B\} \ \forall \ B \ \subset \mathbb{R}^d$$

(b) x_i and x_j are identically distributed if

$$law(x_i) = law(x_j)$$

Definition (*Independence*) $(x_i)_i^\infty$ are independent for every $i_1, ..., i_k$ and every $A_1, ..., A_k \subset \mathbb{R}^d$ Borel

$$P(x_{i1} \ \epsilon \ A_1, ..., x_{ik} \ \epsilon \ A_k) = P(x_{i1} \ \epsilon \ A_1) \cdots P(x_{ik} \ \epsilon \ A_k)$$

Theorem (*Law of Large Numbers*) Let $(Z_n)_{n=1}^\infty$ be a random variable, we say that $(z)_n$ converges to z in probability if $\forall \ \varepsilon > 0$

(weak law)

$$\lim_{n\to\infty} \mathbb{P}\{|z_n - z| \geq \varepsilon\} = 0$$

(strong law)

$$\lim_{n\to\infty} z_n = z \ \ a.e$$

Proof

If $\lim\limits_{n\to\infty} \mathbb{E}[z_n - z] \implies z_{nk} \to z$ a.e for a subsequence $(z_{nk})_k$
Assume we have convergence in probability,

$$\mathbb{E}[z_n - z] = \int_\Omega |z_n - z| d\mathbb{P}$$

$$\implies \mathbb{E}[z_n - z] = \int_{|z_n - z| < \varepsilon} |z_n - z| d\mathbb{P} + \int_{|z_n - z| \geq \varepsilon} |z_n - z| d\mathbb{P} \ \leq \ \varepsilon \, \mathbb{P}|z_n - z| < \varepsilon + ... (missed \ what \ was \ written \ here)$$

$$\implies \mathbb{E}[z_n - z] = \int_{|z_n - z| < \varepsilon} |z_n - z| d\mathbb{P} + \int_\Omega |z_n - z| \chi^2_{A_n^\varepsilon} d\mathbb{P} \ \leq \varepsilon \ \mu\{|z_n - z| \ \leq \ \varepsilon\} + (\int_\Omega |z_n - z|^2 d\mathbb{P})^{\frac{1}{2}} (\int_\Omega \chi^2_{A_n^\varepsilon} d\mathbb{P})^{\frac{1}{2}}$$

We get this from Holder's inequality, which says

$$\int_\Omega |fg| d\mathbb{P} \ \leq \ \sqrt{\int_\Omega f^2 d\mathbb{P}} \sqrt{\int_\Omega g^2 d\mathbb{P}}$$

Now going back to our equation

$$\leq \varepsilon \mathbb{P}(\Omega) + \sqrt{\int_n |z_n - z|^2 d\mathbb{P}} \sqrt{\mathbb{P}[A_n^\varepsilon]}$$

$$\leq \varepsilon + \sqrt{2var(z_n) + 2var(z)}\sqrt{\mathbb{P}[A_n^{\varepsilon}]}$$

From this we can conclude that if $var(z_n) \leq C$ and $z_n \xrightarrow{p} z$, then

$$\mathbb{E}[z_n - z] \ \leq \ \varepsilon + \sqrt{4C} \cdot \mathbb{P}$$

and so,

$$\overline{\lim_{n\to}} |z_n - z| \leq \ \varepsilon \ \forall \ \varepsilon$$

Thus,

$$\overline{\lim_{n\to\infty}} \mathbb{E}[|z_n - z|] = 0$$

<u>**Corollary**</u> If $z_n \xrightarrow{p} z$ the there exists a subsequence $(z_{nk})_{k=1}^{\infty}$ which converges to z in probability a.e.

It is important to note that we don't know if the whole sequence converges in probability a.e, so that is why we consider the subsequence.

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If $x_i : \Omega \to \mathbb{R}^d \quad i = 1, ..., \infty$ are independent and identically distributed (iid) then,

$$\frac{x_1 + ... + x_n}{n} \cong E(X) \ \ for \ n \ large \ enough$$

Additionally, for any two points , w and a,

$$\frac{x_1(w) + ... + x_n(w)}{n} - \frac{x_1(a) + ... + x_n(a)}{n} \to 0 \quad when \ n \to \infty$$

except when $a, w \ \epsilon \ N \ and \ P(N) = 0$

Essentially, for n large enough if we know the expectation at one point we know the expectation at any other point.

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The Law of Large numbers is often used in conjunction with the Central Limit theorem.

<u>**Theorem**</u> *(Central Limit Theorem)* (CLT) If $x_1, ..., x_n$ is a random sample from a distribution with mean μ and variance $\sigma^2 < \infty$ then the limiting distribution of

$$Z_n = \frac{\sum_{i=1}^{n} X_i \ - n\mu}{\sqrt{n}\sigma}$$

is the standard normal, $Z_n \xrightarrow{d} Z \sim N(0, 1)$ as $n \to \infty$.

The key idea behind the CLT is that is can be used to approximate a distribution in cases where the exact distribution is unknown or intractable.

<u>**Remarks**</u>

• n = 30 is sufficiently large for the approximations using the CLT.

• The average of the sample means and standard deviations will equal the population mean and standard deviation.