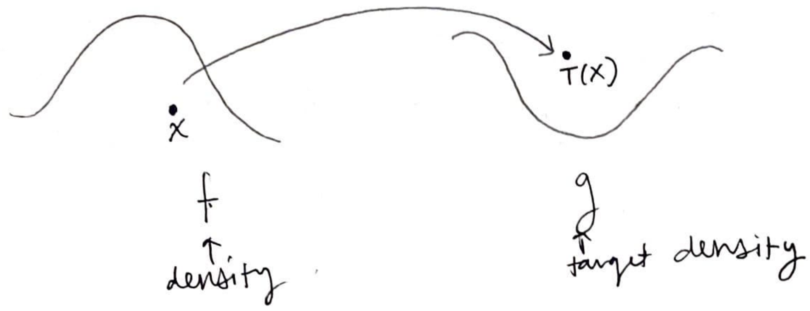
Lecture 1 Intro to Optimal Transport

Monge Problem

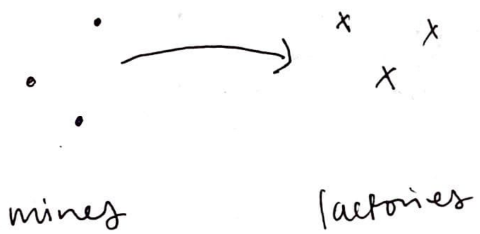


for ea. unit of mass we want to

minimize $\int |x - T(x)| f(x)\, dx$

cost func.
(distance a unit of mass traveled)

eg: mines $\rightarrow$ factory



Consider other costs $c(x, y)$

$$\boxed{\text{minimize } \int c(x, T(x))\, f(x)\, dx}$$

* could have densities that come from an ~~image~~ pixel intensity in an image
  → MRI

can generally set this problem up using measures

→ we have a source measure μ

→ also have target measure ν

- $\mu(E)$ tells us how much mass is in the set E

- require mass balance

→ total mass in source equals total mass in target

$$\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$$

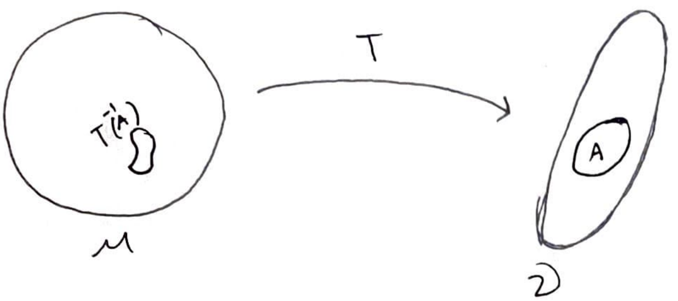
Now we seek a transport map $T(x)$

→ moves mass from the source to the target

- suppose source is supported on set X and target is supported on Y

$T: X \to Y$ where X & Y are subsets of $\mathbb{R}^n$

- want to conserve mass:



want to define the measure that the map T produced

- To define the measure we need to know how it acts on diff. sets

Require ~~[illegible]~~ $\boxed{\mu(T^{-1}(A)) = \nu(A) \quad \forall A \subset Y}$ ← This defines our measure

The amount of mass in the set A on the target measure, ν, is equivalent to that in the set $T^{-1}(A)$ on the source measure, i.e.

$\mu(T^{-1}(A))$ is called the push forward of μ through T, denoted

$T_{\#}\mu$

mass (or measure) conservation: $T_{\#}\mu = \nu$

This is the Monge formulation of OT problem

$$\boxed{\min \left\{ \int_{\mathbb{R}^n} c(x, T(x)) \, d\mu(x) \;\middle|\; \overbrace{T_{\#}\mu = \nu}^{\text{correct map}} \right\}}$$

mass balance

weighting it by how much mass we move (source measure)

* the space that we're minimizing over map that satisfies the constraint

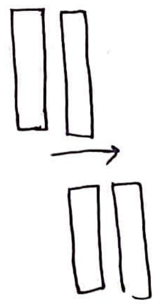
Relevant Questions

- Regularity ???
- does a minimizer actually exist?
- uniqueness?
- stability ???
- feasibility ???

- can we actually compute soln?

Examples

- Book moving problem



Consider a few cost funcs

- Monge cost:

$$c(x, y) = |x - y|$$

There are 2 mass-preserving plans

→ both have cost of 2 in this case

⇒ Non uniqueness

- Quadratic cost: ← commonly used

$$c(x, y) = \frac{1}{2}|x - y|^2$$

- we have unique soln in this case

→ ~~There are cases of quadratic costs where~~

There are cases of quadratic costs where we have non-uniquess

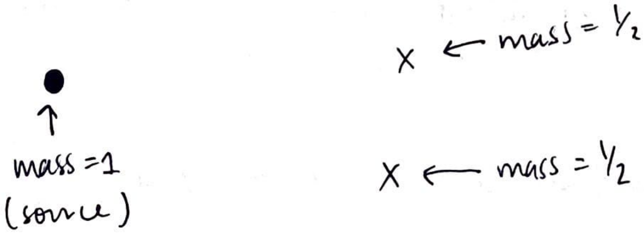
Ex:

either way up rearrange there is no difference in the cost

⇒ non-uniquess

NOTE uniqueness can be an issue w/ point masses

Feasibility



- This can't be done unless we allow mass to split, but in the Monge formulation T as a mapping does not allow splitting of mass

⇒ No feasible mapping

So we need new formulation, so we try to generalize this ~~for~~ via

Kantorovich formulation

→ ideally this formulation should be equivalent to the Monge formulation when it makes sense AND allow for splitting of mass

Now seek transport plan instead of map [redacted] which allows mass to be split.

→ need to be working w/ measures instead of densities

Again we have a source measure, μ, supported on a set X and a target measure ν supported on a set Y.

Given any point x or any set, we need to decide how much mass moves from x to any given point y (need to check this for many points y)

→ we store this mass in a measure π, which must be defined on the product space $X \times Y$

Ex.

mine @ $x = 0$ w/ 1 unit of resource

factories @ $y = 0, 1$ w/ $1/3$, $2/3$ unit respectively

$\pi(0,0) = 1/3$ $\qquad$ $\pi(0,1) = 2/3$ $\qquad$, $\pi(0, \mathbb{R}) = 1$

↑ how much mass goes from $x=0$ to $y=0$

* In general if $A \subset X$ & $B \subset Y$, then $\pi(A,B)$ tells us how much mass moves from $A \to B$

Now we need to formulate OT problem that can solve for this

~~still need mass conserva~~

still need mass conservation (consider what kind of constraint this imposes on set π)

- choose some point $x \in X$
- consider $\pi(x, Y) = \mu(x)$ ← constraint

→ * More generally, if $A \subset X$

$$\pi(A, Y) = \mu(A)$$

~~we say~~

⟺ The amnt. of mass that moves from the set A on the ~~measure~~ space X to ANYWHERE on the ~~measure~~ space Y must be equal to the total amnt of mass that is in set A on the measure μ

* we say μ is the marginal of π on X

→ b/c we integrate away all y dependence

Also we need ν to be the ~~m~~ marginal of π on Y

$\Rightarrow$ If $B \subset Y$ then $\pi(X, B) = \nu(B)$ $\leftarrow$ constraint

* so now we have a new constraint
  - instead of having a condition on T and looking the push forward of the measure under T
  - now we have constraints on the marginals of π, μ & ν

Now how do we measure cost?

- $c(x,y)$ is now weighted by the amount of mass that moves from x to y

This is ~~Katorovich~~ formulation.

$$\boxed{\inf \left\{ \int_{X \times Y} c(x,y)\, d\pi(x,y) \,\middle|\, \pi \in \Pi(\mu,\nu) \right\}}$$

$\uparrow$ *This objective func. is linear in π

looking over measures π that belong to the set $\Pi(\mu,\nu)$. where $\Pi(\mu,\nu)$ consists of measures that have correct marginals.
$\rightarrow$ marginals on X & Y are μ & ν

*This admits a dual formulation that we will * eventually take back to Monge problem

Special Cases (common)

- discrete optimal transport
  $\rightarrow$ mapping a collection of dirac masses to another collection of dirac masses

- continuous OT
  $\rightarrow$ μ & ν are absolutely continuous w/ densities f, g

- ~~discrete~~ Semi-discrete
  $\rightarrow$ μ is absolutely cont. and ν consists of diracs

Let's start in 1 dimension...

· Assume things are "nice enough" for me to work w/

GOAL: ~~FEX~~ Find $T(x)$ to minimize

$$\frac{1}{2}\int_{\mathbb{R}} (x - T(x))\, f(x)\, dx \quad \leftarrow \text{monge}$$

$$\text{s.t.} \quad \int_{T^{-1}(A)} f(x)\, dx = \int_A g(y)\, dy \quad \forall \subset \mathbb{R}$$

- Alternatively can write constraint as:

for all continuous func's on X.

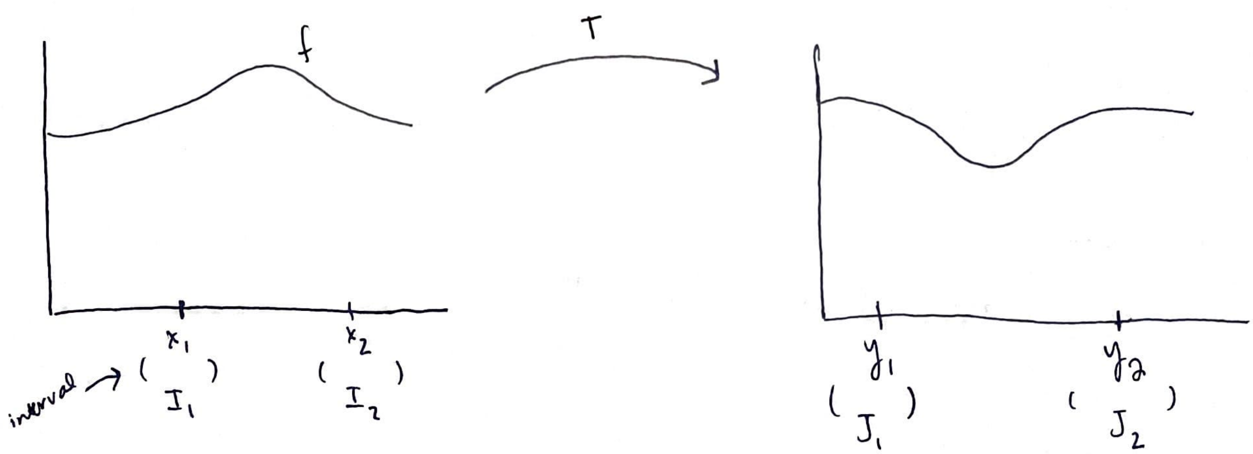
$$\int_X h(T(x))\, f(x)\, dx = \int_Y h(y)\, g(y)\, dy \quad \forall\, h \in C^0(x)$$

$\Rightarrow$ Given any func. h, this map $T(x)$ should preserve measure if we integrate $h(y)g(y)$ over y

Can we learn properties of optimal map if it exists?

· Let's pick two ~~E~~ points x_1 & x_2 when $x_1 < x_2$ & some $\varepsilon > 0$, and make a little open intervals $x_1 \in I_1$ & $x_2 \in I_2$ s.t.

$$\int_{I_1} f(x)\, dx = \varepsilon = \int_{I_2} f(x)\, dx \quad \longleftarrow \text{this means the total amnt of mass in ea. interval is } \varepsilon$$



$y_i = T(x_i)\,,\quad J_i = T(I_i)$

· consider a case where y_1 & y_2 are swapped...

So we permute part of the map and create a new measure-preserving map s.th. $\tilde{T}(x_1) = y_2$ & $\tilde{T}(x_2) = y_1$

$$\tilde{T}(I_1) = J_2 \qquad \tilde{T}(I_2) = J_1$$

elsewhere $\tilde{T}(x) = T(x)$ if $x \notin I_1 \cup I_2$

· T was optimal (for arguments sake)

$$\Rightarrow \frac{1}{2}\int_{\mathbb{R}} (x - T(x))^2 f(x)\,dx \le \frac{1}{2}\int_{\mathbb{R}} (x - \tilde{T}(x))^2 f(x)\,dx$$

NOTE the integral of $T(x)^2$ is the same as that of $\tilde{T}(x)^2$ b/c I have measure preserving map
⇒ only concerned about cross terms

$$\Rightarrow -\int_{I_1} x\,T(x) f(x)\,dx - \int_{I_2} x\,T(x) f(x)\,dx$$

$$\le -\int_{I_1} x\,\tilde{T}(x) f(x)\,dx - \int_{I_2} x\,\tilde{T}(x) f(x)\,dx$$

$$\Rightarrow \frac{1}{\varepsilon}\int_{I_1} x(\tilde{T}(x) - T(x)) f(x)\,dx + \frac{1}{\varepsilon}\int_{I_2} x(\tilde{T}(x) - T(x)) f(x)\,dx \le 0$$

Now we take limit... what happens as $\varepsilon \to 0$?

$\Rightarrow x \to x_1$

$\Rightarrow T(x_1) \to y_1$

$\Rightarrow \tilde{T}(x_1) \to y_2$

$$x_1(y_2 - y_1) + x_2(y_1 - y_2) \le 0$$

$$\Rightarrow (y_2 - y_1)(x_2 - x_1) \ge 0$$ ← This means our map is moving everything in one direction, i.e. it's monotone

↑ ???

* in the case of quadratic cost, the optimal map is monotone

SUMMARY

We start with Monge Problem. Looking for way to minimize cost of transporting mass from one ~~target~~ density to another "target" density. Set the problem up using measures. Introduce the requirement of mass balance, which leads us to seek a measure preserving map. This gives us the constraint $T_{\#}\mu = \nu$. We consider the limitation of the Monge formulation which is the inability to split mass. We seek a new more general formulati that allows for the splitting of mass, and we find the Kantorovich formulation. We now see a transport plan w/ a new constraint $\pi(A, Y) = \mu(A)$. Now instead of having a cond'n on T & looking @ the pushforward, we have constraints on the marginals of the measur

a that contains the amnt. of mass moved from a point/set X/A to Y. We the consider the 1-D case in order to learn about properties of the optimal map. We learn that the optimal map is monotone. (moving everything in one direction)

LEC OUTLINE

- INTRO TO MONGE PROBLEM
- MONGE FORMULATION
  - → LIMITATIONS
- KANTOROVICH FORMULATION
- CONSIDER THE 1-D CASE (DISCRETE)
- PROPERTIES OF OPTIMAL MAP