Derivation and Numerical Solution to the Double Pendulum Trajectories

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[*]www.github.com/dominuszain/DoublePendulumSimulation/tree/main

Abstract

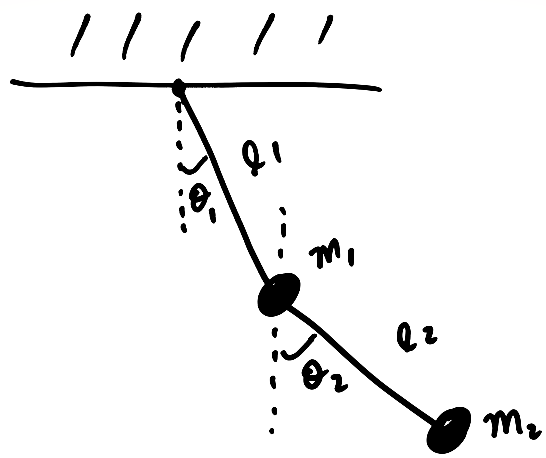
In this study, we first derived the governing equations for the trajectories of the masses in a double pendulum and then solved those differential equations in the Scilab language. Two separate Euler-Lagrange equations were solved symbolically as the degrees of freedom of our system are two. The two obtained differential equations had to be broken down into a total of 4 first order ones to be solved numerically in Scilab. Results were generated for different initial conditions, and were visualized in scatter plots and animations.

1 Introduction

Lagrangian Mechanics is an alternate formulation of Classical Mechanics to Newtonian Mechanics. Instead of dealing with forces, we deal with energies here. The Centeral quantity in it is the Lagrangian, which is defined as the difference of the kinetic and the potential energies. The lagrangian, when substituted into the Euler-Lagrange equation, gives the path that the particle will take classically. This works because the principle of least action holds true. Particles always take the most optimized trajectories. Lagrangian mechanics just capitalizes on this fact. The double pendulum is an example of a chaotic system i.e. a system that is extremely sensitive to initial conditions. Lagrangian Mechanics is extremely helpful in dealing with such system, as it is much easier to work with energies here rather than forces. The derivations were all done symbolically. The obtained differential equations were then solved in the Scilab language. The choice for Scilab for the numerical computations was made because of its rich collection of built-in functions, and it's openness.

2 Results and Discussions

To approach the solution to this problem, first we need to define the quantities. The definitions that we took are captured in the figure attached.



Using the specified definitions, we can define the positional cartesian co-ordinates in the following way:

$$x_1 = l_1 Sin(\theta_1) \tag{1}$$
$$y_1 = -l_1 Cos(\theta_1) \tag{2}$$
$$x_2 = l_1 Sin(\theta_1) + l_2 Sin(\theta_2) \tag{3}$$
$$y_2 = -l_1 Cos(\theta_1) - l_2 Cos(\theta_2) \tag{4}$$

Using these co-ordinates, we can define the kinetic and the potential energies, and then finally the lagrangian:

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \tag{5}$$
$$V = m_1 \times g \times y_1 + m_2 \times g \times y_2 \tag{6}$$
$$L = T - V \tag{7}$$

Since our system had two degrees of freedom, we would need to solve two Euler-Lagrange equations to get the equations of motion of the two masses respectively.

$$\frac{d}{dt} (\frac{\partial}{\partial \dot{\theta}_1}) \ L = \frac{\partial}{\partial \theta_1} \ L \tag{8}$$
$$\frac{d}{dt} (\frac{\partial}{\partial \dot{\theta}_2}) \ L = \frac{\partial}{\partial \theta_2} \ L \tag{9}$$

After solving the both Euler-Lagrange equations, we would get two simultaneous equations containing the terms $\ddot{\theta_1}$ and $\ddot{\theta_2}$. Solving them simultaneously would finally give us the governing differential equations for the two masses. The two obtained differential equations are of the order 2, and thus need to be reduce further, through substitution, to a total of 4 first order ordinary differential equations:

$$\frac{d}{dt}\theta_1 = \omega_1$$
$$\frac{d}{dt}\theta_2 = \omega_2$$

$$\begin{aligned}\frac{d}{dt}\omega_1 &= \frac{\Lambda_1}{l_1(m_1 + m_2 - m_2(Cos^2(\theta_1 - \theta_2)))},\\ \Lambda_1 &= -gm_1Sin(\theta_1) - gm_2Sin(\theta_1) - l_2m_2\omega_2^2Sin(\theta_1 - \theta_2)\\ &-l_1m_2\omega_1^2Sin(\theta_1 - \theta_2)Cos(\theta_1 - \theta_2) + gm_2Cos(\theta_1 - \theta_2)Sin(\theta_2)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\omega_2 &= \frac{\Lambda_2}{l_2(m_1 + m_2 - m_2(Cos^2(\theta_1 - \theta_2)))},\\ \Lambda_2 &= gm_1Cos(\theta_1 - \theta_2)Sin(\theta_1) + gm_2Cos(\theta_1 - \theta_2)Sin(\theta_1)\\ &+l_1m_1\omega_1^2Sin(\theta_1 - \theta_2) + l_1m_2\omega_1^2Sin(\theta_1 - \theta_2)\\ &+l_2m_2\omega_2^2Cos(\theta_1 - \theta_2)Sin(\theta_1 - \theta_2) - gm_1Sin(\theta_2) - gm_2Sin(\theta_2)\end{aligned}$$

Solving the above four differential equations numerically gave the following results, depicted in the figure below:

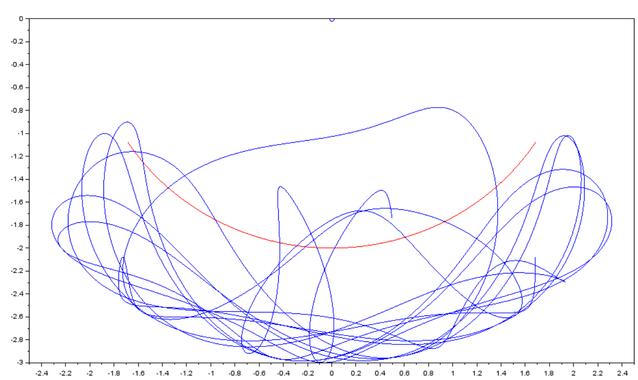


Figure 1: The red line indicates the path of the mass m_1 whereas the blue line indicates the path of the mass m_2. Both masses are anchored at point $(0, 0)$ and the lengths are fixed.