Numerical Solutions to the law of Universal Gravitation for a Two Body system

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Abstract

This study details the procedure for solving the Newton's law of Universal Gravitation for a two body system in two dimensions. A total of 4 second order ordinary differential equations were deduced, which were further broken down into 8 first order ordinary differential equations to be solved in Scilab. The values of the constants, and the initial conditions were set to trivial values, and the results were visualized in a scatter plot.

1 Introduction

The Newton's law of Universal Gravitation provides with a good approximation of the Gravitational Force for everyday use-cases. For a more sophisticated explanation, the Einstein's General theory of Relativity is used. Analytical solutions to the two body problem are possible but are very tideous and not optimal for our use case. Thus numerical methods were used. The differential equations were solved by the built-in ode solver in the Scilab language. Alternate option would be converting the differential equation to a finite difference equation, and using the Euler's Method to solve iteratively. The choice of the Scilab language was made because of it's wide array of built-in functions for complex algorithms, and because the software package is open-source and cross-platform.

2 Results and Discussions

Let us start with the vector form of the Newton's law of universal gravitation:

$$\vec{F_G} = \frac{Gm_1m_2}{r^3}\vec{r} \tag{1}$$

Take two masses, m_1 and m_2, located at co-ordinates (x_1, y_1) and (x_2, y_2) respectively on a two dimensional plane. The $\vec{r}$ is defined to point from the mass m_1 to m_2. Let's now explore the mathematical expression of the force that the mass m_1 exerts on m_2. We know from the newton's second law that force causes a body to accelerate.

$$\vec{F_{12}} = \frac{Gm_1m_2}{r^3}\vec{r} \tag{2}$$

$$m_1\frac{d^2}{dt^2}\vec{r_{m_1}} = \frac{Gm_1m_2}{r^3}\vec{r} \tag{3}$$

Where $\vec{r_{m_1}}$ gives the position vector of the mass m_1. Since our analysis is constrained to two dimensions only, we can expand out the vectors interms of their cartesian co-ordinates.

$$\frac{d^2}{dt^2}(\vec{x_{12}} + \vec{y_{12}}) = \frac{Gm_2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{3}{2}}}((x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y}) \tag{4}$$

Writing the components seperately would give us two second order ordinary differential equations:

$$\frac{d^2}{dt^2}\vec{x_{12}} = \frac{Gm_2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{3}{2}}}(x_2 - x_1)\hat{x} \tag{5}$$

$$\frac{d^2}{dt^2}\vec{y_{12}} = \frac{Gm_2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{3}{2}}}(y_2 - y_1)\hat{y} \tag{6}$$

Similarly, going through the same procedure for the other mass would give us the following differential equation, keeping in mind the fact that $F_{12} = -F_{21}$:

$$\frac{d^2}{dt^2}(\vec{x_{21}} + \vec{y_{21}}) = \frac{Gm_1}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{3}{2}}}((x_1 - x_2)\hat{x} + (y_1 - y_2)\hat{y}) \tag{7}$$

Writing the components seperately into two differential equations is going to segregate the work. This way, the calculations for each of the co-ordinate can be done seperately, and then the results can be superimposed. This fact results directly from the definition of vectors as superpositions of their components.

$$\frac{d^2}{dt^2}\vec{x_{21}} = \frac{Gm_1}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{3}{2}}}(x_1 - x_2)\hat{x} \tag{8}$$

$$\frac{d^2}{dt^2}\vec{y_{21}} = \frac{Gm_1}{[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{3}{2}}}(y_1 - y_2)\hat{y} \tag{9}$$

Solving these 4 second order ordinary differential equations simulatneously would give us the trajectories of both the masses. The differential equation solver in Scilab can only solve a first order differential equation at a time, thus we would need to make substitutions to break each of those second order differential equations into two first order ones. At the end, we would need to solve the following 8 first order ordinary differential equations simulatneously to get the trajectories of both the masses in two dimensions:

$$\frac{d}{dt}\vec{x_{12}} = \vec{V_{x_{12}}} \tag{10}$$
$$\frac{d}{dt}\vec{y_{12}} = \vec{V_{y_{12}}} \tag{11}$$
$$\frac{d}{dt}\vec{x_{21}} = \vec{V_{x_{21}}} \tag{12}$$
$$\frac{d}{dt}\vec{y_{21}} = \vec{V_{y_{21}}} \tag{13}$$

$$\frac{d}{dt}\vec{V_{x_{12}}} = \frac{Gm_2}{[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{3}{2}}}(x_2-x_1)\hat{x} \tag{14}$$
$$\frac{d}{dt}\vec{V_{y_{12}}} = \frac{Gm_2}{[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{3}{2}}}(y_2-y_1)\hat{y} \tag{15}$$
$$\frac{d}{dt}\vec{V_{x_{21}}} = \frac{Gm_1}{[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{3}{2}}}(x_1-x_2)\hat{x} \tag{16}$$
$$\frac{d}{dt}\vec{V_{y_{21}}} = \frac{Gm_1}{[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{3}{2}}}(y_1-y_2)\hat{y} \tag{17}$$

Solving the eight of the above simultaneously with the appropriate constants and initial conditons produced the accurate results.

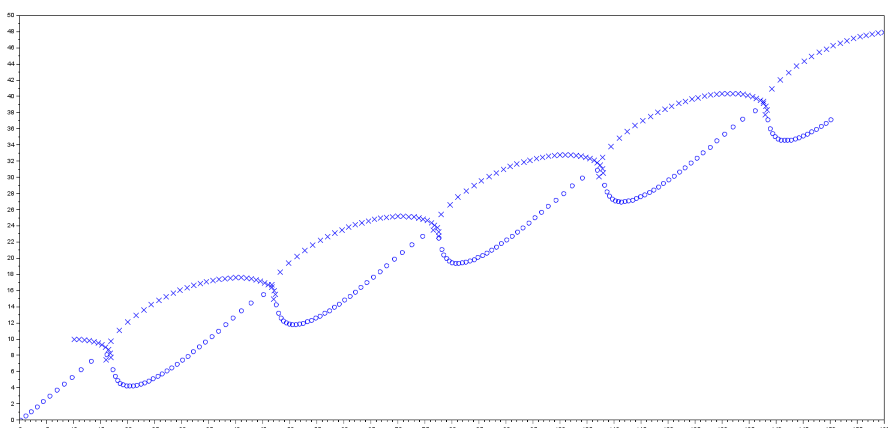

Figure 1: The figure depicts the trajectory of two masses gravitationally bound to each other. The trajectory of the first mass, with initial positon (0, 0) is given by the circles whereas the trajectory of the second mass, with initial position (10, 10) is given by the crosses. Both the masses were given some random but appropriate initial velocities.