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QUANTITATIVE CREDIT RESEARCH

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RESEARCH

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OVERVIEW

The continued growth of the credit derivatives market and the recent spread volatility experienced by most credit products, while offering interesting investment opportunities, call for a deeper understanding of a variety of issues. The valuation and hedging of complex credit products and the non-trivial task of managing credit and liquidity risk represent serious challenges for every investor who actively operates in these markets today. Moreover, as more and more data from new issues and secondary trading of relatively new credit products become available, we are able to uncover interesting empirical regularities that can improve our ability to value and manage credit risk.

To broaden the scope of our *Structured Credit Strategies: Credit Derivatives* quarterly, first issued in October 1999, we are re-launching the publication to include topics relevant to both the cash and the derivatives markets. *Quantitative Credit Research*, also a quarterly publication, will especially focus on modeling and statistical data analysis in the credit arena. In this issue, we deal with a variety of topics.

The first article analyzes the behavior of CDO spreads over the last three years and presents some new evidence on CDO correlations with ABS, CMBS, and corporates. While some empirical results such as the strong relation between AAA CDOs and credit cards may be expected, some findings concerning the relation between CDOs and corporate spreads are less obvious. By revealing some nonintuitive spread co-movements, this correlation study provides the reader with interesting insights for asset allocation in CDOs versus other products.

The second article introduces a simple and easily implementable model for the scenario analysis of spread curve movements. By representing any curve variation as a linear combination of a shift, a twist, and a butterfly movement, this method provides the user with an intuitive way to pre-specify views and then reconstruct the *most likely* curve variation consistent with those views.

Finally, the last article provides a comprehensive review of modeling techniques that are widely used for the valuation and hedging of credit products. The two dominant frameworks, structural modeling and reduced-form modeling, are thoroughly discussed, and the pros and cons of both approaches are highlighted. The message we are trying to convey is that neither approach strictly dominates the other and that the success of any modeling effort will largely depend on the ability to choose the right framework for a given application. For example, while measuring the impact of a proposed change in the capital structure may require modeling the evolution of the firm's asset value (structural approach), the pricing of credit derivatives such as default swaps and default baskets can be more efficiently performed by modeling default times directly (reduced-form approach).

CONTENTS

CDO SPREAD CORRELATION STUDY

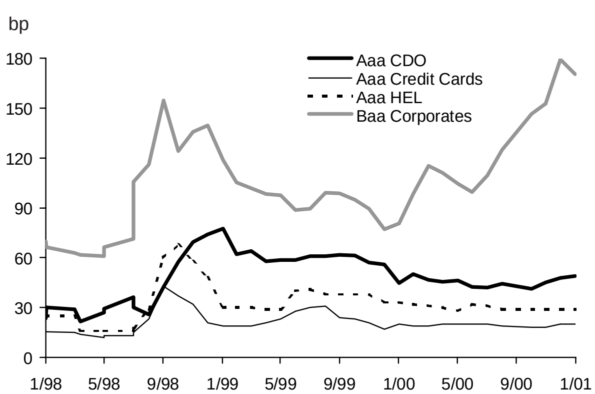
Sunita Ganapati

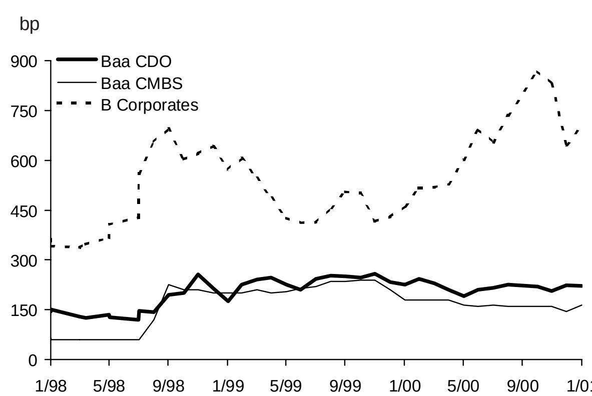
1. CDO Spreads Correlated with ABS/CMBS and with HY Corporates

As CDOs are a hybrid between corporate bonds and structured asset classes, the market continues to wrestle with the relationship between CDO spreads and other sectors. Fall 1998 and fall 2000 provide reference points for CDO spread behavior patterns during times of market stress (Figure 1). During both time frames, CDO spreads appear to have been correlated with ABS and CMBS and appear to have had a limited relationship with corporates. CDOs widened only 4-10 bp during the last two months of 2000, whereas Baa corporate bonds and high yield widened 70-176 bp on a swapped basis. During this time, senior ABS and CMBS were relatively unchanged, and Baa CMBS actually tightened. During the fall 1998 spread sector crisis, CDO spreads (in line with ABS/CMBS) widened 15-150 bp, and lower quality corporates (Baa and below) widened 25-60 bp. CDOs responded more to the liquidity driven widening of fall 1998 than to the credit driven widening of fall 2000, thereby behaving more like a structured asset class. Additionally, ABS and CMBS benchmarks are used as valuation indicators for the asset class, particularly at the investment-grade level, leading one to expect that CDO spreads are highly correlated with structured asset classes.

In order to test the market's intuition that CDOs are highly correlated with ABS/CMBS and not with corporates, we conducted a study of CDO spread correlation with other sectors. We found **statistically significant estimates of correlation, ranging from 0.32 to 0.48, between CDO tranches and ABS/CMBS. Contrary to market intuition, we also found a similar relationship between investment-grade CDO tranches and high-yield corporates**. The correlation estimates are in the range of correlations that studies have generally found for

Figure 1. **Spreads over LIBOR on CDOs relative to Other Asset Classes**





most financial assets.[1] Readers should, however, keep in mind that correlation estimates of 0.32-0.48, while statistically significant, do not explain all the variation in CDO spread changes. In this study, we focus only on quantifying the relationship between CDO spreads and those of other asset classes.

2. CDO Spread Correlation Study

To derive correlation estimates, we use monthly *spread changes* in each asset type and not spread levels, since spreads are auto-correlated. As a result, our estimates are lower, but we believe more accurate, than other recent studies on CDO spread correlation, which have shown estimates of 0.70-0.90. Given that the CDO data series is a history of new issue levels and given the limited liquidity of the asset class, there appears to be a lag effect between the changes in CDO spreads and other asset classes. We found that correlations are maximized at a two-month lag.

Although we found some statistically significant relationships, there are certain shortcomings of this study:

1) The CDO data series covers a 36-month period when there was limited secondary activity; therefore, results could change as the market becomes more liquid and as we have more data.
2) The variability of spread changes is relatively minor and, therefore, the sample estimate of correlation tends to pass the significance level although the values are relatively small.

Figure 2 demonstrates the relationship between various CDO tranches and comparably rated structured securities over the 36-month period January 1998-December 2000. The values in parentheses in the table are the p-values that test the null hypothesis that the correlation between the two asset classes is equal to zero. Whenever the p-values are smaller than the chosen significance level for the test, we can reject the null hypothesis and conclude that the sample correlation is significantly different from zero. The significance level is the complement

[1] For example, a study by Lehman Brothers Mortgage Research group (*Risk and Return in the Mortgage Market*, January 13, 2000, by Arora, Heike, and Mattu) finds the correlation between the Mortgage Index and various parts of the Corporate Index to range from 0.33 to 0.51.

Figure 2. **Correlation of CDOs with ABS and CMBS**
January 1998-December 2000

Like-Rated Securities	**Aaa CDOs r (p)**	**Baa CDOs r (p)**
Credit Cards	**0.48 (0.3%)**	0.16 (39%)
CMBS	**0.32 (6%)**	**0.33 (5%)**
HELs	**0.40 (2%)**	NA

* CDO spreads are lagged by two months. Correlations are computed on like-rated securities—for example, Aaa CDOs and Aaa credit cards. The statistically significant estimates are marked in bold.

of the confidence level. For example, for a 95% confidence level, all estimates with p values less than 5% are statistically significant. In general, 5% and 10% significance levels are widely used for testing. If a p-value is higher than 10%, one should not reject the null hypothesis that the correlation is in fact equal to zero, but should declare the sample estimate (the computed correlation) to be statistically insignificant.

2.1. CDOs—Highest Correlation with Aaa Credit Cards and BBB CMBS

The investor base for Aaa CDO tranches is predominantly ABS CP conduits and structured investment vehicles (SIVs). A majority of their portfolios consist of Aaa ABS and CMBS, and therefore one would naturally expect CDO spreads to be more strongly correlated with these asset types. We find that, in fact, Aaa CDOs and Aaa credit cards have the highest correlation (0.48). Also, the relationship between Aaa HELs and CDOs is strong at 0.40. However, CDOs and CMBS at the Aaa level have a weaker relationship (0.32 at 6% significance).

Credit cards and CDOs constitute a major part of CP conduits and SIV portfolios, and the strong relationship confirms the market's intuition about their relationship. HELs are also used as valuation benchmarks for Aaa CDOs. Their structural likeness (both have wide principal repayment windows and variable cash flows) and the similarity of the collateral quality (non-investment grade borrowers) of both types of assets could be the driver of this relationship.

For Baa-rated CDOs, the relationship that is most statistically significant is that with Baa-rated CMBS (0.33). In the last twelve months, resecuritizers or ABS CDO managers who buy Baa-rated structured securities have driven demand and pricing levels for this part of the capital structure; consequently, this relationship is of no surprise. Unfortunately, we do not have adequate historical data on Baa HELs to analyze their relationship with CDOs. The lack of relationship between Baa credit cards and CDOs is also consistent with the difference in investor bases for the two products. Credit cards typically serve as corporate substitutes and are far more liquid than CDOs. Resecuritizers almost never provide the marginal bid for credit card subordinates.

2.2. CDO Subordinates Tend to Move in Tandem; Aaa CDOs Move with Ba CDOs, But Not with Baa CDOs

Intra-asset class correlation analysis (Figure 3) is useful to investors who participate across the capital structure. We find that Aaa CDOs bear no significant correlation with Baa CDOs, but have had a 0.34 correlation with Ba CDOs.

Figure 3. **Intra-Class Correlation,** January 1998-December 2000

	Aaa CDOs	**Baa CDOs**	**Ba CDOs**
Aaa CDOs	1.0		
Baa CDOs	0.12 (49%)	1.0	
Ba CDOs	**0.34 (4%)**	**0.38 (2%)**	1.0

This would imply that all other things remaining constant, a portfolio of Aaa and Baa CDOs is more diversified that Aaa and Ba CDOs. Additionally, as expected, the two subordinated parts of the capital structure (Baa and Ba) have a statistically significant correlation estimate of 0.38.

2.3. Surprisingly B Corporate Spreads Have Had a Muted Effect on Ba CDOs

If B-rated corporate spreads widen because the market expects a rise in default rates, one would naturally expect that cash flow CDO liability spreads (particularly of lower-rated tranches) should widen in tandem. We find that the correlation of Ba CDOs is statistically insignificant with B corporates, but strong with Ba-rated corporates (0.38 at 2% significance). The adequacy of good data could be playing some role in this computation. For example, the CDO spread data include only new issue transactions and do not incorporate pricing levels on distressed CDOs that have experienced high defaults.

The plausible fundamental explanation for the lack of relationship between Ba CDOs and B corporates is that Ba CDOs trade at significantly wider levels than most Ba securities and already incorporate a higher-than-average default assumption. Therefore, movements in B corporate spreads do not get translated into immediate changes in CDO spreads; the CDO market may wait to see if default rates actually rise. Alternatively, CDO investors may use ratings drift as a measure of credit quality rather than corporate spread changes. Ba CDOs have exhibited reasonable rating stability[2] and, therefore, may not react immediately to changes in credit perception of underlying collateral.

At the investment-grade level, both Aaa and Baa rated CDO tranches show a statistically significant relationship with Ba and B corporates. Part of this relationship could be driven by overall spread sector movements, a study of which is beyond the scope of this report. However, investors in Aaa tranches are naturally more risk averse and, therefore, may react to deteriorating credit quality as represented by changes in corporate spreads. Baa CDOs have shown the largest negative rating drift within CDOs, and, therefore, their sensitivity to corporate spreads (i.e., credit quality changes) may not be surprising.

[2] See *Structured Credit Strategies: Understanding the CDO Downgrade Experience*, August 10, 2000, for a study on CDO Rating Transitions.

Figure 4. **Correlation of CDOs with Corporates,** January 1998-December 2000

	Aaa CDOs	**Baa CDOs**	**Ba CDOs**
B Corporates	**0.32 (6%)**	**0.35 (3%)**	0.23 (17%)
Ba Corporates	**0.40 (2%)**	**0.34 (4%)**	**0.38 (2%)**

The corporate spread series is swapped to LIBOR for computing correlations.

3. Conclusion and Implications for CDO Portfolio Managers

Over the time period of our study, CDOs have tended to behave like a hybrid asset class. Aaas track the ABS/CMBS market. They also respond to spread changes in the high-yield corporate market, possibly as such investors use corporate spreads as a barometer of credit quality. Baa CDO spreads have been so technically driven by resecuritizers that they have naturally tended to move with the structured markets. Additionally, these bonds have a strong relationship with corporates, possibly because they tend to be sensitive to deteriorating collateral credit quality. Ba CDOs have historically exhibited very limited correlation with B-rated corporates but a strong one with Ba-rated corporates. We believe that the relationship between Ba CDOs and B corporates could get stronger going forward, particularly if secondary trading of CDOs improves and if the current elevated default environment persists.

For CDO portfolio managers, correlation estimates can be used to make asset allocation decisions. Some examples are: a portfolio of Aaa and Baa CDOs could provide more spread diversification than Aaa and Ba CDOs. For a high-yield manager who typically purchases B corporates, buying Ba CDOs achieves spread diversification. For a structured product portfolio manager who purchases Ba CDOs, buying Baa CDOs may not achieve diversification. Portfolio managers should put in context the data limitations and the timeframe of the study before making such decisions.

A SIMPLE MODEL FOR CURVE SCENARIOS

Marco Naldi

1. Introduction

In this article, we propose a simple method for creating spread curve scenarios that incorporate a set of pre-specified views.

The data we use in our analysis are the time series of the Lehman Brothers fitted spread curves.[1] The analysis is applicable to any of the available spread curves, whether fitted by rating, sector, rating and sector, or by issuer. In section 2, we show that the dynamics of a fitted spread curve can be accurately approximated as the sum of three orthogonal components representing a shift, a twist, and a change in curvature, respectively. This intuitive representation allows us to build scenarios that incorporate our views in a natural way.

The basic idea of this article is to use our views to back up the realizations of the three components that are most likely to produce them, and then reconstruct the implied movement of the entire curve. By doing so, we are able to measure the intensity of the realizations that are required to achieve the specified targets. Furthermore, once we understand the curve movements represented by the components, we can choose to specify our views directly in terms of their realizations.

Our approach, which we formally characterize as a maximum likelihood problem, is illustrated with examples in section 3, where we also highlight the close relationship between this methodology and multiple regression. In section 4, we evaluate the model's performance, while the algebraic details are presented in the appendix.

2. Characterizing the Principal Movements

The geometry of principal component analysis is formally presented in the appendix. In this section, we apply this technique to study the dynamics of the BBB fitted par spread curve between July 15, 1994 and November 30, 2000.

We start by decomposing the covariance matrix of nine time series representing biweekly changes in the 1-, 2-, 3-, 5-, 7-, 10-, 12-, 15-, and 20-year fitted par spreads for the BBB rating. Figure 1 shows the cumulative percentage of variance explained as a function of the number of components considered. The first three components explain 99.14% of the total variance, suggesting that the time series of spread changes can be well approximated by linear combinations of three common factors.[2]

[1] See "Estimating Credit Spread Curves", Lehman Brothers, *Structured Credit Strategies*, October 1999. The data used in this article can be viewed on our client web site, http://live.lehman.com.

[2] All Lehman Brothers fitted credit curves use the same functional form, a sum of exponentials for the spread discount function. Since each curve has three free parameters, its movements can be described quite well by a three-factor model, the residual error being a consequence of the non-linear relation between par spreads and the estimated parameters.

The first three components can be interpreted as the most important movements of the spread curve and are depicted in Figure 2. We have multiplied them by the volatilities of the respective factors so that they can be interpreted as the basis point movements that will follow a one-standard-deviation realization of the underlying factor.

The first component can be interpreted as a curve shift, since it produces spread changes with the same sign along the curve. It is the type of movement responsible for most of the spread variation (explaining 86% of total variance, see Figure 1). Notice that the shift is not exactly parallel but mildly increases with

Figure 1. **BBB Curve—Explained Variance**

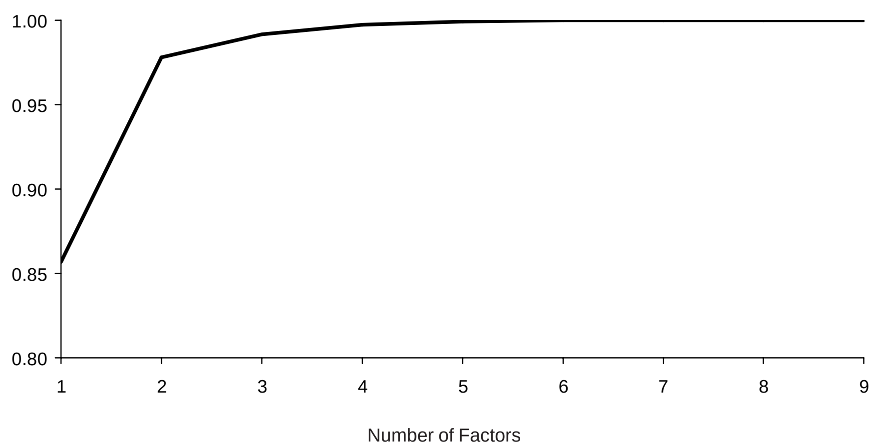
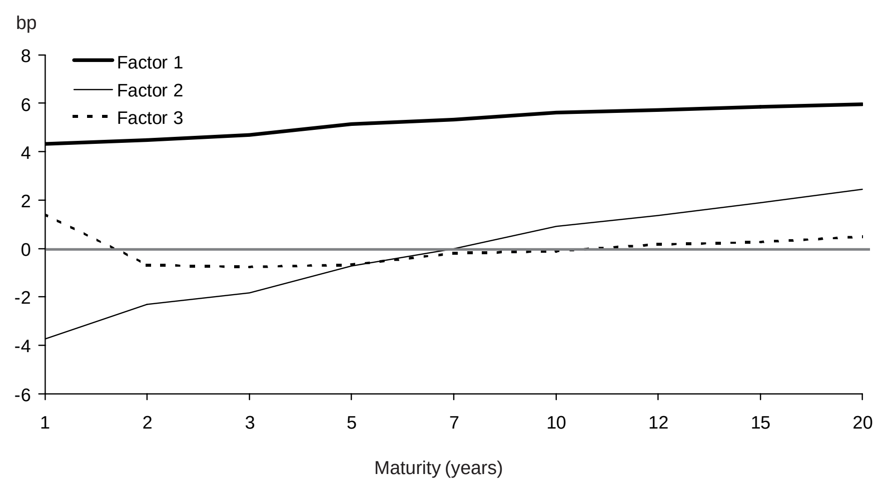


Figure 2. **BBB Prinicipal Movements**



maturity, reflecting the fact that over the period in consideration, BBB long spreads have been slightly more volatile than short spreads. The second component is easily identified as a twist of the curve. A positive realization of the second factor will tighten short spreads and widen long spreads (the opposite will happen in case of a negative realization). The third component is responsible for the curvature of the term structure, increasing or decreasing the hump that characterizes an exponential curve.

For a comparison, Figure 3 depicts the first three components for the BB curve. The shift here is decreasing with maturity, a consequence of the fact that BB short spreads tend to be more volatile than long spreads of the same quality. Another difference between the two rating groups can be seen in the lower relative importance that the curvature (third) component has for the BB curve.

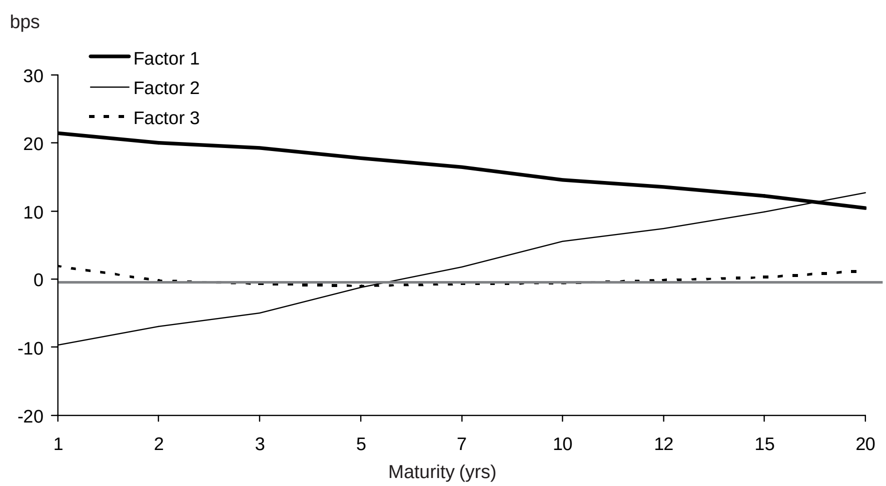
With an understanding of the three major forces that drive the dynamics of a fitted credit curve, we now turn to producing scenarios that satisfy a pre-specified set of views. These views can be expressed as specific rate movements, as changes in spread differences along the curve, as factor realizations, or, more generally, as combinations of all of the above.[3]

3. Scenario Analysis

Using the distributional assumption that factors are conditionally normal, we perform scenario analysis using a maximum likelihood approach, i.e., looking for that combination of factor realizations that maximizes the density function while

[3] Any request that can be expressed as a linear constraint in the factor space can be easily satisfied by a unique analytical solution, as we show in the appendix.

Figure 3. **BB Prinicipal Movements**



meeting the pre-specified constraints. In other words, we try to identify the most likely scenario among all those that are consistent with the stated views.[4]

All scenarios in this section are conditional on the information known as of November 30, 2000, and refer to spread changes for the BBB curve over the following two-week period. Extending the analysis to different horizons is simply a matter of appropriately scaling the factor variances.

3.1. Requesting Spread Movements

Figure 4 presents a situation in which the user requests a 5 bp increase in the 2-year spread. The maximum likelihood scenario that meets this target requires a (0.87, -0.45, -0.13) standard deviation realization of the three factors. In other words, the model is telling us that the most likely way of achieving a 5 bp increase in the 2-year spread is through a significant shift of the whole curve accompanied by a moderate curve flattening and a mild increase in curvature.

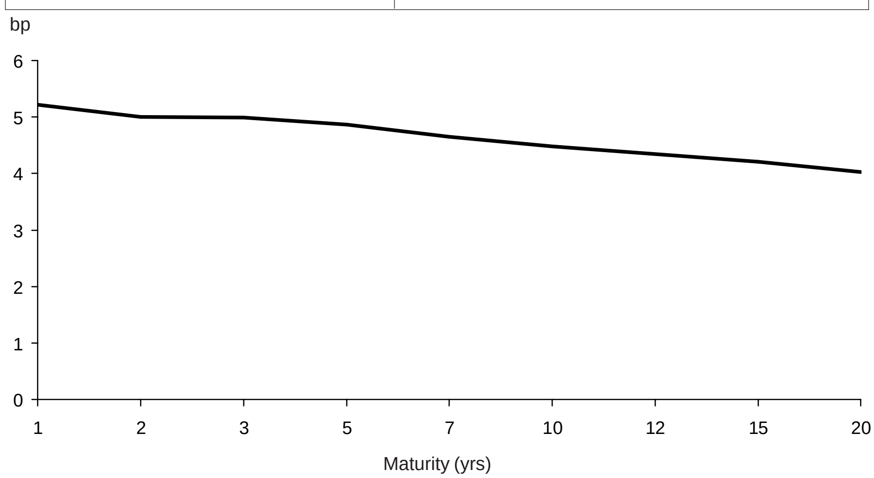
In Figure 5, we present the most likely scenario consistent with an increase of 5 bp in the difference between the 20-year spread and the 2-year spread. This view is immediately accommodated in our framework because the request for a tilt can be formulated as a simple linear constraint.[5] Figure 5 tells us that the

[4] Since we approximate curve movements with linear combinations of three factors, a maximum number of three equality constraints can be pre-specified by the user. In other words, we only have three degrees of freedom because the choice of three factor realizations can satisfy only three linear equations. More constraints may be expressed as inequalities.

[5] Requesting a tilt is geometrically equivalent to requesting one spread movement, even if two spreads are involved in the request. Asking for a scenario in which the difference between the 20-year and the 2-year points increases by 5 bp is less demanding than specifying the exact movements for the two spreads. In fact, the request of a tilt uses only one degree of freedom, not two.

Figure 4. **BBB Spread Change**

Years to Maturity	Target		Factor 1	Factor 2	Factor 3
2	5 bp	Intensity	0.868	-0.446	-0.133



requested steepening is most likely to be produced by a 0.96 bp decrease of the 2-year spread and a 4.04 bp increase in the 20-year spread. Not surprisingly, most of the job is done by the twist component (with a 0.9 standard deviation realization), although a significant upward shift is responsible for the asymmetric result.

A different tilt is assumed in Figure 6. Only the short end of the curve is involved in the request, a 3 bp increase in the difference between the 1-year and 3-year spreads. The model suggests that the most likely way to achieve the specified target is characterized by a significant increase in the curvature of the term structure.

3.2. Constraining the Components

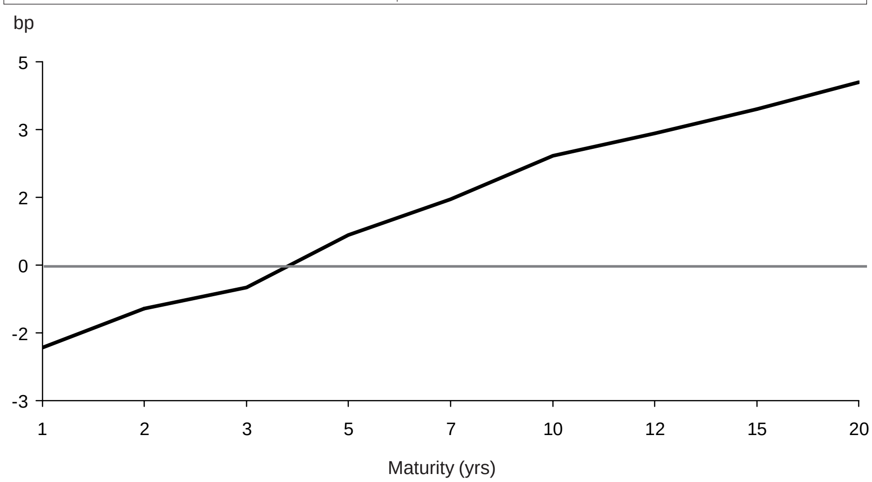
The possibility of expressing views directly on the principal components is particularly desirable in those cases in which the user is able to identify some meaningful relation between the factors driving the curve and other economic and financial variables.[6]

The scenario given in Figure 4 can be modified to add constraints on the principal movements of the curve. For example, we may want to achieve the 5 bp increase in the 2-year spread with an equally intense realization of the shift and twist components. Figure 7 shows that this target can be achieved by a 0.72 standard

[6] For example, it has often been noted that the second principal component of the Treasury par curve becomes significantly more active when Fed interventions are expected.

Figure 5. **BBB Spread Change**

Years to Maturity	TILT Target		Factor 1	Factor 2	Factor 3
2	-5 bp	**Intensity**	0.285	0.907	0.226
20					



deviation realization of the first two factors. Comparing the two outcomes, we now have a scenario in which the curve tilts more and shifts less than before. However, we did not choose the curve position arbitrarily. There are an infinite number of curve changes that achieve a 5 bp increase with an equally intense movement of the first two components. What needs to be determined is the ratio

Figure 6. **BBB Spread Change**

Years to Maturity	TILT Target		Factor 1	Factor 2	Factor 3
1	3 bp	**Intensity**	-0.132	-0.665	0.773
3					

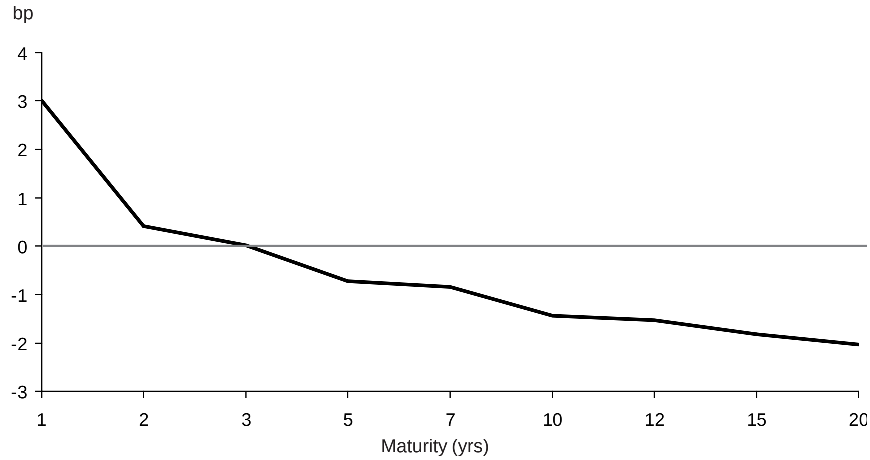
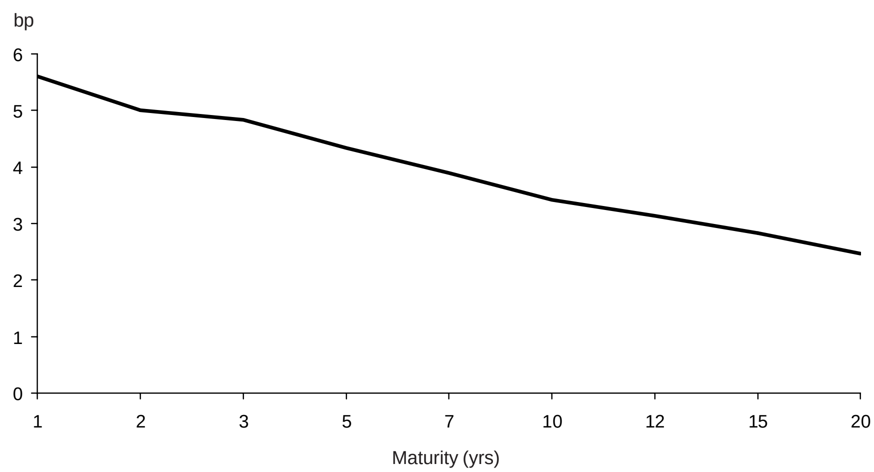


Figure 7. **BBB Spread Change**

Years to Maturity	Target		Factor 1	Factor 2	Factor 3
2	5 bp	**Intensity**	0.724	-0.724	-0.146



between the intensities of the first two factor realizations and the intensity of the curvature effect. Maximizing the likelihood of the curve movement allows us to pin down this ratio.

Figure 8 revisits the tilt example of Figure 5. The user may have a neutral view about the general direction of spreads but still believe in a significant steepening of the curve. Adding the constraint of a null effect of the first component produces the highly symmetric result depicted in Figure 8.

3.3. Maximum Likelihood and Regression

When the user is not interested in recovering the factor intensities required to achieve the specified targets, the model presented above can be further simplified and (almost) equivalently implemented through a multiple regression approach. This is an immediate consequence of the fact that a multivariate normal distribution has a linear regression. In the appendix, we formally show this equivalence.

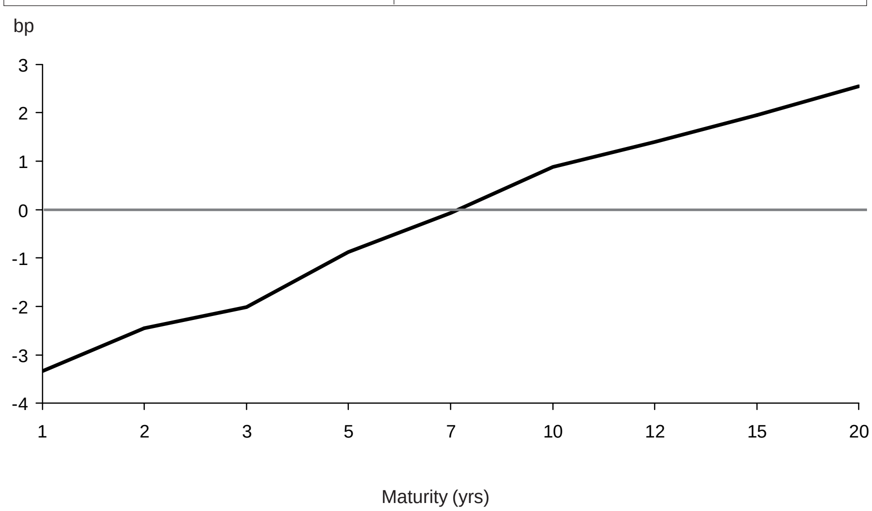
In principle, however, a maximum likelihood approach is more general than linear regression because it can accommodate a non-Gaussian factor distribution. Any distributional assumption can be adopted, although the maximization problem will generally become more involved.

4. Evaluating the Model's Performance

There are two reasons why a curve scenario will turn out to be different from the realized curve movement. First, the specified views will generally contain errors.

Figure 8. **BBB Spread Change**

Years to Maturity	TILT Target		Factor 1	Factor 2	Factor 3
2	-5 bp	**Intensity**	0	0.991	0.247
20					



Second, the model will not be able to forecast the exact curve movement even if the views are correct. In this section, we focus on the second issue.

Using biweekly data from September 30, 1996-November 30, 2000 (100 observations) we perform the following exercise. At time *t*, we forecast the spread curve movement over the period (*t, t+1)* using as our "view" a perfect forecast of the 7-year point on the curve. We forecast the curve movement as described in the previous section and use an increasing window to re-estimate the factor loadings at every step (all available data starting from July 15, 1994, are used at every step). Thus, the factor loadings used to forecast the curve movement in period *(t,t+1)* are fully determined by information available at time *t*.

Figure 9 shows the out-of-sample R^2 for different maturities on several spread curves fitted by rating. As one would expect, the model's forecasting ability decreases as the term gets further away from the 7-year point. Of course, correctly expressing different views would produce different results.

Figure 9. **Out-of-Sample R-Square with Perfect Forecast of the 7-Year Spread Change**

	1-yr	**2-yr**	**3-yr**	**5-yr**	**7-yr**	**10-yr**	**12-yr**	**15-yr**	**20-yr**
AAA	0.46	0.63	0.78	0.96	1.00	0.94	0.87	0.77	0.63
AA	0.60	0.72	0.81	0.96	1.00	0.95	0.89	0.79	0.66
A	0.56	0.70	0.80	0.96	1.00	0.95	0.90	0.81	0.70
BBB	0.55	0.74	0.84	0.97	1.00	0.97	0.94	0.89	0.83
BB	0.64	0.74	0.83	0.96	1.00	0.93	0.85	0.71	0.53

5. Appendix

Principal Component Analysis (PCA)

Our goal is to approximate the *N*-vector of (demeaned) spread changes $\Delta S_t \equiv S_t - S_{t-1}$ with a linear combination of three common factors:[7]

$$\Delta S_t \approx L F_t . \tag{1}$$

Here L is the $N \times 3$ matrix of factor loadings, and F_t is the *3*-dimensional vector of factor realizations during the period (t -1, t). The only available data are a *T*-dimensional sample for ΔS. PCA suggests a sensible way to reconstruct the right-hand side of equation (1). Set the columns of L equal to the first *3* (scaled) eigenvectors of the sample covariance matrix of ΔS, and then recover the factor realizations by least squares projection. Formally, if Ω is the sample covariance matrix of ΔS, set

$$L = \Gamma_3 \Lambda_3^{1/2},$$

where Γ_3 is a $N \times 3$ matrix containing the first *3* eigenvectors of Ω and Λ_3 is a 3×3 diagonal matrix containing the associated eigenvalues. Then, recover F_t as

$$F_t = \left(L'L\right)^{-1} L' \Delta S_t = \Lambda_3^{-1} \Lambda_3^{1/2} \Gamma_3' \Delta S_t = \Lambda_3^{-1/2} \Gamma_3' \Delta S_t ,$$

where the second equality follows from the orthonormality of eigenvectors.[8] Notice that the resulting factors are also orthonormal. In fact, we have

$$E\left[F_t\right] = 0,$$

$$E\left[F_t F_t'\right] = \Lambda_3^{-1/2} \Gamma_3' \Omega \Gamma_3 \Lambda_3^{-1/2} = I_3 ,$$

where I_3 is the 3×3 identity matrix.

The Maximum Likelihood Problem

Formally, we are building scenarios at time t according to the solution of the following program:

$$\begin{aligned} &\max_{F_{t+1}} n(F_{t+1}; 0, I_3) \\ &s.t. \\ &R F_{t+1} = C, \end{aligned} \tag{2}$$

[7] The unconditional means of spread changes will be ignored since none of them is statistically different from zero.

[8] A set of vectors is said to be orthonormal if its elements are orthogonal and have unit length.

where $n(\ .\ ;m,\Theta)$ is the joint density function of a normal random vector with mean m and covariance Θ, I_3 is the 3×3 identity matrix, and F_{t+1} is the 3-dimensional vector of factor realizations in period $(t,t+1)$. Moreover, R is a r x 3 matrix, $r \in \{1,2,3\}$, which describes the linear constraints used to express the user's views and C is a r-dimensional vector containing the target values. For example, to reflect a view expressed as a specific spread movement, a row of R is set equal to the appropriate row of the scaled loading matrix $L=\Gamma_3\Lambda_3^{1/2}$, while the corresponding element of the target vector C is set equal to the hypothesized realization. To constrain the first component to a specific value, just set the corresponding row of R to {1,0,0}. Rewriting program (2) in the equivalent form

$$\min_{F_{t+1}} F'_{t+1}F_{t+1}$$
$$s.t.$$
$$RF_{t+1} = C$$

shows that we are in fact minimizing a strictly quasi-concave function under linear constraints, which guarantees the existence of a unique and well-behaved solution. Solving the Lagrangean problem yields

$$F_{t+1} = R'\left(RR'\right)^{-1}C.$$

Reconstructing the curve movement is now straightforward:

$$\Delta S_{t+1} \approx LF_{t+1} = LR'\left(RR'\right)^{-1}C. \tag{3}$$

Maximum Likelihood and Regression

Suppose views are only expressed in terms of spread variations, and denote with ΔX the sub-vector of ΔS containing the targeted spread changes. Regressing ΔS on ΔX gives the predictive model

$$\Delta S_{t+1} \approx \Omega_{SX}\Omega_{XX}^{-1}\Delta X_{t+1}, \tag{4}$$

where Ω_{SX} is the sample covariance matrix between ΔS and ΔX and Ω_{XX} is the sample covariance matrix of ΔX. Recalling that the R matrix is just the appropriate sub-matrix of L and the factor model provides a close approximation of the sample covariance, we have

$$LR' \approx \Omega_{SX},$$

$$RR' \approx \Omega_{XX},$$

showing that equations (3) and (4) provide an (almost) equivalent conditional forecast of the spread curve variation.

SINGLE-ISSUER MODELS OF DEFAULT

Dominic O'Kane and Lutz Schloegl

1. Introduction

The combination of low government bond yields, low issuance of government debt, and the astonishing growth of the credit derivatives markets has led a significant flow of investors to seek additional yield in corporate and emerging market securities. Banks and other institutions that are in the business of taking on credit exposures are also looking for more sophisticated ways to reduce their credit concentrations, either through diversification or by securitizing their portfolios into first and second-loss products.

As a result, there is a growing need for credit models. These can be used for a variety of purposes—relative value analysis, marking to market of illiquid securities, computing hedge ratios, and portfolio-level risk management. For marking to market, models need to be arbitrage-free to guarantee consistent pricing and must be sufficiently flexible to reprice the current market completely. Furthermore, there is a need for models that can add value by providing some insight into the default process. This is especially so in view of the relative paucity of market data in the credit markets.

In this article, we provide an overview of quantitative models for the pricing and risk management of securities exposed to the credit risk of a single issuer. The important topic of how to model the correlated default risk of multiple issuers will be the subject of a following article.

As a means of classification, credit models can mainly be split into two groups—structural and reduced-form. The former type relates to models that have the characteristic of describing the internal structure of the issuer of the debt, so that default is a consequence of some internal event. The latter type—reduced-form—does not attempt to look at the reasons for the default event. Instead, it models the probability of default itself, or more generally, the probability of a rating transition.

2. Structural Models of Default

The structural approach to modelling default risk attempts to describe the underlying characteristics of an issuer via a stochastic process representing the total value of the assets of a firm or company. When the value of these assets falls below a certain threshold, the firm is considered to be in default.

Historically, this is the oldest approach to the quantitative modelling of credit risk for valuation purposes, originating with the work of Black/Scholes (1973) and Merton (1974). As the fundamental process being described is the value of the firm, these models are alternatively called **firm-value** models. As the name implies, this approach is more suited to the study of corporate issuers, where an actual firm value can be identified, e.g., using balance sheet data. For sovereign issuers, the concept of a total asset value is much less clear-cut, though attempts have been made to adapt this approach to sovereign credit risk using national stock indices as proxies for firm values, c.f. Lehrbass (2000).

2.1. The Merton Model

2.1.1. Framework

Within the Merton model, it is assumed that the firm's capital structure consists of:

- Debt with a notional amount K, in the form of zero coupon bonds with maturity T, and total value today (time t) equal to $B^d(t,T)$.
- Equity with total value today (time t) equal to $S(t)$.

At each time before the bonds mature ($t \le T$), we denote the total market value of the firm's assets by $V(t)$. We shall refer to $V(t)$ as the firm or the asset value interchangeably.

Firms have limited liability. Therefore, by the fundamental balance sheet equation, the firm's total assets must equal the sum of its equity and its liabilities. This means that the stock price and the bond price are linked to the firm value via the equation

$$V(t) = S(t) + B^d(t,T) \cdot \tag{1}$$

The fundamental assumption of the Merton model is that default of the bond can take place only at its maturity, since this is the only date on which a repayment is due. The payoff at maturity is therefore

$$B^d(T,T) = \min(V(T), K) \cdot \tag{2}$$

If the firm value is greater than the redemption value of the debt, then the firm is solvent and the debt is worth par. If the firm value is less than the redemption value, the firm is insolvent and bondholders have first claim on its assets. This means that the shareholders are residual claimants with a payoff at maturity of

$$S(T) = \max(V(T) - K, 0) \cdot \tag{3}$$

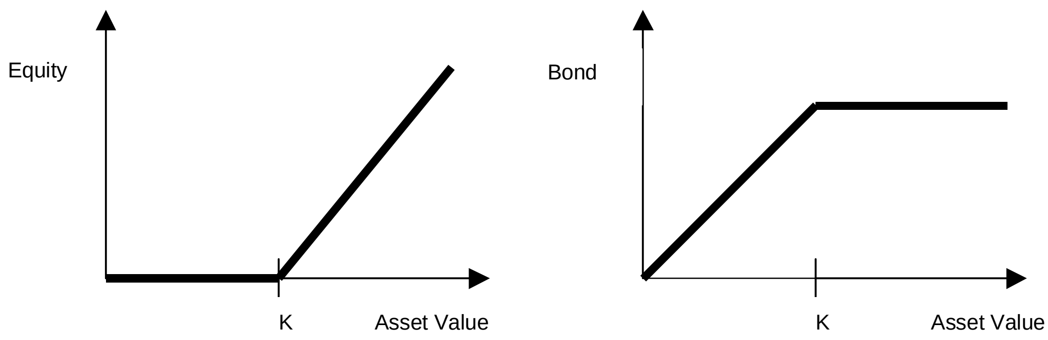
The bond and equity payoffs are shown in Figure 1.

2.1.2. Valuation

In effect, the shareholders hold a European call option on the firm value. Subject to some assumptions,[1] this can be priced just as in the Black/Scholes model. Using equation (1), we can imply out the value of the corporate bond. If $P(t)$

[1] The main assumptions are that the firm value evolves according to a stochastic lognormal process with constant volatility and that interest rates are constant.

Figure 1. **Value of Debt and Equity at Maturity as a Function of Asset Value**



denotes the price of a put option on the firm value with a strike of K and $B(t,T)$ is the price of a non-defaultable zero coupon bond with notional K and maturity T, basic put-call parity implies that

$$B^d(t,T) = V(t) - S(t) = B(t,T) - P(t) \cdot \tag{4}$$

The bondholders have sold the shareholders an option to put the firm back to them for the bond's notional value at maturity T. It is this additional risk that makes them demand a yield spread over the default-free zero coupon bond.

The market price $B^d(0,T)$ of the risky debt is calculated from the Black/Scholes option pricing formula. We introduce the quotient

$$d = \frac{K\exp(-rT)}{V(0)}, \tag{5}$$

where r is the risk-free interest rate. This is the debt-to-assets ratio when the nominal value of the debt is discounted at the market's risk-free interest rate. It is one way of measuring the leverage of the firm. Clearly, a higher value of d leads to a greater degree of risk for the firm. Also, we define

$$h_1 := -\frac{1}{\sigma_F\sqrt{T}}\left(\frac{1}{2}\sigma_F^2 T + \ln d\right) \text{ and } h_2 := h_1 + \frac{2\ln d}{\sigma_F\sqrt{T}}, \tag{6}$$

where σ_F is the volatility of the firm-value. Then, the market value of risky debt in the Merton model is given by

$$B^d(0,T) = K\exp(-rT)\left(N(h_1) + \frac{1}{d}N(h_2)\right), \tag{7}$$

where N denotes the cumulative distribution function of the standard normal distribution. The definition of the credit spread s implies that

$$s = -\frac{1}{T}\ln\left(\frac{B^d(0,T)}{K}\right) - r \cdot \qquad (8)$$

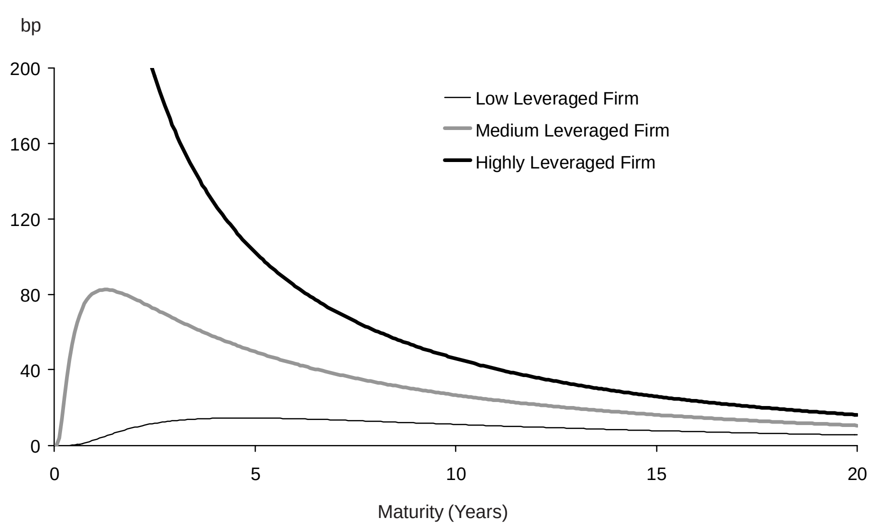
The spread s can be computed using equations (7) and (8) to give the curves shown in Figure 2.

2.1.3. Results

Figure 2 shows the three types of spread curve produced by the Merton model. For a highly leveraged firm, where the face value of outstanding debt is greater than the current firm value, the credit spread is decreasing with maturity and actually explodes as the maturity of the bond goes to zero. Clearly, if the bond were to mature in the next instant, the firm would already be in default. In a sense, this behavior results from the fact that the condition for default is imposed only at the maturity of the bond.

The hump-shaped curve for the firm with medium leverage is typical of the Merton model and can be interpreted as reflecting the fact that the credit quality of such a firm is more likely to deteriorate in the short term. However, should it survive, the credit quality is likely to increase.

Figure 2. **The Credit Spread as a Function of Maturity for Three Bonds Issued by Firms with Different Degrees of Leverage**
Credit Spread Term Structure in Merton Model



Last of all, firms with low leverage, where the assets of the firm can easily cover the debt, are very unlikely to default and can really only become more likely to do so over time. This results in a small but gradual increase in the credit spread until it is almost flat and the asymptotic behavior of the spread becomes dominant.

For reasonable parameters,[2] the credit spread tends to zero as the maturity of the bond goes to infinity. The present value of the outstanding notional of the bond falls in relation to the risk-free growth of the firm's assets, so that the default risk becomes negligible.

2.1.4. Calibration

The decisive pricing inputs of the Merton model are the volatility σ of the firm value and the degree *d* of the firm's leverage. Though the book value of assets and the notional value of outstanding debt can be deduced from a firm's balance sheet, this information is updated on a relatively infrequent basis compared with financial markets data. For pricing purposes, we need the total market value of all the firm's assets. This cannot be observed directly, but must be estimated. Consequently, there is no time series readily available for it, and its volatility must also be estimated.

For a publicly traded company, we can use the model to imply out the firm value and its volatility from the notional of the outstanding debt and stock market data. Recall that the stock is a call option on the firm value. As such, its price is given by the equivalent of the Black/Scholes formula. In the notation of equation (6) this is

$$S = VN(h_2) - K\exp(-rT)\,N(h_1)\,. \tag{9}$$

Also, the stock's delta with respect to the firm value is given by $\Delta = N(h_2)$. A simple calculation then shows that the volatility σ_S of the stock price is given by

$$\sigma_S = \frac{\sigma_F V\Delta}{S}\,. \tag{10}$$

Taking the outstanding notional *K*, as well as the stock price *S* and its volatility σ_S as given, we simultaneously solve equations (9) and (10) for the firm value *V* and its volatility σ_F. This has to be done numerically. We can then use these parameters as inputs for the valuation of debt.

> **Example 1**: Assume that a firm has debt outstanding with a face value of \$100 million and a remaining maturity of three years. The riskless rate is 5%. Total stock market capitalization is \$36 million, with a stock price volatility of 53%.

[2] In particular, in the case where $\sigma_F < \sqrt{2r}$.

Using this information, we can now determine the firm asset value and the asset volatility by performing a two-dimensional root search to solve equations (9) and (10) for V and σ_F. We obtain a total market value V for the firm of \$119.8 million, with an asset volatility σ_F of 17.95%. This implies a debt-to-assets ratio d of 71.85%, where d is computed as in equation (5). The market value of the debt is given by the difference between the firm value and the stock price and is equal to \$83.8 million. The spread s can be calculated from equation (8). It turns out to be 91 bp.

In our previous classification of firms into those with low, medium, and high leverage, the firm in this example qualifies as one with medium leverage. Its total market value is higher than the face value of the outstanding debt, but its credit spread is quite significant. The spread curve is similar to the middle one in Figure 2, with the maximum credit spread of 101 bp being attained for a maturity of about 1.5 years.

For private companies, the whole estimation procedure is more involved, as there are no publicly available equity data.

2.2. Extensions of the Merton Model

The Merton model is the benchmark for all structural models of default risk. However, some of its assumptions pose severe limitations. The capital structure of the firm is very simplistic, as it is assumed to have issued only zero coupon bonds with a single maturity. Geske (1977) and Geske/Johnson (1984) analyze coupon bonds and different capital structures in the Merton framework. In effect, the equity holders are long the option to continue the operation of the firm by servicing the debt at each coupon date. An analytical valuation of bonds is still possible using methods for compound options.

Also, the evolution of the risk-free term structure is deterministic. Several authors have extended the model by combining the mechanism for default with various popular interest rate models. Among these are Shimko/Tejima/van Deventer (1993), who use the Vasicek (1977) specification for the (default) risk-free short rate. Using the techniques well-known from Gaussian interest rate models, credit spreads can be computed. The results are compatible with those in the deterministic case, and credit spreads are generally an increasing function of the short rate volatility and its correlation with the firm value.

2.2.1. The Passage Time Mechanism

It is clearly unrealistic to assume that the default of an issuer becomes apparent only at the maturity of the bond, as there are usually indenture provisions and safety covenants protecting the bondholders during the life of the bond. As an alternative, the time of default can be modelled as the first time the firm value crosses a certain boundary. This specifies the time τ of default as a random variable given by

$$\tau = \min\left\{ t \geq 0 \mid V(t) = K(t) \right\}, \tag{11}$$

where $K(t)$ denotes some time-dependent and possibly stochastic boundary. This passage time mechanism for generating defaults was first introduced by Black/Cox (1976) and has been extended to stochastic interest rates by, among others, Longstaff/Schwartz (1995) and Briys/de Varenne (1997).

The main mathematical difficulty in a passage time model is the computation of the distribution of default times, which is needed for risk neutral pricing. If the dynamics of the firm asset value obey a (continuous) diffusion process, this is a reasonably tractable problem. However, if the paths of the firm asset value are continuous, this has an important practical consequence for the behavior of credit spreads. If the firm value is strictly above the default barrier, then a diffusion process cannot reach it in the next instant—default cannot occur suddenly. Therefore, in a diffusion model, short-term credit spreads must tend toward zero; this is at odds with empirical evidence. One remedy is to allow jumps in the firm value, c.f. Schonbucher (1996) or Zhou (1997). The analytic computation of the passage time distribution, however, becomes much more complicated, and often recourse to simulation is the only option.

Allowing jumps in the firm value also introduces additional volatility parameters. The total volatility of the firm value process is determined by that of the diffusion component, as well as by the frequency and size of jumps. Qualitatively, it can be said that early defaults are caused by jumps in the firm value, whereas defaults occurring later are due primarily to the diffusion component. The additional variables give more freedom in calibrating to a term structure of credit spreads, but also pose the problem of parameter identification.

2.3. Empirical Testing of the Merton Model

There have been a number of empirical tests of the Merton model, which have attempted to analyze both the shape and the level of credit spreads. Compared with equity markets, data issues are much more challenging because of the relative lack of liquidity for many corporate bonds. Most empirical studies have been carried out with relatively small bond samples. At present, the empirical evidence is not wholly conclusive, in particular on the shape of credit spreads for medium and low quality issuers. The main studies are summarized as follows:

- Jones, Mason, and Rosenfeld (1984) have studied monthly prices for the publicly traded debt of a set of 27 companies between 1975-1981 and found that, on the whole, the Merton model does not explain spreads very well, tending to overprice bonds.
- Based on monthly price quotes for corporate zero coupon bonds, Sarig and Warga (1989) have found that credit spreads in the market resemble the shapes produced by the model. However, this evidence is qualified by the small

sample used. Also, zero coupon bonds arguably do not constitute a representative sample of the corporate bond market.

- Helwege and Turner (1999) provide evidence for the fact that the hump-shaped and downward sloping credit spreads observed in empirical studies may just be a consequence of pooling issuers in the same rating class when constructing credit curves. They argue that issuers in the same rating class are not identical with respect to credit risk and that better-quality names tend to issue debt with longer maturities than the lower-quality ones in the same class. When considering individual issuers, they find that high-yield debt exhibits an upward-sloping term structure of spreads in the same manner as investment-grade debt.

An industrial application based on the firm value approach is given by the Expected Default Frequencies® (EDF) provided by the KMV corporation. KMV computes default probabilities for individual issuers. A default boundary is inferred from balance sheet data about the firm's liabilities. An approach based on the Merton model is used to infer the firm value and its volatility from equity prices; i.e., to "delever" the equity price movements. These data give a measure of what is called the "distance to default," which is then mapped to actual default frequencies via a very large proprietary database on corporate defaults. It has been argued by KMV that its model is a better predictor of default than credit ratings, cf. Crosbie (1998). Using EDF's to characterize credit quality, Bohn (1999) does find evidence that the term structure of credit spreads is hump-shaped or downward sloping for high-yield bonds.

2.4. Practical Applications of Firm Value Models

The fact that firm value models focus on fundamentals makes them useful to analysts adopting a bottom-up approach. For example, corporate financiers may find them useful in the design of the optimal capital structure of a firm. Investors, on the other hand, can use them to assess the impact of proposed changes of the capital structure on credit quality. One caveat to this is that it is extremely difficult to apply the firm value model in special situations such as takeovers or leveraged buyouts, where debt might become riskier while equity valuations increase.

The calibration of a firm value model is very data intensive. Moreover, these data are not readily available. It is a non-trivial task to estimate asset values and volatilities from balance sheet data. If one follows the firm value concept to its logical conclusion, then it is necessary to take into account all of the various claims on the assets of a firm—a highly unfeasible task. Furthermore, fitting a term structure of bond prices would require a term structure of asset value volatilities and asset values, which is simply not observable.

In terms of analytical tractability, one has to note that firm value models quickly become cumbersome and slow to compute when we move away from the single zero coupon bond debt structure. If, instead, we introduce a coupon paying bond into the debt structure, then its pricing is dependent on whether the firm value is sufficient to repay the coupon interest on the coupon payment dates. Mathematically, the form of the equations become equivalent to pricing a compound option. Similarly, if the issuer has two zero coupon bonds outstanding, the price of the longer-maturity bond is conditional on whether the company is solvent when the shorter-maturity bond matures. This also makes the pricing formulae very complicated. The pricing of credit derivatives with more exotic payoffs is beyond the limit of this model.

Finally, if a diffusion process is used for the firm value, default is predictable in the sense that we can see it coming as the asset price falls. This means that default is never a surprise. In the real world, it sometimes is. For example, the default of emerging market sovereign bonds is not just caused by an inability to pay, which can be modelled within a firm-value approach, but also by unwillingness to pay, which cannot.

3. Reduced-Form Models

3.1. Introduction

In contrast to structural models, reduced-form credit models do not attempt to explain the occurrence of a default event in terms of a more fundamental process such as the firm value or an earnings stream. Instead, the aim is to describe the statistical properties of the default time as accurately as possible, in a way that allows the repricing of fundamental liquid market instruments and the relative valuation of derivatives. This approach was initiated by Jarrow/Turnbull (1995) and has found wide application since then. The methodology used is closer to that of the actuarial sciences and of reliability theory than to the corporate finance methods used in structural models, and the pricing techniques are similar to those used in traditional models of the term structure, as opposed to the more equity-like structural models.

3.2. Modelling the Default Process

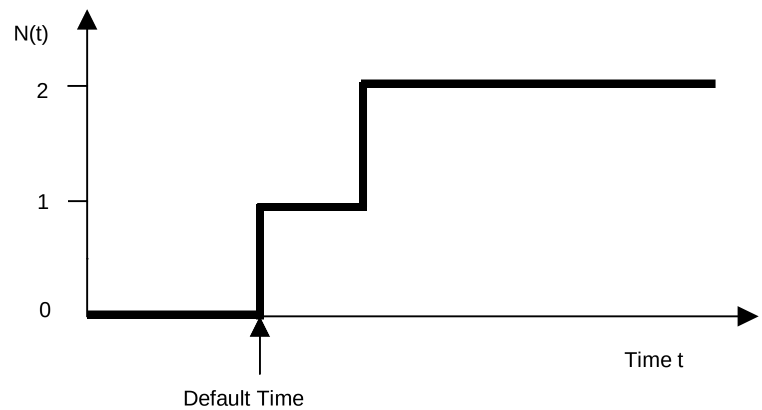
In a reduced-form model, default is treated as an exogenous event. The central object of the modelling procedure is the default counting process N. This is a stochastic process that assumes only integer values. It literally counts default events, with $N(t)$ denoting the number of events that have occurred up to time t. Each such event corresponds to the time of a jump in the value of N; Figure 3 shows a typical path of the process.

In most cases, we will be interested only in the time τ of the first default. This can be written as

$$\tau = \min\left\{ t \geq 0 \mid N(t) \geq 1 \right\}. \tag{12}$$

The simplest example of a default counting process is that of a Poisson process. The stochastic behavior of the process is determined by its **hazard rate** $\lambda(\tau)$. It can be interpreted as a conditional instantaneous probability of default:

Figure 3. **Typical Path of a Default Counting Process**



$$P\left[\tau \leq t + dt \mid \tau > t\right] = \lambda(t)dt\ . \tag{13}$$

Equation (13) states that, conditional on having survived to time *t*, the probability of defaulting in the next infinitesimal instant is proportional to $\lambda(t)$ and the length of the infinitesimal time interval *dt*. The function λ describes the rate at which default events occur, which is why it is called the hazard rate of *N*. Equation (13) can be integrated to give the survival probability for a finite time interval as

$$P\left[\tau > t\right] = \exp(-\int_0^t \lambda(s)ds)\ . \tag{14}$$

Note that the purpose of a reduced-form model is the arbitrage-free valuation of default-linked payoffs. The probability measure *P* is, therefore, a **risk-neutral measure**, meaning that the survival probability under *P* is not directly related to historical default frequencies, but where the default risk can be hedged in the market. Also, the intensity function $\lambda(t)$ governs the behavior of *N* under *P* and must therefore incorporate the risk premium demanded by the market.

3.3. Pricing

In this framework, one uses the risk-neutral approach to compute contingent claims prices. Suppose that *X(T)* is a random payoff made at time *T* if no default event occurs until then. The initial price $C^d(0)$ of this claim is given by the risk neutral expectation of the discounted payoff

$$C^d(0) = E\left[\frac{X(T)}{\beta(T)} 1_{\{\tau > T\}}\right] \tag{15}$$

where $\beta(T) = \exp\left(\int_0^T r(u)du\right)$ is the value of the money market account. If the default time τ is independent of the random payoff *X(T)* and the non-defaultable term structure, we can separate the two terms in the expectation in equation (15) to obtain

$$C^{d}(0) = E\left[\frac{X(T)}{\beta(T)}\right]P\left[\tau > T\right]. \tag{16}$$

The remaining expectation is the price $C(0)$ of a claim to the payoff *X(T)*, which has no default risk. Under the independence assumption, the price of a defaultable claim is obtained by multiplying the price of the equivalent non-defaultable claim by the probability of survival until time *T*.

> **Example 2:** Suppose that the hazard rate has a constant value λ. We consider a zero coupon bond with maturity *T* under the zero recovery assumption; i.e., the bond pays $1 if no default occurs until *T* and nothing otherwise. The survival probability is
>
> $$P\left[\tau > T\right] = \exp(-\lambda T)\,.$$
>
> If *y* is the continuously compounded yield of the corresponding non-defaultable bond, we obtain from equation (17):
>
> $$B^{d}(0,T) = \exp(-(y+\lambda)T)\,.$$
>
> **In the zero recovery scenario, the yield spread of the defaultable bond is exactly equal to the hazard rate of the default process.**

Another type of claim that is often encountered makes a random payment of *X(τ)* at the time of default, if this should occur before some time horizon *T.* Its initial price $\overline{D}(0)$ can be written as

$$\overline{D}(0) = E\left[\frac{X(\tau)}{\beta(\tau)}1_{\{\tau\le T\}}\right]\cdot \tag{19}$$

To compute this expectation, we need the density of the default time. Using the definition of conditional probabilities, equation (13) tells us that the probability of defaulting in the time interval from *t* to *t* + *dt* is given by

$$P[t < \tau \le t+dt] = \lambda(t)P[\tau > t]dt = \lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)dt\cdot \tag{20}$$

We obtain $\overline{D}(0)$ by integrating over the density of τ, so that

$$\overline{D}(0) = \int_0^T E\left[\frac{X(t)}{\beta(t)}\right]\lambda(t)\exp\left(-\int_0^t \lambda(u)du\right)dt\cdot \tag{21}$$

An important special case is the one in which X is constant and equal to one. We denote the price of this claim by $D(0)$. It is an important building block for bonds that recover a fraction of their par amount at the time of default. If $B(0,t)$ denotes the price of a non-defaultable zero coupon bond with maturity t, it follows from equation (21) that

$$D(0) = \int_0^T B(0,t)\lambda(t) \exp\left(-\int_0^t \lambda(u)\, du \right) dt \ . \tag{22}$$

We see that $D(0)$ is just a weighted average of non-defaultable zero coupon bond prices, where the weights are given by the density of t.

> **Example 3:** We price a defaultable zero coupon bond that pays a fraction R = 30% of its notional at the time of default. We assume a maturity T of three years, riskless interest rates are constant at r = 4%, and the hazard rate is λ = 1%. If this were the hazard rate under the objective or real-world probability measure, it would translate into a default probability of 2.96% over the three-year time horizon, which corresponds roughly to that of an issuer with a rating of *BBB*. However, note that we are considering the hazard rate under the risk-neutral measure, so that it also incorporates a risk premium.
>
> The price of the defaultable bond is made up of the recovery payment and the payment at maturity, which is contingent on survival:
>
> $$B^d(0,T) = RD(0) + B(0,T)P[\tau > T].$$
>
> Because the hazard rate and interest rates are constant, we can explicitly calculate the integral in equation (22) to obtain
>
> $$D(0) = \frac{\lambda}{\lambda + r}\left(1 - \exp\left(-(r+\lambda)T\right)\right).$$
>
> Inserting the numbers gives a bond price of $B^d(0,T)$ = \$86.91 on a \$100 notional. Using continuous compounding, this translates into a credit spread of 68 bp. If we compare this spread with actual credit curves, we see that it corresponds roughly to an issuer rating of single *A*. This reflects the risk premium investors demand for holding the bond.

4. Rating-Based Models

4.1. Introduction

Up until now we have focused on the modelling of the default counting process. This approach is sufficient provided we are doing arbitrage-free pricing of defaultable securities that have no explicit rating dependency. The variations in the price of the security that occur as a result of market-perceived deteriorations or improvements in the credit quality of the issuer, and which can lead to changes in credit ratings, can be captured by making the hazard rate stochastic. An increase in the stochastic hazard rate reflects a credit deterioration scenario, and a decrease represents a credit improvement scenario.

However, some securities do have an explicit credit rating dependency. Very recently, a number of telecom bonds have been issued with rating dependent step-ups on their coupons. And though they are a small fraction of the overall market, credit derivative contracts do exist that have payoffs linked to a credit rating.[3] Within the world of derivatives, ratings-linked models are also useful for examining counterparty exposure, especially if collateral agreements are ratings linked. Other potential users of ratings-linked models include certain types of funds that are only permitted to hold assets above a certain ratings threshold. Finally, ratings-linked models have been given a new impetus by the fact that Basle Committee on Bank Supervision has proposed allowing ratings-based methodologies for computing bank regulatory capital.

Moreover, given the wealth of information available from rating agencies, a natural development has been to enrich the binary structure given by the default indicator to one incorporating transitions between different rating classes. The generally used approach is to model these transitions by a Markov chain; it was initiated by Jarrow/Lando/Turnbull (1997) and is described in the following section.

4.2. Description

Suppose that a rating system consists of rating classes $1, \ldots, K$, where K denotes default. The quantity to be modelled is the ratings transition matrix $Q(t,T) = (q_{i,,j}(t,T))_{i,j=1, \ldots, K}$, where the entry $q_{i,,j}(t,T)$ denotes the probability that an issuer in rating class i at time t will be in rating class j at T. The default state is assumed to be absorbing, which means that an issuer never leaves the default state once it has been entered. Economically, this implies that there is no reorganization after a default.

In this setup, the ratings transition process is Markovian; i.e., the current state of the credit rating is assumed to contain all relevant information for future rating changes. This implies that the transition probabilities satisfy the so-called Chapman-Kolmogorov equations. For $t \leq T_1 \leq T_2$, we have

$$Q(t,T_2) = Q(t,T_1)\,Q(T_1,T_2)\,. \tag{27}$$

In other words, the transition matrix between time t and T_2 is the matrix product of the two transition matrices from time t to T_1 and T_1 to T_{21}. If Q is the one-

[3] According to the British Bankers' Association 1998 survey of the credit derivatives market, derivatives conditioning on downgrades is not widely used and has been phased out of the ISDA master documentation.

year transition matrix and the Markov chain is time-homogeneous, the transition probabilities for an n year period are given by $Q(0,n) = Q^n$. In the simplest continuous-time case, the transition matrix is constructed from a time-independent generator $\Lambda = (\lambda(i,j))_{i,j=1,...K}$ via the matrix exponential

$$Q(t,T) = \exp\big(\Lambda(T-t)\big) = \sum_{n=0}^{\infty} \frac{(T-t)^n \Lambda^n}{n!} . \tag{28}$$

For small time intervals, we can consider equation (28) up to first order to obtain

$$Q(t,t+dt) \approx I + \Lambda dt ,$$

and

$$q_{i,j}(t,t+dt) = \begin{cases} \lambda(i,j)\,dt, & i \neq j \\ 1+\lambda(i,i)\,dt, & i = j \end{cases} . \tag{29}$$

Equation (29) gives the natural interpretation of the generator matrix Λ. For $i \neq j$, $\lambda(i,j)$ is the transition rate between the rating classes i and j. Furthermore, $\lambda(i,j)$ is negative and can be interpreted as the exit rate from the ith class. As such, the generator Λ is the natural generalization to a rating class framework of the hazard rate introduced in equation (13).

Example 4: A typical example for a transition matrix is the average one-year matrix provided by Moody's for the period between 1980 and 1999 (Figure 4).

Using the equation $Q(0,5) = Q^5$, we can matrix multiply the transition matrix to obtain the 5-year transition probabilities. By examining the likelihood of ending in the default state, we can compute the following 5-year default probabilities conditional on starting in the corresponding rating category (Figure 5).

Figure 4. **Average One-Year Transition Matrix for 1980-1999 by Moody's Investor Service, Probabilities Are Conditional on Rating's Not Being Withdrawn,** %

	Aaa	**Aa**	**A**	**Baa**	**Ba**	**B**	**Caa-C**	**Default**
Aaa	89.31	10.15	0.50	0.00	0.03	0.00	0.00	0.00
Aa	0.96	88.42	10.04	0.38	0.16	0.02	0.00	0.04
A	0.08	2.34	90.17	6.37	0.81	0.22	0.00	0.02
Baa	0.09	0.39	6.42	84.48	6.92	1.39	0.12	0.20
Ba	0.03	0.09	0.50	4.41	84.25	8.65	0.52	1.54
B	0.01	0.04	0.17	0.58	6.37	82.67	2.98	7.17
Caa-C	0.00	0.00	0.00	1.10	3.06	5.89	62.17	27.77
Default	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Figure 5. **5-Year Default Probabilities for an Issuer Starting in a Rating Category Using Moody's One-Year Transition Matrix**

Aaa	Aa	A	Baa	Ba	B	Caa-C
0.05	0.28	0.62	2.97	11.58	31.23	69.77

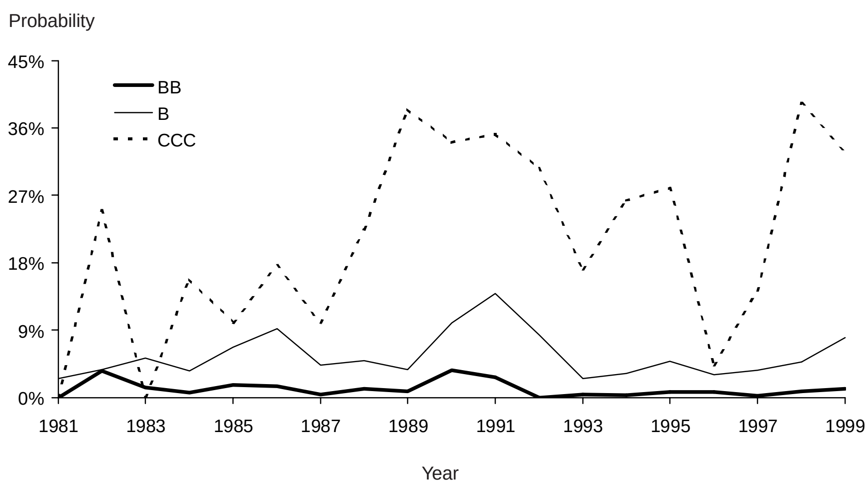
We stress the fact that only risk-neutral probabilities are relevant for pricing. Therefore, the risk premium must be modelled in some way, in order to relate model transition probabilities to historical transition probabilities such as those given above. Typically, one uses a tractable parametric form for a time-dependent generator and attempts to calibrate to the market prices of bonds.

4.3. Discussion

The Markov chain approach to the description of ratings transitions is elegant and tractable, but oversimplifies the actual dynamics, especially in the time-homogeneous formulation. Also, the relatively small amount of data available makes itself felt.

A closer look at empirical transition matrices shows this clearly. Standard and Poor's provides one-year transition matrices for all the years from 1981-1999, c.f. its annual study of long-term defaults and ratings transitions: Standard and Poor's (2000). In theory, if the dynamics of historical ratings changes were described by a homogeneous Markov chain, then all matrices should be the same up to sampling errors. Actually, the one-year default probabilities show huge variations, especially in the sub-investment grade sector (Figure 6).

Figure 6. **One-Year Default Probability Conditional on Rating's Not Being Withdrawn, Based on Standard and Poor's Rating Transitions,** 1981-1999



The small number of actual defaults means that many entries in the empirical transition matrix are zero (Figure 4). This leads to a slight problem when trying to extract a generator of the Markov chain via equation (28). It can be proven mathematically that an exact solution of equation (28), which satisfies the parameter constraints for a generator, does not exist for the matrix in Figure 4. Therefore, one must be satisfied with an approximate solution. For more on this topic, see Israel/Rosenthal/Wei (2000).

On the other hand, the advantage of using a rating-based model for calibration is that it allows the construction of credit curves for issuers of different quality in a unified and consistent manner. This is particularly useful when there are only very few instruments available for calibration in each rating class. However, the calibration procedure is much more involved than with the standard hazard rate approach, due to the larger number of parameters and the internal consistency constraints imposed by the Markov chain framework.

It is well known that market spreads tend to anticipate ratings changes. However, stochastic fluctuations in the credit spread in between ratings changes cannot be modelled with a deterministic generator; it has to be made dependent on additional state variables. In particular, this is crucial for the pricing of payoffs contingent on credit spread volatility. For extensions along these lines, see Lando (1998) and Arvanitis/Gregory/Laurent (1999).

In general, the case for implementing a rating-based model instead of a hazard rate model with a single intensity process is not completely clear due to the tradeoffs mentioned above. Put simply, ratings-linked models are essential for evaluating ratings-linked contingencies in the real-world measure. It is, therefore, an approach which works best for risk managers and investors that have an explicit exposure to downgrade risk as opposed to spread risk. However, for pricing bonds and credit derivatives with no explicit ratings dependency, it is more natural to model the spread or hazard rate, especially if we are pricing within the risk-neutral measure.

5. Conclusions

Modelling credit is a difficult task for a variety of reasons. Nonetheless, credit models have become an essential requirement in the analysis, pricing, and risk management of credit. An understanding of the advantages and disadvantages of the various models is, therefore, necessary to anyone wishing to apply a more quantitative approach.

Structural models of single-issuer default were shown to be best for performing a risk assessment of publicly traded companies or for analyzing the effect of the capital structure of a firm. Structural models are, however, not the preferred choice for pricing and hedging credit derivatives within a risk-neutral framework. This is where reduced form models win out, since they have the flexibility to price even the most complicated exotic credit derivatives, and all within a framework that refits market-observed prices.

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