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QUANTITATIVE CREDIT RESEARCH

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## INTRODUCTION

The structured credit market has been experiencing an explosion in volume over the past few years, particularly of portfolio credit products. More than $60 billion of cash and synthetic CDOs have been issued since the beginning of this year. Basket trades are becoming increasingly common as a risk management tool and as a method of taking/ hedging correlation risks through exposure in specific names, industry sectors, or rating buckets. The growth in structured product impacts valuations not just in the derivatives market, but also in the cash market. For example, a spurt of issuance of cash flow CDOs backed by investment grade collateral has largely driven the 132 bp tightening in the cross-over corporate market since the beginning of the year. The synthetic CDO bid has caused several credits to trade tighter in the default swap market than in the cash market.

In the second issue of our Quantitative Credit Research Quarterly publication we focus on portfolio credit and discuss a wide range of different topics from default modeling, risk and return characteristics of portfolio credit products, to characteristics of basket default swaps. In addition we explore the valuation of esoteric versions of single name credit derivatives—digital default swaps. Such trades, while still a small part of the overall market, are becoming more popular, particularly with European banks, in order to take advantage of a lower regulatory capital charge using digitals. **Investors in cash and synthetic portfolio structures as well as traders and risk managers of credit derivatives positions should find the articles in this issue of relevance for their strategic and tactical decision making.**

The first article in this report takes a new approach to defining default statistics. **The default experience statistic, as measured by the number of defaulted issuers in a given portfolio**, is an alternative way to characterize the probability of default. We define this statistic, show its historical behavior for randomly chosen high yield and investment grade portfolios, and describe its dependence on the portfolio size, and its tendency to have tail risk. We further focus on the rating quality dependence of this statistic, characterizing it by simulated Moody's rating score.

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### CDO Research and Strategies

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The second article discusses **risk return characteristics on leveraged investments in high yield bonds**. We extrapolate into the past to understand the performance of such investments through the decade 1990-2000 in order to put context to CDO equity returns. We conclude that asset managers add significant value and that unmanaged investments can perform well only if an investor is able to choose the "right" vintage. Estimated Sharpe ratios for managed CDO equity compare favorably with those of other fixed income asset classes and the S&P 500.

Following our overview of single-issuer default models in the March 2001 issue of *Quantitative Credit Research*, we present some of the **modeling approaches used to analyze the default risk of multiple issuers**. Default correlation is the essence of portfolio credit modeling. In addition, the number of assets and the credit quality of the assets determine the credit risk profile of a portfolio. Clearly, the more assets in the portfolio, the less exposed is the investor to a single default. Equally, the higher the credit quality of the assets, the less likely a default and the lower the expected loss on the pool of assets. Default correlation plays a significant role in these structures. If the assets in a portfolio have a high default correlation, then, when assets default, they do so in large groups. This can significantly affect the credit risk profile, making large portfolio losses more likely and fattening the tail of the loss distribution.

An *Nth*-to-default basket is significantly more complex than a single-name default swap. The probability of the credit event is jointly determined by the order of protection, the number of reference credits, and their default probabilities and correlations. In the article on **basket default swaps and the credit cycle**, we discuss these complex and sometimes counter-intuitive relationships. For example, we observe that while a general deterioration of the credit quality of the reference assets increases the value of any order of basket protection, the effects of correlation changes are qualitatively different for different orders. Studying the implications of these effects, we show that the return process of a second-to-default investment may be more exposed to market risk than a first-to-default position on the same set of names.

Finally, in the last article, we discuss the **determinants of digital default swap premium**. Digital default swaps are different from conventional (floating recovery) swaps because they transfer different types of risk. Conventional swaps transfer default loss risk, while digitals transfer default event risk. The implicit recovery risk remains unpriced in a hedged digital swap. the digital swap breakeven spread, calibrated to market-observed conventional default swap spread, must contain a premium because of 1) uncertainty about the market average recovery rate and 2) the inability to diversify firm-specific recovery rate deviations completely. We estimate these premia and show examples of pricing for underlying credits of various seniority.

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## DEFAULT EXPERIENCE STATISTIC

The default experience statistic, as measured by the number of defaulted issuers in a given portfolio, is an alternative way to characterize the probability of default, which may have an intuitive appeal for CDO managers and investors in basket structures. We define this statistic, show its historical behavior for randomly chosen high yield and investment grade portfolios, and describe its dependence on the portfolio size. We further focus on the rating quality dependence of the default experience statistic, characterizing it by a simulated Moody's rating score.

### Introduction

Many portfolio managers view default risk in discrete units (*how many bonds in my portfolio may default?*) rather than in continuous units (*what is the probability of default?*). The distinction between the two definitions is largely a matter of interpretation—one can always deduce the distribution of number of defaults from the forecast probability.

However, for small portfolios, the discreteness of the *default experience statistic* (hereafter referred to as *DES*) becomes relevant. In actual life, there will be 1, 2, 3, etc. default experiences, and the average forecast number of defaults, say 2.45, may be less intuitive than the chart of forecast probabilities for each given number of defaults. **This statistic will be of particular interest for CDO collateral managers and investors in basket structures, including both synthetic and cash deals, because the performance of such structured products depends strongly on the realized number of defaults.**

The default experience statistic also highlights the uncertainty around the forecast number of defaults because it shows the whole distribution. As we demonstrate below, the shape of this distribution depends strongly on the number of bonds in the portfolio, and its tails should not be ignored for portfolios of small (less than 100 names) or even medium (200 names) size.

In this article, we show the two sides of the default experience statistic. One is related to the universe of credits from which the portfolio is chosen. We illustrate this with the historical analysis of portfolios chosen from the U.S. high yield and U.S. investment grade issuer universes.

The other is the quality (as measured by Moody's rating score) of the portfolio under consideration. We illustrate this dependence with examples of:

- A small portfolio of investment grade bonds, similar to underlying basket of certain synthetic CDOs, and
- A medium-sized portfolio of high yield bonds, similar to typical collateral of arbitrage CBOs.

**We conclude that knowing the rating score range and the size of the portfolio is often sufficient to estimate the default experience statistic with good accuracy, even if the underlying pool selection is otherwise unknown.**

## Methodology

Assume that at a given point in time, we are given a population of size $N$, from which $N_d$ issuers will default within one year. Let us assume that we randomly select a portfolio of size $n$ from this population. The *default experience statistic (DES)* answers the question: **"What is the probability that there will be exactly $n_d$ defaults in this portfolio within a given time?"**

This question can be answered by a simple counting of possibilities. The total number of various ways a portfolio of $n$ issuers can be selected is given by:

(1) $$K_{\text{total}}(N,n) = C_N^n \text{, where } C_N^n = \frac{N!}{n! \cdot (N-n)!} \text{ is the binomial coefficient.}$$

The number of portfolios that have exactly $n_d$ defaults among all randomly selected ones is given by the combinatorial Equation 2 below. It counts the number of possibilities that $n_d$ defaulted issuers can be selected among the $N_d$ defaulted population and an independent number of possibilities that $n$ - $n_d$ surviving issuers can be selected among the $N$ - $N_d$ non-defaulted population:

(2) $$K_{\text{dflt}}(n_d | N, N_d, n) = C_{N_d}^{n_d} \cdot C_{N-N_d}^{n-n_d}$$

Therefore, the probability of having exactly $n_d$ defaults is given by the following expression:

(3) $$P_{\text{dflt}}(n_d | N, N_d, n) = \frac{K_{\text{dflt}}(n_d | N, N_d, n)}{K_{\text{total}}(N,n)} = \frac{C_{N_d}^{n_d} \cdot C_{N-N_d}^{n-n_d}}{C_N^n}$$

Equation 3 corresponds to a so-called *hypergeometric distribution*. It is consistent with the intuitive notion of the *population default probability* $p = N_d / N$. The expected number of default experiences is simply $\langle n_d \rangle = n \cdot p$.

In the special case of a single issuer portfolio $n = 1$, setting the number of defaults to zero gives the *single issuer survival probability* $q$:

(4) $$q = P_{\text{dflt}}(n_d = 0 | N, N_d, n = 1) = \frac{C_{N_d}^0 \cdot C_{N-N_d}^1}{C_N^1} = \frac{N - N_d}{N} = 1 - p$$

In the limit of large $N$, while keeping the population default probability $p$ constant, the hypergeometric distribution converges to the *binomial distribution* with probability parameter $p$—a commonly used approximation:

(5) $$P_{\text{dflt}}\left(n_d \middle| p = \frac{N_d}{N}, N \to \infty, n\right) = C_n^{n_d} \cdot p^{n_d} \cdot (1-p)^{n-n_d}$$

Thus, while the binomial distribution takes into account only the finite size of the portfolio, the hypergeometric distribution takes into account the finite size of both the portfolio and the selection universe.

### Portfolio Size Effects

Below we plot the probability distribution (3) for various sizes of the sample portfolio. The population universe is chosen to be the U.S. high yield bond universe for January 1997-January 1998.

We show the results for portfolios of various sizes from 50 to 200. One can see that the distributions tend gradually to converge to a certain shape for a sufficiently large size of the portfolio and that the width of the distribution is fairly large. Note that the mass of the distribution is moving to the right simply because, while keeping the probability of the default constant, there are more defaults associated with larger portfolios.

For a relatively large portfolio of 200 issuers, there is a sizeable probability (8.81%) of having seven or more defaults, while the forecast mean is only 4.02 defaults. This highlights our thesis that the tails of the DES distribution are significant. If a high yield portfolio manager were interested in a downside default risk estimate for a blind pool of 200 bonds using this particular historical snapshot, there would be a likelihood of having as many as seven defaults in such a portfolio at the 95% confidence level.

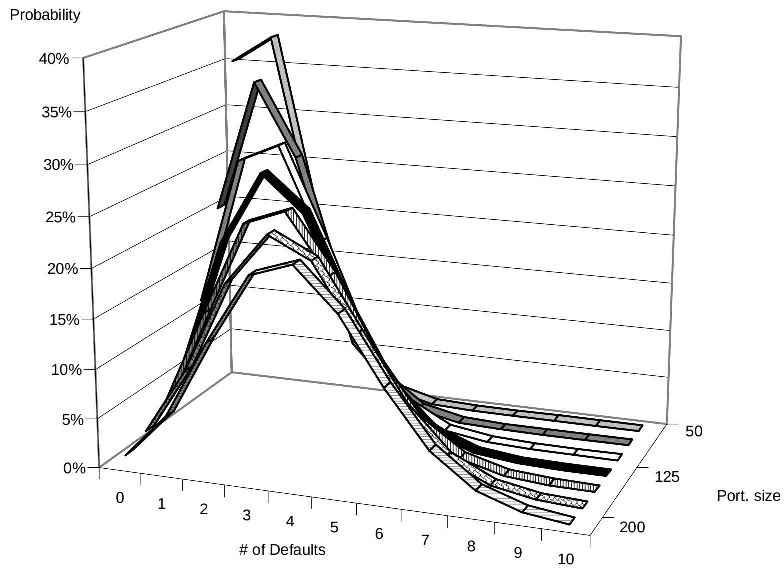
The two charts in Figure 2 depict the historical behavior of the default experience statistic of the high yield universe. For each year from 1987 until 2000, we performed the same computation as outlined above. We estimated the mean, median, top, and bottom quartiles[1] of the distribution of number of defaults, given that particular year's historical population experience and the pre-set size of the portfolio. We examine two portfolios, with 50 and 200 bonds.

We show the data for each cohort year as a stock-style chart. The top of the thin line corresponds to the top quartile, the bottom of thin line corresponds to the bottom quartile. The two ends of the middle box depict the median and the mean of the distribution for that year. If the mean is greater than the median, then the top boundary of the box corresponds to mean, the bottom corresponds to median, and the box is painted white. If the mean is less than the median, the top is the median, the bottom is the mean, and the box is black. Looking at these charts, one can see

[1] Note that the definition of percentiles for hypergeometric distribution is somewhat peculiar due to discreteness of the number of defaults. For each given probability percentile, we choose the *smallest integer* such that the cumulative probability of observing number of defaults equal to or less than that integer is *greater than the chosen percentile*. For a small portfolio size this leads to "jumpy" quartile plots, and even to some cases where the median (50% percentile) coincides with one of the top (75%) or bottom (25%) quartiles. Contrast this with the case of a smooth continuous distribution, where the cutoff value of the input argument corresponds to a strict equality of the cumulative probability to the chosen percentile, and where the different percentile levels can never correspond to the same argument.

Figure 1. **Distribution of Number of Defaults vs. Size of Portfolio**

Total HY issuers: 1044, Total HY defaults: 21, January 1997-January 1998




| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 35.31% | 38.07% | 19.13% | 5.96% | 1.29% | 0.21% | 0.03% | 0.00% | 0.00% | 0.00% | 0.00% |
| 75 | 20.57% | 34.14% | 26.59% | 12.93% | 4.40% | 1.11% | 0.22% | 0.03% | 0.00% | 0.00% | 0.00% |
| 100 | 11.81% | 26.85% | 28.73% | 19.26% | 9.07% | 3.19% | 0.87% | 0.19% | 0.03% | 0.00% | 0.00% |
| 125 | 6.68% | 19.51% | 26.88% | 23.24% | 14.14% | 6.44% | 2.28% | 0.64% | 0.15% | 0.03% | 0.00% |
| 150 | 3.72% | 13.41% | 22.83% | 24.43% | 18.42% | 10.42% | 4.58% | 1.61% | 0.46% | 0.11% | 0.02% |
| 175 | 2.04% | 8.81% | 18.04% | 23.23% | 21.10% | 14.39% | 7.64% | 3.23% | 1.11% | 0.31% | 0.07% |
| 200 | 1.10% | 5.58% | 13.46% | 20.44% | 21.91% | 17.64% | 11.06% | 5.54% | 2.25% | 0.75% | 0.21% |

whether the distribution was wide (quartiles are far from the mean) and whether it was skewed toward a higher or lower number of defaults (judging by the relative position of the mean and the median).

As shown in Figure 2, the relative width of the default experience statistic distribution is much larger for smaller portfolios than for the larger ones. For example, the DES range for a 50-name portfolio drawn from the cohort formed in March 1990 is between 6% and 12% of portfolio size, while the range for a 200-name portfolio is 8.5%-9.5%. This confirms that the tails of DES distribution are significant for small portfolios.

For comparison, Figure 3 shows the investment grade default experience statistic in 5-year cohorts. Each date refers to the start date of cohort life span. The conclusions regarding the distribution width as a function of portfolio size are essentially the same. Note that a shifted pattern of time dependence is due to the longer lifespan of the cohorts—the last cohort spans April 1996 through April 2001.

Please note that in Figures 2 and 3, one must not directly compare the number of defaults (shown on the right Y-axis) for portfolios of different sizes. As we explained, the relevant variable for apples-to-apples comparison is the number of defaults, expressed as the percentage of the portfolio size (shown on the left Y-axis).

Figure 2. **Historical Distribution of High Yield DES in 1-Year Cohorts**

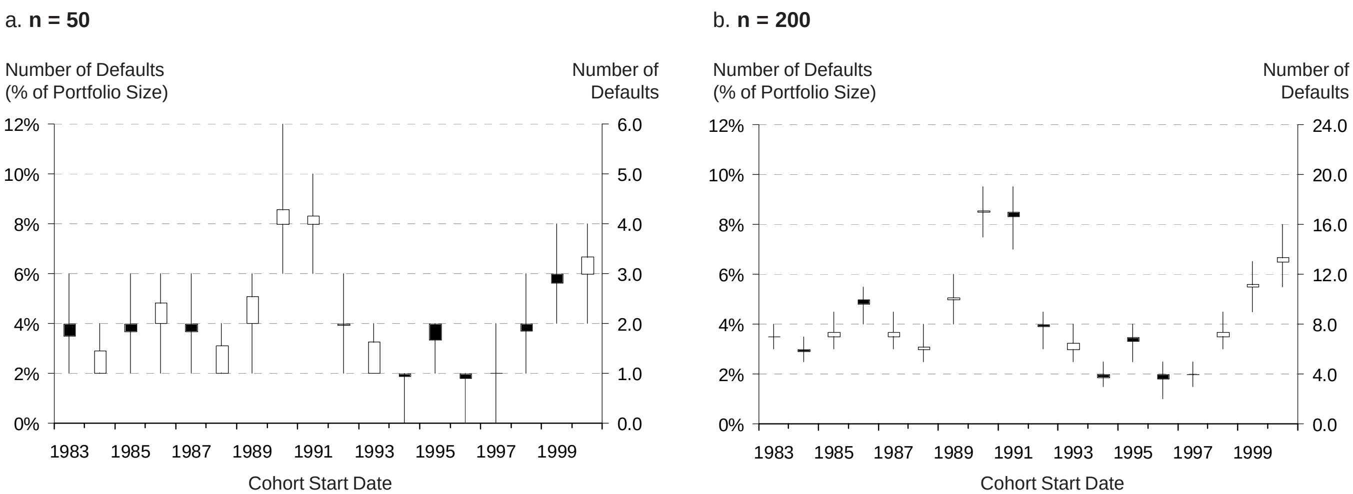
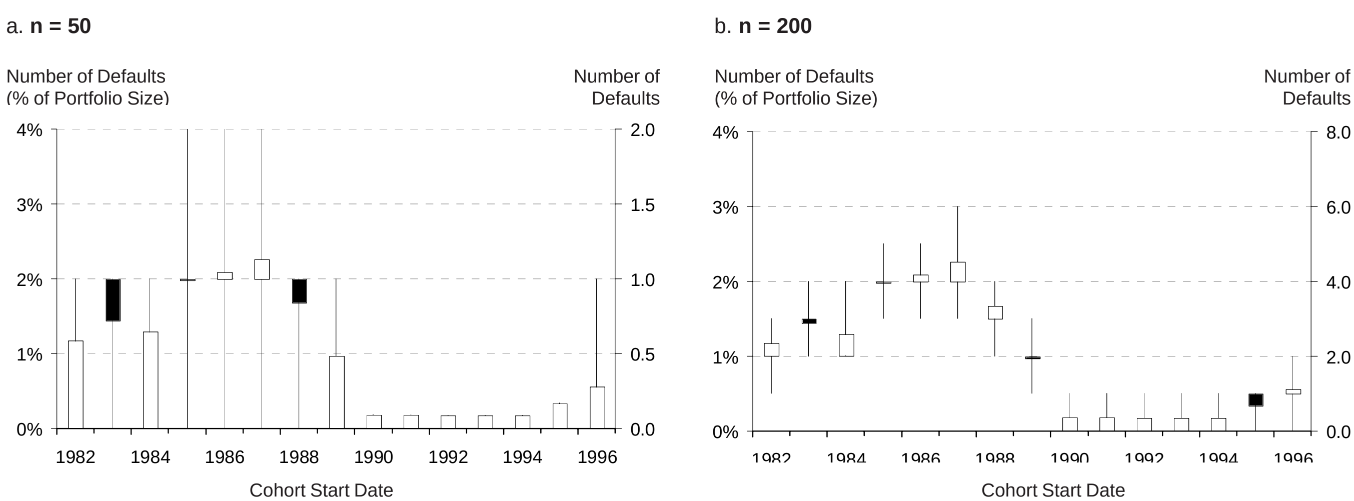



Figure 3. **Historical Distribution of Investment Grade DES in 5-Year Cohorts**



## Portfolio Quality Effects

Let us now focus on the rating composition of the portfolio and the universe. In both synthetic CDO and cash CDO markets, the two main characteristics of the portfolios underlying these structures are the average quality and diversity of the portfolio. We follow the market standard and measure the quality by the Moody's average rating scores. Our goal is to discern whether knowing the rating score and the size of a portfolio is sufficient to be able to characterize the default experience statistic with acceptable accuracy.

### *Small Size Investment Grade Portfolio*

Let us first consider the case of investment grade bonds. Assume that a **portfolio of 40 bonds with a rating score of 240-270** has been selected from the investment grade universe. Such characteristics are typical for many **synthetic CDOs of high grade bonds**. Investors are typically interested in senior and subordinated tranches, which are hit only if sufficiently many (3, 4, etc.) defaults occur within the lifetime of the deal, which is typically 5-10 years.

What can we say about default characteristics of this portfolio if we don't know anything else about its composition? Let us presume that it has been randomly chosen with the rating score range being the only selection criterion.

Figure 4 describes the rating factors used for investment grade bonds. Note that the higher score corresponds to lower credit quality. The portfolio score is calculated as the weighted average of individual security scores. In our investigation, we will always assume equally weighted portfolios; therefore, the portfolio score will be the simple average Moody's score.

Figure 5 shows the comparison of the target range with the score measured on the whole universe (with equal weighting). This characterizes the relative quality of the selected portfolio and will help explain the differences between the estimated default experience statistic for the chosen portfolio and for the whole investment grade universe, shown in Figure 3.

As we can see, the universe's credit quality has deteriorated over the years, with the rating score growing from just above 150 to around 200 (keep in mind that higher score means lower credit quality). The target rating score range is chosen so that it is fairly close to the average score of the investment grade universe in recent years. However, the difference in the early years is somewhat greater, which should lead to a relatively higher number of defaults in the selected portfolio, compared with the average in the universe.

Figure 4. **Moody's Rating Factors for Investment Grade Bonds**

| **Rating** | Aaa | Aa1 | Aa2 | Aa3 | A1 | A2 | A3 | Baa1 | Baa2 | Baa3 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Rating Factor** | 1 | 10 | 20 | 40 | 70 | 120 | 180 | 260 | 360 | 610 |

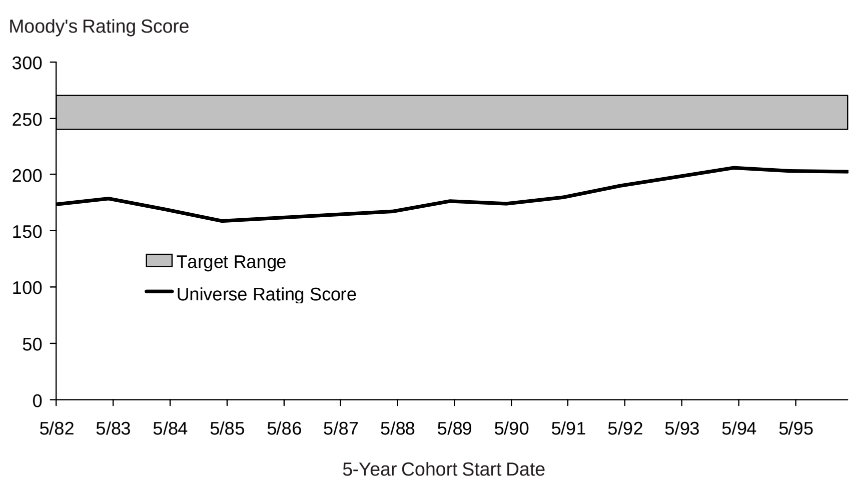
Instead of deriving a complicated and unintuitive formula for the probability of having a given number of defaults under the condition of matching the rating score range, we have estimated the distribution of DES by Monte Carlo simulation.

For each cohort, we randomly selected many portfolios of 40 names from the investment grade universe. Then we selected only those portfolios whose rating score fell in the target range. Since there were typically few such portfolios, we repeated the simulation, adding new ones until the number of randomly selected portfolios with satisfactory scores reached 1000. Then we measured the number of defaulted bonds in each of these portfolios and counted the number of portfolios with 0 defaults, 1 default, 2 defaults, etc. Dividing by the total number of selected portfolios, we obtained the estimate of the probability of having a given number $k$ of defaults in a portfolio of size 40, under the condition of the rating score being in a target range.

The results of this estimation are presented in Figure 6. We show the probability of having 0, 1,…,7 defaults for each of the 5-year cohorts. Note that the X-axis denotes the start date of the cohort, i.e., the cohort life spans five years following the shown date.

As we noted before, the early dates show a higher probability of having one or more default than the investment grade universe, on average (contrast Figure 6 with Figure 3). The recent dates, however, show similar pattern of default experience statistic between the target portfolio and the broad investment grade universe. Figure 7 shows the data corresponding to Figure 6.

Figure 5. **Target Rating Score Range versus Historical Investment Grade Universe Rating Scores**



To characterize the quality of simulated portfolios further, Figure 8 shows the histogram of rating scores for two representative periods, the cohort spanning 1996-2001 and the cohort spanning 1987-1992 (by far the worst period for default experience).

The skew of this distribution to the lower rating scores is not a bias of the simulation. It simply reflects the distribution of the ratings in the investment universe. Moreover, as is clear from a comparison of the two histograms, the skew in the more recent cohort is less than in the cohort formed in 1987. This is because, as we mentioned before, the target rating score range is substantially closer to the

Figure 6. **Historical Default Experience Statistic for a Portfolio of 40 names**
Target Rating Score 240-270

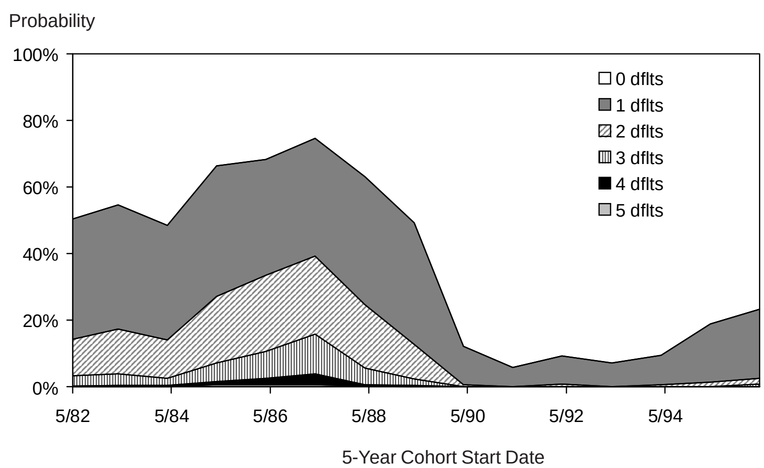



Figure 7. **Historical DES, Portfolio: 40 Names, Rating Score 240-270**

| **Cohort Start Date** | **Cohort End Date** | **Universe Score** | **Port. Mean No. dflts** | **Probability of Having a Given Number of Defaults,** 1000 Trials, % | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
| 1-May-82 | 1-May-87 | 173 | 0.68 | 49.6 | 36.1 | 11.0 | 3.0 | 0.3 | 0.0 | 0.0 | 0.0 |
| 1-Apr-83 | 1-Apr-88 | 179 | 0.77 | 45.3 | 37.3 | 13.5 | 3.4 | 0.5 | 0.0 | 0.0 | 0.0 |
| 1-Apr-84 | 1-Apr-89 | 169 | 0.66 | 51.6 | 34.4 | 11.5 | 2.0 | 0.4 | 0.1 | 0.0 | 0.0 |
| 1-Apr-85 | 1-Apr-90 | 158 | 1.03 | 33.6 | 39.2 | 19.9 | 5.6 | 1.0 | 0.7 | 0.0 | 0.0 |
| 1-Apr-86 | 1-Apr-91 | 162 | 1.16 | 31.8 | 34.7 | 22.9 | 8.0 | 2.0 | 0.6 | 0.0 | 0.0 |
| 1-Apr-87 | 1-Apr-92 | 164 | 1.35 | 25.3 | 35.3 | 23.5 | 11.8 | 3.3 | 0.7 | 0.1 | 0.0 |
| 1-Apr-88 | 1-Apr-93 | 167 | 0.95 | 36.9 | 38.4 | 19.0 | 4.9 | 0.4 | 0.3 | 0.1 | 0.0 |
| 1-Apr-89 | 1-Apr-94 | 176 | 0.65 | 50.8 | 36.5 | 10.4 | 1.9 | 0.3 | 0.1 | 0.0 | 0.0 |
| 1-Apr-90 | 1-Apr-95 | 174 | 0.13 | 87.9 | 11.5 | 0.6 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1-Apr-91 | 1-Apr-96 | 180 | 0.06 | 94.2 | 5.7 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1-Apr-92 | 1-Apr-97 | 190 | 0.10 | 90.8 | 8.4 | 0.8 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1-Apr-93 | 1-Apr-98 | 198 | 0.07 | 92.8 | 7.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1-Apr-94 | 1-Apr-99 | 206 | 0.10 | 90.6 | 8.8 | 0.6 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1-Apr-95 | 1-Apr-00 | 203 | 0.20 | 81.1 | 17.5 | 1.3 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1-Apr-96 | 1-Apr-01 | 202 | 0.27 | 76.7 | 20.7 | 1.8 | 0.8 | 0.0 | 0.0 | 0.0 | 0.0 |

average score of the universe in recent years, while being quite a bit farther off in the early years.

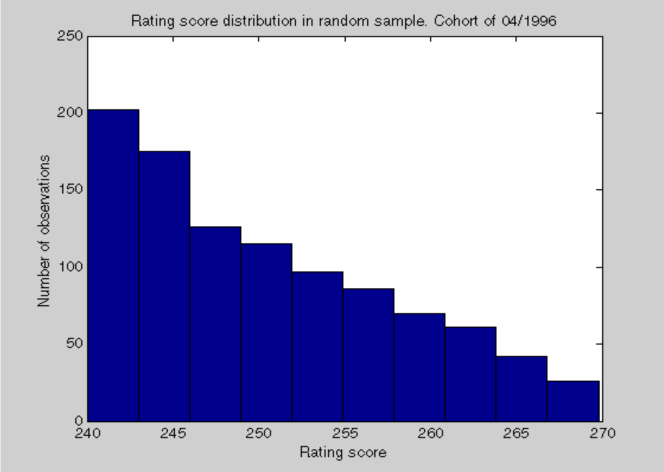
### *Medium-Sized High Yield Portfolio*

Let us now turn to high yield bonds. Assume that a **portfolio of 100 bonds with a rating score of 2220-2720** has been selected from the high yield universe. Such characteristics are typical for many **cash flow CDOs with average collateral of B1/B2 quality**. The average life of typical bonds in collateral of such CDOs is six years. Therefore, to make our numbers more relevant for this particular application, we perform our study with 6-year cohorts.

Again, our goal is to characterize the probability of observing 0, 1, 2, …, 30 defaults in a randomly selected portfolio of 100 names with an initial rating score in the target range.

The rating score table for high yield bonds is presented in Figure 9.

Figure 8. **Histogram of Rating Scores of Simulated Investment Grade Portfolios**



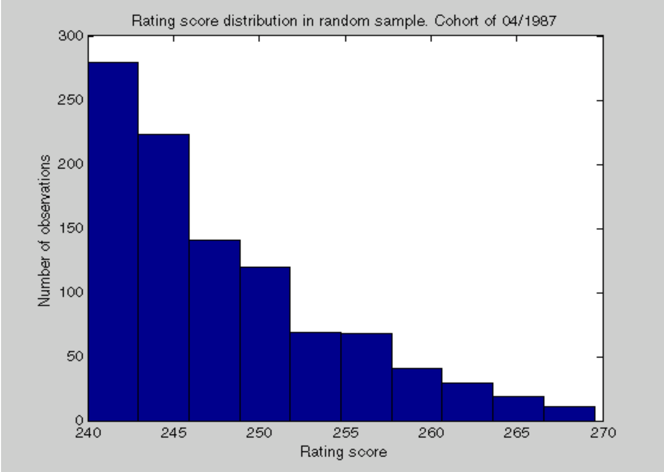




Figure 9. **Moody's Rating Factors for High Yield Bonds**

| **Rating** | Ba1 | Ba2 | Ba3 | B1 | B2 | B3 | Caa1 | Caa2 | Caa3 | Ca |
|---|---|---|---|---|---|---|---|---|---|---|
| **Rating Factor** | 940 | 1350 | 1780 | 2220 | 2720 | 3490 | 4770 | 6500 | 8070 | 10000 |

Figure 10 shows the comparison of the target range with the score measured on the entire high yield universe (with equal weighting). As we can see, the target score range captures the average quality of the selection universe pretty well across all dates.

Figure 10. **Target Rating Score Range versus Historical High Yield Universe Rating Scores**

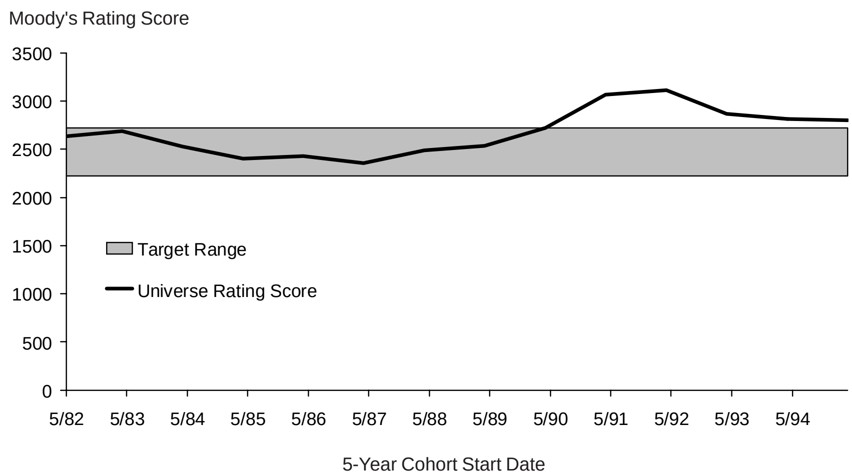
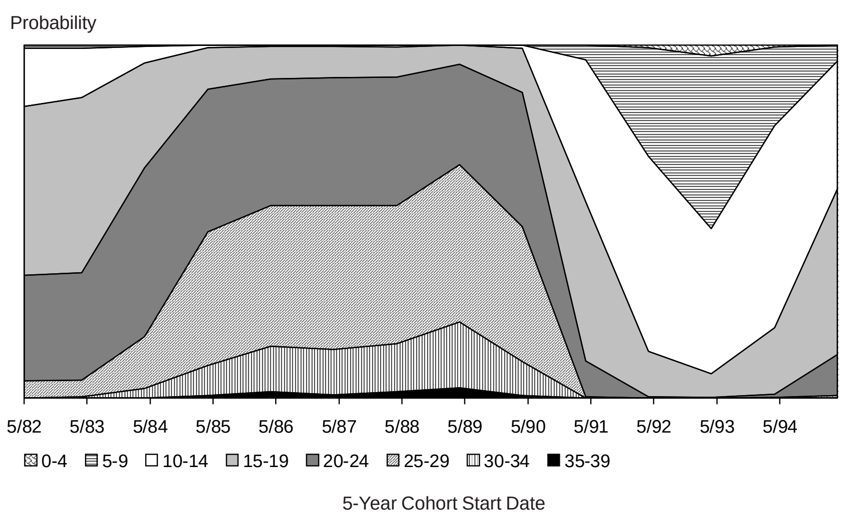



Figure 11. **Historical Default Experience Statistic for a Portfolio of 100 Names**
Target Rating Score 2220-2720



The estimation of the DES distribution for this target score range was done in the same manner described for the investment grade universe. The results for the probability of each given number of defaults are shown in Figure 11. We show the numbers of default grouped in buckets of five (0-4 defaults, 5-9 defaults, etc.).

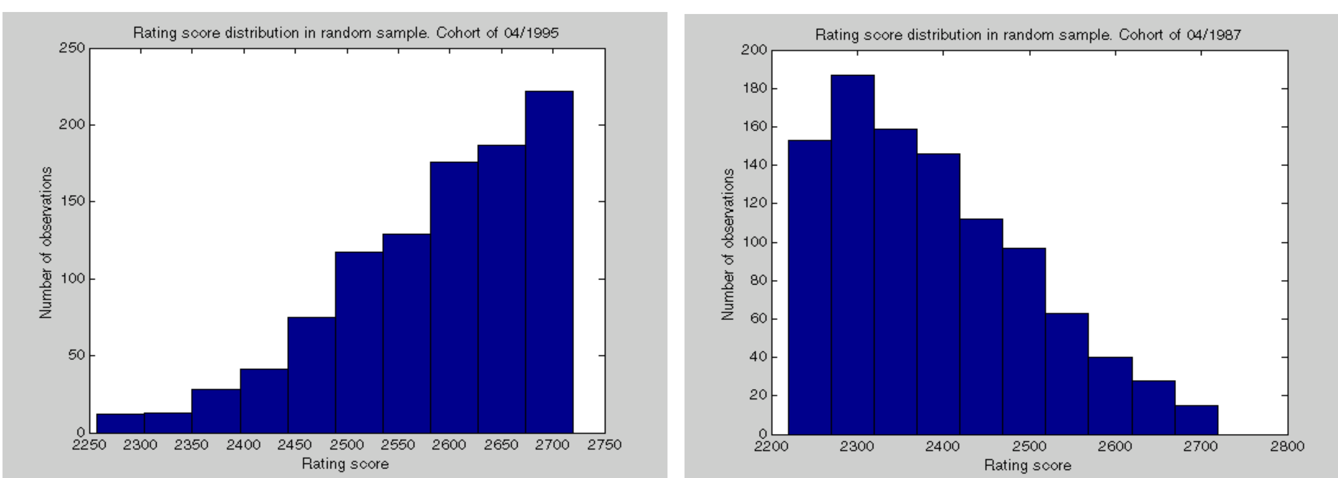
The same data, as well as the universe score and the mean number of defaults, are shown in Figure 12.

Just as in our study of the high grade portfolio, we show in Figure 13 the histogram of the rating scores in our randomly generated sample of portfolios for the most recent cohort (April 1995) and the cohort with worst default characteristics (April 1987).

Figure 12. **Historical DES, Portfolio: 100 Names, Rating Score 2220-2720**

| Cohort Start Date | Cohort End Date | Universe Score | Port. Mean No. dflts | Probability of Having a Given Number of Defaults, 1000 Trials, % | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 0-4 | 5-9 | 10-14 | 15-19 | 20-24 | 25-29 | 30-34 | 35-39 |
| 1-May-82 | 1-May-88 | 2632 | 18.08 | 0.0 | 0.9 | 16.5 | 47.8 | 29.9 | 4.9 | 0.0 | 0.0 |
| 1-Apr-83 | 1-Apr-89 | 2691 | 18.26 | 0.0 | 0.8 | 14.0 | 49.7 | 30.5 | 4.7 | 0.3 | 0.0 |
| 1-Apr-84 | 1-Apr-90 | 2530 | 20.99 | 0.0 | 0.2 | 4.8 | 29.7 | 47.9 | 14.8 | 2.6 | 0.0 |
| 1-Apr-85 | 1-Apr-91 | 2404 | 24.18 | 0.0 | 0.0 | 0.7 | 11.8 | 40.4 | 37.9 | 8.5 | 0.7 |
| 1-Apr-86 | 1-Apr-92 | 2431 | 25.09 | 0.0 | 0.0 | 0.2 | 9.4 | 35.8 | 39.9 | 13.0 | 1.5 |
| 1-Apr-87 | 1-Apr-93 | 2355 | 24.95 | 0.0 | 0.0 | 0.2 | 9.0 | 36.3 | 40.8 | 12.9 | 0.7 |
| 1-Apr-88 | 1-Apr-94 | 2490 | 25.07 | 0.0 | 0.0 | 0.5 | 8.5 | 36.4 | 39.3 | 13.5 | 1.8 |
| 1-Apr-89 | 1-Apr-95 | 2533 | 26.31 | 0.0 | 0.0 | 0.0 | 5.3 | 28.5 | 44.6 | 18.8 | 2.7 |
| 1-Apr-90 | 1-Apr-96 | 2721 | 24.33 | 0.0 | 0.0 | 0.8 | 12.6 | 38.0 | 38.3 | 9.6 | 0.6 |
| 1-Apr-91 | 1-Apr-97 | 3065 | 15.06 | 0.0 | 4.0 | 40.3 | 45.2 | 10.1 | 0.4 | 0.0 | 0.0 |
| 1-Apr-92 | 1-Apr-98 | 3114 | 11.08 | 0.6 | 30.9 | 55.4 | 12.8 | 0.3 | 0.0 | 0.0 | 0.0 |
| 1-Apr-93 | 1-Apr-99 | 2864 | 9.55 | 2.9 | 49.1 | 41.2 | 6.7 | 0.1 | 0.0 | 0.0 | 0.0 |
| 1-Apr-94 | 1-Apr-00 | 2815 | 11.85 | 0.4 | 22.4 | 57.3 | 18.8 | 1.0 | 0.1 | 0.0 | 0.0 |
| 1-Apr-95 | 1-Apr-01 | 2798 | 15.41 | 0.0 | 4.4 | 36.3 | 47.1 | 11.6 | 0.6 | 0.0 | 0.0 |

Figure 13. **Histogram of Rating Scores of Simulated High Yield Portfolios**



Again, it is worth noting that the apparent skew in the rating score distribution is not a bias in our Monte Carlo simulation, but rather the feature of the selection universe. In 1995, the universe happened to have somewhat worse quality on average than the target score range—therefore, we see the natural skew toward higher scores. In 1987, the universe was closer to the lower end of the range, and this is reflected in the histogram's being somewhat skewed toward lower scores.

## Summary

In conclusion, we would like to emphasize again the importance of distribution view for characterizing the default experience statistic in portfolios of limited size (less than several hundred names). The tails of this distribution are wide, and there is a non-negligible statistical chance of having substantially fewer or more defaulted issuers than one might expect naively based on average probability of default. The distribution narrows and approaches a stable shape as the number of securities in the portfolio exceeds roughly 100 names.

We have illustrated this with examples of both the investment grade and the high yield universe. Over time, the mean and the tails of the default experience statistic distribution change substantially, broadly following the mean default rate in the corresponding cohorts.

From a different angle, we have shown that the rating score appears, in many cases, to be a sufficient measure for selection of an otherwise blind portfolio of a fairly small number of issuers. In the case of both investment grade and high yield portfolios, portfolios that were constrained to have the rating score within the target range but were otherwise randomly selected appear to have distribution properties that are close to the typical bond in the universe with a similar rating score. We have ignored the industry diversification effect. While there may be some bias in our results, we believe that the net effect of selecting a random portfolio should be equivalent to setting a fairly high level of diversity score limit.[2]

The authors gratefully acknowledge valuable comments and insights from Sunita Ganapati, Marco Naldi, and Frank Iacono.

[2] Whenever one studies defaults in a portfolio setting, it is important to recognize the effects of default correlation. The diversity score targets are one simple way to control this parameter. While we do not model the dependence of DES upon correlation explicitly, such dependence is implicit in our results due to a certain level of correlation present in the selection universe. By setup, our focus has been on random and independent selection of names from a given cohort. This setup justifies the use of hypergeometric or binomial probability distributions by definition—the independence that we use is not that of forecast defaults but rather that of our choice of names. However, the default characteristics of the cohort itself are in part due to particular diversity and correlation levels, which happen to be realized historically, and one could look at most of the results in this paper as conditional upon those levels.

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## RISK RETURN CHARACTERISTICS OF LEVERAGED INVESTMENTS IN HIGH YIELD DEBT

Leveraged investments in high yield debt have grown in recent years along with the tremendous growth in the arbitrage CDO market. Of the estimated $160 billion of arbitrage high yield/ leveraged loan CDOs issued since 1995, the first loss piece or the equity tranche constitute 8%-10%, or $14 billion. In this article, we explore the factors that drive equity returns on high yield CDO investments and create a benchmarking tool for performance. We combine current CDO capital structure and liability spreads with historical high yield data to extrapolate into the past and to understand the performance of leveraged high yield returns through the entire decade (1990-2000). The lack of public data on CDO equity performance and the relatively recent growth in CDOs make such an analysis, however stylized, a valuable point of reference. We capitalized on the depth of Lehman Brothers Index spread data, Moody's default data, and our database of CDO new issue spreads to overcome some of the data hurdles.

Our study reveals the following:

- Asset managers can add significant value. Only the top third of managers (measured by default rate performance) can produce returns that make CDO equity an attractive asset class.
- Unmanaged investments can perform well only if an investor is able to choose the "right" vintage.
- The Sharpe ratios for managed CDO equity compare favorably with those of other fixed income asset classes and the S&P 500.

CDO equity does make sense from an asset allocation perspective. However, CDO equity investors should evaluate managers carefully before making their investment decisions. Those investors who believe that asset managers cannot consistently deliver good performance can minimize manager risk by timing their investments appropriately.

### Method for Computing Leveraged Returns

We set up a simplified leveraged structure that closely resembles a CDO (see Appendix on page 23). The capital structure is similar to a new issue high yield cash flow CDO with 90% debt and 10% equity. To simulate a CDO monthly cohort series, we assume that such a CDO backed by high yield bonds with average quality B1/B2 is issued every month from 1990 to 2000. The maturity of the CDO is assumed to be six years, which is the mean average life of the B1/B2 Lehman Brothers HY Index over the period.

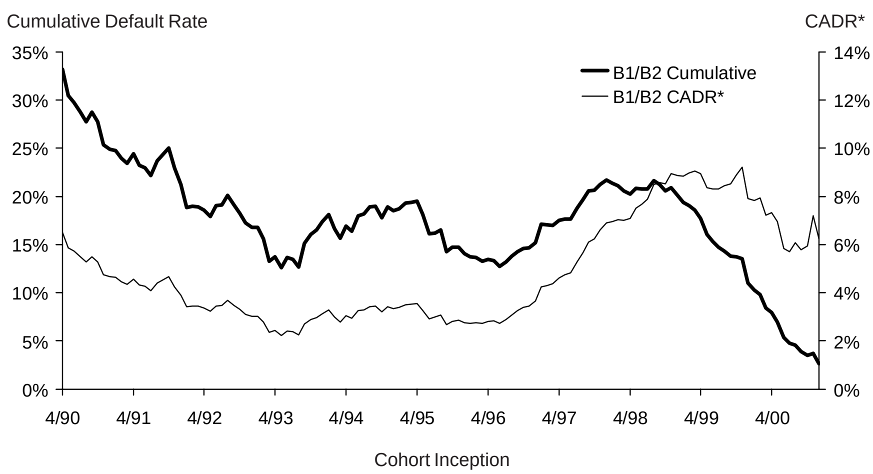
The authors would like to thank Arthur Berd and Marco Naldi for their valuable input. In addition, we appreciate the input from Rich Coffin, Joe Hornback, Mark Howard, Claude Laberge, and Prafulla Nabar.

We assume that each month's CDO purchases collateral at the corresponding monthly average spread on the Lehman Brothers B1/B2 Index and at the average price of the index for that month. The structure incurs the average liability costs (weighted average spread on various debt tranches) corresponding to the "issuance" or cohort month. Asset spreads are swapped to LIBOR to match the liabilities. The structure receives annual coupons on the assets and also pays liabilities annually. Every year, the structure is assumed to experience a default rate equal to the annual cohort default rate. A cohort is defined as all issuers with a rating of B1/B2 at the beginning of that month and is computed using the universe of corporates rated by Moody's Investor Service. The cohort is followed for six years (the life of the deal), and the 6-year cumulative default rate is annualized to compute the annual default rate that the CDO structure experiences (Figure 1). The recovery rate is assumed to be 45%, the historical average for senior unsecured bonds.[1] Using the above data, we compute the cash flows to the equity every year. At the end of six years, we assume that the structure self-liquidates and that all undefaulted assets mature at par. The residual par (if any) flows through to equity holders.

Using equity cash flows, we compute returns using two methods 1) the more common IRR and 2) the more realistic annualized total return. The IRR computation helps compare returns with new issue CDO equity pricing levels. The total

[1] Source: Moody's Investor Service.

Figure 1. **Historical Cumulative Cohort Default Rates and their Equivalent CADRs**




*Constant Annual Default Rate—computed by annualizing the cumulative default rate for that cohort.

Source: Lehman Brothers calculations on Moody's Investors Service data.

return computation helps us compute a Sharpe ratio for comparison with other asset classes. These returns are for an unmanaged CDO, since we are assuming that the CDO experiences the same default rate as the historical average cohort default rate for the universe of high yield bonds. We also compute the returns under different default assumptions (managed CDOs).

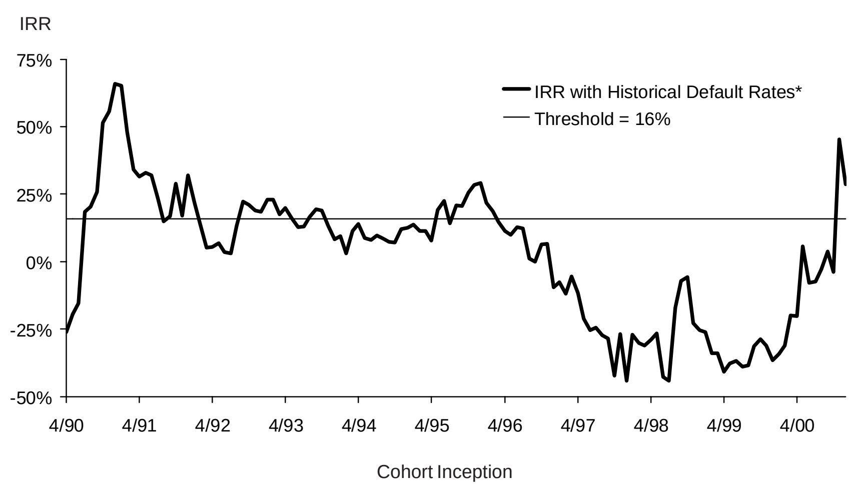
### *Simplifying Assumptions*

Readers should recognize that some simplifying assumptions were necessary for this analysis. Two of our assumptions could result in an overstatement of our equity returns. First, we do not model O/C, I/C ratios and other triggers that result in diversion of cash flows from equity to bondholders. Any such trigger violations would lead to lower equity returns. Second, we assume that par on undefaulted issuers is received at the end of year 6. In reality, a structure amortizes over time, and par is received over a certain window of time. This assumption affects only the timing of cash flows and should have a relatively minor effect. Other assumptions such, as no trading gains/losses, ignoring timing of defaults, and recoveries (i.e., we use constant annual default rates and immediate recovery) are consistent with the manner in which CDO equity is marketed.

## Unmanaged Leveraged Investments Exhibit Volatility

Figure 2 shows the leveraged return (IRR) series based on historical defaults (an unmanaged CDO of B1/B2 credits). Each data point in the graph represents the computed IRR of a 6-year CDO issued in that month, analogous to a monthly cohort. Leveraged investments in unmanaged portfolios exhibit a wide variation in

Figure 2. **IRRs on Leveraged Investments in Lehman Brothers B1/B2 Index**
Origination Cohort April 1990-December 2000




*Historical Default Rates are constant annual default rates (CADRs) for actual default cohorts of 6 or less years.

returns. (An unmanaged CDO is defined as one in which the manager makes no active credit decisions.) An unmanaged leveraged investment in the B1/B2 Index would have resulted in very volatile returns. Estimated IRRs on the stylized CDO range from -44.1% to 65.9% (Figure 3) for the period studied (April 1990-December 2000). **Assuming that investors have a 16% IRR threshold for investing in CDO equity, such a threshold is achieved only one-third of the time for an unmanaged CDO.** If the CDO is issued with no management fee, then the return threshold is achieved 48% of the time.

### Quality of Unmanaged Leveraged Returns Vary by Vintage

We have discussed the vintage effect many times in past publications. CDOs issued at times when collateral quality in the new issue high yield market is solid and when high yield spreads are compensating adequately for default risk naturally tend to perform much better on average than those issued at times of weak primary market credit quality and tight spreads. Figure 2 shows the vintage effect quite clearly. Each data point in represents the IRR of a stylized CDO issued in that month. The IRRs of early 1990s and late 2000 are attractive because the arbitrage during this period was attractive, and although default rates were high, the structures stand up well. The 1995 period was one in which although asset spreads were not very wide, the quality of the collateral originated was strong and the average price of the index was below par, thereby driving good performance on CDOs. 1997-1999 was arguably one of the worst periods for investing in unmanaged portfolios of high yield, given that asset spreads were tight, most bonds were trading at premium dollar prices, and default rates rose thereafter. Most CDOs issued during this period had emerging market concentrations to boost asset spreads (not included in this analysis). The subsequent rise in both U.S. and emerging market default rates was not priced into these spreads, resulting in poor returns.

### Managing Default Rate Produces Superior Leveraged Returns

CDO equity investors are implicitly expecting an asset manager's default performance to be better than the market average. Unlike a total return manager or a market value structure, the primary variables to benchmark cash flow CDOs are default rate, recovery rate, and trading gains/ losses. Of the three, default rate is

Figure 3. **Mean and Volatility of IRRs under Various Scenarios,** %
Default Rate Assumption

| | Historical Cohort Average Default Rate* | 1% | 2% | 3% | 4% |
|---|---|---|---|---|---|
| Mean | 2.4 | 29.1 | 24.9 | 21.3 | 8.5 |
| Std dev | 24.7 | 14.8 | 15.9 | 16.1 | 17.9 |
| Min | -44.1 | 5.6 | -0.7 | -2.6 | -16.7 |
| Max | 65.9 | 78.0 | 76.3 | 74.4 | 67.2 |
| Probability of Outperformance | NA | 17 | 33 | 48 | 61 |

* This default rate varies over time whereas in all other scenarios shown the default rate is constant over time.

arguably the most significant, given the sensitivity of CDO equity returns to default rates. A manager who can choose a portfolio that realizes lower default rates than the historical average can produce superior returns to equity investors in CDOs. We recomputed historical average IRR by changing only the default rate assumption (Figure 3). Instead of using the cohort default rate over time, we use a constant annual default rate (CADR) for each CDO cohort—1%, 2%, 3%, and 4%. **A 3% or less CADR produces much more stable returns for CDO equity.** For example, the downside under a 3% annual default rate scenario is -2.6%, compared with -44.1% for historical average default rate scenarios. Realized recoveries higher than 45% and trading gains can be used to offset default rates, thereby allowing a manager a higher threshold default rate.

One method to measure manager performance is to compute the probability of achieving an annual portfolio default rate lower than 2%. The 2% level is chosen since most new issue CDO equity is marketed at these levels. CADRs of 2% or less produce mean returns that are above the general thresholds of CDO equity investors (16%-18%) and given that the standard deviations, although large (15.9%), are much lower than that of an unmanaged CDO (24.7%).

In the article on default experience statistic (section 1 of this publication), the author discusses the probability of experiencing a given number of defaults in a portfolio of a certain size. If one chooses a 100-bond portfolio randomly from the universe of high yield borrowers, the probability of outperformance can be defined as the probability of experiencing an annual portfolio default rate <=2%. For cohorts originated between April 1990 and December 2000, the average probability of achieving a 1-year default rate of 2% or less is 33%. **This implies that the top one-third of managers can deliver acceptable CDO equity returns over long periods of time.** Managers who do not fall in the top third may still be in a position to deliver acceptable returns if they are taking a view on the bottom of a credit cycle.

The other way we computed the probability of outperformance was to control for rating score i.e., we computed the probability of choosing portfolios with rating scores=B1/B2 (2220-2720 per Moody's CDO scoring scale) that experienced a 6-year cumulative default rate of 11.4% (corresponds to a 2% CADR). The cohort data span April 1990-April 1995 (the last six-year cohort available). **The probability of achieving default rates of 2% or less for six years is 15%.** Achieving a 2% annual default rate over six years is clearly more difficult than achieving it in just one year.

### Historical Total Return Performance of High Yield Mutual Funds

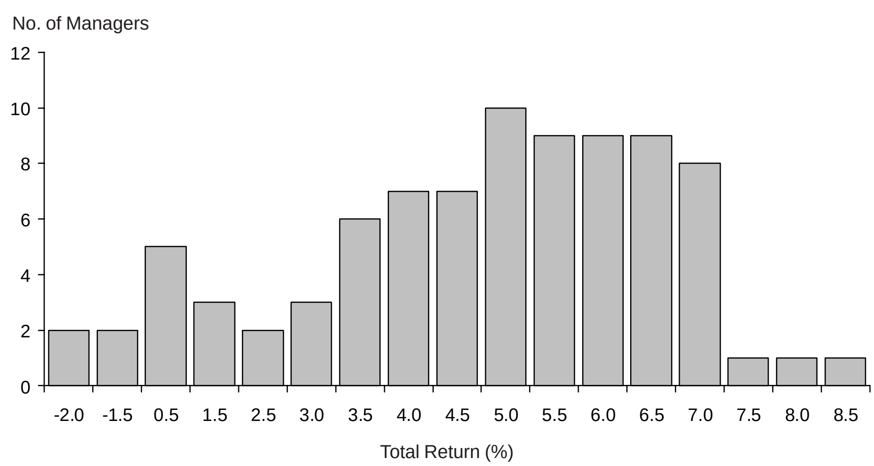
Since there is no publicly available source of information to measure manager performance using default rates, we use total return data from Lipper to understand historical performance of high yield managers. Figure 4 gives CDO investors an idea of the distribution of returns across high yield funds over a long horizon. The 7-year total return data show that the range of the total return distribution is from -2% to 8%, or a total width of 10%. The top one-third of performance corresponds to 6% or greater returns and is populated by 28 of the 85 funds in the data series.

### Sharpe Ratios of Leveraged High Yield Investments versus Other Asset Classes

Using the same CDO construct, we computed annualized total returns on the equity tranches. Total return is computed by adding up the cash flows to the equity and subtracting the initial investment. This number is then annualized.[3] Not surprisingly, the total returns follow the same pattern as the IRRs. However, the magnitude of total returns is clearly lower than IRRs (compare the y-axes of Figure 2 and Figure 5). The difference lies in the reinvestment assumption. IRRs assume reinvestment of interim cash flows at the same yield, whereas total return computations ignore reinvestment of interim cash flows and are similar to cash on cash yield. We also plot total returns on the Lehman Brothers U.S. High Yield Index and on the S&P 500 for comparable holding periods. The return series are 6-year holding periods until April 1996, after which the holding period is lower by one month at a time.

[3] Purists may criticize this method of computing total returns, since we are not marking to market the equity tranche every period. However, CDO equity is a buy-and-hold investment, and assuming interim mark to markets would only cause more noise to the data. In fact, mark to market is inconsistent with the fundamental concept of a cash flow CDO.

Figure 4. **Histogram of 7-Year Total Returns of High Yield Managers**

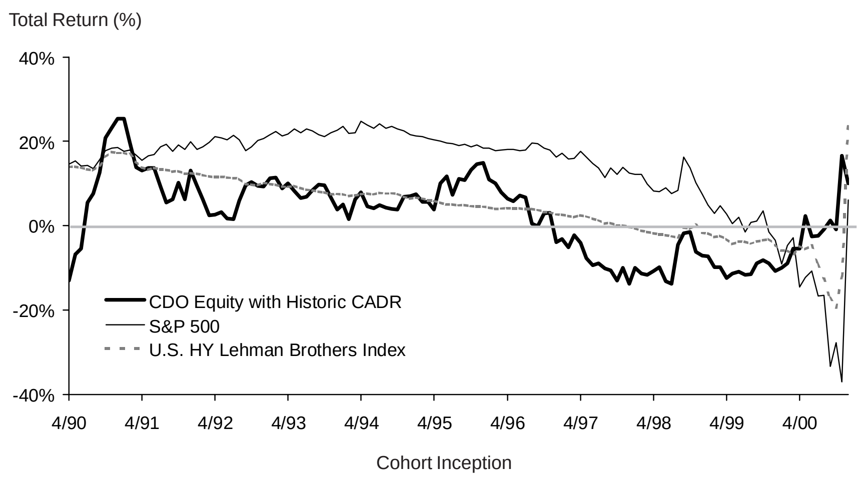



Source: Lipper.

Using the 6-year holding period total return series, we computed Sharpe ratios on CDO equity and compared them with Sharpe ratios of other asset classes. To simulate the concept of cohorts, the investment process analyzed is similar to investing $1 in any of the shown asset classes every month and holding it for a maximum of six years. For cohorts outstanding for less than six years, we use the life of the cohort as the holding period (Figure 6).

In an asset allocation context, it makes very little sense to invest in an unmanaged CDO unless one implicitly takes a timing/ vintage view. This is consistent with U.S. corporate high yield performance; the Sharpe ratios of both are negative. If one chooses a manager who is able to perform in the top third of managers (with a CADR of 2% or less), then the Sharpe ratio (1.06) suggests that CDO equity makes sense in an asset allocation framework.

Figure 5. **Annualized Holding Period Total Returns on CDO Equity**




* Historical Default Rates are constant annual default rates (CADRs) for actual 6 year or less default cohorts

Figure 6. **Estimated Sharpe Ratios of CDO Equity vs. Other Asset Classes**
Cohorts from April 1990-December 2000

| | CDO Equity | | | U.S. Corporate | | | |
|---|---|---|---|---|---|---|---|
| | **@ Historic CADR w/o Mgmt Fee** | **@ Historic CADR incl. Mgmt Fee** | **@ 2% CADR** | **High Yield** | **Invest. Grade** | **U.S. Govt** | **S&P 500** |
| Mean* | -0.16% | -2.88% | 8.67% | -0.43% | 2.50% | 3.16% | 8.19% |
| Std. Dev | 9.24% | 9.58% | 8.18% | 7.86% | 2.79% | 3.22% | 11.95% |
| Sharpe Ratio | -0.02 | -0.30 | 1.06 | -0.05 | 0.90 | 0.98 | 0.69 |

*Excess returns over 91-day T-Bill.

## Conclusion

Through our analysis of historical behavior of spreads and default rates, we have demonstrated several well-discussed anecdotal facts. **Only top-quality managers** can demonstrate superior returns on leveraged high yield investments over the long term. This experience is **similar to many other alternative investments** such as private equity and hedge funds. In an asset allocation framework, unmanaged CDO returns make sense only if an investor is **taking a view on the high yield market and uses CDO equity to execute a leveraged strategy on this market**. Investors should view CDO equity from top-quality managers as an asset class to actively participate in.

## Appendix:
## Description of our Stylized CDO Model and Assumptions

The variables that determine the equity performance of a cash flow CDO are asset spreads, swap spreads, liability costs, other costs (structuring, management fee, residual hedging costs, etc.), capital structure, and loss rates (default and recovery rates). We use the following assumptions to compute a series of monthly leveraged returns:

1) Asset spreads—Historical spreads on the Lehman Brothers B1/B2 Index (April 1990 to December 2000) excluding zero coupons and bonds with maturities less than ten years.
2) Liability costs—From 1997 to 2000, we use monthly average new issue spreads on CDO tranches. Prior to 1997, our pricing data are sparse, and, therefore, we assume that the funding costs are the average cost over the period for which we have data i.e., 1997-2001.
3) Other costs are an annualized estimate of structuring costs, management fees, and any other hedging costs
4) Capital structure: 65.0% Aaa, 8.5% Aa, 11.5% Baa, 5.0% Ba, 10.0% equity.
5) Default rates—We use a monthly cohort default rate series based on the Moody's universe of rated corporates. A cohort is defined as all issuers with a B1/B2 rating as of the beginning of the month. This cohort is tracked from inception and a cumulative default rate is computed. This default rate is annualized.
6) Recovery rates—Historical average recovery of 45% for senior unsecured credits.
7) Assets are purchased at the average price of the Lehman Brothers Index as of that month. So periods of discounted prices tend to push up equity returns, and premium periods have the reverse effect.
8) The structure has a 6-year life, given that the historical mean of the average life of Lehman Brothers B1/B2 Index is six years with a min of 3.96 and a max of 7.15 (a CDO typically has a 5-year call date and a 7-year average life).
9) The structure is self-liquidating at the end of five years i.e., the un-defaulted collateral is assumed to pay off at par at the end of five years.
10) The equity tranche receives the residual cash flows (asset spreads earned less other costs less spreads paid on liabilities) every year. At the end of six years, it also receives the residual par after paying off liabilities.
11) IRRs and total returns are computed on these cash flows. Total returns are then annualized.

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## MULTI-ISSUER MODELS OF DEFAULT

### 1. Introduction

Following our overview of single-issuer default models in the previous issue of *Quantitative Credit Research*, we present some of the modeling approaches used to analyze the default risk of multiple issuers.

Tools for describing correlated defaults are essential. Due to its nature, credit risk should be managed on a portfolio basis, allowing investors to reap the benefits of diversification. From a pricing perspective, derivatives with payoffs conditional on the performance of multiple credits are increasingly being used by investors to create credit risks that match their appetites. Products range from default baskets to tranches of synthetic collateralized debt obligations (CDOs).

In these sorts of structures, the factors that determine the credit risk profile of the portfolio are:

- The number of assets,
- The credit quality of the assets, and
- The default correlation between the assets in the portfolio.

Clearly, the more assets in the portfolio, the less exposed is the investor to a single default. Equally, the higher the credit quality of the assets, the less likely a default and the lower the expected loss on the pool of assets. Default correlation plays a significant role in these structures. In simple terms, the default correlation between two assets is a measurement of the tendency of assets to default together. If the assets in a portfolio have a high default correlation, then, when assets default, they do so in large groups. This can significantly affect the credit risk profile, making large portfolio losses more likely and fattening the tail of the loss distribution.

It is natural to expect that assets issued by companies that have common dependencies would be more likely to default together. For example, companies in the same country are exposed to a common interest rate and exchange rate. Companies within the same industrial sector have the same raw material costs, share the same market, and, so, this could be expected to be even more strongly correlated. On the other hand, the elimination of a competitor may be beneficial to companies in the same sector.

### 2. Default Correlation

When modeling credit risk, we are dealing with probability distributions that diverge significantly from the normal distribution, which is more or less the generally accepted paradigm for market returns. This means that it is not quite as straightforward as one might think to describe the tendency of multiple issuers to default together with a single number such as the linear correlation coefficient. Ultimately, the joint loss distribution of all underlying issuers is what determines the risk profile of a portfolio.

Nevertheless, we obtain a useful starting point by defining the default correlation of two assets A and B. Let us denote by $p_A$ the probability that asset A defaults before some time $T$. Likewise, $p_B$ is the probability that asset B defaults before $T$. The joint probability that both asset A and asset B default before some $T$ is $p_{AB}$. Using the standard definition of the correlation coefficient, we can compute the pairwise default correlation between the two assets A and B as:

$$\rho_D = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1 - p_A)}\sqrt{p_B(1 - p_B)}} \tag{1}$$

Default correlation will always lie between –1 and 1 and will be zero for independent assets. However, depending on the individual default probabilities, it may actually be constrained to lie in a much smaller interval. Also, because it depends on the individual default probabilities, the default correlation of two issuers will invariably depend on the time horizon $T$ considered.

### *2.1. Empirical Evidence of Default Correlation*

Empirical analysis of default correlation is limited by the lack of default events. One study by Lucas (1995), which computes the default correlation between assets in different rating categories, has made two particularly interesting observations. The first is that default correlation increases as we descend the credit rating spectrum. This has been attributed to the fact that lower-rated companies are more vulnerable to an economic downturn than higher-rated companies and, thus, are more likely to default together.

The second observation is that default correlation is horizon dependent and time dependent. It has been postulated that this may be linked to the periodicity of the economic cycle.

As mentioned in Figure 1, computing industry-industry default correlations is difficult due to the shortage of default events. In practice, default correlation is often proxied using some other quantity, such as credit spread correlation or stock

Figure 1. **Empirical Measurement of the 10-Year Default Correlation as a Function of Credit Rating**

| | **10-Year Default Correlations** (%) | | | | | |
|---|---|---|---|---|---|---|
| | **Aaa** | **Aa** | **A** | **Baa** | **Ba** | **B** |
| Aaa | 1 | | | | | |
| Aa | 2 | 0 | | | | |
| A | 2 | 1 | 2 | | | |
| Baa | 2 | 1 | 1 | 0 | | |
| Ba | 4 | 3 | 4 | 2 | 8 | |
| B | 9 | 6 | 9 | 6 | 17 | 38 |

Source: D.J. Lucas, *The Journal of Fixed Income*, March 1995.

price correlation. While this may appear to be a reasonable assumption, strictly speaking, there is no model-independent mathematical relationship that can link the two.

### *2.2. Default Correlation and Basket Default Swaps*

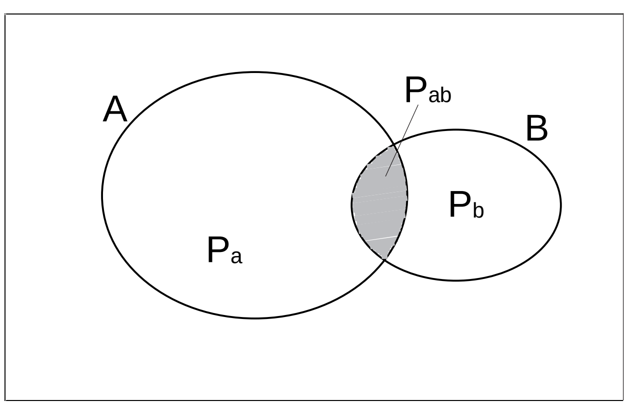
The simplest portfolio credit derivative is the basket default swap. It is like a default swap but for the fact that it is linked to the default of more than one credit. In the particular case of a first-to-default basket, it is the first asset in a basket whose credit event triggers a payment to the protection buyer. As in a default swap, the protection buyer pays a fee to the protection seller as a set of regular accruing cash flows, in return for protection against the first-to-default.

To see how default correlation affects pricing, consider a $T$-maturity first-to-default basket with two assets, A and B, in the basket. Pictorially, one can represent the outcomes of two defaultable assets using a Venn diagram, as shown in Figure 2. Region A encompasses all outcomes in which asset A defaults before the maturity of the basket. Its area equals the probability $p_A$. Similarly, region B encompasses all scenarios in which asset B defaults before the maturity of the basket, and its area equals $p_B$. The shaded overlap region corresponds to all scenarios in which both asset A and asset B default before time $T$ and, so, has an area equal to the probability of joint default $p_{AB}$.

The market-implied individual default probabilities for assets A and B are easily derived from a knowledge of where the individual default swaps trade.

A first-to-default basket pays off as soon as one of the assets in the basket defaults. So in order to value the protection, we need to calculate the probability of this occurring. It is given by the area contained within the outer perimeter of the

Figure 2. **Venn Diagram Representation of Correlated Default for Two Assets**



two regions, since this contains all scenarios in which one or the other or both assets default before the maturity of the basket. This area equals

$$\Omega(\rho_D) = p_A + p_B - p_{AB} \tag{2}$$

To understand the effect of default correlation, consider several scenarios. When the assets are independent, the default correlation is zero and the probability of both assets defaulting before time $T$ is given by $p_{AB} = p_A p_B$. The probability that the first-to-default basket is triggered is then given by

$$\Omega(0) = p_A + p_B - p_A p_B \tag{3}$$

The last term is usually very small compared with the first two since it is the product of two probabilities.

**Example 1:** If $p_A = 5.0\%$ and $p_B = 3.0\%$, then we have $\Omega(0) = 7.85\%$

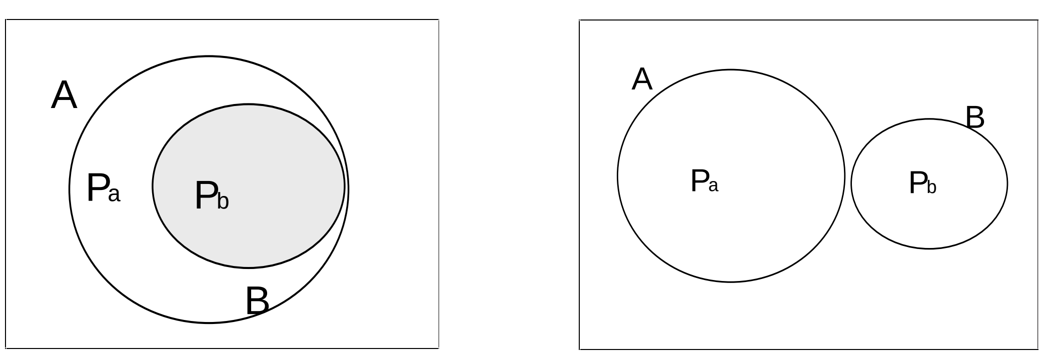
At the highest default correlation, the default of the stronger asset B always results in the default of the weaker asset. The weaker asset A (the one with the higher default probability), can still default by itself. This is shown graphically on the left side of Figure 3. In this limit, the joint default probability is given by the default probability of the stronger asset, so that

$$p_{AB} = \min[p_A, p_B] \,. \tag{4}$$

Figure 3. **At the Maximum Default Correlation (a), Default of the Stronger Asset B Is Always Associated with Default of the Weaker Asset A; At the Minimum Default Correlation (b) the Two Regions Separate, Implying that the Joint Default Probability Has Fallen to Zero; Within a Time Horizon T, The Default Events Are Mutually Exclusive**

a.

b.



As we have chosen A to be the weaker asset, $p_A > p_B$ and the joint probability $p_{AB} = p_B$ so that $\Omega(\rho^{MAX}) = p_A$.

> **Example 2:** Using the above probabilities, we find that at the maximum default correlation, W = max[5.0%, 3.0%} = 5.0%.

As the default correlation falls, there comes a point at which there is a zero probability of both assets defaulting together before time *T*—the default of assets A and B are mutually exclusive. Graphically, there is no intersection between the two regions, and we have $P_{AB} = 0$ (Figure 3b). The probability that the first-to-default basket will be triggered is simply the sum of the two areas A and B.

$$\Omega\left(\rho_D^{MIN}\right) = p_A + p_B \, . \tag{5}$$

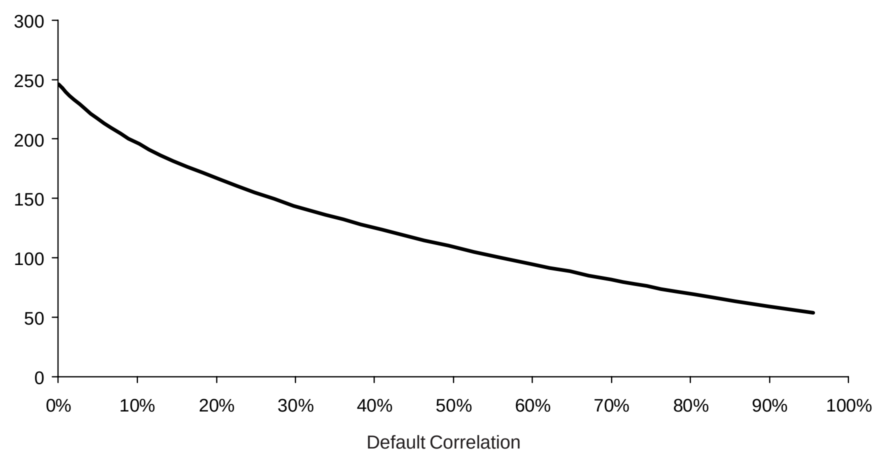
> **Example 3:** Using the above values for the probabilities, we find that at the minimum correlation, $\Omega\left(\rho_D^{MIN}\right) = 5.0\% + 3.0\% = 8.0\%$.

This is when the first-to-default basket is at its riskiest and protection is at its most expensive.

To complete the pricing, we need to discount payoffs, which means that we need to have access to the time at which defaults occur. Also, as we increase the number of assets in the basket, we increase the number of possible default scenarios and the number of inputs required. One approach is to simulate the default times of the assets.

While we do not typically know the form of the joint density distribution for the default times, it is possible to impose one with the correct marginal distributions using the method of copulas. This technique is explained in Li (2000), and we refer the reader to that paper for details. Using such a model, we plot in Figure 4

Figure 4. **First-to-Default Basket Spread as a Function of Default Correlation for a Five-Asset Basket**



the first-to-default basket spread as a function of the default correlation for a 5-asset basket. Each asset in the basket has a 5-year default swap spread of 50 bp. At zero correlation, the basket price is slightly less than 250 bp, which is the sum of the spreads. As the maximum default correlation is approached, the basket spread tends to the maximum of the individual issuer spreads, which (as all spreads are the same) is 50 bp.

Such a model also makes it easy to compute the price of second, third, and *n*th-to-default baskets.

## 3. Modeling Collateralized Debt Obligations

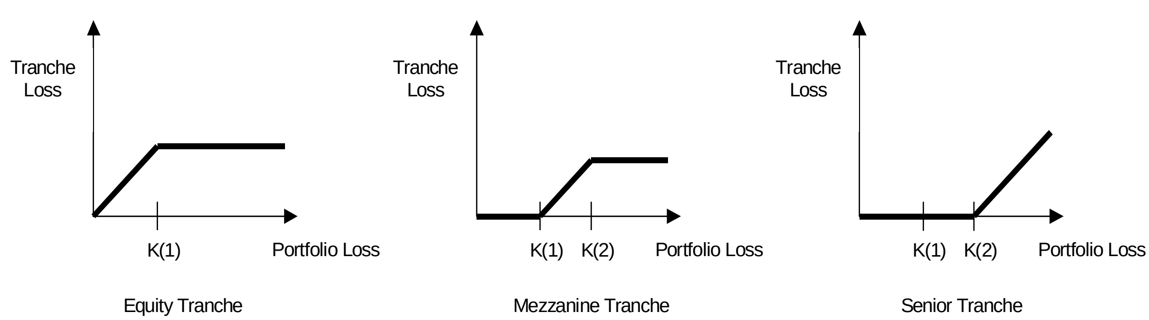
### *3.1 Introduction*

A **collateralized debt obligation (CDO)** is a structure of fixed income securities whose cash flows are linked to the incidence of default in a pool of debt instruments. These debts may include loans, revolving lines of credit, emerging market corporate and sovereign debt, and subordinate debt from structured transactions.

In a CDO, notes are issued in several tranches, with the senior tranches having coupon and principal repayment priority over the mezzanine and equity tranches. The income from the collateral is paid to the most senior tranches first, as interest on the notes. The remaining income is then paid to the notes issued on the mezzanine tranche, followed by payments to those in the equity tranche. As the senior note is paid first, it has a lower credit risk than the other notes and can often achieve a AAA rating.

The likelihood of losses depends on the individual quality of the constituent collateral and the degree of diversification across the pool. The more diversified the collateral, the lower the risk of the senior tranches. Indeed, one of the main determinants of the riskiness of a CDO is the correlation between the collateral in the pool. In this sense, a CDO is closely related to the default basket, in which the

Figure 5. **Tranche Losses in CDO as a Function of Portfolio Loss**



equity tranche is like a first-to-default basket and higher tranches are like second-, third-, and fourth-to-default baskets, respectively.

In each situation, the key issue is modeling the joint default probability given the default probabilities for each individual issuer.

### *3.2 Models for Introducing Default Correlation*

In the simplest case, defaults of issuers occur independently. If a portfolio consists of $m$ bonds with the same notional and the default probability of each issuer over some time horizon is $p$, then the distribution of the percentage loss $L$ of the portfolio is binomial, with

$$P\left[L = \frac{k}{m}\right] = \binom{m}{k} p^k (1-p)^{m-k} . \tag{6}$$

In practically all applications, the defaults of different issuers will be correlated to some extent and, assuming independence, will overestimate the portfolio diversification. In the rating of CDOs, Moody's applies the technique of a "diversity score" to deal with this problem, cf. Moody's (1999). The distribution used to compute the portfolio losses is still the binomial one. However, the number $m$ is not the actual number of credits in the portfolio, but is adjusted downward to take the effects of correlation into account. This is done for credits within an industry sector; different sectors are treated as independent.

Davis and Lo (1999) have developed a model in this framework that explicitly models default correlation. Each issuer can either default idiosyncratically or through contagion from another issuer. The model is static in the sense that only idiosyncratic defaults are transmitted through contagion, i.e., there are no "domino effects." The effect of contagion fundamentally alters the distribution of losses. For a given number of expected defaults, it exhibits fatter tails than the base case binomial distribution, making both very small and very large losses more likely. Unfortunately, the model involves intensive combinatorial calculations for more complex portfolios.

Another approach, based on the firm value model of Merton (1974), is implemented in the third-party system CreditMetrics, cf. Gupton/Finger/Bhatia (1997). It attempts to assess the possible losses to a portfolio due to both defaults and rating changes; the rating system can be either internal or one provided by a rating agency.

**The Merton Approach to Modeling Correlated Default**

Conceptually, the asset return of the $i$th issuer at the model horizon is described by a standard normally distributed random variable $A(i)$. Rating transitions and, in particular, default occur when the asset value $A(i)$ crosses certain thresholds. The default threshold $C(i)$ is implied from the issuer's default probability $p(i)$ with the equation

$$p(i) = P\left[A(i) \le C(i)\right] = N\left(C(i)\right), \tag{7}$$

where, as before, $N$ denotes the cumulative distribution function of the standard normal distribution.

In the single-issuer case, this is merely an exercise in calibration, as the default thresholds are chosen to reproduce default probabilities that reprice market instruments. The concept of asset returns becomes meaningful when studying the joint behavior of more than one issuer. In this case, the returns are described by a multivariate normal distribution. Using the threshold levels determined before, it is possible to obtain the probabilities of joint rating transitions.

As standardized asset returns are used, it is necessary to estimate only the correlation structure of asset values. One reasonable approximation is provided by the correlation between equity prices.

With a completely general correlation structure, the calculation of joint default probabilities becomes computationally intensive, making it necessary to resort to Monte Carlo simulation. However, substantial simplifications can be achieved by imposing more structure on the model. An important concept in this context is that of conditional independence. We assume that the asset return of each of the $m$ issuers is of the form

$$A(i) = \beta(i)\,Z + \sqrt{1-\beta(i)^2}\;Z(i), \tag{8}$$

where $Z, Z(1), \ldots, Z(m)$ are independent standard normal random variables. The variable $Z$ describes the asset returns due to a common market factor, while $Z(i)$ models the idiosyncratic risk of the $i$th issuer and $\beta(i)$ stands for the correlation of $A(i)$ with the market.

The advantage of this setup is that conditional on $Z$, the asset returns are independent. This makes it easy to compute conditional default probabilities. Let us assume that the portfolio is homogeneous, i.e., that $\beta(i)$ and $C(i)$ are the same for all assets. The $i$th issuer defaults in the event that $A(i) \le C$. Using Equation 8, we see that this is equivalent to

$$Z(i) \le \frac{C-\beta Z}{\sqrt{1-\beta^2}}. \tag{9}$$

The conditional default probability $p(Z)$ of an individual issuer is given by

$$p(Z) = N\left(\frac{C-\beta Z}{\sqrt{1-\beta^2}}\right). \tag{10}$$

**Calculating the Loss Distribution**

If we also assume that the exposure to each issuer is of the same notional amount, the probability that the percentage loss $L$ of the portfolio is $k/m$ is equal to the probability that exactly $k$ of the $m$ issuers default, which is given by

$$P\left[L=\frac{k}{m}\right]=\binom{m}{k}N\left(\frac{C-\beta Z}{\sqrt{1-\beta^2}}\right)^k\left(1-N\left(\frac{C-\beta Z}{\sqrt{1-\beta^2}}\right)\right)^{m-k}. \tag{11}$$

In other words, the conditional distribution of *L* is binomial. Again, this distribution becomes computationally intensive for large values of *m*, but we can use methods of varying sophistication to approximate it.

**Large Portfolio Limit**

One very simple and surprisingly accurate method is the "large homogeneous portfolio" approximation, originally due to Vasicek. After conditioning on *Z*, the asset returns are independent and identically distributed. By the law of large numbers, the fraction of issuers defaulting will tend to the probability given in Equation 10. Therefore, one can assume that the percentage loss given *Z* is approximately equal to *p(Z)*. In particular, the expected loss of the portfolio is equal to the individual default probability *p*. Using (10), we see that

$$L \le \theta \Leftrightarrow Z \ge p^{-1}(\theta)\cdot \tag{12}$$

Equation 12 states that the percentage loss of the portfolio will not exceed the level $\theta$ if and only if the market return *Z* is sufficiently high. Because *p(Z)* is a monotonically decreasing function, the upper bound for *L* is translated into a lower bound for *Z*. The unconditional distribution for *L* follows immediately from (12):

$$P[L \le \theta] = N\left[-p^{-1}(\theta)\right]\cdot \tag{13}$$

More involved calculations show that the distribution of *L* actually converges to this limit as *m* tends to infinity. For a given individual default probability, the loss distribution is quite sensitive to the correlation parameter $\beta$. In the limit of $\beta = 0$, the portfolio loss is deterministic, as can be seen from the conditional loss given in Equation 13. As $\beta$ tends to 100%, the loss distribution becomes bimodal, as the portfolio effectively consists of only one credit.

> **Example 4:** Consider a portfolio in which the individual default probability is 7.17%, which corresponds to the default probability of a single-*B* rated issuer over a one-year time horizon. Figure 6 shows the loss distribution if we assume a market correlation of 20%, whereas Figure 7 shows the loss distribution if we assume a market correlation of 55%. At the higher correlation, the tail of the distribution is fatter than for the lower correlation, as there is now a greater tendency for multiple defaults. To ensure that the expected loss remains constant, this means that the likelihood of fewer defaults also increases.

Note that the correlation between issuers is actually the square of the market correlation. Values of 20% and 55% for the market correlation correspond to issuer correlations of 4% and approximately 30%, respectively.

Figure 6. **Loss Distribution of a Large Homogeneous Portfolio with *B*-Rated Issuers and a Market Correlation of 20%**

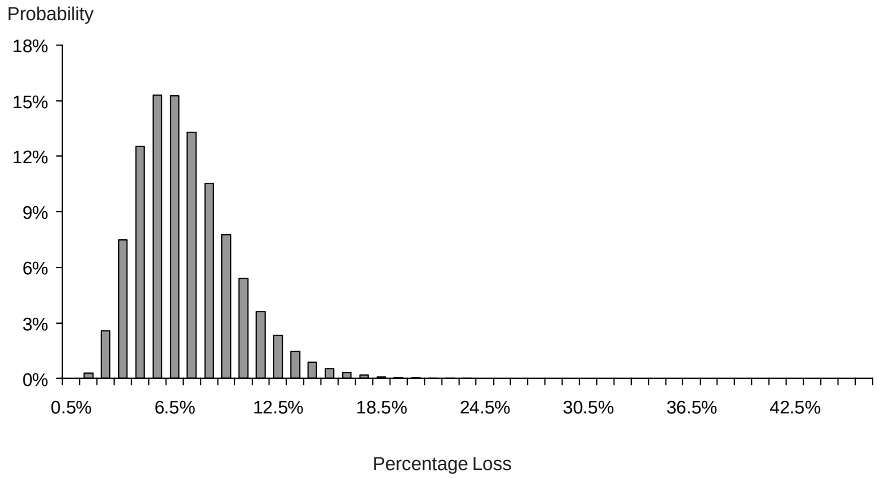
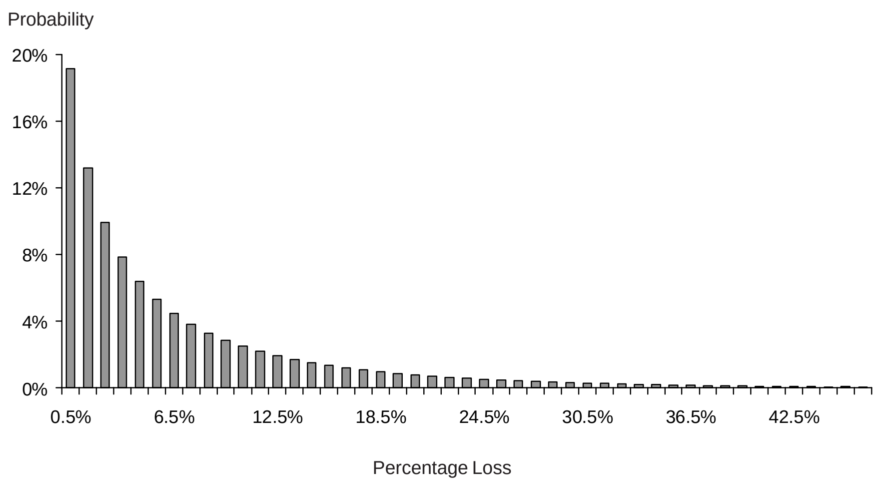



Figure 7. **Loss Distribution of a Large Homogeneous Portfolio with *B*-Rated Issuers and a Market Correlation of 55%**




### The Tranche Loss Distribution

We can use the portfolio distribution to derive the loss distributions of individual tranches. If *L(K(1), K(2))* is the percentage loss of the mezzanine tranche with boundaries *K(1)* and *K(2)*, then

$$L\left(K(1),K(2)\right)=\frac{\max\left(L-K(1),0\right)-\max\left(L-K(2),0\right)}{K(2)-K(1)} \tag{14}$$

Note that the percentage loss is scaled by the width of the tranche. If $\theta < 1$, then

$$L\big(K(1), K(2)\big) \leq \theta \Leftrightarrow L \leq K(1) + \theta\big(K(2) - K(1)\big) \tag{15}$$

Equation 15 shows that we can easily derive the distribution of *L(K(1), K(2))* from that of *L*. The distribution of *L(K(1), K(2))* is discontinuous at the edges of [0,1] due to the probability that the portfolio loss will fall outside the interval *[K(1), K(2)]*. This discontinuity becomes more pronounced when the tranche is narrowed.

> **Example 5:** We consider a mezzanine tranche of the same portfolio of *B*-rated issuers we examined earlier. We suppose that all portfolio losses between 2% and 15% are applied to this tranche. Figure 8 shows the loss distribution assuming a correlation of 20% while, in Figure 9, the assumed correlation is 55%. Again, we see that the loss distributions are fundamentally different.

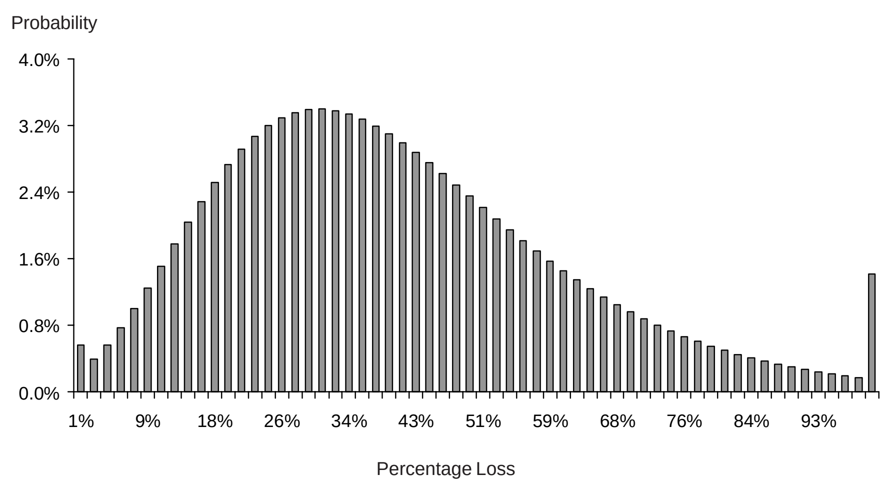
We can compute the expected loss of the tranche analytically. For any *K* in [0,1] , we denote by *L(K)* the payoff

$$L(K) = \max\big(L - K, 0\big) \cdot \tag{16}$$

If we denote the density of the portfolio loss distribution by *g*, the expectation of this payoff is

$$E\big[L(K)\big] = \int_K^1 (s - K) g(s) ds \cdot \tag{17}$$

Figure 8. **Loss Distribution of Mezzanine Tranche Taking Losses between 2% and 15% for a Market Correlation of 20%**



After some algebra, it is possible to rewrite the expectation in terms of the distribution function $N_2$ of a bivariate normal distribution:[1]

$$E[L(K)] = N_2\left(-N^{-1}(K), C, -\sqrt{1-\beta^2}\right) \tag{18}$$

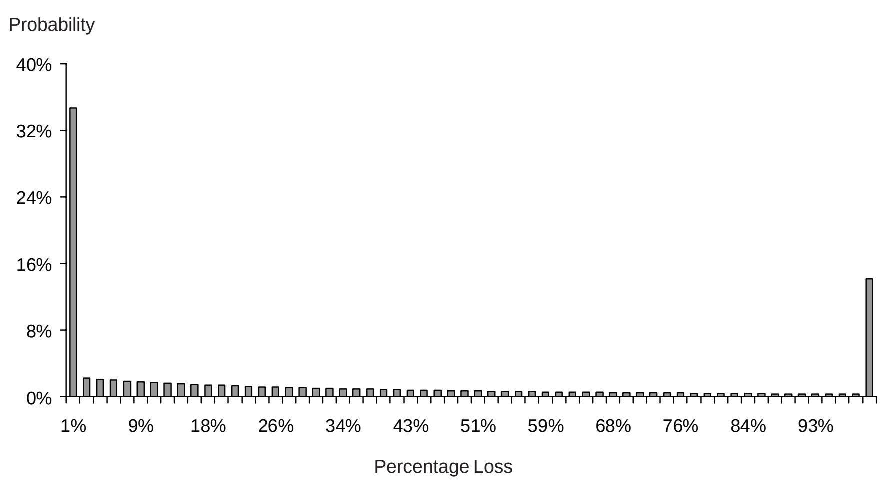
From Equation 17, we immediately conclude that the expected loss of the tranche is given by

$$E[L(K(1),K(2))] = \frac{N_2\left(-N^{-1}(K(1)), C, -\sqrt{1-\beta^2}\right) - N_2\left(-N^{-1}(K(2)), C, -\sqrt{1-\beta^2}\right)}{K(2)-K(1)} \tag{19}$$

> **Example 6:** Again, we consider our portfolio of *B*-rated issuers. We divide the portfolio into three tranches. The equity tranche absorbs the first 2% of all portfolio losses, the mezzanine tranche absorbs all losses between 2% and 15%, and the senior tranche assumes all losses above 15%. Using Equation 19, we compute the expected loss of each tranche as a function of the market correlation parameter. The expected losses are plotted in Figure 10. We see that the equity and the mezzanine tranches actually benefit from a higher market correlation, i.e., less diversification in the portfolio. This is because the expected portfolio loss of 7.17% is well within the range of the mezzanine and beyond the equity tranche. Therefore, these tranches benefit from the

[1] An approximation formula for the cumulative distribution function of the bivariate normal distribution is given in Hull (1997).

Figure 9. **Loss Distribution of Mezzanine Tranche Taking Losses between 2% and 15% for a Market Correlation of 55%**



all-or-nothing gamble that is taken if the correlation becomes very large. Of course, the senior note holders stand to lose only if the diversification of the portfolio is reduced.

### *3.3 Further Modeling Approaches*

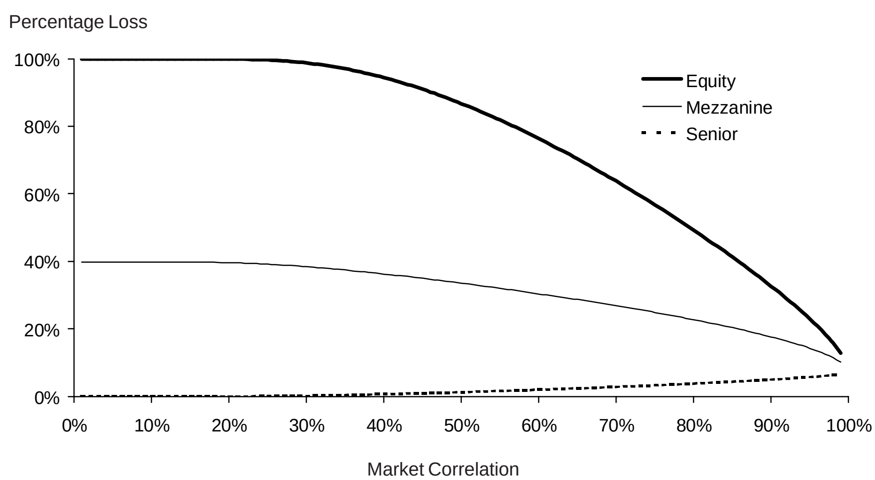
In addition to the asset-based approach presented in detail here, other methods have been used to analyze correlated defaults. Intensity models can be generalized to the multi-issuer case by specifying the joint dynamics of the individual intensities. This is another application of the conditional independence concept; after conditioning on the joint realizations of the intensities, the default counting processes are independent. For the simulation of correlated defaults in this intensity framework, see Duffie and Singleton (1998). An application to the analysis of CDOs is presented in Duffie and Garleanu (1999).

A useful technique for generating correlated default times is the method of copula functions. This effectively lets one separate the modeling of the dependence structure from that of the univariate distributions of the individual default times. For an introduction to the application of copula functions to finance, see Li (2000).

## 4. Summary

Multi-issuer credit derivatives are becoming increasingly important in the market. The pricing of first- and second-loss products requires the use of models that induce a default correlation between assets. In these models, the specification of the dependence structure is of primary importance. This was shown in the discussion of the first-to-default basket and the pricing of CDO tranches.

Figure 10. **Expected Tranche Loss as a Function of Market Correlation**



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# BASKET DEFAULT SWAPS AND THE CREDIT CYCLE

## 1. Introduction

The market for multi-issuer credit derivatives has experienced considerable growth over the last few years. In particular, *Nth*-to-default basket swaps have become popular tools used by investors to achieve exposure to a set of credits.

In an *Nth*-to-default basket swap, two counterparties will agree on a maturity and a set of reference assets (often between 5 and 15), and enter into a contract whereby the protection seller will periodically receive a premium (also called “basket spread”) from the protection buyer. In exchange, the protection seller will stand ready to pay the protection buyer par minus recovery of the *Nth* referenced defaulter in the event that the *Nth* default occurs before the agreed-upon maturity. Portfolio managers often sell protection on a group of names they like in order to enhance their portfolio yields. Conversely, commercial banks generally buy basket protection to hedge their loan portfolios and get capital relief.[1]

Compared with single-name default swaps, basket swaps offer multi-name exposure without modifying the size of the default-contingent payment, which depends only on the notional amount of the swap and the recovery rate of the *Nth* defaulter. On the other hand, the multi-name feature significantly affects the likelihood that the protection will get triggered. In fact, the probability of the credit event is now jointly determined by the order of protection, the number of reference credits, and their default probabilities and correlations.

This complexity raises some interesting questions: How does the probability of the triggering event change as a function of the joint default probabilities of the underlying reference assets? And how is this relation affected by the order of protection and the number of reference assets? **In this article, we will show that while a general deterioration of the credit quality of the reference assets increases the value of any order of basket protection, the effects of correlation changes are qualitatively different for different orders.**

Understanding these relations is essential for choosing a basket swap that is consistent with the desired exposures. For example, the following discussion will show that a protection seller can choose a set of names, a notional amount, and the appropriate order of protection in order to achieve the desired income and simultaneously take a view on the direction of correlation changes. Most important, we will argue that **while the exposure of a basket swap investor to default risk is implicit in the very definition of the contract, her**

I wish to thank Prafulla Nabar, Dominic O’Kane, Lutz Schloegl, Roy Mashal, Sunita Ganapati, Arthur Berd, Jonathan Sandberg, Stefano Risa, and Andrew Tong for many useful comments.

[1] For a general introduction to basket default swaps and other multi-name credit derivatives, see the article by Dominic O’Kane and Lutz Schloegl elsewhere in this issue.

**exposure to systematic spread movements can be quite counterintuitive. In particular, we will see that the return process of a second-to-default investment may be more exposed to market risk than a first-to-default position on the same set of names.**

Since first-to-default (FTD) and second-to-default (STD) baskets are by far the most popular orders of protection traded in the market, we will mostly focus on these two types of contracts. A brief extension of the analysis to third-to-default (TTD) protection is offered to provide more intuition for the results. Section 2 discusses some relevant definitions of correlation and introduces the reader to the modeling methodology, while Section 3 presents our main findings. Finally, Section 4 offers an alternative interpretation for the results in terms of the important concept of "tail dependence."

## 2. Default Correlation and Asset Correlation

Default correlation measures the tendency of two credits to default jointly within a specified horizon. Formally, it is defined as the correlation between two binary random variables that indicate defaults, i.e.,

$$\rho_D = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)}\sqrt{p_B(1-p_B)}}$$

where $p_A$ and $p_B$ are the marginal default probabilities for credits $A$ and $B$ and $p_{AB}$ is the joint default probability. Of course, $p_A$ , $p_B$ , and $p_{AB}$ all refer to a specific horizon. Notice that the default correlation increases linearly with the joint probability of default and is equal to zero if and only if the two default events are independent.

**Default correlations are the fundamental drivers in the valuation of multi-name credit derivatives.** Unfortunately, the scarcity of default data makes joint default probabilities and, thus, default correlations very hard to estimate directly. As a result, researchers have developed alternative methods to calibrate the frequency of joint defaults within their valuation models.

A standard way to simulate correlated defaults relies on the use of copula functions. Generally speaking, copulas are used to link marginal and joint distribution functions. This is extremely useful for the valuation of multi-issuer credit derivatives, since analysts can extract (risk-neutral) marginal default probabilities from liquid single-name products and then value multi-name contracts by simulating correlated survival times.

As an example, consider two credits $A$ and $B$, whose survival times $T_A$ and $T_B$ are exponentially distributed with hazard rate $h$. A joint distribution that correlates $T_A$ and $T_B$ while respecting their marginals can be obtained by means of a bivariate normal copula

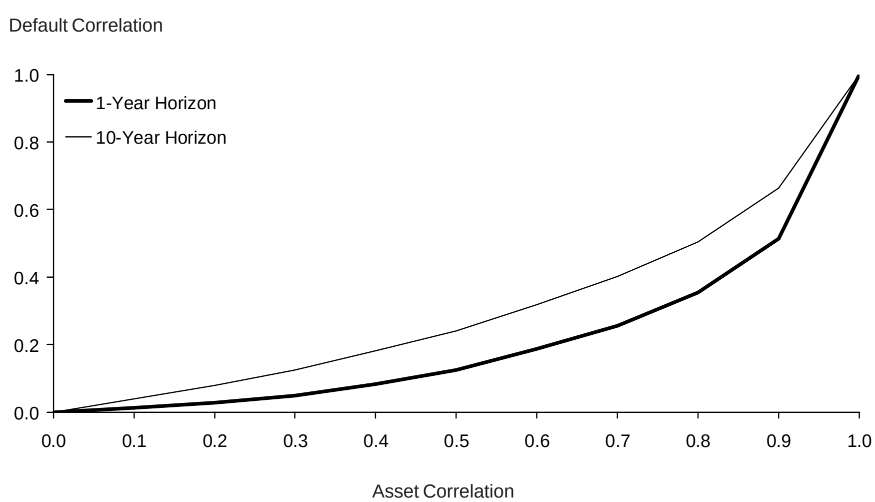
$$P(T_A < x, T_B < y) = \Phi_2(\Phi^{-1}(E_h(x)), \Phi^{-1}(E_h(y)), r),$$

where $\Phi_2(\cdot,\cdot, r)$ denotes a bivariate standard normal distribution with correlation $r$, $\Phi(\cdot)$ is a univariate standard normal distribution and $E_h(\cdot)$ is an exponential distribution with hazard rate $h$. Taking limits, it is straightforward to verify that this joint distribution is perfectly legitimate in that it respects the exponential marginals that we started with.

Notice that this framework can be interpreted as a latent-variable approach where 1) default is caused by a firm's normally distributed asset return falling below the threshold $\Phi^{-1}(E_h(\cdot))$ and 2) the parameter $r$ represents the correlation between the jointly normal asset returns of the obligors. This structural interpretation is useful for two reasons. First, it simplifies calibration by relating the copula parameter $r$ to the correlation between asset returns. This offers a clear advantage since reasonable proxies for asset correlations can be computed from observable data. Second, it reveals that the choice of a particular copula for survival times may be related to an implicit distributional assumption for asset returns. Regarding this last point, we notice here that **multivariate normality is not an innocuous assumption, since it allows only for linear dependence: a correlation matrix is enough to specify the full dependence structure of asset returns.** We will come back to this issue in a later section.

For two credits with a constant hazard rate of 1% and a joint normal distribution of asset returns, Figure 1 plots the relation between asset correlation and default correlation for two different horizons (1 and 10 years). This relation is convex, and its curvature depends on the relevant horizon. For a short period of time, the marginal probabilities of default retrieved from market prices are relatively small. Therefore, the implied default thresholds are low, and joint defaults are going to be infrequent even if asset values are significantly correlated. As a result, default

Figure 1. **Correlation**



correlations are not very sensitive to asset correlations until the latter become relatively large. For a longer horizon, marginal default probabilities are higher, and so are the implied default thresholds. This, in turn, causes default correlations to be more sensitive to changes in asset correlations over the range where asset correlations are relatively low.

## 3. The Value of Protection and the Credit Cycle

In this section, we first explore the relationship between the value of different orders of basket protection, the number of reference credits, and the correlation between asset returns. We then turn our attention to the market exposure of the return process of *Nth*-to-default baskets and use a simple example to highlight the counterintuitive fact that lower orders of protection may actually be more sensitive to market movements.[2]

For the purpose of our discussion, we will consider a basket of $M$ defaultable reference assets with flat default swap spreads at 50 bp and a known recovery rate of 50% of par value. Given these assumptions, hazard rates are flat at 1%.[3] All pairwise asset correlations are equal to $r$, and, without loss of generality, the risk-free (LIBOR) curve is flat at zero. The following results are based on 100k-path Monte-Carlo simulations, where correlated default times are generated using multivariate normal copulas.

### *3.1. FTD Protection*

Imagine selling FTD protection on this basket for a period of 5 years. What premium should you receive? Figure 2 plots the (annual) premium as a function of asset correlation $r$ and the number of reference credits $M$.

The information contained in Figure 2 is by now well understood in the credit derivatives market. A single-name default swap on any of the (individually identical) names in the basket pays a spread equal to 50 bp. For a basket made of $M$ independent names ($r$=0),[4] the premium will be (almost) equal to $M$*50 bp. Roughly speaking, the reference assets independently move in $M$ directions, and each of them can trigger the basket payment, so the probability of the credit event is (almost) $M$ times higher than the probability of the credit event of a single-name swap.[5] On the other extreme, unit correlation implies that the

[2] In this article, we will refer to FTD protection as being of “higher order” than STD protection. Alternatively, we will say that a FTD swap offers a “more senior” order of protection than a STD swap.

[3] Consistent with a flat spread curve is the assumption that the survival time is exponentially distributed with hazard rate $h$. The latter can then be recovered using the simple relation $s=h(1-R)$, where $s$ is the default swap spread and $R$ is the recovery rate.

[4] Notice that linear independence ($r=0$) implies independence of asset returns only because of the joint normality assumption. With a different joint distribution, we may have higher-order dependence of asset returns (and non-zero default correlation) even with zero asset correlation.

[5] To be precise, the probability of drawing a default out of $M$ marginally identical and independent credits is slightly less than $M$ times the marginal default probability. The difference is given by the joint default probabilities, though these are going to be relatively small because of independence.

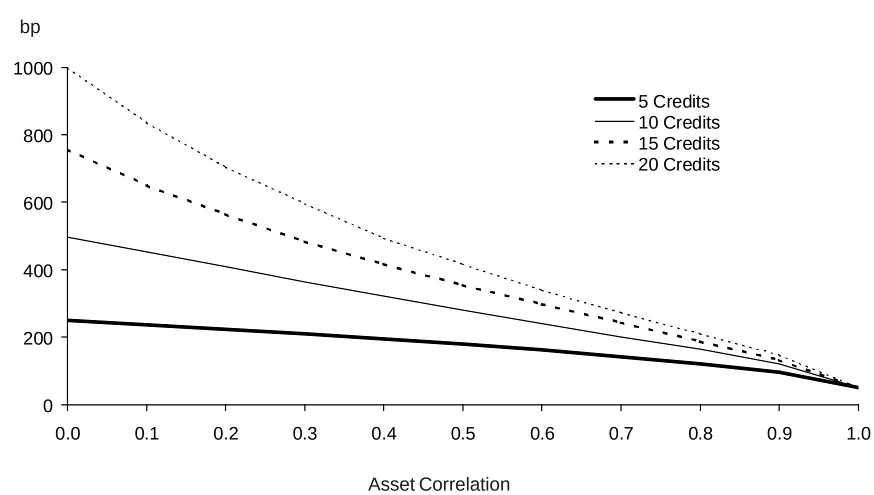
reference assets will either default or survive together. They are really moving in just one direction, and the premium need not be any higher than that paid for single-name protection.

Since the dollar value of a basis point of premium (DV01) varies with the probability of the credit event, the market exposure of a given basket swap is most appropriately analyzed by studying changes in the protection value rather than in the basket premium.[6] Figure 3 plots the value of $1 million FTD protection as a function of $M$ and $r$: the relation is qualitatively analogous to the one seen in Figure 2.

The most important piece of information in Figure 3 is that **the value of FTD protection is a monotonically decreasing function of asset correlation (and therefore of default correlation), since the number of defaulting directions shrinks from *M* to 1 as *r* increases from 0 to 1. This is true irrespective of the number of reference credits *M*, and it implies that a FTD protection seller is always positively exposed to an increase in correlation (and vice-versa for a protection buyer).**

[6] The protection value of a default swap is just the price of an identical default **option** for which the protection buyer makes a single up-front payment.

Figure 2. **5-Year FTD Premium**



### *3.2. STD Protection*

Lower orders of default protection must display a somewhat different sensitivity to correlation changes. In fact, with *M* reference credits, we have that

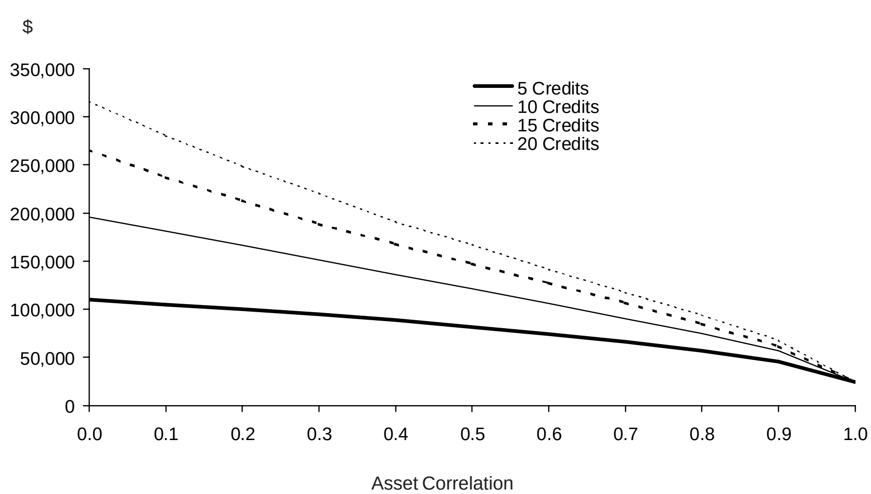
$$P(\text{at least 1 default}) + P(\text{at least 2 defaults}) + \ldots + P(\text{M defaults}) =$$
$$P(\text{1st credit defaults}) + P(\text{2nd credit defaults}) + \ldots + P(\text{Mth credit defaults}).$$

The elements on the left-hand side are the probabilities of the triggering events for each order of protection on the basket. Since the right-hand side is independent of correlation, so must be the left-hand side. Correlation changes simply redistribute the total value of protection among the *M* different orders. We have seen earlier that the probability of having at least one default decreases with correlation. Obviously, the probability that all of the *M* credits will default increases with correlation. The effect of correlation changes on orders of protection in the interval [2, (*M*-1)] is less evident.

Keeping all other parameters as in the previous section, Figure 4 plots the value of $1 million STD protection as a function of *M* and *r*. **For a relatively large number of reference credits, this value decreases with correlation. But for a smaller number of credits, the likelihood of the credit event may increase with correlation.**

When the order of protection is $N \in [2, M-1]$, *N* defaults are needed for the contract to pay out. In this case, varying the correlation parameter can be thought of as varying both a) the correlation between the defaulters needed to trigger the payment and b) the correlation between any other pair of credits. The first effect

Figure 3. **Value of $1 Million 5-Year FTD Protection**



actually increases the probability of the credit event and, therefore, the value of protection. Consistently with the results obtained above, notice that the effect in a) cannot have any impact on the value of FTD protection, since it only takes one credit to default for the payment to be triggered.

If the number of credits referenced by a STD contract is large, there are many pairwise correlations, but there is still only one relevant correlation between defaulters. The effect in b) dominates and the relation between correlation, and protection value is qualitatively similar to that characterizing FTD protection; i.e., it is monotonically decreasing. On the other hand, when the number of reference credits is small, the effect in a) dominates, and increasing correlation increases the probability of the triggering event over a large portion of the correlation range. In the limit, the value of STD protection on a basket of two credits is monotonically increasing in correlation, as the effect described in b) is now switched off. For intermediate numbers of credits in the basket, the two effects give rise to a trade-off that is responsible for the hump-shaped curves plotted in Figure 4.

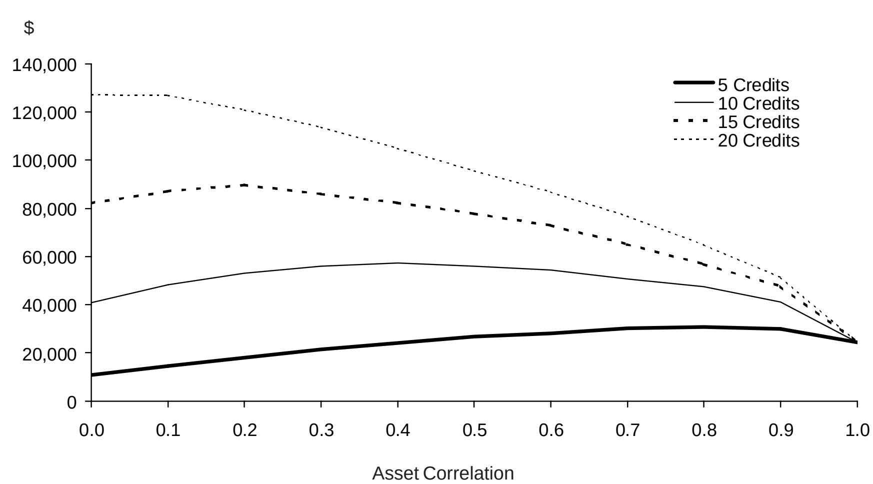
### *3.3. TTD Protection*

From the previous discussion, it follows immediately that decreasing the seniority of default protection will exacerbate the effect described in a). Figure 5 shows that **the value of TTD protection is increasing over a significant portion of the correlation range even with a relatively large number of reference assets.**

### *3.4. Basket Protection and the Credit Cycle: A Simple Example*

There is evidence that a spread widening, defined as a systematic deterioration of the creditworthiness of market borrowers, is generally accompanied by an increase in default correlations. In this section, we analyze the consequences of

Figure 4. **Value of $1 Million 5-Year STD Protection**



such a scenario for basket investors. Here, we model an increase in default correlations with an increase in asset correlations. In the next section, we will argue that default correlation may increase for a different reason.

When spreads widen and asset correlations increase, the counterparties in a FTD basket are subject to two counteracting effects. First, a FTD protection buyer stands to realize a positive mark-to-market because of the increase in market-implied default probabilities. This gain, however, is mitigated by the associated increase in correlations. In other words, a FTD contract offers the counterparties an implicit hedge by providing opposite exposures to positively correlated events.

The value of lower-order protection, on the other hand, may increase with an increase in correlation (Figures 4 and 5). It follows that the return process of basket swaps of different orders may display rather different sensitivities to the credit cycle. A numerical example will clarify this claim.

Take a basket of 10 credits with flat default swap spread curves at 50 bp, recovery rates of 50%, and pairwise asset correlations of 20%. Under the usual assumption of exponentially distributed survival times, market-implied hazard rates are flat at 1%. As in the previous section, LIBOR rates are set equal to zero. Consider first an investor buying five years of FTD protection on this basket for a notional of $1 million. The value of this protection is approximately $166,600, equal to the product of a 409 bp annual premium and a DV01 of $407.33.

To ignore term considerations, imagine that immediately after inception, all spreads widen by 20% and asset correlations increase from 20% to 30%. Every spread curve is now flat at 60 bp, and implied hazard rates are, therefore, flat at 1.2%.

Figure 5. **Value of $1 Million 5-Year TTD Protection**

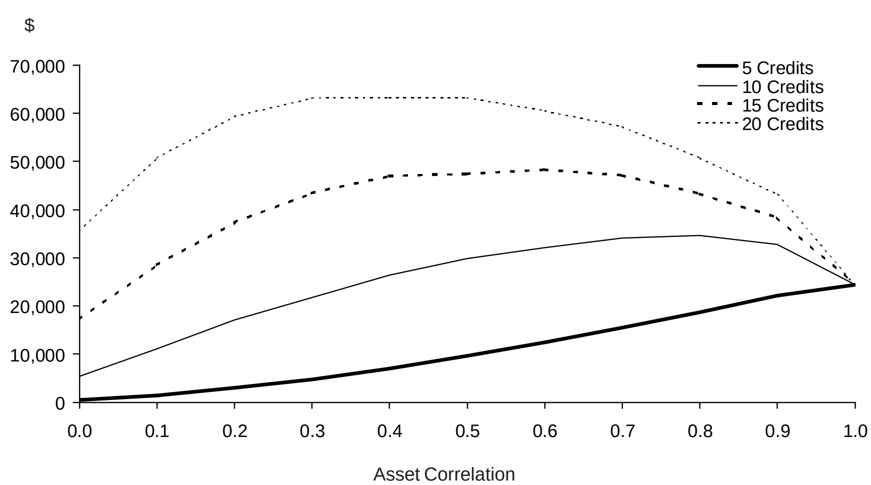


Figure 6 shows the impact of this credit downturn on the value of FTD protection, which increases to approximately $170,500 for a return of 2.34%. The increase in protection value due to the increase in default probabilities is almost completely offset by higher asset (and default) correlations.

Now imagine a second investor buying $1 million of 5-year STD protection on the same basket at the same time. This protection is worth $52,900, and the buyer should pay 111 bp with a $476.58 DV01. Figure 7 shows that after the systematic credit deterioration and the associated increase in asset correlations, the value of STD protection climbs to about $69,000, so that the buyer can book a $16,100 profit for a return of 30.4%. Both the systematic credit deterioration and the correlation increase have worked in her favor.

Figure 6. **$1 Million 5-Year FTD Protection in a Credit Downturn**

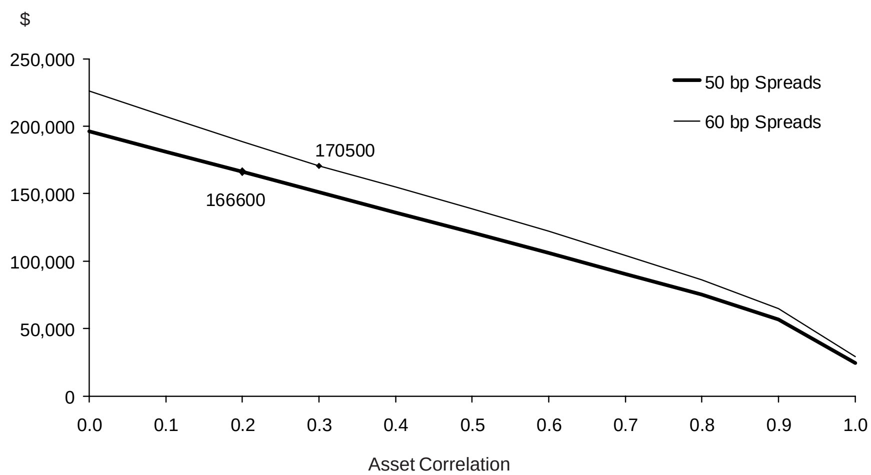
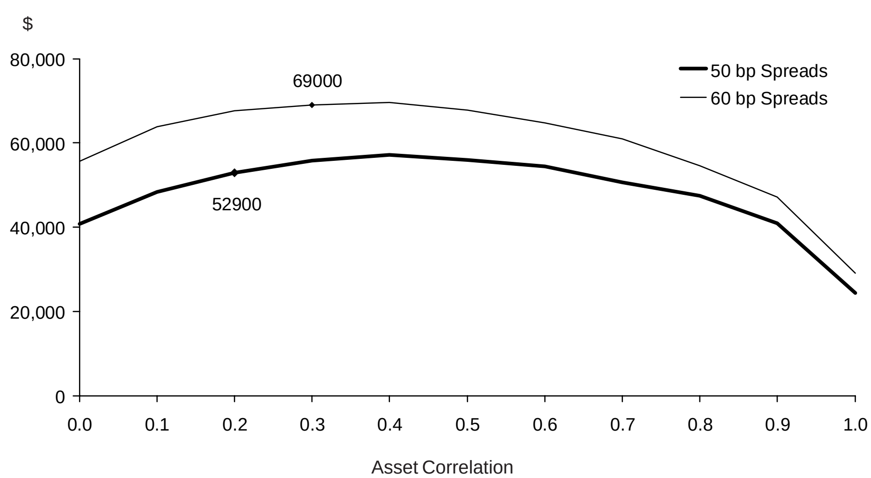



Figure 7. **$1 Million 5-Year STD Protection in a Credit Downturn**



The benefits of the increase in asset correlation are even more pronounced for a TTD protection buyer. Figure 8 shows that the hypothesized market movement would increase the value of this position from $17,100 to $29,800, for a hefty 74.3% return.

**Lower orders of basket protection may get triggered only if senior orders do: they are by definition less exposed to default risk. This does not necessarily imply, however, that they are less exposed to market risk. In fact, we have shown with a simple example that the returns provided by STD and TTD baskets may be more sensitive to a credit downturn than the corresponding FTD contract.**

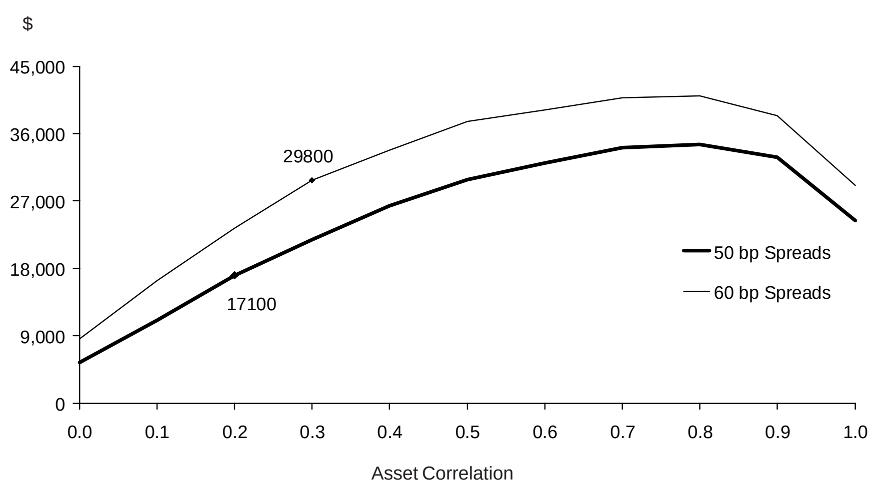
## 4. Linear Dependence and Tail Dependence

The results in the previous section followed from the observation that a spread widening is generally associated with an increase in default correlations. For the purpose of our analysis, we have modeled an increase in default correlations through an increase in the linear dependence (i.e., correlation) of asset returns. In this section, we argue that this is not necessarily the reason why default correlations tend to increase during a credit downturn.

First, one can show that, even holding asset correlation constant, default correlation increases with default probabilities. In fact, this "correlation effect" of quality deterioration is implicitly accounted for in the upward shift of the curves shown in Figures 6, 7, and 8 and is partially responsible for the results obtained above.[7]

[7] See Hans Gersbach and Alexander Nipponen , "The Correlation Effect," 2000, *mimeo*, University of Heidelberg.

Figure 8. **$1 Million 5-Year TTD Protection in a Credit Downturn**



Most important**, default correlations may increase in a credit downturn because of an increase in the tail dependence of asset returns.** When we simulate correlated default times using a normal copula, we are implicitly assuming that the joint behavior of asset returns does not display any tail dependence. However, there is evidence that a credit downturn is generally characterized by an increased frequency of extreme joint movements that cannot be reproduced by a joint normality assumption.

We can modify the simulation of correlated survival times to study the impact of tail dependence on default correlation. For example, we can apply a *t*-copula to the exponential marginals $E_h(\cdot)$ to get

$$P(T_A < x, T_B < y) = t_{2,v}(t_v^{-1}(E_h(x)), t_v^{-1}(E_h(y)), r),$$

where $t_{2,v}(\cdot, \cdot, r)$ is a bivariate standard *t* distribution with *v* degrees of freedom and correlation $r$, and $t_v(\cdot)$ is a univariate standard *t* distribution with *v* degrees of freedom.

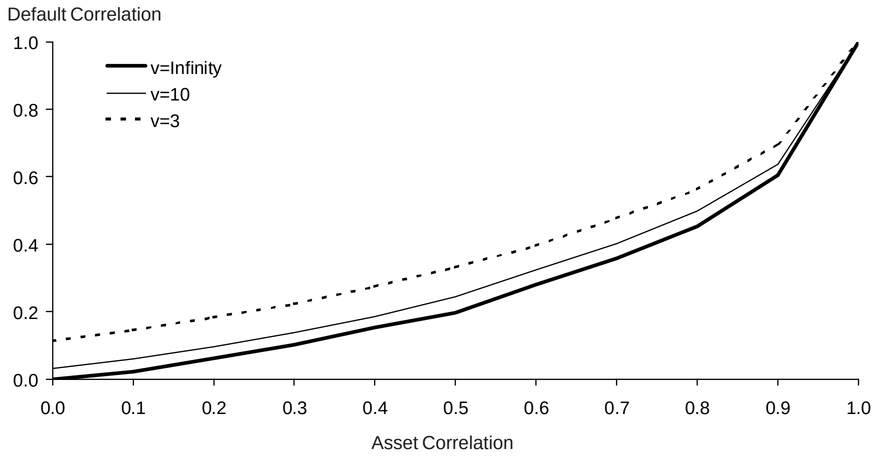
The choice of a *t*-copula for survival times may be viewed as an implicit assumption that asset returns follow a multivariate *t* distribution. Since two jointly $t_{2,v}$ variables are marginally $t_v$ distributed, one can immediately interpret $t_v^{-1}(E_h(\cdot))$ as a default threshold and $t_{2,v}(\cdot, \cdot, r)$ as the joint distribution of asset returns. Compared with a normal, a *t* distribution has an extra parameter: the number of degrees of freedom. As this number goes to infinity, a *t* distribution tends to a normal distribution; i.e., it displays no tail-dependence. But for a finite number of degrees of freedom, a *t* distribution allows for extreme joint realizations.

Using a 5-year horizon and two credits with constant hazard rates of 1%, Figure 9 compares a normal copula and a *t*-copula with 10 and 3 degrees of freedom. **Tail dependence increases default correlation for any value of the asset correlation. In particular, notice that even when asset returns are uncorrelated (i.e., linearly independent), tail dependence can produce a significant amount of default correlation.**[8]

This implies that we have an alternative way to interpret the scenario proposed in the previous section. **Rather than assuming that asset correlation (i.e., linear dependence) increases during a credit downturn, one can think of the joint "fat-tail" behavior becoming more pronounced.** The results following from such an assumption would be analogous to the ones obtained above.

[8] For an exhaustive analysis of the effects of tail-dependence on credit risk, see Mark Nyfeler, "Modeling Dependencies in Credit Risk Management," 2000, Ph.D. Thesis, Federal Institute of Technology, Zurich.

Figure 9. **Correlation**



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# DIGITAL PREMIUM
## Part 1: Practical Considerations

Digital default swaps are different from conventional (floating recovery) swaps because they transfer different types of risk. Conventional swaps protect from default loss risk, while digitals protect from default event risk. The implicit recovery risk remains unpriced in a hedged digital swap. Digital swap break-even spread, calibrated to market-observed conventional default swap spread, must contain a premium because of 1) uncertainty of the market average recovery rate, and 2) the inability to diversify firm-specific recovery rate deviations completely. We estimate these premia and show examples of pricing for underlying credits of various seniorities.

### Introduction

Default swaps are the basic building blocks of the credit derivatives market. They allow an efficient transfer of credit loss risk between counterparties. In a recent review of the credit derivatives market, O'Kane (2000) emphasizes that default swaps are by far the most common type of the traded instrument, accounting for nearly 40% of the outstanding notional as of the end of 1999 (the most recent year with published statistics).

There are three main types of default swap contracts (see O'Kane [2001] and Duffie [1999] for detailed descriptions), differentiated by the type of settlement in the event of default. In all cases, the protection buyer makes regular payments (default swap spread) to the protection seller until the default of the reference issuer or the maturity of the contract, whichever comes first.

- **Floating recovery, physical settlement.** In the event of default on the underlying credit, the protection seller agrees to purchase the reference bond (or the cheapest-to-deliver bond from a pre-specified basket of same-issuer securities) at par value. The protection buyer delivers the accrued amount of the last swap spread payment.
- **Floating recovery, cash settlement.** In the event of default, the protection seller pays par minus the post-default price of the reference security minus the accrued swap premium payment. The post-default price (recovery value) is determined by a dealer poll.
- **Fixed (digital) recovery, cash settlement.** In the event of default, the protection seller pays a pre-specified dollar amount less the accrued swap premium payment.

The first two variants of the default swap contract are by far the most common in the marketplace. However, there is some indication of growing interest from certain clients in fixed recovery swaps, which we call digital swaps, following the market convention.

The difference between the floating recovery and digital swaps is that they are designed to protect from different types of risk. The floating recovery swaps (both physically and cash settled) transfer the risk of *default loss*. The digital swaps transfer the risk of *default event*. The pricing methodologies used to value these two types of swaps need to reflect this difference.

In this article, we will show that the digital default swap break-even spread, calibrated to market-observed conventional default swap spread, must contain a risk premium associated with implicit dependence on the recovery rate. We will estimate this premium using a simple approximation and will decompose it into market and liquidity risk premia, the latter being related to the minimum bid-offer spread for digitals. We will conclude with examples of digital swap pricing.

### Pricing by Replication

Consider the following hedging argument. Assume that a broker sells default protection in a digital swap format and attempts to hedge his exposure with a floating recovery swap on the same underlying credit. The hedge ratio $H$ will be determined by cash flow immunization considerations outlined below.

The broker's cash flows before default are shown below. We denoted the digital swap spread as SD and the floating recovery swap spread as SF. We also assume that the cash flow dates and the swap's maturities are perfectly matched (Figure 1).

When and if default occurs, the cash flows are as shown in Figure 2.

In Figure 2, we denoted the floating default contingent payment value as *CPF* and the fixed (digital) contingent payment as *CPD*. The floating contingent payment will be equal to $CPF = 1 - RF$, where the realized floating recovery value $RF$ will be determined only after the default. The digital swap contingent payoff is

Figure 1. **Hedged Digital Default Swap: Case of No Default**

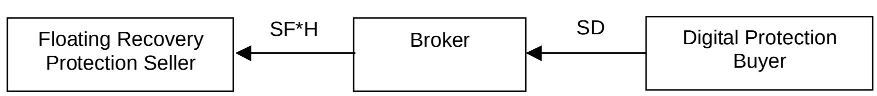
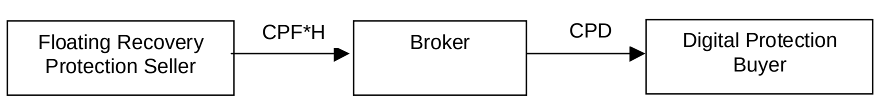



Figure 2. **Hedged Digital Default Swap: Default Case**



frequently expressed in terms of the equivalent choice of fixed recovery $CPD = 1 - RD$. For example, one would say that the digital swap, paying the full amount of notional outstanding in the case of default, corresponds to zero recovery.

The aggregate cash flows for the broker are summarized in Figure 3. (Sign convention: net spread is received and net default contingent amount is paid, as denoted by arrows).

Looking at the Figure 3, one can immediately see the source of the difference between the conventional (floating recovery) and digital default swaps. Let us clarify this picture.

**Assume that the realized recovery rate for the underlying credit is independent of the default timing and interest rates.** Under such assumption, the expected value of the conventional default swap contingent payment *CPF*, as well as its break-even spread *SF*, depends only on the average expected recovery rate and is not sensitive to uncertainty of *RF*. Under the same assumption, the digital contingent payoff *CPD* is, on average, a certain fraction of the floating payoff. Therefore, one can obtain the hedge ratio based on the contingent payment immunization, as:

$$H_{\text{CF-immunization}} = \frac{CPD}{CPF} = \frac{1 - RD}{1 - RF} \tag{1}$$

For example, if the digital payoff is specified as $CPD = 1$ (zero recovery digital swap), then the hedge ratio is $H_{\text{CF - immunization}} = 1/(1 - RF)$ .

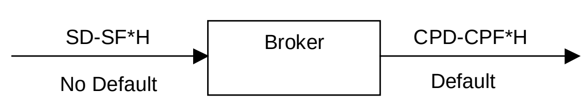
It is apparent from Figure 3 that the scaling of the digital swap spread with respect to conventional spread by the same hedge ratio would achieve a full cash flow immunization hedge, as long as the conventional swap itself was fairly priced.

$$SD_{\text{CF-immunization}} = H_{\text{CF-immunization}} \cdot SF \tag{2}$$

Thus, we define the no-premium (i.e., based on cash flow immunization) break-even spread for the digital swap as:

$$s_{\text{no-premium}}(RF) = \frac{1 - RD}{1 - RF} s_{\text{conventional}}(RF) \tag{3}$$

Figure 3. **Hedged Digital Default Swap: Broker's Aggregate Cash Flows**



We would like to remind the reader that the floating recovery used in Equation 3, is the expected value of the recovery rate, whose realized value will only become known upon default.

### Limitations of the No-Premium Hedge Pricing

In the previous section, we assumed that the expected recovery rates for conventional swaps are exogenously given. However, the recovery rates are notoriously poorly known and are, in fact, widely distributed, as we will discuss in greater detail below. Therefore, **the broker assumes a recovery risk by entering into a hedged digital default swap**.

The risk due to recovery uncertainty is of roughly the same order as the default timing risk.[1] Whether or not this risk must be compensated by a premium depends on whether it can be efficiently diversified away. According to standard finance theory, the diversifiable risk should not be compensated by a risk premium, while non-diversifiable, common risk should.

**Is the recovery risk diversifiable?** Naively, one would argue yes. Indeed, it is a company-specific risk and, therefore, must be diversifiable. Let us examine this answer closer, however. Such a conclusion is based on two implicit assumptions, neither of which actually holds:

- **Assumption 1:** the mean recovery is known and only the deviations from it are company specific.
- **Assumption 2:** there are enough instruments that the uncertainty of the recovery risk can be diversified in the broker's books.

Let us address each of these assumptions in order. The mean recovery, which appears so prominently in the naive estimation of the digital swap spreads, is a poorly known quantity. Figure 4 outlines the recovery statistics from a Moody's study (Hamilton, *et. al.*, 2001) that shows that the recovery rates are extremely widely distributed. The sample mean of such an empirical distribution is rather uncertain. See Appendix A for a detailed model of recovery distributions.

Hamilton and Carty (1999) show the sample mean recovery as a function of time (Figure 5). One can see that recovery rates have varied between 70% and a recent 30%. There have been both secular trends toward lower recoveries and wide fluctuations concurrent with the credit cycle.

The secular trend toward lower recovery rates is easy to understand, since, in latter years, there have been more bankruptcies of high-tech companies with substantially higher levels of intangible assets (R&D), extremely quickly

[1] The standard deviation of the default timing in Poisson process-based models of default is equal to the expected default time itself, which, in turn, is of the same order as the uncertainty due to unknown recovery.

depreciating inventory, etc. Thus, there is much less that can be salvaged by the liquidators. Case in point—Iridium LLC, whose multi-billion dollar assets, including the constellation of satellites, terrestrial networks, and intellectual property, were sold in bankruptcy for only $25 million, resulting in low single-digit recovery for debtholders.

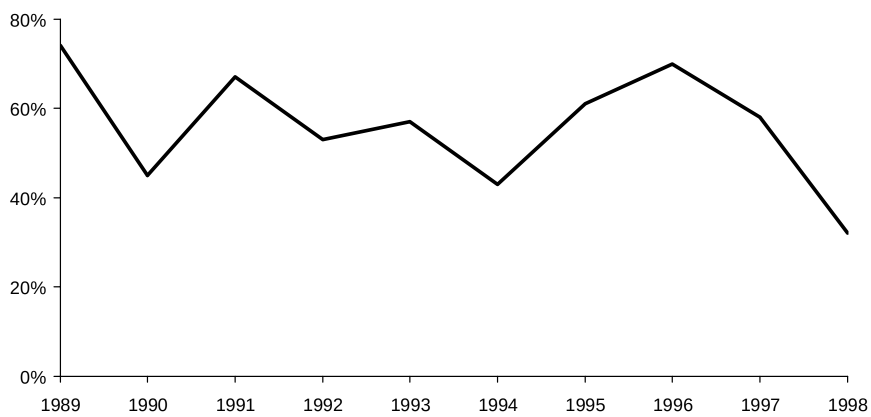
The second assumption is too optimistic as well. Given the extreme width of the recovery distribution shown in Figure 4, it would require many securities in a sample to narrow the effective variance to acceptable levels. There are simply not enough digital default swaps traded at any one time by a single broker to be able to neglect the non-diversified variance of the recovery rates, even if one neglects the important failings of assumption 1.

Figure 4. **Descriptive Statistics for Recovery Given Default,** 1970-2000

| | Recovery Rates | | | | | | |
|---|---|---|---|---|---|---|---|
| | **Median** | **Average** | **St. Dev.** | **Min** | **Max** | **1st Quartile** | **3rd Quartile** |
| **Bank Loans** | | | | | | | |
| Senior Secured | 72.0 | 64.0 | 24.4 | 5.0 | 98.0 | 45.3 | 85.0 |
| Senior Unsecured | 45.0 | 49.0 | 28.4 | 5.0 | 88.0 | 25.0 | 75.8 |
| **Bonds** | | | | | | | |
| Senior Secured | 53.8 | 52.6 | 24.6 | 1.6 | 103.0 | 34.8 | 68.6 |
| Senior Unsecured | 44.0 | 46.9 | 28.0 | 0.5 | 122.6 | 25.0 | 66.8 |
| Senior Subordinated | 29.0 | 34.7 | 24.6 | 0.5 | 123.0 | 15.1 | 50.0 |
| Subordinated | 28.5 | 31.6 | 21.2 | 0.5 | 102.5 | 15.0 | 44.1 |
| Junior Subordinated | 15.1 | 22.5 | 18.7 | 1.5 | 74.0 | 11.3 | 33.0 |
| **Preferred Stock** | 11.1 | 18.1 | 17.2 | 0.1 | 86.0 | 6.4 | 24.9 |

Source: Moody's (Hamilton, *et. al.*, 2001).

Figure 5. **Firm-Level Recovery Rates by Year of Bankruptcy Resolution**




Source: Moody's (Hamilton and Carty, 1999).

### Sensitivity Analysis

How important are the limitations outlined in the previous section? We can get a feel for the answer through a simple sensitivity analysis of conventional and digital default swaps to changes in the recovery rate. We will maintain the assumption of independence of the recovery rates from default time and interest rates.

Before we proceed, we must note that the recovery rate is only one of the relevant variables. As in any such analysis, one must specify the context in which the variable is changing. The most consistent approach is to define a constant market-observable quantity and then allow changes to the variable under consideration while keeping the calibration of the model to the observable quantity intact.

As we mentioned in the introduction, conventional default swaps are by far the most liquid instruments in the credit derivatives market, in some cases even surpassing the actual cash market in trading depth. Therefore, it makes sense to define the conventional default swap spread as the calibration target. Consequently, this spread will be independent of the changing recovery assumption by construction, because we will recalibrate the model to keep the conventional spread constant as we change the expected recovery.

The conventional swap break-even spread depends only on the default loss expectation (the break-even default rate times one minus expected recovery). When the recovery assumption changes, the change is absorbed into the break-even default rate and, consequently, into the distribution of the default times. This new distribution is, by construction, such that the expected present value of the contract payoff does not change and the break-even spread remains constant.

The recovery sensitivity is quite different for digital default swaps when we calibrate them to the conventional swap spreads. While the change in recovery assumption leads to a change in the break-even default rate and, therefore, to a change in the distribution of default times, the contract payoff in case of default remains constant, and, therefore, the expected present value of the payoff and the break-even spread change substantially.

Figure 6 shows the break-even digital default swap spread for each level of digital coupon above flat LIBOR. The payoff is set to 1 (i.e., the protection seller pays the full amount of the notional in case of default). As can be seen, this dependence is quite strong.

### Premium for Uncertain Recovery

An important observation stemming from Figure 6 is that the digital swap break-even spread dependence on the recovery rate is convex. This has important consequences when we recall that the recovery rate is not constant and is, in fact, a random number with a rather wide distribution.

Let us characterize the dependence of the break-even digital default swap spread on the level of implied recovery by the *relative recovery convexity of spread* $\gamma(R)$, i.e., the curvature of the function depicted in Figure 6, divided by the level of the spread at the same recovery. The reason we emphasize the relative convexity rather than the absolute is that this quantity is practically independent of the digital coupon spread level.

The standard Jensen's inequality argument leads to the conclusion that the expected break-even spread is greater than the break-even spread corresponding to the expected value of recovery. We call this difference the **digital premium**.

Figure 6. **Recovery Rate Sensitivity of Digital Swap Break-Even Spread, Calibrated to Conventional Default Swap Spread**

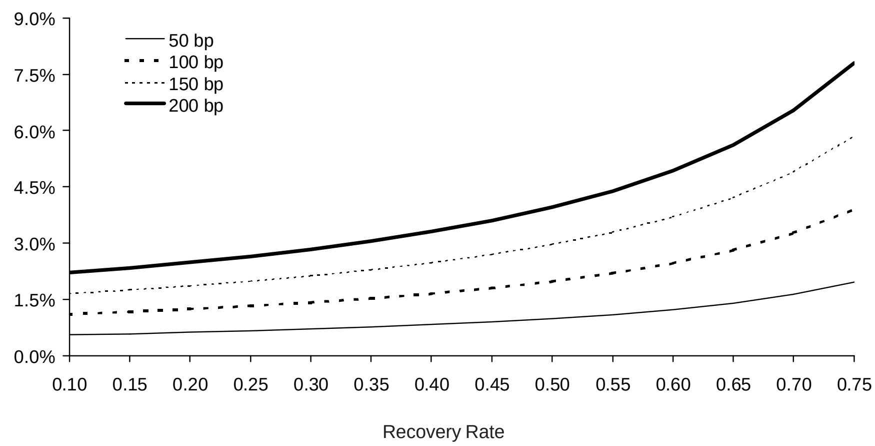



Figure 7. **Relative Convexity of Digital Default Break-Even Spread with Respect to Recovery Rate**

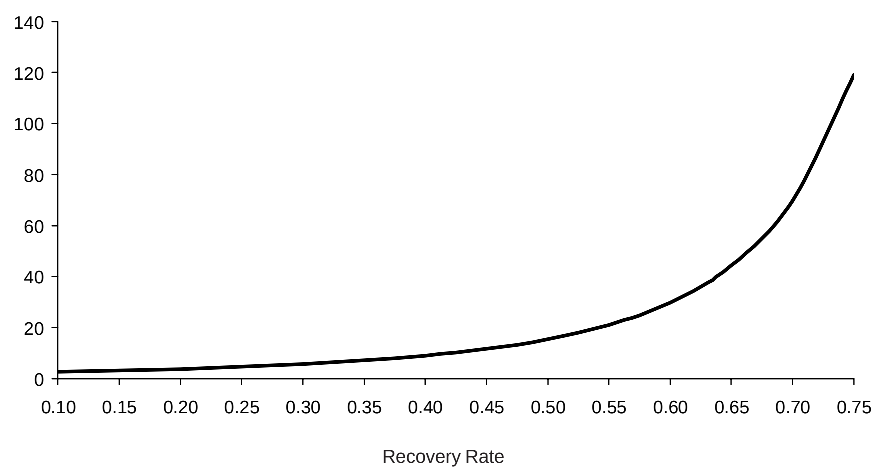


Figure 8 illustrates the emergence of the digital premium. We show the no-premium break-even spread for the digital default swap, similar to the ones depicted in Figure 6. The bell-shaped curve illustrates the recovery distribution in a simplified manner (see Appendix A for more realistic distributions). The straight line approximates the calculation of the expected spread with convexity correction. The ends of this line are set at the characteristic width of the recovery distribution, and the middle point represents the approximate expected value, which is greater than the no-premium spread evaluated at the same value of the expected recovery. The difference between this corrected value and the no-premium spread is the digital premium.

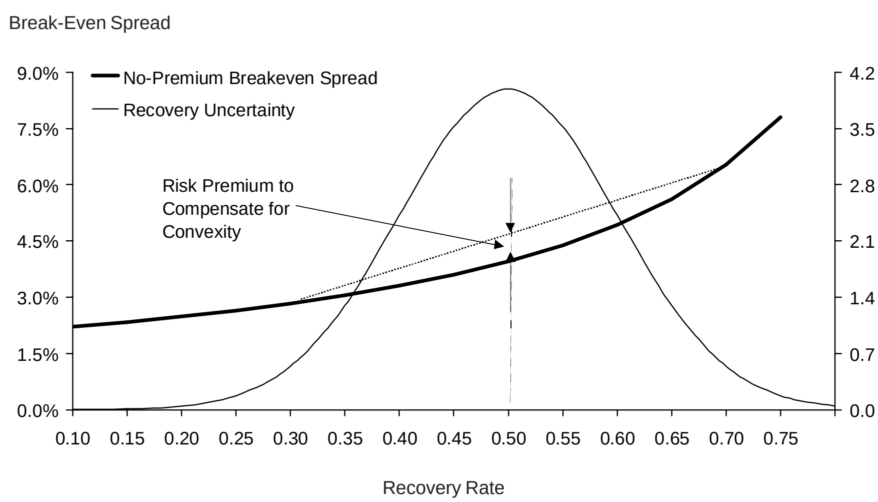
The slope of the spread dependence upon recovery does not matter for the correction calculation. The second-order approximation for the correction in spread is proportional to the variance of recovery and the convexity $C$ of the no-premium spread:

$$\Delta s = C \cdot \frac{\sigma^2}{2} \tag{4}$$

Because the ratio of the convexity $C$ to the spread is independent of the spread itself, it is convenient to work with relative quantities. We define the **relative digital premium** *RDP* as the percentage by which the true digital default swap break-even spread is greater than the no-premium value of this spread given in Equation 3. This relationship is formalized as:

$$RDP = \frac{\Delta s_{\text{premium}}(R)}{s_{\text{no-premium}}(R)} = \gamma(R) \cdot \frac{\sigma^2}{2} \tag{5}$$

Figure 8. **Illustration of Jensen's Inequality as the Source of Risk Premium**



Let us assume that the recovery is random, with market-wide expected value $\overline{R}$ and the variance composed of two components—the uncertainty of market-wide recovery rates and the uncertainty of firm-level (specific) recovery difference from the market-wide average.

(6) $$R = \overline{R} + R'$$

(7) $$\sigma^2(R) = \sigma^2_{\text{market}}(\overline{R}) + \sigma^2_{\text{specific}}(R')$$

As we have argued before, the market-wide uncertainty cannot be diversified away, and, therefore, it will remain roughly the same regardless of the broker's digital default swap book size. The specific variance, to some extent, can be diversified. However, in most cases, there will still be a large residual variance due to the relatively small number of deals in the portfolios.

The typical portfolio includes only a small percentage of digital default swaps. The digitals tend to be a smaller fraction of USD-denominated market compared with the EUR-denominated market. Assuming a typically observed 2%-3% of the portfolio in digitals and assuming several thousand positions in a typical broker book, we estimate the number of concurrently held digital default swaps to be no greater than 100.

Following the variance decomposition in Equation 7, the relative digital premium *RDP* can be further separated into the **market recovery premium**, related to the market uncertainty of expected recovery rates, and the **liquidity premium**, related to the incomplete diversification of firm-specific recovery in the broker's portfolio of *N* hedged digital swaps.

(8) $$RDP = \frac{s_{\text{premium}}(\overline{R}) - s_{\text{no-premium}}(\overline{R})}{s_{\text{no-premium}}(\overline{R})} = \underbrace{\frac{\sigma^2_{\text{market}}}{2} \cdot \gamma(\overline{R})}_{\text{market premium}} + \underbrace{\frac{\sigma^2_{\text{specific}}}{2\sqrt{N}} \cdot \gamma(\overline{R})}_{\text{liquidity premium}}$$

We should note that using the spread convexity to determine the spread premium is only an approximation. The price sensitivity of the swap is a better measure. However, given the linear dependence of the default swap price on the coupon spread and also taking into account the crudeness with which we are able to estimate the empirical parameters, we prefer to use the above relationship because of its appealing simplicity and clear interpretation.

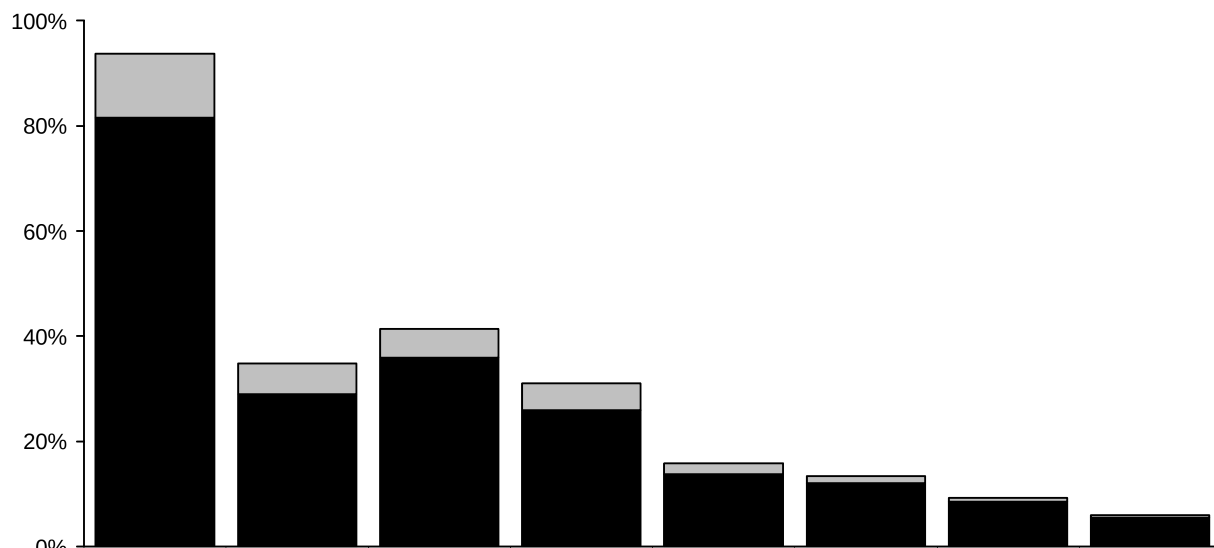
It is important to understand that our arguments are not restricted to cases in which the broker sells hedged digital default swaps. In either the sale or the purchase of digital protection, hedged by conventional default swap, brokers are exposed to pure recovery risk. If they are digital protection sellers, then they are short recovery convexity and should demand the digital premium. If they are digital protection buyers, then they are long recovery convexity and should pay the digital premium. In either case, the liquidity premium goes to the broker and is related to the minimum acceptable bid-offer spread, the full amount of which will often be driven by supply-demand rather than the risk carry outlined here.

Let us emphasize again that conventional default swaps are also exposed to recovery risk. However, it is bundled together with the default rate risk to give default loss rate protection. It is the implied default rate dependence upon recovery, while keeping the conventional default spread constant, that leads to the recovery dependence of the digital swaps.

Figure 9 shows the relative digital premia for various classes of seniority. We assumed the market recovery uncertainty of 20%, the firm-specific uncertainty equal to standard deviation of recovery rates from Figure 4, and the (very optimistic) size of portfolio equal to 100.

It is worthwhile to note that the large relative premium associated with more senior underlying credits is simply a reflection of the fact that these credits have a large expected recovery, and, therefore, the potential downside of the uncertain recovery impact is greatest in this case. On the other extreme, the preferred stocks have very

Figure 9. **Relative Digital Premium, with Breakdown into Market Risk and Liquidity Premia**




| | Loan: Sr Secured | Loan: Sr Unsecured | Bond: Sr. Secured | Bond: Sr. Unsecured | Bond: Sr. Subord | Bond: Subord | Bond: Jr. Subord | Preferred Stock |
|---|---|---|---|---|---|---|---|---|
| Liquidity Premium | 12.1% | 5.8% | 5.4% | 5.1% | 2.1% | 1.3% | 0.7% | 0.4% |
| Market Premium | 81.5% | 29.0% | 35.9% | 26.0% | 13.8% | 12.0% | 8.5% | 5.5% |

low expected recovery, so there is hardly any downside stemming from recovery uncertainty, which leads to low relative convexity and a low relative premium.

To further illustrate our conclusions regarding the pricing of digital default swaps, we present several hypothetical trades in Figure 10. Each row represents a pair of default swaps for the same reference credit. The first three columns define the reference credit. The next two columns show the conventional default swap break-even spread and the corresponding expected recovery, which is simply taken from Figure 4 (the current market expectations for both the spread levels and the recovery rates might be very substantially different from these numbers).

The last five columns show the details of the digital default swap spread. First, we show the contract recovery rate (i.e., one minus the contract's fixed payoff rate in case of default), then the no-premium digital spread calculated according to Equation 3. The next column shows the relative spread convexity with respect to change in recovery, estimated for the level of the corresponding expected recovery rate (compare with Figure 7). The last three columns show the market premium, the liquidity premium, and the total digital default spread (bid/offer). We converted the premia from relative to absolute levels and showed them in basis points to highlight the fact that the total digital spread is simply a combination of the no-premium spread, the market premium, and the liquidity premium.

The assumptions for recovery uncertainty are the same as in Figure 9, i.e., 20% market uncertainty of expected recovery, the firm-specific volatility taken from Figure 4, and the size of the broker's digital portfolio set to 100. The calculations follow Equation 8 (liquidity premium is subtracted to obtain the bid and added to obtain the offer spread).

As we can see, the digital premium can be significant, both in relative terms and in absolute (basis point) terms. Despite the simplifying approximations used in this article, it is encouraging that the results are in the right ball park. For a typical senior subordinated industrial issuer, the digital protection is indeed trading at roughly 10%-15% relative premium, in addition to no-premium scaling given by Equation 3. This is in broad correspondence with the estimates shown in Figure 9 and illustrated in Figure 10.

Figure 10. **Hypothetical 5-Year Default Swap Pricing**

| Reference Credit | | | Conventional Default Swap | | Digital Default Swap | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Sector** | **Qual.** | **Seniority** | **CDS Spread** | **RF** | **RD** | **No-Premium Spread** | $\gamma$ *(R)* | **Market Premium** | **Liquidity Premium** | **DDS Spread** |
| Utility | A3 | Sr. Secd. | 50 | 52.6% | 50% | 53 | 17.9 | 19 | 3 | 69/75 |
| Retail | Baa1 | Sr. Unsecd. | 100 | 46.9% | 0% | 188 | 12.9 | 49 | 10 | 227/247 |
| Ind. | Ba1 | Sr. Subord. | 250 | 34.7% | 25% | 287 | 6.9 | 40 | 6 | 321/333 |

### Summary

We conclude with an emphasis on the main reason for the existence of the digital premium: digital default swaps and conventional swaps protect against different risks. Conventional swaps protect from default loss while the digital swaps protect from default event risk. When conditioned on fitting the market-observable asset swap or conventional default swap rates, the digital default swaps are sensitive to recovery rate assumption. In particular, the digital swap hedged by a conventional swap represents a carry of pure recovery risk.

We have estimated the sensitivity of this hedged position with respect to the uncertainty of the recovery rate and obtained the relative digital spread premium composed of market and liquidity premia. The market premium compensates the writer of the digital protection for convexity exposure with respect to error in market-expected recovery rate estimation. The liquidity premium is associated with incomplete diversification of firm-specific recovery risk in the broker’s books and can be related to the theoretical minimum bid-offer spread.

Our estimates show that these additional market premia are quite large, especially for the most senior underlying credits. The relative digital premium is an increasing function of expected recovery (there is more downside when expected recovery is higher) and an increasing function of recovery uncertainty.

We believe that as the market for credit derivatives matures, there will be more trading of securities that depend on default rate and recovery risk separately, rather than through a default loss combination. The expected recovery dependence will be important for all such instruments. In subsequent publications, we will explore the impact of the recovery assumptions and the pricing of derivatives that depend upon them.

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## Appendix A. Approximating the Recovery Rate Distributions

Figure 4 presents a pattern of wide distribution of recovery rates. One useful description of these empirical statistics uses the so-called *beta distribution*. The main advantage of this distribution is that it is bounded by 0 and 1 and has only two parameters, allowing for an easy fit to the first two moments of the empirical statistics.

Beta distribution is defined by its probability density function:

$$f_R(r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1} \quad , \quad \text{where } r \in [0,1], \alpha > 0, \beta > 0$$

The parameters of this distribution can be fitted to the observed mean and variance:

$$\alpha = \overline{R} \cdot \left( \frac{\overline{R}(1-\overline{R})}{\sigma_R^2} - 1 \right), \quad \beta = (1-\overline{R}) \cdot \left( \frac{\overline{R}(1-\overline{R})}{\sigma_R^2} - 1 \right)$$

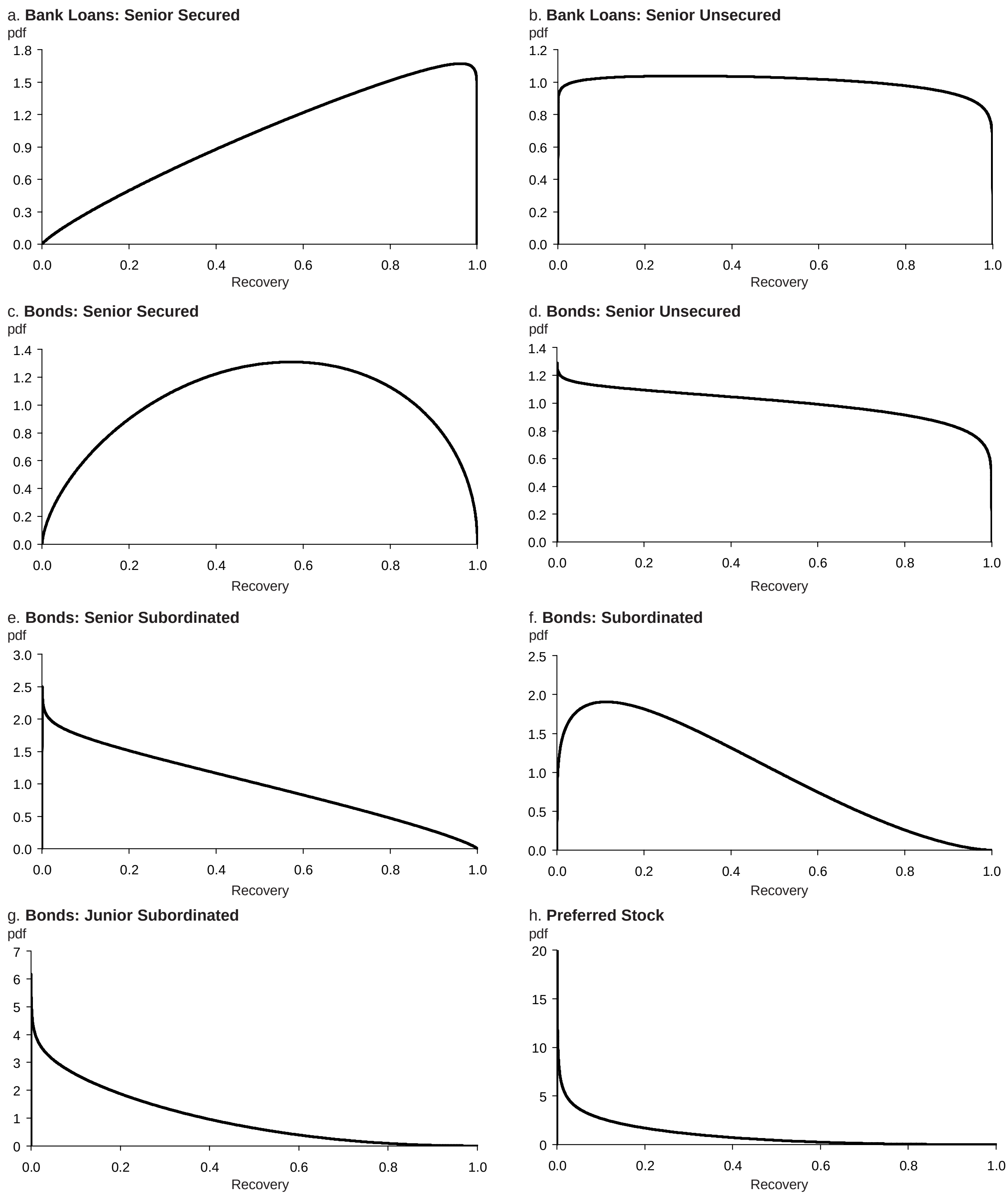
Figure A-1 shows the fitted parameters α, β for each seniority class of corporate debt.

The charts in Figure A-2 show the shapes of the corresponding recovery rate distributions and may give a nice visual image of the extreme uncertainty of the recovery rates, which we noted many times in this article.

Figure A-1. **Fitted Descriptive Statistics for Recovery Given Default**

| | Recovery Rates | | Beta Fit | |
|---|---|---|---|---|
| | **Mean** | **St. Dev.** | **α** | **β** |
| **Bank Loans** | | | | |
| Senior Secured | 64.0 | 24.4 | 1.84 | 1.03 |
| Senior Unsecured | 49.0 | 28.4 | 1.03 | 1.07 |
| **Bonds** | | | | |
| Senior Secured | 52.6 | 24.6 | 1.64 | 1.47 |
| Senior Unsecured | 46.9 | 28.0 | 0.98 | 1.11 |
| Senior Subordinated | 34.7 | 24.6 | 0.95 | 1.79 |
| Subordinated | 31.6 | 21.2 | 1.20 | 2.61 |
| Junior Subordinated | 22.5 | 18.7 | 0.90 | 3.09 |
| **Preferred Stock** | **18.1** | **17.2** | **0.73** | **3.28** |

Figure A-2. **Beta Distribution Approximation of Recovery Rate Statistic**



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