LEHMAN BROTHERS | Fixed Income
Quantitative Credit Research

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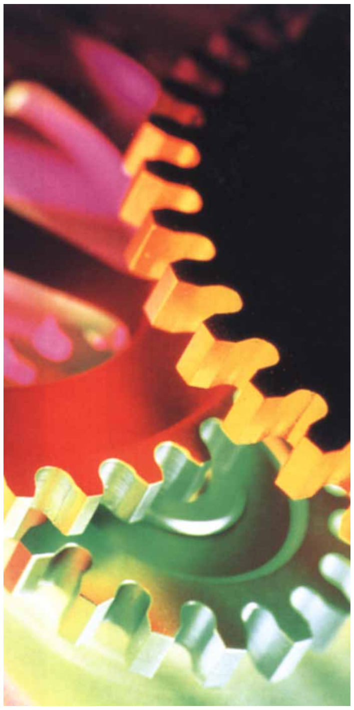
Quarterly

Volume 2004-Q3/Q4



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Base Correlation Explained

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Since the advent of standardised single tranche CDOs on the liquid CDS indices of CDX and iTraxx, there has been a need for a commonly agreed method of quoting the implied correlation between the assets in the respective CDS index. Initially the market chose compound correlation as its quotation convention. More recently, base correlation has become more widely used in the market. In this paper we define, discuss and compare both conventions. We conclude that base correlation possesses a number of desirable properties that make it a more powerful measure of tranche implied correlation. However, we argue that base correlation does not constitute a proper model for correlation skew. [1]

INTRODUCTION

The advent of standard CDO tranches with standard CDS indices as the reference portfolio has greatly enhanced liquidity and transparency in the synthetic CDO market. We are now able to observe daily pricing on a range of tranches linked to US, European and Japanese investment grade and high yield CDS indices. An example of tranche pricing on a selection of these indices is shown in Figure 1.

Figure 1. Indicative pricing for the five standard tranches linked to the CDX Investment Grade NA Series 3 and iTraxx Europe Series 2 indices, for 13 October 2004. [2]

	CDX Investment Grade North America Series 3		iTraxx Europe Series 2	
Tranche	**Lower-Upper strike**	**Upfront / Running Spread (bp)**	**Lower-Upper Strike**	**Upfront / Running Spread (bp)**
Equity	0-3%	37.125% + 500	0-3%	24.25% + 500
Junior Mezzanine	3-7%	259.5	3-6%	137.5
Senior Mezzanine	7-10%	101.0	6-9%	47.5
Senior	10-15%	38.5	9-12%	34.5
Super Senior	15-30%	11.5	12-22%	15.5

Note: On 13 October 2004, the CDX IG NA 3 index traded at 53.5bp and the iTraxx Europe 2 traded at 37bp. Both have a maturity date of 20 March 2010.
Source: Lehman Brothers.

The price of a CDO tranche is a function of the default correlation between the assets in the reference portfolio. See O'Kane, Naldi *et al* [2003] for a discussion. An equity tranche investor can be shown to be long the default correlation between the credits in the underlying CDS index while a senior tranche investor is short this default correlation. Hence an equity tranche will increase in value and a senior tranche will fall in value if the default correlation of the underlying CDS index increases.

Before the advent of standard tranches, dealers looked to historical measures of default correlation. One widespread approach was to proxy the asset return correlation of latent variable models with the correlation of historical equity market returns. For a discussion of such models see O'Kane, Naldi *et al* [2003]. What has changed recently is that by observing the market prices of synthetic CDO tranches, we can begin to extract information about market-implied rather than historical default correlation.

[1] *We would like to thank Wenjun Ruan, Saurav Sen, Minh Trinh and Lutz Schloegl for discussions and comments.*
[2] *Note that the convention for quoting prices is different for equity tranches. From Table 1 we can see that an investor who goes long the credit risk of the 0-3% equity tranche receives an upfront payment of 37.125 percent plus a running annual spread of 500bp. An investor who buys the 3-6% tranche receives an annualised spread of 259.5 bp (paid in quarterly instalments).*

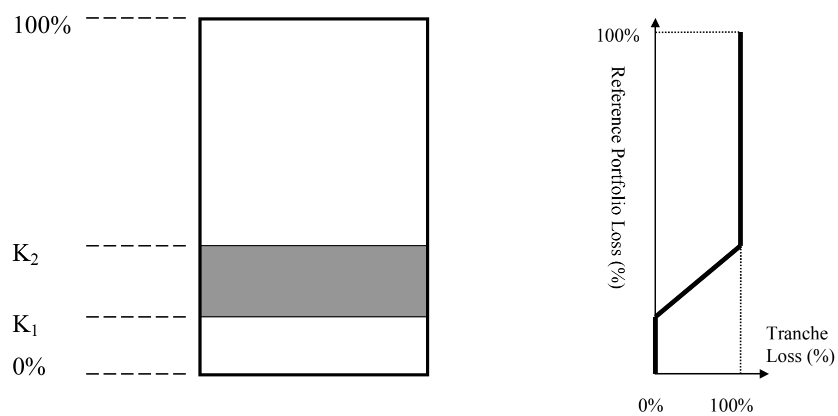
Initially, the market focused on compound correlation as the standard convention. More recently, dealers have begun to use base correlation instead. The main aim of this paper is to define, describe and compare these different measures of implied correlation. We begin with compound correlation.

COMPOUND CORRELATION

The first way to calculate an implied tranche correlation is to calculate the flat correlation that reprices each tranche to fit market prices. This method computes what is known as compound correlation.

We begin by defining a tranche simply in terms of its lower and upper "strikes", which we denote by K_1 and K_2. These are expressed as a percentage of the total notional of the reference portfolio. The lower strike is traditionally referred to as the tranche subordination or attachment point. The upper strike is referred to as the detachment point. The tranche loss is shown as a function of the percentage loss on the reference portfolio notional.

Figure 2. A mezzanine tranche with subordination (lower strike) K_1 and upper strike K_2



Source: Lehman Brothers.

To calculate compound correlation we have to assume a mathematical framework for linking the defaults of all of the different assets in the reference collateral. The standard way of doing this is to use the Large Homogeneous Portfolio (LHP) model. For a derivation see Appendix A. The main modelling assumptions are:

1. The reference portfolio is homogeneous so that all assets share the same pairwise correlation, default probability and recovery rate.
2. The number of assets in the reference portfolio tends to infinity (see discussion below).
3. The default dependency structure is based on a Gaussian copula model.
4. Each tranche is priced off a single flat correlation (the compound correlation of the tranche).

Assumption (1) means that we model the actual reference portfolio as a portfolio of homogeneous assets each with the average spread and recovery rate of the actual reference portfolio. For standard tranches, it means that we assume that the spread and recovery rate of the individual names in the portfolio are the same as the index. This has the advantage that we do not need to exchange information about the individual CDS spreads and recovery rates of each name in the reference portfolio.

Assumption (2) means that the portfolio is infinitely granular so that all idiosyncratic risk has been diversified away. This has the advantage that it enables us to write a simple analytical expression for the tranche price and makes the calculation of the implied correlation very fast.

Assumptions (1), (3) and (4) taken together mean that the dependency structure for each tranche is characterised by a single correlation number. We can therefore solve for the compound correlation from one observed price.

Calculating Compound Correlation

Given a tranche denoted by its lower and upper strikes K_1 and K_2, its present value today, time *t,* is given by:

$$PV_{tranche}\left(K_1, K_2, S_{K_1,K_2}, \rho_{K_1,K_2}\right) = U_{K_1,K_2} + S_{K_1,K_2}\sum_{n=1}^{N} Q_{K_1,K_2}(t_n)\Delta_n Z(t_n) - \sum_{m=1}^{M}\left(Q_{K_1,K_2}(t_{m-1}) - Q_{K_1,K_2}(t_m)\right)Z(t_m) \tag{1}$$

where

U_{K_1,K_2} is the tranche upfront payment,

S_{K_1,K_2} is the tranche contractual spread at issuance,

Δ_n is the accrual period between times t_{n-1} and t_n, usually paid quarterly, Actual 360,

$Z(t)$ is the LIBOR discount factor to time t.

The third term of equation (1) is the present value of the protection leg. Calculation of this involves an integration over time, which is usually discretised to quarterly time steps.

We define the tranche "survival probability" as follows:

$$Q_{K_1,K_2}(t) = 1 - \frac{\mathbf{E}^{LHP}_{\rho(K_1,K_2)}[Min(L(t),K_2)] - \mathbf{E}^{LHP}_{\rho(K_1,K_2)}[Min(L(t),K_1)]}{K_2 - K_1} \tag{2}$$

This survival probability is a measure of the expected percentage notional of the tranche remaining at some time t. The expectation is done using the Gaussian copula LHP model assuming a flat correlation as follows:

$$\mathbf{E}^{LHP}_{\rho}[Min(L(T),K)] = K\Phi(A) + (1-R)N\Phi_{2,-\sqrt{\rho}}(C,-A) \tag{3}$$

where

N is the portfolio notional,

R is the average recovery rate of the reference portfolio,

$C = \Phi^{-1}(p(t))$ is the default threshold for the underlying reference portfolio,

$p(t)$ is the average cumulative default probability to time t of the issuers in the underlying reference portfolio ,

$\Phi(x)$ is the cumulative normal function,

$$A = \frac{1}{\sqrt{\rho}}\left(C - \sqrt{1-\rho}.\Phi^{-1}\left(\frac{K}{N(1-R)}\right)\right)$$

ρ is the average pairwise asset correlation of the issuers in the reference portfolio,

$\Phi_{2,\rho}(x, y)$ is the cumulative bivariate normal with correlation coefficient ρ.

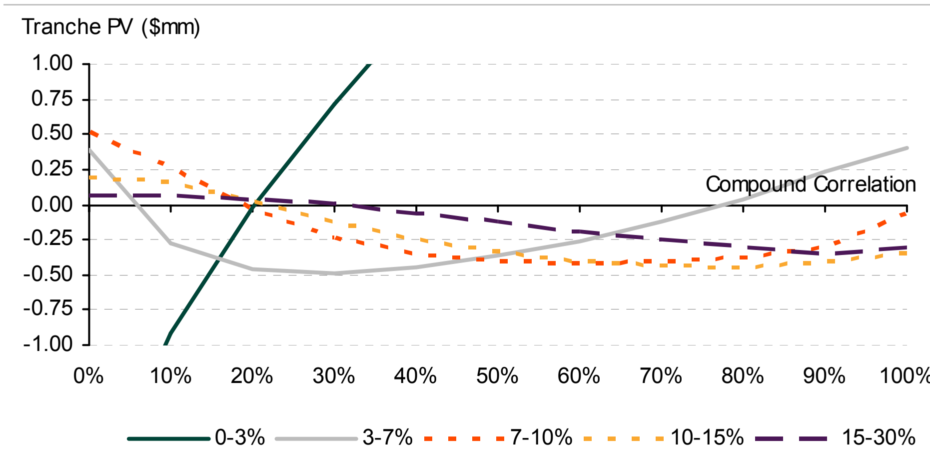
To calculate the compound correlation of a market tranche, we set the contractual spread equal to the observed market quote and, by definition, the present value of the tranche should be equal to zero:

$$PV_{tranche}\left(K_1, K_2, S_{K_1,K_2}, \rho_{K_1,K_2}\right) = 0$$

and we solve for $\rho = \rho_{K_1,K_2}$.

Solving this equation is straightforward, requiring a simple one-dimensional root searching algorithm. This works fine in almost all cases. However, there is sometimes a problem in that either we cannot find a root or that we get two roots. Why this is the case is shown in Figure 3 where we have plotted the present value of the five CDX tranches as a function of the compound correlation.

Figure 3. The present value of the five standard CDX tranches with different compound correlations – from the perspective of a protection seller (investor)



Source: Lehman Brothers.

As expected, we see in Figure 3 that the equity tranche investor is long compound correlation while the senior tranche investor is short compound correlation.

We see that for all tranches, there is a single solution at which the PV is zero, except the 3-7% mezzanine which has two solutions at 5% and 78% compound correlation.

For mezzanine investors, the relationship between changes in the tranche PV and changes in correlation itself changes with correlation. At low correlations, mezzanine tranche investors are short correlation while at high correlations, mezzanine tranche investors are long correlation. Clearly, the two compound correlation solutions of 5% and 78%, while producing the same tranche PV, imply radically different risk profiles. Typically, we choose the lower correlation as it is closer to the other compound correlation solutions for adjacent tranches and because it better fits historical observations of equity return correlation which are widely used as a proxy for asset return correlation. See O'Kane, Naldi *et al* [2003] for a discussion.

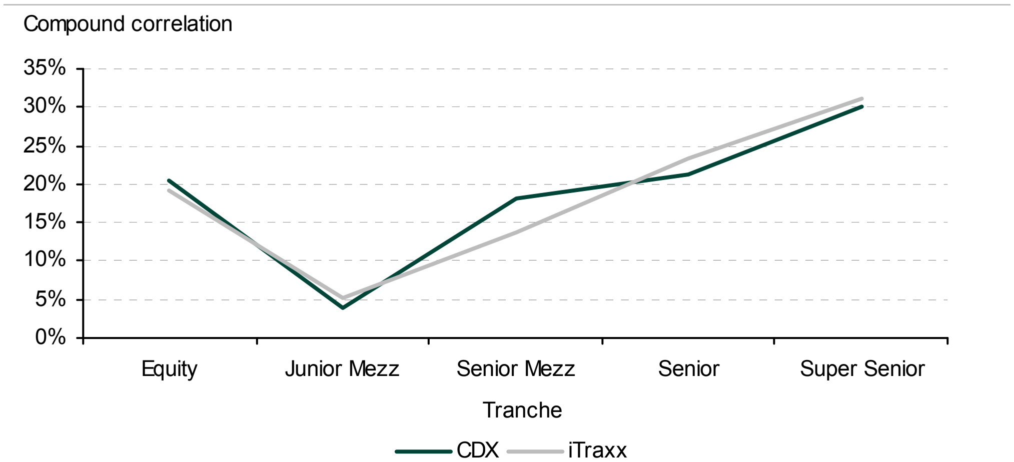
If we keep the reference portfolio spreads and recovery rates fixed and increase observed contractual tranche spreads, all of the curves in Figure 3 are shifted upwards. This causes the compound correlation of the equity tranche to decrease and the compound correlation of the other tranches to increase. If the tranche spreads are sufficiently large, there may not be a solution for the compound correlation of the mezzanine tranche. Equally, if tranche spreads fall, all of the curves in Figure 3 are shifted downwards. This can cause the mezzanine tranche to lose one, and ultimately both, of its solutions for compound correlation.

Explaining the Smile

The compound correlation curve is shown in Figure 4 for both the CDX and iTraxx tranches. The shape of the compound correlation has become known as the correlation "smile". This is because the compound correlation is higher for the equity and senior tranches than it is for the mezzanine tranches.

What is interesting is that this smile shape is common to both CDX and iTraxx tranches and has persisted through the period of the last year during which these tranche prices have been available. Although the similarity in the actual values and the shapes is apparent, care must be taken when comparing CDX and iTraxx since the standard tranches have different attachment points and widths.

Figure 4. The compound correlation curve for CDX Investment Grade NA Series 3 and iTraxx Europe Series 2 indices, for 13 October 2004



Source: Lehman Brothers.

Compound correlation is clearly not the same for all tranches. This simply tells us that a Gaussian copula does not capture the dependency structure implied by market CDO tranche prices. This is not a surprise – indeed, it would be amazing if we could exactly fit the market-implied dependency structure of a portfolio of 125 different credits with a Gaussian copula characterised by a single correlation number.

What is interesting is the smile shape of the compound correlation for CDX IG and iTraxx tranches. This smile is driven by market prices – prices at which buyers and sellers of tranche protection are willing to trade. They therefore contain a mixture of effects, including concerns about systemic versus idiosyncratic credit risk, fear of principal versus mark-to-market losses, liquidity effects, and supply and demand for certain tranches.

Starting with the equity tranche, we note that the compound correlation is typically about 20%. This is actually lower than the 25-30% correlations found using historical equity returns, and since the equity tranche investor is long correlation, this means that the equity investor receives a higher spread than historical correlations would imply. One reason for this effect may be dealer correlation desks paying above the theoretical model price in order to hedge the risks in their correlation books created by selling mezzanine tranches.

At the mezzanine tranche, we see the compound correlation fall below the compound correlation of the equity tranche. As the mezzanine investor has a short correlation position this is simply reflecting the fact that the market is paying a lower spread than historical correlations would imply. This is probably due to the considerable demand for mezzanine tranches in the market. The size of the decline in the compound correlation to values in the range of 5-10% is due mainly to the low correlation sensitivity of the mezzanine tranche, ie, a large reduction in the compound correlation is required in order to fit this lower spread.

The senior tranche compound correlation is the simplest to explain. We see that it has a value similar to the historical average of 25-30%.

The CDO tranche market is segmented, with banks and hedge funds buying equity tranches, retail investors buying mezzanine tranches, and insurance companies focusing on senior tranches. This may explain why there has been little relative movement of tranche compound correlations, ie, few market players are willing or able to put on significant trades across the capital structure in an attempt to take advantage of any perceived richness or cheapness.

We list the advantages and disadvantages of compound correlation in Figure 5 below.

Figure 5. Advantages and disadvantages of compound correlation

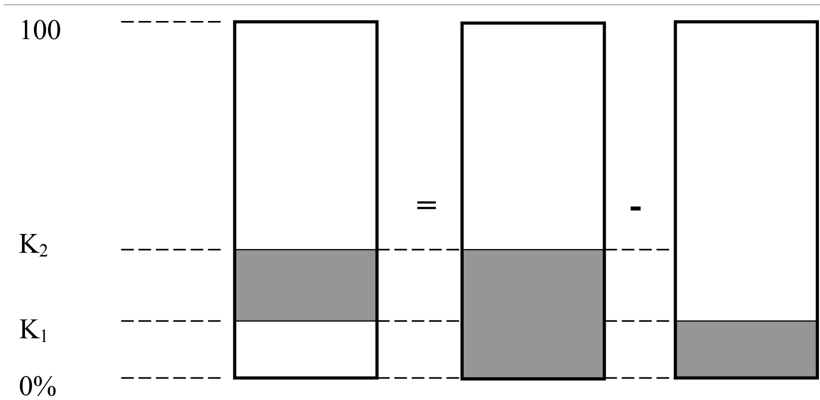
Advantages	Disadvantages
• Only one number per tranche. • Can be compared directly to estimates of historical asset correlation and can also be mapped easily to default correlation. Therefore it is quite intuitive. • We can calculate the compound correlation for a tranche without information about the pricing of the other tranches.	• Sometimes there are two solutions for the mezzanine tranche. One must be chosen as the more economically sensible solution. • Not arbitrage-free across the capital structure, ie, the sum of the protection legs of the tranches does not equal the sum of the protection legs of the underlying CDS portfolio. • Not possible to extend compound correlation to the pricing of tranches on standard indices with non-standard strikes

BASE CORRELATION

The fundamental idea behind the concept of base correlation is that we decompose all tranches into combinations of base tranches, where **a base tranche is simply another name for an equity tranche**. The word “base” comes from the fact that the subordination of a base tranche is always zero, ie, it is attached to the base of the loss distribution.

Consider a mezzanine tranche with lower and upper attachment points K_1 and K_2. This is equivalent to being short the equity tranche with subordination K_1 and long the equity tranche with subordination K_2. This is shown in Figure 6.

Figure 6. A mezzanine tranche with strikes K_1 and K_2 decomposed into long a K_2 strike "base" tranche and short a K_1 strike "base" tranche



Source: Lehman Brothers.

Whereas for compound correlation we calculate the flat correlation required for each tranche to match the market spreads, **for base correlation we value any tranche as the difference between two base tranches**. We then calculate the flat correlation required for each base tranche so that we match the observed market spreads.

Calculating Base Correlation

For a tranche with lower and upper attachment points K_1 and K_2 with a tranche market spread, S, we can compute the net present value of the protection and the premium legs. At inception, this must by definition equal 0.

We can do this by expressing the mezzanine tranche as the difference between two base tranches, as shown in Figure 6, where we allow each base tranche to be priced *using a different flat correlation*. This means that the base correlation is only a function of one parameter, the width of the base tranche. Compare this with compound correlation, which is therefore a tranche-specific function of both the lower and upper strike of the tranche.

We solve for the base correlation using a recursive technique called bootstrapping, ie, we use the information from the first tranche to solve for the second tranche, and so on. Contrast this with the case of compound correlation, where each tranche implied correlation is solved for independently of the other tranches. The procedure for base correlation is:

1. We solve for the equity tranche first by finding the value of ρ_{K_1} which solves the equation:

   $$0 = PV_{tranche}\left(0, K_1, S_{0,K_1}, \rho_{K_1}\right)$$

   where

   $$\begin{aligned} & PV_{tranche}\left(0, K_1, S_{0,K_1}, \rho_{K_1}\right) \\ &= U_{0,K_1} + S_{0,K_1} PV_{premium}\left(0, K_1, \rho_{K_1}\right) - PV_{protection}\left(0, K_1, \rho_{K_1}\right) \\ &= U_{0,K_1} + S_{0,K_1} \sum_{n=1}^{N} Q_{0,K_1}(t_n)\Delta_n Z(t_n) - \sum_{m=1}^{M}\left(Q_{0,K_1}(t_{m-1}) - Q_{0,K_1}(t_m)\right) Z(t_m) \end{aligned}$$

   and the base tranche survival probability is given by:

$$Q_{0,K_1}(t) = 1 - \frac{\mathbf{E}^{LHP}_{\rho_{K_1}}[Min(L(t), K_1)]}{K_1}.$$

This is equivalent to what we did in the calculation of the compound correlation for the equity tranche. Indeed, the base and compound correlation measures are identical for an equity tranche.

2. For the next tranche, we solve for the value of ρ_{K_2} that solves:

$$\begin{aligned} 0 &= PV_{tranche}\left(K_1, K_2, S_{K_1,K_2}, \rho_{K_1}, \rho_{K_2}\right) \\ &= PV_{tranche}\left(0, K_2, S_{K_1,K_2}, \rho_{K_2}\right) - \underbrace{PV_{tranche}\left(0, K_1, S_{K_1,K_2}, \rho_{K_1}\right)}_{\text{known and fixed as } \rho_{K2} \text{ varies}} \end{aligned}$$

The second term is the PV of the first base tranche calculated using the spread from the second tranche. Breaking the tranche PV equation into the premium and protection legs, we have:

$$\begin{aligned} 0 = &\left(S_{K_1,K_2} PV_{premium}\left(0, K_2, \rho_{K_2}\right) - PV_{protection}\left(0, K_2, \rho_{K_2}\right)\right) \\ &- \left(S_{K_1,K_2} PV_{premium}\left(0, K_1, \rho_{K_1}\right) - PV_{protection}\left(0, K_1, \rho_{K_1}\right)\right) \end{aligned}$$

which can be written in terms of tranche survival probabilities as before. We have:

$$S_{K_1,K_2} \sum_{n=1}^{N} Q_{K_1,K_2}(t_n) Z(t_n) - \sum_{m=1}^{M} \left(Q_{K_1,K_2}(t_{m-1}) - Q_{K_1,K_2}(t_m)\right) Z(t_m) = 0$$

where

$$Q_{K_1,K_2}(t) = 1 - \frac{\mathbf{E}^{LHP}_{\rho_{K_2}}[Min(L(t), K_2)] - \mathbf{E}^{LHP}_{\rho_{K_1}}[Min(L(t), K_1)]}{K_2 - K_1} \tag{4}$$

This tranche survival probability is fundamentally different from equation (2), the tranche survival probability for the compound correlation. In equation (2) we use the same correlation for both strikes. In equation (4) we take the expectation for the different base tranches at different correlations. The base correlation is linked to the strike of the base tranche. Since we already know ρ_{K_1} from the previous step, we have one equation with one unknown ρ_{K_2} and we can solve for this by using a one-dimensional root search.

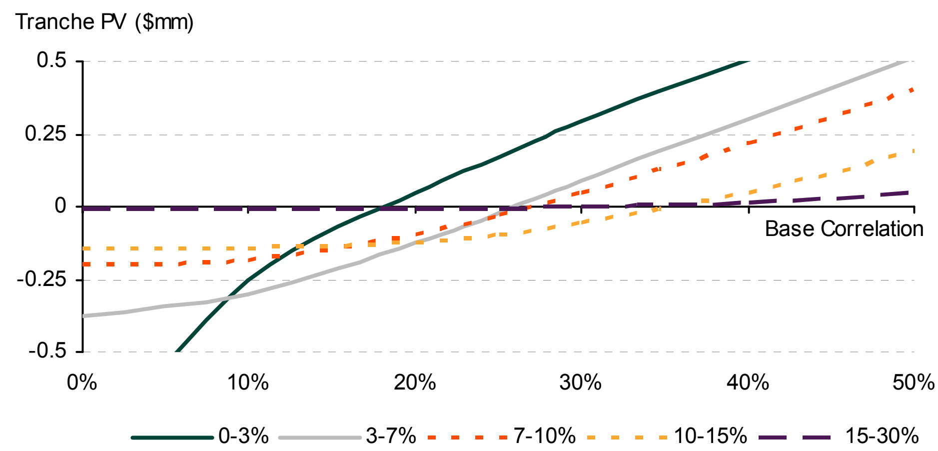
3. We then continue in the same manner through the higher tranches.

Figure 7 shows the tranche PVs for the five standard CDX tranches as a function of base correlation. To generate this graph, we first plotted the PV of the equity tranche by varying the 3% strike base correlation. We then selected the solution for the 3% strike base correlation at which the 0-3% tranche PV is zero. We then calculated the PV of the 3-7% tranche using the solution for the 3% strike base correlation and for different values of the 7% strike base correlation. This produced the line for the 3-7% tranche. The solution for the 7% strike base correlation is the value that gives a 3-7% tranche PV of zero. We then plotted the 7-10% tranche PV using the solution for the 7% tranche by varying the 10% strike base correlation, and so on up the capital structure.

What we find is that all of the tranche PVs are a monotonic and increasing function of the base correlation. This means that there is only one solution, or in certain cases no solution.

We therefore avoid the two-solution problem that we had when determining the compound correlation on the 3-6% mezzanine tranche.

Figure 7. PVs of the five standard CDX tranches as we change the base correlation of the upper base tranche, while using the correct value of base correlation for the lower base tranche



Source: Lehman Brothers.

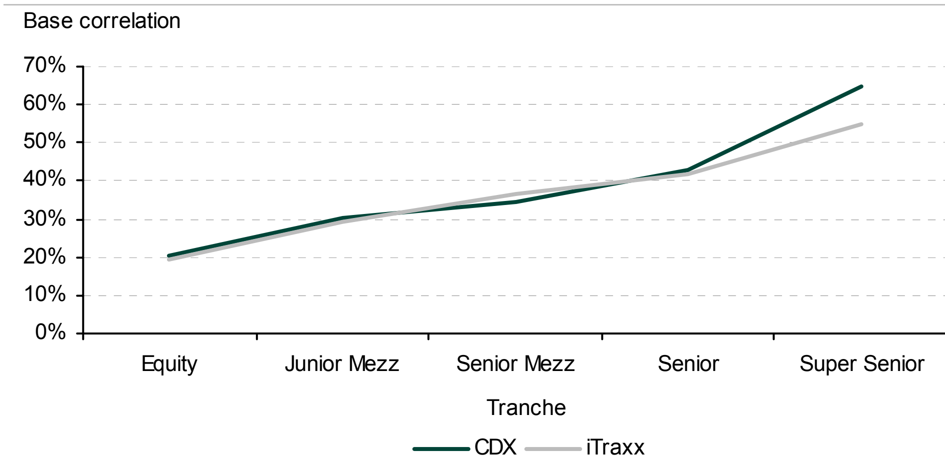
The reason why all the tranche PVs increase as base correlation increases is that for each tranche PV, we are fixing the lower strike base correlation and increasing only that of the upper strike. Hence the only changing component of the tranche PV is due to changes in the PV of the upper base tranche. As all base tranches are equity tranches which are long correlation, all the tranche PVs are increasing functions of their upper strike base correlation.

In some cases, base correlation can have trouble finding a solution for the senior tranche. This may reflect an inconsistency between the market spreads paid on the tranches and the spread paid on the underlying CDS index within the base correlation modelling framework. However it may also reflect a more serious violation of no-arbitrage constraints as discussed later.

Base correlation produces a skew

Figure 8 shows the base correlation calculated for the CDX and iTraxx linked tranches. As with compound correlation, we see considerable similarity between the indices, in both the shapes and levels of the implied base correlations. Instead of a smile shape, we get a "skew". As a result, we find that people who prefer compound correlation speaking of a "smile" and those who favour base correlation speaking of a "skew".

Figure 8. The base correlation curve for CDX Investment Grade NA Series 3 and iTraxx Europe Series 2 indices, for 13 October 2004 for each tranche



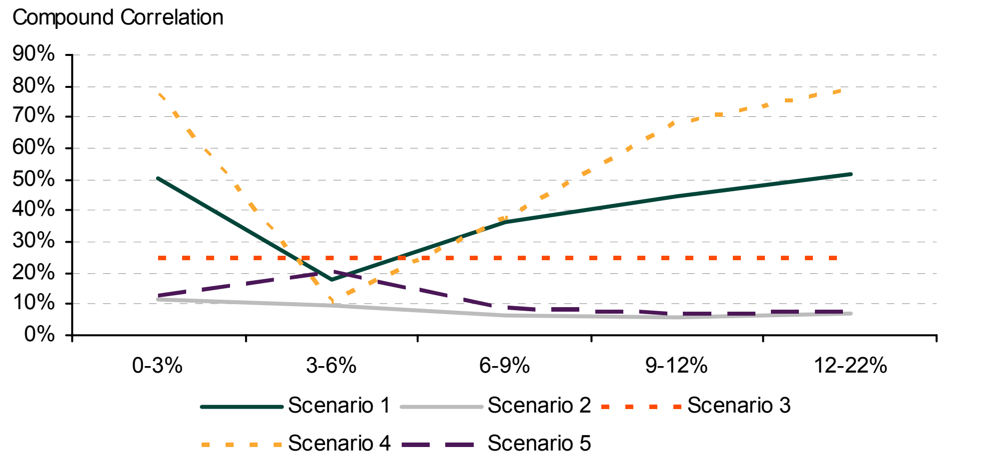
Source: Lehman Brothers.

The base correlation and compound correlation for the 0-3% equity tranche are the same. This follows from the definition of both implied correlation measures.

The next tranche is the 3-7%. If we price the 3-7% junior mezzanine tranche with the 3% strike base correlation implied by the 0-3% tranche spread, then we find the tranche PV is generally negative, ie, the PV of the spread being received on the premium leg is not sufficient to cover the PV of the protection leg. This is due to the apparent low spread paid on mezzanine tranches, as discussed earlier. We can only reduce the value of the 3-7% protection leg by lowering the value of the 0-7% base tranche. As this is an equity tranche, we need to increase the 7% strike base correlation above the 3% strike base correlation. The base correlation for subsequent tranches tends to be higher still because it has to compensate for the previous high values of base correlation on which it depends. As a result, we have an upward sloping base correlation curve.

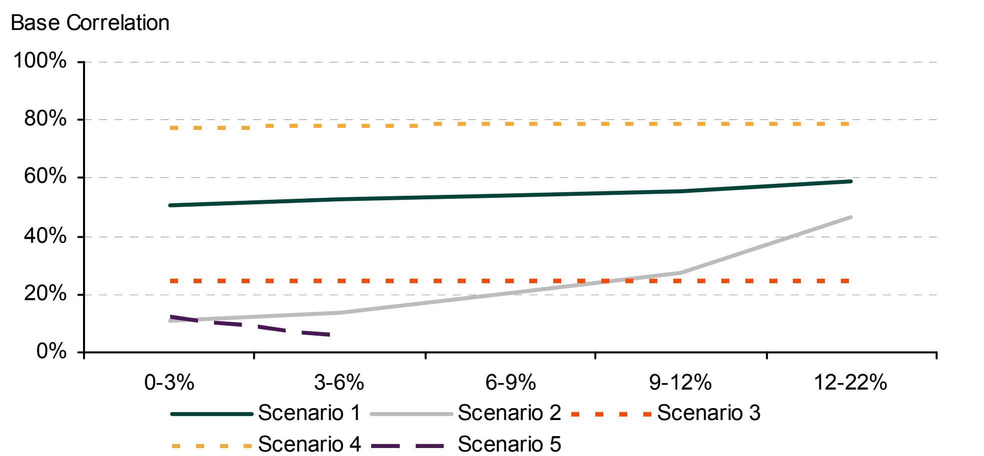
While Figure 8 shows the current shape of the base correlation curve in the market, other shapes are very possible. To demonstrate this, and the mapping to the equivalent compound correlation curve, we generated a number of different tranche spread scenarios. This was done in a way that guaranteed that the resultant spreads were arbitrage-free, see later for a discussion of arbitrage-free skew models. All these scenarios were created for a reference portfolio in which we held the spreads of all of the underlying assets fixed. By changing the underlying dependency structure, we were able to produce different shapes for compound and base correlation. Figures 9 and 10 show the corresponding compound and base correlation for the set of tranche spreads shown in Figure 11. This illustrates the relationship between the two and gives an intuitive feel for how base correlation varies depending on the shape of compound correlation and vice versa.

Figure 9. Compound correlation for the different tranche spread scenarios, based on the iTraxx Europe Series 2 index (see Figure 11 for details)



Source: Lehman Brothers.

Figure 10. Base correlation for the different tranche spread scenarios, based on the iTraxx Europe Series 2 index (see Figure 11 for details)



Source: Lehman Brothers.

Figure 11. The tranche spreads associated with the five different scenarios for iTraxx Europe Series 2, which lead to the different shapes for the compound and base correlation curves shown in Figures 9 and 10

		Running Spread (bp)				
		Scenario				
Tranche	**Lower-Upper strike**	**1**	**2**	**3**	**4**	**5**
Equity[3]	0-3%	646	1396	1101	337	1360
Junior Mezzanine	3-6%	224	160	234	174	233
Senior Mezzanine	6-9%	123	8	86	126	18
Senior	9-12%	82	0.3	36	106	0.6
Super Senior	12-22%	44	0.02	8.2	72	0.03

Source: Lehman Brothers.

While the values calculated for compound and base correlation for the equity tranche are the same, the values for the other tranches vary. For different tranches the change in base correlation tends to be in the opposite direction to that in compound correlation. Let us discuss the different scenarios:

1. The low equity spread and high senior spread mean there is a lot of "smile" in the compound correlation curve. The base correlation curve is upward sloping, starting at a higher correlation due to the low spread of the equity tranche.

2. The equity tranche spread has increased and senior spreads are close to zero. As a result the compound correlation for equity falls and the low senior spreads mean that it does not rise again. In base correlation, we see a steep upward sloping curve starting from a low level.

3. Both compound correlation and base correlation curves are flat as the tranche spreads were generated by the LHP model, using the same correlation for all tranches.

4. The compound correlation curve is extremely "smiley". This is because of the extremely low equity spread combined with significant senior spreads. The base correlation curve has to start off at the high equity compound correlation. The slope of the curve is then squeezed into the remaining gap.

5. High equity spreads and high mezzanine spreads make the compound correlation curve inverted. Base correlation is unable to find a solution for the senior tranches, which illustrates how base correlation can fail to find a solution to reprice the senior tranches even though compound correlation can.

More information: each tranche is characterised by two correlations

The way that base correlation is defined means that the base correlation calculated for each tranche is linked to the base correlation of the tranche below. This is clear from the bootstrapping methodology that we have to employ to calculate base correlation. A 7-10% tranche "knows" the value of the base correlation we computed from the 3-7% tranche, which in turn "knows" the value of the base correlation we computed from the 0-3% tranche.

Indeed, the fact that each tranche survival probability curve is explicitly a function of two base correlations implies that each tranche "knows" more about the shape of the underlying market implied loss distribution than when using compound correlation.

[3] *Here we are quoting the equity tranche with purely a running spread and no upfront payment.*

This does not mean that a base correlation curve contains more information than a compound correlation curve. Taking all points together, both actually contain the same amount of information, ie, the market prices of the different tranches. The point is that individually, each tranche is represented by two base correlation numbers, and so the modelling of that tranche embeds more information about the pricing of the other tranches than in the compound correlation framework.

Base correlation conserves expected loss and delta

A basic arbitrage-free requirement of any CDO pricing model should be that the sum of the protection legs of all of the CDO tranches should equal the sum of the protection legs of all of the underlying credit default swaps.

The easiest way to see why this is an arbitrage-free requirement is to consider what would happen if all the tranches and underlying CDS were quoted in upfront premium terms. A trade in which we sold protection on the whole capital structure of tranches and bought protection on each name in the reference portfolio would give us a risk-free position. Any default on a name in the reference portfolio would incur a loss on a tranche which would be exactly offset by a payment on the CDS linked to that name. Arbitrage requirements mean that the initial cost of this strategy should be zero. As the upfront price of a CDS or tranche is simply the present value of the protection leg, the result follows.

A simple way to show that this arbitrage-free requirement holds for base correlation is to take a simple capital structure consisting of a 0-5% equity tranche, 5-20% mezzanine tranche, and a 20-100% senior tranche. For simplicity we assume that interest rates are zero and that all losses are taken at the maturity date T. Hence, the expected loss on the individual tranches is given by:

$$EL(0,5\%) = E^{LHP}_{\rho(5\%)}[Min(L(T),5\%)] - E^{LHP}_{\rho(0\%)}[Min(L(T),0)]$$

$$EL(5\%,20\%) = E^{LHP}_{\rho(20\%)}[Min(L(T),20\%)] - E^{LHP}_{\rho(5\%)}[Min(L(T),5\%)]$$

$$EL(20\%,100\%) = E^{LHP}_{\rho(100\%)}[Min(L(T),100\%)] - E^{LHP}_{\rho(20\%)}[Min(L(T),20\%)]$$

Summing these, we find that all the intermediate terms cancel out and we get:

$$\begin{aligned} &EL(0,5\%) + EL(5\%,20\%) + EL(20\%,100\%) \\ &= E^{LHP}_{\rho(100\%)}[Min(L(T),100\%)] - E^{LHP}_{\rho(0\%)}[Min(L(T),0)] \end{aligned}$$

Since

$$0\% \leq L(T) \leq 100\%,$$

we have

$$EL(0,5\%) + EL(5\%,20\%) + EL(20\%,100\%) = E[L(T)]$$

We see that the sum of the expected tranche losses is equal to the expected loss of the underlying portfolio which is not a correlation sensitive quantity. It can be shown that this result holds for base correlation even when interest rates are non-zero and losses are taken as they occur.

This important result would not hold for compound correlation because the strike of 5% would be priced at the 0-5% compound correlation in the equity tranche and at the 5-20% compound correlation in the mezzanine tranche. Therefore, the intermediate terms with the

5% strike and the 20% strike would not cancel out and so the sum of the expected tranche losses would not be the same as the expected loss of the underlying portfolio.

The delta of a tranche to any credit is simply the sensitivity of the value of the tranche with respect to a shift in the CDS curve for that specific credit. Since the expected loss is conserved, we would also expect the delta to sum correctly as we look across the entire capital structure, ie, a dealer wishing to hedge a correlation book by building the capital structure around a sold tranche would have more confidence in the base correlation delta than the compound correlation delta.

Interpolating Non-Standard Strikes

Base correlation is a powerful concept because it transforms the compound correlation ρ_{K_1,K_2}, which is a two-dimensional correlation, into the one-dimensional base correlation measure ρ_{K_1}. This reduction of the dimensionality of the correlation parameters enables a simple mechanism for pricing non-standard strikes on the standard indices. Consider the following example.

Suppose we want to price a 6-9% tranche of the CDX portfolio. Using base correlation the price will be the difference between the price of a 0-9% tranche and that of a 0-6% tranche. These base tranches are priced used the base correlation for the 9% and 6% strikes. However, the market information only gives us the base correlations for the 3%, 7% and 10% strikes. It is worth noting that if we plot these base correlations as a function of the strikes, the resulting shape is monotonic, increasing with strike, and close to linear. This suggests that we may be able to interpolate values for the 6% and 9% strike with a reasonable degree of confidence.

For example, suppose the market implied that the base correlation for the 3% strike is 20%, for the 7% strike is 28% and for the 10% strike is 34%. Then using linear interpolation between these values, the base correlation for a 6% tranche is:

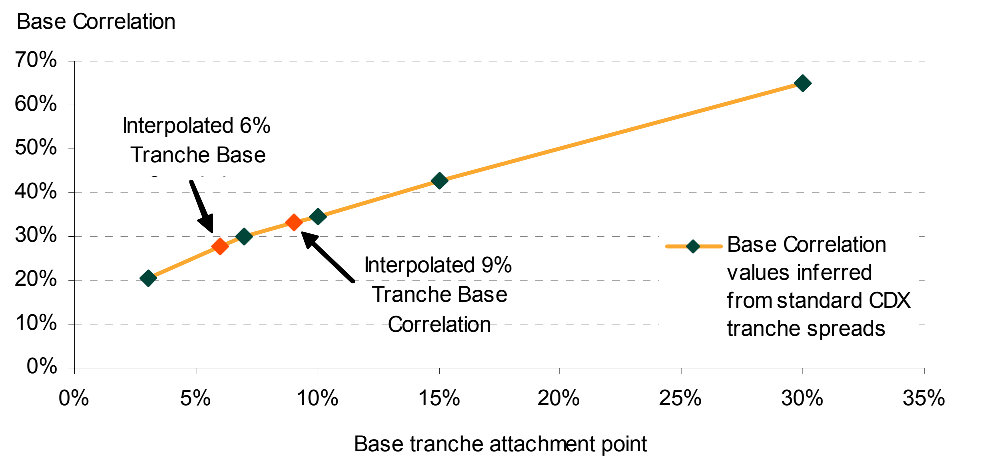
$$\frac{1}{4}\times 20\% + \frac{3}{4}\times 28\% = 26\%$$

and the base correlation for the 9% tranche is:

$$\frac{1}{3}\times 28\% + \frac{2}{3}\times 34\% = 32\%$$

We can then price the 6-9% CDX tranche. Figure 12 shows the values of base correlation that we infer from the market for CDX tranches on 13 October 2004 and the values of base correlation we would interpolate for 6% and 9%.

Figure 12. Base correlation for the five standard CDX tranches and the interpolated base correlations for a 6-9% tranche



Source: Lehman Brothers.

The spread implied by these interpolated base correlations for the 6-9% tranche is 118bp, which compares with 260bp for the 3-7% tranche and 101bp for the 7-10% tranche.

Care must be taken to ensure that the interpolation methodology does not introduce any arbitrage.

Not a proper model of the correlation skew

Although base correlation clearly has more attractive properties than compound correlation in terms of its conservation of the expected losses and tranche deltas, it is not a proper model of the correlation skew. By "proper model", we mean a model that allows us to price and risk-manage all of the tranches on the same CDS index using the same underlying portfolio loss distribution which has been generated in an arbitrage-free manner. An arbitrage-free model satisfies the following conditions:

1. The tranche survival probability, ie, the expected outstanding notional of a tranche must be a monotonically decreasing function of the horizon date. This must be true for all tranches.

2. The absolute value of the expected loss of an equity tranche must be a monotonically increasing function of the width of the equity tranche. This must be true at all horizon dates.

3. The expected loss of a tranche with strikes K_1 and K_2 plus the expected loss of a tranche with strikes K_2 and K_3 must equal the expected loss of a K_1 to K_3 tranche. This must be true at all horizon dates.

Provided that the same correlation is used for different time horizons, both compound and base correlation guarantee that (1) holds. Neither compound nor base correlation guarantees (2), and so both fail the requirements of an arbitrage-free model. However, base correlation does guarantee (3), which compound correlation does not.

In addition to these arbitrage-free requirements, a proper model would also allow us to extend the pricing of standard tranches to the pricing of tranches on non-standard portfolios. For example, with base correlation it is not clear how one would price, say, a CDO on a mixed portfolio of CDX and iTraxx indices. Nor is it clear how base correlation should change as the spread of the CDS index changes. Capturing this cross-dynamic in line with empirical observations would definitely be a desirable feature of a proper model.

Since it satisfies requirement (3), we believe that base correlation is a safer approach than compound correlation. However, we do not believe that it fulfils the requirements of those looking for a model of the correlation skew. This remains an active area of research.

We list the advantages and disadvantages of base correlation in Figure 13.

Figure 13. Advantages and disadvantages of base correlation

Advantages	Disadvantages
• Each tranche is characterised by two base correlations. This embeds more information about the market-implied loss distribution.	• Calculating the base correlation for any tranche requires that the market prices for all of the more junior tranches are known.
• There is always either one solution or no solution. The situation of having two solutions never arises.	• Difficult to build intuition about base correlation and to relate it directly to historical estimates of correlation.
• Conserves expected loss and delta across the capital structure.	• Not an arbitrage-free model of the correlation skew.
• Extends to the pricing of tranches with non-standard strikes.	

CONCLUSION

The Gaussian Copula LHP model has become the Black-Scholes model of the CDO tranche market. However, there are currently two ways of quoting the implied correlation – compound and base correlation. We have defined, discussed and compared these approaches, and shown that base correlation has a number of distinct advantages over compound correlation. These include the fact that it extends naturally to the pricing of non-standard tranches on the liquid indices, and correctly conserves the expected loss and delta across the capital structure. It is also simple to implement. However, we emphasise that base correlation is not a proper model of correlation skew.

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APPENDIX A: LHP FORMULA FOR PRICING A CDO TRANCHE

Under the Gaussian copula LHP model, the assets of the n issuers are modelled by standard normal random variables with a common correlation ρ, probability of default p, notional N and recovery rate R. Default is said to occur if the asset value of an issuer i, Z_i falls below the default threshold C, which is given by $C = \Phi^{-1}(p)$. Under this model we can write the asset value of each issuer as a market factor and a specific component, as follows:

$$Z_i = \sqrt{\rho} Z + \sqrt{1-\rho}\, \varepsilon_i$$

where Z and the ε_i s are independent standard normal random variables. We can then write the conditional probability of asset i defaulting as:

$$\Pi(Z) = \Phi\left(\frac{C - \sqrt{\rho} Z}{\sqrt{1-\rho}}\right)$$

The conditional loss will therefore be the sum of n independent identically distributed random variables with an expected value of $\Pi(Z)(1-R)N$. Under the LHP model, the assumption is that the number of issuers is sufficiently large so that the law of large numbers causes the conditional loss to be exactly $\Pi(Z)(1-R)N$. This is equivalent to saying that all the specific risk of default has been diversified away.

So under the Gaussian copula LHP model, the probability of the loss being greater than some level K is:

$$P[L > K] = \mathbf{E}\left[1_{\{\Pi(Z)(1-R)N > K\}} \mid Z\right] = P[Z < A] = \Phi(A)$$

where

$$A = \frac{1}{\sqrt{\rho}}\left(C - \sqrt{1-\rho}.\Phi^{-1}\left(\frac{K}{N(1-R)}\right)\right)$$

Furthermore, we can calculate $\mathbf{E}[\min(L,K)]$ in a similar way:

$$\mathbf{E}[\min(L,K)] = \mathbf{E}[K1_{\{L>K\}} + L1_{\{L<K\}}] = K\Phi(A) + \mathbf{E}[L1_{\{L<K\}}]$$

and

$$\mathbf{E}[L1_{\{L<K\}}] = \mathbf{E}[\,\mathbf{E}[L1_{\{L<K\}} \mid Z]$$

$$= \mathbf{E}[\,\mathbf{E}[\Pi(Z)(1-R)N.1_{\{L<K\}} \mid Z]$$

$$= (1-R)N\,\mathbf{E}[\,\mathbf{E}[\Pi(Z)1_{\{Z>A\}} \mid Z]$$

$$= (1-R)N \int_A^{\infty} \Phi\left(\frac{C-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)\phi(z)dz$$

$$= (1-R)N \int_{\infty}^{-A} \Phi\left(\frac{C-\left(-\sqrt{\rho}\right)z}{\sqrt{1-\rho}}\right)\phi(z)dz$$

$$= (1-R)N\Phi_{2,-\sqrt{\rho}}(C,-A)$$

So we have:

$$\mathbf{E}[\min(L,K)] = K\Phi(A) + (1-R)N\Phi_{2,-\sqrt{\rho}}(C,-A)$$

This allows us to calculate the tranche survival probability:

$$Q_{K_1,K_2}(t) = 1 - \frac{\mathbf{E}[Min(L(t),K_2)] - \mathbf{E}[Min(L(t),K_1)]}{K_2 - K_1}$$

and the PV of a CDO tranche, which is:

$$V_{K_1,K_2}(t) = S_{K_1,K_2}\sum_{n=1}^{N} Q_{K_1,K_2}(t_n)\delta_n Z(t_n) - \sum_{m=1}^{M}\left(Q_{K_1,K_2}(t_{m-1}) - Q_{K_1,K_2}(t_m)\right)Z(t_m)$$

where

K_1 and K_2 are the attachment point and detachment point of the tranche,

S_{K_1,K_2} is the tranche contractual spread at issuance,

δ_n is the accrual period between times t_{n-1} and t_n,

$Z(t)$ is the LIBOR discount factor to time t.

Introduction to Credit Default Swaptions

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Options on credit default swaps, or default swaptions, are becoming increasingly liquid with ever narrower bid-offer spreads. We present an overview of how to analyze default swaptions and use them to implement specific credit views.

1. INTRODUCTION

Credit default swaptions, or default swaptions, are options to enter into credit default swaps (CDS). Over the past year there has been exponential growth in the notional of default swaptions traded. Most of the notional growth has been in swaptions referencing standardized CDS portfolios such as CDX and iTraxx, but there has also been substantial trading activity in single-issuer default swaptions especially those referencing large issuers such as FMCC (Ford Motor Credit Company) and GMAC (General Motors Acceptance Corporation). Today we see two-way markets in a large number of US credits.

In the next section we present an overview of standard default swaptions and introduce the market terminology. Then we discuss how to evaluate default swaption trades using scenario analysis and breakeven spreads. Section 4 is devoted to the risk measures known as the Greeks. Section 5 gives an overview of default swaption strategies investors can employ to implement their credit views. The appendix contains a discussion of the valuation of default swaptions and how to estimate historical spread volatility.

2. CONTRACTS AND TERMINOLOGY

A default swaption is an option to enter into an underlying CDS at no cost at a given option maturity. Besides the reference credit, the underlying CDS is determined by a maturity, called the swap maturity, and a contractual spread, called the strike spread or just the strike. Default swaptions are generally European with option maturities ranging from one month to one year. The swap maturity is typically five or ten years. The cost of the option is paid up front.

2.1. Payer and receiver swaptions

An option to sell protection is called a receiver swaption or a right-to-receive. An option to buy protection is a payer swaption or a right-to-pay. This terminology is familiar from interest rate swaptions where a payer swaption is an option to become fixed rate payer.

The terminal value of a default swaption depends on the mark-to-market spread on the underlying CDS at option maturity. A buyer of a receiver swaption profits when spreads tighten, because in this case the buyer will exercise the option and sell protection at a spread that is higher than the market spread at which the short protection position can be closed out. Following analogous arguments, a buyer of a payer swaption profits when spreads widen.

2.2. Knockout at default

Unless otherwise specified in the default swaption contract, the buyer of the payer swaption can also exercise the option if the underlying credit has defaulted. Options without that possibility are called knockout swaptions because they are cancelled, or knocked out, if the underlying credit defaults.

A non-knockout payer swaption may have an acceleration clause requiring option exercise immediately following a default. The acceleration clause is particularly relevant for long-term non-knockout swaptions.

The knockout feature is not relevant for receiver swaptions because they will never be exercised after a default.

2.3. Forward starting CDS

A CDS in which the premium payments do not accrue until after a given start date and which cancels, or knocks out, if default occurs before that start date, is called a forward starting CDS, or simply a forward CDS.

The forward CDS spread is the contractual spread in a forward starting CDS that has a market value of zero. It is easy to show that the forward CDS spread is $\alpha \cdot S_2 + (1-\alpha) \cdot S_1$, where $\alpha = PV01_2/(PV01_2 - PV01_1)$, and S_1 and $PV01_1$ are the market spread and PV01 for a standard CDS that matures at the start date of the forward CDS, and S_2 and $PV01_2$ are the spread and PV01 for the standard CDS that matures with the forward CDS. α depends on the entire CDS curve but is relatively insensitive to the spread level. Therefore the forward CDS spread is primarily determined by the CDS spreads to the start and maturity dates of the forward CDS.

As will be discussed in more detail in the appendix, the value of a default swaption is primarily determined by the forward CDS spread from the option maturity to the swap maturity and the volatility of that spread.

2.4. Portfolio swaptions vs single-issuer swaptions

A portfolio swaption (or a portfolio credit default swaption[1]) is an option on a portfolio of CDS. Standardized swaptions referencing the CDX and iTraxx portfolios are liquid and trade with narrow bid-offer spreads.

Standardized portfolio swaptions are non-knockout. After exercise of a portfolio payer swaption, an option buyer holds a long protection CDS on each of the credits in the portfolio, including those that have defaulted between the trade date and the option maturity. The CDS on the defaulted credits can then immediately be settled using standard physical settlement. Similarly, after exercise of a receiver swaption, the option buyer holds short protection CDS on all the credits including those that have defaulted. The option seller can then settle the CDS on the defaulted credits and receive protection payments. As opposed to single-issuer credit default swaps, the knockout feature matters for both payer and receiver portfolio swaptions[2].

2.5. Callable default swaps

A receiver swaption can be embedded into a CDS to produce a callable default swap. The only difference between a callable CDS and a standard CDS is that the protection buyer has the option to cancel the protection at a given call date. The call date corresponds to the option maturity date of the embedded receiver swaption. The strike of the receiver swaption is the callable CDS spread.

The callable CDS spread is chosen so that no upfront exchange of cash is required. Obviously, the callable CDS spread must be higher than the same maturity standard non-callable CDS.

[1] *Portfolio swaptions are also referred to as index options by some market participants.*
[2] *For more on portfolio swaptions see Pedersen (2003).*

2.6. Other terminology

Default swaptions are usually quoted *at-the-money (ATM) spot*. This means that the strike, or the contractual spread in the underlying CDS, is set to the current market spread on the underlying CDS.

Default swaptions can also be quoted *ATM forward*, in which case the strike is the forward CDS spread between the option maturity and the swap maturity as calculated from the current CDS spread curve. The value of a knockout payer swaption with ATM forward strike is equal to the value of the same maturity ATM forward receiver swaption.

A payer swaption is said to be *out-the-money (OTM)* (spot or forward) if the strike is higher than the current market (spot or forward) spread on the underlying CDS. If the strike is lower, the payer swaption is *in-the-money (ITM)*. For a receiver swaption the reverse is the case, for example a receiver swaption with a very low strike would be OTM.

A *straddle* is a portfolio of a payer and receiver swaption both with the same strike, notional, and option and swap maturities. Sometimes straddles are quoted directly with a bid-offer. The meaning of the terminology "an ATM spot straddle" or "an ATM forward straddle" should be clear.

A *synthetic (long protection) forward CDS* is a portfolio of a long payer swaption and a short receiver swaption both with the same strike, notional, and option and swap maturities. If the payer swaption is knockout, the position has the same cashflow as a forward starting CDS.

The *PV01* of a (possibly forward starting) CDS is the value of the premium (or coupon) leg of the CDS divided by the contractual spread of the CDS. In other words, the value of the premium leg if the CDS only paid a 1bp premium. The forward PV01 is the PV01 of a forward starting CDS.

A default swaption is often quoted and traded with *delta exchange*. This means that, together with the option, a certain notional of the underlying standard CDS is traded such that the value of the combined position is insensitive to a small change in the spread on the underlying CDS (see the discussion of delta in section 4 below).

3. EVALUATING DEFAULT SWAPTION TRADES

Most investors trade default swaptions with the intention of holding on to their position until option maturity and thereby limit trading costs, which can be substantial for illiquid credits. In CDX IG and HVOL portfolios and liquid credits such as FMCC and GMAC where swaption bid-offer spreads are tight, investors may consider closing positions before option maturity.

To evaluate default swaption trades, investors should calculate their profit/loss (P/L) under a number of different scenarios. Often it is also useful to summarize the scenario analysis with breakeven spread(s) and value-on-default (VOD).

3.1. Scenario analysis

When the investor plans to hold on to the position to the option maturity, the best horizon for the scenario analysis is usually the option maturity.

It is natural to divide the analysis into:

1) scenarios in which the underlying credit defaults – used to capture VOD risk, and
2) non-default scenarios – used to capture spread risk.

The VOD risk in default swap and swaption positions is usually very easy to describe as a simple function of the recovery rate on the reference credit. Getting a handle on the recovery rate, on the other hand, can be more difficult but new products such as recovery locks[3] can give an indication.

To calculate the P/L at option maturity in a non-default scenario, we need three numbers:

1) the mark-to-market spread for the underlying CDS,

2) the PV01 of the underlying CDS, and

3) the cost of the position (with interest accrued to the option maturity).

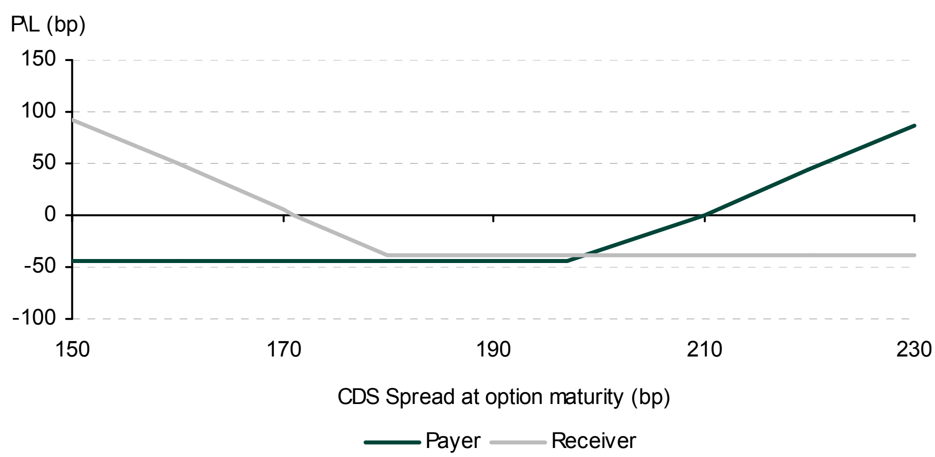
The P\L on long payer and receiver positions at option maturity are:

Payer: max{0, PV01(S)·(S - StrikeSpread)} – PayerCost

Receiver: max{0, PV01(S)·(StrikeSpread - S)} – ReceiverCost

The PV01 is a decreasing function of the mark-to-market spread, denoted by S above. In fact, the PV01 is a function of the entire CDS spread curve for the reference credit. Using a constant PV01 simplifies the P/L calculation. The common choice is the forward PV01 divided by the credit risky discount factor to the option maturity. This is the (risk neutral) expected PV01 at the option maturity conditional on no default. The error from using a constant PV01 is minor for investment grade credits and is common for non-extreme scenarios. If more accuracy is desired, the PV01 for a particular scenario can be calculated using the forward CDS spread curve shifted to match the market spread to the maturity of the underlying CDS for that scenario.

Figure 1. Payoff at Maturity on 180bp and 200bp 20-Dec-2004 FMCC Receiver and Payer swaptions both with swap maturity 20-Dec-2009. Prices as of 1-Nov-2004 with CDS at 185bp



Source: Lehman Brothers.

When all the default swaptions in the analyzed position have the same option and swap maturities we can represent the scenario analysis with the well-known option payoff diagrams (Figure 1).

If the position studied in the scenario analysis contains CDS (or default swaptions on CDS) with different maturities, scenarios for all CDS maturities should be generated. Also, if the

[3] *A recovery lock is a swap between a standard CDS protection leg and a CDS protection leg with a contractually fixed recovery rate. Recovery locks are quoted on the fixed recovery rate.*

position contains bonds on the same credit (eg, if a payer swaption is used together with an interest rate swaption to hedge the downside risk in a corporate bond), the scenario analysis should include different values for the CDS basis at option maturity.

The final and most difficult stage of the scenario analysis is to form a view on the probabilities of the various scenarios occurring. This is especially difficult in multi-dimensional (usually only two-dimensions is feasible) scenarios that, for example, include CDS spreads at different maturities. For multi-dimensional scenarios it is important to a have a view on the correlation between the different simulated variables. Such correlation must usually be estimated empirically.

If VOD risk is different from zero, the investor should evaluate whether the expected P/L in the non-default scenarios appropriately compensates for this risk. This can be done by comparing with short-term credit spreads.

3.2. Scenario analysis for portfolio swaptions

For portfolio swaptions we find it natural to study scenarios determined by both:

1) the number of defaults among the reference credits, and

2) the spread on a portfolio swap referencing only the non-defaulted credits.

In our opinion it is important to have a view on the distribution of the number of defaults before option maturity and how the (mean and variance of the) spread on the portfolio of non-defaulted credits depends on the number of defaults.

See Pedersen (2003) for details on how to calculate the P/L on a portfolio swaption given the number of defaults, the recovery on the defaulted credits, and the spread on the portfolio of non-defaulted credits.

3.3. Breakeven spreads

It is often useful to summarize the scenario analysis with breakeven spread(s) and VOD. The breakeven spread (or spreads) indicates the spread interval for which the analyzed position is profitable. Breakeven calculations are simple when the option maturity is the horizon for the scenario analysis.

A breakeven spread is a spread of the underlying CDS at option maturity that results in a P/L of zero.

Exact calculation of a breakeven spread for a default swaption position must take into account the effect of the spread level on the PV01 for the underlying CDS. However, using a constant PV01 is often acceptable, and it simplifies the calculations (see section 3.1). With a constant PV01, the breakeven spreads for naked payer and receiver swaptions are:

$$BE_{payer} = Strike + Cost_{payer} / PV01$$

$$BE_{receiver} = Strike - Cost_{receiver} / PV01$$

Again assuming that the PV01 is constant, the breakeven spreads for a straddle are:

$$Upper\_BE_{straddle} = Strike + (Cost_{payer} + Cost_{receiver}) / PV01$$

$$Lower\_BE_{straddle} = Strike - (Cost_{payer} + Cost_{receiver}) / PV01$$

A seller of a straddle wants the CDS spread at option maturity to be between $Lower\_BE_{straddle}$ and $Upper\_BE_{straddle}$. A buyer of a straddle wants the spread to be outside that interval.

In section 3.1 we suggested using as a constant PV01 the current forward PV01 divided by the credit risky discount factor, and also incorporating financing cost until option maturity into the cost of the swaption. The cost of the swaption is paid also if default occurs. Imagine we paid for the swaption at option maturity and only if default did not occur. The cost would then be the actual upfront cost divided by the credit risky discount factor to option maturity. If we used this cost for calculating the breakeven spreads, the credit risky discount factors would cancel out in the expression Cost / PV01, and we could simply use the quoted upfront cost and the standard forward PV01 for calculating the breakeven spreads. When separate VOD is not reported, the breakeven spreads are often calculated this way.

4. USING THE GREEKS

Greeks are risk measures that quantify how much the value of an option position changes for small changes in the valuation input parameters. Greeks are used by option traders to manage the risks in an options book but are also useful for designing an option strategy that implements a particular view.

4.1. Delta

Delta measures the change in the swaption value for a (marginal) 1bp increase in the spread of the underlying CDS. Calculating delta requires that the entire CDS spread curve is shifted. Deltas are often reported for a 1bp parallel shift.

The delta of a forward CDS (or equivalently a portfolio of a long knockout payer and a short receiver) is the forward PV01 for that CDS[4]. This implies that the delta of a knockout payer minus the delta of a receiver is equal to the forward PV01.

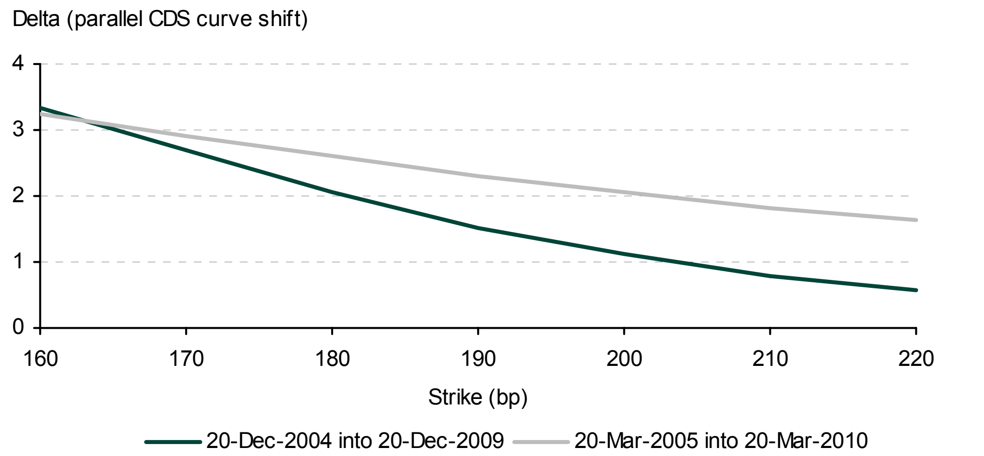
It can be useful to calculate delta term structures in which the CDS spread for only one maturity is increased for the delta calculation. This reveals that a default swaption mainly is exposed to spread risk for two maturities: the option maturity and the swap maturity. Following the discussion in section 2.3, this is expected since the CDS spread that primarily determines the option value is the forward spread from option maturity to swap maturity, which again can be well approximated by the spot CDS spreads to option and swap maturity.

Deltas should not be confused with the *hedge ratios* (confusion can arise from the fact that for stock options hedge ratios and deltas are the same). A hedge ratio is associated with a given (possibly forward starting) CDS. A hedge ratio of 1/2, say, means that to hedge a short position in the swaption, the hedger should buy 1/2 of the swaption notional of protection with the associated hedge CDS. The hedge ratio is calculated as the delta divided by the PV01 of the CDS used as hedge.

Since the delta of a knockout payer minus the delta of a receiver equals the forward PV01, the hedge ratio of the knockout payer minus the hedge ratio of the receiver is equal to 1 when both hedge ratios are with respect to the forward CDS from option to swap maturity. This is consistent with the fact that a long payer and short receiver position has the same cash flow as that forward CDS.

[4] *When the delta is based on a parallel curve shift this is technically only true for flat curves.*

Figure 2. Deltas for FMCC payer swaptions as of 8-Nov-2004

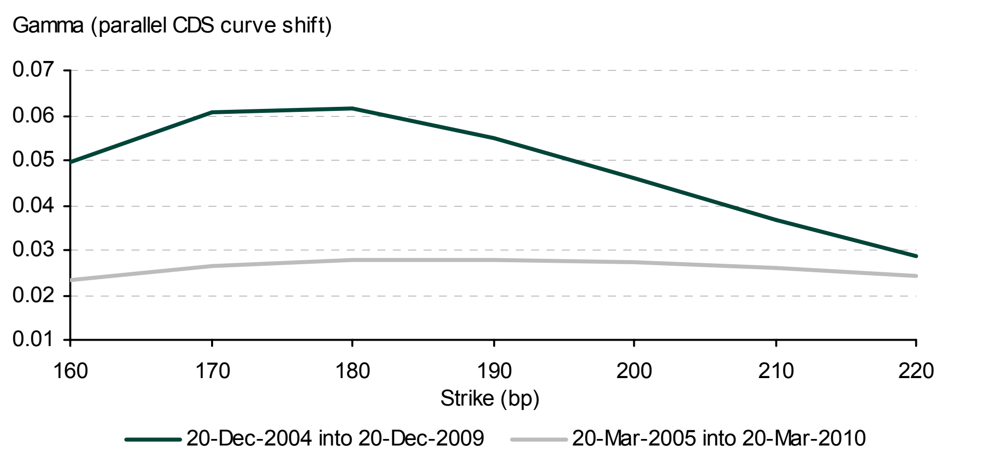


Source: Lehman Brothers.

4.2. Gamma

Gamma is the delta of the delta. It quantifies how much the delta changes for a 1bp increase in spreads. If gamma was always zero, the swaption value would be a linear function of spreads. Gamma is close to zero for deep ITM and OTM swaptions. Gamma is highest when the strike is close to ATM forward. The gamma of a short-term ATM forward swaption is higher than the gamma of a long-term ATM forward swaption.

Figure 3. Gammas for FMCC payer swaptions as of 1-Nov-2004



Source: Lehman Brothers.

We can use delta and gamma to approximate the effect of a spread change on the value of a position. Say the current CDS spread is S_0 and the value of the position is P_0. If spreads suddenly jump to S_1, the value of the position will be approximately

$$P_1 = P_0 + \Delta\cdot(S_1\text{-}S_0) + 1/2\cdot\Gamma\cdot(S_1\text{-}S_0)^2 \tag{4.1}$$

where Δ is the delta of the position and Γ is the gamma of the position.

Notice that if delta is zero (Δ=0) and gamma is positive (Γ>0), any immediate jump (up or down) in spreads will increase the value of the position.

4.3. Theta

Theta quantifies how much the value of the swaption decreases when one day passes. Theta is usually calculated assuming that tomorrow Libor/swap rates and CDS spreads take their forward values implied from today's spot curves.

There is a direct relationship between gamma and theta. We can see this by looking at the effect of a spread change and the passage of 1 day on a delta-neutral position. Using the same notation as above (section 4.2), and assuming Δ=0 and one day passes before the spread changes to S_1, we have

$$P_1 = P_0 + 1/2 \cdot \Gamma \cdot (S_1 - S_0)^2 - \Theta \qquad (4.2)$$

where Θ is theta.

Disregarding the cost of financing the position, the one-day breakeven spread change for the delta-neutral position is found as the value of $S_1 - S_0 = \text{BE}\Delta$ that satisfies the equation for $P_1 = P_0$:

$$\text{BE}\Delta = \pm\sqrt{\frac{2 \cdot \Theta}{\Gamma}} \qquad (4.3)$$

A positive gamma position will make money if the one-day spread change is more than BEΔ and lose money if the spread change is less (disregarding the financing cost). The distribution of the spread change is unrelated to the strike of the swaption which implies a direct relationship between gamma and theta (more on this in section 4.5). We can therefore think of theta as the cost of a positive gamma.

4.4. Vega

Although not a Greek letter, vega is the standard term for the change in the value of an option for 1 percentage point increase in the implied volatility. Vega is highest when the strike is close to ATM forward, although when implied volatilities are increasing in spreads, the strike for which vega is highest may be somewhat above the forward spread. The maximum vega for any strike swaption with a given option maturity is increasing in the option maturity.

In interest rate derivatives, traders of long-term swaptions (longer than 1 year) are sometimes called vega-traders and the traders of short-term swaptions are called gamma-traders. This is due to the different exposures of the short- and long-term swaptions. We can illustrate how the relationship between gamma and vega changes with the time to maturity by calculating the spread jump required for the change in the value of a delta-neutral position to equal 1 vega[5]. Using gammas and vegas from Figures 3 and 4, the required spread change is about 6bp for the 200bp 20-Dec-2004 swaption and 12bp for the 200bp 20-Mar-2005 swaption.

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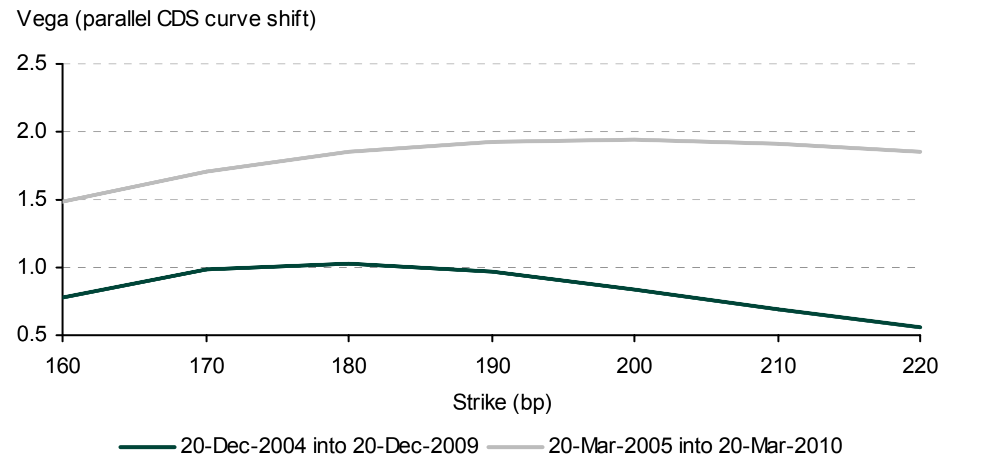
[5] *This number, which is $(2 \cdot V/\Gamma)^{1/2}$, where V is vega, is found using the same logic leading to equation (4.3).*

Figure 4. Vegas for FMCC payer swaptions as of 8-Nov-2004



Source: Lehman Brothers.

4.5. Breakevens using the Greeks

We can use the Greeks to calculate breakeven spreads, but we only recommend using them when analyzing liquid swaptions with tight bid-offer spreads, or when risk managing an options book.

When both delta and theta are different from zero, equations (4.1) and (4.2) can be written as

$$P_1 - P_0 = \Delta\cdot(S_1\text{-}S_0) + 1/2\cdot\Gamma\cdot(S_1\text{-}S_0)^2 - \Theta \tag{4.4}$$

We can use this equation to find the breakeven spread change for the position but we should also take into account the cost of financing the position. Let r denote the one-day interest rate (with daily compounding) and C_1 the cash payout on the position. The carry is then $C_1 - r\cdot P_0$ and the breakeven spread change is found as the value of $S_1 - S_0$ for which $P_1 - P_0 = r\cdot P_0 - C_1$. This leads to the equation

$$\Delta\cdot(S_1\text{-}S_0) + 1/2\cdot\Gamma\cdot(S_1\text{-}S_0)^2 - \Theta = r\cdot P_0 - C_1 \tag{4.5}$$

which can be solved for $S_1 - S_0 = BE\Delta$ to find the breakeven spread change.

If $\Delta = 0$, the breakeven spread change is

$$BE\Delta = \pm\sqrt{\frac{2\cdot(\Theta + r\cdot P_0 - C_1)}{\Gamma}} \tag{4.6}$$

which is a repetition of equation (4.2) but taking into account the cost of financing. We must expect spreads to change by this amount on average.

We can also incorporate vega, denoted by V, and changes in implied volatility into equation 4.5 and find two-dimensional breakevens for spread changes as well as implied volatility changes.

$$\Delta\cdot(S_1\text{-}S_0) + 1/2\cdot\Gamma\cdot(S_1\text{-}S_0)^2 - \Theta + V\cdot(\sigma_1 - \sigma_0) = r\cdot P_0 - C_1 \tag{4.7}$$

The breakeven spread change will then be a function of the implied volatility. Equation (4.7) is the fundamental equation used to risk manage an options book. The left-hand side is the change in the value of the book due to spread changes, time decay and implied volatility changes. The right-hand side is the carry.

Delta, gamma, theta, and vega do not present the full picture of the risks in a swaption position – only the risks associated with small changes in spreads, passing of time, and implied volatilities (the Greeks are therefore sometimes referred to as local risk measures). It is important to complement the Greeks (as well as breakeven spreads) with an analysis of extreme scenarios.

4.6. Delta hedging

Delta hedging is the process of hedging an instrument with gamma different from zero using an instrument with gamma equal to zero (or close to zero). A receiver swaption and a knockout payer swaption should ideally be delta hedged with a forward starting CDS. In practice, delta hedging is done with a standard spot CDS. The spot CDS is not ideal because the forward spread, not the spot spread, is the primary determinant of the value of a default swaption (see appendix). More importantly, however, a dealer hedging a long knockout payer swaption by selling spot CDS protection is exposed to the risk that the credit defaults before the swaption matures, in which case the payer knocks out and the dealer incurs a loss on the hedge CDS. For this reason, dealers often prefer clients selling non-knockout rather than knockout payer swaptions.

For liquid credits and portfolios (such as CDX and iTraxx), a dealer can manage all (gamma and vega, as well as delta) risk with sequential swaption trades. For illiquid credits, a dealer may have to wait a long time before getting the opportunity to enter into an offsetting swaption trade. Until a position can be offloaded, the dealer must delta hedge the position. The costs of delta hedging a default swaption over a long period of time can be substantial and depend on the hedging strategy employed. A hedging strategy involves deciding how frequently to adjust the hedge. This is a difficult problem, giving the skilled trader the opportunity to minimize costs.

5. TRADING STRATEGIES

We discuss a number of generic default swaption trades divided into three types: (1) simple directional trades, (2) volatility trades, and (3) spread trades.

Some of the trades are illustrated by a numerical example. Although the prices used in the examples should be close to the actual market prices on the dates specified, the examples should not in any way be interpreted as actual trade recommendations. For specific trade recommendations we refer to our strategy teams (see eg, Ganapati, Tejwani and Fan (2004)).

5.1. Simple directional trades

The two simplest bullish default swaption trades are to sell a payer swaption or buy a receiver swaption. An investor with a fundamentally bullish view on a credit may also consider hedging the downside risk in a long credit position by buying a payer swaption.

The bullish strategies can be reversed for investors with a bearish credit view.

Yield enhancement with a short payer

An investor who finds a credit attractive at current market levels and sees little risk that the spread will widen should consider selling a payer swaption on the credit.

Example: *On 8-Nov-2004, 5-year FMCC CDS was at 173bp. The 20-Mar-2005 200bp (strike) payer swaption was a bid at 74bp upfront implying a volatility of 49%. The forward PV01 was 4.29. The breakeven was approximately 200bp + 74bp/4.29 = 217bp. If on 20-Mar-2005 the 20-Mar-2010 FMCC spread is below 217bp, selling the payer swaption will be a profitable trade. The main risk is a significant increase in FMCC spreads.*

Hedging the downside risk in a long credit position

Suppose spreads have tightened and an investor has profited from a long credit position. The investor can lock in part of the profit by buying a payer swaption. The strategy is particularly relevant for hedging the downside risk in a credit portfolio with a CDX or iTraxx payer swaption.

A payer swaption can also be used for a single credit, for example, to hedge the downside risk in a corporate bond. The investor may already have hedged the interest rate risk with an asset swap (ASW), in which case the position is exposed to the bond's ASW spread. Ideally the bond should be hedged with a tailored ASW option, but the more liquid default swaption market may provide a more cost-efficient alternative. However, a bond on ASW hedged with a default swaption is also exposed to the risk that the basis between CDS and ASW spreads decreases (the risk that ASW spreads increase more than CDS spreads). This basis risk should be evaluated in the scenario analysis. It may be cost efficient to use a default swaption with a standard swap maturity, although this will introduce curve risk in addition to the basis risk.

Instead of holding on to the long credit position and buying a payer swaption. The investor may also consider closing the position and buying a receiver swaption. The investor will then benefit from spread tightening and be hedged against spread widening.

Direct buying of a receiver swaption is a particularly relevant strategy for an investor with a fundamentally bullish view on a credit but who is concerned about upcoming event risk.

Yield enhancement with a short receiver

An investor who finds a credit overvalued and sees little risk that its spreads will tighten should consider selling a receiver swaption.

Example: *On 8-Nov-2004, 5-year FMCC was at 173bp. The 20-Dec-2004 160bp (strike) receiver swaption was a bid at 12bp upfront corresponding to an annualized carry of 108bp. An investor selling the receiver swaption will profit if on 20-Dec-2004 FMCC is trading above 157bp (= 160 – 12/4.36). The main risk in the trade is a significant tightening in FMCC spreads.*

Hedging the downside risk in a short credit position

An investor who has a short position in a particular credit may consider buying a receiver swaption to hedge the risk of spread tightening. This could be to lock in an existing profit. Alternatively, the investor should consider closing the short position and buying a payer swaption. The payer swaption allows the investor to profit from spread widening with limited exposure to spread tightening.

5.2. Volatility trades

A straddle is the most common vehicle for trading volatility. From the payoff diagram of a straddle (see Figure 5), it is clear that an investor who buys a straddle is buying volatility, that is the investor is taking the view that spreads will be volatile.

There are three basic types of volatility trades/strategies

- Static delta-hedged trade using short term swaptions
- Static delta-hedged trade using long term swaptions
- Dynamically delta-hedged trades

In the current stage of the CDS and default swaption market the most common volatility trades are the first type, but it is useful to understand all three.

To implement the dynamically delta-hedged trade, the CDS market must be liquid with narrow bid-offer spreads. Delta hedging a default swaption is discussed in more detail in section 4.6. In a dynamically delta-hedged trade an investor is taking a view on realized vs implied volatility. It is interesting to note that when the party selling volatility is delta hedging and the party buying volatility is keeping a static trade, it is possible for both parties to profit if there is a large, low volatility (a slow deterioration or improvement in credit quality) change in the credit spread over the life of the swaption position.

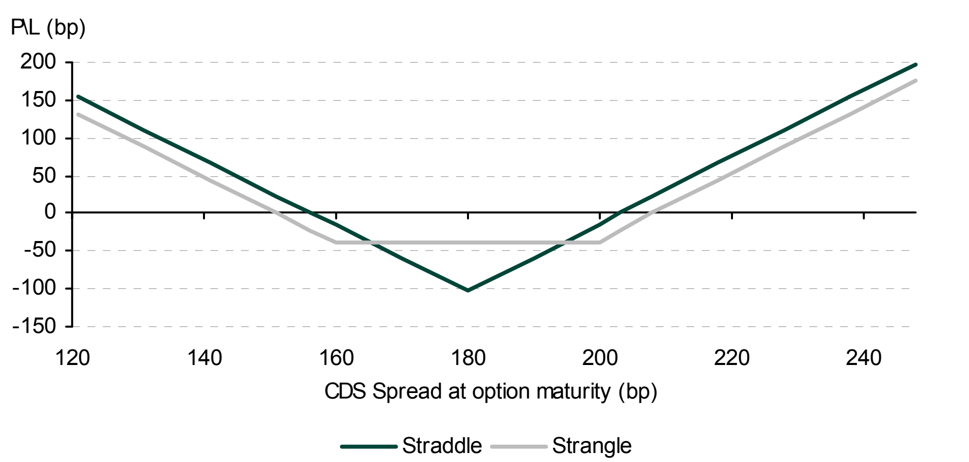
The two other types of volatility trades listed above are also called gamma and vega trades. This is due to the fact that gamma is greater and vega lower for short-term (ATM) swaptions compared with long-term (ATM) swaptions. Short-term swaptions are by far the most liquid and gamma trading, where the investor takes a view on spread movements over the short term, is the prevalent volatility trading strategy.

Short-term volatility trades, or gamma trades, are usually event driven. An investor may expect major news about an issuer within the next few months but be unsure about how the news will affect the credit risk of the issuer. For example, the investor may expect that the issuer will restructure its operations, but sees scenarios that could lead to increased creditworthiness (say an asset sale with proceeds used to pay down debt) or increased credit risk (say an asset sale with proceeds used for stock buyback). If the investor believes that the default swaption market underestimates the likelihood of a large change in the credit spreads of the issuer, the investor should buy volatility.

Straddles and strangles

A strangle is a portfolio of a payer swaption and lower strike receiver swaption both with the same notional and option and swap maturities.

Figure 5. Payoff at Maturity on 20-Dec-2004 straddles and strangles on 20-Dec-2009 FMCC CDS. Prices as of 8-Nov-2004 with CDS at 173bp



Source: Lehman Brothers.

The difference between the strikes on a strangle is mainly a question of leverage. The cost of a wide strangle is lower but a larger spread movement is also required for a long position to be profitable. The strike on a straddle (or strikes on a strangle) is usually chosen so that the delta is close to zero.

The final issue is the option maturity. If the trade is put on in order to profit on a view on the market reaction to an event, the option maturity must obviously be after the event date but would otherwise usually be chosen as close to the event date as possible. If the trade is implementing a general view on spread volatility, the maturity can be looked on as a tradeoff between gamma and vega.

Calendar straddles

A calendar straddle involves two straddles of different maturities and is mainly used to obtain opposite exposure to gamma and vega.

Example: *Assume a macroeconomic announcement is scheduled for release before the standard maturity of the December CDX swaptions. An investor has the view that the announcement will cause a significant move in credit spreads after which spreads will be range bound. The investor believes that this view is not priced into the current CDX swaption prices. How can the investor profit from the view?*

The investor's view can be summarized as delta neutral, gamma positive and vega negative. A possible trade is to buy an ATM December straddle and sell an ATM March straddle. ATM straddles are approximately delta neutral. The gamma of a short-term ATM straddle is higher than the gamma of a long-term ATM straddle. The reverse is the case for vega. Thus the position achieves the desired Greeks.

5.3. Spread trades

Spread trades are directional trades where a long payer or receiver position is partly financed by a short position in a further OTM swaption of the same type. Spread trades are usually employed by investors with a very specific view.

Standard bull and bear spreads

In a standard bear spread the investor buys an ATM or slightly OTM payer swaption and sells a deeper OTM payer. Similarly, in a standard bull spread the investor is long a receiver swaption and short a deeper out of the money receiver. The downside in the trades is limited to the initial cost of the positions. The upside is also limited to a maximum profit obtained when the spread on the underlying CDS are at the strike of the short swaption at option maturity.

Ratio spreads

A ratio spread is similar to a standard bull or bear spread but with the notional on the short swaption two or three times the notional of the long swaption. With this modification there is unlimited downside realized for very large spread movements. However, the cost of the position is reduced or alternatively the maximum profit can be increased (by choosing a deeper OTM strike for the short swaptions).

6. SUMMARY

In this article we have introduced the basic concepts and terminology required to start trading default swaptions. We have discussed methods and tools to analyse, design and evaluate default swaption trades such as scenario analysis, breakeven spreads and local risk measures (Greeks). We have also outlined and discussed particular strategies that are currently being employed by investors, and in the appendix we have summarized the standard market model for valuation and risk analysis.

Whereas portfolio default swaptions have seen tremendous growth in trading volume over the past year, the single-name default swaption market is still, for most credits, a developing market with potentially undiscovered opportunities. Portfolio swaptions allow investors to implement, in a cost-efficient manner, views on the direction and volatility of market-wide credit spreads. Single-name swaptions allow investors to profit from analysis and insight into the credit risk of specific issuers.

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APPENDIX

Valuation of credit default swaptions

The market standard for pricing default swaptions is a modification of the Black formula for interest rate swaptions. The inputs to the valuation are:

- Time to maturity (in years) of the default swaption
- Strike spread
- Implied volatility for the given strike and option and swap maturities
- Forward CDS spread from the option maturity to the swap maturity
- Forward credit risky PV01 from the option maturity to the swap maturity
- The value of credit protection from the trade date to the option maturity (only relevant for non-knockout payer swaptions)

The last three inputs (the forward spread, the forward PV01, and the value of protection to option maturity) are calculated from a curve of CDS spreads and a specified recovery rate based on a standard credit curve model; see O'Kane and Turnbull (2003).

Introducing notation: T_0, T_1 and T_2 are the trade, option maturity and swap maturity dates respectively. $FEP(T_0;T_1)$ is the value of protection from the trade date to the option maturity date. $PV01(T_0;T_1,T_2)$ is the forward PV01 from option maturity to swap maturity as of the trade date. $S(T_0;T_1,T_2)$ is the forward spread from option maturity to swap maturity as of the trade date[6].

The Black formulas for knockout default swaptions are

$$\mathrm{Payer}(T_0) = \mathrm{PV01}(T_0;T_1,T_2)\cdot\left(S(T_0;T_1,T_2)\cdot N(d_1) - K\cdot N(d_2)\right)$$

$$\mathrm{Receiver}(T_0) = \mathrm{PV01}(T_0;T_1,T_2)\cdot\left(K\cdot N(-d_2) - S(T_0;T_1,T_2)\cdot N(-d_1)\right)$$

$$d_1 = \frac{\log(S(T_0;T_1,T_2)/K) + \sigma^2\Delta(T_0,T_1)/2}{\sigma\sqrt{T_0}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\Delta(T_0,T_1)}$$

where $N(\cdot)$ is the standard normal distribution function, $\log(\cdot)$ is the natural logarithm and $\Delta(T_0,T_1)$ the maturity (in years) of the option.

If the payer swaption is non-knockout, $FEP(T_0;T_1)$ must be added to the knockout value given above.

The Black formulas can be derived using standard mathematical finance methods. An appendix in Pedersen and Sen (2004) shows how to use these methods.

American and Bermudan default swaptions can be valued using lattice methods. Pedersen and Sen (2004) show how to implement a stochastic hazard rate model in a lattice and price credit spread volatility products.

6 *Letting $PVP(T_0;T_1,T_2)$ denote the value of the protection leg of a forward CDS that starts at T_1 and matures at T_2,*

$$S(T_0;T_1,T_2) = \frac{PVP(T_0;T_1,T_2)}{PV01(T_0;T_1,T_2)} = \frac{PVP(T_0;T_0,T_2) - PVP(T_0;T_0,T_1)}{PV01(T_0;T_0,T_2) - PV01(T_0;T_0,T_1)} = \frac{PV01(T_0;T_0,T_2)\cdot S(T_0;T_0,T_2) - PV01(T_0;T_0,T_1)\cdot S(T_0;T_0,T_1)}{PV01(T_0;T_0,T_2) - PV01(T_0;T_0,T_1)}$$
$$= \alpha(T_0;T_1,T_2)\cdot S(T_0;T_0,T_2) + (1-\alpha(T_0;T_1,T_2))\cdot S(T_0;T_0,T_1)$$

where $\alpha(T_0;T_1,T_2) = PV01(T_0;T_0,T_2) / (PV01(T_0;T_0,T_2) - PV01(T_0;T_0,T_1))$.

Portfolio swaptions have specific contractual features that are not captured with the Black formulas (see Pedersen (2003) for details). However, today the liquidity in portfolio swaptions has reached a point where the main purpose of a valuation model is to calculate risk measures. It is our opinion that if volatilities are chosen so the Black formula correctly prices the portfolio swaptions, then the risk measures produced by the Black approach are reasonably close to the numbers produced by a more detailed model. In a future publication we will discuss the effects of the positively sloped skew now observed in CDX swaption volatilities.

Example

To illustrate the use of the Black formula we show how to calculate the inputs using a standard default swap calculator such as the CDS calculator on LehmanLive.

We value a 20-Dec-2004 knockout payer swaption on 210bp 20-Dec-2009 FMCC CDS. The valuation is done using the Libor/swap rates at close 27-Oct-2004. Sometime during 27-Oct-2004 when the 20-Dec-2009 CDS was trading at 196bp, our desk quoted (mid) the payer at 116bp upfront. The same strike receiver was quoted at 112bp. Since the value of the knockout payer equals the value of the receiver exactly when the strike is equal to the forward spread we conclude that our desk was using a forward spread slightly above 210bp.

We estimate that a 20-Mar-2005 CDS would trade at 39bp if the 5-year was at 196bp. Using a standard CDS calculator we calculate the PV01 of the two CDS (for the 6-month CDS we input a flat 39bp curve; for the 5-year a flat 196bp curve). We get PV01s of 0.3941 and 4.4850. Using the calculations in the note on the previous page, we find the forward spread and forward PV01 as:

$$\text{Forward PV01} = 4.4850 - 0.3941 = 4.0909$$

$$\text{Forward Spread} = 1.0963 \cdot 196\text{bp} + (1 - 1.0963) \cdot 39\text{bp} = 211.1\text{bp}$$

where 1.0963 is found as 4.4850/4.0909. Notice that the forward spread indeed is slightly above the strike of 210bp.

We choose a volatility of $\sigma = 53\%$ and a maturity in years of $T = 143/365 = 0.3918$ (there are 143 days from the settlement to option maturity). We can now plug the numbers into the Black formula and arrive at the

$$\text{Payer Value} = 115.8\text{bp}$$

$$\text{Receiver Value} = 111.2\text{bp}$$

Estimating spread volatility

The quality of CDS spread data is a main concern when estimating spread volatility. For many credits, trader marked CDS spreads are often constant for several days, and sometimes weeks, in a row[7]. For such credits, standard volatility estimates based on daily spread observations will underestimate true volatilities. Better estimates are obtained when using weekly observations or the weighted changes approach described below.

Estimate based on daily data

Let S_i, $i = 0, 1, \ldots, N$, denote the observed mid-quote on trading day i. We assume that all observations are equally spaced with one trading day in-between.

[7] *Many researchers use Mark-It Partners CDS spreads to track historical spread levels. However, Mark-It Partners data tend to be too smooth compared with trader marked data. Spread volatility estimates should therefore not be based on daily changes in Mark-It Partners spreads. Mark-It Partners data are an average of where a number of dealers have marked their CDS positions and it is well known that some traders only gradually adjust the levels at which they mark their book. This is especially a problem for older data on relatively illiquid credits.*

The first step in calculating standard volatility estimates based on daily data is to find daily spread returns. We use continuous time returns, where

$$d_i = \log(S_i) - \log(S_{i-1}),\ i = 1,...,N$$

is the daily continuously compounded return over day i.

The standard volatility estimate based on daily data is

$$\sigma_{daily} = \sqrt{252}\sqrt{\frac{1}{N}\sum_{i=1}^{N} d_i^{\,2} - \left(\frac{1}{N}\sum_{i=1}^{N} d_i\right)^2}$$

Estimate based on weekly data

Our standard estimate based on weekly data uses only every fifth spread observation. Assume that this leaves us M+1 observations[8]: S_0, S_5, S_{10}, …, S_{5M}. Since we pick exactly every fifth observation our weekly observations will usually not be on the same week day. We calculate the weekly returns

$$w_j = \log(S_{5j}) - \log(S_{5(j-1)}),\ \text{j} = 1, \ldots, \text{M}.$$

The standard volatility estimate based on weekly data is

$$\sigma_{weekly} = \sqrt{252/5}\sqrt{\frac{1}{M}\sum_{j=1}^{M} w_j^{\,2} - \left(\frac{1}{M}\sum_{j=1}^{M} w_j\right)^2}$$

Estimate based on weighted changes

Our third volatility estimate is based on the assumption that we only observe the spread on days when the spread changed from the previous day.

We first identify the days when a change in the spread occurred. Assume that K spread changes occurred on days i(0), i(1), …, i(K). We then find the (annualized) time between the spread changes

$$t_k = \frac{i(k) - i(k-1)}{252},\ \text{k} = 1, \ldots, \text{K}, \qquad (2.1)$$

and the spread returns between changes

$$z_k = \log(S_{i(k)}) - \log(S_{i(k-1)}),\ \text{k} = 1, \ldots, \text{K}. \qquad (2.2)$$

The volatility estimate based on weighted changes is

$$\sigma_{weighted} = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left(\frac{z_k}{\sqrt{t_k}}\right)^2 - \left(\frac{1}{K}\sum_{k=1}^{K}\frac{z_k}{\sqrt{t_k}}\right)^2}$$

[8] *M is the largest integer such that 1+5M is less than or equal to N.*

ORION: A Simple Debt-Equity Model with Unexpected Default

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This article presents enhancements to ORION, a hybrid debt-equity model, introduced in Naik et al. (2003). The enhanced model incorporates the possibility of unexpected default and becomes even more flexible in matching spread curves observed on the market as it allows modelling non-zero short maturity spreads. It offers an analytical solution for pricing a corporate bond, a credit default swap (CDS) and, in case of a fixed barrier, European call and put options. Additionally, the unexpected default partially accounts for the implied volatility skew observed on the equity option markets. The model remains parsimonious and easy to handle. We introduce the enhanced model and discuss its spread curve and volatility skew generating possibilities. Finally, we consider the calibration of the model to bond market spreads and volatility skew of a single company.

1. INTRODUCTION

Credit risk models have been historically divided into two groups: structural credit models, where the time of default event is modeled explicitly as the time when the assets of the firm can no longer pay for its debt; and reduced form credit models, where the default event is modeled exogenously as the first jump of a Poisson process. Structural models were introduced by Merton (1974). While intuitively appealing, the structural approach has a number of disadvantages. First, it is often difficult to observe the capital structure of a firm, i.e. accounting data reported by firms are often noisy. This makes structural models difficult to calibrate. Second, a pure structural credit risk model driven by the evolution in firm's assets (in the Merton 1974 formulation) implies a predictable default event, i.e. the debt holder can see the default coming by observing changes in the capital structure of the firm. This predictability implies zero credit spreads for short maturity debt, which is not usually observed empirically.

The reduced form approach, introduced by Jarrow and Turnbull (1995), is more convenient from an empirical point of view as it allows easier calibration. Default events in the reduced form models are usually unpredictable or semi-predictable. Probabilities of default are modeled through hazard rates (default intensities), which may depend on different factors. The drawback of the reduced form approach is that it does not have a modeled linkage to the firm capital structure.

The ORION model is a hybrid model that incorporates elements of both the structural and reduced form models. It models directly the equity price and defines the default time as the first time the equity price either crosses a stochastic barrier or jumps to zero. The barrier can be derived from the bond or default swap issuer spreads. The spread information is used directly instead of accounting leverage data because the latter is already priced in the spread.

The advantages of the ORION model become apparent in applications where one is interested in the link between the credit quality of the firm and its equity value. The fact that ORION models this link directly allows simultaneous handling of credit and equity derivatives, which is very useful in an analysis of a portfolio of these two financial instruments.

[1] *This report has been written with Simon Polbennikov. We thank Arindam Sandilya and Munkundan Devarajan for their contributions to this report, Marco Naldi and Vasant Naik for their helpful comments and Prof. Stuart Turnbull for his help on the initial ORION model.*

The early version of ORION had difficulty to deal with short maturity credit spreads since the default event was predictable. Predictable default implies that as bond maturity decreases the credit spread goes to zero because the probability of default in a decreasing time interval decreases. Nevertheless, it is often observed on the market that bonds with short maturity still have positive credit spreads. To account for this, we introduce an unexpected default into ORION. This makes the model more flexible in matching spread curves while remaining simple with only a few parameters to calibrate. Moreover, the unexpected defaults modeled are equally able to explain some of the Black-Scholes volatility skew observed on the equity option markets.

The literature on equity-based credit modeling has recently expanded. Das, Sundaram, and Sundaresan (2003) have proposed a model that incorporates default risk, interest risk and equity risk using a lattice framework. As such, they do not propose any closed form solutions. In the equity derivatives and convertible bond pricing literature, some equity jump-diffusion models have been suggested but the focus has been more on calibration and numerical fitting than on developing a simple tractable model of credit risk (see e.g. Andersen and Andreasen (2000) and Andersen and Buffum (2003)).

The article proceeds as follows. Section 2 introduces the framework for default modeling. Section 3 describes the augmented ORION model in detail. In section 4 we address the question of a corporate bond pricing with ORION. Section 5 addresses the question of equity option pricing with ORION. In the case of a fixed default barrier we derive prices of European call and put options in closed form. We investigate credit spread curves and default implied Black-Scholes volatilities of the model in Sections 6 to 9. Section 10 concludes.

2. CAPITAL STRUCTURE AND DEFAULT MODELING

2.1. The firm capital structure

Debt and equity are claims on the firm. Shareholders and debtholders hold these claims in the form of shares, bonds or loans. Within debtholders, some might hold claims more senior than others (bank loans vs. issued bonds, senior vs. subordinates, bonds vs. preferred shares). The capital structure is the collection of these claims on the same underlying asset. Ordering the priority of the claims implies some optionality in the payoffs and modeling the priority rule is important to understand the pricing of the different claims.

2.2. Merton's model

In Merton (1974)'s model, the firm issues a zero coupon debt with fixed time to maturity. The model uses the asset value as the driving variable and defines the default event as the asset value falling under the total value of the firm's liabilities at maturity. Shareholders hold a call option on the firm with the strike price being the face value of the bond and the expiry date equal to the maturity date of the bond, whereas bondholders hold a risk free bond and are short a put option with the same strike price and same expiry date.

Other models in the tradition of Merton's model include the Black and Cox (1976) and Longstaff and Schwartz (1995). These models are called structural models because they explicitly link default with the firm's value and its capital structure.

Typically, in these models, the asset value follows a geometric Brownian motion. A drawback is that the short-term default probability is zero because the firm value dynamic is continuous and cannot suddenly jump to the face value of the debt. This is a property of diffusion processes. Zhou (1997) has extended the model to include jumps in the asset value. With the asset value ability to jump, default is not predictable and short term spreads are typically positive.

2.3. Reduced form model

In contrast to the structural approach, reduced form models directly assume an exogenous default process as a Poisson process without referring to the firm or equity values. The Poisson process is defined by an intensity that makes the probability of defaulting with a jump proportional to the time interval. These approaches include Duffie and Singleton (1994) and Jarrow and Turnbull (1995) among others. The reduced form models have the advantage of being easy to calibrate to market data (e.g. market spreads). In contrast with the structural models (except for the Zhou model), default always occurs as a surprise whenever the Poisson process jumps.

For an application to capital structure arbitrage, the reduced form approach is not very useful because the equity is not modeled at all. The jump process should therefore be present in the firm value process itself such as in Zhou (1997) or in the equity process. We choose the second approach with the ORION model.

3. THE AUGMENTED ORION MODEL

The driving variable of the ORION model is equity price. In that sense, it is close to a structural model. The model assumes that there are two ways a default may happen. First, it happens when the equity value crosses a certain, possibly stochastic, barrier level. Second, the default can be caused by an unexpected jump in equity price to zero level. The first type of default is predictable as a bondholder can anticipate it as the equity value approaches the barrier. The second type of default is unpredictable and happens unexpectedly.

The dynamics of the model under the risk-neutral probability measure is described by the stochastic processes for the equity price and the barrier level:

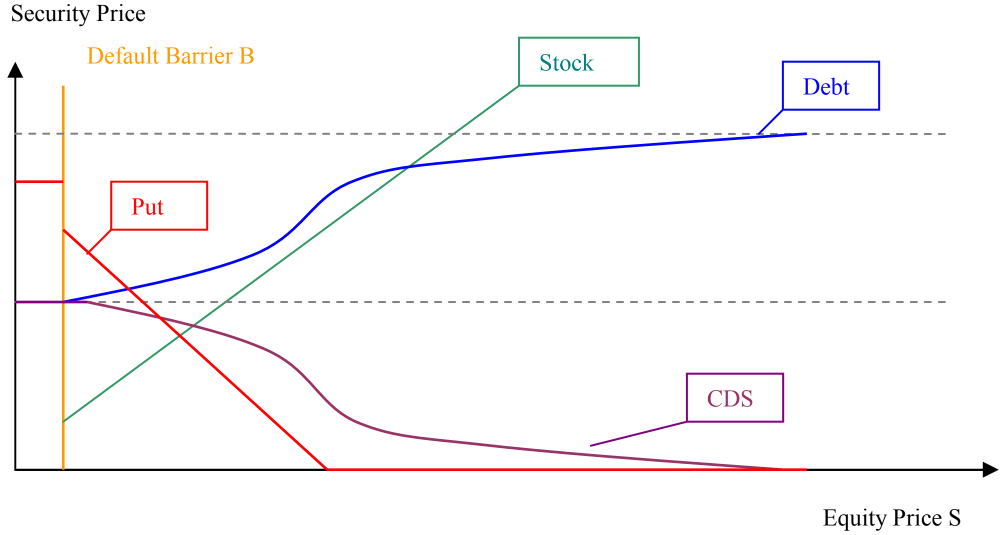
$$\begin{cases} \dfrac{dS_t}{S_t} = (r - \delta + \lambda)dt + \sigma_S dW_{St} - dN_t, \\ \dfrac{dB_t}{B_t} = -\kappa dt + \sigma_B dW_{Bt}, \end{cases}$$

where S_t is the equity price; B_t is the barrier level; N_t is the jump Poisson process, which is equal to 0 before the jump and equals to 1 after the jump; and W_{St}, W_{Bt} are independent standard Brownian motions. Parameters of the model are the risk-free interest rate r, the dividend pay-out ratio δ, the jump intensity λ, the equity volatility σ_S, the barrier drift parameter κ, and the barrier volatility σ_B.

We expect the barrier level to be very often positive. Although in Merton's model, the value of equity is zero upon default, it is empirically observed that the value of the equity is still positive though very small after a default. Equityholders can hope for some limited upside and sometimes the absolute priority rule can be violated (which would reallocate some assets from the bondholders to the equityholders though the latter are holding junior claims). The positive barrier also reflects some positive credit premium not reflected in the equity premium that would thus justify a higher level of spread.

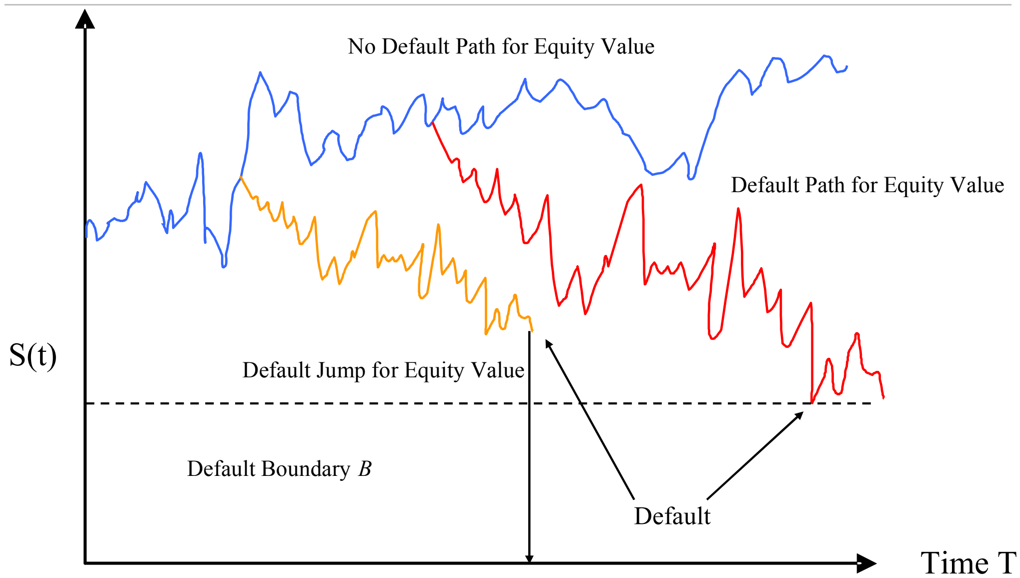
A firm defaults either when its equity price hits the default barrier or when a default jump occurs. Formally we can introduce two stopping times: a barrier crossing stopping time $\Gamma = \inf\{t \mid S_t = B_t\}$ and a jump stopping time $\tau = \inf\{t \mid N_t = 1\}$. Then the default time is defined as the minimum of these two stopping times $\Gamma \wedge \tau$.

Figure 1. Debt, Put and CDS (protection leg) with fixed default barrier



Source: Lehman Brothers.

Figure 2. The equity value is stochastic and default boundary is fixed. Default can be caused by the default boundary moving above the equity value or by a jump of the equity to zero



Source: Lehman Brothers.

The results on the first passage time probabilities are well known in the literature. Here we list the main results, which are the building blocks for credit and equity securities pricing relations.

Hitting survival probability conditional on no-jump:

$$P(\Gamma > t) = \Phi(w_1) - e^{-2bv}\Phi(-w_2).$$

Conditional density of the hitting time:

$$f_\Gamma(t) = \frac{b}{\sigma t^{3/2}} \varphi(w_1).$$

Joint probability for the normalized equity price $Y_t = \ln(S_t / B_t)$ and hitting time Γ:

$$P(Y_t > y; \Gamma > t) = \Phi(u_1) - e^{-2b\nu}\Phi(u_2)$$

where

$$w_{1,2} = \frac{b \pm \mu t}{\sigma\sqrt{t}}, \quad u_{1,2} = \frac{-y \pm b + \mu t}{\sigma\sqrt{t}},$$

$$b = \ln(S_0 / B_0),$$

$$\nu = \mu / \sigma^2,$$

$$\mu = r + \lambda + \kappa - \delta - \sigma_S^2/2 + \sigma_B^2/2,$$

$$\sigma^2 = \sigma_S^2 + \sigma_B^2.$$

For the derivation details see e.g. Bielecki and Rutkowski (2002), chapter 3. These results are readily extended for the unanticipated jump, because the two stopping times τ and Γ are independent.

4. PRICING CORPORATE BOND AND CREDIT DEFAULT SWAP

We can compute the values of the firm's liabilities in the ORION framework. We simply need the survival probabilities, the risk-free discount rates and the stream of liabilities cash flows. The value of a corporate bond can be computed as the discounted sum of coupons and principal payment weighted by survival probabilities and augmented with recovery value in the case of default. Thus, the price of a corporate bond with coupon payments $C(T_j)$, $j = 1,2,\ldots,n$, face value 1 and recovery R is

$$D(0,T) = \sum_{j=1}^{n} C(T_j) Z(0,T_j) P(\Gamma \wedge \tau > T_j) + RG(T),$$

where $Z(0,T_j)$ is the price of a risk-free zero-coupon bond with maturity T_j, and $G(T)$ is the present value of receiving one unit of currency if default occurs on the interval $[0,T]$. We show that $G(T)$ is given by[2]:

$$G(T) = \int_0^T e^{-rt} f_{\Gamma\wedge\tau}(t)dt =$$

$$= \frac{\lambda}{\lambda + r}\left[1 - e^{-(r+\lambda)T} P(\Gamma > T)\right] + \frac{r}{\lambda + r}\left[e^{-b(\nu-\gamma)}\Phi(-a_1) + e^{-b(\nu+\gamma)}\Phi(-a_2)\right],$$

where

[2] *Further details available upon request.*

$$a_{1,2} = \frac{b \pm \beta T}{\sigma \sqrt{T}}, \quad \beta^2 = \mu^2 + 2(r+\lambda)\sigma^2, \gamma = \frac{\beta}{\sigma^2}.$$

Likewise, it is possible to compute the value of CDS spread by valuing the protection leg (cash flow paid by the protection buyer) and the default leg (cash flows paid by the protection seller when there is a default) and making them equal at inception of the CDS contract:

$$RPV01 = \sum_{j=1}^{n} Z(0, T_j) P(\Gamma \wedge \tau > T_j),$$

$$PV_{Default}(0) = (1-R)G(T),$$

$$CDS = \frac{(1-R)G(T)}{RPV01}.$$

5. VALUATION OF THE EUROPEAN CALL AND PUT OPTIONS

In the case of fixed barrier level B_0 we start with the derivation of the price of a European digital call option. Then we present formulas for standard European call and put options.

A European digital call option is a contract which pays one unit of numeraire in case the price of the underlying asset exceeds the strike price. When default of the underlying asset is possible, we assume that the digital option pays nothing in case of default. So, the payoff of the European digital call option is:

$$C_T = 1\{S_T > K\} 1\{\Gamma \wedge \tau > T\}.$$

Since the stopping times are independent, we have

$$1\{\Gamma \wedge \tau > T\} = 1\{\Gamma > T\} 1\{\tau > T\}$$

If we use the normalized equity price Y_t then

$$1\{S_T > K\} = 1\{\ln(S_T) > \ln(K)\} = 1\{Y_T > \ln(K / B_0)\}.$$

The martingale representation theorem gives us the price of the digital option as the discounted expected value of its future payoff under the risk-neutral probability measure:

$$\begin{aligned} C_0 &= e^{-rT} E^Q \{1\{Y_T > \ln(K / B_0)\} 1\{\Gamma > T\}\} P(\tau > T) = \\ &= e^{-rT} P^Q \{Y_T > \ln(K / B_0); \Gamma > T\} P(\tau > T) \end{aligned}$$

Substituting the expression for probabilities given before we obtain the value of the digital call option:

$$C_0 = \begin{cases} e^{-(r+\lambda)T} \left[\Phi(d_1) - e^{-2b\nu} \Phi(d_2)\right] & \text{if } K \geq B_0 \\ e^{-(r+\lambda)T} \left[\Phi(\tilde{d}_1) - e^{-2b\nu} \Phi(\tilde{d}_2)\right] & \text{if } K < B_0 \end{cases}$$

where

$$d_{1,2} = \frac{-\ln(K/B_0) \pm b + \mu T}{\sigma\sqrt{T}}, \tilde{d}_{1,2} = \frac{\pm b + \mu T}{\sigma\sqrt{T}}.$$

To price a standard European put or call option we can use the fact that the payoffs of standard European call and put options can be expressed as a superposition of the payoffs of two digital options under risk-neutral probability measures with different numeraires. In particular, we use stock price as a numeraire to price the part corresponding to the equity option and money market account to price the part corresponding to the digital option. For example, the payoff of a standard European call conditional on no default can be represented as:

$$C_T^E = S_T 1\{S_T > K\} 1\{\Gamma \wedge \tau > T\} - K 1\{S_T > K\} 1\{\Gamma \wedge \tau > T\}.$$

The first part of this expression is the payoff of an asset digital call option, while the second one is the payoff of a cash digital call option, the value of which we derived above. The asset digital call option can be priced analogously to the cash digital call but with the equity price chosen as a numeraire instead the money market account. The resulting price of the European call option is:

$$C_0^E = C_0^S - KC_0,$$

$$C_0^S = \begin{cases} S_0\left[\Phi(d_3) - e^{-2b\tilde{v}}\Phi(d_4)\right] & \text{if } K \geq B_0 \\ S_0\left[\Phi(\tilde{d}_3) - e^{-2b\tilde{v}}\Phi(\tilde{d}_4)\right] & \text{if } K < B_0 \end{cases},$$

$$C_0 = \begin{cases} e^{-(r+\lambda)T}\left[\Phi(d_1) - e^{-2bv}\Phi(d_2)\right] & \text{if } K \geq B_0 \\ e^{-(r+\lambda)T}\left[\Phi(\tilde{d}_1) - e^{-2bv}\Phi(\tilde{d}_2)\right] & \text{if } K < B_0 \end{cases},$$

where

$$d_{3,4} = \frac{-\ln(K/B_0) \pm b + \tilde{\mu}T}{\sigma\sqrt{T}}, \tilde{d}_{3,4} = \frac{\pm b + \tilde{\mu}T}{\sigma\sqrt{T}},$$

$$\tilde{v} = \tilde{\mu}/\sigma^2, \quad \tilde{\mu} = \mu + \sigma^2.$$

We proceed similarly to price a European put option. The main difference is that in case of default, the holder of a put option is in the money and can receive the strike price whereas the holder of a call option receives 0.

We start with the pricing of a digital European put option. A European digital put option is a contract which pays one dollar in case the price of the underlying asset falls below the strike price. When default of the underlying asset is possible we assume that the digital option knocks out and pays nothing. So, the payoff of the European digital put option is:

$$P_T = 1\{S_T < K\} 1\{\Gamma \wedge \tau > T\}.$$

If we use the normalized equity price Y_t then

$$1\{S_T < K\} = 1 - 1\{S_T \geq K\} = 1 - 1\{Y_T \geq \ln(K/B_0)\}.$$

The martingale representation theorem gives us the price of the digital option as the expected value of its future payoff under the risk-neutral probability measure:

$$
\begin{aligned}
P_0 &= e^{-rT} E^Q \{(1 - 1\{S_T \geq K\}) 1\{\Gamma > T\}\} P(\tau > T) = \\
&= e^{-(r+\lambda)T} \left[P^Q \{\Gamma > T\} - P^Q \{Y_T > \ln(K / B_0); \Gamma > T\} \right]
\end{aligned}
$$

Substituting the expressions for the probabilities given earlier in the text, we obtain the price of the European digital put option:

$$
P_0 = \begin{cases} e^{-(r+\lambda)T} \left([\Phi(h_1) - \Phi(d_1)] - e^{-2b\nu} [\Phi(-h_2) - \Phi(d_2)] \right) & \text{if } K \geq B_0 \\ 0 & \text{if } K < B_0 \end{cases},
$$

where

$$
h_{1,2} = \frac{b \pm \mu T}{\sigma \sqrt{T}}.
$$

Further we use the analogy with the European call option case with the only difference that in the case of default the payoff of the put option at maturity is equal to the strike price. We can show that the price of the European put option is:

$$
P_0^E = KP_0 - P_0^S + KI_0,
$$

$$
P_0 = \begin{cases} e^{-(r+\lambda)T} \left([\Phi(h_1) - \Phi(d_1)] - e^{-2b\nu} [\Phi(-h_2) - \Phi(d_2)] \right) & \text{if } K \geq B_0 \\ 0 & \text{if } K < B_0 \end{cases},
$$

$$
P_0^S = \begin{cases} S_0 \left([\Phi(h_3) - \Phi(d_3)] - e^{-2b\tilde{\nu}} [\Phi(-h_4) - \Phi(d_4)] \right) & \text{if } K \geq B_0 \\ 0 & \text{if } K < B_0 \end{cases},
$$

$$
I_0 = e^{-rT} \left(1 - e^{-\lambda T} \left[\Phi(h_1) - e^{-2b\nu} \Phi(-h_2) \right] \right),
$$

where

$$
h_{3,4} = \frac{b \pm \tilde{\mu} T}{\sigma \sqrt{T}}.
$$

6. SPREAD CURVES

In this section we present the credit spread curve generating possibilities of the ORION model with a fixed barrier level. First, we choose a general set of parameters. Then we vary individual parameters of the model to see how they influence the spread curve. In the analysis we assume that the barrier level is fixed and no dividends are paid on the equity. The main parameters of the model are hazard rate λ, equity volatility σ, and distance to barrier, defined as $\ln(S_0) - \ln(B_0)$. Secondary parameters of the model are maturity of the bond, coupon payments, recovery rate, and risk-free rate. The table below contains the general set of parameters we used to analyze spread curves.

Figure 3. Model parameters

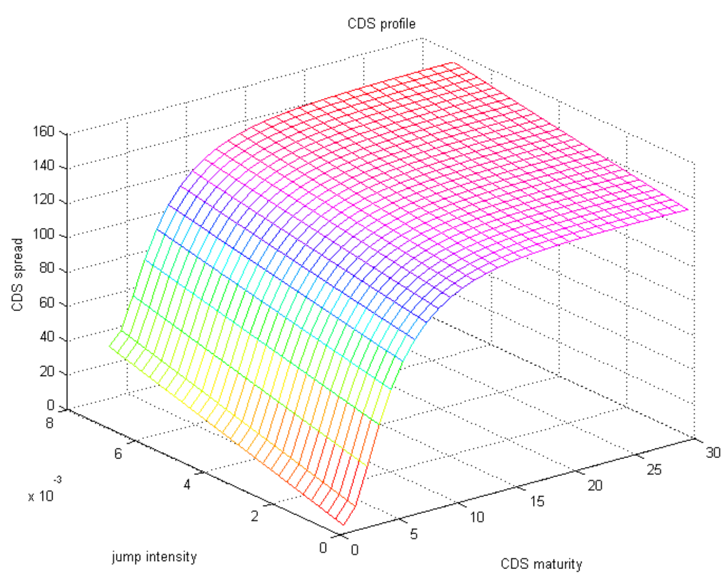
Parameters	Value
Hazard rate	0.7%
Equity volatility	19%
Barrier	3.37
Stock price	7.53
Recovery rate	40%
Risk-free rate	2%
Coupon rate	5%

Source: Lehman Brothers.

Figures 4, 5 and 6 below show different spread curves for different values of jump intensity, equity volatility and barrier level (holding the remaining parameters fixed at values given in Figure 3).

In Figure 4, we see that the effect of the jump is to give a positive spread at the short end of the credit curve: the higher the jump intensity, the higher the spread to compensate for a sudden jump to default must be.

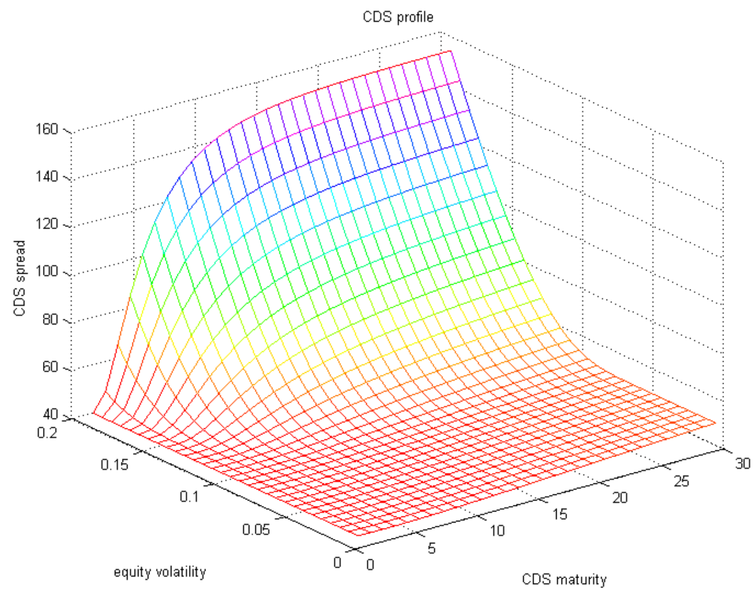
Figure 4. CDS spread curves and jump intensity



Source: Lehman Brothers.

Figure 5 shows spread curves for different volatilities of the equity price. We see that the effect of the equity volatility is to increase the level of the spread curve: the higher the equity volatility, the higher the spread to compensate for a higher default probability must be.

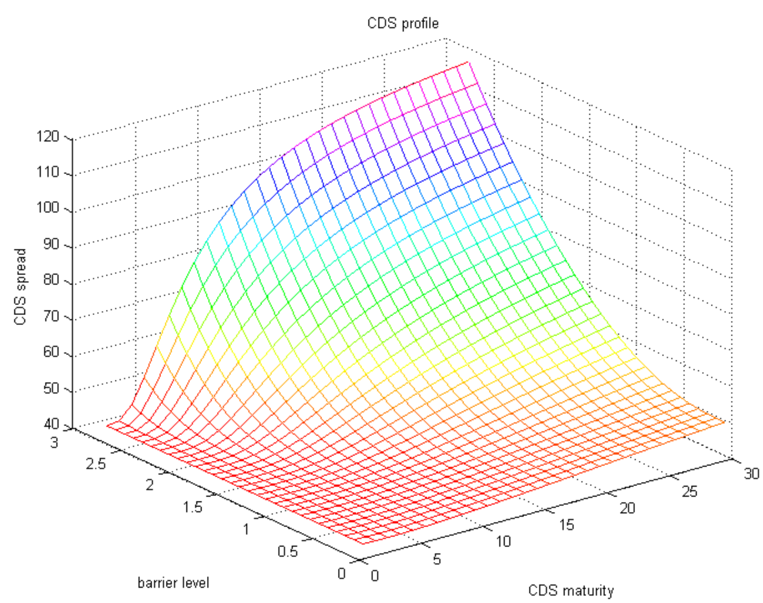
Figure 5. CDS spread curves and equity volatility



Source: Lehman Brothers.

Finally, Figure 6 contains spread curves for different values of the distance to barrier. We see that the effect of the default barrier is also to increase the level of the spread curve: the higher the default barrier, the higher the spread to compensate for a higher default probability must be.

Figure 6. CDS spread curves and default barrier



Source: Lehman Brothers.

7. VOLATILITY SKEW

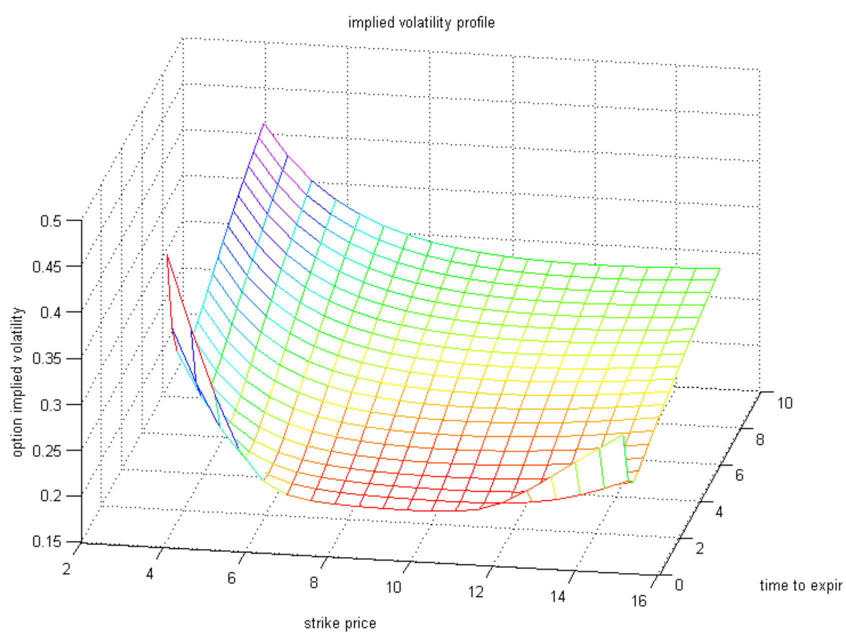
As we have already mentioned, default event and the jump in equity price can induce volatility skew for the Black-Scholes implied volatilities. Here we analyze ORION's capability to produce a Black-Scholes volatility skew.

The volatility skew comes from a violation of the Black and Scholes assumptions, most notably that equity returns follow a log normal distribution. Skewness in the return distribution will typically cause a skew in the volatility curve (Black-Scholes volatility for different strike price that match market prices). The skew has been observed since the crash of 1987, the interpretation being that investors have been factoring a crash probability into the equity return distribution. In that same spirit, our model incorporates an unexpected jump to default that can account for a negative skewness equity in returns[3].

Using derived formula for the price of a European put price, we generate ORION put option prices for a given volatility of the equity value and different strike price ratios of the option. The Black-Scholes formula is applied to the resulting option prices to obtain Black-Scholes implied volatilities. This exercise is done for different values of the model parameters. Specifically, we infer Black-Scholes volatility skews for different values of hazard rates and distances to barrier. The figures below show Black-Scholes implied volatilities for different time to expiry, equity volatility and jump intensity.

We find that for a longer time to expiry of the option, the volatility curve becomes flatter and there is less skew, which is empirically observed (Figure 7). For a higher level of equity volatility, the volatility curve moves upward and becomes flatter (Figure 8). For a higher level of jump intensity, the volatility curve moves upward and becomes more skewed (Figure 9).

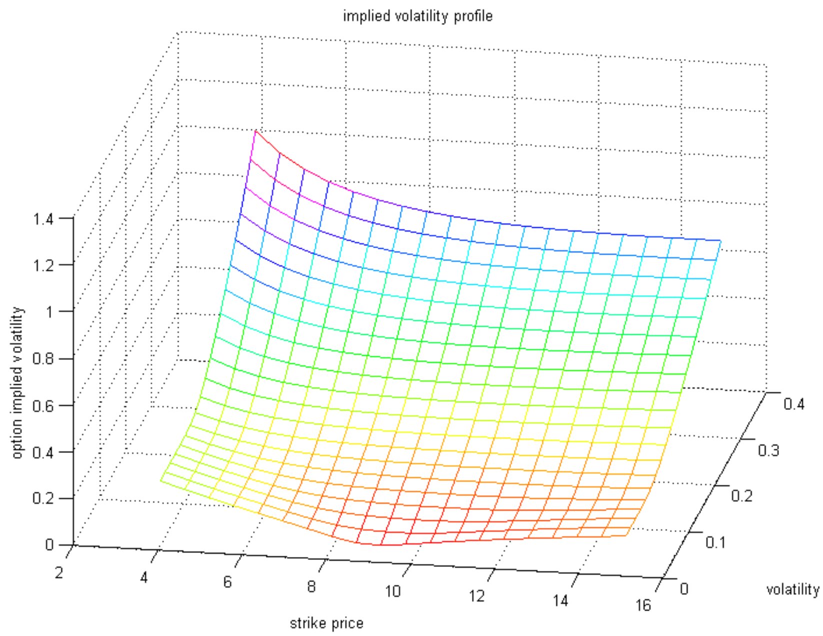
Figure 7. Skew function of time to expiry



Source: Lehman Brothers.

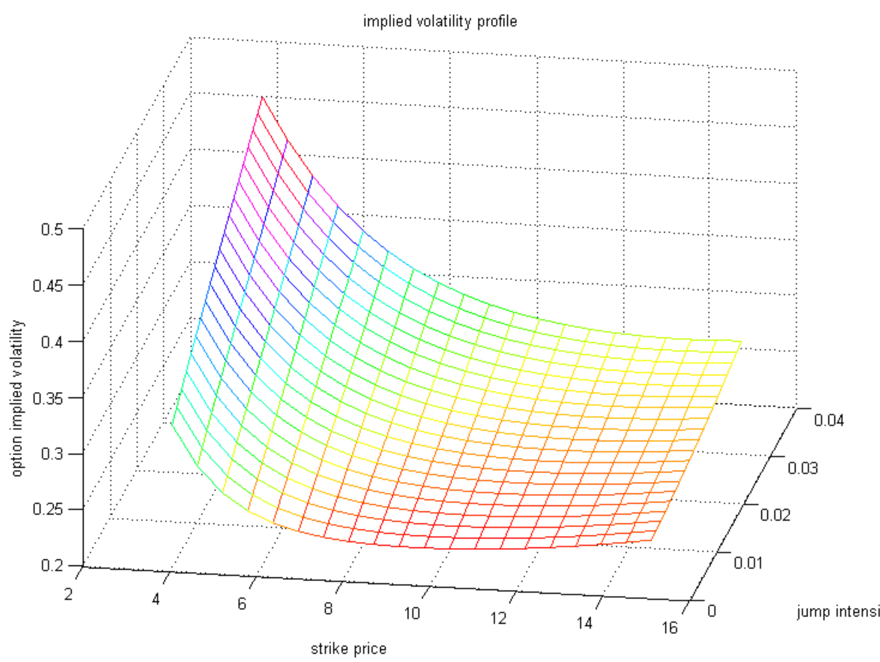
[3] *See Rebonato (2004) for interesting discussions on the equity volatility skew.*

Figure 8. Skew function of volatility



Source: Lehman Brothers.

Figure 9. Skew function of jump intensity



Source: Lehman Brothers.

8. CALIBRATION OF THE MODEL

8.1. Calibration with the credit curve

The calibration of the model can be done using the issuer bond or CDS credit spread curve. The credit spread curve is a plot of the bond spreads against the bond time-to-maturity[4] Inputs for the calibration include:

the Libor rate curve for discounting the cash flows;

current market CDS spreads or bond prices;

the different time to maturity;

recovery rates assumptions;

the current stock price;

We can then minimize the sum of squared deviation between theoretical spreads and market spreads. We can use either the CDS (3y, 5y and 10y CDS spreads) curve or jointly the bond spread curve and the CDS spread curve by backing up the implied default probability and calibrating the model to those.

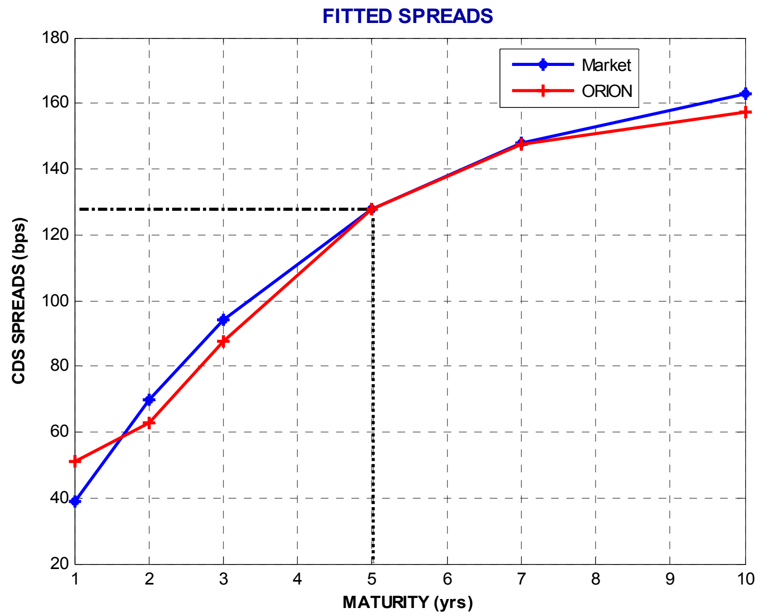
Example: We consider the CDS spread curve for ABB - Currency: CHF - Rating: Ba2/BB+ - Stock Price: CHF 7.58. Date 18 October 2004.

Figure 10. Calibration Results

parameters	value
λ	0.84%
Bo	1.70
σ	28.2%

[4] *See Berd et al. 2003 for some credit curve construction methodology.*

Figure 11. Fitted spreads vs. market spreads



Source: Lehman Brothers.

With these parameters we could infer the relative value between debt and equity. We can compare the equity implied volatility and the credit implied volatility. The credit implied volatility has to be converted into a standard Black-Scholes volatility (without jump in the equity process, the latter will be higher than the former because it encompasses the jump volatility) to be compared with the market implied volatility. If the transformed CDS-implied volatility is higher than the equity-implied volatility, then credit trades cheap vs. equity (the embedded put option in the CDS is too expensive) and the trades should be for instance sell protection and short equity or sell protection and long an equity put option. A direct comparison of the put prices or call prices would lead to similar results. If the model implied ATM put price is higher than the ATM market put price, credit is relatively cheap vs. equity.

8.2. Calibration with the volatility skew

We can also work backward from the volatility skew and calibrate a hazard rate and distance to barrier implied by the equity option market. These two implied parameters define a new equity-implied credit curve which can then be compared with the market credit curve.

With these parameters we could infer the relative value between debt and equity. If the spread curve constructed from the parameters calibrated from the equity volatility market gives on average larger spread than the market credit spread curve then the debt trades cheap vs. equity and the trades should be for instance long debt and short equity or long debt and long an equity put option. This assumes however that the same fundamental factors are driving both the equity and the credit market. We have to be careful that any supply and demand dynamics specific to the equity volatility market could influence the slope of the skew without impacting credit risk.

9. CONCLUSION

We have described the new ORION model with unexpected default. The enhanced model remains parsimonious and easy to handle. Moreover, it becomes even more flexible in matching spread curves and volatility skew observed on the market as it allows modelling non-zero short maturity spreads (for the credit curve) and fatness of the distribution tails of equity (for the skew). The model offers an analytical solution for pricing a corporate bond, a credit default swap (CDS) and, in case of a fixed barrier, European call and put options. This basic set-up is applicable to capital structure arbitrage, not only for hedging but also for relative value between debt and equity. The model can be applied to hedging and the relative value credit-equity trades. We plan to explore these applications in forthcoming publications.

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Consistent Risk Measures for Credit Bonds

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We suggest new definitions of credit bond duration and convexity that remain consistent across all levels of credit quality including deeply distressed bonds, and introduce additional risk measures that are consistent with the survival-based valuation framework. We then show how to use these risk measures for the construction of market neutral portfolios.

INTRODUCTION

This paper continues our investigation of the consistent valuation methodology for credit-risky bonds (see Berd, Mashal, and Wang [2003] and [2004]). In the previous two articles we have developed a set of term structures that are estimated using all the bonds of a given issuer (or sector) as a whole, rather than a specific bond of that issuer. In particular, our primary measure, the term structure of survival probabilities, clearly refers to the issuer and not to any particular bond issued by this issuer. However, when considering a particular bond, investors typically ask three questions:

- Is this bond rich or cheap compared with other bonds of the same issuer or sector?
- How much excess return does this bond provide for taking on the credit risk?
- How can we monetize these relative values, once we measure them?

The answer to the first question lies in the comparison of the observed bond price with the fair value price given by the fitted issuer credit term structures. The OAS-to-Fit measure, introduced in Berd, Mashal, and Wang (2003), gives an unambiguous and consistent answer to this question, free of biases associated with the term to maturity or level of coupon, which plague the conventional spread measures.

The answer to the second question then becomes straightforward, since we have already determined in the previous step the term structure of "fair value" par spreads of the issuer with respect to the underlying credit risk-free market. By adding the consistent issuer-specific and bond-specific spread measures, we are able to give a robust definition of a bond spread and sidestep the ambiguities associated with non-par bond excess return estimation.

In order to answer the last question one must devise a recipe for hedging and risk managing credit bonds, which of course requires calculation of various sensitivity measures. Derivation of such measures and, in particular, the consistent definition of a bond's duration and convexity, as well as the bond's sensitivity to hazard rates and recovery values within the survival-based valuation framework, are the objectives of the present paper.

We will show that the correctly defined duration of credit bonds is often significantly shorter than the widely used modified adjusted duration. This disparity helps explain the fact that high yield bonds do not have quite the same degree of interest rate sensitivity as high grade ones and is especially evident for distressed bonds, for which a 10-year maturity bond may have a duration as low as 1 year. This fact is well known to portfolio managers qualitatively – our paper provides its quantitative formulation.

The flip side of the same coin is the apparent negative correlation between interest rates and the conventionally defined credit spreads (OAS). It results in a similar effect of a dampened effective duration of credit bonds, as explained in Berd and Ranguelova (2003) and Berd and Silva (2004). We argue that a large portion of this negative correlation is "optical" in nature and is due to a misspecification of credit risk by the conventional OAS spread measures.

We also show that what is commonly regarded as a convexity measure for (both credit and Treasury) bonds is also a "duration" measure with respect to interest rate curve steepening/flattening moves. This is an important observation because the so-called "convexity trades" often under- or outperform not due to directional changes in interest rates, but because of the changes in the shape of the curve – as was the case, for example, during the past year and a half.

Finally, we present a concise and simple recipe for setting up well-hedged portfolios of bonds that generalizes the well-known duration-neutral and barbell trading strategies. We discuss how to choose the risk dimensions with respect to which one might wish to be hedged and how to find optimal security weights in the corresponding portfolios.

SURVIVAL-BASED MODELING OF CREDIT-RISKY BONDS

Let us start with a brief reminder of the survival-based valuation methodology, following Berd, Mashal, and Wang (2003). Consider a credit-risky bond that pays fixed cash flows with specified frequency (usually annual or semi-annual). According to the fractional recovery of par assumption, the present value of such a bond is given by the *expected discounted* future cash flows, including the scenarios when it defaults and recovers a fraction of the face value and possibly of the accrued interest, discounted at the risk-free (base) rates.

By writing explicitly the scenarios of survival and default, we obtain the following pricing relationship (see Duffie and Singleton [2003] and Schonbucher [2003] for a detailed discussion of general pricing under both interest rate and credit risk):

$$\begin{aligned} PV \;=\; & \sum_{i=1}^{N} \left(CF^{prin}(t_i) + CF^{\text{int}}(t_i) \right) \cdot \mathrm{E}_t \left\{ Z_{base}(t_i) \cdot I_{\{t_i < \tau\}} \right\} \\ & + \int_0^T \mathrm{E}_t \left\{ \left(R_p \cdot F^{prin}(\tau) + R_c \cdot A^{\text{int}}(\tau) \right) \cdot Z_{base}(\tau) \cdot I_{\{u < \tau \le u+du\}} \right\} \end{aligned} \qquad [1]$$

Here τ denotes the (random) default time, $\mathrm{E}_t\{\bullet\}$ denotes an expectation under the risk-neutral measure at time t, $Z_{base}(\tau)$ is the (random) base discount factor, and $I_{\{X\}}$ denotes an indicator function for a random event *X*.

The first sum corresponds to scenarios in which the bond survives until the corresponding payment dates without default. The total cash flow at each date is defined as the sum of principal $CF^{prin}(t_i)$, and interest $CF^{\text{int}}(t_i)$, payments. The integral corresponds to the recovery cashflows that result from a default event occurring in a small time interval $[u, u+du]$, with the bond recovering a fraction R_p of the outstanding principal face value $F^{prin}(\tau)$ plus a (possibly different) fraction R_c of the interest accrued $A^{\text{int}}(\tau)$.

Following the market convention and Berd, Mashal, and Wang [2003], we assume in the following that recoveries happen discretely only on coupon payment dates. For the case of fixed-coupon bullet bonds with coupon frequency *f* (eg, semi-annual *f=2*) and no recovery of the accrued coupon, this leads to a simplified pricing equation:

$$\begin{aligned} PV \;=\; & Z_{base}(t_N) \cdot Q(t_N) + \frac{C}{f} \cdot \sum_{i=1}^{N} Z_{base}(t_i) \cdot Q(t_i) \\ & + R_p \cdot \sum_{i=1}^{N} Z_{base}(t_i) \cdot D(t_{i-1}, t_i) \end{aligned} \qquad [2]$$

Importantly, the probability $D(t_{i-1}, t_i)$ that default will occur within this time interval, is related to the survival probability $Q(t)$ in a simple well-known fashion, reflecting the conservation of total probability:

[3] $$D(t_{i-1}, t_i) = Q(t_{i-1}) - Q(t_i)$$

In Appendix A we derive a generic continuous-time approximation to the exact formula for the clean price of a fixed-coupon credit bond expressed through the term structure of the instantaneous forward interest rates and hazard (forward default) rates. This formula is quite accurate across all values of coupons and for all shapes and levels of the underlying interest rate and hazard rate curves. While such an approximation is superfluous for numerical computations, it comes in very handy for analytical estimates of bond risk measures, which is why we show it here and will use it in the next section.

RISK MEASURES FOR CREDIT BONDS

In order to risk manage credit bond portfolios one must first calculate various sensitivity measures. Here we define a bond's duration and convexity, as well as its sensitivity to hazard rates and expected recovery values in a manner consistent with the survival-based valuation framework. As it turns out, the newly introduced risk measures are often substantially different from the commonly used ones such as modified adjusted duration, spread duration and convexity (see Fabozzi [2000] and Tuckman [2002] for standard definitions).

Interest Rate Duration

There are many ways to define duration. We can think of duration as the sensitivity to changes in interest rates. We can also think of duration more generally as sensitivity to changes in non-credit related discount rates (such as the issue-specific OAS-to-Fit rate).

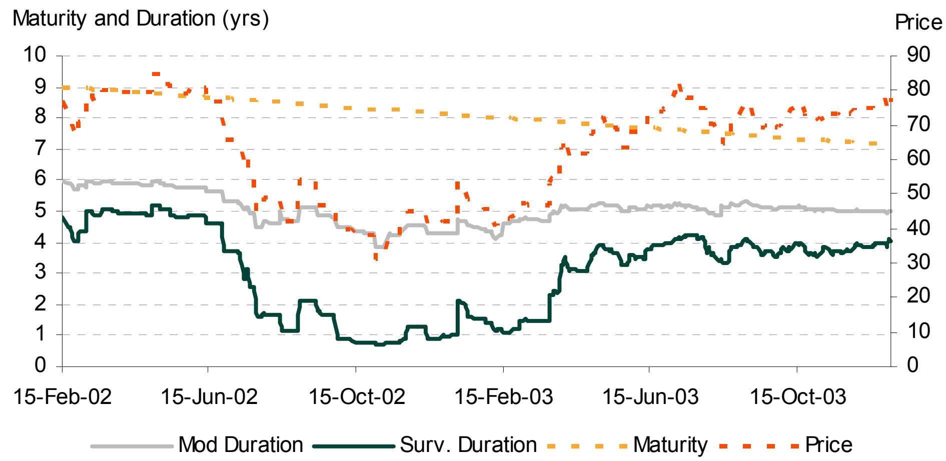
The two definitions become identical if we take the continuously compounded forward rates to be the primary variable with respect to which we measure the sensitivity. A parallel shift in instantaneous forward rates and an equal constant shift in OASF (the non-credit risk related pricing premium/discount) would cause an identical change in the bond's price. Had we defined the interest rate sensitivity with respect to some other measure of rates, such as a parallel shift in par yields, this equality would not hold.

The definition of duration in such terms is a direct modification of the well-known Macaulay duration. If we calculate the survival-based duration as the (minus) log-derivative of the bond's clean price with respect to OASF, it becomes equal to the weighted time to cash flows, where the weights reflect not only the present value of the cash flows (as in the conventional Macaulay duration) *but also the probability of realization of the cash flow*:

[4] $$\begin{aligned} D \;=\; & \left[\sum_{i=1}^{N} t_i \cdot \frac{C}{f} \cdot Z_{base}(t_i) \cdot Q(t_i) \cdot e^{-OASF \cdot t_i} + t_N \cdot Z_{base}(t_N) \cdot Q(t_N) \cdot e^{-OASF \cdot t_N} \right. \\ & \left. + \sum_{i=1}^{N} t_i \cdot R_p \cdot Z_{base}(t_i) \cdot \big(Q(t_{i-1}) - Q(t_i) \big) \cdot e^{-OASF \cdot t_i} \right] \cdot \frac{1}{P} \end{aligned}$$

The survival-based effective duration is always less than the classical Macaulay duration, reflecting the positive probability of receiving earlier (and larger) cash flows in the case of default. Depending on the level of the implied default rates, the differences can be quite large, as shown in Figure 1. It follows the changes in a particular Calpine bond (CPN 8.5 2/15/2011) as the company underwent different levels of distress during the past three years.

Figure 1. Survival-based duration vs. conventional duration, CPN 8.5 2/15/11



Source: Lehman Brothers.

We see in particular that during 2002, as the company was deeply distressed and the bonds were trading at very low prices, the survival-based duration was as low as 1 year, compared with the conventional modified duration which decreased only slightly from the normal levels to around 4-year value. This is a very significant difference for a bond that still had more than eight years to maturity at the time. One could interpret it in terms of a much shorter "expected life" of the distressed security compared with its nominal maturity.

Many high yield portfolio managers are well aware of the propensity of distressed bonds to have much lower interest rate sensitivity than that prescribed by the conventional modified duration, but until recently few have been able to quantify this effect. Equation [4] gives a precise definition to this intuition, and Figure 1 demonstrates how important it can be.

The Apparent Negative Rates-Spreads Correlation

As we just demonstrated, adoption of the survival-based valuation methodology leads to a significant decrease in the forecasted sensitivity to interest rates, even for relatively high grade (low credit risk) credit bonds. This effect should not be entirely surprising to credit portfolio managers. However, it has been usually discussed under the guise of a seemingly unrelated issue of the correlation between interest rates and spreads.

In two previous papers (Berd and Ranguelova [2003] and Berd and Silva [2004]) we have documented the empirical evidence for negative correlation between rates and spreads, and have explained how this results in a lower effective duration of credit bonds. In the framework of commonly used modified durations and OAS durations, the lower effective "duration" of credit bonds comes from the fact that (statistically) spreads tend to get tighter when rates get higher and vice versa. Thus, the movement of spreads tends to offset the movement of interest rates, and the total yield of credit bonds changes by only a fraction of the amount by which the Treasury (or LIBOR) rate changes. This, in turn, means that the expected price impact of a 1bp move in interest rates will be less than what one should expect by simply looking at a bond's conventional (modified adjusted) duration.

The net effect is consistent with that obtained from a fundamentally different model presented in a previous subsection. Why is this the case? Why do spreads, defined in a manner inconsistent with the survival-based valuation framework, conspire to move in such a way as to produce an effect that is similar to the correct model?

To answer this question, let us consider what would the survival-based valuation framework look like if we believed in the spread-discount based methodology (see O'Kane and Sen [2004] for detailed definitions of conventional spread measures). For convenience, we will use the continuous-time approximation defined in Appendix A and ignore the small correction terms. We will also assume for simplicity flat interest rate, hazard rate and spread curves. On the left hand side of equation [5] we write the price of the bond with coupon *C* and maturity *T* using the valuation based on the conventional spread *S*, and on the right hand side the same price expressed under the survival-based methodology.

[5] $$\frac{C}{r+S}\cdot\left(1-e^{-(r+S)\cdot T}\right)+e^{-(r+S)\cdot T}=\frac{C+h\cdot R_p}{r+h}\cdot\left(1-e^{-(r+h)\cdot T}\right)+e^{-(r+h)\cdot T}$$

This relationship may be considered as a parametric definition of the conventional spread *S* as a function of interest rate *r* and hazard rate *h*. This function can be easily obtained numerically by solving the equation for *S,* but it is more elucidating to see the relevant dependencies using the following approximate analytical solution, obtained by solving for the small correction in an expansion $S\approx h\cdot\left(1-R_p\right)+\varepsilon$ in the limit $\left(r+h\right)\cdot T<<1$:

[6] $$S\left(r,h\right)\approx h\cdot\left(1-R_p\right)+\frac{1}{2}\cdot R_p\cdot h\cdot T\cdot\left(C-r-h\cdot\left(1-R_p\right)\right)$$

The correction to the "credit triangle" formula is proportional to the amount by which the coupon differs from the par level, times the total default probability – in other words, the spread bias is related to the risk of losing the bond's price premium. We can see from this formula that if we keep the hazard rates constant, then rising interest rates lead to falling conventional bond spreads. There are of course other, less technical explanations, related to common economic driving factors of interest rates and credit risk, the co-movement of recovery and default rates, etc. But the demonstrated "optical" co-movement induced by the inherent biases of the OAS as a credit measure can account for a large portion of the observed negative correlation between rates and spreads.

Interest Rate Convexity and Twist Duration

Since the survival-based duration is significantly different from the modified duration, it is not surprising that the convexity measure will also be very different. The convexity is defined as the second derivative of the bond price with respect to yield change, expressed as a fraction of the bond's price. We remind the reader that in our definition, the derivative with respect to a parallel shift in forward interest rates is precisely equal to the derivative with respect to a change in OAS-to-Fit.

[7] $$\begin{aligned}\Gamma &= \frac{1}{P}\cdot\frac{\partial^2 P}{\partial r^2}\\ &= \left[\sum_{i=1}^{N} t_i^2\cdot\frac{C}{f}\cdot Z_{base}(t_i)\cdot Q(t_i)\cdot e^{-OASF\cdot t_i}+t_N^2\cdot Z_{base}(t_N)\cdot Q(t_N)\cdot e^{-OASF\cdot t_N}\right.\\ &\quad\left.+\sum_{i=1}^{N} t_i^2\cdot R_p\cdot Z_{base}(t_i)\cdot\left(Q(t_{i-1})-Q(t_i)\right)\cdot e^{-OASF\cdot t_i}\right]\cdot\frac{1}{P}\end{aligned}$$

Survival-based convexity follows a similar pattern to the duration in terms of its deviation from the conventional measure, decreasing rapidly as the implied default rates rise.

We can use the following approximation to estimate the price impact of changes in the bond-specific non-credit related discount/premium encoded in the OAS-to-Fit measure:

[8] $$\frac{P(\Delta OASF)}{P_0} \approx 1 - D \cdot \Delta OASF + \frac{1}{2} \cdot \Gamma \cdot \Delta OASF^2$$

We can also estimate the price impact of parallel shifts and (steepening or flattening) twists in the interest rate curve using its duration and gamma. Assuming a linear change in the forward rates as a function of the term to maturity:

[9] $$\Delta r(t) = \Delta r_{shift} + t \cdot \Delta r_{twist}$$

The price impact can be estimated to the second order in shift and the first order in twist:

[10] $$\frac{P(\Delta r_{shift}, \Delta r_{twist})}{P_0} \approx 1 - D \cdot \Delta r_{shift} + \frac{1}{2} \cdot \Gamma \cdot \Delta r_{shift}^2 - \frac{1}{2} \cdot \Gamma \cdot \Delta r_{twist}$$

The twist duration is equal to half of the shift convexity. Since the shifts and twists explain the majority of the interest rate variability over short horizons, and since our modified definitions for duration and gamma are robust with respect to credit risk levels, this approximation can prove to be very useful across a wide range of market conditions.

In the rest of this paper we will focus on the interest rate dependence and denote these duration and convexity measures as D_r and Γ_r, respectively.

Credit Risk Sensitivity

The duration and convexity measures introduced in the previous subsection deal with bond price sensitivity to factors other than credit risk. The credit risk sensitivity is encoded in the dependence of the bond price on the changes in the hazard rate and recovery rate.

Let us define the hazard rate duration similarly to the interest rate duration D_r – ie, as the (minus) log-derivative of the bond price with respect to a parallel shift of its hazard rate $D_h = -\frac{1}{P} \cdot \frac{\partial}{\partial h} P$. The recovery rate duration $D_R = \frac{1}{P} \cdot \frac{\partial}{\partial R_p} P$ is defined with a plus sign to maintain a positive sign of the duration.

It is convenient to work with the continuous-time approximation to the price of a credit bond (see Appendix A). In Appendix B we calculated the sensitivities of this price to changes in interest rates and hazard rates which we need for our duration measures.

Before we proceed, let us introduce a risk metric borrowed from the CDS market, called “risky PV01”. This measure is instrumental for mark-to-market valuation of default swaps (see O’Kane and Turnbull [2003]), and in the continuous-time approximation it is given by:

[11] $$RPV01(T) = \int_0^T du \cdot e^{-\int_0^u (r(s)+h(s)) \cdot ds}$$

Comparing equations [11], [27] and [29], we obtain the following expression for the bond’s hazard rate duration in terms of interest rate duration and RPV01:

[12] $$D_h = D_r - \left(R_p - (1 - R_c) \cdot \frac{C_f}{2f} \right) \cdot P \cdot RPV01 \approx D_r \cdot (1 - R_p \cdot P)$$

Using the definition of the BCDS term structure in the continuous-time approximation (see Berd, Mashal and Wang [2004]), we obtain an expression for the recovery duration:

[13] $$D_R = \frac{1}{(1-R_p)\cdot P}\cdot RPV01\cdot BCDS \approx \frac{1-P}{(1-R_p)\cdot P}$$

In [12] and [13] we also show "ballpark" approximations which work quite well in many cases, particularly for distressed bonds.

Finally, borrowing from CDS nomenclature again, we define the value-on-default (VOD) risk of a bond which measures the percent loss in case of instantaneous default:

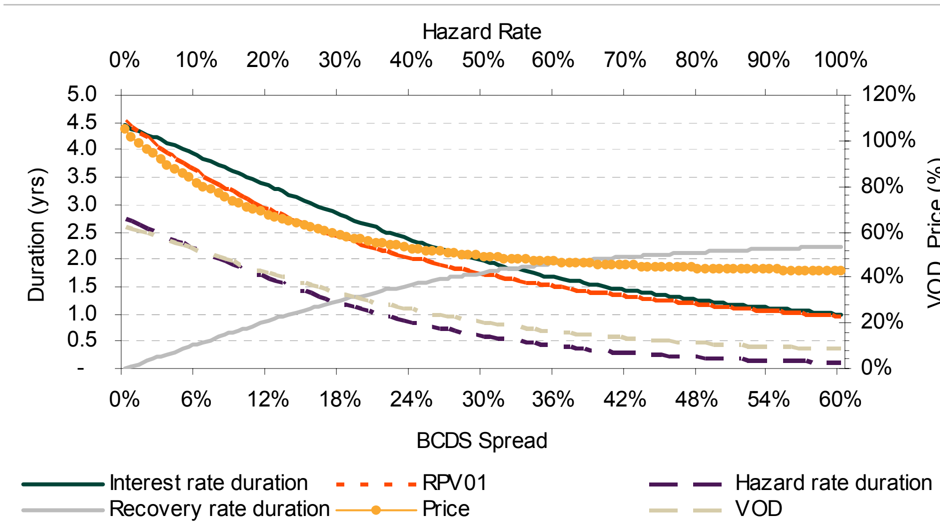
[14] $$VOD = 1 - \frac{R_p}{P}$$

Since RPV01 is manifestly positive (and is usually of the same order of magnitude as the interest rate duration), we conclude that the hazard rate duration is shorter than the interest rate duration. This means that distressed bonds become insensitive not only to interest rates, as demonstrated in the previous subsection, but also to hazard rates. Indeed, once the market prices them to "imminent default", any further increase in hazard rate bears no additional loss for the bond's price. Since the bond price in this scenario is already close to the expected recovery rate, the VOD risk is also very small.

On the other hand, the recovery sensitivity of the bond grows with distress levels proportionally to BCDS. For low levels of credit risk, when BCDS is typically of the order of a few tens or hundreds of basis points, the recovery sensitivity is very small. But for high levels of credit risk BCDS can grow as high as tens of percent, making the recovery rate sensitivity a sizeable number. This should not be surprising since in this case the same change of recovery value is being compared with a much lower initial price level.

Figure 2 shows the dependence of the interest, hazard and recovery rate durations, VOD and RPV01 upon the level of credit risk. We consider a hypothetical 5-year bond with 5% coupon, assuming a flat 4% LIBOR discount rate, flat term structure of hazard rates and 40% recovery rate. The BCDS spread is plotted along the bottom x-axis, and the corresponding hazard rate is plotted along the top x-axis. All durations are plotted against the left y-axis, and the bond price and VOD are plotted against the right y-axis.

Figure 2. Interest, hazard and recovery rate durations, RPV01 and price as functions of hazard rate and BCDS spread



Source: Lehman Brothers.

Let us now turn to the convexity measures. In Appendix B we have derived the second-order derivatives of the bond price with respect to interest rates and hazard rates. Comparing equations [28] and [30] with each other and with the definition of the RiskyPV01 in [11], we obtain the following relationship between the interest and hazard rate convexities:

[15] $$\Gamma_h = \Gamma_r + 2 \cdot \left(R_p - (1 - R_c) \cdot \frac{C_f}{2f} \right) \cdot \frac{1}{P} \cdot \frac{\partial}{\partial h} RPV01$$

Since RPV01 is a decreasing function of the hazard rate, we conclude that the hazard rate convexity is lower than the interest rate convexity for cash bonds.

CONSTRUCTING WELL-HEDGED BOND PORTFOLIOS

Let us now turn to the last question posed in the Introduction. How does one monetize a relative value view? First, in order to avoid inadvertently producing positive or negative returns due to coincidental market timing, one must formulate the strategy in a market-neutral way, using long-short trades. This is necessary while back-testing – in the actual investment process there could be a mixture of directional views (ie, tactical asset allocation) together with relative value views (ie, sector and security selection).

Secondly, when defining the long-short relative value trades one must first determine which risks need to be hedged, or in other words with respect to which market factors do we wish to be “market neutral” and for what time horizon. It is often impossible (or impractical) to be hedged with respect to all market factors. The choice of the hedge will be determined by the expected holding horizon of the relative value trade.

For example, if one expects a fast convergence to fair value then the correct hedge is with respect to sensitivity to the most volatile market factors during the short term – interest rates and spreads. If, however, the trade is not expected to converge fast and could be held to maturity in order to realize the perceived relative value, then the correct hedge is with respect to the long-run risk factors, including the idiosyncratic credit event risk and recovery.

Before we can set up a market-neutral trade, ie, a trade that is well hedged with respect to various risk factors except the one for which we have a rich/cheap signal, we must first clarify the rules for aggregating the sensitivities for a portfolio of bonds. It can be proved that the usual aggregation rules apply and need not be modified when using the survival-based risk measures instead of the conventional ones – the portfolio duration and convexity with respect to a market risk factor are equal to the market value weighted average of constituent security durations and convexities, respectively.

Indeed, the portfolio market value is equal to a sum of market values of constituent bonds, which in turn are just their quantities q_i times their prices P_i. While we discard the strippable cash flow valuation methodology when it comes to a single credit-risky bond, the sum of market values rule still applies to a portfolio of securities. Note that we omit the accrued interest from all calculations because its value is insensitive to all risks (except the risk of immediate default) and may be excluded from duration and convexity calculations.

[16] $$MV_{port} = \sum_{i=1}^{N} q_i \cdot P_i$$

The portfolio duration and convexity with respect to a market risk factor F are defined in the same manner as those for a single bond, which leads to the aggregation rule stated above (the sign κ is equal to -1 for interest and hazard rate durations, and +1 for recovery duration):

[17] $$D_{port} = \frac{\kappa}{MV_{port}} \cdot \frac{\partial MV_{port}}{\partial F} = \sum_{i=1}^{N} \frac{q_i \cdot P_i}{MV_{port}} \cdot \frac{\kappa}{P_i} \cdot \frac{\partial P_i}{\partial F} = \sum_{i=1}^{N} w_i \cdot D_i$$

[18] $$\Gamma_{port} = \frac{\kappa}{MV_{port}} \cdot \frac{\partial^2 MV_{port}}{\partial F^2} = \sum_{i=1}^{N} \frac{q_i \cdot P_i}{MV_{port}} \cdot \frac{\kappa}{P_i} \cdot \frac{\partial^2 P_i}{\partial F^2} = \sum_{i=1}^{N} w_i \cdot \Gamma_i$$

Finally, the VOD risk of the portfolio is also equal to the market value weighted sum of VOD risks of the individual bonds. The aggregated VOD risk has a meaning of a percentage loss in case of simultaneous default of all bonds in the portfolio, and as such it only makes sense for a portfolio consisting of bonds of the same issuer (or perhaps of several issuers all of which are driven to default by the same exogenous factor).

[19] $$VOD_{port} = 1 - \frac{\sum_{i=1}^{N} q_i \cdot R_i}{MV_{port}} = \sum_{i=1}^{N} \frac{q_i \cdot P_i}{MV_{port}} \cdot \left(1 - \frac{R_i}{P_i}\right) = \sum_{i=1}^{N} w_i \cdot VOD_i$$

Let us now derive a generic recipe for setting up market-neutral trades using credit risky bonds. The market-neutral trade is a zero-cost long-short portfolio for which the portfolio durations with respect to the selected set of risk factors are equal to zero.

Generally speaking, such a trade would not be possible to construct unless one also allows some amount of (long or short) cash position in the portfolio. Thus, the portfolio which we consider will contain N bonds plus a cash position. Let us also choose a target set of K risk factors with respect to which we wish to immunize the portfolio. These could be, for example, the interest rate and hazard rate durations, twist sensitivities, VOD, etc. We will assume that all the chosen target risk factors satisfy the market value weighted aggregation rules. Denoting the bond weights as w_i, and the k-th risk sensitivity of the i-th bond as δ_i^k, we can rewrite the market-neutral trade definition in the following fashion:

[20] $$\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} -\left(\sum_{i=1}^{N} \nu_i\right) \Big/ W \\ \nu_1 / W \\ \vdots \\ \nu_N / W \end{pmatrix}, \qquad \text{where} \qquad W = \max\left(\sum_{i=1}^{N} \nu_i^+, \sum_{i=1}^{N} \left(-\nu_i^-\right)\right)$$

Here, we introduced the auxiliary variables ν_i which have the meaning of non-normalized weights of the bond positions, and ν_i^+ and ν_i^- stand for the positive and negative values, respectively. The normalization factor W is defined so that the long-only and short-only sub-portfolios each have a total weight of 1 (ie, the trade is not levered), and the cash weight w_0 is chosen to explicitly solve the zero-cost constraint. The auxiliary variables satisfy the following equation, which guarantees that the portfolio is market-neutral:

[21] $$\begin{pmatrix} \delta_1^1 & \cdots & \delta_N^1 \\ \vdots & \ddots & \vdots \\ \delta_1^K & \cdots & \delta_N^K \\ 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Equation [21] does not have a unique solution unless $N = K + 1$ and the matrix of sensitivities on the left-hand side is not degenerate. However, we can approach the problem in a slightly more general fashion by treating equation [21] not as an exact condition but rather as an approximate equation akin to a regression, where $N \leq K + 1$, and the auxiliary variables v_i are chosen so that they minimize the deviations from each of the target values on the right-hand side. It is useful in this case to take δ_i^k to be "normalized" risk sensitivities, where the normalizing denominator is equal to the target accuracy desired for each risk factor. For example, if we choose the first set of sensitivities δ_i^1 equal to the interest rate durations divided by 0.1 this would correspond to an implicit target accuracy of the duration-neutral solution of approximately 0.1 years of duration. With this in mind, we can write down the solution in a matrix form, denoting the sensitivity matrix as Λ:

[22] $$\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \left(\Lambda' \cdot \Lambda\right)^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Equations [20] and [22] define the optimal zero-cost long-short portfolio which is immunized with respect to a set of chosen market risk factors. If the exact solution is possible, it will be attained by this formula, but even if it is not possible, the formula still remains useful.

How big a difference does the new definition of durations make for setting up market-neutral trades? Consider, for example, the portfolio shown in Figure 3. We have selected three bonds issued by Kraft Foods, and constructed barbell trades using the survival-based methodology and the conventional, spread-based methodology.

For the survival-based methodology, we have immunized the trade with respect to interest rate shifts by targeting zero interest rate duration, with respect to common shifts of the sector-specific hazard rate curve by targeting zero hazard rate duration, and with respect to large-scale issuer credit deterioration by targeting zero VOD. The resulting optimal trade nets 2.56 points upfront in cash plus 30bp running in carry.

For the spread-based methodology, we have targeted zero modified adjusted duration and also balanced the total price. The resulting barbell portfolio weights are different from those obtained in the survival-based methodology, leading to different net relative value assessment (similar carry but no upfront points). This is despite the fact that the bonds under consideration had only moderate price premiums, and the conventional Z-Spreads were close to survival-based par LIBOR spreads (P-Spreads). Had we considered a case of high yield bonds, the discrepancy between the two approaches would have been even greater.

Figure 3. Barbell trades with KFT bonds as of 6/30/2004

			Survival-Based Methodology					Spread-Based Methodology		
Description	**Maturity (yrs)**	**Price**	**P-Spread (bp)**	**Interest Rate Dur.**	**Hazard Rate Dur.**	**VOD**	**Portfolio MV %**	**Z-Spread (bp)**	**Mod. Adj. Dur.**	**Portfolio MV %**
KFT 4.625 11/01/2006	2.34	102.90	4	2.22	1.32	0.61	-59.23%	-1	2.23	-60.04%
KFT 6.25 6/01/2012	7.92	104.90	66	6.24	3.60	0.62	97.44%	64	6.34	100.00%
KFT 6.50 11/01/2031	27.35	100.44	79	11.64	6.76	0.60	-40.77%	80	12.53	-39.96%
Cash	0	100.00	0	0	0	0	2.56%	0	0	0%
Total: Portfolio			30	0.02	-0.03	0.00	0%	33	0.00	0%

Source: Lehman Brothers.

CONCLUSIONS

In this paper we redefined and substantially expanded the set of risk and sensitivity measures for credit bonds. In particular, the new definition of bond duration with respect to interest rates and hazard rate replaces the conventional modified duration and OAS duration measures. We demonstrated that the difference between these measures and the conventional ones can be substantial, resulting in materially different portfolio weights and net relative value estimates when setting up long-short bond trades.

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APPENDIX A: CONTINUOUS-TIME APPROXIMATION FOR CREDIT BOND PRICING

Continuous compounding is a convenient technique which may often simplify the analysis of relative value and forward pricing of credit bonds. It corresponds to coupon payments being made continuously. Consequently, in case of default such a bond would only lose an infinitesimal portion of its interest payments.

The present value for a hypothetical continuously compounded credit-risky bond can be calculated using the instantaneous forward interest rates $r(t)$ and hazard rates $h(t)$.

The base discount function is given by:

$$[23] \qquad Z(t) = \exp\left(-\int_0^t r(s) \cdot ds \right)$$

Under the Poisson model of exogenous default which is common to all reduced-form models, the survival probability is related to the default hazard rate by a similar relationship:

$$[24] \qquad Q(t) = \exp\left(-\int_0^t h(s) \cdot ds \right)$$

The hazard rate, $h(t)$, can in general be stochastic, just as the spreads and the interest rates can. However, throughout we work with "breakeven" hazard rates which are deterministic (while the default event is still random).

Assuming uncorrelated interest, hazard and recovery rates, one can combine equations [12] and [13] to obtain a continuously compounded analog of the bond pricing equation [9]:

$$[25] \quad P(T) = C \cdot \int_0^T du \cdot e^{-\int_0^u (r(s)+h(s)) \cdot ds} + e^{-\int_0^T (r(s)+h(s)) \cdot ds} + R_p \cdot \int_0^T du \cdot h(u) \cdot e^{-\int_0^u (r(s)+h(s)) \cdot ds}$$

This simple formula overestimates the present value of a credit bond for two distinct reasons:

- First, it neglects the expected coupon loss in case of default.
- Second, it overestimates the present value of the regular coupon payments because it presumes that portions of the coupon were paid earlier and it discounts those portions with a correspondingly smaller discount factor (and higher survival probability).

The more accurate approximation which we derive here corrects for these two biases.

The correction for the coupon loss bias can be estimated by noting that the expected timing of the default event under a constant hazard rate assumption is roughly half-way through the payment period. Hence, the expected loss of the accrued interest equals approximately half of the scheduled coupon payment, which in turn is equal to *1/f* fraction of the coupon rate for a bond with frequency *f*. Consequently, to correct for this bias we should explicitly subtract the expected accrued interest loss (assuming a given coupon recovery fraction R_c) from the principal recovery rate R_p in the original formula.

The correction for the early discount bias can be estimated by noting that by distributing the coupon payments evenly between the two coupon dates we get the survival-weighted present value which is roughly half-way between the present value of a bullet coupon payment on the two ends of the coupon period. Thus, for each bullet coupon the continuous-time formula [25] corresponds to a present value bias equal to half of the difference between the "true" present value of the earlier coupon payment and the current coupon payment. When summing up all of these biases, the corrections for all intermediate coupon payments cancel each other, and the total present value bias is simply half of the difference between the present value of the first coupon payment and the last coupon payment. For the valuation date just prior to a coupon payment, this results in a simple estimate since the present value of that impending coupon payment is simply equal to its amount. For other valuation dates the situation is slightly more complicated, but the approximation remains pretty close nevertheless.[1]

We obtain the continuous-time approximation for the clean price of an *f*-frequency credit bond by subtracting these two bias estimates from the original "naïve" formula.

Finally, we should also include the OAS-to-Fit (OASF), an issue-specific discounting measure which we introduced in Berd, Mashal, and Wang (2003), that allows us to use the issuer- or sector-specific hazard rate term structure while exactly fitting the observed price of individual bonds.

The final formula for the clean price of a fixed-coupon credit bond in the continuous-time approximation is:

[26]
$$
\begin{aligned}
P(T|f) \approx\ & C_f \cdot \int_0^T du \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds} + e^{-\int_0^T (r(s)+h(s)+OASF)\cdot ds} \\
& - \frac{C_f}{2f} \cdot \left(1 - e^{-\int_0^T (r(s)+h(s)+OASF)\cdot ds} \right) \\
& + \left(R_p - (1 - R_c) \cdot \frac{C_f}{2f} \right) \cdot \int_0^T du \cdot h(u) \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}
\end{aligned}
$$

This approximation is quite accurate across all values of coupons and for all shapes and levels of the underlying interest rate and hazard rate curves. Both correction terms can be quite important. The coupon loss bias term becomes zero when the coupon recovery is equal to 1. However, in practice we often assume the coupon recovery rate is equal to zero and therefore this correction is not negligible.

[1] *For valuation dates that fall between coupon payment dates, the approximate present value of the first coupon payment is non-trivial because the conventional definition of the "clean" price depends on the linear coupon accrual, while the correct present value calculation involves a discounting which is closer to an exponential formula. We ignore this additional discrepancy in our approximation.*

APPENDIX B: CONTINUOUS-TIME APPROXIMATION FOR SENSITIVITIES

Here we use the continuous-time approximation to the price of the bond defined in Appendix A to derive formulas for first- and second-order price sensitivities to interest rates, hazard rates, and recovery rates. These formulas will allow us to uncover useful relationships between the various "durations" and "convexities" of credit bonds.

[27]

$$\frac{\partial}{\partial r}P(T|f) = -C_f \cdot \int_0^T du \cdot u \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds} - T \cdot \left(1+\frac{C_f}{2f}\right) \cdot e^{-\int_0^T (r(s)+h(s)+OASF)\cdot ds}$$
$$- \left(R_p - (1-R_c)\cdot \frac{C_f}{2f}\right) \cdot \int_0^T du \cdot u \cdot h(u) \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}$$

[28]

$$\frac{\partial^2}{\partial r^2}P(T|f) = C_f \cdot \int_0^T du \cdot u^2 \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds} + T^2 \cdot \left(1+\frac{C_f}{2f}\right) \cdot e^{-\int_0^T (r(s)+h(s)+OASF)\cdot ds}$$
$$+ \left(R_p - (1-R_c)\cdot \frac{C_f}{2f}\right) \cdot \int_0^T du \cdot u^2 \cdot h(u) \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}$$

[29]

$$\frac{\partial}{\partial h}P(T|f) = -C_f \cdot \int_0^T du \cdot u \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds} - T \cdot \left(1+\frac{C_f}{2f}\right) \cdot e^{-\int_0^T (r(s)+h(s)+OASF)\cdot ds}$$
$$- \left(R_p - (1-R_c)\cdot \frac{C_f}{2f}\right) \cdot \int_0^T du \cdot u \cdot h(u) \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}$$
$$+ \left(R_p - (1-R_c)\cdot \frac{C_f}{2f}\right) \cdot \int_0^T du \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}$$

[30]

$$\frac{\partial^2}{\partial h^2}P(T|f) = C_f \cdot \int_0^T du \cdot u^2 \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds} + T^2 \cdot \left(1+\frac{C_f}{2f}\right) \cdot e^{-\int_0^T (r(s)+h(s)+OASF)\cdot ds}$$
$$+ \left(R_p - (1-R_c)\cdot \frac{C_f}{2f}\right) \cdot \int_0^T du \cdot u^2 \cdot h(u) \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}$$
$$- 2\cdot\left(R_p - (1-R_c)\cdot \frac{C_f}{2f}\right) \cdot \int_0^T du \cdot u \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}$$

[31] $$\frac{\partial}{\partial R_p}P(T|f) = \int_0^T du \cdot h(u) \cdot e^{-\int_0^u (r(s)+h(s)+OASF)\cdot ds}$$

[32] $$\frac{\partial}{\partial OASF}P(T|f) = \frac{\partial}{\partial r}P(T|f)$$

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