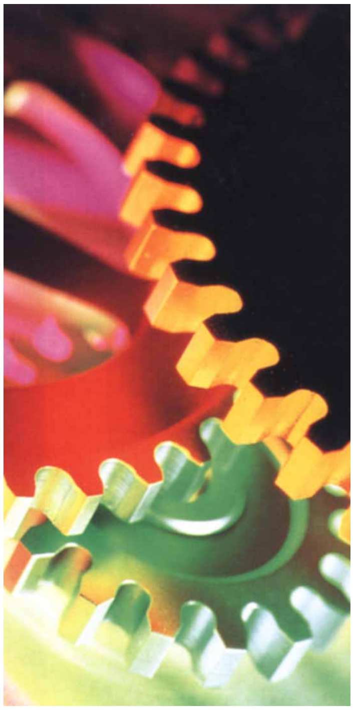
LEHMAN BROTHERS

Fixed Income
Quantitative Credit Research

Quantitative Credit Research

Quarterly

Volume 2003-Q4



CONTACTS

Quantitative Credit Research (U.S.)

Marco Naldi	212-526-1728	mnaldi@lehman.com
Prasun Baheti	212-526-9251	prbaheti@lehman.com
Roy Mashal	212-526-7931	rmashal@lehman.com
Claus M. Pedersen	212-526-7775	cmpeders@lehman.com

Quantitative Credit Research (Europe)

Dominic O'Kane	44-(0)20-7260-2628	dokane@lehman.com
Andrei Greenberg	44-(0)20-7260-1285	andgreen@lehman.com
Lutz Schloegl	44-(0)20-7260-2113	luschloe@lehman.com
Saurav Sen	44-(0)20-7260-2940	sasen@lehman.com

Structured Credit Strategies

Sunita L. Ganapati	415-274-5485	sganapat@lehman.com
Arthur M. Berd	212-526-2629	arthur.berd@lehman.com
Philip Ha	212-526-0319	philipha@lehman.com
Lorenzo Isla	44-(0)20-7260-1482	lisla@lehman.com
Claude A. Laberge	212-526-5450	claberge@lehman.com
Elena Ranguelova	212-526-4507	eranguel@lehman.com
Takahiro Tazaki	81-3-5571-7188	ttazaki@lehman.com
Gaurav Tejwani	212-526-4484	gtejwani@lehman.com

Quantitative Strategies (Europe)

Vasant Naik	44-(0)20-7260-2813	vnaik@lehman.com
Srivaths Balakrishnan	44-(0)20-7260-2180	sbalakri@lehman.com
Albert Desclee	44-(0)20-7260-2474	adesclee@lehman.com
Graham Rennison	44-(0)20-7260-2602	grennis@lehman.com
Jeremy Rosten	44-(0)20-7260-1020	jrosten@lehman.com
Minh Trinh	44-(0)20-7260-1484	mtrinh@lehman.com

Additional Contacts

Mark Howard	Global Credit Strategist	212-526-7777	mhoward@lehman.com
Robert McAdie	Head of European Credit Strategy	44-(0)20-7260-2628	rmcadie@lehman.com
Lev Dynkin	Head of Quantitative Portfolio Strategy	212-526-6302	ldynkin@lehman.com
Hua He	Head of Fixed Income Research, Asia	81-3-5571-7215	hhe@lehman.com

The New Lehman Brothers High Yield Risk Model[1]

Ganlin Chang
1-212-526-5554
gchang@lehman.com

We describe the new Lehman Brothers high yield risk model, which simultaneously models market spread risk and default risk. The market spread risk arises from changes in the level of credit spreads (or yields) in the mark-to-market of a credit product; default risk is the risk that an obligor fails to meet its contractual obligation. Our risk model provides an integrated framework for quantifying both risks.

1. INTRODUCTION

For years, the Lehman Brothers multifactor risk model has served as an effective risk management tool for fixed income money managers. It provides investors with the means to quantify the risk that portfolio returns will deviate from those of their benchmark. Such a risk measure is known as Tracking Error Volatility (TEV) – the "predicted" standard deviation of the relative return of the portfolio with respect to a benchmark. The current version of the risk model covers all the sectors of the U.S. Aggregate Index including Treasury, Agency, Credit, MBS, CMBS and ABS. In this article we introduce a new component of the risk model which covers a non-index asset class: High Yield.

Although high yield is not included in the U.S. Aggregate Index, its importance has grown rapidly in recent years due to the economic downturn and corporate debt overhung from the late 1990s. In the last two years, we have witnessed a steady stream of corporate credit rating downgrades, and the default rate has hit a historic high. Many issuers that had investment grade ratings have been downgraded to high yield or even distressed. Portfolio managers who used to hold investment grade bonds were left with a large amount of high yield bonds in their portfolios. All of this called for a credit risk model which not only covers investment grade, but also the high yield sector.

We begin our analysis with a brief review of our current investment grade risk model, and discuss its limitations when we apply the same methodology to high yield securities. We then lay out our new high yield risk model and illustrate some of the output available on our portfolio analytics platform – POINT.

1.1. Review of Lehman Brothers Investment Grade Credit Risk Model

The current investment grade risk model only covers investment grade corporate bonds – those rated BBB or higher. We will review this model briefly and relate it to the high yield model. A detailed description of the investment grade risk model can be found in Naldi, Chu and Wang (2002).

The investment grade risk model first decomposes the total return of a bond into a deterministic part and different stochastic parts:

$$Tot\ R_t = Carry\ R_t + YC\ R_t + Vol\ R_t + Sprd\ R_t \tag{1}$$

[1] We would like to thank Michael Anderson, Arthur Berd, Lev Dynkin, Michael Guarnieri, Mark Howard, Jay Hyman, Dev Joneja, Alex Kirk, Roy Mashal, Marco Naldi, Claus Pedersen and Stuart Turnbull for their valuable comments and suggestions. Gary Wang and Larry Chen contributed to the development and implementation of the risk model in Lehman Brothers portfolio analytic platform - POINT.

The deterministic part is the carry of the bond, and includes its coupon return and the return due to the passage of time. The stochastic part of the return, which depends on changes in market conditions, is further decomposed into yield curve return, volatility return and spread return. Each component will be modeled as a linear combination of a set of risk factors.

The yield curve return is modeled as depending on changes in six benchmark yields, corresponding to bonds with a maturity of 6 months and 2, 5, 10, 20 and 30 years. The sensitivity of a bond's returns to each of these key rates is measured by its key-rate duration. Therefore, the model of the yield curve return is:

$$YC\ R_t \approx -\sum_{i=1}^{6} KRD_{i,t-1} * \Delta KR_{i,t} + OAC_{t-1} * \left(\overline{\Delta KR}_t\right)^2 \tag{2}$$

where the six key-rate changes, $\Delta KR_{i,t}$, and the squared average key-rate change, $\left(\overline{\Delta KR}_t\right)^2$, represent observable systematic factors, loaded by the bond's key-rate durations (*KRD*) and option-adjusted convexity (*OAC*) respectively.

The volatility return is non-zero only for bonds with embedded options. It is modeled as:

$$Vol\ R_t \approx \frac{100}{Price_{t-1}} * Vega_{t-1} * F_t^{Vol} = -VolDur_{t-1} * F_t^{Vol} \tag{3}$$

where F_t^{Vol} is a latent, non-observable volatility factor with loading as volatility duration.

The spread return for investment grade bonds is modeled primarily on the basis of its industry sector and credit rating. We use nine industry sectors and three credit qualities. Each of the 27 buckets has its own spread factor which can be interpreted as the average OAS change of the bucket. Beyond those 27 bucket factors, we also consider maturity effect, OAS effect and country effect. We have the following setup for spread return:

$$\mathrm{SprdR_t} \approx -OASD_{t-1}\left[F_t^{cell} + \left(TTM_{t-1} - \overline{TTM}_{t-1}^{cell}\right)F_t^{Twist} + \left(OAS_{t-1} - \overline{OAS}_{t-1}^{cell}\right)F_t^{OAS} + F_t^{nonUS}\right] \tag{4}$$

where TTM is the time to maturity and $\overline{x}^{cell}$ represents the median value of variable x in the corresponding issuer group. The realizations of the unobserved latent factors $F_t^{Vol}, F_t^{Cell}, F_t^{Twist}$, F_t^{OAS} and F_t^{nonUS} are all estimated by cross-sectional regressions.

Finally we combine the three individual linear factor models for yield curve, volatility and spread return to provide a grand multi-factor linear risk model for total return as following:

$$R_t^i = Carry_{t-1}^i + L_{t-1}^i F_t' + \varepsilon_t^i \tag{5}$$

where F_t is the (1 x K) vector of the systematic factors while L_{t-1}^i is the corresponding vector of risk exposures; and ε_t^i is the idiosyncratic return. The idiosyncratic component is the part of risk which cannot be explained by the systematic factors: idiosyncratic risks are issuer-specific and independent of each other.

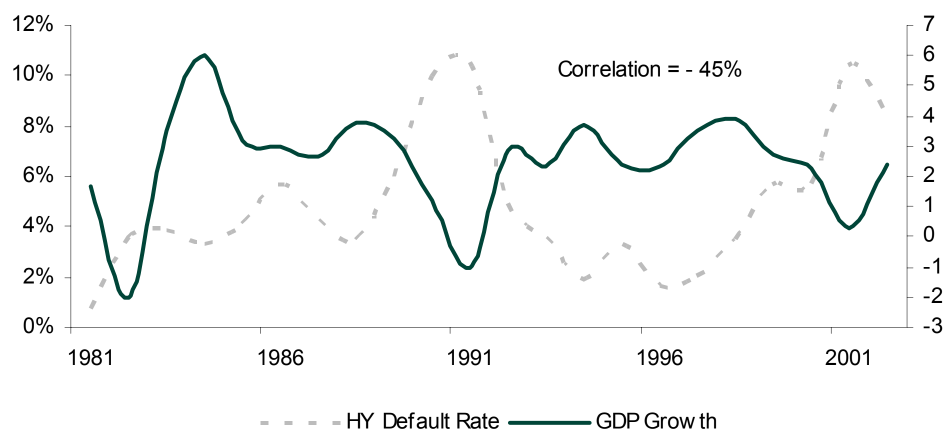
1.2. Motivation behind the New High Yield Risk Model

Having reviewed the current investment grade risk model, we now briefly discuss its limitations when applied to high yield securities. In general, the broad concept of *credit risk* includes *market spread risk* and *default risk*. *Default risk* is the risk that an issuer fails to meet its contractual obligation in a given period of time. *Market spread risk* is the risk due to fluctuation in the level of credit spread, i.e. the price of the bond, supposing that the issuer does not default in the given period. The *market spread risk* depends on investors' assessment of *future* default risk and the liquidity of the underlying bonds.

In addition, when portfolio effects are calculated, we have to take correlation effects into account. Like *market spread risk*, it is widely acknowledged that *default risk* is also a systematic risk, and empirical evidence shows the existence of credit contagion[2]. A number of studies, for example Duffee (1998) and Keenan (2000), have found that aggregate default rates are related to general macro-economic factors and business cycle indicators. The financial distress of one firm can directly trigger the distress of other firms. Intuitively, it is easy to imagine that Global Crossing is more likely to default in a period when WorldCom is in Chapter 11.

Figure 1 shows the aggregate annual default rate for US high yield issues from 1981 to 2002. On the same figure we also plot the annual GDP growth for the US. The negative correlation between default rate and GDP growth is obvious: during boom periods when the economy is doing well, the aggregate default rate is low; during recession periods, the aggregate default rate is high and corporates tend to fall into financial difficulties. For instance, during the recessions of 1990–91 and 2001–03, the aggregate default rate was at historic highs, illustrating that default risk is systematic and relates to macro-economic conditions.

Figure 1. HY Default Rate and GDP Growth (1981 – 2002)



Source: Moody's Investor Service and Lehman Brothers calculations.

Systematic default risk has important implications for high yield credit portfolios. It indicates that default risk is not idiosyncratic and cannot totally be diversified simply by adding new issues to the portfolio. It should be noted that the issue of default is not important when we are calculating the Tracking Error Volatility of investment grade portfolios. The likelihood of

[2] *Credit contagion refers to the propagation of economic distress from one firm to another.*

default is very small for an issuer rated BBB or higher. However, the problems are more serious for high yield bonds, for which the annual probability of default might be as high as 35% for a CCC issuer. Understanding and quantifying such a systematic risk becomes an important issue for credit money managers.

The current investment grade risk model focuses on market spread risk and does not explicitly model default risk. As Naldi, Chu and Wang (2002) mentioned, the investment grade model uses a robust technique to estimate factor realizations and eliminate sample outliers. As a consequence, the investment grade risk model implicitly treats default events as uncorrelated idiosyncratic events and ignores the systematic correlation between different issuers. In this paper, we extend the current investment grade risk model to create a framework which not only accounts for movement of credit spreads, but also takes default risk into account. We develop a very simple conceptual model of default in which issuer default is unpredictable and random, and can occur over the next time period with a probability that depends on current conditions. We keep the traditional multi-factor framework for the component of market spread risk. In the next section, we give a detailed description of the new high yield risk model.

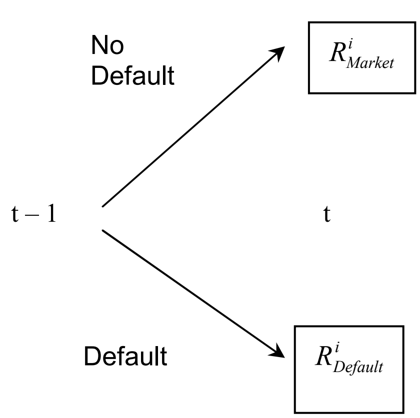
2. AN INTEGRATED HIGH YIELD RISK MODEL

The starting point for the high yield risk model is return decomposition conditional on the event of default, with the return of the bond written as:

$$R_t^i = (1 - I^i) R_{Market}^i + I^i R_{Default}^i \tag{6}$$

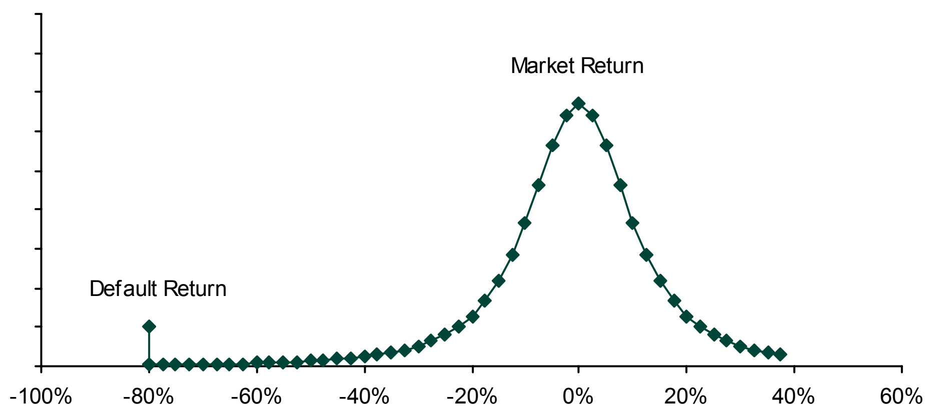
Here I^i is an indicative random variable for default event – it has a value of 1 if the issuer defaults in period t, and 0 otherwise. At the end of the period from t–1 to t, conditional on no default, the return for the bond will be R_{Market}^i. However, if the firm does default during the period, the return will be $R_{Default}^i$. From a continuous time perspective, equation (6) states that the price of a credit can be viewed as the combination of a diffusion process R_{Market}^i and a jump process I^i. Figure 2 illustrates this concept.

Figure 2. Market Spread Return and Default Return



The market return, R^i_{Market}, depends on the movement of interest rates, credit spreads and implied volatility, and the default return, $R^i_{Default}$, depends on the recovery value upon default. Figure 3 illustrates hypothetical distributions for the market spread return and the default return for a single issue.

Figure 3. Distributions of Market Return and Default Return (Single Issue)



We now show the detailed setup for the market spread return and default return separately.

2.1. Market Risk

The setup for market spread return is similar to the investment grade risk model. As we have discussed in section 1, we first decompose the total market return into time return, treasury return, volatility return and spread (residual) return, as in equation (1). The setup for volatility return and treasury return are exactly the same as all other risk models as in equations (2) and (3). However, we should point out that for high yield bonds, only spread (residual) return is significant. The dependence on the Treasury curve or even the volatility surface is relatively small. We retain the yield curve return and volatility return in the split only for the sake of consistency with the other risk models.

The spread return is modeled similarly to that in the investment grade risk model, based on sector and rating. The high yield sector and rating partitions are shown in Figure 4. The partitions are chosen so that there exist enough issues in each bucket along the history (to ensure that the estimation of systematic factors is not dominated by a couple of large issuers) and the issues in each bucket are highly correlated (to ensure that the systematic factors represent the common co-movements for a certain industrial group).[3] In general, the high yield sector partition is different from the investment grade partition.

[3] *We thank Michael Guarnieri for his input on HY partition.*

Figure 4. High-Yield Partition

BB/B	Below B
Basic Industry Cyclical Capital Goods Communication Energy Financial Media Non - Cyclical Technology Transportation Utility	Distressed Group

Bonds rated single-B and above are grouped into 11 sector buckets. The factor loading for each sector factor is simply the option-adjusted spread duration (OASD) of the bond. The corresponding bucket factor can be interpreted as the average OAS change for the sector. Beside the 11 sector factors, we have a twist factor to capture the slope of the spread curve and an OAS factor to differentiate high OAS bonds from low OAS bonds. The loadings for these two factors are the bond's relative time to maturity and OAS with respect to its sector median. The following equation summarizes the details of the spread model for bonds rated B and above:

$$\mathrm{SprdR_t} = -OASD_{t-1}\left[F_t^{Sector} + \left(TTM_{t-1} - \overline{TTM}_{t-1}^{Sector}\right)F_t^{Twist} + \left(OAS_{t-1} - \overline{OAS}_{t-1}^{Sector}\right)F_t^{OAS}\right] + \varepsilon_t \qquad (7)$$

All bonds with a rating of CCC and below are grouped into one single distressed bucket. In general, the spread return for a bond can be roughly approximated by $OASD \times \Delta OAS$. However, such a first order approximation no longer works for the distressed sector because ΔOAS is no longer a small number. In our risk model, we use unit instead of OASD as the loading for the distressed bucket factor, and the factor itself should be interpreted as the average *excess return* of the distressed group.

In addition to the bucket factor, we have four more factors for the distressed sector: a twist factor, a price factor, a leverage factor and a collateral-type factor. The twist factor measures the maturity effect, and its loading is simply the time to maturity of the bond relative to the median of the group. The price factor is similar to the OAS factor for non-distressed bonds and its loading is the relative price of the bond. The leverage factor differentiates between the returns of highly leveraged firms and those of low leveraged firms. Its loading is the relative leverage ratio[4] with respect to its industry peers. Finally, we have a collateral factor which is designed to capture the common movement for subordinated bonds. Only subordinated bonds are exposed to this factor. The following equation summarizes the spread model for distressed bonds:

$$\begin{aligned}\mathrm{SprdR_t} &= F_t^{Distress} + \left(TTM_{t-1} - \overline{TTM}_{t-1}^{Distress}\right)F_t^{Twist} + \left(P_{t-1} - \overline{P}_{t-1}^{Distress}\right)F_t^{\Pr ice} \\ &+ (LEV_{t-1} - \overline{LEV}_{t-1}^{Sector})F_t^{LEV} + I^{Sub}F_t^{Sub} + \varepsilon_t\end{aligned} \qquad (8)$$

4 *Leverage Ratio is defined as:*
(Long-Term Debt + Short-Term Debt) / (Long-Term Debt + Short-Term Debt + Market Value).

As with all other risk models, all spread factors are latent factors and will be estimated by a monthly cross sectional regression. Figure 5 summarizes all high yield market spread factors.

Figure 5. Systematic Market Spread Factors

BB & B		Below B	
Basic Industry	OASD	Distressed	Unit
Cyclical	OASD	Twist Distressed	MAT
Capital Goods	OASD	Price Distressed	Price
Communication	OASD	Leverage	Leverage
Energy	OASD	Subordinated	Unit
Financial	OASD		
Media	OASD		
Non-Cyclical	OASD		
Technology	OASD		
Transportation	OASD		
Utility	OASD		
Twist (BB&B)	OASD*MAT		
OAS (BB&B)	OASD*OAS		

The systematic risk factors (including all yield curve, volatility and spread factors) explain the systematic portion of the market return. The part of the market return which cannot be explained by those systematic factors are idiosyncratic risks. The idiosyncratic risks are issuer-specific and independent of each other. We use the factor regression residuals to estimate the idiosyncratic variances based on issuers' industrial sector and OASD (for non-distressed sectors only). Compared with investment grade bonds, high yield bonds in general have a much higher idiosyncratic variance. Figure 6 shows the idiosyncratic spread volatility for different investment grade and high yield buckets over the period of 1997–2003.

Figure 6. Idiosyncratic Spread Volatility (bp/month) (1997–2003)

	AAA/AA	A	BBB		BB/B
Banking And Brokerage	11	19	25	Basic Industry	93
Financial Companies, Insurance And Reits	10	21	30	Cyclical	87
Basic Industry	8	13	28	Capital Goods	83
Communication And Technology	17	26	33	Communication	144
Consumer Cyclical	8	16	31	Energy	65
Consumer Non-Cyclical	10	13	25	Financial	91
Energy And Transportation	12	13	18	Media	90
Utility	14	19	41	Non – Cyclical	74
Non-Corporate	10	22	32	Technology	126
				Transportation	109
				Utility	86
Average	11	18	29		95

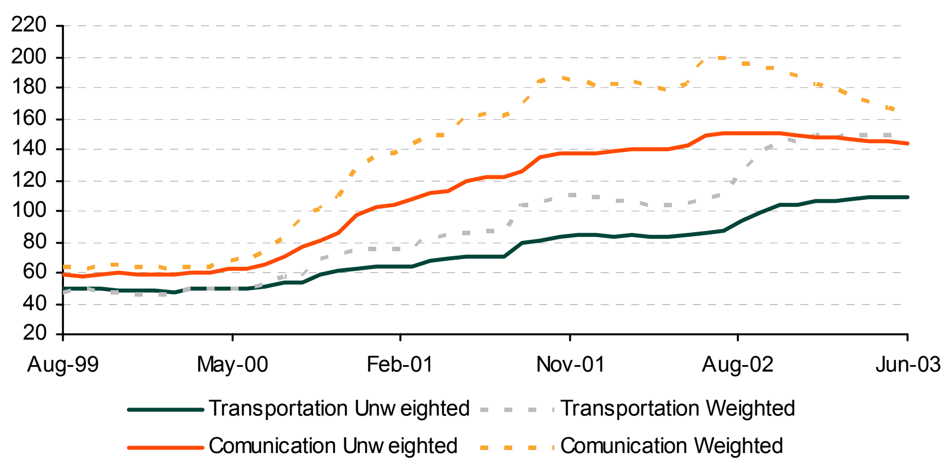
On average, the idiosyncratic spread volatility of high yield (BB/B) bonds is eight times that of AAA/AA bonds (correspondingly, five and three times that of A and BBB bonds). Such a difference has a major impact on portfolio diversification strategies. In general, the idiosyncratic volatility of a portfolio is proportional to $1/\sqrt{N}$, where N is the number of issues in the portfolio[5]. Hence, to diversify away idiosyncratic risk, a BB/B portfolio needs as many as 25 (= 5^2) times the issues as a single-A portfolio.

Another important feature of our high yield risk model is that we offer investors two options to calculate the market spread volatility[6]: a time-weighted scheme and an equal-weighted scheme. The equal-weighted scheme puts equal weight on all historical observations whereas the time-weighted scheme puts more weight on recent observations. Time-weighting is extremely important for idiosyncratic risk because of recent credit blow-ups. In 2001−02, we witnessed a lot of company-specific financial turmoil, for example WorldCom and Enron. As a consequence, the issuer-specific idiosyncratic variance increased dramatically for some industrial sectors. By offering investors two alternative ways to calculate volatility, our risk model is flexible enough to address the issue of time-varying volatility.

In Figures 7a and 7b, we show the time-varying idiosyncratic spread volatility for different industrials. The time variability of volatility is clearly sector and industry dependent. For instance, the time-weighted volatility is much higher than the unweighted volatility for such sectors as Transportation and Communication due to the idiosyncratic events that took place in 2001; whereas for such sectors as Consumer Non-cyclical and Energy, the difference between time-weighted volatility and unweighted volatility is small, which implies relatively stable spread volatility in the last decade.

The time variability for the systematic covariance matrix is not as profound as for the idiosyncratic variance. Nevertheless, we still offer investors two alternative time weighting schemes to calculate the systematic covariance matrix.

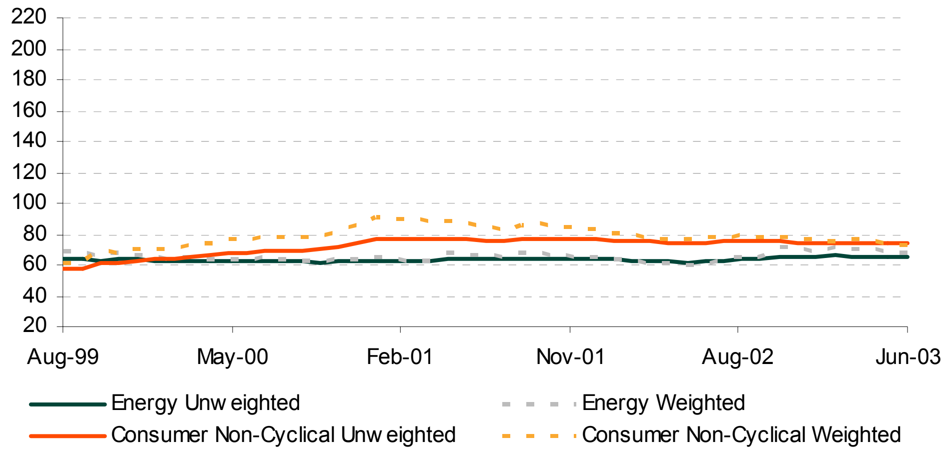
Figure 7a. Idiosyncratic Spread Volatility (Transportation and Communication)



[5] A detailed description of the relationship of idiosyncratic risk variance and number of issues in the portfolio can be found in “Testing the Lehman Brothers Agency Risk Model”, Chang and Naldi (2003).

[6] The time-varying idiosyncratic volatility was first introduced by A. Berd and M. Naldi (2002) in the Lehman Brothers Investment Grade Credit Risk Model.

Figure 7b. Idiosyncratic Spread Volatility (Energy and Consumer Non-Cyclical)



2.2. Default Risk

Having discussed the component of market risk, we will focus on default risk in this section. The default risk for a single corporate issue depends on the default event and the recovery rate upon default. For instance, ignoring the term of market return, R^i_{Market}, in equation (6) yields:

$$R^i_t \approx I^i R^i_{Default} \tag{6a}$$

For default event, we use a simple binominal distribution where the default probability will be calibrated *ex ante* at the beginning of each month. Figure 8 shows the estimated annual default probability for different rating categories.

Figure 8. Annual Default Probability

BB	B	CCC
1.3%	8.7%	32.8%

In general, the recovery rate will also be a random variable at the beginning of the period and its realization depends on the seniority, sector or other firm-specific characteristics[7]. In our model, we will use only a deterministic value based on the issue's seniority and sector for the sake of tractability. The expected recovery rate is calculated as the weighted average of the industry with the same seniority[8]. This recovery model was developed by Arthur Berd and Gaurav Tejwani from the Lehman Brothers Credit Strategy Group. Figure 9 shows the estimated recovery rate for senior unsecured bonds.

7 *Researchers and practitioners typically use a β-distribution to model this risk associated with recovery rate.*

8 *The basic assumption of our recovery model is that the recovery rate is proportional to the par value instead of the market value of the bond. As Duffie and Singleton (1999) and Jarrow, Lando and Turnbull (1997) shows, the assumption of recovery value plays a vital role in credit pricing.*

Figure 9. Recovery Rate (Senior Unsecured)

Banking And Brokerage	64
Financial Companies, Insurance And Reits	64
Basic Industry	27
Communication And Technology	21
Consumer Cyclical	30
Consumer Non-Cyclical	33
Energy And Transportation	32
Utility	51
Non-Corporate	37

Source: Moody's Investor Service and Lehman Brothers calculations.

Given the default probability and expected recovery rate upon default for a single issue, we are then able to approximate the standard deviation of the risk due to default as following:

$$Std \approx \sqrt{\text{Default Prob}} \times \text{Loss Upon Default} \tag{9}$$

To measure the default risk for a portfolio, we still need default correlation among different issues besides default probability and recovery rate. Our setup for default correlation follows a structural framework based on the value of the firm, along the lines of Merton (1974). Under this approach, a default event is triggered whenever the firm's asset value falls below a threshold defined by the firm's liabilities. Dependence of the default of one firm on the default of other firms can be modeled through the correlation among firms' asset value processes.

In reality, we cannot directly observe a firm's asset value. However, in this framework, equity is a call option on the asset value of the firm. Hence we can use equity correlation as a proxy for asset correlation – so the dependence between defaults will be driven by equity correlations. We use historical time-series of equity returns to estimate the equity (asset) correlation for a given pair of credits by building a separate sector-based equity risk model. The equity risk model covers the G-7 equity market and has 70 systematic factors which are based on the seven G-7 countries and 10 MSCI industrial sectors. Using such an equity risk model, we are able to estimate the pairwise equity (asset) correlation for different credits. The correlation structure and the information on default probability enable us to calculate the default correlation using a simulation methodology. Technical details about the simulation can be found in the Appendix.

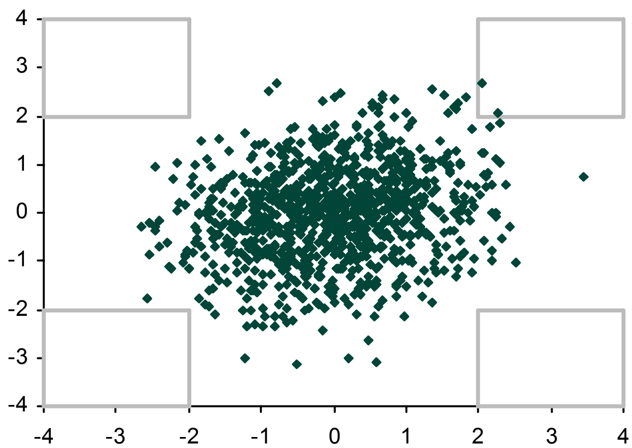
The underlying asset-based structural methodology for default is very popular in standard credit models such as CreditMetrics and KMV, which use a multivariate Gaussian distribution to model the joint equity (asset) return. The unique feature of our setup is that we use a joint Student-t distribution for the underlying equity return.

First of all, the Student-t distribution is a more general form of the Gaussian distribution. Empirical research shows that joint distribution of equity returns more closely follows a Student-t distribution with 10-12 degrees of freedom than a multivariate Gaussian distribution. For instance, R. Mashal and M. Naldi, A. Zeevi (2003) rejected the hypothesis of infinite degrees of freedom (corresponding to Gaussian) using likelihood ratio tests.

Secondly, the joint Student-t distribution has an important feature: *tail-dependence*. Under the joint t-distribution, extreme co-movements are more likely, a property that is crucial for generating default correlation since default is a rare event[9]. In contrast, under a joint Gaussian distribution, the default correlation will be too small for any reasonable correlation between the underlying Gaussian asset value returns. Figure 10 shows the scatter plots of a Gaussian distribution and a Student-t distribution with five degrees of freedom, both with 20% correlation. Clearly, under joint t-distribution, we see more incidents of joint extreme events.

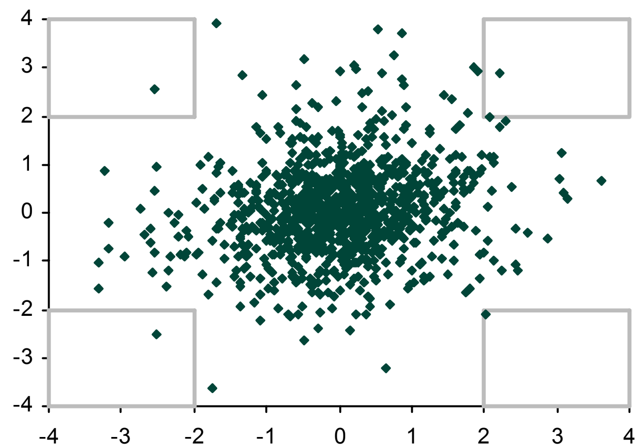
Figure 10. Joint Events under Gaussian and Student-t Distributions

Gaussian



Source: Lehman Brothers.

T



Source: Lehman Brothers.

In summary, we have discussed how to use the default probability and recovery rate to model the default risk for a single issue, and how to use a t-dependence structure to model default correlation among different issuers. However, to get the full picture of the distribution for total return for a credit portfolio, we still need to know the correlation between the market spread return, R^i_{Market}, and the default event, I^i. In reality, it is very natural to believe there is a positive correlation between these two risks: when the economy is doing well, market spread returns tend to be higher and there are few defaults. However, in our risk model, for the sake of tractability we will make a seemingly bold assumption: that these two risks are independent of each other. We would argue that for our purpose, any correlation assumptions will not have material impact on the output of our risk model.

The risk model is typically used over a short to medium time horizon – from a few months to a few years. First, note that the current level of defaults and spreads will be explicit inputs for the risk model. Default risk is primarily driven by these levels. The only remaining concern then is how *changes* in default rates conditional on spread changes are accounted for. The output of the risk model will focus on two quantities: the *Tracking Error Volatility* which is the standard deviation of the return distribution and other tail properties of the distribution. For Tracking Error Volatility, the effect of this correlation is minimal. Intuitively, we can

[9] *For details about extreme events and joint-t distribution, interested readers can refer to R. Mashal and M. Naldi, A. Zeevi (2003), and D. O'Kane and L. Schloegl (2002).*

imagine a Poisson process which governs the default event and a Brownian Motion process which governs market returns. For relatively small time intervals and low default probabilities, it is well understood that the correlation between a Poisson process and Brownian Motion is small even if their underlying factors are highly correlated[10]. For tail properties, results such as Expected Shortfall and VAR are primarily driven by the distribution of default event. Hence, for model tractability, we will assume that default events are independent of the market spread return, R^i_{Market}.

2.3. Tracking Error Volatility

One of the key outputs of the risk model is the Tracking Error Volatility: the *predicted* standard deviation of the relative (or total) return of a portfolio. Under our framework, the total return for a portfolio which includes high yield securities can be written as:

$$
\begin{aligned}
R &= \sum_{i(nonHY)} \theta_i R(nonHY)_i + \sum_{i(HY)} \theta_i R(HY)_i \\
&= \sum_{i(nonHY)} \theta_i (L_i F + \varepsilon_i) + \sum_{i(HY)} \theta_i [(1 - I^i)(L_i F + \varepsilon_i) + I^i \alpha^i] \\
&= \underbrace{\sum_{i(all)} \theta_i (L_i F + \varepsilon_i)}_{R_1(withoutDefault)} + \underbrace{\sum_{i(HY)} \theta_i I^i [\alpha^i - (L_i F + \varepsilon_i)]}_{R_2(Default)}
\end{aligned} \tag{10}
$$

The first component R_1 is identical to our standard multi-factor risk model without default.

Hence, the variance for the return of the portfolio can be written as the following:

$$\operatorname{var}(R) = \operatorname{var}(R_1) + \operatorname{var}(R_2) + 2\operatorname{cov}(R_1, R_2) \tag{11}$$

The first part of the variance is exactly the same as the standard multi-factor risk model excluding default. It will be fully determined by the variance and covariance matrix of the systematic factors, Σ, and idiosyncratic variance, Ω_i. The second and third parts are default related and depend on joint default probability $E(I^i I^j)$ [or default correlation $\rho(I^i, I^j)$][11]. The details of the calculation are shown in the Appendix.

---

[10] *We have tried a simulation to gauge the effect on Tracking Error Volatility of using different correlation assumptions. We are able to show that using reasonable parameters, the Tracking Error Volatility is affected by only 1~3% when we relax the independent assumption.*

[11] *The relationship between the joint default probability,* $E(I^i I^j)$*, and default correlation,* $\rho(I^i, I^j)$*, is given by:*

$$\rho(I^i, I^j) = \frac{E(I^i I^j) - p^i p^j}{\sqrt{(1 - p^i) p^i (1 - p^j) p^j}}$$

where p^i *is the default probability for firm i and* p^j *is the default probability for firm j.*

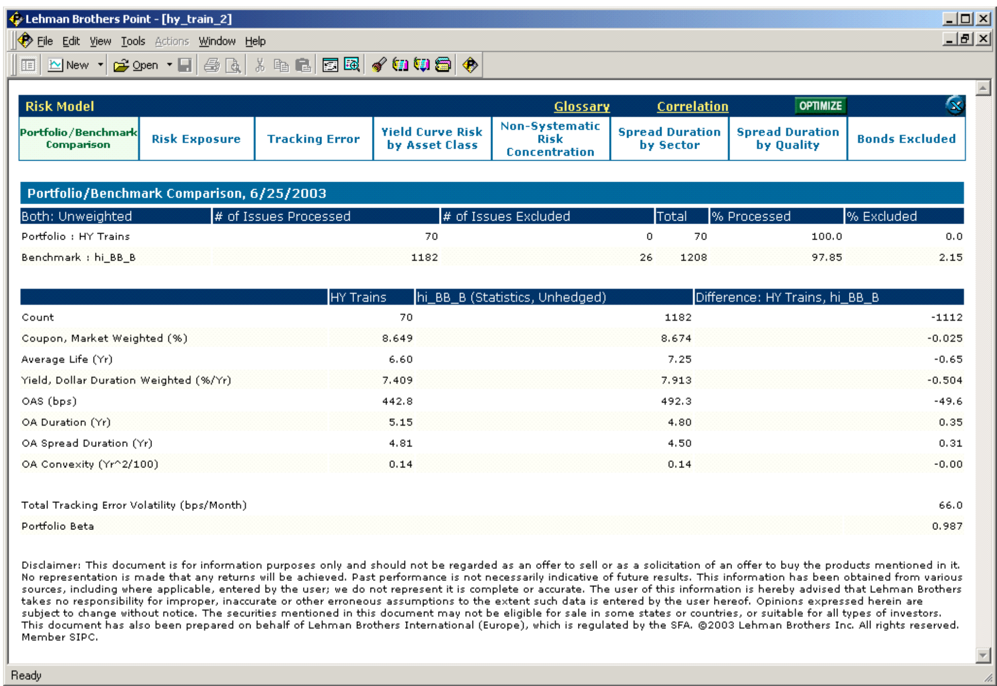
3. POINT IMPLEMENTATION

In this section, we use one example to show the implementation of the high yield risk model on our portfolio analytical platform: POINT. Currently POINT can calculate the total Tracking Error Volatility for a portfolio with high yield issues, and then decompose the total TEV into market component and default component.

The sample portfolio we use in this paper is the Lehman Brothers High Yield TRAINS. TRAINS is a product developed by Lehman Brothers that allows credit investors to invest in a diversified portfolio of credit risk in one single instrument. The high yield TRAINS includes 70 issues across different sectors and different maturity buckets. The following report compares the TRAINS portfolio to the selected benchmark: Lehman Brothers High Yield Index (BB & B). The portfolio has a slightly lower average maturity, lower duration and lower average OAS, but higher spread duration than the index. The bottom of the report displays the total Tracking Error Volatility of 66bp/month for this example.

Figure 11. Portfolio/Benchmark Comparison

Portfolio = Lehman Brothers High Yield TRAIN
Benchmark = US High Yield Index



Portfolio/Benchmark Comparison, 6/25/2003

Both: Unweighted	# of Issues Processed	# of Issues Excluded	Total	% Processed	% Excluded
Portfolio : HY Trains	70	0	70	100.0	0.0
Benchmark : hi_BB_B	1182	26	1208	97.85	2.15

	HY Trains	hi_BB_B (Statistics, Unhedged)	Difference: HY Trains, hi_BB_B
Count	70	1182	-1112
Coupon, Market Weighted (%)	8.649	8.674	-0.025
Average Life (Yr)	6.60	7.25	-0.65
Yield, Dollar Duration Weighted (%/Yr)	7.409	7.913	-0.504
OAS (bps)	442.8	492.3	-49.6
OA Duration (Yr)	5.15	4.80	0.35
OA Spread Duration (Yr)	4.81	4.50	0.31
OA Convexity (Yr^2/100)	0.14	0.14	-0.00
Total Tracking Error Volatility (bps/Month)			66.0
Portfolio Beta			0.987

Disclaimer: This document is for information purposes only and should not be regarded as an offer to sell or as a solicitation of an offer to buy the products mentioned in it. No representation is made that any returns will be achieved. Past performance is not necessarily indicative of future results. This information has been obtained from various sources, including where applicable, entered by the user; we do not represent it is complete or accurate. The user of this information is hereby advised that Lehman Brothers takes no responsibility for improper, inaccurate or other erroneous assumptions to the extent such data is entered by the user hereof. Opinions expressed herein are subject to change without notice. The securities mentioned in this document may not be eligible for sale in some states or countries, or suitable for all types of investors. This document has also been prepared on behalf of Lehman Brothers International (Europe), which is regulated by the SFA. ©2003 Lehman Brothers Inc. All rights reserved. Member SIPC.

The next report decomposes the total TEV into different components. In this example, we see that the biggest contributors for the systematic market TEV are interest rate risk and high yield spread risk, with isolated contributions of 16.6bp/month and 27.7bp/month respectively. The total systematic TEV is 28bp/month, which is smaller than the sum of the yield curve TEV and the high yield spread TEV due to the negative correlation between spread and interest rates. Figure 12 also shows that the idiosyncratic TEV is about 41bp/month in this example. As we have discussed earlier, high yield issues have larger idiosyncratic variance, and investors need more issues to achieve the same diversification goal for a high yield portfolio. This example clearly illustrates this point: although we have 70 issues in TRAINS, the idiosyncratic TEV is still very high and indeed is much larger than the systematic component. Finally, this report also displays the TEV due to default risk which is 43.8bp/month in this example.

Figure 12. Tracking Error Volatility (bp/month)

Portfolio = Lehman Brothers High Yield TRAIN
Benchmark = US High Yield Index

Lehman Brothers Point - [hy_train_2]

Source	Isolated	Cumulative	Change in Cumulative
Key Rates	16.6	16.6	16.6
Convexity Factor	0.0	16.6	-0.0
Treasury Volatility Factor	0.0	16.6	0.0
Agency Volatility Factor	0.0	16.6	0.0
Credit Investment Grade Volatility Facto	0.0	16.6	0.0
Credit High Yield Volatility Factor	0.8	16.2	-0.4
MBS Volatility Factors	0.0	16.2	0.0
Treasury Spread Factors	0.0	16.2	0.0
Agency Spread Factors	0.0	16.2	0.0
Credit Investment Grade Spread Factors	2.9	14.8	-1.4
Credit High Yield Spread Factors	27.7	28.0	13.2
MBS Spread Factors	0.0	28.0	0.0
CMBS Spread Factors	0.0	28.0	0.0
ABS Spread Factors	0.0	28.0	0.0

Systematic Tracking Error Volatility (bps/Month)	28.0
Idiosyncratic Tracking Error Volatility (bps/Month)	41.0
Default Tracking Error Volatility (High Yield Only) (bps/Month)	43.8
Total Tracking Error Volatility (bps/Month)	66.0
Portfolio Expected Return (bps/Month)	62.3
Benchmark Expected Return (bps/Month)	64.7
Difference of Expected Return (bps/Month)	-2.4

Disclaimer: This document is for information purposes only and should not be regarded as an offer to sell or as a solicitation of an offer to buy the products mentioned in it. No representation is made that any returns will be achieved. Past performance is not necessarily indicative of future results. This information has been obtained from various sources, including where applicable, entered by the user; we do not represent it is complete or accurate. The user of this information is hereby advised that Lehman Brothers takes no responsibility for improper, inaccurate or other erroneous assumptions to the extent such data is entered by the user hereof. Opinions expressed herein are subject to change without notice. The securities mentioned in this document may not be eligible for sale in some states or countries, or suitable for all types of investors. This document has also been prepared on behalf of Lehman Brothers International (Europe), which is regulated by the SFA. ©2003 Lehman Brothers Inc. All rights reserved. Member SIPC.

Ready

Figure 13 lists the 20 largest sources of idiosyncratic risk concentration relative to the index. It is interesting to note that the largest contribution comes not from any bond in the TRAINS portfolio, but from the Barrett Resources bonds which form 2.26% of the Lehman Brothers High Yield Index.

Figure 13. Non-Systematic Concentration

Portfolio = Lehman Brothers High Yield TRAIN
Benchmark = US High Yield Index

Lehman Brothers Point - [hy_train_2]

Concentration Source	Issuer Name	Sector	Rating	Issues in Portfolio	% of Portfolio	% of Benchmark	Net Contrb. to OASD	Specific TE Vol (bps/month)
Credit(IG/HY):								

WMB	BARRETT RESOURCES CORP	Other Utilities	B3 B1	0	0.0	2.26	-0.156	13.6
FLEX	FLEXTRONICS INTL LTD	Industrial	BA2	1	1.41	0.24	0.086	11.0
EP	ANR PIPELINE CO.	Other Utilities	B1 B2	0	0.0	1.36	-0.107	9.3
BTU	PEABODY ENERGY CORP	Industrial	BA3	1	1.43	0.17	0.076	7.1
TPC	TRITON PCS INC	Industrial	B3 B2	1	1.48	0.53	0.047	6.8
ABC	AMERISOURCEBERGEN CORP	Industrial	BA3	1	1.43	0.22	0.088	6.5
BYD	BOYD GAMING CORP	Industrial	B1	1	1.42	0.15	0.074	6.4
TCN	TELUS CORPORATION	Industrial	BA1	1	1.46	0.88	0.043	6.3
BLL	BALL CORP	Industrial	BA3 B1	1	1.39	0.23	0.075	6.2
SANM	SANMINA-SCI CORP	Industrial	BA2	1	1.43	0.22	0.048	6.1
DRRA	DURA OPERATING	Industrial	B1 B2	1	1.46	0.2	0.069	6.0
AYE	ALLEGHENY ENERGY INC	Other Utilities	B1	1	1.44	0.39	0.067	5.8
BOW	BOWATER	Industrial	BA1	1	1.41	0.41	0.062	5.8
SPW	SPX CORPORATION	Industrial	BA3	1	1.46	0.22	0.067	5.6
SBGI	SINCLAIR BROADCASTING	Industrial	B2	1	1.45	0.24	0.060	5.5
TRWAUT	TRW AUTOMOTIVE INC	Industrial	B2 B1	1	1.48	0.34	0.061	5.3
RCL	ROYAL CARIBBEAN	Industrial	BA2	1	1.51	0.48	0.058	5.0
DTV	DIRECTV HOLDINGS/FINANCE	Industrial	B1	1	1.44	0.4	0.055	5.0
SQA	SEQUA CORP	Industrial	B1	1	1.45	0.19	0.060	5.0
SPOT	PANAMSAT CORP	Industrial	BA2 BA3	1	1.45	0.34	0.055	5.0

Ready

To assess the effect of our t-dependence structure on the estimation of default TEV, we report the TEV under different structural assumptions for the dependence of defaults: first, assuming that all default events are independent, and next under a Gaussian dependence structure.

Figure 14. Tracking Error Volatility (bp/month) under Different Default Structures

Portfolio = Lehman Brothers High Yield TRAIN
Benchmark = US High Yield Index

	Student – t	Gaussian	Independent
Default TEV	43.8	38.4	25.2
Total TEV	66.0	62.5	55.4

Figure 14 shows that the t-dependence structure produces a higher risk estimate than either the Gaussian or the independence assumption. The default TEV estimate is reduced from 43.8bp/month to 38.4bp/month and to 25.2 bps/month when we use a Gaussian dependence structure and an independent structure respectively. This reduction was solely due to default correlation. Under an independent assumption, the default risk is essentially idiosyncratic because there is no correlation between default events. With a Gaussian assumption, we underestimate the default correlation due to the lack of tail dependence.

4. UPCOMING ENHANCEMENTS

Tracking Error Volatility is a very important measure for portfolio risk, but by no means the only one. From a statistical perspective, TEV only specifies the second moment of the return distribution without considering higher moments. In reality, it is quite possible that the same standard deviation for two different distributions does not necessarily define the same risk profile. For example, consider two distributions with the same standard deviation but different tail properties: one is Gaussian and the other is binominal with a very large probability of a small win and a small probability of a big loss. Even with the same standard deviation, we would argue that the latter has a more risky profile of returns. For credit, and even more so for high yield securities, the return distribution certainly has the flavor of the latter distribution because of default. Hence, risk managers have used other risk metrics beyond Tracking Error Volatility, for example value-at-risk (VAR) and Expected Shortfall (ES)[12]. Those risk measures are designed to capture the tail properties of the return distribution.

To obtain risk metrics such as VAR and Expected Shortfall, we need to specify a certain distribution for the return because different distributions will have different tail properties and different estimates for VAR and Expected Shortfall. The traditional multi-factor risk model cannot handle this issue because of its structural limitation with regard to default event. However, the new high yield risk model gives us a natural framework to estimate risk measures such as VAR and Expected Shortfall. As we have shown in Figure 3, the default component in the return decomposition essentially provides an explicit structural assumption on the left tail of the return distribution.

In this paper, we apply a simulation-based methodology to obtain those risk metrics. In particular, we simulate the default events for issuers in a portfolio based on the correlation structure as discussed in section 2. We then obtain risk measures like VAR and Expected Shortfall for this portfolio. Figure 15 shows the Expected Shortfall for the high yield TRAINS. For comparison, we also report the Expected Shortfall under different dependence assumptions: Gaussian and independent.

Figure 15. Expected Shortfall in bp/month

Portfolio = Lehman Brothers High Yield TRAIN

Confidence Level	5%	1%	0.10%
Student – t	-6.1	-10.88	-21.99
Gaussian	-4.76	-6.31	-8.51
Independent	-4.61	-6.01	-7.4
Relative Difference (Gaussian)	128%	172%	258%
Relative Difference (Independent)	132%	181%	297%

As with the TEV analysis, we see in Figure 15 that our t-dependence structure produces a higher Expected Shortfall across different confidence levels. Furthermore, the difference is greater for lower confidence levels. This qualitative result is expected because the Expected Shortfall mainly depends on the tail of the return distribution of the portfolio, and that is where the t-dependence structural assumption makes a difference.

[12] *VAR at a confidence level of β% is the worst loss in the best β% of all scenarios. For example a VAR of 100 MM at a 99% level of confidence means that the probability of the realized loss wider than 100 MM is only 1%. However, VAR does not distinguish how severe the loss might be in the 1% tail. Unlike VAR, the Expected Shortfall risk measure takes this into account and gives the expected loss in the 1 - β tail.*

5. CONCLUSIONS

In this paper, we described the new Lehman Brothers high yield risk model. The unique feature of the model is that it combines a multi-factor model of market movements with a model of default. In this model, default risk is systematic and default correlation is modeled via a t-dependence structure. Such an integrated framework allows investors to obtain not only the Tracking Error Volatility of the portfolio, but also risk measures such as VAR and Expected Shortfall, which depends on the tail properties of the return distribution.

REFERENCES

A. Berd and M. Naldi (2002), “New Estimation Options for the Lehman Brothers Risk Model: Adjusting to Recent Credit Events”, *Quantitative Credit Research Quarterly*, September 2002.

G. Chang and M. Naldi (2003), “Testing the Lehman Brothers Agency Risk Model”, *Liquid Markets Research Quarterly*, June 2003.

G. Duffee (1998), “The Relation between Treasury Yields and Corporate Yield Spreads”, *Journal of Finance* 53, 2225−2243.

D. Duffie and K. Singleton (1999), “Modeling Term Structures of Defaultable Bonds”, *Review of Financial Studies*, 12, 687–720.

L. Dynkin, J. Hyman, W. Wu (1999), *The Lehman Brothers Multi-factor Risk Model,* Lehman Brothers Publication.

R. Jarrow, D. Lando and S. Turnbull (1997), “A Markov Model for the Term Structure of Credit Spreads”, *Review of Financial Studies*, 10, 481–523.

S. Keenan (2000), “Historical Default Rates of Corporate Bond Issuers 1920–1999”, Moody’s Investors Services.

R. Mashal and M. Naldi, A. Zeevi (2003), “On the Dependence of Equity and Asset Returns”, *Risk*, October 2003, 83−87.

R. Merton (1974), On the Pricing of Corporate Debt: the Risk Structure of Interest Rates, *Journal of Finance*, 29, 449–470.

M. Naldi, K. Chu and G. Wang (2002), “The New Lehman Brothers Credit Risk Model”, *Quantitative Credit Research Quarterly*, May 2002.

D. O’Kane and L. Schloegl (2002), “Tail Dependence and Portfolio Risk”, *Quantitative Credit Research Quarterly*, May 2002.

APPENDIX

Simulation for Default Correlation

In this section, we detail the simulation methodology for calculating the default correlation. Let us look at firm i and j with default probability p^i and p^j. As discussed in section 3, we assume that their underlying asset returns follow a joint student t distribution with degree of freedom ν and correlation ρ_{ij}. The correlation ρ_{ij} and degree of freedom ν will be given by our Equity Risk Model. The simulation procedure works as follows:

1. Derive the thresholds of default $\hat{t}_i$ and $\hat{t}_j$ from default probability p^i and p^j based on the cumulative student t distribution.
2. In simulation path #n, draw $t_i(n)$ and $t_j(n)$ from a joint t distribution with degree of freedom ν and correlation ρ_{ij}. Compare $t_i(n)$ and $t_j(n)$ to the threshold $\hat{t}_i$ and $\hat{t}_j$ to get the default event $I_i(n)$ and $I_j(n)$.
3. Repeat step 2 for N times.
4. Compute the sample correlation of I_i and I_j to get default correlation $\rho(I^i, I^j)$.

Tracking Error Volatility

In this section, we detail the calculation of the Tracking Error Volatility with the presence of default risk. As we have shown in equation (10), the total Tracking Error Volatility can be written as three components:

The first component will be exactly the same as the result of a traditional linear multi-factor risk model:

$$\text{var}(R_1) = L\Sigma L' + \sum_i \theta_i^2 \Omega_i$$

where $L = \sum_i L_i \theta_i$ is the (relative) risk exposure of the portfolio, Σ is the covariance matrix of the systematic factors and Ω_i is the idiosyncratic variance for issuer i.

The second and third components depend on the default risk. The following equations detail the expression for $\text{cov}(R_1, R_2)$ and $\text{var}(R_2)$:

$$\begin{aligned}\text{cov}(R_1, R_2) &= \text{cov}(LF + \sum_{i(HY)} \theta_i \varepsilon_i, \sum_{i(HY)} \theta_i I^i [\alpha^i - (L_i F + \varepsilon_i)]) \\ &= \ldots \\ &= \ldots \\ &= -\sum_{i(HY)} L\Sigma(\theta_i p^i L_i)' - \sum_{i(HY)} \theta^2{}_i p^i \Omega_i\end{aligned}$$

where p^i is the default probability for firm i.

$$\begin{aligned}\mathrm{var}(R_2) &= \mathrm{var}(\sum_{i(HY)} \theta_i I^i[\alpha^i - (L_i F + \varepsilon_i)]) \\ &= \sum_{ij(HY)} \theta_i \theta_j [\alpha^i \alpha^j - (\alpha^i L_i + \alpha^j L_j)\overline{F} + L^i E(FF')L^{j'} + \Omega^i \delta_{ij}] E(I^i I^j) \\ &\quad - [\sum_{i(HY)} \theta_i (\alpha^i - L_i \overline{F}) p^i]^2\end{aligned}$$

where the joint default probability $E(I^i I^j)$ will be derived through simulation as we have discussed earlier. It is evident that the second and third parts will converge to 0 if the default probability p^i go to 0 for all issues in the portfolio (and benchmark). Hence for a portfolio (and the corresponding Benchmark) without High Yield securities, these two components do not contribute to total Tracking Error Volatility.

Hedging Debt with Equity[1]

Vasant Naik
Tel 44 (0) 207 260 2813
vnaik@lehman.com

Minh Trinh
Tel 44 (0) 207 260 1484
mtrinh@lehman.com

Srivaths Balakrishnan
Tel 44 (0) 207 260 2180
sbalakri@lehman.com

Saurav Sen
Tel 44 (0) 207 260 2940
sasen@lehman.com

An important consideration in relative value trading of the debt of a firm against its equity (or options on this equity) is the degree to which the equity exposure hedges the exposure on the debt. In this article, we investigate empirically the effectiveness of hedging debt positions with the issuer's equity. We present and compare two different methodologies for hedging debt with equity. The first methodology (the Lehman Brothers Credit and Equity Statistical Arbitrage model or CAESAR model) is based on the empirically observed co-movements between bond excess returns and equity returns. The second methodology relies on a new equity-based structural model of credit valuation (the Lehman Brothers ORION model) which is used to compute the hedge ratio (or delta) of debt with respect to equity. We also introduce a scenario-based analysis for special situations. Our empirical estimates of the hedge ratios of debt with respect to equity are in the range of 2-4% for A- and BBB-rated issuers and in the range of 12-20% for high yield issuers. We document the presence of a large residual in bond excess returns after removing the effect of equity. This residual is strongly correlated with the performance of the credit market as a whole for investment grade debt. This underscores the importance of a close monitoring of the credit market exposure of debt-equity trades. The effect of overall market movement is lesser in the high yield market, where a pure equity-based hedge performs reasonably well.

1. INTRODUCTION

With the growth and improved liquidity of the corporate and credit derivatives market, capital structure arbitrage and debt-equity relative value trading have recently become popular. Credit hedge funds and banks have become active in such trading. An important consideration in such trades is the degree to which a debt position can be hedged with a position in the issuer's equity, that is, one needs a good estimate of the hedge ratio of debt with respect to equity. This in turn necessitates a good understanding of the co-movement between debt and equity. The objective of this article is to empirically investigate the effectiveness of hedging debt with equity and to provide estimates of the appropriate hedge ratios.

Hedging debt with equity should not be difficult in theory. In a frictionless world, all corporate securities such as debt and equity can be regarded as contingent claims on the same underlying, namely the firm's assets. Indeed, in the world of Merton (1974), a firm's equity is simply a call option on the firm's assets with the strike price being the face value of the debt. The debt is equivalent to a long position in a riskless bond combined with a short position in a put option with the same strike price as the equity. The movements in firm value drive all the uncertainty in the model and debt can be perfectly hedged with equity.

Variations of Merton's model (such as Moody's KMV™, Creditgrades™) are, in fact, being used by many investors in structuring debt-equity trades. It is tempting to use these models at face value, in the same way as the Black-Scholes model (1973) is used for options. In practice, however, there are several reasons to believe that the relationship between debt and equity is not as tight as the Merton model would suggest. Given that fund managers typically

[1] *We would like to thank Stuart Turnbull and Marco Naldi for their suggestions and comments.*

specialize in investing either in fixed-income markets or in equity markets (but rarely both), it is conceivable that the two markets are segmented and that there is only a limited amount of capital allocated to arbitraging any discrepancies between debt and equity markets. The rebalancing and transaction costs in implementing debt-equity trades can also be considerable given the difficulty in shorting debt instruments. It is also the case that the informational cost of implementing debt-equity trades using Merton type models can be high because of the difficulty in obtaining accurate volatility forecasts and capital structure leverage data. It is, therefore, important to investigate the empirical performance of equity-based hedges of debt positions and to compare the performance of empirically-derived and model-based hedging strategies.

We present and compare two methodologies for determining hedge ratios of debt with respect to equity. The first methodology is the CAESAR empirical hedging methodology which uses a regression analysis of bond excess returns on equity returns. This approach is useful in investigating the historical performance of equity-based hedges of debt positions and in estimating the debt-equity hedge ratios. The second methodology is based on the ORION model which is a new equity-based model of credit valuation. This model overcomes some shortcomings of the Merton approach by directly modeling the observed equity value instead of the unobserved firm value. We use the model to determine the delta (hedge ratio) of debt with respect to equity in much the same way as option-pricing models are used to compute deltas of derivatives with respect to the underlying asset.

In addition to the above methodologies, we also present a scenario analysis approach of looking at debt-equity trades. Investors can stress-test a median scenario, their intuitions and predictions by varying the assumptions on spread and equity returns. It is a less systematic approach, but it can prove useful in the implementation of a debt-equity hedging trade.

Our main conclusions are the following. Our empirical estimates (based on CAESAR) of the hedge ratios of debt with respect to equity are in the range of 2-4% for A- and BBB-rated issuers and in the range of 12-20% for high yield issuers. The reduction in the volatility of the position by hedging with the issuers' equity is in the range of only 7-15% for A- and BBB-rated issuers. This implies the presence of a large residual in bond excess returns after removing the effect of equity. This residual is strongly correlated with the performance of the credit market as a whole for investment grade debt. The reduction in volatility by hedging with the issuer's equity and the credit market is in the range of 50-71% for A- and BBB-rated issuers. An implication is that debt-equity trades that are ostensibly hedged against movements in equity values can still have a significant exposure to movements in the credit market as a whole. Thus, a close monitoring and management of the credit market exposure of a book of debt-equity trades is necessary. The effect of overall market movement is reduced in the high yield market, where a pure equity-based hedge performs reasonably well. The reduction in volatility by hedging only with the issuer's equity is in the range of 15-22% for high-yield issuers. This reduction hardly changes (14-20%) when we hedge high-yield issuers with their equity and the credit market.

The article is structured as follows: in section 2, we present the CAESAR empirical hedging methodology for estimating the hedge ratios between debt and equity and for investigating the historical performance of equity-based hedged debt positions. In section 3, we briefly introduce the ORION model and present a similar hedging analysis using the hedge ratios derived from ORION. In section 4, we introduce the scenario analysis approach to looking at the performance of debt-equity trades. We conclude in section 5, with a comparison of our findings from the different methodologies.

2. HEDGING ANALYSIS WITH CAESAR

In this section, we investigate the empirical delta-hedging of debt with equity by using a bond-level regression analysis. We name our regression model CAESAR (Credit and Equity Statistical Arbitrage Model). The empirical betas resulting from the regressions are then used to hedge the debt against the equity movement. We perform three sets of regressions: a single variable regression with equity, a two-variable regression with equity and a Corporate Index, and a three-variable regression with equity, a Corporate Index and an Equity Index. For each of these experiments, we estimate the empirical hedge ratios of debt with respect to equity and report the averages of these hedge ratios by rating and sector. We also quantify the reduction in volatility obtained by implementing the hedging strategies corresponding to the three experiments.

We use monthly bond data from the Lehman Brothers US Investment Grade and High Yield Corporate Indices from January 1990 to August 2003, the Lehman Brothers Euro High Yield and Investment Grade Corporate Indices from January 1999 to August 2003. For our analysis, we consider a subset of these Corporate Indices – more than 4,500 non-callable, non-puttable bonds in USD and around 1,200 bonds in Euro. All the bonds we consider have listed equity. The bond excess returns are monthly excess returns over duration-matched Treasuries; the equity returns are monthly total returns.

2.1. Hedging with Issuers' Equity Alone: CAESAR I

In this experiment, we regress bond excess returns on 1-month equity returns using a 24-month rolling window. At the beginning of the month, for each bond, we compute the beta coefficient of the regression of the bond excess return on the equity return of the issuer using observations in the past 24 month. The estimated beta is an estimate of the hedge ratio for bonds with respect to the issuer's equity. A hedged debt-equity position in our analysis would go long the bond (hedged by duration-matched treasuries) and go short beta times the bond market value of equity.

In Figure 1, we present the average betas and average regression R^2 for the different rating categories in USD for the full sample for 1990-2003 and 1999-2003. The average betas reported in this figure are computed as follows. First, the beta for a particular month for a given rating is computed as the cross-sectional average (par-weighted) of the individual bond betas for the given rating from the above regressions. Then we compute the time-series average of these cross-sectional average betas, by ratings and investment grade sectors and for different sample periods. Figure 1 reports these time-series averages. We also report t-statistics corresponding to these time-series averages. The t-statistics are adjusted for auto-correlation in the series according to the Newey-West procedure. A beta of 0.028 means that to hedge $1,000,000 of bond, one needs $28,000 of stock.

It is seen in Figure 1 that the betas increase as we go down the rating spectrum. In the full sample, the betas range from 0.02 for A-rated bonds to 0.22 for CCC-rated bonds. This observation is consistent with Merton-type models if we consider rating as a proxy for leverage. The lower the leverage, the more the equity is in-the-money and the debt out-of-the-money, thus less sensitive to equity price movements. In the 1999-2003 sample for USD, the betas and R^2 are usually higher: ranging from 0.020 to 0.299 and from 7.3% to 32.5% respectively, reflecting a higher debt-equity correlation in the past few years.

Figure 1 also shows the results for Euro-denominated bonds. As in the USD case, the betas usually increase as we go down the rating spectrum (0.01 for A-rated bonds to 0.32 for CCC-rated bonds). The investment grade betas for Euro AA, A and BBB are smaller than the US betas for the same rating categories. The high yield betas for Euro follow a different pattern: they are higher by more than half than their USD counterparts.

Figure 1. Average beta, average R^2 and average t-stats of the betas by ratings (AA to CC, US and Euro Corporate Indices)

	USD: 1990-2003			USD: 1999-2003			EUR: 1999-2003		
Rating	Average Beta (Equity)	R^2	t-stat (average beta)*	Average Beta (Equity)	R^2	t-stat (average beta)*	Average Beta (Equity)	R^2	t-stat (average beta)*
AA	0.012	10.10%	3.68	0.02	11.00%	6.23	0.003	5.50%	1.73
A	0.019	11.20%	4.87	0.027	12.30%	3.66	0.014	8.90%	1.79
BBB	0.031	10.50%	4.97	0.04	11.50%	2.50	0.029	12.00%	2.04
BB	0.066	11.50%	4.56	0.083	15.80%	2.87	0.116	20.10%	6.18
B	0.105	16.90%	4.51	0.143	23.60%	4.09	0.164	24.40%	1.88
CCC	0.217	19.80%	3.14	0.299	32.50%	8.17	0.323	41.10%	7.81
CC	0.123	22.50%	2.07	0.061	25.00%	0.39	0.32	36.10%	6.14

** the t-statistics correspond to the time-series averages of betas. They are adjusted for auto-correlation in the series according to the Newey-West procedure.*

In Figure 2, we present the betas and R^2 for the different investment grade sectors in USD for the entire sample (1990-2003) and for the more recent period of 1999-2003 and in Euro for 1999-2003. For USD, in the full sample, the betas range from 0.011 for utilities to 0.042 for cyclicals. The average beta is also relatively high for the banking sector; the betas for the other sectors are distributed in a narrow range between 0.018 and 0.020. Since 1999, the betas have increased most dramatically in the telecoms sector (116% higher than in the full sample), financial sector (42%), basic industries sector (39%) and cyclical sector (38%).

The betas in Euro are on average around half the USD betas in magnitude. The beta for utilities in Euro is 0.006 compared with 0.011 in USD; for non cyclicals, it is 0.009 in Euro compared with 0.019 in USD. The most extreme example is for banking, where the beta for Euro is 0.005 compared with 0.027 for USD. In the basic industries, telecoms and energy sectors, the betas are more comparable although smaller than their USD counterparts. R^2 is also the highest for cyclicals and telecoms and the lowest for banking and financials.

Figure 2. Average beta and average R^2 by investment grade sector (US and Euro Corporate Indexes)

	USD: 1990-2003		**USD: 1999-2003**		**EUR: 1999-2003**	
Sector	**Average Beta (Equity)**	**R^2**	**Average Beta (Equity)**	**R^2**	**Average Beta (Equity)**	**R^2**
Banking	0.027	15.6%	0.029	16.1%	0.005	6.3%
Basic Industries	0.018	7.9%	0.025	8.4%	0.018	8.9%
Communications	0.019	8.6%	0.041	11.0%	0.029	12.5%
Cyclicals	0.042	13.9%	0.058	18.5%	0.017	12.6%
Energy	0.020	8.4%	0.024	9.2%	0.021	8.1%
Financials	0.019	11.2%	0.027	9.3%	0.013	6.3%
Non Cyclicals	0.019	8.6%	0.018	6.5%	0.009	7.1%
Utilities	0.011	6.5%	0.012	7.1%	0.006	7.0%

2.2. Hedging with Issuers' Equity and the Corporate Index: CAESAR II

In this experiment, we regress bond excess returns on 1-month equity returns and the corporate market factor (MKT) using a 24-month rolling window. We calculate a different corporate market factor (MKT) for each rating by taking the excess return of a par-weighted portfolio of all bonds in our sample in that rating category. In other words, the excess return of an A-rated bond is regressed on the equity of the issuer and the A-rated corporate market factor. For this reason, the betas against the credit market factors are not comparable across ratings. The beta (equity) and beta (MKT) give the hedge ratios of the debt with respect to the equity and the corporate market factor respectively.

In Figure 3, we present the results for US corporate bonds over the entire sample period. A striking result is the change in the equity betas. The magnitudes of the new betas are now on average half those of the previous betas (without a Corporate Index factor). Among investment grade bonds, the average beta for A-rated bonds is 0.008 compared with 0.019 without the index. Among high yield bonds, the average beta for BB-rated bonds is 0.034 compared with 0.066 without the index.

There is a substantial increase in the R^2 when the corporate market factor is included. These increase by almost three times for investment grade bonds compared with those in experiment 1. This means that a hedging based purely on issuers' equity will leave a significant residual which is strongly correlated with the credit market as a whole. These results are consistent with those documented by Collin-Dufresne *et al* (2001). The importance of the market factor is less for high-yield bonds where the increase in R^2 is about 1.5 times.

The beta on the Corporate Index of same rating is also interesting. It is around 0.8 for A, BBB and B corporate bonds, and close to 1 for BB and CCC-rated bonds. This variation reflects the weight of the systematic component relative to the idiosyncratic component as captured by the equity.

Figure 3 shows a similar drop in equity beta values for euro-denominated bonds. The beta for A-rated bonds is 0.003 – 79% less than the beta without an A-rated Corporate Index. The beta for BBB-rated bonds is 0.016 – 45% less than the beta without a BBB-rated Corporate Index. Among high yield bonds, the beta for BB-rated bonds is 0.081 – 30% less than the beta without a BB-rated Corporate Index. Relative to the USD betas, the Euro-based betas follow a similar pattern to when no Corporate Index is used: they are lower for A-rated bonds and higher for BBB-rated and high yield bonds.

Figure 3. Average beta and average R^2 by ratings (AA to CC, US and Euro Corporate Indexes)

	USD: 1990-2003			EUR: 1999-2003		
Rating	Average Beta (Equity)	Average Beta (MKT)	R^2	Average Beta (Equity)	Average Beta (MKT)	R^2
AA	0.005*	0.698*	39.6%	0.000	0.890*	23.1%
A	0.008*	0.827*	42.7%	0.003	0.701*	27.4%
BBB	0.015*	0.828*	37.8%	0.016*	0.599*	25.0%
BB	0.034*	0.949*	32.2%	0.081*	0.557*	30.5%
B	0.059*	0.779*	38.3%	0.150	0.146*	30.9%
CCC	0.149*	0.988	29.2%	0.254*	0.530*	51.1%
CC	0.041	0.414	29.9%	0.320*	-0.033	38.2%

** corresponds to t-statistics in excess of 2. The t-statistics are for the time-series averages of betas. They are adjusted for auto-correlation in the series according to the Newey-West procedure.*

In Figure 4, we report the results for investment grade US corporate bonds over the full sample by sector. Consistent with the results by rating categories, the equity betas are smaller when a Corporate Index is included. The drop is the largest for telecoms (-63%) and the smallest for utilities (-27%). We also notice a large improvement in R^2 across all sectors.

We present the same results for investment grade Euro corporate bonds by sector. We also see a drop of the equity betas. The drop is the largest for banking (-120%) and utilities (-100%) and the smallest for non-cyclicals (-22%). This means that hedging banking debt and utilities debt in Euro is not efficiently done with equity, but should rather be done with the Corporate Index. As for USD bonds, we notice a large improvement in R^2 across all sectors.

Figure 4. Average beta and average R^2 by investment grade sector (US and Euro Corporate Indexes)

	USD: 1990-2003			EUR: 1999-2003		
Sector	**Average Beta (Equity)**	**Average Beta (MKT)**	**R^2**	**Average Beta (Equity)**	**Average Beta (MKT)**	**R^2**
Banking	0.012	0.943	48.7%	-0.001	0.880	25.0%
Basic Industries	0.008	0.818	35.5%	0.008	0.569	23.1%
Communications	0.007	0.802	34.7%	0.015	1.025	32.2%
Cyclicals	0.017	0.997	44.7%	0.007	0.464	24.2%
Energy	0.009	0.802	37.0%	0.009	0.671	22.6%
Financials	0.009	0.799	45.7%	0.006	1.134	23.5%
Non Cyclicals	0.011	0.711	33.5%	0.007	0.617	25.6%
Utilities	0.008	0.577	34.2%	0.000	0.592	26.5%

2.3. Hedging with Issuers' Equity, and Corporate and Equity Indices: CAESAR III

In this experiment, we regress bond excess returns on 1-month equity returns, the credit market factor (of the same rating, as defined in section 2.2) and the Equity Mirrored Index (EQMKT) using a 24-month rolling window. The EQMKT factor mirrors the Corporate Index of the same rating and is constructed as the equity return on a par-weighted portfolio of the equity of issuers of that particular rating category. The betas from the regression give the hedge ratios with respect to the issuers' equity, the credit market factor (of that rating) and the equity market factor.

In Figure 5, we present the results for US corporate bonds over the full sample. The equity betas for AA-, A-, BBB- and BB-rated bonds are not very different from the equity betas when the Equity Index is not included. Interestingly, the betas on equity indices have a negative sign. This points to a significant correlation in the *idiosyncratic* components of bond excess returns and equity returns. The R^2s tend to improve marginally except for CC-rated bonds, where there is a larger improvement. The pattern is similar in euro-denominated bonds.

Figure 5. Average betas and average R^2 by ratings (AA to CC, US and Euro Corporate Indexes)

	USD: 1990-2003				EUR: 1999-2003			
Rating	Avg Beta (Equity)	Avg Beta (MKT)	Avg Beta (EQMKT)	R^2	Avg Beta (Equity)	Avg Beta (MKT)	Avg Beta (EQMKT)	R^2
AA	0.005*	0.699*	-0.001	42.9%	-0.003*	0.903*	0.002	27.5%
A	0.009*	0.830*	-0.003	46.3%	0.007	0.761*	-0.014*	31.1%
BBB	0.015*	0.850*	-0.008	41.5%	0.021	0.739*	-0.028*	30.2%
BB	0.039*	1.038*	-0.049*	36.2%	0.115*	1.123*	-0.204*	41.4%
B	0.085*	0.921*	-0.100*	41.4%	0.192*	0.516*	-0.198*	37.1%
CCC	0.184*	1.031*	-0.079	31.6%	0.317*	0.759*	-0.236*	53.4%
CC	0.315*	0.642*	-0.325*	41.5%	0.324*	0.067*	-0.081	41.0%

** corresponds to t-statistics in excess of 2. The t-statistics are for the time-series averages of betas. They are adjusted for auto-correlation in the series according to the Newey-West procedure.*

In Figure 6, we report the results for US corporate bonds by investment grade sectors over the full sample. There is a slight improvement in the R^2 and overall the beta coefficients on the issuer equity and the Credit Index have the same order of magnitude as in CAESAR II. A similar pattern exists for euro-denominated bonds.

Figure 6. Average beta and average R^2 by investment grade sector (US and Euro Corporate Indexes)

	USD: 1990-2003				EUR: 1999-2003			
Sector	Avg Beta (Equity)	Avg Beta (MKT)	Avg Beta (EQMKT)	R^2	Avg Beta (Equity)	Avg Beta (MKT)	Avg Beta (EQMKT)	R^2
Banking	0.009	0.897	0.010	52.1%	-0.001	0.905	-0.003	29.1%
Basic Industries	0.008	0.847	-0.006	38.9%	0.014	0.697	-0.026	27.6%
Communications	0.108	0.845	-0.015	38.8%	0.018	1.097	-0.021	35.8%
Cyclicals	0.011	1.037	-0.003	48.1%	0.013	0.560	-0.022	30.1%
Energy	0.011	0.827	-0.008	40.6%	0.009	0.720	-0.007	26.0%
Financials	0.009	0.787	0.000	49.1%	0.006	1.162	-0.006	28.8%
Non Cyclicals	0.006	0.765	-0.004	37.8%	0.009	0.684	-0.013	29.4%
Utilities	0.010	0.596	-0.011	38.4%	0.001	0.634	-0.003	29.8%

2.4. Effectiveness of Hedging with Equity, Credit and Equity Indexes

In this section, we investigate the effectiveness of hedging debt positions using the empirical hedge ratios given by CAESAR I, II and III. For this purpose, we compute the volatility of the hedged and unhedged excess returns for different currencies, ratings and sectors. We also report the percentage reduction in volatility obtained by using hedging strategies corresponding to the three experiments discussed above.

In Figure 7, we present the results by rating for US corporate bonds for the 1990-2003 period.

This figure shows the volatility of monthly excess returns on bonds in our sample (the unhedged volatility) as well as the volatility of returns on hedged positions in bonds. The hedged positions are created according to the three experiments described above using the regression-based hedge ratios. Results corresponding to CAESAR I relate to the case where bond positions in any month are hedged with issuers' equity alone, with the hedge ratios given by betas estimated over the previous 24 months. Results for CAESAR II and III are for the case where the hedge includes the corporate index (CAESAR II) and both Corporate and Equity indices (CAESAR III). Since our starting point is always excess bond returns, all positions are always hedged against interest rate risk.

Hedging only with the equity of the issuer with CAESAR I reduces volatility from an average of 21% for BB-rated bonds to 7% for A-rated bonds and even increases volatility for CC-rated bonds. The inclusion of the Corporate Index of same rating as a hedging instrument improves the hedging performance dramatically for investment grade bonds. There is a 71% reduction in volatility for A-rated bonds in CAESAR II compared with only 7% in CAESAR I, and 57% for BBB-rated bonds in CAESAR II compared with 14% in CAESAR I. The drop in hedging volatility is also significant in the high yield universe for BB- and B-rated bonds. The hedging volatility increases for CCC- and CC-rated corporate bonds. CAESAR III, which incorporates an Equity Index, does not significantly reduce the volatility of hedged returns.

Figure 7. Average volatility and volatility reduction by rating (AA to CC, US Corporate Index) 1990-2003

	UNHEDGED	CAESAR I		CAESAR II		CAESAR III	
Rating	Volatility	Volatility	% Reduction	Volatility	% Reduction	Volatility	% Reduction
AA	0.37	0.35	3.3%	0.12	67.3%	0.12	67.8%
A	0.46	0.43	7.3%	0.13	71.4%	0.12	73.5%
BBB	0.85	0.73	14.1%	0.36	57.3%	0.37	56.8%
BB	1.72	1.35	21.4%	1.08	37.3%	1.08	37.2%
B	2.92	2.32	20.5%	2.05	29.9%	2.01	31.3%
CCC	8.77	7.09	19.1%	7.84	10.6%	7.85	10.4%
CC	12.51	13.98	-11.7%	13.31	-6.4%	14.76	-17.9%

In Figure 8, we report the same results for Euro corporate bonds for the 1999-2003 period. Hedging only with the equity of the issuer with CAESAR I reduces volatility from an average of 15.1% for A-rated bonds to 2.6% for AA-rated bonds and even increases volatility for CCC-rated bonds. Like in the USD case, the inclusion of the Corporate Index of bonds of same rating in CAESAR II also improves the hedging dramatically for investment grade bonds. There is a 74.3% for A-rated bonds compared with 15.1%, and 35.2% for BBB-rated bonds compared with 10.5%. Hedging volatility does not fall in the high yield universe except for B-rated bonds. Interestingly, CAESAR III, which incorporates an Equity Index, improves on the hedging volatility only for BBB-rated bonds, decreasing volatility by 50.5% instead of 35.2%.

Figure 8. Average volatility and volatility reduction by rating (AA to CC, Euro Corporate Index) 1999-2003

	UNHEDGED	CAESAR I		CAESAR II		CAESAR III	
Rating	Volatility	Volatility	% Reduction	Volatility	% Reduction	Volatility	% Reduction
AA	0.11	0.10	2.6%	0.05	52.5%	0.05	54.6%
A	0.37	0.31	15.1%	0.09	74.3%	0.09	74.6%
BBB	0.89	0.80	10.5%	0.58	35.2%	0.44	50.5%
BB	4.25	3.66	13.8%	5.06	-19.2%	6.34	-49.2%
B	14.95	14.37	3.9%	14.49	3.1%	14.32	4.2%
CCC	14.01	14.88	-6.2%	15.48	-10.5%	15.93	-13.7%
CC	12.25	11.73	4.3%	12.40	-1.2%	12.76	-4.1%

In Figure 9, we present the results by investment grade sector for USD corporate bonds for the 1990-2003 period. Hedging only with the equity of the issuer with CAESAR I reduces volatility from an average of 23.2% for cyclical sector bonds to 1.8% for non-cyclical sector bonds. As for ratings, the inclusion of the Corporate Bond Index of same rating in CAESAR II also improves the hedging dramatically for investment grade bonds, from 64.4% to 21.9%. We also see that CAESAR III (hedging including the equity index) does not seem to significantly improve the hedging.

Figure 9. Average volatility and volatility reduction by sector (US Corporate Index) 1990-2003

	UNHEDGED	CAESAR I		CAESAR II		CAESAR III	
Sector	Volatility	Volatility	% Reduction	Volatility	% Reduction	Volatility	% Reduction
Banking	0.42	0.40	4.0%	0.24	42.6%	0.24	43.2%
Basic Industries	0.58	0.55	4.5%	0.32	44.0%	0.33	42.8%
Communications	0.84	0.72	13.5%	0.33	60.3%	0.31	63.3%
Cyclicals	1.00	0.77	23.2%	0.36	64.4%	0.35	64.6%
Energy	0.63	0.60	5.6%	0.34	46.0%	0.35	45.4%
Financials	0.61	0.57	6.7%	0.32	48.3%	0.32	47.3%
Non Cyclicals	0.46	0.45	1.8%	0.30	35.4%	0.30	35.0%
Utilities	1.05	0.94	10.1%	0.82	21.9%	0.81	22.5%

In Figure 10, we report the equivalent results by sector for Euro corporate bonds for the 1999-2003 period. Hedging only with the equity of the issuer with CAESAR I reduces volatility from an average of 16.1% for basic industries sector bonds to 8.6% for cyclical sector bonds. There is even an increase in volatility with non-cyclical sector bonds. Like in the USD case, the inclusion of the Corporate Index of same rating in CAESAR II also improves the hedging dramatically for investment grade bonds, from 55.1% for telecom bonds to 11.3% for non-cyclicals. Finally, CAESAR III seems to improve the hedging for basic industries and cyclical sector bonds and marginally for energy, financial, non-cyclical and utilities sector bonds.

Figure 10. Average volatility and volatility reduction by sector (Euro Corporate Index) 1999-2003

	UNHEDGED	CAESAR I		CAESAR II		CAESAR III	
Sector	Volatility	Volatility	% Reduction	Volatility	% Reduction	Volatility	% Reduction
Banking	0.14	0.13	11.5%	0.08	47.8%	0.08	41.6%
Basic Industries	0.99	0.83	16.1%	0.75	23.6%	0.57	42.2%
Communications	0.94	0.82	12.7%	0.42	55.1%	0.45	52.4%
Cyclicals	0.49	0.45	8.6%	0.38	22.3%	0.29	40.2%
Energy	0.45	0.38	15.4%	0.27	39.1%	0.27	40.1%
Financials	0.45	0.41	10.7%	0.31	32.5%	0.30	34.2%
Non Cyclicals	0.45	0.46	-1.2%	0.40	11.3%	0.40	12.0%
Utilities	0.44	0.39	11.6%	0.32	26.5%	0.32	28.5%

3. HEDGING ANALYSIS WITH ORION

The second hedging methodology is based on our new equity-based credit valuation model, ORION. The structural approach of the model benefits from explicit modeling assumptions and parameter calibration. Its main drawback, however, is model risk, especially if an important variable is not present in the model. In this section, we first describe the ORION model and its implications for hedge ratios. We then look at the performance of equity-based hedging of debt with hedge ratios given by the model.

3.1. The ORION Model

The ORION model is an equity-based credit valuation model. It uses equity price as its main driving variable, and also uses spread information to determine the level of a default barrier. The ORION model avoids the complication of Merton-type models by taking the equity value as the fundamental driving variable instead of the value of the firm's assets. For this reason, ORION does not need to use accounting leverage data explicitly, which may not always be reliably available. With a stochastic barrier perfectly correlated across issuers, ORION can also capture a systematic credit market factor which is not usually modeled in structural credit models.

In the ORION model:

- The firm defaults when the stock price *S(t)* falls below the default barrier, *B(t)*.
- The default barrier evolves stochastically over time.

The barrier, *B(t),* (in comparison to the equity price) can be thought of as a summary measure of the strength of all possible factors that may lead to a credit event (e.g. liquidity shortage, reduction in the ability to service long-term debt or even market-wide financial distress). The barriers could be correlated across firms to reflect a systematic credit market risk factor.

The ORION model assumes that S_t, the stock price of a firm, follows a lognormal Brownian motion:

$$dS_t = S_t(r - \delta_t)dt + \sigma . S_t dW_t$$

where *r* is the default-free interest rate and δ is the dividend pay-out ratio, σ is the volatility of the equity return process, and *Wt* is a Brownian motion mean 0 and standard deviation *dt*.

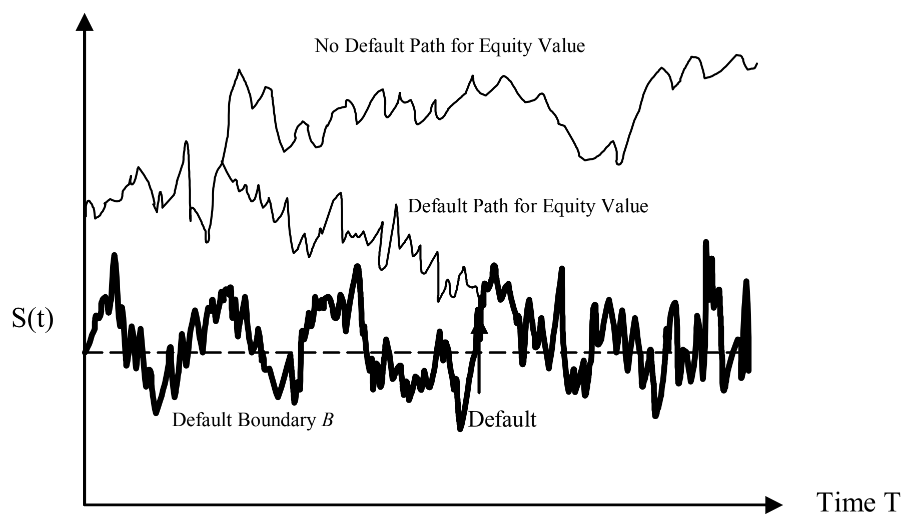
The stochastic barrier follows a diffusion process:

$$dB_t = \sigma_B . B_t dW_{Bt}$$

The initial barrier level is assumed to be B_0.

Default occurs if the equity price falls below the default barrier. Figure 11 illustrates two possible sample paths of the firm's equity.

Figure 11. The equity value and default boundary are both stochastic. Default can be caused by the equity value moving below the default boundary.



Given the above assumption, we can value securities whose cash flows are contingent on default, such as credit bonds and default swaps. Using these valuation equations, we can also compute the model-implied hedge ratios (or deltas) of credit securities with respect to the issuers' equity.

3.2. Pricing of Corporate Debt in ORION

The simple ORION framework allows us to compute the values of the firm's liabilities. For example, the value of a bond is simply the discounted sum of coupons and principal payment weighted by the survival probabilities:

$$D(0,T)) = \sum_{j=1}^{n} C(T_j) Z(0,T_j) P(\Gamma > T_j \mid \Gamma > 0) + RG(T_n)$$

G is the present value of receiving one euro if default occurs over the interval *[0,T]* (see Technical Appendix). $C(T_j)$ is the coupon paid at time T_j, $Z(0,T_j)$ is the discount rate between time 0 and time T_j. $P(\Gamma > T_j \mid \Gamma > 0)$ is the survival probability and R is the bond recovery value. The expressions for survival probabilities are provided in the appendix.

3.3. Derivation of the Hedge Ratios Bond/CDS-Equity

The model implied hedge ratio, $\Delta(t)$, is given by the partial derivative of the bond pricing equation with respect to *S(t)*. The expression for this ratio is provided in the appendix. For hedging a long debt position worth $M, one needs to go short the issuer's equity worth $\$\frac{M\Delta(t)S(t)}{D(t)}$ where *S(t)* is the market value of equity per share and *D(t)* is the market value per unit of debt.

We compare the hedging results obtained using the empirical methodology (CAESAR, described in the previous section) and the model-based methodology (based on ORION). For the model-based hedging, we compute the delta hedge every month after calibration of the model to the equity price, the equity volatility and the spread curve of the issuer. We then go long the bond (hedged by duration-matched Treasuries) and short-sell the appropriate amount of issuer's equity. We compute the returns of the hedged position a month later. We repeat this hedging exercise every month and report the average volatility of the hedged and unhedged excess returns and the percentage reduction in the volatility.

In Figure 12, we present the average hedge ratios (or equity betas) for the top 100 USD issuers (by par amount outstanding) by rating categories for the 1999-2003 period. The model-based hedge ratios are larger than the empirical betas across all the ratings from AA to B. This may be due to the fact that while our model assumes that credit spreads are a fair compensation only for the risk of equity falling below the default barrier, in reality credit spreads contain components unrelated to default risk.

In the same figure, we also present the average equity betas for the top 100 Euro issuers by rating categories. The model-based betas are larger for AA-rated bonds and have the same magnitude for A-rated and below bonds. Interestingly, the empirical and the model-based betas for BB and B-rated bonds are on average very close.

Figure 12. Average betas by ratings (AA to B, Top 100 US and top 100 Euro Corporate Issuers)

	Top 100 USD issuers: 1999-2003		Top 100 EUR issuers: 1999-2003	
	Empirical Hedging With Equity	Model-Based Hedging With Equity	Empirical Hedging With Equity	Model-Based Hedging With Equity
Rating	Average Beta (Equity)	Average Beta (Equity)	Average Beta (Equity)	Average Beta (Equity)
AA	0.029	0.145	0.006	0.068
A	0.034	0.176	0.019	0.099
BBB	0.053	0.194	0.035	0.143
BB	0.082	0.243	0.215	0.212
B	0.200	0.256	0.144	0.148

In Figure 13, we report the average volatility reduction due to empirical and model-based hedging for the top 100 USD issuers by rating categories for the 1999-2003 period. The model-based hedging performs slightly worse than empirical hedging except for B-rated bonds.

Figure 13. Top 100 USD issuers (by par amount outstanding), 1999-2003 (by rating)

	Unhedged	Empirical Hedging With Equity		Model-Based Hedging With Equity	
Rating	Volatility	Volatility	% Reduction	Volatility	% Reduction
AA	0.67	0.61	9.0%	0.74	-11.3%
A	0.93	0.76	19.1%	0.88	6.2%
BBB	1.74	1.26	27.6%	1.33	23.6%
BB	4.01	3.13	22.0%	3.35	16.5%
B	10.65	8.62	19.1%	7.95	25.4%

In Figure 14, we report the average volatility reduction due to hedging for the top 100 Euro issuers by rating categories. Despite the difference in the magnitudes of the hedge ratios, model-based hedging performs better than the empirical hedging for A-, BBB-, BB- and B-rating categories. On average, the volatility drops by a third when an equity hedge is used. It is clearly not a perfect hedge as two-thirds of the volatility remains.

Figure 14. Top 100 Euro issuers (by par amount outstanding), 1999-2003 (by rating)

	Unhedged	Empirical Hedging With Equity		Model-Based Hedging With Equity	
Rating	Volatility	Volatility	% Reduction	Volatility	% Reduction
AA	0.29	0.27	6.6%	0.34	-16.8%
A	0.63	0.54	14.1%	0.39	37.6%
BBB	1.76	1.55	11.9%	1.15	34.8%
BB	6.02	4.51	25.1%	4.48	25.6%
B	2.66	2.84	-6.9%	2.30	13.6%

Finally, it may be noted that the default barrier is also stochastic in ORION: hedging against equity is in theory not sufficient to immunise a bond or CDS portfolio. If we make the additional assumption that the barriers across all the assets are perfectly correlated because the Brownian term is shared across the assets and represents a measure of liquidity or systematic risk, the hedging strategy would consist of hedging against the equity movement and the barrier movement. This will be the subject of future research.

4. SCENARIO ANALYSIS

As shown above, hedging investment grade debt with equity produces mixed results unless wider market factors are taken into account. This section presents a scenario-based approach for analyzing such trades. At the single issuer level, debt-equity trades are highly idiosyncratic in nature. For this reason, the debt-equity relationship is complex and difficult to model accurately. Unpredictable co-movements of debt and equity can have a significant effect on the performance of a debt-equity trade, and so care must be taken to understand the range of possible outcomes. To illustrate how to do this, we present an example below.

Example: Long Bond, Short Equity

Consider an investor who is long €1m face value of a 5.25% November 08 bond (5-year maturity) with a full price of 100.37, and short 15,000 shares with a share price at €10.

For valuation purposes, suppose the LIBOR term structure is flat at 3% and the CDS credit curve is flat at 200bp, with an assumed 40% recovery. The bond is funded at LIBOR flat.

Figure 15 shows the mark-to-market (MTM) of this trade at a 3-month horizon as a function of the spread and stock price on that future horizon date. In this figure, we have allowed the spreads to move in the range of 100-300bp while the share price moves between €8 and €12. Moreover, we have used CDS spreads to characterize the state of the debt market in the future, assuming that bonds and default swaps are consistently priced.

Figure 15 Mark-to-market variation of a long bond, short equity trade. Share price varies across columns and CDS spreads vary down rows

Trade MTM (€ ,000s)		Stock Price								
		8.0	**8.5**	**9.0**	**9.5**	**10.0**	**10.5**	**11.0**	**11.5**	**12.0**
CDS Spread	**100**	+ 80	+ 72	+ 65	+ 57	+ 50	+ 42	+ 35	+ 27	+ 20
	150	+ 57	+ 50	+ 42	+ 35	+ 27	+ 20	+ 12	+ 5	- 3
	200	+ 35	+ 28	+ 20	+ 13	+ 5	- 2	- 10	- 17	- 25
	250	+ 14	+ 7	- 1	- 8	- 16	- 23	- 31	- 38	- 46
	300	- 6	- 14	- 21	- 29	- 36	- 44	- 51	- 59	- 66

As an example, consider what happens if the share price rallies from €10 to €11 and the CDS spread tightens from 200bp to 150bp. First, the MTM of the short equity position is €(11-10) × (-15,000) = - €15,000. If the CDS spread tightens from 200bp to 150bp, then the corresponding model-implied full bond price is 103.82, which means that long bond position has a positive MTM of €(103.82 – 100.37) × 1m = €34,500. The cost of funding the bond over this period is 3% × 0.25 × 100.37 × €1m = €7,500. The net MTM of the trade in this case is therefore a positive €12,000.

Figure 15 shows the wide range of values that the trade MTM can take, depending on the states of the debt and equity markets at the horizon date. If the markets remain static, then the trade accrues carry net of funding, which is shown in the figure as the intersection of the shaded row and column. However, as the table shows, this figure could vary between +€80,000 and - €66,000 depending on the joint realization of spreads and equity prices. Of course, the extreme outcomes are relatively less likely and we should attempt to assign some likelihood to them.

Further, this trade is exposed to default risk, and the VoD (Value on Default) depends strongly on the realised recovery rate, as shown in Figure 16. For example, if the issuer defaults with 30% recovery, then the loss on the bond is €(30.00 – 100.37) × 1m = - €703,700. Assuming conservatively that the equity is worthless after default, the gain on the short equity position is - €10.00 × (-15,000) = + €150,000. The net effect of these is a negative value on default of -€553,700. Figure 16 summarises various outcomes.

Figure 16. Value on default (VoD) as a function of realised recovery rate

Recovery	10%	20%	30%	40%	50%	60%	70%	80%	90%
VoD (‘000)	- 754	- 654	- 554	- 454	- 354	- 254	- 154	- 54	+ 46

This example demonstrates the necessity of stress-testing a debt-equity hedge. A model-based or empirically determined hedge ratio implicitly contains a view on the co-movement of debt and equity markets in the future. If the realised outcomes differ from this view, the MTM of the hedge could deviate significantly from zero. In the remainder of this section, we analyse a specific debt-equity hedge within the framework developed above.

Example: Long Ahold 5.875% May 08 vs Short Equity

Consider an investor who goes long €1m of the Ahold 5.875% 08 bond, paying a full price of 102.52, corresponding to a CDS spread of 200bp, and hedges this by selling Ahold shares. The share price is €8.02. The 5-year EUR swap rate is 3.75%. Suppose the investor has a 3-month horizon.

The hedge implied by the ORION model for this bond is to go short 24.4% of the market value of the bond. This corresponds to selling 31,200 shares at the current price. Figure 17 below shows a sample of how the MTM of this hedge can behave:

Figure 17. Sample MTM of the debt-equity hedge. Share price varies across rows and CDS spreads down columns. All figures are in € '000

	5	6	7	8	9	10	11
50	+ 160	+ 129	+ 98	+ 67	+ 36	+ 4	- 27
75	+ 150	+ 119	+ 87	+ 56	+ 25	- 6	- 37
100	+ 139	+ 108	+ 77	+ 46	+ 15	- 17	- 48
125	+ 129	+ 98	+ 67	+ 36	+ 4	- 27	- 58
150	+ 119	+ 88	+ 57	+ 26	- 6	- 37	- 68
175	+ 109	+ 78	+ 47	+ 16	- 16	- 47	- 78
200	+ 100	+ 68	+ 37	+ 6	- 25	- 56	- 88
225	+ 90	+ 59	+ 28	- 4	- 35	- 66	- 97
250	+ 81	+ 49	+ 18	- 13	- 44	- 75	- 107
275	+ 71	+ 40	+ 9	- 22	- 53	- 85	- 116
300	+ 62	+ 31	- 0	- 31	- 62	- 94	- 125
325	+ 53	+ 22	- 9	- 40	- 71	- 103	- 134
350	+ 45	+ 13	- 18	- 49	- 80	- 111	- 143

Figure 17 shows that the hedge is effective within a certain range of realisations, which represents the implicit view on how the debt-equity relationship is expected to evolve. Outside this range, the hedge can turn out to be a strong bullish or bearish trade. Corporate events such as nationalisation, rights issue, share buyback, merger or a takeover can create unexpected co-movements. As in the previous example, this trade too is exposed to significant default risk, which depends on the realised recovery rate following a credit event.

In summary, the contemporaneous relationship between debt and equity is complex and generating robust hedges consisting only of issuers' equity is difficult. For this reason, scenario analysis is a useful approach to investigate how different outcomes of the market might affect the performance of a given debt-equity trade.

5. CONCLUSIONS

In this article, we have used two different methodologies to hedge debt with equity: (i) the empirical hedging methodology based on a regression analysis (based on our CAESAR model); (ii) model-based hedging based on our equity-based credit valuation model ORION. We have also introduced a scenario-based analysis for special situations. We have presented an empirical investigation of the performance of CAESAR and ORION and discussed the use of scenario analysis of hedging strategies. One important conclusion is that pure investment grade debt-equity hedging needs a Credit Index to perform reasonably well. The reason is that the equity and the credit markets may not be completely integrated: the credit market is subject to a systematic risk factor not directly related to equity. The effect of this overall market factor is lesser for high yield and crossover bonds. The pure equity-based hedging results are better for high yield and crossover bonds both with CAESAR and ORION. It is possible to achieve a reduction in volatility by hedging with equity, more so with high yield and crossover bonds, but it is far from perfect. Debt-equity hedging should be complemented by a better understanding of systematic market effects and through the use of scenario analysis.

REFERENCES

Black and Scholes (1973), "The Pricing of Options and Corporate Liabilities", *The Journal of Political Economy*, 81(3), pp. 637-654.

Collin-Dufresne, Goldstein and Martin (2001), "The Determinants of Credit Spread Changes", *Journal of Finance*, 56(6), pp. 2177-2207.

Merton (1974), "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", R. Merton, *Journal of Finance*, May 1974, or Continuous-Time Finance, Blackwell Publishing, 1990 by R. Merton.

Naik, *et al*. (2003), "Introducing ORION: The Lehman Brothers Hybrid Credit Model", forthcoming.

O'Kane and Schloegl (2001), "Credit Modelling: Theory and Practice", Lehman Brothers Fixed Income Publications, February 2001.

TECHNICAL APPENDIX

We list some results of the ORION model. Full derivation is explained in Naik *et al.* (forthcoming). The notations are:

S :stock price

B :the stochastic barrier

B_0 :the initial level of the barrier

σ :the log volatility of the stock

σ_B :the log volatility of the stock

δ :the dividend pay-out ratio

r :the risk-free interest rate

S follows a diffusion process:

$$dS = S(r-\delta)dt + \sigma.SdW_t$$

where r is the default-free interest rate and δ is the dividend pay-out ratio, σ is the volatility of the equity return process, and Wt is a Brownian motion mean 0 and standard deviation dt.

The stochastic barrier follows a diffusion process:

$$dB = \sigma_B.BdW_{Bt}$$

We define a normalized equity price $X_t = \ln(S_t / B_t)$ which then follows the stochastic process, using Ito's lemma we find:

$$dX_t = (r - \delta_S - \frac{\sigma_S^2}{2} + \frac{\sigma_B^2}{2})dt + \sigma_S.dW_{St} - \sigma_B.dW_{Bt}$$

$$= (r - \delta - \frac{\sigma^2}{2})dt + \sigma.dW_t$$

$$= \mu.dt + \sigma.dW_t$$

Where we use a new Brownian motion:

$$dW_t = \frac{\sigma_S.dW_{St} - \sigma_B.dW_{Bt}}{\sqrt{\sigma_S^2 + \sigma_B^2}}$$

where the new volatility is $\sigma^2 = \sigma_B^2 + \sigma_S^2$ and dividend payout ratio is $\delta = \delta_S - \sigma_B^2$

We define the time of default Γ as the first passage time to the barrier B.

$$\Gamma = \inf\{u \geq 0; S_u < B\}$$

We can carry the formula for the fixed barrier case on survival probabilities, bond prices and CDS spreads using the new variables of volatility σ and dividend ratio δ .

From standard first-passage of time formula, the survival probability $P(\Gamma > T \mid \Gamma > 0)$ can be computed explicitly:

$$P(\Gamma > T \mid \Gamma > 0) = N(h_1) - (B/S)^{2\nu}[1 - N(h_2)]$$

where

$$h_1(T)=\frac{\ln(S/B)+\mu T}{\sigma\sqrt{T}} \qquad h_2(T)=\frac{\ln(S/B)-\mu T}{\sigma\sqrt{T}}$$

$$\mu=r-\delta-\sigma^2/2$$

$$\nu=\mu/\sigma^2$$

The derivative of the survival probability with respect to the stock price is:

$$\frac{\partial P(T)}{\partial S}=\frac{1}{S\sigma\sqrt{T}}[n(h_1(T))+(B/S)^{2\nu}n(h_2(T))]+\frac{2\nu}{S}(B/S)^{2\nu}[1-N(h_2(T))]$$

In the liabilities derivation, we need the present value of one dollar in case of default $G(T)$, the formula for $G(T)$ is:

$$G(T)=(S/B)^{\frac{-\mu+\beta_1}{\sigma^2}}N\left[\frac{\ln(B/S)-\beta_1 T}{\sigma\sqrt{T}}\right]+(S/B)^{\frac{-\mu-\beta_1}{\sigma^2}}N\left[\frac{\ln(B/S)+\beta_1 T}{\sigma\sqrt{T}}\right]$$

$$=(S/B)^{\frac{-\mu+\beta_1}{\sigma^2}}N\left[h_3(T)\right]+(S/B)^{\frac{-\mu-\beta_1}{\sigma^2}}N\left[h_4(T)\right]$$

where $\beta_1^2=\mu^2+2r\sigma^2$

and

$$h_3(T)=\frac{\ln(B/S)-\beta_1 T}{\sigma\sqrt{T}} \qquad h_4(T)=\frac{\ln(B/S)+\beta_1 T}{\sigma\sqrt{T}}$$

We will also need an expression for the derivative of *G* with respect to the equity price is:

$$\frac{\partial G(T)}{\partial S}=\frac{1}{S}\{(\frac{-\mu+\beta_1}{\sigma^2})(S/B)^{\frac{-\mu+\beta_1}{\sigma^2}}N\left[h_3(T)\right]+(\frac{-\mu-\beta_1}{\sigma^2})(S/B)^{\frac{-\mu-\beta_1}{\sigma^2}}N\left[h_4(T)\right]\}$$
$$-\frac{1}{S\sigma\sqrt{T}}\{(S/B)^{\frac{-\mu+\beta_1}{\sigma^2}}n\left[h_3(T)\right]+(S/B)^{\frac{-\mu-\beta_1}{\sigma^2}}n\left[h_4(T)\right]\}$$

The hedge ratio of bond vs equity is:

$$HedgeRatioBond=$$
$$\sum_{j=1}^{n}\left(C(T_j)\frac{Z(0,T_j)}{S\sigma\sqrt{T_j}}[n(h_1(T_j))+(B/S)^{2\nu}n(h_2(T_j))]+\frac{2\nu}{S}(B/S)^{2\nu}[1-N(h_2(T_j))]\right)$$
$$+RN\left(\begin{array}{l}\frac{1}{S}\{(\frac{-\mu+\beta_1}{\sigma^2})(S/B)^{\frac{-\mu+\beta_1}{\sigma^2}}N\left[h_3(T_n)\right]+(\frac{-\mu-\beta_1}{\sigma^2})(S/B)^{\frac{-\mu-\beta_1}{\sigma^2}}N\left[h_4(T_n)\right]\}\\-\frac{1}{S\sigma\sqrt{T}}\{(S/B)^{\frac{-\mu+\beta_1}{\sigma^2}}n\left[h_3(T_n)\right]+(S/B)^{\frac{-\mu-\beta_1}{\sigma^2}}n\left[h_4(T_n)\right]\}\end{array}\right)$$

Forward CDS Spreads

Arthur M. Berd
212-526-2629
arthur.berd@lehman.com

The exact analytical formula for forward CDS spreads is surprisingly simple. We explain its derivation and usage for a variety of spread curve trading strategies.

1. CURVE TRADING IN CDS MADE EASY

As the CDS market matures, more and more names are trading with significant liquidity across the entire range of maturities away from the traditional 5-year point. The trading in shorter maturities is driven primarily by the unwinding of old contracts which was made much easier by the adoption of standardized ISDA maturities, while trading in longer maturities is driven by the hedging needs of synthetic CDOs and other long-term credit portfolios. Such trading leads to price discovery and development of non-trivial spread term structures in CDS. Many investment grade names in the US and Europe are now routinely quoted in the CDS market for maturities extending from 1 through 10 years.

With the rapid development of CDS curve trading, measuring and trading forward CDS spreads is an increasingly popular activity (see McAdie *et. Al.* [2003]). A forward default swap corresponds to buying or selling credit protection that is active for a period of time in the future at a premium whose level is preset today, but payable only during the active period of the contract. For example, a 5-year CDS 2 years forward (i.e. 2x5 forward CDS) refers to protection which starts in year 2 and continues for 5 years afterward, ending in year 7 from today, with premiums paid quarterly during that period. Importantly, the forward CDS knocks out and provides no protection if the credit event occurs prior to the forward start date.

In this note, we derive the following simple and exact formula for the forward CDS spread which should help investors to easily analyze forward CDS trading strategies:

$$S(t,T) = \frac{S(0,T) - \eta(t,T)\cdot S(0,t)}{1-\eta(t,T)}, \qquad \text{where } \eta(t,T) = \frac{\mathrm{RiskyPV01}(0,t)}{\mathrm{RiskyPV01}(0,T)}$$

As we can see, the forward spread depends very simply on the spot spreads to initial $S(0,t)$ and final $S(0,T)$ maturity and on the ratio of risky PV01s to each maturity $\eta(t,T)$. Because the risky PV01 is relatively stable with respect to the interest rate environment and the spread curve term structure, it is often an acceptable approximation to use a set of standard estimates for RPV01 in order to calculate the forward CDS spread. The table in Figure 1 below may serve as a reference for most cases.

Figure 1. Risky PV01 estimates for typical CDS (flat 5% LIBOR rate, 40% recovery)

	Maturity (years)	1	2	3	4	5	6	7	8	9	10
Spread (bp)	50	0.98	1.90	2.77	3.59	4.37	5.10	5.78	6.43	7.04	7.62
	100	0.98	1.89	2.74	3.54	4.28	4.98	5.63	6.24	6.81	7.34
	150	0.97	1.87	2.71	3.48	4.20	4.86	5.48	6.05	6.58	7.07
	200	0.97	1.86	2.68	3.43	4.12	4.75	5.34	5.87	6.37	6.82
	300	0.96	1.83	2.61	3.32	3.96	4.54	5.07	5.54	5.97	6.36
	500	0.95	1.77	2.49	3.13	3.68	4.16	4.59	4.96	5.28	5.56

The forward CDS spreads derived in this fashion can be compared with the spot spreads of similar maturity and, depending on the issuer and market credit outlook, can serve as an indicator of relative value and justification for either long or short forward trades in CDS. Since the ratio of risky PV01s $\eta(t,T)$ is always positive and less than 1, we can deduce several implications for the relationship between the spot and forward CDS curves:

- If the spot spreads to initial and final maturity of the forward range are equal to each other, then the forward spread is also equal to the same number, regardless of the shape of the spread curve at other maturities. This is an unusual feature of CDS spreads: such a statement is definitely not true for cash bonds.
- If the spot CDS differential is positive between the initial and final maturity dates, then the forward CDS spread is greater than the spot spread to the final maturity.
- If the spot CDS differential is negative between the initial and final maturity dates, then the forward CDS spread is less than the spot spread to the final maturity.

To illustrate the estimation procedure for the forward CDS and see whether the approximation using risky PV01s from Figure 1 is sufficient, let us consider the case of Altria (MO), one of the most actively traded names across the entire range of maturities. The recent spread curve in MO happens to exhibit an unusual humped shape, as detailed in Figure 2.

In Figure 2, we show the spread curve as of October 31, 2003 (the values in italics are interpolated based on the rest of the curve), the exact values of risky PV01 for each maturity, and 3- and 5-year forward spreads for various initial dates (the year in this case refers to the start of the forward contract). For the forward spreads we show both the exact calculation and the approximate estimate using the risky PV01 estimates from Figure 1, corresponding to the spread of 200bp. As we can see, the differences between the exact and approximate calculation of the forward CDS spread are negligible.

Given the complex shape of the Altria CDS curve, one can construct many curve trading strategies, depending on the credit outlook. Without taking a specific outlook, we mention below some hypothetical curve strategies for either bullish or bearish views:

- **Bullish:** Go long 5-year CDS starting 5 years forward (5x5 forward CDS), maximize the spread pickup over the spot curve.
- **Bearish:** Go short 3-year CDS starting 2 years forward (2x3 forward CDS). Gain exposure to tightest spread compared with the spot curve.

Using the simple technique outlined here credit investors can easily construct and analyze many such curve strategies.

Figure 2. Spot and forward CDS curves for Altria (MO), as of October 31, 2003

Maturity (years)	1	2	3	4	5	6	7	8	9	10
Spot spread (bp)	215	220	210	200	200	202	205	206	208	210
Risky PV01	0.99	1.92	2.79	3.59	4.33	5.01	5.62	6.19	6.69	7.15
3-yr forward (exact)	194	184	192	214	220	226	228			
3-yr forward (approx)	194	184	192	214	220	226	228			
5-yr forward (exact)	199	197	203	217	225					
5-yr forward (approx)	199	197	203	217	225					

2. DERIVATION OF THE FORWARD CDS SPREAD

In reduced form models of default (see Litterman and Iben [1991], Jarrow and Turnbull [1995], and Duffie and Singleton [1999]), the survival probability is related to the default intensity, *h(t)* (also called hazard rate), by the relationship[1]:

$$Q(t,T) = \exp\left(-\int_t^T h(s)\cdot ds\right) \qquad [1]$$

Here, $Q(t,T)$ denotes the survival probability for the time period (t,T), provided that the credit survived until the beginning of the period. The forward discount function for the riskless rate is given by a similar formula, where $r(s)$ is the instantaneous forward rate:

$$Z(t,T) = \exp\left(-\int_t^T r(s)\cdot ds\right) \qquad [2]$$

Let us now consider a forward contract for default protection during the time period (t,T), which we will enter into on the starting date t if the default has not occurred by that date. With definitions [1] and [2], we can write down the pricing equation for forward CDS by equating the estimated values of the protection and premium leg of the contract as of starting date t under the condition that the debt issuer survived until that time (see O'Kane and Turnbull [2003] for detailed discussion of CDS pricing)[2]. In case of a credit event prior to starting date t, the forward CDS knocks out and the equality is trivial with no payments made on either side. Since there are never any cash flows prior to t, the equality of expected values as of that time implies the equality as of the initial pricing date (time $t = 0$) as well.

$$S(t,T)\cdot \int_t^T du\cdot e^{-\int_t^u (r(s)+h(s))\cdot ds} = (1-R)\cdot \int_t^T du\cdot h(u)\cdot e^{-\int_t^u (r(s)+h(s))\cdot ds} \qquad [3]$$

Let's multiply both sides by $e^{-\int_0^t (r(s)+h(s))\cdot ds}$ and take that expression under the integral, and rewrite the integrals over (t,T) as differences of the same integrals over $(0,T)$ and $(0,t)$:

$$\begin{aligned} &S(t,T)\cdot\left[\int_0^T du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds} - \int_0^t du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}\right] \\ &= (1-R)\cdot\left[\int_0^T du\cdot h(u)\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds} - \int_0^t du\cdot h(u)\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}\right] \end{aligned} \qquad [4]$$

Now, let us divide both sides by the first integral in the difference on the left hand side:

1 *The hazard rate and interest rate can, in general, be stochastic. In this discussion, we restrict our attention to deterministic default intensities and deterministic interest rates.*

2 *This formula ignores the effect of the linear accrual rules for the protection payment, which is known to be small.*

[5]

$$S(t,T)\cdot\left[1-\frac{\int_0^t du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}{\int_0^T du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}\right]=(1-R)\cdot\frac{\int_0^T du\cdot h(u)\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}{\int_0^T du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}$$

$$-(1-R)\cdot\frac{\int_0^t du\cdot h(u)\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}{\int_0^t du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}\cdot\frac{\int_0^t du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}{\int_0^T du\cdot e^{-\int_0^u (r(s)+h(s))\cdot ds}}$$

If we carefully look at each term in this formula, we can see the following familiar patterns:

- The numerator and denominator in the fraction on the left hand side are nothing but the RiskyPV01 to corresponding maturity.
- The loss fraction times the fractional expressions on the right hand side are nothing but the CDS spread to corresponding maturity.
- The last expression on the right hand side contains the same fraction as the left hand side, i.e. the ratio of RiskyPV01s to initial and final maturity.

This brings us to an equivalent equation:

[6]

$$S(t,T)\cdot\left[1-\frac{\text{RiskyPV01}(0,t)}{\text{RiskyPV01}(0,T)}\right]=S(0,T)-\frac{\text{RiskyPV01}(0,t)}{\text{RiskyPV01}(0,T)}\cdot S(0,t)$$

From where we obtain the simple and exact formula for the forward CDS spread, as presented before:

[7]

$$S(t,T)=\frac{S(0,T)-\dfrac{\text{RiskyPV01}(0,t)}{\text{RiskyPV01}(0,T)}\cdot S(0,t)}{1-\dfrac{\text{RiskyPV01}(0,t)}{\text{RiskyPV01}(0,T)}}$$

Note that this formula only applies to CDS spreads – the case of credit bonds is more complicated because of the final maturity payment.

Also note that the formula does not depend explicitly on the recovery rate – the dependence on the recovery is contained in the implied hazard rate. If the hazard rate is calibrated to the market observed spreads using a given recovery rate assumption, then this assumption also influences the estimates of RiskyPV01 (see Berd and Kapoor [2002], and O'Kane and Turnbull [2003] for estimates of recovery dependence).

Acknowledgments: I would like to thank Mahesh Bhimalingam, Marco Naldi, Dominic O'Kane and Ashish Shah for their useful comments and Manish Mehta for helping with calculations.

REFERENCES

Berd, A. and V. Kapoor (2002), "Digital Premium", *Journal of Derivatives*, vol. 10 (3), p. 66.

Duffie, D. and K. Singleton (1999), "Modeling Term Structures of Defaultable Bonds", *Review of Financial Studies*, vol. 12, p. 687.

Jarrow, R. A. and S. M. Turnbull (1995), "Pricing Options on Financial Securities Subject to Default Risk", *Journal of Finance*, vol. 50, p. 53.

Litterman, R. and T. Iben (1991), "Corporate Bond Valuation and the Term Structure of Credit Spreads", *Journal of Portfolio Management*, spring issue, p. 52.

McAdie, R., U. Bhimalingam and S. Sen (2003), "Forward CDS Trades", *European Credit Strategies*, Lehman Brothers Fixed Income Research, November 5, 2003.

O'Kane, D. and S. Turnbull (2003), "Valuation of Credit Default Swaps", *Quantitative Credit Research Quarterly*, vol. 2003-Q1/Q2, Lehman Brothers Fixed Income Research.

Understanding Deltas of Synthetic CDO Tranches

Lutz Schloegl
44-20 7260 2113
luschloe@lehman.com

Andrei Greenberg
44-20 7260 1285
andgreen@lehman.com

Synthetic CDO tranches have become a core part of the credit derivatives market. Because these are portfolio credit products, risks and opportunities present themselves in multiple dimensions. Of primary importance is the sensitivity of tranche values to credit quality shifts in the underlying collateral, popularly known as the tranche delta. We describe the general framework used to analyse this sensitivity, and give intuition regarding the behaviour of tranche deltas. This is particularly relevant, as the dependence of tranche deltas on the various collateral specifications is not always obvious at first glance. This underlines the value of simple approaches that nevertheless describe all relationships and help understand the risks involved.

1. INTRODUCTION

The tranche delta is the fundamental risk measure used to quantify the exposure of tranches to changes in the credit quality of the underlying reference credits. It is closely related to the concept of dynamically risk-managing single tranches of CDO's. Consider a trade whereby an investor sells protection to Lehman Brothers on a synthetic CDO tranche, receiving a fixed coupon on the surviving notional of the tranche over the life of the trade. If spreads were to widen in the future, this would have a negative impact on the investor's position, as the protection which has been sold for a fixed spread would increase in value. One way to hedge against such an outcome is to buy protection on the individual names according to the size of their deltas, thereby immunising the trade to first order against spread movements. On the other side of these trades, dealers dynamically manage their exposures by selling protection on individual names in line with the delta.

Deltas are typically quoted in terms of the default swap notional one would trade to immunise the tranche. If one traded the whole capital structure of a portfolio, in terms of the protection payouts this would be the same as buying protection on all the names contained in it. One can therefore expect that, summed across all tranches, the deltas are close to the notional amount of each name in the portfolio. This is a very useful rule of thumb. However, this view neglects the fact that the premium leg paid on the tranches (as well as on the default swap hedges) is risky. If defaults happen in the collateral, then not only will protection payments be made to the protection buyer, but also the premium payments will be smaller, as the notional of the tranche is written down. Because the coupon paid on equity tranches in particular tends to be very large, this increases the sensitivity to spread movements, and can cause the total sum of the hedge notionals to deviate from the notional of a credit in the portfolio.

Understanding how the risk sensitivity to a particular credit is allocated among the different tranches of a CDO is essential to building intuition for tranche deltas. Two fundamental drivers of tranche valuation are idiosyncratic and systemic risks. An investor in an equity tranche is much more exposed to idiosyncratic risks than investors in the senior parts of the capital structure, since every default causes a capital loss for the former. The main concern of senior investors on the other hand is deterioration in the overall credit quality of the reference names. If the default of an asset is perceived to be a relatively idiosyncratic event, then this asset will affect the equity tranche much more than the senior ones. Conversely, the sensitivity to credits with a high systemic component will be greater for the senior tranches and less for the equity.

A further important aspect determining tranche sensitivities is the timing of defaults. In the widely used times-to-default framework, default timing is determined by the individual credit spreads. High-spread credits will tend to default earlier, and therefore have a greater impact on junior tranches. Credits with low spreads tend to default later, when other credits are expected to have defaulted already. This means that the sensitivity to these credits is concentrated in the senior tranches.

In section 2 we define tranche deltas and describe the model framework used to analyse them. We also introduce a stylised portfolio and a three-tranche CDO structure which we use to demonstrate the delta dependencies in more detail. Sections 3 and 4 focus, respectively, on market (collateral characteristics) and structural (width and subordination of tranches) factors, and how deltas of different tranches are influenced by them. We interpret the results for macro as well as idiosyncratic changes. Conclusions appear in section 5.

2. FRAMEWORK FOR ANALYSIS

Tranche deltas are measured with respect to each particular reference credit and are used to hedge the exposure of the tranche holder to movements in the spread of this credit. To be more specific, we consider the tranche mark-to-market at a certain point in time, *MTM* and look at a specific credit in the collateral, trading at spread s_0. We consider the mark-to-market from the protection buyer's point of view, so that it increases if spreads widen. If the spread for the specific name moves to $s_0 + 1$bp, the mark-to-market of the tranche becomes MTM'. The tranche delta with respect to this credit is the following quantity:

$$\Delta_0 = \frac{\mathrm{MTM}' - \mathrm{MTM}}{\mathrm{PV01}_0 \cdot 1\mathrm{bp}} \qquad (1)$$

where $\mathrm{PV01}_0$ is the present value of a 1bp coupon stream, weighted by the survival probabilities of the credit – in fact, this is the risky PV01 of the corresponding single-name credit default swap. Thus, the tranche delta as given by (1) is expressed in monetary terms as the amount of single-name protection required: to hedge credit risk associated with this particular credit, the investor needs to enter into a CDS contract by buying protection on the notional Δ_0.

In the widely used copula framework (see O'Kane et al for more details) , we compute an implied hazard rate h for each credit from the spread s and the recovery rate R using the credit triangle:

$$h \approx \frac{s}{1-R} \qquad (2)$$

This is then typically combined with a one-factor Gaussian copula. The occurrence of the default time τ is linked to the realisation of correlated Gaussian random variables. In the one-factor specification, these are represented by a market variable Z and an idiosyncratic component ε, with β denoting the correlation with the market factor. The default time is specified as:

$$1 - e^{-h\tau} = \Phi\left(\beta Z + \sqrt{1-\beta^2}\,\varepsilon\right) \qquad (3)$$

Here, Φ denotes the cdf of the standard normal distribution. In the structural interpretation of the model, the Gaussian variable appearing in equation (3) is interpreted as a return on the firm's assets, and β is the correlation of the firm's asset return with the market return as a whole.

From equation (3) we can make several observations which are relevant for our delta discussion. The hazard rate h serves to transform the correlated asset returns into a default time. A higher spread s means that a larger hazard rate h will translate a given realisation of the asset return into a shorter default time. As the spread of a credit increases, its defaults tend to occur earlier.

The second observation concerns the role of the correlation β with the market factor. As β increases, the influence of the market factor on the occurrence of the credit's defaults becomes greater. The probability of that credit defaulting jointly with others becomes larger. However, the relative timing of defaults is still a function of the individual spreads.

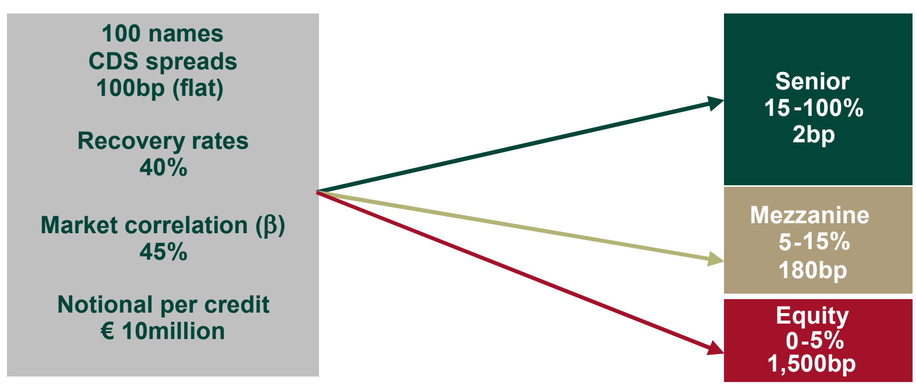
If, for example, in a given portfolio the spread of one credit is 800bp, while the others are trading at 100bp, the likelihood of enough low-spread defaults occurring so that the default of the high-spread credit impacts the senior tranche remains low, even at relatively high levels of correlation. Therefore, the greatest part of the sensitivity to the credit's spread of the 800bp asset remains concentrated in the junior tranche.

It is important to keep in mind that the effect of the spread level is a relative one, it is not only the absolute level of the credit spread which influences the allocation of the risk to the different tranches, but the spread of the asset relative to the portfolio average.

By varying the average spread and β's of the names in the portfolio it is possible to trace the sensitivity of tranche deltas with respect to these parameters on the macro scale. On the other hand, by varying only the spread and β of the individual credit with respect to which the tranche delta is calculated, we measure the credit specific component of the exposure.

We demonstrate our findings using a model CDO trade described below. The collateral consists of 100 credits, all having spreads of 100bp, recovery rates of 40% and market correlation of 45%. The capital structure consists of three tranches: a 5% equity tranche, a 10% mezzanine and a 85% senior tranche. Maturity is five years. The summary of the trade is given in Figure 1.

Figure 1. Collateral and structure of model synthetic CDO



Note that over a period of five years, the spread-implied expected loss on the portfolio is approximately equal to 5%, which is on the border between the equity and the mezzanine tranches.

Among the factors influencing the behaviour of tranche deltas, we separate the market parameters, such as individual spreads and correlations of the underlying names, and the structural factors, pertaining to the characteristics of a tranche, such as its size and attachment point. We analyse each group of factors in succession below.

3. MARKET EFFECTS ON TRANCHE DELTAS

In this section we consider two major drivers of market risk associated with deltas, which come from the characteristics of the collateral: spreads and correlation with the market (β). As was pointed out above, spreads express the relative default timings, while β's represent the effect of asset dependence.

3.1. Spread dependence

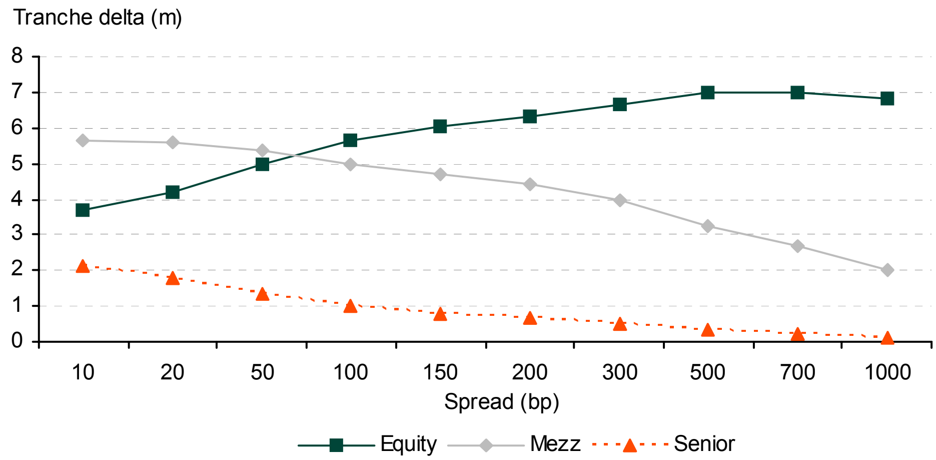
Spread movements change the risk profile of the collateral, and therefore the credit exposure of tranches moves as well. In this section, we trace the dependence of tranche deltas with respect to the spread of the underlying credit.

We start with the stylised portfolio described above and vary the spread of one of the names, measuring the resulting deltas (Figure 2) with respect to this name. A typical shape of the dependence is given in Figure 2(a). For high spreads, the asset in question is likely to be among the first to default, and therefore we see that the largest delta with respect to this name is associated with the equity tranche. Low spreads mean that the immediate risk of default is lower; hence, the risk moves out of the equity into the mezzanine tranche. Recall that the correlation of all assets with the market is 45%; this value is not large enough to trigger a sufficient number of further defaults to concern the senior investor, as we see from the relative sizes of mezzanine and senior deltas. Also unsurprisingly, as the spread of the individual name – and hence, also its default probability – goes up, the deltas of the more senior tranches drop and the equity delta increases.

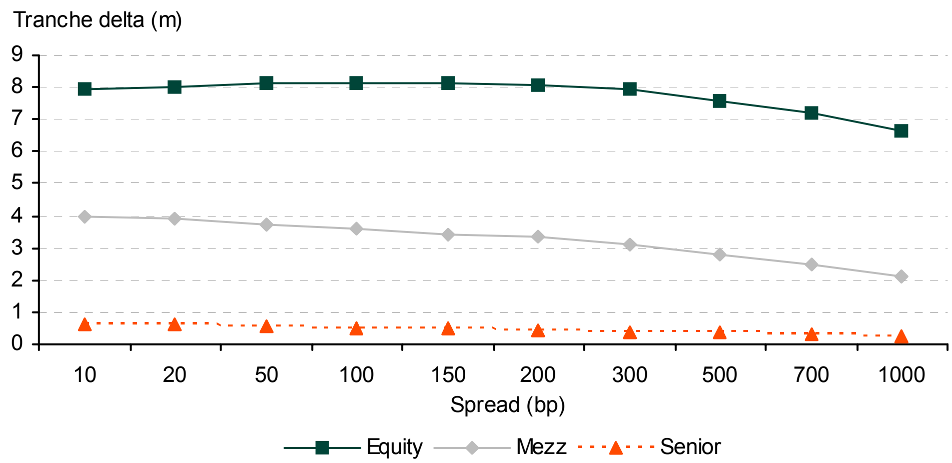
To assess how this pattern is affected by the dependence of assets, we perform the same calculations with different values of market correlation (β) for the individual name (Figure 2(b) and (c)). Low correlation (Figure 2(b)) effectively means that the movement of the spread of this particular asset is virtually independent of what goes on with the rest of the portfolio. This makes the individual name "more idiosyncratic", and we see that the equity delta always stays higher than deltas of other tranches, even when the spread is low and default is less imminent. We can claim that in this case, the correlation effect dominates the default timing effect. When correlation is high (Figure 2(c)), the asset bears a lot of systematic risk, especially when the spread of this name is low: we see that in this case the delta of the senior tranche is largest. As the spread widens, more credit risk associated with this name is absorbed by the mezzanine tranche, resulting in the increase of its delta; finally, as spreads widen further, the default timing effect takes over, and the equity tranche has the largest delta.

Figure 2. Asset spread dependence of tranche deltas

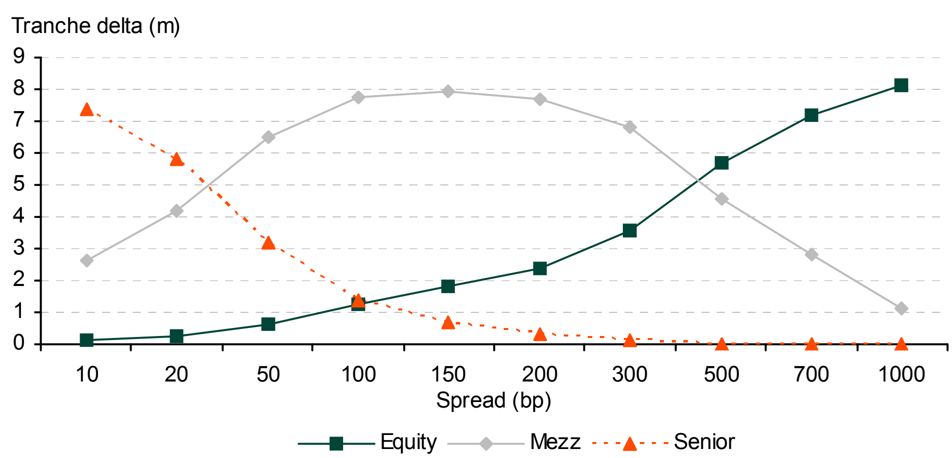
(a) Medium correlation (45%)



(b) Low correlation (10%)



(c) High correlation (90%)



3.2. Correlation dependence

As we have seen in the previous section, tranche delta profiles can be quite different depending on the correlation assumptions. We now look more closely at this dependence.

Once again, we start with our stylised CDO (see Figure 1). We calculate the deltas of all three tranches with respect to one reference name while varying this name's β, or correlation with the market. A typical dependence looks like the plot in Figure 3(a). We see that low β implies larger delta for the equity tranche and smaller deltas for the other tranches, especially the senior tranche. This fits with the earlier reasoning, since low correlation with the market means that the name carries a lot of idiosyncratic risk, which is important for the equity tranche, but virtually insignificant higher in the capital structure. As the β increases, the systematic component of the risk grows, and hence the mezzanine and senior deltas grow, while the equity delta falls. The current level of spreads (100bp) is not sufficiently large to wipe out the mezzanine tranche, so the mezzanine delta remains larger than the senior delta even for very high β's. The pattern also reflects the conventional wisdom, whereby "equity is long correlation, senior is short correlation", as we see how increased correlation moves the risk from the equity tranche up the capital structure.

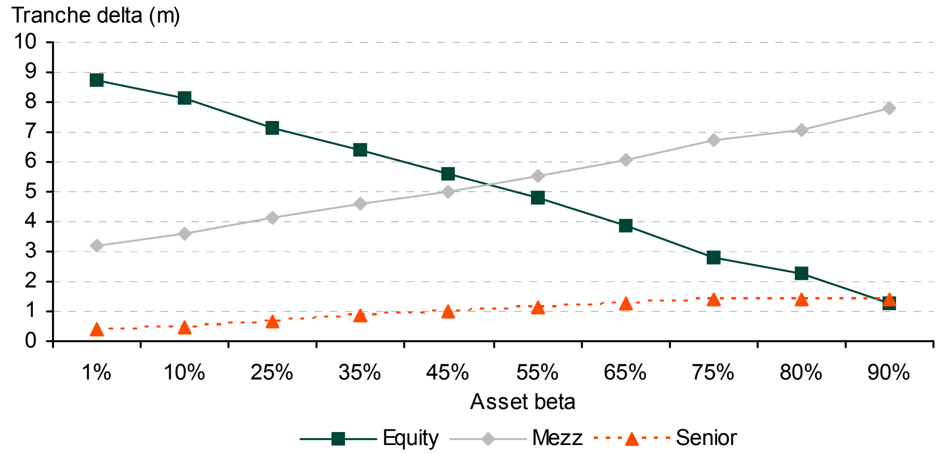
We add the default timing effect on top of the correlation dependence by performing the same analysis for a name trading at tighter and wider spreads. While the pattern is more or less the same when the asset β is low and most of the risk is idiosyncratic and allocated to the equity tranche, the shape is different for higher β. We clearly see the increased sensitivity of the senior tranche at low spreads (Figure 3(b)). Default of a name trading at 10bp is likely to happen quite late compared with others; by this time the junior tranches most probably will already have suffered losses, and therefore the sensitivity of these tranches to the tight name gets smaller with increase of correlation. Conversely, when the given name trades much wider than average, its idiosyncratic risk component due to the likelihood of earlier default remains high for all correlations. Consequently, the equity delta remains high (Figure 3(c)).

3.3. Other factors

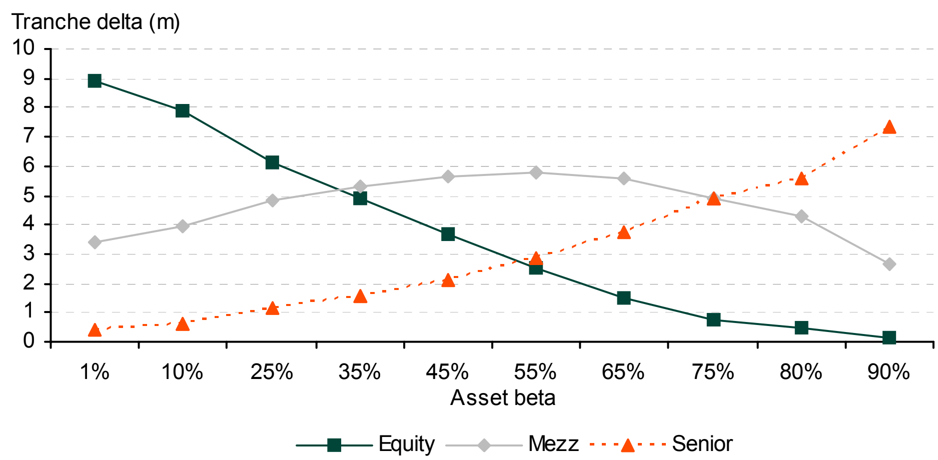
Spreads and correlation are the most important market factors affecting tranche deltas. A few other parameters can also influence the amount of single-name protection required, and we briefly mention them here.

Figure 3. Asset correlation dependence of tranche deltas

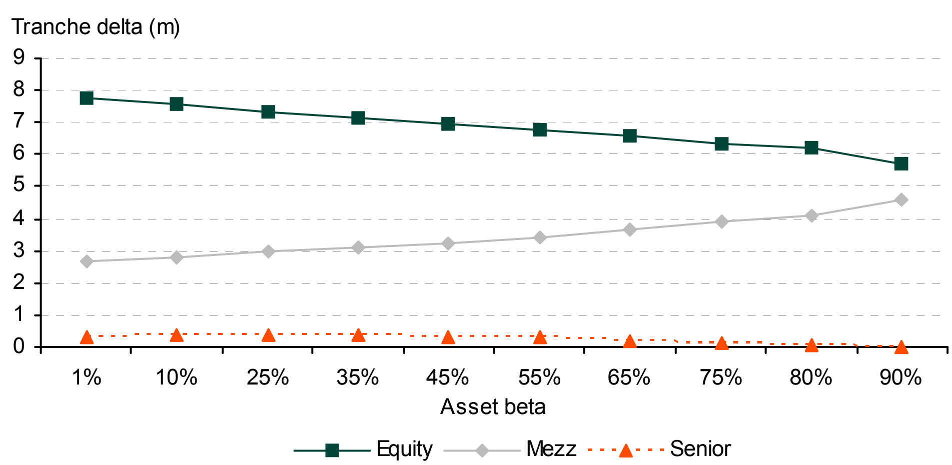
(a) Medium spread (100bp)



(b) Low spread (10bp)



(c) High spread (500bp)



One parameter of the reference names that we have not touched upon is recovery rates. Looking at formula (2), we observe that varying the recovery rate while keeping the spread fixed amounts to changing the default probability of the name, much like it changes when spreads move for fixed recoveries. The shapes of these dependencies are therefore also similar. Since recovery rates are far less likely to change over the life of the trade, except when a default is imminent, it is more natural to monitor the dependence of tranche deltas on spreads than on recovery rates.

So far we have assumed that a hedger enters the offsetting CDS at the market level before the widening. The spread sensitivity of a default swap position increases with the contractual spread. Because the delta is expressed in terms of the default swap notional needed to transact in order to immunize the position, this decreases the delta, all else being equal. If the market spread of the underlying name is lower, the converse situation is observed. On a similar note, we have so far assumed that the maturity of the CDS used for hedging purposes is the same as the maturity of the CDO. Because most of the liquidity in the CDS market is in 5- and 10-year maturities, one may find oneself in a position where, say, one year into a CDO trade, it is necessary to hedge a 4-year tranche exposure with a 5-year CDS, so that the assumption above is no longer fulfilled. To hedge the same exposure with a longer-maturity CDS, one would need a smaller delta due to a larger PV01 of the 5-year CDS.

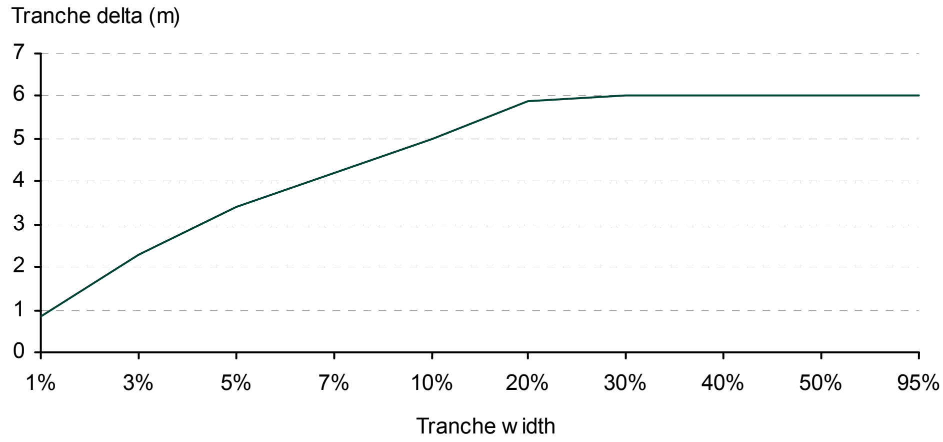
Finally, we note that deltas do change with time, even if the collateral does not suffer any defaults. In fact, if no defaults happen, the deltas of senior tranches go to zero as maturity approaches, since the probability of incurring a loss sufficient to wipe out their subordination is decreasing with time. This observation is important in developing dynamic hedging strategies.

4. STRUCTURAL EFFECTS ON TRANCHE DELTAS

Apart from the characteristics of the collateral, exposure of synthetic tranches to individual credit risk naturally depends on the structure of the CDO trade, most notably on the subordination and width of a given tranche. In this section, we look at a tranche written on the same portfolio and examine how the delta changes as we change the size and attachment point independently.

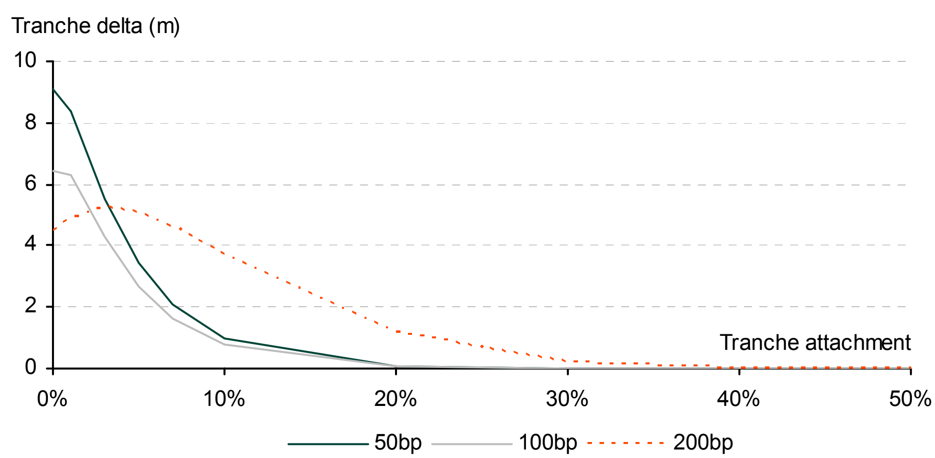
For a fixed level of subordination, a wider tranche is exposed to a larger band of losses and is therefore more risky. (Note that the deltas are expressed in monetary, i.e., absolute units, and not as a percentage.) The growth flattens out once the limit of possible losses is reached, which we see in Figure 4, where the subordination is fixed at 5%.

Figure 4. Tranche width dependence of delta



The dependence on subordination is more subtle and differs depending on the riskiness of the collateral. Intuitively, when the width of a tranche is fixed, we expect the more junior tranches, which are exposed to earlier losses, to exhibit most of the risk sensitivity for a given portfolio. However, this is not always the case in practice. The correct statement is that the tranche which contains the peak of the loss distribution of the collateral is the most sensitive to credit risk. To demonstrate this, we plot the delta of a 10% tranche as it moves up the capital structure for three values of average spread in the underlying portfolio (see Figure 5). We see that the delta does decrease monotonically for relatively low spreads, but when spreads are sufficiently wide, tranche delta is largest for one of the mezzanine tranches, rather than the equity tranche. This can be explained by the fact that the mean of the collateral loss distribution, which is approximately equal to 200bp x 5 years = 10%, is at the edge of a 10% equity tranche; at the same time, a mezzanine tranche with 5% subordination contains most of the loss peak, and is therefore more sensitive and has a larger delta. Note that once the mean of loss distribution is comfortably outside the tranche exposure, the delta decreases to zero at a slower rate.

Figure 5. Tranche attachment dependence of a 10% mezzanine tranche delta



5. CONCLUSIONS

We have developed an intuition behind assessing individual risk exposure of synthetic CDO tranches, expressed in deltas. We have looked at both market-driven and structure-driven factors. CDS spreads observed in the market characterise the relative timings of defaults, which have significant impact on risks of various tranches. On the other hand, the tendency of defaults to occur together, measured by market correlation, or β, represents a different risk factor, which often dominates the timing effect, especially for senior tranches. Our analysis demonstrates the variability of deltas computed with respect to different names, a property which is often neglected in a superficial approach to delta hedging of synthetic tranches. On the structural side, we have observed that depending on the risk profile of the collateral, a mezzanine tranche can be more sensitive to credit risk than the equity tranche.

Overall, we believe that this closer look at tranche deltas, made possible by a carefully designed model, is highly beneficial for understanding risks associated with investing in CDO tranches.

REFERENCES

O'Kane, Naldi, Ganapati, Berd, Pedersen, Schloegl and Mashal (2003), "The Lehman Brothers Guide to Exotic Credit Derivatives", *Risk,* October.

Pricing Multi-Name Default Swaps with Counterparty Risk[1]

Roy Mashal
212-526-7931
rmashal@lehman.com

Marco Naldi
212-526-1728
mnaldi@lehman.com

We present a simple and general methodology for pricing counterparty risk in multi-name default swaps, such as n^{th}-to-default baskets and portfolio loss tranches. The main purpose of our analysis is to derive pricing results without running into the problem of nested valuations, ie, without having to explicitly compute the mark-to-market of the swap in each state where the counterparty defaults. Although the proposed methodology can be used in conjunction with any underlying default model (structural, reduced-form, hybrid), we provide explicit pricing examples for loss tranches of different seniorities by means of a simple and computationally efficient time-to-default simulation, where default dependencies among the reference names and the default-risky counterparty are generated by an appropriately calibrated copula function. We show how fair tranche spreads vary as functions of a) the market-implied default probability of the risky counterparty, b) the asymmetry of the mark-to-market recovery at default, and c) the dependence between the default time of the risky counterparty and the default times of the credits referenced by the swap. Finally, we use our methodology to show that allowing for the possibility of default of the protection seller has a much more significant impact on fair spreads than allowing for the possibility of default of the protection buyer, other things being equal.

1. INTRODUCTION

When an agent enters into an unfunded swap with a default-risky counterparty, she should consider the possibility of her counterparty defaulting before the maturity of the contract. In this case, some or all of the contractual obligations will be left unfulfilled, and she may incur a mark-to-market loss. Examples of such situations are:

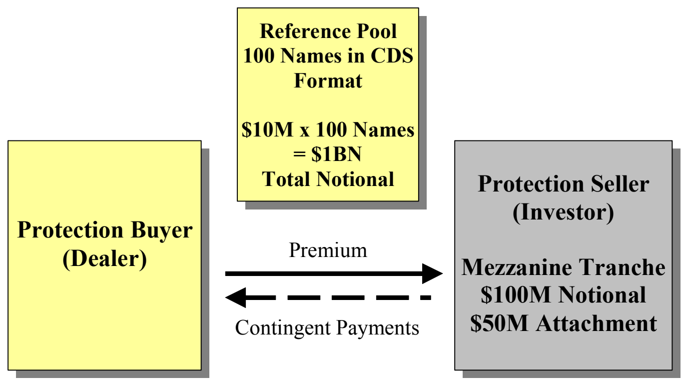
- A first-to-default protection buyer defaulting before maturity and before the default of the reference name.
- A tranche protection seller defaulting before maturity and before the notional of his exposure is completely eroded by defaults of the reference names.

We present a simple and general methodology for pricing counterparty risk in multi-name default swaps, such as n^{th}-to-default baskets and portfolio loss tranches. The main purpose of our analysis is to derive pricing results without running into the problem of nested valuations, ie, without having to explicitly compute the mark-to-market of the swap in each state where the counterparty defaults. In other words, we are looking for a way to price counterparty risk without having to: (a) calculate the conditional distribution of the default times of the surviving reference credits at some future date; and (b) use this conditional joint distribution to perform a forward valuation. The pay-offs of multi-name credit derivatives are generally linked to dozens − and sometimes hundreds − of credits, and nested valuations are highly impractical even with the simplest model.

[1] *We would like to thank Vasant Naik, Dominic O'Kane, Claus Pedersen, Lutz Schloegl and Stuart Turnbull for comments and suggestions, and Sandip Biswas for his precious collaboration in the development of this analysis.*

Both our main discussion and the following application refer specifically to a portfolio loss tranche. In a note, we show that all of our results can be immediately applied to n^{th}-to-default baskets. Portfolio loss tranches are popular contracts in which the protection buyer pays a periodic premium to the protection seller who, in exchange, stands ready to compensate the buyer for a predefined slice of the losses affecting a portfolio of reference names. Figure 1 shows the mechanics of a typical mezzanine tranche swap, where the reference portfolio consists of 100 credits, each with $10 million notional. The protection buyer, typically a dealer, pays an annual premium which is generally quoted as a percentage of the tranche's outstanding notional at each payment date. The investor, on the other hand, agrees to compensate the protection buyer for losses on the reference portfolio above the attachment point of $50 million and up to $150 million, for a total exposure of $100 million.

Figure 1. Mechanics of a typical tranched portfolio swap



The remainder of this article is organized as follows. In section 2, we introduce some notation and discuss the standard pricing equation for a portfolio loss tranche when both counterparties are default-free. In section 3, we introduce counterparty risk and derive our main results. We separately discuss two different cases: a swap between a default-free protection buyer and a default-risky protection seller, and a swap between a default-risky protection buyer and a default-free protection seller. Although the proposed methodology can be used in conjunction with any underlying default model (structural, reduced-form, hybrid), section 4 provides explicit pricing examples for loss tranches of different seniorities by means of a simple and computationally efficient time-to-default simulation, where default dependencies among the reference names and the default-risky counterparty are generated by an appropriately calibrated copula function. Section 5 summarizes our findings and concludes.

2. FAIR DEFAULT SWAP SPREAD WITHOUT COUNTERPARTY RISK

Consider a default-free agent buying protection on a loss tranche from a default-free counterparty. If we denote with A the dollar exposure provided by the tranche, and with $L(t)$ the cumulative dollar loss on the tranche at time t, we can compute the fair compensation for this exposure as the spread S which solves[2]:

$$S \cdot E\left[\sum_{i=1}^{M} B(t_i)\big(A - L(t_i)\big)\right] = E\left[\int_0^{t_M} B(t)dL(t)\right]. \tag{1}$$

This expression simply equates the premium leg to the protection leg of the swap. The premium S is paid on the outstanding notional $(A - L(t_i))$ at a set of predetermined dates t_i, $i=1,2,3\ldots,M$, while protection is paid at the time the loss is incurred. The expectation is taken with respect to the pricing measure, and all payments are discounted using the term structure of risk-free factors $B(t)$.

Notice that equation (1) holds regardless of the assumptions made with respect to the sources of randomness (default times, recoveries and risk-free rates may all be stochastic) and the model used to generate their (risk-neutral) joint distribution.

To simplify the notation in the remainder of the paper, we introduce the following definitions:

$$N(j,k) \equiv \sum_{i=j}^{k} B(t_i)\big(A - L(t_i)\big), \quad j \le k,$$

denotes the sum of discounted outstanding notional of the tranche between the j^{th} and the k^{th} payment dates, and

$$P(u,v) \equiv \int_u^v B(t)dL(t), \quad u \le v,$$

denotes the sum of discounted payments received by the protection buyer between $t=u$ and $t=v$. Notice that the distributions of the random variables N and P depend on the characteristics of the reference credits, such as their market spreads and default correlations, through the dependence on the portfolio loss distribution L.[3]

Using these definitions, equation (1) can be conveniently rewritten as:

$$S \cdot E\big[N(1,M)\big] = E\big[P(0,t_M)\big]. \tag{2}$$

Note: n^{th}-to-default swaps

Notice that equation (1) becomes the standard pricing equation for an n^{th}-to-default basket swap if we substitute:

$\big(A - L(t_i)\big)$ with $\big(1 - P_n(t_i)\big)$, and

$dL(t)$ with $L_n \cdot dP_n(t)$,

---

[2] *To simplify notation, we do not explicitly consider accrual in any of our pricing equations.*

[3] *The risk-neutral expectation of N(j,k) is often called the risky PV01 of a swap with payment dates t_j, t_{j+1}, …, t_{k-1}, t_k.*

where $P_n(t)$ is a point process that jumps from 0 to 1 at the time of default of the n^{th} reference name, and L_n is the loss-given-default of the n^{th} defaulter (notice that, as long as default times are random, L_n is stochastic even if the loss-given-default of each reference name is assumed to be deterministic).

Using these definitions, everything that follows in this article can be immediately applied to n^{th}-to-default basket swaps.

3. FAIR DEFAULT SWAP SPREAD WITH COUNTERPARTY RISK

In this section, we generalize the pricing of a tranched portfolio swap when one of the counterparties is also subject to default risk. For ease of exposition, we develop our arguments assuming that the default time of the risky counterparty is positively correlated with the default times of the names in the reference set. In reality, this is very often the case, since default events clearly have a systematic component. However, everything that follows can be adapted to a general dependence structure of default times.

3.1. Default-free protection buyer, default-risky protection seller

To price a swap between a default-free protection buyer and a default-risky protection seller, we need to specify an assumption regarding the cashflows at the time of default of the risky counterparty. In what follows, we allow for an asymmetric recovery assumption $\{Q,R\ ;\ 0 \le Q \le R \le 1\}$, which indicates that the risk-free protection buyer will recover a fraction Q of the mark-to-market of the trade if this is in her favor, but will pay a possibly higher fraction R of the mark-to-market if this is against her. Notice that we allow for Q and R to be stochastic; therefore, when we write "$Q \le R$" or "$Q=R$", we mean that the (in)equality holds for every element of the state space.[4] We will denote with $S_{Q,R}$ the fair spread paid to a default-risky investor with a $\{Q,R\ ;\ 0 \le Q \le R \le 1\}$ settlement in case of default.

We start with some necessary definitions. First, we introduce the additional notation:

τ^* = default time of the default-risky protection seller, and

$$\tau = \min\{\tau^*, t_M\},$$

$$z = \max\{j \in (0,1,2,\ldots,M) : t_j \le \tau\}.$$

In words, τ denotes the default time of the counterparty or the maturity of the deal, whichever comes first, and t_z is the last payment date before τ.

Next, we define CPL as the (discounted) loss on the swap for the protection buyer as a result of the counterparty's default:

$$CPL(S_{Q,R}) = P(\tau, t_M) - S_{Q,R} N(z+1, M).$$

Finally, we define the (discounted) mark-to-market of the swap for the protection buyer at the time of the counterparty default, MTM, as

$$MTM(S_{Q,R}) = E[CPL(S_{Q,R}) \mid \Im(\tau)] = E[P(\tau, t_M) - S_{Q,R} N(z+1, M) \mid \Im(\tau)], \tag{3}$$

4 *For example, we can model a stochastic mark-to-market recovery with Q≤R if we assume that Q and R are beta-distributed on two non-overlapping supports.*

where, once again, the expectation is taken with respect to the pricing measure, and $\Im(\tau)$ denotes the information available at τ. Equation (3) states that the value of the swap at τ will be given by the difference between the value of the protection leg and the value of the premium leg at τ, and that the valuation will be done conditionally on the information $\Im(\tau)$ available at that time.

With this notation in place, we are now ready to write the pricing equation for a default swap between a default-free protection buyer and a default-risky protection seller:

$$S_{Q,R} \cdot E[N(1,z)] = E[P(0,\tau)] + E\left[\min\left(Q \cdot MTM\left(S_{Q,R}\right), R \cdot MTM\left(S_{Q,R}\right)\right)\right] \tag{4}$$

The expression in (4) equates the present value of the premium paid up to τ to the present value of the protection received up to τ. In addition, the second term on the right-hand side indicates the present value of the (possibly negative) amount recovered by the protection buyer in case of default of the counterparty.

Our goal in this article is to price counterparty risk without explicitly calculating the mark-to-market of the swap in each state where the counterparty defaults. This would require, for each of these states, a derivation of the joint distribution of all stochastic variables conditional on the information set $\Im(\tau)$; for instruments with a large number of underlyings, such as portfolio loss tranches, this would be computationally expensive (see Schonbucher (2002)).

It turns out that our task is very simple if we impose the symmetric restriction $Q=R$, and that the solution to this symmetric problem paves the way for handling the more general (and more realistic) asymmetric recovery. For these reasons, we analyze these two cases separately.

3.1.1. The symmetric case: 0 ≤ Q = R ≤ 1

When $0 \le Q=R=T \le 1$, equation (4) simplifies to:

$$S_{T,T} \cdot E[N(1,z)] = E[P(0,\tau)] + E\left[T \cdot CPL\left(S_{T,T}\right)\right],$$

where we have used the law of iterated expectations and the fact that T is known at τ to conclude that:

$$E\left[T \cdot MTM\left(S_{T,T}\right)\right] = E\left[T \cdot CPL\left(S_{T,T}\right)\right].$$

Substituting for CPL and collecting terms gives:

$$S_{T,T} \cdot E[N(1,z) + T \cdot N(z+1,M)] = E[P(0,\tau) + T \cdot P(\tau,t_M)] \tag{5}$$

Equation (5) shows that the fair spread of this symmetric swap is equivalent to the fair spread on a "step-down" swap whose payments on both legs jump to T times the contractual payments at the time of default of the risky counterparty.

Once we specify the risk-neutral joint distribution of the stochastic variables in the model given the information available today, it is straightforward to compute $S_{T,T}$. More precisely, the expectations in (5) can be obtained by simulating the stochastic variables in the model, calculating discounted premium and protection payments before and after the counterparty default along each path, and averaging across paths.

Notice also that when $Q=R=T=1$, equation (5) simplifies to (2), and $S_{1,1}$ is therefore equal to S, the fair swap spread to be paid to a default-free protection seller. The intuition behind this result should be clear: if any amount due in case of default is fully recovered by the creditor, counterparty risk cannot have any impact on fair pricing.

On the other extreme, when $Q=R=T=0$, the swap knocks out and there is no additional inflow (or outflow) if the risky counterparty defaults. The fair value of the knockout spread $S_{0,0}$ is simply determined by:

$$S_{0,0} \cdot E[N(1,z)] = E[P(0,\tau)].$$

3.1.2. The asymmetric case: 0 ≤ Q ≤ R ≤ 1

An agent entering into a swap with a default-risky protection seller is generally concerned about the asymmetry of the potential mark-to-market settlement: it is likely that only a small fraction of a favorable mark would be recovered if the counterparty defaults, while it is possible that a negative mark would have to be paid in full. Of course, collateral agreements can mitigate the problem, but funding the trade may "kill" the economic rationale for the risky investor. For the pricing to reflect this asymmetry, we need to be able to deal with equation (4) under the assumption that $Q \le R$.

Solving for $S_{Q,R}$ exactly when $Q \le R$ requires the calculation of the mark-to-market of the swap at τ, and this turns out to be computationally expensive even with the simplest model. Our strategy, therefore, will be to find bounds for $S_{Q,R}$ and use their midpoint as an estimate; the examples in the next section will then show that the proposed bounds are in fact very tight, so that the resulting maximum estimation error turns out to be very small.

The next two propositions identify an upper bound $S^U_{Q,R}$ and a lower bound $S^L_{Q,R}$, such that:

$$S^L_{Q,R} \le S_{Q,R} \le S^U_{Q,R}.$$

Proposition 1: An upper bound $S^U_{Q,R}$ is given by $S_{Q,Q}$.

First, observe that any symmetric solution $S_{T,T}$, $Q \le T \le R$, is an upper bound for $S_{Q,R}$. Intuitively, the default-free protection buyer is better off if:

1. the fraction of an unfavorable mark-to-market that she would have to pay in case her counterparty defaults decreases from R to T; and
2. the fraction of a favorable mark-to-market that she would recover increases from Q to T.

Therefore, the fair spread to be paid to the risky protection seller has to increase.

Next, notice that with positive dependence between the default time of the protection seller and the default times of the reference names, the mark-to-market in the event of default of the risky counterparty is expected to be in favor of the risk-free protection buyer. Therefore, $S_{Q,Q}$ is the lowest of all upper bounds $S_{T,T}$, $Q \le T \le R$, and therefore the one of interest to us.

Proposition 2: A lower bound $S^L_{Q,R}$ can be computed as the solution to:

$$S^L_{Q,R} \cdot E[N(1,z)] = E[P(0,\tau)] + E\left[\min\left(Q \cdot CPL\left(S^L_{Q,R}\right), R \cdot CPL\left(S^L_{Q,R}\right)\right)\right]. \qquad (6)$$

Equation (6) is analogous to (4), with the only difference that we have substituted *MTM* with *CPL*. By the law of iterated expectations and Jensen's inequality we have, for any *S*[5]:

$$
\begin{aligned}
&E[\min(Q\cdot CPL(S), R\cdot CPL(S))] = \\
&E[E[\min(Q\cdot CPL(S), R\cdot CPL(S)) \mid \Im(\tau)]] \le \\
&E[\min(Q\cdot E[CPL(S)\mid \Im(\tau)], R\cdot E[CPL(S)\mid \Im(\tau)])] = \\
&E[\min(Q\cdot MTM(S), R\cdot MTM(S))]
\end{aligned}
$$

where we have used the fact that Q and R are known at τ, and the observation that, for $0 \le K_1 \le K_2 \le 1$ and $x \in \Re$, $\min(K_1 x, K_2 x)$ is a concave function of x.

A comparison between equations (4) and (6) is now sufficient to conclude that:

$$S_{Q,R}^{L} \le S_{Q,R}.$$

The crucial advantage of the proposed methodology is that these bounds can be obtained very easily, without explicitly calculating the value of the swap in the states where the risky counterparty defaults. We have shown earlier that the proposed upper bound $S_{Q,R}^{U} = S_{Q,Q}$ can be computed as the solution to equation (5). The lower bound, $S_{Q,R}^{L}$, can be obtained by simulating the stochastic variables in the model, computing discounted premium and protection payments before and after the counterparty default along each path, and numerically solving for the value of $S_{Q,R}^{L}$ that solves equation (6), ie, after substituting for *CPL*,

$$
\begin{aligned}
&S_{Q,R}^{L} E[N(1,z)] = E[P(0,\tau)] + \\
&+ E\left[\min\left(Q\cdot\left(P(\tau,t_M) - S_{Q,R}^{L} N(z+1,M)\right), R\cdot\left(P(\tau,t_M) - S_{Q,R}^{L} N(z+1,M)\right)\right)\right].
\end{aligned}
$$

3.1.3. Window risk

Up to now, we have assumed that if the risky investor defaults, she will do so at a time when she does not owe any money to the protection buyer beyond, possibly, the mark-to-market of the swap. We may want to take into account the fact that, in reality, there is generally a lag between the moment when a loss affecting the tranche is experienced and the moment when the protection seller fulfills the associated obligation. If the counterparty defaults during this period, the protection buyer will suffer a direct loss that must be added to the potential cost of replacing the counterparty for the fulfillment of the remaining contractual obligations.

If we denote with Δ the length of this window, we can see that the fair spread $S_{Q,R}$ now solves:

$$
\begin{aligned}
&S_{Q,R}\cdot E[N(1,z)] = E[P(0,\tau)] + \\
&+ E[\min(Q\cdot MTM(S_{Q,R}), R\cdot MTM(S_{Q,R}))] - E[B(\tau)\cdot(1-Q)\cdot(L(\tau) - L(\tau-\Delta))]
\end{aligned}
\tag{7}
$$

Of course, the longer the window, the lower the fair spread will be. Bounds for $S_{Q,R}$ can be computed as explained earlier.

Notice that modeling window risk in this fashion allows us to account for counterparty risk in cases where the default-risky protection seller does not fund the losses throughout the life of the deal, but only at maturity. In this case, we would simply need to set $\Delta = \tau$ in equation (7) and notice that, by definition of cumulative loss, $L(0)=0$.

5 *An alternative way to look at Proposition 2 is to recognize that, by definition, MTM dominates CPL in the sense of second-order stochastic dominance (see Rothschild and Stiglitz (1970)).*

3.2. Default-risky protection buyer, default-free protection seller

Analogous arguments can be used to find bounds for the fair spread paid by a default-risky protection buyer to a default-free protection seller. Let us slightly modify our previous definitions so that now:

$\tau^* =$ default time of the default-risky protection *buyer*, and

$\tau = \min\{\tau^*, t_M\}$,

$z = \max\{j \in (0,1,2,\ldots,M): t_j \le \tau\}$.

Also, a realistic recovery assumption can be represented by $\{R,Q\ ;\ 0 \le Q \le R \le 1\}$, in which we now denote with Q the fraction of a favorable mark-to-market that the default-free protection *seller* would recover, and with R the possibly higher fraction of an unfavorable mark that she would have to pay in the event of default of her risky counterparty.

With this notation in place, the equation for the fair swap spread $S_{R,Q}$ is now given by

$$S_{R,Q} \cdot E[N(1,z)] = E[P(0,\tau)] + E[\max(Q \cdot MTM(S_{R,Q}), R \cdot MTM(S_{R,Q}))]. \tag{8}$$

The expression in (8) equates the present value of the premium paid up to τ to the present value of the protection received up to τ. In addition, the second term on the right-hand side indicates the present value of the amount recovered (or paid) by the protection buyer in the event she defaults.

Following the same reasoning employed in the previous section, we can immediately identify bounds $S^L_{R,Q}$ and $S^U_{R,Q}$ such that:

$$S^L_{R,Q} \le S_{R,Q} \le S^U_{R,Q}.$$

Proposition 3: A lower bound $S^L_{R,Q}$ is given by $S_{R,R}$.

Proposition 4: An upper bound $S^U_{R,Q}$ can be computed as the solution to:

$$S^U_{R,Q} \cdot E[N(1,z)] = E[P(0,\tau)] + E[\max(Q \cdot CPL(S^U_{R,Q}), R \cdot CPL(S^U_{R,Q}))].$$

We conclude this section with the observation that the results obtained so far do not depend on any assumption made with respect to either the sources of randomness or the particular model used to generate their (risk-neutral) joint distribution.

4. APPLICATION: PRICING A SYNTHETIC LOSS TRANCHE

In this section, we offer a practical application of our methodology by pricing a tranched portfolio default swap. Our goal is to answer a number of questions:

1. What is the impact of counterparty risk on the fair spread of a synthetic loss tranche?
2. How do fair spreads vary in response to a change in the market-implied default probability of the risky counterparty?
3. How do fair spreads vary as default correlations between the risky counterparty and the reference names in the swap change?

4. How do fair spreads vary as we change assumptions regarding the mark-to-market recovery at the time of default of the risky counterparty?
5. How tight are the bounds for the fair spread of a contract with asymmetric mark-to-market settlement? In other words, what is the maximum error that we incur in order to avoid the computationally expensive valuation of the remaining portion of the swap in each state where the risky counterparty defaults?
6. Using the fair spread of a swap between two default-free counterparties as a benchmark, is there any symmetry between: (a) the increase in fair spreads paid by a risky protection buyer to a risk-free protection seller; and (b) the decrease in fair spreads paid by a risk-free protection buyer to a risky protection seller?

To apply our methodology, we now have to specify which variables are random, and choose a model to produce their risk-neutral joint distribution. One simple and popular way to proceed is to employ a Gaussian copula to link the market-implied distributions of the default times of all defaultable credits, and add the simplifying assumptions of deterministic recovery rates and a deterministic risk-free term structure. Li (2000) shows that this approach to modeling default times can be interpreted as an extension of a one-period Merton's model where default times and asset returns share the same correlation matrix.[6]

Following our discussion in section 3, we first consider a tranched portfolio default swap between a risk-free protection buyer and a default-risky protection seller.

4.1. Default-free protection buyer, default-risky protection seller

As highlighted in the previous sections, we can compute bounds for the fair spread of such a swap by simulating default times given only the information available today. We choose the following parameters for our base-case example:

1. We consider a 5-year default swap referring to a pool of 100 names, each with the same notional amount.
2. Each reference credit has a deterministic recovery rate of 35%.
3. Each reference credit, as well as the default-risky protection seller, has a yearly market-implied (risk-neutral) hazard rate of 2%.
4. Asset correlations between any two credits (including the risky counterparty) are all equal to 25%.
5. Q and R are assumed to be deterministic and equal to 35% and 1, respectively. This means that the risk-free protection buyer is going to recover only 35% of a positive mark-to-market but is going to pay in full for a negative mark-to-market if the protection seller defaults.
6. Every default in the reference set is immediately covered by the protection seller. In other words, there is no window risk as defined in section 3.1.3.
7. The risk-free curve is assumed to be deterministic. Risk-neutral cashflows are moved in time using the discount factors implied by the LIBOR curve as of October 2, 2003.

Beyond the proposed bounds for the fair spread of this swap, we calculate the spread that would be fair if both counterparties were default-free. We compute these spreads for three

[6] *Mashal, Naldi and Zeevi (2003) provide empirical evidence supporting a fat-tailed generalization of this model. Boscher and Ward (2002) study the consequences of allowing for stochastic recovery rates in this framework.*

tranches of different seniority: an equity piece covering the first 5% of the portfolio losses, a mezzanine tranche exposing the investor to the 5−10% slice of the losses, and a senior tranche absorbing the remaining defaults. Figure 2 shows the fair spreads for these tranches obtained from a 50,000-path Monte Carlo simulation. The numbers in parenthesis indicate the standard errors as percentages of the estimates, and show that 50,000 paths, requiring only a handful of seconds on a standard PC, are enough to drive down the standard errors to very reasonable levels.

Figure 2. Fair spreads (% std err in parenthesis). 50K-path Monte Carlo simulation

	Equity (0–5%)	Mezzanine (5–10%)	Senior (10–100%)
Risk-Free Fair Spread	22.84% (0.44%)	6.980% (0.73%)	0.285% (1.16%)
Upper Bound for Risky Fair Spread	22.56% (0.44%)	6.627% (0.77%)	0.250% (1.27%)
Lower Bound for Risky Fair Spread	22.50% (0.44%)	6.576% (0.77%)	0.246% (1.27%)

To understand the magnitude of the error introduced solely by the fact that we compute bounds rather than the exact fair spread, we run a 5-million-path simulation to drive the standard errors down to insignificant levels. We then compute the maximum error that we can make by estimating the fair spread using the midpoint of the computed bounds. Figure 3 reports the results and shows that the proposed bounds are indeed quite tight. Intuitively, the adverse effect of the asymmetric recovery is dampened by the positive dependence between the default time of the protection seller and the default times of the reference credits, which implies that the mark-to-market of the swap is going to be in favor of the protection buyer in most states where the risky counterparty defaults.

Figure 3. Maximum error using midpoint. 5M-path Monte Carlo simulation

	Equity (0–5%)	Mezzanine (5–10%)	Senior (10–100%)
Upper Bound for Risky Fair Spread	22.5516%	6.6148%	0.2499%
Lower Bound for Risky Fair Spread	22.4986%	6.5637%	0.2463%
Midpoint	22.5251%	6.5892%	0.2481%
Max Error	2.65 bps	2.55 bps	0.18 bps

We now turn our attention to sensitivity analysis, and study how the bounds for the fair spreads of the three tranches described above change as we vary some of the input parameters.

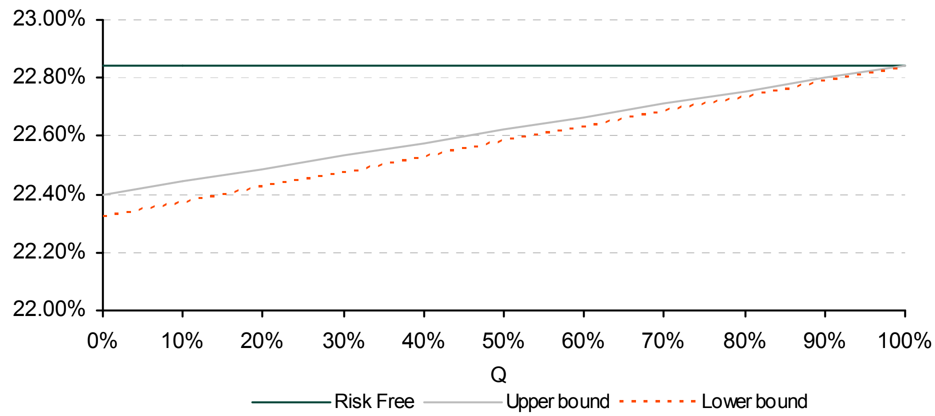
In Figure 4, we let the hazard rate of the protection seller vary between 0% and 4%, while keeping the same parameters for the reference portfolio as well as the same correlations among all credits. When the protection seller is risk-free, both bounds obviously coincide with the risk-free fair spread. As the hazard rate increases, both bounds decrease, and the interval they span becomes larger. Interestingly, the bounds become more convex in the hazard rate of the counterparty as we move up in seniority: since the senior investor will have to cover portfolio losses in very few scenarios, the marginal effect of an increase in her default probability is decreasing.

Figure 4. Fair spreads as functions of counterparty riskiness

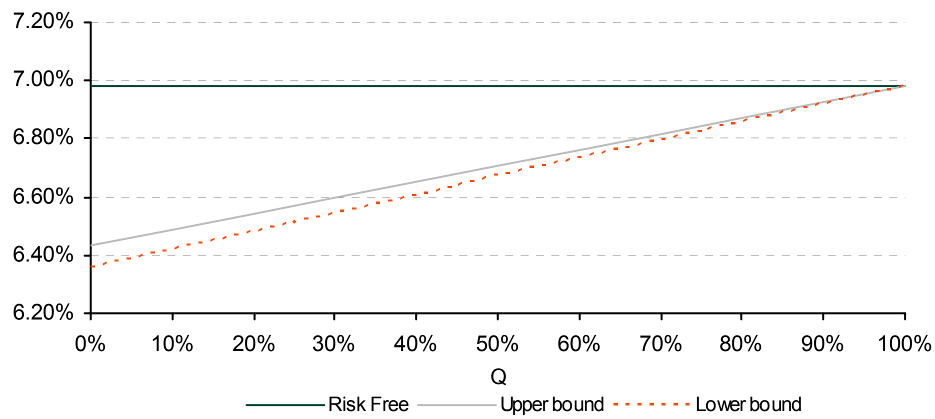
With R set equal to one, Q measures the level of asymmetry of the mark-to-market recovery process. Figure 5 shows the bounds for the fair spreads as we vary Q, while keeping everything else as specified in our base case. With Q equal to one, the bounds once again coincide with the default-free fair spread, since in this case the mark-to-market at τ is always going to be fully paid. As Q decreases to zero, the asymmetry increases, and the fair spread paid to the risky protection seller decreases to compensate the protection buyer for the higher expected loss in case of default. Both bounds increase almost linearly in Q for all tranches and, not surprisingly, the interval they span becomes smaller as the asymmetry of the mark-to-market settlement vanishes.

Figure 5. Fair spreads as functions of asymmetry of mark-to-market recovery

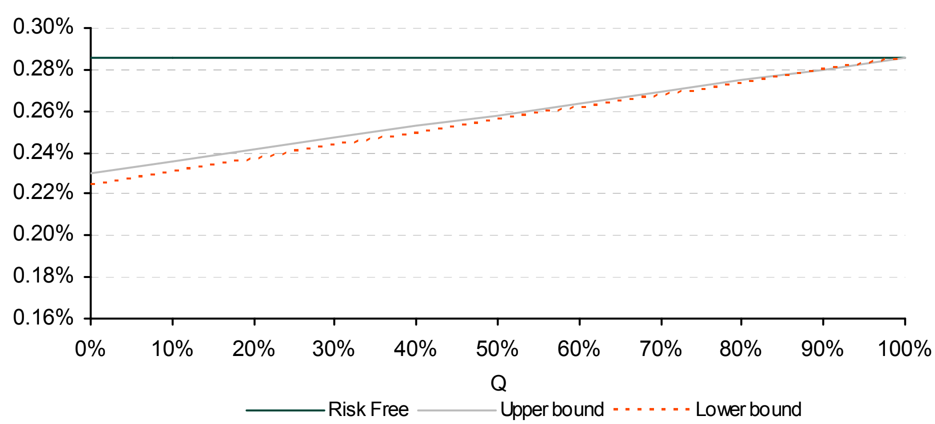
Equity Tranche



Mezzanine Tranche



Senior Tranche



Finally, in Figure 6 we analyze the effect of varying the correlation between the risky protection seller and the credits in the reference portfolio, once again keeping the remaining parameters (including the correlations among the names in the reference set) at the levels fixed in the base case.

The first thing to notice is that our bounds are not tight at low levels of correlation. In particular, when the default time of the counterparty is independent of the default times of the reference credits, the expected mark-to-market of the swap at τ is close to zero (its exact value will generally depend on the shapes of the hazard rates and the risk-free curve). In this case, the only reason for which the spread paid to a risky protection seller should be lower than the one paid to a default-free investor is the asymmetry of the mark-to-market settlement. By definition, this effect cannot be captured by our upper bound, which is the fair spread for a (35%, 35%) symmetric swap, and is therefore very close to the risk-free fair spread.

The second thing to notice is the fact that fair spreads of different tranches behave very differently as we vary the correlation between the risky counterparty and the reference credits. At low levels of correlation, a correlation increase makes the expected mark-to-market at τ more favorable for the protection buyer, since a larger number of defaults are expected to affect the reference portfolio in the states where the counterparty defaults. Because the protection buyer now expects to lose a fraction $(1-Q)$ of a larger mark-to-market, she has to pay a lower spread in order to break even. This holds true for tranches of all seniorities.

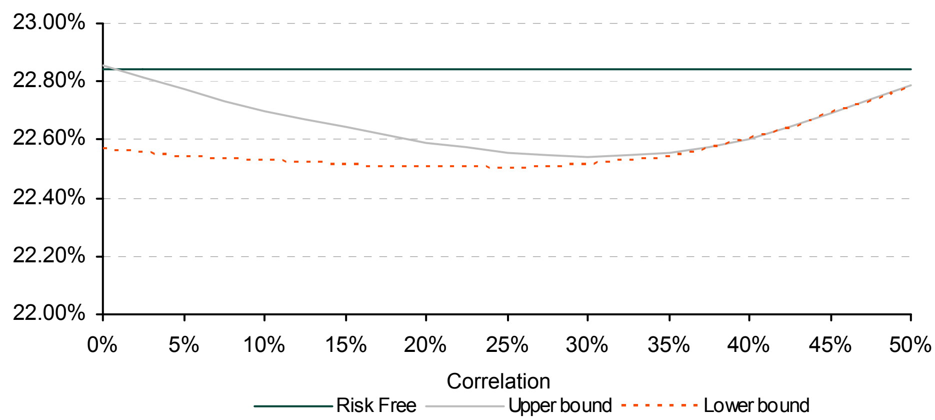
At high levels of correlation, however, an equity tranche is already exhausted at τ in a significant proportion of the states where the protection seller defaults. When this is the case, the mark-to-market of the equity swap at τ is obviously equal to zero. A further increase in correlation increases the frequency of such scenarios, and makes the expected mark-to-market at τ less favorable for the protection buyer. Because she now expects to lose a fraction $(1-Q)$ of a smaller mark-to-market, the break-even spread has to increase.

The same reasoning holds for the mezzanine tranche, although the correlation level at which the fair spread reaches its minimum is higher. As far as the senior tranche goes, the fair spread generally decreases monotonically with the correlation between the risky counterparty and the reference credits.

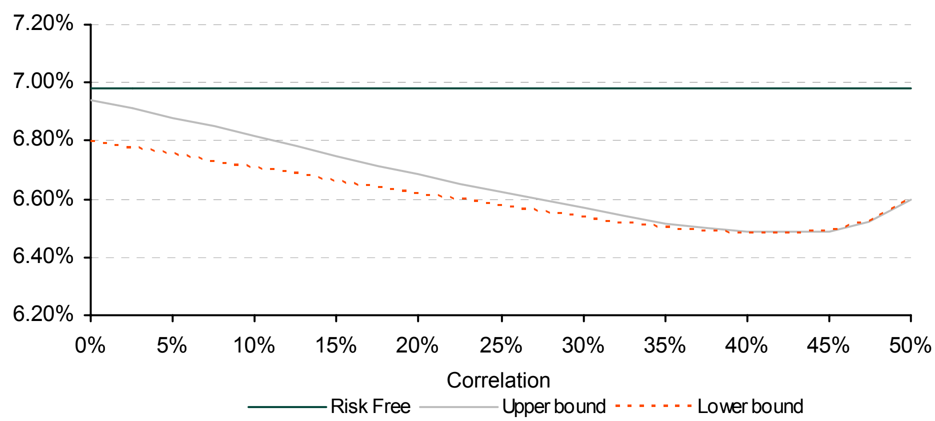
One final observation we can draw from Figure 6 is that the sensitivity of the position of the risk-free protection buyer to an increase in correlation between the risky counterparty and the reference credits is higher for senior tranches than it is for junior tranches.

Figure 6. Fair spreads as functions of correlation between c-p and reference credits

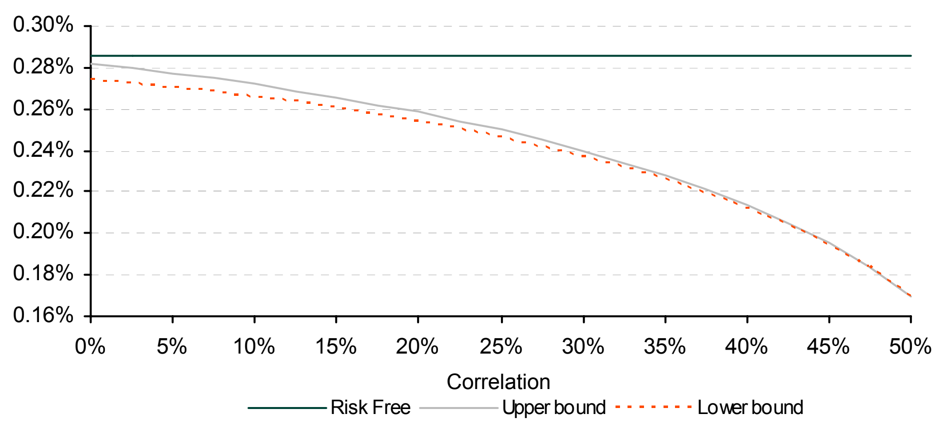
Equity Tranche



Mezzanine Tranche



Senior Tranche



4.2. Default-risky protection buyer, default-free protection seller

We now turn our attention to a swap between a default-risky protection buyer and a default-free protection seller. To compare the bounds for this swap with the ones obtained in the previous section, we keep working with the same base-case parameters on the understanding that:

1. it is now the protection *buyer* who has a yearly (risk-neutral) hazard rate equal to 2%; and
2. Q=35% is now the fraction of a favorable mark-to-market recovered by the default-free protection *seller*, while R=1 is the fraction of a favorable mark-to-market recovered by the default-risky protection *buyer*.

Notice that, from Proposition 3, the lower bound for the fair spread of this swap is given by $S_{1,1}$, which is also equal to the fair spread between two default-free counterparties. Figure 7 shows the bounds for the fair spreads on the same three tranches defined earlier.

Figure 7. Fair spreads (% std err in parenthesis). 50K-path Monte Carlo simulation

	Equity (0−5%)	Mezzanine (5−10%)	Senior (10−100%)
Upper Bound for Risky Fair Spread	22.89% (0.45%)	7.027% (0.74%)	0.289% (1.29%)
Lower Bound for Risky Fair Spread (=Risk-Free Fair Spread)	22.84% (0.44%)	6.980% (0.73%)	0.285% (1.16%)

The main observation we can draw from comparing Figure 2 with Figure 7 is that the default riskiness of the protection buyer produces an increase in fair spreads that is significantly smaller than the reduction in fair spreads resulting from the default riskiness of the protection seller. For example, the fair haircut for a 2% hazard rate protection seller on a 5-year mezzanine tranche is 35−40bp (Figure 2), while the fair surcharge for a 2% hazard rate protection buyer is certainly smaller than 5bp (Figure 7).

This difference is due to two main reasons. First, the distribution of the mark-to-market of a loss tranche at any given horizon is strongly asymmetric around zero, since it is bounded on one side by the outstanding notional of the tranche, and on the other by the present value of the remaining premium payments. Therefore, even with independence between the default time of the risky counterparty and the default times of the reference names, the asymmetry of the mark-to-market recovery has a larger impact on the expected loss due to counterparty default when the risky counterparty is the protection seller. Second, with positive default dependence between the risky counterparty and the reference names in the swap, a risky protection buyer tends to default when the mark-to-market is in her favor, while a risky protection seller tends to default when the mark-to-market is against her. For these reasons, allowing for the possibility of default of the protection seller has a much more significant impact on fair prices than allowing for the possibility of default of the protection buyer, other things being equal. Similar observations were made by O'Kane and Schloegl (2002) for single-name default swaps.

5. SUMMARY

Pricing counterparty risk in multi-name default swaps is a challenging task. In principle, one needs to be able to evaluate the mark-to-market of the trade along each path in which the counterparty defaults, and precisely at the time where this default takes place. This, in turn, requires the calculation of the joint distribution of the default times of the surviving credits conditional on the information available at the time of default of the counterparty. In this article, we have suggested a simple methodology that tackles this computational complexity at its root, without imposing any restriction on the choice of the underlying joint default model.

In summary, we have derived two main results:

1. If the mark-to-market recovery is symmetric, then nested valuations are unnecessary and the contract can be exactly priced as a "step-down" swap, defined as a swap whose payments on both legs step down to a fraction of the contractual payments at the time of default of the risky counterparty.
2. If the mark-to-market recovery is asymmetric, as it is more realistic to expect, then bounds for the fair swap spread can be derived by means of a single simulation of joint defaults which uses only current information. These bounds turn out to be surprisingly tight when we have positive dependence between the default time of the risky counterparty and the default times of the reference names in the swap, which is also generally the case.

Applying these results to synthetic loss tranches of different seniorities, we have illustrated how their fair spreads vary as functions of a) the market-implied default probability of the risky counterparty, b) the asymmetry of the mark-to-market recovery at default, and c) the dependence between the default time of the risky counterparty and the default times of the credits referenced by the swap. Moreover, we have shown that allowing for the possibility of default of the protection seller has a much more significant impact on fair spreads than allowing for the possibility of default of the protection buyer, other things being equal.

We conclude with the observation that the methodology presented in this article can be easily extended to price swaps between two default-risky counterparties; the resulting bounds, however, will generally not be as tight as the ones derived above.

REFERENCES

- Boscher, H. and I. Ward, 2002, "Long or short in CDOs", *Risk*, October, 125-129.
- Li, D. X., 2000, "On default correlation: a copula function approach", *Journal of Fixed Income*, 9, 43-54.
- Mashal R., M. Naldi and A. Zeevi, 2003, "On the dependence of equity and asset returns", *Risk,* October, 83-87.
- O'Kane, D. and L. Schloegl, 2002, "A counterparty risk framework for protection buyers", Lehman Brothers, *Quantitative Credit Research Quarterly*, Q2.
- Rothschild, M. and J. Stiglitz, 1970, "Increasing risk: I. A definition", *Journal of Economic Theory*, 2(3) 225-243.
- Schonbucher, P., 2002, "Copula-dependent default risk in intensity models", Working Paper, ETH Zurich.

Valuation of Portfolio Credit Default Swaptions

Claus M. Pedersen
1-212-526-7775
cmpeders@lehman.com

We discuss the valuation of options on portfolio credit default swaps with a focus on standardized contracts referencing for example the CDX or TRAC-X entities. First, we describe the portfolio swap and swaption contracts. Next, we argue that Black formulas, the standard formulas for pricing single-name default swaptions, are inappropriate for pricing portfolio default swaptions. Finally, we present a simple, easy to implement, alternative model that prices portfolio swaptions using the credit curves of the reference entities and a single volatility parameter.[1]

1. INTRODUCTION

Investors are increasingly finding portfolio credit default swaps (also called portfolio CDS, portfolio default swaps or, in this report, simply portfolio swaps) useful for gaining exposure to market wide credit spreads. Options on portfolio default swaps (portfolio credit default swaptions, portfolio default swaptions, or simply portfolio swaptions) allow investors to leverage this exposure and provide a tool for gaining exposure to market wide credit spread volatility. Standardized portfolio default swaps referencing the CDX.NA.IG (in this report simply CDX) and TRAC-X NA (in this report simply TRAC-X) entities are today trading with bid-offer spreads as low as 1-2bp. Recently, CDX and TRAC-X swaptions have seen increasing trading volume as well. Lehman Brothers is a market maker in CDX and TRAC-X portfolio swaps and swaptions.

The purpose of this research report is to introduce the CDX and TRAC-X portfolio swaptions and present a simple model for their pricing and risk management. Such a model already exists for single-name default swaptions in the form of a modification of Black's formulas for interest rate swaptions. These Black formulas for default swaptions provide the values of swaptions that knock out if the reference entity defaults before swaption maturity. To price an option to buy protection that does not knock out, it is therefore necessary to add to the Black formula price the value of protection until swaption maturity. Portfolio swaptions do not knock out. It has therefore been suggested to price these as single-name non-knockout default swaptions using a credit curve representing the average credit worthiness of the reference entities. However, such an approach will tend to overvalue high strike portfolio swaptions compared with a model that more explicitly models the underlying cashflow.

The easiest way to see that Black formulas give mispricing is by examining the pricing of a deep out-of-the-money payer swaption (this is an option to buy portfolio protection at a very high strike spread). If it is priced as suggested above, its value will be close to the value of protection until swaption maturity. This is incorrect; the price should approach zero as the strike increases. This is because, when the strike is very high, the payer will not be exercised even if a reference entity has defaulted unless the portfolio spread on the non-defaulted entities has widened sufficiently.

We propose to directly model the terminal value of the swaption using a single state variable which we call the default-adjusted forward portfolio spread. As long as no defaults have occurred, this spread can reasonably be called the forward portfolio spread since it is the

[1] *I would like to thank Peter Alpern, Georges Assi, Jock Jones, Roy Mashal, Marco Naldi and Lutz Schloegl for discussions and comments.*

strike for which the payer and receiver portfolio swaptions are equal in value. If no defaults have occurred at swaption maturity, the spread is simply the portfolio spread itself. As we explain in detail below, we propose to incorporate defaults directly into the spread and thereby avoid modeling the number of defaults which would otherwise be required to explicitly model the terminal value of a portfolio swaption.

In section 2, we describe the CDX and TRAC-X portfolio swap and swaption contracts. In section 3, we explain why the Black formulas for default swaptions should not, as suggested by others, be used to price portfolio swaptions. In section 4, we present our alternative valuation model. Section 5 concludes.

2. THE PORTFOLIO SWAP AND SWAPTION CONTRACTS

To value a portfolio swaption it is necessary to understand the details of the swaption contract. The presentation is focused on CDX but, as explained below, TRAC-X works almost identically.

2.1. The CDX swap contract

A buyer of CDX protection is buying credit protection on 125 fixed reference entities. If the notional of the CDX swap is 125 million, say, then buying CDX protection has the same economic effects as buying protection on each reference entity through 125 single-name No-R[2] CDS, each with a notional of 1 million. A buyer of CDX protection is obligated to pay the same premium (called the *fixed rate*) on all the 125 hypothetical underlying CDS. When a default occurs, the CDX protection buyer must physically settle to receive the protection payment for the defaulted entity. After settlement, the relevant hypothetical underlying CDS is eliminated from the CDX swap, which then has a notional that is decreased by 1/125 of the original notional and one less reference entity. The fixed rate remains unchanged. The payment dates are the standard CDS dates (20th of March, June, September and December).

Today there are two CDX swaps with 5- and 10-year maturities. Currently the fixed rate is 60bp for the 5-year contract and 70bp for the 10-year contract. On the 20th of March and September each year (the *roll dates*), new CDX swaps will become *on-the-run* to ensure that the reference entities represent the aggregate market and that the on-the-run contracts have 5- and 10-year maturities[3]. In addition to reference entities and maturities, the fixed rates will also be changed[4]. Changes to new on-the-run CDX swaps have no influence on the existing contracts. The maturity, the fixed rate, and the reference entities are all fixed throughout the life of specific CDX swaps and swaptions.

Reference entities may also be eliminated from the on-the-run CDX swaps between roll dates, for example, if an entity defaults. Eliminated entities will not be replaced between roll dates. If an entity is eliminated, the relevant hypothetical underlying CDS can be split off from existing CDX swap contracts to ensure that they remain on-the-run. This ensures liquidity of the CDX swap contract after a default has been settled.

---

2 *No-R means 'no restructuring' and refers to the fact that only default (bankruptcy and failure to pay) can trigger the CDX contract. In the US, CDS mainly trade under the Mod-R (modified restructuring) clause which includes restructurings as a credit event. For details see O'Kane, Pedersen and Turnbull (2003).*

3 *The reference entities in the new on-the-run swap are chosen by voting among CDX market makers. The voting system is fundamentally different from the rules-based methodology used to determine the constituents of the Lehman Brothers Credit Default Swap Index. See Berd et al. (2003) for more details.*

4 *After publication of the new reference entities, CDX market makers will submit 5- and 10-year quotes, the medians of which will become the new fixed rates.*

The value of a CDX swap is driven by the CDS spreads on the 125 reference entities. When the spreads are high compared with the fixed rate, the swap has a positive value to the protection buyer who therefore pays a *CDX price* to the protection seller at contract initiation (when the swap price is negative its absolute value is paid by the protection seller to the protection buyer). The swap price is quoted in the form of a *CDX spread* which can be readily converted into the price.

The conversion from spread to price can be done via the CDSW calculator on Bloomberg. The calculation is to discount fixed payments equal to the CDX spread minus the fixed rate, as when calculating the mark-to-market on a CDS[5]. The conversion can be written as:

$$P = PV01 \cdot (S - FR)$$

where FR is the fixed rate, S is the quoted CDX spread, P is the corresponding CDX price, and PV01 is the risky PV01 from today to swap maturity, ie, the value of receiving 1bp on the CDX payment dates until maturity of the swap or default, whichever occurs first.

The market standard for quoting CDX spreads assumes that the PV01 is calculated using discount factors that have been calibrated to fit a flat CDS curve with spreads equal to the CDX spread. The calibration uses a recovery-given-default of 40% and default-free interest rates taken from the current Libor curve.

The intrinsic CDX spread

The intrinsic CDX spread is the spread quote that converts into the price of buying the CDX equivalent credit protection through 125 single-name CDS contracts, each with a contractual spread equal to the CDX fixed rate. The intrinsic spread depends on the shape of the individual credit curves, but will be close to the average CDS spread of the appropriate maturity across the CDX reference entities.

Determination of the intrinsic spread is made more difficult by the fact that restructuring is not included as a credit event in the CDX swap contract. No-R CDS spreads are rarely available, but in the US, dealers tend to quote No-R spreads around 5% lower than Mod-R spreads. Intrinsic CDX spreads are usually calculated from Mod-R spreads that have been discounted 5% (see O'Kane, Pedersen and Turnbull (2003)).

Aside from possible error introduced when determining No-R spreads, a CDX spread may differ from its intrinsic value because of specific shorter term demand-supply conditions in CDX and/or CDS markets. Because of the No-R feature, and because of bid-offer spreads in CDS markets, it is generally not feasible to arbitrage the differences between quoted and intrinsic CDX spreads.

Currently, intrinsic CDX spreads tend to be higher than quoted CDX spreads implying that it is cheaper to buy protection through CDX than through the underlying single-name CDS. Part of the difference may be a result of the higher liquidity of CDX compared to CDS. In other words, sellers of CDX protection can be seen as demanding a lower liquidity premium than sellers of single-name CDS protection.

5 *See O'Kane and Turnbull (2003) for details on how to calculate the mark-to-market of a CDS.*

2.2. The CDX swaption contract

A CDX swaption references a specific underlying CDX swap with a specific maturity. Usually the CDX swap was on-the-run when the option was traded. Aside from the underlying CDX swap, the CDX swaption is specified by a swaption maturity, a strike spread, and a swaption type. The standard portfolio swaptions are European. The swaption type is either *payer* or *receiver*. A payer gives the right to become a protection buyer in the underlying CDX swap at the strike spread. A receiver gives the right to become a protection seller.

If the swaption is exercised, the strike spread is converted into an *exercise price* (also called the settlement payment) using the same calculation as when converting a CDX spread quote into a CDX price. For example, if the strike spread is K, then the exercise price is:

$$P(K) = \gamma(K)\left(K - FR\right)$$

where FR is the fixed rate in the underlying CDX swap contract, and $\gamma(K)$ is the risky PV01 calculated at the exercise date using the same assumptions as if converting a CDX spread quote of K into its corresponding CDX price. That is, $\gamma(K)$ is the risky PV01 calculated from a credit curve that has been calibrated to a flat CDS curve with spreads equal to the strike spread, K, using a recovery-given-default of 40% and the Libor curve at the exercise date.

Consider an example. On November 6, 2003, a CDX payer swaption on the 5-year CDX swap with a notional of \$100 million was traded. The swaption maturity is March 22, 2004, the strike spread is K = 55bp, and the fixed rate is FR = 60bp. If we assume that the Libor curve on March 22, 2004 is equal to the forward Libor curve for that date observed on November 6, 2003, then we can find the exercise price. Under this Libor curve, the risky PV01 is $\gamma(K) = 0.0453$ (\$ per \$100 notional). The exercise price is then 0.0453·(55-60) = -0.227, so if the swaption is exercised, the protection seller must pay \$227 thousand to the protection buyer. This cash transfer of \$227 thousand is independent of the number of defaults that may have occurred before swaption maturity and the CDX spread at swaption maturity. The only uncertainty about the amount arises from uncertainty about the Libor curve at the exercise date.

When the swaption is exercised, the protection buyer has in effect bought protection on all reference entities, including those that may have defaulted before swaption maturity. The protection buyer can then immediately settle for protection payment on any defaulted entities. The terminal value of the swaption therefore depends on the recoveries on defaulted entities and the mark-to-market on the CDX swap with the defaulted entities eliminated.

Depending on the maturity of the swaption and the time from the option trade date to the following roll date, the CDX swap with the defaulted entities eliminated may or may not be on-the-run at option maturity. However, given the relatively short swaption maturities usually seen (most swaptions trade with maturities of six months or less), this CDX swap should be liquid throughout the life of the swaption.

Let us extend the example above to determine the terminal value of the swaption. Assume that one reference entity defaulted before March 22, 2004, and the cheapest deliverable obligation issued by the defaulted entity trades at \$45 (per \$100 notional). Also assume that the quoted market CDX spread on March 22, 2004, on the CDX swap with the defaulted entity eliminated is 75bp. Finally, assume as above that the Libor curve on March 22, 2004, is the forward Libor curve for that date observed on November 6, 2003. In this case, the payer swaption will be exercised and the swaption seller must make an initial \$227 thousand cash payment to the swaption buyer at exercise (see details above). After exercise, the swaption buyer can deliver \$800 thousand of notional of the deliverable obligation mentioned above and receive a \$800 thousand cash payment. The cost of the deliverable is \$360 thousand. The last step in determining the terminal value of the swaption is to find the mark-to-market of the CDX swap with the defaulted entity eliminated. The notional of this CDX swap is \$99.2 million and the market value to the swaption buyer of this position turns out to be \$675 thousand (found as 0.0450·(75-60)·\$99.2 million/100, given that the risky PV01 calculated from a flat curve of 75bp is 0.0450). Altogether the payer swaption has a total value of \$1.342 million (227+800-360+675). Notice that for this particular example, the effect of one default is about the same as a 10bp widening in the CDX spread on the non-defaulted entities.

2.3. The TRAC-X swap and swaption contracts

TRAC-X swap and swaption contracts work in the same way as the CDX swap and swaption contracts described above. The differences are the number of reference entities, the identity of the reference entities, the fixed rate, and the procedure for determining new reference entities. We are not aware of any other differences with pricing implications.

There are currently 100 TRAC-X reference entities the composition of which is scheduled for change every three months (around the standard CDS dates, ie, the 20th of March, June, September, and December). A change to new on-the-run swap contracts occurred in September 2003 when 34 entities were replaced. The fixed rate is currently 100bp in both the old and the new TRAC-X swap contracts. On November 6, 2003, TRAC-X was trading at 53 and a TRAC-X protection seller would have to make a significant upfront payment at contract initiation (more than \$2 million on a contract with a \$100 million notional). Currently, there are on-the-run TRAC-X swaps with 5- and 10-year maturities.

The remainder of this report uses generic language, and the issues apply to both CDX and TRAC-X swaptions. We use terminology such as portfolio swap, portfolio swaption and portfolio spread.

3. BLACK FORMULAS FOR DEFAULT SWAPTIONS

Black formulas for default swaptions are simple theoretically consistent formulas for valuing single-name default swaptions. The formulas give values of swaptions that knock out if a default occurs before swaption maturity. The knockout feature is not relevant for receiver swaptions (options to sell protection), as they will never be exercised after a default. For payer swaptions (options to buy protection), it is necessary to add to the Black formula price the value of protection from the swaption trade date to the swaption maturity.

To introduce notation, let $T>0$ be the swaption maturity and let $T_M>T$ be the maturity of the underlying CDS. Let $PV01_t(T,T_M)$ be the value at time $t\leq T$ of a security that pays a 1bp annual flow starting at the first CDS date after time T and ending at T_M or default, whichever occurs first. Similarly, let $PVP_t(T,T_M)$ be the value at time t of a security that pays par minus the recovery-given-default at the time of default if default occurs between time T and time T_M. Given a Libor curve and an issuer CDS curve at time t, $PV01_t(T,T_M)$ and $PVP_t(T,T_M)$ can be found using the Jarrow-Turnbull credit pricing framework with a piecewise linear hazard rate. Furthermore, for $t\leq T$ let:

$$F_t(T,T_M)=\frac{PVP_t(T,T_M)}{PV01_t(T,T_M)}$$

$F_t(T,T_M)$ is the T-forward-starting spread on a CDS that matures at time T_M.

The Black formulas are based on the assumption that:

$$F_T(T,T_M) = F_0(T,T_M)\exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\varepsilon\right)$$

where ε is a standard normal random variable under the risk neutral measure, corresponding to $F_t(T,T_M)$ being lognormally distributed with volatility σ. The formulas are[6]:

$$PS_0^{KO} = PV01_0(T,T_M)\cdot\left(F_0(T,T_M)\cdot N(d_1) - K\cdot N(d_2)\right)$$

$$RS_0 = PV01_0(T,T_M)\cdot\left(K\cdot N(-d_2) - F_0(T,T_M)\cdot N(-d_1)\right)$$

$$d_1 = \frac{\log(F_0(T,T_M)/K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

where K is the strike, PS_0^{KO} is the knockout payer value and RS_0 is the receiver value. The value of a payer that does not knock out is:

$$PS_0 = PS_0^{KO} + FEP_0(0,T)$$

where $FEP_t(0,T)$ (front-end protection) is the value at time t of a contract that pays par minus the recovery-given-default at swaption maturity, T, if default occurs between time 0 and time T. Determining the forward spread, the forward PV01, and the front-end protection, requires a CDS curve for the issuer. The shape of the curve is significant in the valuation.

It has been suggested that the above methodology can be used to value a portfolio swaption, using a CDS curve that represents the average creditworthiness of the reference entities. As an approximation we could, for example, for each maturity calculate the average spread across reference entities and shift the averages to ensure that the 5- and 10-year points match the quoted portfolio spreads. We could also adjust the averages in such a way that if all the reference entities had the adjusted average curve, then the intrinsic portfolio spreads would equal the quoted portfolio spreads.

Although this approach is appealing because of its simplicity, it is not theoretically sound. It is easy to see this by examining the pricing of payer swaptions as the strike increases. For a very high strike, the above approach will give a price close to FEP, the value of the front-end protection. When the strike is infinite, however, the price should be 0, since such a swaption should not be exercised even if most entities defaulted[7].

If all reference entities had the same CDS curve, then the approach would correctly price the difference between the payer and the receiver (this will become clearer in the next section). From that perspective, the front-end protection has to be included in the payer price. However, this causes the decomposition into the payer and receiver prices to become skewed towards high prices for high strike swaptions.

When the reference entities do not have the same CDS curves the pricing of the difference between the payer and the receiver also breaks down. Using the average curve will tend to overvalue a long-payer short-receiver position compared with the approach we suggest in the

6 *For more details on how to derive the formulas, see the section on modeling credit options in the Lehman Brothers Guide to Exotic Credit Derivatives and the references given there.*

7 *Theoretically the swaption could be exercised if all entities defaulted and had a combined recovery of less than 40%.*

next section. This is mainly caused by the concavity in the value of the protection leg of a CDS when seen as a function of the CDS spread level.

4. A MODEL FOR PORTFOLIO DEFAULT SWAPTIONS

To explicitly model the terminal value of a portfolio swaption it is necessary to model both the recovery on defaulted entities and the spread on the portfolio swap with the defaulted entities eliminated. As a simplification, we suggest incorporating defaults directly into the portfolio spread to arrive at a *default-adjusted forward portfolio spread* that can be used as a single state variable to generate the terminal value of the swaption. Alternatively, the terminal value could be modeled directly, but it is not clear which distribution would be reasonable. On the other hand, we find it reasonable to use as a first approximation a lognormal distribution for the default-adjusted forward portfolio spread at swaption maturity.

4.1. The put-call parity

The first step in valuing a portfolio swaption is to identify and value the underlying. We refer to the underlying as the default-adjusted forward portfolio swap. It is different from a regular knockout forward portfolio swap because protection for defaults that occur before swaption maturity are paid at swaption maturity.

The default-adjusted forward portfolio swap can be viewed as a portfolio of non-knockout forward starting CDS on each reference entity. A non-knockout forward starting CDS is a combination of a regular knockout forward starting CDS and front-end protection from today to swaption maturity. The contractual spread in the CDS is the fixed rate in the portfolio swap, denoted FR. The value of such a forward starting CDS at time t≤T is:

$$V_t^i = PVP_t^i(T,T_M) - PV01_t^i(T,T_M) \cdot FR + FEP_t^i(0,T)$$

where an *I* superscript indicates that the values are for the i'th reference entity. T is the swaption maturity and $T_M \geq T$ is the maturity of the CDS. The PVP and PV01 notation was explained in the previous section. $FEP_t^i(0,T)$ is the value at time t of a contract that pays par minus the recovery-given-default if the i'th entity defaults between time 0 and time T. It is important to note that if the i'th entity defaulted between 0 and t, then $FEP_t^i(0,T)$ is par minus recovery discounted on the time t Libor curve from t to T.

The value at time t of the default-adjusted forward portfolio swap is

$$V_t = \sum_{i=1}^{N} \frac{1}{N} V_t^i$$

where N is the number of reference entities. The factor 1/N is used because everywhere the notional of a contract is assumed to be par unless otherwise specified.

If the swaption is exercised, the protection buyer must pay the exercise price to the protection seller. The exercise price is specified by the strike spread, K, but also depends on the Libor curve at option maturity. For valuation purposes, we assume that *Libor rates are deterministic*. We can therefore use today's forward Libor curve at swaption maturity to calculate the exercise price. It is given by:

$$P(K) = \gamma(K)(K - FR) \tag{1}$$

where γ(K) is the risky PV01 from T to T_M calculated on a credit curve that has been fitted to a flat CDS term structure with spreads equal to K, using a recovery-given-default of 40% (see section 2.2 for details).

Because the front-end protection values at T, $FEP_T^i(0,T)$, incorporate defaults between 0 and T, we can write the terminal swaption values as:

$$PS_T(K) = \max\{V_T - P(K), 0\} \tag{2}$$

$$RS_T(K) = \max\{P(K) - V_T, 0\} \tag{3}$$

where PS is the payer and RS is the receiver. We see that $PS_T(K) - RS_T(K) = V_T - P(K)$ and we get the put-call parity:

$$PS_0(K) - RS_0(K) = V_0 - P(K)D(0,T) \tag{4}$$

where D(0,T) is the Libor discount factor from 0 to T.

When pricing the default-adjusted forward portfolio swap, it is important that the portfolio swap itself is priced correctly off the individual credit curves, ie, the intrinsic portfolio spread should equal the current market portfolio spread. In practice, this requires that the individual curves are adjusted as explained in section 2.1.1.

4.2. Pricing the payers and receivers

To price the swaptions, we propose a stochastic model for the value of the default-adjusted forward portfolio swap at swaption maturity, V_T, and thus for the terminal swaption values. We specify the distribution of V_T through what we call the default-adjusted forward portfolio spread, denoted X_T. For any time t between 0 and T, X_t is given by:

$$V_t = \gamma(X_t)(X_t - FR)D(t,T) \tag{5}$$

The function γ and the constant FR were defined in the previous section. D(t,T) is the Libor discount factor from t to T.

When $K = X_0$ then $V_0 = \gamma(K)(K - FR)D(0,T) = P(K)D(0,T)$, and from the put-call parity $PS_0(K) = RS_0(K)$. In words: the default-adjusted forward portfolio spread is the strike for which the value of the payer is equal to the value of the receiver. Because of this relationship we can simply call the default-adjusted forward portfolio spread the *forward portfolio spread* when no defaults have yet occurred. So X_0 is the forward portfolio spread at time 0 for time T.

If no defaults have occurred at T, then the default-adjusted forward portfolio swap is the portfolio swap itself, and V_T is the value to the protection buyer of all the underlying CDS. By the definition of X_T ($V_T = \gamma(X_T)(X_T - FR)$), X_T is the portfolio spread at T. This is consistent with the above terminology of calling X_t the forward portfolio spread at t for T when no defaults have occurred before t.

When specifying the distribution of X_T, we must ensure that the default-adjusted forward portfolio swap is priced correctly in the model. X_T will be specified under the risk neutral measure (where D(t,T) is the numeraire price at time t≤T). Specifically, we must ensure that:

$$V_0 = D(0,T)E[V_T]$$

or equivalently, using (1) and (5):

$$E[P(X_T)] = P(X_0)$$

We assume that X_T is lognormally distributed with:

$$X_T = X_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\varepsilon \right)$$

where ε is a standard normal random variable. μ is the drift and σ the volatility of the default adjusted forward portfolio spread. We choose this specification because of its simplicity and because a lognormal distribution is fairly consistent with empirical evidence. It is not difficult to imagine that the X_t process may contain jumps. X_t may jump, for example, if there is a surprise default of one of the reference entities. On the other hand, an entity usually defaults only when its spreads are already high, in which case the jump in the portfolio spread at the actually time of default may be small. In general, the lognormal distribution in our portfolio context seems more appropriate than assuming a lognormal spread for a single entity.

The drift μ is the free parameter used to ensure that $E[P(X_T)] = P(X_0)$. The function P is well behaved, and can be very well approximated using a polynomial spline. Even a simple second-order Taylor approximation of P around X_0 works well as long as the variance of X_T is not too high. In practice, the drift will be close to 0.

Once the drift has been fixed, and we have the approximation of the function P, it is straightforward to price the swaptions by discounting the expected terminal value using the distribution of X_T. Substituting (5) and (1) into (2) and (3), and using the fact that P is an increasing function, we get:

$$PS_0(K) = D(0,T)E\left[(P(X_T) - P(K))1_{(X_T \geq K)}\right]$$

$$RS_0(K) = D(0,T)E\left[(P(K) - P(X_T))1_{(K \geq X_T)}\right]$$

The equations can be solved in closed form when P is approximated by a polynomial spline.

4.3. Pricing examples

We illustrate by pricing CDX swaptions that mature on March 22, 2004. The underlying CDX swap matures on March 20, 2009. We did the valuation on November 6, 2003, at a time when the CDX spread was 56.

Figure 1. Adjusted CDS spreads (in bp) on the CDX reference entities on November 6, 2003

Maturity	6M	1Y	2Y	3Y	4Y	5Y	7Y
Average	35.6	39.0	44.1	48.8	52.3	55.6	61.3
Maximum	206	228	241	264	277	281	295
Median	22	24	29	33	38	40	46
Minimum	4	4	4	9	9	11	15

The first step in pricing the swaptions is to choose the best possible CDS spread curves for the 125 reference entities. Usually it suffices to base the curves on yesterday's closing levels. The curves can then be adjusted to current market levels by using liquid CDS spreads (eg, parallel shifting based on the 5-year CDS spreads). After the individual curves have been adjusted to the current single-name market, all the curves must be adjusted so that the portfolio of the 125 single-name CDS is priced according to the market CDX spread of 56, ie, so that the intrinsic CDX spread is 56. This adjustment also incorporates the fact that CDX is No-R.

To price the swaptions, we must choose a spread volatility. The spread volatility in our model is not the same as the spread volatility to be used with the Black formula. We model the default-adjusted forward portfolio spread which incorporates the defaults that occur before swaption maturity. The volatility in our model should therefore be higher than the volatility used in the Black formulas. Based on market prices, we choose a default-adjusted forward portfolio spread of 55%. Figure 2 shows the computed payer and receiver values. In parentheses next to a price is the implied Black volatility when the average adjusted CDS spread curve of the reference entities is used in the Black formula.

Figure 2. CDX swaption prices (in $ per $100 par) and implied Black volatilities off the average adjusted CDS spread curve of the reference entities

Strike (in bp)	45	50	55	60	65	70	75
Discounted exercise price	-0.68	-0.45	-0.23	0	0.23	0.45	0.67
Payer price (implied Black vol.)	0.79 (50.4%)	0.63 (48.7%)	0.49 (47.1%)	0.38 (45.3%)	0.29 (42.9%)	0.22 (39.6%)	0.16 (33.7%)
Receiver price (implied Black vol.)	0.08 (49.4%)	0.15 (48.4%)	0.23 (47.0%)	0.35 (45.2%)	0.48 (42.7%)	0.63 (39.0%)	0.80 (32.0%)

There are two main lessons to draw from Figure 2.

- For high strikes, implied Black volatilities for payer swaptions decrease as the strike increases.
- The implied Black volatilities are not the same for payers and receivers with the same strike.

Figure 2 shows that, according to our model, the implied Black volatility for both payer and receiver swaptions should decrease as the strike increases. The effect is very noticeable, especially when comparing payers with strikes of 45 and 75. When the strike is 75, the value of the payer according to our model is very close the value of front-end protection priced off the adjusted average CDS curve (0.16 versus 0.13). When the strike is 80, the payer value is too low for the implied Black volatility to be defined.

We also see that the Black volatilities implied from our model are not the same for payers and receivers with the same strike. This shows that the default-adjusted forward portfolio swap is not priced correctly off the adjusted average CDS spread curve. According to the put-call parity the payer price minus the receiver price plus the discounted exercise price is equal to the value of the default-adjusted forward portfolio swap. In the example above, the default-adjusted forward portfolio swap has a value of 0.03 when the strike is 75. In the Black formulas, the value of the payer minus the value of the receiver does not depend on the volatility. If the discounted exercise price is subtracted from that difference when the strike is 75, we arrive at the default-adjusted forward portfolio swap value priced off the adjusted average CDS curve. This value is 0.005 too low according to the individual curves. It is this mispricing that gives the different implied Black volatilities for the payers and receivers.

5. SUMMARY

We presented the details of the CDX and TRAC-X swap and swaption contracts, and introduced market-consistent terminology to discuss their pricing. We argued that Black's formulas for single-name default swaptions are inappropriate for pricing portfolio swaptions, with the mispricing especially noticeable for deep out-of-the-money payer swaptions. We presented an alternative model that directly models the terminal swaption values through a default-adjusted forward portfolio spread. Our methodology ensures that a combined long-payer short-receiver position, which is insensitive to spread volatility, is priced in a manner consistent with the credit curves of the reference entities. Finally, we illustrated with numerical examples that the price differences between our model and the Black formulas are significant.

REFERENCES

Berd, A., A. Desclée, A. Golbin, D. Munves and D. O'Kane (2003). *The Lehman Brothers Credit Default Swap Index*, Lehman Brothers, Fixed Income Index Group, October 2003.

O'Kane, D, M. Naldi, S. Ganapati, A. Berd, C. Pedersen, L. Schloegl and R. Mashal (2003). *The Lehman Brothers Guide to Exotic Credit Derivatives*, Lehman Brothers, October 2003.

O'Kane, D, C. Pedersen, and S. Turnbull (2003). *Pricing the Restructuring Clause in Credit Default Swaps*, Lehman Brothers, June 2003.

O'Kane, D. and S. Turnbull (2003). "Valuation of Credit Default Swaps." *Quantitative Credit Research Quarterly*, Lehman Brothers, April 2003.

Lehman Brothers Fixed Income Research analysts produce proprietary research in conjunction with firm trading desks that trade as principal in the instruments mentioned herein, and hence their research is not independent of the proprietary interests of the firm. The firm's interests may conflict with the interests of an investor in those instruments.
Lehman Brothers Fixed Income Research analysts receive compensation based in part on the firm's trading and capital markets revenues.
Lehman Brothers and any affiliate may have a position in the instruments or the company discussed in this report.
The views expressed in this report accurately reflect the personal views of Arthur M. Berd, Ganlin Chang, Andrei Greenberg, Roy Mashal, Vasant Naik, Marco Naldi, Claus M. Pedersen, Lutz Schloegl, Minh Trinh, the primary analyst(s) responsible for this report, about the subject securities or issuers referred to herein, and no part of such analyst(s)' compensation was, is or will be directly or indirectly related to the specific recommendations or views expressed herein.
The research analysts responsible for preparing this report receive compensation based upon various factors, including, among other things, the quality of their work, firm revenues, including trading and capital markets revenues, competitive factors and client feedback.

To the extent that any of the views expressed in this research report are based on the firm's quantitative research model, Lehman Brothers hereby certify (1) that the views expressed in this research report accurately reflect the firm's quantitative research model and (2) that no part of the firm's compensation was, is or will be directly or indirectly related to the specific recommendations or views expressed in this report.
`

Any reports referenced herein published after 14 April 2003 have been certified in accordance with Regulation AC. To obtain copies of these reports and their certifications, please contact Larry Pindyck (lpindyck@lehman.com; 212-526-6268) or Valerie Monchi (vmonchi@lehman.com; 44-207-011-8035).

This material has been prepared and/or issued by Lehman Brothers Inc., member SIPC, and/or one of its affiliates ("Lehman Brothers") and has been approved by Lehman Brothers International (Europe), regulated by the Financial Services Authority, in connection with its distribution in the European Economic Area. This material is distributed in Japan by Lehman Brothers Japan Inc., and in Hong Kong by Lehman Brothers Asia. This material is distributed in Australia by Lehman Brothers Australia Pty Limited, and in Singapore by Lehman Brothers Inc., Singapore Branch. This material is distributed in Korea by Lehman Brothers International (Europe) Seoul Branch. This document is for information purposes only and it should not be regarded as an offer to sell or as a solicitation of an offer to buy the securities or other instruments mentioned in it. No part of this document may be reproduced in any manner without the written permission of Lehman Brothers. We do not represent that this information, including any third party information, is accurate or complete and it should not be relied upon as such. It is provided with the understanding that Lehman Brothers is not acting in a fiduciary capacity. Opinions expressed herein reflect the opinion of Lehman Brothers and are subject to change without notice. The products mentioned in this document may not be eligible for sale in some states or countries, and they may not be suitable for all types of investors. If an investor has any doubts about product suitability, he should consult his Lehman Brothers' representative. The value of and the income produced by products may fluctuate, so that an investor may get back less than he invested. Value and income may be adversely affected by exchange rates, interest rates, or other factors. Past performance is not necessarily indicative of future results. If a product is income producing, part of the capital invested may be used to pay that income. Lehman Brothers may make a market or deal as principal in the securities mentioned in this document or in options, futures, or other derivatives based thereon. In addition, Lehman Brothers, its shareholders, directors, officers and/or employees, may from time to time have long or short positions in such securities or in options, futures, or other derivative instruments based thereon. One or more directors, officers, and/or employees of Lehman Brothers may be a director of the issuer of the securities mentioned in this document. Lehman Brothers may have managed or co-managed a public offering of securities for any issuer mentioned in this document within the last three years, or may, from time to time, perform investment banking or other services for, or solicit investment banking or other business from any company mentioned in this document. © 2003 Lehman Brothers. All rights reserved. Additional information is available on request. Please contact a Lehman Brothers' entity in your home jurisdiction.