Paper 2

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1 Divisibility of an exponential sum

Theorem 1. *For all $n \in \mathbb{N}$, $11 \mid (2^{2n+1} + 5^{3n-1})$.*

Proof. When $n = 1$, $2^3 + 5^2 = 33$, $11 \mid 33$. Suppose that for some $n > 1$, then

$$\begin{aligned} 2^{2(n+1)+1} + 5^{3(n+1)-1} &= (2^2)2^{2n+1} + (5^3)5^{3n-1} \\ &= (4)2^{2n+1} + (125)5^{3n-1} \\ &= (4)2^{2n+1} + (4+121)5^{3n-1} \\ &= (4)(2^{2n+1} + 5^{3n-1}) + (121)5^{3n-1} \\ &= (4)(2^{2n+1} + 5^{3n-1}) + (11^2)5^{3n-1} \end{aligned}$$

By the inductive hypothesis, $11 \mid (2^{2n+1} + 5^{3n-1})$. The second term $(11^2)5^{3n-1}$ is also a multiple of 11. Hence, by induction, the theorem holds. □

2 Pararationals

Definition 2. A real number z is *pararational* if there exist $a, b \in \mathbb{Q}$ for which $z = a + b\sqrt{3}$.

Lemma 3. *Suppose $\sqrt{2}$ is pararational. Then there is a rational number b such that $\sqrt{2} = b\sqrt{3}$.*

Incorrect attempted proof of lemma 3. Suppose $\sqrt{2} = b\sqrt{3}$. Then, using $a = 0$, $\sqrt{2} = a + b\sqrt{3}$, so $\sqrt{2}$ is pararational. This is the hypothesis of the lemma, so it is true. □

Criticism Let the hypothesis be the first sentence of Lemma 3 and the claim its second sentence. This argument rests on the assumption that claim is true, meaning the hypothesis is dependent on the claim to be true.

Proof of lemma 3. We suppose that $\sqrt{2} = a + b\sqrt{3}$, where there exist $a, b \in \mathbb{Q}$. If $b = 0$, then $\sqrt{2} = a$. We know that $a \in \mathbb{Q}$. This contradicts irrationality of $\sqrt{2}$ by Lemma 1.7 in Proofs So Far, so $b \neq 0$. If $a \neq 0$, then

$$\begin{aligned} \sqrt{2} &= a + b\sqrt{3} \\ 2 &= a^2 + 2ab\sqrt{3} + 3b^2 \end{aligned}$$

Since $a, b \in \mathbb{Q}$, the sum of the terms a^2 and $3b^2$ is rational. By Theorem 2.12 in Proofs So Far and Theorem 1.20 in Proofs With Hints, the term $2ab\sqrt{3}$ is irrational. So the right hand side must be irrational. This contradicts the left hand side as 2 is rational, so $a = 0$. Since $a = 0$ and $b \neq 0$, there is a rational number b such that $\sqrt{2} = b\sqrt{3}$. □

Proposition 4. *$\sqrt{2}$ is not pararational.*

Proof. Suppose that $\sqrt{2}$ is pararational. By Lemma 3, there is a rational number b such that $\sqrt{2} = b\sqrt{3}$. Then

$$\begin{aligned} \sqrt{2} &= b(\sqrt{3}) \\ 2 &= 3b^2 \end{aligned}$$

Then there are integers p, q, which are both not divisible by 3, such that $b = p/q$. (Otherwise, divide p and q until at least one is not divisible by 3.) Hence $3p^2 = 2q^2$, and $GCD(2, 3) = 1$, so q^2 is divisible by 3. But then q is also divisible by 3, so $q = 3k$ for some $k \in \mathbb{Z}$. Hence $3p^2 = 18k^2$, so $p^2 = 2(3k^2)$, and p^2 is divisible by 3. But then p is also divisible by 3. This contradicts p and q being not both divisible by 3. Hence our supposition was incorrect, and $\sqrt{2}$ cannot be pararational. □

3 Mean spirited

Definition 5. For any $n, p \in \mathbb{N}$, for any sequence $(x_1, x_2, \ldots, x_n)$ of positive real numbers, we define the p *hypermean* of the sequence to be the number

$$\mathrm{HM}_{p,n}(x_1, x_2, \ldots, x_n) = \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n} x_i^p}.$$

In particular, $\mathrm{HM}_{1,n}$ is the arithmetic mean.

Proposition 6. *For any positive real numbers a and b, and any natural number $p > 1$,*

$$\frac{a+b}{2} \leqslant \sqrt[p]{\frac{a^p + b^p}{2}},$$

with equality if and only if $a = b$.

Proof for $p = 2$. Consider the quantity $\Delta = a - b$. As the square of a real number, $\Delta^2 \geqslant 0$. Also, $\Delta = 0$ if and only if $a = b$. Expanding Δ^2, we find

$$0 \leqslant \Delta^2 = (a-b)^2 = a^2 - 2ab + b^2.$$

with equality if and only if $a = b$. Multiplying by $-\frac{1}{4}$ and rearranging the terms, we find that

$$\begin{aligned} 0 \geqslant -\frac{1}{4}(a^2 - 2ab + b^2) &= \frac{2ab - a^2 - b^2}{4} \\ &= \frac{a^2 + 2ab + b^2}{4} - \frac{2a^2 + 2b^2}{4} \\ &= (\frac{a+b}{2})^2 - \frac{a^2 + b^2}{2} \end{aligned}$$

Thus we find that

$$\frac{a+b}{2} \leqslant \sqrt[2]{\frac{a^2+b^2}{2}}$$

with equality if and only if $a = b$. □

Lemma 7. *For any $n, p \in \mathbb{N}$, for any two sequences $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of positive real numbers,*

$$\mathrm{HM}_{p,2n}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = \mathrm{HM}_{p,2}\left[\mathrm{HM}_{p,n}(x_1, x_2, \ldots, x_n), \mathrm{HM}_{p,n}(y_1, y_2, \ldots, y_n)\right]$$

.

Proof. By Definition 5,

$$\mathrm{HM}_{p,2n}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = \sqrt[p]{\frac{1}{2n}(\sum_{i=1}^{n} x_i^p + \sum_{i=1}^{n} y_i^p)}$$

This is the left hand side of the equation. By Definition 5 we also know that the right hand side of the equation would be equivalent to

$$\mathrm{HM}_{p,2}(a, b) = \sqrt[p]{\frac{1}{2}(a^p + b^p)}.$$

Let a be $\mathrm{HM}_{p,n}(x_1, x_2, \ldots, x_n)$ and b be $\mathrm{HM}_{p,n}(y_1, y_2, \ldots, y_n)$, where according to Definition 5

$$a = \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n} x_i^p}$$

,

$$b = \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n} y_i^p}$$

Substituting in a and b, the right hand side of the equation would be

$$\sqrt[p]{\frac{1}{2}(\frac{1}{n}\sum_{i=1}^{n}x_i^p+\frac{1}{n}\sum_{i=1}^{n}y_i^p)}=\sqrt[p]{\frac{1}{2n}(\sum_{i=1}^{n}x_i^p+\sum_{i=1}^{n}y_i^p)}=\mathrm{HM}_{p,2n}(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n)$$

Thus the equation holds. This is for any $n,p\in\mathbb{N}$ and for any two sequences $(x_1,x_2,\dots,x_n)$ and $(y_1,y_2,\dots,y_n)$ of positive real numbers. $\square$

Lemma 8. *For any natural number n, the statements AMHM(n) and AMHM(2) imply the statement AMHM($2n$).*

Proof. Assume the statements AMHM(n) and AMHM(2), i.e., the AMHM inequality for n variables and for 2 variables. To prove the AMHM inequality for $2n$ variables, consider $2n$ positive real numbers and call them $x_1,...,x_n$ and $y_1,...,y_n$. By AMHM(2), we find

$$\mathrm{HM}_{p,2}(a,b)\geqslant\frac{a+b}{2}$$

with equality if and only if $a=b$.

By Definition 5 we know that the left hand side is $\sqrt[2]{\frac{1}{2}(a^p+b^p)}$. By Proposition 6, AMHM(2) has been proven.

Now we apply AMHM(n) to continue,

$$\begin{aligned}\mathrm{HM}_{p,2}\left[\mathrm{HM}_{p,n}(x_1,x_2,\dots,x_n),\mathrm{HM}_{p,n}(y_1,y_2,\dots,y_n)\right]&\geqslant\frac{\frac{1}{n}\sum_{i=1}^{n}x_i+\frac{1}{n}\sum_{i=1}^{n}y_i}{2}\\&=\frac{\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i}{2n}\\&=\frac{x_1+x_2+...+x_n+y_1+y_2+...y_n}{2n}\end{aligned}$$

with equality if and only if $x_1=...=x_n$ and $y_1=...=y_n$. But these equality conditions are equivalent to $x_1=...=x_n=y_1=...=y_n$. By Lemma 7, the left hand side is equivalent to $\mathrm{HM}_{p,2n}(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n)$. And so $\mathrm{HM}_{p,2n}(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n)\geqslant\mathrm{AM}(2n)$. $\square$

Lemma 9. *For any integer $n\geqslant 3$, the statement AMHM(n) implies AMHM($n-1$).*

Proof. Assume AMHM(n). Consider a list of positive real numbers $x_1,...,x_{n-1}$. Define x_n to be their arithmetic mean,

$$x_n=\frac{x_1+...+x_{n-1}}{n-1}.$$

Applying AMHM(n) to the list $x_1, ..x_n$ yields

$$\begin{aligned}\sqrt[p]{\frac{\sum_{i=1}^{n-1} x_i^p}{n} + \frac{x_n^p}{n}} &\geqslant \frac{x_1 + ... + x_{n-1} + x_n}{n} \\ &= \frac{(x_1 + ... + x_{n-1})(1 + \frac{1}{n-1})}{n} \\ &= \frac{(x_1 + ... + x_{n-1})}{n-1} = x_n,\end{aligned}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. By taking exponent p, multiplying by n and rearranging the terms we find

$$\begin{aligned}x_n^p(n-1) &\leqslant \sum_{i=1}^{n-1} x_i^p \\ x_n^p &\leqslant \frac{\sum_{i=1}^{n-1} x_i^p}{n-1}\end{aligned}$$

with equality if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$. Finally, raising both sides to the $\frac{1}{p}$ power, we find

$$x_n \leqslant (\frac{\sum_{i=1}^{n-1} x_i^p}{n-1})^{\frac{1}{p}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_{n-1} = x_n$. Since x_n is the arithmetic mean of $x_1, \ldots, x_{n-1}$, this is the AMHM inequality, i.e. the statement AMHM($n-1$). $\square$

Theorem 10. *For any $n \in \mathbb{N}$, for any sequence of positive numbers $(x_1, x_2, \ldots, x_n)$, and any natural $p > 1$,*

$$\frac{1}{n}\sum_{i=1}^{n} x_i \leqslant \mathrm{HM}_{p,n}(x_1, x_2, \ldots, x_n),$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Proof. When $n = 1$, the statement is tautology. When $n = 2$, the statement is proven in Proposition 6. Lemma 8 implies the statement whenever n is a power of 2. Finally, Lemma 9 implies the statement for all remaining positive integers n. $\square$