```
clear all
sympref('FloatingPointOutput', false)
```

```
ans = logical
   0
```

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Problem 2

Part A

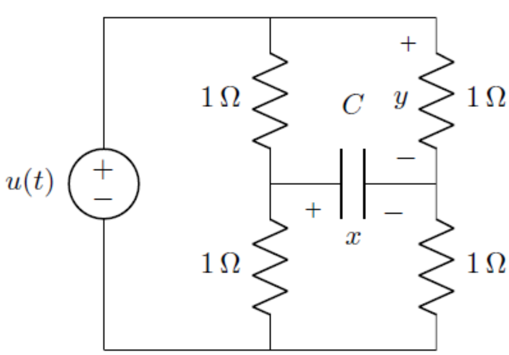


Figure 1: Capacitive bridge.

```
syms i_1 i_2 i_3 u x x_dot y C
%Loop 1 equation
l1eqn = i_1 + i_2 == u
```

l1eqn = $i_1 + i_2 = u$

```
%Node 1 equation
n1eqn = i_1 - i_2 == C*x_dot
```

n1eqn = $i_1 - i_2 = C\,\dot{x}$

```
%Loop 2 equation
l2eqn = -i_1 + y - x == 0
```

l2eqn = $-x + y - i_1 = 0$

```
%Node 2 equation
n2eqn = -y + i_3 == C*x_dot
```

n2eqn = $-y + i_3 = C\,\dot{x}$

```
%Loop 2 equation
l3eqn = -i_2 + x + i_3 == 0
```

l3eqn = $x - i_2 + i_3 = 0$

```
eqns = [l1eqn;n1eqn;l2eqn;n2eqn;l3eqn]
```

eqns =

$$\begin{bmatrix} i_1 + i_2 = u \\ i_1 - i_2 = C\,\dot{x} \\ -x + y - i_1 = 0 \\ -y + i_3 = C\,\dot{x} \\ x - i_2 + i_3 = 0 \end{bmatrix}$$

```
state_eqn = 0 == eliminate(eqns, [i_1 i_2 i_3 y]);
state_eqn = isolate(state_eqn, x_dot)
```

state_eqn =

$$\dot{x} = -\frac{x}{C}$$

```
output_eqn = 0 == eliminate(eqns, [i_1 i_2 i_3 x_dot])
```

output_eqn = $0 = x - 2\,y + u$

```
output_eqn = isolate(output_eqn, y)
```

output_eqn =

$$y = \frac{x}{2} + \frac{u}{2}$$

Part B

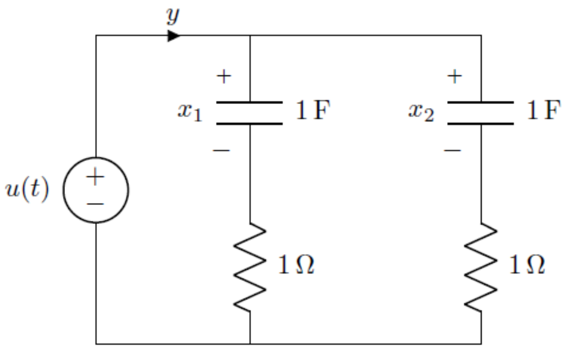


Figure 2: Parallel circuit.

```
syms x_1 x_2 x_dot_1 x_dot_2

%Loop 1 equation
l1eqn = x_1 + i_1 - u == 0
```

l1eqn = $x_1 - u + i_1 = 0$

```
%Loop 2 equation
l2eqn = x_2 + i_2 - u == 0
```

l2eqn = $x_2 - u + i_2 = 0$

```
%Node 1 equation
n1eqn = y == x_dot_1 + x_dot_2
```

n1eqn = $y = \dot{x}_1 + \dot{x}_2$

```
%Node 2 equation
n2eqn = i_1 == x_dot_1
```

n2eqn = $i_1 = \dot{x}_1$

```
%Node 3 equation
n3eqn = i_2 == x_dot_2
```

n3eqn = $i_2 = \dot{x}_2$

```
eqns = [l1eqn;l2eqn;n1eqn;n2eqn;n3eqn]
```

eqns =

$$\begin{bmatrix} x_1 - u + i_1 = 0 \\ x_2 - u + i_2 = 0 \\ y = \dot{x}_1 + \dot{x}_2 \\ i_1 = \dot{x}_1 \\ i_2 = \dot{x}_2 \end{bmatrix}$$

```
solved_state_eqns = eliminate(eqns, [i_1 i_2 y])
```

solved_state_eqns = $\left[-x_2 - \dot{x}_2 + u, x_1 - x_2 + \dot{x}_1 - \dot{x}_2\right]$

```
solved_state_eqns = [0;0] == [solved_state_eqns(1);solved_state_eqns(2)];
solved_state_eqns = [solved_state_eqns(1)-solved_state_eqns(2);solved_state_eqns(1)];
solved_state_eqns = [isolate(solved_state_eqns(1), x_dot_1);isolate(solved_state_eqns(2), x_dot
```

solved_state_eqns =

$$\begin{bmatrix} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + u \end{bmatrix}$$

```
solved_output_eqn = 0 == eliminate(eqns, [i_1 i_2 x_dot_1 x_dot_2]);
solved_output_eqn = isolate(solved_output_eqn, y)
```

solved_output_eqn = $y = -x_1 - x_2 + 2\,u$

Problem 3

Part A

State space model of spring mass system:

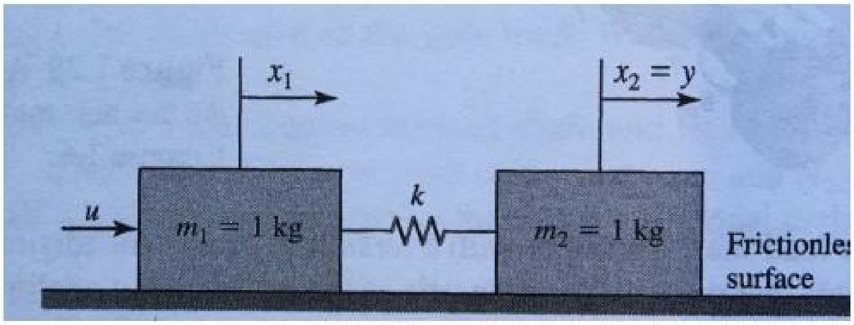


Figure 3: Two-mass-spring system. Assume a frictionless surface. Source: [Za] Fig. 1.31.

Here, the state space model is $\dot{x} = Ax + Bu$ where $x = [\ x_1\ \ x_2\ \ \dot{x}_1\ \ \dot{x}_2\ ]$

```
m1 = 1;
m2 = 1;
k = 1;

A = [0      0   1  0
```

```
         0      0   0  1
       -k/m1  k/m1  0  0
        k/m2 -k/m2  0  0];

B = [ 0
      0
     1/m1
      0 ];

C = [ 0 1 0 0];

D = 0;

sys = ss(A,B,C,D);
```

Controllability Gramian W_c

$$W_c(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$$

```
syms Tau t n
```

Symbolic Integrand for W_c

```
Wc_integrand = expm(A*Tau)*(B*B.')*expm(A.'*Tau);
```

Symbolically Integrate (too large to display)

```
Wc(t) = int(Wc_integrand,Tau, 0, t);
```

Evaluate W_c at the time values $t_1 = 10,\ t_2 = 20,\ and\ \ t_3 = 100$ and determine wether or not $W_c(t_1), W_c(t_2), W_c(t_3)$ are convergent.

Via a proof shown in the hand calculations, any real diagonalizable matrix is convergent if and only if all of its eigenvalues are less than 1. The controllability grammians of this system are all real and diagonalizable and are shown to each have at least one eigenvalue greater than 1.

```
t1 = 10;
t2 = 20;
t3 = 100;
Wc1 = double(Wc(t1))
```

```
Wc1 = 4×4
   84.1477   82.7081   14.3302   10.9210
   82.7081   83.7694   13.9540   10.7948
   14.3302   13.9540    4.1031    1.2504
   10.9210   10.7948    1.2504    3.3960
```

```
Wc2 = double(Wc(t2))
```

```
Wc2 = 4×4
  672.9142  665.4171   49.9649   50.5351
  665.4171  662.9182   49.4649   50.0351
   49.9649   49.4649    7.4974    2.4991
   50.5351   50.0351    2.4991    7.5044
```

```
Wc3 = double(Wc(t3))
```

```
Wc3 = 4×4
10^4 ×
    8.3365    8.3327    0.1249    0.1251
    8.3327    8.3315    0.1249    0.1251
    0.1249    0.1249    0.0037    0.0012
    0.1251    0.1251    0.0012    0.0038
```

t1 = 10 sec

```
% V is the matrix of eigenvectors and D is a diagonal matrix of eigenvalues
[V1, Wc_bar_1] = eig(Wc1);
%Make sure they are ordered from largest to smallest
[eigvals1,ind1] = sort(diag(Wc_bar_1), 'descend');
disp(["Reordered Eigenvalue of Wc1 = " + eigvals1 "Index = " + ind1]);
```

```
    "Reordered Eigenvalue of Wc1 = 170.5095"     "Index = 4"
    "Reordered Eigenvalue of Wc1 = 2.458196"     "Index = 3"
    "Reordered Eigenvalue of Wc1 = 1.456741"     "Index = 2"
    "Reordered Eigenvalue of Wc1 = 0.9917897"    "Index = 1"
```

```
Wc_bar_s1 = Wc_bar_1(ind1,ind1)
```

```
Wc_bar_s1 = 4×4
  170.5095         0         0         0
         0    2.4582         0         0
         0         0    1.4567         0
         0         0         0    0.9918
```

Since the matrix $\overline{W}_{c_1}$, which is in the eigenbasis of W_{c_1}, has at least one eigenvalue greater than 1,

$\lim_{n\to\infty} W_{c_1}{}^n \neq \tilde{0}$. I.e. W_{c_1} does not converge (see hand calculations for proof).

```
%Reordered Eigenvectors
Vs1 = V1(:,ind1)
```

```
Vs1 = 4×4
    0.7000    0.0540   -0.4444   -0.5565
    0.6981   -0.0605    0.5900    0.4011
    0.1195    0.6300   -0.4824    0.5967
    0.0917   -0.7723   -0.4708    0.4164
```

t2 = 20 sec

```
% V is the matrix of eigenvectors and D is a diagonal matrix of eigenvalues
[V2, Wc_bar_2] = eig(Wc2);
%Make sure they are ordered from largest to smallest
[eigvals2,ind2] = sort(diag(Wc_bar_2), 'descend');
disp(["Reordered Eigenvalue of Wc2 = " + eigvals2 "Index = " + ind2]);
```

```
    "Reordered Eigenvalue of Wc2 = 1340.8664"    "Index = 4"
    "Reordered Eigenvalue of Wc2 = 5.0021324"    "Index = 3"
    "Reordered Eigenvalue of Wc2 = 2.6068916"    "Index = 2"
    "Reordered Eigenvalue of Wc2 = 2.3587624"    "Index = 1"
```

```
Wc_bar_s2 = Wc_bar_2(ind2,ind2)
```

```
Wc_bar_s2 = 4×4
10^3 ×
    1.3409         0         0         0
         0    0.0050         0         0
         0         0    0.0026         0
         0         0         0    0.0024
```

Since the matrix $\bar{W}_{c_2}$, which is in the eigenbasis of W_{c_2}, has at least one eigenvalue greater than 1,

$\lim_{n\to\infty} W_{c_2}{}^{n} \neq \tilde{0}$. I.e. W_{c_2} does not converge (see hand calculations for proof).

```
%Reordered Eigenvectors
Vs2 = V2(:,ind2)
```

```
Vs2 = 4×4
    0.7078   -0.0006    0.4551    0.5403
    0.7025    0.0017   -0.5345   -0.4699
    0.0527   -0.7181    0.4948   -0.4866
    0.0533    0.6959    0.5122   -0.5005
```

t3 = 100 sec

```
% V is the matrix of eigenvectors and D is a diagonal matrix of eigenvalues
[V3, Wc_bar_3] = eig(Wc3);
%Make sure they are ordered from largest to smallest
[eigvals3,ind3] = sort(diag(Wc_bar_3), 'descend');
disp(["Reordered Eigenvalue of Wc3 = " + eigvals3 "Index = " + ind3]);
```

```
    "Reordered Eigenvalue of Wc3 = 166704.1732"    "Index = 4"
    "Reordered Eigenvalue of Wc3 = 25.00875972"    "Index = 3"
    "Reordered Eigenvalue of Wc3 = 12.61979239"    "Index = 2"
    "Reordered Eigenvalue of Wc3 = 12.36925083"    "Index = 1"
```

```
Wc_bar_s3 = Wc_bar_3(ind3,ind3)
```

```
Wc_bar_s3 = 4×4
10^5 ×
    1.6670         0         0         0
         0    0.0003         0         0
         0         0    0.0001         0
         0         0         0    0.0001
```

Since the matrix $\bar{W}_{c_3}$, which is in the eigenbasis of W_{c_3}, has at least one eigenvalue greater than 1,

$\lim_{n\to\infty} W_{c_3}{}^{n} \neq \tilde{0}$. I.e. W_{c_3} does not converge (see hand calculations for proof).

```
%Reordered Eigenvectors
Vs3 = V3(:,ind3)
```

```
Vs3 = 4×4
    0.7071    0.0000    0.4870   -0.5126
    0.7069   -0.0000   -0.5023    0.4979
    0.0106    0.7080    0.5045    0.4940
    0.0106   -0.7062    0.5059    0.4952
```

Which States Are Most Controllable?

The degree of controllability of any state in this system is determined, roughly, by how much of the eigenvector associated with the dominant eigenvalue lies in the direction of that state. Since for each of these controllibility Gramian matrices, there is one eigenvalue that is orders of magnitude larger than the others, this is a good criteria.

For $t_1 = 10s$, the dominant eigenvector is:

```
Vs1(:,1)
```

```
ans = 4×1
    0.7000
    0.6981
    0.1195
    0.0917
```

Which has the associated eigenvalue:

```
disp(eigvals1(1))
```

```
  170.5095
```

The eigenvalue points largely in the directions of x_1 and x_2, but is slightly more in x_1 than it is in x_2. These states are the most controllable.

For $t_2 = 20s$, the dominant eigenvector is:

```
Vs2(:,1)
```

```
ans = 4×1
    0.7078
    0.7025
    0.0527
    0.0533
```

Which has the associated eigenvalue:

```
disp(eigvals2(1))
```

```
   1.3409e+03
```

The eigenvalue points largely in the directions of x_1 and x_2, but is slightly more in x_1 than it is in x_2. These states are the most controllable.

For $t_3 = 100s$, the dominant eigenvector is:

```
Vs3(:,1)
```

```
ans = 4×1
    0.7071
    0.7069
    0.0106
    0.0106
```

Which has the associated eigenvalue:

```
disp(eigvals3(1))
```

```
   1.6670e+05
```

The eigenvalue points largely in the directions of x_1 and x_2, but is slightly more in x_1 than it is in x_2. These states are the most controllable.

The clear trend is that the position states tend to be much more controllable than the velocity states and x_1, where the input most direclty affects the mass, is slightly more controllable than x_2.

Observability Gramian W_o

$$W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

Symbolic Integrand for W_o

```
Wo_integrand = expm(A'*Tau)*(C'*C)*expm(A*Tau);
```

Symbolically Integrate (too large to display)

```
Wo(t) = int(Wo_integrand,Tau, 0, t);
```

Evaluate W_o at the time values $t_1 = 10,\ t_2 = 20,\ and\ \ t_3 = 100$ and determine wether or not $W_o(t_1), W_o(t_2), W_o(t_3)$ are convergent.

Via a proof shown in the hand calculations, any real diagonalizable matrix is convergent if and only if all of its eigenvalues are less than 1. The observability grammians of this system are all real and diagonalizable and are shown to each have at least one eigenvalue greater than 1.

```
Wo1 = double(Wo(t1))
```

```
Wo1 = 4×4
    3.3960    1.2504   10.7948   10.9210
```

```
     1.2504    4.1031   13.9540   14.3302
    10.7948   13.9540   83.7694   82.7081
    10.9210   14.3302   82.7081   84.1477
```

```
Wo2 = double(Wo(t2))
```

```
Wo2 = 4×4
     7.5044    2.4991   50.0351   50.5351
     2.4991    7.4974   49.4649   49.9649
    50.0351   49.4649  662.9182  665.4171
    50.5351   49.9649  665.4171  672.9142
```

```
Wo3 = double(Wo(t3))
```

```
Wo3 = 4×4
10^4 ×
    0.0038    0.0012    0.1251    0.1251
    0.0012    0.0037    0.1249    0.1249
    0.1251    0.1249    8.3315    8.3327
    0.1251    0.1249    8.3327    8.3365
```

t1 = 10 sec

```
% V is the matrix of eigenvectors and D is a diagonal matrix of eigenvalues
[V1, Wo_bar_1] = eig(Wo1);
%Make sure they are ordered from largest to smallest
[eigvals1,ind1] = sort(diag(Wo_bar_1), 'descend');
disp(["Reordered Eigenvalue of Wo1 = " + eigvals1 "Index = " + ind1]);
```

```
    "Reordered Eigenvalue of Wo1 = 170.5095"     "Index = 4"
    "Reordered Eigenvalue of Wo1 = 2.458196"     "Index = 3"
    "Reordered Eigenvalue of Wo1 = 1.456741"     "Index = 2"
    "Reordered Eigenvalue of Wo1 = 0.9917897"    "Index = 1"
```

```
Wo_bar_s1 = Wo_bar_1(ind1,ind1)
```

```
Wo_bar_s1 = 4×4
  170.5095         0         0         0
         0    2.4582         0         0
         0         0    1.4567         0
         0         0         0    0.9918
```

Since the matrix $\bar{W}_{o_1}$, which is in the eigenbasis of W_{o_1}, has at least one eigenvalue greater than 1,

$\lim_{n\to\infty} W_{o_1}{}^{n} \neq \tilde{0}$. I.e. W_{o_1} does not converge (see hand calculations for proof).

```
%Reordered Eigenvectors
Vs1 = V1(:,ind1)
```

```
Vs1 = 4×4
    0.0917    0.7723   -0.4708    0.4164
    0.1195   -0.6300   -0.4824    0.5967
    0.6981    0.0605    0.5900    0.4011
    0.7000   -0.0540   -0.4444   -0.5565
```

t2 = 20 sec

```
% V is the matrix of eigenvectors and D is a diagonal matrix of eigenvalues
```

```
[V2, Wo_bar_2] = eig(Wo2);
%Make sure they are ordered from largest to smallest
[eigvals2,ind2] = sort(diag(Wo_bar_2), 'descend');
disp(["Reordered Eigenvalue of Wo2 = " + eigvals2 "Index = " + ind2]);
```

```
"Reordered Eigenvalue of Wo2 = 1340.8664"    "Index = 4"
"Reordered Eigenvalue of Wo2 = 5.0021324"    "Index = 3"
"Reordered Eigenvalue of Wo2 = 2.6068916"    "Index = 2"
"Reordered Eigenvalue of Wo2 = 2.3587624"    "Index = 1"
```

```
Wo_bar_s2 = Wo_bar_2(ind2,ind2)
```

```
Wo_bar_s2 = 4×4
10^3 ×
    1.3409         0         0         0
         0    0.0050         0         0
         0         0    0.0026         0
         0         0         0    0.0024
```

Since the matrix $\bar{W}_{o_2}$, which is in the eigenbasis of W_{o_2}, has at least one eigenvalue greater than 1,

$\lim_{n\to\infty} W_{o_2}{}^{n} \neq \tilde{0}$. I.e. W_{o_2} does not converge (see hand calculations for proof).

```
%Reordered Eigenvectors
Vs2 = V2(:,ind2)
```

```
Vs2 = 4×4
    0.0533    0.6959   -0.5122    0.5005
    0.0527   -0.7181   -0.4948    0.4866
    0.7025    0.0017    0.5345    0.4699
    0.7078   -0.0006   -0.4551   -0.5403
```

t3 = 100 sec

```
% V is the matrix of eigenvectors and D is a diagonal matrix of eigenvalues
[V3, Wo_bar_3] = eig(Wo3);
%Make sure they are ordered from largest to smallest
[eigvals3,ind3] = sort(diag(Wo_bar_3), 'descend');
disp(["Reordered Eigenvalue of Wo3 = " + eigvals3 "Index = " + ind3]);
```

```
"Reordered Eigenvalue of Wo3 = 166704.1732"    "Index = 4"
"Reordered Eigenvalue of Wo3 = 25.00875972"    "Index = 3"
"Reordered Eigenvalue of Wo3 = 12.61979239"    "Index = 2"
"Reordered Eigenvalue of Wo3 = 12.36925083"    "Index = 1"
```

```
Wo_bar_s3 = Wo_bar_3(ind3,ind3)
```

```
Wo_bar_s3 = 4×4
10^5 ×
    1.6670         0         0         0
         0    0.0003         0         0
         0         0    0.0001         0
         0         0         0    0.0001
```

Since the matrix $\overline{W}_{o_3}$, which is in the eigenbasis of W_{o_3}, has at least one eigenvalue greater than 1,

$\lim_{n\to\infty} {W_{o_3}}^n \neq \widetilde{0}$. I.e. W_{o_3} does not converge (see hand calculations for proof).

```
%Reordered Eigenvectors
Vs3 = V3(:,ind3)
```

```
Vs3 = 4×4
    0.0106    0.7062   -0.5059   -0.4952
    0.0106   -0.7080   -0.5045   -0.4940
    0.7069    0.0000    0.5023   -0.4979
    0.7071   -0.0000   -0.4870    0.5126
```

Which States Are Most Observable?

The degree of observability of any state in this system is determined, roughly, by how much of the eigenvector associated with the dominant eigenvalue lies in the direction of that state. Since for each of these observability Gramian matrices, there is one eigenvalue that is orders of magnitude larger than the others, this is a good criteria.

For $t_1 = 10s$, the dominant eigenvector is:

```
Vs1(:,1)
```

```
ans = 4×1
    0.0917
    0.1195
    0.6981
    0.7000
```

Which has the associated eigenvalue:

```
disp(eigvals1(1))
```

```
  170.5095
```

The eigenvalue points largely in the directions of $\dot{x}_1$ and $\dot{x}_2$, but is slightly more in $\dot{x}_2$ than it is in $\dot{x}_1$. These states are the most observable.

For $t_2 = 20s$, the dominant eigenvector is:

```
Vs2(:,1)
```

```
ans = 4×1
    0.0533
    0.0527
    0.7025
    0.7078
```

Which has the associated eigenvalue:

```
disp(eigvals2(1))
```

```
1.3409e+03
```

The eigenvalue points largely in the directions of $\dot{x}_1$ and $\dot{x}_2$, but is slightly more in $\dot{x}_2$ than it is in $\dot{x}_1$. These states are the most observable.

For $t_3 = 100s$, the dominant eigenvector is:

```
Vs3(:,1)
```

```
ans = 4×1
    0.0106
    0.0106
    0.7069
    0.7071
```

Which has the associated eigenvalue:

```
disp(eigvals3(1))
```

```
1.6670e+05
```

The eigenvalue points largely in the directions of $\dot{x}_1$ and $\dot{x}_2$, but is slightly more in $\dot{x}_2$ than it is in $\dot{x}_1$. These states are the most observable.

The clear trend is that the velocity states tend to be much more observable than the position states and $\dot{x}_2$, is slightly more observable than $\dot{x}_1$.

Part B

Three Linear Quadratic Regulators

Linear quadratic regulators are used to choose stabilizing k matrices for full state feedback. Bryson's rule for various values of the maximum acceptable state and input are utilized to get three different K matrices.

```
%Bryson's rule for xmax = 3 meters, x_dot_max = 10 m/s, and Umax = 2 newtons - because why
%not?
Q1 = [1/3 0 0 0; 0 1/3 0 0; 0 0 1/10 0; 0 0 0 1/10];
R1 = 1/2;

[K1,S1,e1] = lqr(A,B,Q1,R1);
K1
```

```
K1 = 1×4
    1.1488    0.0059    1.5804    0.9373
```

```
%Bryson's rule for xmax = 1 meters, x_dot_max = 15 m/s, and Umax = 20 newtons
Q2 = [1/1 0 0 0; 0 1/1 0 0; 0 0 1/15 0; 0 0 0 1/15];
R2 = 1/20;

[K2,S2,e2] = lqr(A,B,Q2,R2);
K2
```

```
K2 = 1×4
    6.4807   -0.1562    3.7809    4.5822
```

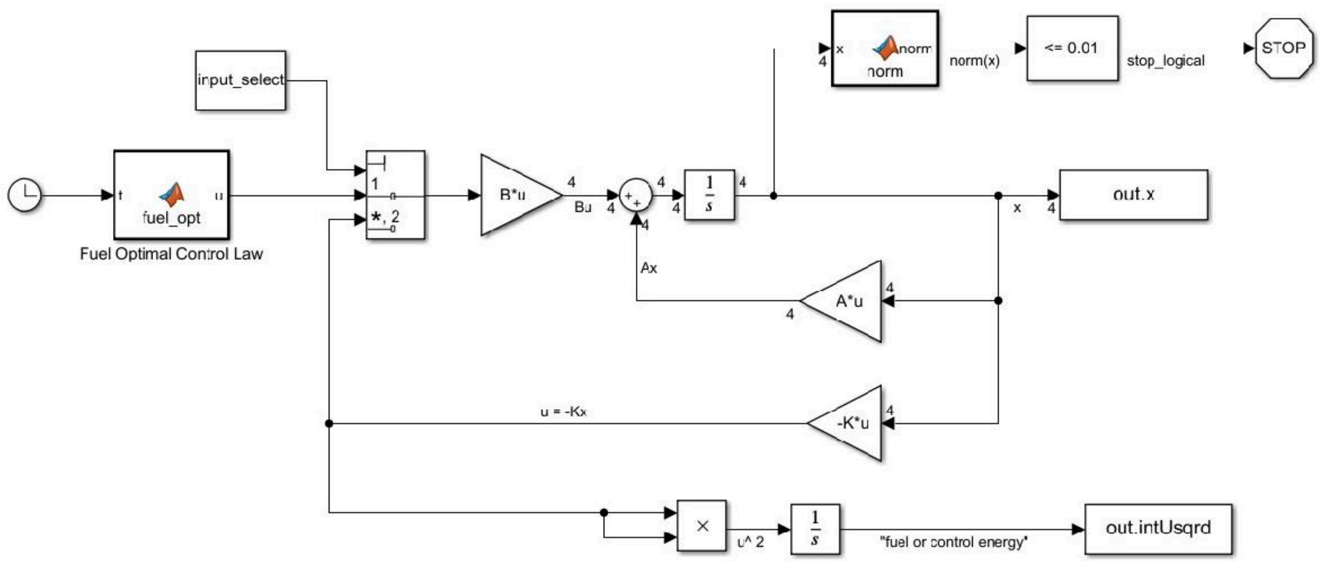
```
%Bryson's rule for xmax = .05 meters, x_dot_max =2 m/s, and Umax = 3 newtons
Q3 = [1/.05 0 0 0; 0 1/.05 0 0; 0 0 1/2 0; 0 0 0 1/2];
R3 = 1/3;

[K3,S3,e3] = lqr(A,B,Q3,R3);
K3
```

```
K3 = 1×4
   10.6433    0.3111    4.7735    7.8756
```

Simulate the Full State Feedback Controllers from $\widetilde{x}_0 = [1\ \ 0\ \ 2\ \ 0]$

The full state feedback state space model was implemented in Simulink to allow for easy calculation of integrated controller fuel and determination of steady state from the euclidean norm of the state vector. The final simulation time is the time when the state vector enters a unit hypersphere of radius 0.01.

$\dot{x} = (A - BK)x$



Run 'initial' simulations and compare the energy cost and time to steady state

```
x0 = [1 0 2 0]'; %Initial state vector
tstop = 25;
input_select = 2; %Full state feedback input NOT fuel optimal control input
t_end = 3; %Not used now but need to initialize the variable so it's not undefined
Wc_sim = double(Wc(5)); %Grammian for 5 seconds
```

```
```

LQR 1

```
K = K1;
out1 = sim('FSF');
x1 = out1.x;
tsim1 = out1.tout;
energy1 = out1.intUsqrd(end);
ss_time1 = tsim1(end);
```

LQR 2

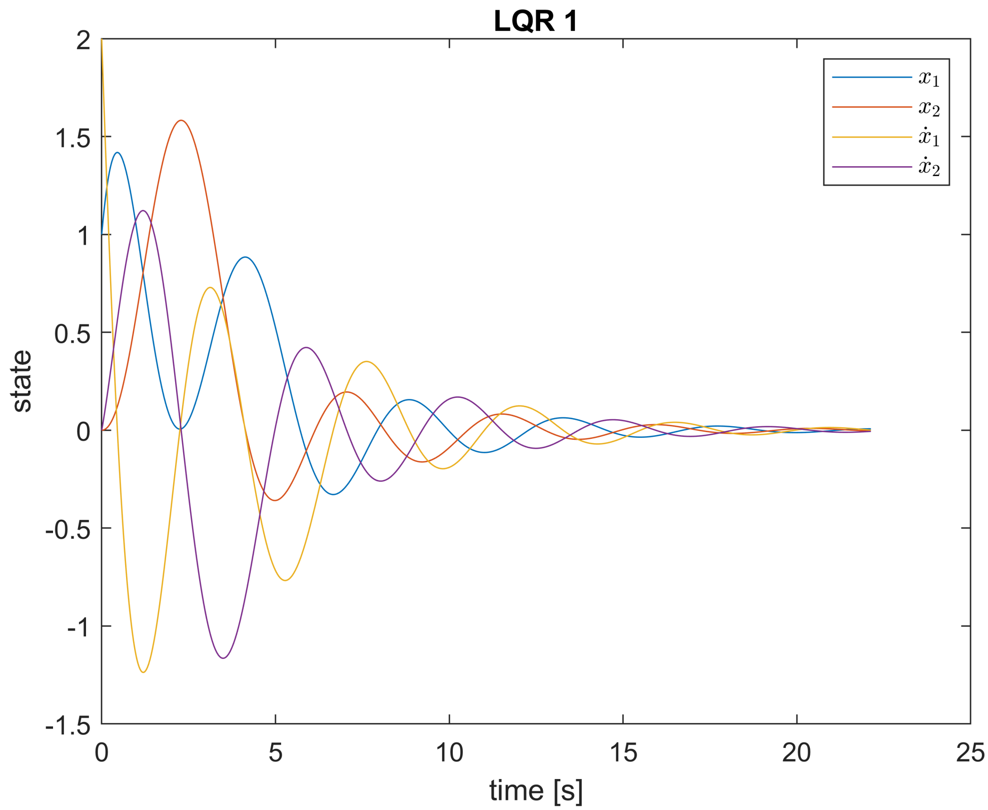
```
K = K2;
out2 = sim('FSF');
x2 = out2.x;
tsim2 = out2.tout;
energy2 = out2.intUsqrd(end);
ss_time2 = tsim2(end);
```

LQR3

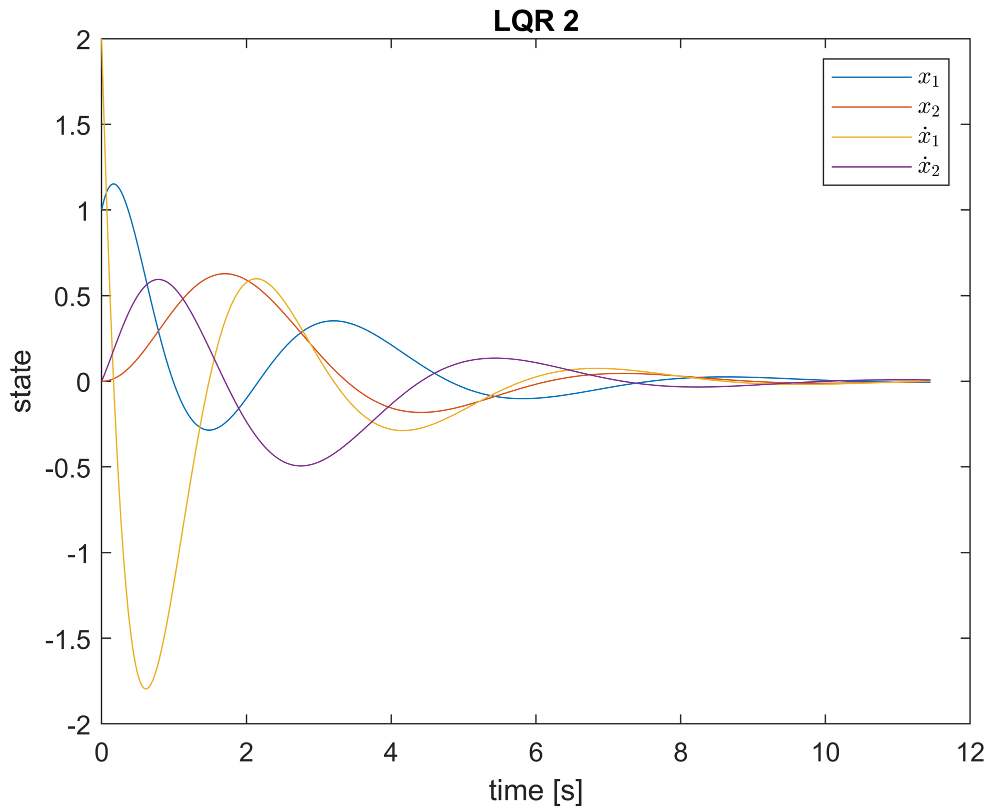
```
K = K3;
out3 = sim('FSF');
x3 = out3.x;
tsim3 = out3.tout;
energy3 = out3.intUsqrd(end);
ss_time3 = tsim3(end);
```

Plot results

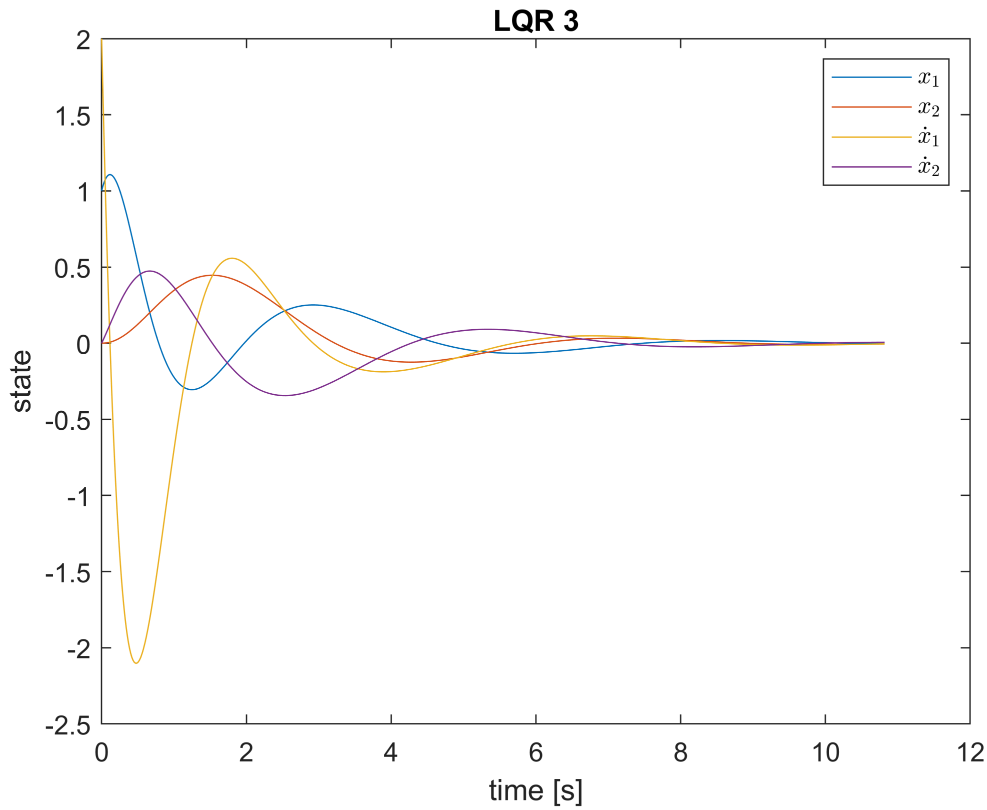
```
figure
plot(tsim1,x1)
legend('$x_1$','$x_2$', '$\dot{x}_1$','$\dot{x}_2$','interpreter','latex')
xlabel('time [s]')
ylabel('state')
title('LQR 1')
```




```
figure
plot(tsim2,x2)
legend('$x_1$','$x_2$', '$\dot{x}_1$','$\dot{x}_2$','interpreter','latex')
xlabel('time [s]')
ylabel('state')
title('LQR 2')
```




```
figure
plot(tsim3,x3)
legend('$x_1$','$x_2$', '$\dot{x}_1$','$\dot{x}_2$','interpreter','latex')
xlabel('time [s]')
ylabel('state')
title('LQR 3')
```



Discussion

Controller 1 utilized the least energy but took the most amount of time. Controller 3 utilized the most energy but took the least amount of time. Controller 2 landed between those two scenarios.

```
disp("Energy cost controller 1 was " + num2str(energy1,2) + ".")
```

```
Energy cost controller 1 was 6.9.
```

```
disp("Time to steady state controller 1 was " + num2str(ss_time1,3) + " seconds.")
```

```
Time to steady state controller 1 was 22.1 seconds.
```

```
disp("Energy cost controller 2 was " + num2str(energy2,3) + ".")
```

```
Energy cost controller 2 was 31.2.
```

```
disp("Time to steady state controller 2 was " + num2str(ss_time2,3) + " seconds.")
```

```
Time to steady state controller 2 was 11.5 seconds.
```

```
disp("Energy cost controller 3 was " + num2str(energy3,3) + ".")
```

```
Energy cost controller 3 was 51.1.
```

```
disp("Time to steady state controller 3 was " + num2str(ss_time3,3) + " seconds.")
```

```
Time to steady state controller 3 was 10.8 seconds.
```

Part C

Control law is now the open loop law

$$u(t) = -B^T e^{A^T(t_1 - t)} W_c^{-1}(t_1)[e^{At}x_0 - x_1]$$

Instead of the closed loop law

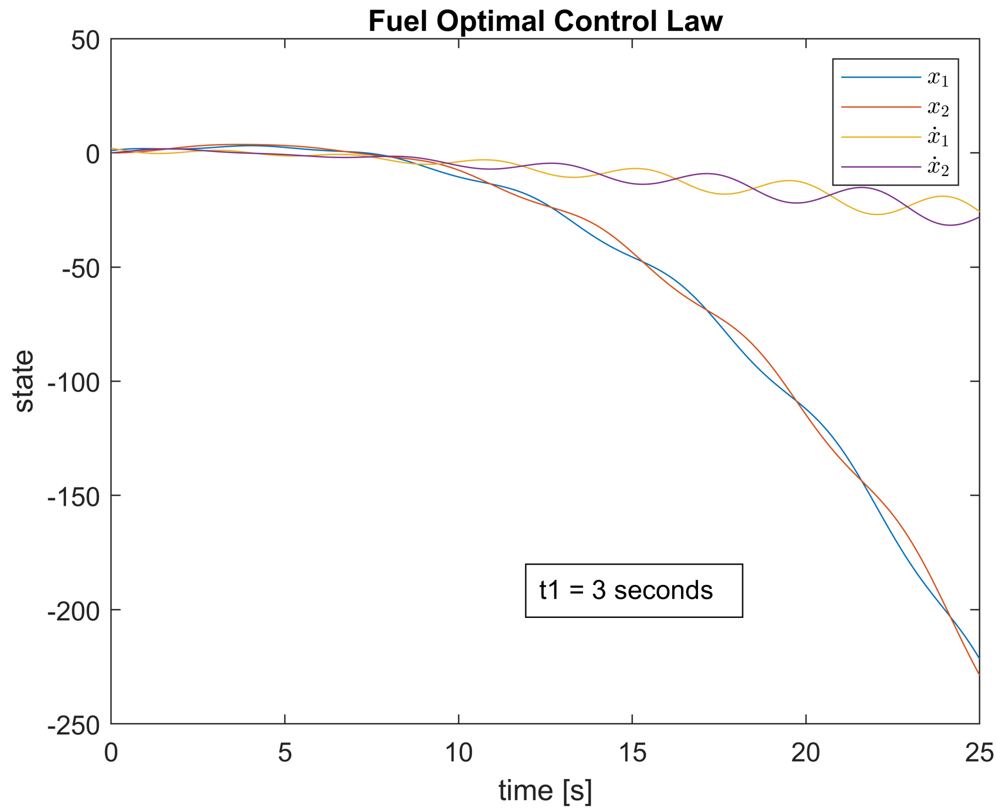
$$u(t) \quad = \quad - Kx(t)$$

Simulate Fuel Optimal Control Law

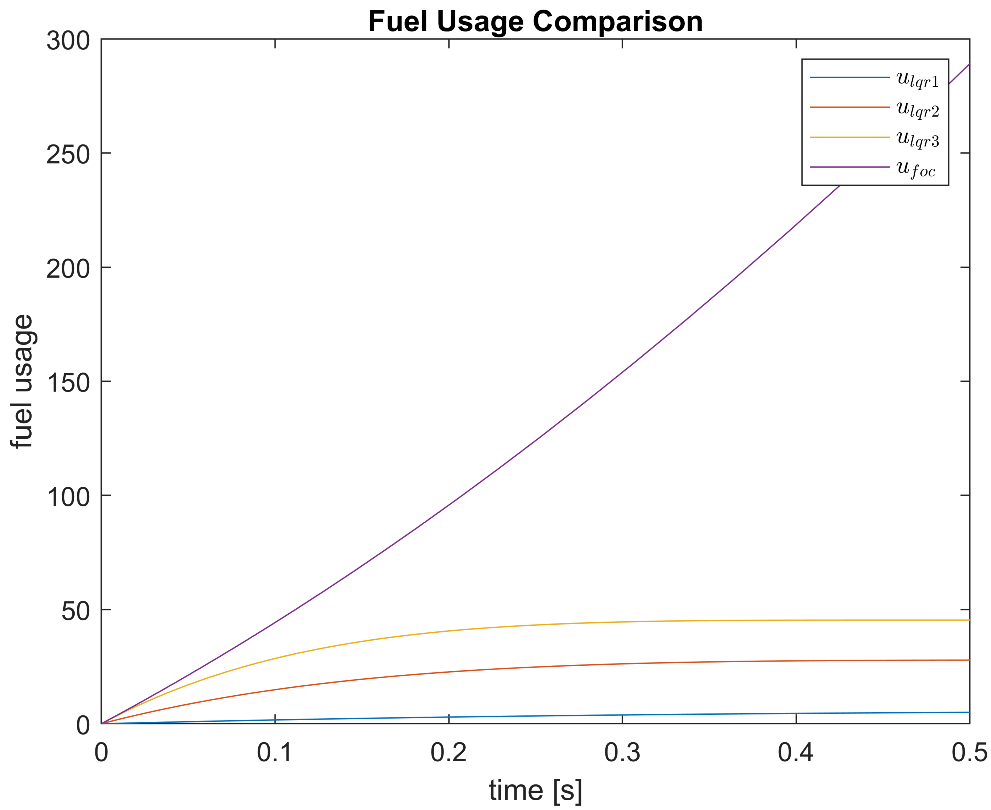
```
x0 = [1 0 2 0]'; %Initial state vector
tstop = 25;
input_select = 1; %Fuel optimal control law
%t1 from the control law (named differently because "t1" is already used  elsewhere)
t_end = 3; %Not used now but need to initialize the variable so it's not undefined
Wc_sim = double(Wc(5)); %Grammian for 5 seconds

out4 = sim('FSF');
x4 = out4.x;
tsim4 = out4.tout;
energy4 = out4.intUsqrd(end);
ss_time4 = tsim4(end);
```

Plot Results

```
figure
plot(tsim4,x4)
legend('$x_1$','$x_2$', '$\dot{x}_1$','$\dot{x}_2$','interpreter','latex')
xlabel('time [s]')
ylabel('state')
title('Fuel Optimal Control Law')
annotation('textbox', [0.5, 0.2, 0.1, 0.1],'string', ['t1 = ' num2str(t_end) ' seconds'])
```




```
figure
plot(tsim1,out1.intUsqrd,tsim2,out2.intUsqrd,tsim3,out3.intUsqrd,tsim4,out4.intUsqrd)
legend('$u_{lqr1}$','$u_{lqr2}$', '$u_{lqr3}$','$u_{foc}$','interpreter','latex')
xlabel('time [s]')
ylabel('fuel usage')
title('Fuel Usage Comparison')
xlim([0 .5])
```



Discussion

Since the system is not open loop stable, the fuel optimal control law u(t) does not take the state to zero let alone within time t_1 in an energy minimizing manner - the system is actually unstable. The fuel consumption is plotted over a very small time interval to see where the unstable controller's fuel consumption diverges from the three stable controller's fuel consumption.