

Random variate generation for the generalized inverse Gaussian distribution

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Abstract We provide a uniformly efficient and simple random variate generator for the entire parameter range of the generalized inverse Gaussian distribution. A general algorithm is provided as well that works for all densities that are proportional to a log-concave function φ , even if the normalization constant is not known. It requires only black box access to φ and its derivative.

Keywords Random variate generation · Simulation · Monte Carlo method · Expected time analysis

1 Introduction

The two-parameter form of the generalized inverse Gaussian distribution (or GIG) has density proportional to

$$f(x) = x^{\lambda-1} \exp\left(-\frac{\omega}{2}\left(x + \frac{1}{x}\right)\right), \quad x > 0.$$

There are two parameters, $\lambda \in \mathbb{R}$, and $\omega > 0$. It seems to have been introduced for the first time by a French statistician, Etienne Halphen, in Halphen (1941) (see Seshadri 1997, and Morlat 1956). Barndorff-Nielsen and Halgreen (1977) coined the name generalized inverse Gaussian distribution. It is also known as the Sichel distribution, named after Herbert Sichel, who studied its properties. Its statistical properties are discussed by Jørgensen (1982) and Eberlein and von Hammerstein (2002).

There is a three-parameter version that is equivalent up to a change of scale. The density can be written explicitly as

$$\frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} \exp\left(-\frac{ax + b/x}{2}\right), \quad x > 0,$$

where $a, b > 0$ and $p \in \mathbb{R}$ are the parameters, and K_p is a modified Bessel function of the second kind. The case $b = 0$ is excluded, since then we must have $a > 0$ and $p > 0$ which corresponds to a standard gamma law. The case $a = 0$ is also excluded because then we must have $p < 0$ and $b > 0$, and this reduces to the inverse of a gamma distribution. We will only consider the two-parameter version, because if X follows the three-parameter law with parameters (p, a, b) , then $\sqrt{a/b}X$ follows the two parameter law with $\lambda = p$ and $\omega = 2\sqrt{b/a}$.

The inverse Gaussian (Folks and Chhikara 1978) and gamma distributions are special cases of the generalized inverse Gaussian distribution for $p = -1/2$ and $b = 0$, respectively. The moment generating function and all moments are expressed in function of the Bessel functions of the second kind.

For the inverse Gaussian, there is a clever trick by Michael et al. (1976) which replaces $x + 1/x$ by y , and makes use of the fact that the transformed density can be transformed back if one is careful. That idea does not quite work for GIG variate generation, but it is almost possible to do so. Indeed, there is a simple uniformly efficient generator. Recently, other competing approaches have emerged for the inverse Gaussian law—see, e.g., Lesosky and Horrocks (2003).

2 A transformed GIG distribution

The following “trick” helps in a large number of examples. If X is GIG or modified GIG with parameters (λ, ω) , then

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$Y = \log X$ has a log-concave density proportional to

$$\varphi(y) \stackrel{\text{def}}{=} \exp\left(\lambda y - \omega \frac{e^y + e^{-y}}{2}\right) = \exp(\lambda y - \omega \cosh y),$$

$$y \in \mathbb{R}.$$

From this, we rediscover that if X is a GIG (λ, ω) random variable then $1/X$ is a GIG $(-\lambda, \omega)$ random variable. Hence, one only needs to consider $\lambda \geq 0$.

This density has a unique mode at the solution m of

$$\sinh m = \frac{\lambda}{\omega},$$

which can be written explicitly as a solution of a quadratic equation,

$$m = \log\left(\frac{\lambda}{\omega} + \sqrt{1 + \frac{\lambda^2}{\omega^2}}\right).$$

It is convenient to normalize. We define

$$\psi(x) = \log\left(\frac{\varphi(m+x)}{\varphi(m)}\right), \quad x \in \mathbb{R}.$$

Inversely,

$$\varphi(m+x) = \varphi(m)e^{\psi(x)}, \quad x \in \mathbb{R}.$$

We note that $\psi(0) = 0$, $\psi(x) \leq 0$, and ψ is concave. More explicitly,

$$\begin{aligned} \psi(x) &= \lambda x - \omega(\cosh(m+x) - \cosh m) \\ &= \lambda x - \omega(\cosh m \cosh x + \sinh m \sinh x - \cosh m) \\ &= -\lambda(\sinh x - x) - \sqrt{\omega^2 + \lambda^2}(\cosh x - 1), \end{aligned}$$

and

$$\psi'(x) = -\lambda(\cosh x - 1) - \sqrt{\omega^2 + \lambda^2} \sinh x.$$

In summary, if we can generate Y with density proportional to e^ψ [and such Y will be called a transformed GIG random variable in this paper], then

$$\exp(m+Y) = \left(\frac{\lambda}{\omega} + \sqrt{1 + \frac{\lambda^2}{\omega^2}}\right)e^Y$$

has the GIG distribution with parameter (λ, ω) when $\lambda \geq 0$.

3 History

Rather than doing an individual job on this particular family of densities, we recall that all log-concave densities for which the density is available in black box format, and for which the location of the mode (or a mode) is known, one can generate random variates by the rejection method thanks to a universal inequality given in Devroye (1984). That method breaks down when one only knows the density up to a normalization constant. In the cases of a GIG distri-

bution, we know that

$$\frac{1}{2K_\lambda(\omega)}\varphi(y), \quad y \in \mathbb{R},$$

is a bona fide density. Given that one has access to the Bessel function of the second kind in a computer library, this could be used, and the discussion would end here.

However, this is not a wise decision if one does not know the error tolerance of the program that computes the Bessel function. Others have attempted to deal with the GIG distribution directly, such as Atkinson (1979, 1982) and Dagpunar (1989). Knowledge of the normalization constant was not necessary. Adaptive rejection sampling (Gilks 1992; Gilks and Wild 1992, 1993) or its modifications (Leydold and Hörmann 2000, 2001, or Hörmann et al. 2004) can lead to fast algorithms, but a generally applicable complexity analysis is often lacking. The most recent method of Botts et al. (2011) mentions a generator based on transformed density rejection, which seems to be the industry standard today. It is called Tinflex. The fundamental question remains whether expected complexity of the algorithm is uniformly bounded over all choices of the parameter.

Even as recently as 2009, approximations of the inverse of the GIG distribution function were advocated (Lai 2009). Others point out a myriad of ways of obtaining the GIG distribution as a limit of a stochastic process (see, e.g. Eberlein and von Hammerstein 2002). In this respect, it is useful that the GIG law is infinitely divisible. Generating random variates as a limit of a process is a time-honored way of doing things, but it is precisely this that we wish to avoid. A survey of the software available for the GIG distribution was published by Breymann and Lüthi (2011).

We will work out, by way of example, how one can approach the design in general in the absence of a normalization constant, and give useful theorems for further applications. Some ideas were already implicitly mentioned in Sect. 7.2.6 of Devroye (1986), and in particular in Theorem 2.6 (page 299) and the algorithm on page 301. The coverage in the present paper is tighter, incorporates the modern transformation ideas, and has a considerably improved analysis. In particular, when applied to the (transformed) GIG distribution, the algorithm is particularly solid and has uniformly bounded expected complexity over the entire parameter range.

4 The design of a rejection method for a log-concave density

Given a log-concave function that is proportional to a density, normalize it as in the previous section by centering at the mode m and making the value there 1. This yields a log-

concave density with mode at 0 that is proportional to the function $\exp(\psi(x))$, $x \in \mathbb{R}$, with $\psi(0) = 0$. After generating a random variate X with the latter density, it suffices to return $m + X$. It is assumed that we have ψ and ψ' available in black box format. We assume also that ψ is continuous.

Let $\rho > 0$ be a design parameter. We will see why a value in the range $[1/2, 2]$ is good. Let $s, t > 0$ be chosen in such a way that

$$\psi(-s) = \psi(t) = -\rho.$$

Since ψ is continuous and concave and $\exp(\psi)$ is integrable, such s and t exist and are unique. By concavity of ψ , the following bound applies in general:

$$e^{\psi(x)} \leq \begin{cases} 1 & \\ e^{\psi(t)+(x-t)\psi'(t)} & \\ e^{\psi(-s)+(x+s)\psi'(-s)} & \end{cases}.$$

We apply the right exponential tail inequality on $[t', \infty)$, and the left exponential tail inequality on $(-\infty, -s']$, where

$$\psi(t) + (t' - t)\psi'(t) = 0, \quad \psi(-s) + (-s' + s)\psi'(-s) = 0,$$

i.e.,

$$t' = t - \frac{\psi(t)}{\psi'(t)}, \quad s' = s + \frac{\psi(-s)}{\psi'(-s)}.$$

We summarize:

$$e^{\psi(x)} \leq \chi(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } -s' \leq x \leq t', \\ e^{\psi(t)+(x-t)\psi'(t)} & \text{if } x > t', \\ e^{\psi(-s)+(x+s)\psi'(-s)} & \text{if } x < -s'. \end{cases}$$

This design method has been known and applied in numerous papers, such as in the references mentioned earlier. See Figs. 1, 2, and 3 for examples.

We first recall the rejection method with two exponential caps (see Devroye 1986 or Hörmann et al. 2004). To apply it, we need to compute $\int \chi$ over the three intervals of interest:

$$p \stackrel{\text{def}}{=} \int_{-\infty}^{-s'} \chi(u) du = \frac{1}{\psi'(-s)},$$

$$q \stackrel{\text{def}}{=} \int_{-s'}^{t'} \chi(u) du = t' + s',$$

$$r \stackrel{\text{def}}{=} \int_{t'}^{\infty} \chi(u) du = \frac{1}{-\psi'(t')}.$$

[Algorithm for log-concave densities of the form $\exp(\psi)$]

Input: functions ψ , ψ' must be available.

[Set-up]

Choose $\rho > 0$.

Let $s, t > 0$ be such that $\psi(-s) = \psi(t) = -\rho$.

Compute $(\eta, \zeta, \theta, \xi) = (-\psi(t), -\psi'(t), -\psi(-s), \psi'(-s))$

Set $(p, r) = (1/\xi, 1/\zeta)$

Compute $t' = t - r\eta$

Compute $s' = s - p\theta$

Compute $q = t' + s'$.

[Generation]

Repeat Generate U, V, W uniformly on $[0, 1]$.

If $U < q/(p+q+r)$ then set $X = -s' + qV$

else if $U < (q+r)/(p+q+r)$

then set $X = t' + r \log(1/V)$

else set $X = -s' - p \log(1/V)$

Until $W\chi(X) \leq \exp(\psi(X))$, where

$$\chi(x) = \mathbb{1}_{[-s', t']}(x) + \mathbb{1}_{(t', \infty)}(x)e^{-\eta - \zeta(x-t)}$$

$$+ \mathbb{1}_{(-\infty, -s')}(x)e^{-\theta + \xi(x+s)}$$

Return X (X has density proportional to e^{ψ} .)

We prove the uniform performance bound in Theorem 1 below. Note that the special case $\rho = 1$ has already been dealt with in the literature. See, e.g., Hörmann et al. (2004), or Hörmann (1994, 1995). For every log-concave density, the rejection method requires not more than about 1.58 iterations on average. As with the method of Devroye (1984), where a bound of 4 was obtained, we assume that the mode m of the log-concave density is known and that the density can be computed in black box format. The requirement that the normalization constant of the density is known is now dropped, but on the other hand, it is assumed that the derivative of the density is also available in black box format. It is the latter condition that permits us to improve the uniform rejection constant.

Theorem 1 *The expected number of iterations for the algorithm above, designed with parameter $\rho > 0$ is bounded from above by*

$$\frac{\max(1, \rho)}{1 - e^{-\rho}}.$$

The minimal value of this bound, attained at $\rho = 1$, is

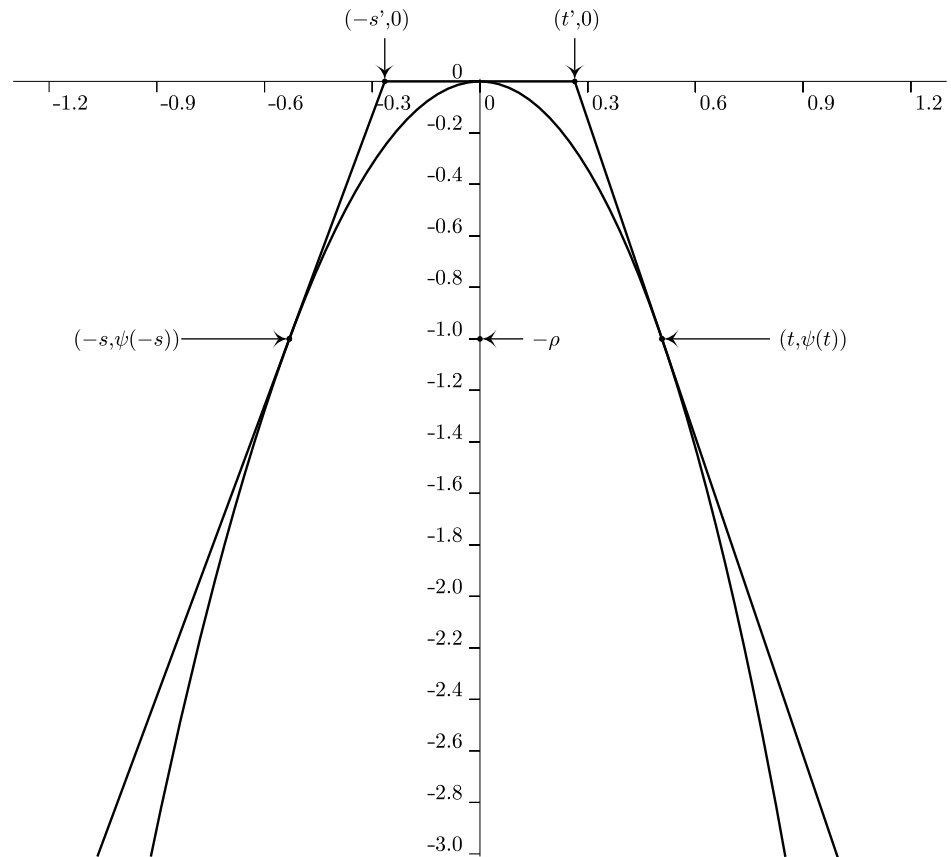
$$\frac{e}{e-1} = 1.5819767068693264 \dots$$

Proof Since ψ is concave and $\psi(0) = 0$, we have

$$\psi(x) \geq \frac{x}{t} \psi(t), \quad 0 \leq x \leq t \quad \text{and}$$

$$\psi(x) \geq \frac{x}{-s} \psi(-s), \quad -s \leq x \leq 0,$$

Fig. 1 The function ψ and its three-piece hat function are shown. Note that the concave function ψ has a peak at 0 and is normalized to have $\psi(0) = 0$. The example is the transformed GIG density with $(\lambda, \omega) = (2, 7)$



and

$$\psi'(t) \leq \frac{\psi(t)}{t} = -\frac{\rho}{t} \quad \text{and} \quad \psi'(-s) \geq \frac{\psi(-s)}{s} = \frac{\rho}{s}.$$

Thus,

$$\begin{aligned} \int_0^t e^{\psi(x)} dx &\geq \int_0^t e^{x\psi(t)/t} dx = \frac{t}{-\psi(t)} \times (1 - e^{\psi(t)}) \\ &= \frac{t}{\rho} \times (1 - e^{-\rho}). \end{aligned}$$

A similar set of bounds applies to the left of the origin:

$$\int_{-s}^0 e^{\psi(x)} dx \geq \frac{s}{\rho} \times (1 - e^{-\rho}).$$

We also have

$$\begin{aligned} \int \chi(x) dx &= p + q + r \\ &= \frac{1}{\psi'(-s)} + t' + s' + \frac{1}{-\psi'(t)} \\ &= \frac{1}{\psi'(-s)} + t - \frac{\rho}{-\psi'(t)} + s - \frac{\rho}{\psi'(-s)} \\ &\quad + \frac{1}{-\psi'(t)} \\ &= \frac{1 - \rho}{\psi'(-s)} + t + s + \frac{1 - \rho}{-\psi'(t)} \end{aligned}$$

$$\leq \begin{cases} \frac{s(1-\rho)}{\rho} + t + s + \frac{t(1-\rho)}{\rho} & \text{if } \rho \leq 1, \\ t + s & \text{if } \rho \geq 1 \end{cases} = (t + s) \max(1, 1/\rho).$$

The expected number of iterations is

$$\frac{\int \chi(x) dx}{\int e^{\psi(x)} dx} \leq \frac{(t + s) \max(1, 1/\rho)}{\frac{t+s}{\rho} \times (1 - e^{-\rho})} = \frac{\max(1, \rho)}{1 - e^{-\rho}}. \quad \square$$

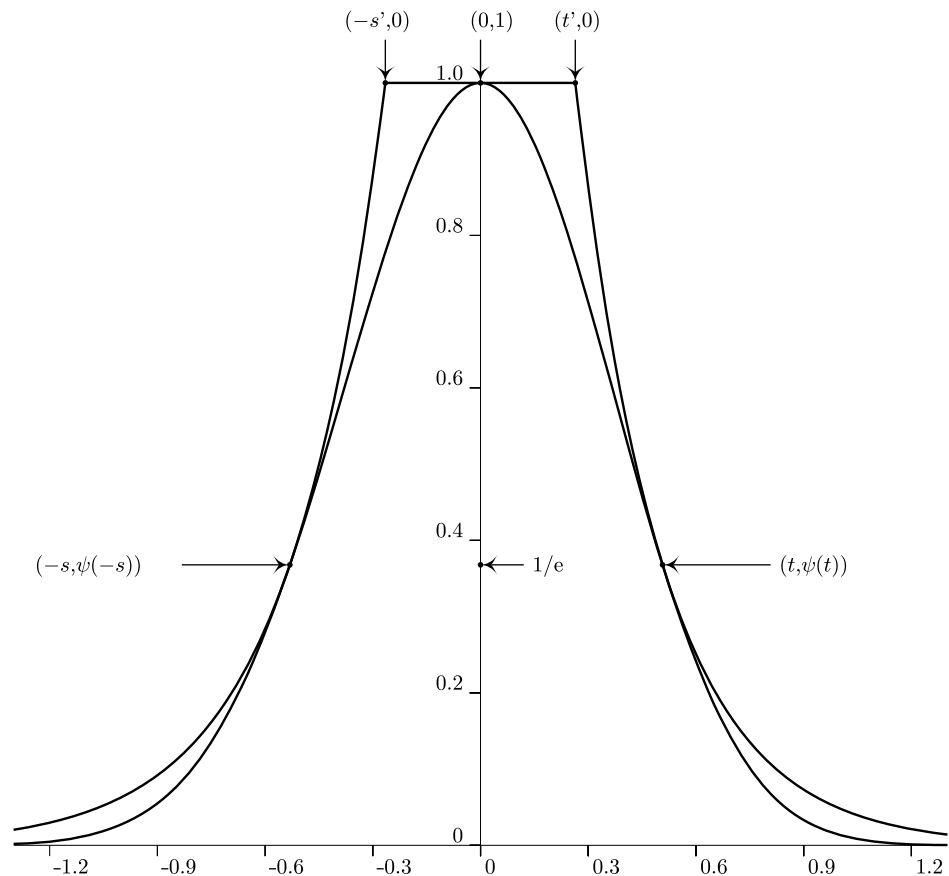
5 Discussion

The algorithm of the previous section can be automated and improved. Automation has been the theme in the work of Hörmann et al. (2004). Their software is capable of generating random variates in many situations, and includes the log-concave family, but also numerous other interesting families of densities.

Improvements come from using table methods and gridding to create finer partitions for bounding, either statically or dynamically (adaptively). In that sense, the algorithm of the previous section gives an upper bound for the most primitive set-up. One can in general get the rejection constant to converge to one.

For particular distributions, the main problem in the application is the determination of $s, t > 0$ such that

Fig. 2 The function $\exp(\psi)$ and its three-piece hat function are shown on real scales. Note that the log-concave function $\exp(\psi)$ has a peak at 0 and is normalized to have $\psi(0) = 1$. The example is the transformed GIG density with $(\lambda, \omega) = (2, 7)$



$$\psi(-s) = \psi(t) = -\rho.$$

It is rare that the inverse of ψ would be available. An example would be the transformed GIG random variable with parameter $\lambda = 0$. In that case, $m = 0$, and $t > 0$ is the solution of

$$\psi(x) = -\omega(\cosh x - 1), \quad x \in \mathbb{R}.$$

We obtain

$$t = -s = 1 + \frac{\rho}{\omega} + \sqrt{(1 + \rho/\omega)^2 - 1}.$$

In many cases, the Newton-Raphson algorithm will converge quickly. For example, for a solution $t > 0$ of $\psi(t) = -\rho$, start with $t_0 > 0$. Then iterate using

$$t_{n+1} = t_n - \frac{\psi(t_n) + \rho}{\psi'(t_n)}.$$

If $\psi'(t) < 0$ for all $t > 0$, then the concavity and continuity of ψ imply that $t_n \rightarrow t$ as $n \rightarrow \infty$. For the solution $s > 0$ of $\psi(-s) = -\rho$, start with $s_0 < 0$. Then iterate using

$$s_{n+1} = s_n + \frac{\psi(-s_n) + \rho}{\psi'(-s_n)}.$$

However, it is more useful to have explicit values for s and t . The onus then is to show that with these choices, one

has uniformly bounded rejection constants over all densities in the family. We will illustrate this first on the transformed GIG distribution and then on the gamma distribution.

6 Application to the GIG distribution

For the transformed GIG law with general (λ, ω) , the equation

$$\psi(t) = -\lambda(\sinh t - t) - \sqrt{\omega^2 + \lambda^2}(\cosh t - 1) = -\rho$$

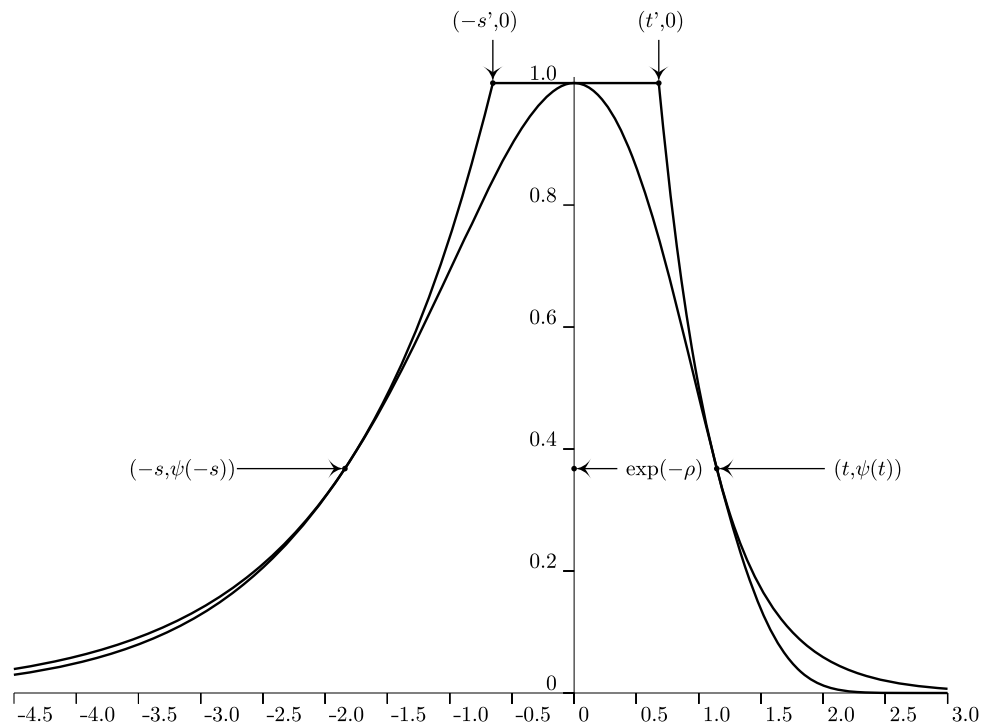
is nonlinear and has no explicit solution. It is convenient to rewrite it by grouping terms differently:

$$\begin{aligned} \psi(t) &= -(\sqrt{\omega^2 + \lambda^2} - \lambda)(\cosh t - 1) - \lambda(e^t - 1 - t) \\ &\stackrel{\text{def}}{=} -\alpha(\cosh t - 1) - \lambda(e^t - 1 - t). \end{aligned}$$

Here $\alpha = \sqrt{\omega^2 + \lambda^2} - \lambda$ is a positive parameter, and in fact, both terms in the last representation are nonpositive for all values of t and the parameters, and are concave in t with mode at 0.

The choice of $s > 0$ and $t > 0$ we propose is as follows:

Fig. 3 The function $\exp(\psi)$ and its three-piece hat function are shown for the gamma (1) (exponential) density



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If  $-\psi(1) \in [1/2, 2]$ ,
  then set  $t = 1$ 
  else if  $-\psi(1) > 2$ , then set  $t = \sqrt{2/(\alpha + \lambda)}$ 
  else [if  $-\psi(1) < 1/2$ ],
    then set  $t = \log(4/(\alpha + 2\lambda))$ 
If  $-\psi(-1) \in [1/2, 2]$ ,
  then set  $s = 1$ 
  else if  $-\psi(-1) > 2$ ,
    then set  $s = \sqrt{4/(\alpha \cosh 1 + \lambda)}$ 
  else [if  $-\psi(-1) < 1/2$ ],
    then set  $s = \min(\frac{1}{\lambda}, \log(1 + \frac{1}{\alpha} + \sqrt{\frac{1}{\alpha^2} + \frac{2}{\alpha}}))$ 

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The proof of the uniform speed of the rejection algorithm with this choice of (s, t) (Theorem 2) is given in the Appendix.

Theorem 2 (Uniform boundedness) *With the above choice for (t, s) , the expected number of iterations in the rejection method is not more than*

$$\max\left(\frac{1}{1 - \exp(-0.34114777\dots)}, \frac{e}{1 - e^{-e}}\right) = 3.459655\dots$$

For the sake of completeness, we summarize the random variate generator in its entirety.

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[Generator for the GIG distribution with
parameters  $\omega > 0, \lambda \geq 0$ ]
[Functions needed]
  Define  $\psi(x) = -\alpha(\cosh x - 1) - \lambda(e^x - x - 1)$ 
  Define  $\psi'(x) = -\alpha \sinh x - \lambda(e^x - 1)$ 
[Set-up]
Set  $\alpha = \sqrt{\omega^2 + \lambda^2} - \lambda$ 
If  $-\psi(1) \in [1/2, 2]$ ,
  then set  $t = 1$ 
  else if  $-\psi(1) > 2$ , then set  $t = \sqrt{2/(\alpha + \lambda)}$ 
  else [if  $-\psi(1) < 1/2$ ],
    then set  $t = \log(4/(\alpha + 2\lambda))$ 
If  $-\psi(-1) \in [1/2, 2]$ ,
  then set  $s = 1$ 
  else if  $-\psi(-1) > 2$ ,
    then set  $s = \sqrt{4/(\alpha \cosh 1 + \lambda)}$ 
  else [if  $-\psi(-1) < 1/2$ ],
    then set  $s = \min(\frac{1}{\lambda}, \log(1 + \frac{1}{\alpha} + \sqrt{\frac{1}{\alpha^2} + \frac{2}{\alpha}}))$ 
Compute  $(\eta, \zeta, \theta, \xi) = (-\psi(t), -\psi'(t), -\psi(-s), \psi'(-s))$ 
Set  $(p, r) = (1/\xi, 1/\zeta)$ 
Compute  $t' = t - r\eta$ 
Compute  $s' = s - p\theta$ 
Compute  $q = t' + s'$ 
[Generation]
Repeat Generate  $U, V, W$  uniformly on  $[0, 1]$ .
  If  $U < q/(p + q + r)$  then set  $X = -s' + qV$ 
  else if  $U < (q + r)/(p + q + r)$ 
    then set  $X = t' + r \log(1/V)$ 
  else set  $X = -s' - p \log(1/V)$ 
Until  $W\chi(X) \leq \exp(\psi(X))$ , where

$$\chi(x) = \mathbb{1}_{[-s', t']}(x) + \mathbb{1}_{(t', \infty)}(x)e^{-\eta - \zeta(x - t)} + \mathbb{1}_{(-\infty, -s')}(x)e^{-\theta + \xi(x + s)}$$

Return  $(\frac{\lambda}{\omega} + \sqrt{1 + \frac{\lambda^2}{\omega^2}})e^X$ 

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7 Another example: the gamma density

It is a daring undertaking to attempt to publish a gamma random variate generator in 2011, considering that tens if not hundreds of generators have been published world-wide since the 1970s. Most of the recent generators have uniformly bounded expected complexities. The gamma distribution of parameter $a > 0$ has density

$$f(x) = \frac{x^{a-1} e^{-x}}{\Gamma(a)}, \quad x > 0.$$

It is log-concave if and only if $a \geq 1$. This bifurcation has also resulted, historically, in a dichotomy in generation methods. The case $a < 1$ is relatively unimportant because of the property (see Devroye 1986) that for all $a > 0$,

$$G_a \stackrel{\mathcal{L}}{=} U^{\frac{1}{a}} G_{a+1},$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution, U is uniform $[0, 1]$, and G_a is a gamma (a) random variable. It is nevertheless refreshing to know that there is one method that works for all cases using the log-concave method explained in the present note. While not the fastest method, it offers the advantage of simple error-free coding and best of all, a design that is not gamma-specific but rather more general.

Set

$$Y = \log(G_a/a),$$

or, inversely, $G_a = ae^Y$. This is the exponential transformation that also worked fine for the genetically close GIG distribution. The objective is to generate Y and return $G_a = ae^Y$. The random variable Y is supported on \mathbb{R} , and has density given by

$$\varphi(y) = \frac{\exp(ay + a \log a - ae^y)}{\Gamma(a)}.$$

This log-concave function reaches a peak at the origin, and thus fits in our framework. We have

$$\frac{\varphi(y)}{\varphi(0)} \stackrel{\text{def}}{=} e^{\psi(y)} = e^{a(1+y-e^y)}.$$

This is a particularly fine formula, to which we could apply our method with the ideal design constant $\rho = 1$. The nonlinear equation

$$\psi(y) = a(1 + y - e^y) = -\rho$$

is easy to solve numerically by the Newton-Raphson method. Alternatively, one can try to propose explicit values for t and s that give respectable ρ values.

We propose the choices

$$(t, s) = \begin{cases} (\sqrt{\frac{2}{a}}, \sqrt{\frac{2}{a}}) & \text{if } a \geq 1, \\ (\log(1 + \frac{1}{a}), \frac{1}{a}) & \text{if } a < 1. \end{cases}$$

These lead to a rejection constant that is uniformly bounded over all values of the parameter a (see Theorem 3 below).

For the sake of completeness, the gamma algorithm is given in its entirety—this is just a replay of the general algorithm with ψ replaced by $a(1 + y - e^y)$ and ψ' replaced by $a(1 - e^y)$.

[Set-up]

If $a \geq 1$ then set $s = t = \sqrt{\frac{2}{a}}$
 else set $s = 1/a$, $t = \log(1 + \frac{1}{a})$

Set $p = \frac{1}{a(1 - \exp(-s))}$.

Set $r = \frac{1}{a(\exp(t) - 1)}$.

Set $t' = t + ra(1 + t - \exp(t))$.

Set $s' = s - pa(-1 + s + \exp(-s))$.

Set $q = t' + s'$.

[Generation]

Repeat Generate U, V, W uniformly on $[0, 1]$.

If $U < q/(p + q + r)$ then set $Y = -s' + qV$

else if $U < (q + r)/(p + q + r)$

then set $Y = t' + r \log(1/V)$

else set $Y = -s' - p \log(1/V)$

Until $Y \in [-s', t']$ and $\log W \leq a(1 + Y - \exp(Y))$,

or $Y > t'$ and

$\log W \leq a((-t + Y + 1) \exp(t) - \exp(Y))$,

or $Y < -s'$ and

$\log W \leq a((s + Y + 1) \exp(-s) - \exp(Y))$

Return $X = a \exp(Y)$ (Y has density proportional to e^{ψ} , and $X \stackrel{\mathcal{L}}{=} G_a$.)

Theorem 3 For the given choices of t and s we have the following inequalities: if $a \geq 1$, then

$$-2.96 \leq \psi(t) \leq -1, \quad -1 \leq \psi(-s) \leq -0.76,$$

and if $a \leq 1$,

$$-1 \leq \psi(t) \leq -1 + \log 2 \leq -0.306,$$

$$-1 \leq \psi(-s) - 1 + 1/e \leq -0.632.$$

Therefore, the maximal expected number of iterations over all values of a is not more than

$$\max\left(\frac{2.96}{1 - e^{-2.96}}, \frac{1}{1 - e^{-0.306}}\right) = 3.7934 \dots$$

Proof Theorem 3 consists of eight inequalities. For $a \geq 1$, the choice $t = s = \sqrt{2/a}$ means that the maximal value taken by these parameters is at most $\sqrt{2}$. Taylor's series with remainder implies that for $0 \leq x \leq \sqrt{2}$,

$$e^x - 1 - x - \frac{x^2}{2} \in \left[0, \frac{x^3 e^{\sqrt{2}}}{6}\right] \subseteq [0, 0.69x^3],$$

and thus,

$$\begin{aligned}\psi(t) &= a(1 + t - \exp(t)) \in a\left[-\frac{t^2}{2} - 0.69t^3, -\frac{t^2}{2}\right] \\ &= [-1 - 0.69\sqrt{2a}, -1] \subseteq [-1 - 1.38t, -1] \\ &\subseteq [-2.96, -1].\end{aligned}$$

For $-\sqrt{2} \leq x \leq 0$,

$$e^x - 1 - x - \frac{x^2}{2} \in \left[\frac{x^3}{6}, 0\right],$$

and thus,

$$\begin{aligned}\psi(-s) &= a(1 - s - \exp(-s)) \in a\left[-\frac{s^2}{2}, -\frac{s^2}{2} + \frac{s^3}{6}\right] \\ &= [-1, -1 + t/6] \subseteq [-1, -0.76].\end{aligned}$$

Now consider $a < 1$. The choice $t = \log(1 + 1/a)$ yields

$$\begin{aligned}\psi(t) &= a(1 + t - \exp(t)) = a(\log(1 + 1/a) - 1/a) \\ &= -1 + a \log(1 + 1/a) \in [-1, -1 + \log 2].\end{aligned}$$

Finally, setting $s = 1/a$ yields

$$\begin{aligned}\psi(-s) &= a(1 - 1/a - \exp(-1/a)) \\ &= -1 + a(1 - \exp(-1/a)) \in [-1, -1 + 1/e]. \quad \square\end{aligned}$$

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