Assignment 3 [30 marks]

*Code must be written in Python3 and submitted a **working** script for each sub-question.*

Please use the file names specified in the question e.g. [A1Q1.py]

The deadline for this assignment is **2018-12-05 Wednesday at 11:00**

You **must** answer the sheet showing **your student number**.

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Kapitza oscillator [A3Q1.py]

A rigid pendulum hanging from fixing which moves up and down following equation $y(t)$ can be described by the differential equation

$$\ddot{\theta} + \Gamma\dot{\theta} + \kappa \sin\theta = A(t)\sin\theta$$

where $A(t)$ is the vertical acceleration of the fixing, Γ is a damping rate, and $\sqrt{\kappa} = \omega_0$ is the natural frequency. θ is the angle with the vertical and $\theta = 0$ means pointing down.

Simulate the motion with a stationary fixing (i.e. $A(t) = 0$) with $\Gamma = 1\,\mathrm{s}^{-1}$ and $\omega_0 = 2\pi \times 1\,\mathrm{Hz}$. Start with the pendulum released from rest at $\theta = 0.9\pi$ (i.e. nearly straight up). [4 marks]

Plot time evolution $\theta(t)$ for the first $20\,\mathrm{s}$ [2 marks]

Simulate the motion with an oscillating fixing using $A(t) = A_0 \sin(100\omega_0 t)$ with $A_0 = 10^4\,\mathrm{s}^{-2}$, and the other conditions the same as above. [2 marks]

Plot time evolution $\theta(t)$ and comment on your result. [2 marks]

Penning traps [A3Q2.py]

A Penning trap for charged particles has a uniform magnetic field $\mathbf{B}$ along $\hat{\mathbf{z}}$ and electrodes which repel the particle from $z = \pm z_0$ and attract it to a ring electrode of radius r_0 in the $z = 0$ plane. When conditions are right, the particle cannot reach this ring electrode, because the magnetic field makes it follow a curved path.

The electric potential is $\Phi(\mathbf{x}) = \Phi_0(2z^2 - x^2 - y^2)$, where x, y, z are the components of position vector $\mathbf{x}$. This gives an electric field $\mathbf{E}(\mathbf{x}) = \kappa(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}})$. The differential equation describing the motion is:

$$\mathbf{F} = m\ddot{\mathbf{x}} = q\left(\mathbf{E}(\mathbf{x}) + \dot{\mathbf{x}} \times \mathbf{B}\right)$$

n.b. the electric field is a *function of position*.

Simulate this scenario with $\kappa = 10^4\,\mathrm{V/m^2}$ and a 1 Tesla magnetic field for a calcium ion (atomic mass 40) with one electron removed (i.e. $q = +e$ where e is the fundamental charge). Start the ion at position $(1\,\mathrm{mm}, 0, 1\,\mathrm{mm})$ with initial speed $100\,\mathrm{m/s}$ along $\hat{\mathbf{x}}$.

Simulate the first millisecond. [8 marks]

Plot (a) the trajectory (x, y) and (b) the time-domain plot $z(t)$ vs t. [2 marks]

***Hint**: The state vector now contains 6 elements: 3 for position and 3 for speed.*

Assignment 3 continues on the next page.

Stochastic Oscillator [A3Q3.py]

Real oscillators are subject to thermal fluctiations and this is most apparent on microscopic systems.

We can simulate this using a **Stochastic** Differential Equation:

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = \nu(t)$$

where $\nu(t)$ is a Wiener process: the values of $\nu(t)$ are random, Gaussian distributed with zero mean, and the value at time t is completely uncorrelated with the value at any other time.

This can be solved numerically using a modified version of Euler's method:

$$X_{n+1} = X_n + f(t_n, X_n)\delta t + dW(\delta t)$$

where $X_n = (x_n, v_n)$ is the state vector and $dW(\delta t)$ is a function of δt which describes the thermal fluctiatons. In this case,

$$dW(\delta t) = \sqrt{\frac{\Gamma\, k_B T}{M}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sqrt{\delta t} \times \mathrm{randn}()$$

where k_B is Boltzmann's constant, T is the temperature, and M is the mass of the oscillator.

You are provided with a modified Euler method called sde:

```
def sde(f, x0, dt, dW):
    tn = 0
    xn = x0

    while True:
        yield tn,xn
        xn = xn + f(tn,xn)*dt + dW(dt)
        tn = tn + dt
```

Make a function dW(dt) implementing the above (it **must** give a different random number each time) and simulate the stochastic harmonic oscillator, with initial conditions $X_0 = (0, 0)$, for $\omega_0 = 2\pi \times 100\,\mathrm{kHz}$, $\Gamma = 10^4\,\mathrm{s}^{-1}$, $T = 300\,\mathrm{K}$, and $M = 10^{-18}\,\mathrm{kg}$.

Simulate with time-step $\delta t = 10\,\mathrm{ns}$ and up to at least $10\,\mathrm{ms}$. [6 marks]

Plot the time-domain plot $x(t)$. [2 marks]

Plot a 2D **histogram** of the phase space $(x, v/\omega_0)$, using plt.hist2d and around 50 bins. [2 marks]

End of Assignment 3.