Linear Algebra Done Right

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Contents

1 Vector Spaces

1.1 $\mathbb{R}^n, \mathbb{C}^n$

Definition (Complex Numbers)**.** An ordered pair $(a, b) \in \mathbb{R}$, denoted $a + bi$.

- The set of all complex numbers is denoted $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- Addition and multiplication are defined:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

  where $a, b, c, d \in \mathbb{R}$.

$\mathbb{R}$ and $\mathbb{C}$, with the usual operations of addition and multiplication, are fields. All definitions, proofs and theorems for $\mathbb{F}$ denoting fields *(except for inner product spaces)* apply for arbitrary fields.

Definition (List)**.** For $n > 0$, a list of length n is an ordered collection of n elements. Also known as a n-tuple. Two lists are equal $\iff$ they have the same length and same elements in order. By definition, each list has finite non-negative integer length.

Definition ($\boldsymbol{F^n}$)**.** The set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_k \in \mathbb{F} \text{ for } k = 1, \ldots, n\}.$$

For $(x_1, \ldots, x_n \in \mathbb{F}^n$, x_k is the k-th coordinate of $x_1, \ldots, x_n)$.

1.2 Vector Spaces

Definition (Addition and Scalar Multiplication)**.** To define a vector space, we define the operation on a set:

- An *addition* on a set V is a function $+ : (u, v) \in V^2 \mapsto u + v \in V$
- A *scalar multiplication* on a set V is a function $\times : (\lambda, v) \in \mathbb{F} \times V \mapsto \lambda v \in V$

Definition (Vector Space)**.** A set V equipped with addition and scalar multiplication on V satisfying the following properties:

- Commutative. $\forall u, v \in V$,

$$u + v = v + u$$

- Associative. $\forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}$,

$$(u + v) + w = u + (v + w) \text{ and } (ab)w = a(bw).$$

- Additive Identity. $0 \in V$. $\forall v \in V$,

$$v + 0 = v$$

- Additive Inverse. $\forall v \in V, \exists!\, w \in V$,

$$v + w = 0.$$

- Multiplicative Identity. $1 \in \mathbb{F}$. $\forall v \in V$,
$$1v = v$$
- Distributive Properties. $\forall u, v \in V$ and $\forall a, b \in \mathbb{F}$,
$$a(u + v) = au + av \text{ and } (a + b)v = av + bv.$$

Satisfying these properties, we say that V is a vector space over $\mathbb{F}$.

Notation ($\mathbb{F}^S$). The set of functions $S \mapsto \mathbb{F}$.
Suppose the sum $f + g \in \mathbb{F}^S$ is the function defined by $(f + g)(x) = f(x) + g(x), \forall x \in S$, and the product $\lambda f \in \mathbb{F}^S$ is the function defined by $(\lambda f)(x) = \lambda f(x), \forall x \in S$

Example ($\boldsymbol{F^S}$ is a vector space). If $S \neq \varnothing \implies F^S$ is a vector space over F.

- The additive identity of $\mathbb{F}^S$ is the function $0 : S \mapsto \mathbb{F}$ defined by
$$0(x) = 0, \forall x \in S.$$
- For $f \in \mathbb{F}^S$, the additive inverse of f is the function $-f : S \mapsto \mathbb{F}$ defined by
$$(-f)(x) = -f(x), \forall x \in S.$$

The vector space $\mathbb{F}^n$ is a special case of the vector space $\mathbb{F}^S$, and can be thought of as $\mathbb{F}^{\{1,2,\ldots,n\}}$. Similarly, $\mathbb{F}^\infty$ is analogous to $\mathbb{F}^{\{1,2,\ldots\}}$

Identity and Inverse Uniqueness

Proposition (Unique Additive Identity)**.** *A vector space has a unique additive identty.*

Proof. Let 0 and $0'$ be unique identities for some vector space V. Then
$$0' = 0' + 0 = 0 + 0' = 0.$$
The first equality holds because 0 is an additive identity, the second equality comes from commutativity, and the third equality holds because $0'$ is an additive identity. Hence $0 = 0'$, and the additive identity is unique.

Proposition (Unique Additive Inverse)**.** *Every element in the vector space has a unique additive inverse.*

Proof. Suppose V is a vector space. Let $v \in V$. Suppose w and w' are additive inverses of v. Then
$$w = w + 0 = w + (v + w') = (w + v) + w' = (v + w) + w' = 0 + w' = w'.$$
Hence $w = w'$, and the additive inverse is unique.

Notation (Additive Inverse). Let $v, w \in V$. We denote the additive inverse of v, $-v$. We define $w - v$ to be $w + (-v)$.

1.3 Subspaces