1 Limits and Continuity

$\lim_{x\to a} f(x)$ exists$\Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Continuity at x = c$\Leftrightarrow \lim_{x\to a} f(x) = L = f(c)$

Continuity at endpoints only one-sided limit.

Continuity at an interval$\Leftrightarrow \forall c \in I(f$ is continuous at $c)$

Squeeze Theorem

Let $I = (a,c)\cup(c,b)$. $\forall x \in I(f(x) \le g(x) \le h(x))$

$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L \Rightarrow \lim_{x\to c} g(x) = L$

Special Limit

$\lim_{x\to c} g(x) = 0 \Rightarrow \lim_{x\to c} \frac{\sin(g(x))}{g(x)} = \lim_{x\to c} \frac{\tan(g(x))}{g(x)} = 1$

Indeterminate Limit

$\lim_{x\to c} \frac{f(x)}{g(x)} \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

2 Theorems

Intermediate Value Theorem

f is continuous on $[a,b]$, $k \in [f(a), f(b)] \Rightarrow \exists c \in [a,b](f(c) = k)$

Rolle's Theorem

f is differentiable on (a,b), $f(a) = f(b) \Rightarrow \exists c \in (a,b)(f'(c) = 0)$

Mean Value Theorem

f is differentiable on (a,b), $\exists c \in (a,b)(f'(c) = \frac{f(b)-f(a)}{b-a})$

3 Derivative

Limit Definition

$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$

Differentiability on an Interval$\Leftrightarrow$ Continuous on the interval

Quotient Rule

$\frac{d}{dx}(\frac{u}{v}) = \frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2}$

Inverse Derivative

$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$

Derivative of Parametric

$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

$\frac{d^2y}{d^2x} = \frac{f'(t)\cdot g''(t) - f''(t)\cdot g'(t)}{(f'(t))^3}$

Critical Point

Not end point, $f'(c) = 0 \vee f'(c)$ does not exist.

Absolute Extrema by First Derivative

$\forall x < c(f'(x) > 0) \wedge \forall x > c(f'(x) < 0) \Rightarrow f$ has an absolute maximum at c

$\forall x < c(f'(x) < 0) \wedge \forall x > c(f'(x) > 0) \Rightarrow f$ has an absolute minimum at c

Local Extrema by First Derivative

f is differentiable on $(a,c)\cup(c,b)\wedge$ continuous at c

f' changes from + to − at $x = c \Rightarrow$ max

f' changes from − to + at $x = c \Rightarrow$ min

Second Derivative Test

$f'(c) = 0 \wedge f''(c) < 0 \Rightarrow$local max at c

$f'(c) = 0 \wedge f''(c) > 0 \Rightarrow$local min at c

4 Integration

Partial Fractions Simplify $\frac{P(x)}{Q(x)}$

Factors of $Q(x)$	
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$

Trigo Substitution

Expression	Substitution
$\sqrt{a^2-(x+b)^2}$	$x+b = a\sin\theta$
$\sqrt{(x+b)^2+a^2}$	$x+b = a\tan\theta$
$\sqrt{(x+b)^2-a^2}$	$x+b = a\sec\theta$

Domains: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \theta < \frac{\pi}{2}$ OR $\pi \le \theta < \frac{3\pi}{2}$

Integration by Parts

$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$

(diff) Log, Inverse, Alge, Trig, Expo (int)

Riemann Sum

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} (\frac{b-a}{n}) f(a + k(\frac{b-a}{n}))$

Fundamental Theorem of Calculus

$\int_a^b f(x)dx = F(b) - F(a)$

Second Fund. Theorem of Calc

$F(x) = \int_a^x f(t)dt, a \le x \le b, F(a) = 0$

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Improper Integrals

$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx$

$\int_a^b f(x)dx = \lim_{c\to d^-}\int_a^c f(x)dx + \lim_{c\to d^+}\int_c^b f(x)dx$

5 Integration Applications

Area between curves

$\int_a^b |f(x) - g(x)|dx, \int_a^b \left|f(y) - g(y)\right|dy$

Volume of Solid by Disk

$\pi\int_a^b f(x)^2dx - \pi\int_a^b g(x)^2dx$

$\pi\int_a^b f(y)^2dy - \pi\int_a^b g(y)^2dy$

Volume of Solid by Cyindrical Shell

$2\pi\int_a^b x|f(x) - g(x)|dx$

$2\pi\int_a^b y\left|f(y) - g(y)\right|dy$

Length of curve y=f(x)

$\int_a^b \sqrt{1+f'(x)^2}dx, \int_p^q \sqrt{1+f'(y)^2}dy$

6 Equations

Ellipse $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$

$x = a\cos(t) + x_0, y = b\sin(t) + y_0$

Circle $(x-x_0)^2 + (y-y_0)^2 = r^2$

$x = r\cos(t) + x_0, y = r\sin(t) + y_0$

Horizontal Hyperbola

$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$

$x = a\sec(t) + x_0, y = b\tan(t) + y_0$

Vertical Hyperbola

$\frac{(y-y_0)^2}{b^2} - \frac{(x-x_0)^2}{a^2} = 1$

$x = a\tan(t) + x_0, y = b\sec(t) + y_0$

7 Series

Convergence of series $\lim_{n\to\infty} a_n = L$

Absolute Convergence $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Geometric Series Convergent for $|r| < 1$

$$\sum_{i=1}^{n} ar^{i-1} = \begin{cases} \frac{a(1-r^n)}{1-r} & if\, r \ne 1 \\ an & if\, r = 1 \end{cases}$$

p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ convergent $\Leftrightarrow p > 1$

8 Tests for Convergence

n-th Term Test

$\lim_{n\to\infty} a_n \ne 0 \vee \lim_{n\to\infty} a_n DNE \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

Convergence by Partial Sum

$\forall n \in \mathbb{N}\ \exists k \in \mathbb{R}(S_n < k) \Leftrightarrow \sum_{n=1}^{\infty} a_n$ of non-negative terms converge

Integral Test

$\forall n \in \mathbb{N}(a_n = f(n)), \forall x \ge 1(f(x)$ is continuous, positive, decreasing)

$\sum_{n=1}^{\infty} a_n$ is convergent$\Leftrightarrow \int_1^{\infty} f(x)$ is convergent

Comparison Test

$\forall n \in \mathbb{N}(0 \le a_n \le b_n)$

$\sum_{n=1}^{\infty} a_n$ is divergent$\Rightarrow \sum_{n=1}^{\infty} b_n$ is divergent.

$\sum_{n=1}^{\infty} b_n$ is convergent$\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

Ratio/Root Test

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ OR $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$,

$0 \le L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

$L = 1$ is inconclusive.

Alternating Series Test

$\forall n \in \mathbb{N}(b_n > 0 \wedge b_n \ge b_{n+1}) \wedge \lim_{n\to\infty} b_n = 0 \Rightarrow$ Alternating series

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots$ and

$\sum_{n=1}^{\infty}(-1)^{n}b_n = -b_1 + b_2 - b_3 + b_4 - \ldots$ are convergent.

9 Power Series

Deriving Radius of Convergence:

Using Ratio or Root Test!

Radius of Convergence

$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \frac{1}{R}$ OR $\lim_{n\to\infty}\sqrt[n]{|c_n|} = \frac{1}{R}$

For $\sum_{n=0}^{\infty} c_n(x-a)^n$,

$|x-a| < R$: Absolute Convergence

$|x-a| = R$: Check with Convergence Tests

Power Series Representation

$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

$\int f(x) = \sum_{n=0}^{\infty}\frac{c_n(x-a)^{n+1}}{n+1} + C$

Taylor Series of f at $x = a$

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n,\ |x-a| < R$$

Maclaurin Series of f = Taylor Series of f at $a = 0$:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x-a)^n,\ |x-a| < R$$

Common Maclaurin Series

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$

10 3D Coordinate System

Distance between points

$D = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

Length of Vector

$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Dot Product

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta$

$\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

Cross Product

Area of a parallelogram with sides $\vec{a}$ and $\vec{b}$:

$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

Distance from Point Q to Line PR:

$\frac{\|\vec{PQ}\times\vec{PR}\|}{\|\vec{PR}\|}$

Angle between Vectors

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$

$\sin\theta = \frac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\,\|\vec{b}\|}$

Projection of $\vec{b}$ on $\vec{a}$

Vector Projection $proj_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$

Scalar Projection $comp_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$

Distance between Point and Plane

Plane Equation: $d = ax + by + cz$, Point: (x_0, y_0, z_0)

$Distance = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$

Cylinder

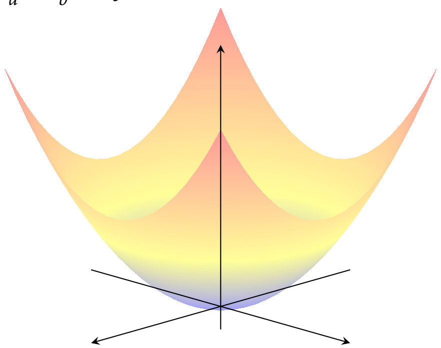
A surface where $\exists$ Plane P such that all the planes parallel to P intersect the surface in the same curve.

11 Calculus but 3D :D

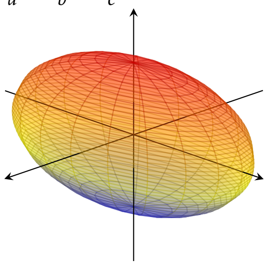
Vector-Valued Function $f:\mathbb{R}\to\mathbb{V}_n$
Derivative of Vector-valued Function
$r'(t)=\langle f'(t),g'(t),h'(t)\rangle$
Arc Length of Curve
$\int_a^b \sqrt{f'(t)^2+g'(t)^2+h'(t)^2}dt=\int_a^b \|r'(t)\|dt$
Tangent line to Curve
$r'(a)$ is the tangent vector to the curve at a.
Form a line with the origin point at $t=a$ and the tangent vector.
Differentiability of f(x,y)
Tangent Plane at $x=a,y=b$ is a good approximation to f at points close to (a,b)
Chain Rule but 3D
$z=f(g(t),h(t))\Rightarrow \frac{dz}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$
$z=f(g(s,t),h(s,t))\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$
$z=f(g(s,t),h(s,t))\Rightarrow \frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$
Increment/Differential of z
$\triangle z=f(x+\triangle x,y+\triangle y)-f(x,y)$
$dz=f_x(x,y)dx+f_y(x,y)dy$
Gradient of f(x,y)
$\nabla f=\langle f_x,f_y\rangle$ is normal to the level curve $f(x,y)=k$
$\nabla f=\langle f_x,f_y,f_z\rangle$ is normal to any curve C on surface S $f(x,y,z)=k$
Tangent Plane to Surface $z=f(x,y)$
$z=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$
Tangent Plane to Level Surface
$\nabla f(x_0,y_0,z_0)\cdot\langle x-x_0,y-y_0,z-z_0\rangle=0$
$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$
Direction Derivative of f(x,y)
At (x_0,y_0), in the direction of unit vector $\langle a,b\rangle$:
$D_uf(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ha,y_0+hb)-f(x_0,y_0)}{h}$
$D_uf(x,y)=\nabla f\cdot\langle a,b\rangle$
Maximum Rate of Change of f at point P
∇f is maximum ($\|\nabla f(P)\|$) in direction $\nabla f(P)$
∇f is minimum ($-\|\nabla f(P)\|$) in direction $-\nabla f(P)$
Critical Points of f(x,y)
$f_x(a,b)=0=f_y(a,b)$ OR f_x or f_y does not exist at (a,b)
Local Maxima, Minima, Saddle Points
$\forall(x,y)$ in a disk with center (a,b), $f(x,y)\le f(a,b)\Rightarrow(a,b)$ is a local maximum point, $f(a,b)$ is a local maximum value
$\forall(x,y)$ in a disk with center (a,b), $f(x,y)\ge f(a,b)\Rightarrow(a,b)$ is a local minimum point, $f(a,b)$ is a local minimum value
Critical point, $\forall$ open disk centered (a,b), $(x,y)\in\mathbb{D},f(x,y)<f(a,b)$ and $(x,y)\in\mathbb{D},f(x,y)>f(a,b)\Rightarrow(a,b)$ is a saddle point of f
Second Derivative Test for f(x,y)
Discriminant $D=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$
$D>0\wedge f_{xx}(a,b)>0\Rightarrow$ Local Minima
$D>0\wedge f_{xx}(a,b)<0\Rightarrow$ Local Maxima
$D<0\Rightarrow$ Saddle Point

12 Equations but 3D :)

Line
$\overrightarrow{r}=\overrightarrow{r_0}+t\overrightarrow{v},\ t\in\mathbb{R}$
$x=x_0+at,y=y_0+bt,z=z_0+ct,\ t\in\mathbb{R}$
Plane
$\overrightarrow{n}\cdot\overrightarrow{r}=\overrightarrow{n}\cdot\overrightarrow{r_0}$
$ax+by+cz=-(ax_0+by_0+cz_0)=d$
Sphere
$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$
Elliptic Paraboloid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$



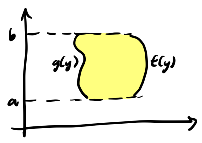
Ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



13 Double Pain D:

Double Riemann Sum
$\lim_{m,n\to\infty}\sum_{i=1}^m\sum_{j=1}^n f(x_{ij}^*,y_{ij}^*)\triangle A$
Volume of Solid above Rectangle below z=f(x,y)
$V=\int\int_R f(x,y)dA$
Split Double Integral: Special Case
$f(x,y)=g(x)h(y),\ R=[a,b]\times[c,d]$
$V=\int\int_R f(x,y)dA\Rightarrow\int_a^b g(x)dx\int_c^d h(y)dy$
Double Integral: Region Type

Type I



Type II

Type I: $\int_a^b\int_{g(x)}^{f(x)}f(x,y)dydx$
Type II: $\int_a^b\int_{f(y)}^{g(y)}f(x,y)dxdy$
Decomposition of Domains
$\int\int_D f(x,y)dA=\int\int_{D_1}f(x,y)dA+\ldots+\int\int_{D_n}f(x,y)dA$
Area/Volume by Iterated Integrals
$A(D)=\int\int_D 1dA$
$V(S)=\int\int\int_S 1dV$
Rectangle$\mapsto$Polar Coordinates
$r^2=x^+y^2\Rightarrow x=r\cos\theta,y=r\sin\theta$
If f is continuous on a polar rectangle
$R=\{(r,\theta):0\le a\le r\le b,\alpha\le\theta\le\beta\}\wedge 0\le\beta-\alpha\le 2\pi$
$\int\int_R f(x,y)dA=\int_\alpha^\beta\int_a^b f(r\cos\theta,r\sin\theta)rdrd\theta$
Surface Area of z=f(x,y)
$\int\int_D dS=\int\int_D\sqrt{f_x^2+f_y^2+1}dA$

14 Ordinary Differential Equations

Reading Comprehension
Separable First Order ODE
$\frac{dy}{dx}=f(x)g(y)$
$\int\frac{1}{g(y)}dy=\int f(x)dx+C$
Non-Separable First Order ODE
Type 1:
$\frac{dy}{dx}=g(\frac{y}{x})$
Let $v=\frac{y}{x}$. Then $y'=v+xv'$.
$v'=\frac{g(v)-v}{x}$ Type 2:
$\frac{dy}{dx}=f(ax+by)$
Let $u=ax+by$. Then $u'=a+by'$, $y'=\frac{u'-a}{b}$.
$\frac{u'-a}{b}=f(u)\Rightarrow u'=bf(u)+a$
Linear First Order ODE
$\frac{dy}{dx}+P(x)y=Q(x)$
Let $I(x)=e^{\int P(x)dx}$.
$y\cdot I(x)=\int Q(x)\cdot I(x)dx$
Bernoulli Equations
$\frac{dy}{dx}+p(x)y=q(x)y^n,n\in\mathbb{R}\backslash\{0,1\}$
Let $u=y^{1-n}$. $u'=(1-n)y^{-n}y'$.
Multiply both sides by $(1-n)y^{-n}$.
$(1-n)y^{-n}y'+(1-n)y^{-n}yp(x)=(1-n)y^{-n}y^nq(x)$
$u'+(1-n)p(x)u=(1-n)q(x)$