# Set Theory

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# 1 Introduction

*Notation* (Set Notation and Definitions). To specify sets, we denote:

- $\varnothing$ for the empty set, the set having no elements.
- $\{a, b, c, d\}$ for small sets by listing its members; otherwise
- $\{x|\phi(x)\}$ the set of things satisfying the condition $\phi$. We read: the set of all $x$ such that $\phi(x)$.

By convention, we denote sets with capital letters $A, B, \ldots$, and objects or elements belonging to those sets with lowercase letters $a, b, \ldots$.

- If an object $a$ belongs to a set $A$, we denote $a \in A$ with the set-membership relation $\in$.
- If every element in $A$ is also an element of $B$, we say $A$ is a subset of $B$, and denote $A \subset B$ (Inclusion relation $\subset$).
- If $A \subseteq B \wedge A \neq B$, we say $A$ is a proper subset of $B$, and denote $A \subsetneq B$ (Proper inclusion relation $\subsetneq$).
- $\mathcal{P}(A)$ denotes the powerset of $A$, the set of all subsets of $A$.
- $A \cup B = \{x|x \in A \vee x \in B\}$
- $A \cap B = \{x|x \in A \wedge x \in B\}$.

**Definition** (Ordered Pairs)**.** Sets are in themselves unordered. To implement ordered pairs such that $\langle a, b\rangle = \langle a', b'\rangle \iff a = a' \wedge b = b'$, we define: $\langle a, b\rangle = \{\{a, b\}, \{a\}\}$, and $\langle a, b, c\rangle = \langle\langle a, b\rangle, c\rangle, \ldots$

**Definition** (Cartesian Product)**.** $\mathbb{A} \times \mathbb{B} = \{\langle x, y\rangle |x \in \mathbb{A} \wedge y \in \mathbb{B}\}$. The set whose members are all the ordered pairs whose first member is in $\mathbb{A}$ and whose second member is in $\mathbb{B}$.

**Relations** For a binary relation $R$ between members of set $\mathbb{A}$ and $\mathbb{B}$, the extension of $R$ is the set of ordered pairs $\{\langle x, y\rangle |x \text{ is } R \text{ to } y\} \subseteq \mathbb{A} \times \mathbb{B}$. In general, the extension of a $n$-place relation is the set of $n$-tuples standing in that relation. The unary property $P$ defined over some set $\mathbb{A}$ is the set of members of $\mathbb{A}$ which are $P$.

**Functions** For a unary function $f : \mathbb{A} \mapsto \mathbb{B}$, the extension (or graph) of $f$ is the set of ordered pairs $\{\langle x, y\rangle |f(x) = y\}$. Similarly for $n$-place functions.
For most purposes, we can identify a relation or function with its extension.

**Cardinality** Two sets have the same cardinality $\iff \exists$ exists a bijection between the set. A set is countably infinite if they have the same cardinality with $\mathbb{N}$.

**Postulate** (Axiom of Choice)**.** *Given an infinite family of sets, there exists a choice function, which chooses a single member from each set in the family.*

### 1.1 Naivety

**Definition** (Normal)**.** A set that is not a member of itself is normal.

Is there a set $R$ whose members are all, and only the normal sets? No!
**Proof.** Suppose not. The set $R$ is normal $\iff$ $R \in R$. Hence $R$ is not normal. Then $R \in R$. A contradiction. There is no set $R$ whose members are all, and only the normal sets.

Naive set theory is a theory which makes the assumption that all properties has a extension, or set theory developed without rigorous axiomatisation. We build our foundations in naive set theory, first and foremost.

## 2 Naive Set Theory