Real Analysis

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Contents

1 Real Numbers

1.1 Building up to $\mathbb{R}$

We start with the natural numbers $\mathbb{N}$. To define the operations of addition, subtraction, with the additive inverse of 0, we extend our system to the integers $\mathbb{Z}$. To further define multiplication, division, and the multiplicative inverse 1, we extend the system again to the rational numbers $\mathbb{Q}$.

Definition (Fields)**.** A field is a set containing at least two distinct elements, called 0 and 1, along with operations of addition and multiplication satisfying the following properties:

- Commutative. $\forall \alpha, \beta \in \mathbb{F}$,
$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha.$$
- Associative. $\forall \alpha, \beta, \gamma \in \mathbb{F}$,
$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \text{ and } (\alpha\beta)\gamma = \alpha(\beta\gamma).$$
- Identities. $0, 1 \in \mathbb{F}$. $\forall \lambda \in \mathbb{F}$,
$$\lambda + 0 = \lambda \text{ and } \lambda 1 = \lambda.$$
- Inverses. $\forall \alpha \in \mathbb{F}, \exists! \, \beta \in \mathbb{F}$,
$$\alpha + \beta = 0 \text{ and } \alpha\beta = 1.$$
- Multiplication is distributive over addition. $\forall \lambda, \alpha, \beta \in \mathbb{F}$,
$$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta.$$

$\mathbb{Q}$ is a field, since it has all of these properties. $\mathbb{N}, \mathbb{Z}$ are not fields. Additionally, $\mathbb{Q}$ is an ordered field, with the natural order $\leq$.

These properties allow algebraic operations of addition, subtraction, multiplication and division, but not always square roots ($\sqrt{2}$). It is possible to approximate $\sqrt{2}$ to any finite degree of precision, but there exists a "hole" in the number line where it ought to be. If we want every length along the number line to correspond to an actual number, we require inclusion of the irrational numbers, constructing $\mathbb{R}$. Informally, $\mathbb{R}$ is obtained by "filling in the gaps" in $\mathbb{Q}$.

1.2 Preliminaries

Sets

We adopt the foundations of naive set theory and intuition, which is adequate for most intents and purposes of conducting real analysis.

Functions

Definition (Function)**.** Given two sets A, B, a function from A to B is a mapping that takes each element $x \in A$ and associates with it a *single* element of B. We denote $f : A \mapsto B$, and given an element $x \in A$, the expression $f(x)$ represents the element in B associated with x by f.

Definition (Domain, Codomain, and Range)**.** Given $f : A \mapsto B$, A is known as the *domain* of f, B the *codomain* of f, and the *range* of f is defined $\{y \in B : \exists x \in A(y = f(x))\} \subseteq B$.

Proposition (Triangle Inequality)**.** $|a+b| \leq |a| + |b|$. *Proof by considering the cases where a, b, and $a+b$ are positive and negative.*

Theorem (Equality of real numbers)**.** *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$, it follows that $|a-b| < \epsilon$.*

Proof.

1. ($\Rightarrow$) If $a = b$, then for evey real number $\epsilon > 0$ it follows that $|a-b| = 0 < \epsilon$.
2. ($\Leftarrow$)
   Suppose not. For every real number $\epsilon > 0, |a-b| < \epsilon$ and $a \neq b$. Assuming $a \neq b$, we choose

   $$|a-b| = \epsilon_0.$$

   Then $\epsilon > 0$ by definition, and

   $$|a-b| < \epsilon_0 \text{ and } |a-b| = \epsilon_0.$$

   This cannot both be true, and hence the initial assumption is false. Therefore, $a = b$.