Computational Assignment Project 2

MIE377: Financial Optimization Models

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1 Introduction

Preface

The following paragraphs in *Introduction,* and some paragraphs in *Methodology* cover topics that are taken directly from our previous report, but are included here for the reader's convenience.

1.1 Portfolio Optimization

In the realm of asset management, Modern Portfolio Theory provides excellent insights into which assets should be included in an investor's portfolio. Portfolio Optimization is a widely utilized mathematical tool for finding an optimal asset distribution for investment. In finance, an investor's objective is typically to minimize risk subject to a target return, although this has many variations. Nevertheless, mathematical programming is evoked to solve the optimization problem and find the best weights for each asset.

1.2 Overview

The purpose of this project is to both design an automated asset management system and devise an investment strategy that optimizes a portfolio using Mean-Variance Optimization (MVO), using estimated parameters from regularization. A dataset consisting of twenty U.S. assets and eight factors were used in the development of the algorithm. The frequency of observations is monthly, ranging from December 1991 to December 2016. The assets and factors used throughout the report are summarized in Table 1.2.1 and Table 1.2.2.

F	CAT	DIS	MCD	KO	PEP	WMT	C	WFC	JPM
AAPL	IBM	PFE	JNJ	XOM	MRO	ED	T	VZ	NEM

Table 1.2.1 - List of Assets by Ticker Symbol

Market ('Mkt_RF')	Size ('SMB')	Value ('HML')	Short-term reversal ('ST_Rev')
Profitability ('RMW')	Investment ('CMA')	Momentum ('Mom')	Long-term reversal ('LT_Rev')

Table 1.2.2 - List of Factors

2 Methodology

In this section, the strategy with respect to designing our automated asset management system is discussed, followed by detailed descriptions of our implementation.

2.1 Strategy

Initially, the problem was broken down into two different optimization problems. Firstly, estimates for asset returns and covariances were obtained using the factor returns data. Then, an optimal portfolio was constructed using the estimates for asset returns and covariances that were obtained earlier.

To obtain these estimates, various multi-factor models were used. For instance, the Fama-French (FF) model was first implemented. The factor returns dataset contains data for excess market returns, size, and value, which are the three factors used in the Fama-French model. As such, multiple linear regression was performed using those three factors to determine the regression coefficients and then the estimates for the asset returns and covariances. The Fama-French Five Factor (FF 5) model was implemented as an extension to the Fama-French model. This model adds two more additional factors: ‘profitability’ and ‘investment’. Furthermore, CAPM was implemented since the factor returns dataset also contains values for the risk-free rate. In this case, simple linear regression was performed with the excess asset returns as the response vector, and the excess market returns as the predictor. This provided the regression coefficients needed to estimate the asset returns and covariances.

Sparse factor models were also obtained via Best Subset Selection (BSS), LASSO, and ridge regression. These regression models were implemented on MATLAB, and then fed different cardinality constraint parameters and penalty parameters, based on the regression model being used, to estimate the asset returns and covariances. Moreover, for BSS, cardinality constraint parameters between one and seven were selected in order to balance complexity and interpretability of our model. All the relevant data have been presented in Appendix A.

Different versions of MVO were considered to determine the optimal portfolio using the obtained estimates of asset returns and covariances. Version 1 of MVO is simply minimizing the portfolio variance subject to meeting a target return, which is the average of the estimate of asset returns. Version 2 maximizes return subject to an upper bound on variance. Version 3 of MVO considers the risk-return trade-off of the portfolio, wherein the objective function is penalized with the risk-aversion coefficient, ψ.

Robust models were also implemented during testing, as robustness addresses the issue of parameter uncertainty. In other words, any small change in any of the input parameters could result in a big change of optimal portfolio selection. By explicitly measuring this uncertainty and incorporating it as a deterministic measure within the model, ‘robustified’ models account for more situations and often produce better results. The following models only apply an uncertainty set on the expected returns, μ, whilst variances were unmodified: MVO Version 3, and Max Sharpe.

A variety of other models were additionally implemented to test their performance: Risk Parity - a model that equalizes the risk contribution per asset, CVaR (Conditional Value at Risk) - a model that aims to minimize expected losses exceeding VaR, and Max Sharpe - a model that maximizes ex ante Sharpe ratios.

2.2 Time Frame

The algorithm was trained using 60 months of data. The longest available time frame (i.e. five years) was used in order to minimize estimation variance and maximize predictive power. Optimization & machine learning researchers typically use a five-year time frame when retraining reinforcement models (Zhang, Zohren, & Roberts, 2020), and five-years of data is considered a "*modest number of estimations*" in the framework of some CVaR-based optimization problems (Ban, Karoui, & Lim, 2018). Overall, using 60 months of data produced the best results, in terms of Sharpe ratio, and average turnover rates.

2.3 Mean Variance Optimization (MVO)

Three different versions of MVO are used throughout the optimization phase of the project. The objective of version one of MVO is to merely minimize risk subject to meeting the target return, which for this project is the average of the asset returns. Similarly, the objective of version three of MVO is to minimize risk, but simultaneously penalize the return by introducing a risk-aversion parameter (ψ). In other words, version three of MVO optimizes the portfolio based on the risk-return trade-off, minimizing the risk for the same return. Lastly, version two of MVO maximizes the portfolio excess return subject to an upper bound constraint on the portfolio variance. In order to determine a suitable upper bound for the portfolio variance, the square of the standard deviation of annual returns of SPY over the last five years was used, which is approximately 0.0225 (Yahoo Finance, n.d.).

2.4 Factor Models

In this section, some well-known factor models are discussed that have been used to estimate the parameters to generate portfolio weights.

2.4.1 Capital Asset Pricing Model (CAPM)

The CAPM, which forms the basis of the Modern Portfolio Theory, contains an excess market return as a single factor in the model. In fact, the CAPM describes asset i's risk as a function of its market exposure, β_{iM}, which is a measure of the systematic risk of the asset i. The CAPM is implemented as below:

$$\mathbf{r}_i - \mathbf{r}_f = \boldsymbol{\beta}_{i,m}(\mathbf{f}_m - \mathbf{r}_f) + \epsilon_i$$

Note that $\alpha_i = 0$ in CAPM.

2.4.2 Fama-French Model

The FF model is a well-known multi-factor model that contains 3 factors: excess market return, size, and value. Unlike the assumptions made in the class for the derivation under an ideal environment, these three factors are, in fact, correlated. The FF model is implemented as below:

$$\mathbf{r}_i - \mathbf{r}_f = \alpha_i + \boldsymbol{\beta}_{i,m}(\mathbf{f}_m - \mathbf{r}_f) + \boldsymbol{\beta}_{i,s}\mathbf{f}_s + \boldsymbol{\beta}_{i,v}\mathbf{f}_v + \epsilon_i$$

Note that the FF model is calibrated using ordinary least squares (OLS) regression.

2.4.3 Fama-French 5 Factor Model

$$\mathbf{r}_i - \mathbf{r}_f = \alpha_i + \boldsymbol{\beta}_{i,m}(\mathbf{f}_m - \mathbf{r}_f) + \boldsymbol{\beta}_{i,s}\mathbf{f}_s + \boldsymbol{\beta}_{i,v}\mathbf{f}_v + \boldsymbol{\beta}_{i,p}\mathbf{f}_p + \boldsymbol{\beta}_{i,inv}\mathbf{f}_{inv} + \epsilon_i$$

The FF 5 factor model is an extension of the FF model, as it contains two new factors: ‘profitability’ and ‘investment’. Profitability refers to the concept that companies reporting higher future earnings have higher returns in the stock market (Hayes, Scott, & Rathburn, 2021). Investment captures the idea that there is a trade-off in average return when a company invests conservatively or aggressively (Fama & Kenneth, 2015).

2.5 Regularization Models

Regularization is a crucial technique used in the project to promote sparsity and reduce statistical overfitting by introducing an additional penalty term in the error function. Sparsity is essential in optimization as it is a data-driven approach to the factor selection process for the regression model, making the model more interpretable with fewer factors. In addition, all factors are taken into account by promoting sparsity, yet the optimization selects the best-fitting subset.

2.5.1 Best Subset Selection (BSS)

The l_0 norm is equivalent to a cardinality constraint, naturally promoting sparsity and controlling the number of non-zero coefficients. Since l_0 norm is discontinuous and non-convex, BSS is also non-convex, and must be formulated as a Mixed Integer Program with p+1 auxiliary binary variables y_j , where $j = 1, .., p + 1$. Therefore, the penalized version of BSS is transformed into the constrained version as below:

$$
\begin{aligned}
\min_{\mathbf{B}_i,\mathbf{y}} \quad & \|\mathbf{r}_i - \mathbf{X}\mathbf{B}_i\|_2^2 \\
\text{s.t.} \quad & \mathrm{L}\mathbf{y} \le \mathbf{B}_i \le \mathrm{U}\mathbf{y} \\
& \mathbf{1}^T\mathbf{y} \le \mathrm{K} \\
& \mathrm{y}_j \in \{0,1\}, \mathrm{j} = 1, ..., \mathrm{p}{+}1
\end{aligned}
$$

2.5.2 Least Absolute Shrinkage and Selection Operator (LASSO)

The l_1 norm is the sum of absolute values of the elements of a vector. Although It is continuous and convex, it is not smooth, requiring to be reformulated using p+1 auxiliary variables to incorporate the absolute value function. This creates an optimization problem with 2p+2 constraints:

$$
\begin{aligned}
\min_{\mathbf{B}_i,\mathbf{y}} \quad & \|\mathbf{r}_i - \mathbf{X}\mathbf{B}_i\|_2^2 + \lambda\mathbf{1}^T\mathbf{y} \\
\text{s.t.} \quad & \mathbf{y} \ge \mathbf{B}_i \\
& \mathbf{y} \ge -\mathbf{B}_i
\end{aligned}
$$

2.5.3 Ridge Regression

The l_2 norm is known as the Euclidean norm, the length of a vector. It is a continuous, convex, and smooth function, allowing to find a closed-form solution. Although it shrinks the coefficients to zero, it does not promote true sparsity. The penalized version of ridge regression is as below:

$$
\begin{aligned}
\min_{\mathbf{B}_i} \quad & \|\mathbf{r}_i - \mathbf{X}\mathbf{B}_i\|_2^2 \\
\text{s.t.} \quad & \|\mathbf{B}_i\|_2^2 \le s
\end{aligned}
$$

2.6 Optimization Models

In this section, some well-known financial optimization models are discussed that are used to determine optimal portfolio weights based on specific goals.

2.6.1 MVO Version 1: Minimize Risk

The standard version of MVO, as presented in Harry Markowitz' 1952's 'Portfolio Selection' (Markowitz, 1952). The optimal portfolio will lie on the efficient frontier - a set of portfolios designed to give the highest returns given a level of risk.

$$\begin{aligned}
\min_{\mathbf{x}} \ & \mathbf{x}^T \mathbf{Q} \mathbf{x} \\
\text{s.t.} \ & \boldsymbol{\mu}^T \mathbf{x} \geq \mathbf{R} \\
& \mathbf{1}^T \mathbf{x} = 1 \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

2.6.2 MVO Version 2: Maximize Return

Version 2 of MVO involves maximizing return subject to an upper bound on the portfolio variance.

$$\begin{aligned}
\max_{\mathbf{x}} \ & \boldsymbol{\mu}^T \mathbf{x} \\
\text{s.t.} \ & \mathbf{x}^T \mathbf{Q} \mathbf{x} \leq \epsilon^2 \\
& \mathbf{1}^T \mathbf{x} = 1 \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

2.6.3 MVO Version 3: Risk-Return Trade-Off

Version 3 of MVO involves minimizing the risk-return trade-off.

$$\begin{aligned}
\min_{\mathbf{x}} \ & \boldsymbol{x}^T \mathbf{Q} \mathbf{x} - \psi \boldsymbol{\mu}^T \mathbf{x} \\
\text{s.t.} \ & \mathbf{1}^T \mathbf{x} = 1 \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

2.6.4 Maximize the Ex Ante Sharpe Ratio

The Sharpe ratio is a performance metric that allows us to compare the return of an asset through a reward-risk ratio. Ex Ante refers to using estimated parameters to compute the Sharpe ratio, in contrast with Ex Post which uses realized values to compute the Sharpe ratio. This gives an idea of what the future expectation of the portfolio should look like. The ratio is given below:

$$SR_p = \frac{\boldsymbol{\mu}^T \mathbf{x} - \mathbf{r}_f}{\sqrt{\mathbf{x}^T \mathbf{Q} \mathbf{X}}}$$

The SR is highly non-linear and difficult to solve in a standard maximization problem. However, applying two key transformations: $\kappa = \frac{1}{(\mu - r_f)^T x}$ and $y = \kappa x$ allow for a convex optimization problem in the following form:

$$
\begin{aligned}
\min_{\mathbf{y},\kappa} \quad & \mathbf{y}^T\mathbf{Q}\mathbf{y} \\
\text{s.t.} \quad & (\boldsymbol{\mu} - \mathbf{r}_f)^T\mathbf{y} = 1 \\
& \mathbf{1}^T\mathbf{y} = \kappa \\
& \mathbf{A}\mathbf{y} \leq \mathbf{b}\kappa \\
& \kappa \geq 0 \\
& \mathbf{y} \geq 0
\end{aligned}
$$

It is important to note that this minimizing $y^T Q y$ is equivalent to maximizing $\frac{1}{\sqrt{y^T Q y}}$ which indeed, after algebraic manipulation, is equivalent to maximizing $\frac{\mu^T x - r_f}{\sqrt{x^T Q x}}$. Additionally, it is necessary to recover the optimal portfolio weights since a budget constraint was not explicitly specified:

$$
\mathbf{x}_j^* = \frac{\mathbf{y}_j^*}{\kappa^*} = \frac{\mathbf{y}_j^*}{\sum_{i=1}^{n} \mathbf{y}_i^*}
$$

2.6.5 Risk Parity

Risk parity introduces the idea that all assets ought to have an equal risk contribution, thus diversifying an investor's risk as much as possible. It becomes clear that the risk parity optimization problem solves the portfolio optimization problem by assigning wealth in such a way that the marginal risk of each individual asset is equalized. The optimization setup, in its convex scheme, is presented below:

$$
\begin{aligned}
\min_{\mathbf{y}} \quad & \frac{1}{2}\mathbf{y}^T\mathbf{Q}\mathbf{y} - c\sum_{i=1}^{n} ln(y_i), \ where\ c > 0 \\
\text{s.t.} \quad & \mathbf{y} \geq \mathbf{0}
\end{aligned}
$$

Where c is a positive arbitrary constant, and y is the decision variable used to recover optimal portfolio weights.

2.6.6 Conditional Value at Risk (CVaR)

Value-at-Risk (VaR) is a statistical 'tail-based' risk measure that quantifies the extent and probabilities of potential financial losses over a specific time frame. More specifically, VaR is the minimum loss in a portfolio with a probability of greater than or equal to $1 - \alpha$, where α is a confidence interval. Since VaR optimization is a non-convex activity, Conditional Value-at-Risk (CVaR) is introduced to measure the expected value of losses greater than or equal to VaR. Hence, the formal definition of CVaR is of the following form:

$$CVaR_\alpha(\mathbf{x}) = \frac{1}{1-\alpha}\int_{f(\mathbf{x},\mathbf{r})\geq VaR_\alpha(\mathbf{x})}^{\infty} f(\mathbf{x},\mathbf{r})p(\mathbf{r})d\mathbf{r}$$

Where $p(r)$ is the probability density function of the vector of random asset returns, and $f(x,\ r)$ is the loss of portfolio for a realization of random asset returns. This definition implies that the CVaR is the average loss in excess of VaR.

By strictly minimizing variance, a portfolio is subject to aversions to any deviations in variance - regardless of whether it may actually be in favor. Hence, in the following form, the CVaR optimization is transformed into a convex form to minimize the downside risk only:

$$\begin{aligned}
\min_{\mathbf{x},\mathbf{z},\gamma} \quad & \gamma + \frac{1}{(1-\alpha)S}\sum_{s=1}^{S} z_s \\
\text{s.t.} \quad & z_s \geq 0, \qquad\qquad\quad s = 1,\ldots,S \\
& z_s \geq -\hat{\mathbf{r}}_s^T\mathbf{x} - \gamma, \;\; s = 1,\ldots,S \\
& \boldsymbol{\mu}^T\mathbf{x} \geq R \\
& \mathbf{1}^T\mathbf{x} = 1
\end{aligned}$$

Where an auxiliary variable z_s is introduced to deal with the non-smoothness in the objective function.

2.6.7 Minimum Average Turnover Rate

This model minimizes the average turnover rate and penalizes the portfolio volatility using the risk-aversion parameter, subject to a target return.

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \|\mathbf{x}-\mathbf{x}_0\|_2 + \psi\sqrt{\mathbf{x}^T\mathbf{Q}\mathbf{x}} \\
\text{s.t.} \quad & \boldsymbol{\mu}^T\mathbf{x} \geq R \\
& \mathbf{1}^T\mathbf{x} = \mathbf{1} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

2.7 Robust Optimization Models

Robust optimization deals with uncertainty sets, rather than taking on a stochastic approach. The robust framework acknowledges that the inputs are subject to errors and thus belong to an uncertainty set around their nominal estimates. It is related to similar optimization procedures such as stochastic programming. However, the latter considers solutions that are feasible in a probabilistic sense whereas the robust formulation provides solutions that are feasible for all realizations of the uncertain parameters (Georgantas, Doumpos, & Zopounidis, 2021).

2.7.1 Box Uncertainty Set Robust MVO

The simplest approach is to define a 'box uncertainty' set, where the vector of true expected returns lies somewhere within a 'box' centered around the estimate μ. The uncertainty set is described as follows:

$$\mu^{true} \in \mathcal{U}_\mu = \left\{ \mu^{true} \in \mathbb{R}^n : \left| \mu_i^{true} - \hat{\mu}_i \right| \leq \delta_i, \ i = 1, \ldots, n \right\}$$

Where δ_i is the maximum distance between the estimate and true expected returns. The optimization problem is defined below:

$$\begin{aligned} \min_{\mathbf{x}} \ & \mathbf{x}^T \mathbf{Q} \mathbf{x} \\ \text{s.t.} \ & \mu^T \mathbf{x} - \delta^T |\mathbf{x}| \geq R \\ & \mathbf{1}^T \mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0} \\ & where\ \delta_i = \varepsilon_1 \Theta_{ii}^{1/2} = \varepsilon_1 \sigma_i / \sqrt{T} \end{aligned}$$

The specification of $\varepsilon_1 = 1.96$ is adopted for a 95 % confidence level that the true expected return vector lies within the uncertainty set. $|x|$ can be replaced with an auxiliary variable to simplify the implementation.

2.7.2 Ellipsoidal Uncertainty Set Robust MVO

Another approach is to model the uncertainty set as an ellipsoid. This formulation allows the uncertainty set to consider correlations among assets. This uncertainty set is defined as follows:

$$\mu^{true} \in \mathcal{U}_\mu = \left\{ \mu^{true} \in \mathbb{R}^n : (\mu^{true} - \hat{\mu})^T \Theta^{-1} (\mu^{true} - \hat{\mu}) \leq \varepsilon_2^2 \right\}$$

And the optimization problem is transformed as such:

Here, **Θ** represents the diagonals of the covariance matrix divided by the total number of observations, T. By construction, we do need to take the absolute value of x since we are using the sum of squared deviations.

2.7.3 Distributionally Robust MVO

The distributionally robust MVO model is based on work done by Jose Blanchet, Lin Chen, and Xun Yu Zhou (Blanchet, Chen, & Zhou, 2018). It involves usage of the *Wasserstein Distance*, also known as the "earth's movers distance", which is the minimum cost required to transform one probability distribution, P, to another distribution P_n. More specifically, P_n is the empirical probability derived from historical information. The goal is to robustify against worst-case realizations of P_n , such that the Wasserstein distance is less than or equal to δ. The uncertainty set is as follows:

$$\mathcal{U}_\delta(P_n) := \{P : D_c(P, P_n) \leq \delta\}$$

The optimization problem is provided below:

$$\begin{aligned} \min_{\mathbf{x}} \ & (\sqrt{\mathbf{x}^T \mathbf{Q} \mathbf{x}} + \sqrt{\delta} \, || \, \mathbf{x} \, ||_{\,2})^2 \\ \text{s.t.} \ & \mathbf{1}^T \mathbf{x} = 1 \\ & \boldsymbol{\mu}^T \mathbf{x} \geq R + \sqrt{\delta} \, || \, \mathbf{x} \, ||_{\,2} \end{aligned}$$

2.7.4 Box Uncertainty Set Robust Sharpe

A 'robustified' version of the Max Sharpe Ratio model with a box uncertainty set was also implemented. The optimization problem is of the following form:

$$\begin{aligned} \min_{\mathbf{y},\mathbf{z}} \ & \mathbf{y}^T \mathbf{Q} \mathbf{y} \\ \text{s.t.} \ & (\boldsymbol{\mu} - \mathbf{r}_f)^T \mathbf{y} - \boldsymbol{\delta}^T \mathbf{z} \geq 1 \\ & \mathbf{z} \geq \mathbf{y} \\ & \mathbf{z} \geq -\mathbf{y} \\ & \mathbf{1}^T \mathbf{y} \geq \mathbf{0} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Recover the optimal portfolio weights since we did not explicitly specify an asset weighting constraint.

$$x_i^* = \frac{y_i^*}{\sum_{i=1}^{n} y_i^*}$$

2.7.5 Ellipsoidal Uncertainty Set Robust Sharpe

A 'robustified' version of the Max Sharpe Ratio model with an ellipsoidal uncertainty set was also implemented. The optimization problem is of the following form:

$$\begin{aligned} \min_{\mathbf{y},\kappa} \quad & \mathbf{y}^T \mathbf{Q} \mathbf{y} \\ \text{s.t.} \quad & (\boldsymbol{\mu} - \mathbf{r}_f)^T \mathbf{y} - \varepsilon_2 \left\| \boldsymbol{\Theta}^{1/2} \mathbf{y} \right\|_2 \geq 1 \\ & \mathbf{1}^T \mathbf{y} \geq 0 \\ & \mathbf{y} \geq 0 \end{aligned}$$

Recover the optimal portfolio weights since we did not explicitly specify an asset weighting constraint.

$$x_i^* = \frac{y_i^*}{\sum_{i=1}^{n} y_i^*}$$

2.8 The Risk-Aversion Parameter

Selection of a risk-aversion parameter, ψ, is dependent on a multitude of factors - namely, the cost of risk. ψ measures a trade-off between expected return and risk, and thus is different according to an investor's perceptions and objective. Grinold & Kahn provide a formula for selecting ψ:

$$\psi = \frac{\mu_B}{2\sigma_B^{\,2}} \quad \text{Eqn. (1)}$$

where μ_B and σ_B represent the expected excess return and benchmark risk, respectively. These parameters are usually calibrated from a benchmark index, such as the S&P 500. However, $\psi = 1.5$ was chosen for version 3 of MVO since, through testing, values of $\sigma_P \approx 0.03$, $\mu \approx 0.012$, and $\frac{\mu}{2\sigma_B^{\,2}} \approx 1.5$ were obtained through multiple trials. A risk-aversion parameter of 100 was chosen for the minimize average turn-over rate model, as this allowed for a strong balance between high sharpe ratio and low turnover. Other values of the risk-aversion parameter were tested, but had little or negative impact on the metrics of the optimal portfolio.

2.9 LASSO/Ridge Penalty Parameters

The LASSO tuning parameter (also known as the penalty parameter[1]), λ, functions similarly to the risk-aversion parameter, as it determines a trade-off between bias and variance of the model. Chichignoud, Lederer, & Wainwright state that λ is "*unknown in practice*", although there are several advanced methods to estimate values. Various values of λ were tested with both versions of MVO, and the goal was to find limiting λ's such that any further increase/decrease in λ would not make any significant changes to the Sharpe ratio and average turnover rate. After multiple tests, an upper/lower bound of $\lambda = 100$ and $\lambda = 0.0001$ were obtained. Initially, a value of $\lambda = 100$ was tested with each algorithm, and then was successively halved by two (roughly) until the lower bound of $\lambda = 0.001$ was met. This essentially allowed for a wide range of both λ values and observations of Sharpe ratio and average turnover rate, whilst minimizing computation time and resources.

3 Analysis

In this section, the decision-making process is explored by identifying the goals associated with selecting a model, followed by an analysis of the trends observed in the data.

3.1 Decision-Making Process

The team collected data concerning the Sharpe ratio and the average turnover rate for different models and datasets from different time periods as presented in Appendix A and Appendix B. When selecting a model, annual returns are an important metric to consider. However, it is often beneficial to consider annual returns per unit of risk given the length of the investment horizon. Given a longer time horizon, consistently high portfolio volatility in the pursuit of higher returns can put the invested capital at risk. As such, a preference will be given for a higher Sharpe ratio, and a lower average turnover rate when selecting a model. Moreover, a requirement that the average turnover rate be under 30% is imposed (Boyte-White, 2020).

3.2 Results

Some relevant data from different models and datasets from different time periods that was used to guide the decision-making process is presented in Table 2 below. The penalty parameter was set to 3 when implementing LASSO. Also, a confidence level of 95% was used for the ellipsoidal uncertainty set, which is often used as the standard.

Table 2: Sharpe Ratio and Average Turnover Rate For a Few Models and the 3 Different Datasets

Models	Dataset 1	Dataset 2	Dataset 3

[1] "Penalty parameter" is used for the remainder of the text

	Sharpe Ratio	Average Turnover	Sharpe Ratio	Average Turnover	Sharpe Ratio	Average Turnover
1. Risk Parity with LASSO	0.1751	0.1078	0.1214	0.09133	0.17836	0.12752
2. Max. Sharpe Ratio with LASSO	0.23175	0.51586	0.10148	0.56627	0.17059	0.74276
3. Robust (Ellipsoidal) Max. Sharpe Ratio with LASSO	0.23052	0.45171	0.11829	0.3429	0.16617	0.60957
4. MVO Version 1 with LASSO	0.20042	0.14272	0.12451	0.109	0.17887	0.1576
5. Robust (Ellipsoidal) MVO Version 3 with LASSO	0.18299	0.12282	0.11907	0.10697	0.17366	0.14065
6. Minimize Average Turnover Rate with a risk-aversion parameter of 100 with LASSO*	0.19622	0.08196	0.12839	0.05312	0.1801	0.10745
* Selected model is highlighted.						

3.3 Model Selection

Referring to Table 2, there are large fluctuations in the average turnover rate for model 2 and model 3. Furthermore, the average turnover rate for these models is consistently above 0.3. As such, these models will not be considered. Model 1 and model 5 are discarded as well since model 6 posts a higher Sharpe ratio and lower average turnover rate when compared to them. Model 4 has a higher Sharpe ratio than model 6 only with the first data set. However, model 6 consistently posts significantly lower average turnover rates when compared to model 4, and has a higher Sharpe ratio when using dataset 2 and 3. Due to the substantially lower average turnover rate, model 6 will be selected for the automated asset management system.

Additionally, the wealth evolution plot, along with the portfolio weights for dataset 1, 2, and 3 are shown in Figure 1, 2, and 3, respectively.

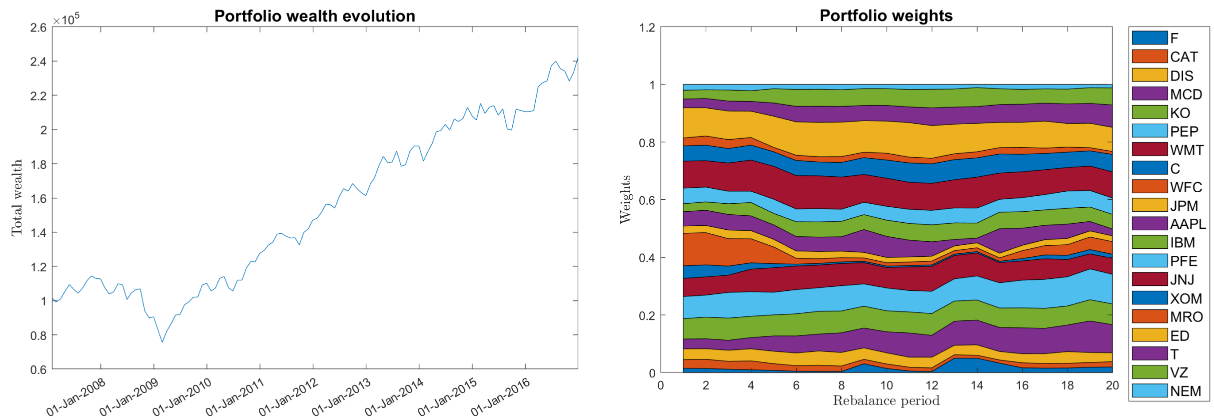

Figure 1: Dataset 1 with minimizing average turnover rate (risk-aversion parameter of 100) with LASSO (penalty parameter of 3)

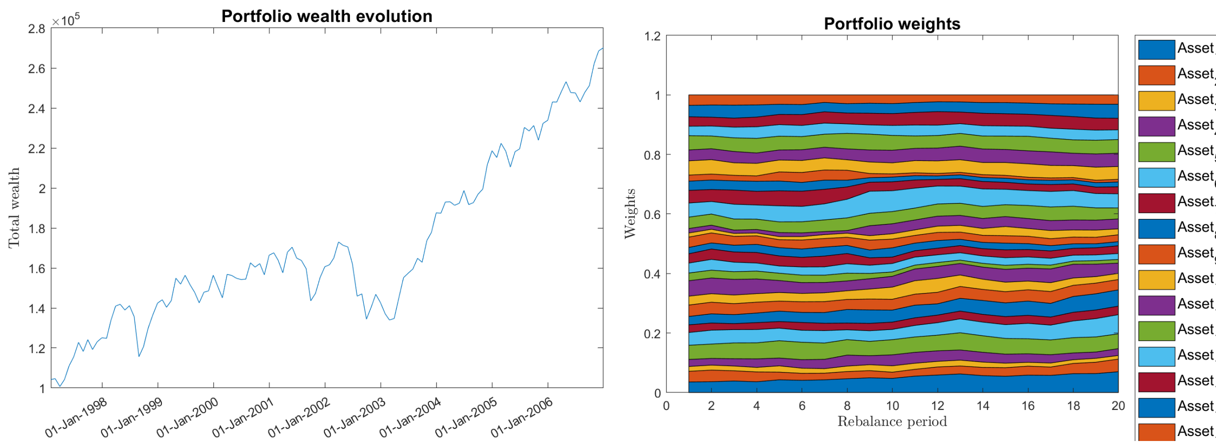

Figure 2: Dataset 2 with minimizing average turnover rate (risk-aversion parameter of 100) with LASSO (penalty parameter of 3)

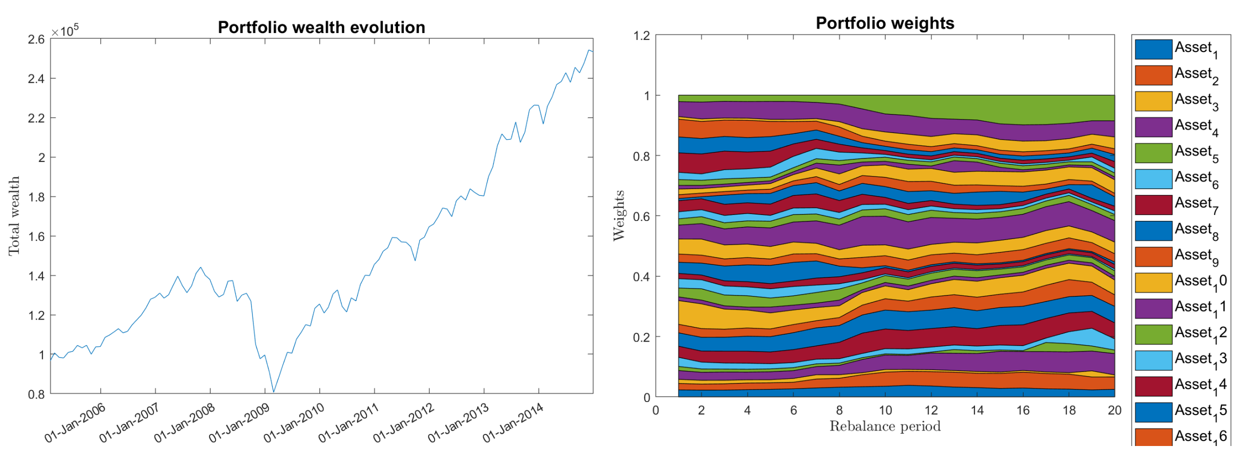

Figure 3: Dataset 3 with minimizing average turnover rate (risk-aversion parameter of 100) with LASSO (penalty parameter of 3)

4 Discussion and Conclusion

In this section, the advantages and disadvantages associated with the selected model are discussed based on considerations of the time period, annual returns, and average turnover rate.

4.1 Sharpe Ratio and Average Turnover Rate

Annual returns are possibly the most important metric when it comes to evaluating the performance of a portfolio. However, it is often advantageous to consider returns per unit of risk because the pursuit of higher returns with no regard for risk can lead to the loss of the invested capital. This is especially true for longer investment horizons. As such, the team valued having a higher Sharpe ratio.

Average turnover rate is a measure of how frequently assets in a fund or portfolio are replaced in a given period of time (Chen & Mansa, 2021). By nature, neither a high average turnover rate is necessarily bad, nor a low average turnover rate is necessarily good. However, in reality, a high average turnover rate often incurs greater costs, such as trading fees and commissions. Furthermore, more frequent asset replacements generate short-term capital gains, which is taxable income, thereby ultimately diminishing the overall portfolio return. Hence, the model with one of the lowest average turnover rates was selected.

4.2 Investment Horizon

Real markets are exposed to varying degrees of both idiosyncratic and unsystematic risks - due to a constantly changing economic and financial landscape. Instead of attempting to profit from precise market-timing, a more conservative and passive investing approach was rationalized. Models with high turnovers (> 30%) were ultimately rejected since frequent replacement of assets incur higher trading cost and potentially lead to more risk. In “Algorithmic Trading and Direct Market Access”, Johnson visualizes the market impact – timing risk tradeoff between aggressive and passive trading approaches.

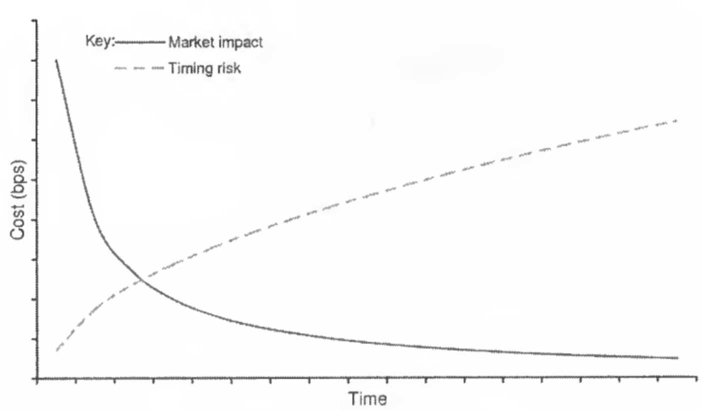

Figure 4: Market Impact - Timing Risk Tradeoff (pg 133)

The chosen model is ‘cost-driven[2]’ whereby it attempts to mitigate market risk. The model was shown to have a stable turnover rate and Sharpe ratio in various market scenarios. However, for a longer investment horizon that includes changing market cycles, lower and consistent turnover rates are preferred as this will reduce transaction costs, have a lesser market impact, and is more tax efficient. Although the portfolio would be more exposed to timing risk, this is neutralized by the fact that financial markets are constantly growing and many investors agree that it is favorable to have “time in the market” than “time the market” (RBC Global Asset Management, 2022). Lastly, short-selling was disallowed due to additional external costs (i.e. margin account and the cost to borrow) and the potential for unlimited losses.

4.3 Advantages and Disadvantages of Selected Model

In the real world, when considering the return of a portfolio, transaction costs are first deducted. However, when evaluating the Sharpe ratio in this project, the transaction costs were not subtracted from the portfolio returns because they were not specified. While the team achieved Sharpe ratios of more than 0.23 in some cases, the average turnover rates were also very high in those cases. Generally, a higher average turnover rate would translate into higher transaction costs, which would lower returns, and subsequently, the post ante Sharpe ratio. Thus, the model selected based on the approach of minimizing the average turnover rate has the advantage of posting very low average turnover rates while penalizing portfolio volatility subject to a target return constraint. Furthermore, this approach has the advantage of valuing higher Sharpe ratios as well.

A disadvantage of the selected model is the wealth tradeoff that becomes apparent when comparing Figure 1 and Figure 5 below. Figure 5 displays a portfolio value of approximately one million dollars at the end of the investment period with an initial invested amount of $100,000. However, the selected model results in a portfolio value of roughly $240,000 at the end of the investment period with an initial investment of $100,000. Thus, the automated asset management system that has been designed is ideal for a risk-averse investor since priority was given to maximizing the Sharpe ratio, which involves maximizing excess returns per unit of risk.

[2] An algorithm that seeks to minimize overall transaction costs including implicit costs like market impact and timing risk

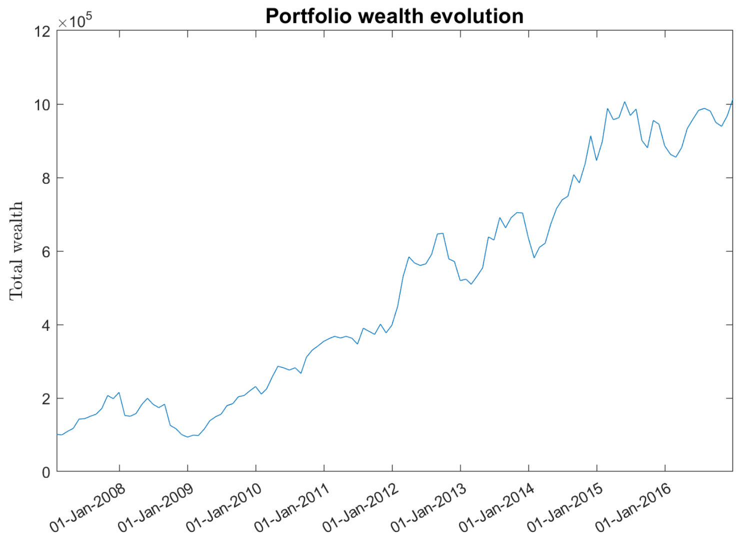

Figure 5: Portfolio value over time with dataset 1 with an approach on maximizing the Sharpe ratio with LASSO (penalty parameter of 3).

4.4 Conclusion

The aim of this project was to compare and contrast different types of optimization models in order to design an automated asset management system. Expanding on what was conducted previously, a number of approaches, such as Risk Parity, maximizing the Sharpe ratio, and robust MVO, were implemented for analysis. An assortment of factor and regression models such as Fama-French, BSS, and LASSO, were tested with various optimization methods to find an acceptable balance between the Sharpe ratio and average turnover. A multitude of portfolios, including their asset distribution and projected wealth evolution, were studied and ratiocinated. The optimal portfolio chosen was a minimum average turnover rate optimization model ($\psi = 100$) with a LASSO-based regression ($\lambda = 3$). Although this goal was realized, there are many opportunities for further analyses. Firstly, a more fitting approach to finding λ would be required to implement a grid-search algorithm. By dividing the domain, every plausible value of λ (which can be initialized randomly) can be calculated along the grid, according to the optimization constraints. This could result in a better value for λ in a more efficient way. Transaction costs and round lots were not explicitly considered in the optimization problem. Accounting for these additional variables would create a more complex problem, but would yield more realistic, delicate solutions. Further analyses may include more re-balancing periods with a larger set of historical data, generating factor returns with principal component analysis, and more stochastic approaches such as stock prices modeled as semi-martingales with Stochastic Portfolio Theory (SPT).

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Appendix A - Additional Data

Note: The last sixty months (five years) of data from “AssetPrices_1” and “FactorReturns_1” was used in the tables below. The risk-aversion parameter for version 3 of MVO is 1.5 consistently.

Table 1: Sharpe Ratio and Average Turnover Using OLS with Version 1 and Version 3 of MVO

Version 1 of MVO		Version 3 of MVO	
Sharpe Ratio	Average Turnover	Sharpe Ratio	Average Turnover
0.20246	0.39943	0.19902	0.24519

Table 2: Sharpe Ratio and Average Turnover Using Best Subset Selection and Version 1 of MVO for Different Cardinality Constraints

Cardinality Constraint	1	2	3	4	5	6	7
Sharpe Ratio	0.14762	0.18841	0.17946	0.19168	0.19342	0.19342	0.19342
Average Turnover	0.27611	0.29373	0.28722	0.27859	0.27156	0.27156	0.27156

Table 3: Sharpe Ratio and Average Turnover Using Best Subset Selection and Version 3 of MVO for Different Cardinality Constraints

Cardinality Constraint	1	2	3	4	5	6	7
Sharpe Ratio	0.01847	0.16586	0.16586	0.17107	0.17338	0.17337	0.17337
Average Turnover	1.2528	0.44873	0.44873	0.42137	0.39540	0.39583	0.39583

Table 4: Sharpe Ratio and Average Turnover Using Fama-French and CAPM Factor Model with Version 1 and Version 3 of MVO

	Version 1 of MVO		Version 3 of MVO	
	Sharpe Ratio	Average Turnover	Sharpe Ratio	Average Turnover
Fama-French	0.18933	0.39242	0.20063	0.22690
CAPM	0.18377	0.36122	0.20106	0.23643

Table 5: Sharpe Ratio and Average Turnover Using LASSO with Version 1 and Version 3 of MVO

Penalty Parameter (λ)	Version 1 of MVO		Version 3 of MVO	
	Sharpe Ratio	Average Turnover	Sharpe Ratio	Average Turnover
100	0.17953	0.12204	0.17953	0.12204
50	0.17953	0.12204	0.17953	0.12204
25	0.17953	0.12204	0.17953	0.12204
12	0.17953	0.12204	0.17953	0.12204
6	0.17344	0.1244	0.18215	0.37315
3	0.20042	0.14274	0.18215	0.37315
1	0.19909	0.16818	0.17586	0.38646
0.5	0.19337	0.16876	0.17582	0.38598
0.25	0.19248	0.16781	0.17646	0.3828
0.12	0.18611	0.18778	0.17766	0.36926
0.06	0.17996	0.21663	0.17737	0.37764
0.03	0.18462	0.28747	0.17678	0.37958
0.01	0.18968	0.25433	0.17641	0.38591
0.001	0.19329	0.26856	0.17372	0.39464
0.0001	0.1934	0.27106	0.1734	0.3957

Table 6: Sharpe Ratio and Average Turnover Using OLS with Version 2 of MVO

Upper bound on Portfolio Volatility	Sharpe Ratio	Average Turnover
0.05	0.17438	0.50869
0.10	0.18196	0.34624
0.15	0.20447	0.36177
0.20	0.19693	0.48233
0.25	0.19314	0.52632
0.30	0.19314	0.52632
0.35	0.19314	0.52632
0.40	0.19314	0.52632

Note: The highlighted row is to indicate the results from using an upper bound on portfolio volatility of 0.15, which is based on volatility of SPY over the last five years. Moving forward, an upper bound on portfolio volatility of 0.15 will be used consistently for version 2 of MVO.

Table 7: Sharpe Ratio and Average Turnover Using BSS with Version 2 of MVO

Cardinality Constraint	Sharpe Ratio	Average Turnover
1	-0.045017	1.5082
2	0.14599	0.8783
3	0.17764	0.27709
4	0.20265	0.4062
5	0.19949	0.41245
6	0.19861	0.4191
7	0.19861	0.4191

Table 8: Sharpe Ratio and Average Turnover Using BSS with Version 3 of MVO with a Penalty Parameter of 1.5

Cardinality Constraint	Sharpe Ratio	Average Turnover
1	0.018465	1.2528
2	0.12529	0.60618
3	0.16586	0.44873
4	0.17107	0.42137
5	0.17338	0.3954
6	0.17337	0.39583
7	0.17337	0.39583

Table 9: Sharpe Ratio and Average Turnover Using Ridge Regression with Version 1 of MVO

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.19035	0.16429
50	0.19017	0.16366
25	0.19006	0.16314
12	0.19	0.16271
6	0.18991	0.16242
3	0.18958	0.16278
1	0.18713	0.17219
0.5	0.18392	0.19319
0.25	0.18048	0.22235
0.12	0.18324	0.25717
0.06	0.18508	0.29205
0.03	0.18753	0.32363
0.01	0.19324	0.35752

0.001	0.20104	0.39264
0.0001	0.20242	0.39722

Table 10: Sharpe Ratio and Average Turnover Using Ridge Regression with Version 2 of MVO

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.20148	0.37727
50	0.20144	0.37887
25	0.2014	0.38053
12	0.20135	0.38282
6	0.20128	0.38644
3	0.20116	0.39274
1	0.20091	0.41007
0.5	0.2011	0.42292
0.25	0.20136	0.4294
0.12	0.20158	0.42182
0.06	0.20219	0.40774
0.03	0.20277	0.39443
0.01	0.20344	0.38031
0.001	0.20285	0.36373
0.0001	0.20273	0.35918

Table 11: Sharpe Ratio and Average Turnover Using Ridge Regression with Version 3 of MVO

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.1664	0.41793
50	0.17936	0.39045
25	0.18873	0.36409
12	0.19688	0.32315
6	0.20156	0.29292
3	0.20247	0.25956
1	0.20055	0.23353
0.5	0.1988	0.22599
0.25	0.19862	0.22423
0.12	0.19907	0.22099
0.06	0.19845	0.22793
0.03	0.19778	0.23653
0.01	0.19722	0.24979
0.001	0.19839	0.25955
0.0001	0.19849	0.25849

Table 11: Sharpe Ratio and Average Turnover Using LASSO with Version 1 of MVO

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.17953	0.12204
50	0.17953	0.12204
25	0.17953	0.12204
12	0.17953	0.12204
6	0.17344	0.1244

3	0.20042	0.14274
1	0.19909	0.16818
0.5	0.19337	0.16876
0.25	0.19243	0.16781
0.12	0.18611	0.18778
0.06	0.17996	0.21663
0.03	0.18462	0.23747
0.01	0.18968	0.25433
0.001	0.19329	0.26856
0.0001	0.1934	0.27106

Table 12: Sharpe Ratio and Average Turnover Using LASSO with Version 2 of MVO

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.16533	0.11697
50	0.16533	0.11697
25	0.16533	0.11697
12	0.16533	0.11697
6	0.037979	0.28805
3	0.24301	0.44397
1	0.20153	0.37674
0.5	0.20149	0.38758
0.25	0.20244	0.39733
0.12	0.20289	0.40203
0.06	0.20292	0.40367
0.03	0.20273	0.40504
0.01	0.20265	0.40632

0.001	0.1995	0.41246
0.0001	0.1987	0.41828

Table 13: Sharpe Ratio and Average Turnover Using LASSO with Version 3 of MVO

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.17953	0.12204
50	0.17953	0.12204
25	0.17953	0.12204
12	0.17953	0.12204
6	0.16184	0.14712
3	0.18215	0.37315
1	0.17586	0.38646
0.5	0.17582	0.38598
0.25	0.17646	0.3828
0.12	0.17766	0.36926
0.06	0.17737	0.37764
0.03	0.17678	0.37958
0.01	0.17641	0.38591
0.001	0.17372	0.39464
0.0001	0.1734	0.3957

Table 14: Sharpe Ratio and Average Turnover Using OLS with Robustified Version 3 of MVO using Box Method

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
0.001	0.17574	0.84211
0.05	0.17802	0.79275

0.10	0.17825	0.74297
0.25	0.16187	0.7509
0.50	0.11922	0.72214
1	0.1281	0.67729
1.5	0.13642	0.6926
5	0.12134	0.65807
10	0.14589	0.4929
20	0.16761	0.37864
50	0.18851	0.3152
500	0.20058	0.29739

Table 15: Sharpe Ratio and Average Turnover Using BSS with Robustified Version 3 of MVO using Box Method

Cardinality Constraint	Sharpe Ratio	Average Turnover
1	0.1781	0.15985
2	0.17511	0.16229
3	0.17401	0.16533
4	0.17663	0.16161
5	0.17626	0.16109
6	0.17625	0.16103
7	0.17625	0.16103

Table 16: Sharpe Ratio and Average Turnover Using LASSO with Robustified Version 3 of MVO using Box Method

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.17981	0.1236
50	0.17981	0.1236

25	0.17981	0.1236
12	0.17981	0.1236
6	0.17977	0.12359
3	0.1802	0.12355
1	0.18061	0.12108
0.5	0.18018	0.12001
0.25	0.18127	0.11998
0.12	0.17586	0.12395
0.06	0.16755	0.13445
0.03	0.1703	0.14281
0.01	0.17416	0.15046
0.001	0.17605	0.15956
0.0001	0.17623	0.16089

Table 16: Sharpe Ratio and Average Turnover Using Ridge Regression with Robustified Version 3 of MVO using Box Method

Penalty Parameter (λ)	Sharpe Ratio	Average Turnover
100	0.17993	0.12002
50	0.18001	0.11899
25	0.18011	0.11816
12	0.18018	0.11747
6	0.18016	0.11701
3	0.17985	0.11684
1	0.17731	0.12057
0.5	0.1733	0.12963
0.25	0.16932	0.14335

0.12	0.16829	0.16002
0.06	0.17091	0.18365
0.03	0.177	0.20885
0.01	0.18546	0.25385
0.001	0.19843	0.29262
0.0001	0.20038	0.29683