COMPLEX ANALYSIS: SOLUTIONS 5

1. Find the poles and residues of the following functions

$$\frac{1}{z^4+5z^2+6}, \qquad \frac{1}{(z^2-1)^2}, \qquad \frac{\pi\cot(\pi z)}{z^2}, \qquad \frac{1}{z^m(1-z)^n} \quad (m,n\in\mathbb{Z}_{>0})$$

Solution: Throughout we use the following formula for calculating residues: If $f(z)$ has a pole of order k at $z=z_0$ then

$$\operatorname{res}(f,z_0)=\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\Big((z-z_0)^k f(z)\Big)\Big|_{z=z_0}.$$

In particular, if $f(z)$ has a simple pole at z_0 then the residue is given by simply evaluating the non-polar part: $(z-z_0)f(z)$, at $z=z_0$ (or by taking a limit if we have an indeterminate form).

Let

$$f(z):=\frac{1}{z^4+5z^2+6}=\frac{1}{(z^2+2)(z^2+3)}=\frac{1}{(z+i\sqrt{2})(z-i\sqrt{2})(z+i\sqrt{3})(z-i\sqrt{3})}.$$

This has simple poles at $z=\pm i\sqrt{2},\pm i\sqrt{3}$ with residues

$$\operatorname{res}(f,i\sqrt{2})=(z-i\sqrt{2})\frac{1}{z^4+5z^2+6}\Big|_{z=i\sqrt{2}}=\frac{1}{(z+i\sqrt{2})(z^2+3)}\Big|_{z=i\sqrt{2}}=\frac{1}{2i\sqrt{2}},$$

$$\operatorname{res}(f,-i\sqrt{2})=(z+i\sqrt{2})\frac{1}{z^4+5z^2+6}\Big|_{z=-i\sqrt{2}}=-\frac{1}{2i\sqrt{2}},$$

$$\operatorname{res}(f,i\sqrt{3})=-\frac{1}{2i\sqrt{3}},$$

$$\operatorname{res}(f,-i\sqrt{3})=\frac{1}{2i\sqrt{3}}.$$

For the second one let

$$f(z) = \frac{1}{(z^2-1)^2} = \frac{1}{(z+1)^2(z-1)^2}.$$

This has double poles at ± 1. From the formula we get

$$\operatorname{res}(f,1) = \frac{d}{dz}\frac{1}{(z+1)^2}\Big|_{z=1} = -1/4,$$

$$\operatorname{res}(f,-1) = \frac{d}{dz}\frac{1}{(z-1)^2}\Big|_{z=-1} = 1/4.$$

For the third let

$$f(z) = \frac{\pi\cot(\pi z)}{z^2}.$$

Now, $\cot(\pi z)$ has poles wherever $\sin(\pi z) = 0$, so at $z = n \in \mathbb{Z}$. About these points we have

$$\begin{aligned}\sin(\pi z) =& \sin(\pi n) + \pi\cos(\pi n)(z-n) - \pi^2\sin(\pi n)\frac{(z-n)^2}{2!} - \pi^3\cos(\pi n)\frac{(z-n)^3}{3!} + \cdots \\ =&(-1)^n\pi(z-n)\big[1 - \pi^2\frac{(z-n)^2}{3!} + O((z-n)^4)\big]\end{aligned}$$

and

$$\begin{aligned}\cos(\pi z) =& \cos(\pi n) - \pi\sin(\pi n)(z-n) - \pi^2\cos(\pi n)\frac{(z-n)^2}{2!} + \pi^3\sin(\pi n)\frac{(z-n)^3}{3!} + \cdots \\ =&(-1)^n\big[1 - \pi^2\frac{(z-n)^2}{2!} + O((z-n)^4)\big].\end{aligned}$$

Hence, for z close to $n \in \mathbb{Z}$, we have

$$\begin{aligned}\cot(\pi z) =& \frac{1-\pi^2(z-n)^2/2 + O((z-n)^4)}{\pi(z-n)\big[1-\pi^2(z-n)^2/6 + O((z-n)^4)\big]} \\ =& \frac{1-\pi^2(z-n)^2/2 + O((z-n)^4)}{\pi(z-n)}\big[1+\pi^2(z-n)^2/6 + O((z-n)^4)\big] \\ =& \frac{1}{\pi(z-n)} - \frac{\pi}{3}(z-n) + O((z-n)^3)\end{aligned}$$

Therefore, $f(z) = \pi\cot(\pi z)/z^2$ has simple poles at $z = n \neq 0$ and a triple pole at $z = 0$. For the simple poles we have

$$\operatorname{res}(f,n) = (z-n)\frac{\pi\cot(\pi z)}{z^2}\Big|_{z=n} = \frac{1}{n^2}.$$

For the triple pole at at $z = 0$ we have

$$f(z) = \frac{1}{z^3} - \frac{\pi^2}{3}\frac{1}{z} + O(z)$$

so the residue is $-\pi^2/3$.

Finally, the function

$$f(z) = \frac{1}{z^m(1-z)^n}$$

has a pole of order m at $z = 0$ and a pole of order n at $z = 1$. From the formula for residues we have

$$\begin{aligned}
\operatorname{res}(f, 0) =& \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{1}{(1-z)^n}\right)\bigg|_{z=0} \\
=& \frac{n(n+1)\cdots(n+m-2)}{(m-1)!} \\
=& \frac{(n+m-2)!}{(n-1)!(m-1)!}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{res}(f, 1) =& \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{(-1)^n}{z^m}\right)\bigg|_{z=1} \\
=& \frac{-m(m+1)\cdots(m+n-2)}{(n-1)!} \\
=& -\frac{(m+n-2)!}{(m-1)!(n-1)!} \\
=& -\operatorname{res}(f, 0).
\end{aligned}$$

2. Use the substitution $e^{i\theta} = z$ along with the residue theorem to show that

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

Solution: As suggested we let $e^{i\theta} = z$ so that $d\theta = dz/(iz)$ and the integral becomes

$$\frac{1}{i}\int_{|z|=1} \frac{dz}{z(2 + (z + z^{-1})/2)} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2 + 4z + 1}.$$

Now $z^2 + 4z + 1$ has zeros of order 1 at $z = z_\pm = -2 \pm \sqrt{3}$ and so the integrand has simple poles at z_+ and z_-. Only z_+ lies in the unit disk and therefore by the residue theorem

$$\begin{aligned}
\frac{2}{i}\int_{|z|=1}\frac{dz}{z^2+4z+1} =&\frac{2}{i}\times 2\pi i\times \operatorname{res}\Big(\frac{1}{z^2+4z+1}, z_+\Big)\\
=&4\pi(z-z_+)\frac{1}{z^2+4z+1}\Big|_{z=z_+}\\
=&4\pi(z-z_+)\frac{1}{(z-z_+)(z-z_-)}\Big|_{z=z_+}\\
=&4\pi\frac{1}{z_+-z_-}\\
=&4\pi\frac{1}{2\sqrt{3}}\\
=&\frac{2\pi}{\sqrt{3}}.
\end{aligned}$$

3. Evaluate the following integrals via residues. Show all estimates.

(i)

$$\int_0^\infty \frac{x^2}{x^4+5x^2+6}dx$$

(ii)

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}dx,\quad a \text{ real}$$

(iii)

$$\int_0^\infty \frac{\log x}{1+x^2}dx$$

Solution: (i) Since the integrand is an even function the integral in question is equal to $I/2$ where

$$I=\int_{-\infty}^\infty \frac{x^2}{x^4+5x^2+6}dx.$$

As a function of a complex variable, the integrand has simple poles at $\pm i\sqrt{2}$, $\pm i\sqrt{3}$. We will be considering a semicircular contour in the upper half plane so we only need calculate the residues at $z=i\sqrt{2}, i\sqrt{3}$. A slight modification of the first calculation in question 1 gives

$$\operatorname{res}\Big(\frac{z^2}{z^4+5z^2+6}, i\sqrt{2}\Big)=\frac{(i\sqrt{2})^2}{2i\sqrt{2}}=\frac{i\sqrt{2}}{2}$$

and
$$\operatorname{res}\left(\frac{z^2}{z^4+5z^2+6}, i\sqrt{3}\right) = -\frac{(i\sqrt{3})^2}{2i\sqrt{3}} = -\frac{i\sqrt{3}}{2}.$$

Now, Consider the semicircular contour Γ_R, which starts at R, traces a semicircle in the upper half plane to $-R$ and then travels back to R along the real axis. Then, on taking R large enough, by the residue theorem
$$\int_{\Gamma_R} \frac{z^2}{z^4+5z^2+6}dz = 2\pi i \times \sum \text{residues inside } \Gamma_R = 2\pi i(\frac{i\sqrt{2}}{2} - \frac{i\sqrt{3}}{2}) = \pi(\sqrt{3}-\sqrt{2}).$$
On the other hand
$$\lim_{R\to\infty}\int_{\Gamma_R} \frac{z^2}{z^4+5z^2+6}dz = I + \lim_{R\to\infty}\int_{\gamma_R} \frac{z^2}{z^4+5z^2+6}dz$$
where γ_R is the semicircle in the upper half plane. But by the Estimation Lemma
$$\Big|\int_{\gamma_R} \frac{z^2}{z^4+5z^2+6}dz\Big| \leqslant \pi R \max_{z\in\gamma_R}\Big|\frac{z^2}{z^4+5z^2+6}\Big| \ll \frac{1}{R} \to 0$$
as $R \to \infty$. Hence, $I = \pi(\sqrt{3}-\sqrt{2})$ and so
$$\int_0^\infty \frac{x^2}{x^4+5x^2+6}dx = \frac{\pi}{2}(\sqrt{3}-\sqrt{2}).$$
The last estimate in the inequality for the integral over γ_R should be clear since the dominant term is the z^4 term in the denominator, but for completeness:
$$\begin{aligned}\pi R \max_{z\in\gamma_R}\Big|\frac{z^2}{z^4+5z^2+6}\Big| &= \frac{\pi}{R}\max_{z\in\gamma_R}\Big|\frac{1}{1+5z^{-2}+6z^{-4}}\Big| \\ &\leqslant \frac{\pi}{R}\max_{z\in\gamma_R}\frac{1}{1-5|z|^{-2}-6|z|^{-4}} \\ &= \frac{\pi}{R}\frac{1}{1-5R^{-2}-6R^{-4}}\end{aligned}$$
where we have used $|z+w| \geqslant |z|-|w|$ in the second line. On taking $R \geqslant 5$, say, this is $\leqslant 2\pi/R$ and the constant 2π is absorbed into the $\ll$ symbol.

(ii). We have
$$\int_0^\infty \frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\Im\left(\int_{-\infty}^\infty \frac{xe^{ix}}{x^2+a^2}dx\right).$$
Denote this last integral by J. Again, we will consider J as the horizontal section of the contour Γ_R from part (i).

In the upper half plane the integrand has a simple pole at $z = ia$ with residue

$$\operatorname{res}\Big(\frac{ze^{iz}}{z^2+a^2}, ia\Big) = (z-ia)\frac{ze^{iz}}{z^2+a^2}\bigg|_{z=ia} = \frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}.$$

Hence, by the residue theorem

$$\pi i e^{-a} = \lim_{R\to\infty}\int_{\Gamma_R}\frac{ze^{iz}}{z^2+a^2}dz = J + \lim_{R\to\infty}\int_{\gamma_R}\frac{ze^{iz}}{z^2+a^2}dz.$$

Thus it remains to show that this last integral vanishes in the limit. This is similar to question 7 (ii) of Problems 3; a trivial estimate of the integrand is $\ll 1/R$ which is not enough for the Estimation Lemma. Instead we apply integration by parts which is probably the quickest way (see Problems 3). Integrating e^{iz} and differentiating the rest gives

$$\int_{\gamma_R}\frac{ze^{iz}}{z^2+a^2}dz = \frac{ze^{iz}}{i(z^2+a^2)}\bigg|_R^{-R} - \frac{1}{i}\int_{\gamma_R}\Big(\frac{1}{z^2+a^2} - \frac{2z^2}{(z^2+a^2)^2}\Big)e^{iz}dz.$$

The first term on the right is $-2R\cos R/i(R^2+a^2) \ll 1/R$. For the integrals we use the Estimation Lemma to give

$$\Big|\int_{\gamma_R}\frac{e^{iz}}{z^2+a^2}dz\Big| \leqslant \pi R\max_{z\in\gamma_R}\Big|\frac{e^{iz}}{z^2+a^2}\Big| \leqslant \pi R\frac{1}{R^2-a^2} \ll \frac{1}{R},$$

$$\Big|\int_{\gamma_R}\frac{z^2e^{iz}}{(z^2+a^2)^2}dz\Big| \leqslant \pi R\max_{z\in\gamma_R}\Big|\frac{z^2e^{iz}}{(z^2+a^2)^2}\Big| \leqslant \pi R^3\frac{1}{(R^2-a^2)^2} \ll \frac{1}{R},$$

as $R\to\infty$. Hence, $J = \pi i e^{-a}$ and so

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\Im(J) = \frac{\pi}{2}e^{-a}.$$

(iii) This is quite hard and, as I discovered recently, the solution is in Conway's book anyway (pg. 117–118). My bad. It's still instructive to attempt this before reading Conway though.

We see that, as a function of a complex variable, the integrand has a branch cut and simple poles at $z = i, -i$. Taking the branch of the log with $-\pi < \arg(z) < \pi$, we would like to choose a contour which lies just above and below the cut and that also picks up the residues at $i, -i$. A natural choice is the contour we used in the lectures. This consisted of a small circle about $z = 0$, horizontal lines just above and below the negative real axis, and a large circle completing the contour.

Unfortunately, with this choice the integrals over the horizontal lines L_1, L_2 are given by

$$\begin{aligned}&\int_{L_1} \frac{\log z}{1+z^2}dz + \int_{L_2} \frac{\log z}{1+z^2}dz\\ =&\int_{-\infty}^{0} \frac{\log(x+i\delta)}{1+(x+i\delta)^2}dx + \int_{0}^{-\infty} \frac{\log(x-i\delta)}{1+(x-i\delta)^2}dx\\ \to&\int_{-\infty}^{0} \frac{\log|x|+\pi i}{1+x^2}dx + \int_{0}^{-\infty} \frac{\log|x|-\pi i}{1+x^2}dx\\ =&2\pi i\int_{-\infty}^{0} \frac{1}{1+x^2}dx\end{aligned}$$

and the integral we're after has disappeared.

This motivates the choice of a new contour: we want something with a horizontal line over the whole real axis, since then the integral over this line is given by

$$\int_{-\infty}^{\infty} \frac{\log|x|}{1+x^2}dx = 2\int_{0}^{\infty} \frac{\log x}{1+x^2}dx$$

and we avoid the problem of cancellation. Consequently, we choose a branch of $\log z$ with a branch cut along the negative imaginary axis. To avoid $z=0$ and the branch cut we indent our contour with a small circle in the upper half plane. We complete the contour with a large circle.

Thus, let $\Gamma_{r,R}$ be the contour consisting of a line from r to R, a semicircle in the upper half plane traced from R to $-R$, a line from $-R$ to $-r$, and a semicircle in the upper half plane traced from $-r$ to r. Follwing the details in Conway we find

$$\int_{0}^{\infty} \frac{\log x}{1+x^2}dx = 0.$$

4. Let Γ_N be the square which crosses the real axis at $\pm(N+1/2)$ with $N \in \mathbb{N}$.

(i) Show that $\cot(\pi z)$ is bounded on Γ_N and hence show that

$$\lim_{N\to\infty} \int_{\Gamma_N} \frac{\pi\cot(\pi z)}{z^2}dz = 0.$$

(ii) For a given N compute the above integral via residues. Conclude something interesting.

Solution: (i) We have

$$\cot(\pi z) = i\frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}} = i\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = i\frac{e^{2\pi i x}e^{-2\pi y}+1}{e^{2\pi i x}e^{-2\pi y}-1}.$$

Thus, if $y \geqslant 1$

$$|\cot(\pi z)| \leqslant \frac{1+e^{-2\pi y}}{1-e^{-2\pi y}} \leqslant \frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

Since $\cot(\overline{\pi z}) = \overline{\cot(\pi z)}$ we obtain the same bound for $y \leqslant -1$. It remains to show that $\cot(\pi z)$ is bounded for $z = \pm(N+1/2)+iy$ with $|y|<1$. But in this case

$$|\cot(\pi z)| = \left|\frac{e^{\pm 2\pi i(N+1/2)}e^{-2\pi y}+1}{e^{\pm 2\pi i(N+1/2)}e^{-2\pi y}-1}\right| = \frac{|-e^{-2\pi y}+1|}{|-e^{-2\pi y}-1|} < \frac{e^{2\pi}-1}{e^{-2\pi}+1} =: C.$$

Since C is bigger than the bound for $|y| \geqslant 1$, we may say that $|\cot(\pi z)| < C$ on all of Γ_N.

(ii) By the estimation lemma

$$\left|\int_{\Gamma_N} \frac{\pi\cot(\pi z)}{z^2}dz\right| \leqslant 8\pi(N+1/2)\max_{z\in\Gamma_N}\left|\frac{\cot(\pi z)}{z^2}\right| < 8\pi(N+1/2)\frac{C}{(N+1/2)^2} = \frac{8\pi C}{N+1/2}$$

and this tends to zero as $N \to \infty$.

On the other hand, for a fixed N this integral is given by $2\pi i \times$ the sum of residues inside Γ_N. Using the results from question 1 then gives

$$\int_{\Gamma_N} \frac{\pi\cot(\pi z)}{z^2}dz = 2\pi i\Big(\sum_{\substack{-N\leqslant n\leqslant N\\ n\neq 0}} \frac{1}{n^2} - \frac{\pi^2}{3}\Big).$$

Therefore, on letting $N \to \infty$ we have

$$0 = \sum_{\substack{-\infty\leqslant n\leqslant\infty\\ n\neq 0}} \frac{1}{n^2} - \frac{\pi^2}{3} = 2\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{\pi^2}{3}$$

and so

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

which is interesting.

5. (i) Let $f(z) = z^6 + \cos z$. Find the change in argument of $f(z)$ as z travels once around the circle of radius 2, center zero, in the positive direction.

(ii) How many solutions does $3e^z - z = 0$ have in the disk $|z| \leqslant 1$?

(iii) Use Rouche's Theorem to prove that a polynomial of degree n has n roots in $\mathbb{C}$.

Solution: (i) By the argument principle, the change in argument of $f(z)$ as z travels around the circle is equal to $(2\pi \times)$ the number of zeros minus poles of $f(z)$ inside the circle, so just the number of zeros then.

To find the number of zeros of $f(z)$ inside the circle we compare $f(z)$ with its dominant term z^6 and apply Rouche's Theorem. So let $g(z) = z^6$. On the circle we have

$$|g(z)| = |z|^6 = 2^6 = 64$$

and

$$|f(z) - g(z)| = |\cos(z)| \leqslant \frac{|e^{iz}| + |e^{-iz}|}{2} \leqslant e^{|z|} = e^2 < 64.$$

Thus, Rouche's Theorem applies and $f(z)$ has the same number of zeros inside the circle as z^6, which is 6. Hence, the change in argument is $2\pi \times 6 = 12\pi$.

(ii). Rephrasing the question we ask for the number of zeros of $f(z) = 3e^z - z$ in the closed unit disk. We apply Rouche's Theorem again. This time the dominant term is $g(z) = 3e^z$. On the unit circle we have

$$|g(z)| = 3e^{\Re(z)} \geqslant 3e^{-1} > 1$$

and

$$|f(z) - g(z)| = |z| = 1.$$

By Rouche's Theorem $3e^z - z$ has the same number of zeros in the unit disk as $3e^z$, which is none. So the answer is no solutions.

(iii) In the lectures we showed that the polynomial $z^6 + z + 2$ has 6 zeros in the disk $|z| \leqslant 2$ by comparing it with z^6 and applying Rouche's Theorem. We generalise this by comparing a general polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \qquad a_n \neq 0$$

with its leading term

$$g(z) = a_n z^n$$

on arbitrarily large circles. On the circle $|z| = R > 1$ we have

$$|g(z)| = |a_n| R^n$$

and

$$\begin{aligned}|f(z)-g(z)| =&|a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\cdots+a_0|\\ \leqslant&|a_{n-1}|R^{n-1}+|a_{n-2}|R^{n-2}+\cdots+|a_0|\\ <&n\max_{0\leqslant j\leqslant n-1}|a_j|R^{n-1}\\ =&\Big(\frac{n\max_{j=0}^{n-1}|a_j|}{|a_n|R}\Big)|a_n|R^n\end{aligned}$$

Therefore, on a circle of radius $R \geqslant n\max_{j=0}^{n-1}|a_j|/|a_n|$ we have $|f(z)-g(z)|<|g(z)|$. By Rouche's Theorem $f(z)$ has the same number of zeros inside the circle as $g(z)$, which is n.

6. Prove that the function

$$f(z)=\sum_{n\in\mathbb{Z}}\frac{1}{(n+z)^2}$$

is meromorphic on $\mathbb{C}$.

Solution: Clearly, $f(z)$ has poles only at the integers so we need to show that $f(z)$ is analytic on $\mathbb{C}\backslash\mathbb{Z}$. By the results given in the lectures, this will follow if we can show that the series is uniformly convergent on compact subsets of said region. Since

$$f(z)=\frac{1}{z^2}+\sum_{n\neq 0}\frac{1}{(n+z)^2}$$

it suffices to show the uniform convergence of this last sum.

So let $|z|\leqslant R$, let z be fixed away from integers: $|z+n|\geqslant\rho>0\ \forall n\in\mathbb{Z}$, and let $-M$ be the nearest integer to z. Note that the series looks like $2\sum_{n\neq 0}n^{-2}$ which is convergent. So we first pull out the convergent part by writing

$$\sum_{\neq 0}\left|\frac{1}{(n+z)^2}\right|=\sum_{n\neq 0}\frac{1}{n^2|1+z/n|^2}.$$

Then if $|n|\leqslant|M|$

$$|z+n|\geqslant|z+m| \implies |1+z/n|\geqslant(|M|/|n|)|1+z/m|\geqslant|1+z/M|\geqslant\rho/|M|.$$

If $|n|>|M|+1$

$$|1+z/n|\geqslant|1+\Re(z)/n|\geqslant 1-\frac{|\Re(z)|}{|n|}>1-\frac{|M|+1}{|M|+2}=\frac{1}{|M|+2}.$$

Therefore,

$$\begin{aligned}\sum_{n\neq 0}\left|\frac{1}{(n+z)^2}\right| \leqslant & \frac{|M|^2}{\rho^2}\sum_{0\neq|n|\leqslant|M|}\frac{1}{n^2}+(|M|+2)^2\sum_{|n|>|M|+1}\frac{1}{n^2}+\frac{1}{|z\pm(M\pm 1)|^2}\\ \leqslant & 2\zeta(2)\frac{(R+1)^2}{\rho^2}+2\zeta(2)(R+3)^2+\frac{4}{\rho^2}\\ < & \infty.\end{aligned}$$

Hence, by the Weierstrass M test with $M_n = n^{-2}\max((R+3)^2,(R+1)^2/\rho^2)$ the series is uniformly convergent on compact subsets of $\mathbb{C}\backslash\mathbb{Z}$.

7. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function whose derivatives $\varphi^{(k)}(t)$, $k \geqslant 0$, are of rapid decay at ∞ i.e.

$$\lim_{t\to\infty} t^A\varphi^{(k)}(t) = 0$$

for all $A \in \mathbb{R}$ and all $k \geqslant 0$. The *Mellin transform* of φ is defined as

$$\tilde{\varphi}(z) = \int_0^\infty \varphi(t)t^{z-1}dt.$$

(i) Prove that $\tilde{\varphi}(z)$ is analytic in the region $\Re(z) > 0$.

(ii) Use integration by parts to show that $\tilde{\varphi}(z)$ can be analytically continued to $\mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$, with possible simple poles at $z = -n$.

(iii) Find the residues at these poles and compute the formal sum

$$\sum_{n=0}^{\infty}\mathrm{res}(\tilde{\varphi}(z)t^{-z}, -n).$$

What does this look like?

(iv) Given part (iii), suggest an argument that would prove

$$\frac{1}{2\pi i}\int_{\Re(z)=c}\tilde{\varphi}(z)t^{-z}dz = \varphi(t)$$

where the integral is over the vertical line from $c - i\infty$ to $c + i\infty$ with $c > 0$.

Solution: (i). Since φ is smooth on $\mathbb{R}$ it is bounded in a neighbourhood of $t = 0$: $|\varphi(t)| \leqslant C$ for $t \in [0, t_0]$, say. Then

$$\int_0^{t_0}|\varphi(t)t^{z-1}|dt \leqslant C\int_0^{t_0} t^{\Re(z)-1}dt = Ct_0^{\Re(z)}/\Re(z) < \infty.$$

Also, since φ is of rapid decay, for any z there exists a $t_1 = t_1(\Re(z))$ such that $|\varphi(t)t^{z-1}| \leqslant Dt^{-2}$ for all $t \geqslant t_1$. Then

$$\int_{t_1}^{\infty} |\varphi(t)t^{z-1}|dt \leqslant D/t_1 < \infty.$$

Since the remaining integral $\int_{t_0}^{t_1} |\varphi(t)t^{z-1}|dt$ is finite the integral is absolutely convergent in the region $\Re(z) > 0$.

Let γ be a closed curve in the region $\Re(z) > 0$. By absolute convergence and Fubini's Theorem

$$\int_\gamma \tilde{\varphi}(z)dz = \int_\gamma \int_0^\infty \varphi(t)t^{z-1}dtdz = \int_0^\infty \varphi(t)t^{-1}\int_\gamma t^z dzdt.$$

But since t^z is analytic in the region $\Re(z) > 0$ for all $t \in [0, \infty)$ this last inner integral is zero by Cauchy's Theorem. Hence,

$$\int_\gamma \tilde{\varphi}(z)dz = 0.$$

The continuity of $\tilde{\varphi}(z)$ follows exactly as it does for the Gamma function, hence it is analytic by Morera's Theorem.

(ii). Integrating by parts once gives

$$\tilde{\varphi}(z) = \frac{\varphi(t)t^z}{z}\bigg|_{t=0}^{\infty} - \frac{1}{z}\int_0^\infty \varphi'(t)t^z dt = -\frac{1}{z}\int_0^\infty \varphi'(t)t^z dt.$$

By the same reasoning used in part (i), the integral $\int_0^\infty \varphi'(t)t^z dt$ is analytic in the region $\Re(z) > -1$. Hence, the above expression provides a meromorphic continuation of $\tilde{\varphi}(z)$ to said region. At $z = 0$ we have a residue of

$$z \cdot -\frac{1}{z}\int_0^\infty \varphi'(t)t^z dt\bigg|_{z=0} = -\int_0^\infty \varphi'(t)dt = \varphi(0).$$

In particular, if $\varphi(0) = 0$ then there is no pole at $z = 0$.

More generally, integrating by parts n times gives

$$\tilde{\varphi}(z) = \frac{(-1)^n}{z(z+1)\cdots(z+n-1)}\int_0^\infty \varphi^{(n)}(t)t^{z+n-1}dt. \tag{1}$$

Again, this expression is seen to be analytic in the region $\Re(z) > -n$, with the exception of possible simple poles at $z = 0, -1, \ldots, -n+1$. Since n is arbitrary, we're done.

(iii). From equation (1) we see that

$$\begin{aligned}\operatorname{res}(\tilde{\varphi}(z), z=-n) =&(z+n)\frac{(-1)^{n+1}}{z(z+1)\cdots(z+n)}\int_0^\infty \varphi^{(n+1)}(t)t^{z+n}dt\Big|_{z=-n}\\ =&\frac{(-1)^{n+1}}{(-n)(-n+1)\cdots(-1)}\int_0^\infty \varphi^{(n+1)}(t)dt\\ =&-\frac{1}{n!}\int_0^\infty \varphi^{(n+1)}(t)dt\\ =&\frac{1}{n!}\varphi^{(n)}(0).\end{aligned}$$

In particular, if $\varphi(t)$ is of rapid decay at zero as well as at infinity, then $\tilde{\varphi}(z)$ is entire.

Since all the poles are simple, an extra factor of t^{-z} in the above residue calculations gives a factor of $t^{-z}|_{z=-n} = t^n$ and hence

$$\sum_{n=0}^\infty \operatorname{res}(\tilde{\varphi}(z)t^{-z}, -n) = \sum_{n=0}^\infty \frac{1}{n!}\varphi^{(n)}(0)t^n.$$

This, of course, looks like $\varphi(t)$ in the form of a Taylor expansion about $t=0$.

(iv). Assuming that the above series converges and equals $\varphi(t)$ (so φ is analytic in the real analysis sense) then we would like a contour integral which captures all these residues. As suggested, we should look at

$$\frac{1}{2\pi i}\int_{\Re(z)=c}\tilde{\varphi}(z)t^{-z}dz.$$

We would like to truncate this integral at heights $z = c \pm iT$ and then consider the truncated integral as part of a rectangular contour which encloses some of the residues. To estimate these integrals we need bounds on $\tilde{\varphi}(z)$.

From equation (1) we see that for fixed $\Re(z)$, $\tilde{\varphi}(z) \to 0$ as $|\Im(z)| \to \infty$. More precisely, as $\Im(z) \to \infty$ equation (1) gives

$$|\tilde{\varphi}(z)| \leqslant \frac{1}{|z(z+1)\cdots(z+n-1)|}\int_0^\infty |\varphi^{(n)}(t)|t^{\Re(z)+n-1}dt \leqslant \frac{C_{\Re(z),n}}{|\Im(z)|^n},$$

for some constant $C_{\Re(z),n}$. Consequently,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Re(z)=c}\tilde{\varphi}(z)t^{-z}dz =&\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\tilde{\varphi}(z)t^{-z}dz + \frac{1}{2\pi}\Big[\int_T^\infty + \int_{-\infty}^{-T}\Big]\tilde{\varphi}(c+iy)t^{-c-iy}dy\\ =&\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\tilde{\varphi}(z)t^{-z}dz + O(T^{-n+1})\end{aligned}$$

We now consider this last integral as part of a rectangular contour Γ whose left edge crosses the real axis at $-N-1/2$. Then,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\tilde{\varphi}(z)t^{-z}dz &= \frac{1}{2\pi i}\int_{\Gamma}\tilde{\varphi}(z)t^{-z}dz - \frac{1}{2\pi i}\int_{\text{other edges of }\Gamma}\tilde{\varphi}(z)t^{-z}dz \\ &= \sum_{n=0}^{N}\frac{1}{n!}\varphi^{(n)}(0)t^{n} - \frac{1}{2\pi i}\int_{\text{other edges of }\Gamma}\tilde{\varphi}(z)t^{-z}dz\end{aligned}$$

by the residue theorem. Using our bound on $|\tilde{\varphi}(z)|$ we can estimate the remaining integrals. The largest contribution comes from the integral over the left edge of Γ, and this is seen to be $O(t^{M+1/2})$. The other integrals are small in size. On letting $T\to\infty$ we get

$$\frac{1}{2\pi i}\int_{\Re(z)=c}\tilde{\varphi}(z)t^{-z}dz = \sum_{n=0}^{N}\frac{1}{n!}\varphi^{(n)}(0)t^{n} + O(t^{N+1/2}) = \varphi(t).$$