

Introduction to Nuclear Reactor Theory

NUEN 301

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Based upon Dr. Adams' notes.

with contributions by Dr. Ragusa (2005, 2007, 2019, 2020) and Dr. Till (2018)

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Disclaimer: These notes are not substitutes for attending class! They are intended to allow you to do less writing and **more listening & thinking** during class, and to help you keep your notes organized.

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Part I

Overview and Review

Chapter 1

Overview

1.1 Introduction

This course is about the fundamentals of nuclear fission chain reactions and how these chain reactions can be harnessed in nuclear reactors. Recall that neutrons can induce fission in certain heavy nuclides. Recall further that when a heavy nucleus fissions, it releases particles (including neutrons) and energy. These facts suggest the possibility of a self-sustaining energy-releasing chain reaction: neutrons cause fissions, which produce neutrons, which cause more fissions, etc. This, of course, is the idea behind fission reactors.

It should be no surprise that nuclear engineers need to understand a bit more detail than this. How does one build a reactor whose chain reaction is controllable? How does one know whether a given reactor design (or fission system) will work? If a known design is changed in a certain way, how will the reactor/system behavior change? These and many other questions land on nuclear engineers' desks.

Answering them requires knowledge of where the neutrons are, what they are doing, and what happens when neutrons do what neutrons do. This course is about where the neutrons go, what they do, and what happens when they do what they do.

1.2 Life History of a Neutron

A neutron in a reactor ultimately faces limited options:

1. Leak, or
2. Get absorbed.

(A neutron could also decay into a proton, electron, and neutrino, but this is rare enough in a reactor that we ignore it. How do we know we can ignore it?)

If a neutron leaks, we pay it no further attention. If the neutron is absorbed, it can either

- a) Cause fission, or
- b) not.

If the absorbing nucleus fissions, it releases neutrons, gammas, and fission fragments.¹ Most fission fragments are radioactive, and as they and their daughters decay they release more gammas as well as betas and anti-neutrinos. Fission fragments and their radioactive-decay products are collectively called “fission products”.

If the absorbing nucleus does not fission, it is in an excited state, because the neutron absorption adds “binding energy” that puts the new nucleus above its ground state. It gets rid of the excess energy by

emitting stuff,

where “stuff” can be photons (gamma rays), particles (possibly including one or more neutrons), or both. The absorbing nucleus becomes a new kind of nucleus. (Its A -number increases by 1 when it absorbs the neutron, and its A and Z numbers may change further when it relieves itself (de-excites).)

On the way to its ultimate destiny (either leakage or absorption), a neutron may

1. Move in \approx a straight line arbitrarily far without collision;
2. Elastically scatter off of a heavy nucleus, losing little energy;
3. Inelastically scatter off of a heavy nucleus, losing more energy;
4. Elastically scatter off of a light nucleus, maybe losing significant energy;
5. Do any or all of the above any number of times.

¹ “Fission fragments” are the new, smaller nuclei formed by the splitting of a larger nucleus.

1.3 Behavior of Nuclear Reactors/Systems

The overall behavior of a fission reactor/system depends on the balance between neutron loss and neutron production:

neutron production rate $>$ loss rate \Rightarrow **neutron population and power increase**

neutron production rate $=$ loss rate \Rightarrow population and power **stay steady**

neutron production rate $<$ loss rate \Rightarrow population and power **decrease**

Thus, nuclear engineers must be able to estimate the neutron

production rates

and

loss rates

in reactors.

Estimating neutron production and loss rates in a reactor requires knowledge of:

1. How neutrons **interact with matter**,
2. Properties of **the matter (cross sections)**,
3. The neutron **distribution** (“scalar flux” and “net current density”), and
4. **How to use this information.**

If we know these things we can figure out anything we want, such as the **power density/distribution** at a certain point in the reactor. If we don’t know them, we cannot figure out much of anything.

In this course we will discuss all of these things. Our **main focus** will be on **obtaining and using neutron distributions**.

1. We will devote considerable time to techniques for obtaining estimates of the **scalar flux and current**.
2. We will use these quantities to compute **production rates**, **loss rates**, and other items of interest in reactors.
3. We will also explore how **changes** in a reactor’s composition **affect the reactor’s behavior**.

- (a) For example, what happens when fission products (the new smaller atoms formed by splitting the heavy ones) build up?
- (b) What happens when a reactor heats up?
- (c) What would happen if we changed the size or shape of the reactor, or inserted a control rod?

1.3.1 Material Properties and Neutron-Nucleus Interactions

We need to know how the material in our reactor interacts with neutrons. This information is contained in

macroscopic cross sections

A **macroscopic cross section** for a reaction of type “ x ” in a given material (where x represents absorption or fission or elastic scattering or ...) is the

$$\frac{\text{expected number of reactions of type “}x\text{”}}{\text{distance traveled by neutrons in the material.}}$$

The macroscopic cross section is composed of atom densities and “microscopic” cross sections.

Atom densities vary with location, of course, and can also vary with time.

Microscopic cross sections depend on the type of atom, the material temperature, and the neutron kinetic energy. The math is:

$\Sigma_x(\vec{r}, E, T, t)$ = macroscopic cross section for reaction “ x ” at position \vec{r} and time t
for a neutron of kinetic energy E

$$= \sum_{i=1}^I \underbrace{N_i(\vec{r}, t)}_{\text{type-}i \text{ atoms/cm}^3} \underbrace{\sigma_x^i(E, T)}_{\text{microscopic cross section}}, \quad (1.1)$$

The rate per unit volume at which a reaction of type “ x ” occurs is usually given by a simple formula, which we illustrate here using absorption as our example reaction type and letting E denote the neutron kinetic energy:

$$\begin{aligned} \text{ARRD} &= \left[\frac{\text{expected absorptions}}{\text{cm}^3 - s} \right] \text{ at position } \vec{r} \text{ and time } t \\ &= \int_0^\infty \underbrace{\left[\frac{\text{exp'd absorptions of energy-}E \text{ neutrons}}{\text{neutron path length at } E} \right]}_{\Sigma_a(\vec{r}, E, T, t)} \underbrace{\left[\frac{\text{distance traveled by energy-}E \text{ neutrons}}{\text{cm}^3 - s - [\text{unit energy}]} \right]}_{\text{"scalar flux," } \phi(\vec{r}, E, t)} dE \\ &= \int_0^\infty \Sigma_a(\vec{r}, E, T, t) \phi(\vec{r}, E, t) dE . \end{aligned} \quad (1.2)$$

where ARRD = Absorption Reaction Rate Density. Every word is important: rate because you have per s in the units; density because you have per cm^3 . Caveat: the notion of density depends on what you are talking about; it is not always per cm^3 ; for instance, inhabitant density is not per cm^3 but per mile² or per km² ...

1.3.2 Neutron Distributions

Recall that on the way to its ultimate destiny (leakage or absorption) a neutron may

1. Zip along arbitrarily far without collision;
2. Elastically scatter off of a heavy nucleus, losing little energy;
3. Inelastically scatter off of a heavy nucleus, losing much of its energy;
4. Elastically scatter off of a light nucleus, losing much of its energy;
5. Do any or all of the above any number of times.

The distribution of neutrons in a reactor depends on the details of each of these events. To calculate this distribution, we will need to know the probability of each event (as a function of the neutron’s location and energy) and what each event does to the neutron population. Thus, we must learn the details of important neutron reactions, including absorption, scattering, and fission.

Once we know about neutron-material interactions, we will be able to write a statement of **conservation of free neutrons** in precise mathematical language. In English, this statement will say:

$$\left[\begin{array}{c} \text{Rate of Change} \\ \text{of neutron population} \end{array} \right] = [\text{Gain Rate}] - [\text{Loss Rate}] . \quad (1.3)$$

This is a simple equation, based on the simple concept of **conservation**. Although simple, this equation and concept are very important. We will see that

There is only one possible neutron distribution that satisfies this statement
in complete detail in any given reactor!

In this course we will explore what our conservation equation tells us about the **time dependence** of the total neutron population in a reactor. Everyone can probably guess that the population in a supercritical reactor increases with time, for example, but our exploration will allow us to quantify exactly how it increases. That is, we will figure out a mathematical expression for the neutron population as a function of t .

We will make the diffusion approximation to the neutron conservation equation, giving us the neutron diffusion equation. We will spend a lot of time on the diffusion equation. This will give us insight into how neutrons distribute themselves **spatially** in reactors.

We will also study how neutrons distribute themselves **in energy**. The distribution in position and energy is quantified by the “energy-dependent scalar flux,” $\phi(\vec{r}, E)$. We will spend significant time trying to estimate this distribution in the four-dimensional position-energy space.

We will not spend much time on how neutrons distribute themselves in **direction**. This more complicated topic will be deferred to a later point in your studies. The neutron distribution in position, energy, and direction is quantified by the “energy-dependent angular flux.”

We will finally put everything together to gain insight into how neutrons distribute themselves in space, energy, and time in homogeneous reactors. We will calculate the multiplication factor of simple reactors given their sizes, shapes, and material compositions. We will design critical reactors that meet desired characteristics.

This, of course, is very cool.

1.3.3 Using Material Properties & Neutron Distributions

Our main use of material properties and neutron distributions is to calculate neutron **production and loss rates in reactors**.

1. **Production** comes either from neutron sources in or near the reactor or from neutron-nucleus reactions such as fission.
2. **Loss** comes from either leakage out of the reactor or from neutron absorption by nuclei.

Thus, we must know how to compute neutron-nucleus reaction rates and how to compute leakage rates.

1. **Reaction rates** follow Eq. (1.2) above (or something similar), which requires knowledge of the neutron scalar flux, ϕ .
2. **Leakage rates** are given by a simple formula involving the neutron net current density (often referred to as simply the “current”), which we will discuss in Chapter 6.

1.4 Summary

Our basic goal is to gain an understanding of how reactors behave and why they behave that way.

This will come from knowledge of :

1. Neutron interactions with matter,
2. Properties (such as cross sections) of reactor materials,
3. Neutron distributions in reactors,
4. How to use cross sections and neutron distributions to calculate neutron production and loss rates.

Chapter 2

Nuclear Reactors: Some History

2.1 A Brief History

Before we mention the first human-made¹ nuclear reactor, the Chicago Pile number 1 (CP-1), and its reaching criticality in 1942, we will briefly look back over about 50 years of research that led to the criticality of CP-1. It is customary to choose 1896 as the starting point of the nuclear era.

2.1.1 Before 1896

Elements are the building blocks of matter. The smallest particle of an element that still retains the identity of that element is the atom. All atoms of a given element are identical to one another, but differ from the atoms of other elements. Ancient Greeks (**Democrite**) first predicted the existence of the atom around 500 BC. They named the predicted particle ‘*atomos*,’ meaning “indivisible.”

In 1803, John **Dalton** (1766-1844) proposed a systematic set of postulates to describe the atom. Dalton’s work paved the way for modern-day acceptance of the atom. Similarly, in 1808, Louis Joseph **Gay-Lussac** (1778-1850) published his memoir on the combination of gaseous substances with each other. Even though his work supported Dalton’s atomic theory, Dalton didn’t accept Gay-Lussac’s work. But the Italian Amedeo **Avogadro** (1776-1856) saw it as the key to a better understanding of molecular constituency. Atomic theories of Dalton, Gay-Lussac and others were the only theories that permitted people to understand

¹Nature assembled critical nuclear reactors on earth long before humans did. Search the internet for the Oklo reactors, for example. Fascinating story!

the periodic table, proposed in 1869 by Dmitri **Mendeleiev** (1834-1907). Nonetheless, many other scientists of the 18th and 19th centuries considered the atom to be merely a subordinate player in chemical reactions, an uninteresting, homogeneous, positively charged “glob” that contained scattered electrons (the plum-pudding model of the atom, aka Thomson’s model). That premise remained unchallenged until the end of the 19th century, when a series of brilliant discoveries opened the door to the atomic science of the twentieth century. Working concurrently and often collaboratively, three pioneering scientists helped release the genie of the atom.

2.1.2 Early Breakthroughs

2.1.2.1 The fortuitous discovery of radioactivity

Henri **Becquerel** (1852-1908), a French physicist, was the son and grandson of physicists. Becquerel was familiar with the work of Wilhelm Conrad Roentgen, who on December 22, 1895 “photographed” his wife’s hand, revealing the unmistakable image of her skeleton, complete with wedding ring. Roentgen’s wife had placed her hand in the path of X-rays which Roentgen created by beaming an electron ray energy source onto a cathode tube. Roentgen’s discovery of these “mysterious” rays capable of producing an image on a photographic plate excited scientists of his day, including Becquerel. Becquerel chose to study the related phenomena of fluorescence and phosphorescence. In March of 1896, quite by accident, he made a remarkable discovery.

Becquerel found that, while the phenomena of fluorescence and phosphorescence had many similarities to each other and to X-rays, they also had important differences. While fluorescence and X-rays stopped when the initiating energy source was halted, phosphorescence continued to emit rays some time after the initiating energy source was removed. However, in all three cases, the energy was derived initially from an outside source.

In March of 1896, during a time of overcast weather, Becquerel found he couldn’t use the sun as an initiating energy source for his experiments. He put his wrapped photographic plates away in a darkened drawer, along with some crystals containing uranium. Much to Becquerel’s surprise, the plates were exposed during storage by invisible emanations from the uranium. The emanations did not require the presence of an initiating energy source—the crystals emitted rays on their own! Although Becquerel did not pursue his discovery of radioactivity, others did and, in so doing, changed the face of both modern medicine and modern science.

2.1.2.2 Po and Ra: Marie Skłodowska-Curie (1867-1934) and Pierre Curie (1859-1906)

Working in the Becquerel lab, **Marie Curie** and her husband, **Pierre**, began what became a life-long study of radioactivity. It took fresh and open minds, along with much dedicated work, for these scientists to establish the properties of radioactive matter. Marie Curie wrote, “The subject seemed to us very attractive and all the more so because the question was entirely new and nothing yet had been written upon it.”

Becquerel had already noted that uranium emanations could turn air into a conductor of electricity. Using sensitive instruments invented by Pierre Curie and his brother, Pierre and Marie Curie measured the ability of emanations from various elements to induce conductivity. On February 17, 1898, the Curies tested an ore of uranium, pitchblende, for its ability to turn air into a conductor of electricity. The Curies found that the pitchblende produced a current 300 times stronger than that produced by pure uranium. They tested and recalibrated their instruments, and yet they still found the same puzzling results. The Curies reasoned that a very active unknown substance in addition to the uranium must exist within the pitchblende. In the title of a paper describing this hypothesized element (which they named polonium after Marie’s native Poland), they introduced the new term: “radio-active.”

After much grueling work, the Curies were able to extract enough polonium and another radioactive element, radium, to establish the chemical properties of these elements. Marie Curie, first with her husband and then continuing after his death, established the first quantitative standards by which the rate of radioactive emission of charged particles from elements could be measured and compared. In addition, she found that there was a decrease in the rate of radioactive emissions over time and that this decrease could be calculated and predicted. But perhaps Marie Curie’s greatest and most unique achievement was her realization that radiation is an atomic property of matter rather than a separate independent emanation.

Despite the giant step forward which science had now taken in it’s understanding of radioactivity, scientists still understood little of the structure of the atom. This understanding awaited the work of Ernest Rutherford.

2.1.2.3 Ernest Rutherford’s model of the atom

In 1911, **Rutherford** (1871-1937) conducted a series of experiments in which he bombarded a piece of gold foil with positively charged (alpha) particles emitted by radioactive material. Most of the particles passed through the foil undisturbed, suggesting that the foil was made up mostly of empty space rather than of a sheet of solid atoms. Some alpha particles, however, “bounced back,” indicating the presence of solid matter. Atomic particles, Rutherford’s

work showed, consisted primarily of empty space surrounding a well-defined central core called a nucleus. His atomic model superseded quickly the plum-pudding model that had been proposed by J.J. Thomson (who was his former boss!). (Lest you think Thomson was not an outstanding scientist, remember that he discovered the electron.)

In a long and distinguished career, Rutherford laid the groundwork for the determination of atomic structure. In addition to defining the planetary model of the atom, he showed that radioactive elements undergo a process of decay over time. And, in experiments which involved what newspapers of his day called “splitting the atom,” Rutherford was the first to artificially transmute one element into another.

Rutherford eventually coined the terms for some of the most basic principles in the field: alpha, beta, and gamma rays, the proton, the neutron, half-life, and daughter atoms. Several of the century’s giants in physics studied under him, including Niels Bohr, James Chadwick, and Robert Oppenheimer.

2.1.3 On the road to fission

2.1.3.1 The neutron

In 1932, James **Chadwick** (1891-1974) discovered the neutron.

2.1.3.2 Artificial radioactivity

In 1934, Frederic **Joliot** (1900-1958) and his wife Irene **Joliot-Curie** (1897-1956), discovered the artificial radioactivity using an alpha-source. When bombarded with alpha particle, aluminum 27 (usual aluminum) produces one neutron and phosphorous 30. The latter beta-decays in 2.5 minutes.

2.1.3.3 Neutron induced reactions

After the discovery of the neutron by Chadwick in 1932, nuclear physicists used neutrons to bombard many elements. It was easy to get the neutron to the nuclei because of the neutral charge of the neutron. If the neutron is captured, an isotope of the initial target nucleus is obtained. The new isotope may be radioactive, in which case it decays, often by beta-decay. Enrico **Fermi** (1901-1954), among others, performed research on neutron-induced reactions. Fermi was mostly interested in discovering and filling the blanks in the periodic table. He bombarded uranium (which at that time was the last known element) with neutrons. He also wanted to explore the domain past uranium, the transuranics element. To his surprise, he obtained many more radioactive elements than he was hoping for.

2.1.3.4 Fission

Fermi's experiment was reproduced in many laboratories. It took 4 years for an answer to come. In 1938, Otto **Hahn** and Fritz **Strassman** showed the presence of barium as a result of bombardment of uranium with neutrons. Since Ba is an element of intermediate mass, the conclusion is obvious: uranium had fissioned after absorbing a neutron. Lisa **Meitner** calculated that fission released a **huge amount of energy**.

Researchers around the world reproduced the fission experiments. Soon, it was discovered that fission releases **3 secondary neutrons** (a figure that was a little bit optimistic!), which led to the idea of creating **a self-sustained chain reaction**.

Francis **Perrin** was the first to mention the idea of **critical** size (equivalent to critical mass).

2.1.4 WW II

The next parts of the story are well known:

- the fear of Nazi Germany,
- Szilard's visit to Einstein and Einstein's letter to Pres. Roosevelt,
- the Manhattan project (which we will discuss more in detail when we discuss the CP-1 reactor)
- the uranium bomb, Hiroshima, 8/6/1945
- the plutonium bomb, Nagasaki, 8/9/1945

2.2 The Chicago Pile -1

2.2.1 Details and pictures

Construction of CP-1, or Chicago Pile Number One, was done under the football stadium in an abandoned squash court. On December 2, 1942, mankind first harnessed the energy of the atom. Fermi's pile produced only 1/2 watt of power. But that was all the power the United States needed to start the next phase of the bomb's development.

The pile contained 771,000 pounds of graphite, 80,590 pounds of uranium oxide and 12,400 pounds of uranium metal when it went "critical." It cost about \$1 million to produce and build. The pile took the form of a flattened ellipsoid which measured 25 feet wide and 20 feet high.

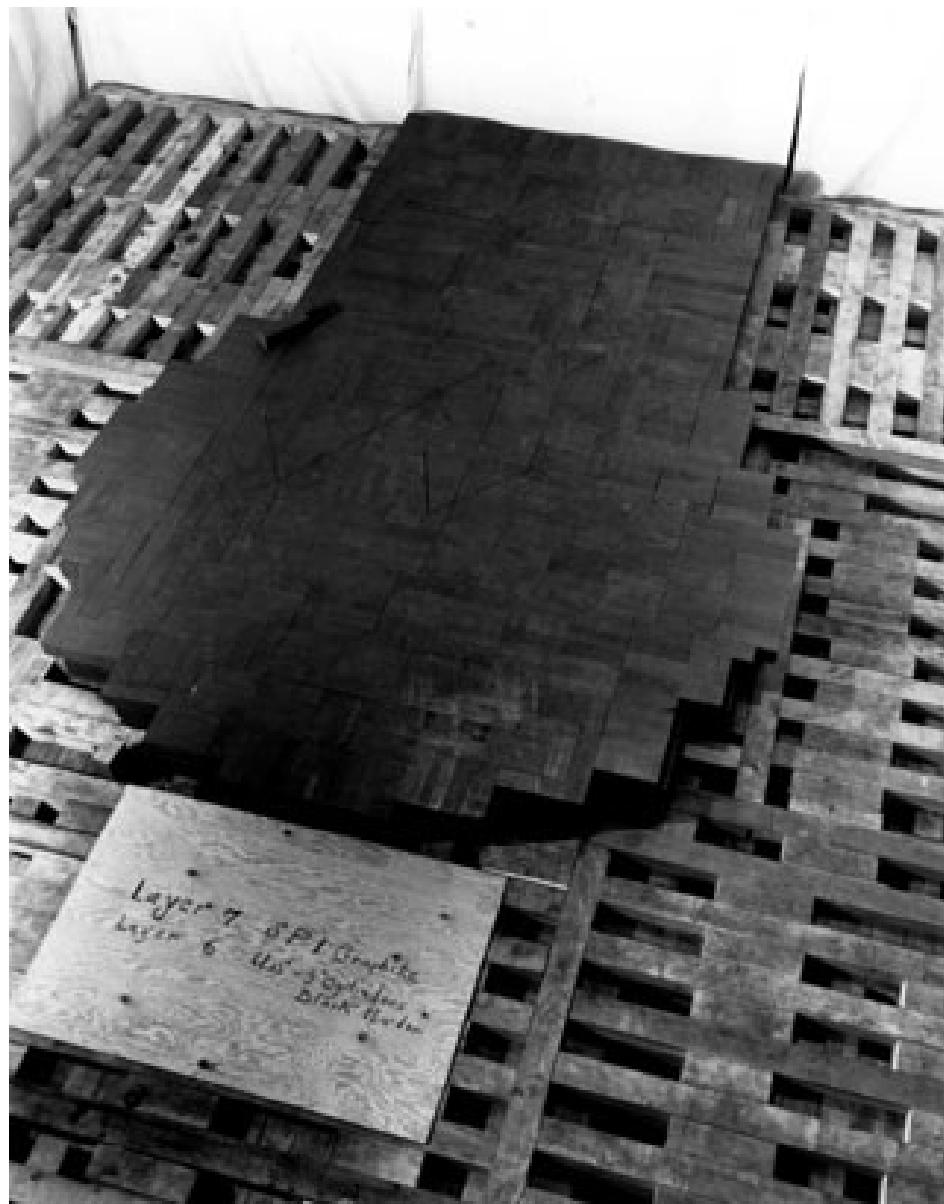


Figure 2.1: The 7th layer of graphite and edges of 6th layer containing 3-inch pseudospheres of black uranium oxide. Beginning with layer 6, alternate courses of graphite containing uranium metal and/or uranium oxide fuel were separated by layers of solid graphite blocks.



Figure 2.2: Tenth layer of graphite blocks containing pseudospheres of black and brown uranium oxide. The brown briquets, slightly richer in uranium, were concentrated in the central area. In the foreground and on either side are cavities filled with graphite, now presumed to have been an expedient measure dictated by shortage of fuel and, possibly, a last minute change in the lattice arrangement.



Figure 2.3: Construction detail, Chicago Pile

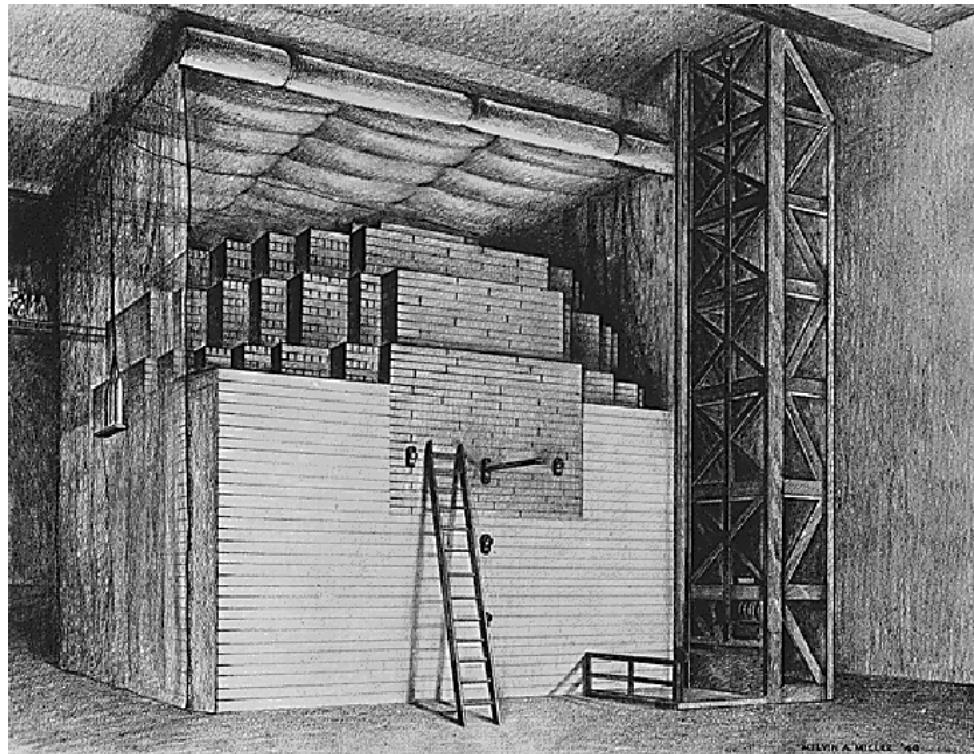


Figure 2.4: Drawing of CP-1



Figure 2.5: A painting of the CP-1 scene

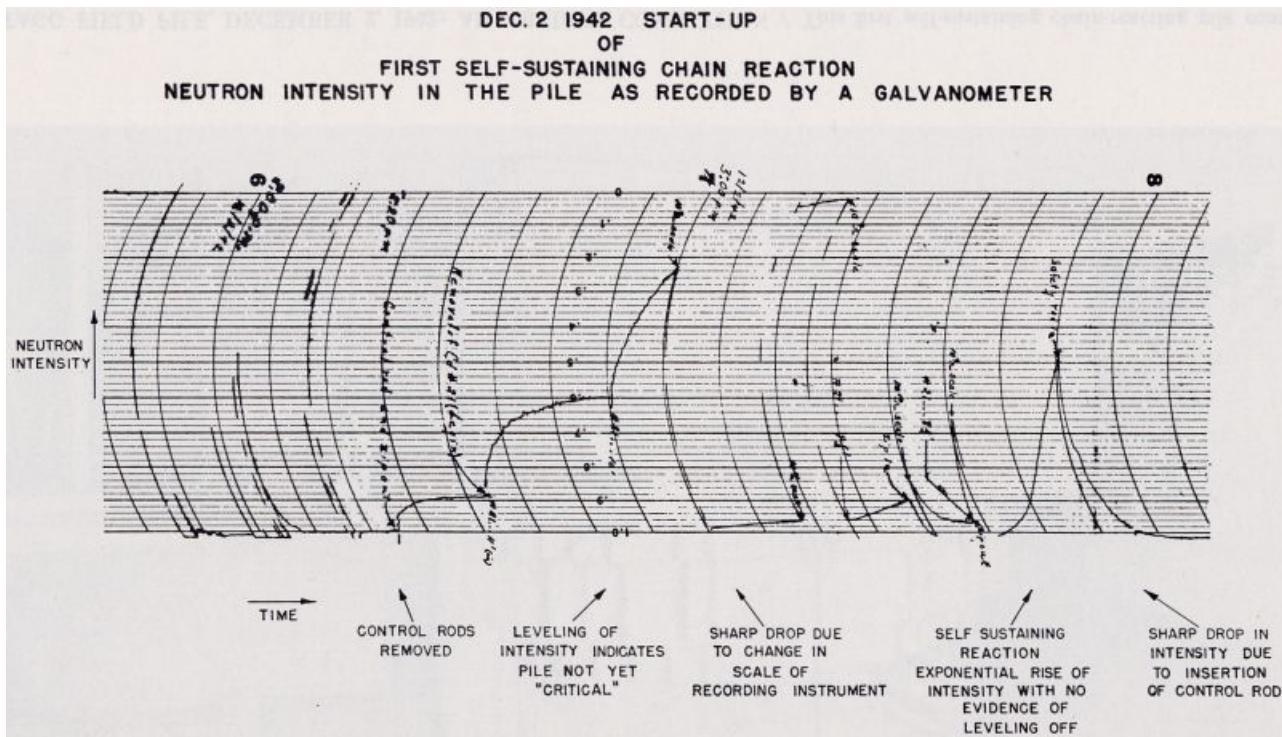


Figure 2.6: Log chart

2.2.2 Already many lessons learned

Developments occurred with amazing speed in the early days of nuclear engineering. Recall that the neutron was discovered in 1932, fission in 1938 (published in 1939), and then Fermi had a critical reactor operating before the end of 1942. Equally amazing was how quickly the pioneers developed an understanding of concepts and developed procedures for dealing with fission reactors. For example:

1. Heterogeneous core configuration:

Fissions occur in the uranium, neutrons stream out into the graphite, where they lose energy (slow down) by scattering and have a good chance of escaping parasitic absorption in ^{238}U while they are slowing down. Most neutrons eventually reach thermal equilibrium (low energy), and then most of them migrate back into the uranium, where slow neutrons are likely to cause fission. If the graphite and uranium were uniformly mixed, too many neutrons would be parasitically absorbed in U-238 while they were slowing down, and even an infinitely large reactor could not go critical unless the uranium were enriched in ^{235}U .

2. Instrumentation and control (I&C) systems were present:

One operator was keeping an eye on the “console” while the other one was in charge on the Cadmium control rod. Cadmium is a neutron absorber. When the control rod is inserted, more neutrons are absorbed, stopping the chain reaction; when it is withdrawn, more neutrons are available for additional fissions, enhancing the chain reaction. The communication between instrumentation and control allows for the desired operation functions. (Here that communication was verbal!)

3. Safety:

Besides I&C, additional safety measures are needed. At CP-1, in case of an emergency, a safety rod, maintained outside of the core through a rope that ran up to the balcony, would be released by cutting the rope with an ax. This rod would be released with the “Safety Control Rod Ax Man” (SCRAM) cut the rope. The last line of defense consisted of a “liquid-control squad” that stood on a platform, ready to flood the pile with a cadmium-salt solution.

4. Radiation protection:

The detector (galvanometer) suspended outside of the reactor would measure the ambient radiation.

2.2.3 Detailed Story of The First Pile

By Corbin Allardice and Edward R. Trapnell

On December 2, 1942, man first initiated a self-sustaining nuclear chain reaction, and controlled it.

Beneath the West Stands of Stagg Field, Chicago, late in the afternoon of that day, a small group of scientists witnessed the advent of a new era in science. History was made in what had been a squash-rackets court.

Precisely at 3:25 p.m., Chicago time, scientist George Weil withdrew the cadmium-plated control rod and by his action man unleashed and controlled the energy of the atom.

As those who witnessed the experiment became aware of what had happened, smiles spread over their faces and a quiet ripple of applause could be heard. It was a tribute to Enrico Fermi, Nobel Prize winner, to whom, more than to any other person, the success of the experiment was due.

Fermi, born in Rome, Italy, on September 29, 1901, had been working with uranium for many years. In 1934 he bombarded uranium with neutrons and produced what appeared to

be element 93 (uranium is element 92) and element 94. However, after closer examination it seemed as if nature had gone wild; several other elements were present, but none could be fitted into the periodic table near uranium where Fermi knew they should have fitted if they had been the transuranic elements 92 and 94. It was not until five years later that anyone, Fermi included, realized he had actually caused fission of the uranium and that these unexplained elements belonged back in the middle part of the periodic table.

Fermi was awarded the Nobel Prize in 1938 for his work on transuranic elements. He and his family went to Sweden to receive the prize. The Italian Fascist press severely criticized him for not wearing a Fascist uniform and failing to give the Fascist salute when he received the award. The Fermis never returned to Italy.

From Sweden, having taken most of his personal possessions with him, Fermi proceeded to London and thence to America where he has remained ever since.

The modern Italian explorer of the unknown was in Chicago that cold December day in 1942. An outsider looking into the squash court where Fermi was working would have been greeted by a strange sight. In the center of the 30- by 60-foot room, shrouded on all but one side by a gray balloon cloth envelope, was a pile of black bricks and wooden timbers, square at the bottom and a flattened sphere on top. Up to half of its height, its sides were straight. The top half was domed, like a beehive. During the construction of this crude appearing but complex pile (the name which has since been applied to all such devices) the standing joke among the scientists working on it was: "If people could see what we're doing with a million-and-a-half of their dollars, they'd think we are crazy. If they knew why we are doing it, they'd know we are."

In relation to the fabulous atomic bomb program, of which the Chicago Pile experiment was a key part, the successful result reported on December 2nd formed one more piece for the jigsaw puzzle which was atomic energy. Confirmation of the chain reactor studies was an inspiration to the leaders of the bomb project, and reassuring at the same time, because the Army's Manhattan Engineer District had moved ahead on many fronts. Contract negotiations were under way to build production-scale chain reactors, land had been acquired at Oak Ridge, Tennessee, and millions of dollars had been obligated.

Three years before the December 2nd experiment, it had been discovered that when an atom of uranium was bombarded by neutrons, the uranium atom sometimes was split, or fissioned. Later, it had been found that when an atom of uranium fissioned, additional neutrons were emitted and became available for further reaction with other uranium atoms. These facts implied the possibility of a chain reaction, similar in certain respects to the reaction which is the source of the sun's energy. The facts further indicated that if a sufficient quantity of uranium could be brought together under the proper conditions, a self-sustaining chain reaction would result. This quantity of uranium necessary for a chain reaction under given conditions is known as the critical mass, or more commonly, the "critical size" of the particular pile.

For three years the problem of a self-sustaining chain reaction had been assiduously studied. Nearly a year after Pearl Harbor, a pile of critical size was finally constructed. It worked. A self-sustaining nuclear chain reaction was a reality.

Construction of the Pile

Construction of the main pile at Chicago started in November. The project gained momentum, with machining of the graphite blocks, pressing of the uranium oxide pellets, and the design of instruments. Fermi's two "construction" crews, one under Zinn and the other under Anderson, worked almost around the clock. V.C. Wilson headed up the instrument work.

Original estimates as to the critical size of the pile were pessimistic. As a further precaution, it was decided to enclose the pile in a balloon cloth bag which could be evacuated to remove the neutron-capturing air.

This balloon cloth bag was constructed by Goodyear Tire and Rubber Company. Specialists in designing gasbags for lighter-than-air craft, the company's engineers were a bit puzzled about the aerodynamics of a square balloon. Security regulations forbade informing Goodyear of the purpose of the envelope and so the Army's new square balloon was the butt of much joking.

The bag was hung with one side left open; in the center of the floor a circular layer of graphic bricks was placed. This and each succeeding layer of the pile was braced by a wooden frame. Alternate layers contained the uranium. By this layer-on-layer construction a roughly spherical pile of uranium and graphite was formed.

Facilities for the machining of graphite bricks were installed in the West Stands. Week after week this shop turned out graphite bricks. This work was done under the direction of Zinn's group, by skilled mechanics led by millwright August Knuth. In October, Anderson and his associates joined Zinn's men.

Describing this phase of the work, Albert Wattenberg, one of Zinn's group, said, "We found out how coal miners feel. After eight hours of machining graphite, we looked as we were made up for a minstrel. One shower would remove only the surface graphite dust. About a half-hour after the first shower the dust in the pores of your skin would start oozing. Walking around the room where we cut graphite was like walking on a dance floor. Graphite is a dry lubricant, you know, and the cement floor covered with graphite dust was slippery."

Before the structure was half complete, measurements indicated that the critical size at which the pile would become self-sustaining was somewhat less than had been anticipated in the design.

Computations Forecast Success

Day after day the pile grew toward its final shape. And as the size of the pile increased, so did the nervous tension of the men working on it. Logically and scientifically they knew this pile would become self-sustaining. It had to. All the measurements indicated that it would. But still the demonstration had to be made. As the eagerly awaited moment drew nearer, the scientists gave greater and greater attention to details, the accuracy of measurements, and exactness of their construction work.

Guiding the entire pile construction and design was the nimble-brained Fermi, whose associates described him as “completely self-confident but wholly without conceit.”

So exact were Fermi’s calculations, based on the measurements taken from the partially finished pile, that days before its completion and demonstration on December 2, he was able to predict almost to the exact brick the point at which the reactor would become self-sustaining.

But with all their care and confidence, few in the group knew the extent of the heavy bets being placed on their success. In Washington, the Manhattan District had proceeded with negotiations with I duPont de Nemours and Company to design, build, and operate a plant based on the principles of the then unproved Chicago pile. The \$350,000,000 Hanford Engineer Works¹⁰ at Pasco, Washington, was to be the result.

At Chicago during the early afternoon of December 1st, tests indicated that critical size was rapidly being approached. At 4:00 p.m. Zinn’s group was relieved by the men working under Anderson. Shortly afterwards, the last layer of graphite and uranium bricks was placed on the pile. Zinn, who remained, and Anderson made several measurements of the activity within the pile. They were certain that when the control rods were withdrawn, the pile would become self-sustaining. Both had agreed, however, that should measurements indicate the reaction would become self-sustaining when the rods were withdrawn, they would not start the pile operating until Fermi and the rest of the group could be present. Consequently, the control rods were locked and further work was postponed until the following day.

That night the word was passed to the men who had worked on the pile.

Assembly for the Test

About 8:30 on the morning of Wednesday, December 2nd, the group began to assemble in the squash court.

At the north end of the squash court was a balcony about ten feet above the floor of the court. Fermi, Zinn, Anderson, and Compton were grouped around instruments at the east end of the balcony. The remainder of the observers crowded the little balcony. R. G. Nobles,

one of the young scientists who worked on the pile, put it this way: "The control cabinet was surrounded by the "big wheels"; the "little wheels" had to stand back."

On the floor of the squash court, just beneath the balcony, stood George Weil, whose duty it was to handle the final control rods. In the pile were three sets of control rods. One set was automatic and could be controlled from the balcony. Another was an emergency safety rod. Attached to one of this rod was a rope running through the pile and weighted heavily on the opposite end. The rod was withdrawn from the pile and tied by another rope to the balcony. Hilberry was ready to cut this rope with an axe should something unexpected happen, or in case the automatic safety rods failed. The third rod, operated by Weil, was the one which actually held the reaction in check until withdrawn the proper distance.

Since this demonstration was new and different from anything ever done before, complete reliance was not placed on mechanically operated control rods. Therefore, a "liquid-control squad," composed of Harold Lichtenberger, W. Nyer, and A. C. Graves, stood on a platform above the pile. They were prepared to flood the pile with cadmium-salt solution in case of mechanical failure of the control rods.

Each group rehearsed its part of the experiment.

At 9:45 Fermi ordered the electrically operated control rods withdrawn. The man at the controls threw the switch to withdraw them. A small motor whined. All eyes watched the lights which indicated the rod's position.

But quickly, the balcony group turned to watch the counters, whose clicking stepped up after the rods were out. The indicators of these counters resembled the face of a clock, with 'hands' to indicate neutron clock. Nearby was a recorder, whose quivering pen traced the neutron activity within the pile.

Shortly after ten o'clock, Fermi ordered the emergency rod, called "Zip," pulled out and tied.

"Zip out," said Fermi. Zinn withdrew "Zip" by hand and tied it to the balcony rail. Weil stood ready by the "vernier" control rod which was marked to show the number of feet and inches which remained within the pile.

At 10:37 Fermi, without taking his eyes off the instruments, said quietly:

"Pull it to 13 feet, George." The counters clicked faster. The graph pen moved up. All the instruments were studied, and computations were made.

"This is not it," said Fermi. "The trace will go to this point and level off." He indicated a spot on the graph. In a few minutes the pen came to the indicated point and did not go above that point. Seven minutes later Fermi ordered the rod out another foot.

Again the counters stepped up their clicking, the graph pen edged upwards. But the clicking was irregular. Soon it leveled off, as did the thin line of the pen. The pile was not self-sustaining—yet.

At eleven o'clock, the rod came out another six inches; the result was the same: an increase in rate, followed by the leveling off.

Fifteen minutes later, the rod was further withdrawn and at 11:25 it was moved again. Each time the counters speeded up, the pen climbed a few points. Fermi predicted correctly every movement of the indicators. He knew the time was near. He wanted to check everything again. The automatic control rod was reinserted without waiting for its automatic feature to operate. The graph line took a drop, the counters slowed abruptly.

At 11:35, the automatic safety rod was withdrawn and set. The control rod was adjusted and "Zip" was withdrawn. Up went the counters, clicking, clicking, faster and faster. It was the clickety-click of a fast train over the rails. The graph pen started to climb. Tensely, the little group watched, and waited, entranced by the climbing needle.

Whrrrump! As if by a thunder clap, the spell was broken. Every man froze—then breathed a sigh of relief when he realized the automatic rod had slammed home. The safety point at which the rod operated automatically had been set too low.

"I'm hungry," said Fermi. "Let's go to lunch."

Time Out for Lunch

Perhaps, like a great coach, Fermi knew when his men needed a "break."

It was a strange "between halves" respite. They got no pep talk. They talked about everything else but the "game." The redoubtable Fermi, who never says much, had even less to say. But he appeared supremely confident. His "team" was back on the squash court at 2:00 p.m. Twenty minutes later, the automatic rod was reset and Weil stood ready at the control rod.

"All right, George," called Fermi, and Weil moved the rod to a predetermined point. The spectators resumed their watching and waiting, watching the counters spin, watching the graph, waiting for the settling down and computing the rate of rise of reaction from the indicators.

At 2:50 the control rod came out another foot. The counters nearly jammed, the pen headed off the graph paper. But this was not it. Counting ratios and the graph scale had to be changed.

"Move it six inches," said Fermi at 3:20. Again the change—but again the leveling off. Five minutes later, Fermi called: "Pull it out another foot."

Weil withdrew the rod.

"This is going to do it," Fermi said to Compton, standing at his side. "Now it will become self-sustaining. The trace will climb and continue to climb. It will not level off."

Fermi computed the rate of rise of the neutron counts over a minute period. He silently, grim-faced, ran through some calculations on his slide rule.

In about a minute he again computed the rate of rise. If the rate was constant and remained so, he would know the reaction was self-sustaining. His fingers operated the slide rule with lightning speed. Characteristically, he turned the rule over and jotted down some figures on its ivory back.

Three minutes later he again computed the rate of rise in neutron count. The group on the balcony had by now crowded in to get an eye on the instruments, those behind craning their necks to be sure they would know the very instant history was made. In the background could be heard Wilcox Overbeck calling out the neutron count over an annunciator system. Leona Marshall (the only female present), Anderson, and William Sturm were recording the readings from the instruments. By this time the click of the counters was too fast for the human ear. The clickety-click was now a steady brrrr. Fermi, unmoved, unruffled, continued his computations.

The Curve is Exponential

"I couldn't see the instruments," said Weil. "I had to watch Fermi every second, waiting for orders. His face was motionless. His eyes darted from one dial to another. His expression was so calm it was hard to read. But suddenly, his whole face broke into a broad smile."

Fermi closed his slide rule—

"The reaction is self-sustaining," he announced quietly, happily. "The curve is exponential."

The group tensely watched for twenty-eight minutes while the world's first nuclear chain reactor operated.

The upward movement of the pen was leaving a straight line. There was no change in indicate a leveling off. This was it.

"O.K., 'Zip' in," called Fermi to Zinn who controlled that rod. The time was 3:53 p.m. Abruptly, the counters slowed down, the pen slid down across the paper. It was all over.

Man had initiated a self-sustaining nuclear reaction—and then stopped it. He had released the energy of the atom's nucleus and controlled that energy.

Right after Fermi ordered the reaction stopped, the Hungarian-born theoretical physicist Eugene Wigner presented him with a bottle of Chianti wine. All through the experiment Wigner had kept this wine hidden behind his back.

Fermi uncorked the wine bottle and sent out for paper cups so all could drink. He poured a little wine in all the cups, and silently, solemnly, without toasts, the scientists raised the cups to their lips—the Canadian Zinn, the Hungarians Szilard and Wigner, the Italian Fermi, the Americans Compton, Anderson, Hilberry, and a score of others. They drank to success—and to the hope they were the first to succeed.

A small crew was left to straighten up, lock controls, and check all apparatus. As the group filed from the West Stands, one of the guards asked Zinn:

“What’s going on, Doctor, something happen in there?”

The guard did not hear the message which Arthur Compton was giving James B. Conant at Harvard, by long-distance telephone. Their code was not prearranged.

“The Italian navigator has landed in the New World,” said Compton. “How were the natives?” asked Conant. “Very friendly.”

2.3 World nuclear power reactors as of 2013

The following info was downloaded from <http://www.wano.org.uk/>.
WANO = World Association of Nuclear Operators.

KEY TO MAP SYMBOLS, COLOUR CODES AND ABBREVIATIONS					Unit Country/ Area
Plant status category					
As of 1 August 2013					
Operational					
Construction					
Planned					
Permanent Shutdown					
WANO Regional Centres					
AC = Atlanta	PC = Paris				
MC = Moscow	TC = Tokyo				
WANO Offices					
London & Hong Kong					
BRAZIL					
06. Angra	PC / AC				
Map location number	Plant/Station name	Primary Regional Affiliation	Secondary Regional Affiliation	No. of Reactors, type and plant status category	
Reactor type symbols					
PWR	Pressurised light water moderated and cooled reactor (VVER for former USSR)				
BWR	Boiling light water cooled and moderated reactor				
PHWR	Pressurised heavy water moderated and cooled reactor				
FBR	Fast breeder reactor				
LWGR	Light water cooled, graphite moderated reactor				
GCR	Gas cooled, graphite moderated reactor				
REP	Reprocessing facility				
HTGR	High temperature gas reactor				
FSUE	(Atomflot) with 8 atomic icebreakers and 1 atomic container ship				
Floating nuclear power station					
Chalk River National Research Universal (NRU) reactor at the Chalk River Laboratories					

Figure 2.7: Explanation of symbols and abbreviations in the World lists of nuclear reactors.

ARGENTINA					
01. Atucha	PC				
02. Embalse	PC				
ARMENIA					
03. Armenia	MC				
BELGIUM					
04. Doel	PC				
05. Tihange	PC				
BRAZIL					
06. Angra	PC / AC				
BULGARIA					
07. Kozloduy	MC				
CANADA					
08. Bruce	AC				
09. Chalk River NRU	AC				
10. Darlington	AC				
11. Gentilly	AC				
12. Pickering	AC				
13. Point Lepreau	AC				
CHINA					
14. Changjiang	TC				
15. Fangchenggang	PC				
16. Fanglijshan	TC				
17. Fuging	TC				
18. Daya Bay	PC				
19. Haiyang	AC				
20. Hongyanhe	PC				
21. Ling Ao	PC				
22. Ningde	PC				
23. Qinshan	TC				
24. Sanmen	TC				
25. Shidao Bay	AC				
26. Taishan	PC				
27. Tianwan	MC				
28. Yangjiang	PC				
CZECH REPUBLIC					
29. Dukovany	MC				
30. Temelin	MC				
FINLAND					
31. Lovisa	MC				
32. Olkiluoto	PC				
FRANCE					
33. Belleville	PC				
34. Blayais	PC				
35. Bugey	PC				
36. Cattenom	PC				
37. Chinon B	PC				
38. Choob Z	PC				
39. Civaux	PC				
40. Cruas	PC				
41. Dampierre	PC				
42. Fessenheim	PC				
43. Flamanville	PC				
44. Golfech	PC				
45. Gravelines	PC				
46. La Hague	PC				
47. Nogent	PC				
48. Paluel	PC				
49. Penly	PC				
50. Saint-Alban	PC				
51. Saint-Laurent B	PC				
52. Tricastin	PC				
GERMANY					
53. Biblis	PC				
54. Brokdorf	PC				
55. Brunsbuettel	PC				
56. Emsland	PC				
57. Grafenheinfeld	PC				
58. Grohnde	PC				
59. Gundremmingen	PC				
60. Isar	PC				
61. Kruemmel	PC				
KOREA					
62. Neckarwestheim	PC				
63. Philippsburg	PC				
64. Unterweser	PC				
HUNGARY					
65. Paks	MC				
INDIA					
66. Kaiga	TC				
67. Kakrapar	TC				
68. Kalpakkam	TC				
69. Kudankulam	MC				
70. Madras	TC				
71. Narora	TC				
72. Rajasthan	TC				
73. Tarapur	TC				
IRAN					
74. Bushehr	MC				
JAPAN					
75. Fukushima Daiichi	TC				
76. Fukushima Daini	TC				
77. Genkai	TC				
78. Hamaoka	TC				
79. Higashidori	TC				
80. Ikata	TC				
81. Kaminosaki	TC				
82. Kashiwazaki Kariwa	TC				
83. Mihama	TC				
84. Monju	TC				
85. Ohi	TC				
86. Ohma	TC				
87. Onagawa	TC				
88. Sendai	TC				
89. Shika	TC				
90. Shimane	TC				
91. Takahama	TC				
92. Tokai-Daini	TC				
93. Tomari	TC				
94. Tsuruga	TC				
MEXICO					
102. Laguna Verde	AC				
PAKISTAN					
103. Chashma	TC				
104. Karachi	TC				
ROMANIA					
105. Cernavoda	AC				
RUSSIAN FEDERATION					
106. Akademik Lomonosov	MC				
107. Atomflot	MC				
108. Balakovo	MC				
109. Baltic	MC				
110. Beloyarsk	MC				
111. Bilibino	MC				
112. Kalinin	MC				
113. Kola	MC				
114. Kursk	MC				
115. Leningrad	MC				
116. Novovoronezh	MC				
117. Rostov	MC				
118. Smolensk	MC				
SLOVAKIA					
119. Bohunice	MC				
120. Mochovce	MC				
SLOVENIA					
121. Krsko	PC				
SOUTH AFRICA					
122. Koeberg	AC				

Figure 2.8: World list of nuclear reactors, part 1.

SPAIN			
123. Almaraz	PC	■■	
124. Asco	PC	■■	
125. Cofrentes	PC	●	
126. Santa-Maria De Garona	PC	●	
127. Trillo	PC	■	
128. Vandelllos 2	PC	■	
SWEDEN			
129. Forsmark	PC	●●●	
130. Oskarshamn	PC	●●●	
131. Ringhals	PC	■■■■	
SWITZERLAND			
132. Beznau	PC	■■	
133. Goesgen	PC	■	
134. Leibstadt	PC	●	
135. Muehleberg	PC	●	
TAIWAN			
136. Chinshan	TC	●●	
137. Kuosheng	TC	●●	
138. Lungmen	TC	●●	
139. Maanshan	TC	■■	
THE NETHERLANDS			
140. Borssele	PC	■	
UKRAINE			
141. Khmelnitski	MC	■■■■■	
142. Rovno	MC	■■■■■	
143. South Ukraine	MC	■■■■■	
144. Zaporozhye	MC	■■■■■■■■	
UNITED ARAB EMIRATES			
145. Barakah	AC / TC	■■■■■	
UNITED KINGDOM			
146. Chapelcross	PC	▼▼▼▼	
147. Dungeness B	PC	▼▼	
148. Hartlepool	PC	▼▼	
149. Heysham	PC	▼▼▼▼	
150. Hinkley Point R	PC	▼▼	
UNITED STATES OF AMERICA			
151. Hunterston B	PC	▼▼	
152. Oldbury	PC	▼▼	
153. Sellafield	PC	○	
154. Sizewell A	PC	▼▼	
154. Sizewell B	PC	■	
155. Torness	PC	▼▼	
156. Wyfia	PC	▼▼	
185. Keweenaw	AC	■	
186. LaSalle County	AC	●●	
187. Limerick	AC	●●	
188. McGuire	AC	■■	
189. Millstone	AC	■■	
190. Monticello	AC	●	
191. Nine Mile Point	AC	●●	
192. North Anna	AC	■■	
193. Oconee	AC	■■■■	
194. Oyster Creek	AC	●	
195. Palisades	AC	■	
196. Palo Verde	AC	■■■■	
197. Peach Bottom	AC	●●	
198. Perry	AC	●	
199. Pilgrim	AC	●	
200. Point Beach	AC	■■	
201. Prairie Island	AC	■■	
202. Quad Cities	AC	●●	
203. River Bend	AC	●	
204. Robinson	AC	■	
205. Salem	AC	■■	
206. Seabrook	AC	■	
207. Sequoyah	AC	■■	
208. Shearon Harris	AC	■	
209. South Texas Project	AC	■■	
210. St Lucie	AC	■■	
211. Surry	AC	■■	
212. Susquehanna	AC	●●	
213. Three Mile Island	AC	■	
214. Turkey Point	AC	■■	
215. Vermont Yankee	AC	●	
216. Virgil C. Summer	AC	■■■■■	
217. Vogtle	AC	■■■■■■■	
218. Waterford 3	AC	■	
219. Watts Bar	AC	■■	
220. Wolf Creek	AC	■	

Figure 2.9: World list of nuclear reactors, part 2.

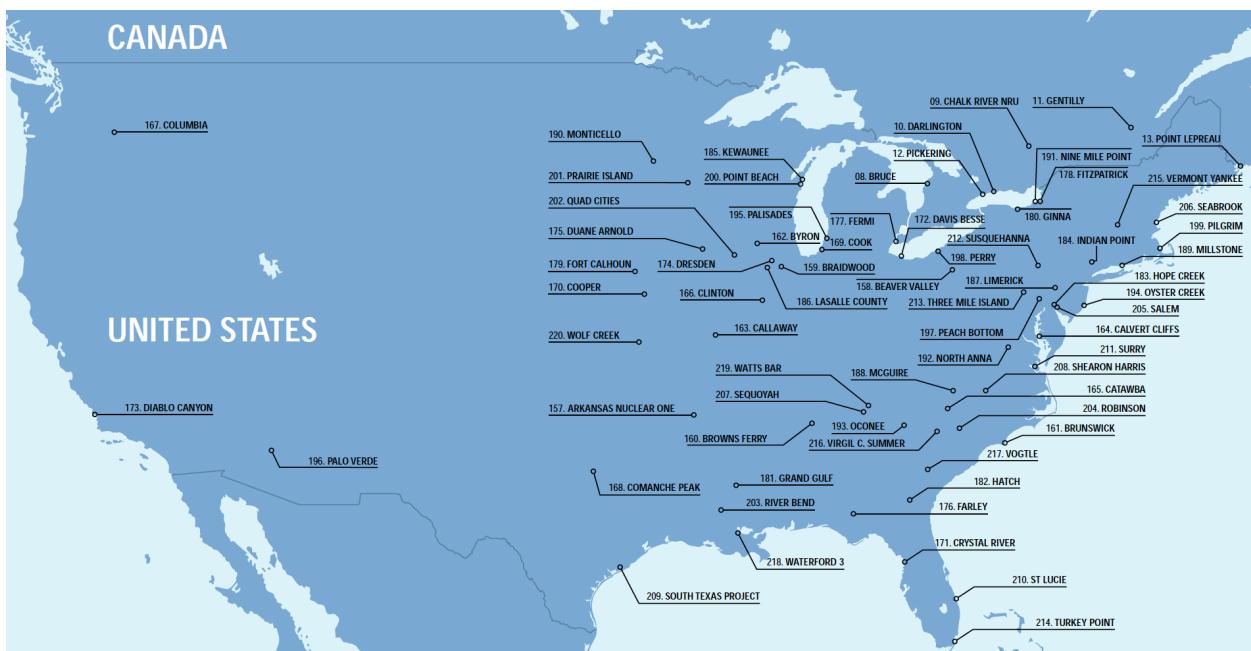


Figure 2.10: Locations of nuclear power plants in the U.S. and Canada. Some locations have multiple reactors.

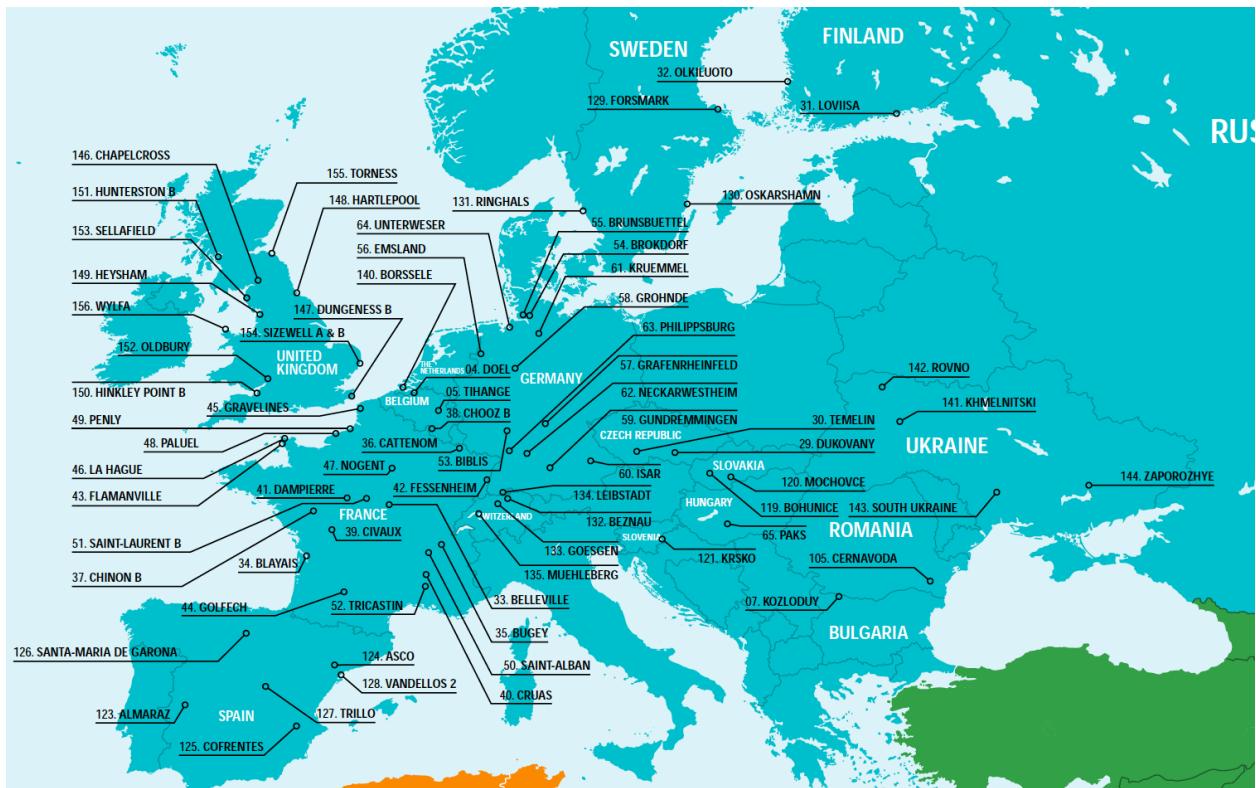


Figure 2.11: Locations of nuclear power plants in Europe. Some locations have multiple reactors.

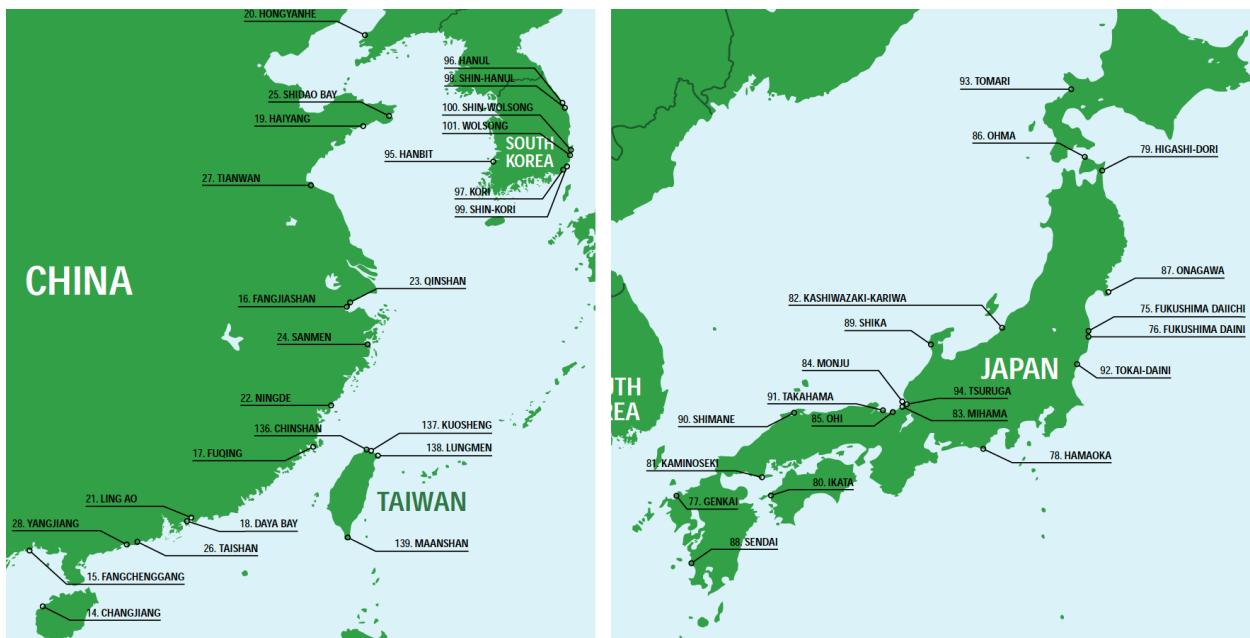


Figure 2.12: Locations of nuclear power plants in SE Asia. Some locations have multiple reactors.

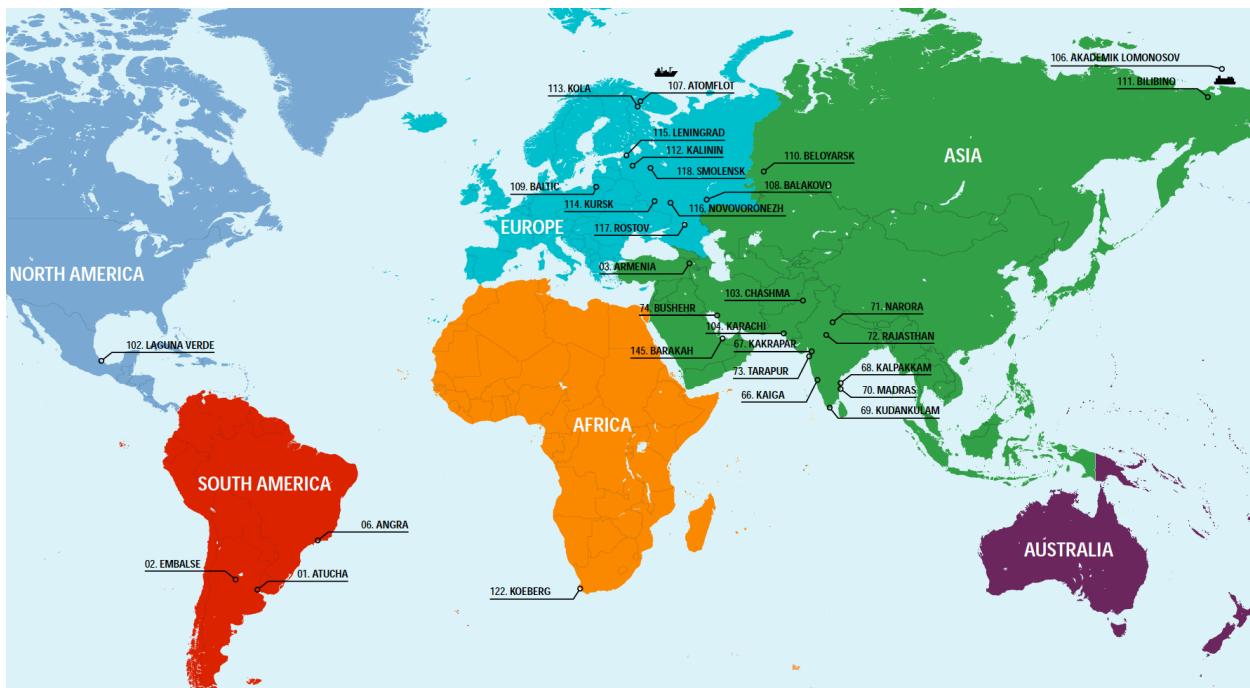


Figure 2.13: Locations of nuclear power plants in other parts of the world. Some locations have multiple reactors.

Chapter 3

Neutron-Nucleus Interactions

3.1 Introduction

The main goal of this course is for each student to develop an understanding of how reactors behave and to develop the skills to quantify reactor behavior.

- We know that reactor behavior fundamentally depends on the production and loss rates of neutrons.
- We also know that these rates depend on interactions between neutrons and nuclei . In this chapter we review these interactions.

3.2 A Simple Experiment

We consider a beam of neutrons, all traveling at the same speed in the same direction, perpendicular to a thin sheet of some material. See Fig. 3.1. The material contains only a single kind of nuclide, and it is so cold that all nuclei can be considered at rest in the lab frame. We assume that the sheet is so thin that all nuclei see the same beam

intensity—no nucleus is shielded by other nuclei. We define:

$$\begin{aligned} I &\equiv \text{beam intensity} \\ &= nv \end{aligned} \quad (3.1)$$

$$= \text{rate per unit area at which neutrons reach target} \left[\frac{\text{neutrons}}{\text{cm}^2 \text{ s}} \right], \quad (3.2)$$

$$n \equiv \text{neutron density in beam} \left[\frac{\text{neutrons}}{\text{cm}^3} \right], \quad (3.3)$$

$$v \equiv \text{neutron speed (not velocity)} \left[\frac{\text{cm}}{\text{s}} \right], \quad (3.4)$$

$$N \equiv \text{number density of target nuclei} \left[\frac{\text{nuclei}}{\text{cm}^3} \right], \quad (3.5)$$

$$A \equiv \text{target - beam interaction area} [\text{cm}^2], \quad (3.6)$$

$$\Delta x \equiv \text{target thickness} [\text{cm}]. \quad (3.7)$$

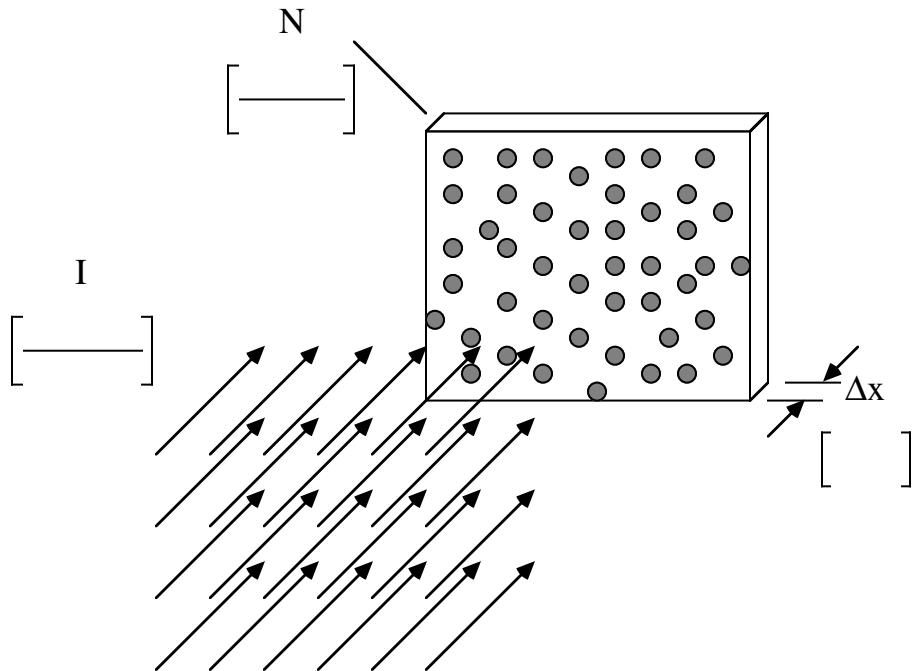


Figure 3.1: Mono-energetic mono-directional beam of particles incident on a thin target containing a single kind of nuclide, with all nuclei at rest (“cold” target). **Write units on the figure: nuclei/cm³, n/(cm²-s), cm**

We reason that the rate at which neutrons collide with nuclei must be proportional to IA (the rate at which neutrons strike) and to $N\Delta x$ (the areal density of target nuclei). Putting this statement into mathematical language, and using the symbol σ_t to represent the “constant” of proportionality, we get the following

$$\text{Reaction Rate} = \sigma_t IA N\Delta x \quad (3.8)$$

or

$$\left[\frac{\text{colliding particles}}{\text{s}} \right] = \left[\frac{\text{cm}^2}{\text{nucleus}} \right] \left[\frac{\text{particles}}{\text{cm}^2 - \text{s}} \right] \left[\text{cm}^2 \right] \left[\frac{\text{nuclei}}{\text{cm}^3} \right] [\text{cm}] \quad (3.9)$$

(rate \rightarrow per unit time.)

3.2.1 Microscopic Cross Sections

The “constant” of proportionality, σ_t , is called the **microscopic total cross section**. Its units are

area per nucleus

If neutrons were classical point particles and nuclei were classical spheres, σ_t would be just the cross-sectional area of the nucleus. (Think about it.) In reality neutron-nuclear interactions obey quantum-mechanical laws; nevertheless, you can think of σ_t as

the “effective cross-sectional area”

that a nucleus presents to a neutron.

When dealing with microscopic cross sections we use the following area unit:

$$1 \text{ barn} = 1 \text{ b} \equiv 10^{-24} \text{ cm}^2 \quad (3.10)$$

We have described σ_t as a “constant” of proportionality. However, if we changed anything about our simple experiment, we might expect our “constant” to change. This is, in fact, the case:

Microscopic cross sections depend on:

the relative speed between the neutron and nucleus

and on

the nuclide (type of nucleus)

Friendly advice: don’t forget this.

Exercise: Consider a neutron (mass m , lab-frame velocity \vec{v}) colliding with a nucleus (mass M , lab-frame velocity \vec{V}). Prove that there is a one-to-one relationship between the:

- relative speed between the neutron and nucleus, and
- total kinetic energy of the neutron and nucleus in the center-of-mass reference frame.

Example

Consider a neutron beam impinging on a thin, cold sheet as shown in Fig. 3.2. The beam has an intensity of $10^{10} \text{ n}/(\text{cm}^2\text{-s})$. The neutrons all have the same speed; each has a kinetic energy of 1 MeV. The beam completely covers the target area. The microscopic cross section for 1-MeV neutrons is 5 barns (per atom), the target area is 3 cm^2 , the target thickness is 0.01 cm, and the atom density is $4 \times 10^{22} \text{ atoms/cm}^3$. At what rate do the incident neutrons collide with target atoms?

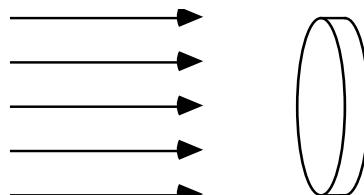


Figure 3.2: Neutron beam on thin sheet.

Solution: By Eq. (3.8) we have

$$\begin{aligned}\text{Rate} &= \sigma_t I A N \Delta x \\ &= \left[\frac{5 \text{ b}}{\text{atom}} \right] \left[\frac{10^{10} \text{ n}}{\text{cm}^2 \cdot \text{s}} \right] [3 \text{ cm}^2] \left[\frac{4 \times 10^{22} \text{ atoms}}{\text{cm}^3} \right] [0.01 \text{ cm}] \left[\frac{10^{-24} \text{ cm}^2}{\text{b}} \right] \\ &= 6 \times 10^7 \text{ n/s}\end{aligned}$$

3.2.2 Microscopic Cross Sections for Different Reactions

In the preceding discussion we were looking at the total rate at which neutrons interact with nuclei, without distinguishing among different kinds of interactions (elastic scattering, fission, inelastic scattering, capture, etc.). The σ_t that we used above is called the microscopic **total** cross section. It is the sum of microscopic cross sections for various specific interactions:

$$\sigma_t \equiv \sigma_s + \sigma_a$$

$$= \overbrace{\left(\sigma_e + \sigma_{e,r} + \sigma_p + \sigma_{in} \right)}^{\sigma_s} + \overbrace{\left(\sigma_\gamma + \sigma_f + \sigma_{n,2n} + \sigma_{n,3n} + \sigma_{n,\alpha} + \sigma_{n,p} + \dots \right)}^{\sigma_a} \quad (3.11)$$

where

$$\begin{aligned} \sigma_t &= \text{microscopic } \mathbf{total} \text{ cross section} \\ \sigma_s &= \text{microscopic } \mathbf{scattering} \text{ cross section} \\ \sigma_a &= \text{microscopic } \mathbf{absorption} \text{ cross section} \end{aligned} \quad (3.12)$$

$$\begin{aligned} \sigma_e &= \text{microscopic } \mathbf{elastic \ scattering} \text{ cross section} \\ \sigma_{e,r} &= \text{microscopic } \mathbf{resonance \ elastic \ scattering} \text{ cross section} \\ \sigma_p &= \text{microscopic } \mathbf{potential \ scattering} \text{ cross section} \\ \sigma_{in} &= \text{microscopic } \mathbf{inelastic \ scattering} \text{ cross section} \end{aligned} \quad (3.13)$$

$$\begin{aligned} \sigma_\gamma \text{ or } \sigma_c &= \text{microscopic } \mathbf{radiative \ capture} \text{ cross section (gamma(s) emitted)} \\ \sigma_f &= \text{microscopic } \mathbf{fission} \text{ cross section} \\ \sigma_{n,2n} &= \text{microscopic cross section for } \mathbf{2 \ neutrons} \text{ emitted} \\ \sigma_{n,3n} &= \text{microscopic cross section for } \mathbf{3 \ neutrons} \text{ emitted} \\ \sigma_{n,\alpha} &= \text{microscopic cross section for } \mathbf{alpha \ particle} \text{ emitted} \\ \sigma_{n,p} &= \text{microscopic cross section for } \mathbf{proton} \text{ emitted} \\ \dots & \end{aligned} \quad (3.14)$$

The microscopic cross section for a certain kind of reaction can be thought of as the “effective area” of the nucleus for that particular kind of reaction. Note that the ratio

$$\sigma_x / \sigma_t = \text{probability that, given a collision, a reaction of type } x \text{ occurred.} \quad (3.15)$$

Example

In the previous example, the total microscopic cross section was 5 barns/atom. Suppose the scattering cross section (σ_s) was 0.5 barns/atom. What was the absorption rate in the target?

Solution:

We know the collision rate was 6×10^7 n/s. (Do we really know this?) We also know that

$$\sigma_a/\sigma_t = (\sigma_t - \sigma_s)/\sigma_t = (5 - 0.5)/5 = 0.9$$

$$\implies \text{90\% of collisions were absorptions} \implies 5.4 \times 10^7 \text{abs/s} \quad (3.16)$$

Question: Did we miss anything?

scattering ...

3.2.3 Macroscopic Cross Sections

We notice that the microscopic cross section for reaction “ x ”, σ_x , is usually multiplied by the number density, N , of the element in question. [See Eq.eq3.8.] We therefore find it convenient to give the product $N\sigma_x$ a name. If a material contains only one kind of nuclide, then

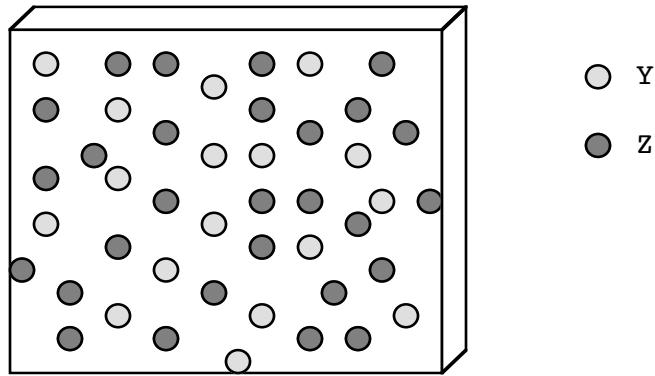
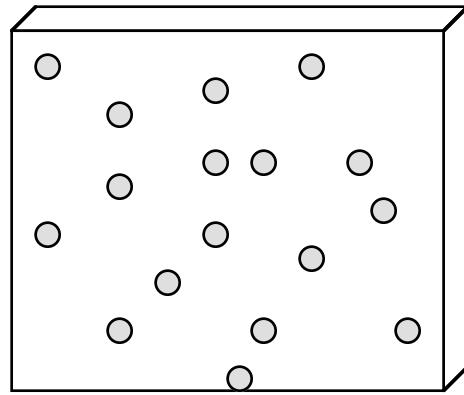
$$\begin{aligned} \Sigma_x &= \text{macroscopic cross section for reaction of type } x \\ &\equiv N\sigma_x \end{aligned} \quad (3.17)$$

for each different type, $x = t, a, s, \gamma, f, e, \text{in}, \dots$ of interaction. Macroscopic cross sections have units of inverse length:

$$[\text{nuclei}/\text{cm}^3] \times [\text{cm}^2/\text{nucleus}] = \text{cm}^{-1}.$$

3.2.4 Mixtures of Nuclides

In the preceding discussion the “target” we considered was composed purely of a single nuclide. The generalization to mixtures is straightforward. Consider a material composed of two nuclides, Y and Z , as shown in Fig. 3.3. Let us pretend for a moment that the “ Z ” nuclei are not present, as shown in Fig. 3.4.

Figure 3.3: Thin target with two nuclides, Y and Z .Figure 3.4: Thin target with two nuclides, Y and Z , with only Y shown.

We know the reaction rate for the Y -only case—it is the single-nuclide case we already analyzed:

$$\text{Rate for case } Y = IA\Sigma_t^Y \Delta x \left[\frac{n}{cm^2 - s} \right] [cm^2][cm^{-1}][cm], \quad (3.18)$$

where

$$\Sigma_t^Y \equiv N^Y \sigma_t^Y. \quad (3.19)$$

Likewise, if we pretend now that the “ Y ” nuclei are not present, we find the Z -only rate:

$$\text{Rate for case } Z = IA\Sigma_t^Z \Delta x \left[\frac{n}{s} \right], \quad (3.20)$$

where

$$\Sigma_t^Z \equiv N^Z \sigma_t^Z. \quad (3.21)$$

The important point to realize is that the two nuclides do not interfere with each other's reactions with neutrons. It follows that the total reaction rate is the sum of the Y and Z rates:

$$\text{Rate for mixture} = IA\Sigma_t^{\text{mixture}}\Delta x \left[\frac{n}{s} \right], \quad (3.22)$$

where

$$\Sigma_t^{\text{mixture}} \equiv \Sigma_t^Z + \Sigma_t^Y .$$

This was the two-nuclide case. It is easy to generalize:

$$\Sigma_t \equiv \sum_{i=1}^I \Sigma_t^i = \sum_{i=1}^I N^i \sigma_t^i , \quad (3.23)$$

where i denotes a nuclide index and I is the number of different kinds of nuclides in the mixture.

Important Exception to Eq. (3.23). If the nuclei in the mixture are bound to each other, as in a molecule or a solid, then the nuclei might interfere with each other's reactions with low-energy neutrons. This is most important for scattering of low-energy neutrons, especially when a relatively light nucleus is bound to a heavier one.

Example:

We uniformly mix 50 grams of helium (${}^4\text{He}$) and 6 grams of hydrogen (${}^1\text{H}$) in a 200-cm³ container. What is the macroscopic total cross section at a neutron speed of 2200 m/s?

Data:

$$\begin{aligned}\sigma_t^{\text{He}} &= 0.8 \text{ b} \\ \sigma_t^{\text{H}} &= 38 \text{ b}\end{aligned}$$

atomic mass of He ≈ 4 amu $\implies 4$ g/mol

atomic mass of H ≈ 1 amu $\implies 1$ g/mol .

Solution:

1. We know that $\Sigma_t = \Sigma_t^{\text{He}} + \Sigma_t^{\text{H}}$.

2. Compute Σ_t for each component:

$$\begin{aligned}\Sigma_t^{He} &= N^{He} \sigma_t^{He} \\ &= \left[\frac{50 \text{ g}}{200 \text{ cm}^3} \right] \left[\frac{1 \text{ mol}}{4 \text{ g}} \right] \left[\frac{0.6022 \times 10^{24} \text{ nuclei}}{\text{mol}} \right] \left[\frac{0.8 \text{ b}}{\text{nucleus}} \right] \left[\frac{10^{-24} \text{ cm}^2}{\text{b}} \right] \\ &\approx 0.03 \text{ cm}^{-1}\end{aligned}$$

$$\begin{aligned}\Sigma_t^H &= N^H \sigma_t^H \\ &= \left[\frac{6 \text{ g}}{200 \text{ cm}^3} \right] \left[\frac{1 \text{ mol}}{1 \text{ g}} \right] \left[\frac{0.6022 \times 10^{24} \text{ nuclei}}{\text{mol}} \right] \left[\frac{38 \text{ b}}{\text{nucleus}} \right] \left[\frac{10^{-24} \text{ cm}^2}{\text{b}} \right] \\ &\approx 0.69 \text{ cm}^{-1}\end{aligned}$$

3. Thus $\Sigma_t^{\text{mixture}} \approx 0.72 \text{ cm}^{-1}$.

Naturally occurring elements: Frequently, the number of atoms of a particular isotope need to be computed. *Example:* If natural iron is used, many isotopes naturally present will play a role in the neutron-iron interaction process (Fe-54, Fe-56, Fe-57 and Fe-58 are naturally present). But because they are made of different numbers of neutrons, their nuclear structure is different, and so are their respective **microscopic cross sections**. So let us suppose I need to compute the total macroscopic cross section of a piece of natural iron. This will be, by definition,

$$\begin{aligned}\Sigma_t &= \sum_{i=1}^I \Sigma_t^i \quad \text{where } i = \text{Fe-54, Fe-56, Fe-57, and Fe-58} \\ &= \sum_{i=1}^I N^i \sigma_t^i\end{aligned}$$

You can get the microscopic cross sections from evaluated databases. I still need to compute the atom densities of each naturally occurring isotopes:

$$N^i = \gamma^i \times \frac{\rho N_a}{M}$$

where

γ^i is **atom fraction**, often given in atom percent so do not forget to divide by 100. For naturally occurring isotopes, the atom fraction is also called the isotopic abundance.

isotope	Fe-54	Fe-56	Fe-57	Fe-58
γ_i	5.8%	91.72%	2.2%	0.28%

Chemical compounds:

If you have a **chemical compound**, say H_2O (omitting the presence of deuterium), the macroscopic XS is:

$$\Sigma_t^{\text{H}_2\text{O}} = N^{\text{H-1}}\sigma_t^{\text{H-1}} + N^{\text{O}}\sigma_t^{\text{O}} = N^{\text{H}_2\text{O}}(2\sigma_t^{\text{H-1}} + \sigma_t^{\text{O}})$$

because

$$\begin{aligned}N^{\text{H-1}} &= 2N^{\text{H}_2\text{O}} \\N^{\text{O}} &= N^{\text{H}_2\text{O}}\end{aligned}$$

What if you want to consider ordinary water where there is a small amount of **deuterium**?

$$\Sigma_t^{\text{H}_2\text{O}} = N^{\text{H-1}}\sigma_t^{\text{H-1}} + N^{\text{D}}\sigma_t^{\text{D}} + N^{\text{O}}\sigma_t^{\text{O}}$$

where, this time,

$$\begin{aligned}N^{\text{H-1}} + N^{\text{D}} &= 2N^{\text{H}_2\text{O}} \\N^{\text{O}} &= N^{\text{H}_2\text{O}} \\N^{\text{H-1}} &= \gamma^{\text{H-1}}(N^{\text{H-1}} + N^{\text{D}}) = 2\gamma^{\text{H-1}}N^{\text{H}_2\text{O}} \\N^{\text{D}} &= \gamma^{\text{D}}(N^{\text{H-1}} + N^{\text{D}}) = 2\gamma^{\text{D}}N^{\text{H}_2\text{O}}\end{aligned}$$

Weight percents, w/o or wt%:

Finally, in many nuclear applications, the isotopic composition of an element is artificially modified (think of the enrichment of uranium for instance). In this case, the usual practice is to quantify the enrichment in **weight percent** (percent by weight), abbreviated as w/o, rather than atom percent. For this case, the atom density of the i -th isotope in an element of density ρ is then given by:

$$N^i = w^i \times \frac{\rho N_a}{M_i}$$

where

w^i is the weight percent and M_i is the atomic mass of the i -th isotope.

The weight percent is mathematically related to the atom fraction as

$$w^i = \gamma_n^i \frac{M_i}{M}$$

where γ_n^i is the fraction of nuclei of type i in a given element (atom fraction).

Note: In my expressions, $0 \leq w^i \leq 1$, so I have already divided by 100.

Let's take the example of a standard light water reactor fuel: UO₂, which is usually enriched in U-235. The enrichment is often given in w/o. Let's compute the macroscopic *absorption* XS for the mixture, assuming we only have U-235, U-238, and O-16. By definition,

$$\Sigma_a^{\text{UO}_2} = N^{235}\sigma_a^{235} + N^{238}\sigma_a^{238} + N^{16}\sigma_a^{16}$$

We have also

$$\begin{aligned} N^{16} &= 2N^{\text{UO}_2} \\ N^{235} &= \gamma^{235}N^{\text{UO}_2} \\ N^{238} &= \gamma^{238}N^{\text{UO}_2} \end{aligned}$$

but we do not know the γ 's (atom-percent) values; we only know the weight-percent values. However, we can compute the following:

$$N^{\text{UO}_2} = \frac{\rho_{\text{UO}_2}\mathcal{N}_a}{M_{\text{UO}_2}}$$

with

$$M_{\text{UO}_2} = M_U + 2M_O$$

If we figure out M_U , we are done because we would know everything to compute $\Sigma_a^{\text{UO}_2}$:

$$\boxed{\Sigma_a^{\text{UO}_2} = \frac{\rho_{\text{UO}_2}\mathcal{N}_a}{M_{\text{UO}_2}} \left(w^{235} \frac{M_U}{M_{235}} \sigma_a^{235} + w^{238} \frac{M_U}{M_{238}} \sigma_a^{238} + 2\sigma_a^{16} \right)}$$

We get M_U as follows:

$$\boxed{\sum_{i=235,238} w^i \frac{M_i}{M_i} = \sum_{i=235,238} \frac{\gamma^i}{M_U} = \frac{1}{M_U} \sum_{i=235,238} \gamma^i = \frac{1}{M_U}}$$

Done.

3.3 Attenuation, Mean Free Paths, . . .

We return to our uniform beam of mono-energetic neutrons incident normally upon a target, as in the Fig. 3.1. This time we allow the target to be arbitrarily thick. We define:

$$\begin{aligned} I(x) &\equiv \text{intensity of uncollided neutrons at distance } x \text{ into target} \left[\frac{n}{cm^2 - s} \right] \\ &= \text{intensity of neutrons that reach } x \text{ without interacting with nuclei.} \end{aligned} \quad (3.24)$$

Conservation of neutrons in the beam tells us (we are neutron accountants, remember):

$$\begin{bmatrix} \text{beam} \\ \text{intensity} \\ \text{at } x_2 \end{bmatrix} = \begin{bmatrix} \text{beam} \\ \text{intensity} \\ \text{at } x_1 \end{bmatrix} + \begin{bmatrix} \text{intensity} \\ \text{added} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix} - \begin{bmatrix} \text{intensity} \\ \text{removed} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix}$$

In the problem we've described there is no mechanism for adding to the uncollided intensity, so the second term on the right-hand side is zero. If we assume that $x_2 - x_1$ is very small, with no "hidden" nuclei, then the intensity removed is given by our Eq.(3.8)—the thin-target collision rate—divided through by the area A . This equation gives removal because

every collision removes a neutron from the "uncollided" beam.

We must divide the equation by A because intensity removal is removal rate per unit area. Our conservation statement, written mathematically, is therefore:

$$I(x_2) = I(x_1) + 0 - I(x_1) \times \Sigma_t \times (x_2 - x_1) \quad (3.25)$$

or

$$\frac{I(x_2) - I(x_1)}{x_2 - x_1} = -I(x_1) \Sigma_t, \quad x_2 - x_1 \text{ small.} \quad (3.26)$$

Taking the limit as x_2 approaches x_1 , we obtain a differential equation:

$$\frac{dI}{dx} = -I(x) \Sigma_t. \quad (3.27)$$

We can easily solve this equation using an integrating factor. We obtain:

$$I(x) = I_0 e^{-\Sigma_t x}, \quad (3.28)$$

where I_0 is the "uncollided" intensity at $x = 0$. Thus, uncollided neutrons are

exponentially attenuated

as they try to pass through matter. This is analogous to the exponential decay of radioactive nuclides as they try to pass through time.

It is often useful to know the average path length that a neutron travels between collisions. We give it a name:

mean free path \equiv average path length between collisions

Let us compute it. We first need the probability of traveling an arbitrary distance x before colliding within dx about x :

$$\begin{aligned}
 p(x)dx &\equiv \text{probability that a neutron's first collision is in } dx \text{ about } x \\
 &= (\text{first-collision rate in } dx \text{ about } x) / (\text{incident rate}) \\
 &= \frac{\Sigma_t I(x)dx}{I_0} \\
 &= \Sigma_t e^{-\Sigma_t x} dx . \tag{3.29}
 \end{aligned}$$

Now we want the average (“mean”) of the distance x . Note that in general, the average of a function $f(x)$ over the interval (x_1, x_2) is:

$$\text{mean (average) value of } f = \langle f \rangle = \frac{\int_{x_1}^{x_2} w(x)f(x)dx}{\int_{x_1}^{x_2} w(x)dx} , \tag{3.30}$$

where $w(x)$ is some given “weight function.” In our case the weight function is the probability density function $p(x)$; thus:

$$\begin{aligned}
 &\text{mean distance to first collision} \\
 &= \text{mean free path} \\
 &= \frac{\int_0^{\infty} xp(x)dx}{\int_0^{\infty} p(x)dx} \\
 &= \int_0^{\infty} xp(x)dx = \int_0^{\infty} \Sigma_t x e^{-\Sigma_t x} dx \quad \text{note that denominator}=1 \\
 &= \dots = \frac{1}{\Sigma_t} . \tag{3.31}
 \end{aligned}$$

A large cross section means a short mean free path, as one would expect. Of course, mean free path has units of

length.

We can define mean free paths for various individual reactions as well. For example,

mean path length (not straight – line distance) to absorption
= **absorption mean free path**

$$= \frac{1}{\Sigma_a} . \quad (3.32)$$

Another commonly-used mean free path is the **scattering mean free path**, $1/\Sigma_s$.

Example:

How far do 2200-m/sec neutrons travel, on average, between collisions in the gas mixture of our previous example?

Solution:

The question asks for the neutron mean free path. The answer is

$$\text{mean free path} = 1/\Sigma_t \approx 1/(0.72\text{cm}^{-1}) \approx 1.39\text{cm} . \quad (3.33)$$

3.4 Monte Carlo Programming For Uncollided Flux In A Slab Under Normal Incidence

The goal here is to compute, via a Monte Carlo technique, the uncollided intensity in a 1D homogeneous slab of material. A particle beam is normally incident on the left side of the slide, with initially intensity I_0 . The macroscopic total cross section is denoted by Σ_t . The slab has a thickness L .

As we have seen before, the exact analytical solution is simply

$$I(x) = I_0 \exp(-\Sigma_t x).$$

Read the supplementary information given the the presentation (filename **Introduction-to-the-Monte-Carlo-Method.pdf**). That presentation covers the following aspects

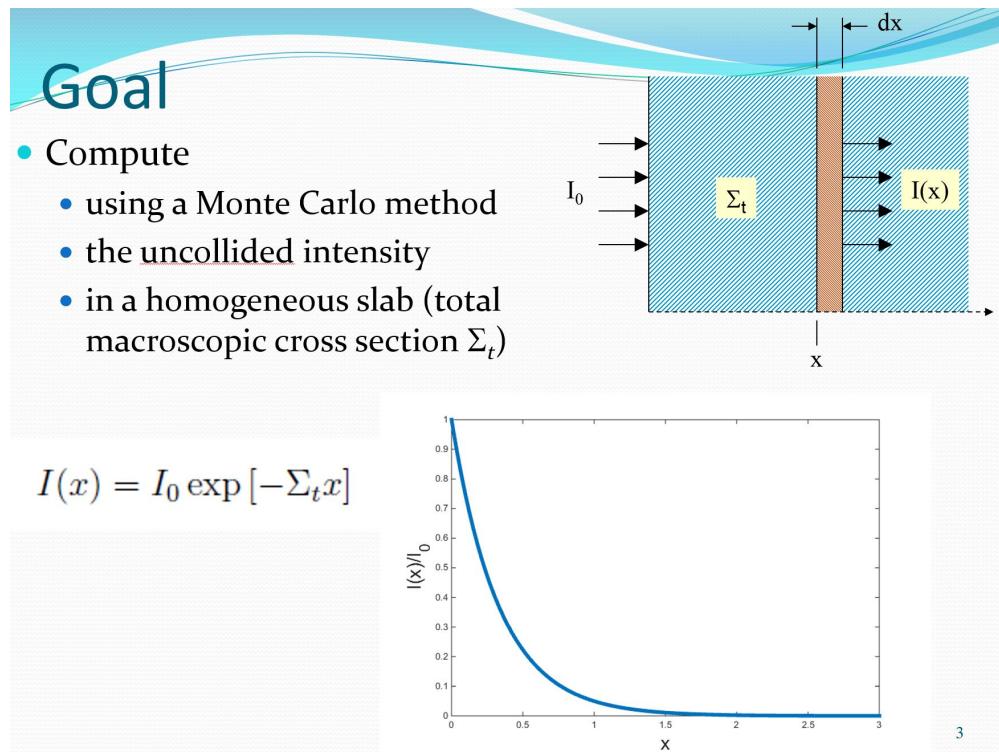


Figure 3.5: Monte Carlo Programming For Uncollided Flux In A Slab Under Normal Incidence.

1. a brief overview Computing and the Manhattan Project in order to make you aware how scientific computing has been a key part of nuclear engineering since its beginning;
2. a discussion about “density” (it’s not always per unit volume!) and how to go between density distributions (I use the example of car density on game day and explain how to obtain the car density on the highway, expressed in cars/km, if given the car density expressed in cars/mile);
3. once the above point has been assimilated, we can finally start talking about how to set up a Monte Carlo process. Here, I show how to relate distance traveled with a random number. To relate these two quantities, I need to know
 - (a) the probability density of my random number (super easy, they are usually uniformly distribution over the $[0, 1]$ interval), and
 - (b) the probability density for interaction (that should be easy too, this was seen in NUEN 302);
4. after that, I explain how to collect track lengths in various sub-portions of the slab (the slab has been sliced like bread and we care about the track length left by each particle

in each slide of the slab); the collection of track lengths will help us compute the track length density in each slide once a lot of particles have been created and followed;

5. finally, I give an outline of a simple code we will write in class and some sample results.

3.5 Temperature Dependence

To this point we have considered “cold” targets —so cold that the nuclei are not moving. We have also noted that microscopic cross sections depend on the

relative speed between the neutron and nucleus.

In the cold-target case, this relative speed is just the neutron’s lab-frame speed.

Let us now consider reaction rates in the more realistic case in which nuclei are moving, which will be the case when the material is not at absolute zero temperature. To set the stage, first recall our thin-target experiment, and first suppose that all the nuclei are at rest in the lab frame (which means a very cold target). Let

\vec{v} \equiv lab-frame velocity of each **neutron in the beam**

\vec{V} \equiv lab-frame velocity of **a given nucleus in the target**

$\xrightarrow{\text{cold}} \vec{0}$ in this “cold-target” case

Then we have

$v_r \equiv$ relative speed between neutron and nucleus

$$= |\vec{v} - \vec{V}| \xrightarrow{\text{cold}} |\vec{v}| = v = \text{lab-frame neutron speed in this cold-target case} . \quad (3.34)$$

and the reaction-rate density (reactions per unit volume per unit time) is

$$\begin{aligned} \text{RRD} &\xrightarrow{\text{cold target}} IN\sigma_{t,cold}(v) && \text{in this cold-target case} . \\ &\xrightarrow{I=nv} nvN\sigma_{t,cold}(v) && \text{in this cold-target case} . \end{aligned} \quad (3.35)$$

Suppose we used our simple cold-target experiment to measure the cross section for a range of neutron lab-frame kinetic energies, with the result shown in Fig. 3.6. (We have chosen to plot $\sigma_{t,cold}$ as a function of neutron kinetic energy, not speed. This is okay because there is a one-to-one relation between neutron speed and neutron kinetic energy in any given reference frame.)

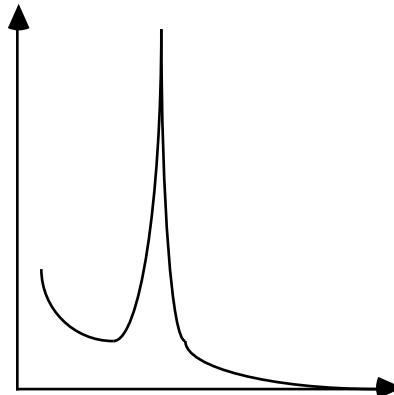


Figure 3.6: “Cold” cross section, $\sigma_{t,cold}$, as a function of neutron kinetic energy in the lab reference frame. “Cold” means nuclei are at rest in the lab frame. Vertical axis label $\sigma_{t,cold}(E)$; horizontal $E = E_{lab}$.

Now consider the same material with **vibrating** nuclei. This is the real-life case, of course. Now there is a **range** of relative speeds seen by the incident neutrons, as indicated in Fig. 3.7. To compute reaction-rate densities in this case we have to know something about

the **distribution** of nucleus velocities.

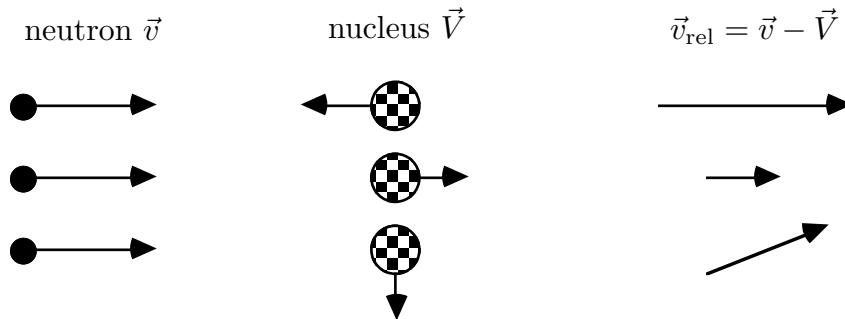


Figure 3.7: Schematic of relative speed between incident neutrons and three nuclei selected from a vibrating population.

That is, we must know how many nuclei have this or that velocity. Let us define the target nuclei distribution, $\mathbb{N}(\vec{V})$:

$$\mathbb{N}(\vec{V})dV_x dV_y dV_z = \text{nuclei/cm}^3 \text{ whose velocities are}$$

in the “velocity cube” $dV_x dV_y dV_z$ at \vec{V} . (3.36)

Now if we consider only the nuclei whose velocities are in the “velocity cube” $dV_x dV_y dV_z$, we see that the relative speed between the neutrons and those particular nuclei is:

$$v_{\text{rel}} = \left| \vec{v} - \vec{V} \right| .$$

This relative speed matters in two ways:

1. It affects the cross section: the cross section for this set of nuclides is the same as the “cold” cross section evaluated at a speed of v_{rel} , not v .
2. It affects the effective beam intensity, because the rate at which neutrons impinge on this set of target nuclei is determined by v_{rel} , not v . This means the effective intensity is

$$I_{\text{eff}} = nv_{\text{rel}} = I \times \frac{v_{\text{rel}}}{v} .$$

It follows that the reaction-rate density between the neutrons and the nuclei with this particular range of velocities is

RRD with only the nuclei

$$\text{whose veloc. are in cube} = I \frac{\left| \vec{v} - \vec{V} \right|}{v} \left\{ \mathbb{N}(\vec{V}) dV_x dV_y dV_z \right\} \times \sigma_{t,\text{cold}} \left(\left| \vec{v} - \vec{V} \right| \right) . \quad (3.37)$$

The total reaction-rate density is the “sum” (integral) over all of the velocity cubes:

$$\text{RRD} = I \times \iiint_{\text{nucleus velocities}} \mathbb{N}(\vec{V}) \sigma_{t,\text{cold}} \left(\left| \vec{v} - \vec{V} \right| \right) \frac{\left| \vec{v} - \vec{V} \right|}{v} dV_x dV_y dV_z , \quad (3.38)$$

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or (multiply and divide by the integral of \mathbb{N}):

$$\text{RRD} = I \times \left[\frac{\iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) \sigma_{t,\text{cold}} \left(\left| \vec{v} - \vec{V} \right| \right) \left| \vec{v} - \vec{V} \right| dV_x dV_y dV_z}{v \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z} \right] \times \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z . \quad (3.39)$$

Note that

$$\iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z = N , \text{ the ordinary number density.} \quad (3.40)$$

The ratio in brackets defines an averaged cross section (or more precisely, [an averaged $\sigma \times v_{rel}$] divided by $v_{lab} \equiv v$):

$$\sigma_t(\vec{v}, \mathbb{N}) \equiv \frac{\iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) \sigma_{t,cold}(|\vec{v} - \vec{V}|) |\vec{v} - \vec{V}| dV_x dV_y dV_z}{v \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z} . \quad (3.41)$$

Notice something important: $|\vec{v} - \vec{V}|$ is sometimes greater than and sometimes less than the neutron speed, v , which is $|\vec{v}|$. This means the averaging process includes values of the cold cross section from all speeds in the neighborhood of the neutron's lab-frame speed, v . To say it another way, if we want a simple formula to give the correct reaction rate, the cross section we must use is

an average of the “cold” cross section

over a range of speeds.

If the material is sufficiently cold and the cross section sufficiently smooth, so that $\sigma_{t,cold}(|\vec{v} - \vec{V}|) \approx \sigma_{t,cold}(|\vec{v}|)$ for all of the nucleus velocities \vec{V} , then the nucleus velocities are unimportant and the averaged cross section becomes the “cold” cross section:

$$\sigma_t(\vec{v}, \mathbb{N}) \xrightarrow{\text{cold limit}} \frac{\sigma_{t,cold}(|\vec{v}|) v \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z}{v \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z} = \sigma_{t,cold}(v) . \quad (3.42)$$

But in general the correct averaged cross section depends on the distribution of nucleus velocities. If the nuclei are vibrating isotropically—that is, if there is no preferred direction along which the nuclei are wiggling—then our averaged cross section cannot depend on the neutron's direction. That is, in this case

the material looks the same from any direction,

and thus the reaction rate must be the same for a neutron entering the material from any direction. So in this **special case of isotropic nucleus motion**, the average cross section depends on the neutron's speed (or kinetic energy), not on its direction, and we can write it as $\sigma_t(v, \mathbb{N})$ or $\sigma_t(E, \mathbb{N})$, instead of $\sigma_t(\vec{v}, \mathbb{N})$:

$$\sigma_t(\vec{v}, \mathbb{N}) \xrightarrow{\text{isotropic } \mathbb{N}(\vec{V})} \underbrace{\sigma_t(|\vec{v}|, \mathbb{N})}_{\text{no dependence on neutron direction}} \equiv \sigma_t(v, \mathbb{N}) \equiv \sigma_t(E, \mathbb{N}) . \quad (3.43)$$

How do we characterize the distribution of nucleus velocities? In many cases of practical interest, including almost all cases of interest in nuclear-reactor analysis, the nucleus velocity distribution is well approximated by a

Maxwellian

distribution, which is easy to describe mathematically:

$$\begin{aligned} \mathbb{N}(\vec{V}) &\xrightarrow{\text{Maxwellian}} N\left(\frac{M}{2\pi kT}\right)^{3/2} \exp\left[-\frac{M|\vec{V}|^2}{2kT}\right] \\ &= N\left(\frac{M}{2\pi kT}\right)^{3/2} \exp\left[-\frac{M(V_x^2 + V_y^2 + V_z^2)}{2kT}\right], \end{aligned} \quad (3.44)$$

where M is the mass of one molecule of the material. We can see from this expression that

- \mathbb{N} does not depend on nucleus direction (i.e., it is an **isotropic** distribution), and
- \mathbb{N} is completely determined by the material's temperature.

To summarize: in many cases of practical interest, the nuclei vibrate isotropically, and their distribution is well approximated by a **Maxwellian**, which depends only on **temperature**. In such cases the microscopic cross section of Eq. (3.41) depends only on the neutron speed (not direction) and the material temperature:

$$\sigma_t(\vec{v}, \mathbb{N}) \xrightarrow{\text{Maxwellian } \mathbb{N}(\vec{V})} \sigma_t(v, T) = \sigma_t(E, T), \quad (3.45)$$

where T is the material temperature.

It follows that for a thin-target experiment in which the nuclei vibrate in a Maxwellian distribution, we recover our original thin-target formula as long as we use a microscopic cross section that has been appropriately averaged over nucleus velocities:

$$\begin{aligned} \text{RRD} &= I \times \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) \sigma_{t,\text{cold}}(|\vec{v} - \vec{V}|) \frac{|\vec{v} - \vec{V}|}{v} dV_x dV_y dV_z \\ &= I \times \left[\frac{\iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) \sigma_{t,\text{cold}}(|\vec{v} - \vec{V}|) |\vec{v} - \vec{V}| dV_x dV_y dV_z}{v \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z} \right] \times \iiint_{\text{nucl.vels.}} \mathbb{N}(\vec{V}) dV_x dV_y dV_z \\ &= I \sigma_t(\vec{v}, \mathbb{N}) N \end{aligned}$$

$$\xrightarrow{\text{If Maxwellian } \mathbb{N}(\vec{V})} I \sigma_t(v, T) N = I \sigma_t(E, T) N = nv \sigma_t(E, T) N . \quad (3.46)$$

This generalizes to a thin target composed of a mixture of nuclides:

$$\begin{aligned} \text{RRD} &\xrightarrow{\text{Maxwellian } \{\mathbb{N}^j(\vec{V}^j)\}} I \sum_{j=1}^{\text{number of nuclides}} \sigma_t^j(v, T) N^j \\ &= I \Sigma_t(v, T) = I \Sigma_t(E, T) , \quad (3.47) \end{aligned}$$

where Σ_t is the macroscopic cross section for the mixture. We have recognized that there is a one-to-one relation between the neutron's lab-frame speed (v) and its lab-frame kinetic energy (E), which means we can use either v or E to represent the dependence of the averaged cross section on the neutron's lab-frame speed. It is common practice to use E .

It is common practice to drop the T argument in the cross section. **You must know from context whether or not the cross sections being used depend on the material temperature!**

Consider the averaged cross section for a given nuclide and recall that it is an average of the "cold" cross section for that nuclide over a **range of energies** around the neutron's lab-frame energy. The range over which the averaging takes place increases with increasing temperature, because higher temperature produces a wider range of vibrational velocities of the nuclei. We illustrate this in Fig. 3.8, which illustrates how the averaging process smooths out the cross section more and more as material temperature increases.

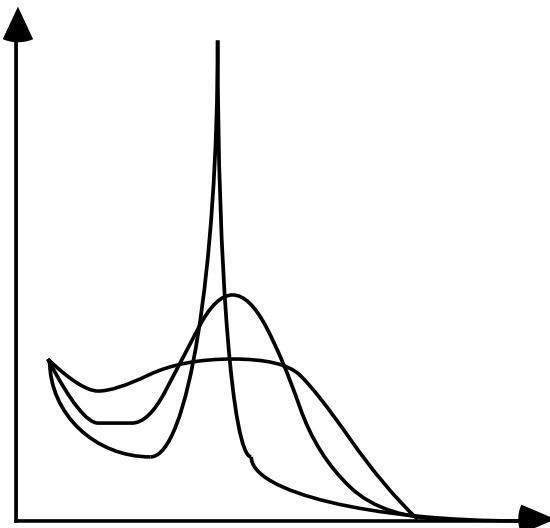


Figure 3.8: Illustration of averaged cross section for three temperatures. $\sigma(E, T)$ vs. E for T_{cold} , $T_1 > T_{cold}$, and $T_2 > T_1$.

The most important consequence of this is

Doppler broadening

of cross-section **resonances**. This effect, pictured in Fig. 3.8, leads to increased parasitic capture of neutrons as fuel temperature increases, at least in commercial reactors. [This is not obvious at this point in our discussion, but we will discuss it later in more detail.] This important

negative-feedback

mechanism adds considerably to the safety of large power reactors.

Even away from resonances, the cross sections we use are

temperature-dependent,

for the same reason. The change in the averaged cross section with changing temperature is just larger near a resonance.

3.6 Differential and Double-Differential Scattering Cross Sections

3.6.1 Differential scattering cross section in energy

Often we care not only about the rate at which a certain reaction occurs, but also about the results of the reaction. For example, suppose we know the rate at which 1-MeV neutrons scatter off of hydrogen in our reactor. What happens to the scattered neutrons? At what energy or energies do they emerge? To compute what happens next, we will have to know this sort of thing.

Thus, we need not only the cross section for scattering, but also the

distribution of scattered neutron energies

that results from the scattering events.

Let us define this distribution:

$P(E_i \rightarrow E_f) dE_f$ = probability that a neutron scattering with **initial energy E_i**

emerges with energy in dE_f about E_f .

We could measure $P(E_i \rightarrow E_f)$ by returning to our thin-target experiment and placing energy-sensitive detectors all around the target. If we did this for neutrons of a certain energy E_i , we might find something like the plot shown in Fig. 3.9.



Figure 3.9: Illustration of $P(E_i \rightarrow E_f)$.

The function $P(E_i \rightarrow E_f)$ is a **probability density function**. The area under the curve is **1**, and P has units of **inverse energy** (MeV^{-1}). Recall the isotropic elastic scattering from NUEN 302 where $E_f \in [\alpha E_i, E_i]$, with $\alpha = (\frac{A-1}{A+1})^2$.

Exercise:

Assume that the function $P(E_i \rightarrow E_f)$ is known. What fraction of neutrons emerges in the interval $E_1 < E_f < E_2$?

Solution:

From the definition of $P(E_i \rightarrow E_f)$ it follows that:

$$\text{fraction} = \int_{E_1}^{E_2} dE_f P(E_i \rightarrow E_f)$$

From the definition of P it also follows that:

$$\sigma_s(E_i)P(E_i \rightarrow E_f) dE_f = \begin{aligned} &\text{microscopic cross section for scattering} \\ &\text{from energy } E_i \text{ into an interval } dE_f \text{ about } E_f \end{aligned}$$

We give the following product its own symbol:

$$\sigma_s(E_i \rightarrow E_f) \equiv \sigma_s(E_i)P(E_i \rightarrow E_f) \quad (3.48)$$

and call it a **microscopic differential scattering cross-section**. Its units are

area per unit energy per nucleus,

commonly expressed in barns/(MeV-nucleus). Naturally, we also define a macroscopic differential cross section for nuclide j :

$$\Sigma_s^j(E_i \rightarrow E_f) = N^j \sigma_s^j(E_i \rightarrow E_f) \quad (3.49)$$

and for a mixture:

$$\Sigma_s(E_i \rightarrow E_f) = \sum_{j=1}^{\text{number of nuclides}} N^j \sigma_s^j(E_i \rightarrow E_f) . \quad (3.50)$$

The units of the macroscopic differential scattering cross section are inverse length per unit energy, commonly expressed in $\text{cm}^{-1} \text{ MeV}^{-1}$.

The name “differential” cross section makes some sense if you notice that the integral of the differential scattering cross section is the regular scattering cross section:

$$\begin{aligned} \int_0^\infty dE_f \sigma_s(E_i \rightarrow E_f) &= \int_0^\infty dE_f \sigma_s(E_i) P(E_i \rightarrow E_f) \\ &= \sigma_s(E_i) \int_0^\infty dE_f P(E_i \rightarrow E_f) \\ &= \sigma_s(E_i) . \end{aligned} \quad (3.51)$$

3.6.2 Differential scattering cross section in direction

The discussion above defines the machinery—the differential scattering cross section in energy—for quantifying how scattering changes a neutron’s energy. Scattering also changes a neutron’s direction. If we care about the directional distribution of neutrons, we must be able to quantify such changes. For this we use the differential scattering cross section in direction, $\sigma_s(E_i, \vec{\Omega}_i \rightarrow \vec{\Omega}_f)$, which is defined such that:

$$\begin{aligned} \sigma_s(E_i) P(\vec{\Omega}_i \rightarrow \vec{\Omega}_f) d\Omega_f &= \text{microscopic cross section for scattering from} \\ &\text{direction } \vec{\Omega}_i \text{ to a direction in the cone } d\Omega_f \text{ around } \vec{\Omega}_f \\ &= \sigma_s(E_i, \vec{\Omega}_i \rightarrow \vec{\Omega}_f) d\Omega_f \end{aligned} \quad (3.52)$$

Of course there is a corresponding macroscopic differential scattering cross section in direction, obtained by multiplying the microscopic cross section by the nuclide number density.

Once again, the integral of the differential scattering cross section is the scattering cross section:

$$\begin{aligned} \int_{\text{all directions}} d\Omega_f \sigma_s(E_i, \vec{\Omega}_i \rightarrow \vec{\Omega}_f) &= \int_{\text{all directions}} d\Omega_f \sigma_s(E_i) P(\vec{\Omega}_i \rightarrow \vec{\Omega}_f) \\ &= \sigma_s(E_i) \int_{\text{all directions}} d\Omega_f P(\vec{\Omega}_i \rightarrow \vec{\Omega}_f) \\ &= \sigma_s(E_i) . \end{aligned} \quad (3.53)$$

3.6.3 Double-differential scattering cross section

When a neutron scatters off of a nucleus, its change in direction is usually correlated with its change in energy. That is, we cannot treat directional changes and energy changes as if they were independent. It follows that a complete description of the scattering process treats both together, using $\sigma_s(E_i \rightarrow E_f, \vec{\Omega}_i \rightarrow \vec{\Omega}_f)$, which is defined such that:

$$\begin{aligned} \sigma_s(E_i)P\left(E_i \rightarrow E_f, \vec{\Omega}_i \rightarrow \vec{\Omega}_f\right)dE_f d\Omega_f &= \text{microscopic cross section for scattering from} \\ &\quad \text{direction } \vec{\Omega}_i \text{ to a direction in the cone } d\Omega_f \text{ around } \vec{\Omega}_f \\ &\quad \text{and from energy } E_i \text{ to an energy in } dE_f \text{ around } E_f \\ &= \sigma_s(E_i \rightarrow E_f, \vec{\Omega}_i \rightarrow \vec{\Omega}_f)dE_f d\Omega_f \end{aligned} \quad (3.54)$$

Once again, the integral of this differential scattering cross section is the scattering cross section:

$$\begin{aligned} \int_0^\infty dE_f \int_{\text{all directions}} d\Omega_f \sigma_s(E_i \rightarrow E_f, \vec{\Omega}_i \rightarrow \vec{\Omega}_f) \\ &= \int_0^\infty dE_f \int_{\text{all directions}} d\Omega_f \sigma_s(E_i)P\left(E_i \rightarrow E_f, \vec{\Omega}_i \rightarrow \vec{\Omega}_f\right) \\ &= \sigma_s(E_i) \int_0^\infty dE_f \int_{\text{all directions}} d\Omega_f P\left(E_i \rightarrow E_f, \vec{\Omega}_i \rightarrow \vec{\Omega}_f\right) \\ &= \sigma_s(E_i) \int_0^\infty dE_f P(E_i \rightarrow E_f) \\ &= \sigma_s(E_i) . \end{aligned} \quad (3.55)$$

3.6.4 Neutrons emerging from fission

Fission is also a reaction from which neutrons emerge. We could define a differential fission cross-section much like the differential scattering cross section in energy:

$$\sigma_f(E_i \rightarrow E_f) = \sigma_f(E_i)P_{\text{fission}}(E_i \rightarrow E_f) \quad (3.56)$$

In most practical cases, the energies of the neutrons that emerge from fission are almost independent of the incident neutron energy, which means $P_{\text{fission}}(E_i \rightarrow E_f)$ is a function of only E_f . We give this function its own symbol (χ , the Greek letter “chi”) and name:

$$P_{\text{fission}}(E_i \rightarrow E_f) \xrightarrow{\text{usually}} \approx \chi(E_f) \equiv \text{fission spectrum.}$$

We provide plots of the fission spectrum in Figs. 3.10. We also provide the associated cumulative distribution function, $X(E)$ in Fig. 3.11, where:

$$X(E) \equiv \int_0^E dE' \chi(E') . \quad (3.57)$$

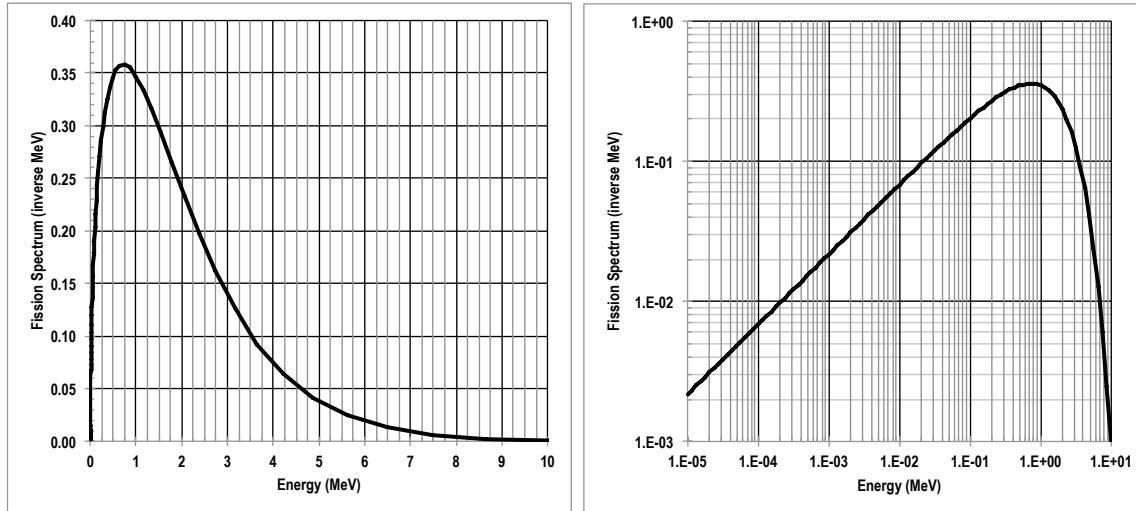


Figure 3.10: Fission Spectrum, $\chi(E)$, on linear-linear (left) and log-log (right) scales.

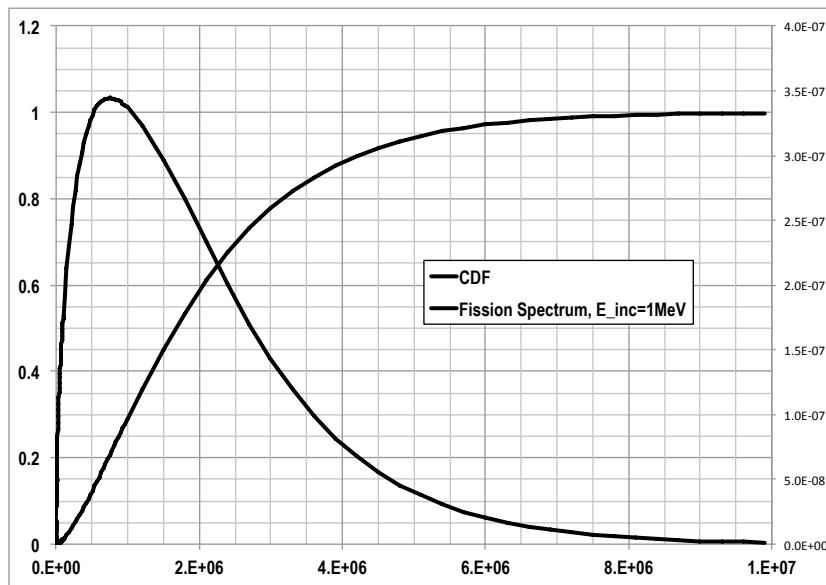


Figure 3.11: Cumulative distribution function (CDF), $\int_0^E \chi(E') dE'$ shown along with the fission-spectrum, (χ) , which is a probability density function (PDF). The CDF scale is on the left axis and PDF scale is on the right. Example of meaning: 88% of neutrons born from fission are born with energies < 4 MeV.

Regarding the directional distribution of emerging neutrons, in nuclear-reactor analysis it is very nearly

isotropic.

This means there is no need to define a function that quantifies the directional distribution of neutrons that emerge from fission.

3.7 Reaction Rate Densities

3.7.1 Preliminaries

We have quantified reaction-rate densities (RRDs) for mono-directional, mono-energetic beams of neutrons incident perpendicularly on both thin and thick targets. Here we generalize to cases in which neutrons are not traveling in single-velocity beams (i.e., the real world).

Recall our expression for RRD given a beam neutrons of intensity I and lab-frame kinetic energy E , incident perpendicularly on a very thin target whose nuclei are vibrating in a Maxwellian distribution at temperature T :

$$\text{RRD} = I \times \Sigma_t(E, T). \quad (3.58)$$

(Remember: for this equation to be correct, Σ_t must have been appropriately averaged over all nucleus velocities, and the nuclei must be wiggling isotropically.) If we recall the physical interpretation of Σ_t , then this equation implies a certain physical interpretation for the beam intensity:

$$\frac{\text{expected reactions}}{\text{cm}^3\text{-s}} = \underbrace{\left[\frac{\text{neutrons' distance traveled}}{\text{cm}^3\text{-s}} \right]}_{\text{"path-length rate density"}} \left[\frac{\text{expected reactions}}{\text{neutrons' distance traveled}} \right]. \quad (3.59)$$

It is easy to compute path-length rate (total-distance-traveled rate) from the density. Consider some volume, V , in which the density of neutrons is n and all neutrons are moving with speed v (possibly in different directions). What is the total distance traveled by all the neutrons in the volume in time dt ?

$$\begin{aligned} \text{total distance traveled} &= [\text{distance traveled per neutron}] [\text{number of neutrons}] \\ &= [v dt] [n V]. \end{aligned} \quad (3.60)$$

To get path-length rate we divide by dt . To get path-length rate density, we also divide by the volume, V . The result:

$$\frac{\text{neutron path length}}{\text{cm}^3\text{-s}} = \frac{\text{total path length}}{V dt} = \frac{v dt n V}{V dt} = nv . \quad (3.61)$$

This is an important, general result! We restate it:

$\text{path-length rate per unit volume} = \text{neutron density} \times \text{neutron speed}. \quad (3.62)$

We give this quantity a name and its own symbol:

$\text{path-length rate per unit volume} = nv = \phi = \text{scalar flux} . \quad (3.63)$

We will constantly use this quantity, the scalar flux, to calculate reaction rates of interest in nuclear reactors. Important: Eq. (3.63) is correct even if neutrons are all moving in different directions.

Terminology: “Path-length rate” \equiv total distance traveled by neutrons per unit time.

How about the beam case?

Given neutrons moving in a mono-velocity beam, then scalar flux $= \phi = I$.

3.7.2 Distribution of neutron velocities

What if neutrons have a distribution of velocities in the lab frame? What is the correct expression for reaction-rate densities? To answer this, we first define a velocity-dependent neutron distribution function, \mathbf{n} , analogous to the nucleus distribution function, \mathbb{N} , that we used previously:

$$\mathbf{n}(\vec{v})dv_xdv_ydv_z = \text{neutrons/cm}^3 \text{ whose velocities are}$$

in the “velocity cube” $dv_xdv_ydv_z$ at \vec{v} . (3.64)

Given only the neutrons with velocities in this range, what would be the RRD with nuclei whose velocities are distributed according to \mathbb{N} ? If we apply what we have developed so far we find the answer:

RRD with neutrons

$$\text{in veloc. cube} = \mathbf{n}(\vec{v}) dv_x dv_y dv_z \times \iiint_{\text{nucl. vels.}} \mathbb{N}(\vec{V}) |\vec{v} - \vec{V}| \sigma_{t,cold}(|\vec{v} - \vec{V}|) dV_x dV_y dV_z . \quad (3.65)$$

show units on this line show units on this line

The total RRD is the “sum” (integral) over all neutron velocities:

$$\text{RRD} = \iiint_{\text{neutron vels.}} dv_x dv_y dv_z \mathbf{n}(\vec{v}) \iiint_{\text{nucl. vels.}} dV_x dV_y dV_z \mathbb{N}(\vec{V}) |\vec{v} - \vec{V}| \sigma_{t,cold}(|\vec{v} - \vec{V}|) . \quad (3.66)$$

This is the correct expression for a neutron-nucleus interaction rate density, in general. This six-dimensional integral simplifies quite a bit, though, if the nuclei are wiggling isotropically and we compute an averaged cross section as detailed previously.

If the nuclei are in a Maxwellian distribution, then as before we can generate an averaged cross section and use it to simplify the RRD expression:

$$\text{RRD} = \iiint_{\text{neutron vels.}} dv_x dv_y dv_z \mathbf{n}(\vec{v}) [Nv \sigma_t(v, T)] . \quad (3.67)$$

The term in brackets depends only on the neutron speed, not direction. Because of this, it is convenient to express the neutron’s velocity distribution in terms of neutron speed and direction. Then the direction integration can be carried out, leaving only an integral over neutron speed:

$$\text{RRD} = \int_0^\infty dv n_{\text{speed}}(v) v \Sigma_t(v, T) . \quad (3.68)$$

Here the “speed” subscript specifies that n_{speed} is a neutron density function that is a density in regular xyz space and also in speed. We recognize that speed times density is scalar flux:

$$\text{RRD} = \int_0^\infty dv \phi_{\text{speed}}(v) \Sigma_t(v, T) . \quad (3.69)$$

Usually we use neutron energy rather than speed:

$$\text{RRD} = \int_0^\infty dE \phi(E) \Sigma_t(E, T) . \quad (3.70)$$

Remark: $\phi_{\text{speed}}(v) \neq \phi(E)$ for $E = \text{kinetic energy of a neutron of speed } v$. The correct equality is $\phi_{\text{speed}}(v)dv = \phi(E)dE$.

Remember: Equation (3.70) is correct only if nuclei are vibrating in approximately a Maxwellian distribution and the cross section has been appropriately averaged over this distribution.

It should be no surprise that there is a similar expression for each different kind of reaction:

$$\begin{aligned} \text{fission rate density} &= \int_0^\infty dE \phi(E) \Sigma_f(E, T) , \\ \text{absorption rate density} &= \int_0^\infty dE \phi(E) \Sigma_a(E, T) , \\ \text{scattering rate density} &= \int_0^\infty dE \phi(E) \Sigma_s(E, T) , \end{aligned} \quad (3.71)$$

etc.

3.8 Summary

We know that reactor behavior fundamentally depends on the production and loss rates of neutrons. Many production and loss mechanisms are neutron-nucleus reactions; thus, computation of production & loss rates requires computation of reaction rates. The following summarizes a few key facts about reaction rates:

1. reaction rates are volume integrals of reaction rate densities.
2. reaction rate densities are (usually) integrals over energy of $\phi(E)\Sigma(E)$.
3. $\phi(E)$ is path-length rate per unit volume per MeV.
4. $\Sigma_x(E)$ is expected reactions of type x per unit path length.
5. $\Sigma_x(E)$ is $\sum_i N^i \sigma_x^i(E)$.
6. There is a different σ for each different kind of reaction.

7. Each σ fundamentally depends on relative speed between the neutron & nucleus. Almost always, we can average this “fundamental” σ over the distribution of nucleus velocities and obtain a cross section that is a function of the neutron lab-frame speed. Buried in this averaged cross section, of course, are the details of the wiggles of the nuclei; however, we can usually characterize these wiggles by the material’s temperature. Thus, the cross sections that look like functions of the neutron lab-frame speed (or kinetic energy) are also functions of the material temperature.
8. The mean free path for reactions of type x , which is the average distance a neutron of energy E travels before having a reaction of type x , is $1/\Sigma_x(E)$. This average distance is independent of the neutron’s history—it holds for any neutron of energy E in the material whose cross section is $\Sigma_x(E)$, even if that neutron has already traveled some distance in the material! (The neutron has no memory of its previous travels.)

Part II

Principles of Chain Reactions

Chapter 4

Fission

4.1 Introduction

Can you think of any discovery in the history of science that changed the world as quickly and dramatically as the discovery that nuclear fission chain reactions are possible? It was not long ago (very late 1930s) that scientists realized that:

- fission can be induced by neutron absorption,
- fission releases neutrons (which can go on to cause other fissions), and
- fission releases enormous quantities of energy ($\approx 10^8 \times$ as much as a chemical reaction).

These facts imply the possibility of a chain reaction that could release enormous energy from small quantities of matter. Soon after this came to light, scientists and engineers had built a critical reactor (1942) and exploded a nuclear weapon (1945). Nuclear weapons have changed the world dramatically, and they continue to play significant roles in international relations.

Nuclear reactors have also had a significant impact. Research reactors have led to discoveries and power reactors have generated enormous quantities of electrical energy. Unfortunately, nuclear reactors have also had accidents with substantial releases of radioactive material. One goal of nuclear engineering is to make such accidents far less likely, with much milder consequences. Remember, nuclear engineering is a young discipline, with fission discovered only a few decades ago. There remain opportunities for improvement.

Here we explore some fundamentals of fission. We know that fission of a heavy nucleus releases energy—a lot of energy for something so small. Oddly enough, fusion of two light

nuclei also releases a relatively large quantity of energy. Fissioning a medium-mass nucleus or fusing two of them would, on the other hand, consume energy. Why is this? In this chapter we will explore this. We will also explore what comes from a fissioning atom, and on what time scale, in some detail.

Fission reactors actually produce new fuel as they operate. If they produce more fuel than they consume, they are called breeder reactors. Otherwise, they are converters. Commercial reactors in operation today are converters. In this chapter we will briefly discuss and quantify this.

4.2 Brief Look at Key Fission Physics

4.2.1 Binding energy

The mass of a nucleus is less than the sum of the masses of the neutrons and protons contained in it. This mass difference is called the **mass defect**. The mass defect, Δm , is given by

$$\Delta m = Zm_p + Nm_n - M_{\text{nucleus}}(Z, A) , \quad (4.1)$$

where Z is the number of protons in the nucleus, A is the number of nucleons (protons + neutrons) in the nucleus, m_p is the mass of the proton ($m_p = 1.007227 \text{ u}^1$), m_n is the mass of the neutron ($m_n = 1.008665 \text{ u}$), and $M_{\text{nucleus}}(Z, A)$ is the mass of the nucleus. A is often called the “mass number.” When we express Δm in energy units, we have the nuclear binding energy,

$$\text{BE}_{\text{nuclear}} = \Delta m \times c^2 = [Zm_p + Nm_n - M_{\text{nucleus}}(Z, A)] \times c^2 ,$$

i.e., the energy that would be required to break the nucleus into its constitutive nucleons. This energy is called the binding energy since it represents the energy that “binds” the nucleus together. On the other hand, when Z protons and N neutrons are brought together to form a nucleus, an energy of the amount $\Delta m \times c^2$ is released. If $\Delta m \times c^2$ is re-supplied, the Z protons and N neutrons can be separated again.

Electron binding energy. When discussing nuclear (not chemical) reactions we usually neglect the energy that binds the electrons to the nucleus, because this energy is many orders of magnitude smaller than the energy with which nucleons are bound to each other. This small electron binding energy is as follows for a nucleus of mass $M_{\text{nucleus}}(Z, A)$ and its Z electrons of mass m_e :

$$\text{BE}_e(Z, A) = [Zm_e + M_{\text{nucleus}}(Z, A) - M_a(Z, A)] \times c^2 , \quad (4.2)$$

¹“u” is the abbreviation for atomic mass unit, defined to be $1.6605402 \times 10^{-27} \text{ kg} = 931.5016 \text{ MeV}/c^2$

where $M_a(Z, A)$ is the mass of the neutral atom including its Z bound electrons. The atom's total binding energy is the sum of the nucleons' binding energy and the electrons' binding energy, which can be expressed as:

$$\begin{aligned} \text{BE}_{\text{atom}}(Z, A) &= \text{BE}_{\text{nucleus}}(Z, A) + \text{BE}_{\text{electrons}}(Z, A) \\ &= \Delta m_{\text{total}} \times c^2 \\ &= [Z(m_p + m_e) + Nm_n - M_a(Z, A)] \times c^2. \end{aligned} \quad (4.3)$$

But because $\text{BE}_{\text{electrons}}(Z, A) \ll \text{BE}_{\text{nucleus}}(Z, A)$ we often make the approximation

$$\text{BE}_{\text{nucleus}}(Z, A) \approx \text{BE}_{\text{atom}}(Z, A) = [Z(m_p + m_e) + Nm_n - M_a(Z, A)] \times c^2. \quad (4.4)$$

This approximate formula is useful because tables of $M_a(Z, A)$ are easily obtainable (charts of nuclides, reference books, the internet, e.g., <http://ie.lbl.gov/toimass.html> and within it, the nice search site <http://ie.lbl.gov/toi2003/MassSearch.asp>, ...).

The total binding energy is an increasing function of the atomic mass number A , but it does not increase at a constant rate. A more interesting curve is binding energy **per nucleon** (i.e., $\text{BE}_{\text{nuclear}}/A$) as a function of A . Let us sketch this in Fig. 4.1.

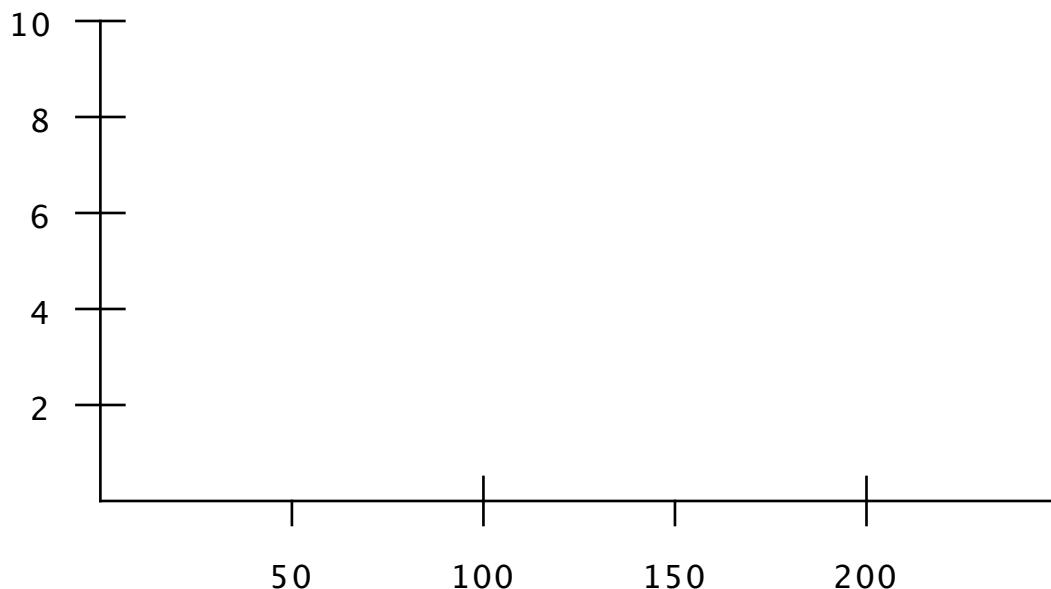


Figure 4.1: Binding energy per nucleon as a function of mass number. Nuclides with mass numbers in the 50-70 range are the most tightly bound. 7.6 at $A=12$ and $A=240$. 1 at $A=2$. 8.8 at $A=60$. 7 at $A=4$.

For low A 's, there are a few deviations from the curve, which are linked to “magic numbers” of nucleons that form unusually stable (tightly bound) nuclei. Examples are ${}^4\text{He}$ and ${}^{12}\text{C}$.

At $A \approx 60$, the curve reaches its maximum (Ni, Fe) and then for $A \gtrsim 60$, the curve is smoothly decreasing.

Nuclei with high average binding energy per nucleon are more tightly bound, so a larger amount of energy must be used to break a nucleon away, whereas nuclei with low average binding energy per nucleon are less tightly bound and can be broken apart more easily.

Therefore, whenever a more tightly bound nucleus is formed by combining two less tightly bound lighter nuclei, energy is released in the process. This describes the **fusion** principle. On the other end of the curve, whenever a heavy nucleus can be split into two lighter parts, which are more tightly bound, energy is released in the process. This describes the **fission** principle.

Since fission of heavy nuclei is favored energetically, we might ask why it does not occur spontaneously. First, because of short-range nuclear forces there is a

potential-energy barrier

(of several MeV) that must be overcome before a nucleus is free to fly apart. That is, the nucleus must climb a several-MeV “hill” before it can fall off of a many-MeV cliff.

Second, spontaneous fission does occur, because of quantum-mechanical “tunneling” through the potential barrier. This is a rare event in most heavy nuclides; for example, the half-life for spontaneous fission in ${}^{238}\text{U}$ is 6.5×10^{15} years. However, there are exceptions: Californium-252 (${}^{252}\text{Cf}$) has a half-life for spontaneous fission of only 66 years. Because of this, ${}^{252}\text{Cf}$ is often used as a neutron source.

We must design our reactors so that heavy nuclei can routinely overcome the several-MeV fission barrier. There are at least two ways to overcome the barrier:

1. **slam the nucleus with a multi-MeV particle, using its kinetic energy**
2. **let the nucleus absorb a neutron, thereby releasing binding energy**

We use the second approach in our reactors, partly because multi-MeV particles are not available in sufficient quantity to use the first, and partly because neutrons are available—they are emitted from fission, which suggests the potential of a self-sustaining chain reaction.

We can compute the energy released by any particular fission event. For example,

1n	+	${}^{235}U$	\longrightarrow	${}^{95}Zr$	+	${}^{138}Te$	+	$3 {}^1n$
BE/A [MeV/nucleon]	(*)	-	7.5	8.6	8.3	-	-	-
BE [MeV]	-	1762	817	1146	-	-	-	-

(*) estimate the value of BE_n/A from Fig. 4.1

In this example, we have a net gain of $\approx 1146 + 817 - 1762 = 201$ MeV per fission. This estimate, which we obtained just by looking at a plot, is quite close to the actual value. If you want the accurate figures for this reaction, you must look up the exact atomic-mass or binding-energy values.

4.3 Fissile, Fissionable, and Fertile Nuclides

If a nuclide is reasonably likely to fission after absorbing a very slow neutron, it is called

fissile.

Fissile nuclei surpass the fission barrier with only the

binding energy of another neutron.

Some fissile nuclei:

U-233, U-235, Pu-239, Pu-241 .

If a nuclide is reasonably likely to fission after absorbing a neutron with kinetic energy of an MeV or two, but not after absorbing a slow neutron, it is called

fissionable.

Fissionable nuclei need the binding energy of another neutron, plus

some more energy,

to make it over the fission barrier.

Some fissionable nuclei:

Th-232, U-234, U-236, U-238, Pu-240, Pu-242 .

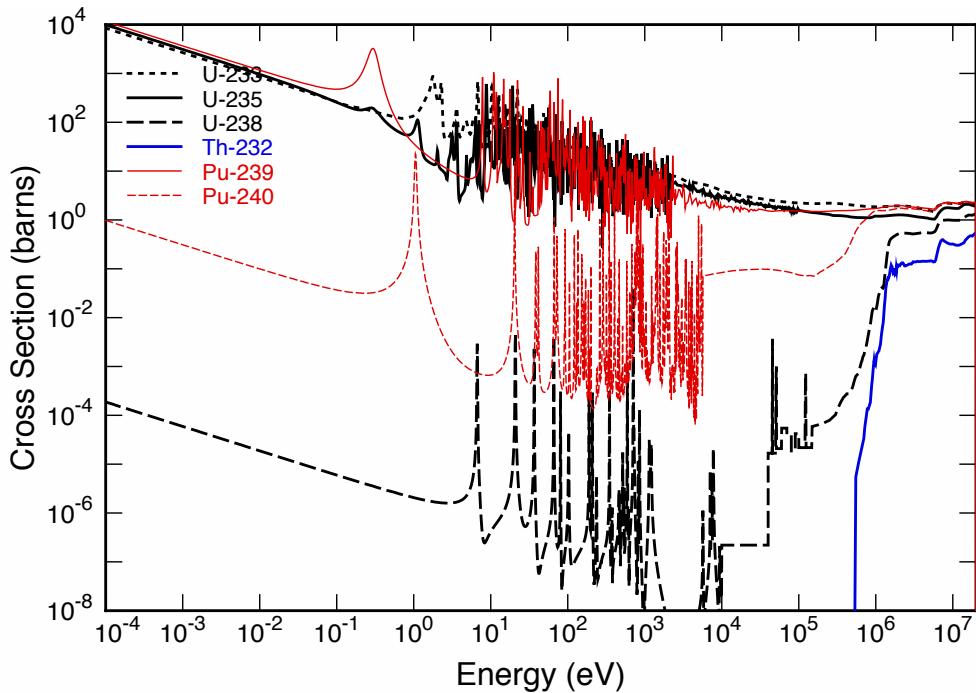


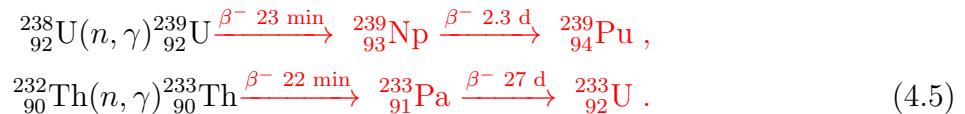
Figure 4.2: Microscopic fission cross sections for several fissile and fissionable nuclides. Note that the vertical log scale spans many orders of magnitude. Note further that although the ^{238}U fission cross section becomes quite small for energies below ≈ 1 MeV, there is no threshold below which it goes to zero. The same is true of ^{232}Th , although the vertical axis is not shown at low enough energies to see this.

Fission in fissionable nuclei is

not a true **threshold reaction**,

in the following sense: the cross section is not zero below any threshold neutron energy, because fission (which you will remember is energetically favored!) can occur by quantum-mechanical “tunneling” even when there is not enough energy to get over the barrier. It’s just less likely. Figure 4.2 shows the fission cross sections for some fissile and fissionable nuclides.

Fertile nuclides are those that are non-fissile but can produce a fissile nuclide after absorbing a neutron. Two important examples:



4.4 Conversion and Breeding

Reactors that contain fertile material convert some of it to fissile material. They are called converter reactors.

All reactors convert to some extent, and most commercial reactors convert a significant amount of ^{238}U to ^{239}Pu , as indicated in the first line of Eq. (4.5). Other actinides (elements with $Z > 88$, some of which are fissile) are also produced. These new nuclides may produce a significant portion of a reactor's power, especially at the end of a fuel cycle. It is possible to extract this material from spent reactor fuel and use it in later fuel cycles.

For any reactor, we define:

$$\text{conversion ratio} = \frac{\text{fissile atoms produced}}{\text{fissile atoms consumed}} \quad (4.6)$$

If this ratio is greater than unity (which is quite possible!), the reactor is called a breeder

reactor. In this case the conversion ratio is called the breeding ratio.

If we define "fuel" to mean "**fissile** atoms," then breeder reactors produce more fuel than they consume. This cannot go on indefinitely; at some point the world's supply of fertile material would run out. However, the world's supply of fertile material is far, far greater than its supply of fissile material. Because of this, breeder reactors could extend the lifetime of fission power into the distant future (many, many centuries).

Breeding in a reactor is possible only if the following inequality is true .

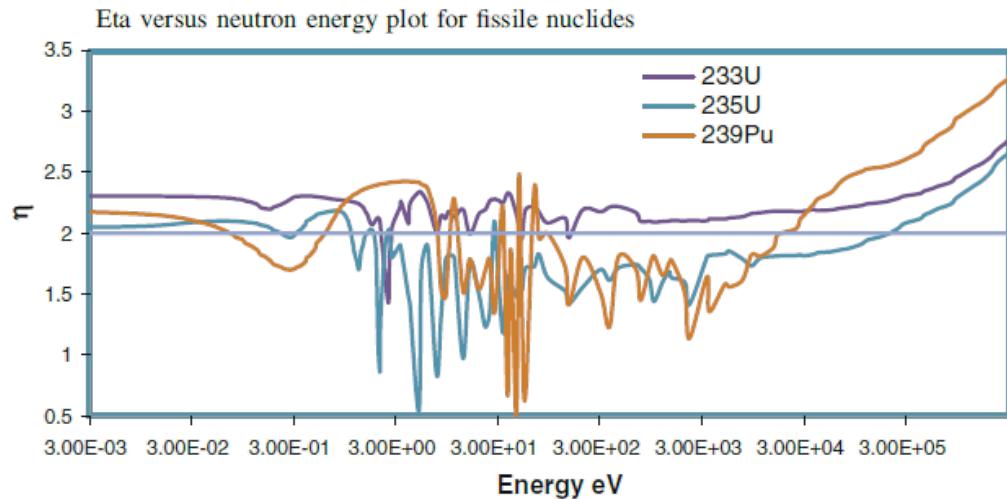
$\eta \equiv$ reproduction factor

$$= \frac{\# \text{ of neutrons produced}}{\# \text{ of neutrons absorbed in fuel}} = \frac{\sum_{i=1}^{\# \text{ of fuel nuclides}} [\nu^i \Sigma_f^i]}{\sum_{i=1}^{\# \text{ of fuel nuclides}} [\Sigma_a^i]} > 2. \quad (4.7)$$

Note that η will depend on

the energies of the neutrons getting absorbed.

In fact, η tends to increase substantially at neutron energies above 100 keV (0.1 MeV) for most fissile materials. For this reason, breeders are usually “fast” rather than “thermal” reactors. This means their neutrons are mostly kept at high energies rather than being slowed down to thermal (low) energies. However, it is possible to design a thermal breeder reactor using ^{232}Th as the fertile fuel and ^{233}U as the fissile fuel. Figure 4.3 illustrates why this is possible.



Source: Aqueous Reprocessing by THOREX Process, 2013, by P. V. Achuthan & A. Ramanujam, page 281

Figure 4.3: Reproduction factor, η , as a function of the energy of the incident neutron, for ^{233}U , ^{235}U , and ^{239}Pu .

We close this section with more detailed plots of binding energy per nucleon, shown in Fig. 4.4.

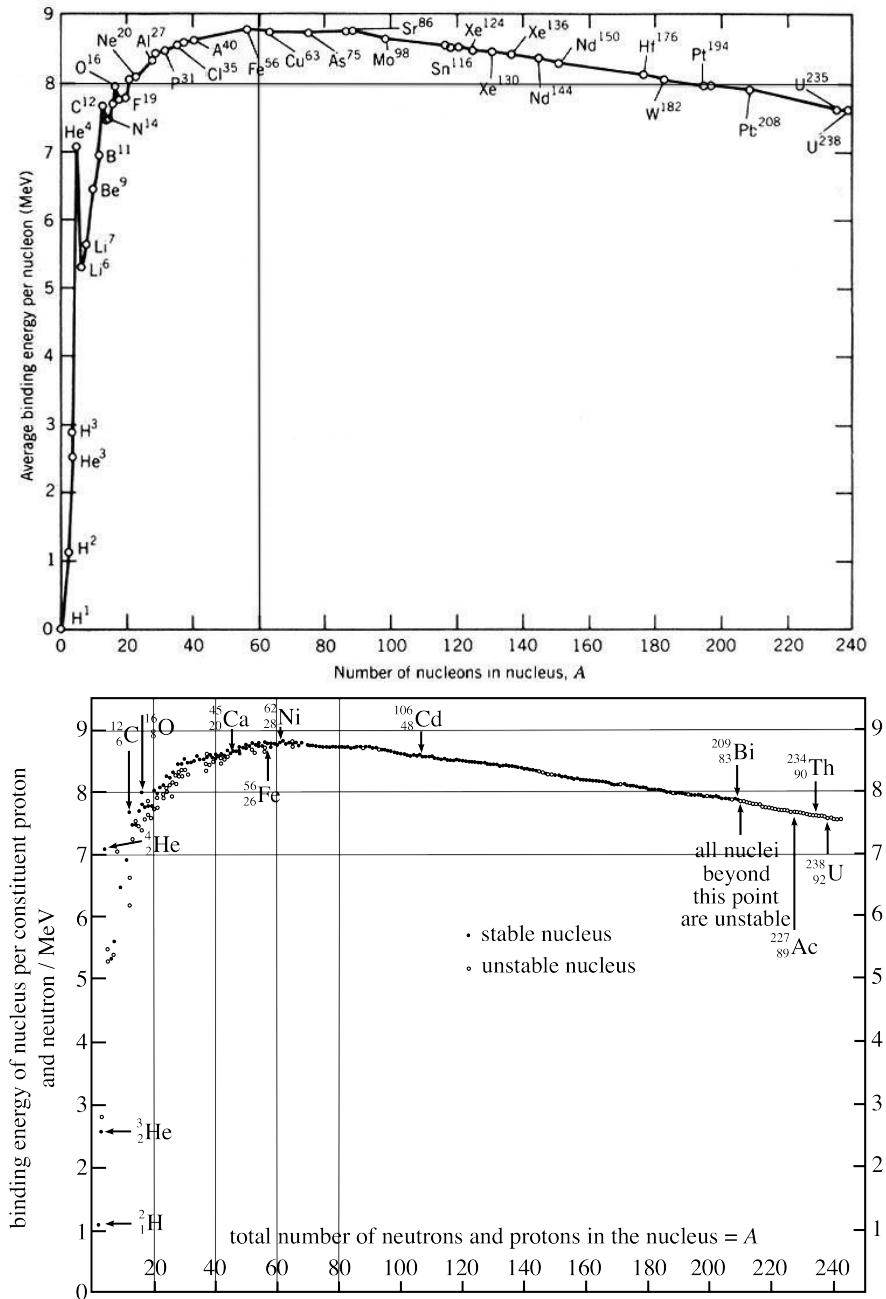


Figure 4.4: Binding energy per nucleon as a function of mass number. Top figure uses the most stable isobar (most tightly bound nuclide) for each value of A and identifies a number of nuclides of interest, with straight lines drawn between data points. Bottom figure shows values for more nuclides, not just the most stable isobars, and identifies several of them, with no lines drawn between data points.

4.5 Fission Process and Results

We have seen that the absorption of a neutron puts the resulting compound nucleus in an excited state, mostly from the binding energy added by absorption of the neutron but also including the center-of-mass-frame kinetic energy. We have also seen that sometimes this excitation energy is enough to cause a heavy nucleus to **fission**. Fission is illustrated in Fig. 4.5.

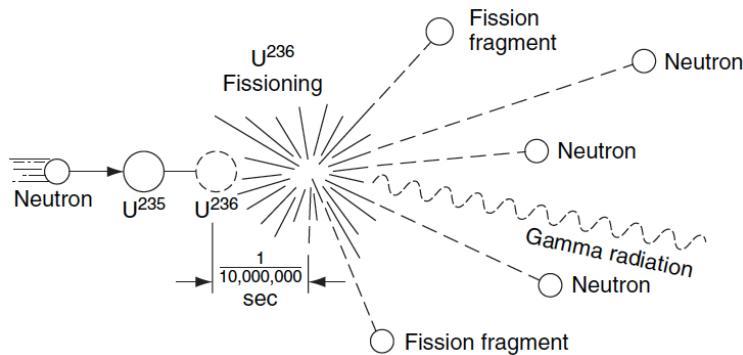


Figure 4.5: Schematic of a fission event in ^{235}U .

The following are all part of the fission process. For each event we give the time scale and the amount of energy that is released to the particles that are emitted.

1. scission (splitting into two or more fragments): ≈ 0.01 picoseconds (ps)
2. prompt neutrons emitted: $\approx 0.01 - 1$ ps; ≈ 5 MeV
3. prompt gammas emitted: $\approx 0.01 - 1$ ps; ≈ 7 MeV
4. fission fragments slow down: ≈ 10 ps; $\approx 165 - 170$ MeV to matter
5. fission products and daughters decay: $\ll 1$ s to \gg centuries; $\approx 25 - 30$ MeV

Note that the 25 – 30 MeV from fission-product decay includes

- $\approx 12 - 15$ MeV carried away by antineutrinos
- $\approx 7 - 8$ MeV carried by β^- 's
- $\approx 6 - 7$ MeV carried by γ 's
- small amount carried by delayed neutrons

One immediate result of fission (10-20 femtoseconds after scission) is

fission fragments (usually two)

that rush away from each other with total kinetic energy of roughly 165 MeV, which is most of the energy that is ultimately released by the fission process (roughly 200 MeV). The fission fragments give up their kinetic energy quickly in typical reactor materials:

They come to rest ≈ 10 picoseconds (ps) after scission.

They lose energy via

Coulomb interactions with electrons and surrounding nuclei,

thereby converting their kinetic energy to

heating of the material within a small distance from the fission event.

Note: Even if we restrict ourselves to a particular nuclide, such as ^{235}U , we can talk only about average values for the energy carried by different kinds of particles. The fission process is stochastic—there are lots of different ways that the compound nucleus can split, and lots of different paths that the excited fission fragments can follow to de-excite.

Question: Does all of the energy released by fission go into heating the reactor material? That is, is all of this energy “recoverable?” Explain.

Answer: No. Some of the particles escape, carrying away their kinetic energy:

- The fission-fragment energy is deposited locally.
- The betas are deposited locally.
- In a large reactor, only a few percent of the neutrons and gammas escape.
- The neutrinos escape.

Question: Suppose a reactor has been operating at 100% power for a long time. If we “scram” the reactor and thus stop fissions from occurring, have we stopped the heating of the fuel and structural materials? Explain.

Answer: No. The fission products continue to decay, releasing betas and gammas that deposit energy in the reactor material.

Question: What fraction of reactor power comes from the decay of fission products?

Answer: $(\approx 13 - 15 \text{ MeV}) / (\approx 200 \text{ MeV}) = \approx 6-7\%$

What does this tell you about cooling systems for reactors? Do we still need them after we've scrammed the reactor? **Yes. Decay heat is more than sufficient to melt the reactor.**

Question: What other mechanisms (reactions) produce recoverable energy in reactors?

Answer: Radiative capture—the binding energy of the neutron is emitted in γ 's

Neutron scattering also heats the reactor material, because the neutron transfers KE to material. However, this is not additional energy on top of what is released from fission—it is part of the 5 MeV carried from the fission event by the prompt neutrons.

4.6 Fission-fragment mass distribution

One might guess that the most likely way a compound nucleus would split would be into two equal-sized halves. It turns out that a 1/3 - 2/3 split is favored significantly over a half-and-half split, at least for fission that is initiated mostly by the binding energy of a neutron. (If we cause fission by bombarding heavy nuclei with sufficiently energetic particles, say hundreds of MeV, then the half-and-half split becomes most favored.) For fissions in nuclear reactors we have approximately the distribution of fission products given in Fig. 4.6 below.

The top left figure compares, on a logarithmic scale, the mass distribution of fission fragments from fission of three different fissile nuclides when fission is caused by slow neutrons. Note that there are two distinct high-probability humps with a much lower-probability valley in between. This is seen more clearly on the linear scale in the figure in the top right. Note further that when a higher-mass nuclide fissions, it is the low-mass hump that moves to the right (compare ^{233}U against ^{239}Pu). The bottom figure shows how the valley gets filled in when fission is caused by higher-energy neutrons, shown here for fission of ^{235}U induced by thermal neutrons and by 14-MeV neutrons.

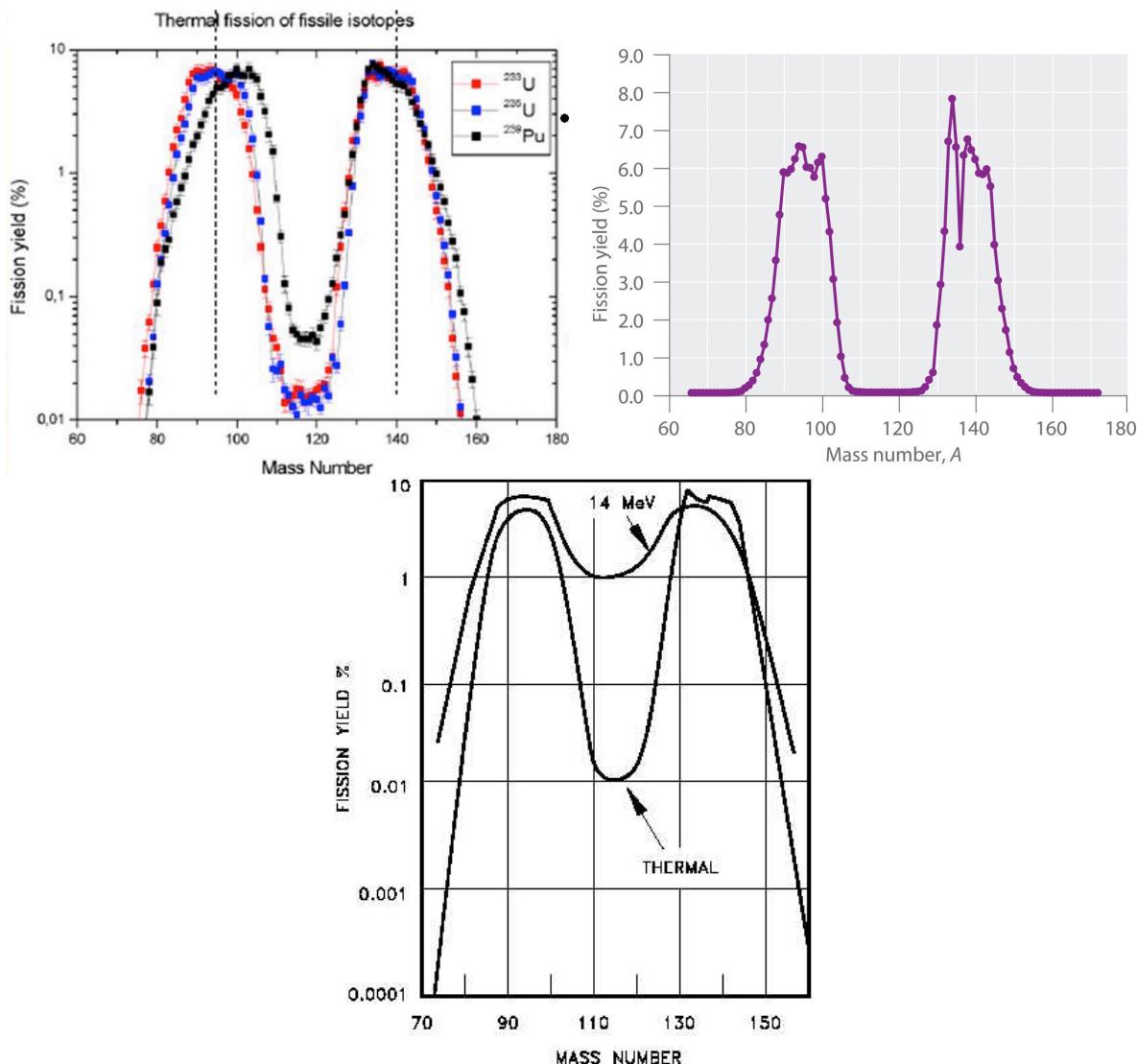


Figure 4.6: Fission-fragment mass-number distribution. Top left: fission induced by absorption of low-energy neutrons, for ^{233}U , ^{235}U , and ^{239}Pu . Top right: same thing for ^{235}U but on a linear scale. Bottom: same thing, but for different incident neutron energies.

Chapter 5

Chain Reactions

5.1 Introduction

We know that neutron production and loss rates determine the behavior of a nuclear reactor. In this chapter we introduce some terms that help us describe production rates, loss rates, and how well they are balanced.

5.2 Multiplication Factor: Definitions

Suppose we could remove all the neutrons from a source-free reactor, then initiate exactly enough fissions to create N fission neutrons. Let us define these to be a

“**generation**” of neutrons.

Some neutrons in any given generation will cause fission. The neutrons resulting from such fissions are defined to constitute the next generation.

We introduce a common definition of the multiplication factor:

$$\begin{aligned} k &= \text{multiplication factor} \\ &= \frac{\# \text{ of neutrons in a generation}}{\# \text{ of neutrons in the previous generation}}. \end{aligned} \tag{5.1}$$

Equivalently, k = number of “children” neutrons born per “parent” neutron. **Terminology:**

$k < 1$	\iff	neutron population is decreasing	\iff	subcritical
$k = 1$	\iff	neutron population is steady	\iff	critical
$k > 1$	\iff	neutron population is increasing	\iff	supercritical

Another common definition of multiplication factor is:

$$\begin{aligned} k &= \frac{\text{production rate of neutrons}}{\text{loss rate of neutrons}} \\ &= \frac{P(t)}{L(t)}. \end{aligned} \quad (5.2)$$

Now that we have defined k , we note that

these “definitions” are both wrong

unless the neutrons are in their “fundamental-mode” distribution for the reactor in question. In the case of a fundamental-mode distribution, the two “definitions” given above are both correct. Later we will discuss this fundamental mode (and other modes) and develop a more rigorous definition of k . For now we offer the following teaser:

k is a property of the **system** being studied,
not of the **neutrons** in the system.

To understand this, first understand that a critical reactor is, by definition, a reactor that could sustain a steady-state fission chain reaction without help from sources. If we place a neutron source in some ordinary object (like a desk, for example), the neutrons in that object will quickly attain a steady-state distribution, with the production rate equaling loss rate. This does not mean the desk is a critical reactor! It could not sustain a steady-state chain reaction without help from sources.

5.3 A Simple Problem

Consider an **infinite uniform** “sea” of some material. Pretend that

all neutrons have **the same speed**,

and that

all cross sections (appropriately averaged over nuclear motion) are known.

Let us assume that at the “beginning” (whenever that is), neutrons are introduced in to the medium, uniformly and isotropically. What will happen?

Any time we want to learn something about a problem like this, we should turn to our favorite fundamental truth (we are neutron accountants, remember):

conservation!

In this problem, we write our conservation statement as:

$$\text{rate of change of neutron population} = \text{rate of gain} - \text{rate of loss}$$

or

$$\text{change rate} = \nu \times [\text{fission rate}] - [\text{absorption rate}], \quad (5.3)$$

where

$$\nu \equiv \text{average number of neutrons emitted per fission.}$$

Question: Why do we not consider loss from leakage?

Answer: There's no leakage from an infinite medium.

Let us consider some fixed finite volume, V , of our infinite medium. Is there any **net** leakage out of this volume?

No, everything is uniform, so just as many neutrons come in as go out.

Thus, the above conservation statement holds in every sub-volume of the infinite medium.

Let us write our conservation statement in mathematical notation. First, let

$$\begin{aligned} n &= \text{neutron density [n/cm}^3\text{]} \\ v &= \text{neutron speed [cm/s]} \end{aligned}$$

Note that our conservation statement involves reaction rates, for which we need the scalar flux. What is the scalar flux in this problem? Answer: Note ϕ is always $v \times n$

$$\begin{aligned} \phi &= \text{path-length rate per unit volume} \\ &= [\text{n's per unit volume}] \times [\text{path length per n per s}] \\ &= [n][v] [\text{n}/(\text{cm}^2\text{-s})] \end{aligned}$$

Now we can write our conservation statement mathematically, replacing each set of words with the appropriate mathematical expression:

$$\frac{dn}{dt} V = \nu \Sigma_f n v V - \Sigma_a n v V \quad (5.4)$$

Now that we have a differential equation, we can figure out things through mathematical manipulation and analysis. First, note that the volume divides out of each term, and we have:

$$\frac{dn}{dt} = \nu \Sigma_f n v - \Sigma_a n v . \quad (5.5)$$

which means, in words, that:

$$\text{change-rate density} = \text{production-rate density} - \text{absorption-rate density} .$$

We know how to solve this equation for the neutron density, n . We get

$$n(t) = n_0 \exp((\nu \Sigma_f - \Sigma_a) vt) \quad (5.6)$$

Pretty remarkable: all we did was recognize conservation, and now we've figured out everything about the problem!

Question: What is k in the infinite-medium one-speed problem? Hint: the fundamental-mode distribution in this infinite homogeneous medium is just the spatially uniform (constant) distribution.

Answer: Let's use both of our "definitions" from before. Here's one:

$$k = \frac{\text{production rate of neutrons}}{\text{loss rate of neutrons}} = \frac{\nu \Sigma_f n v}{\Sigma_a n v} = \frac{\nu \Sigma_f}{\Sigma_a} . \quad (5.7)$$

Here's the other:

$$k = \frac{\text{children birth rate}}{\text{parent death rate}} = \frac{\nu \Sigma_f n v}{\Sigma_a n v} = \frac{\nu \Sigma_f}{\Sigma_a} . \quad (5.8)$$

Question: What must be true about the material cross sections to make a steady-state neutron population possible, given one-speed neutrons in a source-free infinite medium? That is, what would make the medium critical?

Answer: We must have

$$\nu \Sigma_f = \Sigma_a .$$

Question: Does the multiplication factor depend on the neutron population?

Answer: No! Only on material properties!

5.4 Multiplication Factor: Formulas

We shall develop here a formula for the multiplication factor, k , in a reactor. We do so by considering the life history of a neutron from birth to death, describing each fork in the road with a mathematical statement of probability. (We assume that the neutrons are in the fundamental-mode distribution.) First we define:

P_{FNL} = fast non-leakage probability
= probability that a fast neutron **does not leak**

P_{TNL} = thermal non-leakage probability
= probability that a thermal neutron **does not leak**

p = resonance-escape probability
= probability that a fast neutron is not absorbed while slowing to thermal,
given that it didn't leak

u_T = thermal utilization (sometimes given the symbol f)
= probability that thermal neutron is absorbed in fuel, given that **it was absorbed**

u_F = fast utilization
= probability that fast neutron is absorbed in fuel, given that it was **absorbed while fast**

P_{TAF} = prob. that thermal absorption in fuel **causes fission**

P_{FAF} = probability that fast absorption in fuel causes fission

ν_T = average number of neutrons emitted per fission caused by a thermal neutron
i.e., per “thermal fission”

ν_F = average number of neutrons emitted per fission caused by a fast neutron
i.e., per “fast fission”

Exercise: What would P_{TAF} be in terms of Σ 's? How about u_T ?

Solution:

$$P_{\text{TAF}} = \Sigma_{f,\text{th,fuel}} / \Sigma_{a,\text{th,fuel}}$$

$$u_T = \frac{\sum_{a,\text{th,fuel}} \phi_{\text{th,fuel}} V_{\text{fuel}}}{\sum_{a,\text{th,fuel}} \phi_{\text{th,fuel}} V_{\text{fuel}} + \sum_{a,\text{th,mod}} \phi_{\text{th,mod}} V_{\text{mod}}}$$

In Fig. 5.1 we sketch the possible neutron life histories in a reactor.

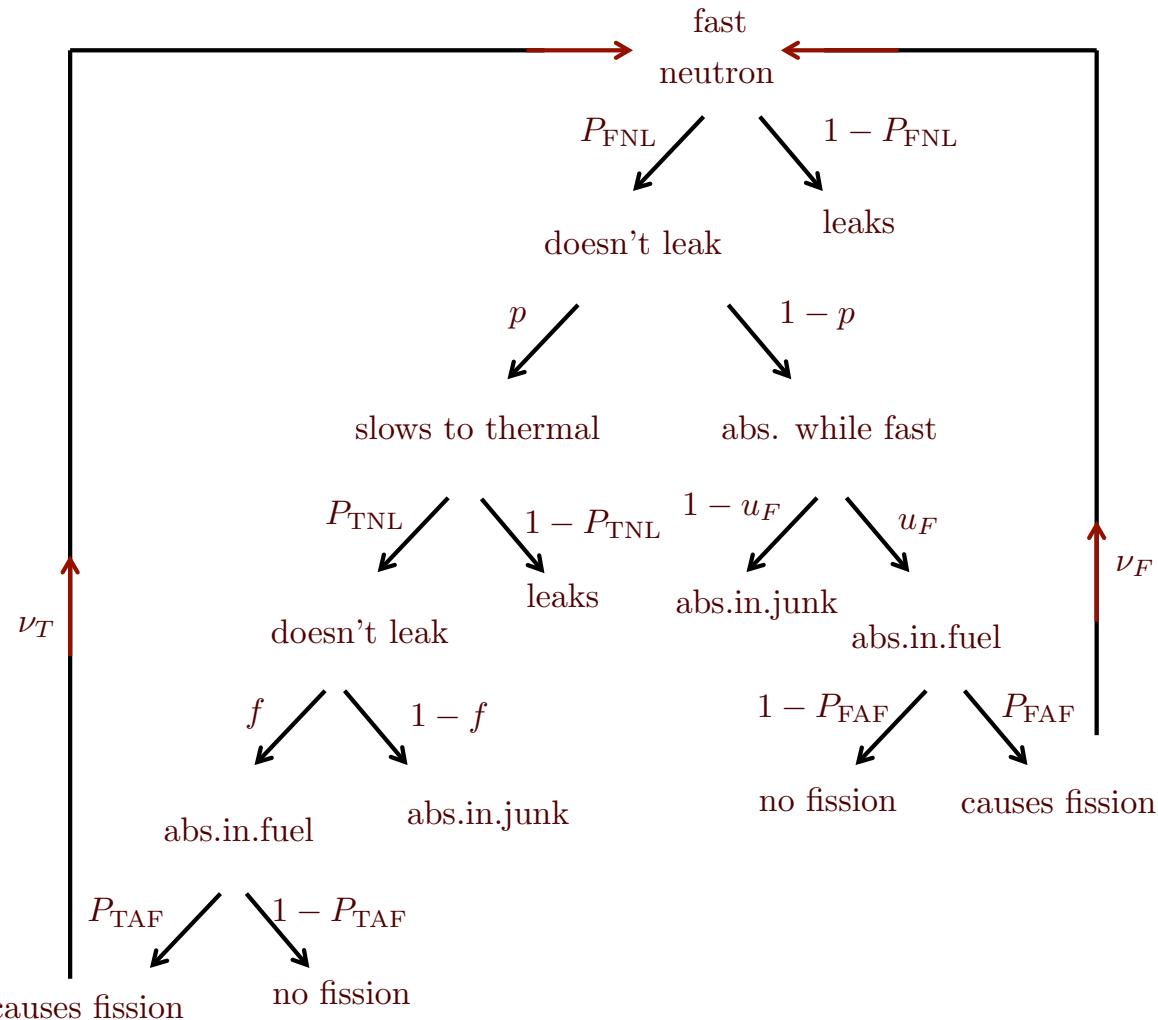


Figure 5.1: Possible histories of neutrons in a reactor. Manipulation of the probabilities of each event produces the well-known six-factor formula.

It follows that the multiplication factor must be:

$$k = P_{\text{FNL}} p P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T + P_{\text{FNL}} (1-p) u_F P_{\text{FAF}} \nu_F . \quad (5.9)$$

It is customary to define

$$\begin{aligned} \epsilon &= \text{fast-fission factor} \\ &= \frac{\text{expected } \# \text{ of n's from all fissions}}{\text{expected } \# \text{ of n's from thermal fissions}} \\ &= \frac{P_{\text{FNL}} p P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T + P_{\text{FNL}} (1-p) u_F P_{\text{FAF}} \nu_F}{P_{\text{FNL}} p P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T} \\ &= 1 + \frac{1-p}{p} \frac{u_F P_{\text{FAF}} \nu_F}{P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T} \end{aligned}$$

and

$$\begin{aligned} \eta_T &= \text{thermal reproduction factor} \\ &= \text{expected } \# \text{ of n's produced per thermal absorption in fuel} \\ &= P_{\text{TAF}} \nu_T \end{aligned}$$

It is not customary, but we shall also define:

$$\begin{aligned} \eta_F &= \text{fast reproduction factor} \\ &= \text{expected } \# \text{ of n's produced per fast absorption in fuel} \\ &= P_{\text{FAF}} \nu_F \end{aligned}$$

This gives us:

$$k = P_{\text{FNL}} [p P_{\text{TNL}} u_T \eta_T + (1-p) u_F \eta_F] \quad (5.10)$$

$$= P_{\text{FNL}} P_{\text{TNL}} \epsilon p u_T \eta_T . \quad (5.11)$$

This last equation is the well-known

six-factor formula.

In an infinite medium, nothing leaks and we have the **four-factor formula**:

$$k_\infty = \epsilon p u_T \eta_T \quad (5.12)$$

Beware: While p , u_T , and η_T depend only on material properties, ϵ depends also on P_{TNL} and thus is different in an infinite medium than in a finite medium. See example next page!

Note that

k_∞ is an upper bound for k .

Example:

Recall that a neutron in a reactor will ultimately either leak or get absorbed. Suppose that in a certain reactor, for every 1000 neutrons that are born from fission, the following is how many leak and how many get absorbed:

- 20 leak after they become thermal
- 80 leak while they are still fast
- 200 are absorbed in fuel while they are fast
- 400 are absorbed in fuel after they become thermal
- 100 are absorbed in non-fuel materials while fast
- 200 are absorbed in non-fuel materials while thermal

Suppose further that of every 400 thermal neutrons that are absorbed in fuel, 320 actually cause fission, and that on average there are 2.5 neutrons emitted from these thermally-induced fissions. Likewise, suppose that of every 200 fast neutrons absorbed in fuel, 60 actually cause fission, and that on average there are 3.0 fast neutrons emitted from these fast fissions.

Question: What are the six factors in the six-factor formula, and what is the multiplication factor? You should use the empty neutron life history figure below before looking at the solutions.

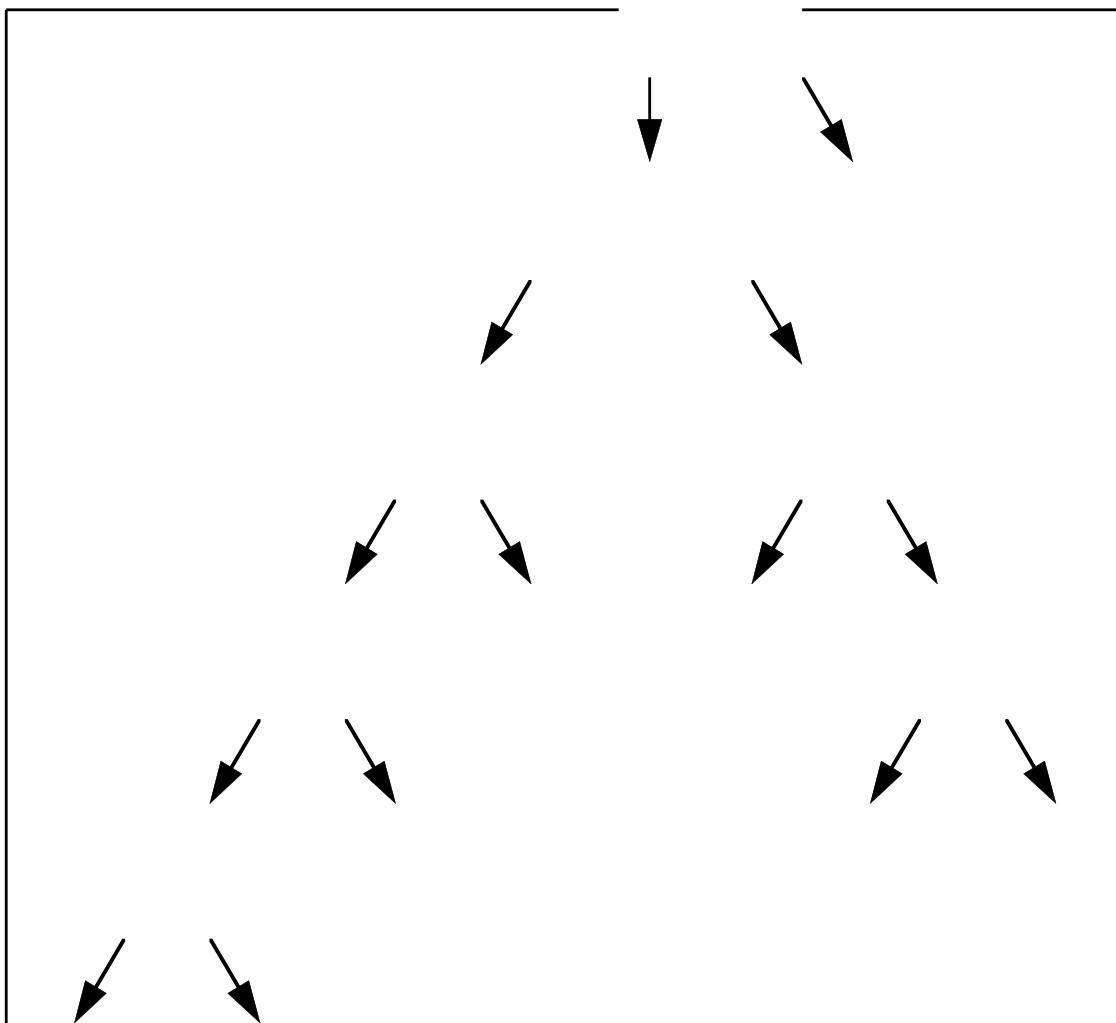


Figure 5.2: Possible histories of neutrons in a reactor.

Solution: First, forget all the formula stuff and use common sense to get the multiplication factor. Then you can check later to make sure your factors give the right answer. The common-sense approach to k goes like this: How many fission neutrons will result from 1000 initial fission neutrons? Well, we know that there are 320 thermal fissions, with 2.5 neutrons emitted from each, which gives us $(2.5)(320) = 800$ new neutrons. Also, there are 60 fast fissions, producing $(3)(60) = 180$ more new neutrons. So the multiplication factor is $(800+180)/1000 = 0.98$.

We now calculate the six factors:

$$\begin{aligned} P_{\text{FNL}} &= (1000-80)/1000 = 920/1000 \\ P_{\text{TNL}} &= (620-20)/620 = 600/620 \\ p &= (920-200-100)/920 = 620/920 \\ u_T &= 400/(400+200) = 400/600 \\ \eta_T &= 2.5 \times 320/400 = 800/400 \end{aligned}$$

That's five of them. For the fast-fission factor, ϵ , we first obtain some intermediate results:

$$\begin{aligned} 1-p &= (920-620)/920 = 300/920 \\ u_F &= 200/(200+100) = 200/300 \\ \eta_F &= 3.0 \times 60/200 = 180/200 \end{aligned}$$

Now we can get the fast-fission factor:

$$\begin{aligned} \epsilon &= \frac{P_{\text{FNL}} p P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T + P_{\text{FNL}} (1-p) u_F P_{\text{FAF}} \nu_F}{P_{\text{FNL}} p P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T} \\ &= \frac{800/1000 + 180/1000}{800/1000} = 1 + \frac{180}{800} \\ &= \frac{980}{800} \end{aligned}$$

It is easy to see that if we multiply the six factors, we obtain 980/1000, as we should. Note that because the probabilities are **conditional**, the numerator of one factor typically cancels the denominator of another!

Tricky example

Suppose the reactor in the previous example were made infinite, so that $P_{\text{FNL}} = P_{\text{TNL}} = 1$. What is the new k , which is k_∞ ?

Solution: To get the answer we might be tempted to blindly apply the four-factor formula using the factors calculated above. This would produce:

$$k_\infty = \epsilon \eta_T u_T p \xrightarrow{\text{tempting but wrong}} \frac{980}{800} \frac{800}{400} \frac{400}{600} \frac{620}{920} = \frac{980}{920} \frac{620}{600} = 1.1007 \quad (5.13)$$

The problem is that changing the leakage changes (reduces, in fact) the value of the fast-fission factor, ϵ , so we cannot use the value we had in the finite reactor. Recall that

$$\begin{aligned}\epsilon &= \frac{P_{\text{FNL}} p P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T + P_{\text{FNL}} (1-p) u_F P_{\text{FAF}} \nu_F}{P_{\text{FNL}} p P_{\text{TNL}} u_T P_{\text{TAF}} \nu_T} \\ &\xrightarrow{\infty \text{ reactor}} \frac{p u_T P_{\text{TAF}} \nu_T + (1-p) u_F P_{\text{FAF}} \nu_F}{p u_T P_{\text{TAF}} \nu_T} = 1 + \frac{(1-p) u_F \eta_F}{p u_T \eta_T} \\ &= 1 + \frac{(300/920)(200/300)(180/200)}{(620/920)(400/600)(800/400)} = 1 + \frac{180}{800} \frac{600}{620} = 1 + \frac{180}{800} \frac{30}{31}\end{aligned}$$

Using this value of the fast-fission factor we find the correct k -infinity:

$$k_\infty = \epsilon \eta_T u_T p = 1.0942 \quad (5.14)$$

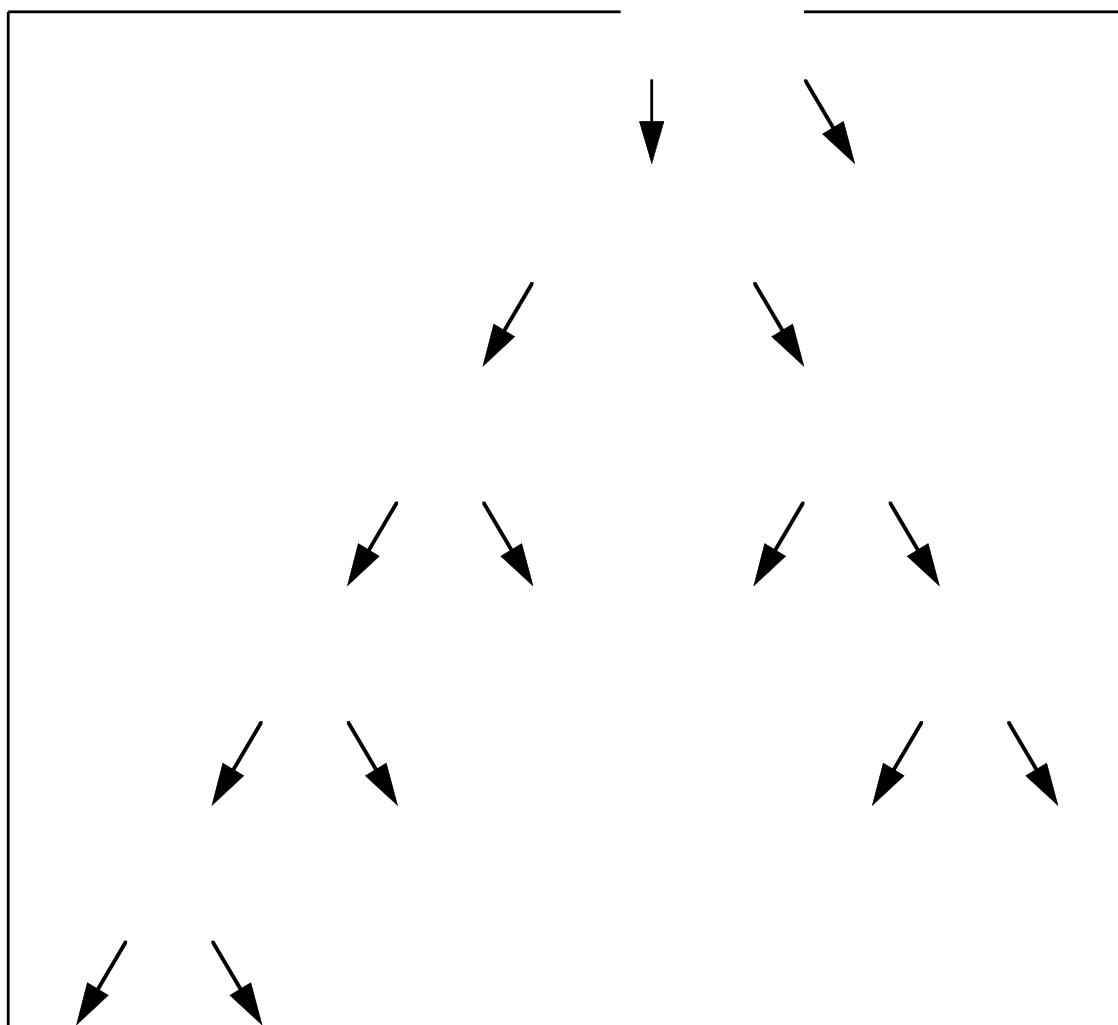


Figure 5.3: Possible histories of neutrons in a reactor.

Part III

Conservation of neutrons: transport and diffusion equations

Chapter 6

Flux, Current, Etc.

6.1 Introduction

A reactor's behavior depends on the production rates and loss rates of neutrons in the reactor. We also know that some production and loss is due to neutron-nucleus reactions, and that some is due to neutron leakage.

Reaction rates usually can be calculated from the neutron scalar flux. Net leakage rates (net flow rates across surfaces) can be calculated from the neutron net current density (often called "current" for short). One-way leakage rates can be calculated from neutron partial current densities. All of these can be calculated from the angular flux. In this chapter we study the scalar flux, angular flux, net current density, and partial current densities, and we explore how to use them to calculate quantities of interest.

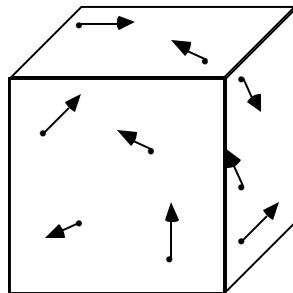
6.2 Scalar flux

6.2.1 One-speed (mono-energetic) case

Consider some volume, V , in which the density of neutrons is n neutrons/cm³, with all neutrons moving at the same speed v but in many different directions (see Fig. 6.1).

What is the rate at which neutrons make tracks in this volume? To say it differently, what is the total distance traveled by neutrons in this volume per unit time? In time Δt , each neutron travels:

$$\text{distance} = v \times \Delta t \quad \text{units: } [\text{cm}/\text{s}] \times [\text{s}] = [\text{cm}] \quad (6.1)$$

Figure 6.1: One-speed neutrons in a volume V .

and there are $n \times V$ neutrons in the volume. Thus,

path-length in V in Δt is $n \times V \times v \times \Delta t$.

$$[\text{n/cm}^3] \times [\text{cm}^3] \times [\text{cm/s}] \times [\text{s}] = [\text{n-cm}] \quad (6.2)$$

If we divide by the volume (to get a density) and by the time interval (to get a rate), and include arguments to acknowledge that the neutron density may be different at different locations and times, we obtain:

$$\begin{aligned} \phi(\vec{r}, t) &= n(\vec{r}, t) \times v = \text{path-length rate density} \\ &\text{in units of: } [\text{n/cm}^3] \times [\text{cm/s}] = [\text{n}/(\text{cm}^2\text{-s})] \end{aligned} \quad (6.3)$$

This is a general result:

Scalar flux is always speed times density.

Be careful, though: “density” is context-dependent. It is always some number of things per unit “volume,” but “volume” may not be the physical x - y - z volume. There are many cases in which density is number per something else. For example, population density is inhabitants per unit **area**, car density on the highway is cars per unit **length**, etc.

Note:

Every density is a limit. Mathematically, the density of things at some point \vec{x} in some “phase space” is

$$d(\vec{x}) = \lim_{V \rightarrow \text{small}} \left[\frac{\text{expected number of things in } V \text{ around } \vec{x}}{V} \right] \quad (6.4)$$

where \vec{x} is a point in a phase space and V is a phase-space volume that contains the point \vec{x} .

We will see examples of various densities later in this chapter. Also see Appendix B for more information.

6.2.2 Energy-Dependent Scalar Flux: distribution of speeds (energies)

Neutrons do not bounce around with a single energy in nuclear reactors; they have a distribution of energies.

Question: How many neutrons in a commercial reactor have exactly 1 MeV kinetic energy?

Answer: Exactly **ZERO**.

If we asked how many neutrons have energies between 0.999 and 1.001 MeV, instead of exactly 1.0000000000000000000000000000000... MeV, the answer would be different. This chapter will give you the tools to find such an answer.

We define the **energy-dependent neutron density** $n(\vec{r}, E, t)$ such that:

$$\begin{aligned} n(\vec{r}, E, t) dx dy dz dE = & \text{ number of neutrons in volume } dx dy dz \text{ at position } \vec{r} = (x, y, z) \\ & \text{whose energies are in the interval } dE \text{ at energy } E \\ & \text{at time } t . \end{aligned} \tag{6.5}$$

What are the units of $n(\vec{r}, E, t)$? The phase space volume here is x, y, z , and E , so the units are:

energy-dependent neutron density units = neutrons / (length³ × energy).

(Note that $n \times dx \times dy \times dz \times dE$ is number of neutrons, $d^3r \equiv dx dy dz$ is spatial volume, dE is energy.) Typically, the energy is expressed in MeV (or keV or eV). We will mainly use MeV but be sure to check every time, because it does make a difference!

Likewise, we define the **energy-dependent scalar flux**:

$$\phi(\vec{r}, E, t) \equiv n(\vec{r}, E, t) \times v(E) \tag{6.6}$$

= path-length rate density

$$\text{Typical units: } \frac{\text{n}}{\text{cm}^3\text{-MeV}} \times \frac{\text{cm}}{\text{s}} = \frac{\text{n-cm}}{\text{cm}^3\text{-MeV-s}} = \frac{\text{n}}{\text{cm}^2\text{-MeV-s}}$$

Example

Suppose that in some reactor, the energy-dependent scalar flux in the moderator is given by:

$$\phi(\vec{r}, E, t) = \frac{\phi_0}{E} \text{ for } E \in (10 \text{ eV}, 10 \text{ keV})$$

Question: What are the units of ϕ and ϕ_0 ?

Answer: ϕ has units $\text{n}/(\text{cm}^2\text{-s-MeV})$, so ϕ_0 must have units $\text{n}/(\text{cm}^2\text{-s})$.

Question: What is the density of neutrons that have energies between 1 and 10 keV? That is, how many neutrons/ cm^3 have energies in this range?

Answer: Since $v(E) = \sqrt{2E/m}$, where m is neutron mass, we have [see Eq.(6.3)]:

$$n(\vec{r}, E, t) = \frac{\phi(\vec{r}, E, t)}{v(E)} = \frac{\phi_0}{E\sqrt{2E/m}}$$

Thus,

$$\begin{aligned} \text{answer} &= \int_{1 \text{ keV}}^{10 \text{ keV}} n(\vec{r}, E, t) dE = \frac{\phi_0}{\sqrt{2/m}} \int_{1 \text{ keV}}^{10 \text{ keV}} \frac{dE}{E\sqrt{E}} = \frac{\phi_0}{\sqrt{2/m}} \int_{1 \text{ keV}}^{10 \text{ keV}} E^{-3/2} dE \\ &= \frac{\phi_0}{\sqrt{2/m}} (-2) [E^{-1/2}]_{1 \text{ keV}}^{10 \text{ keV}} \\ &= 2\phi_0 \left[\frac{1}{\sqrt{2[1 \text{ keV}]/m}} - \frac{1}{\sqrt{2[10 \text{ keV}]/m}} \right] \\ &= 2\phi_0 \left[\frac{1}{v(E)} \Big|_{E=1 \text{ keV}} - \frac{1}{v(E)} \Big|_{E=10 \text{ keV}} \right] \end{aligned}$$

6.3 Angular Flux

Often we require knowledge of not only the distance traveled by neutrons per unit time per unit spatial volume per unit energy [which is $\phi(\vec{r}, E, t)$], but also of their distribution in direction. Let us define the neutron **angular density** as follows.

$$\begin{aligned} n(\vec{r}, E, \vec{\Omega}, t) d^3 r dE d\Omega &= \text{number of neutrons in volume } d^3 r \text{ at position } \vec{r} \\ &\quad \text{whose energies are in the interval } dE \text{ at energy } E \\ &\quad \text{and whose directions are in the cone } d\Omega \text{ at direction } \vec{\Omega} \\ &\quad \text{at time } t . \end{aligned} \tag{6.7}$$

Note that

$$\begin{aligned} d^3 r dE d\Omega &= dx dy dz dE d\varphi \sin \theta d\theta \\ &= dx dy dz dE d\varphi d\mu , \end{aligned}$$

where $\mu \equiv \cos \theta$. That is, this is a differential “volume” in a “space” with **six** independent axes. That is, the space of position-energy-direction is a **six-dimensional phase-space volume**. The neutron angular density is a density in this 6-D phase space.

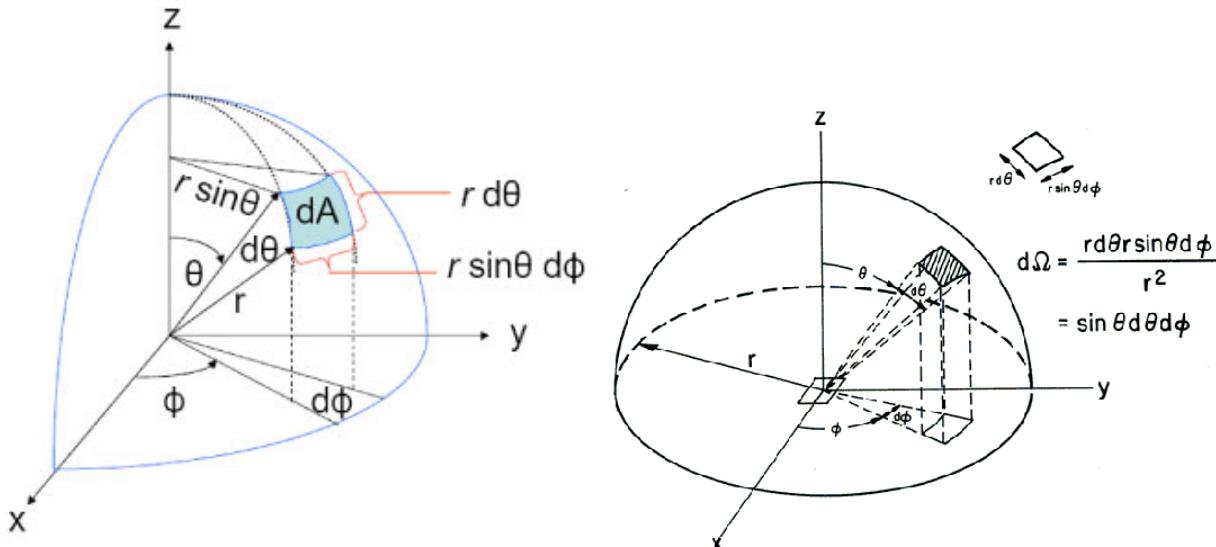


Figure 6.2: Solid angle

Note: do not confuse φ -the azimuthal angle when dealing with direction- with ϕ -the scalar flux.

Using the angles θ and φ from Fig. 6.2, we can actually define the coordinates of $\vec{\Omega}$

$$\vec{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} \quad (6.8)$$

The neutron **angular flux** is defined to be speed times angular density:

$$\psi(\vec{r}, E, \vec{\Omega}, t) \equiv n(\vec{r}, E, \vec{\Omega}, t) \times v(E)$$

= path-length rate density

$$\text{Typical units: } \frac{\text{n}}{\text{cm}^3\text{-MeV-ster}} \times \frac{\text{cm}}{\text{s}} = \frac{\text{n-cm}}{\text{cm}^3\text{-MeV-ster-s}} = \frac{\text{n}}{\text{cm}^2\text{-MeV-ster-s}}$$

Note the relations among the three neutron densities we have encountered:

$$\begin{aligned} n(\vec{r}, E, t) &= \iint_{4\pi} d\Omega n(\vec{r}, E, \vec{\Omega}, t) \\ &= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta n(\vec{r}, E, \theta, \varphi, t) \\ &= \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu n(\vec{r}, E, \mu, \varphi, t) \end{aligned} \quad (6.9)$$

$$\begin{aligned} n(\vec{r}, t) &= \int_0^\infty dE n(\vec{r}, E, t) \\ &= \int_0^\infty dE \iint_{4\pi} d\Omega n(\vec{r}, E, \vec{\Omega}, t) \end{aligned} \quad (6.10)$$

The following are direct results of the relations above:

$$\begin{aligned}\phi(\vec{r}, E, t) &= \iint_{4\pi} d\Omega \psi(\vec{r}, E, \vec{\Omega}, t) \\ &= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \psi(\vec{r}, E, \theta, \varphi, t) \\ &= \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \psi(\vec{r}, E, \mu, \varphi, t)\end{aligned}\quad (6.11)$$

$$\begin{aligned}\phi(\vec{r}, t) &= \int_0^\infty dE \phi(\vec{r}, E, t) \\ &= \int_0^\infty dE \iint_{4\pi} d\Omega \psi(\vec{r}, E, \vec{\Omega}, t)\end{aligned}\quad (6.12)$$

$\phi(\vec{r}, t)$, which is the scalar flux due to neutrons of all energies, is called the **total scalar flux**.

Note that we are using the same symbol, n , for three different functions: the angle- and energy-dependent neutron density, the energy-dependent neutron density, and the total neutron density. This is not ideal, but it is standard practice. The same holds for the symbol ϕ , which we use for two different scalar-flux functions. Be careful—you must deduce which function it is from the context. For example, if you encounter a ϕ that has both a position and energy argument, and it is inside an integral over energy, then you deduce that it is the energy-dependent scalar flux, which has units of neutron distance traveled per unit spatial volume per unit energy per unit time, and not the total scalar flux, which has no per-unit-energy units.

6.4 Reaction Rates

We saw in Chapter 3 that the general expression for the rate per unit volume at which a reaction of type “ x ” occurs at position \vec{r} at time t can be written as

$$\text{RRD}_x(\vec{r}, t) = \iiint_{\text{neutron vels.}} dv_x dv_y dv_z n(\vec{r}, \vec{v}, t) \iiint_{\text{nucl. vels.}} dV_x dV_y dV_z \mathbb{N}(\vec{r}, \vec{V}, t) |\vec{v} - \vec{V}| \sigma_{x,cold}(|\vec{v} - \vec{V}|)$$

We can rewrite this in terms of energy- and direction-dependent densities, noting that (for example) $\vec{v} = v(E_n)\vec{\Omega}_n$:

$$\begin{aligned} \text{RRD}_x(\vec{r}, t) &= \int_0^\infty dE_n \iint_{4\pi} d\Omega_n \left\{ n(\vec{r}, E_n, \vec{\Omega}_n, t) \right. \\ &\quad \times \int_0^\infty dE_{nuc} \iint_{4\pi} d\Omega_{nuc} \mathbb{N}(\vec{r}, E_{nuc}, \vec{\Omega}_{nuc}, t) \left| v(E_n) \vec{\Omega}_n - V(E_{nuc}) \vec{\Omega}_{nuc} \right| \right. \\ &\quad \times \left. \sigma_{x,cold} \left(\left| v(E_n) \vec{\Omega}_n - V(E_{nuc}) \vec{\Omega}_{nuc} \right| \right) \right\} \end{aligned}$$

As we saw in Chapter 3, we can define an averaged σ_x such that this expression becomes

$$\begin{aligned} \text{RRD}_x(\vec{r}, t) &= \int_0^\infty dE_n \iint_{4\pi} d\Omega_n \left\{ n(\vec{r}, E_n, \vec{\Omega}_n, t) N(\vec{r}, t) v(E_n) \sigma_x(v(E_n), \vec{\Omega}_n) \right\} \\ &= \int_0^\infty dE_n \iint_{4\pi} d\Omega_n \left[n(\vec{r}, E_n, \vec{\Omega}_n, t) v(E_n) \right] \left[N(\vec{r}, t) \sigma_x(v(E_n), \vec{\Omega}_n) \right] \\ &= \int_0^\infty dE_n \iint_{4\pi} d\Omega_n \psi(\vec{r}, E_n, \vec{\Omega}_n, t) \Sigma_x(\vec{r}, v(E_n), \vec{\Omega}_n, t) \end{aligned} \quad (6.13)$$

We also saw in Chapter 3 that we can almost always take the nuclei to be vibrating **isotropically**, in which case the averaged cross section no longer depends on the neutron direction, but only on the neutron speed, or equivalently energy:

$$\begin{aligned} \text{RRD}_x(\vec{r}, t) &\xrightarrow{\text{isotropic nuclei}} \int_0^\infty dE_n \underbrace{\iint_{4\pi} d\Omega_n \psi(\vec{r}, E_n, \vec{\Omega}_n, t)}_{\phi(\vec{r}, E_n, t)} \Sigma_x(\vec{r}, E_n, t) \\ &= \int_0^\infty dE \phi(\vec{r}, E, t) \Sigma_x(\vec{r}, E, t) \end{aligned} \quad (6.14)$$

The averaged σ_x , which is contained within Σ_x , depends on the distribution of nucleus speeds, which we can often characterize by the material temperature. We often omit temperature from the argument list, but the dependence is implied by the form of the equation. That is, we know that if we write Eq. (6.14) for RRD, then the cross section must have been averaged over nucleus velocities, and those velocities depend on temperature.

We can also define an **energy-dependent reaction-rate density** for reactions of type “ x ”:

$$\text{RRD}_x(\vec{r}, E, t) = \Sigma_x(\vec{r}, E, t) \times \phi(\vec{r}, E, t). \quad (6.15)$$

Units:

$$\left[\frac{\text{reactions}}{\text{cm}^3\text{-MeV-s}} \right] = \left[\frac{\text{reactions}}{\text{cm of n travel}} \right] \times \left[\frac{\text{cm of n travel}}{\text{cm}^3\text{-MeV-s}} \right]$$

Important!

1. $\phi(\vec{r}, E, t)$ is a density in space and in energy. It is neutron path length rate per cm^3 per MeV.
2. It does not make sense to ask what the path-length rate is at a point in phase space, such as \vec{r} and/or E . It does make sense to ask what the path-length rate **density** is at a point.
3. $\Sigma(\vec{r}, E, t)$ is **not a density**
in space or in energy. It does make sense to ask what Σ is at \vec{r} and at E .
4. $\Sigma_s(\vec{r}, E_i \rightarrow E_f, t)$ is not a density in space or in incident energy, E_i .
It is a density in outgoing energy, E_f .

How many neutrons have outgoing energy of exactly 100 keV after a scattering event?
Exactly zero.

6.5 Examples

I could have put this entire section in yellow because they are only examples.

Suppose you have been analyzing a reactor and you have computed the energy-dependent scalar flux, $\phi(\vec{r}, E)$, everywhere in the reactor and for all neutron energies. You also know all the macroscopic cross sections everywhere in the reactor.

Question #1: How many neutrons are in the reactor?

Answer: First, the number of neutrons is the volume integral of the neutron density:

$$n_{tot} \text{ [neutrons]} = \iiint_{\text{volume}} d^3r \ n(\vec{r}) \quad \text{Note: } n(\vec{r}) \text{ units are [n/cm}^3\text{]}$$

Second, the “regular” neutron density is the energy integral of the energy-dependent neutron density:

$$n(\vec{r}) \text{ [n/cm}^3\text{]} = \int_0^\infty dE \ n(\vec{r}, E) \quad \text{Note: } n(\vec{r}, E) \text{ is in [n/(cm}^3\text{-MeV)]}$$

Third, the energy-dependent neutron density is the energy-dependent scalar flux divided by speed:

$$n(\vec{r}, E) \text{ [n/(cm}^3\text{-MeV)]} = \frac{\phi(\vec{r}, E)}{v(E)} = \frac{\phi(\vec{r}, E)}{\sqrt{2E/m}}$$

Note: It is usually OK to use the classical expression to relate kinetic energy and speed for neutrons in reactors. Recall that very few neutrons have energies as high as 10 MeV—most have energies that are much lower. For a 10-MeV neutron, $v/c \approx 1/7$ and the error in the classical expression for kinetic energy is around 1.5%. The error for lower-energy neutrons is smaller.

Putting everything together, we obtain an expression for the total neutron population in the reactor, in terms of the energy-dependent scalar flux function:

$$n_{tot} \text{ [neutrons]} = \iiint_{\text{volume}} d^3r \int_0^\infty dE \frac{\phi(\vec{r}, E)}{\sqrt{2E/m}} .$$

Question #2: Assuming that each fission produces 200 MeV of energy on average and 190 MeV of this gets deposited in the reactor (with the other 10 MeV carried away by leaking neutrinos, gammas, and neutrons), what is the reactor power?

Answer: Take things a step at a time:

$$\text{power} = \frac{\text{energy}}{\text{s}} = \frac{\text{energy deposited}}{\text{fission}} \times \text{fission rate} ,$$

$$\frac{\text{energy deposited}}{\text{fission}} = 190 \text{ MeV/fission} ,$$

$$\text{fission rate} = \iiint_{\text{volume}} d^3r \text{ (fission rate density)}$$

$$\text{fission rate density} = \int_0^\infty dE \Sigma_f(\vec{r}, E) \phi(\vec{r}, E)$$

Combine everything:

$$\text{power [MeV/s]} = \frac{190 \text{ MeV}}{\text{fission}} \times \iiint_{\text{volume}} d^3r \int_0^\infty dE \Sigma_f(\vec{r}, E) \phi(\vec{r}, E)$$

$$\frac{\text{MeV}}{\text{s}} = \frac{\text{MeV}}{\text{fission}} \times \text{cm}^3 \times \text{MeV} \times \frac{\text{fissions}}{\text{n-cm}} \times \frac{\text{n-cm}}{\text{cm}^3 \cdot \text{s} \cdot \text{MeV}}$$

Question #3: At what rate are neutrons produced by fission in the reactor?

Answer: First, a rate is the volume integral of a rate density (rate per unit volume):

$$\text{rate } \left[\frac{\text{n}}{\text{s}} \right] = \iiint_{\text{volume}} d^3r \text{ (production rate density)}$$

Second, a rate density involving neutrons is the integral over energy of an energy-dependent rate density:

$$\text{production rate density } \left[\frac{\text{n}}{\text{cm}^3\text{-s}} \right] = \int_0^\infty dE \nu(E) \Sigma_f(\vec{r}, E) \phi(\vec{r}, E)$$

$$\frac{\text{n}}{\text{cm}^3\text{-s}} = \text{MeV} \times \frac{\text{n}}{\text{fission}} \times \frac{\text{fissions}}{\text{n-cm}} \times \frac{\text{n-cm}}{\text{cm}^3\text{-s-MeV}}$$

Combine everything:

$$\text{rate [n/s]} = \iiint_{\text{volume}} d^3r \int_0^\infty dE \nu(E) \Sigma_f(\vec{r}, E) \phi(\vec{r}, E)$$

Question #4: What is the rate at which neutrons scatter from energies above 1 MeV to energies below 0.5 MeV in the reactor?

Answer: First, a rate is the volume integral of a rate density (rate per unit volume):

$$\text{rate [scatters/s]} = \iiint_{\text{volume}} d^3r \text{ (downscattering rate density)}$$

$$\frac{\text{scatters}}{\text{s}} = \text{cm}^3 \times \frac{\text{scatters}}{\text{cm}^3\text{-s}}$$

Second, the downscattering rate density that was requested requires integration over initial neutron energies **and** integration over final neutron energies. The quantity that is integrated is the scalar flux times a differential scattering cross section:

$$\text{downscattering rate density} = \int_{E_i=1 \text{ MeV}}^\infty dE_i \int_{E_f=0}^{0.5 \text{ MeV}} dE_f \Sigma_s(\vec{r}, E_i \rightarrow E_f) \phi(\vec{r}, E_i)$$

$$\frac{\text{scatters}}{\text{cm}^3\text{-s}} = \text{MeV} \times \text{MeV} \times \frac{\text{scatters}}{\text{n-cm-MeV}} \times \frac{\text{n-cm}}{\text{cm}^3\text{-s-MeV}}$$

Combine everything:

$$\text{rate [n/s]} = \iiint_{\text{volume}} d^3r \int_{E_i=1 \text{ MeV}}^\infty dE_i \phi(\vec{r}, E_i) \int_{E_f=0}^{0.5 \text{ MeV}} dE_f \Sigma_s(\vec{r}, E_i \rightarrow E_f)$$

$$= \iiint_{\text{volume}} d^3r \int_{E_i=1 \text{ MeV}}^{E_i=\infty} dE_i \phi(\vec{r}, E_i) \Sigma_s(\vec{r}, E_i) \int_{E_f=0}^{E_f=0.5 \text{ MeV}} dE_f P(\vec{r}, E_i \rightarrow E_f)$$

6.6 Neutron Net Current Density and Leakage Rates

6.6.1 One-speed Net Current Density

In this section we will find that the net leakage of neutrons across a surface can be written in terms of a vector quantity called

neutron net current density.

We will find that the net leakage rate, across a flat surface whose unit normal is \vec{e}_n , is just

$$\vec{e}_n \cdot \vec{J} A \quad (6.16)$$

where A is the area and \vec{J} is the neutron net current density [n/cm²-s]. Sometimes we will simply call this the “neutron current density” or the “net current density” or the “net current” or even just the “current.” All of these terms refer to the same thing, which most precisely is called the “neutron net current density.”

We begin with a beam of mono-energetic, mono-directional neutrons crossing a surface that is perpendicular to the x axis (see Fig. 6.3). Here $\vec{\Omega}$ is defined to be a unit vector in the direction of the beam. The number of neutrons crossing the surface in time Δt is:

$$\begin{aligned} \text{number crossing area } A \text{ in time } \Delta t &= n \times v \times \cos \alpha_x \times A \times \Delta t, \\ \text{neutrons} &= \frac{n}{cm^3} \times \frac{cm}{s} \times - \times cm^2 \times s \end{aligned} \quad (6.17)$$

where α_x is the angle between $\vec{\Omega}$ and \vec{e}_x .

Now, because

$$\cos \alpha_x = \vec{\Omega} \cdot \vec{e}_x \quad (6.18)$$

and because

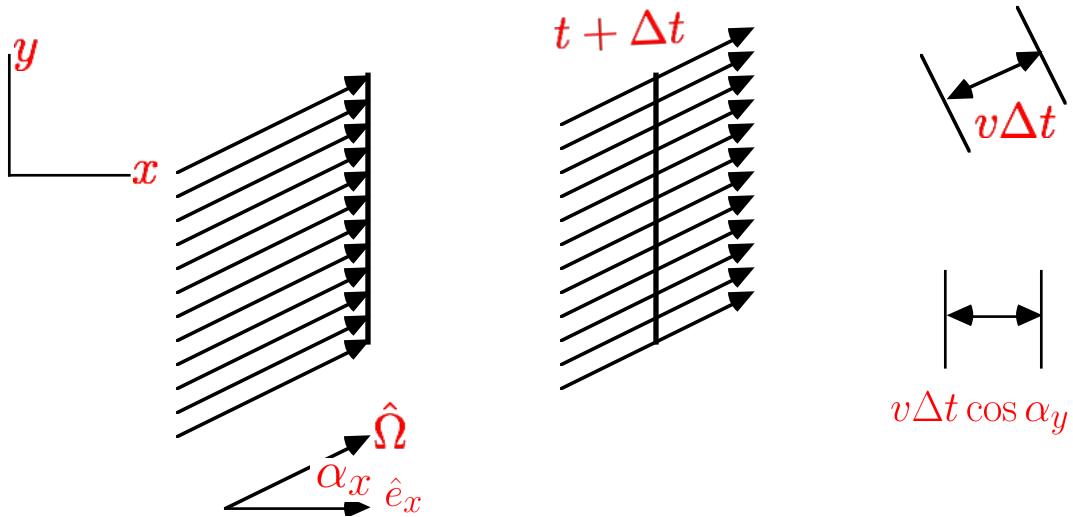
$$n \times v = I = \text{beam intensity} \quad (6.19)$$

we have

$$\text{number crossing area } A \text{ in time } \Delta t = I \vec{\Omega} \cdot \vec{e}_x A \Delta t \quad (6.20)$$

Thus, the crossing rate per unit area is:

$$\text{crossing rate per unit area} = I \vec{\Omega} \cdot \vec{e}_x \quad (6.21)$$

Figure 6.3: Beam crossing surface perpendicular to x axis.

Question: What if $\alpha_x > 90^\circ$?

Answer: Then $\cos \alpha_x < 0$ and the crossing rate is negative.

This means neutrons flow from the $+\vec{e}_x$ side of the surface to the $-\vec{e}_x$ side.

Now consider the same beam again, but with a different surface (see Fig. 6.4). The only difference between this case and the previous case is the orientation of the surface. In particular, the unit vector normal to the surface is now \vec{e}_y instead of \vec{e}_x ; thus,

$$\cos \alpha_y = \vec{\Omega} \cdot \vec{e}_y \quad (6.22)$$

Everything else is the same, so

$$\text{crossing rate per unit area} = I \vec{\Omega} \cdot \vec{e}_y \quad (6.23)$$

In fact, given a surface with any orientation (see Fig. 6.5), we have

$$\text{crossing rate per unit area} = I \vec{\Omega} \cdot \vec{e}_n \quad (6.24)$$

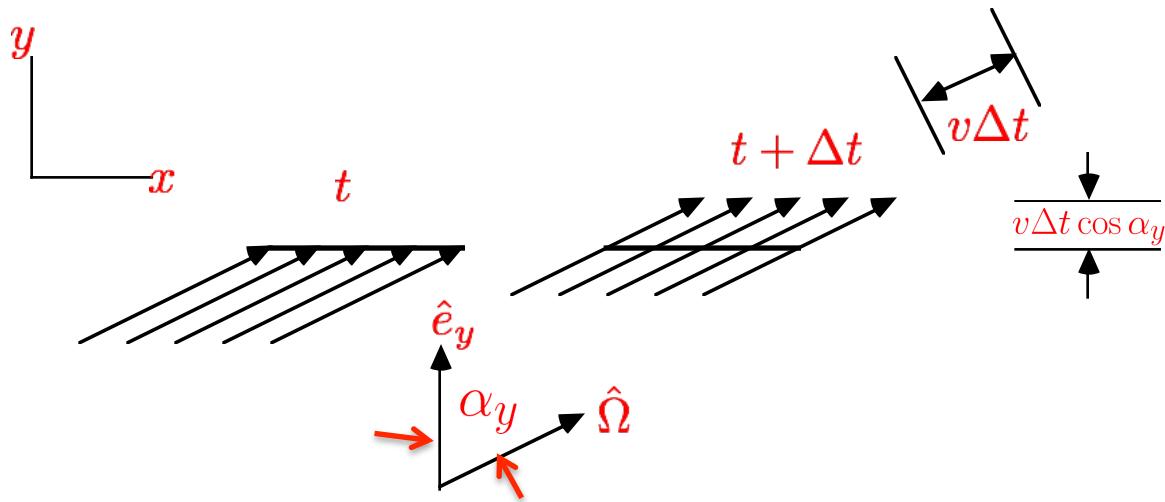
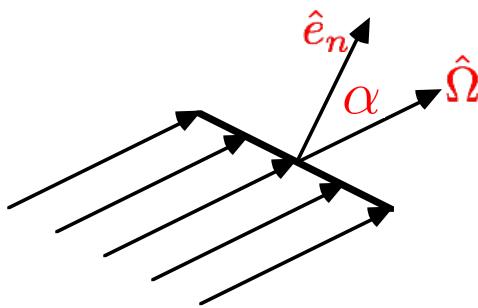
Figure 6.4: Neutron beam crossing surface perpendicular to \vec{e}_y .

Figure 6.5: Neutron beam crossing surface with arbitrary orientation.

Notice that no matter what the orientation of the surface was, we needed the vector quantity

$$I \vec{\Omega} \quad (6.25)$$

in order to compute the crossing rate. In our simple beam example, this product of the beam intensity (a scalar) times the direction vector (a vector) is the net current density (a vector):

$$\vec{J} = \text{net current density} = I \vec{\Omega} \quad [\text{only for a beam !!!!}] \quad (6.26)$$

This formula is valid **only for mono-directional beams!**

Notes:

1. The net current density has the **same units as the scalar flux**.

2. It does **not** have the same physical meaning!!
3. This may seem confusing at first. I'm sorry. That's just the way it is.
4. You can use scalar flux to calculate reaction rates. You can use net current density to calculate net surface-crossing rates.

6.6.2 Energy-dependent Neutron Net Current Density

We know that neutrons don't travel about mono-energetically—neutrons in reactors are distributed in energy. As we did with the scalar flux, we define an energy-dependent net current density:

$$\vec{J}(\vec{r}, E, t) dE = \text{net current density of neutrons at position } \vec{r} \\
\text{whose energies are in the interval } dE \text{ at energy } E \\
\text{at time } t \quad (6.27)$$

We see that the product $\vec{J}(\vec{r}, E) dE$ has units of net current density, which are:

$$\text{n}/(\text{cm}^2\text{-s}).$$

Thus, the energy-dependent net current density, $\vec{J}(\vec{r}, E)$, must have units of

$$\text{n}/(\text{cm}^2\text{-s-MeV}).$$

Similar to our definition of total scalar flux, we define the **total net current density** as:

$$\begin{aligned} \vec{J}(\vec{r}, t) &= \text{total net current density} \\ &= \int_0^\infty dE \vec{J}(\vec{r}, E, t) \end{aligned} \quad (6.28)$$

6.6.3 Relation to Beam Intensities and Angular Flux

Consider some volume in which several different mono-directional beams are crossing. In region where they cross, the neutron net current density is the vector sum of the net current density from each beam:

$$\vec{J}(\vec{r}, E, t) = \sum_{\ell=1}^{\# \text{ of beams}} I_\ell(\vec{r}, E, t) \vec{\Omega}_\ell \quad [\text{only for beams !!!!}] \quad (6.29)$$

Similarly, in a region where multiple beams are crossing, the neutron scalar flux is the scalar sum of the beam intensities:

$$\phi(\vec{r}, E, t) = \sum_{\ell=1}^{\text{\# of beams}} I_\ell(\vec{r}, E, t) \quad [\text{only for beams !!!!}] \quad (6.30)$$

In the completely general case, when neutrons are not moving around in beams, the expression for \vec{J} is an integral over all directions:

$$\begin{aligned} \vec{J}(\vec{r}, E, t) &= \iint_{4\pi} d\Omega \vec{\Omega} \psi(\vec{r}, E, \vec{\Omega}, t) \\ &= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \vec{\Omega} \psi(\vec{r}, E, \theta, \varphi, t) \\ &= \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \vec{\Omega} \psi(\vec{r}, E, \mu, \varphi, t) \end{aligned} \quad (6.31)$$

$$\begin{aligned} \vec{J}(\vec{r}, t) &= \int_0^\infty dE \vec{J}(\vec{r}, E, t) \\ &= \int_0^\infty dE \iint_{4\pi} d\Omega \vec{\Omega} \psi(\vec{r}, E, \vec{\Omega}, t) \end{aligned} \quad (6.32)$$

$\vec{J}(\vec{r}, t)$, which is the net current density due to neutrons of all energies, is called the **total net current density**.

More explicitly for each component:

$$J_x(\vec{r}, E, t) = \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \underbrace{(1 - \mu^2)^{1/2} \cos \varphi}_{\Omega_x} \psi(\vec{r}, E, \mu, \varphi, t) \quad (6.33)$$

$$J_y(\vec{r}, E, t) = \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \underbrace{(1 - \mu^2)^{1/2} \sin \varphi}_{\Omega_y} \psi(\vec{r}, E, \mu, \varphi, t) \quad (6.34)$$

$$J_z(\vec{r}, E, t) = \int_0^{2\pi} d\varphi \int_{-1}^1 d\mu \underbrace{\mu}_{\Omega_z} \psi(\vec{r}, E, \mu, \varphi, t) \quad (6.35)$$

Note that we are using the same symbol, \vec{J} , for two different functions: the energy-dependent net current density and the total net current density. This is not ideal, but it is standard practice.

Example of cancellation effect

In the many-beam case, the net current density from some beams may cancel the net current density from others. For example, given two beams of the same intensity but opposite directions, we would have:

$$I_1 = I_2 \quad \text{and} \quad \vec{\Omega}_2 = -\vec{\Omega}_1 , \quad (6.36)$$

which means

$$\begin{aligned} \vec{J} &= I_1 \vec{\Omega}_1 + I_2 \vec{\Omega}_2 = I_1 \times (\vec{\Omega}_1 + \vec{\Omega}_2) \\ &= I_1 \times (\vec{\Omega}_1 - \vec{\Omega}_1) = \vec{0} \quad [\text{opposing beams of same } I] \end{aligned} \quad (6.37)$$

Thus, \vec{J} is related to the **net** rate at which neutrons cross surfaces, which is why we call it the

net current density

Multi-beam example

Consider some volume in which several different mono-directional beams are crossing (see Fig. 6.6). In this case, the neutron current density is the **vector sum** of currents from each beam:

$$\vec{J}(x, y, z) = I_1(y) \vec{\Omega}_1 + I_2(y) \vec{\Omega}_2 + I_3(z) \vec{\Omega}_3 \quad [\text{only for 3 beams !!!!}] \quad (6.38)$$

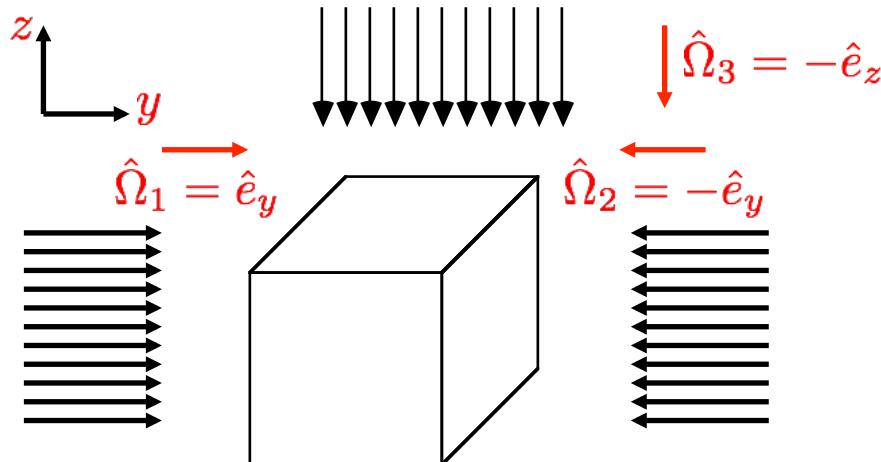


Figure 6.6: Three-beam example. Label the three intensities and three direction vectors

Here we have noted that beams 1 and 2 move only along the y axis and thus their attenuations depend only on y within the block—because their attenuation distances depend only on y —while beam 3 moves in the $-z$ direction and thus its intensity depends only on z .

6.6.4 “Leakage” Rates

We need often need to use the net current density to calculate

net leakage rates

across surfaces. Here's how it goes. Consider a point \vec{r}_s on some surface, with outward unit normal $\hat{e}_n(\vec{r}_s)$ (see Fig. 6.7). Then, given what we learned previously for net leakage rates across surfaces,

the net leakage rate density

$$\text{outward through surface at point } \vec{r}_s = \hat{e}_n(\vec{r}_s) \cdot \vec{J}(\vec{r}_s) \quad \left[\frac{\text{n}}{\text{cm}^2 \cdot \text{s}} \right] \quad (6.39)$$

Note that this is an

areal rate density (rate per square centimeter)

To get the net leakage rate through some surface, we *integrate* the net leakage rate density over the area. If we denote an infinitesimal surface area as $d^2 r$, then

net leakage rate

$$\text{outward through surface } S = \iint_S d^2 r \hat{e}_n(\vec{r}_s) \cdot \vec{J}(\vec{r}_s) \quad (6.40)$$

It really is that simple!

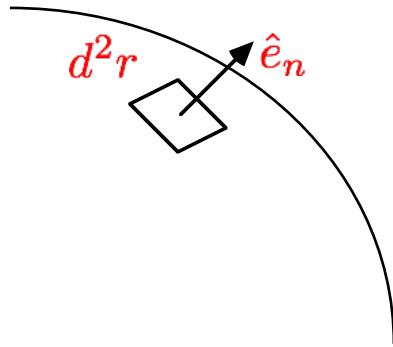


Figure 6.7: A point on some surface, with outward unit normal e_n .

6.6.5 Examples

I could have put this entire sub-section in yellow because they are only examples.

You have been analyzing a parallelepiped reactor, whose spatial domain is given by:

$$\begin{aligned} -\frac{a}{2} &\leq x \leq \frac{a}{2} \\ -\frac{b}{2} &\leq y \leq \frac{b}{2} \\ -\frac{c}{2} &\leq z \leq \frac{c}{2} \end{aligned}$$

You have somehow figured out that the total net current density (i.e., energy-dependent net current density integrated over all energies) is:

$$\vec{J}(\vec{r}) = \vec{J}(x, y, z) = J_0 \left[\vec{e}_x \sin \frac{\pi x}{a} + \vec{e}_y \sin \frac{\pi y}{b} + \vec{e}_z \sin \frac{\pi z}{c} \right], \quad (6.41)$$

units of \vec{J} : $\left[\frac{n}{cm^2 \cdot s} \right]$

Note: This means that the three components of the net current density are:

$$\begin{aligned} \vec{J}(\vec{r}) \cdot \vec{e}_x &= J_x(x, y, z) = J_x(x) = J_0 \sin \frac{\pi x}{a}, \\ \vec{J}(\vec{r}) \cdot \vec{e}_y &= J_y(x, y, z) = J_y(y) = J_0 \sin \frac{\pi y}{b}, \\ \vec{J}(\vec{r}) \cdot \vec{e}_z &= J_z(x, y, z) = J_z(z) = J_0 \sin \frac{\pi z}{c}. \end{aligned}$$

Question 1: What are the units of the constant J_0 ?

Answer: J_0 must have units of $n/(cm^2 \cdot s)$, because the sin functions are dimensionless.

Question 2: Which of the following is a vector and which is a scalar?

$$\vec{J}(\vec{r}), J_x(\vec{r}), J_y(\vec{r}), J_z(\vec{r}), J_0$$

Answer: vector, scalar, scalar, scalar, scalar

Question 3: At what net rate do neutrons leak outward through the top ($z = +c/2$) surface of the reactor?

Answer:

$$\begin{aligned}
 & \text{net leakage rate} \\
 \text{outward through top surface} &= \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \vec{e}_z \cdot \vec{J}\left(x, y, \frac{c}{2}\right) \\
 &= J_0 \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \sin \frac{\pi}{2} = J_0[a][b][1] \\
 &= J_0 a b \quad \text{units: } \left[\frac{n}{s}\right]
 \end{aligned}$$

Question 4: What is the total net rate at which neutrons leak out of the reactor?

Answer:

We will compute leakage out of the bottom ($z = -c/2$) surface, then write the others by inspection.

$$\begin{aligned}
 & \text{net leakage rate outward} \\
 \text{through bottom surface} &= \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy (-\vec{e}_z) \cdot \vec{J}\left(x, y, -\frac{c}{2}\right) \\
 &= J_0 \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy (-1) \sin \frac{-\pi}{2} = J_0[a][b][1] \\
 &= J_0 a b \quad \text{units: } \left[\frac{n}{s}\right]
 \end{aligned}$$

The other surfaces are similar; the final answer is

$$\begin{aligned}
 \text{net leakage rate out of reactor} &= [\text{top} + \text{bottom} + \text{back} + \text{front} + \text{right} + \text{left}] \text{ leakage rates} \\
 &= \dots = 2J_0(ab + bc + ac)
 \end{aligned}$$

Question 5: At what **net** rate do neutrons leak from back to front across the surface at $x=-a/4$?

Answer:

Since we said back to front, the unit normal we need is $+\vec{e}_x$. Thus,

$$\begin{aligned}
 & \text{net leakage rate} \\
 & \text{from back to front across surface} \\
 & \text{at } x = -a/4 \text{ is:} \quad \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz \vec{e}_x \cdot \vec{J}\left(-\frac{a}{4}, y, z\right) \\
 & = J_0 \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz \sin \frac{-\pi}{4} = J_0[b][c][-1/\sqrt{2}] \\
 & = -\frac{J_0 b c}{\sqrt{2}} \quad \text{units: } \left[\frac{\text{n}}{\text{s}}\right]
 \end{aligned}$$

Note that this leakage rate

is negative!

This is okay. It just means that more neutrons are flowing toward the back than toward the front across this particular surface.

If the question had asked for the net rate of leakage from front to back across the surface $x=-a/4$, we would have gotten

$$\text{rate} = +\frac{J_0 b c}{\sqrt{2}} .$$

because we would have used the unit normal $-\vec{e}_x$ instead of $+\vec{e}_x$. Make sure you understand this!

Another Example

You have been analyzing a parallelepiped reactor, whose spatial domain is given by:

$$\begin{aligned}
 -\frac{a}{2} & \leq x \leq \frac{a}{2} \\
 -\frac{b}{2} & \leq y \leq \frac{b}{2} \\
 -\frac{c}{2} & \leq z \leq \frac{c}{2}
 \end{aligned}$$

and you have found that the energy-dependent net current density is:

$$\vec{J}(\vec{r}, E) = \vec{J}(x, y, z, E) = J_0 \frac{E}{E_0^2} \exp\left(-\frac{E}{E_0}\right) \left[\vec{e}_x \sin \frac{\pi x}{a} + \vec{e}_y \sin \frac{\pi y}{b} + \vec{e}_z \sin \frac{\pi z}{c} \right],$$

with units of $\left[\frac{n}{cm^2 \cdot s \cdot MeV}\right]$

Question 1: What is the net neutron loss rate due to leakage from the reactor?

Answer: First let's compute net loss rate out of the back face ($x=-a/2$) of the reactor:

$$\begin{aligned} \text{net leakage back} &= \int_{\text{back surf}} d^2 r \int_0^\infty dE \vec{e}_n(\vec{r}) \cdot \vec{J}(\vec{r}, E) \\ &= \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz \int_0^\infty dE (-\vec{e}_x) \cdot \vec{J}\left(\frac{-a}{2}, y, z, E\right) \\ &= J_0 \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz (-1) \sin \frac{-\pi}{2} \int_0^\infty dE \frac{E}{E_0^2} \exp\left(-\frac{E}{E_0}\right) \\ &= \frac{J_0}{E_0^2} [b][c](-1)[-1] \int_0^\infty dE E \exp\left(-\frac{E}{E_0}\right) \\ &= \frac{J_0}{E_0^2} b c \left[-(E_0 E + E_0^2) e^{-E/E_0}\right]_{E=0}^{E=\infty} \\ &= J_0 b c \\ \text{units: } &\left[\frac{n}{s}\right] \end{aligned}$$

Similarly,

$$\begin{aligned} \text{net leakage front} &= \int_{\text{front surf}} d^2 r \int_0^\infty dE \vec{e}_n(\vec{r}_s) \cdot \vec{J}(\vec{r}_s, E) \\ &= \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz (+\vec{e}_x) \cdot \vec{J}\left(\frac{+a}{2}, y, z, E\right) \\ &= J_0 \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz (+1) \sin \frac{+\pi}{2} \int_0^\infty dE \frac{E}{E_0^2} \exp\left(-\frac{E}{E_0}\right) \\ &= \dots = J_0 b c , \quad \text{with units} = \left[\frac{n}{s}\right] \end{aligned}$$

The other faces are handled the same way. The final answer is

$$\begin{aligned}\text{net leakage rate out of reactor} &= [\text{top} + \text{bottom} + \text{back} + \text{front} + \text{right} + \text{left}] \text{ leakage rates} \\ &= 2J_0(ab + bc + ac)\end{aligned}$$

Question: What are the units of J_0 ? How about E_0 ?

Answers:

1. We know from our answer above that (area)(J_0) must be neutrons/s; thus,
 J_0 units must be $\text{n}/(\text{cm}^2\text{-s})$
2. We know that exponents are dimensionless; thus
 E_0 must have the same units as E , i.e., energy units

6.7 Summary

Our basic goal in this course is to gain an understanding of how reactors behave and why they behave that way. This requires, at the very least, an understanding of neutron production and loss rates in reactors.

In this chapter we discussed neutron-nucleus reaction rates (RRs). We found the following.

1.

$$\text{reaction rate [units = n/s]} = \iiint_{\text{volume}} d^3r \text{ (reaction rate density at } \vec{r})$$

2.

$$\begin{aligned} \text{RR density at } \vec{r} & \quad \left[\text{units} = \frac{\text{reactions}}{\text{cm}^3 \cdot \text{s}} \right] \\ &= \int_0^\infty dE \text{ (energy-dependent RR density at } \vec{r} \text{ and } E) \end{aligned}$$

3.

$$\text{energy-dependent RR density at } \vec{r}, E \quad \left[\text{units} = \frac{\text{reactions}}{\text{cm}^3 \cdot \text{s} \cdot \text{MeV}} \right] = \Sigma_x(\vec{r}, E) \phi(\vec{r}, E)$$

Here

$\phi(\vec{r}, E)$ = energy-dependent scalar flux $[\text{n}/(\text{cm}^2 \cdot \text{s} \cdot \text{MeV})] = n(\vec{r}, E) \times v(E)$,

$n(\vec{r}, E)$ = energy-dependent neutron density $[\text{n}/(\text{cm}^3 \cdot \text{MeV})]$,

Σ_x = macroscopic cross section for reaction of type x $[\text{cm}^{-1}]$, and

$v(E)$ = speed of neutron whose kinetic energy is E .

Note that

- Σ_x is the sum of the macroscopic cross sections for all of the nuclides.

- Σ_x for a given nuclide is $N\sigma_x$ for that nuclide.
- σ_x for a given nuclide and a given neutron lab-frame energy **must be pre-averaged over nucleus motion** in order for the expressions above to give the correct rates.

Production of neutrons in reactors is mainly by fission. The production rate due to fission, by the above equations, is

$$\text{production rate } \left[\frac{\text{n}}{\text{s}} \right] = \iiint_{\text{volume}} d^3r \int_0^\infty dE \nu(E) \Sigma_f(\vec{r}, E) \phi(\vec{r}, E)$$

where $\nu(E)$ is the average number of neutrons emitted from a fission that is caused by absorption of a neutron of energy E .

Loss of neutrons in reactors is by absorption and leakage. By the above equations,

$$\text{absorption rate } \left[\frac{\text{n}}{\text{s}} \right] = \iiint_{\text{volume}} d^3r \int_0^\infty dE \Sigma_a(\vec{r}, E) \phi(\vec{r}, E)$$

The other neutron-loss mechanism in reactors is leakage, which we also discussed in this chapter. We found that

1.

$$\text{net leakage rate outward through surface } S \text{ in } \left[\frac{\text{n}}{\text{s}} \right] \text{ is: } \iint_S d^2r \vec{e}_n(\vec{r}_s) \cdot \vec{J}(\vec{r}_s)$$

where $\vec{e}_n(\vec{r}_s)$ = outward unit normal at position \vec{r}_s on the surface,

2.

$$\begin{aligned} \vec{J}(\vec{r}) &= \text{total net current density} \\ &= \int_0^\infty dE \vec{J}(\vec{r}, E) \end{aligned}$$

$$3. \vec{J}(\vec{r}, E) = \text{energy-dependent net current density [n/(cm}^2\text{-s-MeV)] .}$$

The loss rate due to leakage is just the net outleakage rate (or net leakage rate outward, or outleakage rate minus inleakage rate—all the same thing).

Thus, if we know the energy-dependent scalar flux, the energy-dependent net current density, and all macroscopic cross sections, we can compute neutron production and loss rates in a reactor.

Chapter 7

Establishing Neutron Conservation Laws

7.1 Introduction

We know that a reactor's behavior depends on the gain and loss rates of neutrons in the reactor. We also know that some gain and loss is due to neutron-nucleus reactions, and that some is due to neutron leakage. In the previous chapter we developed mathematical expressions for reaction rates and leakage rates. In this chapter we put these expressions into statements of conservation to see what we can learn about how the neutron population changes with time in reactors and how the neutrons distribute themselves in position and energy.

To do this we will examine conservation statements for different neutron populations in different settings, ranging from simplistic to realistic. The populations and settings we consider are as follows.

- Conservation in an Infinite Uniform Medium:
 - Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.
Setting = infinite uniform medium, pretending that all fission neutrons are emitted promptly.
Goal: Establish time-dependent and energy-dependent balance of a population of neutrons, ignoring delayed neutrons. Later, we will slightly simplify this to study in depth neutron slowing-down,

- Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.

Setting = infinite uniform medium, accounting for delayed neutrons (which are emitted during the decay of fission products and their daughters).

Goal: Establish the point reactor kinetics equations (time-dependent balance of a population of neutrons), **accounting** for delayed neutrons.

- Neutron Conservation in Finite Reactors:

- Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.

Setting = finite reactor, accounting for delayed neutrons.

Goal: The time-dependent, energy-dependent, space-dependent neutron conservation law (time-space-energy dependent).

- Population = all neutrons in some chosen volume whose energies are in some chosen energy interval and whose directions are in some chosen directional range.

Setting = finite reactor, accounting for delayed neutrons.

Goal: The most detailed conservation law: the **neutron transport** equation (time-space-energy-direction dependent).

In every case we will follow the same procedure. **First** we will note that

$$\text{change rate} = \text{gain rate} - \text{loss rate}$$

for whatever population we have chosen. **Second**, for whatever neutron population we are considering, we will construct a mathematical expression for each of these three terms, consistent with the setting under consideration. **Third**, we will try to solve the resulting mathematical equation and learn what it tells us about the behavior of the neutron population.

7.2 Neutron Conservation in an Infinite Uniform Medium

In this section we consider the first two settings, which involve infinite homogeneous reactors. In these settings we do not have to worry about leakage, so we can focus our attention solely on production and loss from neutron-nucleus interactions. We also do not have to worry about spatial distributions, because everything is uniformly distributed throughout the infinite medium.

7.2.1 Case 1: Population = n's with positions in V and energies $E \in \Delta E$; Setting = energy-dependent neutrons and no delayed neutrons

Here we study an infinite medium and pretend that there are no delayed neutrons. Now that we are writing our conservation statement only for neutrons in a certain energy range— $=(E_1, E_2)$ or “ ΔE ” for short —. Our gain mechanisms are:

1. Emission of neutron with $E \in \Delta E$ from the fixed (extraneous) source ,
2. Emission of **prompt** neutrons with $E \in \Delta E$ from **neutron-induced fission** ,
3. Scattering into ΔE from pre-scatter energies outside ΔE .

Our loss mechanisms are:

1. Absorption of neutron with $E \in \Delta E$,
2. Scattering of a neutron whose pre-scatter energy is in ΔE and whose post-scatter energy is not in ΔE .

There is no net leakage from any volume in our infinite medium, because the neutrons are distributed uniformly in position and direction, so gain from inleakage cancels with loss from outleakage, i.e. net leakage is zero.

We define:

$n(E, t)$ = energy-dependent neutron density (see previous chapter) ,

$v(E)$ = speed of neutron whose kinetic energy is E ,

$S_{\text{ext}}(E)$ = energy-dependent fixed source-rate density ,

$\chi(E)dE$ = fraction of fission neutrons emitted with energies in dE at E

The probability density function $\chi(E)$ is called the

fission spectrum .

It quantifies the energy distribution of the neutrons that are emitted from fission. (See Fig. 3.10.)

Our conservation statement for neutrons that are in the volume \mathbf{V} and have energies in the interval ΔE is:

$$\frac{d}{dt} \underbrace{\left[\mathbf{V} \int_{\Delta E} dE n(E, t) \right]}_{\text{neutrons in } \mathbf{V} \text{ and } \Delta E} = \underbrace{\left[\int_{\Delta E} dE \chi(E) \right]}_{\text{frac. of fission n's born in } \Delta E} - \underbrace{\mathbf{V} \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t)}_{\text{n/s born from fission}}$$

$$- \underbrace{\mathbf{V} \int_{\Delta E} dE \Sigma_a(E) v(E) n(E, t)}_{\text{absorptions/s}} + \underbrace{\mathbf{V} \int_{\Delta E} dE S_{\text{ext}}(E)}_{\text{n/s from fixed source}}$$

$$+ \underbrace{\mathbf{V} \int_{\Delta E} dE \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t)}_{\text{n/s scattered into and within } \Delta E}$$

$$- \underbrace{\mathbf{V} \int_{\Delta E} dE \Sigma_s(E) v(E) n(E, t)}_{\text{n/s scattered out of and within } \Delta E} . \quad (7.1)$$

Remark: There is a kind of “error” in the inscattering term and a similar “error” in the outscattering term. The “inscattering” term, as written, includes scattering from **all** possible initial energies, including initial energies that were already in ΔE . That is, as written it includes “within-interval” scattering as well as inscattering. The outscattering term has a similar error—it includes “within-interval” scattering as well as outscattering from the interval ΔE . Both terms contain exactly the same “within-interval” error term, one with a plus sign and one with a minus sign. So the two “error” terms cancel each other, and our equation is correct. If we had written the correct terms in the first place it would have looked complicated, and in the end we would have canceled out the complicated part and gotten the same result.

The volume is constant and cancels out of the equation. Also, we know that $\Sigma_a(E) + \Sigma_s(E) = \Sigma_t(E)$, which allows us to combine the absorption and outscattering integrals:

$$\begin{aligned} \frac{d}{dt} \left[\int_{\Delta E} dE n(E, t) \right] &= \left[\int_{\Delta E} dE \chi(E) \right] \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t) \\ &\quad - \int_{\Delta E} dE \Sigma_t(E) v(E) n(E, t) + \int_{\Delta E} dE S_{\text{ext}}(E) \\ &\quad + \int_{\Delta E} dE \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t) . \end{aligned} \quad (7.2)$$

We recognize that every term involves an integral over the energy interval ΔE , so we group the terms into a single integral:

$$\int_{\Delta E} dE F(E) = 0 , \quad (7.3)$$

where we have defined a shorthand notation for the following combination of various rate densities:

$$\begin{aligned} F(E) &\equiv \frac{\partial}{\partial t} n(E, t) + \Sigma_t(E) v(E) n(E, t) - S_{\text{ext}}(E) \\ &\quad - \chi(E) \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t) \\ &\quad - \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t) . \end{aligned} \quad (7.4)$$

Note that Eq. (7.3) holds for every possible ΔE that we might pick—we did not say that ΔE needed to be anything special. This means that the function $F(E)$ integrates to zero over every possible range of integration.

Q: What does that tell us about $F(E)$?

A: It must be zero for all E ! This means that the integrand should always be zero

Given this important observation, we can rearrange Eq. (7.4) to obtain:

$$\begin{aligned} \frac{\partial}{\partial t} n(E, t) + \Sigma_t(E) v(E) n(E, t) &= S_{\text{ext}}(E) + \chi(E) \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t) \\ &\quad + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t) . \end{aligned} \quad (7.5)$$

Because reaction-rate terms always involve the product of speed times density, which we know is the **scalar flux**, it is usually more convenient to write our conservation equations in terms of the scalar flux, $\phi(E, t) = v(E)n(E, t)$, instead of the neutron density, $n(E, t)$. This is a simple substitution:

$$\frac{1}{v(E)} \frac{\partial}{\partial t} \phi(E, t) + \Sigma_t(E) \phi(E, t) = S_{\text{ext}}(E) + \chi(E) \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) \phi(E_i, t) . \quad (7.6)$$

This is an

integro-differential equation

for the energy-dependent scalar flux in an infinite uniform medium. This equation is well-posed and has a unique solution. If the source is known and the cross sections are known, then this problem can be solved to whatever accuracy we require.

That is, since we have now explicitly recognized conservation for every sub-population of neutrons that affects gain and loss (i.e., the neutrons in every spatial volume in every energy interval), we are not forced to make approximations about neutron distributions. However, there is a second source of neutrons in a reactor, which are the neutrons produced by the (β^- or n) decay of neutron precursors. This one is analyzed in the next section.

7.2.2 Case 2: Adding neutron precursors to Case 1

We add to our gain mechanism:

Emission of delayed neutrons from neutron-induced fission

We define:

$S_{dn}(E, t)$ = energy-dependent delayed-neutron source-rate density,
 $\chi_p(E)$ = fission spectrum of prompt neutrons,
 $\nu_p(E)$ = average number of prompt neutrons emitted,
from a fission caused by a neutron of energy E .

Everything proceeds just as in the previous subsection, and we end up with:

$$\begin{aligned} \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(E, t) + \Sigma_t(E) \phi(E, t) &= S_{\text{ext}}(E) + S_{\text{dn}}(E, t) \\ &+ \chi_p(E) \int_0^\infty dE_i \nu_p(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ &+ \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) \phi(E_i, t). \end{aligned} \quad (7.7)$$

The big difference here is that the delayed-neutron source, $S_{\text{dn}}(E, t)$, is not known—it is something we have to solve for, just like the scalar flux $\phi(E, t)$. How do we do this? First we need to understand how delayed neutrons come about.

7.2.2.1 Delayed-neutron physics and math

Let us review the processes that lead to emission of delayed neutrons. Consider, for example, the decay scheme for Bromine-87, a long long-lived fission product with a half-life of 55 seconds, as shown in Fig. 7.1. The ^{87}Br atoms **that eventually decay to metastable ^{87}Kr** (and thus yield a neutron) are called

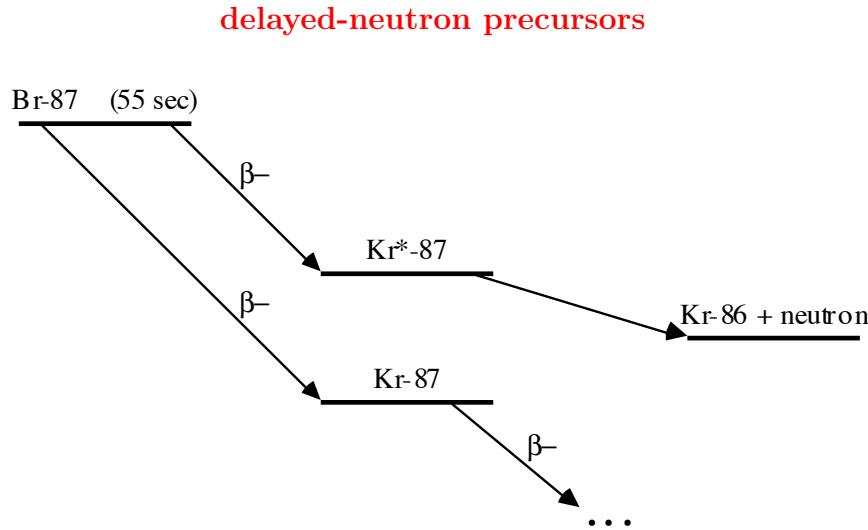
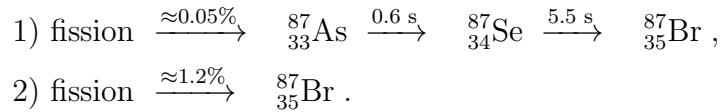


Figure 7.1: Decay scheme for ^{87}Br , 2.6% of the atoms of which are **delayed-neutron precursors**. (Frequency of outcomes for Br-87 decay: 2.6% to Kr^* -87 and 97.4% to Kr -87.)

(These are only $\approx 2.6\%$ of the ^{87}Br atoms.) Notice that the precursor does not actually expel a neutron. Instead, it decays to an excited state of another nuclide, which then emits the neutron. Also note that the precursor does not have to be a direct fission fragment, but may simply be in the decay chain of a fission fragment. Two ways to produce Bromine-87, for example, are:



The delayed neutron in question is delayed by the sum of the decay times of the precursor and its parents, in the whole chain all the way back to the fission event whose fission product is the start of the chain.

Q: Why are delayed neutrons important in nuclear reactors? **A:** Because they reduce the response time of the reactor.

Approximately 270 delayed-neutron precursors have been identified to date. Of these, only a few dozen are of much practical significance.

We seek an equation or set of equations that will give us $S_{\text{dn}}(E, t)$, the energy-dependent delayed-neutron emission rate density, which appears in the equation for the energy-dependent scalar flux in our infinite-medium problem. Let us define some helpful quantities:

$C_i(t)$ = expected number density of type- i precursors at time t .

λ_i = decay constant of type- i precursors.

$\chi_i(E)$ = energy spectrum of neutrons emitted after decay of type- i precursors.

From these definitions it follows that

$$S_{\text{dn}}(E, t) = \sum_{i=1}^{\# \text{ of precursor types}} \chi_i(E) \lambda_i C_i(t) \quad (7.8)$$

This is fine, but it doesn't help much unless we know $C_i(t)$ for each precursor type, i . How do we find these precursor densities?

As usual, we turn to our favorite equation:

$$\text{Change rate} = \text{gain rate} - \text{loss rate}.$$

For what population of things should we write this equation?

$$\text{population} = \text{type-}i \text{ precursors in volume } V$$

We already know the loss rate from decay of the precursors—it is λ_i times the precursor population. But what is the gain rate?

Note that each precursor had its ultimate origin in fission. That is, each precursor is either a fission product or originated from the decay of a fission product. So we know that the production rate density of precursors will be tied to

the fission-rate density

Also note that

each precursor ultimately produces exactly one delayed neutron

It is useful and customary to define

$$\nu_{di}(E) \equiv \text{expected number of type-}i \text{ precursors produced from a fission that is caused by a neutron of energy } E . \quad (7.9)$$

Because each precursor ultimately produces exactly one delayed neutron, ν_{di} is also the expected number of delayed neutrons that will ultimately be emitted from type- i precursors, per fission caused by a neutron of energy E .

With these definitions we can now write our conservation equation for precursors of type i :

$$\frac{d}{dt}C_i(t) = \int_0^\infty dE \nu_{di}(E) \Sigma_f(E) \phi(E, t) - \lambda_i C_i(t) . \quad (7.10)$$

Q: Why is this a conservation equation?

A: Because it is describing: “change rate = gain rate – loss rate”

7.2.2.2 Back to Finishing Case 2...

Let us collect what we have so far:

$$\begin{aligned} \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(E, t) + \Sigma_t(E) \phi(E, t) &= S_{\text{ext}}(E) + \sum_{i=1}^{\# \text{ of precursor types}} \chi_i(E) \lambda_i C_i(t) \\ &+ \chi_p(E) \int_0^\infty dE_i \nu_p(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ &+ \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) \phi(E_i, t) . \end{aligned} \quad (7.11)$$

$$\frac{d}{dt}C_i(t) = \int_0^\infty dE \nu_{di}(E) \Sigma_f(E) \phi(E, t) - \lambda_i C_i(t) . \quad (7.12)$$

We see that

- The scalar flux depends on the delayed-neutron precursor concentrations .
- The delayed-neutron precursors concentrations depend on the scalar flux .

Thus, we have a

coupled system

of equations to solve. While this may look like a complicated system that could be difficult to solve, it is at least a well-posed set of equations that has a unique solution (given initial conditions for the scalar flux and the precursor concentrations).

So once again conservation has come through for us! That is, we have written “change rate = gain rate – loss rate” in mathematical terms, for both the neutron population and the precursor populations, and arrived at a well-posed system of equations with a unique solution. The solution, if we can find it, tells us in detail how the neutrons and precursors behave in the setting we have considered.

I hope you see that conservation is a powerful tool with far-reaching implications!

7.3 Reactor Kinetics Equations

Let us see what we can learn about solutions of Eqs. (7.11) and (7.12). Reminder: these are for an infinite homogeneous reactor. We begin by integrating the first equation over all neutron energies:

$$\begin{aligned} \frac{d}{dt} \left[\int_0^\infty dE \frac{\phi(E, t)}{v(E)} \right] + \int_0^\infty dE \Sigma_t(E) \phi(E, t) &= \int_0^\infty dE S_{\text{ext}}(E) + \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) \\ &\quad + \int_0^\infty dE_i \nu_p(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ &\quad + \int_0^\infty dE_i \Sigma_s(E_i) \phi(E_i, t) . \end{aligned} \quad (7.13)$$

Here we have recognized that the integral of probability distribution functions, such as $\chi_p(E)$, $\chi_{di}(E)$, or $P(E_i \rightarrow E)$ (which is buried in $\Sigma_s(E_i \rightarrow E)$), equals 1. Now we subtract the scattering-rate density from both sides of the equation and change dummy variables E_i to E :

$$\begin{aligned} \frac{d}{dt} \left[\int_0^\infty dE \frac{\phi(E, t)}{v(E)} \right] + \int_0^\infty dE \Sigma_a(E) \phi(E, t) &= \int_0^\infty dE S_{\text{ext}}(E) + \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) \\ &\quad + \int_0^\infty dE \nu_p(E) \Sigma_f(E) \phi(E, t) . \end{aligned} \quad (7.14)$$

Rewrite in terms of neutron density:

$$\frac{d}{dt} \left[\int_0^\infty dE n(E, t) \right] + \int_0^\infty dE \Sigma_a(E) v(E) n(E, t) = \int_0^\infty dE S_{\text{ext}}(E)$$

$$+ \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) + \int_0^\infty dE \nu_p(E) \Sigma_f(E) v(E) n(E, t). \quad (7.15)$$

Let us introduce some definitions:

$$n_{\text{tot}}(t) \equiv \int_0^\infty dE n(E, t), \quad (7.16)$$

$$S_{\text{tot}}(t) \equiv \int_0^\infty dE S_{\text{ext}}(E), \quad (7.17)$$

$$\langle v\Sigma_a \rangle \equiv \frac{\int_0^\infty dE [v(E) \Sigma_a(E)] n(E, t)}{\int_0^\infty dE n(E, t)}, \quad (7.18)$$

$$\langle v\nu\Sigma_f \rangle \equiv \frac{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE n(E, t)}. \quad (7.19)$$

and add another:

$$\langle v\nu_p\Sigma_f \rangle \equiv \frac{\int_0^\infty dE [v(E) \nu_p(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE n(E, t)}. \quad (7.20)$$

Q: What do we conserve in the $\langle \cdot \rangle$ averaging of the cross sections?

A: The reaction rates.

Define the

Delayed-neutron fraction, β ("beta"),

as follows:

$$\beta \equiv 1 - \frac{\int_0^\infty dE [v(E) \nu_p(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)} = \frac{\int_0^\infty dE [v(E) \nu_d(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)} \quad (7.21)$$

$$= \frac{\langle v \nu_d \Sigma_f \rangle}{\langle v \nu \Sigma_f \rangle}$$

= fraction of fission neutrons that are born delayed.

Remark: It follows from previous definitions that

$$\nu(E) = \nu_p(E) + \sum_{i=1}^{\# \text{ of precursor types}} \nu_{di}(E) \quad (7.22)$$

Rewrite the conservation equation in terms of these defined quantities:

$$\frac{d}{dt} [n_{\text{tot}}(t)] = [\langle v \nu_p \Sigma_f \rangle - \langle v \Sigma_a \rangle] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) . \quad (7.23)$$

(Remember: this is for an infinite homogeneous reactor.) Now define

$\ell_p \equiv$ average time from prompt-neutron birth to absorption

$$= \frac{1}{\langle v \Sigma_a \rangle} . \quad \text{only for infinite reactor!} \quad (7.24)$$

That is,

ℓ_p is the prompt-neutron lifetime

Also define

$k \equiv$ multiplication factor

$$= \frac{\langle v\nu\Sigma_f \rangle}{\langle v\Sigma_a \rangle} \xrightarrow{\text{Eq. (7.21)}} = \frac{\langle v\nu_p\Sigma_f \rangle}{(1-\beta)\langle v\Sigma_a \rangle} . \quad (7.25)$$

In terms of these quantities, our equation for n_{tot} can be written:

$$\begin{aligned} \frac{d}{dt} [n_{\text{tot}}(t)] &= \left[\frac{(1-\beta)k\langle v\Sigma_a \rangle - \langle v\Sigma_a \rangle}{1} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \\ &= \left[\frac{(1-\beta)k - 1}{\ell_p} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \\ &= \left[\frac{k - \beta k - 1}{\ell_p} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \\ &= \left[\frac{\frac{k-1}{k} - \beta}{\ell_p/k} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \end{aligned} \quad (7.26)$$

We define

$$\rho \equiv \text{reactivity} = \frac{k-1}{k} , \quad (7.27)$$

$$\Lambda \equiv \text{mean generation time} = \frac{\ell_p}{k} . \quad (7.28)$$

= expected time for N neutrons to have N descendants .

With these definitions our equation for the neutron density can be written in a form that has become standard (and famous, or perhaps infamous from a student's point of view) for time-dependent neutronics calculations:

$$\frac{d}{dt} [n_{\text{tot}}(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \quad (7.30)$$

Similarly, we can write our precursor equation in standard form:

$$\frac{d}{dt}C_i(t) = \frac{\beta_i}{\Lambda}n_{\text{tot}}(t) - \lambda_i C_i(t) , \quad (7.31)$$

where we have defined:

$$\beta_i \equiv \frac{\int_0^\infty dE [v(E) \nu_{di}(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)} \quad (7.32)$$

= fraction of fission neutrons that are type-*i* delayed.

We have now written our equations in the form of

coupled first-order ODEs .

We can learn a lot about the behavior of the neutron population by studying and solving these equations!

We observe that:

- The quantities ρ , β , β_i , and Λ are all defined in terms of $n(E, t)$, which of course depends on time. Thus, in general, these quantities all depend on time.
- It turns out that in many interesting problems, ρ , β , β_i , and Λ are all approximately constant. Thus, it is useful for us to study the solution of our coupled first-order ODEs in this interesting constant-parameter case. This case is simple enough to permit analytic solutions that are enormously helpful in building an understanding of nuclear-reactor behavior.

We repeat these very important equations here, Eqs. (7.30) and (7.31), and introduce the name by which they are known:

THE POINT-REACTOR KINETICS EQUATIONS (PRKEs):

$$\frac{d}{dt} [n_{\text{tot}}(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) , \quad (7.33)$$

$$\frac{d}{dt} C_i(t) = \frac{\beta_i}{\Lambda} n_{\text{tot}}(t) - \lambda_i C_i(t) . \quad (7.34)$$

Remember these are just two couple equations of conservation for the neutron density and the number density of neutron precursors. The PRKEs are used extensively to study the time-dependent behavior of nuclear reactors. In Chapter 8, we will study them.

Motivate them with an example.

7.4 Neutron Conservation in Finite Reactors

In this section we consider the Cases 3 and 4. All of these involve **finite** reactors. In these settings we must address leakage as a loss or gain mechanism. We also must worry about spatial distributions, because the reactor may have different materials in different places, which means reaction rates will depend on where the neutrons are in addition to how many there are.

7.4.1 Case 3: Population = all n's in a given sub-volume of the reactor;

Setting = finite reactor with proper treatment of delayed neutrons

For this population our gain mechanisms are:

- Emission from the fixed source in sub-volume ΔV
- Prompt emission from neutron-induced fission in sub-volume ΔV
- Delayed emission from neutron-induced fission in sub-volume ΔV
- Inleakage through the sub-volume surface, $\partial\Delta V$

Our loss mechanisms are:

- Absorption in sub-volume ΔV
- Outleakage through the sub-volume surface , $\partial\Delta V$

For a characteristic volume ΔV in the reactor, the leakage term captures all neutrons streaming in and out of the boundary surface of this volume. For simplicity, we combine (out-leakage – inleakage) into

net outleakage

We then state a conservation equation for a volume ΔV in the reactor:

$$\begin{aligned} \frac{d}{dt} \left[\iiint_{\Delta V} d^3r \int_0^\infty dE n(\vec{r}, E, t) \right] &= \iiint_{\Delta V} d^3r \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &+ \iiint_{\Delta V} d^3r \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i \iiint_{\Delta V} d^3r C_i(\vec{r}, t) \\ &- \iiint_{\Delta V} d^3r \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) - \iint_{\partial\Delta V} d^2r \int_0^\infty dE \vec{e}_n(\vec{r}) \cdot \vec{J}(\vec{r}, E, t) . \quad (7.35) \end{aligned}$$

We can use a mathematical “trick” to write the surface integral of the sub-volume as a volume integral. Recall Gauss’s divergence theorem:

$$\iiint_{\Delta V} d^3r \vec{\nabla} \cdot \vec{u} = \iint_{\partial\Delta V} d^2r \vec{e}_n \cdot \vec{u} . \quad (7.36)$$

Let's use this on the leakage term in our conservation equation:

$$\begin{aligned}
 \frac{d}{dt} \left[\iiint_{\Delta V} d^3r \int_0^\infty dE n(\vec{r}, E, t) \right] &= \iiint_{\Delta V} d^3r \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\
 &+ \iiint_{\Delta V} d^3r \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i \iiint_{\Delta V} d^3r C_i(\vec{r}, t) \\
 &- \iiint_{\Delta V} d^3r \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) - \iiint_{\Delta V} d^3r \int_0^\infty dE \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) . \quad (7.37)
 \end{aligned}$$

We recognize that

every term is an integral over the sub-volume .

Thus, we can collect everything into a single integral over the sub-volume:

$$\begin{aligned}
 &\iiint_{\Delta V} d^3r \left\{ \frac{\partial}{\partial t} \int_0^\infty dE n(\vec{r}, E, t) - \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \right. \\
 &- \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) - \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_i(\vec{r}, t) \\
 &\left. + \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) + \int_0^\infty dE \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) \right\} = 0 . \quad (7.38)
 \end{aligned}$$

This equation says that the integral of some function over some sub-volume of the reactor equals zero. This must be true for any sub-volume of the reactor, because we did not choose any special sub-volume for our conservation statement. Thus, we see that the integral of the function inside the { } integrates to zero for any range of integration. Thus, the function must be zero, and we must have:

$$\begin{aligned} \frac{\partial}{\partial t} \left[\int_0^\infty dE n(\vec{r}, E, t) \right] + \int_0^\infty dE \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) + \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ = \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ + \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_i(\vec{r}, t) \end{aligned} \quad (7.39)$$

In this equation, every term has units of:

$$\text{n/cm}^3\text{-s}$$

That is, every term is a “something” rate density, where every “something” is either change or some kind of gain or loss, and where the “density” is in ordinary 3-D space.

We can already recognize a problem that we have encountered previously with our conservation equations: Even if we knew

the conserved quantity, $\int dE n(\vec{r}, E, t)$,

it would not give us enough information to calculate

the absorption and fission rate densities

that appear in the equation, because these rates depend on the

energy distribution of the neutrons

and not just on the total neutron density. We recognize the source of this problem:

we have not stated conservation in enough detail.

Nevertheless, as we have done before, we shall press on by defining some average quantities. We shall proceed slightly differently than before, just to illustrate that there are many ways to attack these problems. This time we choose the **scalar flux** instead of the **neutron density** as the fundamental unknown. In this case the neutron speed will be included in the weight function in the averaged cross sections, $\langle \Sigma \rangle$, instead of being part of the averaged quantity, as with our previous $\langle v\Sigma \rangle$ averages. That is, we define:

$$\left\langle \frac{1}{v} \right\rangle \equiv \frac{\int_0^\infty dE \frac{\phi(\vec{r}, E, t)}{v(E)}}{\int_0^\infty dE \phi(\vec{r}, E, t)}, \quad (7.40)$$

$$\langle \Sigma_a(\vec{r}, t) \rangle \equiv \frac{\int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t)}{\int_0^\infty dE \phi(\vec{r}, E, t)}, \quad (7.41)$$

$$\langle \nu_p \Sigma_f(\vec{r}, t) \rangle \equiv \frac{\int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t)}{\int_0^\infty dE \phi(\vec{r}, E, t)}. \quad (7.42)$$

Q: What do we conserve in the $\langle \cdot \rangle$ averaging of the cross sections?

A: The reaction rates.

and we recall the definitions of “total” scalar flux and “total” net current density from a previous chapter:

$$\phi(\vec{r}, t) \equiv \int_0^\infty dE \phi(\vec{r}, E, t), \quad (7.43)$$

$$\vec{J}(\vec{r}, t) \equiv \int_0^\infty dE \vec{J}(\vec{r}, E, t). \quad (7.44)$$

We make a similar definition for the fixed source rate density:

$$S_{\text{ext}}(\vec{r}, t) \equiv \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t). \quad (7.45)$$

With these definitions we can rewrite our conservation equation as:

$$\begin{aligned}
 & \left\langle \frac{1}{v} \right\rangle \frac{\partial}{\partial t} \phi(\vec{r}, t) + \vec{\nabla} \cdot \vec{J}(\vec{r}, t) + \langle \Sigma_a(\vec{r}, t) \rangle \phi(\vec{r}, t) \\
 & = \langle \nu_p \Sigma_f(\vec{r}, t) \rangle \phi(\vec{r}, t) \\
 & + S_{\text{ext}}(\vec{r}, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_i(\vec{r}, t)
 \end{aligned} \tag{7.46}$$

We also need equations for the delayed-neutron precursor concentrations, C_i , which appear in this neutron conservation equation. If we follow the same kind of procedure as before, but expressing precursor production in terms of the total scalar flux, we obtain:

$$\frac{\partial}{\partial t} C_i(\vec{r}, t) = \langle \nu_{di} \Sigma_f(\vec{r}, t) \rangle \phi(\vec{r}, t) - \lambda_i C_i(\vec{r}, t) . \tag{7.47}$$

IMPORTANT:

Understand the physical meaning of each term in these equations!

Equation (7.46) is an exact conservation equation for the neutrons in a reactor, provided that the terms in angle brackets, $\langle \rangle$, are averaged with the exact energy-dependent scalar flux. Of course, we don't know the energy-dependent scalar flux in advance, and this equation does not give us the opportunity to calculate it. Further, we don't have any equations for

the net current density, \vec{J} ,

nor do we have any obvious way to express the divergence of \vec{J} in terms of the scalar flux. So we have an exact equation but cannot solve it without some additional knowledge. We can take two approaches at this point. First, we could introduce an approximation for the current as a function of the scalar flux. This is done in the next chapter in the framework of diffusion theory. Other choice is to examine the conservation equation in further detail, i.e. considering the dependencies in energy and in **direction**. This leads to the

transport equation

So let's try conservation in even more detail.

7.4.2 Case 4: Adding the dependence in energy and direction to Case 3

Our setting is the **real world** (finite reactor, non-uniform structure, neutrons distributed in energy, delayed neutrons present). Our conservation statement will give us information about the spatial and energy and **directional** distribution of the neutrons. This will produce a set of equations whose solution tells us all we need to know about neutron behavior in reactors! The problem is that the equation is complicated enough that it is difficult to solve, even approximately.

For this population our gain mechanisms are:

- Emission from the fixed source
- Prompt emission from neutron-induced fission
- Delayed emission from neutron-induced fission
- Inleakage through the sub-volume surface
- “Inscattering” from energies outside the energy sub-interval and/or directions outside the directional cone to energies in the sub-interval and directions in the cone.

Our loss mechanisms are:

- Absorption
- Outleakage through the sub-volume surface
- “Outscattering” from energies in the energy sub-interval and directions in the directional cone to energies outside the sub-interval and/or directions outside the cone.

As we have done before, we combine (outleakage – inleakage) into net outleakage.

Before we can write mathematical expressions for the gain and loss rates, we must develop a way to describe the neutron distribution in direction, and to describe how scattering changes a neutron’s direction. We define the direction- and energy-dependent neutron density:

$$n(\vec{r}, E, \vec{\Omega}, t) d^3r dE d\Omega \equiv \text{expected number of neutrons in the volume } d^3r \text{ at } \vec{r}, \\ \text{with lab-frame kinetic energy in the interval } dE \text{ at } E, \\ \text{and with direction in the solid-angle cone } d\Omega \text{ around the direction } \vec{\Omega}.$$

This is a density in a six-dimension phase space: 3 spatial dimensions, 1 energy dimension, and 2 direction dimensions. (It takes two numbers—say a polar angle and an azimuthal angle—to define a direction.) The units are neutrons per unit volume per unit energy per unit solid angle. The usual unit of solid angle is the **steradian**. So in our usual units we have $n/(cm^3\text{-MeV-ster})$.

Analogous to the energy-dependent scalar flux, which is speed times the energy-dependent neutron density, we define the energy-dependent **angular flux**:

$$\psi(\vec{r}, E, \vec{\Omega}, t) \equiv v(E)n(\vec{r}, E, \vec{\Omega}, t) . \quad (7.48)$$

Note the identity relating angular flux to scalar flux:

$$\phi(\vec{r}, E, t) \equiv v(E) \iint_{4\pi} d\Omega \psi(\vec{r}, E, \vec{\Omega}, t) . \quad (7.49)$$

Now recall the double-differential scattering cross section:

$\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t)dE d\Omega \equiv$ expected scatters per unit neutron path length,
 for incident neutrons of energy E' and direction $\vec{\Omega}'$,
 that produce scattered neutrons with kinetic energy in the interval dE at E
 and with direction in the solid-angle cone $d\Omega$ around the direction $\vec{\Omega}$.

This has units of inverse [cm-MeV-ster]. It is a scattering cross section at the incident neutron energy, multiplied by a distribution function in outgoing neutron energy and direction.

Now our conservation equation is

$$\begin{aligned}
& \frac{d}{dt} \left[\iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega n(\vec{r}, E, \vec{\Omega}, t) \right] \\
&= \left[\iint_{\Delta \Omega} d\Omega \frac{1}{4\pi} \right] \left[\int_{\Delta E} dE \chi_p(E) \right] \iiint_{\Delta V} d^3r \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
&+ \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega S_{\text{ext}}(\vec{r}, E, \vec{\Omega}, t) \\
&+ \sum_{i=1}^{\# \text{ dnp types}} \left[\iint_{\Delta \Omega} d\Omega \frac{1}{4\pi} \right] \left[\int_{\Delta E} dE \chi_{di}(E) \right] \lambda_i \iiint_{\Delta V} d^3r C_i(\vec{r}, t) \\
&+ \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E', \vec{\Omega}', t) \\
&- \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E \rightarrow E', \vec{\Omega} \rightarrow \vec{\Omega}', t) \psi(\vec{r}, E', \vec{\Omega}', t) \\
&- \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \Sigma_a(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
&- \iint_{\partial \Delta V} d^2r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \vec{e}_n(\vec{r}) \cdot \vec{\Omega} \psi(\vec{r}, E, \vec{\Omega}, t) . \tag{7.50}
\end{aligned}$$

We use the divergence theorem to convert the surface integral to a volume integral, and then every term is an integral over the six-dimensional volume [$\Delta V \Delta E \Delta \Omega$]. We recognize that the integral of a double-differential scattering cross section is just a scattering cross section, much as we did in the previous case. We recognize that $\Sigma_t = \Sigma_a + \Sigma_s$. We collect all the terms into one giant integral, then recognize (as before) that the integrand must be zero. This gives:

$$\begin{aligned}
& \frac{d}{dt} \left[n(\vec{r}, E, \vec{\Omega}, t) \right] + \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}, t) + \Sigma_t(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
&= \frac{\chi_p(E)}{4\pi} \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
&+ S_{\text{ext}}(\vec{r}, E, \vec{\Omega}, t) + \sum_{i=1}^{\# \text{ dnp types}} \frac{1}{4\pi} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) \\
&+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E', \vec{\Omega}'t) . \tag{7.51}
\end{aligned}$$

We can rewrite the neutron density, n , as the angular flux divided by the speed, ψ/v :

THE TIME-DEPENDENT NEUTRON TRANSPORT EQUATION:

$$\begin{aligned}
& \frac{1}{v(E)} \frac{d}{dt} \psi(\vec{r}, E, \vec{\Omega}, t) + \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}, t) + \Sigma_t(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
&= \frac{\chi_p(E)}{4\pi} \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
&+ S_{\text{ext}}(\vec{r}, E, \vec{\Omega}, t) + \sum_{i=1}^{\# \text{ dnp types}} \frac{1}{4\pi} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) \\
&+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E', \vec{\Omega}'t) . \tag{7.52}
\end{aligned}$$

This is a single integro-differential equation for the single unknown function ψ . We can show that it has a unique solution, given appropriate boundary and initial conditions. Thus,

considering conservation in detail (i.e., in every part of a six-dimensional phase space that includes position, direction, and energy) has eliminated the difficulty we found in the previous cases!

Equation (7.52) is the **neutron transport equation**. It is an essentially exact description of the behavior of neutrons in reactors. It is a relatively complicated equation that is relatively difficult to solve. Fortunately, we can gain a great deal of insight into reactor behavior by using equations that contain simplifying approximations. We do so in the next chapter. Later in your studies you will return to the transport equation to get a more precise picture of some neutronics details.

7.5 Summary

In this chapter we have delved into the details of writing conservation equations for various defined populations in various settings.

We first started with an infinite reactor and derived the conservation equations in a sub-volume first considering only neutrons emitted by fission and a extraneous source. These equations are exact and no closure models are needed. However, in the next section, we added as a source the delayed neutrons emitted by neutron precursors, which may be fission products and or belong to the decay-chain of these ones. A new conservation equation for the number density of neutron precursors was necessary to close our system. Integrating the system over the energy range, the point kinetics equation was derived.

Then, we considered a finite reactor, in which net leakage plays a role in the conservation equations. Now, the energy dependence balance is not enough to model the conservation of neutrons in the reactor, due to the presence of the neutron current density in the leakage term. Two different roads can be followed. First, to introduce an approximate model for the neutron current density, which is done in the next chapter. Second, to derive a balance considering the angular dependence of the neutron flux, which leads to the neutron transport equation.

Some important remarks are:

- We found that in every case that treated delayed neutrons we could cast our conservation equations, along with the equations for the delayed-neutron-precursor concentrations, into the form of **Point-Reactor Kinetics Equations** (PRKEs). Even when we did not know how to calculate the coefficients in these equations, we could study the characteristics of their solutions in all scenarios of interest.
- In particular, if the coefficients in the PRKEs—namely ρ , the $\{\beta_i\}$, and Λ —are constant (which they often are for interesting problems), then we can write down equations for $n_{\text{tot}}(t)$ and the $\{C_i(t)\}$. This is a natural segway to Chapter 8.
- **Important:** We derived the PRKEs using an infinite problem. You will obtain exactly the same form for the equations if the domain is finite. That is, the PRKEs are quite general! So our work here applies far beyond the infinite uniform medium that we started with. In case of a finite domain, the only differences are buried in the definitions of the coefficients:
 1. The prompt-neutron lifetime, ℓ_p is **shortened** by the possibility of leakage.
 2. The multiplication factor, k , is **reduced** by loss from leakage.

If you want additional details, please see Appendix E.

- In Case 3 we derived an essentially exact statement of neutron conservation in a reactor. The only difficulty we found with our conservation equation is that it contains the unknown net current density, \vec{J} , in addition to the unknown scalar flux, ϕ .
- Note that this difficulty is not dependent on the energy or time. Hence, the same problem will be found if we considered a mono-energetic population of neutrons that is steady in time.

Part IV

Neutron distribution in time

Chapter 8

Time Dependence via PRKE

Foreword: Read Appendix D for a review of mathematical techniques to solve ODEs.

8.1 Introduction

In this Chapter, we study the time dependence of a neutron population. We do this using the Point Reactor Kinetic Equations, PRKEs, recalled below.

This model, although simplified because it does not address the interaction between the space+energy distribution of neutrons and their temporal variation, is a widely used model to understand the temporal behavior of nuclear reactors (and fissile systems, in general).

THE POINT-REACTOR KINETICS EQUATIONS (PRKEs):

$$\frac{d}{dt} [n(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n(t) + S(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) , \quad (8.1)$$

$$\frac{d}{dt} C_i(t) = \frac{\beta_i}{\Lambda} n(t) - \lambda_i C_i(t) . \quad (8.2)$$

8.2 Analytical solution of the PRKE equations

We consider the case in which ρ , β , β_i , Λ , and S are constant. (It turns out that for much of the time during many transients of interest, this is a very good approximation, so this is an important case to understand.) In this case, which has constant coefficients, the Point-Reactor Kinetics Equations (PRKEs) are a set of coupled **first-order ODEs with constant coefficients**. We know from our previous mathematical studies that:

1. The solution of each equation is the sum of
 - a “particular” solution plus a “homogeneous” (no-source) solution.
2. The no-source solution of each equation is a **sum of exponentials**.

Given our assumption that S and the coefficients are constants, it is easy to show that constant particular solutions satisfy the PRKEs (with the exception of one tricky case). You should verify that the following solutions do satisfy the equations:

$$C_i^{\text{particular}} = \frac{\beta_i}{\Lambda \lambda_i} n^{\text{particular}} , \quad (8.3)$$

$$n^{\text{particular}} = - \left[\frac{\Lambda}{\rho} \right] S . \quad (8.4)$$

You can see that there is a problem with this particular solution if the reactivity, ρ , is zero. This is the “tricky” case—the case of a critical reactor—which we will address later.

Now let us address the “homogeneous” solution—the solution that satisfies the equations with S removed. If we define

$$I = \text{number of types of delayed-neutron precursors that we are tracking}, \quad (8.5)$$

then the PRKEs are a set of $I + 1$ coupled ODEs (with constant coefficients in the case we are studying right now). We know from our previous mathematics studies that each “homogeneous” solution (n , C_1 , C_2 , \dots , C_I) will be

a sum of $I + 1$ exponentials.

and that the time constants in the exponentials for a given function (such as $C_2(t)$) are

the same as the time constants in the other functions.

Thus, our PRKE solutions have the following form (excluding the tricky $\rho = 0$ case and continuing to assume that S_{tot} is constant):

$$n(t) = \sum_{j=1}^{I+1} A_j e^{s_j t} - \frac{\Lambda}{\rho} S . \quad (8.6)$$

$$\begin{aligned} C_i(t) &= \sum_{j=1}^{I+1} C_{i,j} e^{s_j t} + \frac{\beta_i}{\Lambda \lambda_i} n^{\text{particular}} \\ &= \sum_{j=1}^{I+1} C_{i,j} e^{s_j t} - \frac{\beta_i}{\Lambda \lambda_i} \frac{\Lambda}{\rho} S . \end{aligned} \quad (8.7)$$

But how do we find the constants $s_j, j = 1, \dots, I+1$? This is not terribly complicated: we insert the solutions into the equations and find the s values that allow the solutions to be non-trivial (i.e., the solutions that can have non-zero A_j and $C_{i,j}$ coefficients in front of the exponentials in the solutions). After a bit of algebra (okay, more than just a bit) we find that each of the s values must satisfy the following equation:

THE INHOUR EQUATION:

$$\rho = \Lambda s + \sum_{i=1}^I \frac{\beta_i s}{s + \lambda_i} . \quad (8.8)$$

There are exactly $I + 1$ distinct values of s that satisfy this equation, which is called the “**inhour equation.**” This equation gives us the s values that go into Eqs (8.6)-(8.7). These values determine how quickly or slowly each part of the solution changes with time. Note that the s values have units of

inverse time

In the early days of reactor analysis, people often used “inverse hours” as the unit for the s values. This led to the name “in-hour,” which the equation still bears.

It is not difficult to show, for example by plotting the left-hand side and right-hand side of the inhour equation on the same plot as in Fig. 8.1, that the $I+1$ values of s have the following properties:

- I of them are always negative, and in fact $< -\lambda_{\min}$.
- The other one (call it s_1) has the same sign as the reactivity, ρ .
- If $\rho = 0$, then $s_1 = 0$.
- If $\rho < 0$, then s_1 is negative but is closer to 0 than the other s values:

$$s_1 \in (-\lambda_{\min}, 0) \quad \text{if } \rho < 0 .$$

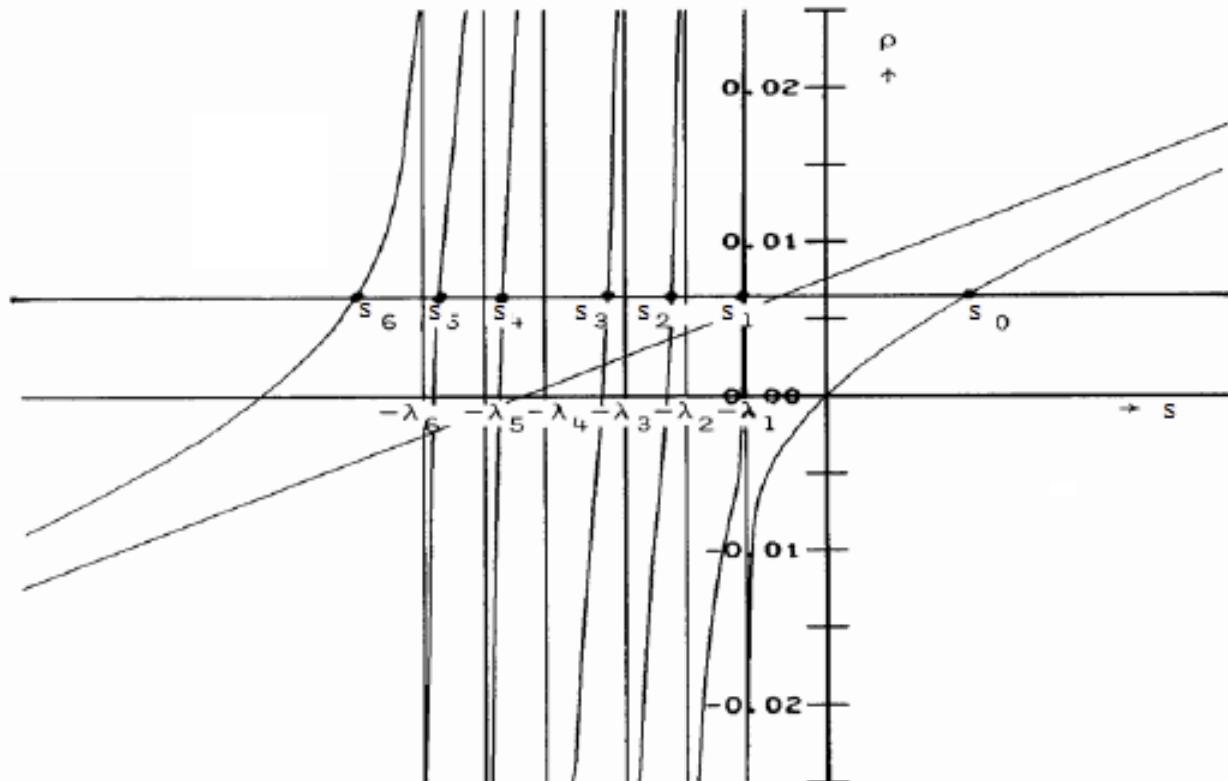


Figure 8.1: Graphical “solution” of inhour equation with $I = 6$ groups of delayed neutron precursors. There is already a horizontal constant line that is the value of $\rho > 0$; look at the signs of the $I + 1$ roots s_i . Now add another horizontal line for $\rho < 0$ and look at the signs of the $I + 1$ roots s_i in that case.

That is, I values of s will always be real and negative; the other value, which is the largest algebraically, is real and has the same sign as the reactivity, ρ . If $\rho < 0$, there is one s value that can never be more negative than $-\lambda_{\min}$, where λ_{\min} is the shortest of the precursor decay constants. This corresponds to the precursor with the longest half-life, which is ^{87}Br , with $T_{1/2} = 55$ s. Thus,

$$\lambda_{\min} \approx \frac{1}{80s}$$

This simple information about the s values tells us a great deal about the time variation of the neutron population in a reactor. We explore this for the six possible cases in the subsections that follow.

8.2.1 Subcritical Reactor with Fixed Source

If the reactor is subcritical, then $\rho < 0$ and, according to the discussion above, all exponential time coefficients $\{s_j, j = 1 \dots I + 1\}$ are negative. It follows that for large t , the exponentials become vanishingly small, and we are left with only the “particular” part of the solution:

$$n(t) \xrightarrow{\text{large } t} -\frac{\Lambda}{\rho} S = -\frac{\ell_p/k}{(k-1)/k} S = \frac{\ell_p}{1-k} S. \quad (8.9)$$

That is, we have discovered that

In a **source-driven subcritical system**,
the neutron population eventually reaches a steady state .

The steady-state neutron density is proportional to:

- the source strength,
- the prompt-neutron lifetime, ℓ_p ,
- $1/(1-k)$.

The factor of $(1-k)$ in the denominator describes the phenomenon of

“subcritical multiplication.”

We see that because of subcritical multiplication, if k is arbitrarily close to 1, then the steady-state neutron population can be arbitrarily large, even with a “small” source and a subcritical reactor.

Note further that for k close to unity, ρ is close to zero and thus s_1 will be close to zero. This means that one exponential in the expression for n will be slow to decay away, which means it will take a **long time** for the neutron population to attain its steady-state level.

8.2.2 Supercritical Reactor with Fixed Source

This case is relatively simple. The particular solution is a constant, I of the exponentials decay with time, and one of the exponential terms has a positive s value. The increasing exponential eventually dominates all other terms, and

$$n(t) \xrightarrow{\text{large } t} A_1 e^{s_1 t}, \quad \text{with } s_1 > 0. \quad (8.10)$$

That is, we have discovered that

In a **source-driven supercritical system**,
the neutron population eventually increases exponentially with time .

8.2.3 Critical Reactor with Fixed Source

If the reactor is critical, then $\rho = 0$ and, according to the discussion above, all exponential time constants $\{s_j\}$ are negative except for one that is zero. **That is, one of the exponential functions is actually a constant function.** However, we stated earlier that a constant “particular” solution was also chosen. This means, that for this particular case, the particular solution needs to be something else. We try the next-simplest function we can imagine for the particular solution, namely one that is linear in t . We find then that the following particular solution does satisfy the equations with $\rho=0$:

$$n^{\text{particular}}(t) = tC_p , \quad (8.11)$$

and

$$C_i^{\text{particular}}(t) = \frac{\beta_i}{\lambda_i \Lambda} \left(t - \frac{1}{\lambda_i} \right) C_p , \quad (8.12)$$

where

$$C_p \equiv \frac{\Lambda}{\Lambda + \sum_{i=1}^I \frac{\beta_i}{\lambda_i}} S = \text{ a positive constant.} \quad (8.13)$$

Equation (8.11) says the “particular” portion of the solution for n is a linearly increasing function of time. What about the “homogeneous” portion? It has I exponentials that decay away in time, so they soon become unimportant. The other exponential is $\exp(0)$, because

$$\underline{s_1=0 \text{ when } \rho=0.}$$

Of course, $\exp(0)=1$. We therefore have

$$n(t) \xrightarrow{\text{large } t} A_1 + t \left[\frac{\Lambda}{\Lambda + \sum_{i=1}^I \frac{\beta_i}{\lambda_i}} S \right] . \quad (8.14)$$

Thus, we have now found that

In a **source-driven critical system**,
the neutron population eventually increases linearly with time .

8.2.4 Subcritical Reactor with No Fixed Source

In a source-free reactor the particular solution is **zero** for all cases. If the reactor is subcritical, then $\rho < 0$ and, as discussed above, all exponential time constants $\{s_j\}$ are negative. If we continue to let s_1 represent the algebraically largest s (the one closest to zero, in this case), then the other exponential terms decay more rapidly than the s_1 term, and we have

$$n(t) \xrightarrow{\text{large } t} A_1 e^{-|s_1|t}. \quad (8.15)$$

That is,

In a **source-free subcritical system**,
the neutron population eventually decreases exponentially with time .

Important: At the beginning of a transient (for example, immediately after rapid insertion of control rods) the population may drop **much faster than this single exponential**, because of the other rapidly-varying exponentials that are part of the solution. But the population eventually approaches this single decaying exponential with time constant s_1 .

Recall that in a subcritical system,

$$s_1 \in (-\lambda_{\min}, 0). \quad (8.16)$$

Also note that λ_{\min} is associated with the longest-living delayed-neutron precursors, which have half-lives of around **55 seconds**, so

$$\lambda_{\min} = \frac{\ln 2}{T_{1/2,\max}} \approx \frac{1}{80 \text{ s}}. \quad (8.17)$$

This means it takes the neutron population at least 80 seconds to decrease by a factor of e . This is not a very fast decrease! But remember:

the population may decrease
much faster than this in the early stages of a transient !

8.2.5 Supercritical Reactor with No Fixed Source

In this case one of the exponential terms has a positive s value while all other exponential terms are decaying in time. The increasing exponential eventually dominates the others, and

$$n(t) \xrightarrow{\text{large } t} A_1 e^{s_1 t}. \quad (8.18)$$

That is,

In a **source-free supercritical system**,
the neutron population eventually increases exponentially with time .

At the beginning of a transient the population may grow more slowly or more rapidly than this, or even decrease for a little while (!), but it eventually approaches this single increasing exponential.

8.2.6 Critical Reactor with No Fixed Source

If the reactor is critical, then $\rho=0$ and, as discussed above, all exponential time constants $\{s_j\}$ are negative except for one that is zero. It follows that

$$n(t) \xrightarrow{\text{large } t} A_1 . \quad (8.19)$$

That is,

In a **source-free critical system**,
the neutron population eventually reaches a steady state .

This is just what we would expect—the definition of a critical reactor is that it is **able to sustain** a steady chain reaction, without help from “extra” sources of neutrons. However, note that the population could change significantly while it is on its way to the steady value. That is,

even in a source-free critical reactor, the neutron population
may change for a while before it settles into steady state.

8.2.7 First Summary of PRKE Solutions

We have studied PRKE solutions for situations in which S (fixed source), ρ (reactivity), β (delayed-neutron fraction), β_i (type- i delayed-neutron fraction), and Λ (mean generation time) are all constants. This is a realistic case that does arise in nature! We studied six sub-cases, which are the combinations of subcritical, supercritical, and critical reactors with and without fixed sources.

Interestingly, we find that while the presence of delayed neutrons can cause interesting and complicated behavior early in a transient, eventually the neutron population behaves quite predictably:

1. If there is **no fixed source**, then the neutron population:

- (a) eventually grows exponentially if $\rho > 0$;
- (b) eventually falls exponentially if $\rho < 0$;
- (c) eventually stays steady if $\rho = 0$;

2. If there **is a fixed source**, then the neutron population:

- (a) eventually grows exponentially if $\rho > 0$;
- (b) eventually reaches a steady value if $\rho < 0$;
- (c) eventually grows linearly in time if $\rho = 0$;

If you compare this against what we found in Case 1 of the previous chapter, you see similarities, and you may be tempted to conclude that delayed neutrons are not important. Not so—**delayed neutrons are extremely important!** They have a dramatic effect on the time constants that appear in the exponentials mentioned above!

If there were no delayed neutrons, then it is easy to see (by looking at the inhour equation with $\beta=0$) that

$$s \xrightarrow{\beta=0} \frac{k-1}{\Lambda}. \quad (8.20)$$

In a typical commercial reactor, Λ is on the order of 0.0001 s. Now consider a slightly supercritical reactor, with $k=1.001$. How much would the neutron population (and thus the power) change in 1 second if there were no delayed neutrons?

$$n(t)|_{t=1 \text{ s}} = n(0) \exp \left[\frac{0.001}{0.0001 \text{ s}} (1 \text{ s}) \right] = n(0) e^{10} > 22,000 n. \quad (8.21)$$

Imagine how difficult it would be to control a reactor if its power could change this quickly!

Delayed neutrons slow this down dramatically. They make it relatively easy to control reactors that are slightly supercritical or subcritical. There will be exercises in which you will quantify this for yourself.

Included here are a series of important sketches that summarize what we found above. These are Figs. 8.2-8.6. **Study these!** Be able to **explain** them to a high-school student. Practice explaining them to each other. Work all of this into your intuition about the way neutrons behave in reactors.

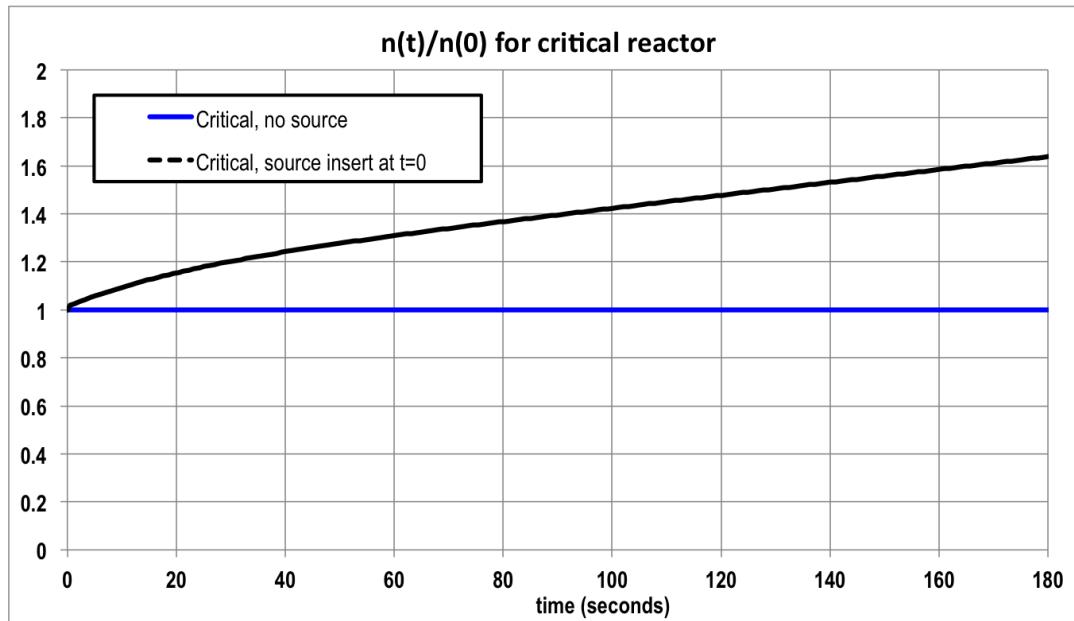


Figure 8.2: Behavior of neutron population in a **CRITICAL** reactor. The with-fixed-source figure assumes the source is inserted at $t=0$. Note the jump before the smooth increase that becomes linear in time.

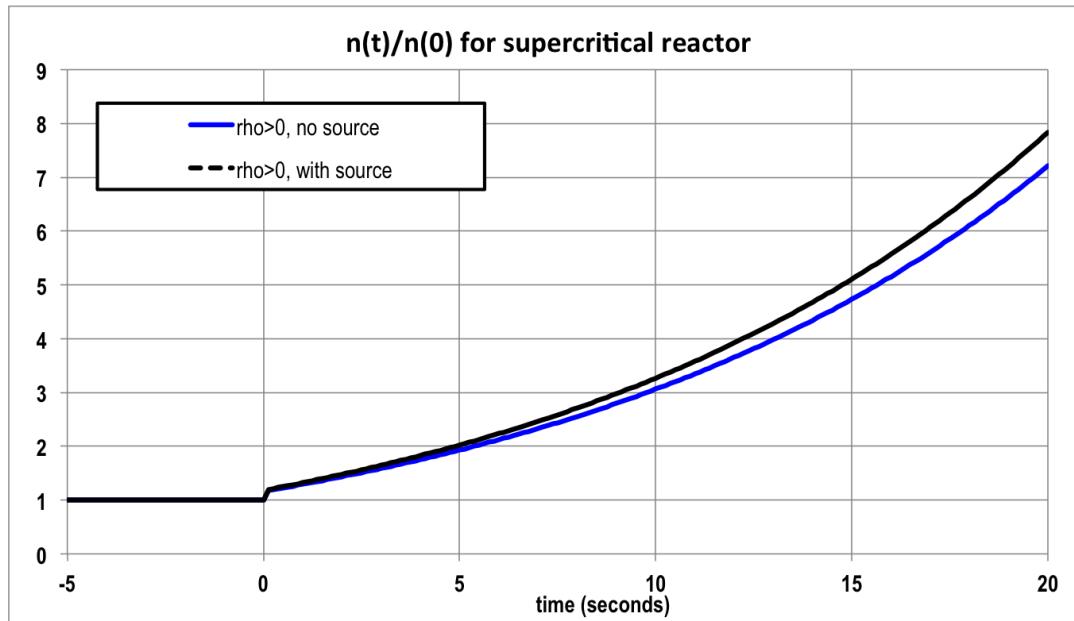


Figure 8.3: Behavior of neutron population in a **SUPERCRITICAL** reactor. The reactor is at steady state until time zero, when there is an insertion of positive reactivity (perhaps by withdrawal of control rods). Note the rapid jump in population prior to the slower exponential increase.

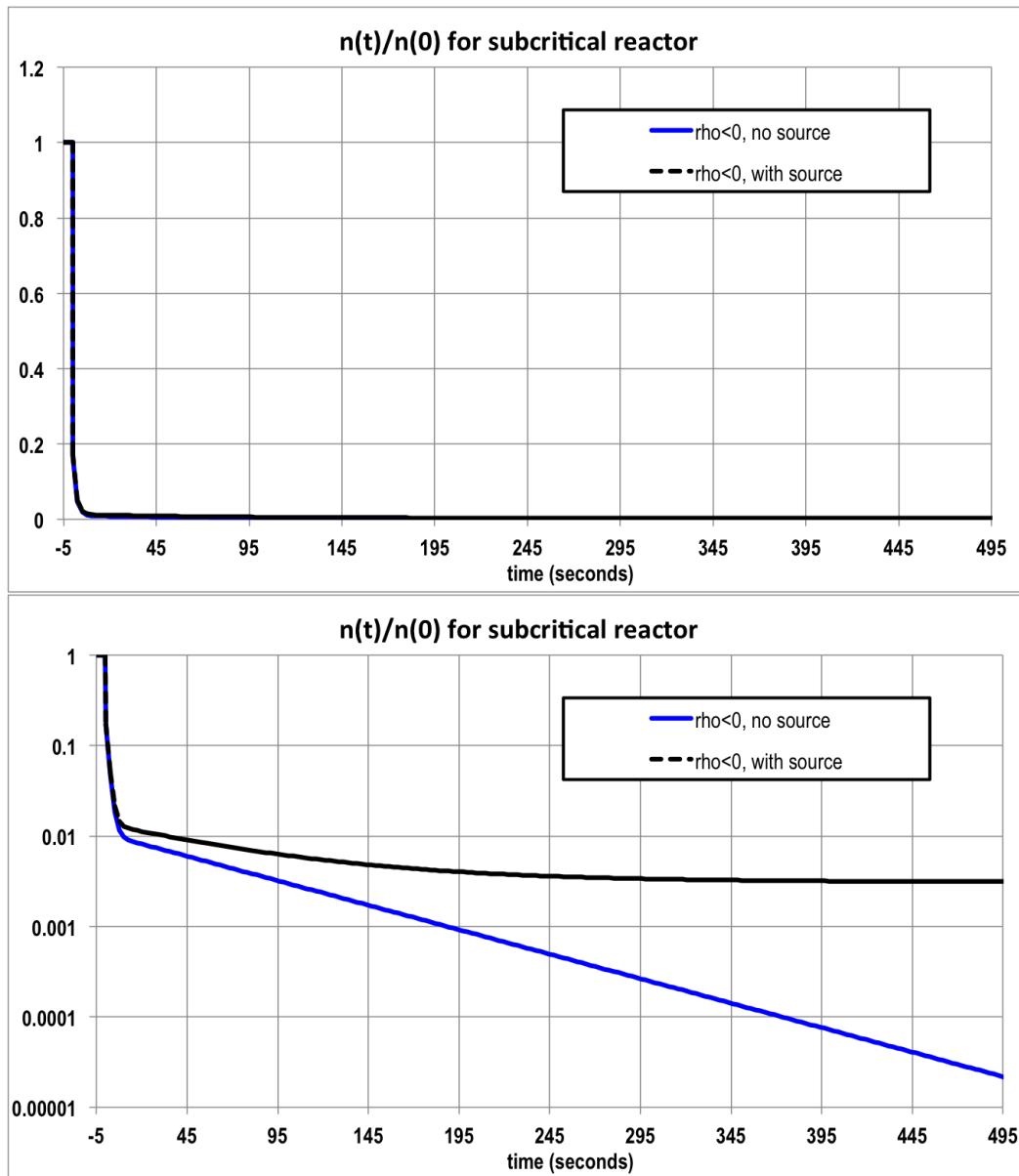


Figure 8.4: Behavior of neutron population in a **SUBCRITICAL** reactor (linear and semi-log). The reactor is at steady state until time zero, when there is an insertion of negative reactivity (perhaps by insertion of control rods). Note the rapid drop in population prior to the slower exponential decay to either zero (if no source) or a new steady value (if source is present).

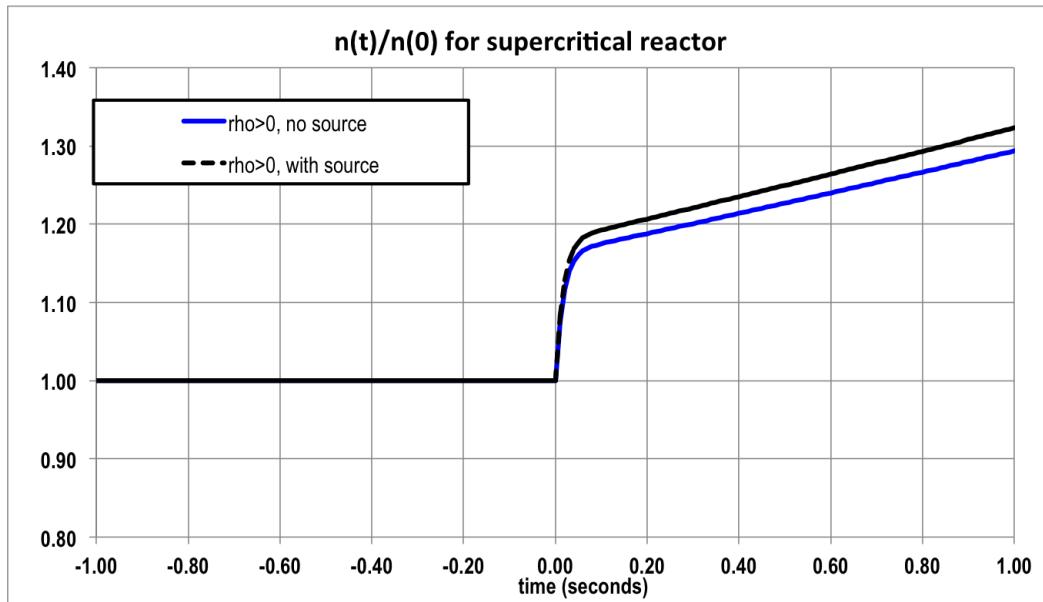


Figure 8.5: Behavior of neutron population in a **SUPERCRITICAL** reactor. The reactor is at steady state until time zero, when there is an insertion of positive reactivity (perhaps by withdrawal of control rods). This zoom-in view of early time more clearly shows the rapid jump in population prior to the slower exponential increase.

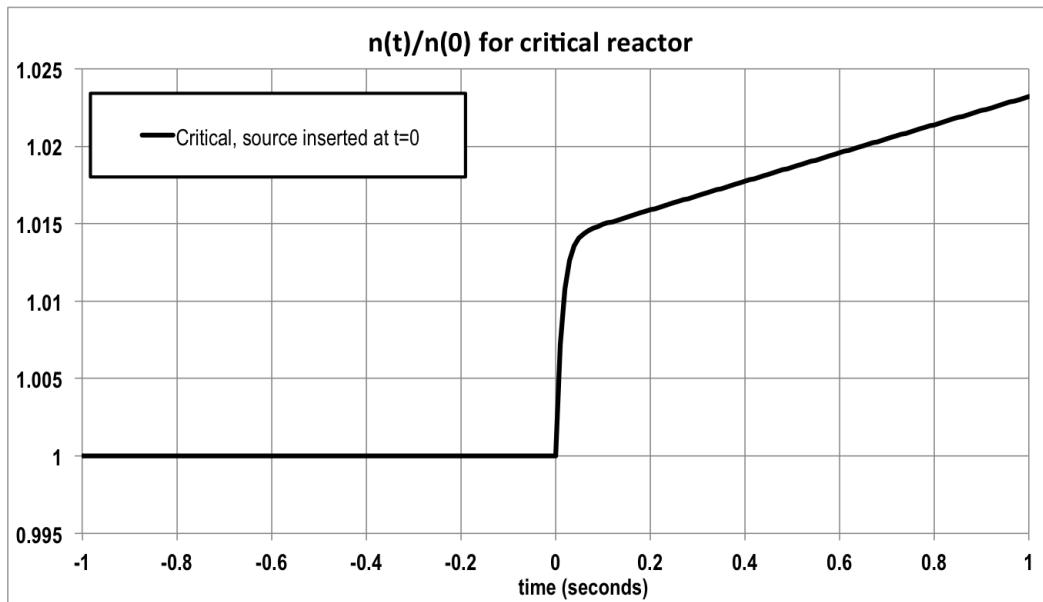


Figure 8.6: Behavior of neutron population in a **CRITICAL** reactor with source is inserted at $t=0$. This zoom-in view of early time more clearly shows the rapid jump before the smooth increase that becomes linear in time).

8.3 A simpler setting: only one delayed neutron precursor group

8.3.1 Analytical solution with no external source present

Let us solve the PRKEs, but with one simplification to make the Math easier: lump all delayed-neutron precursors in one group.

The simplified PRKEs are:

$$\frac{d}{dt} [n(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n(t) + \lambda C(t) , \quad (8.22)$$

$$\frac{d}{dt} C(t) = \frac{\beta}{\Lambda} n(t) - \lambda C(t) . \quad (8.23)$$

with initial conditions (reactor initially critical):

$$n(0) = n_{\text{init}} = \text{given} , \quad (8.24)$$

$$C(0) = C_{\text{init}} = \frac{\beta}{\lambda \Lambda} n_{\text{init}} = \text{given} . \quad (8.25)$$

Assume the following exponential forms:

$$n(t) = n_0 e^{st} \quad C(t) = C_0 e^{st}$$

Plug back in Eqs. (8.22)-(8.23) (and simplify throughout by e^{st}):

$$n_0 s = \left[\frac{\rho - \beta}{\Lambda} \right] n_0 + \lambda C_0 ,$$

$$C_0 s = \frac{\beta}{\Lambda} n_0 - \lambda C_0 .$$

We get from the second equation:

$$C_0 = \frac{1}{\lambda + s} \frac{\beta}{\Lambda} n_0 \quad (8.26)$$

Plug back in the first equation:

$$\nu_0 s = \left[\frac{\rho - \beta}{\Lambda} \right] \nu_0 + \lambda \frac{1}{\lambda + s} \frac{\beta}{\Lambda} \nu_0$$

After some algebra, we get the in-hour equation

$$\Lambda s + \frac{\beta s}{\lambda + s} = \rho \quad (8.27)$$

Eq (8.27) is also a simple quadratic equation in s and you can solve for the two roots s_1 and s_2 . As before,

s_1 is of the sign of ρ

s_2 is always negative

Now, we have our final form of the solution:

$$n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad (8.28)$$

$$C(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}. \quad (8.29)$$

When invoking Eq (8.26), there is a relationship between A_i and C_i so that

$$C_i = \frac{1}{\lambda + s_i} \frac{\beta}{\Lambda} A_i$$

so that the final form is

$$n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad (8.30)$$

$$C(t) = \frac{\beta}{\Lambda} \left(\frac{A_1}{\lambda + s_1} e^{s_1 t} + \frac{A_2}{\lambda + s_2} e^{s_2 t} \right). \quad (8.31)$$

Now, we only need to find two constants, A_1 and A_2 . We use initial conditions:

$$\begin{aligned} n(0) &= A_1 + A_2 = n_{\text{init}} \\ C(0) &= \frac{\beta}{\Lambda} \left(\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2} \right) = \frac{\beta}{\lambda \Lambda} n_{\text{init}} \end{aligned}$$

or, after some simplifications,

$$A_1 + A_2 = n_{\text{init}} \quad (8.32)$$

$$\left(\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2} \right) = \frac{n_{\text{init}}}{\lambda} \quad (8.33)$$

Solve for A_1 and A_2 . Done. Plot for various values of ρ .

8.3.2 Analytical solution with external source present

The PRKE are:

$$\frac{d}{dt} [n(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n(t) + \lambda C(t) \textcolor{red}{+ S} , \quad (8.34)$$

$$\frac{d}{dt} C(t) = \frac{\beta}{\Lambda} n(t) - \lambda C(t) . \quad (8.35)$$

with initial conditions (reactor initially subcritical):

$$n(0) = n_{\text{init}} = -\frac{\Lambda}{\rho_{\text{init}}} S , \quad (8.36)$$

$$C(0) = C_{\text{init}} = \frac{\beta}{\lambda \Lambda} n_{\text{init}} = -\frac{\beta}{\lambda \rho_{\text{init}}} S . \quad (8.37)$$

Note that ρ in Eq (8.34) is the final reactivity. So, you add or subtract some amount of reactivity to the initial reactivity ρ_{init} in order to get the final reactivity ρ .

8.3.2.1 Final reactivity is not zero

If the final reactivity is not zero, then all of the arguments of the exponentials in the homogeneous solution are non-zero. s_1 is of the sign of ρ . $s_2 < 0$. We just need to add to the homogeneous solution a particular solution. We seek it as a constant and finally obtain:

$$n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} - \frac{\Lambda}{\rho} S , \quad (8.38)$$

$$C(t) = \frac{\beta}{\Lambda} \left(\frac{A_1}{\lambda + s_1} e^{s_1 t} + \frac{A_2}{\lambda + s_2} e^{s_2 t} \right) - \frac{\beta}{\lambda \rho} S . \quad (8.39)$$

Apply initial conditions to get the constants (solve the following system):

$$A_1 + A_2 - \frac{\Lambda}{\rho} S = -\frac{\Lambda}{\rho_{\text{init}}} S , \quad (8.40)$$

$$\frac{\beta}{\Lambda} \left(\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2} \right) - \frac{\beta}{\lambda \rho} S = -\frac{\beta}{\lambda \rho_{\text{init}}} S . \quad (8.41)$$

or, after some simplifications, solve:

$$A_1 + A_2 = -\Lambda S \left(\frac{1}{\rho_{\text{init}}} - \frac{1}{\rho} \right) , \quad (8.42)$$

$$\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2} = -\frac{\Lambda S}{\lambda} \left(\frac{1}{\rho_{\text{init}}} - \frac{1}{\rho} \right) . \quad (8.43)$$

Solve for A_1 and A_2 . Done. Plot for various values of $\rho \neq 0$.

8.3.2.2 Final reactivity is zero

In this case, $s_1 = 0$ and $s_2 < 0$. Because $s_1 = 0$, the associated exponential function is actually a constant function. As a consequence, the particular cannot be a constant function but the next-simplest function (let's try a linear-in-time function again). **We are going to leave it at this ... NUEN 304 will take over.**

8.3.3 Numerical solution

Here's the coolest part. You can view the PRKEs in matrix notation as

$$\frac{dX}{dt} = A(t)X(t) + b(t)$$

where

$$X = \begin{bmatrix} n \\ c \end{bmatrix} \quad A(t) = \begin{bmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{bmatrix} \quad b(t) = \begin{bmatrix} S(t) \\ 0 \end{bmatrix}$$

Next, we select equally spaced times at which we wish to compute the solution: $t = \Delta t, 2\Delta t, \dots, N\Delta t$. Now, using Taylor series expansion, we can approximate the derivative of X over the time interval $[t_n, t_{n+1}]$ as:

$$\frac{dX}{dt} \approx \frac{X(t_{n+1}) - X(t_n)}{t_{n+1} - t_n} = \frac{X(t_{n+1}) - X(t_n)}{\Delta t}.$$

Note that if we assumed X to be a straight line in the interval $[t_n, t_{n+1}]$, then our expression for the derivative would be exact! If this were the case, we could say that the derivative has this value for any time in $[t_n, t_{n+1}]$. For example, we could decide whether this derivative applies at t_n , t_{n+1} or any other time in between:

$$\left. \frac{dX}{dt} \right|_{t_n} \text{ or } \left. \frac{dX}{dt} \right|_{t_{n+1}} \text{ or ...}$$

For reasons you will see in MATH, we pick t_{n+1} . Therefore, the other side of the PRKEs needs to be evaluated at that time too:

$$\left. \frac{dX}{dt} \right|_{t_{n+1}} = \frac{X(t_{n+1}) - X(t_n)}{\Delta t} = A(t_{n+1})X(t_{n+1}) + b(t_{n+1}).$$

We introduce the short-cut notation $X_n = X(t_n)$ and so forth. This yields:

$$\frac{X_{n+1} - X_n}{\Delta t} = A(t_{n+1})X_{n+1} + b(t_{n+1}).$$

Re-arranging:

$$(I - \Delta t A(t_{n+1})) X_{n+1} = X_n + \Delta b(t_{n+1}),$$

where I is the identity matrix. This is a small 2×2 linear system that one can solve for solution vector X_{n+1} , knowing vector X_n . We can do this many times. Well, not you, but a computer because solving 100,000 times such systems gets tiring quickly. Once you have X_{n+1} , you move to the next time step and compute X_{n+2} , knowing X_{n+1} and so forth. We jump start the process by computing X_1 , knowing X_0 , the initial condition values. Pretty cool.

In gory details, the 2×2 linear system matrix on the left-hand side is:

$$\begin{bmatrix} 1 - \Delta t \left(\frac{\rho(t_{n+1}) - \beta}{\Lambda} \right) & -\Delta t \lambda \\ -\Delta t \frac{\beta}{\Lambda} & 1 + \Delta t \lambda \end{bmatrix}$$

and the right-hand side vector is

$$\begin{bmatrix} n(t_n) + \Delta t S(t_{n+1}) \\ c(t_n) \end{bmatrix}$$

Or, all in all, solve this system for $n(t_{n+1})$ and $c(t_{n+1})$:

$$\begin{bmatrix} 1 - \Delta t \left(\frac{\rho(t_{n+1}) - \beta}{\Lambda} \right) & -\Delta t \lambda \\ -\Delta t \frac{\beta}{\Lambda} & 1 + \Delta t \lambda \end{bmatrix} \begin{bmatrix} n(t_{n+1}) \\ c(t_{n+1}) \end{bmatrix} = \begin{bmatrix} n(t_n) + \Delta t S(t_{n+1}) \\ c(t_n) \end{bmatrix}$$

Honestly, this is so much simpler than brainstorming about analytical solutions ... you should embrace computing ... engineers do a lot of this every single day ...

8.4 Feedback

A few words are in order about an important phenomenon that we have not yet described:

feedback

In a nuclear reactor, feedback happens when changes in the neutron population cause changes in the reactor properties. These changes in properties then alter the way that the neutron population behaves. Most of the feedback in a nuclear reactor is caused by

temperature changes

in the reactor material. Remember that temperature can be strongly influenced by the

heat-generation rate

which of course is determined by the

fission rate

(and other exothermic-reaction rates, to a lesser extent) which of course are determined by the

neutron population

Also recall that

cross sections

depend upon the

material temperature

and that

reactivity (or multiplication factor)

is a function of the cross sections. That is, we need to know the temperature to get the cross sections that allow us to calculate the neutron population, but we need to know the neutron population to calculate the fission source that drives the temperature.

Illustration:

1. Consider a reactor that is critical at a very low power level, such that fission is not causing the fuel to be noticeably hotter than its surroundings.
2. Now suppose control rods are pulled out so that the reactivity becomes positive (because the absorption cross section has been reduced). As we have seen, the neutron population will soon be on an exponential increase. Thus, the fission rate will increase exponentially with time.
3. Eventually this will cause the fuel to heat up significantly. This will change the cross sections in the fuel (remember—the cross sections are averaged over nucleus motion, and temperature determines the motion). In fact, in almost all reactors this will cause the absorption cross section to increase more than the fission cross section, which causes the multiplication factor to **decrease**.
4. As temperature increases further, the multiplication factor decreases further. This continues until k reaches 1 and the reactor settles into steady state.

The above is an example of

negative feedback .

This is what we want. Negative feedback gives us

stability.

which means that fluctuations in neutron population are damped out in time. Positive feedback, on the other hand, would amplify changes. Imagine a reactor in which increased temperature caused an increase in k (and thus ρ). In such a reactor, a small increase in neutron population would produce an increase in reactivity, which would cause the population to increase, which would further increase reactivity, etc. The reactor power would spiral upward until something (such as melting fuel) introduced negative feedback. Not good.

So we design our reactors with negative feedback.

8.5 Summary

Previously, we found that in every case that treated delayed neutrons we could cast our conservation equations, along with the equations for the delayed-neutron-precursor concentrations, into the form of **Point-Reactor Kinetics Equations** (PRKEs). Even when we did not know how to calculate the coefficients in these equations, we could study the characteristics of their solutions in all scenarios of interest.

In particular, if the coefficients in the PRKEs—namely ρ , the $\{\beta_i\}$, and Λ —are constant (which they often are for interesting problems), then we can write down equations for $n(t)$ and the $\{C_i(t)\}$. These equations contain exponentials that contain time constants (the $\{s_i\}$), which can be obtained from the “**INHOUR Equation**.”

There are six scenarios that illustrate a wide variety of possible reactor behaviors. We studied each of these under the assumption that the coefficients We found that

1. If there is no fixed source, then the neutron population:
 - (a) eventually grows exponentially if $\rho > 0$;
 - (b) eventually falls exponentially if $\rho < 0$;
 - (c) eventually stays steady if $\rho = 0$.
2. If there is a fixed source, then the neutron population:
 - (a) eventually grows exponentially if $\rho > 0$;
 - (b) eventually reaches a steady value if $\rho < 0$;
 - (c) eventually grows linearly in time if $\rho = 0$.

We discussed the important phenomenon of **feedback**. We did not go into detail, but we noted that we design our reactors with **negative feedback**, which is a stabilizing influence. Feedback changes the behavior of the neutron population with time because it changes the reactivity, ρ , which of course strongly influences the population’s behavior.

Part V

Neutron distribution in space

Chapter 9

Diffusion Theory

9.1 Introduction

We are by now well aware that a reactor's behavior depends on the gain and loss rates of neutrons in the reactor. We also know that some gain and loss is due to neutron-nucleus reactions, and that some is due to neutron leakage (meaning neutrons crossing surfaces). We have learned how to compute reaction rates using the neutron scalar flux (ϕ) and how to compute net leakage rates using the neutron net current density (\vec{J}). We have studied the implications of various neutron conservation equations, considering different settings and writing conservation statements for different neutron populations.

In the real-world setting of a finite heterogeneous reactor, when we applied conservation to the sub-population of neutrons in some sub-volume of the reactor, some interval in energy, and some cone of directions, we obtained the following transport equation for the “angular flux,” ψ :

$$\begin{aligned}
 & \frac{1}{v(E)} \frac{\partial}{\partial t} \psi(\vec{r}, E, \vec{\Omega}, t) + \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}, t) + \Sigma_t(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
 &= S_{\text{ext}}(\vec{r}, E, \vec{\Omega}, t) + \frac{\chi_p(E)}{4\pi} \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
 &+ \sum_{i=1}^{\# \text{ dnp types}} \frac{1}{4\pi} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) \\
 &+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E', \vec{\Omega}', t) . \tag{9.1}
 \end{aligned}$$

Equation (9.1) is the neutron **transport** equation. It is essentially exact, and it has a unique solution that tells us everything we need to know about the neutron distribution, but it is difficult to solve. Its solution is the

$$\text{angular flux, } \psi(\vec{r}, E, \vec{\Omega}, t).$$

Recall the relation between the angular flux and the scalar flux, and between the angular flux and the net current density:

$$\phi(\vec{r}, E, t) \equiv \iint_{4\pi} d\Omega \psi(\vec{r}, E, \vec{\Omega}, t), \quad (9.2)$$

$$\vec{J}(\vec{r}, E, t) \equiv \iint_{4\pi} d\Omega \vec{\Omega} \psi(\vec{r}, E, \vec{\Omega}, t). \quad (9.3)$$

Here $\vec{\Omega}$ is a unit vector pointing in the direction of particle travel, and $d\Omega$ is an element of solid angle. Recall that if neutrons were zipping around in mono-directional beams, the scalar flux would be the sum of the intensities from all the beams, and the net current density would be the sum of (beam intensity \times direction vector) from all the beams. The integrals above are the generalization of this result to the case of infinitely many infinitesimally small beams, with $\psi(\vec{r}, E, \vec{\Omega}, t)d\Omega$ representing the intensity the beam in direction $\vec{\Omega}$.

Thus, if we could solve the transport equation for the angular flux (ψ), we could then directly obtain the scalar flux (ϕ) and the net current density (\vec{J}). We could then calculate any reaction rate and any net leakage rate.

The neutron transport equation written here contains the delayed-neutron source, which depends on the delayed-neutron precursor concentrations. Recall that we have derived equations for these:

$$\frac{\partial}{\partial t} C_i(\vec{r}, t) = \int_0^\infty dE' \nu_{di}(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) - \lambda_i C_i(\vec{r}, t), \\ i = 1, \dots, \# \text{ of d.n.p. types.} \quad (9.4)$$

That is, we know how to deal with the precursor concentrations, at least in principle. So if we could solve the transport equation along with the precursor-concentration equations, we would know everything about the neutrons in our reactor.

Because the transport equation is relatively difficult to solve, for now we seek a simpler (but approximate) path toward figuring out what the neutrons are doing in a given reactor. Recall that before we derived the neutron transport equation, we derived a slightly less detailed conservation statement by looking at the neutron population in some spatial sub-volume and

some energy interval, but not distinguishing among neutrons traveling in different directions. This produced the following conservation equation, which is just as exact as Eq. (9.1) but does not express conservation in as much detail :

$$\begin{aligned}
 & \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(\vec{r}, E, t) + \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) + \Sigma_t(\vec{r}, E, t) \phi(\vec{r}, E, t) \\
 &= S_{\text{ext}}(\vec{r}, E, t) + \chi_p(E) \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
 &+ \sum_{i=1}^{\# \text{ dmp types}} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) + \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E, t) \phi(\vec{r}, E', t). \quad (9.5)
 \end{aligned}$$

It is perhaps not surprising that we obtain exactly this equation if we integrate the transport equation over all directions—that is, if we operate on Eq. (9.1) with $\iint_{4\pi} d\Omega$.

We recognized that while the conservation equation, Eq. (9.5), is simpler than the transport equation, it is not solvable, because it has four unknown functions: ϕ , J_x , J_y , and J_z . (Each of these is a function of x , y , z , and E .) Thus, to make use of this equation, we will need to augment it with equations that relate J_x , J_y , and J_z to other functions (such as ϕ) and parameters (such as cross sections).

In this chapter we do just that: we develop an approximate equation that relates the net current density (\vec{J}) to the scalar flux (ϕ). These approximate equations lead to

the neutron diffusion equation,

which is an approximate but very useful equation. After we derive the diffusion equation we will solve it for various problems that will help us understand how neutrons distribute themselves in reactors.

Remark:

Equations (9.1) and (9.5), our transport and conservation equations, do not account for neutron-emitting reactions other than fission and scattering. If other neutron-emitting reactions—such as $(n,2n)$, $(n,3n)$, $(n,n+p)$, $(n,n+\alpha)$, etc.—are important, then additional terms must be added to the right-hand sides of the equations to account for the neutrons gained via these reactions.

9.2 Fick's Law and the Neutron Diffusion Equation

In this section we obtain approximate equations that relate the neutron current density \vec{J} to the scalar flux ϕ . The full derivation is given in Appendix F. These equations can be written as a single vector equation, which takes the form:

$$\vec{J}(\vec{r}, E, t) \approx -D(\vec{r}, E, t) \vec{\nabla} \phi(\vec{r}, E, t), \quad (9.6)$$

This equation, known as Fick's Law, is

an approximation!! not exact!!

It would be more appropriately called Fick's Approximation (or perhaps "Fick's Lie" if it helps you remember that it is not really a law.) However, in many cases of practical interest, it is a reasonable approximation. For a particularly relevant example, this approximation works quite well for neutrons in

large homogeneous reactors.

It is intuitively appealing that the net flow of neutrons (described by \vec{J}) might be in the direction opposite the gradient of the scalar flux (i.e., $-\vec{\nabla}\phi$). If you imagine a box of neutrons, with a higher concentration in one corner than elsewhere, you can easily imagine a net flow from the higher-concentration area to the lower. See Fig. 9.1.

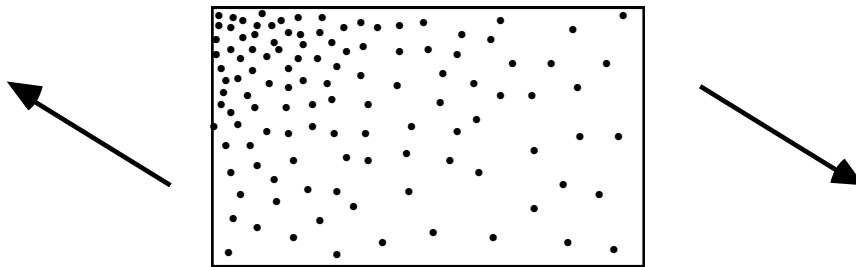


Figure 9.1: Illustration of neutrons flowing from high-concentration region to low-concentration region. Label $\vec{\nabla}\phi$ and \vec{J} .

Fick's law first originated in chemistry. (Fick was a chemist.) It states that chemical species migrate from high-concentration regions to low-concentration regions.

A similar relation for heat conduction is named after the French physicist Joseph Fourier:

$$q''(\vec{r}) = -k(\vec{r}) \vec{\nabla} T(\vec{r})$$

This says that heat flows by conduction from high-temperature regions to lower-temperature regions, at a rate proportional to the gradient of the temperature.

9.2.1 Mono-energetic case

Recall that Fick's law (**approximation!**) is derived in Appendix F.

$$\vec{J}(\vec{r}) \approx -D(\vec{r}) \vec{\nabla} \phi(\vec{r}) . \quad (9.7)$$

Fick's law ("approximation")

Here we have defined

$$D(\vec{r}) = \text{diffusion coefficient [units = cm]}$$

$$= \frac{1}{3\Sigma_{tr}(\vec{r})} . \quad (9.8)$$

where

$$\begin{aligned} \Sigma_{tr}(\vec{r}) &= \Sigma_{\text{transport}} \\ &= \Sigma_t(\vec{r}) - \langle \mu_0(\vec{r}) \rangle \Sigma_s(\vec{r}) , \end{aligned} \quad (9.9)$$

and where

$$\langle \mu_0(\vec{r}) \rangle = \text{"mu-zero-bar"} = \text{average scattering-angle cosine (lab frame)} . \quad (9.10)$$

Note: If you have no information about $\langle \mu_0 \rangle$, then make, and state, the approximation $\Sigma_{tr} \approx \Sigma_t$.

Equations (9.7) and (9.8) define the

diffusion approximation

for the one-speed or one-energy-group case. (The term “one-group” means we have grouped all of the neutrons together regardless of their energies. More on such things later.)

THE ONE-GROUP NEUTRON DIFFUSION EQUATION:

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \phi(\vec{r}, t) - \vec{\nabla} \cdot [D(\vec{r}, t) \vec{\nabla} \phi(\vec{r}, t)] + \Sigma_a(\vec{r}, t) \phi(\vec{r}, t) \\ = S_{\text{ext}}(\vec{r}, t) + \nu_p(\vec{r}, t) \Sigma_f(\vec{r}, t) \phi(\vec{r}, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_i(\vec{r}, t). \end{aligned} \quad (9.11)$$

9.2.2 Energy-dependent case

The generalization of the diffusion approximation to the energy-dependent case is simple to write down:

$$\vec{J}(\vec{r}, E, t) = -D(\vec{r}, E) \vec{\nabla} \phi(\vec{r}, E), \quad (9.12)$$

Energy-dependent diffusion approximation

Here we have defined

$$D(\vec{r}, E) = \text{energy-dependent diffusion coefficient [units = cm].} \quad (9.13)$$

In the one-speed (or one-group) case we had a simple definition for D in terms of Σ_{tr} , which itself had a simple definition in terms of Σ_t , Σ_s , and $\langle \mu_0 \rangle$. The best definition for $D(\vec{r}, E, t)$ is not so clear in the energy-dependent case. We can still use the following expression for the diffusion coefficient, D :

$$D(\vec{r}, E, t) = \frac{1}{3\Sigma_{tr}(\vec{r}, E, t)}. \quad (9.14)$$

However, the best definition for $\Sigma_{tr}(\vec{r}, E, t)$ is not so clear. This is a loose end that will hopefully be tied up later in your studies.

If you do not have any information about $\Sigma_{tr}(\vec{r}, E, t)$, the following can be a reasonable approximation:

$$\Sigma_{tr}(\vec{r}, E, t) \approx \Sigma_t(\vec{r}, E, t) - \langle \mu_0(\vec{r}, E, t) \rangle \Sigma_s(\vec{r}, E), \quad (9.15)$$

where

$$\begin{aligned} \langle \mu_0(\vec{r}, E, t) \rangle &= \text{the average lab-frame scattering-angle cosine} \\ &\text{for a particle with final energy } E . \end{aligned} \quad (9.16)$$

If you are not told anything about $\langle \mu_0(\vec{r}, E, t) \rangle$, just state that you are assuming isotropic scattering in the lab frame, which means $\langle \mu_0(\vec{r}, E, t) \rangle = 0$ and $\Sigma_{tr}(\vec{r}, E, t) = \Sigma_t(\vec{r}, E, t)$. But be aware that this may not be a very accurate approximation.

The balance equation, which we derived previously, is essentially exact. (Remember, if neutron-emitting reactions other than fission and scattering are important, then corresponding gain terms should be included to keep our balance equation “essentially exact.”) We repeat the balance equation here:

$$\begin{aligned} &\frac{1}{v(E)} \frac{\partial}{\partial t} \phi(\vec{r}, E, t) + \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) + \Sigma_t(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &= S_{\text{ext}}(\vec{r}, E, t) + \chi_p(E) \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\ &+ \sum_{i=1}^{\# \text{ dnp types}} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) + \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E, t) \phi(\vec{r}, E', t) . \end{aligned} \quad (9.17)$$

The true scalar flux ϕ and the true net current density \vec{J} in a reactor

always satisfy the conservation equation [Eq. (9.17)] !!

However, Fick’s Law (the “diffusion approximation”) is not exact. That is, in general the true scalar flux and net current density

$$\text{do not really satisfy } \vec{J}(\vec{r}, E, t) = -D(\vec{r}, E, t) \vec{\nabla} \phi(\vec{r}, E, t),$$

even for the best possible value of $D(\vec{r}, E, t)$. But even though it is not perfect, in many cases of practical interest this diffusion approximation is

close enough to the truth to give accurate, informative results.

We will use the diffusion approximation extensively in this course.

If we insert the diffusion approximation (9.12) into our balance equation (9.17), we obtain:

THE ENERGY-DEPENDENT NEUTRON DIFFUSION EQUATION:

$$\begin{aligned}
 & \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(\vec{r}, E, t) - \vec{\nabla} \cdot \left[D(\vec{r}, E, t) \vec{\nabla} \phi(\vec{r}, E, t) \right] + \Sigma_t(\vec{r}, E, t) \phi(\vec{r}, E, t) \\
 &= S_{\text{ext}}(\vec{r}, E, t) + \chi_p(E) \int_0^{\infty} dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
 &+ \sum_{i=1}^{\# \text{ dnp types}} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) + \int_0^{\infty} dE' \Sigma_s(\vec{r}, E' \rightarrow E, t) \phi(\vec{r}, E', t) . \tag{9.18}
 \end{aligned}$$

It is not exact, but for most of this course, it is the model that we will use to describe the behavior of neutrons in reactors.

The diffusion equation is not exact, but it is reasonably accurate in many cases, including the case of a large, homogeneous reactor. We will use the diffusion equation extensively to develop an understanding of how neutrons distribute themselves spatially in reactors.

9.2.3 Partial Current Densities

Fick's law (approximation!) is a vector equation, which is the same thing as three scalar equations—one for each of the three vector components:

$$J_x(x, y, z) = -D(x, y, z) \frac{\partial \phi(x, y, z)}{\partial x}, \quad (9.19)$$

$$J_y(x, y, z) = -D(x, y, z) \frac{\partial \phi(x, y, z)}{\partial y}, \quad (9.20)$$

$$J_z(x, y, z) = -D(x, y, z) \frac{\partial \phi(x, y, z)}{\partial z}. \quad (9.21)$$

We can express each component of the current in terms of “**partial currents**,” which describe one-way flow rates. For example:

$$(n/s \text{ crossing upward} - n/s \text{ crossing downward}) = (J_z^+ - J_z^-) dx dy$$

The scalar functions J_z^+ and J_z^- are called “partial currents” or “one-way currents” in the $+$ and $-z$ directions. Partial currents can be used to describe one-way (not net) flow rates across surfaces. Recall that when using Fick's law, the net or one-way flow rates of neutrons are approximate.

Next, we give expressions for these **one-way flow rates**. The functions that contain this information are called

partial current densities.

For example, given the diffusion approximation, the partial current density in the $-z$ direction is

$$J_z^-(\vec{r}, E, t) = \frac{\phi(\vec{r}, E, t)}{4} + \frac{D(\vec{r}, E, t)}{2} \frac{\partial \phi(\vec{r}, E, t)}{\partial z}. \quad \text{in the diffusion approximation} \quad (9.22)$$

There are similar expressions for other partial current densities. All J, D , and ϕ functions below have arguments (\vec{r}, E, t) , but we omit the arguments for simplicity:

$$J_z^+ = \frac{\phi}{4} - \frac{D}{2} \frac{\partial \phi}{\partial z}, \quad \text{in the diffusion approximation} \quad (9.23)$$

$$J_x^+ = \frac{\phi}{4} - \frac{D}{2} \frac{\partial \phi}{\partial x}, \quad \text{in the diffusion approximation} \quad (9.24)$$

$$J_x^- = \frac{\phi}{4} + \frac{D}{2} \frac{\partial \phi}{\partial x}, \quad \text{in the diffusion approximation} \quad (9.25)$$

$$J_y^+ = \frac{\phi}{4} - \frac{D}{2} \frac{\partial \phi}{\partial y}, \quad \text{in the diffusion approximation} \quad (9.26)$$

$$J_y^- = \frac{\phi}{4} + \frac{D}{2} \frac{\partial \phi}{\partial y}. \quad \text{in the diffusion approximation} \quad (9.27)$$

There is nothing magical about our choice of coordinate axes or the unit vectors that point along those axes. Partial current densities can be defined relative to any direction:

$$J_n^\pm(\vec{r}, E, t) = \frac{\phi(\vec{r}, E, t)}{4} \mp \frac{D(\vec{r}, E, t)}{2} \vec{e}_n \cdot \vec{\nabla} \phi(\vec{r}, E, t). \quad \text{in the diffusion approximation} \quad (9.28)$$

Partial current densities can tell us the one-way (**not just net!**) flow rate from one side of a surface to the other:

$$\iint_{\mathbf{S}} d^2 r J_n^+(\vec{r}) = \text{rate at which neutrons cross the surface } \mathbf{S} \\ \text{from the } -\vec{e}_n \text{ side to the } +\vec{e}_n \text{ side.} \quad (9.29)$$

A similar expression gives the one-way (not net!) flow rate from in the other direction:

$$\iint_{\mathbf{S}} d^2 r J_n^-(\vec{r}) = \text{rate at which neutrons cross the surface } \mathbf{S} \\ \text{from the } +\vec{e}_n \text{ side to the } -\vec{e}_n \text{ side.} \quad (9.30)$$

Remember, these are **one-way, not net** flow rates. If you want the net flow rate, you need to subtract the two one-way flow rates:

$$\iint_{\mathbf{S}} d^2 r [J_n^+(\vec{r}) - J_n^-(\vec{r})] = \text{net rate at which neutrons cross the surface } \mathbf{S} \\ \text{from the } -\vec{e}_n \text{ side to the } +\vec{e}_n \text{ side.} \quad (9.31)$$

Let us insert the diffusion approximation for the two partial current densities into this expression:

$$\iint_{\mathbf{S}} d^2 r \left[\left(\frac{\phi}{4} - \frac{D}{2} \vec{e}_n \cdot \vec{\nabla} \phi \right) - \left(\frac{\phi}{4} + \frac{D}{2} \vec{e}_n \cdot \vec{\nabla} \phi \right) \right] \\ = \iint_{\mathbf{S}} d^2 r \vec{e}_n \cdot \underbrace{[-D \vec{\nabla} \phi]}_{\text{diffusion approx. for } \vec{J}}. \quad (9.32)$$

So we see that the diffusion approximation for the partial and net current densities satisfies the identity

$$\vec{e}_n \cdot \vec{J} = J_n^+ - J_n^- , \quad (9.33)$$

as it should.

Note:

Every **partial** current density is a **NON-NEGATIVE SCALAR** function.
Net current density is a **VECTOR** function.

9.2.4 Two-Group Case

Suppose we divide all the neutrons in a reactor into two energy “groups” :

group 1 = “**fast**” ;

group 2 = “**slow**.”

What would the conservation equation look like for each group?

Group 1:

$$\begin{aligned}
 & \frac{1}{v_1} \frac{\partial \phi_1(\vec{r}, t)}{\partial t} + \underbrace{\vec{\nabla} \cdot \vec{J}_1(\vec{r}, t)}_{\text{fast net O.L.}} + \left[\underbrace{\Sigma_{a,1}(\vec{r}, t)}_{\text{fast abs.}} + \underbrace{\Sigma_{s,1 \rightarrow 2}(\vec{r}, t)}_{\text{fast to thermal scat.}} \right] \phi_1(\vec{r}, t) \\
 &= \underbrace{\chi_{p,1}(\vec{r}, t)}_{\text{frac. born. in fast grp}} \left[\underbrace{\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t)}_{\text{n/s from fisns caused by fast n's}} + \underbrace{\nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)}_{\text{n/s from fisns caused by thermal n's}} \right] \\
 &+ \underbrace{S_{ext,1}(\vec{r}, t)}_{\text{fast-n birth, fixed src}} + \underbrace{S_{d,1}(\vec{r}, t)}_{\text{fast-n delayed birth}} + \underbrace{\Sigma_{s,2 \rightarrow 1}(\vec{r}, t) \phi_2(\vec{r}, t)}_{\text{thermal to fast scatter}}, \tag{9.34}
 \end{aligned}$$

Group 2:

$$\begin{aligned}
 & \frac{1}{v_2} \frac{\partial \phi_2(\vec{r}, t)}{\partial t} + \underbrace{\vec{\nabla} \cdot \vec{J}_2(\vec{r}, t)}_{\text{thermal net O.L.}} + \left[\underbrace{\Sigma_{a,2}(\vec{r}, t)}_{\text{thermal abs.}} + \underbrace{\Sigma_{s,2 \rightarrow 1}(\vec{r}, t)}_{\text{thermal to fast scat.}} \right] \phi_2(\vec{r}, t) \\
 &= \underbrace{\chi_{p,2}(\vec{r}, t)}_{\text{frac. born. in thermal grp}} \left[\underbrace{\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t)}_{\text{n/s from fisns caused by fast n's}} + \underbrace{\nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)}_{\text{n/s from fisns caused by thermal n's}} \right] \\
 &+ \underbrace{S_{ext,2}(\vec{r}, t)}_{\text{thermal-n birth, fixed src}} + \underbrace{S_{d,2}(\vec{r}, t)}_{\text{thermal-n delayed birth}} + \underbrace{\Sigma_{s,1 \rightarrow 2}(\vec{r}, t) \phi_1(\vec{r}, t)}_{\text{fast to thermal scatter}}. \tag{9.35}
 \end{aligned}$$

Here we have defined “group-averaged” absorption cross sections ($\Sigma_{a,1}$ and $\Sigma_{a,2}$), group-averaged speeds (v_1 and v_2), group-to-group scattering cross sections ($\Sigma_{s,1 \rightarrow 2}$ and $\Sigma_{s,2 \rightarrow 1}$), and the fraction of fission neutrons born in each group (χ_1 and χ_2). We have also defined a short-hand notation for the delayed-neutron source ($S_{d,g}$).

Suppose we make the diffusion approximation for the net current density in each group:

$$\vec{J}_g(\vec{r}, t) = -D_g(\vec{r}, t) \vec{\nabla} \phi_g(\vec{r}, t), \text{ for } g = 1, 2. \tag{9.36}$$

Then what would the resulting 2-group diffusion equations look like? Like this:

Group 1:

$$\begin{aligned} \frac{1}{v_1} \frac{\partial \phi_1(\vec{r}, t)}{\partial t} - \vec{\nabla} \cdot [D_1(\vec{r}, t) \vec{\nabla} \phi_1(\vec{r}, t)] + [\Sigma_{a,1}(\vec{r}, t) + \Sigma_{s,1 \rightarrow 2}(\vec{r}, t)] \phi_1(\vec{r}, t) \\ = \chi_{p,1}(\vec{r}, t) [\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)] \\ + S_{ext,1}(\vec{r}, t) + S_{d,1}(\vec{r}, t) + \Sigma_{s,2 \rightarrow 1}(\vec{r}, t) \phi_2(\vec{r}, t), \end{aligned} \quad (9.37)$$

Group 2:

$$\begin{aligned} \frac{1}{v_2} \frac{\partial \phi_2(\vec{r}, t)}{\partial t} - \vec{\nabla} \cdot [D_2(\vec{r}, t) \vec{\nabla} \phi_2(\vec{r}, t)] + [\Sigma_{a,2}(\vec{r}, t) + \Sigma_{s,2 \rightarrow 1}(\vec{r}, t)] \phi_2(\vec{r}, t) \\ = \chi_{p,2}(\vec{r}, t) [\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)] \\ + S_{ext,2}(\vec{r}, t) + S_{d,2}(\vec{r}, t) + \Sigma_{s,1 \rightarrow 2}(\vec{r}, t) \phi_1(\vec{r}, t). \end{aligned} \quad (9.38)$$

We will worry later about how to obtain the group-averaged cross-sections. For now, we simply note that:

The 2-group diffusion equations are widely used in reactor analysis

9.3 When Can You Trust Diffusion Theory?

Remember: Diffusion theory is based on the diffusion **approximation**. It is not perfect. However, we have argued that it is reasonably accurate in many practical problems. But how do you know when to trust it and when not to?

You can trust diffusion theory to give an accurate solution if all of the following are true:

1. The region of interest is several mean-free paths from boundaries or material interfaces.
2. The region of interest is several mean-free paths from strong localized sources.
3. Scattering and/or fission are the dominant reactions in the region of interest.

In particular: in a large, homogeneous nuclear reactor, diffusion theory is pretty good!

9.4 Math: the gradient ($\vec{\nabla}$), divergence ($\vec{\nabla} \cdot$) and Laplacian (∇^2) Operators

In various coordinate systems the derivative operators appearing in the diffusion equation take different forms, as shown in Table 9.1, where we use f to denote any scalar function (such as scalar flux) and \vec{u} to denote any vector function (such as net current density).

Table 9.1: Gradient, Divergence, and Laplacian operators, various coordinate systems.

Coord. Sys.	$\vec{\nabla}f$	$\vec{\nabla} \cdot \vec{u}$	$\vec{\nabla} \cdot \vec{\nabla}f = \nabla^2 f$
3D Cartesian	$\frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$	$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
1D Cartesian	$\frac{df}{dx}\vec{e}_x$	$\frac{du_x}{dx}$	$\frac{d^2 f}{dx^2}$
1D spherical	$\frac{df}{dr}\vec{e}_r$	$\frac{1}{r^2} \frac{d(r^2 u_r)}{dr}$	$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df}{dr} \right)$
1D cylindrical	$\frac{df}{dr}\vec{e}_r$	$\frac{1}{r} \frac{d(r u_r)}{dr}$	$\frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right)$
2D cylindrical	$\frac{\partial f}{\partial r}\vec{e}_r + \frac{\partial f}{\partial z}\vec{e}_z$	$\frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{\partial u_z}{\partial z}\vec{e}_z$	$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{\partial^2 f}{\partial z^2}$

9.5 Boundary Conditions and Interface Conditions for the Neutron Diffusion Equation

The steady-state diffusion equation is a

second-order partial differential equation.

We know that differential equations are satisfied by

families of solutions.

To determine which solution from the family is correct for a given problem, we must apply

boundary conditions.

With our steady-state diffusion equation, we must specify at each point on the problem boundary either:

1. $\phi(\vec{r})$, or
2. the normal derivative of ϕ , $\vec{e}_n(\vec{r}) \cdot \vec{\nabla} \phi(\vec{r})$, or
3. a linear combination of $\phi(\vec{r})$ and $\vec{e}_n(\vec{r}) \cdot \vec{\nabla} \phi(\vec{r})$.

Here $\vec{e}_n(\vec{r})$ is the outward unit normal at position \vec{r} on the problem boundary. You will sometimes see $\vec{e}_n \cdot \vec{\nabla} \phi$ written as $\partial \phi / \partial n$. The two expressions mean the same thing—the derivative of ϕ in the outward-normal direction.

9.5.1 Vacuum boundary condition

If part of our problem domain is surrounded by something we are pretending is a perfect vacuum, we usually will use an “extrapolated boundary” condition. That is, we assume that the scalar flux is decreasing as we move toward such a boundary, and that it is decreasing at such a rate that, if we extrapolated it, it would extrapolate to zero at “ d ” cm from the boundary:

1. General extrapolated boundary condition (BC): $\phi(\vec{r}_s + d \vec{e}_n) = 0$.
2. 1D slab extrapolated BC at right face ($x = a/2$): $\phi(a/2 + d) = 0$.
3. 1D slab extrapolated BC at left face ($x = -a/2$): $\phi(-a/2 - d) = 0$.
4. 1D sphere extrapolated BC at outer surface ($r = R$): $\phi(R + d) = 0$

Of course, the scalar flux does not really go to zero out there. We are just saying that it is decreasing fast enough that it would hit zero if we extrapolated it. See Fig. 9.2.

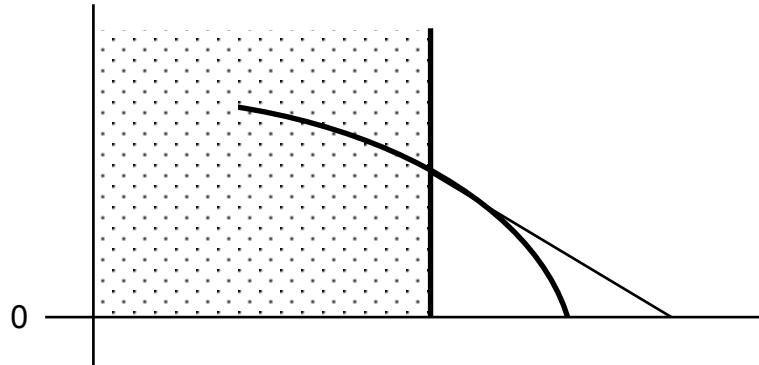


Figure 9.2: Illustration of scalar-flux extrapolations at a vacuum boundary. d d_{linear}

When we say that the scalar-flux **function** goes to zero at some distance outside the boundary, we are

extrapolating with curvature.

The alternative is to extrapolate **linearly**, using the scalar-flux slope at the boundary. Both kinds of extrapolations are shown in Fig. 9.2. You can see that the two kinds of extrapolations can describe the same interior solution if their extrapolation distances are chosen just right. If the solution is curving downward, the linear extrapolation distance is larger than the with-curvature distance, for a given interior solution. This is reversed if the solution is concave upward.

When we are performing analytic calculations, extrapolation with curvature

simplifies the algebra and the form of the solution.

This is what we will usually do in this course.

How do we know what the distance should be? There are several ways to figure this out. For one, rigorous analyses using the transport equation tell us that the best **linear extrapolation distance** to use, at least in certain simple problems, is

$$d_{\text{linear}} = (0.71044\dots) \times 3D = (2.1313\dots)D. \quad (9.39)$$

Using the diffusion approximation for partial current density we can also derive a linear extrapolation distance of $2D$. Because the boundary is a vacuum, we know that there are no incoming neutrons, right? So let's write that the diffusion approximation's expression for the incoming partial current density on the boundary equals zero:

$$J_n^-(\vec{r}_s) = \frac{\phi(\vec{r}_s)}{4} + \frac{D(\vec{r}_s)}{2} \vec{e}_n \cdot \vec{\nabla} \phi \Big|_{\vec{r}_s} = 0 \text{ (because of vacuum BC)} \quad (9.40)$$

Multiply through by 4:

$$[\phi(\vec{r}_s)] + [2D] \times \left[\frac{\partial \phi}{\partial n} \right]_{\vec{r}_s} = 0 \quad (9.41)$$

or

$$[\text{value}] + [\text{distance}] \times [\text{slope}] = 0.$$

This equation says that if we extrapolate the scalar-flux function **linearly** outside the boundary, the function would hit zero at a distance $2D$ outside the boundary. That is, the diffusion approximation to the incoming partial current density (which must =0 at a vacuum boundary) produces a linear extrapolation distance of $d_{\text{linear}} = 2D$.

Notice that the rigorous transport-theory result from Eq. (9.39) is $d_{\text{linear}} = 2.13 \times D$. Further observe that for concave-downward solutions, the extrapolation-with-curvature distance should be smaller than the linear extrapolation distance. Thus, it appears that $2D$ is a reasonably good extrapolation-with-curvature distance if the solution is concave downward. $2D$ might be smaller than the optimal distance for problems with concave-upward solutions, but for simplicity we will usually use the value of $2D$ for our approximate calculations.

So for our purposes, the extrapolation distance $d(\vec{r}_s)$, for extrapolation with curvature, will usually be taken to be

$$d(\vec{r}_s) = 2D(\vec{r}_s) \quad (9.42)$$

at each point \vec{r}_s on the surface, where $D(\vec{r}_s)$ is the diffusion coefficient inside the problem domain at the boundary position \vec{r}_s .

9.5.2 Reflecting (or “mirror”) boundary.

There is no net flow of neutrons across a reflecting boundary. No net flow means

$$\vec{e}_n \cdot \vec{J} = 0 \quad (9.43)$$

In the diffusion approximation, this means

$$\vec{e}_n \cdot D \vec{\nabla} \phi = 0 \Rightarrow \vec{e}_n \cdot \vec{\nabla} \phi = 0. \quad (9.44)$$

This is the diffusion-theory reflecting boundary condition. Some specific examples:

1. 1D slab reflecting BC at right face ($x = a/2$, e.g.): $\frac{d\phi}{dx} \Big|_{x=a/2} = 0$.
2. 1D slab reflecting BC at left face ($x = 0$, e.g.): $\frac{d\phi}{dx} \Big|_{x=0} = 0$.
3. 3D brick-shaped reactor with reflecting BC at $x = 0$: $\frac{\partial \phi}{\partial x} \Big|_{x=0} = 0$, all y and z .

9.5.3 Interface between different materials.

At the interface between materials we must have continuity of two quantities:

1. neutron density (\Rightarrow scalar flux too)
2. net flow across interface (\Rightarrow normal component of net current density)

This translates to:

1. General interface at \vec{r}_i , with normal $\vec{e}_{n,i}$:

- (a) ϕ is continuous;
- (b) $\vec{e}_{n,i} \cdot D \vec{\nabla} \phi$ is continuous;

2. 1D slab interface at x_i :

- (a) $\phi(x_i + \epsilon) = \phi(x_i - \epsilon)$
- (b) $[D \frac{d\phi}{dx}]_{x=x_i+\epsilon} = [D \frac{d\phi}{dx}]_{x=x_i-\epsilon}$,

where ϵ is vanishingly small. Note that because the diffusion coefficient is not necessarily identical on both sides of the interface, the slope of the scalar flux is

not necessarily continuous.

3. 3D interface at y_i :

- (a) $\phi(x, y_i + \epsilon, z) = \phi(x, y_i - \epsilon, z)$, all x, z
- (b) $[D \frac{\partial \phi}{\partial y}]_{y=y_i+\epsilon} = [D \frac{\partial \phi}{\partial y}]_{y=y_i-\epsilon}$, all x, z .

where ϵ is vanishingly small.

9.5.4 Other helpful truths

When properly specified, the boundary conditions of a diffusion problem uniquely determine the solution. However, sometimes we can use common-sense shortcuts to save us a bit of algebra. That is, we know that the scalar flux:

1. is real and non-negative
everywhere that the diffusion equation is valid;
2. is finite everywhere that the diffusion equation is valid,
except possibly at singular points of a source distribution.

Recognizing these facts often allows us to eliminate certain potential solutions before we even apply the boundary conditions.

Note that we stated “everywhere the diffusion equation is valid” in the above truths. For example, we might find in some problem that $\phi(x) = \exp(x/L)$ for x in the interval $(0, 10)$. This function goes to infinity as x goes to infinity, but that doesn’t matter, because we want to use it only for $x \in (0, 10)$.

9.6 Steady-State One-Group Diffusion Solutions in Homogeneous Media

In a previous section we wrote down the **two-group** diffusion equations, in which neutrons were rather crudely divided according to their energies into only two groups: a fast group and a slow group. Here we lump them all even more crudely, into a **single** group, and assume that we are given cross sections and diffusion coefficients that have been appropriately averaged over all neutron energies (in addition to being averaged over all nucleus velocities).

In this section we will use the diffusion equation to compute the scalar flux (and the net current density) as a function of position \vec{r} for several simple geometrical configurations. In each problem we shall assume that

material properties are constant.

(That's what **homogeneous media** means.) We will also assume steady state—**no time dependence**. Finally, for now we shall consider only materials with $k_\infty \leq 1$. In an infinite homogeneous medium, with $\nu\Sigma_f$ and Σ_a properly averaged over nucleus velocities and neutron energies, recall that the multiplication factor is $\nu\Sigma_f/\Sigma_a$. That is,

$$k_\infty \leq 1 \Rightarrow \Sigma_a - \nu\Sigma_f \geq 0.$$

We define the one-group “net removal” cross section as follows:

$$\Sigma_r \equiv \Sigma_a - \nu\Sigma_f = \Sigma_a \times (1 - k_\infty) \quad (9.45)$$

The restriction $k_\infty \leq 1$ means $\Sigma_r \geq 0$. We impose this restriction for simplicity in this initial introduction to diffusion solutions. When $k_\infty > 1$ the general form of the solution changes, and no steady-state solution exists unless there is sufficient leakage to balance the excess collisional gain. For now we shall avoid these details and focus on basic properties of diffusion solutions in materials with $k_\infty \leq 1$.

Note: If there is no fissile or fissionable material in the problem, then colorhide $\Sigma_r = \Sigma_a$.

Under these assumptions our diffusion equation is (for **steady-state, one-group, homogeneous material**)

$$-D \nabla^2 \phi(\vec{r}) + \Sigma_r \phi(\vec{r}) = S_{ext}(\vec{r}) . \quad (9.46)$$

Note that we have taken D outside the divergence operator and recognized that $\text{div}(\text{grad}(f))$ is the Laplacian on f :

$$\vec{\nabla} \cdot (D \vec{\nabla} \phi) \xrightarrow{D=\text{constant}} D \vec{\nabla} \cdot \vec{\nabla} \phi = D \nabla^2 \phi . \quad (9.47)$$

We often divide through by D and rewrite Eq. (9.46) as

$$-\nabla^2 \phi(\vec{r}) + \frac{1}{L^2} \phi(\vec{r}) = \frac{S_{ext}(\vec{r})}{D} . \quad (9.48)$$

where we have introduced the **diffusion length**, L :

$$L \equiv \sqrt{\frac{D}{\Sigma_a}} \equiv \text{diffusion length [units = cm]} . \quad (9.49)$$

and a related function, \tilde{L}^2 :

$$\tilde{L} \equiv \sqrt{\frac{D}{\Sigma_r}} \equiv \sqrt{\frac{D}{\Sigma_a(1 - k_\infty)}} \quad [\text{units = cm}] . \quad (9.50)$$

A note regarding syntax: some people prefer to use the symbol Δ for ∇^2 .

9.6.1 Steady-State One-Group Diffusion Equation: Various Coordinate systems

Different transport and diffusion problems have different symmetries and other characteristics that lend themselves to description in different coordinate systems. The main difference this makes in the transport equation is in the form of the $\vec{\Omega} \cdot \vec{\nabla}\psi$ term. The main difference in the diffusion equation is the form of the $\vec{\nabla} \cdot D\vec{\nabla}\phi$ term. Here we catalog some interesting examples.

9.6.1.1 Cartesian

Given Cartesian coordinates with a solution that depends on all three variables, we have the following for our steady-state one-group diffusion equation:

$$-\frac{\partial}{\partial x} \left(D(x, y, z) \frac{\partial \phi(x, y, z)}{\partial x} \right) - \frac{\partial}{\partial y} \left(D(x, y, z) \frac{\partial \phi(x, y, z)}{\partial y} \right) - \frac{\partial}{\partial z} \left(D(x, y, z) \frac{\partial \phi(x, y, z)}{\partial z} \right) + \Sigma_r(x, y, z) \phi(x, y, z) = S(x, y, z), \quad \text{3D Cart., heterogeneous} \quad (9.51)$$

$$- D \left[\frac{\partial^2 \phi(x, y, z)}{\partial x^2} + \frac{\partial^2 \phi(x, y, z)}{\partial y^2} + \frac{\partial^2 \phi(x, y, z)}{\partial z^2} \right] + \Sigma_r \phi(x, y, z) = S(x, y, z), \quad \text{3D Cart., homogeneous} \quad (9.52)$$

Given Cartesian coordinates with a solution that depends on only x and y (infinite and uniform in z), we have the following for our steady-state one-group diffusion equation:

$$\begin{aligned} & -\frac{\partial}{\partial x} \left(D(x, y) \frac{\partial \phi(x, y)}{\partial x} \right) - \frac{\partial}{\partial y} \left(D(x, y) \frac{\partial \phi(x, y)}{\partial y} \right) \\ & + \Sigma_r(x, y) \phi(x, y) = S(x, y) , \end{aligned} \quad \text{2D Cart., heterogeneous} \quad (9.53)$$

$$\begin{aligned} & -D \left[\frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} \right] \\ & + \Sigma_r \phi(x, y) = S(x, y) , \end{aligned} \quad \text{2D Cart., homogeneous} \quad (9.54)$$

Given Cartesian coordinates with a solution that depends on only x (infinite and uniform in y and z), we have the following for our steady-state one-group diffusion equation in “slab geometry”:

$$-\frac{d}{dx} \left(D(x) \frac{d\phi(x)}{dx} \right) + \Sigma_r(x) \phi(x) = S(x) , \quad \text{1D slab, heterogeneous} \quad (9.55)$$

$$-D \frac{d^2 \phi(x)}{dx^2} + \Sigma_r \phi(x) = S(x) , \quad \text{1D slab, homogeneous} \quad (9.56)$$

9.6.1.2 Spherical

Given a symmetric problem in which everything depends only on distance from the origin, spherical coordinates are most convenient, and only one spatial coordinate matters: r , the distance from the origin. This is sometimes called “1D spherical geometry”:

$$-\frac{1}{r^2} \frac{d}{dr} \left(r^2 D(r) \frac{d\phi(r)}{dr} \right) + \Sigma_r(r) \phi(r) = S(r) , \quad \text{1D spherical, heterogeneous} \quad (9.57)$$

$$-D \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi(r)}{dr} \right) + \Sigma_r \phi(r) = S(r) , \quad \text{1D spherical, homogeneous} \quad (9.58)$$

9.6.1.3 Cylindrical

Given a symmetric problem in which everything depends only on distance from an axis of symmetry (uniform in azimuthal coordinate; infinite and uniform in z), cylindrical coordinates are most convenient, and only one spatial coordinate matters: r , the perpendicular distance from the axis. This is sometimes called “1D cylindrical geometry”:

$$-\frac{1}{r} \frac{d}{dr} \left(r D(r) \frac{d\phi(r)}{dr} \right) + \Sigma_r(r) \phi(r) = S(r) , \quad \text{1D cylindrical, heterogeneous (9.59)}$$

$$-D \frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi(r)}{dr} \right) + \Sigma_r \phi(r) = S(r) , \quad \text{1D cylindrical, homogeneous (9.60)}$$

Given a symmetric problem in which things depends on both distance from an axis of symmetry and the position along the axis (z coordinate), but not on the azimuthal coordinate around the axis, cylindrical coordinates are most convenient, and only two spatial coordinates matter: r , the perpendicular distance from the axis, and z , the “height” parallel to the axis. This is sometimes called “2D cylindrical” or “ r - z ” geometry:

$$-\frac{1}{r} \frac{\partial}{\partial r} \left(r D(r, z) \frac{\partial\phi(r, z)}{\partial r} \right) - \frac{\partial}{\partial z} \left(D(r, z) \frac{\partial\phi(r, z)}{\partial z} \right) + \Sigma_r(r, z) \phi(r, z) = S(r, z) , \quad \text{2D cylindrical, heterogeneous (9.61)}$$

$$-D \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\phi(r, z)}{\partial r} \right) + \frac{\partial^2\phi(r, z)}{\partial z^2} \right] + \Sigma_r \phi(r, z) = S(r, z) , \quad \text{2D cylindrical, homogeneous (9.62)}$$

The solution of the diffusion equation is

[the diffusion approximation of] the scalar flux.

Once we have the scalar flux, we can get

[the diffusion approximation of] the net current density

by using

$$\text{Fick's Approximation, } \vec{J}(\vec{r}, E, t) = -D(\vec{r}, E, t) \vec{\nabla} \phi(\vec{r}, E, t) . \quad (9.63)$$

Remember, we need the scalar flux and net current density because:

- reactor behavior depends on neutron **production & loss rates**,
- production & loss rates depend on **reaction rates & net leakage rates**,
- reaction-rate densities depend on **scalar flux**,
- net leakage rates depend on **net current density**.

We will now solve the steady-state one-speed diffusion equation in a few homogeneous problems, to illustrate how its done.

9.6.2 Plane Source, infinite medium

9.6.2.1 Equation to solve

Suppose that at $x = 0$ in an infinite homogeneous medium, there is an infinitesimally thin plane source of neutrons emitting

$$S_0 \text{ [n/cm}^2\text{-s]}$$

This is a 1D problem—nothing varies in the y or z directions—so Eq. (9.56) applies, as long as we can figure out how to write the fixed source, $S(x)$. Note that $S(x) = 0$ for all x except $x = 0$, but that its integral from a negative x value to a positive x value must be finite and positive. It turns out that there is a way to write such a function:

$$-\frac{d^2\phi(x)}{dx^2} + \frac{1}{L^2} \phi(x) = \frac{S_0}{D} \delta(x) \quad (9.64)$$

where

$$\delta(x) = \text{Dirac delta function.}$$

Properties of the Dirac delta:

1. $\delta(x - x_0) = 0$ except at $x = x_0$, where it $\rightarrow +\infty$. It has zero width and infinite height such that the area under it exactly equals 1.
2. $\delta(x - x_0)dx$ is dimensionless. [$\delta(x)$ is inverse length if x is length.]
3. $\int_a^b \delta(x - x_0)f(x)dx = \begin{cases} f(x_0), & x_0 \in [a; b] \\ 0, & \text{otherwise} \end{cases}$
4. $\delta(x)$ is your friend !

This delta-function information is not required for this problem. It is introduced here because you'll need it later, so it is good to make its acquaintance.

Any time you have a singular source—one that is not distributed in space but has zero volume (as is the case here, where the source doesn't extend at all in the x direction)—it is easier to pose a separate diffusion equation in each region delimited by the source discontinuities. In our current problem, for every x location except $x = 0$ there is no source, and we have

$$-\frac{d^2\phi(x)}{dx^2} + \frac{1}{\tilde{L}^2}\phi(x) = 0 , \quad x \neq 0 . \quad (9.65)$$

We should actually solve two problems: 1) $x > 0$, and 2) $x < 0$.

It is easy to verify that $\exp(x/\tilde{L})$ and $\exp(-x/\tilde{L})$ both satisfy the equation. Thus, the general form of the solution in the right half-space ($x > 0$) is:

$$\phi_R(x) = A_R e^{-x/\tilde{L}} + C_R e^{x/\tilde{L}} , \quad x > 0 . \quad (9.66)$$

Here R denotes the right half-space. We also have the left half-space solution:

$$\phi_L(x) = A_L e^{-x/\tilde{L}} + C_L e^{x/\tilde{L}} , \quad x < 0 . \quad (9.67)$$

There are four unknowns coefficients: A_L , A_R , C_L , and C_R . These must be determined by boundary (or similar) conditions.

9.6.2.2 Boundary conditions

Technically, there are no “boundaries” in this problem (the domain is infinite), but there are similar conditions that serve the same purpose: a finiteness condition and a source/interface condition.

The finiteness condition says that the scalar flux must remain finite throughout the domain on which we solve the diffusion equation. In this problem this means everywhere except $x = 0$. If we look at the right half-space solution, we immediately see that $\exp(x/\tilde{L})$ goes to infinity for large x . Thus we must have

$$C_R = 0 . \quad (9.68)$$

Similarly, the left half-space solution remains finite only if

$$A_L = 0 . \quad (9.69)$$

This leaves us with

$$\phi_R(x) = A_R e^{-x/\tilde{L}} , \quad x > 0 , \quad (9.70)$$

$$\phi_L(x) = C_L e^{x/\tilde{L}} = C_L e^{-|x|/\tilde{L}} , \quad x < 0 . \quad (9.71)$$

To find the remaining constants, we take a close look at

neutron leakage rates

i.e., the rates at which neutrons cross defined surfaces, near the source plane. Let us look at the net neutrons per cm^2 per s crossing the plane at $x = \epsilon$, where ϵ is a small positive number. We know that the **net** rate of particles moving to the right is just

$$\text{net rate right} = \vec{e}_x \cdot \vec{J} \Big|_{x=\epsilon} , \text{ or } J_x \Big|_{x=\epsilon} \quad (9.72)$$

We also know from the definition of the **net** rate that this can be written as:

$$\begin{aligned} & \text{crossing rate/cm}^2 \text{ to right} - \text{crossing rate/cm}^2 \text{ to left} \\ &= \frac{S_0}{2} + \text{rate from scattering events left of } (x = \epsilon) \\ & \quad - \text{rate from scattering events right of } (x = \epsilon) \end{aligned} \quad (9.73)$$

As $\epsilon \rightarrow 0$, the second and third terms become equal in magnitude, by symmetry. Thus,

$$\lim_{x \rightarrow 0^+} J_x(x) = \frac{S_0}{2}. \quad (9.74)$$

This is often called a “source condition.”

Similarly, we also have

$$\lim_{x \rightarrow 0^-} J_x(x) = -\frac{S_0}{2}. \quad (9.75)$$

Alternate version: we can also write the diffusion equation as follows

$$\frac{dJ}{dx} + \Sigma_r \phi(0) = S_0 \delta(x)$$

Integration from $x = -\epsilon$ to $x = +\epsilon$

$$J(\epsilon) - J(-\epsilon) = \Sigma_r \int_{-\epsilon}^{\epsilon} \phi(x) dx = \underbrace{\int_{-\epsilon}^{\epsilon} S_0 \delta(x) dx}_{=S_0 \text{ by definition}}$$

By symmetry, $J(-\epsilon) = -J(\epsilon)$. The reaction rate integral goes to 0 as $\epsilon \rightarrow 0$. So, as $\epsilon \rightarrow 0$:

$$2J(0) = S_0$$

(same source condition as before!)

9.6.2.3 Solving for the constants

We apply our “source-condition” equations to our right- and left-half solutions to find the remaining constants. Begin with the right half:

$$\begin{aligned} \frac{S_0}{2} &= \lim_{x \rightarrow 0^+} J_x(x) = \lim_{x \rightarrow 0^+} \left[-D \frac{d\phi_R}{dx} \right] \\ &= \lim_{x \rightarrow 0^+} \left[\frac{D}{\tilde{L}} A_R e^{-x/\tilde{L}} \right] \\ &= \frac{DA_R}{\tilde{L}} \Rightarrow A_R = \frac{S_0 \tilde{L}}{2D}. \end{aligned} \quad (9.76)$$

Thus, for $x > 0$ our diffusion solution is

$$\phi_R(x) = \frac{S_0 \tilde{L}}{2D} \exp(-x/\tilde{L}), \quad \text{for } x > 0. \quad (9.77)$$

If we consider $x < 0$ and follow the same procedure we find

$$\begin{aligned} \frac{S_0}{2} &= -\lim_{x \rightarrow 0^-} J_x(x) = \lim_{x \rightarrow 0^-} \left[+D \frac{d\phi_R}{dx} \right] = \lim_{x \rightarrow 0^+} \left[\frac{D}{\tilde{L}} C_L e^{x/\tilde{L}} \right] \\ &= \frac{DC_L}{\tilde{L}} \Rightarrow C_L = \frac{S_0 \tilde{L}}{2D}. \end{aligned} \quad (9.78)$$

and

$$\phi_L(x) = \frac{S_0 \tilde{L}}{2D} \exp(+x/\tilde{L}), \quad \text{for } x < 0. \quad (9.79)$$

This is symmetric with the right-half solution, as it should be. We can combine Eqs. (9.77) and (9.79):

$$\phi(x) = \frac{S_0 \tilde{L}}{2D} \exp(-|x|/\tilde{L}) \quad \text{for } x \neq 0. \quad (9.80)$$

Diffusion solution, plane source in infinite homogeneous medium

Figure 9.3 sketches the scalar flux as a function of x in this problem.

Note: We did not solve the diffusion equation **at** $x = 0$, and we do not have an expression for the scalar flux **at** $x = 0$. This is an example of a general truth: we will not solve for the scalar flux at locations where the source rate density is infinite (as it is when we say a finite number of neutrons are being emitted from an object that has zero volume, such as our “plane source”).

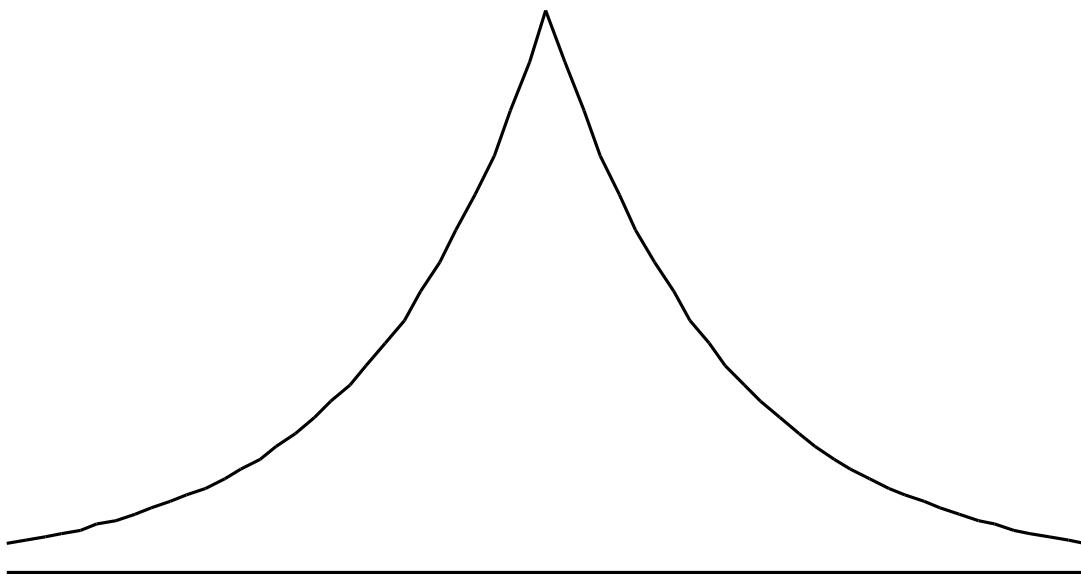


Figure 9.3: Diffusion-theory scalar flux given a plane source at $x = 0$ in an infinite medium.

9.6.3 Alternate derivation of the “source” condition at $x = 0$

Let us integrate our diffusion equation over a small interval centered around the source (because the interval contains the source, the equation now has the delta-function source term in it):

$$\int_{-\epsilon}^{+\epsilon} dx \left[-D \frac{d^2\phi(x)}{dx^2} + \Sigma_a \phi(x) \right] = \int_{-\epsilon}^{+\epsilon} dx S_0 \delta(x) \quad (9.81)$$

Looking at it term by term:

1. The leakage term yields:

$$\begin{aligned}
 \int_{-\epsilon}^{+\epsilon} dx \left[-D \frac{d^2\phi(x)}{dx^2} \right] &= -D \left. \frac{d\phi}{dx} \right|_{+\epsilon} + D \left. \frac{d\phi}{dx} \right|_{-\epsilon} \\
 &= J_x(+\epsilon) - J_x(-\epsilon) \\
 &\xrightarrow[\epsilon \rightarrow 0]{} J_x(0^+) - J_x(0^-) \\
 &= D \frac{A_R}{\tilde{L}} + D \frac{C_L}{\tilde{L}}
 \end{aligned} \quad (9.82)$$

2. The absorption term yields:

$$\int_{-\epsilon}^{+\epsilon} dx \Sigma_a \phi(x) \xrightarrow{\epsilon \rightarrow 0} 2\epsilon \Sigma_a \phi(0) \xrightarrow{\epsilon \rightarrow 0} 0 \quad (9.83)$$

3. The source term yields:

$$\int_{-\epsilon}^{+\epsilon} dx S_0 \delta(x) \xrightarrow{\epsilon \rightarrow 0} S_0 \quad [\text{property of the Dirac delta-function}] \quad (9.84)$$

4. Put it all together:

$$J_x(0^+) - J_x(0^-) = S_0 \quad (9.85)$$

or

$$A_R + C_L = \frac{S_0 \tilde{L}}{D} \quad (9.86)$$

From here we can invoke the symmetry of the problem:

$$\phi_R(x) = \phi_L(-x) \quad (9.87)$$

which, for the derivatives, yields

$$\frac{d}{dx} \phi_R(x) = -\frac{d}{dx} \phi_L(-x) \quad (9.88)$$

i.e.,

$$J_{x,R}(x) = -J_{x,L}(x) \quad (9.89)$$

which translates into $A_R = C_L$. Hence, we have:

$$A_R = C_L = \frac{S_0 \tilde{L}}{2D}, \quad (9.90)$$

which is what we found previously using the “source condition.”

9.6.4 Point Source in Infinite Homogeneous Medium

9.6.4.1 Equation to solve

Suppose at $r = 0$ in an infinite homogeneous medium there is an infinitesimally small source of neutrons emitting

$$S_0 \quad [\text{units} = \text{n/s}].$$

This is a 1D problem in spherical coordinates. Away from $r = 0$, the problem is source-free, and we have:

$$-\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\phi(r)}{dr} \right] + \frac{1}{\tilde{L}^2} \phi(r) = 0 \quad \text{for } r \neq 0. \quad (9.91)$$

The general solution of this homogeneous ODE is

$$\phi(r) = \frac{A \exp(-r/\tilde{L}) + C \exp(r/\tilde{L})}{r} \quad (9.92)$$

You should verify by plugging this into the equation!!! This solution was found by making the substitution

$$w(r) = r\phi(r)$$

into Eq. (9.91), which results in a slab-like diffusion equation for w . This means w satisfies the slab-geometry diffusion equation, which means it is the sum of two exponentials. Then $\phi(r)$ is just $1/r$ times that solution, because of the substitution equation.

9.6.4.2 Boundary Conditions

The “boundary” conditions are as follows:

1. as $r \rightarrow +\infty$, the flux must remain finite
2. at $r = 0$, we need a condition related to the point source

By our finiteness at $+\infty$ requirement we see that

$$C = 0.$$

Thus, all that remains is finding A . To do this, we take a close look at the neutron leakage rates (i.e., the rate at which neutrons cross defined surfaces) near the source point. Consider the net neutrons per s crossing a spherical shell of radius r , in the outward direction. We know that this is just

$$4\pi r^2 \left[\vec{e}_r \cdot \vec{J}(\vec{r}) \right], \text{ or } 4\pi r^2 J_r(\vec{r}). \quad (9.93)$$

We also know that this is just

$$\text{crossing rate outward} - \text{crossing rate inward}.$$

As r approaches 0, the second term vanishes, because the surface area gets vanishingly small. But all of the source neutrons must still cross outward. Thus,

$$\lim_{r \rightarrow 0} 4\pi r^2 J_r(r) = S_0 . \quad (9.94)$$

We can use this to determine A . In spherical coordinates we have

$$J_r(r) = - [\vec{e}_r \cdot D \vec{\nabla} \phi(\vec{r})] = -D \frac{d\phi(\vec{r})}{dr} \quad (9.95)$$

Thus,

$$\begin{aligned} S_0 &= \lim_{r \rightarrow 0} 4\pi r^2 J_r(r) = \lim_{r \rightarrow 0} \left[-D 4\pi r^2 \frac{d}{dr} \left(\frac{A \exp(-r/\tilde{L})}{r} \right) \right] \\ &= \lim_{r \rightarrow 0} \left[D 4\pi r^2 A \exp(-r/\tilde{L}) \left(\frac{1}{r^2} + \frac{1}{r\tilde{L}} \right) \right] \\ &= D 4\pi A \end{aligned} \quad (9.96)$$

Thus,

$$\phi(r) = \frac{S_0}{4\pi D} \frac{\exp(-r/\tilde{L})}{r} , \quad r \neq 0 . \quad (9.97)$$

Diffusion solution, point source in infinite homogeneous medium

Comparing this to the plane-source result, you will see that the scalar flux decreases more rapidly in the point-source case—there is a $1/r$ factor in addition to the decreasing exponential. This makes physical sense: the neutrons are now

spreading out

as well as getting absorbed.

Also note that, as $r \rightarrow 0$, the flux behaves like $1/r$ and **goes to infinity at $r = 0$!!!.**

Question: How can this be? Did we make an error?

Answer: This physically impossible flux is caused by the physically impossible source, which emits finite neutrons from 0 volume.

Even with the scalar flux $\rightarrow \infty$ at $r = 0$, we still obtain physically reasonable answers for observable quantities, such as total number of neutrons in a finite volume or total path-length rate in a finite volume. For example, how many neutrons are within a distance R of the source?

$$\begin{aligned} \text{number} &= \int_0^R 4\pi r^2 \frac{\phi(r)}{v} dr = \frac{S_0}{vD} \int_0^R r \exp(-r/\tilde{L}) dr \\ &= \frac{S_0}{v\Sigma_a} \left[1 - (1 + R/\tilde{L}) \exp(-R/\tilde{L}) \right] \end{aligned} \quad (9.98)$$

If we let $R \rightarrow \infty$ we find a total of

$$\frac{S_0}{v\Sigma_a} \text{ neutrons}$$

in the entire infinite medium. This is the number of particles that are emitted during one mean lifetime, which makes sense.

If we consider small R , so that $R/\tilde{L} = d \ll 1$, then

$$\text{number} = \frac{S_0}{v\Sigma_a} \left[1 - (1 + R/\tilde{L}) \underbrace{\exp(-R/\tilde{L})}_{1 - \frac{R}{\tilde{L}} + \frac{1}{2}\left(\frac{R}{\tilde{L}}\right)^2 + \dots} \right] \xrightarrow{R/\tilde{L}=d \ll 1} \frac{S_0}{v\Sigma_a} \frac{d^2}{2} \quad (9.99)$$

That is, the number of neutrons within a small distance $d\tilde{L}$ of the source is proportional to d^2 , which shrinks to zero for small d even though the scalar flux becomes infinite at the source point.

9.6.5 Plane Source centered in a Finite Slab Surrounded by Vacuum

9.6.5.1 Equations to Solve

Consider now a plane source at $x = 0$ in a slab that goes from $x = -a$ to $x = +a$, with a plane source at $x = 0$ emitting S_0 neutrons per unit area per unit time. The diffusion equation for this problem is:

$$-\frac{d^2\phi(x)}{dx^2} + \frac{1}{\tilde{L}^2}\phi(x) = \frac{S_0 \delta(x)}{D}, \text{ for } x \in (-a; +a) . \quad (9.100)$$

This problem is much like the plane-source infinite-medium problem, except that the finiteness conditions at $\pm\infty$ are now replaced by boundary conditions. We still split the domain in two subregions: $x > 0$ and $x < 0$.

For $x > 0$ we still have

$$\phi_R(x) = A_R \exp(-x/\tilde{L}) + C_R \exp(+x/\tilde{L}) . \quad (9.101)$$

For $x < 0$ we still have

$$\phi_L(x) = A_L \exp(-x/\tilde{L}) + C_L \exp(x/\tilde{L}) . \quad (9.102)$$

9.6.5.2 Boundary and other conditions

At vacuum boundaries we use “extrapolated” boundary conditions. In this problem, this translates to:

1. vacuum at $x = a$:

$$\phi_R(a + d) = 0 \quad (9.103)$$

where d = extrapolation distance. (This is extrapolation with curvature.) Note that this time we cannot say $C_R = 0$. The function for $\phi_R(x)$ may become infinite as $x \rightarrow \infty$, but this doesn't matter because the function describes the scalar flux only for $x \in (-a, a)$. For any finite value of C_R the function remains finite in that range.

2. vacuum at $x = -a$:

$$\phi_L(-a - d) = 0 \quad (9.104)$$

where d = extrapolation distance. (This is extrapolation with curvature.) This time we cannot say $A_L = 0$, for the reasons described just above.

At the interface with the plane source we will again have a “source condition” that is derived by considering net flow rates across planes just next to the source planes. This proceeds exactly as in the infinite-medium problem:

1. source condition for $x > 0$ is

$$\lim_{x \rightarrow 0^+} \left[-D \frac{d\phi_R}{dx}(x) \right] = \frac{S_0}{2} \quad (9.105)$$

2. source condition for $x < 0$ is

$$\lim_{x \rightarrow 0^-} \left[-D \frac{d\phi_R}{dx}(x) \right] = -\frac{S_0}{2} \quad (9.106)$$

Equations (9.103) and (9.105) are two equations for the two constants A_R and C_R ; a few steps of algebra produce them.

The $x < 0$ half of the problem is handled in the same way (one vacuum boundary condition at $x = -a - d$ and one source condition for $x < 0$). Because of symmetry, the four conditions are not coupled, i.e., you can solve the two equations (9.103) and (9.105) for A_R and C_R and two similar equations for A_L and C_L . **Important:** in absence of symmetry (e.g., if the planar source is not centered in the slab), then the four equations for A_R , C_R , A_L and C_L would need to be solved all together. **Suggestion:** see if you can write them down and verify for yourself this statement.

Finally, the solution for the planar source centered in a finite slab surrounded by vacuum is:

$$\phi(x) = \frac{S_0 \tilde{L}}{2D} \frac{\exp\left(-\frac{|x|}{\tilde{L}}\right) - \exp\left(-\frac{|x| - 2(a+d)}{\tilde{L}}\right)}{1 + \exp\left(-\frac{2(a+d)}{\tilde{L}}\right)} \quad (9.107)$$

Diffusion solution, plane source at center of finite homogeneous slab

This solution is shown in Fig. 9.4, with the infinite-medium solution shown also for comparison.

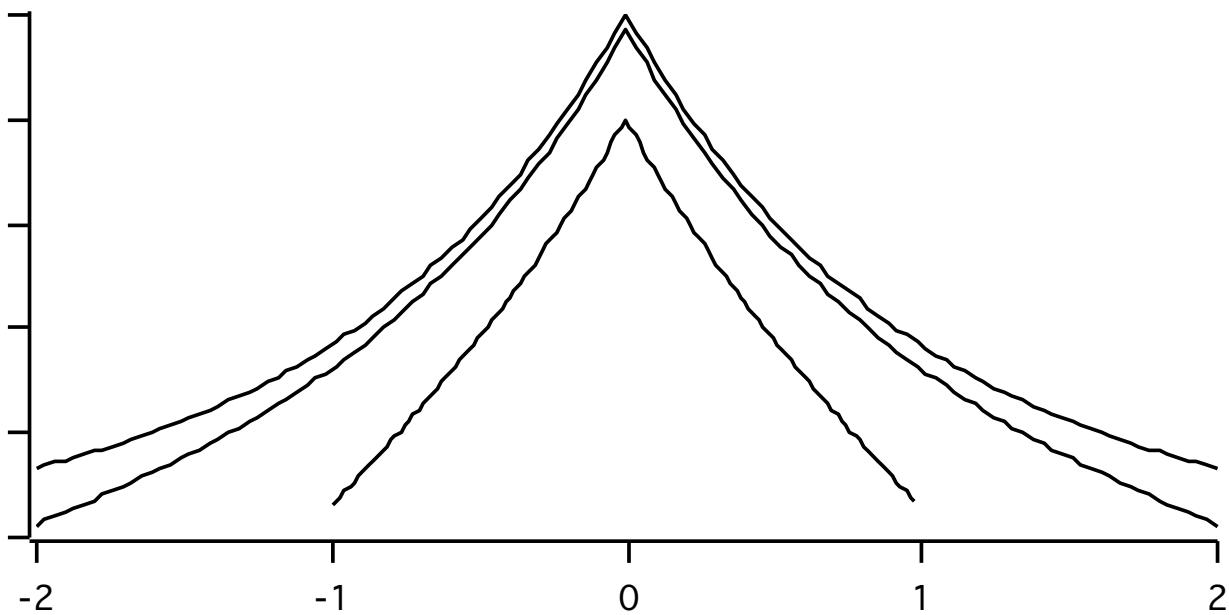


Figure 9.4: Diffusion scalar flux, plane source, infinite and finite homogeneous slabs. Horiz. axis x/L , vert axis $\phi(x)$, 0 to $S_0 \tilde{L}/2D$.

9.6.6 Hints on 1D Diffusion Solutions:

When solving 1D slab diffusion equations with a given extraneous source, we will generally have in each region:

a “particular solution” and a “homogeneous” (no-source) solution.

The no-source solution will be a linear combination of

two linearly independent functions.

These two functions can be written, for example in the slab-geometry case, as

$$\phi(x) = A \exp\left(-\frac{x}{\tilde{L}}\right) + C \exp\left(\frac{x}{\tilde{L}}\right) \quad (9.108)$$

or

$$\phi(x) = \tilde{A} \cosh\left(\frac{x}{\tilde{L}}\right) + \tilde{C} \sinh\left(\frac{x}{\tilde{L}}\right) \quad (9.109)$$

or

$$\phi(x) = A' \cosh\left(\frac{x - x_0}{\tilde{L}}\right) + C' \sinh\left(\frac{x - x_0}{\tilde{L}}\right) \quad (9.110)$$

Given values for A and C in Eq. (9.108), it is easy to find values for \tilde{A} and \tilde{C} in Eq. (9.109) that make the two expressions equal. The converse is also true—for any \tilde{A} and \tilde{C} there is an A and a C that make the expressions equal. The same is also true for A' and C' in Eq. (9.110). This is a long-winded way of saying that the exponential functions span the same space as the hyperbolic functions, and adding or subtracting constants to/from x in the functions does not change this.

So you can choose either form and if you do everything correctly you will get the same answer. However, the form that you choose (exponentials or hyperbolics) can **dramatically** affect how much algebra you do to find your constants.

Here are some tips on choosing. In general,

Which form you choose can **dramatically** affect how much algebra you do to find your constants.

Here are some tips on choosing. In general,

1. **Use cosh & sinh in finite regions.** For one thing, sinh has a zero, which is very nice for dealing with extrapolated boundary conditions. For another, cosh is an even function of x , which is nice in problems whose solutions are symmetric.
2. **Use $\exp(\pm x/\tilde{L})$ in infinite regions.** One exponential goes to zero as x goes to infinity, and the other blows up - this can make it very easy to determine that one constant is zero, for example.
3. $\sinh(c+x/\tilde{L})$ and $\cosh(c+x/\tilde{L})$ are just as valid as $\sinh(x/\tilde{L})$ and $\cosh(x/\tilde{L})$. In some problems you can use this to great advantage. For example, if you have a reflecting boundary at $x = b$, you might try $\sinh[(x - b)/\tilde{L}]$ and $\cosh[(x - b)/\tilde{L}]$.

As an example, let us rework the example of a planar source centered in a finite slab. We have, for $x > 0$:

$$-\frac{d^2\phi_R(x)}{dx^2} + \frac{1}{\tilde{L}^2}\phi_R(x) = 0 \quad (9.111)$$

$$\phi_R(\tilde{a}) = 0 \text{ [first BC, } \tilde{a} = a + d] \quad (9.112)$$

$$J_{x,R}(0) = \frac{S_0}{2} \text{ [second BC]} \quad (9.113)$$

We could seek the solution using hypertrigonometric functions, i.e.,

$$\phi_R(x) = A_R \cosh(x/\tilde{L}) + C_R \sinh(x/\tilde{L}), \quad (9.114)$$

but it would be smarter to let the first boundary condition guide us, and seek the solution as:

$$\phi_R(x) = A_R \cosh\left(\frac{\tilde{a}-x}{\tilde{L}}\right) + C_R \sinh\left(\frac{\tilde{a}-x}{\tilde{L}}\right). \quad (9.115)$$

Then the first boundary condition tells us $A_R = 0$, and the second BC yields:

$$(-D)\left(-\frac{C_R}{\tilde{L}}\right) \cosh\left(\frac{\tilde{a}}{\tilde{L}}\right) = \frac{S_0}{2} \implies C_R = \frac{S_0 \tilde{L}}{2D} \frac{1}{\cosh\left(\frac{\tilde{a}}{\tilde{L}}\right)} \quad (9.116)$$

and thus

$$\phi_R(x) = \frac{S_0 \tilde{L}}{2D} \frac{\sinh\left(\frac{\tilde{a}-x}{\tilde{L}}\right)}{\cosh\left(\frac{\tilde{a}}{\tilde{L}}\right)}. \quad (9.117)$$

If you use the definitions of cosh and sinh you can recast the above solution in terms of $\exp(x/\tilde{L})$ and $\exp(-x/\tilde{L})$:

$$\phi(x) = \frac{S_0 \tilde{L}}{2D} \frac{\exp\left(-\frac{|x|}{\tilde{L}}\right) - \exp\left(-\frac{|x|-2(a+d)}{\tilde{L}}\right)}{1 + \exp\left(-\frac{2(a+d)}{\tilde{L}}\right)} \quad (9.118)$$

but this is not as elegant as the form that uses the sinh and cosh functions. The hypertrigonometric functions (another term for hyperbolic trig functions) are much easier to use for a problem of finite dimensions!!! I hope to have convinced you that the **hypertrigonometric functions** cosh & sinh **can be of great help in simplifying the required algebra for finite problems.**

9.6.7 Uniformly distributed source in a finite slab

9.6.7.1 Equations to Solve, including the BC

Consider a fixed volumetric source, S (n/cm³-s), in a slab of extrapolated width $2\tilde{a}$.

The diffusion equation and boundary conditions are

$$-\frac{d^2\phi(x)}{dx^2} + \frac{1}{\tilde{L}^2}\phi(x) = \frac{S}{D} \quad (9.119)$$

$$\phi(\tilde{a}) = 0 \text{ [first BC]} \quad (9.120)$$

$$J_x(0) = 0 \text{ [second BC]} \quad (9.121)$$

We have chosen to solve in the right half of the problem and use a symmetry condition at $x = 0$ = the centerline.

9.6.7.2 Solution

We can seek the solution using hypertrigonometric functions, i.e.,

$$\phi(x) = \frac{S}{\Sigma_r} + A \sinh\left(\frac{x}{\tilde{L}}\right) + C \cosh\left(\frac{x}{\tilde{L}}\right). \quad (9.122)$$

The 1st BC yields:

$$\frac{S}{\Sigma_r} + A \sinh\left(\frac{\tilde{a}}{\tilde{L}}\right) + C \cosh\left(\frac{\tilde{a}}{\tilde{L}}\right) = 0 \quad (9.123)$$

The 2nd BC yields:

$$-\frac{D}{\tilde{L}}(A \cosh(0) + C \sinh(0)) = 0 \implies A = 0. \quad (9.124)$$

Hence, from the first BC,

$$C = -\frac{S}{\Sigma_r} \frac{1}{\cosh\left(\frac{\tilde{a}}{\tilde{L}}\right)} \quad (9.125)$$

Finally, we have the solution:

$$\phi(x) = \frac{S}{\Sigma_r} \left[1 - \frac{\cosh\left(\frac{x}{\tilde{L}}\right)}{\cosh\left(\frac{\tilde{a}}{\tilde{L}}\right)} \right]. \quad (9.126)$$

It is easy to see that this solution satisfies the extrapolated boundary condition. (Verify this yourself!) It is also easy to see that the solution is symmetric—the scalar flux is the same for a given negative value of x as for the same-magnitude positive value. Also, if $a \gg \tilde{L}$, the solution in the center of the slab is approximately the infinite-medium solution, S/Σ_r . A helpful exercise would be to sketch this solution for such a “thick” slab.

9.6.8 Uniformly distributed source in a sphere

9.6.8.1 Equations to Solve, including the BC

Consider a fixed source, S (n/cm³-s), in a sphere of extrapolated radius \tilde{R} .

$$-\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\phi(r)}{dr} \right] + \frac{1}{\tilde{L}^2} \phi(r) = \frac{S}{D} \quad (9.127)$$

$$\phi(\tilde{R}) = 0 \text{ [first BC]} \quad (9.128)$$

$$J_r(0) = 0 \text{ [second BC]} \quad (9.129)$$

9.6.8.2 Solution

We can seek the solution using hypertrigonometric functions, i.e.,

$$\phi(r) = \frac{S}{\Sigma_r} + \frac{A \sinh\left(\frac{r}{\tilde{L}}\right) + C \cosh\left(\frac{r}{\tilde{L}}\right)}{r} \quad (9.130)$$

The 1st BC yields:

$$\frac{S}{\Sigma_r} + \frac{A \sinh\left(\frac{\tilde{R}}{\tilde{L}}\right) + C \cosh\left(\frac{\tilde{R}}{\tilde{L}}\right)}{\tilde{R}} = 0 \quad (9.131)$$

For the 2nd BC yields, we need to compute the flux derivative:

$$\begin{aligned} \frac{d\phi}{dr} &= \frac{\frac{r}{\tilde{L}} \left[A \cosh\left(\frac{r}{\tilde{L}}\right) + C \sinh\left(\frac{r}{\tilde{L}}\right) \right] - A \sinh\left(\frac{r}{\tilde{L}}\right) - C \cosh\left(\frac{r}{\tilde{L}}\right)}{r^2} \\ &= \frac{\cosh\left(\frac{r}{\tilde{L}}\right) \left[-C + \frac{r}{\tilde{L}} A \right] + \sinh\left(\frac{r}{\tilde{L}}\right) \left[-A + \frac{r}{\tilde{L}} C \right]}{r^2} \end{aligned}$$

As $r \rightarrow 0$, $\cosh\left(\frac{r}{L}\right) \rightarrow 1$ and $\sinh\left(\frac{r}{L}\right) \rightarrow \frac{r}{L}$. Hence,

$$\frac{d\phi}{dr} \rightarrow \frac{-C + \frac{r}{L}A + \frac{r}{L}\left[-A + \frac{r}{L}C\right]}{r^2} = \frac{-C + \left(\frac{r}{L}\right)^2 C}{r^2} = \frac{-C}{r^2} + \frac{C}{\tilde{L}^2}$$

Hence, the 2nd BC requires $C = 0$.

Finally,

$$\phi(r) = \frac{S}{\Sigma_r} \left[1 - \frac{\tilde{R}}{r} \frac{\sinh\left(\frac{r}{\tilde{L}}\right)}{\sinh\left(\frac{\tilde{R}}{\tilde{L}}\right)} \right]. \quad (9.132)$$

It is easy to see that this solution satisfies the extrapolated boundary condition. (Verify this yourself!) Also, if $R \gg \tilde{L}$, the solution in the center of the slab is approximately the infinite-medium solution, S/Σ_r . A helpful exercise would be to sketch this solution for such a “thick” sphere.

9.7 The Diffusion Length (L)

The diffusion length L , defined as

$$\begin{aligned} L &= \sqrt{\frac{D}{\Sigma_a}}, \quad [\text{cm}] \\ &= \text{diffusion length} \end{aligned} \quad (9.133)$$

has a useful physical interpretation. It is proportional to an average straight-line distance between neutron birth point and absorption point, at least in an infinite non-fissioning medium.

Recall that if S_0 neutrons/second are born at a point in an infinite non-fissioning medium, the scalar flux a distance r always is (in the diffusion approximation) given by:

$$\phi(r) = \frac{S_0}{4\pi D} \frac{\exp(-r/L)}{r}, \quad (9.134)$$

a result we derived a few pages ago. (Recall that in a non-fissioning medium $\tilde{L} = L$.) The absorption-rate density (ARD) a distance r from the source is therefore

$$\text{ARD}(r) = \frac{S_0}{4\pi} \frac{\Sigma_a}{D} \frac{\exp(-r/L)}{r} \quad (9.135)$$

and the absorption rate between r and $r + dr$ is the ARD times the volume of the spherical shell between r and $r + dr$:

$$\frac{S_0}{4\pi} \frac{\Sigma_a}{D} \frac{\exp(-r/L)}{r} \times 4\pi r^2 dr = S_0 \frac{1}{L^2} \exp(-r/L) r dr \quad (9.136)$$

Thus, the probability of absorption between r and $r + dr$, where r is the distance from a point source in an infinite medium, is just the absorption rate divided by the source rate:

$$p(r) dr = \frac{1}{L^2} \exp(-r/L) r dr . \quad (9.137)$$

Recall: General definition of an average:

$$\langle f \rangle = \frac{\int w(x)f(x) dx}{\int w(x) dx} \quad (9.138)$$

In our case, the variable is r , not x , and the weight function is the probability density function for absorption, $p(r)$:

$$\langle f \rangle = \frac{\int_0^\infty p(r)f(r) dr}{\int_0^\infty p(r) dr} \quad (9.139)$$

The denominator = 1: the probability that a neutron gets absorbed somewhere in the infinite medium is unity:

$$\int_0^\infty p(r) dr = \int_0^\infty \frac{1}{L^2} \exp(-r/L) r dr = 1 \quad (9.140)$$

If we let $f(r) = r^2$, so we are computing the **mean squared distance from birth point to absorption point**, we get

$$\langle r^2 \rangle = \int_0^\infty \frac{1}{L^2} \exp(-r/L) r^3 dr = \dots = 6L^2 . \quad (9.141)$$

This $\langle r^2 \rangle$ is the

average squared straight-line distance (not path traveled)
between birth point and absorption point in an infinite homogeneous medium.

We see from Eq. (9.141) that

$$L^2 = \frac{1}{6} \langle r^2 \rangle . \quad (9.142)$$

We conclude that:

The diffusion length, L , is related to how far neutrons migrate from birth point to absorption point, on average.

If the diffusion length is long, neutrons tend to travel a long way. If it is short, they don't. If you have a material with a large diffusion length, you will need a lot of it to keep the neutrons from escaping!

9.8 Return to Solutions of Homogeneous Problems

9.8.1 Infinite line source in an infinite medium

9.8.1.1 Equation to Solve

We have solved problems with plane sources and point sources. Now suppose a line source is placed on the z -axis. It emits

$$S_0 \text{ [n/cm-s]}$$

The source function will be represented by:

$$S_0 \delta(x) \delta(y) [\text{n/cm}^3\text{-s}]$$

The diffusion equation is:

$$-D \nabla^2 \phi(\vec{r}) + \Sigma_r \phi(\vec{r}) = S_{ext}(\vec{r}) . \quad (9.143)$$

We should write ∇^2 using cylindrical coordinates (r, θ, z) . Here, $\phi(\vec{r})$ means $\phi(r, \theta, z)$. Because there is symmetry of revolution around the axis of the line source there is no dependence on the θ spatial coordinate, and because the source is infinitely long there is no dependence on z .

Away from the source, i.e. for $r > 0$, the diffusion equation in cylindrical coordinates is:

$$- D \frac{1}{r} \frac{d}{dr} \left[r \frac{d\phi(r)}{dr} \right] + \Sigma_r \phi(r) = 0 \quad (9.144)$$

or

$$\frac{d^2}{dr^2} \phi(r) + \frac{1}{r} \frac{d}{dr} \phi(r) - \frac{1}{\tilde{L}^2} \phi(r) = 0 . \quad (9.145)$$

This is a very special differential equation, known as a

modified Bessel's equation of order zero.

Its solution is the combination of two functions, denoted by $K_0(r/\tilde{L})$ and $I_0(r/\tilde{L})$ in the literature (just like we had $\exp(x/\tilde{L})$ and $\exp(-x/\tilde{L})$ for 1-D slabs). Hence, the flux is given by:

$$\phi(r) = A \times K_0 \left(\frac{r}{\tilde{L}} \right) + C \times I_0 \left(\frac{r}{\tilde{L}} \right) \quad (9.146)$$

where A and C are constants. For more information about Bessel's functions, see the Appendix on special functions.

9.8.1.2 Boundary Conditions

We need to require 2 conditions:

1. It is an infinite problem, so we need the flux to remain finite as $r \rightarrow +\infty$.
2. The problem has a singular source, so we need a source condition (obtained by integrating the diffusion equation over a small volume around that discrete source)

9.8.1.3 Solution

- Because $I_0(r) \rightarrow \infty$ as $r \rightarrow \infty$, we see that we must have

$$C = 0$$

for the flux to remain finite at infinity.

- The source condition at $r = 0$ will give us the other constant A , as follows.

The leakage from a cylinder of unit height and radius ϵ centered around the line source is obtained by integrating the diffusion equation over that volume:

$$\begin{aligned} S_0 &= 2\pi r J(r) \\ &= -2\pi r D \frac{d\phi}{dr} = -2\pi r D A K'_0 \left(\frac{r}{\tilde{L}} \right) = 2\pi r \frac{D}{\tilde{L}} A K_1 \left(\frac{r}{\tilde{L}} \right) \end{aligned} \quad (9.147)$$

As $r \rightarrow 0$, we have

$$2\pi r \frac{D}{\tilde{L}} A K_1 \left(\frac{r}{\tilde{L}} \right) \xrightarrow[r \rightarrow 0]{} 2\pi r A \frac{D}{\tilde{L}} \frac{1}{\frac{r}{\tilde{L}}} = 2\pi D A$$

If we put this all together we have the solution:

$$\phi(r) = \frac{S_0}{2\pi D} K_0 \left(\frac{r}{\tilde{L}} \right) \quad (9.148)$$

9.9 Green's functions

We have solved for the flux distribution in space for the following simple cases:

- plane source in an infinite homogeneous medium;
- line source in an infinite homogeneous medium;
- point source in an infinite homogeneous medium.

We can write the point-source solution in terms of Cartesian coordinates, with the point \vec{r}' defined as the location of the point source:

$$\phi(\vec{r}) = \underbrace{S_0}_{n/s} \frac{e^{-|\vec{r}-\vec{r}'|/\tilde{L}}}{4\pi D |\vec{r}-\vec{r}'|} . \quad (9.149)$$

The flux due to a superposition of sources is the superposition of the individual fluxes due to each source. A distributed source can be viewed as a superposition of point sources. If $S(\vec{r}')$ is source rate density at position \vec{r}' , then $S(\vec{r}') d^3 r'$ is effectively a point source. So the solution given a distributed source in an infinite medium is:

$$\phi(\vec{r}) = \iiint \underbrace{d^3 r' S(\vec{r}')}_{n/s} \frac{e^{-|\vec{r}-\vec{r}'|/\tilde{L}}}{4\pi D |\vec{r}-\vec{r}'|} \quad (9.150)$$

The function that relates a source at one point to the flux at another is called a “Green’s function,” $G(\vec{r}, \vec{r}')$. For an infinite homogeneous medium, we see that the Green’s function is:

$$G(\vec{r}, \vec{r}') = \frac{e^{-|\vec{r}-\vec{r}'|/\tilde{L}}}{4\pi D |\vec{r}-\vec{r}'|}. \quad (9.151)$$

A line source or plane source can be viewed as a collection of infinitesimally small sources. As a result, we could use Eq. (9.150) to generate the plane-source and line-source solutions we found previously. It would be helpful for you to try this yourself!

9.10 Steady-State One-Group Diffusion Solutions in Heterogeneous Media

In this section we consider problem domains composed of more than one homogeneous region. Figure 9.5 shows, for example, a two-region slab that is infinite in the y and z directions.

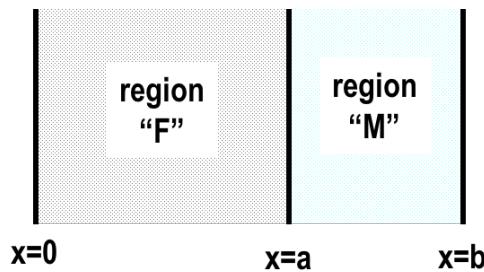


Figure 9.5: Two-region slab.

We still use the diffusion equation in each region. In a problem with regions F and M , for example, we will have (given steady state and the one-group assumption, and given no fissioning material in either region):

$$-D^F \nabla^2 \phi^F(\vec{r}) + \Sigma_a^F \phi^F(\vec{r}) = S^F(\vec{r}), \quad \vec{r} \in \text{region } F, \quad (9.152)$$

and

$$-D^M \nabla^2 \phi^M(\vec{r}) + \Sigma_a^M \phi^F(\vec{r}) = S^M(\vec{r}), \quad \vec{r} \in \text{ region } M. \quad (9.153)$$

Note that within a given region, D is constant and thus comes outside the divergence operator, which allows us to use the Laplacian in the leakage term. We write down the general solution in each region; each region's solution will contain unknown constants. We will have to use boundary conditions at problem boundaries and

interface (continuity) conditions

at each interface between regions. In a two-region slab, this means

4 unknown constants and 4 conditions.

Example

Consider the two-region slab shown in Fig. 9.5. In the left region let S_{ext} be S^F n/(cm³-s); in the right let S_{ext} be S^M n/(cm³-s). Let the boundaries at $x = 0$ and $x = b$ be reflecting ("mirror"). Compute the (diffusion approximation of the) scalar flux in the slab.

Our diffusion equations are:

$$-D^F \frac{d^2}{dx^2} \phi^F(x) + \Sigma_a^F \phi^F(x) = S^F, \quad x \in [0; a] \quad (9.154)$$

and

$$-D^M \frac{d^2}{dx^2} \phi^M(x) + \Sigma_a^M \phi^F(x) = S^M, \quad x \in [a; b]. \quad (9.155)$$

Our general solutions can be written in many forms, using exponentials and/or hyperbolic sines and cosines, and with arguments that add or subtract constants from x . We choose the following form:

$$\phi^F(x) = \frac{S^F}{\Sigma_a^F} + A^F \cosh(x/L^F) + C^F \sinh(x/L^F) \quad (9.156)$$

$$\phi^M(x) = \frac{S^M}{\Sigma_a^M} + A^M \cosh\left(\frac{x-b}{L^M}\right) + C^M \sinh\left(\frac{x-b}{L^M}\right) \quad (9.157)$$

You will soon see why we've chosen this form for our general solutions. Note that all of the above means that our scalar flux is

$$\phi(x) = \begin{cases} \phi^F(x), & x \in (0, a) , \\ \phi^M(x), & x \in (a, b) . \end{cases} \quad (9.158)$$

Our boundary conditions are:

1. reflecting at $x = 0$:

$$J_x(0) = -D^F \frac{d\phi^F}{dx} \Big|_{x=0} = 0 \quad (9.159)$$

2. reflecting at $x = b$:

$$J_x(b) = -D^M \frac{d\phi^M}{dx} \Big|_{x=b} = 0 \quad (9.160)$$

Our interface conditions are:

1. continuity of scalar flux:

$$\phi^F(a) = \phi^M(a) \quad (9.161)$$

2. continuity of normal component of current:

$$-D^F \frac{d\phi^F}{dx} \Big|_{x=a} = -D^M \frac{d\phi^M}{dx} \Big|_{x=a} \quad (9.162)$$

Now let us determine our four constants $\{A^F, C^F, A^M, C^M\}$ using our four conditions (two boundary, two interface).

Reflecting at $x = 0$ tells us that $C^F = 0$.

Reflecting at $x = b$ tells us that $C^M = 0$.

Thus, so far we have:

$$\phi^F(x) = \frac{S^F}{\Sigma_a^F} + A^F \cosh \left(x/L^F \right) \quad (9.163)$$

$$\phi^M(x) = \frac{S^M}{\Sigma_a^M} + A^M \cosh \left((x-b)/L^M \right) \quad (9.164)$$

Now we apply the interface conditions. Continuity of scalar flux at $x = a$ tells us that:

$$\frac{S^F}{\Sigma_a^F} + A^F \cosh(a/L^F) = \frac{S^M}{\Sigma_a^M} + A^M \cosh((a-b)/L^M) \quad (9.165)$$

Continuity of J_x tells us that:

$$A^F \frac{1}{L^F} \sinh(a/L^F) = A^M \frac{1}{L^M} \sinh((a-b)/L^M) \quad (9.166)$$

These are two equations for our two unknown constants A^F and A^M . Note that all the cosh and sinh terms in these two equations are

just numbers—just constants!

That is, there is no “ x ” anywhere. This is just a simple system of two equations (9.165, 9.166) and two unknowns (A^F and A^M). You know how to solve this! It has the form

$$\left\{ \begin{array}{l} \alpha_1 A^F + \beta_1 A^M = q_1 \\ \alpha_2 A^F + \beta_2 A^M = q_2 \end{array} \right\} \quad (9.167)$$

Don’t let the ugly α ’s and β ’s intimidate you—they are just numbers!

9.11 Criticality (k -eigenvalue) Problems

9.11.1 k -eigenvalue equation from physical arguments

It would be a good idea to review the physics of fission, for example as covered in Chapter 4 of these notes.

Suppose a reactor is critical and operating in steady state. The energy-dependent diffusion equation that describes this situation is:

$$\begin{aligned} -\vec{\nabla} \cdot \left(D(\vec{r}, E) \vec{\nabla} \phi(\vec{r}, E) \right) + \Sigma_t(\vec{r}, E) \phi(\vec{r}, E) &= \chi(\vec{r}, E) \int_0^\infty dE_i \nu \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ &\quad + \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E) \phi(\vec{r}, E_i) \end{aligned} \quad (9.168)$$

Rewrite:

$$\begin{aligned}
 & -\vec{\nabla} \cdot \left(D(\vec{r}, E) \vec{\nabla} \phi(\vec{r}, E) \right) + \Sigma_t(\vec{r}, E) \phi(\vec{r}, E) - \underbrace{\int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E) \phi(\vec{r}, E_i)}_{\text{net loss rate density from leakage and collisions}} \\
 & = \chi(\vec{r}, E) \underbrace{\int_0^\infty dE_i \nu \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i)}_{\text{production rate density from fission}} \quad (9.169)
 \end{aligned}$$

Suppose now that a similar reactor is **not critical**, which means loss rate is not equal to gain rate. Suppose we multiply the loss term by some number so that

$$\begin{aligned}
 & \text{loss rate density} \times \text{number} = \text{production rate density} \\
 & \Rightarrow \text{loss rate density} = \frac{1}{\text{number}} (\text{production rate density}) .
 \end{aligned}$$

Then:

- If (number) < 1, the reactor is **subcritical**;
- If (number) > 1, the reactor is **supercritical**.
- If (number) = 1, the reactor is **critical**.

If we call this number “ k ” then our energy-dependent diffusion equation becomes:

$$\begin{aligned}
 & -\vec{\nabla} \cdot \left(D(\vec{r}, E) \vec{\nabla} \phi(\vec{r}, E) \right) + \Sigma_t(\vec{r}, E) \phi(\vec{r}, E) = \frac{1}{k} \chi(\vec{r}, E) \int_0^\infty dE_i \nu \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\
 & \quad + \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E) \phi(\vec{r}, E_i) \quad (9.170)
 \end{aligned}$$

This equation says that for some scalar-flux function $\phi(\vec{r})$ **and** some number k ,

$$(\text{loss rate}) = (\text{production rate}) / k .$$

Equation (9.170) is an **EIGENVALUE EQUATION**. Both k (the **eigenvalue**) and $\phi(\vec{r})$ (the **eigenfunction**) are unknowns. A solution of the eigenvalue problem is a (k, ϕ) pair that satisfies the equation and its boundary conditions.

Question: Can a k -eigenvalue problem have a time variable or a time-derivative term?

Answer: No. It seeks (k, ϕ) pairs that cause loss rate to equal [production rate]/ k .

Question: Can a k -eigenvalue problem have a fixed source or a prescribed intensity of neutrons entering through the boundary?

Answer: No. Eigenvalue problems always look at how a system behaves **without driving terms**.

9.11.2 Mathematical aspects of eigenvalue problems and criticality

Equation (9.170) is an eigenvalue equation. When combined with boundary conditions it becomes an eigenvalue problem. Here we summarize a few things you need to understand about eigenvalue problems.

The number “ k ” is an

eigenvalue

and the function $\phi(\vec{r}, E)$ is an

eigenfunction.

Note that eigenvalue and eigenfunction mean

“characteristic” value and “characteristic” function.

There is a large set of {eigenvalue, eigenfunction} pairs that satisfy (9.170) and its associated boundary conditions. In fact, there are

infinitely many!

Only one eigenfunction in the set can be

positive throughout the whole domain V for all neutron energies E .

We give this eigenfunction its own name. It is called the

fundamental mode

The eigenvalue corresponding to this special eigenfunction is, it turns out, the

largest

eigenvalue of all. We, in nuclear engineering, give it a special name:

k_{eff} .

This value, k_{eff} , is the

multiplication factor

of the reactor. To see this, let's assume we know the eigenfunction that goes with the eigenvalue k_{eff} —call it $\phi_{\text{fund}}(\vec{r})$ for fundamental-mode scalar flux—and then let's integrate the equation over all energies and over the entire reactor volume:

$$\begin{aligned} & \iiint_V \left[-\vec{\nabla} \cdot \left[\int_0^\infty dE D(\vec{r}, E) \vec{\nabla} \phi_{\text{fund}}(\vec{r}, E) \right] + \int_0^\infty dE \Sigma_t(\vec{r}, E) \phi_{\text{fund}}(\vec{r}, E) \right] d^3r \\ &= \frac{1}{k_{\text{eff}}} \int_0^\infty dE \chi(E) \iiint_V \left[\int_0^\infty dE_i \nu \Sigma_f(\vec{r}, E_i) \phi_{\text{fund}}(\vec{r}, E_i) \right] d^3r \\ &+ \iiint_V \left[\int_0^\infty dE \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E) \phi(\vec{r}, E_i) \right] d^3r \end{aligned} \quad (9.171)$$

or

$$\begin{aligned} k_{\text{eff}} &= \frac{\iiint_V d^3r \int_0^\infty dE [\nu \Sigma_f \phi_{\text{fund}}]}{\iiint_V d^3r \int_0^\infty dE [-\vec{\nabla} \cdot D \vec{\nabla} \phi_{\text{fund}} + \Sigma_a \phi_{\text{fund}}]} \\ &= \frac{\text{production rate}}{\text{loss rate}} \end{aligned} \quad (9.172)$$

This agrees with an earlier “definition” of multiplication factor. It is also straightforward to show that the other “definition” of multiplication factor—the population in one generation divided by the population in the previous generation—is also satisfied by the fundamental-mode eigenfunction (ϕ_{fund}) and its eigenvalue (k_{eff}).

The **actual definition** of multiplication factor, according to energy-dependent diffusion theory, is

$$\text{multiplication factor} \equiv k_{\text{eff}} \equiv \text{largest eigenvalue of Eq. (9.170).}$$

We can extend this beyond the diffusion description. (Remember, the diffusion equation contains an approximation.) First we define two operators:

$$\mathbf{L} = \text{loss operator: } \mathbf{L}\phi \equiv \text{net loss rate density from leakage and collisions ,} \quad (9.173)$$

$$\mathbf{P} = \text{production operator: } \mathbf{P}\phi \equiv \text{production rate density from fission .} \quad (9.174)$$

With these definitions the k -eigenvalue problem can be written as

$$\mathbf{L}\phi = \frac{1}{k}\mathbf{P}\phi , \text{ or } \mathbf{L}^{-1}\mathbf{P}\phi = k\phi . \quad (9.175)$$

This makes it clear that

$$\begin{aligned} \text{multiplication factor} &\equiv k_{\text{eff}} \\ &\equiv \text{largest eigenvalue of the [Loss}^{-1}\text{ Production] operator.} \end{aligned} \quad (9.176)$$

This is the fundamental definition of the multiplication factor. Note that it is defined in terms of production and loss operators, not in terms of any neutron population. It is a property of the materials and size and shape of the reactor, and nothing else! This definition also holds no matter what approximation you might choose for the loss and production operators—whatever approximation you choose, your corresponding approximate multiplication factor will satisfy Eq. (9.176) with your approximate loss and production operators. Nature's multiplication factor satisfies this equation with nature's true loss and production operators. The thermal-group diffusion approximate multiplication factor satisfies it with diffusion theory's approximate loss and production operators.

Remember:

eigenvalue problems **never contain fixed sources!**

Eigenvalue problems tell us

characteristics of the system itself.

They couldn't do this if sources were present, because sources "force" the system.

Please do not forget this, or bad things can happen!

One last note: when solving a fixed-source problem, the magnitude of the flux is always proportional to the magnitude of the external source (see all of the examples solved previously). But now that we are solving an eigenproblem (which, again, does not contain an external source), we cannot give the magnitude of the solution. Suppose a function $\phi(\vec{r})$ is solution of the following equation:

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) = \frac{\nu\Sigma_f}{k}\phi(\vec{r}). \quad (9.177)$$

Then any multiple of $\phi(\vec{r})$ (say $C_0\phi(\vec{r})$ where C_0 is a constant) is also solution. We can see this because we can multiply the equation by C_0 and obtain:

$$-D\nabla^2C_0\phi(\vec{r}) + \Sigma_aC_0\phi(\vec{r}) = \frac{\nu\Sigma_f}{k}C_0\phi(\vec{r}) \quad (9.178)$$

9.12 Diffusion k -Eigenvalue Problems: Homogeneous Reactors

Consider now a **homogeneous** reactor, in which the material is the same everywhere in the reactor. This means D , χ , ν , and all the Σ 's are independent of position. For such a **homogeneous reactor** our **diffusion-theory k -eigenvalue equation** simplifies to:

$$\begin{aligned} -D(E)\nabla^2\phi(\vec{r}, E) + \Sigma_t(E)\phi(\vec{r}, E) &= \frac{1}{k}\chi(E)\int_0^\infty dE_i\nu\Sigma_f(E_i)\phi(\vec{r}, E_i) \\ &\quad + \int_0^\infty dE_i\Sigma_s(E_i \rightarrow E)\phi(\vec{r}, E_i) \end{aligned} \quad (9.179)$$

If we consider a "bare" reactor (one that is surrounded by vacuum), then the following extrapolated boundary condition is reasonable:

$$\phi(\vec{r}_s + d\vec{e}_n(\vec{r}_s), E) = 0, \quad \vec{r}_s \in \partial V. \quad (9.180)$$

In reality, it would be best to let the extrapolation distance, d , be different for different neutron energies. (Remember, we have seen that the best extrapolation distance to use is related to the diffusion coefficient, D , which varies with neutron energy.) But to keep the math tractable we shall declare the extrapolation distance to be independent of energy, for purposes of our explorations in this chapter. This is not a bad approximation if the reactor is much larger than the diffusion coefficient.

How do we attack our energy-dependent diffusion problem? The solution depends on both spatial variables and an energy variable. Let us try **separation of variables**:

$$\phi(\vec{r}, E) \stackrel{?}{=} \Phi(\vec{r})f(E) \quad (9.181)$$

(Here we use the symbol Φ to represent the spatial part of the separated solution. We will see later that we can interpret this as the total scalar flux, which is the integral of the energy-dependent scalar flux, $\phi(\vec{r}, E)$.) If we insert this guess into the energy-dependent diffusion equation, divide through by the product $[D(E)\Phi(\vec{r})f(E)]$, and rearrange, we obtain

$$-\frac{1}{\Phi(\vec{r})}\nabla^2\Phi(\vec{r}) = -\frac{\Sigma_t(E)}{D(E)} + \frac{\frac{1}{k}\chi(E)\int_0^\infty dE_i\nu\Sigma_f(E_i)f(E_i) + \int_0^\infty dE_i\Sigma_s(E_i \rightarrow E)f(E_i)}{f(E)D(E)} \quad (9.182)$$

Observe:

1. The right-hand side: does not depend on **position**
2. The left-hand side: does not depend on **energy**

It follows that each side must equal a constant. For historical reasons we shall call this constant “ B^2 .” We then obtain two separate problems:

Spatial problem:

$$-\frac{1}{\Phi(\vec{r})} \nabla^2 \Phi(\vec{r}) = B^2 \quad \text{or} \quad \nabla^2 \Phi(\vec{r}) + B^2 \Phi(\vec{r}) = 0 \quad \text{or} \quad -\nabla^2 \Phi(\vec{r}) = B^2 \Phi(\vec{r}) \quad (9.183)$$

$$\Phi(\vec{r}_s + d \vec{e}_n(\vec{r}_s)) = 0, \quad \vec{r}_s \in \partial V. \quad (9.184)$$

Energy problem:

$$B^2 = -\frac{\Sigma_t(E)}{D(E)} + \frac{\frac{1}{k}\chi(E) \int_0^\infty dE_i \nu \Sigma_f(E_i) f(E_i) + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i)}{f(E) D(E)} \quad (9.185)$$

or

$$[D(E)B^2 + \Sigma_t(E)] f(E) = \frac{1}{k}\chi(E) \int_0^\infty dE_i \nu \Sigma_f(E_i) f(E_i) + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i) \quad (9.186)$$

9.12.1 Important notes about “separated” problems

1. If we solve the separated problems, the spatial-shape function $\Phi(\vec{r})$ tells us how the neutrons distribute themselves spatially in the reactor, and the energy-shape function $f(E)$ tells us how the neutrons distribute themselves in energy. $f(E)$ is called the “neutron spectrum.”
2. The spatial equation, $-\nabla^2 \Phi(\vec{r}) = B^2 \Phi(\vec{r})$, is itself an eigenvalue equation. It is widely known as the “Helmholtz equation.”
 - When boundary conditions are included, it becomes a well-posed eigenvalue problem.
 - The solutions are the $\{B^2, \Phi\}$ pairs that satisfy the equation and its boundary conditions.
 - For such a pair, B^2 is an eigenvalue of the $-\nabla^2$ operator and $\Phi(\vec{r})$ is the associated eigenfunction.
 - Infinitely many $\{B^2, \Phi\}$ pairs satisfy the equation and its boundary conditions.
3. Suppose B^2 is an eigenvalue of the $-\nabla^2$ operator and $\Phi(\vec{r})$ is the associated eigenfunction. If $-D(E)\nabla^2[\Phi(\vec{r})f(E)]$ is “net outleakage rate density,” then we see that

$D(E)B^2[\Phi(\vec{r})f(E)]$ also is net outleakage rate density. It follows that $D(E)B^2$ is a kind of “leakage cross section.”

4. If we integrate the energy problem, Eq. (9.186), over all energies, we obtain

$$\begin{aligned} B^2 \int_0^\infty dE f(E) D(E) + \int_0^\infty dE f(E) \Sigma_t(E) &= \frac{1}{k} \underbrace{\left[\int_0^\infty dE \chi(E) \right]}_{=1} \int_0^\infty dE_i \nu \Sigma_f(E_i) f(E_i) \\ &\quad + \int_0^\infty dE_i \underbrace{\left[\int_0^\infty dE \Sigma_s(E_i \rightarrow E) \right]}_{=\Sigma_s(E_i)} f(E_i) \end{aligned} \quad (9.187)$$

If we now recognize that $\Sigma_a = \Sigma_t - \Sigma_s$ and divide each term in the equation by the integral of $f(E)$, we obtain:

$$B^2 \frac{\int_0^\infty dE f(E) D(E)}{\int_0^\infty dE f(E)} + \frac{\int_0^\infty dE f(E) \Sigma_a(E)}{\int_0^\infty dE f(E)} = \frac{1}{k} \frac{\int_0^\infty dE_i \nu \Sigma_f(E_i) f(E_i)}{\int_0^\infty dE f(E)} \quad (9.188)$$

Note that we have not made approximations—we just integrated. We recognize that each ratio is an average and use angle brackets to denote averages. Then:

$$B^2 \langle D \rangle + \langle \Sigma_a \rangle = \frac{1}{k} \langle \nu \Sigma_f \rangle \quad (9.189)$$

and

$$B^2 = \frac{\frac{1}{k} \langle \nu \Sigma_f \rangle - \langle \Sigma_a \rangle}{\langle D \rangle} \quad \Rightarrow k = \frac{\langle \nu \Sigma_f \rangle}{\langle \Sigma_a \rangle + \langle D \rangle B^2} \quad (9.190)$$

We see that there is a one-to-one correspondence between k and B^2 .

How do we solve these energy-dependent diffusion-theory k -eigenvalue problems for bare homogeneous reactors? Three basic steps:

1. Solve the Helmholtz problem for the given reactor's size and shape. This gives B^2 values and $\Phi(\vec{r})$ functions. $\Phi(\vec{r})$ describes the neutrons' spatial distribution.
(There are infinitely many solution pairs. We are usually most interested in the one with the smallest B^2 eigenvalue, which corresponds to the largest k eigenvalue, which is the multiplication factor.)
2. Solve the energy problem, Eq. (9.186), with B^2 known from the first step. This gives the spectrum $f(E)$, which describes the neutrons' energy distribution.
(There is a different $f(E)$ for each different B^2 eigenvalue. We are usually most interested in the smallest B^2 eigenvalue, as noted above.)
3. Use the spectrum to generate averaged D , Σ_a , and $\nu\Sigma_f$ values, then use Eq. (9.190) to obtain k . If all of this was done using the smallest B^2 eigenvalue, then the resulting k is the multiplication factor of the reactor.

In the remainder of this chapter we shall focus on solving the spatial (Helmholtz) problem for the B^2 eigenvalues and $\Phi(\vec{r})$ eigenfunctions, for a variety of reactor shapes. We defer solution of the energy problem to a later chapter. That is, we will solve the following problem:

$$-\nabla^2\Phi(\vec{r}) = B^2\Phi(\vec{r}) \quad (9.191)$$

$$\Phi(\vec{r}_s + d \vec{e}_n(\vec{r}_s)) = 0, \quad \vec{r}_s \in \partial V, \quad (9.192)$$

for various reactor shapes. Note that from Eq. (9.190) we can rewrite our equation as:

$$-\langle D \rangle \nabla^2 \Phi(\vec{r}) + \langle \Sigma_a \rangle \Phi(\vec{r}) = \frac{1}{k} \langle \nu \Sigma_f \rangle \Phi(\vec{r}). \quad (9.193)$$

Here the $\langle \rangle$ quantities are averages over energy, with weight function $f(E)$ (the neutron spectrum in the reactor). We will solve for $f(E)$ in a subsequent chapter. For now we just acknowledge that if we knew $f(E)$ we could obtain the $\langle \rangle$ quantities, and the above equation would be satisfied by the $\Phi(\vec{r})$ function that satisfies the Helmholtz problem.

In what follows we get tired of writing the angle brackets, so we shall drop those for now and just agree among ourselves that in the remainder of this chapter, the quantities D , Σ_a , and $\nu\Sigma_f$ are short-hand notation for $\langle D \rangle$, $\langle \Sigma_a \rangle$, and $\langle \nu \Sigma_f \rangle$

9.12.2 k -Eigenvalue Solution: Bare Homogeneous Slab

We consider now a **bare, homogeneous** slab reactor (infinite in y and z), of width a in the x direction. Bare means it is

surrounded by nothing but **vacuum**.

Homogeneous means

all material properties are **constant**.

We want to know whether this reactor is critical, and also what the neutrons are doing in it. To find out, we must solve the following eigenvalue problem:

$$-D \frac{d^2\phi(x)}{dx^2} + \Sigma_a \phi(x) = \frac{1}{k} \nu \Sigma_f \phi(x) , \quad x \in \left(-\frac{a}{2}; \frac{a}{2}\right) , \quad (9.194)$$

$$\phi(\tilde{a}/2) = \phi(-\tilde{a}/2) = 0 . \quad (9.195)$$

where

$$\tilde{a} = \text{extrapolated slab width} \equiv a + 2d$$

with d is the extrapolation distance. Note that the extrapolated slab width has two additional distances added to it—one on the left and one on the right. (Unless you are told otherwise, use $d = 2D$. Remember this! So in this problem you would use $\tilde{a} = a + 4D$.)

9.12.2.1 Solving the eigenproblem

Let us rewrite our equation as follows:

$$\frac{d^2\phi(x)}{dx^2} + \frac{\frac{\nu\Sigma_f}{k} - \Sigma_a}{D} \phi(x) = 0 , \quad x \in \left(-\frac{a}{2}; \frac{a}{2}\right) \quad (9.196)$$

and **define** the number B^2 :

$$B^2 \equiv \frac{\frac{\nu\Sigma_f}{k} - \Sigma_a}{D} = \frac{\frac{1}{k} \frac{\nu\Sigma_f}{\Sigma_a} - 1}{L^2} = \frac{\frac{k_\infty}{k} - 1}{L^2} \quad (9.197)$$

This gives us

$$\frac{d^2\phi(x)}{dx^2} + B^2 \phi(x) = 0 , \quad x \in \left(-\frac{a}{2}; \frac{a}{2}\right) \quad (9.198)$$

Note and remember: there is a simple one-to-one relationship between B^2 and the eigenvalue k . In fact, we can think of B^2 as an eigenvalue itself. Equation (9.198) shows, in fact, that B^2 is an eigenvalue of the negative of the Laplacian operator. That is, B^2 is an eigenvalue of $-\nabla^2$.

Our equation cannot be satisfied by just any old value of B^2 ; only certain characteristic values will work. We must find those values.

Let us solve the problem. Note:

- If $B^2 > 0$, then $A \sin(Bx) + C \cos(Bx)$ satisfies the equation.
- If $B^2 < 0$ then $A \sinh(Bx) + C \cosh(Bx)$ satisfies the equation.
- If $B^2 = 0$ then $A + C x$ satisfies the equation.
- Only the $B^2 > 0$ solution can satisfy the boundary conditions, which say that the function equals zero at two different values of x , namely $\pm\tilde{a}/2$.

So the only values of B^2 that satisfy the eigenproblem are $B^2 > 0$, and the most general ϕ that satisfies the equation is a linear combination of sines and cosines:

$$\phi(x) = A \sin(Bx) + C \cos(Bx) \quad (9.199)$$

Let us apply the boundary conditions:

$$A \sin\left(-B\frac{\tilde{a}}{2}\right) + C \cos\left(-B\frac{\tilde{a}}{2}\right) = 0 \quad (9.200)$$

$$A \sin\left(B\frac{\tilde{a}}{2}\right) + C \cos\left(B\frac{\tilde{a}}{2}\right) = 0 \quad (9.201)$$

If these equations had to be true for all constants B , the only solution would be

the trivial solution: $A = C = 0$.

(To see this, add the equations. The sine terms cancel. If the result had to be true for all B this would mean $C = 0$. If you subtract the equations you find that if the result had to be true for all B , then it would mean $A = 0$.)

We are not interested in zero solutions. Instead, we seek those characteristic values of B that allow nontrivial solutions. If we add the boundary-condition equations (9.200 and 9.201), we obtain

$$2C \cos\left(B\frac{\tilde{a}}{2}\right) = 0 \quad (9.202)$$

Either C or the cosine term must be zero. The cosine term = 0 for the following values of B :

$$B_n = \frac{n\pi}{\tilde{a}}, \quad n = 1, 3, 5, \dots \quad (9.203)$$

These are some of the eigenvalues of our slab problem. The associated eigenfunctions are

$$C_n \cos(B_n x) = C_n \cos\left(\frac{n\pi x}{\tilde{a}}\right), \quad n = 1, 3, 5, \dots \quad (9.204)$$

where C_n is any constant (and n is odd).

If we subtract the boundary-condition equations (9.200 and 9.201) we obtain

$$2A \sin\left(B \frac{\tilde{a}}{2}\right) = 0. \quad (9.205)$$

Either A or the sine term must be zero. The sine term will be zero for the following values of B :

$$B_n = \frac{n\pi}{\tilde{a}}, \quad n = 2, 4, 6, \dots$$

These are the rest of the eigenvalues of our slab problem. The associated eigenfunctions are

$$A_n \sin(B_n x) = A_n \sin\left(\frac{n\pi x}{\tilde{a}}\right), \quad n = 2, 4, 6, \dots \quad (9.206)$$

where A_n is any constant (and n is even).

That's it. We have completely solved the eigenvalue problem specified by Eq. (9.196). Now we will discuss the solution at some length.

9.12.2.2 “Modes”

The n^{th} eigenfunction is often called the

n^{th} mode.

There is only one eigenfunction that can be positive throughout the problem domain. It is called the

fundamental mode.

In our problem the fundamental mode is

$$\text{fundamental mode} = C_1 \cos(B_1 x) = C_1 \cos\left(\frac{\pi x}{\tilde{a}}\right)$$

Sometimes we are given information that allows us to rule out some eigensolutions. For example, we may be told the solution is symmetric; if so, then we would know that

there were no sine terms present in our solution.

A very important special case is source-free steady-state. Remember this:

In a source-free steady state reactor, only **the fundamental mode is present.**

Note that eigenvalue problems tell us only the shapes of the modes. Even if we were told that our reactor is source-free and steady-state, we would still need more information to determine the **amplitude** or magnitude of the solution—that is, to determine

the value of C_1 .

The “additional information” might be the total reactor power, the value of the peak scalar flux, the total net outleakage rate, etc.—something that nails down the amplitude of the flux.

In Figure 9.6 we sketch the first few eigenfunctions (“modes”) of our bare homogeneous slab reactor. As noted previously, only the fundamental mode is positive throughout the domain. Note that the fundamental mode is “less leaky” than the others: it has a

smaller gradient at the boundaries.

9.12.3 Important Concepts, Terms, and Definitions

In this subsection we define some terms that nuclear engineers find very useful in analyzing, designing, and discussing reactors. You should understand them all.

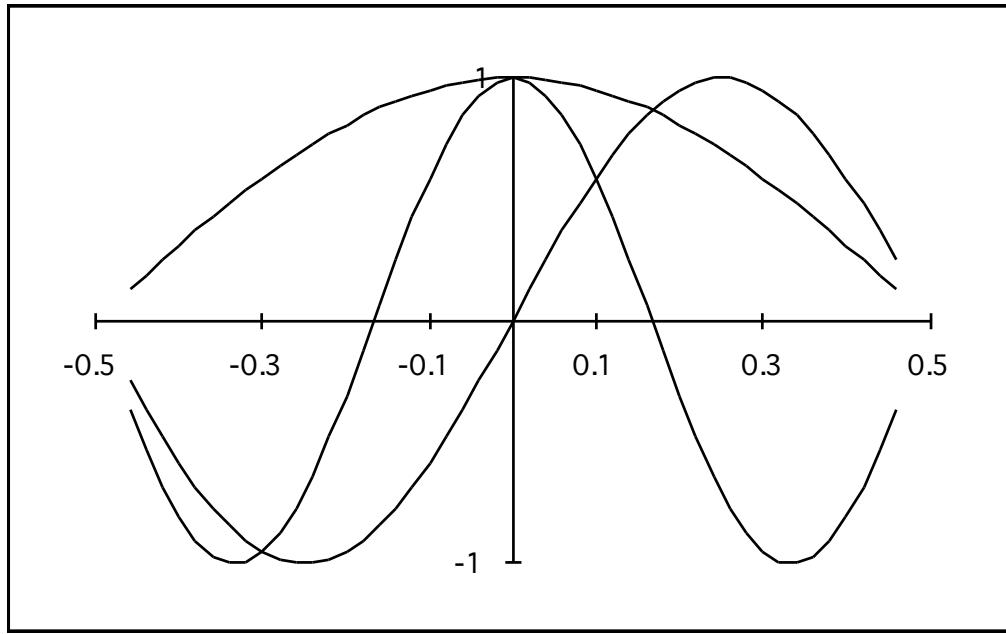


Figure 9.6: Sketch of first 3 eigenmodes for a 1D slab centered at the origin.

9.12.3.1 Geometric buckling

Recall the eigenvalue equation that we have been working on, and write it for the n^{th} {eigenvalue, eigenfunction} pair:

$$\frac{d^2\phi_n(x)}{dx^2} + B_n^2 \phi_n(x) = 0 \quad (9.207)$$

Divide through by $\phi_n(x)$:

$$B_n^2 = -\frac{1}{\phi_n(x)} \frac{d^2\phi_n(x)}{dx^2} \quad (9.208)$$

Note:

1. B_n^2 is a measure of solution curvature.
2. Every $B_n^2 > 0$. Thus, where ϕ_n is positive it is concave down, and where ϕ_n is negative it is concave up. Check this by examining Fig. 9.6.

We have a name for the B_n^2 associated with the fundamental mode:

$$\text{Fundamental-mode } B_n^2 \equiv B_g^2 \equiv \text{ "geometric buckling."} \quad (9.209)$$

The geometric buckling, B_g^2 , is a measure of how much the fundamental mode is curved, or “buckled.” In a bare slab we have

$$B_g^2 = \left(\frac{\pi}{\tilde{a}}\right)^2 \quad \text{geometric buckling, bare homogeneous slab} \quad (9.210)$$

If you forget this, bad things are likely to happen (for example, to your grade)! (Build your intuition: how does curvature change as the reactor width increases? What is the curvature of the fundamental mode in an infinite medium?)

9.12.3.2 Connection with k

The problem we wrote down originally was

$$-D \frac{d^2\phi_n(x)}{dx^2} + \Sigma_a \phi_n(x) = \frac{\nu \Sigma_f \phi_n(x)}{k_n} \quad (9.211)$$

We have written it here explicitly denoting the n^{th} $\{k_n, \phi_n\}$ pair. We then wrote it in a slightly simpler-looking form by defining

$$B_n^2 = \frac{\frac{1}{k_n} \frac{\nu \Sigma_f}{\Sigma_a} - 1}{L^2} \quad (9.212)$$

This is the relationship between the B -eigenvalues and the k -eigenvalues. Solve for k_n :

$$k_n = \frac{\frac{\nu \Sigma_f}{\Sigma_a}}{1 + L^2 B_n^2} = \frac{k_\infty}{1 + L^2 B_n^2} = \frac{\nu \Sigma_f}{\Sigma_a + D B_n^2} \quad (9.213)$$

The k_n associated with the fundamental mode has a special name:

Fundamental $k_n \equiv k_{\text{eff}} = \text{“k-effective”} = \text{multiplication factor .}$ (9.214)

It is obvious that we have:

$$k_{\text{eff}} = \frac{\frac{\nu \Sigma_f}{\Sigma_a}}{1 + L^2 B_g^2} = \frac{\nu \Sigma_f}{\Sigma_a + D B_g^2}, \quad (9.215)$$

because B_g^2 is always the smallest of the B_n^2 eigenvalues.

9.12.3.3 Materials Buckling

If our reactor is **critical**, then $k_{\text{eff}} = 1$ and

$$B_g^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} \quad (9.216)$$

The term on the left is geometric buckling, which depends only on reactor geometry (shape and (extrapolated) size). (In a bare slab, the only “geometry” information is the extrapolated slab width.) The term on the right depends only on

material properties.

We assign this term a special name:

$$\frac{\nu\Sigma_f - \Sigma_a}{D} = B_m^2 = \text{“materials buckling,”} \quad (9.217)$$

If the two bucklings are equal—that is if the material properties and the reactor geometry are a perfect match for each other—then:

the reactor is critical.

In general,

$$\begin{aligned} B_m^2 &< B_g^2 \Leftrightarrow \text{subcritical} \\ B_m^2 &= B_g^2 \Leftrightarrow \text{critical} \\ B_m^2 &> B_g^2 \Leftrightarrow \text{supercritical} \end{aligned}$$

The critical relationship is easy to remember. To easily remember the others, imagine what must happen when Σ_f gets larger or smaller (which makes the materials buckling larger or smaller).

The above relations apply to any bare homogeneous reactor, regardless of shape! Only the definition of B_g^2 , which of course depends on geometry, will change.

9.12.4 Bare Homogeneous Sphere

9.12.4.1 Solving the eigenproblem

We consider next a bare homogeneous spherical reactor. Bare means it is surrounded by nothing but vacuum. Homogeneous means all material properties are constant. We want

to know whether or not this reactor is critical, and also what the neutrons are doing in it. We have to solve the following eigenvalue problem (the expression of ∇^2 is slightly more complicated, but we can look it up in the appendix, or in a math book, or a table in these notes, or the web, or . . .):

$$-D \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi_n(r)}{dr} \right) + \Sigma_a \phi_n(r) = \frac{1}{k_n} \nu \Sigma_f \phi_n(r), \quad r \in [0; R] \quad (9.218)$$

$$\phi_n(\tilde{R}) = 0 \quad [\text{that's 1 BC}] \quad (9.219)$$

$$\left. \frac{d\phi_n}{dr} \right|_{r=0} = 0 \quad \text{or} \quad \phi_n(r)|_{r=0} < \infty \quad [\text{that's the 2nd BC}] \quad (9.220)$$

where

$$\tilde{R} = \text{extrapolated radius} = R + d,$$

with d is the extrapolation distance. (Unless you are told otherwise, use $d = 2D$.) Note that you have a choice of conditions you can impose at $r = 0$. In a symmetric sphere we know that the derivative of the scalar flux must equal zero at the origin; this is one condition you could impose. Alternatively, since there are no singular sources in this problem **we know that the scalar flux remains finite everywhere in the sphere**, including at $r = 0$. Sometimes it is easier to impose this condition, and it gives the same answer.

We have recognized, by using the subscript n , that our eigenvalue problem will have many {eigenvalue, eigenfunction} pairs as solutions.

If we bring all terms to the left side of Eq. (9.218) and divide through by $(-D)$, we obtain

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi_n(r)}{dr} \right) + B_n^2 \phi_n(r) = 0 \quad \text{for } r \in [0; R]$$

where we have defined:

$$B_n^2 = \frac{\frac{\nu \Sigma_f}{k_n} - \Sigma_a}{D} = \frac{\frac{\nu \Sigma_f}{\Sigma_a k_n} - 1}{L^2} \quad (9.221)$$

exactly as we did before. Note that there is a one-to-one relationship between B_n^2 and the eigenvalue k_n . In fact, we can think of B_n^2 as an eigenvalue itself. Eigenvalue means characteristic value. Our equation can't be satisfied by just any old B^2 ; only certain characteristic values will work.

Let's solve the problem. As before, we can figure out that B^2 must be > 0 , because the solutions associated with negative or zero values cannot satisfy the boundary conditions. You can verify (by plugging it in) that the following satisfies our equation for $B^2 > 0$:

$$\phi_n(r) = \frac{A_n \sin(B_n r) + C_n \cos(B_n r)}{r} \quad (9.222)$$

(You can derive this solution by making the following substitution: $\phi_n(r) = w(r)/r$, which will produce a slab-like equation for $w(r)$. Try it!)

Now we apply boundary conditions. First consider $r = 0$. As $r \rightarrow 0$, $\sin(B_n r)/r \rightarrow B_n$, so the term multiplied by A_n remains finite. However, as $r \rightarrow 0$, $\cos(B_n r)/r \rightarrow \infty$. Thus, the solution can remain finite at $r = 0$ only if

$$C_n = 0 . \quad (9.223)$$

Now the vacuum boundary condition looks like:

$$\frac{A_n \sin(B_n \tilde{R})}{\tilde{R}} = 0 \quad (9.224)$$

If this equation had to be true for all constants B_n , the only solution would be the trivial solution, $A_n = 0$. We are not interested in zero solutions. Instead, we seek those certain values of B —characteristic values for this geometry—that allow nontrivial solutions. The sine term will be zero for the following values of B_n :

$$B_n = \frac{n\pi}{\tilde{R}} \quad \text{for } n = 1, 2, 3, \dots$$

The squares of these values, B_n^2 , are the eigenvalues of our spherical Laplacian eigenvalue problem. The associated eigenfunctions are:

$$\phi_n(r) = \frac{A_n \sin(B_n r)}{r} \quad (9.225)$$

where A_n is any constant.

That's it. We have completely solved the diffusion k -eigenvalue problem for a bare homogeneous sphere. Now we will discuss the solution.

The one eigenfunction that does not change sign anywhere in the problem domain is called the **fundamental mode**. In the bare-sphere problem the fundamental mode is

$$\text{fundamental mode} = \phi_{\text{fund}}(r) = A_1 \frac{1}{r} \sin\left(\frac{\pi r}{\tilde{R}}\right) \quad (9.226)$$

The following provides some insight into what the “fundamental mode” means. Suppose the fissioning nuclides in a bare homogeneous reactor could be altered such that $\nu(E)$ became $\nu(E)/k$, where k is the multiplication factor of the reactor. The altered reactor would then be critical. If this altered reactor were left alone for a while with no fixed sources, the neutron population would reach steady state and the scalar flux would have the “shape” (distribution in x, y, z , and E) of the fundamental mode.

An eigenfunction just describes the

shape

of the flux. If this is the shape of the flux in an actual physical problem, the magnitude of the flux must be determined by other conditions. For example, we may be told that a spherical reactor is operating in steady state at a certain power level. If Eq. (9.226) described the total scalar flux as a function of position, and we are told that 190 MeV of energy is deposited per fission, then we can write an expression that we can solve for the constant A_1 :

$$P = \text{known power [energy per unit time]}$$

$$\approx A_1 \Sigma_{f,th} \left[\frac{190 \text{ MeV}}{\text{fission}} \right] \int_0^R \frac{1}{r} \sin \left(\frac{\pi r}{\tilde{R}} \right) 4\pi r^2 dr , \quad (9.227)$$

where we have recognized that the volume element in spherical coordinates= $4\pi r^2 dr$.

What the modes look like. Let us sketch the first few eigenfunctions of our spherical reactor (see Figure 9.7). Note that the fundamental mode is the only one that doesn’t change sign in the reactor volume, and note that the modes become increasingly oscillatory as n increases. That is, higher- n modes oscillate positive-negative-positive over shorter distances.

9.12.4.2 Geometric buckling for bare homogeneous sphere

Recall that:

The B_n^2 associated with the fundamental mode $\equiv B_g^2$ = “geometric buckling.”

It is a measure of how much the fundamental mode is curved, or “buckled.” In a bare sphere we have

$$B_g^2 = \left(\frac{\pi}{\tilde{R}} \right)^2 \quad \text{geometric buckling, bare sphere} \quad (9.228)$$

If you forget this, bad things could happen.

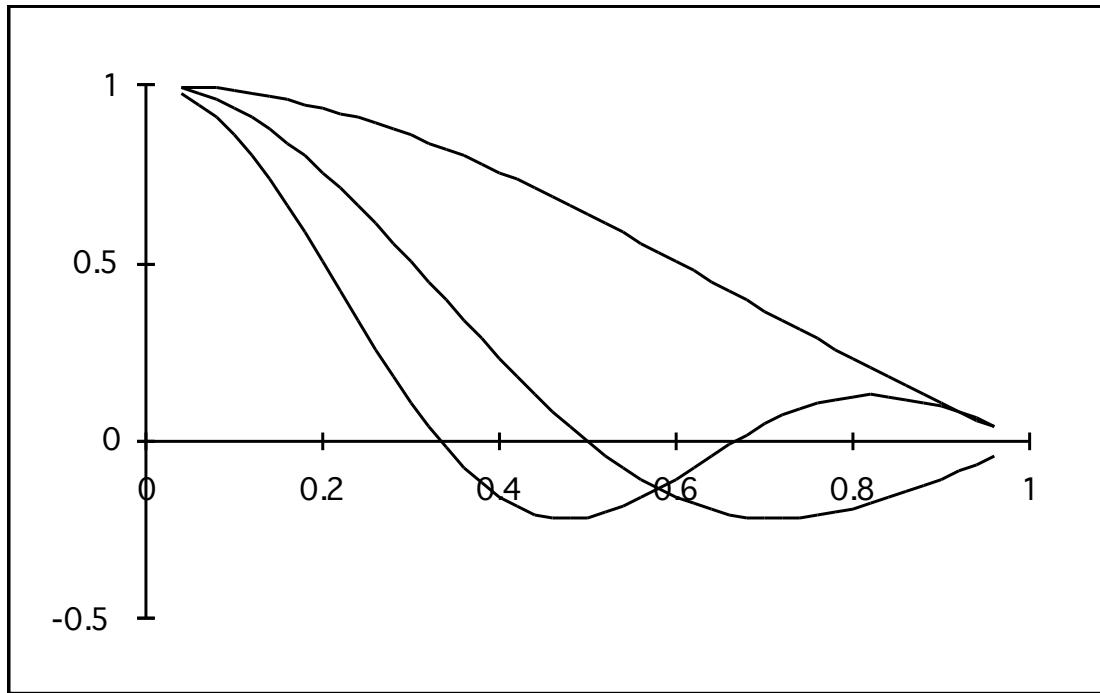


Figure 9.7: Sketch of the first 3 eigenmodes for a bare homogeneous spherical reactor.

9.12.4.3 Connection with k_{eff} .

This is just like in slab geometry, or any other bare geometry for that matter. We still have:

$$B_n^2 = \frac{\frac{\nu \Sigma_f}{\Sigma_a k_n} - 1}{L^2} \Rightarrow k_n = \frac{\nu \Sigma_f}{\Sigma_a + D B_n^2} \quad (9.229)$$

And of course, we still have:

$$k_{\text{eff}} = k_{\text{largest}} = \frac{\nu \Sigma_f}{\Sigma_a + D B_g^2} = \frac{\frac{\nu \Sigma_f}{\Sigma_a}}{1 + L^2 B_g^2} \quad (9.230)$$

9.12.5 Bare Shoebox (= Brick = Rectangular Parallelepiped)

We have seen that the spatial problem we must solve in a bare homogeneous reactor is $-\nabla^2\phi = B^2\phi$. In a bare homogeneous rectangular parallelepiped (brick- or shoebox-shaped) reactor, this becomes:

$$-\left[\frac{\partial^2\phi(x,y,z)}{\partial x^2} + \frac{\partial^2\phi(x,y,z)}{\partial y^2} + \frac{\partial^2\phi(x,y,z)}{\partial z^2}\right] = B^2\phi(x,y,z) \quad (9.231)$$

$$\text{for } \{x,y,z\} \in \left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right] \times \left[-\frac{c}{2}, \frac{c}{2}\right]. \quad (9.232)$$

The extrapolated boundary conditions are defined over the entirety of each of the six surfaces:

$$\phi\left(\frac{\tilde{a}}{2}, y, z\right) = \phi\left(-\frac{\tilde{a}}{2}, y, z\right) = 0, \quad y \in \left(-\frac{\tilde{b}}{2}, \frac{\tilde{b}}{2}\right), \quad z \in \left(-\frac{\tilde{c}}{2}, \frac{\tilde{c}}{2}\right), \quad (9.233)$$

$$\phi\left(x, \frac{\tilde{b}}{2}, z\right) = \phi\left(x, -\frac{\tilde{b}}{2}, z\right) = 0, \quad x \in \left(-\frac{\tilde{a}}{2}, \frac{\tilde{a}}{2}\right), \quad z \in \left(-\frac{\tilde{c}}{2}, \frac{\tilde{c}}{2}\right), \quad (9.234)$$

$$\phi\left(x, y, \frac{\tilde{c}}{2}\right) = \phi\left(x, y, -\frac{\tilde{c}}{2}\right) = 0, \quad x \in \left(-\frac{\tilde{a}}{2}, \frac{\tilde{a}}{2}\right), \quad y \in \left(-\frac{\tilde{b}}{2}, \frac{\tilde{b}}{2}\right), \quad (9.235)$$

where the \sim quantities are extrapolated widths & heights & lengths:

$$\tilde{a} = a + 2d, \quad \tilde{b} = b + 2d, \quad \tilde{c} = c + 2d, \quad (9.236)$$

and d is the extrapolation distance. (Unless you are told otherwise, use $d = 2D$.) **Note that the y and z ranges must be spelled out on the boundaries where x is constant (i.e., at $x = \pm\tilde{a}/2$).** A similar comment holds for each of the other surfaces.

There is a fundamental difference between this and our previous eigenvalue problems: our equation is now a

partial

differential equation. How do we solve it? A standard approach to partial differential equations—one that works in many problems of interest—is to guess a **separable** solution. That is, we guess that the unknown function, which in our case is $\phi(x, y, z)$, is the product of functions of single variables:

$$\phi(x, y, z) \xrightarrow{\text{guess}} X(x) \times Y(y) \times Z(z). \quad (9.237)$$

We insert this hypothesis into our equation (noting that the derivatives become full derivatives because each function depends on a single variable):

$$-\left[Y(y)Z(z)\frac{d^2X(x)}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} \right] = B^2X(x)Y(y)Z(z) \quad (9.238)$$

and then divide through by $-X(x)Y(y)Z(z)$:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -B^2 \quad (9.239)$$

Note:

- the first term depends only on x ,
- the second term depends only on y ,
- the third term depends only on z ,
- the right-hand side doesn't depend on x , y , or z .

If the second, third, and fourth terms don't change when we vary x , the first term doesn't change either (because they are in an equation together). This (and similar reasoning for y and z) means that every term is a constant! We give these constants names:

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \text{constant} = -B_x^2, \quad (9.240)$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = \text{constant} = -B_y^2, \quad (9.241)$$

$$\frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = \text{constant} = -B_z^2. \quad (9.242)$$

If we add these three equations we see that we must have the following relationship between these constants and the B^2 defined above:

$$B^2 = B_x^2 + B_y^2 + B_z^2. \quad (9.243)$$

Equations 9.240 through 9.242 are three separate eigenvalue problems of the type we have already solved. For example, the X problem looks like:

$$\frac{d^2 X(x)}{dx^2} + B_x^2 X(x) = 0$$

with

$$X\left(\pm\frac{\tilde{a}}{2}\right) = 0$$

The Y and Z problems are similar. We have already solved these problems! Each is just a slab-geometry problem! The solutions are:

$$B_{x,i} = \frac{i\pi}{\tilde{a}}, \quad i = 1, 2, 3, \dots \quad (9.244)$$

$$X_i(x) = \begin{cases} C_{x,i} \cos\left(\frac{i\pi x}{\tilde{a}}\right), & i = 1, 3, 5, \dots \\ A_{x,i} \sin\left(\frac{i\pi x}{\tilde{a}}\right), & i = 2, 4, 6, \dots \end{cases} \quad (9.245)$$

$$B_{y,j} = \frac{j\pi}{\tilde{b}}, \quad j = 1, 2, 3, \dots \quad (9.246)$$

$$Y_j(y) = \begin{cases} C_{y,j} \cos\left(\frac{j\pi y}{\tilde{b}}\right), & j = 1, 3, 5, \dots \\ A_{y,j} \sin\left(\frac{j\pi y}{\tilde{b}}\right), & j = 2, 4, 6, \dots \end{cases} \quad (9.247)$$

and

$$B_{z,\ell} = \frac{\ell\pi}{\tilde{c}}, \quad \ell = 1, 2, 3, \dots \quad (9.248)$$

$$Z_\ell(z) = \begin{cases} C_{z,\ell} \cos\left(\frac{\ell\pi z}{\tilde{c}}\right), & \ell = 1, 3, 5, \dots \\ A_{z,\ell} \sin\left(\frac{\ell\pi z}{\tilde{c}}\right), & \ell = 2, 4, 6, \dots \end{cases}$$

Recall that our original equation was

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi = 0$$

where

$$B^2 \equiv \frac{\frac{1}{k} \frac{\nu \Sigma_f}{\Sigma_a} - 1}{L^2} \quad (9.249)$$

We see now that $\phi_{ij\ell}(x, y, z)$ and $B_{ij\ell}^2$ satisfy this equation if

$$(i, j, \ell)\text{-th mode} = \phi_{ij\ell}(x, y, z) = X_i(x)Y_j(y)Z_\ell(z) \quad (9.250)$$

$$(i, j, \ell)\text{-th } B^2 \text{ eigenvalue} = B_{ij\ell}^2 = \left(\frac{i\pi}{\tilde{a}}\right)^2 + \left(\frac{j\pi}{\tilde{b}}\right)^2 + \left(\frac{\ell\pi}{\tilde{c}}\right)^2 \quad (9.251)$$

We have solved the problem for the bare homogeneous brick! So: What is the fundamental mode, and what is the geometric buckling?

$$\text{fundamental mode: } \phi_{\text{fund}}(x, y, z) = \phi_{111}(x, y, z) = [\text{constant}] \times \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \cos\left(\frac{\pi z}{c}\right)$$

$$\text{geometric buckling: } B_g^2 = B_{111}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2$$

**Fundamental Mode and Geometric Buckling,
Bare Homogeneous Rectangular Parallelepiped**

What is k_{eff} for a bare homogeneous brick-shaped reactor, according to diffusion theory?

$$k_{\text{eff}} = \frac{\nu \Sigma_f}{\Sigma_a + DB_g^2} = \frac{\nu \Sigma_f}{\Sigma_a + D \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2 \right]} \quad (9.252)$$

9.12.6 Bare Homogeneous Infinite Cylinder

We consider next a bare, homogeneous infinitely-long cylindrical reactor of radius R . We want to know whether or not this reactor is critical, and also what the neutrons in it are doing. We have to solve the same kind of eigenvalue problem we have already solved for slabs, spheres, and bricks. The only difference is the form of the Laplacian operator. In an infinitely long homogeneous cylinder we have the following eigenvalue problem:

$$-\frac{1}{r} \frac{d}{dr} \left[r \frac{d\phi_n(r)}{dr} \right] = B_n^2 \phi_n(r) \text{ for } r < R, \quad (9.253)$$

$$\phi_n(\tilde{R}) = 0 \quad (9.254)$$

$$\left. \frac{d\phi_n}{dr} \right|_{r=0} = 0 \quad \text{or} \quad \left. \phi_n(r) \right|_{r=0} < \infty, \quad (9.255)$$

where \tilde{R} = extrapolated radius = $R + d$, with d is the extrapolation distance. (Use $d = 2D$ unless told otherwise.) Note that the Laplacian has $1/r$ and an r factors that the Cartesian Laplacian does not have. Also note that there is a choice of conditions to impose at the origin. You may use whichever is more convenient. (The finiteness condition is fair game because there are no singular sources in this problem.)

Let us rewrite our equation as we have done before:

At this stage of our slab-geometry analysis we recognized that the solutions of our differential equations were sines and cosines. At this stage of our spherical-geometry analysis we recognized that we could perform a substitution, $\phi(r) = w(r)/r$, to find that the solutions were $(1/r) \times$ sines and cosines. But cylindrical geometry is different. Its solution will not involve sines and cosines.

To proceed, we carry out the derivatives in the Laplacian term and rewrite our equation in a form that you may recognize:

$$\frac{d^2\phi_n}{dr^2} + \frac{1}{r} \frac{d\phi_n(r)}{dr} + (B_n^2 - 0) \phi_n(r) = 0 \text{ for } r < R \quad (9.256)$$

We now recognize that this is

Bessel's Equation

(but he'll let you use it). If $B^2 > 0$ (which it is, as we showed earlier for slab geometry), the solution is a linear combination of

“ordinary” Bessel functions

of the first and second kinds (see Appendix II of this set of notes, or see any reference about Bessel functions):

$$\phi_n(r) = A_n J_0(B_n r) + C_n Y_0(B_n r) \quad (9.257)$$

Please understand: J_0 and Y_0 are names of functions, like sin and cos and ln. They are not constants—that's not what the 0 subscript means here!

If we were solving fixed-source steady-state subcritical problems, then $-1/L^2$ would play the role of B^2 in our equations, and our solution would be a combination of modified Bessel functions. This is much like the slab and sphere cases in which the eigenvalue problem involves sines and cosines but the fixed-source problem involves hyperbolic sines and cosines. That is, the modified Bessel functions can be thought of as the “hyperbolic” versions of Bessel functions, and in fact this terminology has been used in the past.

Back to our problem. The boundary condition at $r = 0$ requires that the derivative of the scalar flux be 0 or that the scalar flux be finite. The function $Y_0(r)$ is unbounded when $r \rightarrow 0$, whereas $J_0(r)$ remains bounded and has a zero derivative as $r \rightarrow 0$. It follows that we must have

$$C_n = 0 .$$

and therefore

$$\phi_n(r) = A_n J_0(B_n r) . \quad (9.258)$$

Our extrapolated boundary condition tells us that either

$$A_n = 0 \text{ [which doesn't interest us]} \\ \text{or}$$

$$J_0\left(B_n \tilde{R}\right) = 0$$

Thus, the eigenvalues B_n are the numbers such that

$$B_n \tilde{R} = z_n , \text{ where } z_n = \text{the } n\text{-th “zero” of } J_0.$$

This is analogous to B_n being the n -th ‘zero’ of the sine function in the spherical problem. The zeroes of the sine are simple: the sine function = 0 when its argument = $n\pi$ for

$n = 1, 2, 3\dots$ The zeroes of J_0 are not this easy to write down, but the concept is the same and the zeroes are known.

The smallest ‘zero’ of J_0 is

$$z_{\text{smallest}} \equiv \zeta_0 = 2.405\dots \quad (9.259)$$

The geometric buckling of a bare, infinitely long homogeneous cylinder, according to diffusion theory, is therefore

$$B_g^2 = \left(\frac{2.405\dots}{\tilde{R}} \right)^2 \quad \text{geometric buckling, infinitely long cylinder} \quad (9.260)$$

The fundamental mode is always the eigenfunction associated with the smallest B^2 (which is the geometric buckling):

$$\phi_{\text{fund}}(r) = A_0 J_0 \left(\frac{2.405\dots}{\tilde{R}} r \right) \quad \text{fund. mode, infinitely long cylinder} \quad (9.261)$$

9.13 Homogeneous-Reactor Eigenvalues and Eigenfunctions

Table 9.2 collects geometric buckling and fundamental-mode scalar-flux functions for various bare homogeneous reactor shapes. Recall that the multiplication factor is

$$k_{\text{eff}} = \frac{\nu \Sigma_f}{\Sigma_a + DB_g^2} \xrightarrow{\text{previous definitions}} \frac{\langle \nu \Sigma_f \rangle}{\langle \Sigma_a \rangle + \langle D \rangle B_g^2} \quad (9.262)$$

Table 9.2: Geometric Bucklings & Fundamental Modes, Bare Homogeneous Reactors.

Geometry	Extrap'd dims.	B_g^2	ϕ_{fund}
Slab	$\tilde{a} = a + 4D$	$\left(\frac{\pi}{\tilde{a}}\right)^2$	$A \cos\left(\frac{\pi x}{\tilde{a}}\right)$
Sphere	$\tilde{R} = R + 2D$	$\left(\frac{\pi}{\tilde{R}}\right)^2$	$A \frac{1}{r} \sin\left(\frac{\pi r}{\tilde{R}}\right)$
∞ cylinder [†]	$\tilde{R} = R + 2D$	$\left(\frac{\zeta_0}{\tilde{R}}\right)^2$	$A J_0\left(\frac{\zeta_0 r}{\tilde{R}}\right)$
Finite Cyl. [†]	$\tilde{R} = R + 2D$ $\tilde{H} = H + 4D$	$\left(\frac{\zeta_0}{\tilde{R}}\right)^2 + \left(\frac{\pi}{\tilde{H}}\right)^2$	$A J_0\left(\frac{\zeta_0 r}{\tilde{R}}\right) \cos\left(\frac{\pi z}{\tilde{H}}\right)$
Brick	$\tilde{a} = a + 4D$ $\tilde{b} = b + 4D$ $\tilde{c} = c + 4D$	$\left(\frac{\pi}{\tilde{a}}\right)^2 + \left(\frac{\pi}{\tilde{b}}\right)^2 + \left(\frac{\pi}{\tilde{c}}\right)^2$	$A \cos\left(\frac{\pi x}{\tilde{a}}\right) \cos\left(\frac{\pi y}{\tilde{b}}\right) \cos\left(\frac{\pi z}{\tilde{c}}\right)$

[†]Here $\zeta_0 \equiv 2.405\dots$ = the smallest “zero” of the J_0 Bessel Function.

9.14 Summary

Our basic goal in this course is to gain an understanding of how reactors behave and why they behave that way. This requires, at the very least, an understanding of neutron production

and loss rates in reactors. In this chapter we went a long way toward that. We found that:

1. The neutron balance equation (either one-speed or energy-dependent) comes from simply counting neutrons. It is essentially exact - the things it ignores are not important in reactors.
2. The neutron balance equation has too many unknowns - another equation is needed to relate the net current density (\vec{J}) and the scalar flux (ϕ).
3. Fick's Law, or the diffusion approximation, can be used to relate \vec{J} and ϕ :

$$\vec{J}(\vec{r}, E, t) = -D(\vec{r}, E, t) \vec{\nabla} \phi(\vec{r}, E, t) \quad (9.263)$$

It is an approximation. It is reasonably accurate if all of the following conditions are met:

- (a) a few mean free paths from material interfaces (including problem boundaries),
- (b) a few mean free paths from strong localized sources,
- (c) when $\Sigma_\gamma \ll \Sigma_s + \nu\Sigma_f$,

You must know these conditions under which the diffusion approximation is accurate!

4. If we approximate \vec{J} in the balance equation using the diffusion approximation, we get the diffusion equation. (Either one-speed or multi-group or energy-dependent.)
5. Each term in the diffusion equation is a physically-meaningful “rate density.” For example, in the one-group diffusion equation:
 - (a) $-\vec{\nabla} \cdot D(\vec{r}, t) \vec{\nabla} \phi(\vec{r}, t)$ = net outleakage rate density [n/(cm³-s)],
 - (b) $\Sigma_a(\vec{r}, t) \phi(\vec{r}, t)$ = absorption rate density [n/(cm³-s)],
 - (c) $\nu\Sigma_f(\vec{r}, t) \phi(\vec{r}, t)$ = fission-neutron production rate density [n/(cm³-s)],
 - (d) $S_{ext}(\vec{r}, t)$ = “extraneous” or “fixed” source rate density [n/(cm³-s)].

Similar interpretations hold for the energy-dependent diffusion equation, although in that case each term is a density in energy as well as space, and thus has units n/(cm³-s-MeV).

6. On the boundary of a diffusion problem we must specify:
 - (a) ϕ ,
 - (b) $\vec{e}_n \cdot \vec{\nabla} \phi$
 - (c) or a combination of ϕ and $\vec{e}_n \cdot \vec{\nabla} \phi$

in order to have a completely specified ('well-posed') problem.

7. At an interface between different materials, we have continuity of
 - (a) scalar flux,
 - (b) normal component ($\vec{e}_n \cdot \vec{J} = -D \vec{e}_n \cdot \vec{\nabla}\phi$) of net current density.
8. Here is a (long-ish) review on boundary/interface conditions
 - (a) The diffusion is a second-order differential equation (it has second derivatives in spatial variables).
 - (b) As a consequence, in one-dimensional problems (those that have only a single spatial variable), its solution, for each homogeneous material zone, is expressed as a linear combination of two independent solutions, plus the particular solution.
 - (c) Thus, we need two constraints per homogeneous material zone. Eg., for a 3-zone reactor, we need 6 constraints.
 - (d) In heterogeneous media, we have at each interface, the **continuity of the scalar flux** and **continuity of the normal component of the current** (2 interface conditions per interface). That is, for a 3-zone slab or spherical reactor, there are 2 interfaces, hence 4 constraints. We still need 2 more conditions to solve (next item below).
 - (e) No matter what configuration, you always need to apply 2 boundary conditions (per dimension). Let us summarize them in detail:
 - i. **Infinite** medium. The flux must remain finite [unless there are singular sources, as in the point-source problem where the flux goes to infinity at the origin].
 - ii. **Vacuum** : use the extrapolated boundary condition or use "no incoming partial current of neutrons."
 - iii. The presence of **localized singular** sources can also lead to the derivation of a special type boundary condition known as the **source condition**, which signifies that the **net leakage rate** [not the current density] of a small volume surrounding the source **is equal to the source intensity**.
9. We applied the diffusion equation to many problems of interest. First we studied fixed-source diffusion problems:
 - (a) with localized or distributed sources,
 - (b) for both homogeneous and multi-region problems.
10. To solve a fixed-source diffusion problem, we

- (a) write the diffusion equation for each homogeneous region,
- (b) write all interface and boundary conditions,
- (c) find a particular solution and the no-source solution, then add them to get the general solution,
- (d) use interface and boundary conditions to solve for the constants in the general solution.
11. We also solved k -eigenvalue diffusion problems. To write down a k -eigenvalue diffusion equation you can follow these steps:
- (a) Write down the diffusion equation that would be true if the reactor were critical and operating in steady state. This means there is no fixed source and no time variable. It also means you don't have to distinguish between prompt and delayed neutrons.
- (b) Multiply the fission term by $1/k$.
12. To solve a k -eigenvalue diffusion problem for a bare homogeneous reactor, we
- (a) write the diffusion k -eigenvalue equation, which contains the Laplacian for the particular geometry of your reactor,
- (b) write appropriate boundary conditions,
- (c) guess the separable solution $\phi(\vec{r}, E) = \Phi(\vec{r}) f(E)$, obtaining separate problems for $\Phi(\vec{r})$ (the energy-integrated scalar flux, which describes the neutron spatial distribution) and $f(E)$ (the neutron spectrum, which describes the neutron energy distribution),
- (d) solve the eigenvalue problem $-\nabla^2\Phi = B^2\Phi$ to obtain B^2 and $\Phi(\vec{r})$, recognizing that the smallest B^2 eigenvalue is B_g^2 = “geometric buckling” and the associated $\Phi(\vec{r})$ eigenfunction is $\Phi_{\text{fund}}(\vec{r})$ = fundamental-mode spatial distribution,
- (e) insert B_g^2 into the energy problem and solve for $f(E)$, the fundamental-mode spectrum (to be done in a later chapter),
- (f) use this $f(E)$ to generate the energy-averaged values $\langle D \rangle$, $\langle \Sigma_a \rangle$, and $\langle \nu \Sigma_f \rangle$ (to be done in a later chapter).
- (g) recognize that $k_{\text{eff}} = \text{multiplication factor} = \frac{\langle \nu \Sigma_f \rangle}{\langle \Sigma_a \rangle + \langle D \rangle B_g^2}$.

We end with the usual disclaimer: This was not a complete list of everything you need to know from this chapter.

Part VI

Neutron distribution in energy

Chapter 10

Moderation (Slowing Down)

10.1 Introduction

10.1.1 Where we are and where we are going

We understand that a reactor's behavior depends on the gain and loss rates of the neutrons in the reactor. We also know that some gain and loss is due to neutron-nucleus reactions, and the rest is due to neutron leakage (meaning neutrons crossing surfaces). We have learned how to compute reaction rates using the neutron scalar flux (ϕ) and how to compute net leakage rates using the neutron net current density (\vec{J}).

We have studied the implications of various neutron conservation equations, considering different settings and writing conservation statements for different neutron populations. We have introduced the diffusion approximation and studied what the diffusion equation says about the way neutrons distribute themselves spatially in a reactor. So far, though, we have not studied how neutrons distribute themselves in energy. We will begin this study in this chapter.

In a previous chapter we began our study of the diffusion k -eigenvalue problem for bare homogeneous reactors:

$$\begin{aligned} -D(E)\nabla^2\phi(\vec{r}, E) + \Sigma_t(E)\phi(\vec{r}, E) &= \frac{1}{k}\chi(E)\int_0^\infty dE_i\nu\Sigma_f(E_i)\phi(\vec{r}, E_i) \\ &\quad + \int_0^\infty dE_i\Sigma_s(E_i \rightarrow E)\phi(\vec{r}, E_i) \end{aligned} \quad (10.1)$$

$$\phi(\vec{r}_s + d\vec{e}_n(\vec{r}_s), E) = 0, \quad \vec{r}_s \in \partial V. \quad (10.2)$$

We found that the problem “separates” into a spatial problem and an energy problem:

$$\phi(\vec{r}, E) \rightarrow \Phi(\vec{r})f(E) \quad (10.3)$$

where

$$-\nabla^2\Phi(\vec{r}) = B^2\Phi(\vec{r}) \quad (10.4)$$

$$\Phi(\vec{r}_s + d\vec{e}_n(\vec{r}_s)) = 0, \quad \vec{r}_s \in \partial V. \quad (10.5)$$

and

$$[D(E)B^2 + \Sigma_t(E)]f(E) = \frac{1}{k}\chi(E)\int_0^\infty dE_i\nu\Sigma_f(E_i)f(E_i) + \int_0^\infty dE_i\Sigma_s(E_i \rightarrow E)f(E_i) \quad (10.6)$$

The spatial problem depends only on the **geometry** (size and shape) of the reactor and on an extrapolation distance (d). It is an eigenvalue problem with infinitely many solutions for a given geometry. Each solution is an (eigenvalue,eigenfunction) pair, (B^2, Φ) . The most important solution for our purposes is the “fundamental-mode” solution, with eigenvalue = **geometric buckling** = B_g^2 and eigenfunction = $\Phi_{\text{fund}}(\vec{r})$. In the previous chapter we solved the spatial problem for various reactor shapes, thereby learning what B_g^2 and $\Phi_{\text{fund}}(\vec{r})$ are for those reactor geometries (slab, sphere, cylinder, rectangular parallelepiped).

In this chapter we study the energy problem, which looks like the k -eigenvalue problem for an infinite homogeneous medium except that it has a $D(E)B^2$ term added to $\Sigma_t(E)$. This term plays the role of an “escape cross section.” The B^2 in this term comes from the spatial problem, and it is the only connection to the reactor geometry.

We are most interested in the neutron energy distribution, $f(E)$, that corresponds to the fundamental mode. This means **we are most interested in the energy problem that contains the B^2 associated with the fundamental mode**, which of course is B_g^2 :

$$[D(E)B_g^2 + \Sigma_t(E)] f(E) = \frac{1}{k} \chi(E) \int_0^\infty dE_i \nu \Sigma_f(E_i) f(E_i) + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i) \quad (10.7)$$

**Energy problem, fundamental mode,
diffusion k -eigenvalue problem for bare homogeneous reactor**

We divide the energy problem into two energy ranges:

1. “**FAST**” energy range: $E > E_{th}$,
2. “**THERMAL**” energy range: $E < E_{th}$,

where

$$\begin{aligned} E_{th} &\equiv \text{Energy boundary between fast and thermal “groups” of neutrons,} \\ &= \text{energy below which essentially no neutrons are born from fission} \\ &\Rightarrow \int_0^{E_{th}} \chi(E) dE \ll 1, \\ &= \text{energy above which there are essentially no “upsattered” neutrons.} \end{aligned} \quad (10.8)$$

Values of E_{th} that meet these conditions are in the range of ≈ 0.5 eV to a few eV.

Given an E_{th} that meets the conditions stated above, the energy-problem equation has no upscattering when $E > E_{th}$, which means the lower bound in the inscattering integral becomes E instead of 0 in the “fast” group:

$$[D(E)B_g^2 + \Sigma_t(E)] f(E) = \frac{\chi(E)}{k} \int_0^\infty dE_i \nu \Sigma_f(E_i) f(E_i) + \int_E^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i), \\ E > E_{th} \quad (\text{“fast group”}). \quad (10.9)$$

Also, there is no fission source in the “thermal group” equation:

$$[D(E)B_g^2 + \Sigma_t(E)] f(E) = \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i), \quad E < E_{th} \quad (\text{“thermal group”}). \\ = \underbrace{\int_0^{E_{th}} dE_i \Sigma_s(E_i \rightarrow E) f(E_i)}_{\text{thermal-to-thermal}} + \underbrace{\int_{E_{th}}^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i)}_{\text{fast-to-thermal}}, \quad E < E_{th} \quad (10.10)$$

In this chapter we focus on the “fast” energy range. We address the thermal energy range in a subsequent chapter. Before we solve our equations for $f(E)$, we observe how those solutions will help us calculate the multiplication factor. If we integrate the thermal-group equation over all thermal energies we obtain (after swapping the integrations in E and E_i on the RHS):

$$\int_0^{E_{th}} dE [D(E)B_g^2 + \Sigma_t(E)] f(E) = \int_0^{E_{th}} dE_i \left[\int_0^{E_{th}} dE \Sigma_s(E_i \rightarrow E) \right] f(E_i) \\ + \int_{E_{th}}^\infty dE_i \left[\int_0^{E_{th}} dE \Sigma_s(E_i \rightarrow E) \right] f(E_i) \\ \xrightarrow{\text{no upscattering to } E > E_{th}} \int_0^{E_{th}} dE_i [\Sigma_s(E_i)] f(E_i) \\ + \int_{E_{th}}^\infty dE_i \left[\int_0^{E_{th}} dE \Sigma_s(E_i \rightarrow E) \right] f(E_i) \quad (10.11)$$

We can subtract the thermal scattering rate density (right side) from the thermal collision

rate density (left side) to make $\Sigma_a(E)$ appear:

$$\int_0^{E_{th}} dE [D(E)B_g^2 + \Sigma_a(E)] f(E) = \int_{E_{th}}^{\infty} dE_i \left[\int_0^{E_{th}} dE \Sigma_s(E_i \rightarrow E) \right] f(E_i) \quad (10.12)$$

Define “Fast” and “thermal” quantities:

$$f_{th} \equiv \int_0^{E_{th}} dE f(E) \quad (10.13)$$

$$f_F \equiv \int_{E_{th}}^{\infty} dE f(E) \quad (10.14)$$

$$D_{th} \equiv \frac{\int_0^{E_{th}} dE D(E) f(E)}{\int_0^{E_{th}} dE f(E)} \quad (10.15)$$

$$\Sigma_{a,th} \equiv \frac{\int_0^{E_{th}} dE \Sigma_a(E) f(E)}{\int_0^{E_{th}} dE f(E)} \quad (10.16)$$

$$\Sigma_{F \rightarrow th} = \frac{\int_{E_{th}}^{\infty} dE_i \left[\int_0^{E_{th}} dE \Sigma_s(E_i \rightarrow E) \right] f(E_i)}{\int_{E_{th}}^{\infty} dE_i f(E_i)} \quad (10.17)$$

The definitions for $D_{th}, \Sigma_{a,th}, \Sigma_{F \rightarrow th}$ are the the standard definition of a weighted average quantity.

Then we have

$$[D_{th}B_g^2 + \Sigma_{a,th}] f_{th} = \Sigma_{F \rightarrow th} f_F \quad (10.18)$$

Note that this gives us the fast-to-thermal flux ratio:

$$\frac{f_F}{f_{th}} = \frac{D_{th}B_g^2 + \Sigma_{a,th}}{\Sigma_{F \rightarrow th}} \quad (10.19)$$

Similarly, if we integrate the “fast” equation from E_{th} to ∞ , and we define:

$$D_F \equiv \frac{\int_{E_{th}}^{\infty} dED(E)f(E)}{\int_{E_{th}}^{\infty} dEf(E)}, \quad (10.20)$$

$$\Sigma_{a,F} \equiv \frac{\int_{E_{th}}^{\infty} dE\Sigma_a(E)f(E)}{\int_{E_{th}}^{\infty} dEf(E)}, \quad (10.21)$$

$$\nu\Sigma_{f,F} \equiv \frac{\int_{E_{th}}^{\infty} dE\nu(E)\Sigma_f(E)f(E)}{\int_{E_{th}}^{\infty} dEf(E)}, \quad (10.22)$$

$$\nu\Sigma_{f,th} \equiv \frac{\int_0^{E_{th}} dE\nu(E)\Sigma_f(E)f(E)}{\int_0^{E_{th}} dEf(E)}, \quad (10.23)$$

then we obtain

$$[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F = \frac{1}{k} [\nu\Sigma_{f,F} f_F + \nu\Sigma_{f,th} f_{th}]. \quad (10.24)$$

We can now solve for k in terms of averaged material properties and B_g^2 . The fast-group equation tells us

$$\begin{aligned} k &= \frac{\nu\Sigma_{f,th} f_{th} + \nu\Sigma_{f,F} f_F}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} \\ &= \frac{\nu\Sigma_{f,th} f_{th}}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} + \frac{\nu\Sigma_{f,F} f_F}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} \end{aligned} \quad (10.25)$$

Manipulate the first term:

$$\frac{\nu \Sigma_{f,th} f_{th}}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} = \frac{\nu \Sigma_{f,th} f_{th}}{\Sigma_{a,th} f_{th}} \times \frac{\Sigma_{a,th} f_{th}}{[D_{th} B_g^2 + \Sigma_{a,th}] f_{th}} \times \frac{[D_{th} B_g^2 + \Sigma_{a,th}] f_{th}}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} \quad (10.26)$$

Identify terms from our “six-factor-formula” derivation of k :

$$\frac{\nu \Sigma_{f,th} f_{th}}{\Sigma_{a,th} f_{th}} = \frac{\nu \Sigma_{f,th}}{\Sigma_{a,th}} = \frac{\Sigma_{a,th}^{\text{fuel}}}{\Sigma_{a,th}} \frac{\nu \Sigma_{f,th}^{\text{fuel}}}{\Sigma_{a,th}^{\text{fuel}}} = u_{\text{th}} \times \eta_{\text{th}} \quad (10.27)$$

$$\frac{\Sigma_{a,th}}{D_{th} B_g^2 + \Sigma_{a,th}} = \frac{1}{1 + L_{\text{th}}^2 B_g^2} = P_{\text{TNL}} \quad (10.28)$$

$$\frac{[D_{th} B_g^2 + \Sigma_{a,th}] f_{th}}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} \xrightarrow{\text{Eq. (10.19)}} = \frac{\Sigma_{F \rightarrow th} f_F}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} = P_{\text{FNL}} p \quad (10.29)$$

Similarly,

$$\frac{\nu \Sigma_{f,F} f_F}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}] f_F} = \underbrace{\frac{\nu \Sigma_{f,F}}{\Sigma_{a,F}}}_{\eta_F u_F} \times \underbrace{\frac{\Sigma_{a,F}}{[D_F B_g^2 + \Sigma_{a,F} + \Sigma_{F \rightarrow th}]}}_{P_{\text{FNL}}(1-p)} \quad (10.30)$$

Our expression for k can now be written in a form we have seen before:

$$k_{\text{eff}} = p P_{\text{FNL}} P_{\text{TNL}} u_{\text{th}} \eta_{\text{th}} + P_{\text{FNL}}(1-p) u_F \eta_F . \quad (10.31)$$

In this chapter we develop useful approximate expressions for the factors that are related to the neutron slowing-down process:

- P_{FNL} , the fast non-leakage probability,
- p , the resonance-escape probability, and
- $u_F \eta_F$, the fast utilization \times the fast reproduction factor.

To do this we must explore how neutrons slow down from their original high energies (which they have when they are born from fission) to much lower energies corresponding to approximate thermal equilibrium with the reactor materials. As we do this we will learn how neutrons distribute themselves in energy as they slow down.

In a subsequent chapter we shall address neutron thermalization—what happens to neutrons once they have slowed down—and address the factors related to thermal neutrons.

10.2 Some Physics

The vast majority of today's reactors are thermal reactors. In a thermal reactor most fissions are caused by “thermal” neutrons, i.e., neutrons that are essentially in thermal equilibrium with the reactor materials. Roughly put, this happens when neutrons' kinetic energies are comparable to the nuclei's kinetic energies. This corresponds to neutron energies below 1 eV. Since neutrons emerge from fission with energies on the order of 10^6 eV, they must lose **a lot** of their energy (more than 99.9999% of it!) before they are “thermalized.” In this chapter, we examine

neutron moderation (slowing down),

which is the process by which neutrons lose energy. This will help us understand:

1. what materials are best for moderators, and
2. the effect of resonances on neutron absorption rates (which is significant!)

In thermal reactors, neutron energies range from almost zero to more than 10^7 eV (10 MeV). We can divide this energy range into three segments, each characterized by different kinds of neutron-nucleus interactions. (The boundaries between segments are not precise.)

In this chapter we will briefly explore the highest-energy regime and then focus mostly on the **slowing-down** regime. We discuss the low-energy (“thermal”) regime in a subsequent chapter.

Table 10.1: **Interaction physics in different energy ranges**

Energy Range:	0 to ≈ 1 eV	≈ 1 eV to $\approx 10^5$ eV	$\approx 10^5$ eV to $\approx 10^7$ eV
Dominant process	Up- and down-scattering		Emission from fission
Interaction Physics:	Elastic scattering from bound nuclei	Elastic scattering from free nuclei, isotropic in CM frame (s-wave)	Elastic scattering from free nuclei; anisotropic in CM frame (p-wave & higher)
Interaction Physics:	Inelastic scattering from bound nuclei		Inelastic scattering from free nuclei
Interaction Physics:	wave effects (diffraction) (very low energies)		
Interaction Physics:	$1/v$ absorbers	Resonance Absorption (resolved resonances)	Resonance Absorption (unresolved resonances)
Interaction Results:	Thermalization of neutrons	Neutron Moderation (slowing down)	Fast Fission; Moderation

10.3 Elastic Scattering

This is a recall from NUEN 302.

10.3.1 Two-Body Kinematics

We need to know how much energy is lost by neutrons when they scatter elastically off of a nucleus. Let's figure it out (see Fig. 10.1).

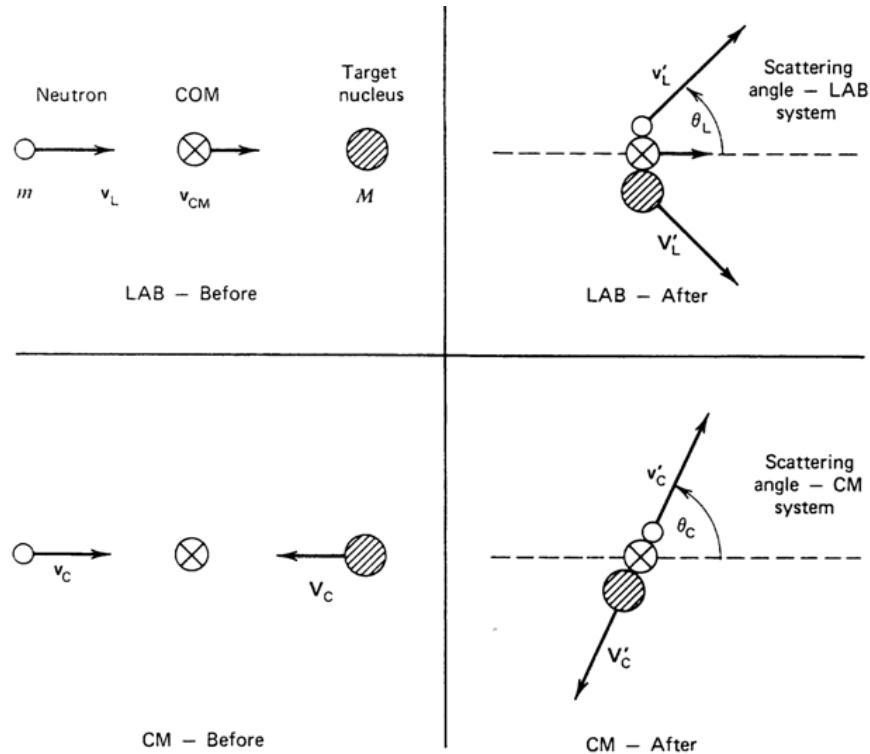


Figure 10.1: Schematics of neutron-nucleus scattering. Lab frame top left and right; Center-of-Mass (CM) frame bottom left and right.

Consider a neutron incident upon a **nucleus that is at rest** in the laboratory reference frame (hereafter called the “lab frame”). Let m be the neutron mass and M the nucleus mass. We will also define:

$$A \equiv \frac{M}{m} \quad (10.32)$$

Note that A is the ratio of nucleus and neutron masses, which is roughly (but not exactly) the same as the mass number. For example, A for ^{16}O is $15.9949146/1.0086649 = 15.8575$, which is ≈ 16 .

We collect our notational definitions in Table 10.2.

By definition of the center of mass (CoM), its position is given by

$$\vec{r}_{CoM} \equiv \frac{1}{m+M} (m\vec{r} + M\vec{R}) \quad (10.33)$$

Table 10.2: Notation for neutron-nucleus scattering kinematics

	neutron	nucleus	Center of Mass
mass	m	M	—
position in lab frame	\vec{r}	\vec{R}	\vec{r}_{CoM}
Initial velocity, lab frame	\vec{v}_L	\vec{V}_L	\vec{v}_{CoM}
Initial velocity, CoM frame	\vec{v}_C	\vec{V}_C	$\vec{0}$
Final velocity, lab frame	\vec{v}'_L	\vec{V}'_L	\vec{v}_{CoM}
Final velocity, CoM frame	\vec{v}'_C	\vec{V}'_C	$\vec{0}$

We differentiate this with respect to time to obtain an equation for the speed, \vec{v}_{CoM} , of the CoM:

$$\begin{aligned}\vec{v}_{CoM} &= \frac{1}{m+M} (m\vec{v}_L + M\vec{V}_L) \\ &= \frac{m}{m+M} \vec{v}_L \quad (\text{Note: nucleus initially at rest } \Rightarrow \vec{V}_L = 0)\end{aligned}\tag{10.34}$$

$$= \frac{1}{1+A} \vec{v}_L \tag{10.35}$$

Let us write the neutron and nucleus initial velocities in the CoM frame:

$$\vec{v}_C = \vec{v}_L - \vec{v}_{CoM} = \frac{A}{1+A} \vec{v}_L \tag{10.36}$$

$$\vec{V}_C = \vec{V}_L - \vec{v}_{CoM} = \frac{1}{1+A} \vec{v}_L \tag{10.37}$$

It is easy to show that the neutron and nucleus speeds in the CoM frame are the same before and after the collision. That is,

$$\vec{v}'_C = \vec{v}_C = \frac{A}{1+A} \vec{v}_L \quad \text{and} \quad \vec{V}'_C = \vec{V}_C = -\frac{1}{1+A} \vec{v}_L \tag{10.38}$$

[To show this, write down conservation of the momentum and kinetic energy **in the CoM reference frame.**]

Let us put both of the “after” pictures on in the same figure (Fig. 10.2). From the law of cosines, we have:

$$v_L'^2 = v_C'^2 + v_{CoM}^2 + 2v_C'v_{CoM} \cos(\theta_C) \quad (10.39)$$

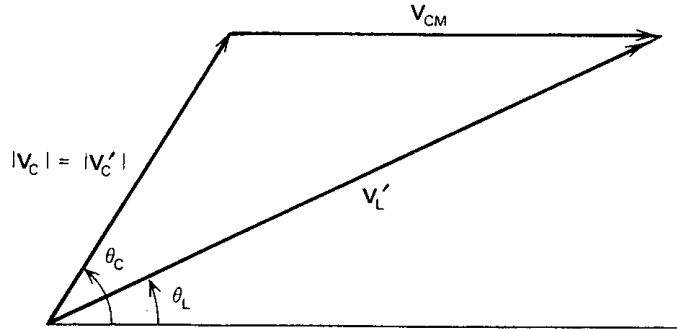


Figure 10.2: Neutron velocity after an elastic scattering collision. Relation between lab-frame and CoM-frame velocities is shown, along with relation between lab-frame and CoM-frame scattering angles. Figure uses slightly different notation than in our equations: Figure’s \mathbf{V} with subscript C or L is our \vec{v} with subscript C or L , and figure’s \mathbf{V}_{CM} is our \vec{v}_{CoM} .

The preceding equations, and some algebra, ultimately give the final **lab-frame** neutron kinetic energy in terms of the initial **lab-frame** neutron kinetic energy and the **CoM-frame** scattering angle:

$$\begin{aligned} \left. \frac{E_f}{E_i} \right|_{LAB !!!} &= \frac{\frac{1}{2}mv_L'^2}{\frac{1}{2}mv_L^2} \\ &= \frac{v_C'^2 + v_{CoM}^2 + 2v_C'v_{CoM} \cos(\theta_C)}{v_L^2} \\ &= \frac{v_C^2 + v_{CoM}^2 + 2v_Cv_{CoM} \cos(\theta_C)}{\left(\frac{1+A}{A}v_C\right)^2} \\ &= \frac{A^2 + 1 + 2A \cos(\theta_C)}{(A+1)^2} \end{aligned} \quad (10.40)$$

We can rewrite this important expression as follows:

$$E_f = E_i \frac{1}{2} [(1+\alpha) + (1-\alpha) \cos(\theta_C)] \quad (10.41)$$

where the parameter α is defined as

$$\alpha = \left(\frac{A-1}{A+1} \right)^2 \quad (10.42)$$

Please do not forget this definition or Eq. (10.41)! Note that α depends only on the ratio of the nucleus mass to the neutron mass. Note that the whole term is squared; it is **not**, for example, $(A^2 - 1)/(A^2 + 1)$.

A reminder of what the terms mean:

- E_f = neutron kinetic energy after the elastic scattering event, **lab frame**
- E_i = neutron kinetic energy before the elastic scattering event, **lab frame**
- θ_C = scattering angle, **center-of-mass** reference frame

Please do not forget these equations or definitions!

Equation (10.42) tells us a great deal that we need to know. For example, what is the maximum possible neutron energy after an elastic scatter, and at what angle does it occur?

$$E_{f,\max} = E_i \quad , \text{ and it occurs at } \theta_C = 0 \quad (\text{a graze}) \quad (10.43)$$

What is the minimum possible neutron energy after an elastic scatter, and at what angle does it occur?

$$E_{f,\min} = \alpha E_i \quad , \text{ and it occurs at } \theta_C = \pi = 180^\circ \quad (\text{a back-scatter}) \quad (10.44)$$

This is particularly important to our study of neutron moderation in thermal reactors. The smaller the α of a nucleus, the more energy it can cause a neutron to lose in an elastic scattering event. Table 10.3 gives approximate values of α for a few nuclides of interest in reactor engineering.

Table 10.3: Values of the scattering parameter “ α ” for a few nuclides of interest

Nucleus	${}^1\text{H}$	${}^2\text{H=D}$	${}^9\text{Be}$	${}^{12}\text{C}$	${}^{16}\text{O}$	${}^{56}\text{Fe}$	${}^{238}\text{U}$
A	≈ 1	≈ 2	≈ 9	≈ 12	≈ 16	≈ 56	≈ 238
α	≈ 0	≈ 0.111	≈ 0.640	≈ 0.716	≈ 0.779	≈ 0.931	≈ 0.983

Remark: In this course we often make the approximation that A is the mass number of the nuclide in question. In “real” calculations one should use the actual ratio of the nucleus mass to the neutron mass. For example, for the ${}^{16}\text{O}$ nucleus, if we say $A \approx 16$ we obtain $\alpha \approx 15^2/17^2 = 225/289 \approx 0.7785$. The correct value of A is $\approx 15.9949146/1.0086649 \approx 15.8575$, which gives $\alpha \approx 0.7768$. The approximate value is close enough for the purposes of this course but there is no reason not to use the correct value in important calculations.

10.3.2 Differential Scattering Cross Section for Elastic Scattering

This is an abbreviated version on this topic. See the extra-material section at the end of this Chapter, Section 10.13 for even more details!!

Our goal in this section is to develop an expression for the differential scattering cross section $\Sigma_s(E_i \rightarrow E)$, which, as you undoubtedly recall, is the scattering cross section at energy E_i multiplied by a probability distribution describing the energies of scattered neutrons:

$$\Sigma_s(E_i \rightarrow E_f) = \Sigma_s(E_i)P(E_i \rightarrow E_f)$$

Recall that:

$P(E_i \rightarrow E_f)dE_f$ = fraction of neutrons, scattered from energy E_i ,
that exit from the scattering event with energies in dE_f at E_f .

We have already found out something about $P(E_i \rightarrow E_f)$ for elastic scattering:

It is zero if $E_f < \alpha E_i$ or $E_f > E_i$

Now we want to find what it is for $\alpha E_i < E_f < E_i$.

Note: Neutrons that emerge with energies between E_f and $E_f + dE_f$ must have scattered through a CM angle between $\theta_C - d\theta_C$ and θ_C , where E_f is related to θ_C by Eq. (10.41).

Because there is a simple 1-to-1 relation between the final energy E_f and the deviation angle θ_C , we can easily relate $P(E_i \rightarrow E_f)$ and $p(\theta_C)$ (the probability density function of scattering with deviation angle θ_C). The main result is:

$$P(E_i \rightarrow E_f) = \frac{4\pi\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \frac{1}{(1-\alpha)E_i} \quad \text{if } \sigma_s(E_i, \theta_C) \text{ is in barn/(ster)} \quad (10.45)$$

$$= \frac{2\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \frac{1}{(1-\alpha)E_i} \quad \text{if } \sigma_s(E_i, \theta_C) \text{ is in barn/(cosine)} \quad (10.46)$$

Recall that all of this manipulation was for $\alpha E_i < E_f < E_i$; we had already found that $P(E_i \rightarrow E_f)$ is zero for E_f outside this range. Thus, we finally have the following expression for the **distribution of energies of elastically-scattered neutrons**:

$$P(E_i \rightarrow E_f) = \begin{cases} \frac{2\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \frac{1}{(1-\alpha)E_i}, & \text{for } E_f \in [\alpha E_i; E_i] \\ 0 & \text{otherwise} \end{cases} \quad (10.47)$$

An important special case is **isotropic** scattering in the CM frame, which means

$$\sigma_s(E_i, \theta_C) = \frac{\sigma_s(E_i)}{2} \quad (10.48)$$

Isotropic scattering in the CM frame is called

s-wave scattering

In this case, we have the very simple result

$$P(E_i \rightarrow E_f) = \begin{cases} \frac{1}{(1-\alpha)E_i}, & \text{for } E_f \in [\alpha E_i, E_i] \\ 0 & \text{otherwise} \end{cases} \quad (10.49)$$

Figure 10.3 illustrates the distribution of scattered-neutron energies in the lab frame, given s-wave elastic scattering.

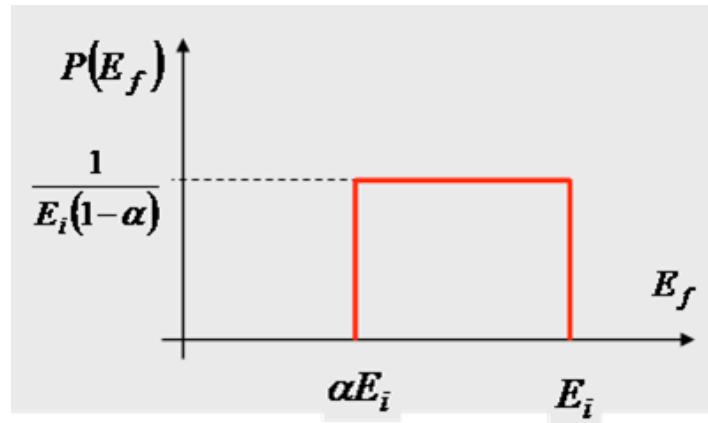


Figure 10.3: Distribution of scattered-neutron energies in the lab frame, following s-wave elastic scattering of neutrons with initial energies E_i .

Note several things about this distribution of exiting energies **from s-wave scattering**:

1. It is a constant in the allowed energy range; i.e., it doesn't depend on the exiting energy for $E_f \in [\alpha E_i, E_i]$
2. If we integrate $P(E_i \rightarrow E_f)$ over exiting energies we get 1 (which makes sense!)
3. If α is close to 1, the distribution is very **narrow**,
If α is close to zero, the distribution is very **broad**

Another important special case is **linearly anisotropic** scattering in the CM frame, which means

$$\sigma_s(E_i, \theta_C) = \frac{\sigma_s(E_i)}{2} [1 + \beta \cos(\theta_C)] \quad (10.50)$$

This is called

p-wave scattering.

In this case, our picture of the exiting-neutron energy distribution is illustrated in Fig. 10.4.

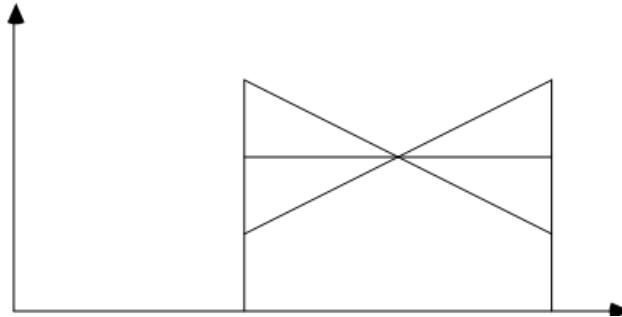


Figure 10.4: Distribution of scattered-neutron energies in the lab frame, following p-wave elastic scattering of neutrons with initial energies E_i .

Note several things about these distributions:

1. They depend linearly on the exiting energy E_f for $E_f \in [\alpha E_i, E_i]$
2. If we integrate $P(E_i \rightarrow E_f)$ over exiting energies we get 1 (which makes sense!)
3. If α is close to 1, the distribution is very **narrow**,
If α is close to zero, the distribution is very **broad**

Measurements and theory both show that

1. Given neutrons energies between ≈ 1 eV and ≈ 50 keV, and given relatively light isotopes, scattering is essentially isotropic in the CM frame.
2. As neutrons energies increase or as the mass of the nucleus increases, p-wave effects become important. Further increases in nucleus mass and/or neutron energy place us in the regime of more complicated scattering laws.

As you have seen, we know $P(E_i \rightarrow E_f)$ if we have elastic scattering and we know $\sigma_s(E_i, \theta_C)$. Section 10.4 below provides pictures showing $\sigma_s(E_i, \theta_C)$ as a function of E_i and $\cos(\theta_C)$ for a few interesting isotopes. Observe the trends. To get your own pictures like this, for whatever isotopes interest you, go to <http://t2.lanl.gov/data/ndviewer.html>. This is a site operated by the data group at Los Alamos National Laboratory. It allows you to download postscript files of cross section plots. You control what isotope you get, which data file they use, which interaction the cross section is for, log or linear for each scale, and the ranges displayed on each axis. Try it!

10.4 Cross Sections of Some Nuclides of Interest

In Appendix A we provide plots of some cross sections, highlighting behavior that has a significant impact on neutron distributions in energy, for several nuclides of interest to nuclear engineers. Here we comment on some of the plots.

- ^1H (hydrogen) cross sections features: **A)** Except for $E < \approx 0.1$ eV, the total and elastic-scattering cross sections are almost the same. That is, except for very slow neutrons, ^1H is essentially a pure elastic scatterer. **B)** The ^1H σ_s is remarkably large, ≈ 20 barns. Compare this to other low-Z nuclides, such as He-4, C-12, and O-16. **C)** ^1H is a “ $1/v$ ” absorber, which means σ_a is proportional to $1/(\text{neutron speed}) = \text{proportional to } 1/\sqrt{E}$. (Observe that every decrease in energy of $\times 100$ results in an increase in absorption cross section of $\times 10$.) This is a common feature of light (low-A) nuclides. **D)** $\sigma_s \approx \text{constant}$ for $E < 10$ keV.
- ^2H (deuterium, or hydrogen-2) cross sections features: **A)** Except for $E > \approx 3$ MeV and $E < \approx 10^{-5}$ eV, the deuterium is essentially a pure elastic scatterer. **B)** ^2H is a “ $1/v$ ” absorber, with σ_a proportional to $1/(\text{neutron speed}) = \text{proportional to } 1/\sqrt{E}$. This absorption cross section is more than a factor of 500 smaller than that of ^1H ! This is why D_2O (“heavy water”) can be used as a moderator for natural-uranium reactors whereas ordinary water cannot.

- ^3He (helium-3) features: **A**) ^3He is a strong absorber, with $\sigma_a \gg \sigma_s$ for $E \lesssim 10^5$ eV. The absorption reaction is not capture, but is (n,p), with reaction products of a proton and a triton. The combination of a high cross section and charged-particle products make ^3He attractive for neutron detectors. **B**) $\sigma_s \approx \text{constant}$ for $E \in (0.01\text{eV}, 50\text{keV})$. **C**) ^3He is a “ $1/v$ ” absorber, $\Rightarrow \sigma_a \approx \text{proportional to } 1/v = \text{proportional to } 1/\sqrt{E}$.
- ^4He (helium-4) features: **A**) For $E \in (10^{-3}, 10^6$ eV), ^4He simply scatters neutrons elastically. **B**) $\sigma_s \approx \text{constant} = 0.75$ b for $E < 10^5$ eV. **C**) The cross section grows significantly (factor of ten!) for $E > 10^5$ eV, because of a resonance just above 1 MeV.
- ^{10}B (boron-10) features: **A**) ^{10}B is a strong neutron absorber, with $\sigma_a \gg \sigma_s$ for $E < 10^5$ eV. The absorption interaction is not capture, but rather $n + ^{10}\text{B} \rightarrow \alpha + ^7\text{Li}$. The high cross section and relative abundance makes boron a common choice for control rods/blades/drums in reactors. The charged-particle products make ^{10}B an attractive choice for some kinds of neutron detectors. **B**) $\sigma_s \approx \text{constant} = 2.2$ b for all E below the resonance that peaks at ≈ 300 keV. **C**) ^{10}B is a “ $1/v$ ” absorber.
- ^{12}C (carbon-12) features: **A**) For $E < 5$ MeV, the ^{12}C simply scatters neutrons elastically. **B**) $\sigma_s \approx \text{constant} = 4.7$ b for all E below ≈ 20 keV. **C**) For $E >$ a few MeV, inelastic scattering comes into play in addition to elastic scattering. For $E = 14$ MeV the inelastic cross section is ≈ 0.43 barns.
- ^{16}O (oxygen-16) features: **A**) For $E < 5$ MeV, simply scatters neutrons elastically. (There is $1/v$ absorption, but it is tiny for $E > 10^{-5}$ eV.) **B**) $\sigma_s \approx \text{constant} \approx 4$ b for $E < \approx 20$ keV. **C**) For $E >$ a few MeV, inelastic scattering comes into play in addition to elastic scattering. For $E = 14$ MeV the inelastic cross section is ≈ 0.5 barns.
- ^{135}Xe (xenon-135) cross sections. The most striking feature of ^{135}Xe is how enormous its cross section is for slow neutrons—**more than a million barns!** ^{135}Xe is a fission product, and it is also the decay product of another fission product (^{135}I = iodine-135). In fact, more than 6% of fissions ultimately produce this neutron-hungry ^{135}Xe nuclide. We must design our reactors to have enough “excess reactivity” to stay critical even after ^{135}Xe builds up.
- ^{235}U (uranium-235) features: **A**) The fission cross section is significantly larger than the capture cross section, and it is most of the total for thermal neutrons. Quantitatively, the 2200-m/s fission cross section ($\Rightarrow E=0.0253$ eV) is a bit below 600b and the capture cross section is only ≈ 100 b, so fission is by far the most likely interaction for a thermal neutron with U-235. **B**) There is a low-lying resonance (at a few tenths of an eV) that strongly perturbs the $1/v$ functional form for thermal neutrons. **C**) We see the typical behavior of a fissile nuclide, with a $1/v$ region at very low E , a resolved-resonance region at intermediate energies, and a fall-off of the capture cross section for higher E .

- ^{238}U (uranium-238) cross sections. Compare against ^{235}U . Note several features: **A)** For thermal neutrons the ^{238}U cross section is much lower than that of a fissile nuclide, and fission does not become important until $E > \approx 1 \text{ MeV}$. **B)** There is a low-lying resonance (at 6.6 eV) that strongly perturbs the neutron energy distribution just above the thermal range. **C)** This is typical behavior for a fissionable (not fissile) nuclide: a $1/v$ region at very low E , a resolved-resonance region at intermediate energies, and a fall-off of the capture cross section for higher E .
- ^{239}Pu (plutonium-239) cross sections. Looks a lot like ^{235}U ! Main differences: its low-lying resonance is much stronger and its cross section is significantly higher for thermal neutrons. Its capture/fission ratio is also higher. It is easy to see that this could be a reasonable substitute for ^{235}U as a fissile fuel.
- ^1H (hydrogen-1) probability density function for scattering angle in the center-of-mass frame. The functions plotted are $P(E_i, \mu_{0C}) = \sigma_s(E_i, \mu_{0C})/\sigma_s(E_i)$, as a function of μ_{0C} for several values of E_i . A flat plot with value 0.5 defines **isotropic scattering in the center-of-mass frame**, which is called **s-wave** scattering. You can see that H-1 is a perfect s-wave scatterer.
- ^2H (deuterium) probability density function (pdf) for scattering angle in the center-of-mass frame. Deuterium begins to deviate from s-wave scattering for neutron energies $\gtrsim 100 \text{ keV}$.
- ^{12}C (carbon-12) pdf for scattering angle in the center-of-mass frame. Carbon begins to deviate from s-wave scattering for neutron energies $\gtrsim 2 \text{ MeV}$. The pdf becomes complicated at high energies and that at some high energies it is forward-peaked (probability of small-angle scattering is greater) while at other energies it is backward-peaked (probability of backscattering is greater).
- ^{16}O (oxygen-16) pdf for scattering angle in the center-of-mass frame. O-16 begins to deviate from s-wave scattering for neutron energies $\gtrsim 200 \text{ keV}$. The pdf becomes complicated and that at some high energies it is forward-peaked (probability of small-angle scattering is greater) while at other energies it is backward-peaked (probability of backscattering is greater).
- ^{238}U (uranium-238) pdf for scattering angle in the center-of-mass frame. U-238 begins to deviate from s-wave scattering for neutron energies $\gtrsim 50 \text{ keV}$. The pdf is forward-peaked (probability of small-angle scattering is greater) at high energies and becomes extremely forward-peaked for neutron energies above 1 MeV!

For convenience we collect plots of a few of these nuclides here—the ones that play the largest roles in the vast majority of reactors in the world (reactors that use regular “light” water, H_2O , as their moderators and coolants) in Fig. 10.5.

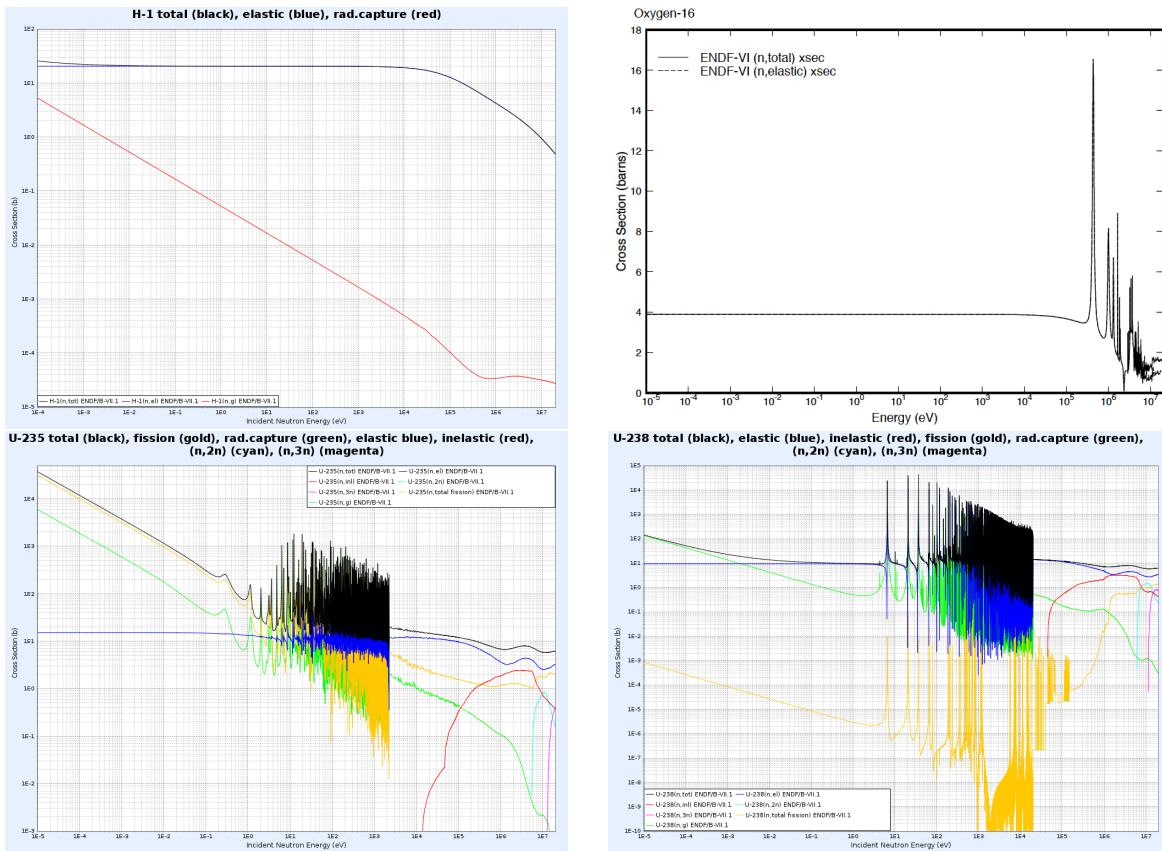


Figure 10.5: Cross sections of key nuclides in light-water reactors: ${}^1\text{H}$, ${}^{16}\text{O}$, ${}^{235}\text{U}$, ${}^{238}\text{U}$. Hydrogen cross sections are smooth, its σ_a is proportional to $1/v$ and is small except for low-energy neutrons, and its σ_s is almost constant at ≈ 20 barns for $E < 10^4$ eV. The oxygen cross sections are smooth and flat (≈ 4 barns) for neutron energies below roughly 10^5 eV, with resonances above that range. The ${}^{235}\text{U}$ σ_f is large and proportional to $1/(v)$ at low energies, has resolved resonances from ≈ 0.3 eV to ≈ 20 keV, and drops to 1-2 barns for neutron energies above 10^5 eV. The ${}^{235}\text{U}$ σ_γ has similar behavior but smaller, especially at high energies. The ${}^{238}\text{U}$ σ_f is tiny for neutron energies below about 1.3 MeV. Its σ_γ behaves much like the ${}^{235}\text{U}$ σ_γ except that it is smaller for low-energy neutrons and larger in the resonance range.

10.5 Energy Distribution in Highest-Energy Range

Before we tackle the slowing-down range, which Table 10.1 approximates as $E \in \approx(1 \text{ eV}, 100 \text{ keV})$, we address the fast range, which we loosely define as $E \gtrsim 100 \text{ keV}$. Recall our equation for the neutron energy distribution function, $f(E)$, which came from “separating” the spatial and energy variables in our energy-dependent k -eigenvalue diffusion problem:

$$\begin{aligned} [D(E)B^2 + \Sigma_t(E)] f(E) &= \chi(E) + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i) \\ &= \chi(E) + S_{\text{inscat}}(E). \end{aligned} \quad (10.51)$$

No, I did not forgotten to divide χ by k . Short answer, it is renormalization trick. Long answer: see the extra-material section at the end of this Chapter, Section 10.13.

10.5.1 Formal Solution

Consider the fundamental-mode distribution. If we ignore for the moment that the in-scattering source rate density, $S_{\text{inscat}}(E)$, depends on the solution, $f(E)$, then we can “solve” Eq. (10.51) for the solution:

$$f(E) = \frac{\chi(E) + S_{\text{inscat}}(E)}{D(E)B_g^2 + \Sigma_t(E)}. \quad (10.52)$$

In almost every reactor, and certainly in large reactors, the probability that a neutron of energy E will have a(nother) collision is much greater than the probability that it will leak without a(nother) collision. That is, $\Sigma_t(E) \gg D(E)B_g^2$. Thus, in a large reactor, we can understand the neutron energy distribution in the high-energy range if we understand three functions:

1. The fission spectrum, $\chi(E)$,
2. The in-scattering source-rate density, $S_{\text{inscat}}(E)$, and
3. The macroscopic total cross section, $\Sigma_t(E)$.

10.5.2 The Fission Spectrum, $\chi(E)$

Figure 10.6 illustrates the fission spectrum, $\chi(E)$, on both linear-linear and log-log scales. The linear-scale plot makes it clear that the vast majority of the neutrons emerging from fission do so with energies > 100 keV. In fact, more than 99% of fission neutrons emerge with energies > 100 keV. The log-scale plot shows a low-energy “tail” of the distribution that scales as $E^{1/2}$ and a high-energy tail that drops steeply for energies > 5 MeV.

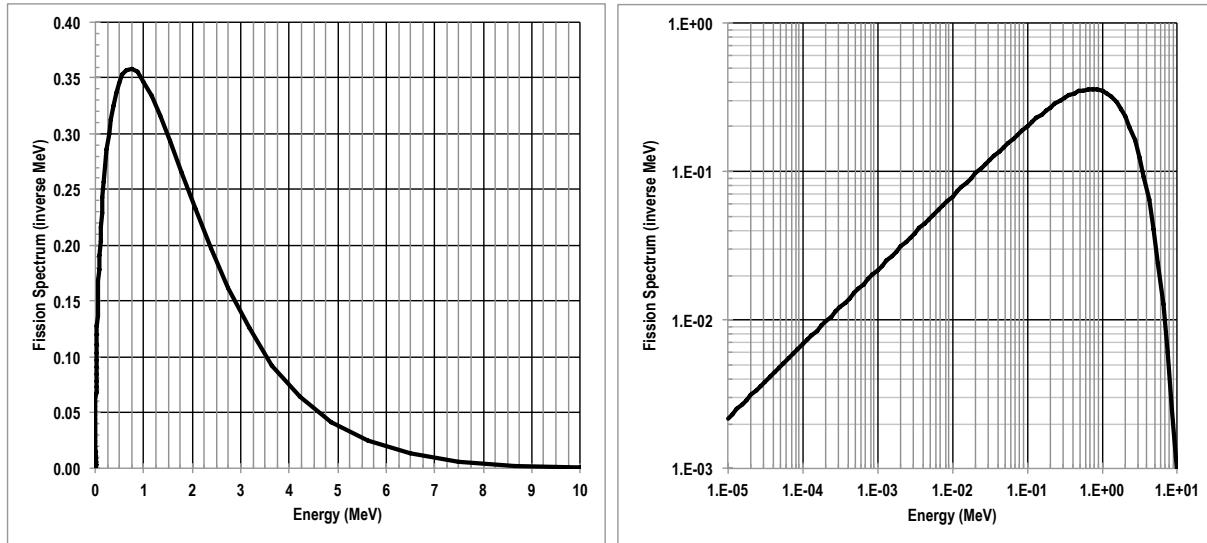


Figure 10.6: The fission spectrum, $\chi(E)$, which gives the energy distribution of neutrons as they emerge from the fission process. The same function is shown on the left (linear-linear scale) and right (log-log). The peak is near 750-800 keV. The average energy of a neutron from fission is near 2 MeV. More than 99% emerge from fission with energy > 100 keV.

10.5.3 The In-Scattering Source Rate Density, $S_{\text{inscat}}(E)$

Observe the following about the scattering processes for high-energy neutrons:

- Each elastic scattering off of a heavy nucleus causes the neutron to
lose very little energy

and thus has little effect on the energy distribution. From the point of view of the energy distribution, it is almost as if the event never happened.

- Each inelastic scattering event causes the neutron to
lose substantial energy.

This almost always removes the neutron from the energy range we are considering (which is $E > \approx 100$ keV). That is, inelastic scattering contributes relatively little to the scattering source in this energy range.

- Each elastic scattering off of a light nucleus causes the scattered neutron to emerge with energy

between αE_i and E_i ,

where E_i was the pre-scatter neutron energy, $\alpha = [(A - 1)/(A + 1)]^2$ for that nuclide, and α has a value well below unity.

The qualitative result of all of this is:

In the high-energy range ($E > \approx 100$ keV),
 $S_{\text{inscat}}(E)$ is relatively low at the highest energies and increases as E decreases.

It follows from the discussion above that

1. $\chi(E) \gg S_{\text{inscat}}(E)$ for the highest energies ($E >$ a few MeV);
2. $S_{\text{inscat}}(E)$ becomes comparable to $\chi(E)$ for $E \in$ (few 100 keV, few MeV);
3. For $E <$ a few hundred keV, the scattering source dominates: $S_{\text{inscat}}(E) \gg \chi(E)$.

10.5.4 The $1/\Sigma_t(E)$ Multiplier

We turn next to the $1/\Sigma_t(E)$ term in the solution. Consider reactors moderated by H_2O or D_2O , which includes almost all of the commercial reactors built to date. Observe the following about the total cross sections of the main materials in such reactors, for E ranging from 100 keV to 10 MeV:

- The $\sigma_t(E)$'s for the uranium and plutonium isotopes decrease smoothly as E increases from 10^5 eV to 10^7 eV.
- $\sigma_t(E)$ for ${}^1\text{H}$ decreases smoothly from > 10 b to < 1 b as E increases from 10^5 eV to 10^7 eV.
- $\sigma_t(E)$ for ${}^2\text{D}$ decreases smoothly from ≈ 3 b to ≈ 1 b as E increases from 10^5 eV to 10^7 eV.
- $\sigma_t(E)$ for ${}^{16}\text{O}$ has resonances in the energy range of interest here, superimposed on a smoothly decreasing function that ranges from ≈ 4 b to ≈ 1.5 b as E increases from 10^5 eV to 10^7 eV. The resonances produce a peak cross section of ≈ 17 b and also produce a sharp minimum of a fraction of a barn and an energy between 2 and 3 MeV.

It follows that there are two main effects of the $1/\Sigma_t(E)$ multiplier:

1. The multiplier is higher at high energies and lower at lower energies, in the energy range considered here.
2. The multiplier has sharp minima at energies at which oxygen has resonances, and sharp maxima at energies at which oxygen has sharp minima between its resonances. (See Fig. A.11.)

If we combine what we have learned about the uncollided and scattered neutrons, we conclude the following about the distribution of neutron energies in a large homogeneous reactor:

For $E > \approx 100$ keV, the neutron energy distribution can be viewed as a modified fission spectrum. The modifications include

1. The addition of the in-scattering source, which has little effect for the highest energies but becomes stronger and stronger at lower and lower E .
2. The overall effect of the $1/\Sigma_t(E)$ multiplier, which is to raise the spectrum at the higher energies in the range and reduce it at the lower energies in the range.
3. The detailed effects of resonances on the $1/\Sigma_t(E)$ multiplier, which cause sharp down-spikes in $f(E)$ at resonance energies and up-spikes at energies where resonance effects cause $\Sigma_t(E)$ to have sharp minima.

This will be illustrated in later sections. Before we proceed, though, we give a sneak preview of the actual energy spectrum in a pressurized light-water reactor, in Fig. 10.7.

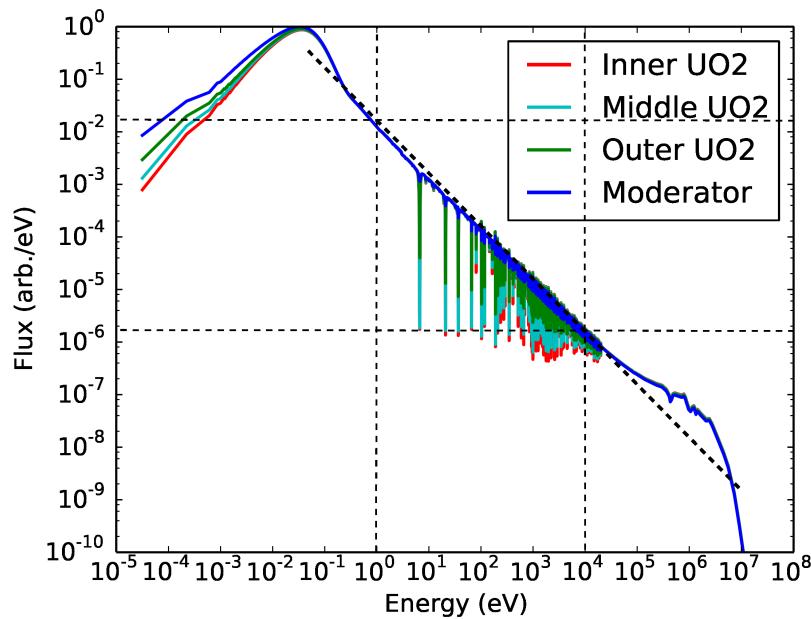


Figure 10.7: Neutron energy spectrum at four different spatial points in simple PWR lattice. For $E \lesssim 0.5$ eV the spectrum is \approx Maxwellian. For $E \in (1\text{eV}, 10^5 \text{ eV})$ the spectrum is nearly proportional to $1/E$ (the diagonal dashed line) but with sharp downspikes at energies of uranium resonances. For $E > 10^5$ eV the fission spectrum is superimposed on the $1/E$ shape and there are downspikes at energies of oxygen resonances.

10.6 Slowing Down in Hydrogen: Infinite Medium, No Absorption

Recall the energy equation that we obtain via separation of variables, for the fundamental-mode energy distribution:

$$[D(E)B_g^2 + \Sigma_t(E)] f(E) = \chi(E) + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i) \quad (10.53)$$

We will now solve this equation under the assumptions:

1. pure hydrogen ($A = 1$) (unrealistic but will teach us a lot)
2. no absorption [$\Sigma_t(E) = \Sigma_s(E)$] (\approx true for ${}^1\text{H}$ in slowing-down range)
3. no inelastic scattering (true for ${}^1\text{H}$)
4. isotropic scattering in CM frame (true for ${}^1\text{H}$)
5. no neutrons above some upper energy E_0 . (essentially correct for reactors)
6. infinite medium ($B_g^2 \rightarrow 0$) (let's start simple)

Under these assumptions the slowing-down equation becomes:

$$[0 + \Sigma_s(E)] f(E) = \chi(E) + \int_E^{E_0} dE' \frac{\Sigma_s(E')}{E'} f(E') \quad (10.54)$$

(Do you see why? We have replaced Σ_t by Σ_s because we are pretending there is no absorption. We have inserted the formula for the differential scattering cross section for hydrogen, $A \approx 1$ hence $\alpha \approx 0$, under the assumption of s-wave elastic scattering, which is correct for ${}^1\text{H}$.)

We define the **collision-rate density** $F(E)$ as follows. (This is just a convenient change of variables to simplify the math.)

$$F(E) \equiv \Sigma_t(E)f(E) \xrightarrow{\text{assuming } \Sigma_a=0} \Sigma_s(E)f(E). \quad (10.55)$$

In terms of $F(E)$, our equation is

$$F(E) = \chi(E) + \int_E^{E_0} dE' \frac{1}{E'} F(E') \quad (10.56)$$

We can solve this equation analytically; the solution is

$$F(E) = \chi(E) + \frac{1}{E} \int_E^{E_0} dE' \chi(E') \quad (10.57)$$

which we can rewrite in terms of the separated energy-dependent portion of the scalar flux, $f(E)$, by dividing through by Σ_s :

$$f(E) = \frac{\chi(E)}{\Sigma_s(E)} + \frac{1}{E\Sigma_s(E)} \int_E^{E_0} dE' \chi(E') \quad (10.58)$$

Let us examine this result under the assumption:

- $\chi(E)$ is nonzero only between E^* and E_0 . (Reasonable approximation of the fission source, which generates almost all neutrons between $E^* \approx 100$ keV and $E_0 \approx 10$ MeV.)

In this case, the complete slowing-down solution in an infinite medium of ${}^1\text{H}$ becomes:

$$f(E) = \begin{cases} 0, & \text{for } E > E_0 \\ \frac{\chi(E)}{\Sigma_s(E)} + \frac{1}{E\Sigma_s(E)} \int_E^{E_0} dE' \chi(E'), & \text{for } E^* < E < E_0 \\ \frac{1}{E\Sigma_s(E)} \int_{E^*}^{E_0} dE' \chi(E') = \frac{1}{E\Sigma_s(E)} & \text{for } E < E^* \end{cases} \quad (10.59)$$

Note that if $\Sigma_s(E) \approx \text{constant}$, which it is for hydrogen in much of the slowing-down regime, then, below the source minimum energy E^* ,

the scalar flux is proportional to $\frac{1}{E}$.

This was the solution of a very simple problem. However, we will see this approximate $1/E$ behavior again in the solution of more realistic problems.

If you remember nothing else about slowing down,
remember $1/E!$

10.7 Lethargy and Worm (ξ)

In this section we briefly introduce some standard variables, nomenclature, and terminology.

10.7.1 Lethargy u

A variable that is often used in place of neutron kinetic energy is “lethargy.” Its definition is:

$$u = \text{lethargy} = \ln \left[\frac{E_0}{E} \right] \quad (10.60)$$

where E_0 is just some reference energy, usually chosen to be the high-energy cutoff. Note that as E decreases, the lethargy **increases**. (Slower neutrons are more lethargic!)

The lethargy variable is used extensively in slowing-down studies, chiefly because it simplifies the equations. Notice:

- As E decreases, u increases,
- As $E \rightarrow 0$, $u \rightarrow \infty$
- If $E < E_0$, $u > 0$.
- $u_f - u_i = \ln \left[\frac{E_0}{E_f} \right] - \ln \left[\frac{E_0}{E_i} \right] = \ln \left[\frac{E_i}{E_f} \right]$
- $E = E_0 e^{-u}$

We often perform a change of variable, and talk about densities in lethargy instead of densities in energy. For example, the lethargy-dependent collision density, $F(u)$, is related to the energy-dependent collision density, $F(E)$, as follows:

$$F(u)du = -F(E)dE. \quad (10.61)$$

The minus sign is because u increases when E decreases. This is like a change of variable in an integral.

Another way to understand this is to consider something more familiar. Suppose the density of cars on the highway is $f(x)$ with x in miles and f in cars/mile; and say that the density of cars on the highway is also $g(y)$ with y in km and g in cars/km; then we must have $f(x)dx = g(y)dy$.

We are abusing notation here because we shouldn't use the same letter F for both $F(u)$ and $F(E)$. However, this is common practice in nuclear engineering, so we should get used to it.

From the definition of u ,

$$du = d \ln \left[\frac{E_0}{E} \right] = -\frac{dE}{E} \quad (10.62)$$

Thus,

$$F(u) = EF(E). \quad (10.63)$$

The lethargy-dependent scalar flux, $\phi(u)$, satisfies a similar relation:

$$\phi(u) = E\phi(E) \quad (10.64)$$

But note that $\Sigma_t(u) = \Sigma_t(E)$ (**no factor of E**), because Σ is **not a density** in energy or lethargy!!

Question: What is $\phi(u)$ in the purely-scattering ^1H problem, for lethargies higher than the source cutoff $u^* = \ln \left[\frac{E_0}{E_s} \right]$?

Answer: $\phi(u) = E\phi(E) = \frac{S_{tot}}{\Sigma_s(E)}$, where $E(u) = E_0 \exp(-u)$

Q2: What is $F(u)$ in that problem?

A2: $F(u) = S_{tot}$

Q3: What are the units of $\phi(u)$ and $\phi(E)$?

A3: $\phi(E)$ has units $\text{n}/(\text{cm}^2\text{-s-MeV})$; $\phi(u)$ has units $\text{n}/(\text{cm}^2\text{-s})$

10.7.2 ξ (“Worm,” or if you wish to be unsophisticated, “squiggle”)

A quantity of considerable interest to us is the

average lethargy gain per scattering collision.

Lethargy gain is the same thing as **logarithmic energy decrement**:

$$u_f - u_i = \ln(E_0/E_f) - \ln(E_0/E_i) = \ln(E_i/E_f) = \ln(E_i) - \ln(E_f) \quad (10.65)$$

We give the average lethargy gain per scattering collision the symbol ξ , which we lovingly call “worm.” (It is really the Greek letter “xi,” pronounced “ksee.” Some nuclear engineers call it “squiggle,” but we regard this as unsophisticated.) Its definition is:

$$\begin{aligned} \xi(E_i) &= \int_0^\infty dE_f [u(E_f) - u(E_i)] P(E_i \rightarrow E_f) \\ &= \int_0^\infty dE_f \ln \left[\frac{E_i}{E_f} \right] P(E_i \rightarrow E_f) \end{aligned} \quad (10.66)$$

We cannot evaluate this analytically, in general. However, given the special case of s-wave elastic scattering, we can:

$$\xi(E_i) = \int_0^\infty dE_f \ln \left[\frac{E_i}{E_f} \right] p(E_i \rightarrow E_f) \quad (10.67)$$

$$\xrightarrow[\alpha E_i]{\text{s-wave elastic}} \int_{\alpha E_i}^{E_i} dE_f \ln \left[\frac{E_i}{E_f} \right] \frac{1}{1-\alpha} \frac{1}{E_i} \quad (10.68)$$

$$\xrightarrow{x=E_f/E_i} \frac{1}{1-\alpha} \frac{1}{E_i} \int_{\alpha}^1 x dx \ln \left[\frac{1}{x} \right] \quad (10.69)$$

$$= \frac{1}{1-\alpha} [1 + \alpha \ln \alpha - \alpha] \quad (10.70)$$

Hence,

$$\xi = 1 + \frac{\alpha \ln \alpha}{1 - \alpha} \quad (10.71)$$

Only if scattering is elastic and isotropic in the CM frame

If we have a mixture of different nuclides, we can define an average ξ , denoted $\bar{\xi}$ (“worm-bar”):

$$\bar{\xi}(E) = \frac{\sum_{i=1}^{\#nuclides} \Sigma_s(E) \xi_i(E)}{\sum_{i=1}^{\#nuclides} \Sigma_s(E)} \quad (10.72)$$

Note: Because in general the scattering cross sections depend on energy, so in general will $\bar{\xi}(E)$ worm-bar. However, for energies in the slowing-down range (≈ 1 eV to ≈ 100 keV), the scattering cross sections of most light nuclides are nearly constant. Thus, in back-of-the-envelope calculations, we often ignore the energy dependence of worm-bar and simply use a suitable average value over the slowing-down energy range.

10.8 How good is a given moderator?

Clearly, worm-bar is an important number in the evaluation of moderators. However, it is not the entire story.

A good moderator should have

1. a high ξ (or low α) so that neutrons loose a lot of kinetic energy per collision;
2. a high scattering cross section, so that the moderating collisions will occur with high frequency;
3. a low absorption cross section, so it doesn't cause neutron losses.

This reasoning leads to two figures of merit for moderators:

$$\text{moderating power} = \bar{\xi} \Sigma_s \quad (10.73)$$

$$\text{moderating ratio} = \bar{\xi} \frac{\Sigma_s}{\Sigma_a} \quad (10.74)$$

Table 10.4 provides these figures of merit for a few different moderators.

Table 10.4: Various figures of merit for various moderators of interest.

Moderator	A	$\bar{\xi}$	avg. # of collisions from 2 MeV to 1 eV	$\bar{\xi} \Sigma_s$	$\bar{\xi} \frac{\Sigma_s}{\Sigma_a}$
H ₂ O	1 and 16	0.920	16	1.35	71
D ₂ O	2 and 16	0.509	29	0.176	5670
He	4	0.425	43	n/a	83
C	12	0.158	91	0.06	192
U-238	238	0.008	1730	0.003	0.009

You will notice that we have included “average number of collisions from 2 MeV to 1 eV” in our table. What we have actually shown is the average number of collisions to give the collided neutrons an **average lethargy** equal to that of a 1-eV neutron. This interesting quantity is easy to calculate once we know $\bar{\xi}$:

$$\langle \# \text{ collisions} \rangle = \frac{\text{total lethargy gain}}{\langle \frac{\text{lethargy gain}}{\text{collision}} \rangle} = \frac{\ln \left[\frac{1 \cdot 10^6}{1} \right]}{\bar{\xi}} = \frac{14.5}{\bar{\xi}}$$

10.9 Slowing Down in Hydrogen: Finite Medium With Absorption

Recall that in this chapter we have found that for **bare homogeneous** reactors, with extrapolation distance independent of neutron energy, the solution separates into the product of a spatial function and an energy function: $\phi(\vec{r}, E) = \Psi(\vec{r})f(E)$. In a previous section we found that the energy dependence of the scalar flux in an **infinite medium** of hydrogen, given a source emitting neutrons with energies greater than E^* , is

$$f(E) \xrightarrow{\text{infinite medium } {}^1\text{H}} \begin{cases} 0, & E > E_0 \\ \frac{\chi(E)}{\Sigma_s(E)} + \frac{1}{E\Sigma_s(E)} \int_E^{E_0} dE' \chi(E'), & E \in (E^*, E_0) \\ \frac{1}{E\Sigma_s(E)} \int_{E^*}^{E_0} dE' \chi(E') = \frac{1}{E\Sigma_s(E)}, & E < E^*. \end{cases} \quad (10.75)$$

Now consider **finite reactors**, which introduce the leakage term, $D(E)B_g^2$, and consider the presence of **absorption**, which means a nonzero $\Sigma_a(E)$. We can still solve this problem analytically if we assume that all of the energy loss is from scattering off of ${}^1\text{H}$. (The vast majority of the energy loss in a light-water-moderated reactor from scattering off of ${}^1\text{H}$, so this is not a bad approximation for water-moderated reactors.) The result is:

$$f(E) \approx \begin{cases} 0, & E > E_0 \\ \frac{1}{D(E)B_g^2 + \Sigma_t(E)} \left[\chi(E) + \frac{1}{E} \int_E^{E_0} dE' \chi(E') C(E') p_{re}(E' \rightarrow E) p_{nl}(E' \rightarrow E) \right], & E \in (E^*, E_0) \\ \frac{S_{\text{eff}}}{E} \frac{p_{nl}(E^* \rightarrow E) p_{re}(E^* \rightarrow E)}{[D(E)B_g^2 + \Sigma_t(E)]}, & E < E^* \end{cases} \quad (10.76)$$

Here we have defined

$$C(E') \equiv \frac{\Sigma_s(E')}{D(E')B_g^2 + \Sigma_t(E')} \quad (10.77)$$

= probability that a neutron born with energy E'
scatters before it is absorbed or leaks

$$p_{re}(E' \rightarrow E) \equiv \exp \left\{ - \int_E^{E'} dE'' \frac{1}{E''} \frac{\Sigma_a(E'')}{D(E'')B_g^2 + \Sigma_t(E'')} \right\} \quad (10.78)$$

\approx probability that a neutron that scatters at energy E'
does not get absorbed before reaching energy E

$$p_{nl}(E' \rightarrow E) \equiv \exp \left\{ - \int_E^{E'} dE'' \frac{1}{E''} \frac{D(E'')B_g^2}{D(E'')B_g^2 + \Sigma_t(E'')} \right\} \quad (10.79)$$

\approx probability that a neutron that scatters at energy E'
does not leak before reaching energy E

$$S_{\text{eff}} \equiv \int_{E^*}^{E_0} dE' \chi(E') C(E') p_{re}(E' \rightarrow E^*) p_{nl}(E' \rightarrow E^*) \quad (10.80)$$

= effective source \approx fraction of fission neutrons that survive to energy E^*

Recall that in these expressions, E^* is the energy below which almost no fission neutrons are born, and recall that ≈ 100 keV is a reasonable value for this. Also recall that E_0 is the energy above which essentially no neutrons exist in the reactor, and that $\approx 10\text{-}12$ MeV is a reasonable value for this.

We can use this solution to generate approximate expressions for the factors that we seek. Recall that these factors are:

- P_{FNL} (fast non-leakage probability),
- p (resonance-escape probability), and
- $u_F \eta_F$ (fast utilization times fast reproduction factor).

Before we do this, though, we generalize the solution to allow for slowing down caused by nuclides other than hydrogen.

10.10 Slowing Down: Realistic Case

Here we treat the case of a **finite reactor** with **arbitrary moderator** (not just $A=1$), **with absorption**. This case is too difficult to solve analytically, but we can create a reasonably accurate approximate solution for large homogeneous reactors. This approximate solution for the scalar flux shape in energy is:

$$f(E) = \frac{1}{\bar{\xi}(E) [D(E)B_g^2 + \Sigma_t(E)]} \left[\chi(E) + \frac{1}{E} \int_E^{E_0} dE' \chi(E') C(E') p_{re}(E' \rightarrow E) p_{nl}(E' \rightarrow E) \right]. \quad (10.81)$$

For the main energy range of interest ($E < E^*$), this reduces to

$$f(E) = \frac{S_{\text{eff}} p_{nl}(E^* \rightarrow E) p_{re}(E^* \rightarrow E)}{E \bar{\xi}(E) [D(E)B_g^2 + \Sigma_t(E)]} \quad \text{for } E < E^*. \quad (10.82)$$

Here we have defined

$$C(E') \equiv \frac{\Sigma_s(E')}{\bar{\xi}(E) [D(E')B_g^2 + \Sigma_t(E')]} \quad (10.83)$$

= probability that a neutron born with energy E'
scatters before it is absorbed or leaks

$$p_{re}(E' \rightarrow E) \equiv \exp \left\{ - \int_E^{E'} dE'' \frac{1}{E''} \frac{\Sigma_a(E'')}{\bar{\xi}(E'') [D(E'')B_g^2 + \Sigma_t(E'')]} \right\} \quad (10.84)$$

= probability that a neutron that scatters at energy E'
does not get absorbed before reaching energy E

$$p_{nl}(E' \rightarrow E) \equiv \exp \left\{ - \int_E^{E'} dE'' \frac{1}{E''} \frac{D(E'')B_g^2}{\bar{\xi}(E) [D(E'')B_g^2 + \Sigma_t(E'')]} \right\} \quad (10.85)$$

= probability that a neutron that scatters at energy E'
does not leak before reaching energy E

Note that the only difference in the solution with non-hydrogen moderator is the inclusion of the factor $\bar{\xi}$ in the denominator of several terms. This factor is

$\bar{\xi} \equiv$ average “logarithmic energy decrement” per scattering event

= average value of $[\ln(E_i) - \ln(E_f)]$

= average “lethargy” gain per scattering event.

If you compared to the previous solution, which assumed that hydrogen was the only nuclide that could cause neutrons to lose energy, you would conclude that:

for pure ^1H , $\bar{\xi}$ must be exactly 1.

This is correct. Only hydrogen-1 has such a large value of worm-bar. Because water is mostly hydrogen and its cross section is larger than oxygen’s, worm-bar for H_2O is also large: ≈ 0.92 . Worm-bar for some other moderators is given in a table in a previous section, along with some other interesting measures of moderator performance.

10.11 Obtaining the factors P_{FNL} , p , and $u_F\eta_F$

Now we shall use what we have learned to create useful approximate expressions for the factors that we need.

10.11.1 Fast Non-Leakage Probability (P_{FNL})

We are interested in the probability that a neutrons slows to the “thermal” energy range without leaking or getting absorbed. We shall define an energy,

$$E_{\text{thermal}} \equiv E_{th},$$

to be the thermal “cutoff” energy. If a neutron slows to E_{th} , we say that it has become “thermal.” The value for E_{th} is typically in the range of 0.5 to 2 eV.

From Eq. (10.85) we have:

$$p_{nl}(E' \rightarrow E_{th}) \equiv \exp \left\{ -B_g^2 \int_{E_{th}}^{E'} dE \frac{1}{E} \frac{D(E)}{\bar{\xi}(E) [D(E)B_g^2 + \Sigma_t(E)]} \right\} \quad (10.86)$$

We shall not prove the following, but it turns out that the integral in the exponent is related to a “slowing-down distance”:

$$\int_{E_{th}}^{E'} dE \frac{1}{E} \frac{D(E)}{\bar{\xi}(E) [D(E)B_g^2 + \Sigma_t(E)]} \approx \tau(E' \rightarrow E_{th}) \quad (10.87)$$

where we have defined

$$\tau(E' \rightarrow E_{th}) \equiv \frac{1}{6} \text{ of the average squared distance}$$

from the point where a neutron **is born with energy E'**

to the point where **it first slows to an energy $\leq E_{th}$** . (10.88)

We are often very interested in the distance from the neutrons birthpoint from a fission event to the point where it becomes thermal. We define:

$$\tau_{th} \equiv \frac{1}{6} \text{ of the average squared distance}$$

from the point where a neutron **is born from fission**

to the point where **it first slows to thermal**. (10.89)

With this definition and the previous equations, we obtain our useful approximate formula for the fast non-leakage probability:

$$P_{\text{FNL}} \approx \exp \{-B_g^2 \tau_{th}\} \quad (10.90)$$

This is the approximation you should use for the fast non-leakage probability in a large, bare, homogeneous reactor!

Beware: If you use this expression (correctly) and it tells you that P_{FNL} is far from unity (say, less than 0.8), then your reactor is probably not large enough for this to be a very accurate approximation.

For historical reasons (going back to Fermi and a very clever approximation that he made to do the first nuclear reactor analysis),

τ is called “Age”

and

τ_{th} is called “Age to thermal.”

Remember that “Age” is proportional to a mean-squared slowing-down distance, and thus has units of area (cm^2). “Age to thermal” characterizes distance from fast-neutron birth point to thermalization point, in the same way that L_{th}^2 characterizes distance from thermal-neutron birth point to absorption point. The two can be added to form the total “migration area,” which is $1/6$ the mean squared distance from birth point (the location of the birthing fission event) to the point of absorption as a thermal neutron:

$$M^2 \equiv \tau_{th} + L_{th}^2 = \text{“migration area”} \Rightarrow M = \sqrt{\tau_{th} + L_{th}^2} = \text{“migration length”} \quad (10.91)$$

One more observation about P_{FNL} : If P_{FNL} is close to 1, then its exponent is small, and

$$P_{\text{FNL}} \approx e^{-B_g^2 \tau_{th}} = \frac{1}{e^{B_g^2 \tau_{th}}} \approx \frac{1}{1 + B_g^2 \tau_{th}} \approx 1 - B_g^2 \tau_{th} . \quad [\text{if } B_g^2 \tau_{th} \ll 1] \quad (10.92)$$

10.11.2 Resonance-Escape Probability (p)

We found earlier that

$$p_{re}(E' \rightarrow E_{th}) \approx \exp \left\{ - \int_E^{E'} dE \frac{1}{E} \frac{\Sigma_a(E)}{\bar{\xi}(E) [D(E)B_g^2 + \Sigma_t(E)]} \right\} \quad (10.93)$$

This is the probability that a neutron escapes absorption all the way from birth at energy E' until it slows down to \leq the “thermal” energy E_{th} . Because of the presence of E in the denominator of the integrand and because of the relatively low value of the absorption cross section for high-energy neutrons, this probability is relatively insensitive to the value chosen

for E' as long as E' is in the energy range where fission neutrons appear. We can therefore pick a representative value for E' (say 1 MeV, for example) and then use the resulting expression as the resonance-escape probability for all neutrons born from fission.

We now take a closer look at the denominator of the integrand that appears in the exponent. If we multiplied Σ_t by the scalar flux, we would obtain the collision-rate density. If we multiplied DB_g^2 by the scalar flux, we would obtain the net outleakage rate density. It follows that

$$\frac{D(E)B_g^2}{\Sigma_t(E)} = \frac{\text{probability of leakage}}{\text{probability of collision}} \quad (10.94)$$

for a neutron of energy E . In a large reactor, this ratio will be

extremely small!

Thus, we shall ignore the leakage term in the denominator and use the approximation:

$$p \approx \exp \left\{ - \int_E^{E_{\text{fission}}} dE \frac{1}{\bar{\xi}(E)\Sigma_t(E)} \frac{\Sigma_a(E)}{E} \right\} \quad (10.95)$$

where E_{fission} is a representative energy of a neutron born from fission. We recall now that in the slowing-down energy range, the dominant interaction between neutrons and light (low- A) nuclei (which are the main ingredients of all moderators) is

potential scattering

This is elastic scattering off of the “potential” of the nucleus, which means the neutron does not actually penetrate the nucleus. The potential scattering cross section is essentially independent of the energy of the neutron, and we give it the symbol Σ_p .

We multiply and divide by this cross section, recognizing that it is independent of energy, to obtain:

$$p \approx \exp \left\{ - \frac{1}{[\bar{\xi}\Sigma_p]_{\text{mod}}} \int_E^{E_{\text{fission}}} dE \frac{\Sigma_p^{\text{mod}}}{\Sigma_t(E)} \frac{\Sigma_a(E)}{E} \right\} \quad (10.96)$$

We define the **resonance integral for absorption** for nuclide i in some given mixture of nuclides as follows:

$$I_{r,A,i} \equiv \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\Sigma_p^{\text{mod}}}{\Sigma_t^{\text{mix}}(E)} \frac{\sigma_a^i(E)}{E} . \quad (10.97)$$

There is a corresponding resonance integral for **fission**. In the general case of an arbitrary mixture of fuel and moderator nuclides, this integral would need to be evaluated on a case-by-case basis, because it would depend upon the details of the mixture. However, there is a limiting case that simplifies things considerably:

the “infinitely dilute” limit

in which there are lots, lots more moderator nuclei than fuel (or other absorbing) nuclei. In this limit, note that

$$\Sigma_t^{\text{mix}}(E) = \Sigma_p^{\text{mod}} + \Sigma_t^{\text{fuel}}(E) \xrightarrow{\text{dilute}} \Sigma_p^{\text{mod}} . \quad (10.98)$$

In this limit, the resonance integral that comes into play is one that depends only on the properties of the absorbing nuclides in the reactor:

$$I_{r,A,i}^\infty \equiv \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\sigma_a^i(E)}{E}$$

= infinitely dilute resonance integral for absorption, nuclide *i* . (10.99)

Note the ∞ superscript that denotes the “infinitely dilute” limit.

Thus, in the dilute limit (lots more moderating atoms than absorbing atoms), the resonance integral can be replaced by the infinitely-dilute resonance integral, and our expression for the resonance-escape probability becomes:

$$p \approx \exp \left\{ -\frac{1}{[\bar{\xi} \Sigma_p]_{\text{mod}}} \sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}^\infty \right\} \quad (10.100)$$

If there are lots more moderator atoms than absorber atoms, this is the approximation to use for resonance-escape probability

Beware: If you use this expression correctly and it tells you that p is very small (say, less than 0.5), then your mixture is not dilute enough for it to be very accurate!

10.11.3 Fast Utilization times Fast Reproduction Factor

The product of these two factors is

the number of fission neutrons emitted
per fast-neutron absorption in the reactor.

It is easy to write an expression for this in terms of the energy shape of the scalar flux:

$$u_F \eta_F = \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \nu \sigma_f^i(E) f(E)}{\sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \sigma_a^i(E) f(E)} \quad (10.101)$$

Most of the absorptions and fissions will take place at energies at which most of the neutrons present are neutrons that have scattered, not neutrons that are freshly born from fission. (Most fission neutrons are born with energies above 100 keV; most absorption takes place below this energy.) For these energies, our approximate solution for $f(E)$ is given by Eq. (10.82), reproduced here and then approximated to reflect the usual large-reactor case that $D(E)B_g^2 \ll \Sigma_t(E)$:

$$\begin{aligned} f(E) &= \frac{S_{\text{eff}} p_{nl}(E^* \rightarrow E) p_{re}(E^* \rightarrow E)}{E \bar{\xi} [D(E)B_g^2 + \Sigma_t(E)]}, \quad E < E^* \\ &\approx \frac{S_{\text{eff}} p_{nl}(E^* \rightarrow E) p_{re}(E^* \rightarrow E)}{E \bar{\xi} \Sigma_t(E)}, \quad E < E^*. \end{aligned} \quad (10.102)$$

We insert this into Eq. (10.101) to obtain:

$$u_F \eta_F = \frac{\frac{S_{\text{eff}}}{\bar{\xi}} \sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\nu \sigma_f^i(E)}{E \Sigma_t^{\text{mix}}(E)} p_{nl}(E^* \rightarrow E) p_{re}(E^* \rightarrow E)}{\frac{S_{\text{eff}}}{\bar{\xi}} \sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\sigma_a^i(E)}{E \Sigma_t^{\text{mix}}(E)} p_{nl}(E^* \rightarrow E) p_{re}(E^* \rightarrow E)} \quad (10.103)$$

The $S_{\text{eff}}/\bar{\xi}$ terms cancel. We recognize that $p_{\text{rep}_{nl}}$ can be replaced by an average value in each integral, and we assume that the fission-integral average is approximately the same as the absorption-integral average. We also multiply numerator and denominator by Σ_p^{mod} . Then

$$\begin{aligned} u_F \eta_F &\approx \frac{\langle p_{nl} p_{\text{re}} \rangle \sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\nu \sigma_f^i(E)}{E} \frac{\Sigma_p^{\text{mod}}}{\Sigma_t^{\text{mix}}(E)}}{\langle p_{nl} p_{\text{re}} \rangle \sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\sigma_a^i(E)}{E} \frac{\Sigma_p^{\text{mod}}}{\Sigma_t^{\text{mix}}(E)}} \\ &= \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\nu \sigma_f^i(E)}{E} \frac{\Sigma_p^{\text{mod}}}{\Sigma_t^{\text{mix}}(E)}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i \int_{E_{th}}^{E_{\text{fission}}} dE \frac{\sigma_a^i(E)}{E} \frac{\Sigma_p^{\text{mod}}}{\Sigma_t^{\text{mix}}(E)}} = \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F I_{r,F,i}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}} \end{aligned} \quad (10.104)$$

Note that resonance integrals have appeared again!

Recall that the only resonance integrals we have readily available to us are the

infinitely dilute

resonance integrals, in which the ratio of Σ_p^{mod} to Σ_t^{mix} is taken to be unity. That is, in the “infinitely dilute” resonance integrals we assume that there are lots more moderator atoms than other atoms. Of course, the resonance integrals approach this limit when the ratio of fuel atoms to moderator atoms is **very small**.

To summarize, we have developed an approximate expression for $u_F \eta_F$ that depends on resonance integrals, which means that if our reactor has a high ratio of moderator to fuel atoms we can estimate $u_F \eta_F$ in terms of infinitely-dilute resonance integrals:

$$u_F \eta_F \approx \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F I_{r,F,i}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}} \xrightarrow{\text{dilute}} \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F I_{r,F,i}^{\infty}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}^{\infty}} \quad (10.105)$$

If there are lots more moderator atoms than absorber atoms, this is the approximation to use for $u_F \eta_F$

10.12 Summary

In this chapter we studied how neutrons distribute themselves in energy as the slow down after they are born with high energy during the fission process. For energies between ≈ 1 eV and ≈ 100 keV we found that the fundamental-mode scalar flux approximately satisfies:

$$\phi_{\text{fund}}(\vec{r}, E) \approx \Psi_{\text{fund}}(\vec{r}) \frac{S_{\text{eff}} p_{nl}(E^* \rightarrow E) p_{re}(E^* \rightarrow E)}{E \bar{\xi}(E) [D(E) B_g^2 + \Sigma_t(E)]}. \quad (10.106)$$

Here $\Psi_{\text{fund}}(\vec{r})$ is the spatial shape of the fundamental-mode scalar flux, which was studied in detail in a previous chapter. Actual energy-dependent scalar fluxes will be plotted in detail in a later chapter.

We emphasized that for energies **in between resonances**, the cross sections are roughly constant and ϕ is proportional to $1/E$. At resonance energies, where $\Sigma_t(E)$ is large, the scalar flux can dip sharply in general. However, if there are lots more moderator atoms than fuel atoms, the scalar-flux dips can be mild.

We used our approximate expression for scalar flux to develop several useful formulas that will help us estimate the multiplication factor for any **bare homogeneous** reactor with a high ratio of moderator atoms to fuel atoms. They are summarized here:

$$P_{\text{FNL}} \approx \exp \{-B_g^2 \tau_{th}\} \quad (10.107)$$

$$p \approx \exp \left\{ -\frac{1}{[\bar{\xi} \Sigma_p]_{\text{mod}}} \sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}^\infty \right\} \quad (10.108)$$

$$u_F \eta_F \approx \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F I_{r,F,i}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}} \xrightarrow{\text{dilute}} \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F I_{r,F,i}^\infty}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}^\infty} \quad (10.109)$$

The quantities τ_{th} , $\bar{\xi}$, σ_p , $I_{r,A}^\infty$, and $I_{r,F}^\infty$ are tabulated for nuclides and mixtures that are commonly used in reactors. See the Appendix on “Cross Sections and Other Data.”

10.13 Extra Material

10.13.1 Energy problem

The energy-dependent equation, Eq. (10.7), looks exactly like an

infinite-medium

k -eigenvalue equation except that it has an additional

$D(E)B^2$ term.

This term plays the same role in the equation as

the absorption cross section, $\Sigma_a(E)$.

It accounts for **outleakage** from the reactor, and can be thought of as a kind of

net-outleakage or escape cross section.

(Recall that we are considering bare reactors, which means that net outleakage through the reactor surface is the same as outleakage, because inleakage is zero.)

Note that if $\tilde{f}(E)$ is any solution of Eq. (10.7), then so is any constant times $\tilde{f}(E)$. That is, this equation says nothing about \tilde{f} 's amplitude. It gives information only about $\tilde{f}(E)$'s shape (which describes the energy distribution of the neutrons in the reactor). Suppose \tilde{f} and k satisfy the equation. Then define the constant C_{int} as follows:

$$C_{int} \equiv \frac{1}{k} \int_0^\infty dE_i \nu \Sigma_f(E_i) \tilde{f}(E_i). \quad (10.110)$$

This is just a number—a constant. Divide the equation by this constant:

$$[D(E)B^2 + \Sigma_t(E)] \frac{\tilde{f}(E)}{C_{int}} = \chi(E) + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) \frac{\tilde{f}(E_i)}{C_{int}} \quad (10.111)$$

This shows that the function $f(E) \equiv \tilde{f}(E)/C_{int}$, which is just a constant times the original solution function \tilde{f} (and thus is also a solution), satisfies the following:

$$[D(E)B^2 + \Sigma_t(E)] f(E) = \chi(E) + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) f(E_i) \quad (10.112)$$

This has the form of

a fixed-source steady-state infinite-medium problem.

The solution will tell us the energy shape of the eigenfunction that is associated with whichever B^2 eigenvalue is put into the equation.

10.13.2 Differential Scattering Cross Section for Elastic Scattering: Demonstration

Neutrons that emerge with energies between E_f and $E_f + dE_f$ must have scattered through a CM angle between $\theta_C - d\theta_C$ and θ_C , where E_f is related to θ_C by Eq. (10.41). Thus, the probability of scattering into $(E_f, E_f + dE_f)$ is equal to the probability of scattering through $(\theta_C - d\theta_C, \theta_C)$.

We have already named the probability of scattering into $(E_f, E_f + dE_f)$:

$$\text{prob of scattering into } (E_f, E_f + dE_f) = p(E_i \rightarrow E_f) dE_f \quad (10.113)$$

Now we will name the probability of scattering into $(\theta_C - d\theta_C, \theta_C)$:

First, we recall the probability of scattering from $\vec{\Omega}_i$ to $\vec{\Omega}_f$ [directions are given in the CoM, $\vec{\Omega}_i \cdot \vec{\Omega}_f = \theta_C$] is given by:

$$p(\vec{\Omega}_i \rightarrow \vec{\Omega}_f) d\Omega_f = \frac{\sigma_s(E_i, \vec{\Omega}_i \cdot \vec{\Omega}_f)}{\sigma_s(E_i)} d\Omega_f$$

where the final solid angle is $d\Omega_f = d\varphi \sin \theta_C d\theta_C$. $p(\vec{\Omega}_i \rightarrow \vec{\Omega}_f)$ has units of 1/ster.

Recall that scattering is axi-symmetric (independent of φ),

$$p(\vec{\Omega}_i \rightarrow \vec{\Omega}_f) = \frac{p(\theta_C)}{2\pi}$$

$$\sigma_s(E_i, \vec{\Omega}_i \rightarrow \vec{\Omega}_f) = \frac{\sigma_s(E_i, \theta_C)}{2\pi}$$

You can think of this as the definition of $\sigma_s(E_i, \theta_C)$, the differential scattering cross section for scattering through the center-of-mass angle θ_C . It has units of

barns/(unit cosine).

Hence,

$$p(0 \rightarrow \theta_C) \sin \theta_C d\theta_C = \frac{\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \sin \theta_C d\theta_C$$

The probability of scattering into ($\theta_C, \theta_C + d\theta_C$) is thus:

$$p(0 \rightarrow \theta_C) \sin \theta_C d\theta_C = \frac{\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \sin \theta_C d\theta_C$$

The probability of scattering into ($\theta_C - d\theta_C, \theta_C$) is thus:

$$p(0 \rightarrow \theta_C) \sin \theta_C (-d\theta_C) = -\frac{\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \sin \theta_C d\theta_C$$

Why do we care about into ($\theta_C - d\theta_C, \theta_C$)? well, when the final energy E_f increases, then the final CoM angle θ_C should decrease.

Now we set these two probabilities equal to each other, obtaining:

$$\text{prob of scattering into } (E_f, E_f + dE_f) = p(E_i \rightarrow E_f) dE_f \quad (10.114)$$

$$\text{prob of scattering into } (\theta_C - d\theta_C, \theta_C) = -\frac{\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \sin \theta_C d\theta_C \quad (10.115)$$

or

$$p(E_i \rightarrow E_f) dE_f = -\frac{\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \sin \theta_C d\theta_C \quad (10.116)$$

Recall the relationship (10.41) which gives us:

$$E_f = E_i \frac{(1 + \alpha) + (1 - \alpha) \cos(\theta_C)}{2} \quad (10.117)$$

We can differentiate this to obtain:

$$dE_f = -\frac{E_i}{2} (1 - \alpha) \sin \theta_C d\theta_C \quad (10.118)$$

Now we can rewrite Eq. (10.116) as

$$p(E_i \rightarrow E_f) \left[-\frac{E_i}{2} (1 - \alpha) \sin \theta_C d\theta_C \right] = -\frac{\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \sin \theta_C d\theta_C \quad (10.119)$$

or

$$p(E_i \rightarrow E_f) = \frac{2\sigma_s(E_i, \theta_C)}{\sigma_s(E_i)} \frac{1}{(1 - \alpha)E_i} \quad (10.120)$$

Chapter 11

Thermal Neutrons

11.1 Introduction

The vast majority of today's reactors are thermal reactors. In a thermal reactor, most fissions are caused by "thermal" neutrons - neutrons that are essentially in thermal equilibrium with the reactor materials. In this chapter we study the physics of low-energy neutron interactions, the energy distribution ("spectrum") of neutrons in the thermal range, and thermal reaction rates.

11.2 Infinite Medium, No Sources or Absorption - the Maxwellian Distribution

Consider an infinite medium, in which nothing depends on position (homogeneous). In such a medium the exact balance equation, which by now we know very well, is:

$$\Sigma_t(E)\phi(E) = S(E) + \int_0^\infty dE' \Sigma_s(E' \rightarrow E)\phi(E') \quad (11.1)$$

Now consider the case in which there is no source and no absorption:

$$\Sigma_s(E)\phi(E) = \int_0^\infty dE' \Sigma_s(E' \rightarrow E)\phi(E') \quad (11.2)$$

Note that our neutrons will behave just like one (dilute) component in a mixture of gases—they will just bounce around forever. What distribution will they attain? The answer is the

Maxwellian distribution,

which is given by the following equation

$$\phi_M(E) = n_{th} \frac{2}{\sqrt{\pi}} \frac{1}{kT} \left(\frac{2kT}{m} \right)^{1/2} \left(\frac{E}{kT} \right) \exp \left(-\frac{E}{kT} \right) \quad (11.3)$$

Energy-dependent scalar flux if neutrons are in Maxwellian

Here,

$$\begin{aligned} n_{th} &= \text{thermal-neutron density [n/cm}^3\text{]} \\ m &= \text{neutron mass} \\ k &= \text{Boltzmann's constant} \\ &\approx 8.617065 \times 10^{-5} \text{ eV/K} \simeq \frac{1 \text{ eV}}{11605 \text{ K}} \\ T &= \text{temperature of mixture (K)} \end{aligned}$$

The rather amazing answer, Eq. (11.3),

is true for ANY differential scattering cross section, $\Sigma_s(E' \rightarrow E)$,

that is physically realizable. The proof of this is by statistical mechanics and is beyond the scope of our study. In a nutshell: This is the distribution that has the highest entropy, which is what the second law of thermodynamics says the neutrons must attain. Do not forget the following main result, which we now restate:

In the absence of absorption, leakage, and sources,
neutrons attain the Maxwellian distribution,
which is given by Eq. (11.3).

(You do not need to memorize all of the constants in this distribution, but you should know that the Maxwellian distribution for the scalar flux is of the form: constant $\times E e^{-E/kT}$.)

Notes on units and meanings:

1. kT has units of energy. $\frac{E}{kT}$ is a dimensionless energy variable.
2. The energy dependence of the Maxwellian scalar flux is of the form $\epsilon e^{-\epsilon}$, where $\epsilon = E/(kT)$. The other terms do not contain the E variable. So if you want to see the shape of the Maxwellian scalar flux, plot $\epsilon e^{-\epsilon}$ as a function of ϵ .
3. The term $(2kT/m)^{1/2}$ has units of speed, and $1/(kT)$ has units of 1/energy. So the Maxwellian flux has units of (neutron density) \times speed divided by energy, or (n-cm)/(cm³-MeV-s), as it should.

We plot the Maxwellian scalar flux for two different temperatures in Fig. 11.1.

You can observe in the figure something that is easy to prove: the peak of the Maxwellian scalar-flux distribution is at the following energy:

$$E_{peak} = kT \quad (11.4)$$

[You can prove this by setting $d\phi_M/dE = 0$ and solving for E .] That is, the “most probable” energy is simply kT . Recall that at room temperature, kT is approximately

$$k \times 293.15 \text{ K} = 0.0253 \text{ eV.}$$

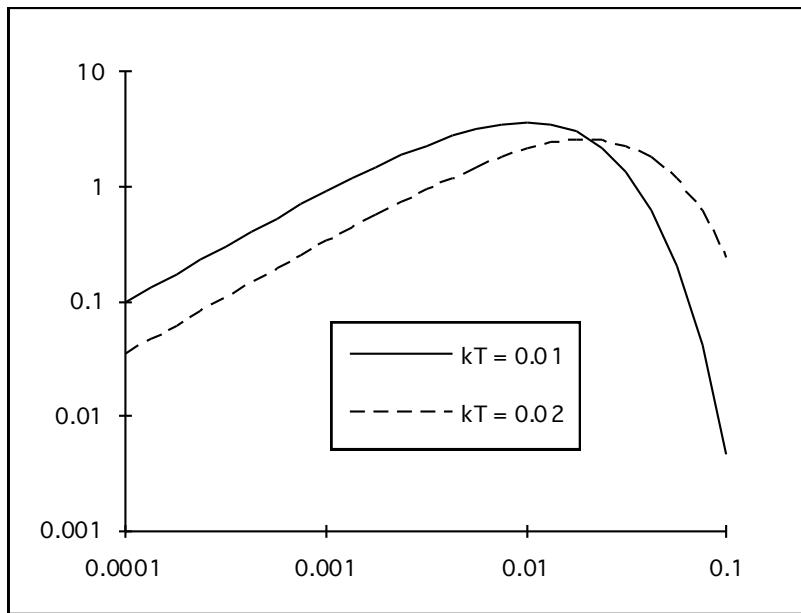


Figure 11.1: Energy-dependent scalar flux for neutrons in Maxwellian distributions. Labels: $\phi_M(E)$ and E . Drop dotted lines from peaks, showing they're at kT .

A neutron with this $E = 0.0253$ eV has approximately the following speed:

$$E = 0.0253 \text{ eV} \Rightarrow \text{speed} = 2200 \text{ m/s} = 220,000 \text{ cm/s.}$$

When we wish to compute thermal-neutron reaction rates, we will of course need thermal-neutron cross sections. These cross sections depend on the relative speed between the neutron and nucleus, of course. It turns out (as we shall see) that in many cases all we really need to know is the cross section evaluated at only one relative speed. In the beginning days of nuclear engineering, the speed **2200 m/s** was chosen to be the relative speed at which to tabulate cross sections. The chart of the nuclides provides 2200-m/s cross sections, and most books about reactor theory contain appendices that tabulate such cross sections for nuclides that are commonly used in reactors.

In our work with thermal neutrons we will often use the subscript 0 to denote quantities associated with room temperature, the associated most-probable energy, and the associated speed:

$$T_0 \equiv 293.15 \text{ K} , \quad (11.5)$$

$$E_0 \equiv 0.0253 \text{ eV} \approx \frac{1}{40} \text{ eV} , \quad (11.6)$$

$$v_0 \equiv 2200 \frac{m}{s} = 220,000 \frac{cm}{s} . \quad (11.7)$$

Do not confuse this definition of E_0 with the one used in our neutron-slowing-down studies. In those studies it was around 10-12 MeV. Here it is much, much smaller!

11.3 Real-world Case: $\phi(E) \approx$ Perturbed Maxwellian

In a real reactor, which has absorption, leakage, and the down-scattering source, the following effects need to be included:

- Absorption** Recall from your observation of cross sections that in the low-energy range, the lower the neutron speed, the higher the absorption cross section. (See plots in the cross-section appendix.) This means that for thermal neutrons, lower-energy neutrons are absorbed more strongly than higher-energy ones. As a consequence, the neutron spectrum is shifted to higher energies. Unless the absorption is very strong, the perturbed spectrum looks somewhat like a Maxwellian at a higher temperature. (Recall that a higher temperature shifts the Maxwell spectrum towards the right). This phenomenon of shifting the spectrum to higher energies is sometimes called

absorption heating
 or
absorption hardening

(A spectrum that has more high-energy neutrons is often called “harder.”) We define the “neutron temperature” as follows:

$$T_n \equiv \text{“neutron temperature”} \quad (11.8)$$

≡ temperature of the Maxwellian that best fits the actual neutron spectrum. (11.9)

- finiteness** (the reactor is not infinite). This is a smaller effect. It is understood as follows: more energetic neutrons have lower interaction probability with the medium (their cross section is lower). Hence they leak more easily, and the spectrum is slightly shifted to the left. This is sometimes called “**diffusion cooling**.” Its effects are accounted for in the choice of the “neutron temperature.”
- presence of sources.** The neutrons incoming in the thermal range (typically the thermal cut-off energy is 0.025 eV), come from the slowing down range, where they behaved as $\approx 1/E$. The effect is that the Maxwellian spectrum for $E >$ a few times kT , the Maxwellian tail of the thermal spectrum blends into the $1/E$ shape of the slowing-down spectrum.
- presence of low-lying resonances.** Some heavy nuclides have resonances at the upper end of the thermal-neutron-energy range. Neutrons are heavily absorbed in their vicinity. This distorts the Maxwellian spectrum—the flux is depressed near the resonance energy. See Fig. 11.2 for an example of how the Pu in Mixed-Oxide (MOX) fuel perturbs the spectrum, relative to Uranium-Oxide (UOX) fuel.

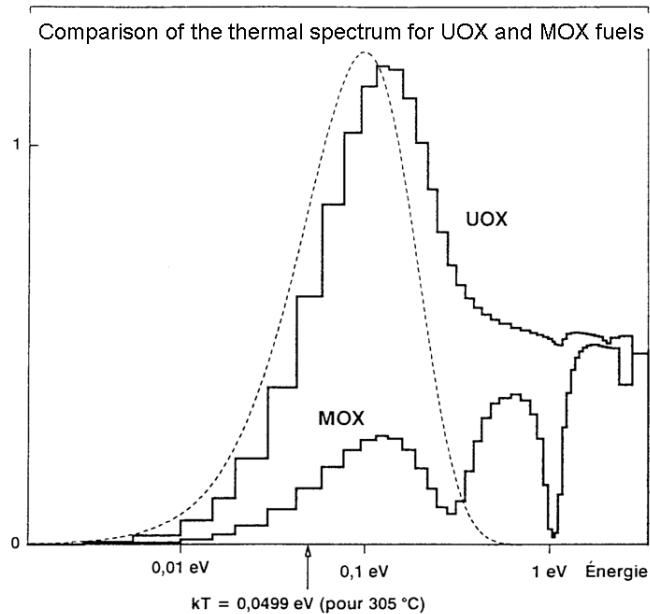


Figure 11.2: Comparison of the Maxwellian distribution with the thermal spectrum in UOX fuel and MOX fuel.

11.4 Reaction rates in the thermal range

We have often used the expression

$$x\text{-reaction rate density at position } \vec{r} = \int_0^{\infty} dE \Sigma_x(\vec{r}, E) \phi(\vec{r}, E) \quad (11.10)$$

where “ x ” could be scattering, fission, absorption, or any other kind of reaction, and where E is the neutron kinetic energy in the lab frame. This expression seems to imply that the cross section depends only on the neutron’s lab-frame speed. But we know that in reality, the cross section for a given reaction depends fundamentally on the

relative speed, v_r ,

between the neutron and the nucleus, and **not** on the lab speed of the neutron only. Thus, when we write something like Eq. (11.10), we are tacitly assuming that $\Sigma_x(\vec{r}, E)$ has been appropriately averaged over the velocity distribution of the nuclei. This was covered in detail in Chapter 3 of these notes. Recall from that chapter that the absorption reaction rate is really given by:

$$\text{Abs. rate density at } \vec{r} = \iiint_{\text{all } \vec{v}} d^3v \ n(\vec{r}, \vec{v}) \iiint_{\text{all } \vec{V}} d^3V \sum_{i=1}^{\# \text{ of nuclides}} N_i(\vec{r}, \vec{V}) v_r \sigma_a^i(v_r) \quad (11.11)$$

where $v_r \equiv |\vec{V} - \vec{v}|$.

11.4.1 $1/v$ Absorbers

Now consider the special (but common) case of a “ $1/v$ absorber.” In this case σ_a is:

$$\sigma_a(v_r) = \frac{\text{constant}}{v_r} \Rightarrow v_r \sigma_a(v_r) = v_0 \sigma_a(v_0). \quad (11.12)$$

If we insert this into the general expression for absorption rate we find that all dependence on v_r vanishes and we are left with a **very simple result**:

$$\begin{aligned} \text{Abs. rate density at } \vec{r} &= \iiint_{\text{all } \vec{v}} d^3v \ n(\vec{r}, \vec{v}) \iiint_{\text{all } \vec{V}} d^3V \sum_{i=1}^{\# \text{ of nuclides}} N_i(\vec{r}, \vec{V}) v_r \sigma_a^i(v_r) \\ &= \left[\iiint_{\text{all } \vec{v}} d^3v \ n(\vec{r}, \vec{v}) \right] \sum_{i=1}^{\# \text{ of nuclides}} \left[\iiint_{\text{all } \vec{V}} d^3V \ N_i(\vec{r}, \vec{V}) \right] [v_0 \sigma_a^i(v_0)] \\ &= n_{th} v_0 \sum_{i=1}^{\# \text{ of nuclides}} N_{tot}^i \sigma_a^i(v_0) \\ &= \Sigma_a(E_0) n_{th} v_0 \end{aligned} \quad (11.13)$$

where E_0 is the neutron energy corresponding to the speed v_0 , n_{th} is the thermal-neutron density [n/cm^3], and N_{tot}^i is the density of nuclides of type i [nuclei/cm^3].

Equation (11.13) is an important and remarkable result. Given a $1/v$ cross section, the reaction rate is

1. **independent** of the velocity distribution of the neutrons, and
2. **independent** of the velocity distribution of the nuclei!!!

Given a $1/v$ absorber, we can find the thermal absorption rate if we know only a few simple things:

1. the thermal-neutron density n_{th} [n/cm³],
2. the nucleus density N_{tot}^i [nuclei/cm³] for each nuclide present,
3. the cross section at some specific relative speed v_0 .

It is common practice to tabulate cross sections at this specific speed:

$$v_0 = 2200 \text{ m/s}$$

It is also conventional to define

$$\phi_0 = \text{"2200 m/s flux"} = n_{tot} v_0$$

Important note:

$$\phi_0 = \text{"2200 m/s flux"} = n_{tot} v_0 \neq \phi(E_0) \neq \phi_{th} !!!$$

The units aren't even the same! Be careful with this. Note that ϕ_0 is not the "thermal flux," and it does not have the physical meaning of path-length rate per unit volume. It is simply a convenient product of two numbers!

Example:

A few cm away from some H₂O-pool-type reactor, the density of thermal neutrons is found to be 10⁵ n/cm³. At this location what is the thermal absorption rate density?

Solution:

Since hydrogen and oxygen are 1/v absorbers, all we need to know is:

- Σ_a evaluated at some known relative speed,
- the neutron density.

The neutron density is given. Cross sections evaluated at 2200 m/s are tabulated in an appendix of your textbook, an appendix of these notes, on the chart of the nuclides, and in many other places. From any of those we find

$$\Sigma_a^{H20} \text{ at } (2200 \text{ m/s}) = 0.0222 \text{ cm}^{-1} .$$

(This assumes a water density of approximately 1 g/cm³.) Thus,

$$\begin{aligned} \text{Thermal abs. rate density} &= \Sigma_a(v_0)n_{th}v_0 \\ &= [0.0222 \times 220000 \times 10^5] \frac{\text{abs}}{\text{cm}^3\text{-s}} \end{aligned}$$

11.4.2 Non-1/v Absorbers

A survey of cross sections shows that

light nuclei are all 1/v absorbers !

However, some intermediate and heavy nuclei are not 1/v absorbers throughout the entire thermal-energy range, usually because of resonances at low neutron energies. We have an incredibly simple expression for absorption and fission rate densities for nuclei that are 1/v absorbers—nature is very kind to nuclear engineers in this case—but what do we do about the others?

The first thing to note is that even intermediate and heavy nuclei have absorption cross sections that are $1/v$ or nearly $1/v$ over much of the thermal range. This suggests that an approximate correction to our simple $1/v$ expression for absorption or fission rate density might be all we need. Such a correction, called a

non- $1/v$ factor

was in fact pursued in the early days of nuclear engineering. In 1962, C. H. Westcott published a set of non- $1/v$ factors under the following assumptions:

- neutrons have Maxwellian distribution at the “neutron temperature” T_n ,
- nucleus motion in lab frame is negligible.

While both of these seem a bit severe, especially the second one, it is important to remember that:

- the cross sections are almost $1/v$;
- when cross sections are $1/v$ the velocity distributions of the neutrons and nuclei don’t matter at all;
- the correction we need is not very big.

This is the “simple correction” factor we were after. If we know this correction factor for a given nuclide then we can calculate absorption rate densities in non- $1/v$ absorbers almost as easily as in $1/v$ absorbers:

$$\text{Thermal abs. rate dens.} = [n_{tot}v_0] \sum_{i=1}^{\# \text{ of nuclides}} \Sigma_a^i(v_0) g_a^i(T_n) \quad (11.14)$$

There is a similar expression for fission:

$$\text{Thermal fission rate dens.} = [n_{tot}v_0] \sum_{i=1}^{\# \text{ of nuclides}} \Sigma_f^i(v_0) g_f^i(T_n) \quad (11.15)$$

Note that the non- $1/v$ factors for absorption and fission are different in general!

11.5 Summary

In this chapter we studied thermal neutrons—neutrons whose energies are comparable to the energies of the background nuclei. We found:

1. In an ∞ medium with $\Sigma_a = 0 = S$, $\phi(E) = \text{Maxwellian at } T$, where $T = \text{temperature of medium}$.
2. With small absorption, $\phi(E)$ Maxwellian at “neutron temperature” $T_n > T$.
3. In finite medium, “diffusion cooling” can slightly lower T_n . (Small effect in large reactors).
4. In finite media with large absorption, $\phi(E)$ in the thermal range deviates from a Maxwellian in predictable ways.
5. We also studied thermal interaction rates. Scattering of thermal neutrons is complicated because nuclei are bound to other nuclei in solids. The same is true in fluids if there are multi-atom molecules. But recall that we developed a simple expression for absorption rate density:

$$\text{Abs. rate density} = [n_{tot}v_0] \sum_{i=1}^{\# \text{ of nuclides}} \Sigma_a^i(v_0) g_a^i(T_n)$$

Here we defined some terms:

$$\phi_0 = \text{“2200 m/s flux”} \equiv n_{tot}v_0$$

$$\phi_{th} = \text{“thermal flux”} \equiv \int_0^{E_{th}} dE \phi(E)$$

and noted that $\Sigma_a(E_0)$ = absorption cross section evaluated at a relative speed of 2200 m/s, and $g_a(T_n)$ is the absorption non- $1/v$ factor for a neutron distribution that is approximately a Maxwellian at temperature T_n . The precise formula for a non- $1/v$ absorber is really more complicated—it depends in general on the actual velocity distributions of the neutrons and nuclei—but for this course we will use the formula given above, which is usually a reasonable approximation.

6. Finally, remember:

For every absorption formula there is a similar fission formula.

Part VII

Putting it all together

Chapter 12

Assimilation

12.1 Introduction

Our goal all along has been to gain an understanding of nuclear reactors. As we've noted many times, this requires knowledge of how neutrons are produced and lost. We've also noted that production and loss rates depend on a multitude of factors, most of which we have discussed in some detail.

Perhaps the most basic thing we'd like to know about any given reactor is its criticality state: is it subcritical, supercritical, or critical? The multiplication factor, k_{eff} , tells us the answer. In Chapter 5 we developed a 6-factor formula for k_{eff} , and in later chapters we learned that k_{eff} is an eigenvalue of the $[\text{Loss}]^{-1}[\text{Production}]$ operator. If we accept the diffusion approximation, this means k_{eff} is an eigenvalue of a differential (or integro-differential) equation.

We have now developed solutions of our differential and integro-differential equations, using the diffusion approximation and sometimes using further approximations, all of which are accurate for large homogeneous reactors with lots more moderator than fuel. These solutions now enable us to estimate each of the six factors in the six-factor formula for k_{eff} using what we know along with tabulated data.

We also found, in Chapter 9 of these notes, solutions for how neutrons distribute themselves spatially in a bare homogeneous reactor. We now know this for basic shapes such as spheres, cylinders, and rectangular parallelepipeds. In Chapter 10 we learned how neutrons distribute themselves in energy as they slow to thermal energies in infinite reactors, discovering that the dominant behavior is a $1/E$ distribution in the slowing-down range. We found that absorption perturbs this distribution, causing downward spikes at resonances and a lowering of the flux just below each resonance (compared to the value it would have had without

the resonance). We developed an approximate expression for P_{FNL} that is accurate for large reactors and one for p that is accurate for dilute concentrations of fuel mixed with moderator.

In Chapter 11 we found that neutrons in the thermal range tend toward a Maxwellian distribution in energy. We saw that absorption can perturb this distribution, but that the perturbed distribution looks a lot like a Maxwellian at a higher temperature, which we call the “neutron temperature.” We also discovered that for $1/v$ absorbers, which include all low-Z nuclides, it is very easy to compute thermal absorption rates, and that for non- $1/v$ absorbers it is easy to get approximate absorption (and fission) rates by using a non- $1/v$ factor.

In the present chapter we put all of this together and show step by step how to use it to estimate the multiplication factor of a bare homogeneous reactor. We provide tables of needed data (such as thermal diffusion length and age to thermal for various moderators). We also show that we can calculate any reaction rate in any part of the reactor, because we know the spatial and energy distribution of the neutrons.

12.2 Multiplication Factor of Bare Homogeneous Reactor

For the remainder of this chapter let us agree that “ k ” is the multiplication factor of the reactor. That is, we drop the “eff” subscript. We are not concerned in this chapter with any of the other (smaller) k -eigenvalues, so there should be no confusion.

The six-factor formula says

$$k = P_{\text{FNL}} \epsilon p P_{\text{TNL}} u_T \eta_T \quad (12.1)$$

In Chapter 5 we noted that we could write an expression without the fast-fission factor, ϵ :

$$k = P_{\text{FNL}} p P_{\text{TNL}} u_T \eta_T + P_{\text{FNL}} (1 - p) u_F \eta_F \quad (12.2)$$

In Chapters 9, 10, and 11 we obtained approximate expressions for P_{TNL} , $(\eta_T u_T)$, and $(u_F \eta_F)$:

$$\begin{aligned} k &\approx P_{\text{FNL}} p \frac{1}{1 + L_{th}^2 B_g^2} \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_{th}^i \sigma_f^i(v_0) g_f^i(T_n)}{\sum_{i=1}^{\# \text{ of nuclides}} N_i \sigma_a^i(v_0) g_a^i(T_n)} \\ &+ P_{\text{FNL}} (1 - p) \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F I_{r,F,i}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}} \end{aligned} \quad (12.3)$$

Note: the resonance integrals in the 2nd term play the same role as the $\sigma(v_0)g(T_n)$ quantities in the 1st term.

Our goal in this chapter is to review how to estimate each term in this equation using what we have learned throughout the course. In the following subsections we will develop or recall approximate expressions for each needed quantity:

- P_{FNL} , which depends on τ_{th} ,
- p ,
- ν_{th} and ν_F , and
- L_{th}^2 .

12.2.1 Fast Non-Leakage Probability

In Chapter 10 we agreed to use

$$P_{\text{FNL}} \approx e^{-B_g^2 \tau_{th}} \left(\xrightarrow{B_g^2 \tau_{th} \ll 1} \approx 1 - B_g^2 \tau_{th} \quad \text{or} \quad \frac{1}{1 + B_g^2 \tau_{th}} \right) \quad (12.4)$$

But where do we find the age to thermal, τ_{th} ? This quantity is tabulated for different moderators. In Table 12.1 we reproduce data from a table given in the Lamarsh text. (This is based in part on data given in L. S. Kothari and V. P. Duggal, “Scattering of Thermal Neutrons from Solids and Their Thermalization Near Equilibrium,” Advances in Nuclear Science and Technology, Vol. 2. New York: Academic Press, 1964.) Note that we have provided two H₂O numbers at two different densities. We do this to illustrate that a mean squared distance must be inversely proportional to the square of the density. If the density changes by a factor (0.9/1.0), then τ (and also L_{th}^2) must change by a factor of $(1.0/0.9)^2 = 1/0.81$. Note that $27/0.81 \approx 33.3$.

Table 12.1: Approximate “Age” from fission-to-thermal for various moderators. Also approximate average logarithmic energy decrements ($\bar{\xi}$) and potential scattering cross sections (σ_p).

Moderator	at ρ of	Approximate τ_{th}	$\bar{\xi}$	σ_p
H ₂ O	1.0 g/cm ³	27 cm ²	0.920	44 b
H ₂ O	0.9 g/cm ³	33 cm ²	0.920	44 b
D ₂ O	1.1 g/cm ³	131 cm ²	0.509	10.5 b
Be	1.85 g/cm ³	102 cm ²	0.209	6.1 b
Graphite (C)	1.70 g/cm ³	368 cm ²	0.158	5 b

What about a mixture of fuel with one of these moderators? We noted in Ch. VIII that an approximation for age to thermal is

$$\tau(E_{\text{fission}} \rightarrow E_{th}) \approx \int_{E_{th}}^{E_{\text{fission}}} dE \frac{D(E)}{E\bar{\xi}(E)\Sigma_t(E)} = \int_{E_{th}}^{E_{\text{fission}}} dE \frac{1}{3E\bar{\xi}(E)\Sigma_{tr}(E)\Sigma_t(E)} \quad (12.5)$$

If the mixture is dilute (lots more moderator atoms than fuel atoms), then only at resonance energies will the fuel's presence significantly change the cross sections. These are very narrow ranges compared to the full range of integration. In these ranges the cross sections will be higher than the moderator-only value, and thus in these tiny ranges the contribution to τ_{th} will decrease. We conclude that:

In a dilute mixture of fuel and moderator,
the age to thermal is only slightly lower than in the pure moderator.

Thus, in our calculations of large reactors with dilute homogeneous mixtures,

We will approximate $\tau_{th} \approx \tau_{th}^{\text{mod}}$.

12.2.2 Resonance-Escape Probability

In Chapter 10 we agreed to use

$$p \approx \exp \left\{ -\frac{1}{[\bar{\xi} \Sigma_p]_{\text{mod}}} \sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}^{\infty} \right\} \quad (12.6)$$

Here Σ_p is the potential scattering cross section of the moderator. If you explore how we got Eq. (12.6) you will see that Σ_p is the scattering cross section evaluated at energies where most of the resonance absorption is taking place, which means from around 1 eV to a few keV.

How to tell whether to trust infinitely-dilute approximation in calculating p : if Eq. (12.6) produces a value of p that is too far below 1, then the mixture is not dilute enough to trust the approximation $I \approx I^\infty$. The closer p is to 1, the better the approximation:

- If the approximation gives a $p < 0.5$, **don't trust it.**
- If it gives $p \in (0.5, 0.7)$ it is not terribly accurate, but maybe not horrible.
- If it gives $p > 0.85$, it's probably as accurate as the other approximations we are making.
- I_r^∞ is always **larger** than the real I_r for a given problem, which means our approximation for p is always **smaller** than the real p .

12.2.3 Resonance Integrals

Where do we find the resonance integrals for the absorbing nuclides? If we are in the dilute limit (lots more moderator atoms than fuel atoms), we can get them from the chart of the nuclides. Table 12.2 provides infinitely dilute resonance integrals (for both absorption [which includes capture + fission] and fission) for a few interesting nuclides.

12.2.4 Neutrons Emitted per Fission (ν_T and ν_F)

Experiments and theory agree that a particular fission event may release 0 or 1 or 2 or 3 or 4 or more neutrons. In reactor analysis we are rarely interested in how many fissions release 0, how many release 1, etc.; all we usually care about is how many are released on average from fissions of a given nuclide, caused by the absorption of neutrons of a given energy. Experiments find that this value fits a simple formula over wide ranges of incident neutron energy:

$$\nu(E) \approx \nu_0 + a E \quad (12.7)$$

where ν_0 and a are constants for a given nuclide and a given range of E , and

E = energy of the neutron that caused the fission.

Values of these constants are given in Table 12.3 for various interesting nuclei, sometimes in two energy ranges to accurately fit the measured data. With one exception, the numbers are from Keepin (G. R. Keepin, Physics of Nuclear Kinetics, Reading, Mass.: Addison-Wesley, 1965). The exception is for ^{238}U , where I have taken the numbers from the plot shown in Fig. 12.1.

Table 12.2: Infinitely-Dilute Resonance Integrals [barns]. (From <http://atom.kaeri.re.kr>)

Nuclide	Capture + Fission	Fission
^{235}U	411	278
^{238}U	280	2
^{239}Pu	484	303
^{240}Pu	8112	9
Fe (natural)	1.4	0
Zr (natural)	1.2	0
^{16}O	0.0004	0
^1H	0.15	0
^2H (D)	0.0003	0
H_2O	0.3004 $(= 0.15 + 0.15 + 0.0004)$	0
D_2O	0.0010 $(= 0.0003 + 0.0003 + 0.0004)$	0

Note that this function varies so slowly in energy that the correct value to use for thermal neutrons is

$$\nu_{th} = \nu_0 \quad (12.8)$$

Note in particular that this is **2.43 for U-235**. You should remember this number!

What about ν_F , the average number of neutrons released per fission caused by a non-thermal (fast) neutron? In a thermal reactor, recall that most non-thermal fissions occur in resonances the slowing-down energy range, where the flux is roughly proportional to $1/E$. Because the resolved-resonance range for the important fissile and fissionable nuclides is roughly from 1 eV to a few tens of keV, and because of the $1/E$ flux, a reasonable average to use for ν_F would be the ν value evaluated at a few keV or so. This would give

$$\nu_F \approx \nu(E)|_{E=1\text{keV}} \approx \nu_0 + a \times 0.001 \text{ MeV.} \quad (12.9)$$

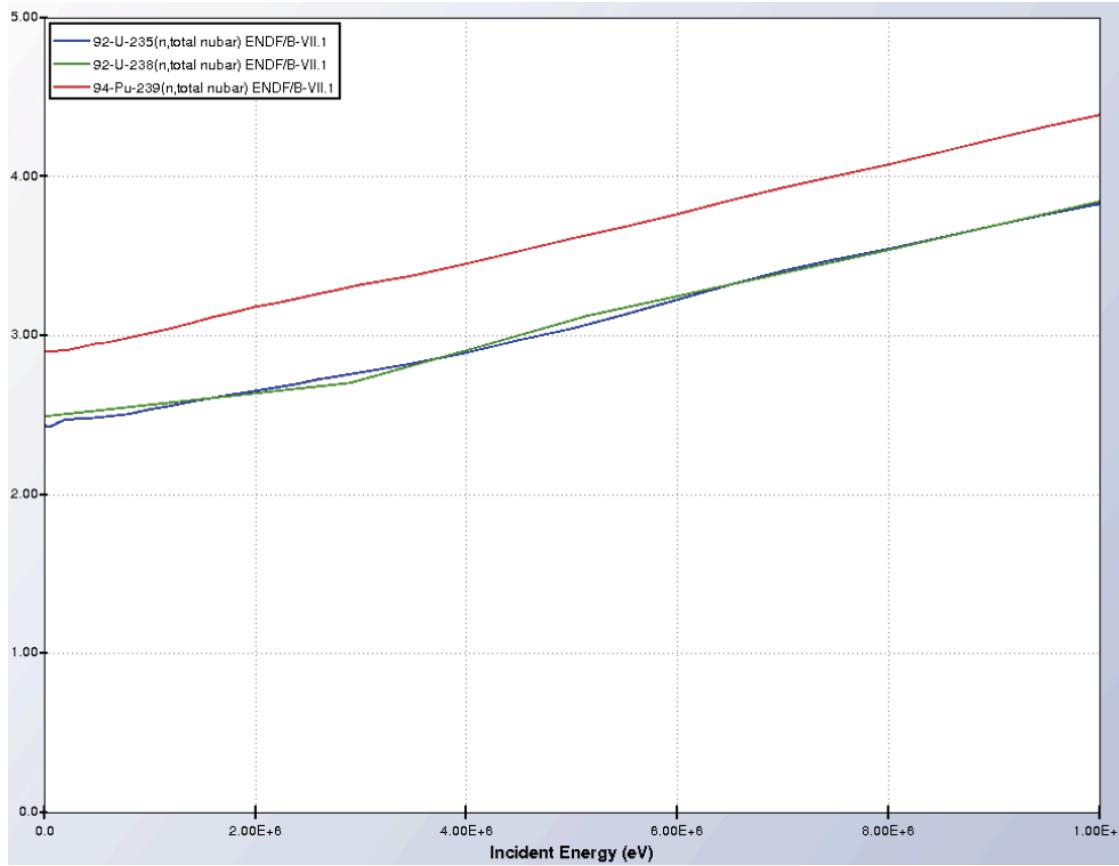


Figure 12.1: $\nu(E)$ = expected number of neutrons (prompt+delayed) emitted per fission as a function of energy (E) for ^{235}U , ^{238}U , and ^{239}Pu .

We see from Table 12.3 that the constant a is always $< 1 \text{ MeV}^{-1}$. This means the term added to ν_0 is < 0.001 , which is negligibly small compared to ν_0 . We conclude that:

In a thermal reactor, we can use $\nu_F \approx \nu_{\text{th}} = \nu_0$.

Table 12.3: Values of the constants ν_0 and a in Eq. (12.7).

Nuclide	$\nu_0 = \nu_{th}$	a , [MeV $^{-1}$]	Energy Range
^{232}Th	1.87	0.164	all E
^{233}U	2.48	0.075	$E \in (0, 1)$ MeV
	2.41	0.136	$E \geq 1$ MeV
^{235}U	2.43	0.065	$E \in (0, 1)$ MeV
	2.35	0.150	$E \geq 1$ MeV
^{238}U	2.50	0.0074	$E \in (0, 3)$ MeV
	2.242	0.160	$E \geq 3$ MeV
^{239}Pu	2.87	0.148	$E \in (0, 1)$ MeV
	2.91	0.133	$E \geq 1$ MeV

12.2.5 Thermal Non-Leakage Probability (P_{TNL})

We now have approximations fully defined for everything except the thermal-neutron non-leakage probability in our equation for k . This term is:

$$P_{\text{TNL}} = \frac{1}{1 + L_{th}^2 B_g^2} \quad (12.10)$$

We can evaluate this term if we know L_{th} , the thermal diffusion length. We recall its definition and rearrange that definition:

$$L_{th}^2 = \frac{D_{th}}{\Sigma_{a,th}} = \frac{D_{th}^{\text{mod}}}{\Sigma_{a,th}^{\text{mod}}} \frac{D_{th}}{D_{th}^{\text{mod}}} \frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}} = \textcolor{red}{L_{th,\text{mod}}^2} \frac{D_{th}}{D_{th}^{\text{mod}}} \frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}} = \textcolor{red}{L_{th,\text{mod}}^2} \frac{\Sigma_{tr,th}^{\text{mod}}}{\Sigma_{tr,th}} \frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}} \quad (12.11)$$

This shows that L_{th}^2 for the mixture is L_{th}^2 for the moderator altered by two multiplicative correction factors. An example will help clarify the validity of the approximations we are about to propose. Consider the example of a mixture of ^{235}U and H_2O , with 1 ^{235}U atom per 100 H_2O molecules. Begin with the absorption ratio that appears in the previous equation:

$$\begin{aligned}
 \frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}} &= \frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}^{\text{mod}} + \Sigma_{a,th}^{\text{fuel}}} = \frac{[N\sigma_a(v_0)g_a]^{\text{mod}} \sqrt{\frac{T_0}{T_n} \frac{\sqrt{\pi}}{2}}}{[N\sigma_a(v_0)g_a]^{\text{mod}} \sqrt{\frac{T_0}{T_n} \frac{\sqrt{\pi}}{2}} + [N\sigma_a(v_0)g_a]^{\text{fuel}} \sqrt{\frac{T_0}{T_n} \frac{\sqrt{\pi}}{2}}} \\
 &= \frac{[N\sigma_a(v_0)g_a]^{\text{mod}}}{[N\sigma_a(v_0)g_a]^{\text{mod}} + [N\sigma_a(v_0)g_a]^{\text{fuel}}} \\
 &= \frac{N_{\text{H}_2\text{O}} [0.66 \text{ barns}]}{N_{\text{H}_2\text{O}} [0.66 \text{ barns}] + \frac{N_{\text{H}_2\text{O}}}{100} [680 \text{ barns}] g_a^{235}(T_n)} \\
 &= \frac{0.66}{0.66 + 6.80g_a^{235}(T_n)} \approx \frac{1}{11}. \tag{12.12}
 \end{aligned}$$

(In this equation we have noted that the thermal cross sections, $\sigma_{a,th}$, are related to the 2200-m/s cross sections by a factor of $g_a(T_n)\sqrt{T_0/T_n}\sqrt{\pi}/2$, where $T_0 \equiv 293.15$ K. This follows from assuming that the neutrons are in a Maxwellian at T_n . Note that the $\sqrt{T_0/T_n}\sqrt{\pi}/2$ factor divides out of every term in a ratio of absorption cross sections, so it does not play a role here.)

Obviously we cannot ignore a factor that is so far from 1.0! We conclude that **we must correct L_{th} for absorption by fuel atoms!** (Note that even if the $\text{U}/\text{H}_2\text{O}$ ratio had been 1/1000, this correction factor would still be around 1/2.)

Now consider the transport-cross-section ratio. To get an estimate of the magnitude of this ratio, we shall look at the ratio of total cross sections, which is almost the same as the ratio of the transport cross sections. We found above that the temperature and $\sqrt{\pi}/2$ factors divide out of our cross-section ratios, so here we will jump straight to 2200-m/s values. We also recognize that g values tend to be around 1 or 2 and that ignoring them does not affect the rough conclusion we are trying to reach. We note that the 2200-m/s scattering cross section in U-235 is around 15 b, whereas it is around 100 b for H₂O. Thus:

$$\begin{aligned} \frac{\Sigma_{tr,th}^{\text{mod}}}{\Sigma_{tr,th}} &= \frac{\Sigma_{tr,th}^{\text{mod}}}{\Sigma_{tr,th}^{\text{mod}} + \Sigma_{tr,th}^{\text{fuel}}} \approx \frac{\Sigma_{t,th}^{\text{mod}}}{\Sigma_{t,th}^{\text{mod}} + \Sigma_{t,th}^{\text{fuel}}} \approx \frac{[N\sigma_t(v_0)]^{\text{mod}}}{[N\sigma_t(v_0)]^{\text{mod}} + [N\sigma_t(v_0)]^{\text{fuel}}} \\ &= \frac{N_{H_2O}(103 + 0.66) [\text{b}]}{N_{H_2O}(103 + 0.66) [\text{b}] + \frac{N_{H_2O}}{100}(15 + 680) [\text{b}]} \\ &\approx \frac{104}{104 + 6.95} \approx 0.93. \end{aligned} \quad (12.13)$$

This is close enough to unity that we can ignore it, especially since for a large reactor it is going to be multiplied by the small number B_g^2 and added to unity (see expression for P_{TNL}). So we can use D_{th} for the moderator as an approximation for D_{th} for the moderator-fuel mixture.

We conclude that for reasonably dilute mixtures of fuel and moderator, we can use

$$L_{th}^2 \approx L_{th,\text{mod}}^2 \left[\frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}^{\text{mod}} + \Sigma_{a,th}^{\text{fuel}}} \right] \quad (12.14)$$

Note that the mixture's diffusion length is always smaller than the moderator's if we have a dilute mixture.

We need a table of diffusion lengths for common moderators. Our Table 12.4 takes its numbers from Lamarsh, who takes them from an AEC report (Reactor Physics Constants, U.S. Atomic Energy Commission Report ANL, 2nd ed., 1963, Section 3.3. The diffusion properties of D₂O are sensitive to the amount of H₂O present. Graphite often contains impurities that affect its diffusion properties; the values given here are for highly-purified

Table 12.4: Thermal Neutron Diffusion Parameters of Common Moderators at 20 °C.

Moderator	Density [$\frac{\text{g}}{\text{cm}^3}$]	D_{th} [cm]	$\Sigma_{\text{a},\text{th}}$ [cm $^{-1}$]	L_{th}^2 [cm 2]	L_{th} [cm]
H ₂ O	1.00	0.16	0.0197	8.1	2.85
D ₂ O	1.10	0.87	2.9×10^{-5}	30,000	170
Be	1.85	0.50	1.04×10^{-3}	480	22
BeO	2.96	0.47	6.0×10^{-4}	790	28
Graphite (C)	1.70	0.84	2.4×10^{-4}	3,500	59
Graphite (C)	2.20	0.65	3.1×10^{-4}	2,090	46

graphite.) The exception is the final row of the table, which simply translates the AEC report values to higher-density graphite.

As the parenthetical remark above indicates, for materials with minuscule absorption cross sections (like graphite or heavy water), the addition of even a small amount of absorber has a dramatic effect on the diffusion length. You can see this from the first part of this subsection, in which we studied how the presence of fuel alters the L_{th}^2 for the moderator-fuel mixture. The correction we derived was for the presence of fuel, but it is equally valid to correct for the presence of an impurity. In our problems, impurities won't make a difference unless the impurity's macroscopic absorption cross section is comparable to the fuel's. This is exceedingly unlikely!

12.2.6 Non-1/v Factors (g_a and g_f)

Appendix 1 of this set of notes provides 2200-m/s cross sections for many nuclides of interest. Any chart of the nuclides also provides these. We also need non-1/v factors as functions of neutron temperature for heavy nuclides of interest. Some of these are provided in Table 12.5.

12.3 Spectrum in a Homogeneous Thermal Reactor

Continuing the theme of putting pieces together, let's assemble our estimate of the neutron flux spectrum (i.e., the neutron distribution in energy) that we would expect to see in a large thermal reactor.

Table 12.5: Non-1/v factors

$T_n, ^\circ\text{C}$	Xe ¹³⁵	Sm ¹⁴⁹	U ²³³		U ²³⁵		U ²³⁸	Pu ²³⁹		Pu ²⁴⁰	
	g_a	g_a	g_a	g_f	g_a	g_f	g_a	g_a	g_f	g_a	g_f
20	1.158	1.617	0.998	1.000	0.978	0.976	1.002	1.072	1.049	1.028	1.025
100	1.210	1.887	0.997	1.001	0.961	0.958	1.003	1.161	1.115	1.028	1.025
200	1.236	2.090	0.997	1.003	0.946	0.941	1.005	1.339	1.253	1.028	1.025
400	1.186	2.185	1.001	1.007	0.929	0.921	1.009	1.891	1.690	1.028	1.025
600	1.091	2.085	1.007	1.013	0.923	0.911	1.012	2.532	2.204	1.028	1.025
800	0.989	1.925	1.015	1.020	0.918	0.904	1.016	3.101	2.660	1.028	1.025
1000	0.886	1.757	1.023	1.028	0.912	0.896	1.020	3.535	3.008	1.028	1.025

We know that in the high-energy range (**a few hundred keV to 10 MeV**), the dominant phenomenon is that neutrons are born from fission and then scatter down to lower energies. The result is that in this energy range, the scalar-flux spectrum has a “hump” from the **fission spectrum**. (In Chapter 10, we saw that the $\chi(E)$ term on the right-hand side of the slowing-down equation dominates in the highest part of this energy range, while the in-scattering term dominates in the lower part, and there is also a $1/\Sigma_t(E)$ factor that alters the shape somewhat.)

In the slowing-down range (around **an eV to a few hundred keV**), the $\chi(E)$ term of Chapter 10’s slowing-down equation is negligible, and the other term is very nearly proportional to $1/E$, with sharp drops at resonances (where the Σ_t term in the denominator has sharp increases). These drops are in such narrow energy ranges that they are narrower than the width of the plot line in a plot that covers a wide energy range, except for the few resonances at the lowest-energy part of the range.

In the thermal range (**below approximately 1 eV**), we learned in Chapter 11 that the scalar flux is essentially Maxwellian, although at an effective “neutron temperature” that will usually be slightly higher than the temperature of the reactor material.

All of this comes together in Figure 12.2. This figure comes from a calculation of a large homogeneous reactor composed of UO_2 mixed with H_2O , in proportions typical of a large commercial pressurized-water reactor. The uranium is enriched to 2.4% ^{235}U . The mixture was at room temperature, close to 300 Kelvin. Dr. Andrew Till performed the detailed calculation (while he was at TAMU as a Ph.D. candidate) using the SCALE code system, developed and maintained by Oak Ridge National Lab, in fall 2014. The calculation was at high enough fidelity that it can be considered an essentially exact solution of the slowing-down equation that we have been studying, as in Chapter 10.

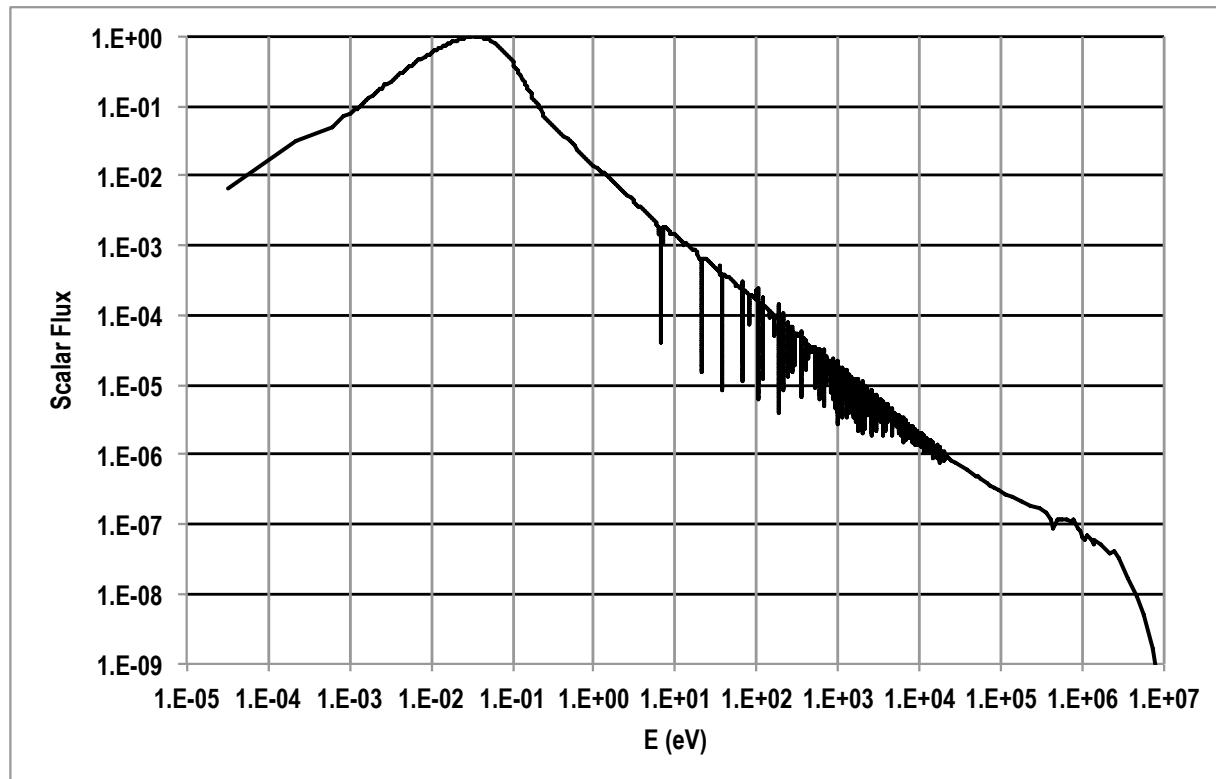


Figure 12.2: Detailed calculation of flux spectrum in a large UO_2 -fueled H_2O -moderated homogeneous reactor operating near room temperature. Calculated by A. Till using the SCALE code system.

To get a clearer picture of the effect of resonances on the flux shape, we zoom in on the energy range from 10 to 1000 eV in Fig. 12.3. Note that resonances get closer and closer together, at least on the log-scale plot, as energy increases, and they also get relatively narrower. If you look closely you will see that the flux at 100 eV is slightly lower than $10 \times$ the flux at 1000 eV. It would be exactly $10 \times$ larger if there were no leakage or absorption, because it would be proportional to $1/E$. But leakage and absorption do have an effect, causing the flux to be slightly lower than a $1/E$ proportionality would dictate.

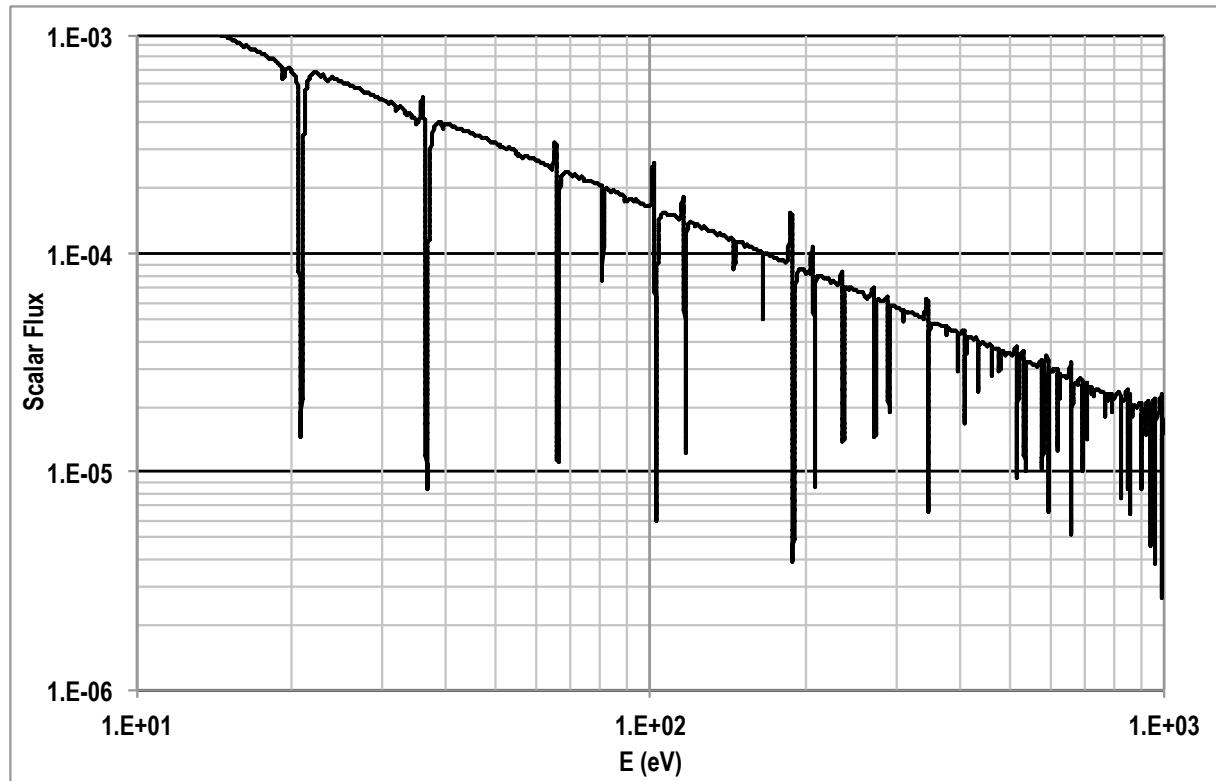


Figure 12.3: Detailed calculation of flux spectrum in a large UO₂-fueled H₂O-moderated homogeneous reactor operating near room temperature, zooming in on part of resonance region. Calculated by A. Till using the SCALE code system.

To illustrate how close the thermal flux is to a perturbed Maxwellian, we zoom in on the thermal energy range, below 1 eV, in Fig. 12.4. The calculated scalar flux looks much like the Maxwellian shifted to the right—to higher temperature, illustrating “absorption heating”—except that it blends into a $1/E$ shape at the higher end of the thermal energy range.

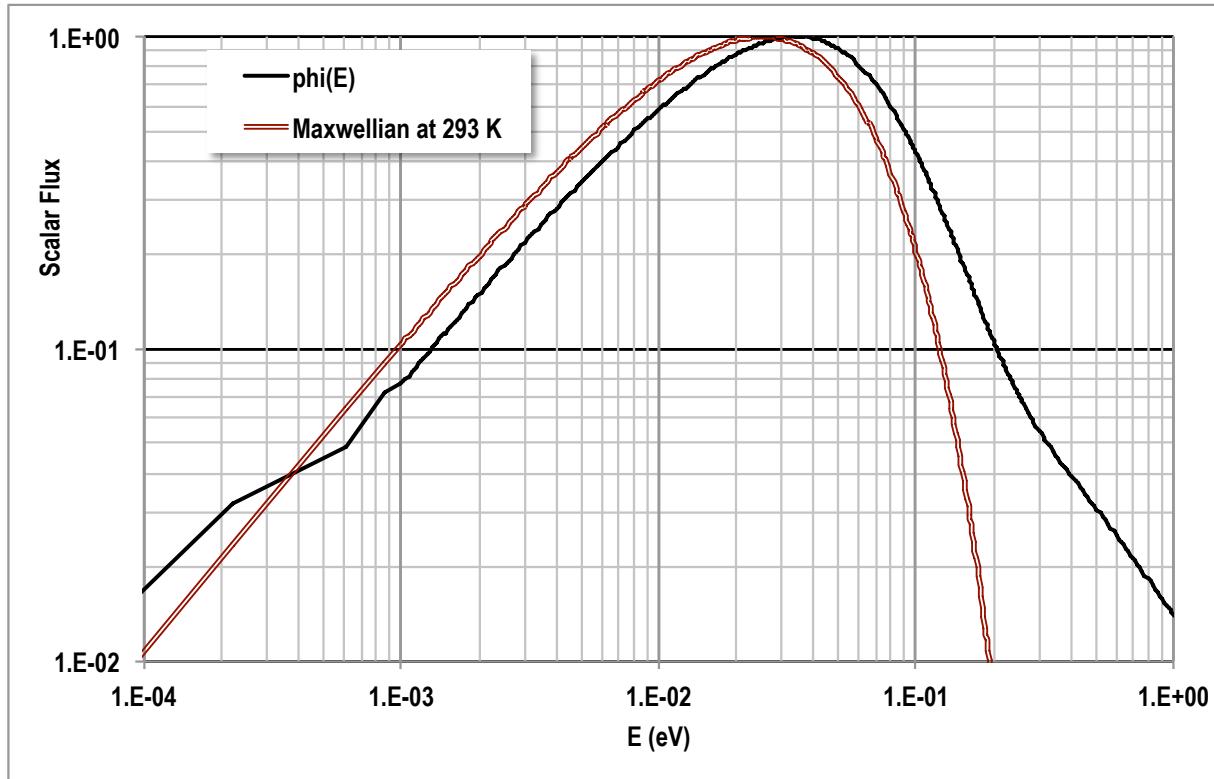


Figure 12.4: Detailed calculation of flux spectrum in a large UO_2 -fueled H_2O -moderated homogeneous reactor operating near room temperature, zooming in on thermal region. The (red) double line is a Maxwellian at room temperature; the solid line is from the detailed calculation. Calculated by A. Till using the SCALE code system..

12.3.1 Spectrum in a Heterogeneous Thermal Reactor

In the preceding sections we have figured out how neutrons should distribute themselves in energy inside a large homogeneous “thermal” reactor (e.g., a reactor designed to slow down most of the neutrons before they cause fission). Here we test our understanding by examining detailed solutions of the neutron transport equation in a simple problem: An infinite square lattice in which each square in the x - y plane contains one infinitely tall circular UO_2 fuel rod (or “pin”) surrounded by water. This is not a homogeneous reactor, so the solution will be different at different positions.

Figure 12.5 shows a high-fidelity calculation of the scalar flux at four different positions in the unit cell, usually called a “pin cell.” The locations are: in the water between fuel pins, inside the fuel but near the fuel/moderator interface, near the center of the fuel pin, and between the center and outside of the fuel pin. The temperature of each material was taken to be ≈ 300 K in this example.

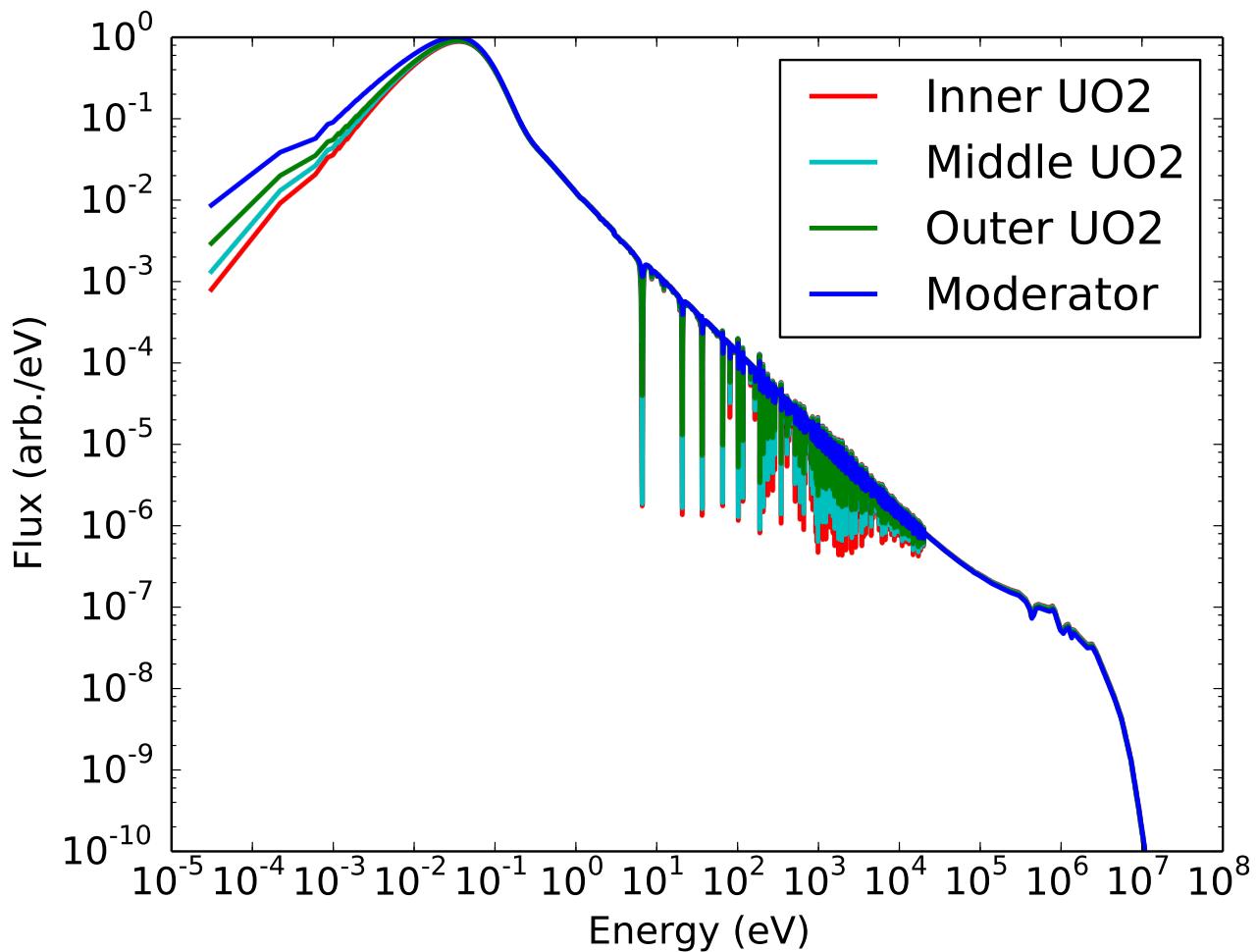


Figure 12.5: Energy-dependent scalar flux at various positions in an infinite square lattice of circular UO_2 pins in H_2O . Inner, Middle, and Outer UO_2 refer to positions in near the center of the fuel pin, about halfway between the center and surface, and inside the pin near the pin/water interface. “Moderator” is in the water between fuel pins. All materials were at room temperature (near 300 K). (Plot courtesy of Andrew Till, who generated it using the CENTRM code in the SCALE code system.)

There are many features worth noting in the spectrum plots of Fig. 12.5. The **low-energy** (“thermal”) spectrum is

roughly Maxwellian

with a peak between 0.01 and 0.1 eV. In the moderator it is not far from $kT_{material} \approx 0.025$ eV. Detailed inspection reveals that the peak becomes lower and moves to the right as we move from the moderator to the outer, middle, and inner portions of a fuel pin. This is an example of

“absorption hardening” or “**absorption heating**.”

(Absorption of thermal neutrons is strong in the fuel and weak in the water.) This is as predicted in our discussion of the scalar flux in the thermal energy range.

Moving to slightly higher energies we see a **steep drop** in the spectrum just to the right of the peak, as is characteristic of a Maxwellian. However, whereas a pure Maxwellian drops by a factor of ≈ 1000 when energy goes from $E = E_{peak} = kT$ to $E = 10 \times E_{peak}$, the figure shows a less dramatic drop, because the exponential “tail” of the Maxwellian blends into the $\approx 1/E$ shape in the slowing-down range. This was predicted in our discussion of the scalar flux in the thermal range.

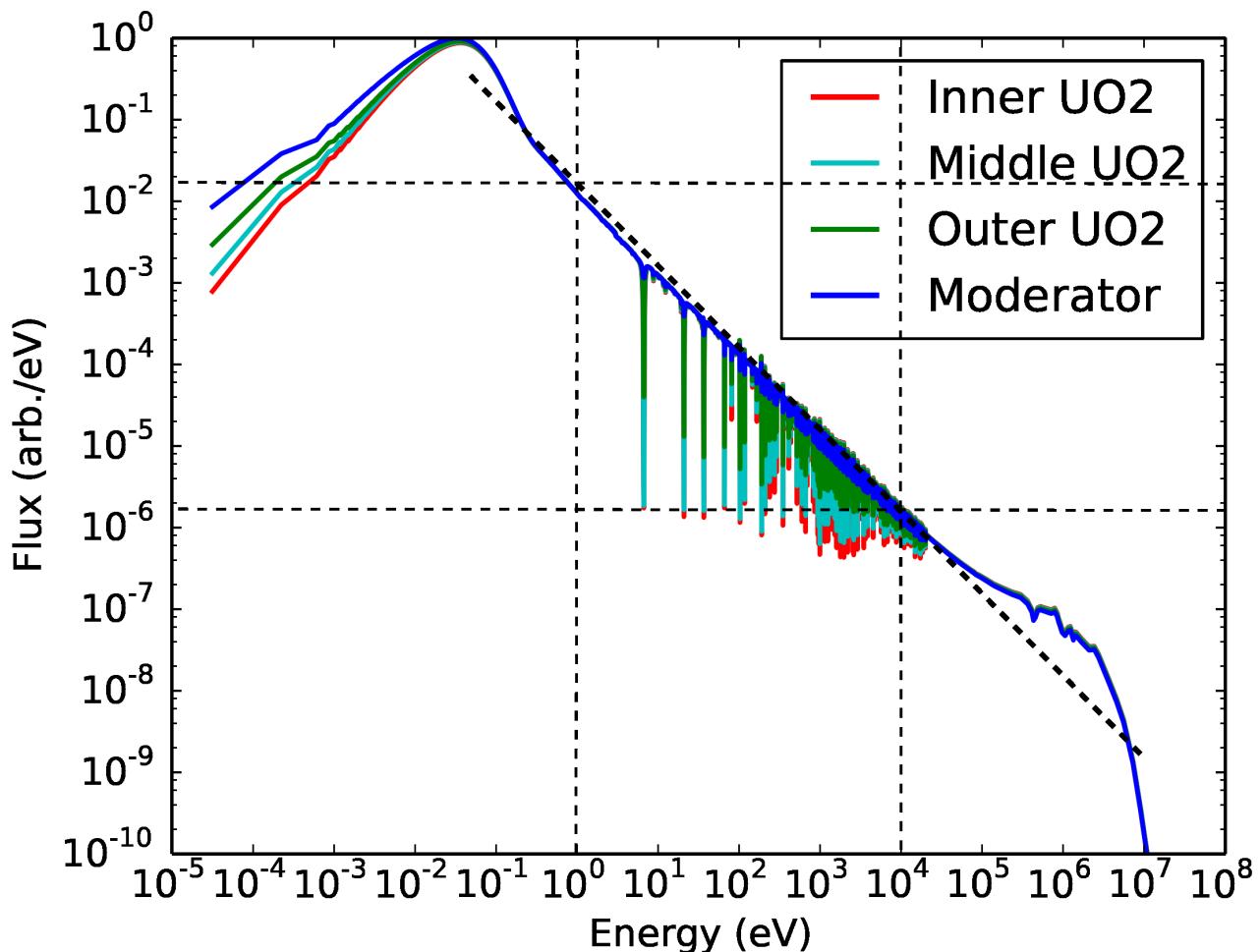


Figure 12.6: Energy-dependent scalar flux at various positions in an infinite square lattice of circular UO₂ pins in H₂O. Inner, Middle, and Outer UO₂ refer to positions in near the center of the fuel pin, about halfway between the center and surface, and inside the pin near the pin/water interface. “Moderator” is in the water between fuel pins. This figure has some dashed lines superimposed to help with quantitative interpretation. The vertical and horizontal lines are four orders of magnitude apart. The function $constant/E$ passes diagonally through two of the intersections, as shown by the diagonal dashed line.

We turn next to the **slowing-down energy range**, which in this room-temperature reactor extends from below 1 eV to $\approx 10^5$ eV (100 keV). See figure 12.6, which is the same as the previous figure except that some dashed lines have been added to help us quantify the behavior of the scalar flux. Observe that in this range:

1. The overall shape of the scalar flux is

very nearly = **constant/ E** ,

as predicted by the slowing-down analysis in a preceding section.

2. The solution at each spatial point shows

sharp down-spikes

in the scalar flux at energies for which either ^{238}U or ^{235}U has a “resolved” resonance (which means a resonance that can be well characterized because it is well enough known and its extent is not too strongly overlapped by a neighboring resonance). The resolved resonance range in this calculation ends at approximately 20 keV (2×10^4 eV); this is why the sharp down-spikes in the flux plots do not appear above that energy. (There are plenty of resonances above 20 keV, but they are

“unresolved”

and are not individually modeled in this calculation.)

3. The down-spikes are deepest for the inner and middle portions of the fuel pins (red and light-blue plot lines, with red tips visible below some light-blue tips). They are not as deep at the outer portion of the fuel pins, near the water (green curve, plotted on top of the light-blue). They are not deep at all (but still visible) in the scalar flux in the water, between fuel pins (dark-blue curve, plotted on top of the green), but they are still easily visible even on this log-scale plot.
4. Close inspection shows that the scalar flux at all spatial points (in both fuel and water) is “on” the dashed constant/ E line at $E \approx 20$ keV, but below this dashed line at $E = 1$ eV. If there were no losses, the flux would remain on the constant/ E line. The ratio of the actual flux to the constant/ E line at $E = 1\text{eV}$ is $p_{surv}(20 \text{ keV}, 1 \text{ eV})$, which is the probability that a neutron survives the downscattering journey from 20 keV to 1 eV. See Eq. (10.82).
5. The plot comes from an infinite-medium calculation. If there were leakage, then the scalar flux would fall even farther below the constant/ E curve, by an amount determined by the probability of neutrons leaking during the slowing-down process. In a large reactor this probability is low, and the effect of leakage is virtually undetectable in a plot of the energy-dependent scalar flux.
6. Above the resolved-resonance range, the scalar flux is almost the same in the water as in each part of the fuel: the red, light-blue, green, and dark-blue curves very nearly coincide. This is because the fuel and water segments are not very many mean-free paths thick except for neutron energies at which the fuel cross section is high, which means neutrons whose energies are either low or at uranium resonances.³³²

Finally we turn to the **high-energy range**, $E \gtrsim 10^5$ eV. Here we see the effects of the fission source, with energy distribution $\chi(E)$, superimposed on the constant/ E line. We also see the influence of the resonances in oxygen cross section in the range from ≈ 200 keV to a few MeV, with down- and up-spikes in the scalar flux corresponding to up- and down-spikes in the oxygen cross section. (See oxygen cross-section plot in Appendix A.) This is just as predicted in the discussion of the high-energy range.

12.4 Example: k_∞ of Uranium + D₂O

Consider a mixture with Q uranium atoms per D₂O molecule. Assume a neutron temperature of 100 °C. Suppose the U is entirely ²³⁸U and ²³⁵U, with W ²³⁵U atoms per total uranium atom. **Find k_∞ of this mixture as a function of Q and W .** Plot this for a few interesting values of W .

We begin with our k equation, Eq. (12.3), with the non-leakage probabilities set to 1. This is:

$$k = p \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_{th}^i \sigma_f^i(v_0) g_f^i(T_n)}{\sum_{i=1}^{\# \text{ of nuclides}} N_i \sigma_a^i(v_0) g_a^i(T_n)} + (1 - p) \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F^i I_{r,F,i}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}} \quad (12.15)$$

We begin with our expression for p and start filling in data:

$$\begin{aligned} p &\approx \exp \left\{ -\frac{1}{[\bar{\xi} \Sigma_p]_{\text{mod}}} \sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i} \right\} \\ &\xrightarrow{\text{dilute}} \exp \left\{ -\frac{1}{\bar{\xi} \sigma_p^{D_2O} N_{D_2O}} [N_{D_2O} I_{r,A,D_2O}^\infty + N_U (W I_{r,A,235}^\infty + [1 - W] I_{r,A,238}^\infty)] \right\} \\ &= \exp \left\{ -\frac{1}{\bar{\xi} \sigma_p^{D_2O}} [I_{r,A,D_2O}^\infty + Q (W I_{r,A,235}^\infty + [1 - W] I_{r,A,238}^\infty)] \right\} \\ &= \exp \left\{ -\frac{0.001 + Q (W 411 + [1 - W] 280)}{0.509 \times 10.5} \right\} \end{aligned} \quad (12.16)$$

We plot this in Fig. 12.7 for natural uranium ($W = 0.007$) and also for a typical value of enrichment ($W = 0.035$) as a function of Q . As you can see, the enrichment doesn't make much difference in p , because for low enrichments p is dominated by absorption in ²³⁸U.

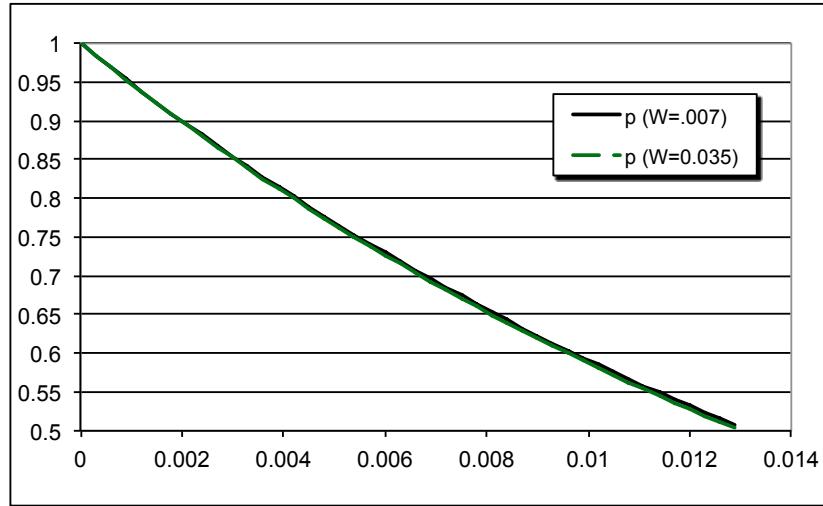


Figure 12.7: Resonance-escape probability as a function of $Q = U/D_2O$ ratio for two different values of $W = {}^{235}U/(\text{total U})$ ratio \approx enrichment level.

Next we recognize that

$$\nu_{th}^{235} = \nu_F^{235} = 2.43 \quad \text{and} \quad \nu_{th}^{238} = \nu_F^{238} = 2.30 . \quad (12.17)$$

Now for the resonance-integral ratio that is our approximation for $u_F\eta_F$:

$$\begin{aligned} & \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F^i I_{r,F,i}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}} \xrightarrow{\text{dilute}} \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F^i I_{r,F,i}^\infty}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}^\infty} \\ &= \frac{N_U [W \times 2.43 \times 278[b] + (1 - W) \times 2.30 \times 2[b]]}{N_U [W \times 411[b] + (1 - W) \times 280[b]]} \\ &= \frac{W(2.43)(278) + (1 - W)(2.30)(2)}{W411 + (1 - W)280} \end{aligned} \quad (12.18)$$

Note that we have ignored the resonance integral in D₂O. From our table of resonance integrals we see that the resonance integral in D₂O is ≈ 0.001 b. This is too small to make a significant contribution, so we don't worry about it. Be careful, though—sometimes in a dilute mixture the moderator's resonance integral may matter! This is often true with regular “light” water.

Now for the thermal ratio that is our approximation for $u_T \eta_T$:

$$\begin{aligned} & \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_{th}^i \sigma_f^i(v_0) g_f^i(T_n)}{\sum_{i=1}^{\# \text{ of nuclides}} N_i \sigma_a^i(v_0) g_a^i(T_n)} \\ &= \frac{N_U [2.43 \times W \times 580 \times 0.958 + \text{negligible}]}{N_{D_2O}(2 \times 0.0005 + 0.0002) + N_U [W(680)(0.961) + (1 - W)(2.72)(1.003)]} \\ &= \frac{W(2.43)(580)(0.958)}{\frac{1}{Q}0.0012 + W(680)(0.961) + (1 - W)(2.72)(1.003)} \end{aligned} \quad (12.19)$$

Let us find numbers for these ratios under the assumption of natural uranium ($W \approx 0.007$):

$$\begin{aligned} u_F \eta_F : & \frac{W(2.43)(278) + (1 - W)(2.30)(2)}{W411 + (1 - W)280} \xrightarrow{W=0.007} \approx 2.43 \times 0.0139 \\ u_T \eta_T : & \frac{W(2.43)(580)(0.958)}{\frac{1}{Q}0.0012 + W(680)(0.961) + (1 - W)(2.72)(1.003)} \xrightarrow{W=0.007} \approx 2.43 \times \frac{3.89}{7.28 + 0.0012/Q} \end{aligned}$$

With these, and with $W = 0.007$ plugged into our expression for p , we have:

$$k \xrightarrow{W=0.007} \approx 2.43 \left\{ e^{-(2.81/5.35)Q} \frac{3.89}{7.28 + 0.0012/Q} + (1 - e^{-(2.81/5.35)Q}) (0.0139) \right\} \quad (12.20)$$

This is plotted as a function of $Q = U/D_2O$ ratio in Fig. 12.8. Note that there is a wide range of U/D_2O ratios for which $k_\infty > 1$, which means it should be possible to create a bare homogeneous reactor from natural uranium and heavy water.

12.5 Example: k of Uranium + D₂O cylinder

Consider a finite cylindrical reactor composed of natural uranium mixed with D₂O. Again assume a neutron temperature of 100 °C, and assume a density of 1.1 g/cm³. Consider a bare cylinder of this material of radius 2 m and height 3 m, and assume the near-optimal ratio of 1 uranium atom for every 500 D₂O molecules ($Q = 0.002$). What is the multiplication factor?

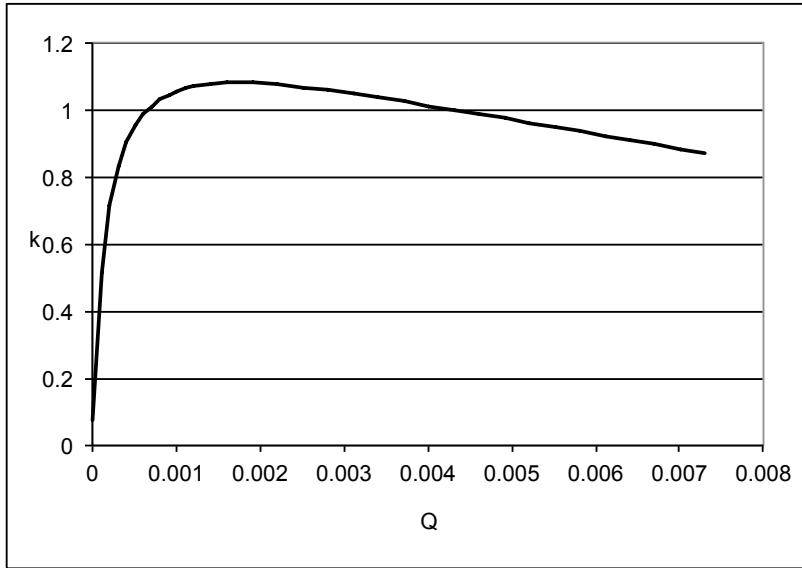


Figure 12.8: Infinite-medium multiplication factor as a function of U/D₂O ratio for natural uranium.

From the previous section we have $k_\infty = 1.0785$. Now we need the non-leakage probabilities. For each of them we will need the geometric buckling, so let's get it:

$$\begin{aligned} B_g^2 &= \left(\frac{2.405}{R+d} \right)^2 + \left(\frac{\pi}{H+2d} \right)^2 \approx \left(\frac{2.405}{R} \right)^2 + \left(\frac{\pi}{H} \right)^2 \\ &= \left(\frac{2.405}{200\text{cm}} \right)^2 + \left(\frac{\pi}{300\text{ cm}} \right)^2 \approx 0.00025 \text{ cm}^{-2}. \end{aligned} \quad (12.21)$$

For the fast non-leakage probability we will need the age to thermal. The age to thermal in D₂O is approximately

$$\tau_{th}^{D_2O} \approx 131 \text{ cm}^2. \quad (12.22)$$

according to our table above. We will also need the square of the thermal diffusion length. Our formula for this was derived above:

$$L_{th}^2 \approx L_{th,\text{mod}}^2 \left[\frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}^{\text{mod}} + \Sigma_{a,th}^{\text{fuel}}} \right] \quad (12.23)$$

According to our table,

$$L_{th,D_2O}^2 = 3.0 \times 10^4 \text{ cm}^2. \quad (\text{That's pretty big!}) \quad (12.24)$$

The other term is:

$$\begin{aligned} \left[\frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}^{\text{mod}} + \Sigma_{a,th}^{\text{fuel}}} \right] &= \frac{\frac{1}{Q} 0.0012}{\frac{1}{Q} 0.0012 + W(680)(0.961) + (1 - W)(2.72)(1.003)} \\ &= \frac{(500)(0.0012)}{(500)(0.0012) + (0.007)(680)(0.961) + (0.993)(2.72)(1.003)} \\ &\approx 0.076 . \end{aligned} \quad (12.25)$$

Putting everything together, we have:

$$\begin{aligned} P_{\text{FNL}} &\approx e^{-B_g^2 \tau_{th}} \approx e^{-0.00025 \times 131} \approx 0.968 \\ P_{\text{TNL}} &\approx \frac{1}{1 + L_{th}^2 B_g^2} \approx \frac{1}{1 + (3 \times 10^4 \text{ cm}^2)(0.076)(0.00025 \text{ cm}^{-2})} \approx 0.637 \end{aligned} \quad (12.26)$$

Note that the huge diffusion length in D₂O (which is the result of having almost no absorption in D₂O) causes lots of thermal neutrons to leak out! This makes the reactor quite subcritical, as we see when we insert our numbers:

$$\begin{aligned} k &\approx 2.43 \left\{ P_{\text{FNL}} P_{\text{TNL}} e^{-(2.81/5.35)Q} \frac{3.89}{7.28 + 0.0012/Q} + P_{\text{FNL}} (1 - e^{-(2.81/5.35)/500}) (0.0136) \right\} \\ &= 2.43 \left\{ (0.968)(0.637)e^{-\frac{2.81/5.35}{500}} \frac{3.89}{7.28 + 0.0012 \times 500} + (0.968) \left(1 - e^{-\frac{2.81/5.35}{500}}\right) (0.0136) \right\} \\ &\approx 0.67 \end{aligned} \quad (12.27)$$

We conclude that while it is possible to make a critical reactor out of a homogeneous mixture of natural uranium and heavy water, the reactor must be really large! This is why CANDU reactors (Canadian reactors fueled by natural uranium and moderated by heavy water) are much larger than light-water reactors. See Table 4.1 from the Lewis book for a comparison of volumes of reactors of different types, and note that the CANDU reactor volume is more than 10 times the volume of a PWR.

12.6 Example: Finding Critical Size

Suppose you have a mixture for which you have calculated the following numbers:

$$\begin{aligned}
 p &\approx \exp \left\{ -\frac{1}{[\bar{\xi} \Sigma_p]_{\text{mod}}} \sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}^\infty \right\} = 0.9 , \\
 u_F \eta_F &\approx \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_F I_{r,F,i}}{\sum_{i=1}^{\# \text{ of nuclides}} N_i I_{r,A,i}} = 0.85 , \\
 u_T \eta_T &\approx \frac{\sum_{i=1}^{\# \text{ of nuclides}} N_i \nu_{th}^i \sigma_f^i(v_0) g_f^i(T_n)}{\sum_{i=1}^{\# \text{ of nuclides}} N_i \sigma_a^i(v_0) g_a^i(T_n)} = 1.2 , \\
 \tau_{th} &\approx 27 \text{ cm}^2 , \\
 L_{th}^2 &\approx L_{th,\text{mod}}^2 \left[\frac{\Sigma_{a,th}^{\text{mod}}}{\Sigma_{a,th}^{\text{mod}} + \Sigma_{a,th}^{\text{fuel}}} \right] = 5 \text{ cm}^2 . \tag{12.28}
 \end{aligned}$$

Assignment: Find a reactor size and shape that would be critical if surrounded by vacuum.

Solution: The first step is to recognize that the only geometric parameter that enters into our approximate expression for k is

the geometric buckling, B_g^2 .

So the first part of the solution is to find a B_g^2 that satisfies the following equation:

$$\begin{aligned}
 k = 1 &= P_{\text{FNL}} p P_{\text{TNL}} u_T \eta_T + P_{\text{FNL}} (1-p) u_F \eta_F \\
 &\approx e^{-B_g^2 \tau_{th}} (0.9) \frac{1}{1 + L_{th}^2 B_g^2} (1.2) + e^{-B_g^2 \tau_{th}} (1-0.9)(0.85) \\
 &= e^{-B_g^2 (27 \text{ cm}^2)} \left[\frac{1.08}{1 + B_g^2 (5 \text{ cm}^2)} + 0.085 \right] . \tag{12.29}
 \end{aligned}$$

We now encounter an algebraic difficulty: This equation for B_g^2 is

transcendental

This is because it involves B_g^2 in an exponent as well as in a polynomial. If one had a modern calculator or spreadsheet this would not be an issue—one could just tell the software to “solve” for B_g^2 .

Even with pencil and paper there are tricks that can make this tractable. We can see from Eq. (12.29) that in our problem, $k_\infty = 1.08 + 0.085 = 1.165$. This means the non-leakage probabilities must be within 5-15% of unity. This in turn means that neither $B_g^2 \tau_{th}$ nor $L_{th}^2 B_g^2$ can be greater than about 0.1 or maybe 0.15. If $B_g^2 \tau_{th}$ is this small, then the following approximation is reasonably accurate:

$$e^{-B_g^2 \tau_{th}} = 1 - B_g^2 \tau_{th} + \frac{1}{2} (B_g^2 \tau_{th})^2 - \frac{1}{3!} (B_g^2 \tau_{th})^3 \dots \approx 1 - B_g^2 \tau_{th} \quad (12.30)$$

If we use this approximation we obtain a quadratic equation for B_g^2 :

$$\begin{aligned} k = 1 \Rightarrow 1 &\approx (1 - B_g^2 \tau_{th}) \left[\frac{1.08}{1 + L_{th}^2 B_g^2} + 0.085 \right] \\ &\Rightarrow 1 + L_{th}^2 B_g^2 \approx (1 - B_g^2 \tau_{th}) [1.08 + (1 + L_{th}^2 B_g^2) 0.085] \\ &\Rightarrow (B_g^2)^2 [0.085 \tau_{th} L_{th}^2] + B_g^2 [L_{th}^2 + 1.165 \tau_{th} - 0.085 L_{th}^2] + [1 - 1.165] \approx 0 \\ &\Rightarrow (B_g^2)^2 [0.085 \tau_{th} L_{th}^2] + B_g^2 [0.915 L_{th}^2 + 1.165 \tau_{th}] + [-0.165] \approx 0 \end{aligned} \quad (12.31)$$

This can be solved using the quadratic formula:

$$B_g^2 = \frac{1}{(2)(0.085)\tau_{th}L_{th}^2} \left\{ -[0.915L_{th}^2 + 1.165\tau_{th}] \pm \sqrt{[0.915L_{th}^2 + 1.165\tau_{th}]^2 + (4)(0.085)\tau_{th}L_{th}^2(0.165)} \right\} \quad (12.32)$$

The + sign will give the correct (positive) solution.

Part VIII

Appendices

Appendix A

Selected Cross-Section Plots

In this appendix we provide plots of some cross sections, highlighting behavior that has a significant impact on neutron distributions in energy, for several nuclides of interest to nuclear engineers.

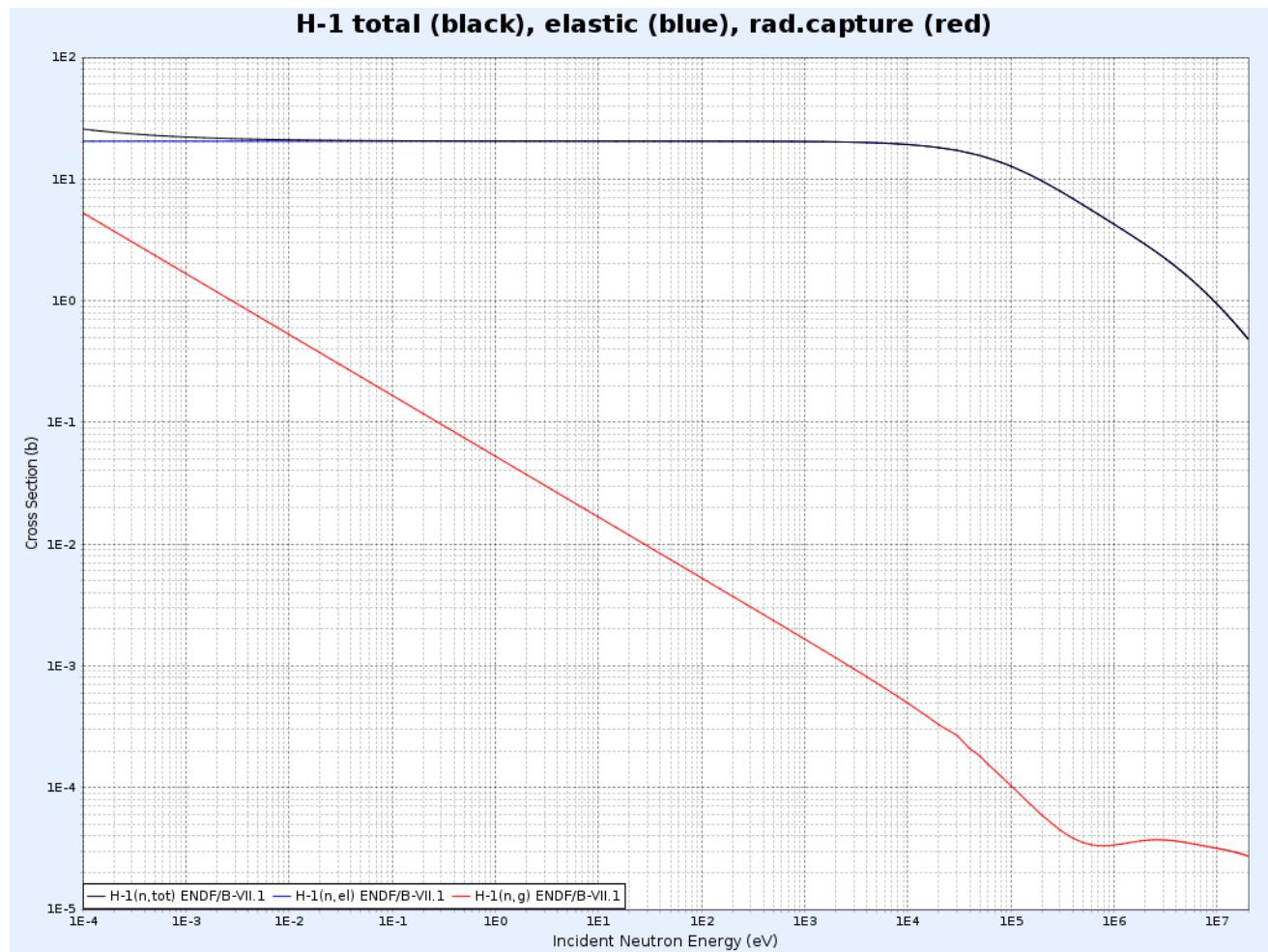


Figure A.1: ${}^1\text{H}$ (hydrogen) cross sections. Note several features: **A)** Except for $E \lesssim 0.1$ eV, the total and elastic-scattering cross sections are almost the same. That is, except for very slow neutrons, ${}^1\text{H}$ is essentially a pure elastic scatterer. **B)** The ${}^1\text{H}$ scattering cross section is remarkably large, ≈ 20 barns. Compare this to other low-Z nuclides, such as ${}^4\text{He}$, ${}^{12}\text{C}$, and ${}^{16}\text{O}$. **C)** ${}^1\text{H}$ is a “ $1/v$ ” absorber, which means its absorption cross section is proportional to $1/(\text{neutron speed})$, which means proportional to $1/\sqrt{E}$. (Verify that every decrease in energy of $\times 100$ results in an increase in absorption cross section of $\times 10$.) This is a common feature of light (low-A) nuclides. **D)** The elastic-scattering cross section is almost constant for $E < 10$ keV. **(E)** The cross section shown here is for ${}^1\text{H}$ in a “free gas,” that is, under the assumption that each hydrogen atom is not bound to any other atom. For low-energy neutrons the cross section is significantly different if the hydrogen is bound, as it almost always is in real materials (such as water or polyethylene).

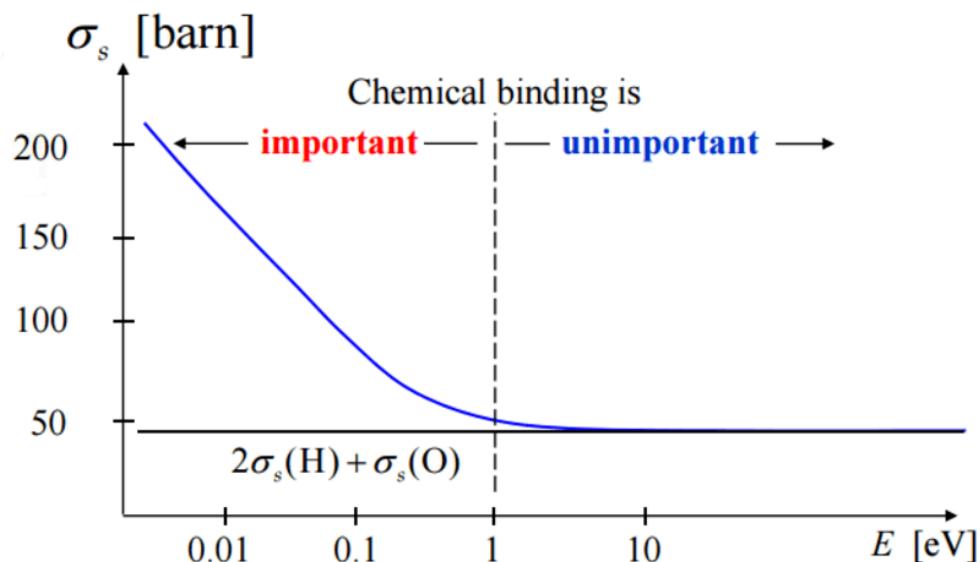


Figure A.2: H_2O (regular water) scattering cross section, illustrating the substantial effect of molecular bonds on the scattering of low-energy neutrons. The curve that looks flat [marked as $2\sigma_s(\text{H}) + \sigma_s(\text{O})$] is the sum of the cross sections for two **unbound** hydrogen nuclei plus one **unbound** oxygen nucleus. The curve that increases as neutron energy decreases is the sum of the cross sections for two hydrogen nuclei plus one oxygen nucleus when the atoms are bound in an H_2O molecule. The difference is mainly in the scattering cross section for the bound hydrogen, not the oxygen. That is, the scattering cross section for “bound” hydrogen is much higher than the cross section for “free” hydrogen, for low-energy neutrons. Note that the strength of the bond that holds one hydrogen atom to the oxygen atom in a water molecule is ≈ 4.4 eV, which is about where the bound and free cross sections become noticeably different.

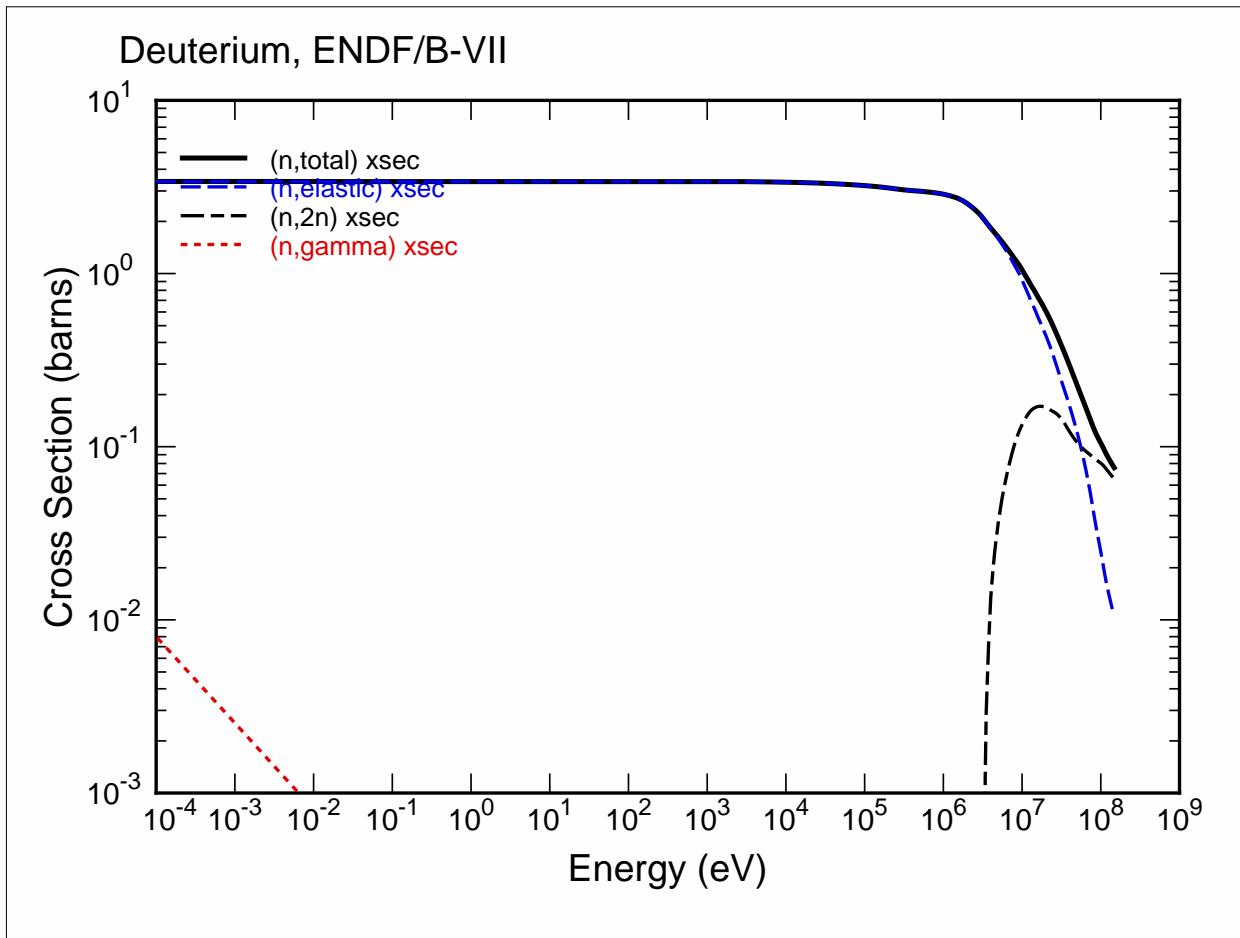


Figure A.3: ${}^2\text{H}$ (deuterium, or hydrogen-2) cross sections. Note several features: **A)** Except for $E > \approx 3$ MeV and $E < \approx 10^{-5}$ eV, the total and elastic-scattering cross sections are almost the same. That is, except for very slow and very fast neutrons, deuterium is essentially a pure elastic scatterer. **B)** ${}^2\text{H}$ is a “ $1/v$ ” absorber, which means its absorption cross section is proportional to $1/(neutron speed)$, which means proportional to $1/\sqrt{E}$. **C)** Fast neutrons can convert deuterium to ordinary hydrogen via the $(n,2n)$ reaction, although its cross section is not large.

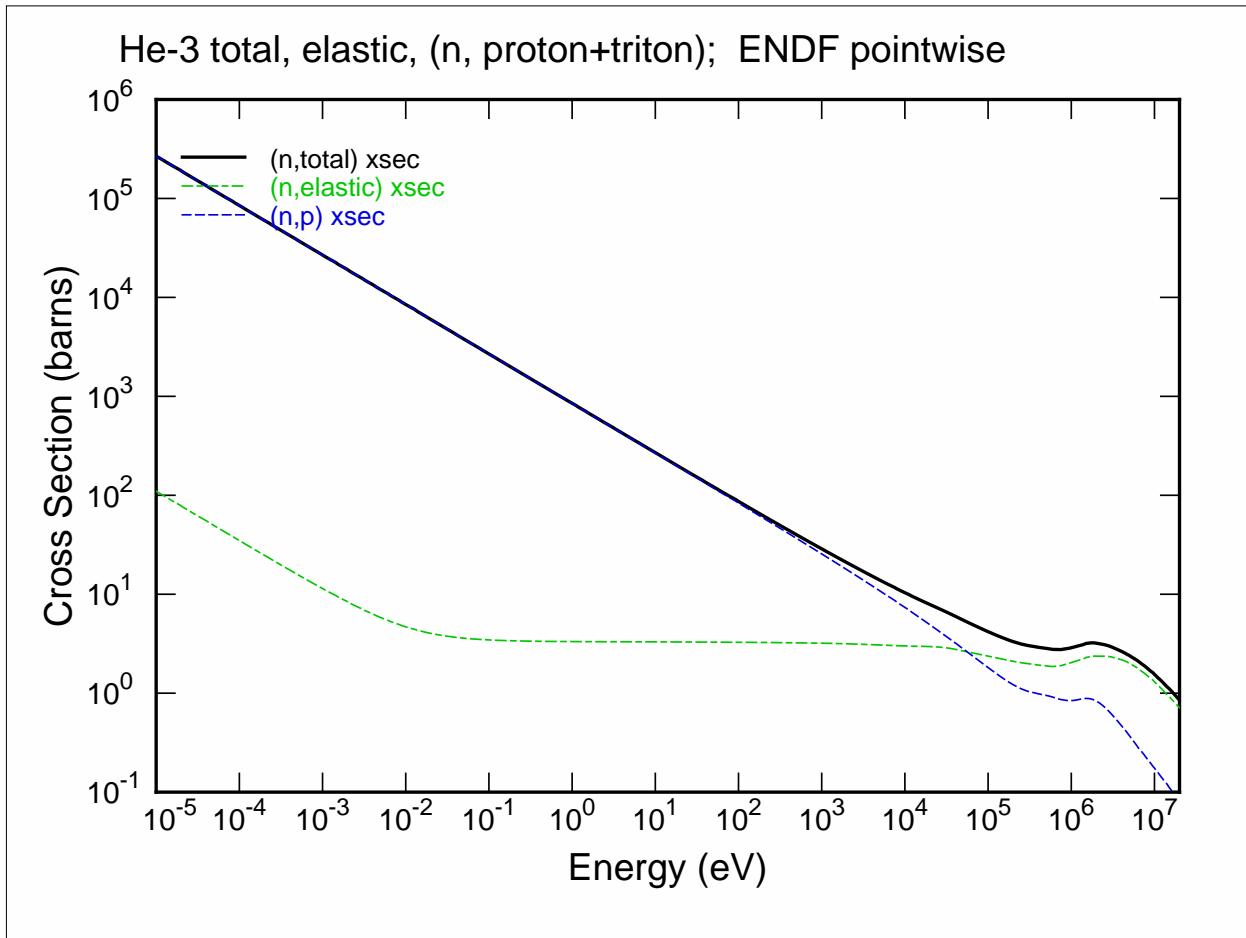


Figure A.4: ${}^3\text{He}$ (helium-3) cross sections. Note several features: **A)** Helium-3 is a strong neutron absorber, with $\sigma_a \gg \sigma_s$ for $E < \approx 10^5$ eV. Interestingly, the absorption reaction is not capture, but is (n,p), with reaction products of a proton and a triton. The combination of a high cross section and charged-particle products make ${}^3\text{He}$ an attractive material for neutron detectors. **B)** The scattering cross section is almost constant for all E below ≈ 50 keV. **C)** ${}^3\text{He}$ is a “ $1/v$ ” absorber, which means its absorption cross section is proportional to $1/(\text{neutron speed})$, which means proportional to $1/\sqrt{E}$.

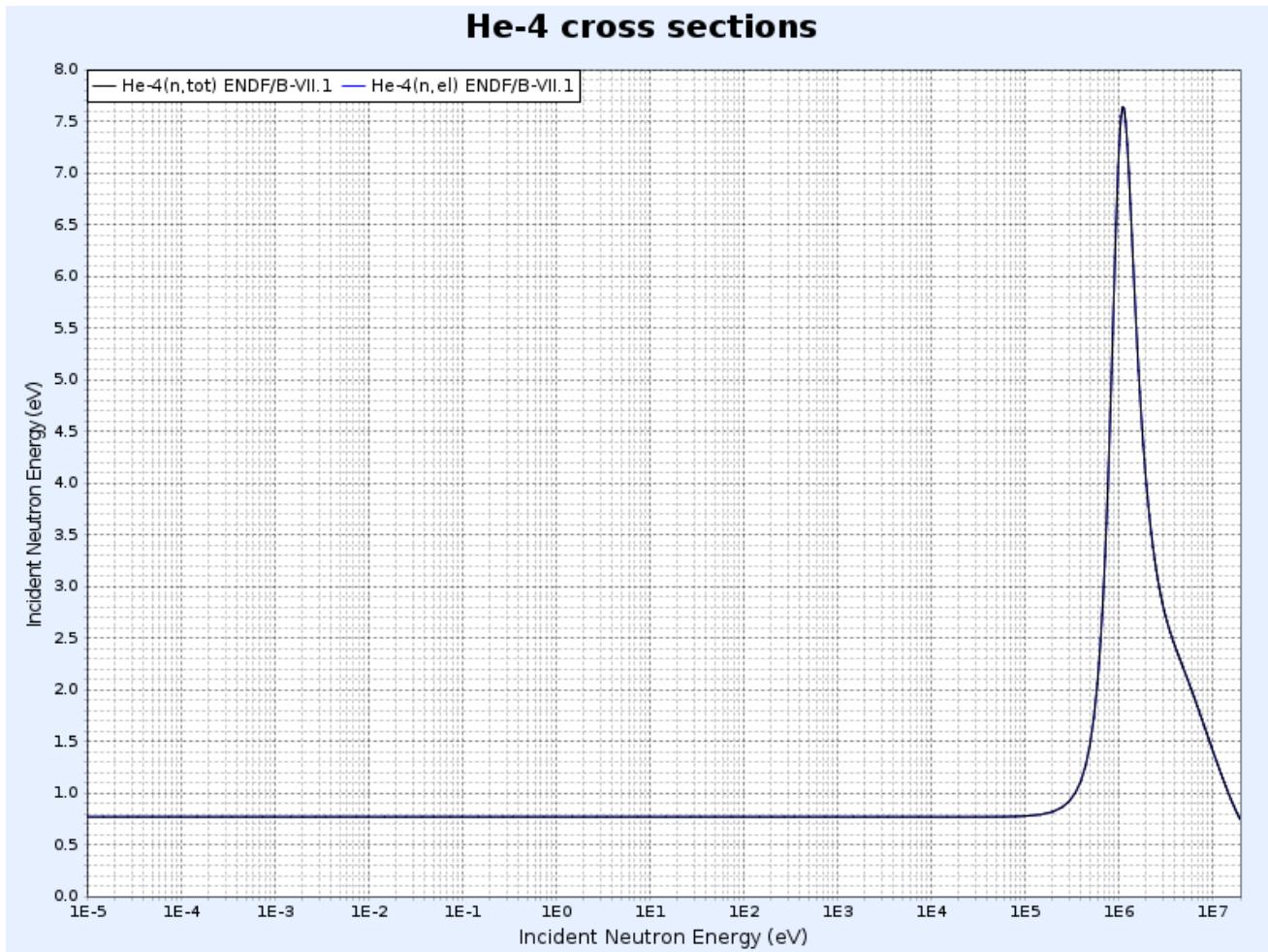


Figure A.5: ${}_2^4\text{He}$ (helium-4) cross sections. Note several features: **A)** For $E \in (10^{-5}, 2 \times 10^7 \text{ eV})$, the total and elastic-scattering cross sections are identical. That is, helium-4 prefers to simply scatter neutrons elastically. This makes it an excellent choice for a coolant if one wishes to build a gas-cooled reactor. **B)** The cross section is constant for $E < 10^5 \text{ eV}$. **C)** The cross section grows significantly (factor of 10!) for $E > 10^5 \text{ eV}$, because of a resonance just above 1 MeV.

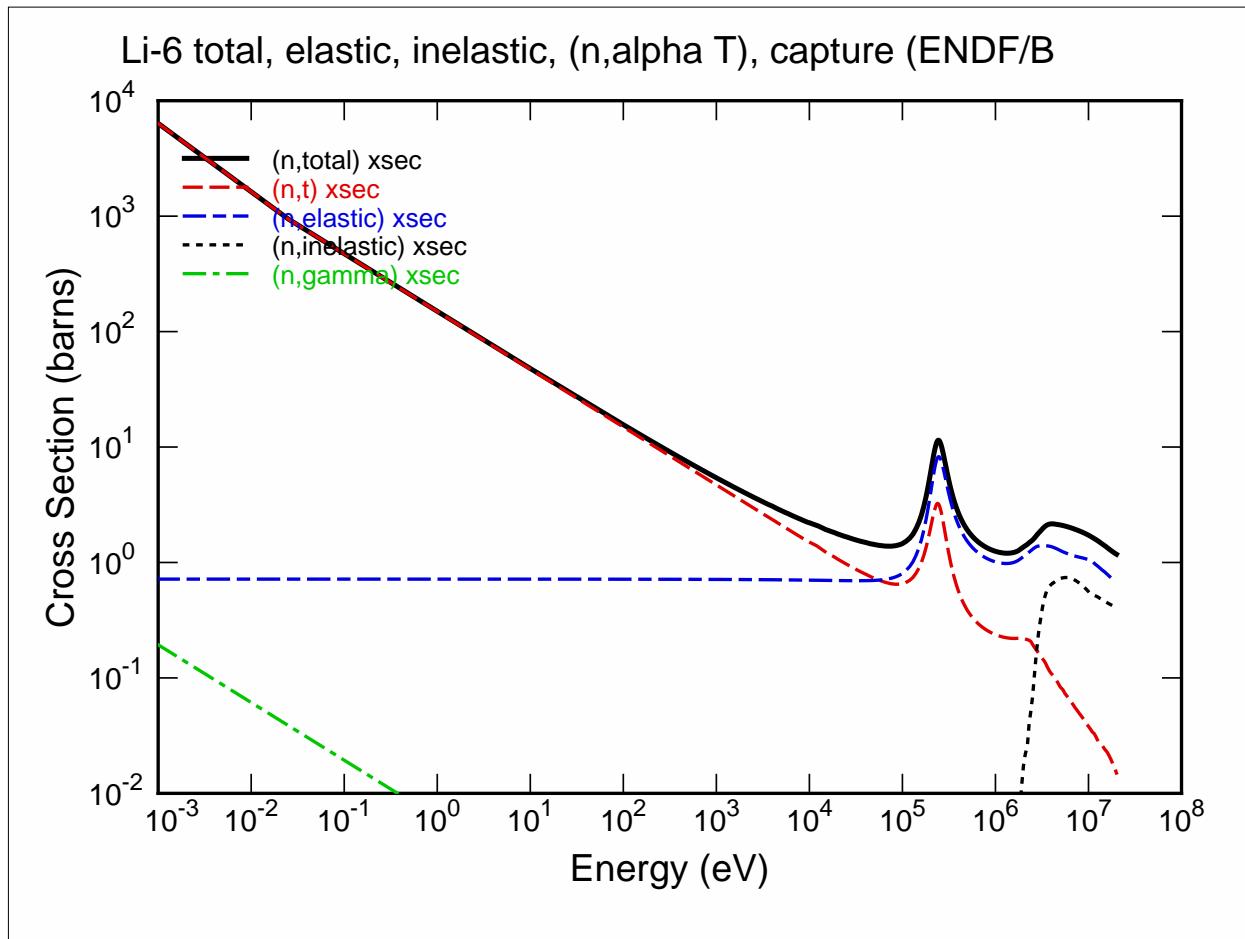


Figure A.6: ${}^6\text{Li}$ (lithium-6) cross sections. Note several features: **A)** Lithium-6 is a strong neutron absorber, with $\sigma_a \gg \sigma_s$ for $E < \approx 10^4$ eV. Interestingly, the dominant absorption reaction is not capture, but ${}_0^1\text{n} + {}^6\text{Li} \rightarrow {}^3_1\text{T} + {}^4_2\text{He}$, often called the “ $n, \alpha T$ ” reaction. This is the reaction that it most often used to create tritium, which is a key ingredient for D+T fusion reactions and is therefore needed for experiments that study fusion and also for some nuclear weapons. **B)** The scattering cross section is almost constant for all E below ≈ 100 keV. **C)** ${}^6\text{Li}$ is a “ $1/v$ ” absorber, which means its absorption cross section is proportional to $1/(\text{neutron speed})$, which means proportional to $1/\sqrt{E}$, for energies below the resonance that is just over 100 keV. This is also true individually for the capture reaction and the $(n, \alpha T)$ reaction, which together make up the absorption cross section in that energy range.

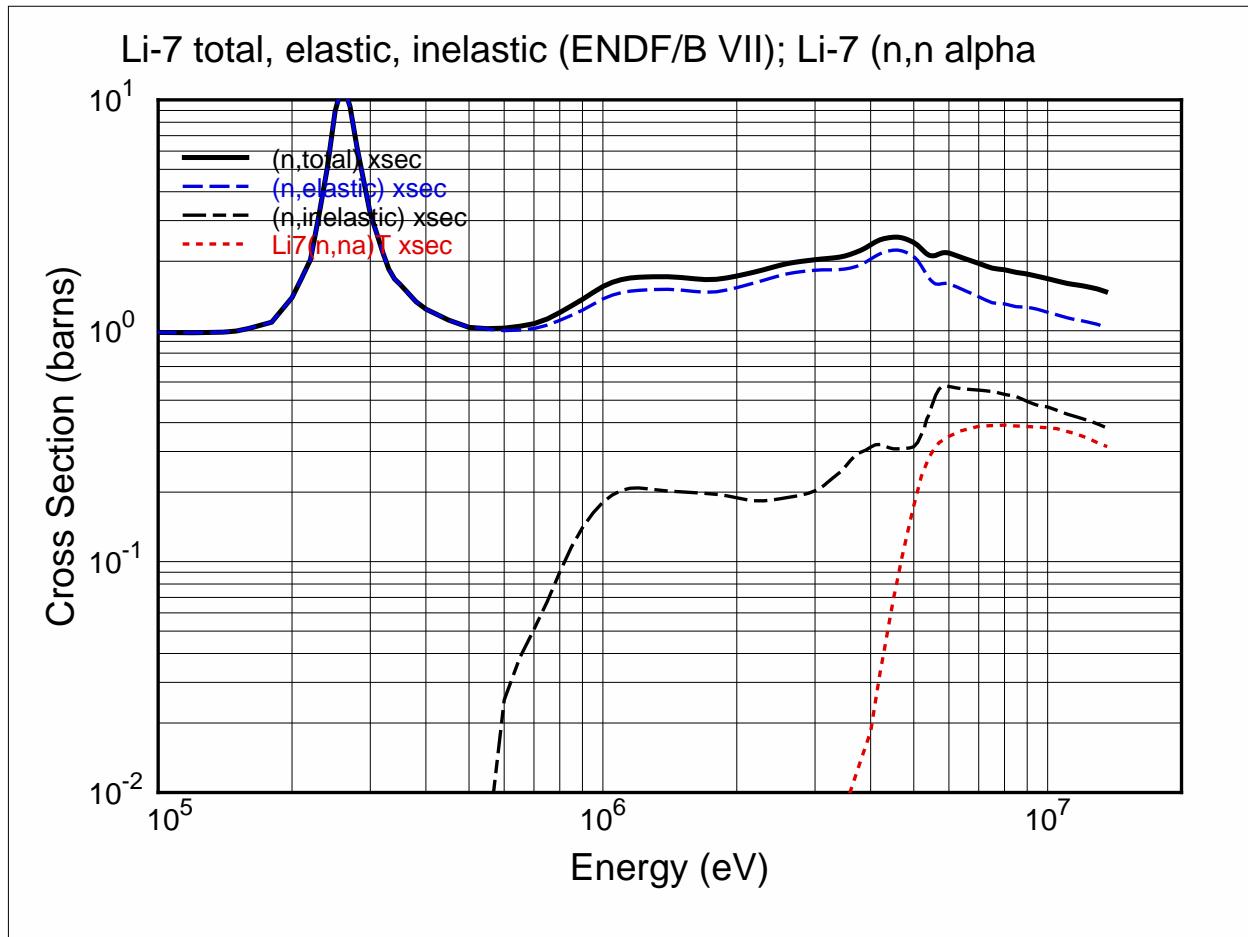


Figure A.7: ${}^7\text{Li}$ (lithium-7) cross sections in the high-energy range. Note: For neutron energies above 5 MeV there is a cross section of a few tenths of a barn for an interesting reaction: ${}_0^1\text{n} + {}^7_3\text{Li} \rightarrow {}^3_1\text{T} + {}^4_2\text{He} + {}_0^1\text{n}$, or $(\text{n}, \alpha \text{T n})$. Note that this reaction produces a tritium nucleus and a low-energy neutron. If ${}^6\text{Li}$ is also present (which it usually is if ${}^7\text{Li}$ is present), this low-energy neutron could produce a second tritium nucleus! Thus, there is a way to produce two tritium atoms from a single neutron if that neutron had energy > 5 MeV. Without some possibility of generating more than one tritium atom from one high-energy neutron, large-scale power production from D+T fusion would be impractical, because there would not be enough tritium to keep the reactors running.

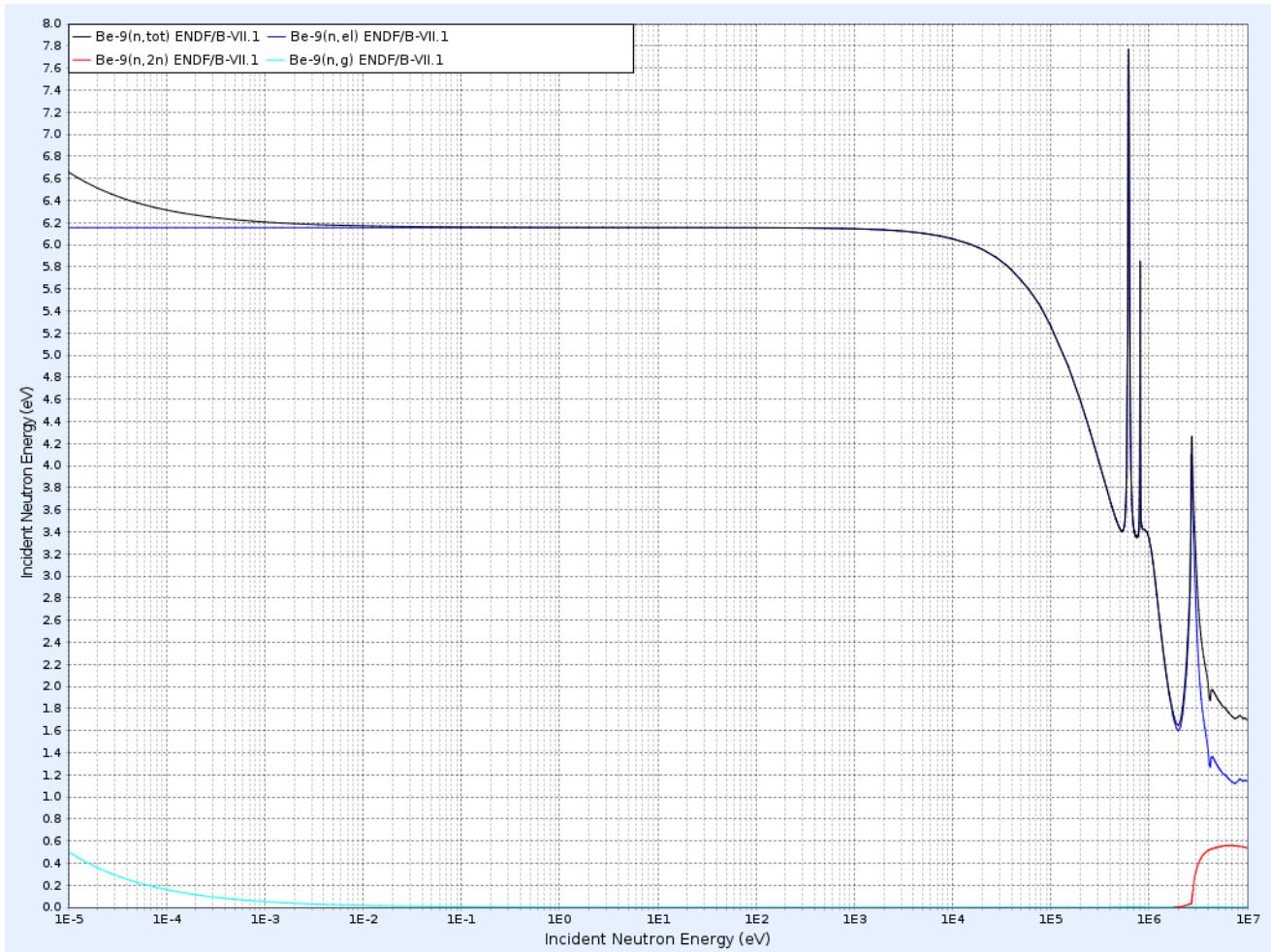


Figure A.8: ^{9}Be (beryllium-9) cross sections. Note several features: **A)** Beryllium-9 is a metal and is not a strong neutron absorber—its capture cross section at 0.025 eV is less than 0.01 barn—so it is often used as a neutron reflector or as a structural material that doesn't eat neutrons. **B)** Be-9 can act as a neutron multiplier, for it has a non-negligible ($n,2n$) cross section for neutron energies > 3 MeV.

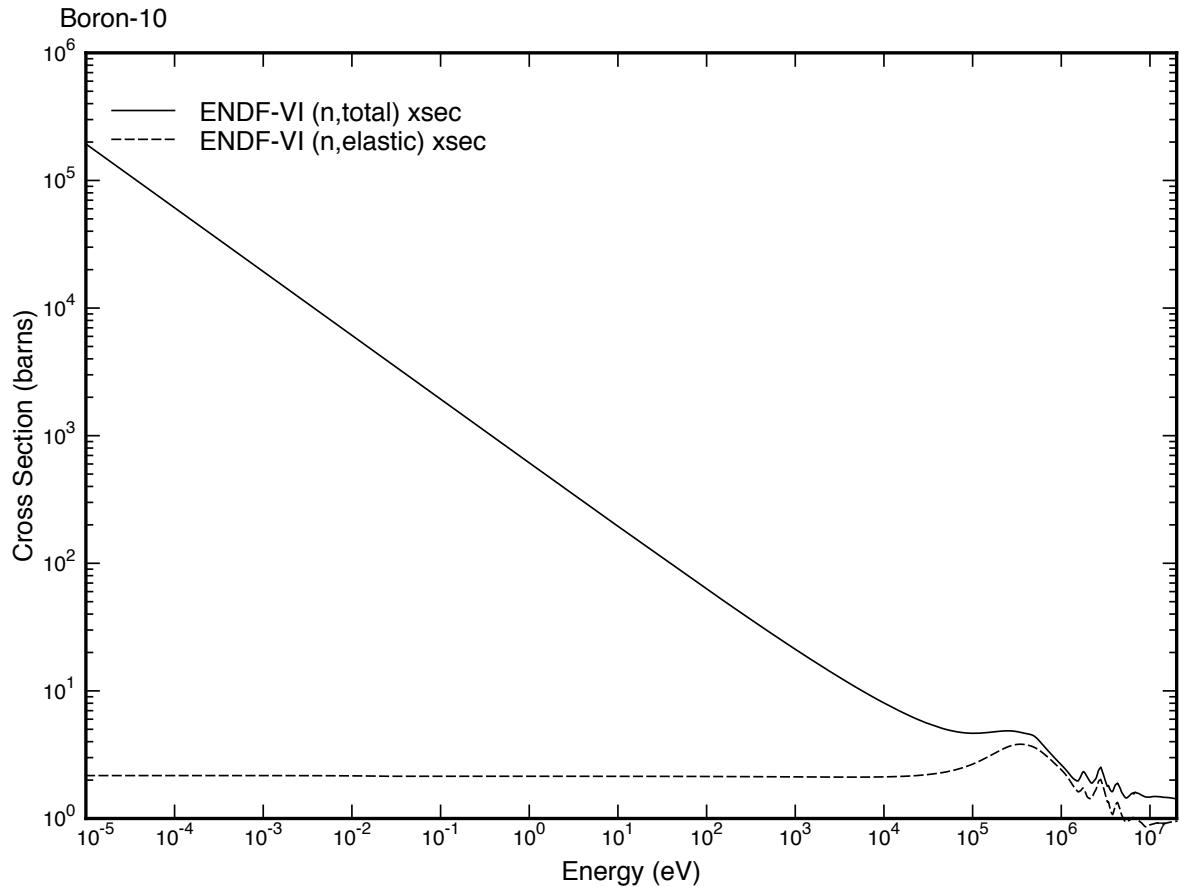


Figure A.9: $^{10}_5\text{B}$ (boron-10) cross sections. Note several features: **A)** Boron-10 is a strong neutron absorber, with $\sigma_a \gg \sigma_s$ for $E < 10^5$ eV. The absorption interaction is not capture, but rather (n,α) , in which $n + {}^{10}\text{B} \rightarrow \alpha + \text{lithium-7}$. The high cross section and relative abundance makes boron a common choice for control-rod or soluble-poison material in reactors. The charged-particle products make B-10 an attractive choice for some kinds of neutron detectors. **B)** The scattering cross section is almost constant for all E below the resonance that peaks at ≈ 300 keV. **C)** ${}^{10}\text{B}$ is a “ $1/v$ ” absorber, which means its absorption cross section is proportional to $1/(\text{neutron speed})$, which means proportional to $1/\sqrt{E}$. **D)** Boron-10 is $\approx 20\%$ of natural boron, with the other $\approx 80\%$ being boron-11.

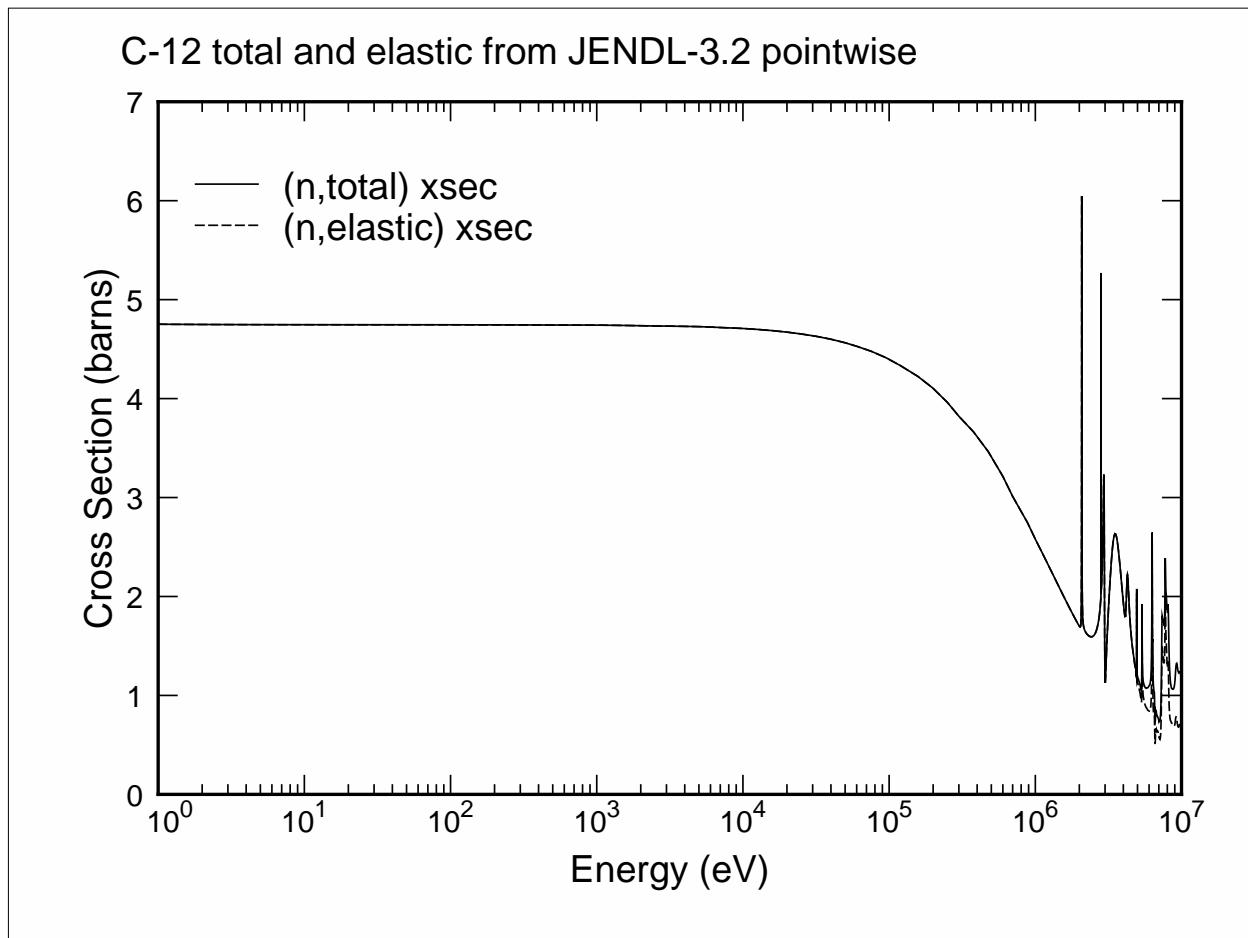


Figure A.10: ^{12}C (carbon-12) cross sections. Note several features: **A**) For $E < 5 \text{ MeV}$, the total and elastic-scattering cross sections are almost the same. That is, carbon-12 prefers to simply scatter neutrons elastically. **B**) The scattering cross section is almost constant for all E below $\approx 20 \text{ keV}$. **C**) For $E >$ a few MeV, inelastic scattering is the main reaction that comes into play in addition to elastic scattering. For $E = 14 \text{ MeV}$ the inelastic cross section is $\approx 0.43 \text{ barns}$.

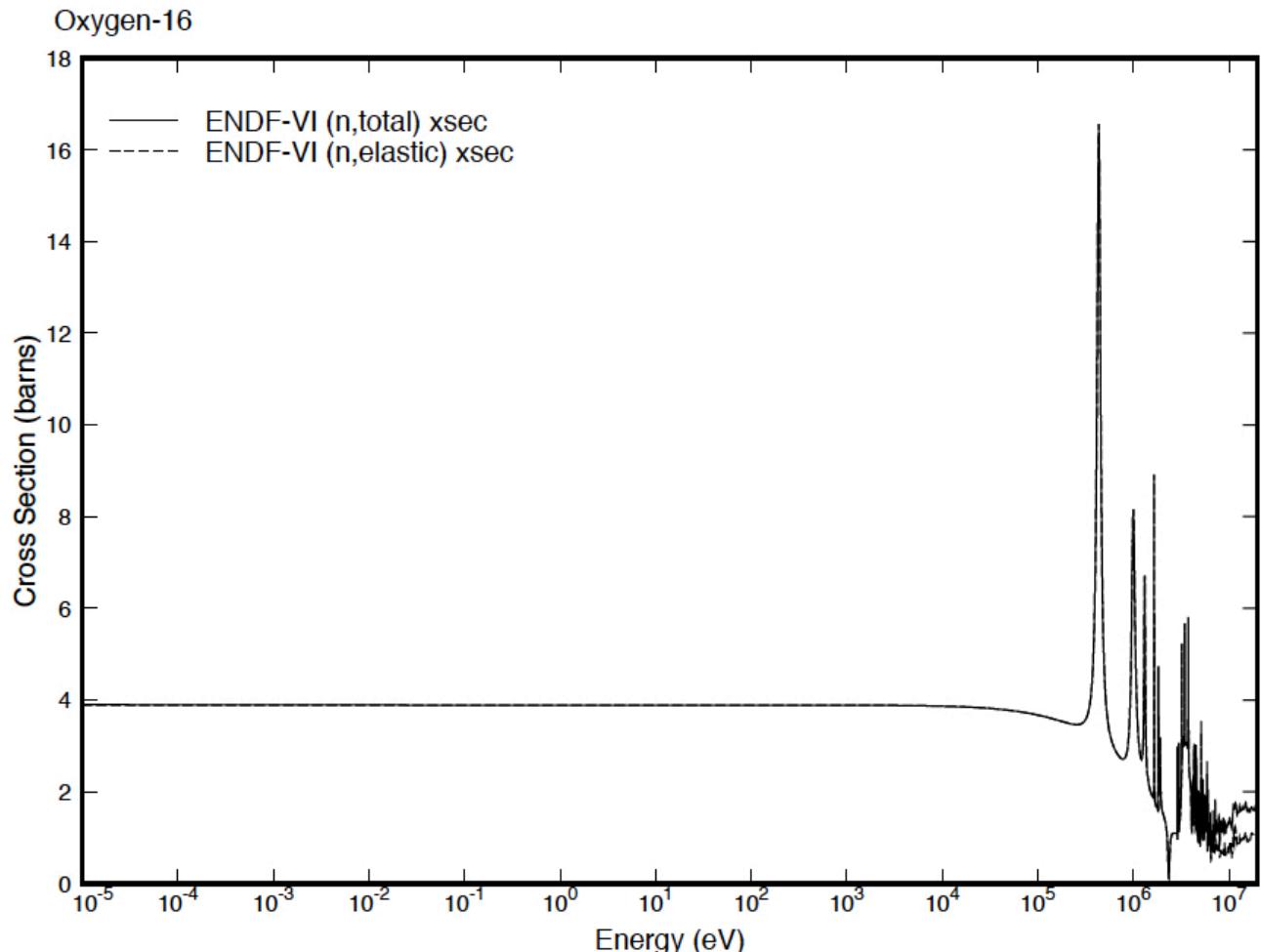


Figure A.11: ^{16}O (oxygen-16) cross sections. Note several features: **A**) For $E < 5 \text{ MeV}$, the total and elastic-scattering cross sections are almost the same. That is, oxygen-16 prefers to simply scatter neutrons elastically. (There is $1/v$ absorption, but it is quite small for $E > 10^{-5} \text{ eV}$.) **B**) The scattering cross section is almost constant for all E below $\approx 20 \text{ keV}$. **C**) For $E >$ a few MeV, inelastic scattering is the main reaction that comes into play in addition to elastic scattering. For $E = 14 \text{ MeV}$ the inelastic cross section is $\approx 0.5 \text{ barns}$.

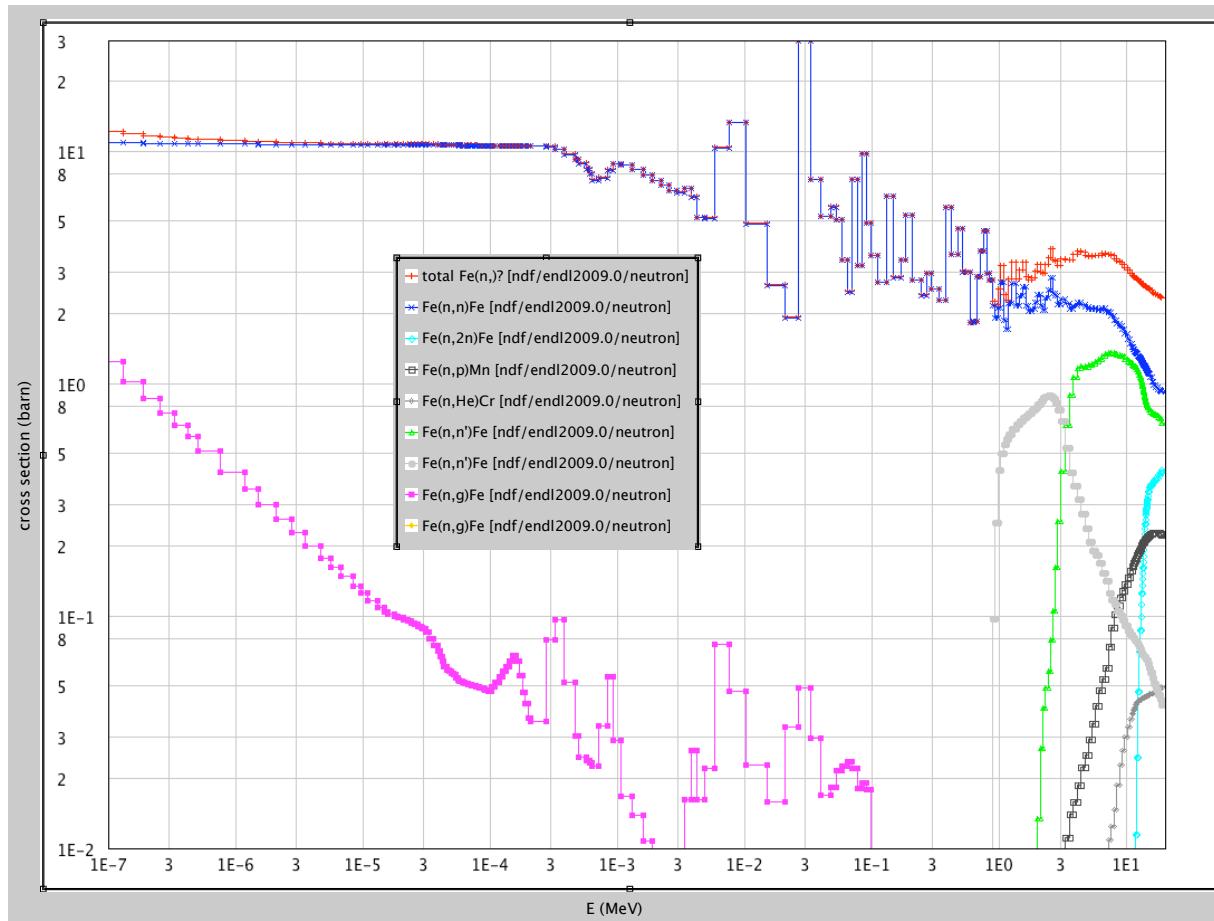


Figure A.12: ^{26}Fe (natural isotopic mix) cross sections. Note several features: **A)** For $E \in (10 \text{ eV}, 1 \text{ MeV})$, the total and elastic-scattering cross sections are almost the same and are mostly in the range of 3 to 11 barns. That is, in this energy range, iron is almost a pure scatterer. **B)** There is $1/v$ capture, with cross section of roughly 1 barn at 0.1 eV (and thus 10 barns at 0.001 eV, which is off the scale to the left). **C)** There are lots of threshold reactions that come into play for neutron energies above 1 MeV.

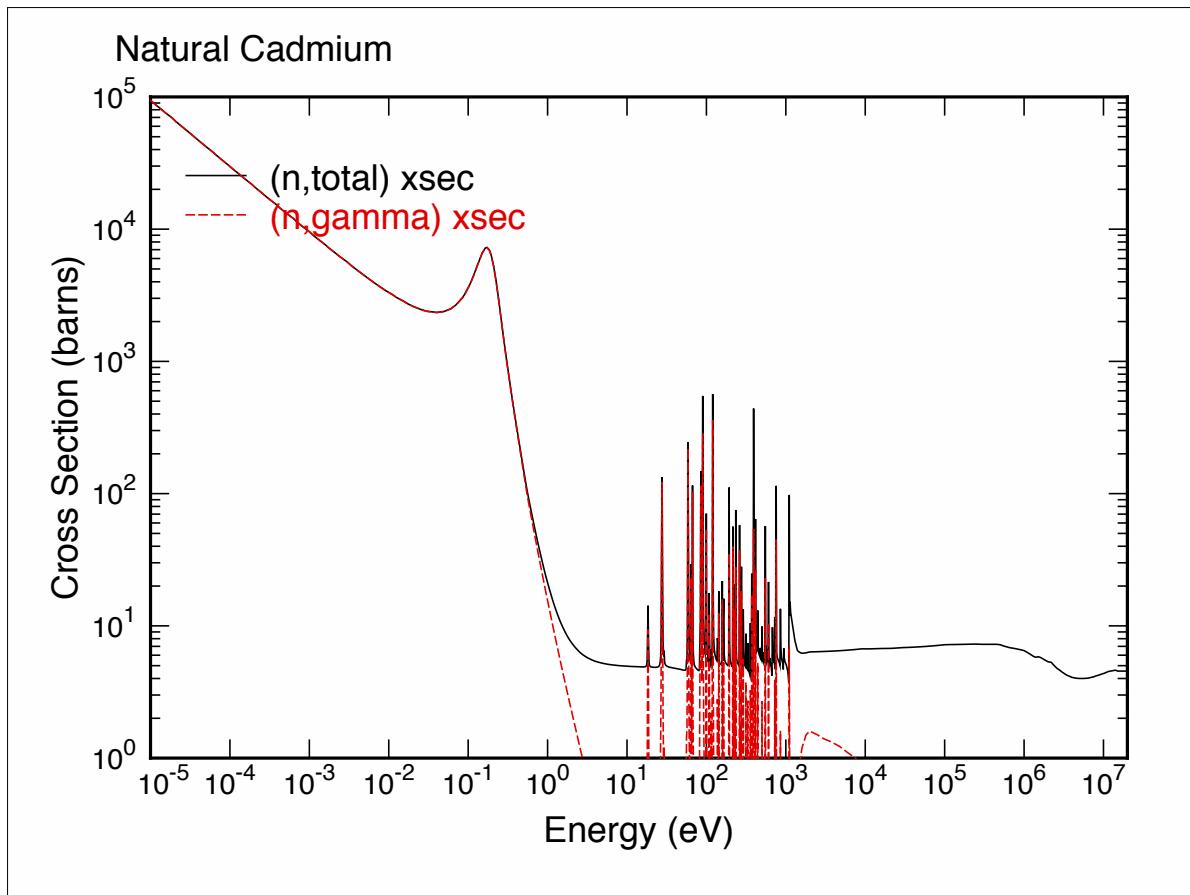


Figure A.13: Natural cadmium (^{48}Cd) cross sections. Cadmium's cross section jumps by a factor of 1000 from $E > 1\text{eV}$ to $E < 1\text{eV}$. This huge low-lying resonance makes Cd a non- $1/v$ absorber (although you can see that even this becomes $1/v$ at very low energies). This huge jump makes Cd a very useful material for screening out thermal neutrons. Cd covers are often placed on foils for this purpose in neutron-absorption experiments.

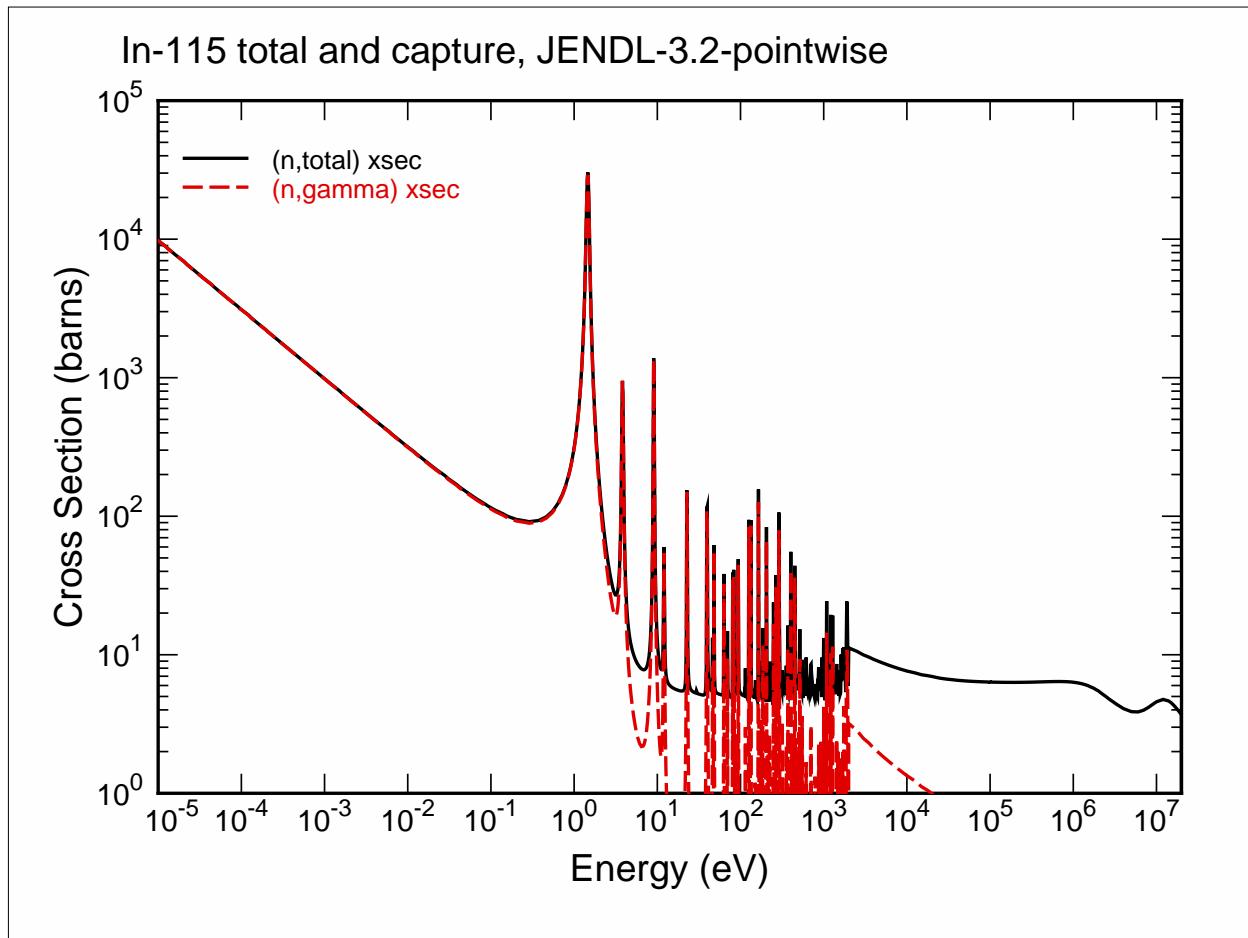


Figure A.14: $^{115}_{49}\text{In}$ (indium-115) cross sections. Indium-115, which is 95.7% of natural indium, has a big fat resonance at 1.45 eV, which makes it good for absorbing neutrons that are almost thermal. If you cover an In foil with Cd, the In will not see thermal neutrons and will absorb mostly neutrons of energies close to 1.45 eV. The resulting product, In-116, has a metastable state that β -decays with half-life 54 minutes, which makes it relatively straightforward to “count” it in a detector and infer how many absorptions took place.

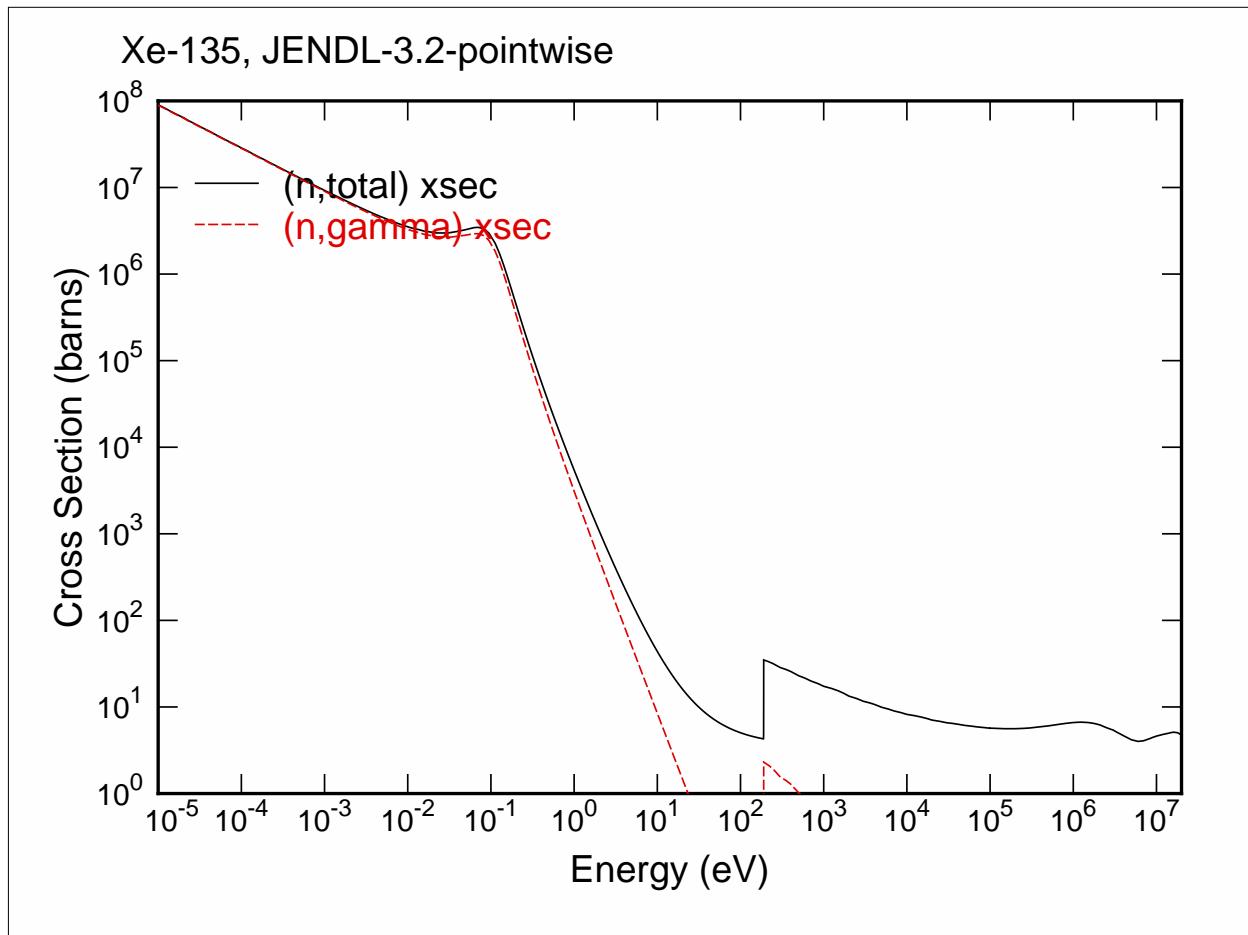


Figure A.15: $^{135}_{54}\text{Xe}$ (xenon-135) cross sections. The most striking feature of xenon-135 is how enormous its cross section is for slow neutrons—**more than a million barns!** Xe-135 is a fission product, and it is also the decay product of another fission product (Iodine-135). In fact, > 6% of fissions ultimately produce this neutron-hungry nuclide. We must design our reactors to have enough “excess reactivity” to stay critical even after Xe-135 builds up.

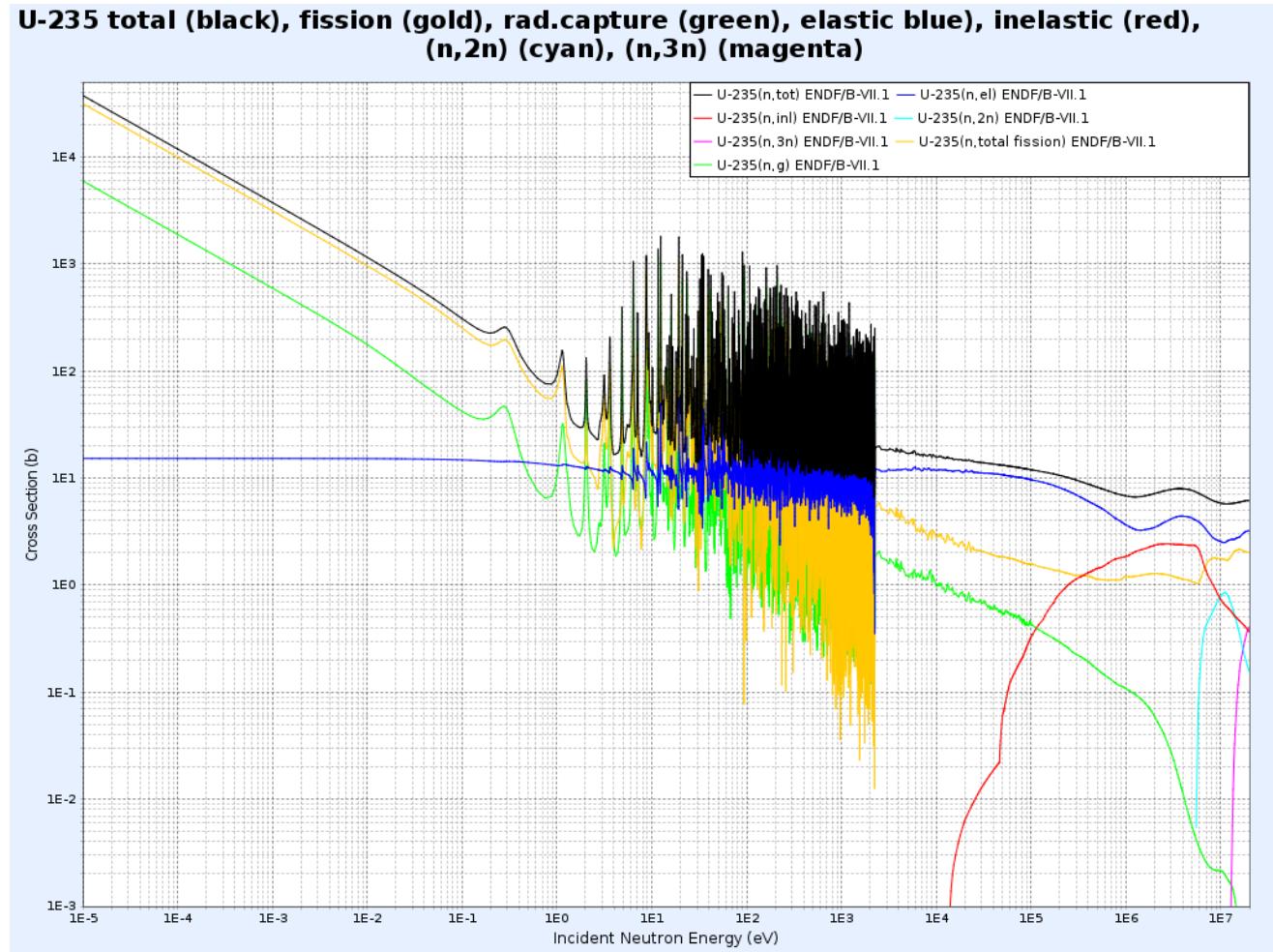


Figure A.16: $^{235}_{92}\text{U}$ (uranium-235) cross sections. Note several features: **A)** The fission cross section is significantly larger than the capture cross section, and it is most of the total for thermal neutrons. Quantitatively, the 2200-m/s fission cross section ($\Rightarrow E=0.0253$ eV) is a bit below 600b and the capture cross section is only ≈ 100 b, so fission is by far the most likely interaction for a thermal neutron with U-235. **B)** There is a low-lying resonance (at about 0.3 eV) that perturbs the $1/v$ functional form for thermal neutrons. **C)** We see the typical behavior of a fissile nuclide: a $1/v$ region at very low E , a resolved-resonance region from $< 1\text{eV}$ to ≈ 20 keV, a fall-off of the capture cross section for $E > 1$ MeV, inelastic scattering being significant in the energy range where neutrons are born from fission, and $(n,2n)$ reactions being possible for neutrons of several MeV.

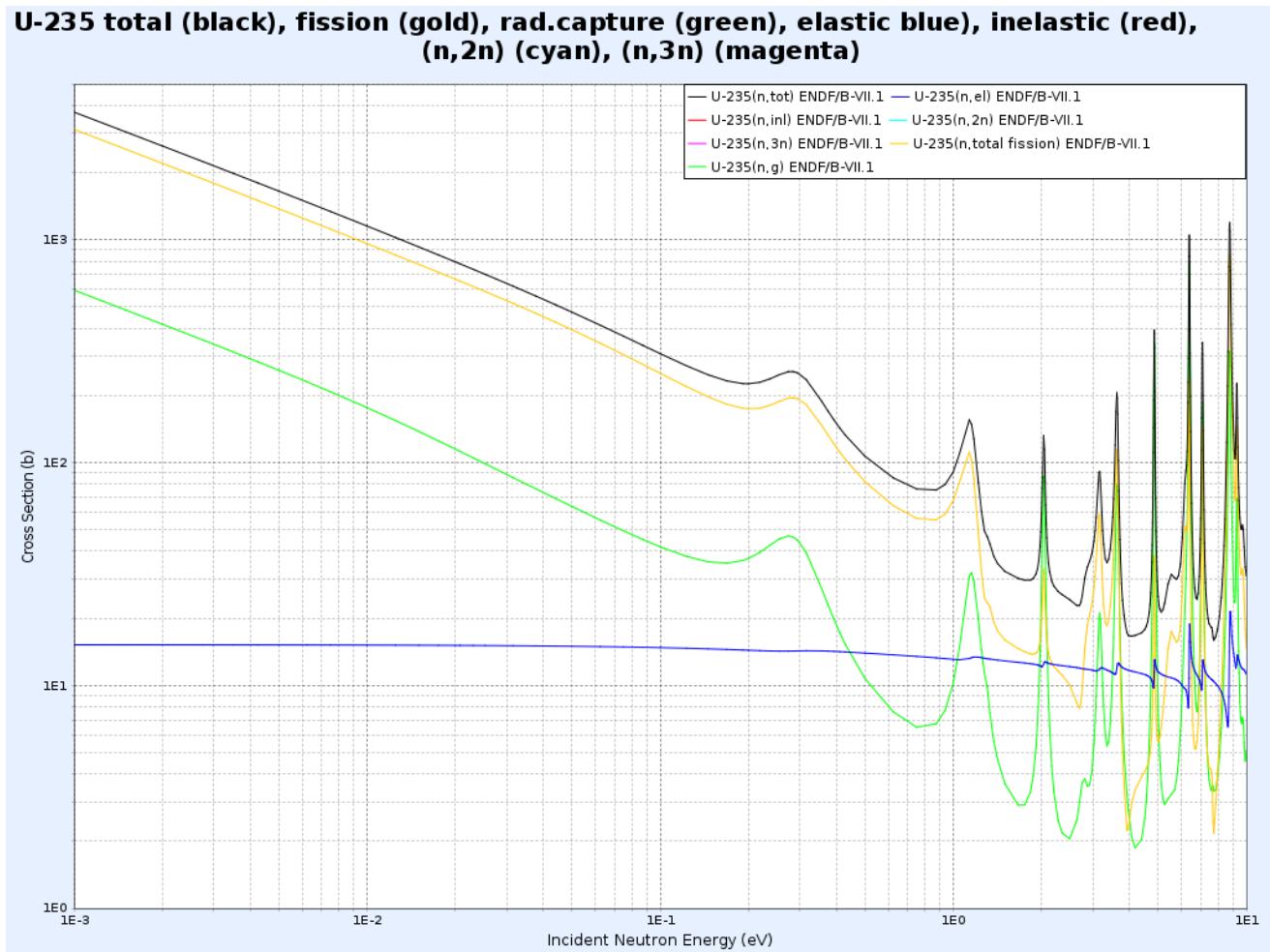


Figure A.17: $^{235}_{92}\text{U}$ (uranium-235) cross sections below 10 eV, showing the details of low-lying resonances.

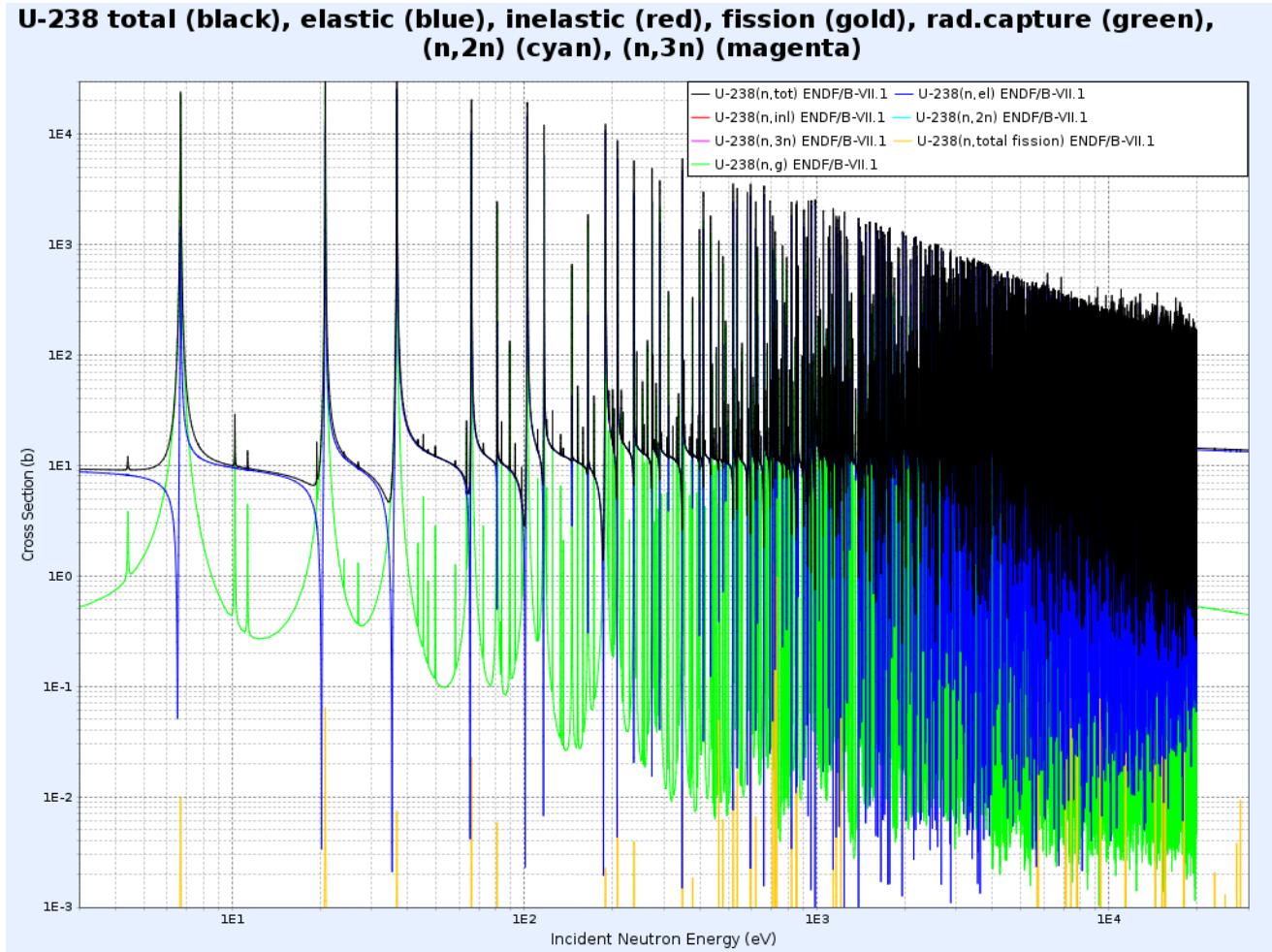


Figure A.18: $^{235}_{92}\text{U}$ (uranium-235) cross sections from 3 eV to 30 keV, showing details of the resolved-resonance range.

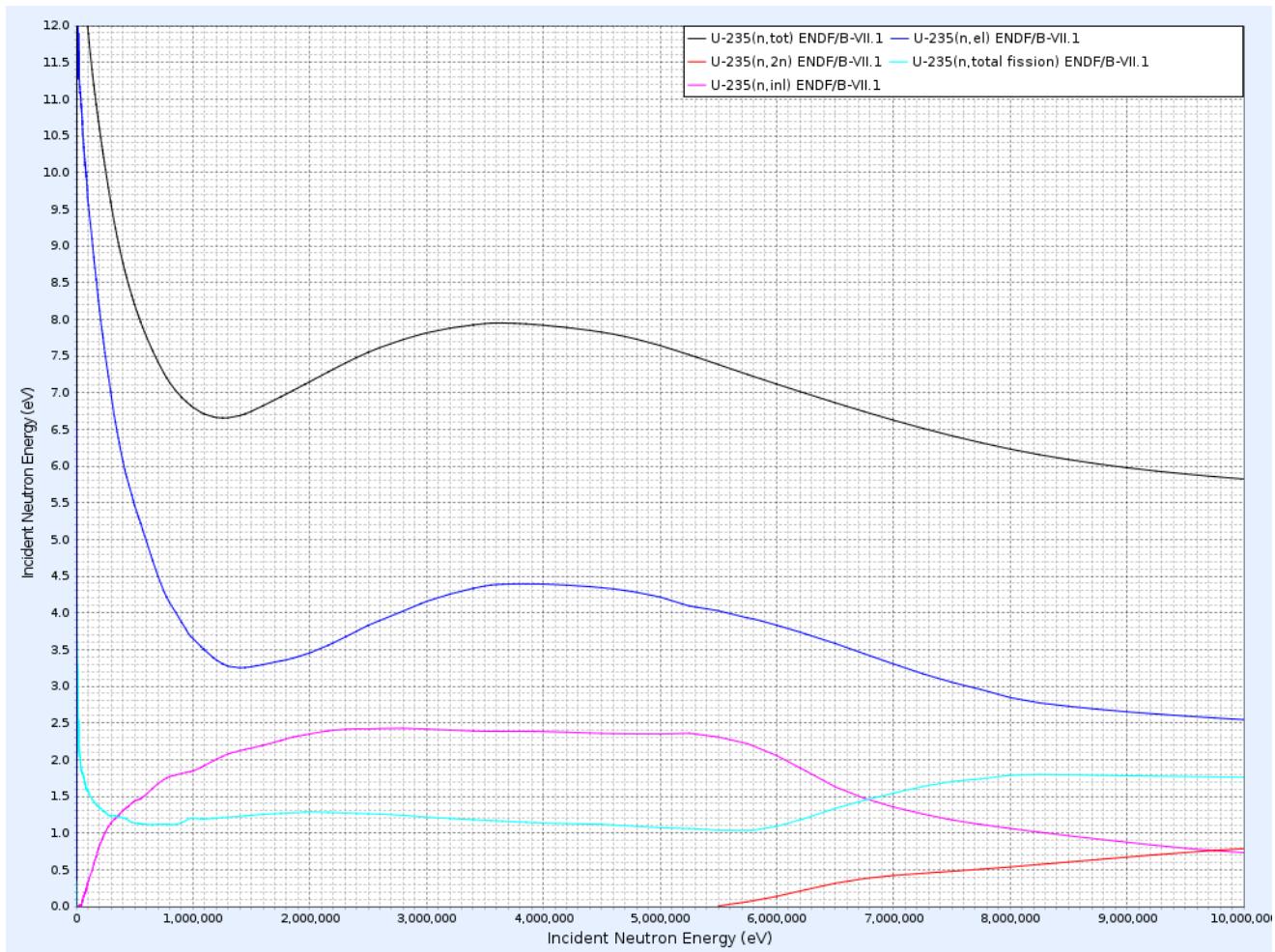


Figure A.19: $^{235}_{92}\text{U}$ (uranium-235) cross sections on a linear-linear scale. This view is useful for determining what happens to neutrons if they interact with U-235 after they are first born from fission.

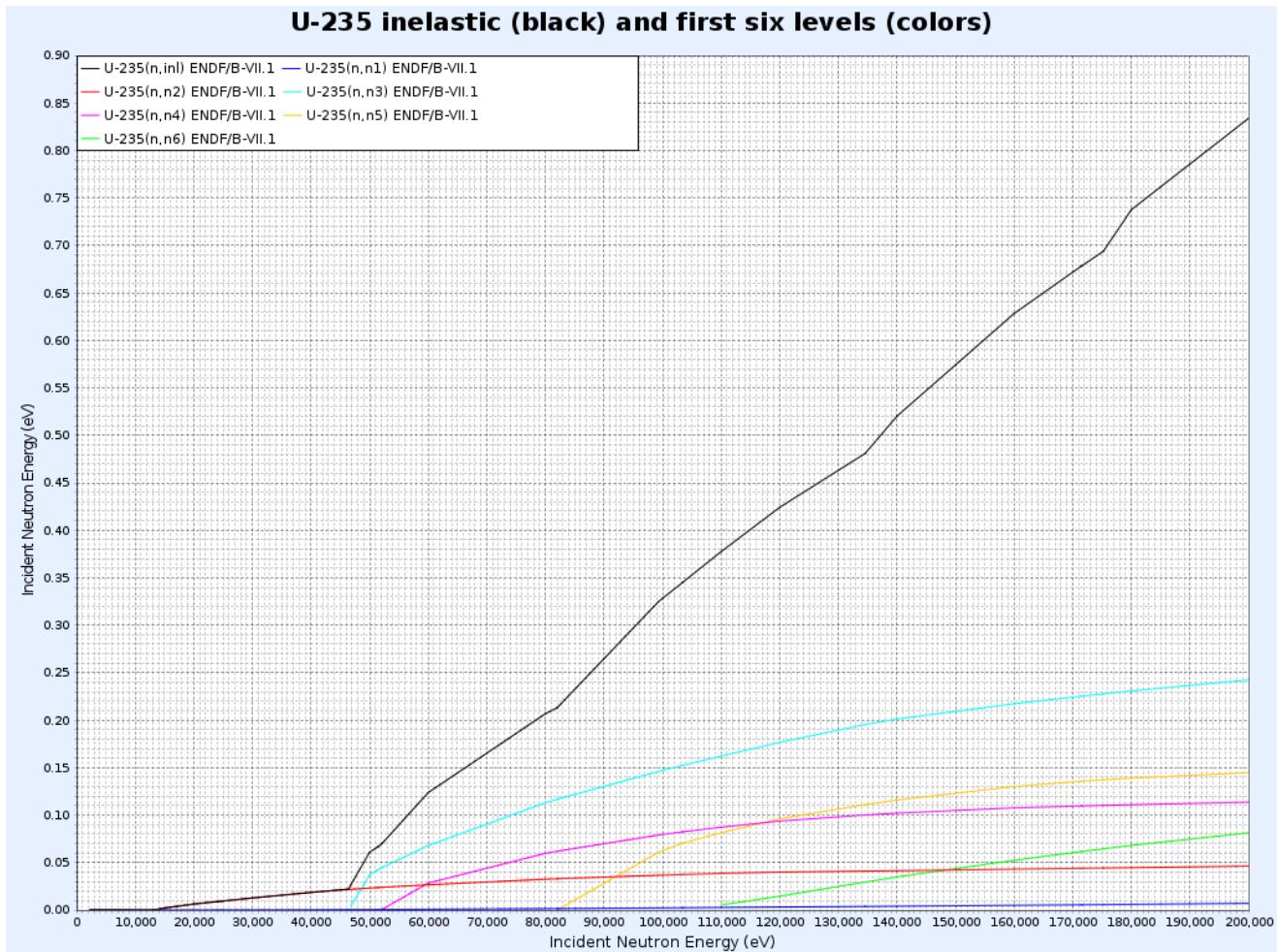


Figure A.20: $^{235}_{92}\text{U}$ (uranium-235) inelastic scattering cross section (black) and the portions of the inelastic scattering cross section that are due to the nucleus being left in the first six excited states (the six colored curves), on a linear-linear scale. You can see that the threshold for exciting each level is higher than that for exciting the previous level. For example, the fourth level has a threshold of about 52 keV and the fifth level has a threshold of about 82 keV. You can see that the overall inelastic scattering cross section is the sum of the cross sections for these six levels, until the 7th level kicks in somewhere around 135 keV.

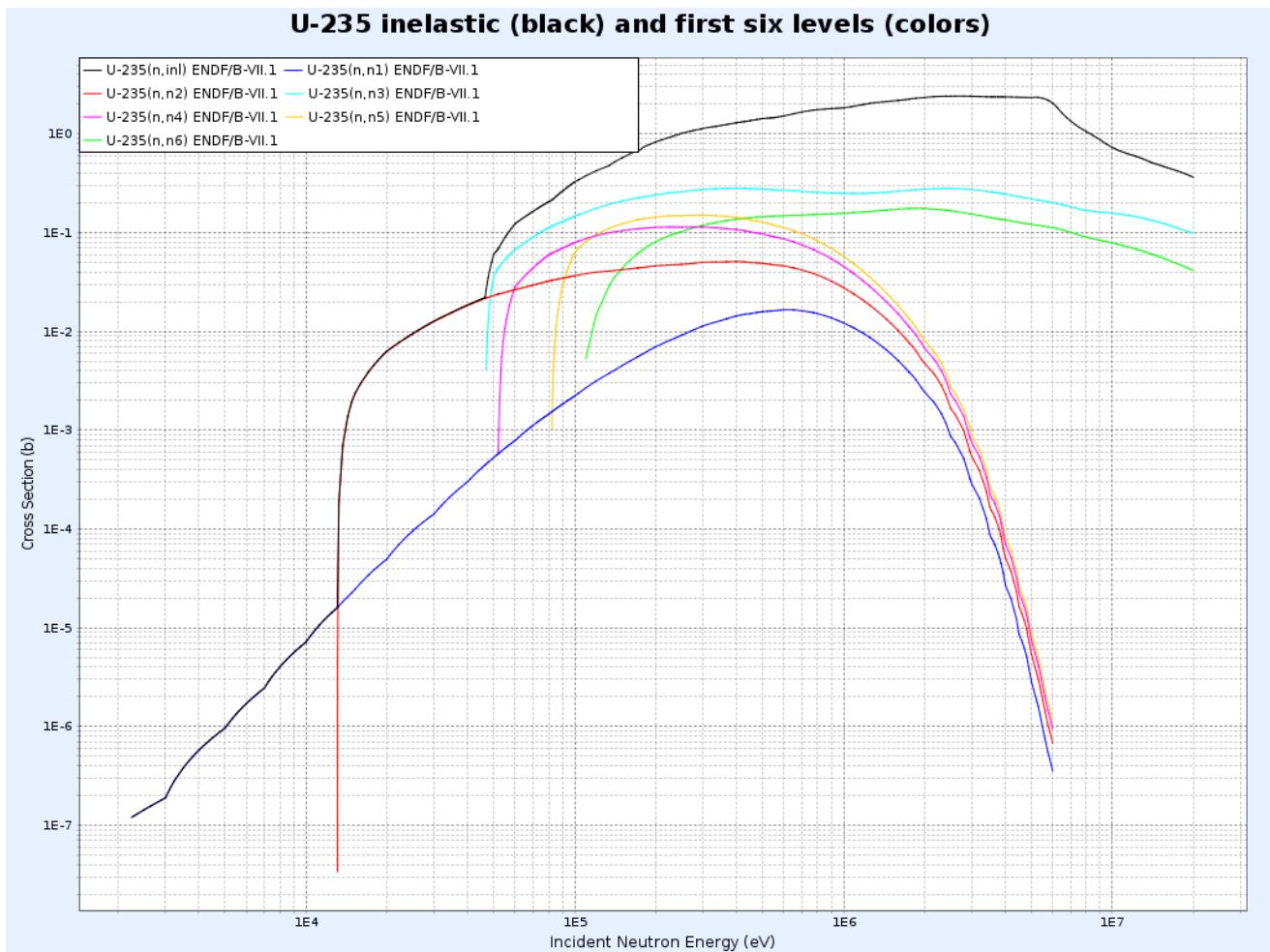


Figure A.21: $^{235}_{92}\text{U}$ (uranium-235) inelastic scattering cross section (black) and the portions of the inelastic scattering cross section that are due to the nucleus being left in the first six excited states (the six colored curves), on a log-log scale. The log-log scale makes it easier to see the threshold nature of each interaction.

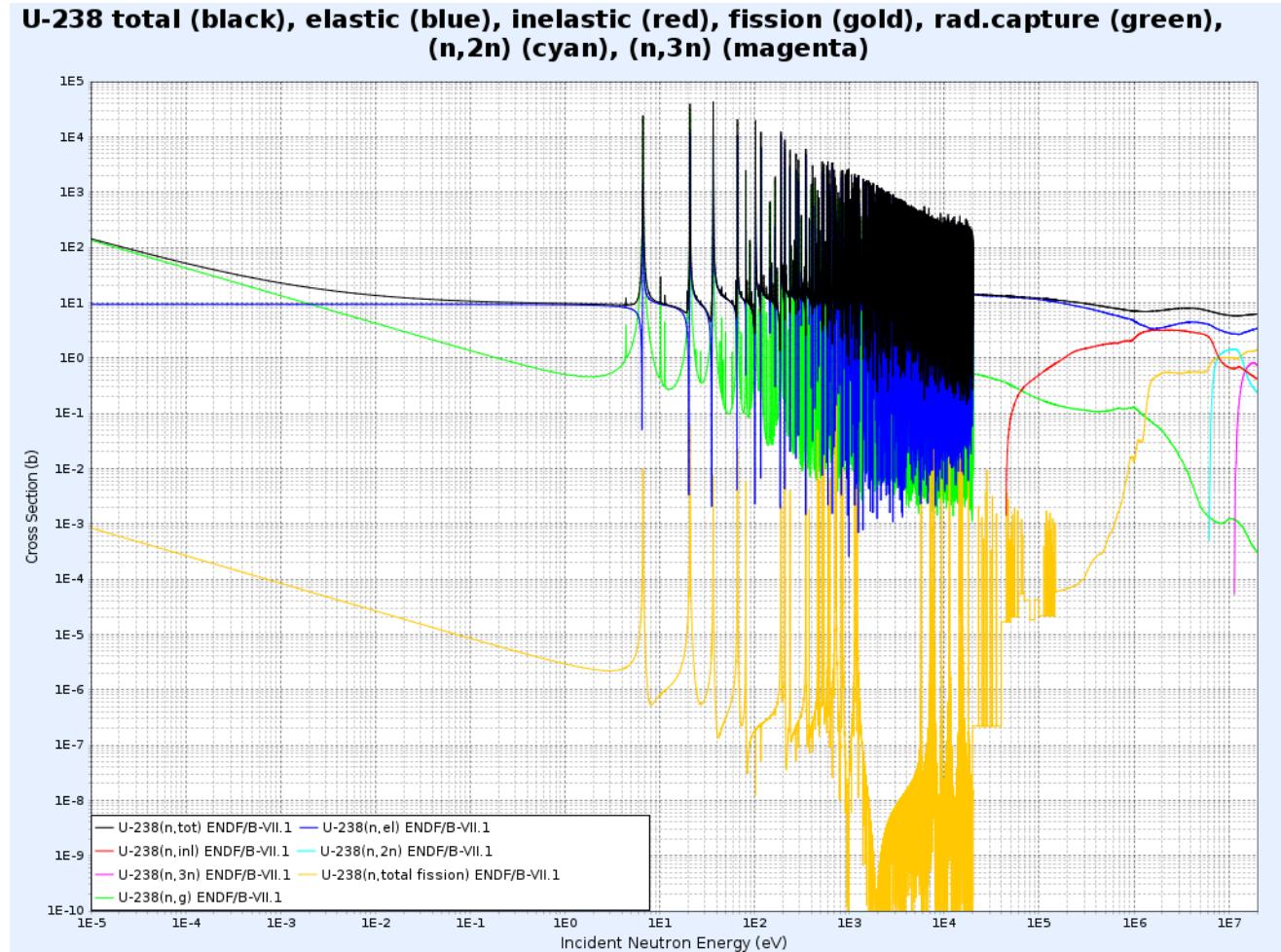


Figure A.22: ^{238}U (uranium-238) cross sections. Compare against U-235. Note several features: **A)** The U-238 cross section is much lower than that of a fissile nuclide for thermal neutrons, and fission does not become important until $E \gtrsim 1 \text{ MeV}$. **B)** There is a low-lying resonance (at 6.6 eV) that strongly perturbs the $1/v$ functional form for thermal neutrons. **C)** This is typical behavior for a fissionable (not fissile) nuclide: a $1/v$ region at very low E , a resolved-resonance region at intermediate energies, a fall-off of the capture cross section for higher E , inelastic scattering being significant in the energy range where neutrons are born from fission, and (n,2n) reactions being possible for neutrons of several MeV.

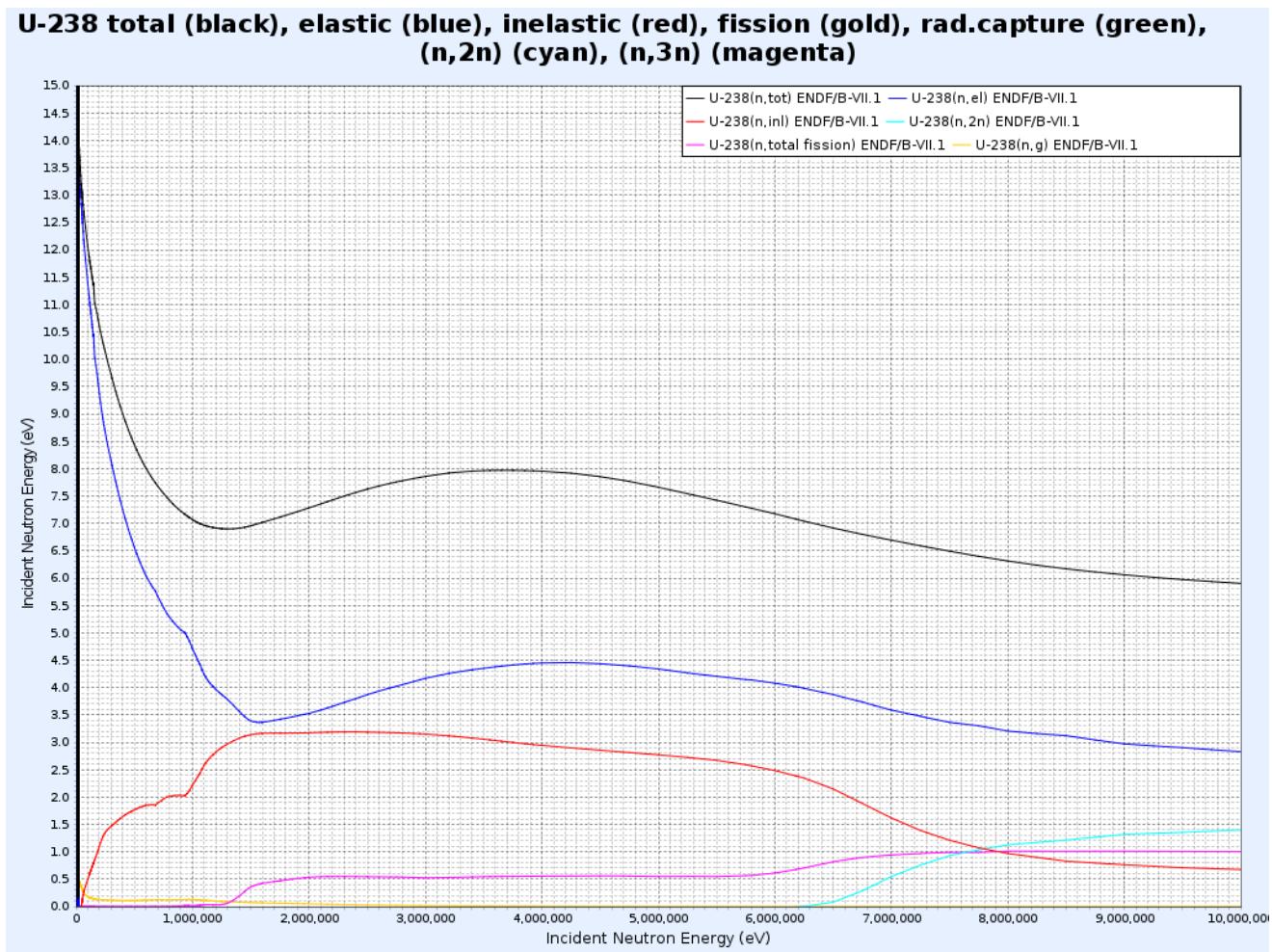


Figure A.23: $^{238}_{92}\text{U}$ (uranium-238) cross sections on a linear-linear scale. This view is useful for determining what happens to neutrons if they interact with U-238 after they are first born from fission.

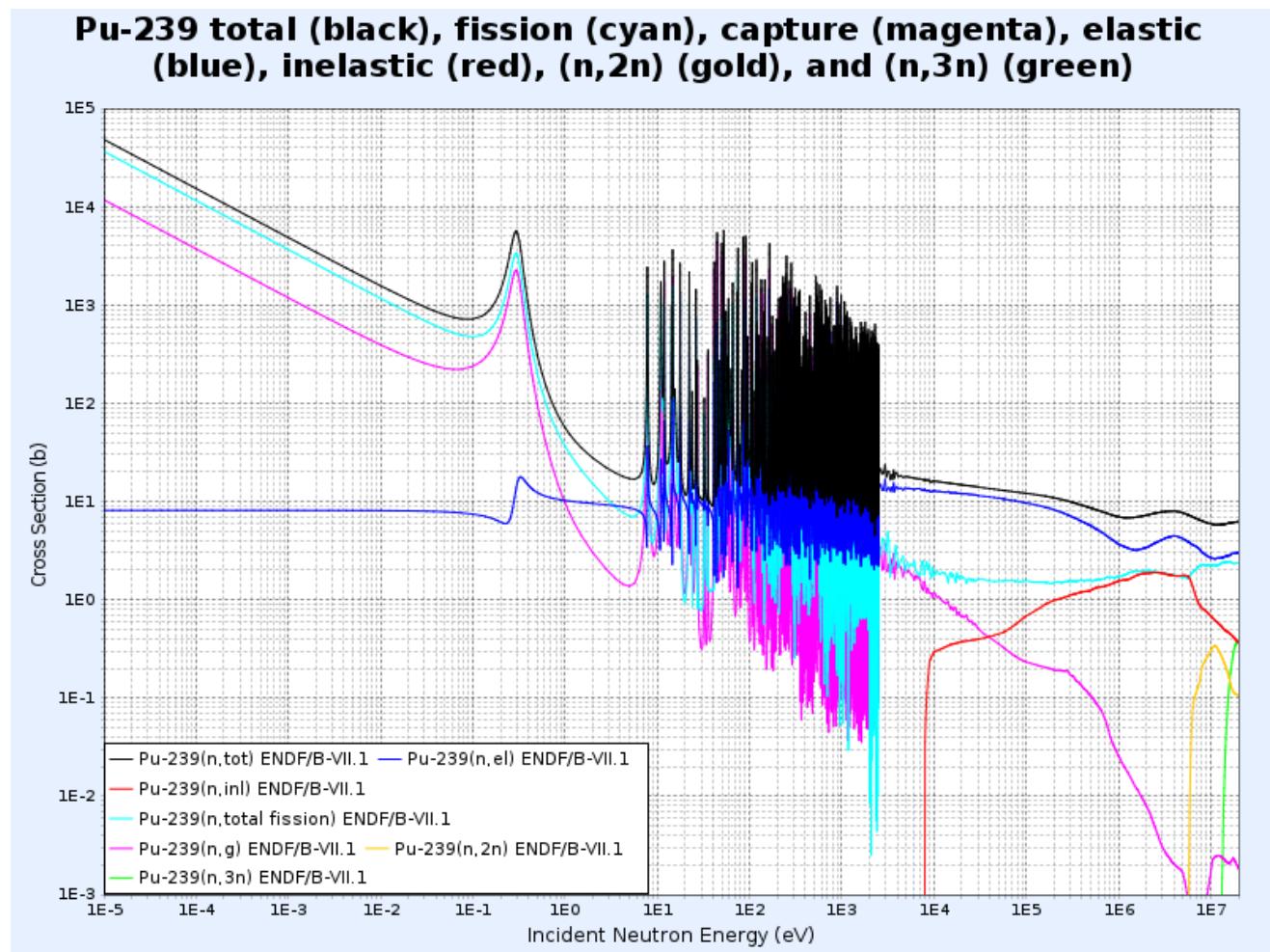


Figure A.24: ^{239}Pu (plutonium-239) cross sections. Looks a lot like U-235, because both are fissile. Main differences: its low-lying resonance is stronger and its cross section is higher for thermal neutrons. Its capture/fission ratio is also higher.

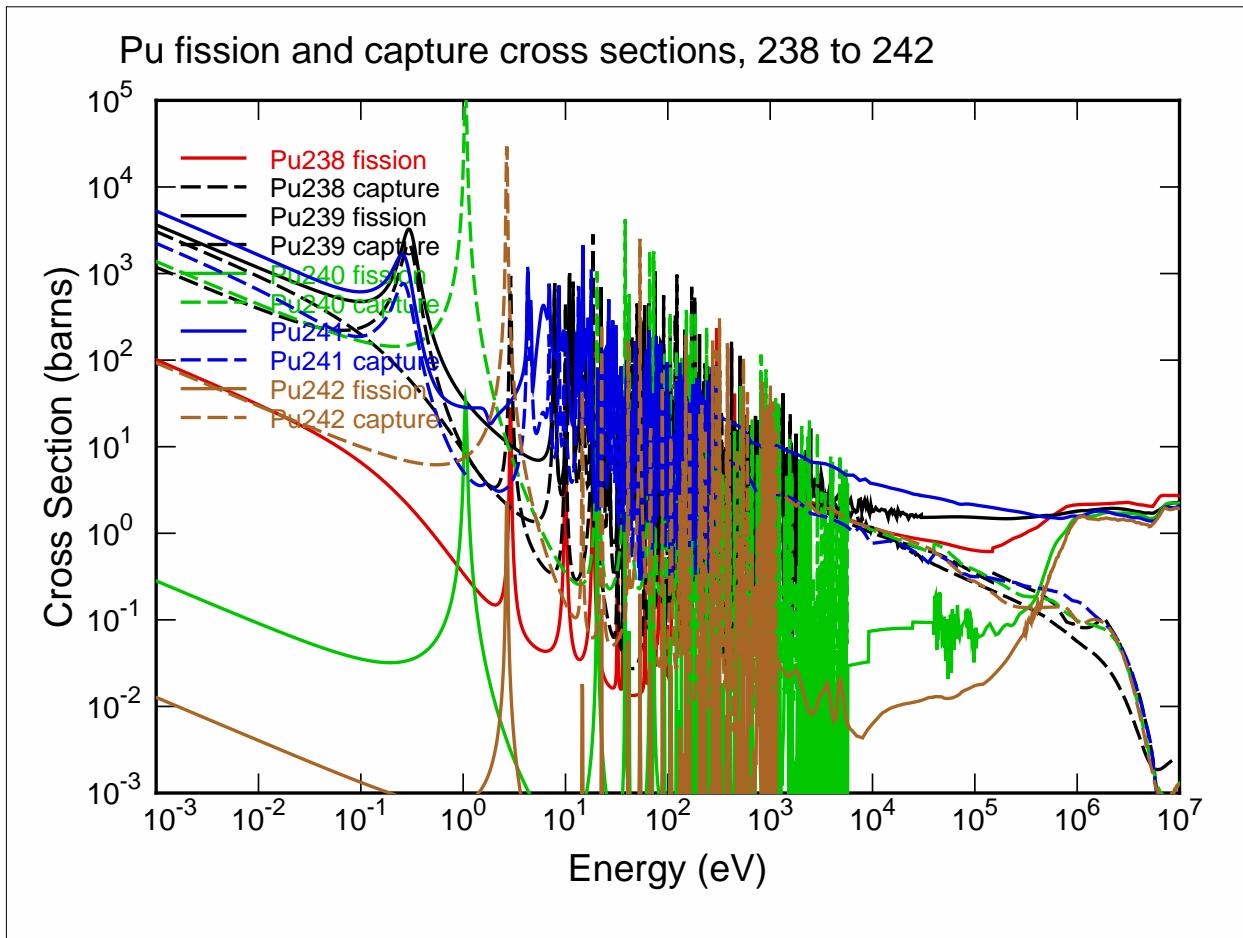


Figure A.25: Cross sections for several plutonium isotopes. Pu-239 and 241 are fissile, and exhibit typical fissile behavior. Pu-240 and 242 are fissionable, and exhibit typical fissionable behavior. Pu-238 is somewhere in between. Note that if the fissile nuclides are present in a reactor, then capture reactions will produce the fissionable nuclides. Note that all fission cross sections become comparable for $E \approx 1$ MeV, and all capture cross sections fall off in that range. This is why fast-spectrum reactors can be designed to consume more actinides than they create.

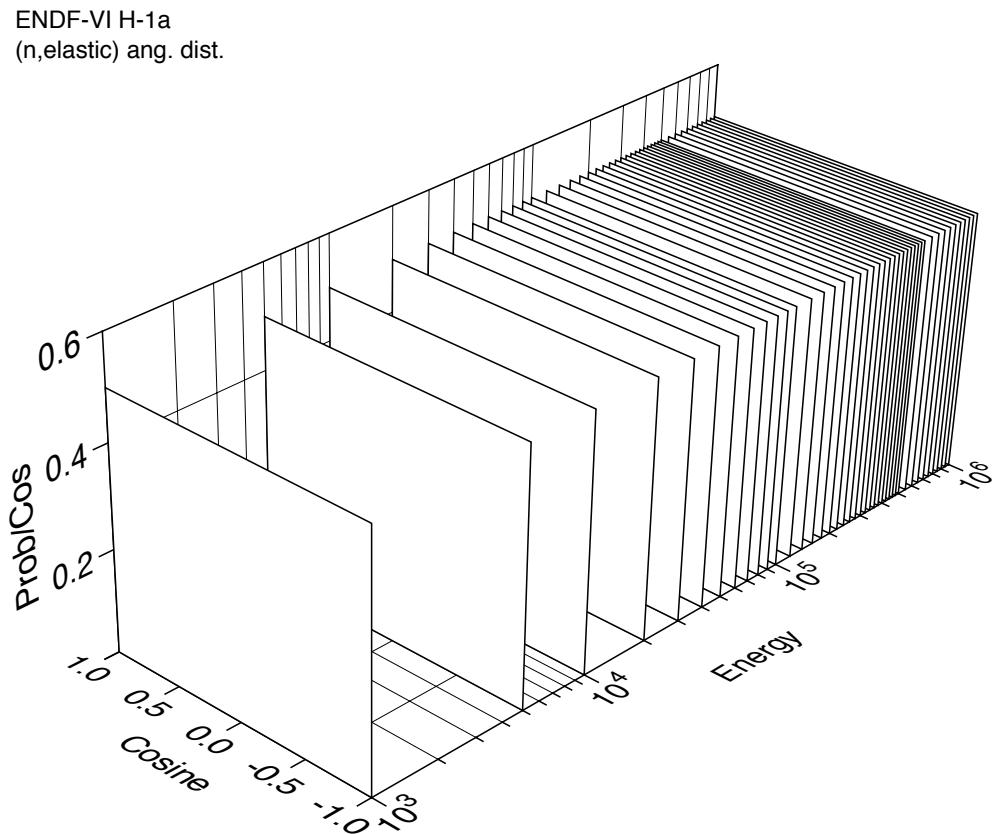


Figure A.26: ${}^1\text{H}$ (hydrogen-1) probability density function for scattering angle in the center-of-mass frame. The functions plotted are $P(E_i, \mu_{0C}) = \sigma_s(E_i, \mu_{0C})/\sigma_s(E_i)$, as a function of μ_{0C} for several values of E_i . A flat plot with value 0.5 defines **isotropic scattering in the center-of-mass frame**, which is called **s-wave** scattering. You can see that H-1 is a perfect s-wave scatterer.

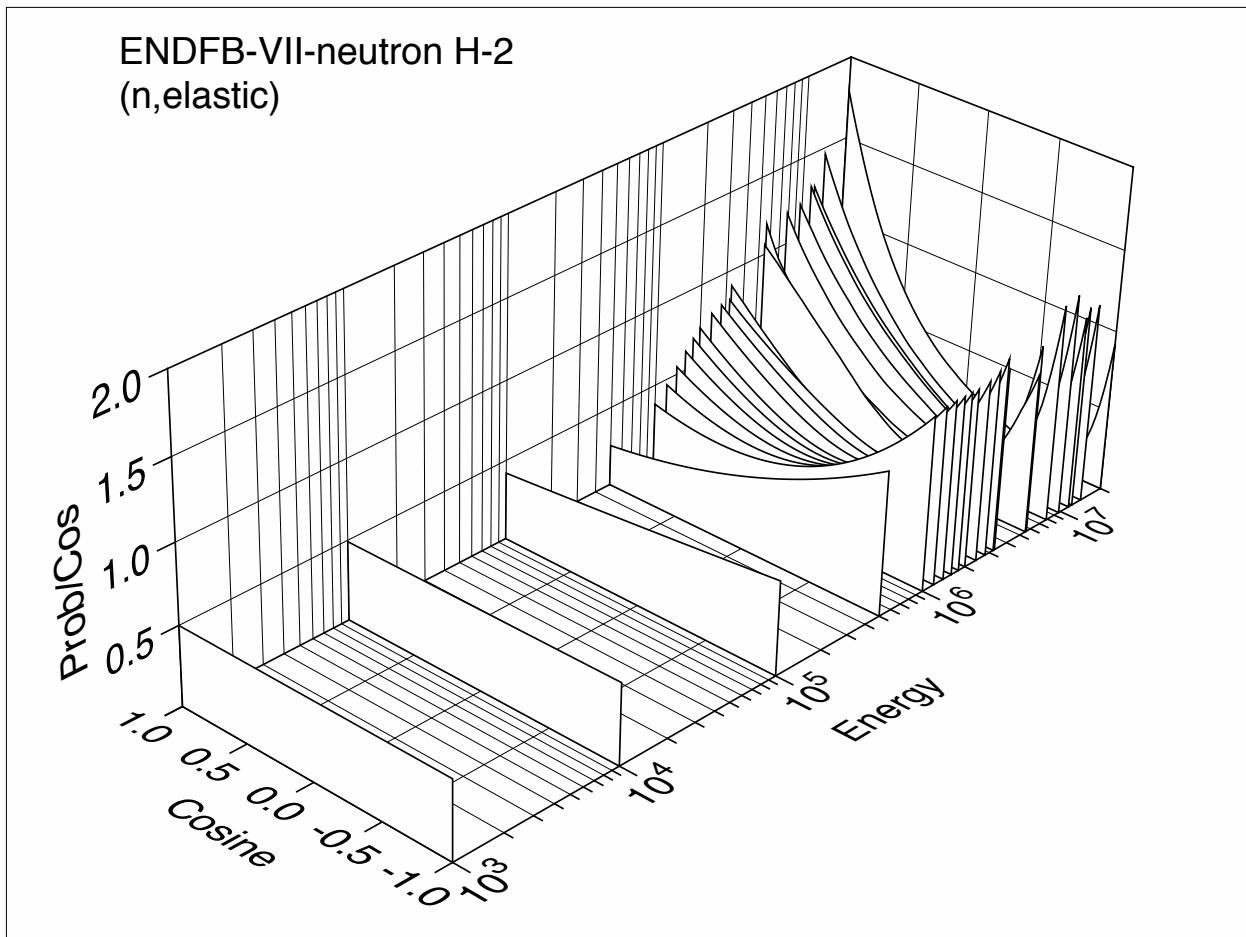


Figure A.27: ${}^2\text{H}$ (deuterium) probability density function for scattering angle in the center-of-mass frame. The functions plotted are $P(E_i, \mu_{0C}) = \sigma_s(E_i, \mu_{0C})/\sigma_s(E_i)$, as a function of μ_{0C} for several values of E_i . You can see that deuterium begins to deviate from s-wave scattering for neutron energies $\geq \approx 100$ keV.

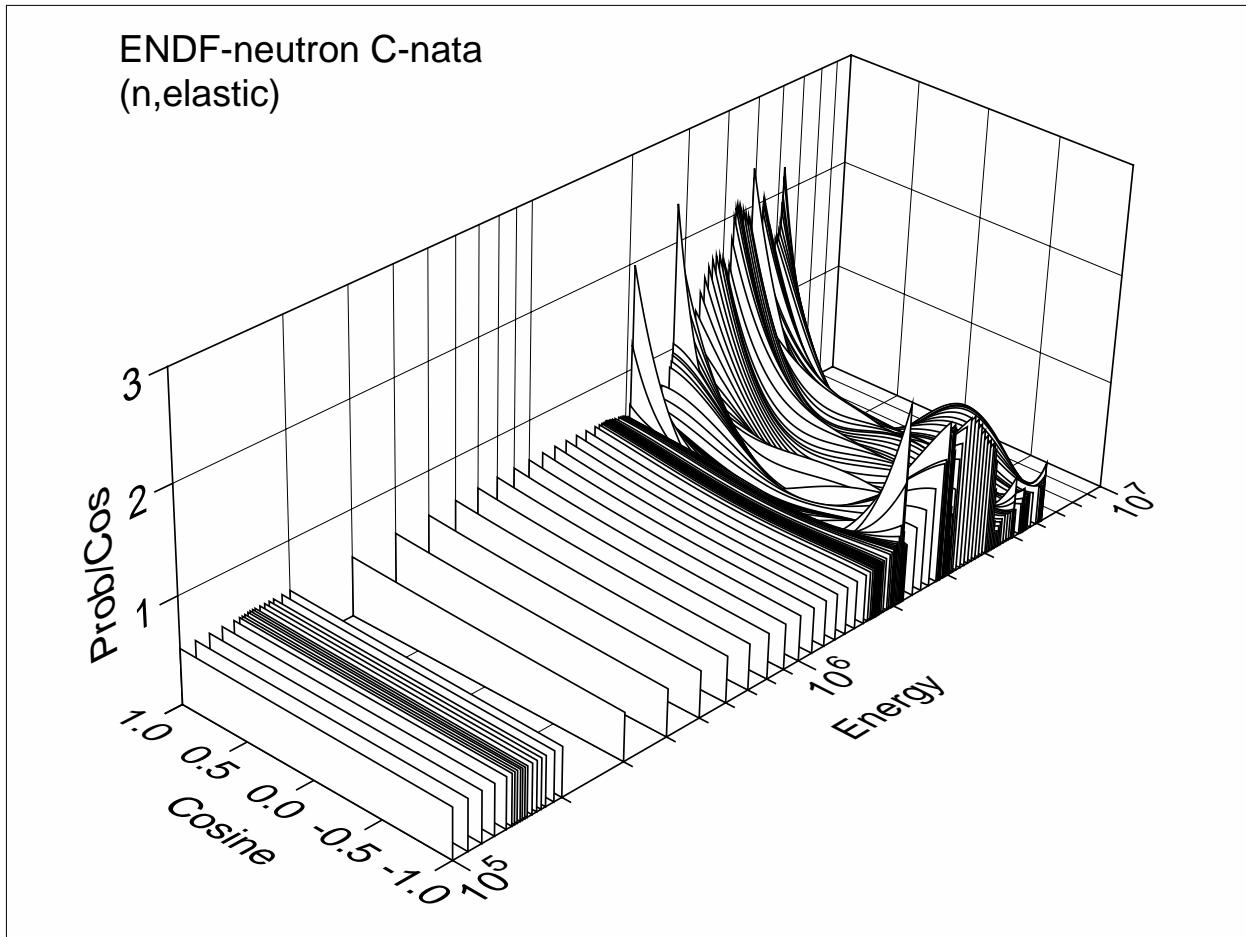


Figure A.28: ^{12}C (carbon-12) probability density function for scattering angle in the center-of-mass frame. The functions plotted are $P(E_i, \mu_{0C}) = \sigma_s(E_i, \mu_{0C})/\sigma_s(E_i)$, as a function of μ_{0C} for several values of E_i . You can see that carbon begins to deviate from s-wave scattering for neutron energies $>\approx 2$ MeV. You can also see that the pdf becomes complicated at high energies and that at some high energies it is forward-peaked (probability of small-angle scattering is greater) while at other energies it is backward-peaked (probability of backscattering is greater).

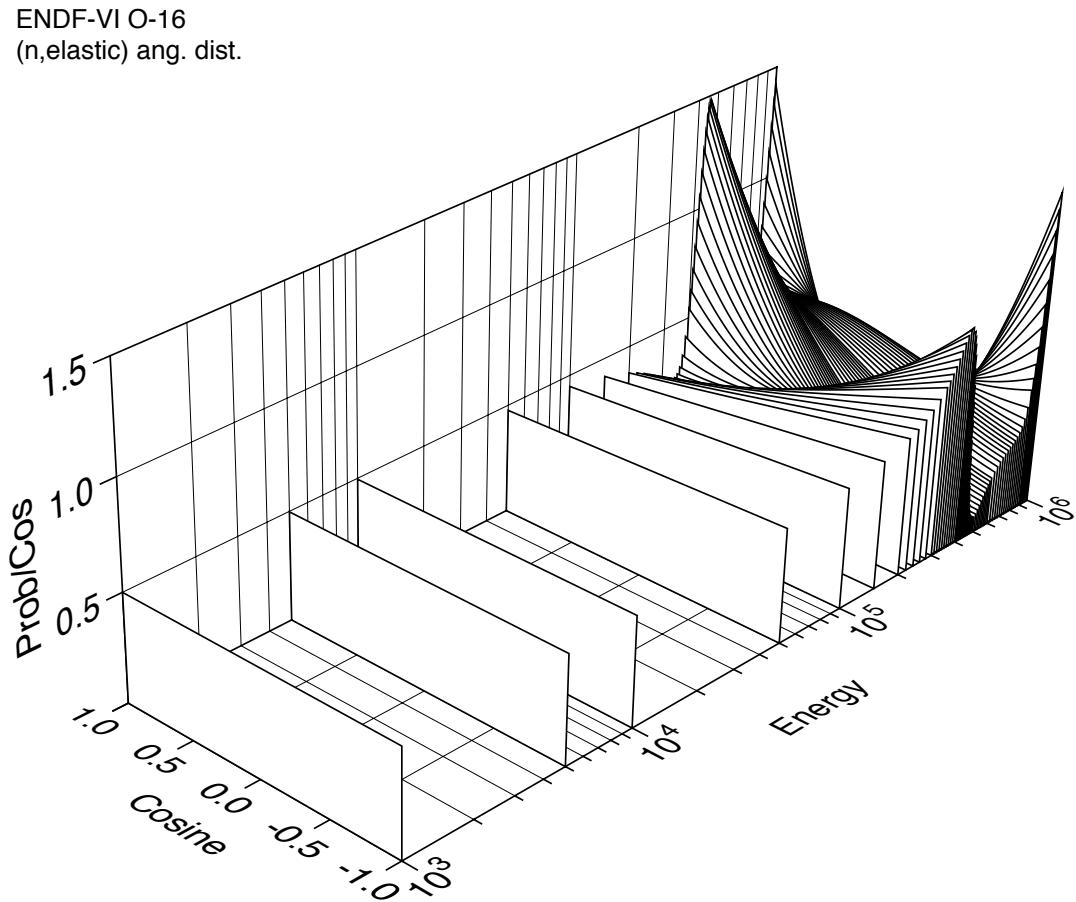


Figure A.29: ^{16}O (oxygen-16) probability density function for scattering angle in the center-of-mass frame. The functions plotted are $P(E_i, \mu_{0C}) = \sigma_s(E_i, \mu_{0C})/\sigma_s(E_i)$, as a function of μ_{0C} for several values of E_i . You can see that O-16 begins to deviate from s-wave scattering for neutron energies $\geq \approx 200$ keV. You can also see that the pdf becomes complicated and that at some high energies it is forward-peaked (probability of small-angle scattering is greater) while at other energies it is backward-peaked (probability of backscattering is greater).

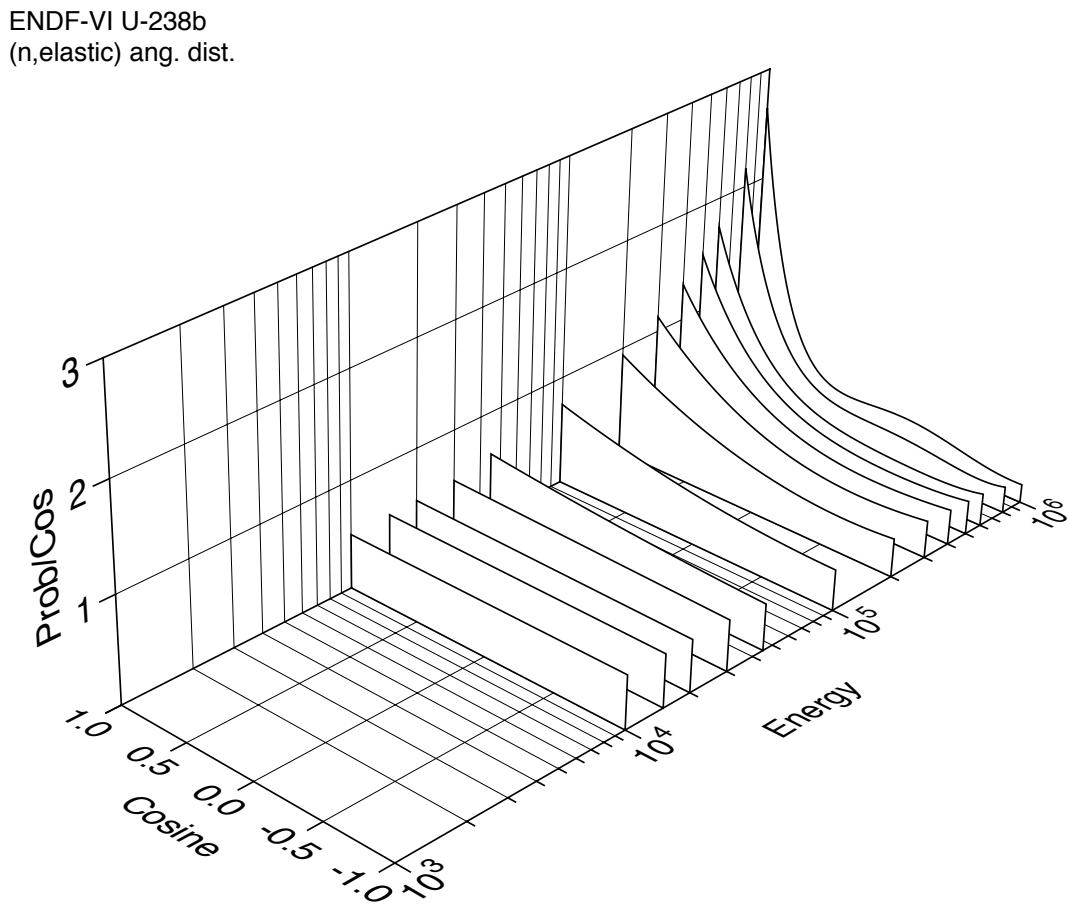


Figure A.30: ^{238}U (uranium-238) probability density function for scattering angle in the center-of-mass frame. The functions plotted are $P(E_i, \mu_{0C}) = \sigma_s(E_i, \mu_{0C})/\sigma_s(E_i)$, as a function of μ_{0C} for several values of E_i . You can see that U-238 begins to deviate from s-wave scattering for neutron energies $\geq \approx 50$ keV. You can also see that the pdf forward-peaked (probability of small-angle scattering is greater) at high energies.

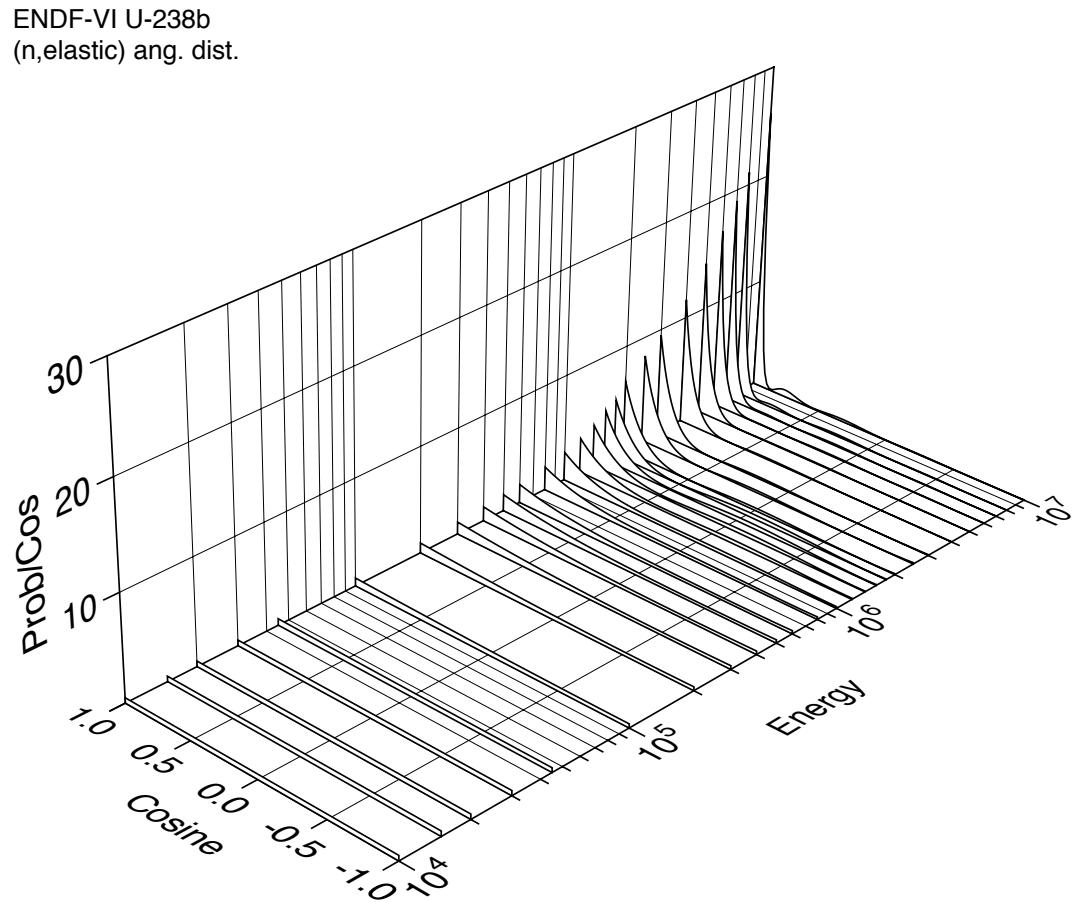


Figure A.31: ^{238}U (uranium-238) probability density function for scattering angle in the center-of-mass frame. The functions plotted are $P(E_i, \mu_{0C}) = \sigma_s(E_i, \mu_{0C})/\sigma_s(E_i)$, as a function of μ_{0C} for several values of E_i . Note the vertical scale—scattering becomes extremely forward-peaked in U-238 for neutron energies above 1 MeV!

Appendix B

Densities and Expected Values

B.1 Introduction

Our expressions for various gain and loss rates are written in terms of various densities that must be integrated over various generalized (phase-space) volumes to describe the rates that we want. This appendix is intended to clarify a few points about densities.

We are familiar with mass density expressed as mass per unit volume, for example in units of g/cm³. All densities have units of

“something per unit volume,”

at least if we are liberal in our interpretation of the term “volume”. The “volume” in question is usually called the

phase-space volume.

B.2 Simple Examples

Figure B.1 is a schematic illustration of a string with knots tied along its length. If you were asked about the knot density in the string, you would probably quickly assess that it is about 3 knots/cm. Would it bother you that the density is “number per unit length” instead of number per unit (length³)? (Probably knot.) Because the string is essentially a one-dimensional object, the “volume” in our knot density is a length.

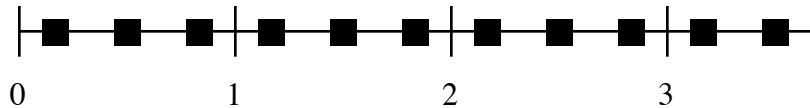


Figure B.1: Knots in a string. (What kind of knots?)

This illustrates an important concept:

The “volume” in a density may not be a 3D spatial volume and thus may not have units of [length]³. It could be simpler (just length) or more complicated (with an arbitrarily large number of dimensions, not limited to 3).

Let us continue to explore our knots-in-string example. If we plotted knot density as a function of position along our string, and we were faithful to the details shown in Fig. B.1, we would obtain the plot shown in Fig. B.2.

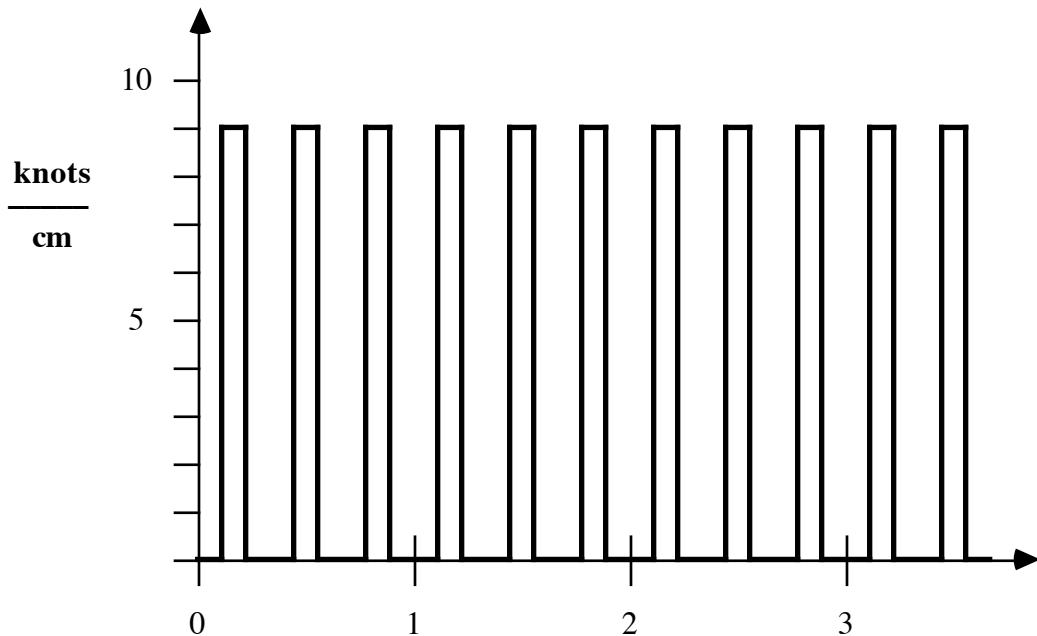


Figure B.2: Detailed knot density as a function of position along the string of Fig. B.1

This is a highly detailed plot of knot density. It shows where each individual knot is, and you can deduce that each knot is 1/9 cm long. We would not normally need such a detailed picture of a density. (Imagine the analogous picture of mass density in a material, knowing that most of the volume of an atom is essentially empty space!) We usually want densities

that are really averages over small volumes. For example, Fig. B.3 might be a more useful picture of knot density in our string.



Figure B.3: Knot density as a function of position along the string of Fig. B.1, using averages over small portions of the phase-space volume.

Figure B.3 tells us that the knot density is uniform, with a value of 3 knots/cm. It is obtained by averaging the detailed plot over small intervals.

If interpreted literally, our second picture of knot density implies that the knots are a continuum. We know this is not true, just as we know that matter is not a continuum. However, provided we never need to know exactly how many knots are in a very small interval, the simpler picture is adequate. For example, if we only cared how many knots were in one-kilometer intervals of string, our simpler picture would give the same answer as the detailed and complicated one.

As a second example of a density, let us consider population density:

$$\text{population density} = \text{people}/\text{mi}^2$$

In this example, the “volume” is spatial area. Again, we rarely need to know a detailed, microscopic picture of population density. Such a picture would show a zero density almost everywhere, with a spike at each person’s current location, and it would change constantly.

A third example is number density of atoms:

$$\text{atom number density} = \text{atoms}/\text{cm}^3$$

Here the volume is the spatial volume that we all know and love. Number density is much like mass density.

B.3 Definition

DEFINITION

Every density is a limit. We define the density of things at some point \vec{x} in some phase space as follows:

$$d(\vec{x}) = \lim_{V \rightarrow \text{small}} \left[\frac{\text{expected number of things in } V \text{ containing } \vec{x}}{V} \right] \quad (\text{B.1})$$

where \vec{x} is a point in phase space and V is a portion of phase-space volume that contains \vec{x} .

B.4 More Examples

Consider gas molecules bouncing about in a room. Suppose we do not care where in the room they are, but we want to know about their velocities. This is described by a density in three-dimensional velocity space:

$$\text{molecular velocity density} = \text{molecules} / (\text{cm}^3/\text{s}^3)$$

See Figure B.4.

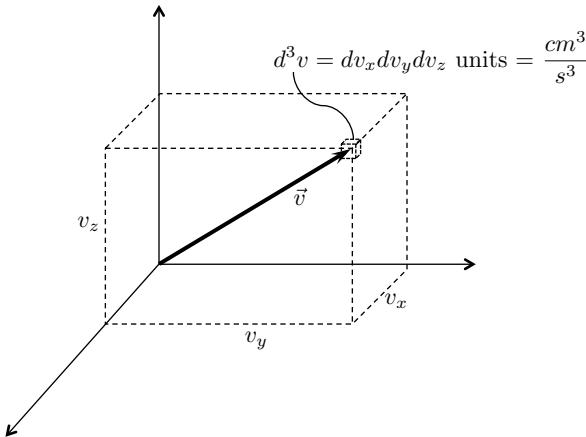


Figure B.4: A velocity vector (\vec{v}) defines a point (with coordinates (v_x, v_y, v_z)) in a 3D “velocity space.” A differential element of phase-space volume is a “velocity cube,” $dv_x dv_y dv_z$, which has units of length³/time³.

Define the molecular velocity density function as follows:

$$D(v_x, v_y, v_z) dv_x dv_y dv_z \equiv \text{number of molecules whose velocities are in the intervals } dv_x \text{ containing } v_x, dv_y \text{ containing } v_y, \text{ and } dv_z \text{ containing } v_z. \quad (\text{B.2})$$

In more compact notation:

$$D(\vec{v}) d^3v \equiv \text{number of molecules whose velocities are in } d^3v \text{ containing } \vec{v}. \quad (\text{B.3})$$

To find how many molecules are going upward, we could integrate this density over all x velocities, all y velocities, and all positive z velocities:

$$\text{upward-going molecules} = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_0^{\infty} dv_z D(\vec{v}). \quad (\text{B.4})$$

Note how the units work out:

$$\text{molecules} = \frac{cm}{s} \frac{cm}{s} \frac{cm}{s} \frac{\text{molecules}}{cm^3/s^3}. \quad (\text{B.5})$$

We encounter densities in other “volumes” as well. For example, the angular neutron density is a density in position and velocity. We often replace velocity by the equivalent variables energy and direction. In this case:

$$\text{neutron density} = \text{neutrons} / (\text{cm}^3 \cdot \text{MeV. ster}) .$$

We define this energy- and direction-dependent neutron density function as follows:

$$n(x, y, z, \theta, \gamma, E) dx dy dz \sin \theta d\theta d\gamma dE \equiv \text{number of neutrons whose position coordinates are in the intervals } dx \text{ containing } x, dy \text{ containing } y, \text{ and } dz \text{ containing } z, \text{ whose direction coordinates are in the intervals } d\theta \text{ containing } \theta \text{ and } d\gamma \text{ containing } \gamma, \text{ and whose energies are in the interval } dE \text{ containing } E. \quad (\text{B.6})$$

In shorthand,

$$n(\vec{r}, \vec{\Omega}, E) d^3r d\Omega dE \equiv \begin{aligned} &\text{number of neutrons with positions} \\ &\text{in } d^3r \text{ containing } \vec{r}, \\ &\text{directions in } d\Omega \text{ containing } \vec{\Omega}, \text{ and} \\ &\text{energies in } dE \text{ containing } E. \end{aligned} \quad (\text{B.7})$$

The phase-space volume here is a six-dimensional quantity: three dimensions in position, two in direction, one in energy. Integrating this density over all energies and directions gives an “ordinary” neutron density—the number of neutrons per cm³:

$$n_{\text{tot}}(\vec{r}) = \int_0^\infty dE \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\gamma n(\vec{r}, \vec{\Omega}, E). \quad (\text{B.8})$$

Note how the units work out:

$$\frac{n}{\text{cm}^3} = \text{MeV. ster.} \frac{n}{\text{cm}^3 \cdot \text{MeV. ster}}. \quad (\text{B.9})$$

B.5 Expected Values

Our general definition of a density uses the term “expected number.” This deserves discussion. Imagine a small volume in a nuclear reactor—say a few cubic angstroms—that is operating in “steady state.” At a given instant it would be unlikely that a neutron would be in this tiny volume. However, if we observed this volume over a long time we would find that the average number of neutrons in the volume is not zero. This average number would be the “expected number” of neutrons in the volume. It would be less than 1.0, but this should not bother us.

Notice that “steady state” does not apply microscopically. In any phase-space volume we can expect fluctuations about the expected value of the neutron population. The fluctuations become insignificant relative to the expected value if the expected value is large, but become large relative to the expected value if the expected value is small. “Steady state” refers to the macroscopic behavior of the system (e.g., reactor). It is an ideal state that is never realized perfectly in practice.

Now consider the same volume in the same reactor but remove the steady-state restriction, so that even macroscopically we allow things to be changing significantly in the reactor. Now imagine a large number of macroscopically identical reactors, all macroscopically doing the same thing. Then take an “ensemble average”—an average over all of the reactors—of the

number of neutrons in the tiny volume. This average is the “expected number” of neutrons in the volume.

When we write down our conservation equations, they will be in terms of densities that describe expected values. A solution is perhaps best thought of as representing an average over a large number of systems. If all we care about is integral quantities for which the expected value involves a large number of neutrons (such as the power generated by an entire fuel rod in a reactor operating at full power), then we can expect the fluctuations (either temporal or system-to-system) to be small compared to the expected value, and the integral quantity for a particular reactor will be about the same as the average over a large number of reactors.

B.6 Changes of Variables

Densities are functions of independent variables. The independent variables are coordinates that define points in phase space. We can use different coordinate systems to describe the same phase space, and often a problem can be greatly simplified by choosing a coordinate system that is well matched to it. We therefore must understand the relationships among density functions that describe the same physical quantity in different coordinate systems.

Changing coordinates (independent variables) in a density function is closely related to changing variables in an integral, because the integral of a density over a phase-space subvolume is the number of “things” in that subvolume, which is a physical quantity that must be independent of the coordinate system that is used.

B.6.1 Single independent variable (one-dimensional phase spaces)

Consider the knot density function in a previous example. This density must have units of knots per unit length, but any units can be chosen for the length variable. Let $d(x)$ be the density expressed in knots per inch, with x expressed in inches, and $s(\ell)$ be the density in knots per cm, with ℓ expressed in cm. Then we must have:

$$d(x)dx = s(\ell)d\ell . \quad (\text{B.10})$$

This illustrates a fundamental truth in changes of variables for density functions: the number of “things” must be the same no matter what variables are used to describe the density. That is, we must begin with a statement of equality in which the terms express a number of things. This means we will always have differential elements of phase-space volume, expressed in the different variables, in our statement that relates two densities in different variables.

What is the relation between d and s ? It is easy to find:

$$d(x) = s(\ell) \left| \frac{d\ell}{dx} \right| \approx 2.54s(\ell) , \quad (\text{B.11})$$

because there are ≈ 2.54 cm/inch. Of course, this case is simple enough that you could have written down the relation from the beginning, but it illustrates a process that can be used in more complicated cases. Remark: the absolute value operation is included because densities are defined to be non-negative, whereas it is possible to choose coordinate systems such that the derivative in question is negative.

Consider now a density in a speed variable, and call the speed-dependent density $f(v)$. Suppose we wish to use kinetic energy instead of speed, and we define the energy-dependent density to be $g(E)$, where $E = mv^2/2$. Remember, these densities are describing the same physical quantity (which could be neutrons in this case). The relation between the two densities is as before:

$$f(v)dv = g(E)dE \quad \Rightarrow \quad f(v) = g(E) \left| \frac{dE}{dv} \right| = mv g(E) = \sqrt{2Em} g(E) . \quad (\text{B.12})$$

Remark: We often commit a notational sin and use the same symbol for a density in two different independent variables, even though the density functions are obviously different. That is, we might write

$$n(v) = \sqrt{2Em} n(E) . \quad (\text{B.13})$$

We do this even though the n function on the left is different from the n function on the right. This has been going on for decades in studies of neutron transport, so I have no hope of reversing it. If we are going to use the same symbol, n , for these two different functions, it would be less confusing if we at least added descriptive subscripts, such as:

$$n_{\text{speed}}(v)dv = \sqrt{2Em} n_{\text{energy}}(E) . \quad (\text{B.14})$$

But neither we nor other authors will consistently do this. Therefore, one must always be careful to figure out from context exactly what density function (i.e., what coordinate system) is being used in any given set of equations.

Remark: Densities expressed in different coordinate systems have different units. In the example above, the speed-dependent neutron density contains per-(length/time) units, whereas the energy-dependent neutron density contains per-energy units.

B.6.2 Multiple independent variables (multi-dimensional phase spaces)

As is the case in multi-dimensional integrals with changes of variables, the relation between densities in multi-dimensional phase spaces involves a Jacobian. Consider a density function $f(\vec{x})$ in an N -dimensional phase space, with coordinates x_1, x_2, \dots, x_N , and another density function $g(\vec{y})$ describing the same physical quantity in terms of coordinates y_1, y_2, \dots, y_N . Then we have

$$f(\vec{x})dx_1dx_2\dots dx_N = g(\vec{y})dy_1dy_2\dots dy_N , \quad (\text{B.15})$$

and

$$f(\vec{x}) = g(\vec{y}) |\mathbb{J}| , \quad (\text{B.16})$$

where

$$\begin{aligned} \mathbb{J} &\equiv \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_N}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_N} & \cdots & \frac{\partial y_N}{\partial x_N} \end{bmatrix} \\ &= \text{the "Jacobian" of the transformation} \end{aligned} \quad (\text{B.17})$$

and $|\mathbb{J}|$ is the absolute value of the determinant of the Jacobian matrix.

Consider for example a mass density expressed in terms of Cartesian coordinates, $\rho_{xyz}(x, y, z)$, and the same density expressed in terms of polar spatial coordinates, $\rho_{\text{polar}}(r, \theta, \gamma)$. Then we have

$$\rho_{\text{polar}}(r, \theta, \gamma) = \rho_{xyz}(x, y, z) |\mathbb{J}| , \quad (\text{B.18})$$

where

$$|\mathbb{J}| \equiv \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \gamma} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \gamma} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \gamma} \end{vmatrix} . \quad (\text{B.19})$$

Observe that

$$x = r \sin \theta \cos \gamma , \quad (\text{B.20})$$

$$y = r \sin \theta \sin \gamma , \quad (\text{B.21})$$

$$z = r \cos \theta . \quad (\text{B.22})$$

Here we have defined the polar angle to be θ and the azimuthal angle to be γ . Now we turn the crank:

$$|\mathcal{J}| \equiv \begin{vmatrix} \sin \theta \cos \gamma & r \cos \theta \cos \gamma & -r \sin \theta \sin \gamma \\ \sin \theta \sin \gamma & r \cos \theta \sin \gamma & r \sin \theta \cos \gamma \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} = \dots = r^2 \sin \theta \quad (\text{B.23})$$

It follows that

$$\rho_{\text{polar}}(r, \theta, \gamma) = \rho_{xyz}(x, y, z) r^2 \sin \theta = \rho_{xyz}(x, y, z) (x^2 + y^2 + z^2) \sin \theta . \quad (\text{B.24})$$

B.7 Observations

We conclude our discussion of densities (distributions) with some summary observations:

- It makes no sense to ask how many objects exist at a point in phase space. It does make sense to ask what the density is at a point in phase space.
- To get a number, integrate a density over a phase-space volume.
- The “volume” is not always in x - y - z space.
- We treat densities much as we would if the things they describe were continuous. (See the two different plots of knot density in a string. We use the second view.) This does not mean that we ignore the fact that particles exist as discrete packets. We are simply using “expected values” in our definition of densities.
- A density has units of things per unit phase-space volume. The units will differ if we choose different coordinates to represent points in phase space. The transformation of a density from one coordinate system to another involves the determinant of a Jacobian matrix.

Appendix C

Neutronics: Problems, Models, & Conditions

C.1 Introduction

You have encountered what may seem to be a bewildering array of equations that purport to represent what may seem to be a bewildering array of different kinds of neutronics problems. Here we attempt to collect and organize these, hopefully providing a helpful reference and helping you organize all of this in your mind.

There are three basic types of neutronics problems:

1. **Time-dependent**
2. **Steady-state**
3. **Eigenvalue**

The most commonly used “mathematical models” for any of these types of problems are

1. Neutron **transport** equations for the **angular flux**.
2. Neutron **diffusion** equations for the **scalar flux**.

For any combination of problem type (time-dependent, steady-state, or eigenvalue) and basic mathematical model (transport or diffusion), an additional modeling approximation is usually made with the energy variable. This additional approximation is called

the **multigroup** approximation.

The multigroup approximation converts equations that have a continuous energy variable (which essentially means the solution is sought at an infinite number of energy values) to a finite set of equations, with one equation for each “group” of neutrons, where neutrons are defined to be in one “group” if their kinetic energies are in that group’s energy interval.

The multigroup approximation may use any number of groups. This could be a single energy group, as in our equations for the thermal flux in Chapter 9, or two energy groups (which is fairly common), or tens of thousands of energy groups (which some neutronics codes routinely do), or even more (which is rare in neutronics, but possible). Using more energy groups usually gives a more accurate solution but requires the solution of more coupled equations (one equation per group). The equations are coupled via scattering and fission terms—the in-scattering and fission sources in a given group depend on fluxes in other groups.

C.2 Time-Dependent Problems

If any nuclide in a time-dependent neutronics problem can fission, then a complete description of the problem must separately account for neutrons that are emitted promptly from fission and those that are delayed, appearing only when certain fission products or their daughters decay by neutron emission.

C.2.1 Time-Dependent Energy-Dependent Transport

The time-dependent continuous-energy neutron transport equation is:

$$\begin{aligned}
 & \frac{1}{v(E)} \frac{\partial \psi(\vec{r}, E, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}, t) + \Sigma_t(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
 &= S_{\text{fixed}}(\vec{r}, E, \vec{\Omega}, t) + \frac{1}{4\pi} \sum_{i=1}^{\# \text{ of dnp types}} \chi_i(E) \lambda_i C_i(\vec{r}, t) \\
 &+ \frac{\chi_p(E)}{4\pi} \int_0^\infty dE_i \nu_p(\vec{r}, E_i, t) \Sigma_f(\vec{r}, E_i, t) \phi(\vec{r}, E_i, t) \\
 &+ \int_0^\infty dE_i \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, E_i \rightarrow E, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E_i, \vec{\Omega}_i, t). \tag{C.1}
 \end{aligned}$$

Here ψ is the angular flux:

$$\psi \text{ units} = (\text{n-cm})/(\text{cm}^3\text{-MeV-ster-s}),$$

where “ster” stands for steradian, which is a unit of “solid angle.” A per-steradian quantity is a density in direction space. The angular flux is speed times a density in position (3 coordinates), energy (one), and direction (two). That is, it is a density in a six-dimensional phase space.

C.2.2 Time-Dependent Energy-Dependent Diffusion

If we integrate the time-dependent energy-dependent transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the following time-dependent energy-dependent **neutron conservation equation**, without approximation:

$$\begin{aligned}
 & \frac{1}{v(E)} \frac{\partial \phi(\vec{r}, E, t)}{\partial t} + \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) + \Sigma_t(\vec{r}, E, t) \phi(\vec{r}, E, t) \\
 &= S_{\text{fixed},0}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ of dnp types}} \chi_i(E) \lambda_i C_i(\vec{r}, t) \\
 &+ \chi_p(E) \int_0^\infty dE_i \nu_p(\vec{r}, E_i, t) \Sigma_f(\vec{r}, E_i, t) \phi(\vec{r}, E_i, t) \\
 &+ \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E, t) \phi(\vec{r}, E_i, t) . \quad [\text{conservation equation}] \quad (\text{C.2})
 \end{aligned}$$

Here $S_{\text{fixed},0}$ is the direction-integrated source. [S_{fixed} in the transport equation is neutrons per ($\text{cm}^3\text{-s-MeV-ster}$). $S_{\text{fixed},0}$ is neutrons per ($\text{cm}^3\text{-s-MeV}$).] If we then replace \vec{J} using the diffusion approximation, we obtain the time-dependent energy-dependent neutron diffusion equation:

$$\begin{aligned} & \frac{1}{v(E)} \frac{\partial \phi(\vec{r}, E, t)}{\partial t} - \vec{\nabla} \cdot [D(\vec{r}, E, t) \vec{\nabla} \phi(\vec{r}, E, t)] + \Sigma_t(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &= S_{\text{fixed},0}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ of dnp types}} \chi_i(E) \lambda_i C_i(\vec{r}, t) \\ &+ \chi_p(E) \int_0^\infty dE_i \nu_p(\vec{r}, E_i, t) \Sigma_f(\vec{r}, E_i, t) \phi(\vec{r}, E_i, t) \\ &+ \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E, t) \phi(\vec{r}, E_i, t) . \end{aligned} \quad (\text{C.3})$$

Notice that the only difference between the diffusion equation and the (essentially exact) neutron conservation equation is in the leakage term, which in the diffusion equation contains the (approximate) replacement of $-D\vec{\nabla}\phi$ for \vec{J} . This is true for the time-dependent, steady-state, and eigenvalue problems, and it is true for continuous-energy equations or the multigroup equations.

C.2.3 Time-Dependent Multigroup Transport

The time-dependent multigroup neutron transport equations are:

$$\begin{aligned}
 & \frac{1}{v_g} \frac{\partial \psi_g(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} \psi_g(\vec{r}, \vec{\Omega}, t) + \Sigma_{t,g}(\vec{r}, t) \psi_g(\vec{r}, \vec{\Omega}, t) \\
 &= S_{\text{fixed},g}(\vec{r}, \vec{\Omega}, t) + \frac{1}{4\pi} \sum_{i=1}^{\# \text{ of dnp types}} \chi_{i,g} \lambda_i C_i(\vec{r}, t) \\
 &+ \frac{\chi_{p,g}}{4\pi} \sum_{g'=1}^G \nu_{p,g'}(\vec{r}, t) \Sigma_{f,g'}(\vec{r}, t) \phi_{g'}(\vec{r}, t) \\
 &+ \sum_{g'=1}^G \iint_{4\pi} d\Omega_i \Sigma_{s,g' \rightarrow g}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_{g'}(\vec{r}, \vec{\Omega}_i, t), \quad g = 1, \dots, G. \quad (\text{C.4})
 \end{aligned}$$

Here ψ_g and ϕ_g are the angular and scalar fluxes integrated over the g -th energy interval:

$$\begin{aligned}
 \psi_g(\vec{r}, \vec{\Omega}, t) &\equiv \int_{E_g}^{E_{g-1}} dE \psi(\vec{r}, E, \vec{\Omega}, t) \quad [\text{units} = (\text{n-cm})/(\text{cm}^3\text{-ster-s})]. \\
 \phi_g(\vec{r}, t) &\equiv \int_{E_g}^{E_{g-1}} dE \phi(\vec{r}, E, t) \quad [\text{units} = (\text{n-cm})/(\text{cm}^3\text{-s})].
 \end{aligned}$$

Notice that the group-integrated fluxes are not densities in energy, but are densities in the other dimensions: five-dimensional phase space (position and direction) for angular flux, and three-dimensional phase space (position) for scalar flux.

The multigroup equations contain cross sections and other parameters that have been averaged over the energy ranges associated with the neutron groups. This includes cross sections, speeds (v_g), and ν values.

The g -th multigroup equation is coupled to the other equations through scattering and fission terms. This makes physical sense—scattering can cause a neutron to leave one group and become a member of another, and fission can be caused by a neutron in one group but then emit neutrons into other groups.

C.2.4 Time-Dependent Multigroup Diffusion

If we integrate the time-dependent multigroup transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the following time-dependent multigroup **neutron conservation equations**, without approximation:

$$\begin{aligned}
 & \frac{1}{v_g} \frac{\partial \phi_g(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot \vec{J}_g(\vec{r}, t) + \Sigma_{t,g}(\vec{r}, t) \phi_g(\vec{r}, t) \\
 &= S_{\text{fixed},g,0}(\vec{r}, t) + \sum_{i=1}^{\# \text{ of dnp types}} \chi_{i,g} \lambda_i C_i(\vec{r}, t) \\
 &+ \chi_{p,g} \sum_{g'=1}^G \nu_{p,g'}(\vec{r}, t) \Sigma_{f,g'}(\vec{r}, t) \phi_{g'}(\vec{r}, t) \\
 &+ \sum_{g'=1}^G \Sigma_{s,g' \rightarrow g}(\vec{r}, t) \phi_{g'}(\vec{r}, t) , \quad g = 1, \dots, G . \quad [\text{conservation equations}] \quad (\text{C.5})
 \end{aligned}$$

Here $S_{\text{fixed},0}$ is the direction-integrated source. [$S_{\text{fixed},g}$ in the transport equation is neutrons per ($\text{cm}^3\text{-s-ster}$). $S_{\text{fixed},g,0}$ is neutrons per ($\text{cm}^3\text{-s}$).] If we then replace \vec{J}_g using the diffusion approximation, we obtain the time-dependent multigroup neutron diffusion equations:

$$\begin{aligned}
 & \frac{1}{v_g} \frac{\partial \phi_g(\vec{r}, t)}{\partial t} - \vec{\nabla} \cdot [D_g(\vec{r}, t) \vec{\nabla} \phi_g(\vec{r}, t)] + \Sigma_{t,g}(\vec{r}, t) \phi_g(\vec{r}, t) \\
 &= S_{\text{fixed},g,0}(\vec{r}, t) + \sum_{i=1}^{\# \text{ of dnp types}} \chi_{i,g} \lambda_i C_i(\vec{r}, t) \\
 &+ \chi_{p,g} \sum_{g'=1}^G \nu_{p,g'}(\vec{r}, t) \Sigma_{f,g'}(\vec{r}, t) \phi_{g'}(\vec{r}, t) \\
 &+ \sum_{g'=1}^G \Sigma_{s,g' \rightarrow g}(\vec{r}, t) \phi_{g'}(\vec{r}, t) , \quad g = 1, \dots, G . \tag{C.6}
 \end{aligned}$$

Notice that the only difference between the diffusion equations and the (essentially exact) neutron conservation equations is in the leakage term, where the approximation $\vec{J} \approx -D\vec{\nabla}\phi$ has been made.

C.2.5 Time-Dependent Two-Group Transport

The time-dependent two-group neutron transport equations are just a particular instance of the multigroup equations. The following two equations are nothing more than the multigroup equations given above, spelled out for each group for the particular case of $G = 2$.

$$\begin{aligned} & \frac{1}{v_1} \frac{\partial \psi_1(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} \psi_1(\vec{r}, \vec{\Omega}, t) + \Sigma_{t,1}(\vec{r}, t) \psi_1(\vec{r}, \vec{\Omega}, t) \\ &= S_{\text{fixed},1}(\vec{r}, \vec{\Omega}, t) + \frac{1}{4\pi} \sum_{i=1}^{\# \text{ of dnp types}} \chi_{i,1} \lambda_i C_i(\vec{r}, t) \\ &+ \frac{\chi_{p,1}}{4\pi} [\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)] \\ &+ \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_1(\vec{r}, \vec{\Omega}_i, t) + \Sigma_{s,2 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_2(\vec{r}, \vec{\Omega}_i, t) \right], \quad (\text{C.7}) \end{aligned}$$

$$\begin{aligned} & \frac{1}{v_2} \frac{\partial \psi_2(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} \psi_2(\vec{r}, \vec{\Omega}, t) + \Sigma_{t,2}(\vec{r}, t) \psi_2(\vec{r}, \vec{\Omega}, t) \\ &= S_{\text{fixed},2}(\vec{r}, \vec{\Omega}, t) + \frac{1}{4\pi} \sum_{i=1}^{\# \text{ of dnp types}} \chi_{i,2} \lambda_i C_i(\vec{r}, t) \\ &+ \frac{\chi_{p,2}}{4\pi} [\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)] \\ &+ \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_1(\vec{r}, \vec{\Omega}_i, t) + \Sigma_{s,2 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_2(\vec{r}, \vec{\Omega}_i, t) \right]. \quad (\text{C.8}) \end{aligned}$$

The two energy intervals in a two-group problem are usually chosen to be:

- interval 1: $E \in (E_{th}, E_0)$, where $E_0 \approx 10\text{-}12 \text{ MeV}$,
- interval 2: $E \in (0, E_{th})$, where $E_{th} \approx 0.5\text{-}4 \text{ MeV}$.

With these choices, essentially all fission neutrons are born in group 1, the “fast” group, and there is very little scattering from group 2 to group 1 (very little “upscattering” from the “thermal” group to the “fast” group.) We rewrite the equations explicitly making the approximations that upscattering is ignored and all neutrons emerging from fission emerge in the fast group:

$$\begin{aligned}
 & \frac{1}{v_1} \frac{\partial \psi_1(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} \psi_1(\vec{r}, \vec{\Omega}, t) + \Sigma_{t,1}(\vec{r}, t) \psi_1(\vec{r}, \vec{\Omega}, t) \\
 &= S_{\text{fixed},1}(\vec{r}, \vec{\Omega}, t) + \frac{1}{4\pi} \sum_{i=1}^{\# \text{ of dnp types}} \lambda_i C_i(\vec{r}, t) \\
 &+ \frac{1}{4\pi} [\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)] \\
 &+ \iint_{4\pi} d\Omega_i \Sigma_{s,1 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_1(\vec{r}, \vec{\Omega}_i, t), \tag{C.9}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{v_2} \frac{\partial \psi_2(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} \psi_2(\vec{r}, \vec{\Omega}, t) + \Sigma_{t,2}(\vec{r}, t) \psi_2(\vec{r}, \vec{\Omega}, t) \\
 &= S_{\text{fixed},2}(\vec{r}, \vec{\Omega}, t) \\
 &+ \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_1(\vec{r}, \vec{\Omega}_i, t) + \Sigma_{s,2 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi_2(\vec{r}, \vec{\Omega}_i, t) \right]. \tag{C.10}
 \end{aligned}$$

These are the **time-dependent two-group transport equations** with the usual approximations made because of the usual choice of the energy boundary between the two groups.

C.2.6 Time-Dependent Two-Group Diffusion

If we integrate the time-dependent two-group transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the following time-dependent two-group **neutron conservation equations**, without approximation:

$$\begin{aligned}
 & \frac{1}{v_1} \frac{\partial \phi_1(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot \vec{J}_1(\vec{r}, t) + \Sigma_{t,1}(\vec{r}, t)\phi_1(\vec{r}, t) \\
 &= S_{\text{fixed},1,0}(\vec{r}, t) + \sum_{i=1}^{\# \text{ of dnp types}} \chi_{i,1}\lambda_i C_i(\vec{r}, t) \\
 &+ \chi_{p,1} [\nu_{p,1}(\vec{r}, t)\Sigma_{f,1}(\vec{r}, t)\phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t)\Sigma_{f,2}(\vec{r}, t)\phi_2(\vec{r}, t)] \\
 &+ [\Sigma_{s,1 \rightarrow 1}(\vec{r}, t)\phi_1(\vec{r}, t) + \Sigma_{s,2 \rightarrow 1}(\vec{r}, t)\phi_2(\vec{r}, t)] , \quad \text{[conservation equation]} \tag{C.11}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{v_2} \frac{\partial \phi_2(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot \vec{J}_2(\vec{r}, t) + \Sigma_{t,2}(\vec{r}, t)\phi_2(\vec{r}, t) \\
 &= S_{\text{fixed},2,0}(\vec{r}, t) + \sum_{i=1}^{\# \text{ of dnp types}} \chi_{i,2}\lambda_i C_i(\vec{r}, t) \\
 &+ \chi_{p,2} [\nu_{p,1}(\vec{r}, t)\Sigma_{f,1}(\vec{r}, t)\phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t)\Sigma_{f,2}(\vec{r}, t)\phi_2(\vec{r}, t)] \\
 &+ [\Sigma_{s,1 \rightarrow 2}(\vec{r}, t)\phi_1(\vec{r}, t) + \Sigma_{s,2 \rightarrow 2}(\vec{r}, t)\phi_2(\vec{r}, t)] . \quad \text{[conservation equation]} \tag{C.12}
 \end{aligned}$$

If we choose the two energy intervals in the two-group problem as usual:

- interval 1: $E \in (E_{th}, E_0)$, where $E_0 \approx 10\text{-}12 \text{ MeV}$,
- interval 2: $E \in (0, E_{th})$, where $E_{th} \approx 0.5\text{-}4 \text{ MeV}$,

and we make the diffusion approximation, then we obtain the usual form of the **time-dependent two-group diffusion equations**:

$$\begin{aligned} & \frac{1}{v_1} \frac{\partial \phi_1(\vec{r}, t)}{\partial t} - \vec{\nabla} \cdot \left[D_1(\vec{r}, t) \vec{\nabla} \phi_1(\vec{r}, t) \right] + [\Sigma_{t,1}(\vec{r}, t) - \Sigma_{s,1 \rightarrow 1}(\vec{r}, t)] \phi_1(\vec{r}, t) \\ &= S_{\text{fixed},1,0}(\vec{r}, t) + \sum_{i=1}^{\# \text{ of dnp types}} \lambda_i C_i(\vec{r}, t) \\ &+ [\nu_{p,1}(\vec{r}, t) \Sigma_{f,1}(\vec{r}, t) \phi_1(\vec{r}, t) + \nu_{p,2}(\vec{r}, t) \Sigma_{f,2}(\vec{r}, t) \phi_2(\vec{r}, t)] , \end{aligned} \quad (\text{C.13})$$

$$\begin{aligned} & \frac{1}{v_2} \frac{\partial \phi_2(\vec{r}, t)}{\partial t} - \vec{\nabla} \cdot \left[D_2(\vec{r}, t) \vec{\nabla} \phi_2(\vec{r}, t) \right] + \Sigma_{a,2}(\vec{r}, t) \phi_2(\vec{r}, t) \\ &= S_{\text{fixed},2,0}(\vec{r}, t) + \Sigma_{s,1 \rightarrow 2}(\vec{r}, t) \phi_1(\vec{r}, t) . \end{aligned} \quad (\text{C.14})$$

Notice that the within-group scattering terms have been subtracted from both sides of each equation.

C.2.7 Time-Dependent One-Group Transport

The time-dependent one-group neutron transport equation is just a particular instance of the multigroup equations for the particular case of $G = 1$. In this case we do not need a “group” subscript because there are not multiple groups to keep track of:

$$\begin{aligned} & \frac{1}{v} \frac{\partial \psi(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, \vec{\Omega}, t) + \Sigma_t(\vec{r}, t) \psi(\vec{r}, \vec{\Omega}, t) \\ &= S_{\text{fixed}}(\vec{r}, \vec{\Omega}, t) + \frac{1}{4\pi} \sum_{i=1}^{\# \text{ of dnp types}} \lambda_i C_i(\vec{r}, t) \\ &+ \frac{1}{4\pi} \nu_p(\vec{r}, t) \Sigma_f(\vec{r}, t) \phi(\vec{r}, t) + \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}, t) \psi(\vec{r}, \vec{\Omega}_i, t) . \end{aligned} \quad (\text{C.15})$$

C.2.8 Time-Dependent One-Group Diffusion

If we integrate the time-dependent one-group transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the following time-dependent one-group **neutron conservation equation**, without approximation:

$$\begin{aligned} & \frac{1}{v} \frac{\partial \phi(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot \vec{J}(\vec{r}, t) + \Sigma_a(\vec{r}, t) \phi(\vec{r}, t) \\ &= S_{\text{fixed},0}(\vec{r}, t) + \sum_{i=1}^{\# \text{ of dnp types}} \lambda_i C_i(\vec{r}, t) + \nu_p(\vec{r}, t) \Sigma_f(\vec{r}, t) \phi(\vec{r}, t) . \quad [\text{conservation}] \end{aligned} \quad (\text{C.16})$$

Notice that we have subtracted the scattering term from both sides of the equation.

If we replace \vec{J} using the diffusion approximation, we obtain the **time-dependent one-group neutron diffusion equation**:

$$\begin{aligned} & \frac{1}{v} \frac{\partial \phi(\vec{r}, t)}{\partial t} - \vec{\nabla} \cdot \left[D(\vec{r}, t) \vec{\nabla} \phi(\vec{r}, t) \right] + \Sigma_a(\vec{r}, t) \phi(\vec{r}, t) \\ &= S_{\text{fixed},0}(\vec{r}, t) + \sum_{i=1}^{\# \text{ of dnp types}} \lambda_i C_i(\vec{r}, t) + \nu_p(\vec{r}, t) \Sigma_f(\vec{r}, t) \phi(\vec{r}, t) . \end{aligned} \quad (\text{C.17})$$

C.2.9 Precursor Equations for Time-Dependent Problems

All of the time-dependent transport, conservation, and diffusion equations contain delayed-neutron emission terms, which contain precursor concentrations ($C_i(\vec{r}, t)$) at every position and time of interest. The equation describing the concentration (atoms per unit volume) of the i -th type of delayed-neutron precursor is:

$$\frac{\partial C_i(\vec{r}, t)}{\partial t} = -\lambda_i C_i(\vec{r}, t) + \begin{cases} \int_0^\infty dE \nu_{d,i}(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t), & \text{continuous-energy} \\ \sum_{g=1}^G \nu_{d,i,g}(\vec{r}, t) \Sigma_{f,g}(\vec{r}, t) \phi_g(\vec{r}, t), & \text{multigroup} \end{cases} \quad (\text{C.18})$$

This holds for $i = 1, \dots$ number of precursor types.

C.2.10 Initial Conditions for Time-Dependent Problems

The equations for the delayed-neutron precursor concentrations, $\{C_i(\vec{r}, t), i = 1, \dots, I\}$, where I = number of precursor types, have first-order derivatives with respect to time. Well-posed time-dependent problems therefore require “initial conditions” for each precursor type at each spatial location. That is, the precursor concentrations must be known at some initial time, t_0 . We use subscript 0 to denote the known concentration at time t_0 :

$$C_i(\vec{r}, t_0) = C_{i,0}(\vec{r}) = \text{known function}, \quad i = 1, \dots, I, \quad \vec{r} \in V. \quad (\text{C.19})$$

The time-dependent transport and diffusion problems also contain first derivatives with respect to time. Well-posed time-dependent problems therefore require that the solution be known at some initial time for all points in the relevant phase space. We use subscript 0 to denote the known concentration at time t_0 :

$$\psi(\vec{r}, E, \vec{\Omega}, t_0) = \psi_0(\vec{r}, E, \vec{\Omega}) = \text{known}, \quad \vec{r} \in V, \text{ all } E, \text{ all } \vec{\Omega} \quad [\text{en-dep. transp.}] \quad (\text{C.20})$$

$$\psi_g(\vec{r}, \vec{\Omega}, t_0) = \psi_{g,0}(\vec{r}, \vec{\Omega}) = \text{known}, \quad \vec{r} \in V, \quad g = 1, \dots, G, \text{ all } \vec{\Omega} \quad [\text{MG transp.}] \quad (\text{C.21})$$

$$\phi(\vec{r}, E, t_0) = \phi_0(\vec{r}, E) = \text{known}, \quad \vec{r} \in V, \text{ all } E, \quad [\text{en-dep. difn.}] \quad (\text{C.22})$$

$$\phi_g(\vec{r}, t_0) = \phi_{g,0}(\vec{r}) = \text{known}, \quad \vec{r} \in V, \quad g = 1, \dots, G \quad [\text{MG difn.}] \quad (\text{C.23})$$

C.3 Steady-State Problems

In steady-state problems there are several simplifications relative to time-dependent problems:

1. The time variable disappears and all time derivatives disappear.
2. We do not have to separately keep up with prompt and delayed neutrons—time delays have no significance in steady state.

The first simplification is obvious, and the second should be clear on physical grounds. Delayed neutrons still appear with delays, but the rate at which they appear is exactly the rate at which precursors are formed. Thus, the equation is the same as if they appeared immediately. To see this mathematically recall the precursor equation, Eq. (C.18), and write it for the steady-state case:

$$\lambda_i C_i(\vec{r}) = \begin{cases} \int_0^\infty dE_i \nu_{d,i}(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) , & \text{continuous-energy} \\ \sum_{g'=1}^G \nu_{d,i,g'}(\vec{r}) \Sigma_{f,g'}(\vec{r}) \phi_{g'}(\vec{r}) , & \text{multigroup} \end{cases} \quad (\text{C.24})$$

Now add the prompt + delayed emission terms in the time-dependent continuous-energy transport or diffusion equation, dropping the time variable since we are considering the steady-state case, then substitute for the precursor decay rates using Eq. C.24:

$$\begin{aligned} & \sum_{i=1}^{\# \text{ of dnp types}} \chi_i(E) \lambda_i C_i(\vec{r}) + \chi_p(E) \int_0^\infty dE_i \nu_p(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ &= \sum_{i=1}^{\# \text{ of dnp types}} \chi_i(E) \int_0^\infty dE_i \nu_{d,i}(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ & \quad + \int_0^\infty dE_i \nu_p(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ &= \int_0^\infty dE_i \underbrace{\left[\chi_p(E) \nu_p(\vec{r}, E_i) + \sum_{i=1}^{\# \text{ of dnp types}} \chi_i(E) \nu_{d,i}(\vec{r}, E_i) \right]}_{\equiv \chi(E) \nu(\vec{r}, E_i)} \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ &= \chi(E) \int_0^\infty dE_i \nu(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \end{aligned} \quad (\text{C.25})$$

Similar manipulations hold for the multigroup cases. Thus, we see that in steady state, the delayed- and prompt-neutron emission terms add to form the total emission term from fission. All of the various forms of steady-state transport and diffusion equations now follow directly from the time-dependent equations. We catalog all of them below for completeness.

C.3.1 Steady-State Energy-Dependent Transport

The steady-state continuous-energy neutron transport equation is:

$$\begin{aligned}
 & \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}) + \Sigma_t(\vec{r}, E) \psi(\vec{r}, E, \vec{\Omega}) \\
 &= S_{\text{fixed}}(\vec{r}, E, \vec{\Omega}) + \frac{\chi(E)}{4\pi} \int_0^\infty dE_i \nu(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\
 &+ \int_0^\infty dE_i \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, E_i \rightarrow E, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi(\vec{r}, E_i, \vec{\Omega}_i) . \tag{C.26}
 \end{aligned}$$

C.3.2 Steady-State Energy-Dependent Diffusion

If we integrate the steady-state energy-dependent transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the time-dependent energy-dependent **neutron conservation equation**, without approximation. If we then insert the diffusion approximation we obtain the steady-state energy-dependent diffusion equation:

$$\begin{aligned}
 & -\vec{\nabla} \cdot [D(\vec{r}, E) \vec{\nabla} \phi(\vec{r}, E)] + \Sigma_t(\vec{r}, E) \phi(\vec{r}, E) \\
 &= S_{\text{fixed},0}(\vec{r}, E) + \chi(E) \int_0^\infty dE_i \nu(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\
 &+ \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E) \phi(\vec{r}, E_i) . \tag{C.27}
 \end{aligned}$$

C.3.3 Steady-State Multigroup Transport

The steady-state multigroup neutron transport equations are:

$$\begin{aligned}
 & \vec{\Omega} \cdot \vec{\nabla} \psi_g(\vec{r}, \vec{\Omega}) + \Sigma_{t,g}(\vec{r}) \psi_g(\vec{r}, \vec{\Omega}) \\
 &= S_{\text{fixed},g}(\vec{r}, \vec{\Omega}) + \frac{\chi_g}{4\pi} \sum_{g'=1}^G \nu_{g'}(\vec{r}) \Sigma_{f,g'}(\vec{r}) \phi_{g'}(\vec{r}) \\
 &+ \sum_{g'=1}^G \iint_{4\pi} d\Omega_i \Sigma_{s,g' \rightarrow g}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_{g'}(\vec{r}, \vec{\Omega}_i) , \quad g = 1, \dots, G . \quad (\text{C.28})
 \end{aligned}$$

C.3.4 Steady-State Multigroup Diffusion

If we integrate the steady-state multigroup transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the time-dependent multigroup **neutron conservation equations**, without approximation. If we then replace \vec{J}_g using the diffusion approximation, we obtain the steady-state multigroup neutron diffusion equations:

$$\begin{aligned}
 & -\vec{\nabla} \cdot [D_g(\vec{r}) \vec{\nabla} \phi_g(\vec{r})] + \Sigma_{t,g}(\vec{r}) \phi_g(\vec{r}) \\
 &= S_{\text{fixed},g,0}(\vec{r}) + \chi_g \sum_{g'=1}^G \nu_{g'}(\vec{r}) \Sigma_{f,g'}(\vec{r}) \phi_{g'}(\vec{r}) \\
 &+ \sum_{g'=1}^G \Sigma_{s,g' \rightarrow g}(\vec{r}) \phi_{g'}(\vec{r}) , \quad g = 1, \dots, G . \quad (\text{C.29})
 \end{aligned}$$

C.3.5 Steady-State Two-Group Transport

The steady-state two-group neutron transport equations are just a particular instance of the multigroup equations. The following two equations are nothing more than the multigroup equations given above, spelled out for each group for the particular case of $G = 2$.

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi_1(\vec{r}, \vec{\Omega}) + \Sigma_{t,1}(\vec{r}) \psi_1(\vec{r}, \vec{\Omega}) \\ = S_{\text{fixed},1}(\vec{r}, \vec{\Omega}) + \frac{\chi_1}{4\pi} [\nu_1(\vec{r}) \Sigma_{f,1}(\vec{r}) \phi_1(\vec{r}) + \nu_2(\vec{r}) \Sigma_{f,2}(\vec{r}) \phi_2(\vec{r})] \\ + \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_1(\vec{r}, \vec{\Omega}_i) + \Sigma_{s,2 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_2(\vec{r}, \vec{\Omega}_i) \right], \quad (\text{C.30}) \end{aligned}$$

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi_2(\vec{r}, \vec{\Omega}) + \Sigma_{t,2}(\vec{r}) \psi_2(\vec{r}, \vec{\Omega}) \\ = S_{\text{fixed},2}(\vec{r}, \vec{\Omega}, t) + \frac{\chi_2}{4\pi} [\nu_1(\vec{r}) \Sigma_{f,1}(\vec{r}) \phi_1(\vec{r}) + \nu_2(\vec{r}) \Sigma_{f,2}(\vec{r}) \phi_2(\vec{r})] \\ + \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_1(\vec{r}, \vec{\Omega}_i) + \Sigma_{s,2 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_2(\vec{r}, \vec{\Omega}_i) \right]. \quad (\text{C.31}) \end{aligned}$$

If the energy boundary between the two groups is chosen as usual, then $\chi_1 = 1$, $\chi_2 = 0$, and the $\Sigma_{s,2 \rightarrow 1}$ term is often dropped because it is small relative to other terms in the equations.

C.3.6 Steady-State Two-Group Diffusion

If we integrate the steady-state two-group transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the steady-state two-group **neutron conservation equations**, without approximation. If we then choose the two energy intervals in the two-group problem as usual:

- interval 1: $E \in (E_{th}, E_0)$, where $E_0 \approx 10\text{-}12$ MeV,
- interval 2: $E \in (0, E_{th})$, where $E_{th} \approx 0.5\text{-}4$ MeV,

then $\chi_1 = 1$, $\chi_2 = 0$, and the $\Sigma_{s,2 \rightarrow 1}$ term is often dropped because it is small relative to other terms in the equations. If we do this and then make the diffusion approximation, then we obtain the usual form of the **steady-state two-group diffusion equations**:

$$\begin{aligned} -\vec{\nabla} \cdot [D_1(\vec{r}) \vec{\nabla} \phi_1(\vec{r})] + [\Sigma_{t,1}(\vec{r}) - \Sigma_{s,1 \rightarrow 1}(\vec{r})] \phi_1(\vec{r}) \\ = S_{\text{fixed},1,0}(\vec{r}) + [\nu_1(\vec{r}) \Sigma_{f,1}(\vec{r}) \phi_1(\vec{r}) + \nu_2(\vec{r}) \Sigma_{f,2}(\vec{r}) \phi_2(\vec{r})] , \end{aligned} \quad (\text{C.32})$$

$$-\vec{\nabla} \cdot [D_2(\vec{r}) \vec{\nabla} \phi_2(\vec{r})] + \Sigma_{a,2}(\vec{r}) \phi_2(\vec{r}) = S_{\text{fixed},2,0}(\vec{r}) + \Sigma_{s,1 \rightarrow 2}(\vec{r}) \phi_1(\vec{r}) . \quad (\text{C.33})$$

C.3.7 Steady-State One-Group Transport

The steady-state one-group neutron transport equation is just a particular instance of the multigroup equations for the particular case of $G = 1$. In this case we do not need a “group” subscript because there are not multiple groups to keep track of:

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, \vec{\Omega}) + \Sigma_t(\vec{r}) \psi(\vec{r}, \vec{\Omega}) \\ = S_{\text{fixed}}(\vec{r}, \vec{\Omega}) + \frac{1}{4\pi} \nu(\vec{r}) \Sigma_f(\vec{r}) \phi(\vec{r}) + \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi(\vec{r}, \vec{\Omega}_i) . \end{aligned} \quad (\text{C.34})$$

C.3.8 Steady-State One-Group Diffusion

If we integrate the steady-state one-group transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the time-dependent one-group **neutron conservation equation**, without approximation. If we then replace \vec{J} using the diffusion approximation, we obtain the **steady-state one-group neutron diffusion equation**:

$$-\vec{\nabla} \cdot [D(\vec{r})\vec{\nabla}\phi(\vec{r})] + \Sigma_a(\vec{r})\phi(\vec{r}) = S_{\text{fixed},0}(\vec{r}) + \nu(\vec{r})\Sigma_f(\vec{r})\phi(\vec{r}) . \quad (\text{C.35})$$

C.4 k -Eigenvalue Problems

Consider any steady-state neutronics equation from the previous section. Suppose we collect the leakage, absorption, and scattering terms into an operator \mathbf{L} operating on the solution (where the solution is either ψ or $\{\psi_g\}$ or ϕ or $\{\phi_g\}$), then collect the fission terms into an operator \mathbf{P} operating on the solution. Then we can write any of the steady-state equations in the following form:

$$\mathbf{L}f = \mathbf{P}f + q , \quad (\text{C.36})$$

where f is the solution (either ψ or $\{\psi_g\}$ or ϕ or $\{\phi_g\}$) and q is the fixed source (either S_{fixed} or $\{S_{\text{fixed},g}\}$ or $S_{\text{fixed},0}$ or $\{S_{\text{fixed},0,g}\}$).

To form a k -eigenvalue equation, whether transport or diffusion and whether energy-dependent or multigroup, we begin with a steady-state equation, remove all fixed sources, and simply multiply each fission term by the factor $1/k$. In the operator notation introduced above, this means

$$\mathbf{L}f = \frac{1}{k} \mathbf{P}f . \quad (\text{C.37})$$

We can rearrange this:

$$[\mathbf{L}^{-1}] [\mathbf{P}] f = k f , \quad (\text{C.38})$$

which shows that k is an eigenvalue of the $\{[\text{Loss}]^{-1} [\text{Production}]\}$ operator—reminiscent of production rate divided by loss rate.

Important notes:

1. Eigenvalue problems never have fixed sources or any other “forcing” terms (such as prescribed angular fluxes or partial currents incident on a boundary).
2. Eigenvalue problems do not provide information about solution magnitude—only about the distribution, or “shape” of the solution.
3. Neutronics eigenvalue problems never have time variables.
4. A solution of an eigenvalue problem is both an eigenvalue and the associated eigenfunction. That is, it is an (eigenvalue, eigenfunction) pair.
5. For any finite reactor, there are infinitely many solutions of a given neutronics eigenvalue problem. That is, there will always be an infinite number of (eigenvalue, eigenfunction) pairs that satisfy the eigenvalue equation and its associate boundary conditions.
6. The largest of the k -eigenvalues is the **multiplication factor** of the reactor. The associated eigenfunction gives the **fundamental-mode** k -eigenproblem neutron distribution in the reactor.

C.4.1 k -Eigenvalue Energy-Dependent Transport

The k -eigenvalue continuous-energy neutron transport equation is:

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}) + \Sigma_t(\vec{r}, E) \psi(\vec{r}, E, \vec{\Omega}) \\ = \frac{1}{k} \frac{\chi(E)}{4\pi} \int_0^\infty dE_i \nu(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ + \int_0^\infty dE_i \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, E_i \rightarrow E, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi(\vec{r}, E_i, \vec{\Omega}_i) . \end{aligned} \quad (\text{C.39})$$

C.4.2 k -Eigenvalue Energy-Dependent Diffusion

If we integrate the k -eigenvalue energy-dependent transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the time-dependent energy-dependent **neutron conservation equation**, without approximation. If we then insert the diffusion approximation we obtain the k -eigenvalue energy-dependent diffusion equation:

$$\begin{aligned} -\vec{\nabla} \cdot [D(\vec{r}, E) \vec{\nabla} \phi(\vec{r}, E)] + \Sigma_t(\vec{r}, E) \phi(\vec{r}, E) \\ = \frac{1}{k} \chi(E) \int_0^\infty dE_i \nu(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ + \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E) \phi(\vec{r}, E_i) . \end{aligned} \quad (\text{C.40})$$

C.4.3 k -Eigenvalue Multigroup Transport

The k -eigenvalue multigroup neutron transport equations are:

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi_g(\vec{r}, \vec{\Omega}) + \Sigma_{t,g}(\vec{r}) \psi_g(\vec{r}, \vec{\Omega}) = \frac{1}{k} \frac{\chi_g}{4\pi} \sum_{g'=1}^G \nu_{g'}(\vec{r}) \Sigma_{f,g'}(\vec{r}) \phi_{g'}(\vec{r}) \\ + \sum_{g'=1}^G \iint_{4\pi} d\Omega_i \Sigma_{s,g' \rightarrow g}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_{g'}(\vec{r}, \vec{\Omega}_i) , \quad g = 1, \dots, G . \end{aligned} \quad (\text{C.41})$$

C.4.4 k -Eigenvalue Multigroup Diffusion

If we integrate the k -eigenvalue multigroup transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the time-dependent multigroup **neutron conservation equations**, without approximation. If we then replace \vec{J}_g using the diffusion approximation, we obtain the k -eigenvalue multigroup neutron diffusion equations:

$$\begin{aligned} & -\vec{\nabla} \cdot [D_g(\vec{r})\vec{\nabla}\phi_g(\vec{r})] + \Sigma_{t,g}(\vec{r})\phi_g(\vec{r}) \\ &= \frac{1}{k}\chi_g \sum_{g'=1}^G \nu_{g'}(\vec{r})\Sigma_{f,g'}(\vec{r})\phi_{g'}(\vec{r}) + \sum_{g'=1}^G \Sigma_{s,g' \rightarrow g}(\vec{r})\phi_{g'}(\vec{r}), \quad g = 1, \dots, G. \end{aligned} \quad (\text{C.42})$$

C.4.5 k -Eigenvalue Two-Group Transport

The k -eigenvalue two-group neutron transport equations are just a particular instance of the multigroup equations. The following two equations are nothing more than the multigroup equations given above, spelled out for each group for the particular case of $G = 2$.

$$\begin{aligned} & \vec{\Omega} \cdot \vec{\nabla}\psi_1(\vec{r}, \vec{\Omega}) + \Sigma_{t,1}(\vec{r})\psi_1(\vec{r}, \vec{\Omega}) \\ &= \frac{1}{k} \frac{\chi_1}{4\pi} [\nu_1(\vec{r})\Sigma_{f,1}(\vec{r})\phi_1(\vec{r}) + \nu_2(\vec{r})\Sigma_{f,2}(\vec{r})\phi_2(\vec{r})] \\ &+ \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega})\psi_1(\vec{r}, \vec{\Omega}_i) + \Sigma_{s,2 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega})\psi_2(\vec{r}, \vec{\Omega}_i) \right], \end{aligned} \quad (\text{C.43})$$

$$\begin{aligned} & \vec{\Omega} \cdot \vec{\nabla}\psi_2(\vec{r}, \vec{\Omega}) + \Sigma_{t,2}(\vec{r})\psi_2(\vec{r}, \vec{\Omega}) \\ &= \frac{1}{k} \frac{\chi_2}{4\pi} [\nu_1(\vec{r})\Sigma_{f,1}(\vec{r})\phi_1(\vec{r}) + \nu_2(\vec{r})\Sigma_{f,2}(\vec{r})\phi_2(\vec{r})] \\ &+ \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega})\psi_1(\vec{r}, \vec{\Omega}_i) + \Sigma_{s,2 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega})\psi_2(\vec{r}, \vec{\Omega}_i) \right]. \end{aligned} \quad (\text{C.44})$$

If the energy boundary between the two groups is chosen as usual, then $\chi_1 = 1$, $\chi_2 = 0$, and the $\Sigma_{s,2 \rightarrow 1}$ term is often dropped because it is small relative to other terms in the equations.

C.4.6 k -Eigenvalue Two-Group Diffusion

If we integrate the k -eigenvalue two-group transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the k -eigenvalue two-group **neutron conservation equations**, without approximation. If we then choose the two energy intervals in the two-group problem as usual:

- interval 1: $E \in (E_{th}, E_0)$, where $E_0 \approx 10\text{-}12$ MeV,
- interval 2: $E \in (0, E_{th})$, where $E_{th} \approx 0.5\text{-}4$ MeV,

then $\chi_1 = 1$, $\chi_2 = 0$, and the $\Sigma_{s,2 \rightarrow 1}$ term is often dropped because it is small relative to other terms in the equations. If we do this and then make the diffusion approximation, then we obtain the usual form of the **k -eigenvalue two-group diffusion equations**:

$$\begin{aligned} -\vec{\nabla} \cdot [D_1(\vec{r}) \vec{\nabla} \phi_1(\vec{r})] + [\Sigma_{t,1}(\vec{r}) - \Sigma_{s,1 \rightarrow 1}(\vec{r})] \phi_1(\vec{r}) \\ = \frac{1}{k} [\nu_1(\vec{r}) \Sigma_{f,1}(\vec{r}) \phi_1(\vec{r}) + \nu_2(\vec{r}) \Sigma_{f,2}(\vec{r}) \phi_2(\vec{r})] , \end{aligned} \quad (\text{C.45})$$

$$-\vec{\nabla} \cdot [D_2(\vec{r}) \vec{\nabla} \phi_2(\vec{r})] + \Sigma_{a,2}(\vec{r}) \phi_2(\vec{r}) = \Sigma_{s,1 \rightarrow 2}(\vec{r}) \phi_1(\vec{r}) . \quad (\text{C.46})$$

C.4.7 k -Eigenvalue One-Group Transport

The k -eigenvalue one-group neutron transport equation is just a particular instance of the multigroup equations for the particular case of $G = 1$. In this case we do not need a “group” subscript because there are not multiple groups to keep track of:

$$\vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, \vec{\Omega}) + \Sigma_t(\vec{r}) \psi(\vec{r}, \vec{\Omega}) = \frac{1}{k} \frac{1}{4\pi} \nu(\vec{r}) \Sigma_f(\vec{r}) \phi(\vec{r}) + \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi(\vec{r}, \vec{\Omega}_i) . \quad (\text{C.47})$$

C.4.8 *k*-Eigenvalue One-Group Diffusion

If we integrate the *k*-eigenvalue one-group transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the time-dependent one-group **neutron conservation equation**, without approximation. If we then replace \vec{J} using the diffusion approximation, we obtain the ***k*-eigenvalue one-group neutron diffusion equation**:

$$-\vec{\nabla} \cdot [D(\vec{r})\vec{\nabla}\phi(\vec{r})] + \Sigma_a(\vec{r})\phi(\vec{r}) = \frac{1}{k}\nu(\vec{r})\Sigma_f(\vec{r})\phi(\vec{r}) . \quad (\text{C.48})$$

C.5 α -Eigenvalue Problems

Recall the discussion in the k – eigenvalue section that makes use of the \mathbf{L} and \mathbf{P} operators. We can use exactly the same operators to introduce the general form of the α -eigenvalue problem. To do so, we add the term α/v to the \mathbf{L} operator, where α is an eigenvalue and v is the neutron speed (either $v(E)$ in the energy-dependent cases or v_g in the multigroup cases):

$$\left[\mathbf{L} + \frac{\alpha}{v} \right] f = \mathbf{P} f , \quad (\text{C.49})$$

or

$$v [\mathbf{P} - \mathbf{L}] f = \alpha f . \quad (\text{C.50})$$

In the latter form we see that α is an eigenvalue of the operator: speed \times [production – loss].

Important notes:

1. Eigenvalue problems never have fixed sources or any other “forcing” terms (such as prescribed angular fluxes or partial currents incident on a boundary).
2. Eigenvalue problems do not provide information about solution magnitude—only about the distribution, or “shape” of the solution.
3. Neutronics eigenvalue problems never have time variables.
4. A solution of an eigenvalue problem is both an eigenvalue and the associated eigenfunction. That is, it is an (eigenvalue, eigenfunction) pair.
5. For any finite reactor, there are infinitely many solutions of a given neutronics eigenvalue problem. That is, there will always be an infinite number of (eigenvalue, eigenfunction) pairs that satisfy the eigenvalue equation and its associate boundary conditions.
6. The α eigenvalues are often called “time-absorption” eigenvalues. If you insert the form $\psi(\vec{r}, E, \vec{\Omega}, t) \rightarrow f(\vec{r}, E, \vec{\Omega})e^{\alpha t}$ into the time-dependent transport equation, then cancel the common $e^{\alpha t}$ factor from all terms, you obtain the α -eigenvalue transport equation. A similar statement holds for $\phi(\vec{r}, E, t) \rightarrow f(\vec{r}, E)e^{\alpha t}$ in the time-dependent diffusion equation.
7. The algebraically largest of the α -eigenvalues (not largest in magnitude, but farthest to the right on the real axis) is associated with the fast-growing or most slowly-decaying mode in a source-free time-dependent system. The associated eigenfunction gives the **fundamental-mode** α -eigenproblem neutron distribution in the system.

C.5.1 α -Eigenvalue Energy-Dependent Transport

The α -eigenvalue continuous-energy neutron transport equation is:

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}) + \left[\Sigma_t(\vec{r}, E) + \frac{\alpha}{v(E)} \right] \psi(\vec{r}, E, \vec{\Omega}) \\ = \frac{\chi(E)}{4\pi} \int_0^\infty dE_i \nu(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ + \int_0^\infty dE_i \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, E_i \rightarrow E, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi(\vec{r}, E_i, \vec{\Omega}_i). \end{aligned} \quad (\text{C.51})$$

C.5.2 α -Eigenvalue Energy-Dependent Diffusion

If we integrate the α -eigenvalue energy-dependent transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the time-dependent energy-dependent **neutron conservation equation**, without approximation. If we then insert the diffusion approximation we obtain the α -eigenvalue energy-dependent diffusion equation:

$$\begin{aligned} -\vec{\nabla} \cdot \left[D(\vec{r}, E) \vec{\nabla} \phi(\vec{r}, E) \right] + \left[\Sigma_t(\vec{r}, E) + \frac{\alpha}{v(E)} \right] \phi(\vec{r}, E) \\ = \chi(E) \int_0^\infty dE_i \nu(\vec{r}, E_i) \Sigma_f(\vec{r}, E_i) \phi(\vec{r}, E_i) \\ + \int_0^\infty dE_i \Sigma_s(\vec{r}, E_i \rightarrow E) \phi(\vec{r}, E_i). \end{aligned} \quad (\text{C.52})$$

C.5.3 α -Eigenvalue Multigroup Transport

The α -eigenvalue multigroup neutron transport equations are:

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi_g(\vec{r}, \vec{\Omega}) + \left[\Sigma_{t,g}(\vec{r}) + \frac{\alpha}{v_g} \right] \psi_g(\vec{r}, \vec{\Omega}) \\ = \frac{\chi_g}{4\pi} \sum_{g'=1}^G \nu_{g'}(\vec{r}) \Sigma_{f,g'}(\vec{r}) \phi_{g'}(\vec{r}) \\ + \sum_{g'=1}^G \iint_{4\pi} d\Omega_i \Sigma_{s,g' \rightarrow g}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_{g'}(\vec{r}, \vec{\Omega}_i), \quad g = 1, \dots, G. \end{aligned} \quad (\text{C.53})$$

C.5.4 α -Eigenvalue Multigroup Diffusion

If we integrate the α -eigenvalue multigroup transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the time-dependent multigroup **neutron conservation equations**, without approximation. If we then replace \vec{J}_g using the diffusion approximation, we obtain the α -eigenvalue multigroup neutron diffusion equations:

$$\begin{aligned} -\vec{\nabla} \cdot \left[D_g(\vec{r}) \vec{\nabla} \phi_g(\vec{r}) \right] + \left[\Sigma_{t,g}(\vec{r}) + \frac{\alpha}{v_g} \right] \phi_g(\vec{r}) \\ = \chi_g \sum_{g'=1}^G \nu_{g'}(\vec{r}) \Sigma_{f,g'}(\vec{r}) \phi_{g'}(\vec{r}) + \sum_{g'=1}^G \Sigma_{s,g' \rightarrow g}(\vec{r}) \phi_{g'}(\vec{r}), \quad g = 1, \dots, G. \end{aligned} \quad (\text{C.54})$$

C.5.5 α -Eigenvalue Two-Group Transport

The α -eigenvalue two-group neutron transport equations are just a particular instance of the multigroup equations. The following two equations are nothing more than the multigroup equations given above, spelled out for each group for the particular case of $G = 2$.

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi_1(\vec{r}, \vec{\Omega}) + \left[\Sigma_{t,1}(\vec{r}) + \frac{\alpha}{v_1} \right] \psi_1(\vec{r}, \vec{\Omega}) \\ = \frac{\chi_1}{4\pi} [\nu_1(\vec{r}) \Sigma_{f,1}(\vec{r}) \phi_1(\vec{r}) + \nu_2(\vec{r}) \Sigma_{f,2}(\vec{r}) \phi_2(\vec{r})] \\ + \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_1(\vec{r}, \vec{\Omega}_i) + \Sigma_{s,2 \rightarrow 1}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_2(\vec{r}, \vec{\Omega}_i) \right], \end{aligned} \quad (\text{C.55})$$

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi_2(\vec{r}, \vec{\Omega}) + \left[\Sigma_{t,2}(\vec{r}) + \frac{\alpha}{v_2} \right] \psi_2(\vec{r}, \vec{\Omega}) \\ = \frac{\chi_2}{4\pi} [\nu_1(\vec{r}) \Sigma_{f,1}(\vec{r}) \phi_1(\vec{r}) + \nu_2(\vec{r}) \Sigma_{f,2}(\vec{r}) \phi_2(\vec{r})] \\ + \iint_{4\pi} d\Omega_i \left[\Sigma_{s,1 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_1(\vec{r}, \vec{\Omega}_i) + \Sigma_{s,2 \rightarrow 2}(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi_2(\vec{r}, \vec{\Omega}_i) \right]. \end{aligned} \quad (\text{C.56})$$

If the energy boundary between the two groups is chosen as usual, then $\chi_1 = 1$, $\chi_2 = 0$, and the $\Sigma_{s,2 \rightarrow 1}$ term is often dropped because it is small relative to other terms in the equations.

C.5.6 α -Eigenvalue Two-Group Diffusion

If we integrate the α -eigenvalue two-group transport equations over all directions—that is, if we operate on them with $\iint d\Omega$ —we obtain the α -eigenvalue two-group **neutron conservation equations**, without approximation. If we then choose the two energy intervals in the two-group problem as usual:

- interval 1: $E \in (E_{th}, E_0)$, where $E_0 \approx 10\text{-}12$ MeV,
- interval 2: $E \in (0, E_{th})$, where $E_{th} \approx 0.5\text{-}4$ MeV,

then $\chi_1 = 1$, $\chi_2 = 0$, and the $\Sigma_{s,2 \rightarrow 1}$ term is often dropped because it is small relative to other terms in the equations. If we do this and then make the diffusion approximation, then we obtain the usual form of the **α -eigenvalue two-group diffusion equations**:

$$\begin{aligned} -\vec{\nabla} \cdot [D_1(\vec{r}) \vec{\nabla} \phi_1(\vec{r})] + \left[\Sigma_{t,1}(\vec{r}) - \Sigma_{s,1 \rightarrow 1}(\vec{r}) + \frac{\alpha}{v_1} \right] \phi_1(\vec{r}) \\ = \nu_1(\vec{r}) \Sigma_{f,1}(\vec{r}) \phi_1(\vec{r}) + \nu_2(\vec{r}) \Sigma_{f,2}(\vec{r}) \phi_2(\vec{r}) , \end{aligned} \quad (\text{C.57})$$

$$-\vec{\nabla} \cdot [D_2(\vec{r}) \vec{\nabla} \phi_2(\vec{r})] + \left[\Sigma_{a,2}(\vec{r}) + \frac{\alpha}{v_2} \right] \phi_2(\vec{r}) = \Sigma_{s,1 \rightarrow 2}(\vec{r}) \phi_1(\vec{r}) . \quad (\text{C.58})$$

C.5.7 α -Eigenvalue One-Group Transport

The α -eigenvalue one-group neutron transport equation is just a particular instance of the multigroup equations for the particular case of $G = 1$. In this case we do not need a “group” subscript because there are not multiple groups to keep track of:

$$\begin{aligned} \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, \vec{\Omega}) + \left[\Sigma_t(\vec{r}) + \frac{\alpha}{v} \right] \psi(\vec{r}, \vec{\Omega}) \\ = \frac{1}{4\pi} \nu(\vec{r}) \Sigma_f(\vec{r}) \phi(\vec{r}) + \iint_{4\pi} d\Omega_i \Sigma_s(\vec{r}, \vec{\Omega}_i \rightarrow \vec{\Omega}) \psi(\vec{r}, \vec{\Omega}_i) . \end{aligned} \quad (\text{C.59})$$

C.5.8 α -Eigenvalue One-Group Diffusion

If we integrate the α -eigenvalue one-group transport equation over all directions—that is, if we operate on it with $\iint d\Omega$ —we obtain the time-dependent one-group **neutron conservation equation**, without approximation. If we then replace \vec{J} using the diffusion approximation, we obtain the **α -eigenvalue one-group neutron diffusion equation**:

$$-\vec{\nabla} \cdot [D(\vec{r})\vec{\nabla}\phi(\vec{r})] + \left[\Sigma_a(\vec{r}) + \frac{\alpha}{v}\right]\phi(\vec{r}) = \nu(\vec{r})\Sigma_f(\vec{r})\phi(\vec{r}) . \quad (\text{C.60})$$

C.6 Boundary Conditions

Time-dependent, steady-state, and eigenvalue problems, whether transport or diffusion, all involve spatial derivatives. Well-posed problems require not only the correct equations but also boundary conditions in well-posed form. All transport boundary conditions specify the incoming angular flux at each point on the problem boundary, for all incoming directions and all energy unknowns (either all E for continuous-energy problems or all groups in multi-group problems). All diffusion boundary conditions approximately specify the rate at which neutrons enter the volume through its boundary, although it may not be obvious from the form of the boundary condition that it has this underlying meaning.

Appendix D

Ordinary Differential Equation Primer

D.1 Introduction

In this course, but more generally in many engineering and scientific disciplines, physical laws (aka governing laws) are expressed using a first-order and second-order derivatives.

For example, for functions of one variable, a first-order derivative may be expressed as follows:

$$f'(x) = \frac{df}{dx} = f^{(1)}(x)$$

and a second-order derivative as follows:

$$f''(x) = \frac{d^2f}{dx^2} = f^{(2)}(x)$$

People may prefer one notation over another but they are all equivalent.

D.1.1 Example-1: Radioactive decay

For example, the rate of decay of a radioactive nuclide may be expressed as follows:

$$\frac{dN}{dt} = -\lambda N(t) \tag{D.1}$$

This problem is not well-posed because there are an infinite number of solutions to it. All of the solutions have the following form

$$N(t) = Ae^{-\lambda t}$$

Yes, you can verify that by plugging this generic solution in Eq. (D.1), the equality is satisfied. Indeed,

$$\frac{dN}{dt} = -\lambda A e^{-\lambda t}$$

and

$$-\lambda N(t) = -\lambda A e^{-\lambda t}$$

To well pose, or close, the problem, we need here the initial condition. We only need one such condition and we usually only know values at the initial time, here $N_0 = N(t_0)$. Using this, we now have a unique solution:

$$N(t) = N_0 e^{-\lambda(t-t_0)} \quad \text{for } t \geq t_0$$

Eq. (D.1) is an example of first-order Ordinary Differential Equation (ODE). Mathematicians would write it as follows (including its initial conditions to make the problem well posed):

$$\begin{cases} u' + au = b & \text{for } x \geq x_0, \quad \text{with } (a, b) \in \mathbb{R}^2 \\ u(x_0) = u_0 & \text{with } u_0 \in \mathbb{R} \end{cases} \quad (\text{D.2})$$

Here, I have purposefully change time t for another symbol x (we use meaningful symbols for the indepedent variable but I just wanted to show you that mathematically, it makes no difference whatsoever). u is now the sought-after solution. $a = \lambda$ (caveat sign, this is on the left-hand side now). I added a source term b in case there are a constant **external** source generating the nuclide under consideration, here $b = 0$ but we will keep that term our most general formulation.

In the next section, we will see how to solve such ODEs.

D.1.2 Example-2: Neutron diffusion in a homogeneous slab

Another example is neutron diffusion in a homogeneous slab, with a constant source present, and with zero-flux boundary conditions applied on both extremities of the slab:

$$-D \frac{d^2\phi}{dx^2} + \Sigma_a \phi = S \quad \text{for } x_1 \leq x \leq x_2 \quad (\text{D.3})$$

This equation has a second-order derivative. For the problem to be well posed, it **needs two** boundary conditions:

$$\phi(x_1) = 0 \quad \text{and} \quad \phi(x_2) = 0$$

Eq. (D.1) is an example of second-order Ordinary Differential Equation. Mathematicians would write it as follows (including its two boundary conditions to make the problem well posed):

$$\begin{cases} au'' + bu = c & \text{with } (a, b, c) \in \mathbb{R}^3 \\ u(x_1) = u_1 & \text{with } u_1 \in \mathbb{R} \\ u(x_2) = u_2 & \text{with } u_2 \in \mathbb{R} \end{cases} \quad (\text{D.4})$$

In a subsequent section, we will see how to solve such ODEs.

D.2 Solving first-order ODEs

D.2.1 Using an integrating factor

To solve Eq. (D.1), recalled below,

$$\begin{cases} u' + au = b & \text{for } t \geq t_0, \quad \text{with } (a, b) \in \mathbb{R}^2 \\ u(t_0) = u_0 & \text{with } u_0 \in \mathbb{R} \end{cases}$$

we use an integrating factor. We multiply both sides by e^{at} and get

$$e^{at} \frac{du}{dt} + ae^{at}u(t) = be^{at}$$

Next, we note that

$$\frac{d(e^{at}u(t))}{dt} = e^{at} \frac{du}{dt} + ae^{at}u(t)$$

so that the equation can be re-written as

$$\frac{d(e^{at}u(t))}{dt} = be^{at}$$

We then integrate this equation from $t = t_0$ to $t = T$:

$$\int_{t=t_0}^{t=T} dt \frac{d(e^{at}u(t))}{dt} = \int_{t_0}^{t=T} dt be^{at}$$

Carrying out the integration yields:

$$e^{at}u(T) - e^{at_0}u(t_0) = \frac{b}{a} (e^{at} - e^{at_0})$$

Multiplying throughout by $e^{-(aT)}$ (caveat, negative sign) and noting that $u(t_0) = u_0$, we have

$$u(T) - u_0 e^{-a(T-t_0)} = \frac{b}{a} (1 - e^{-a(T-t_0)})$$

or

$$u(T) = u_0 e^{-a(T-t_0)} + \frac{b}{a} (1 - e^{-a(T-t_0)})$$

Since T can take any value, I revert back to a notation that uses t , just because it's more pleasing to the eye.

$$u(t) = u_0 e^{-a(t-t_0)} + \frac{b}{a} (1 - e^{-a(t-t_0)}) \quad (\text{D.5})$$

Some remarks:

- if $a = 0$, we cannot divide by a . We start for the governing law

$$u' = b$$

which states that the derivative of u is a constant b . Thus $u(t) = bt + A$. The constant A is obtained from the initial condition so that the solution is

$$u(t) = u_0 + b(t - t_0)$$

- if $b = 0$, we just have a decaying solution

$$u(t) = u_0 e^{-(a(t-t_0))}$$

- for very long time $t \gg t_0$, if an equilibrium is to be reached (this happens when $a > 0$), one can neglect the time derivative in the governing law, so that

$$\frac{dy}{dt}^0 + au = b$$

that is, the solution tends to $u_\infty \rightarrow b/a$. Note that our solution does exactly this:

$$u(t \rightarrow \infty) = \cancel{u_0 e^{-a(t-t_0)}}^0 + \frac{b}{a} \left(1 - \cancel{e^{-a(t-t_0)}}^0 \right)$$

D.2.2 Seeking the sum of a homogeneous solution and a particular solution

Another approach consists in seeking the solution as the sum of [a homogeneous solution and a particular so

$$u(t) = u_h(t) + u_p(t)$$

The homogeneous solution satisfies the governing law **without** the source term b present:

$$u'_h + au_h = 0$$

We know that a solution to this equation is of the form

$$u_h(t) = U e^{-at}$$

We then need to find **one** particular solution that satisfies the governing law (with the right-hand side present, i.e., $u'_p + au_p = b$). Here, you to try your luck. We only need **one** such solution. The easiest thing to try is see if a constant solution $u_p(t) = C$ could work. Indeed,

$$0 + aC = b \quad \rightarrow C = b/a$$

is a valid particular solution.

Thus, our solution is of the form

$$u(t) = u_h(t) + u_p(t) = U e^{-at} + \frac{b}{a}$$

Only now should you invoke the initial condition to find U :

$$u(t_0) = U e^{-at_0} + \frac{b}{a} = u_0 \quad \rightarrow U = \left(u_0 - \frac{b}{a} \right) e^{at_0}$$

and

$$u(t) = \left(u_0 - \frac{b}{a} \right) e^{-a(t-t_0)} + \frac{b}{a}$$

I let you check this is the same as before ...

D.3 Solving second-order ODEs

To solve Eq. (D.3), recalled below,

$$\begin{cases} au'' + bu = c & \text{with } (a, b, c) \in \mathbb{R}^3 \\ u(x_1) = u_1 & \text{with } u_1 \in \mathbb{R} \\ u(x_2) = u_2 & \text{with } u_2 \in \mathbb{R} \end{cases}$$

we looking for the solution as a sum of a homogeneous solution and a particular solution

$$u(x) = u_h(x) + u_p(x)$$

The homogeneous solution satisfies the governing law **without** the source term c present:

$$au''_h + bu_h = 0 \quad (\text{D.6})$$

We seek u_h of the form

$$u_h(x) = Ue^{rx}$$

where r is to be determined. Plugging this in Eq. (D.6), we get

$$ar^2 + b = 0$$

(the e^{rx} can cancel because we do not have a right-hand side. That's the beauty of it). now

$$r = \pm \sqrt{-\frac{b}{a}}$$

In our case, $-b/a > 0$, so we have two real solutions, $k = \sqrt{-\frac{b}{a}}$ and $-k$. Thus, we need to have the two types of fuctions in our homogeneous solution:

$$u_h(x) = Ae^{kx} + Be^{-kx}$$

Next, we need to find **one** particular solution that satisfies the governing law (with the right-hand side present, i.e., $au''_p + bu_p = c$). Here, you to try your luck. We only need **one** such solution. The easiest thing to try is see if a constant solution $u_p(x) = C$ could work. Indeed,

$$0 + bC = c \rightarrow C = c/b$$

is a valid particular solution.

Thus, our solution is of the form

$$u(x) = u_h(x) + u_p(x) = Ae^{kx} + Be^{-kx} + \frac{c}{b}$$

Only now should you invoke the boundary conditions to find A and B :

$$\begin{cases} u_1 = u(x_1) = Ae^{kx_1} + Be^{-kx_1} + \frac{c}{b} \\ u_2 = u(x_2) = Ae^{kx_2} + Be^{-kx_2} + \frac{c}{b} \end{cases}$$

This is a little 2×2 linear system to find A and B

$$\begin{bmatrix} e^{kx_1} & e^{-kx_1} \\ e^{kx_2} & e^{-kx_2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} u_1 - \frac{c}{b} \\ u_2 - \frac{c}{b} \end{bmatrix}$$

Appendix E

Establishing Neutron Conservation Laws

Foreword: Read Appendix D for a review of mathematical techniques to solve ODEs.

E.1 Introduction

We know that a reactor's behavior depends on the gain and loss rates of neutrons in the reactor. We also know that some gain and loss is due to neutron-nucleus reactions, and that some is due to neutron leakage. In the previous chapter we developed mathematical expressions for reaction rates and leakage rates. In this chapter we put these expressions into statements of conservation to see what we can learn about how the neutron population changes with time in reactors and how the neutrons distribute themselves in position and energy.

To do this we will examine conservation statements for different neutron populations in different settings, ranging from simplistic to realistic. The populations and settings we consider are as follows. **Do not go over the cases in details. This is just for them to understand why there are so many cases when they read their notes again.**

1. Population = all neutrons in some chosen volume.
Setting = infinite uniform medium, pretending that all neutrons have the same energy and that all fission neutrons are emitted promptly.
- Goal:** Establish time evolution of a population of neutrons, ignoring delayed neutrons.

2. Population = all neutrons in some chosen volume.

Setting = infinite uniform medium, pretending that all fission neutrons are emitted promptly but allowing neutrons to have different energies.

Goal: Establish time evolution of a population of neutrons, ignoring delayed neutrons, but showing how spectrally averaged cross sections are used. Basically a slight improvement over case #1.

3. Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.

Setting = infinite uniform medium, pretending that all fission neutrons are emitted promptly.

Goal: Establish time-dependent and energy-dependent balance of a population of neutrons, ignoring delayed neutrons. Later, we will slightly simplify this to study in depth neutron slowing-down,

4. Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.

Setting = infinite uniform medium, accounting for delayed neutrons (which are emitted during the decay of fission products and their daughters).

Goal: Establish the point reactor kinetics equations (time-dependent balance of a population of neutrons), **accounting** for delayed neutrons.

5. Population = all neutrons.

Setting = finite reactor, pretending that all fission neutrons are emitted promptly.

Goal: Establish time evolution of a population of neutrons, accounting for neutron leakage, ignoring delayed neutrons. Basically, a slight improvement over case #2.

6. Population = all neutrons.

Setting = finite reactor, accounting for delayed neutrons.

Goal: Establish time evolution of a population of neutrons, accounting for neutron leakage, accounting for delayed neutrons. Basically, we will obtain again the point reactor kinetics equations (case #4) where only small modifications to the coefficients are needed to account for leakage.

7. Population = all neutrons in some chosen volume.

Setting = finite reactor, accounting for delayed neutrons.

Goal: Establish a time-dependent and space-dependent balance of neutrons. We use spectrally averaged cross sections again. The main reason for this case is just to build up for the next case, case #8.

8. Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.

Setting = finite reactor, accounting for delayed neutrons.

Goal: The time-dependent, energy-dependent, space-dependent neutron conversation law (time-space-energy dependent).

9. Population = all neutrons in some chosen volume whose energies are in some chosen energy interval and whose directions are in some chosen directional range.

Setting = finite reactor, accounting for delayed neutrons.

Goal: The most detailed conservation law: the **neutron transport** equation (time-space-energy-direction dependent).

In every case we will follow the same procedure. **First** we will note that

$$\text{change rate} = \text{gain rate} - \text{loss rate}$$

for whatever population we have chosen. **Second**, for whatever neutron population we are considering, we will construct a mathematical expression for each of these three terms, consistent with the setting under consideration. **Third**, we will try to solve the resulting mathematical equation and learn what it tells us about the behavior of the neutron population.

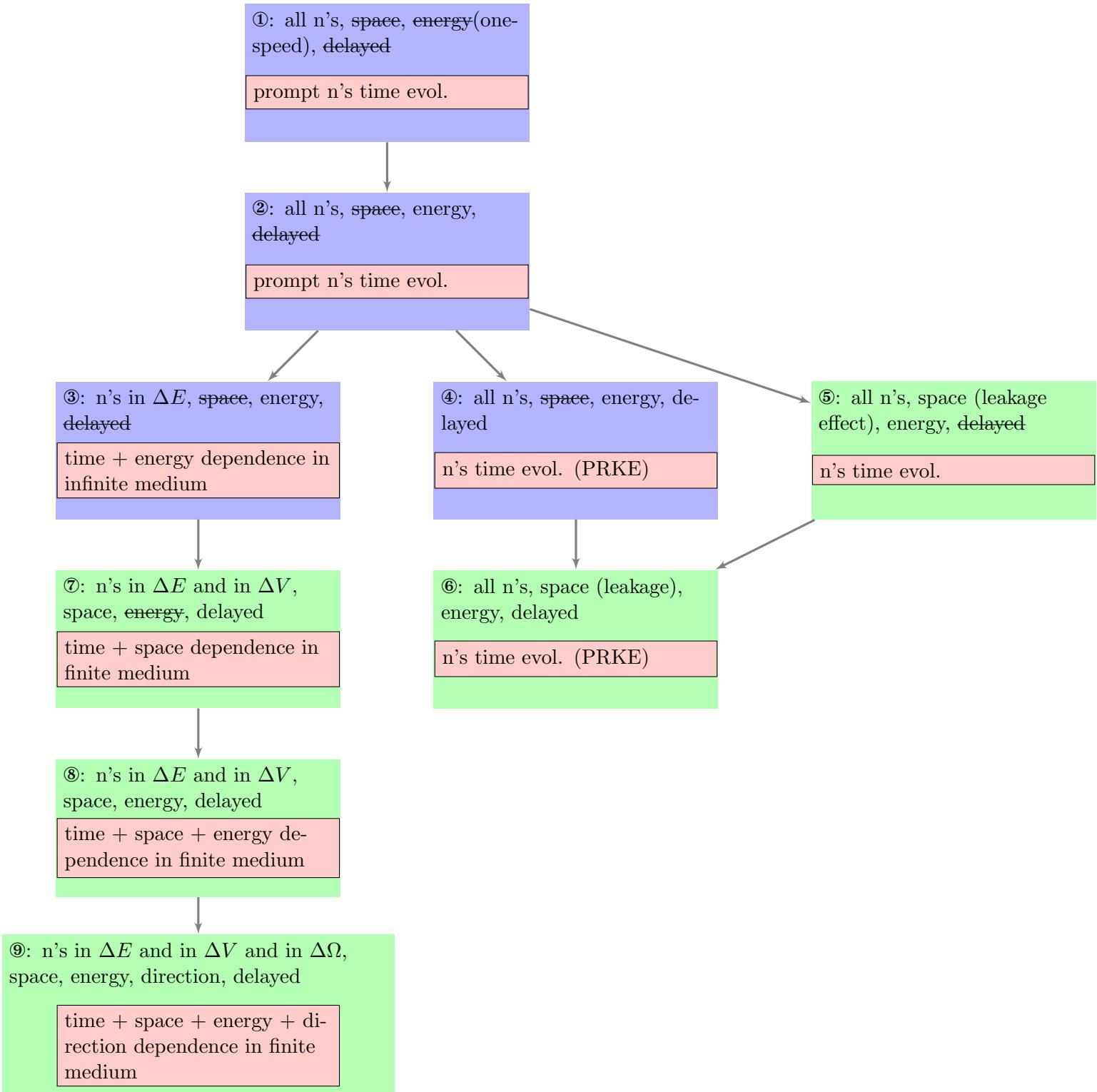


Figure E.1: Flowchart of the cases studied in this Chapter (blue=infinite medium; green=finite medium).

E.2 Neutron Conservation in an Infinite Uniform Medium

In this section we consider the first four settings, all of which involve infinite homogeneous reactors. In these settings we do not have to worry about leakage, so we can focus our attention solely on production and loss from neutron-nucleus interactions. We also do not have to worry about spatial distributions, because everything is uniformly distributed throughout the infinite medium.

E.2.1 Case 1: Population = all n's in V; Setting = one-speed neutrons and no delayed neutrons

We solved this problem in Chapter III of these notes for the case in which there were no fixed (or “extraneous”) sources spontaneously emitting neutrons into the medium. Here we will allow for the possibility of such sources. An example of a “fixed” source would be $^{252}\text{Californium}$, which emits neutrons as it spontaneously fissions, not needing to absorb a neutron first to trigger the fission. Another example is an alpha-emitter (such as $^{241}\text{Americium}$) combined with beryllium—an Am-Be source—in which alpha particles are absorbed by Be nuclei and the resulting compound nuclei release neutrons.

We assume that the neutrons are distributed uniformly and isotropically throughout the medium, and we focus on a fixed volume in the medium, which we call \mathbf{V} . Now our gain mechanisms are:

1. Emission from **the fixed source**.
2. Emission from **fissions caused by neutron absorption**.

Our only loss mechanism is:

1. **absorption**.

Technically, leakage contributes to both gain and loss in the volume, because neutrons do leak into and out of the volume. However, the net leakage from any volume in our infinite medium is zero, because the neutrons are distributed uniformly in position and direction and thus when a neutron enters the volume it has a “twin” that leaves the volume. So inleakage and outleakage cancel each other in our conservation statement.

We define:

$$n(t) = \text{neutron density} \quad [\text{units} = \text{n/cm}^3]$$

$$v = \text{neutron speed} \quad [\text{units} = \text{cm/s}]$$

$$S_{\text{ext}} = \text{“fixed” or “extraneous” source-rate density} \quad [\text{units} = \text{n}/(\text{cm}^3 \cdot \text{s})]$$

Then our conservation statement for neutrons in the volume \mathbf{V} is:

$$\frac{dn}{dt}\mathbf{V} = \nu\Sigma_f n v\mathbf{V} + S_{\text{ext}}\mathbf{V} - \Sigma_a n v\mathbf{V}. \quad (\text{E.1})$$

The volume is constant and cancels out of the equation, leaving:

$$\frac{dn}{dt} = (\nu\Sigma_f - \Sigma_a) v n + S_{\text{ext}}. \quad (\text{E.2})$$

We can solve this equation (by using an integrating factor, for example). We obtain:

$$n(t) = n(0)e^{(\nu\Sigma_f - \Sigma_a)vt} + S_{\text{ext}} \left[\frac{e^{(\nu\Sigma_f - \Sigma_a)vt} - 1}{(\nu\Sigma_f - \Sigma_a)v} \right] \quad (\text{E.3})$$

Once again, conservation plus a bit of mathematics has told us a great deal about the behavior of the neutron population of interest. Note:

1. If there is no fixed source, then the neutron population:

- (a) grows exponentially if $\nu\Sigma_f > \Sigma_a$,
- (b) stays steady if $\nu\Sigma_f = \Sigma_a$,
- (c) falls exponentially if $\nu\Sigma_f < \Sigma_a$.

2. If there is a fixed source, then the neutron population:

- (a) eventually grows exponentially if $\nu\Sigma_f > \Sigma_a$,
- (b) grows linearly in time if $\nu\Sigma_f = \Sigma_a$,
- (c) reaches a steady value if $\nu\Sigma_f < \Sigma_a$.

One case—a critical reactor with a fixed source—is slightly tricky mathematically. In our current setting (infinite uniform medium with one-speed neutrons), “critical” corresponds to $\nu\Sigma_f = \Sigma_a$. There are two ways to find the correct result. One way is to return to Eq. (E.2) and note that the fission and absorption terms cancel, leaving a simple equation that says the time-derivative of n is the constant S_{ext} . Positive constant time derivative means linearly increasing with time. The other way is to use L’Hospital’s rule to evaluate the troublesome 0/0 term in Eq. (E.3).

**E.2.2 Case 2: Population = all n's in V;
Setting = energy-dependent neutrons and no delayed neutrons**

In this case we have the same conservation statement as before, but now that we are addressing the reality that neutrons have a whole spectrum of energies, each term in the equation involves an integral over neutron energies:

$$\frac{d}{dt} \left[\mathbf{V} \int_0^\infty dE n(E, t) \right] = \mathbf{V} \int_0^\infty dE \nu(E) \Sigma_f(E) v(E) n(E, t) - \mathbf{V} \int_0^\infty dE \Sigma_a(E) n(E, t) v(E) + \mathbf{V} \int_0^\infty dE S_{\text{ext}}(E) . \quad (\text{E.4})$$

As before, the volume is constant and cancels out of the equation.

When we try to solve this equation we encounter a problem: the fission and absorption terms are not just

constants times the function in the time-derivative term.

That is, this equation is not of the form we had previously, which was

$$\frac{df}{dt} = C_1 f(t) + C_2 . \quad (\text{E.5})$$

What can we do? There are at least two choices:

1. We can **make approximations** to the fission and absorption terms.
2. We can **examine conservation in more detail**, applying it to each energy interval as well as each spatial volume.

Let us take the first approach first. (We will explore the second approach in the next subsection.) **Without approximation**, we can multiply and divide the absorption term by the energy-integrated neutron density (which is the quantity in the time-derivative term). That is, we have the identity:

$$\int_0^\infty dE \Sigma_a(E) n(E, t) v(E) = \left[\frac{\int_0^\infty dE \Sigma_a(E) n(E, t) v(E)}{\int_0^\infty dE n(E, t)} \right] \left[\int_0^\infty dE n(E, t) \right] . \quad (\text{E.6})$$

Now we recognize that the term in brackets is a

weighted average

of $v(E)\Sigma_a(E)$ over all energies, with

weight function $n(E, t)$.

We use a bracket notation to denote this average:

$$\langle v \Sigma_a \rangle \equiv \frac{\int_0^\infty dE [v(E)\Sigma_a(E)] n(E, t)}{\int_0^\infty dE n(E, t)}. \quad (\text{E.7})$$

We define a similar average for the fission term:

$$\langle v \nu \Sigma_f \rangle \equiv \frac{\int_0^\infty dE [v(E)\nu(E)\Sigma_f(E)] n(E, t)}{\int_0^\infty dE n(E, t)}. \quad (\text{E.8})$$

(Be careful with symbols that look similar! $v(E)$ = “vee” = speed of a neutron whose energy is E . $\nu(E)$ = Greek letter “nu” = average number of neutrons emitted from fission if the fissions are caused by absorption of a neutron of energy E .)

Finally, we define:

$$n_{\text{tot}}(t) \equiv \int_0^\infty dE n(E, t), \quad (\text{E.9})$$

$$S_{\text{ext,tot}} \equiv \int_0^\infty dE S_{\text{ext}}(E). \quad (\text{E.10})$$

With these definitions, our conservation equation can be rewritten as:

$$\frac{d}{dt} [n_{\text{tot}}(t)] = [\langle v \nu \Sigma_f \rangle - \langle v \Sigma_a \rangle] n_{\text{tot}}(t) + S_{\text{ext,tot}}. \quad (\text{E.11})$$

This appears to be of the same form as Eq. (E.2), which we have already solved. If the angle-bracket terms can be taken to be constant in time (a shaky assumption, but let us go with it for the moment), then the solution of Eq. (E.11) is:

$$\begin{aligned} n_{\text{tot}}(t) &= n_{\text{tot}}(0) e^{[\langle v \nu \Sigma_f \rangle - \langle v \Sigma_a \rangle]t} + S_{\text{tot,ext}} \left[\frac{e^{[\langle v \nu \Sigma_f \rangle - \langle v \Sigma_a \rangle]t} - 1}{\langle v \nu \Sigma_f \rangle - \langle v \Sigma_a \rangle} \right] \\ &= n_{\text{tot}}(0) e^{\alpha t} + S_{\text{tot,ext}} \left[\frac{e^{\alpha t} - 1}{\alpha} \right] \end{aligned} \quad (\text{E.12})$$

So, ... have we managed to solve the problem despite the issue identified above?

No, not really.

The issue that keeps Eq. (E.12) from being a “real” solution is that

We don’t know $\langle v\nu\Sigma_f \rangle$ or $\langle v\Sigma_a \rangle$,

because they depend on the unknown $n(E, t)$.

Even if we knew the averaged cross section and solved Eq. (E.12), we would know only n_{tot} , not the energy-dependent neutron density $n(E, t)$, which is what we would need to calculate $\langle v\nu\Sigma_f \rangle$ and $\langle v\Sigma_a \rangle$.

Still, we have made progress. Even though we don’t know the values of $\langle v\nu\Sigma_f \rangle$ and $\langle v\Sigma_a \rangle$, we see from our “formal” solution that the total neutron population, $n_{\text{tot}}(t)$, in this case has

the same functional form

as the solution in case 1. That is, we see that

1. If there is no fixed source, then the neutron population:
 - (a) grows exponentially if $\langle v\nu\Sigma_f \rangle > \langle v\Sigma_a \rangle$,
 - (b) stays steady if $\langle v\nu\Sigma_f \rangle = \langle v\Sigma_a \rangle$,
 - (c) falls exponentially $\langle v\nu\Sigma_f \rangle < \langle v\Sigma_a \rangle$,
2. If there is a fixed source, then the neutron population:
 - (a) eventually grows exponentially if $\langle v\nu\Sigma_f \rangle > \langle v\Sigma_a \rangle$,
 - (b) grows linearly in time if $\langle v\nu\Sigma_f \rangle = \langle v\Sigma_a \rangle$,
 - (c) reaches a steady value if $\langle v\nu\Sigma_f \rangle < \langle v\Sigma_a \rangle$.

We know this even though we may not know exactly what are the values of $\langle v\nu\Sigma_f \rangle$ and $\langle v\Sigma_a \rangle$.

But perhaps we can go farther. Suppose we learn enough about neutron distributions in reactors that we can make a **guess** for the energy distribution of the neutron density. For example, suppose we guess that:

$$n(E, t) \approx n_{\text{tot}}(t)f(E) , \quad (\text{E.13})$$

and we also make a guess for the function $f(E)$. Then our guess would produce answers for the needed averaged quantities:

$$\langle v\Sigma_a \rangle \approx \frac{\int_0^\infty dE [v(E)\Sigma_a(E)] n_{\text{tot}}(t)f(E)}{\int_0^\infty dE n_{\text{tot}}(t)f(E)} = \text{known and constant } [\cancel{n_{\text{tot}}}] , \quad (\text{E.14})$$

$$\langle v\nu\Sigma_f \rangle \approx \frac{\int_0^\infty dE [v(E)\nu(E)\Sigma_f(E)] n_{\text{tot}}(t)f(E)}{\int_0^\infty dE n_{\text{tot}}(t)f(E)} = \text{known and constant} . \quad (\text{E.15})$$

This would make our “formal” solution “real” in the sense that we could now compute $n_{\text{tot}}(t)$ for any time t , using the known cross-section averages in Eq. (E.12). However, this “real” solution would normally be

only approximate,

because the equation used for $n(E, t)$ in the cross-section averages is only approximate.

There are several lessons to learn from the case we just examined:

1. Even though we don’t know the exact values of the $\langle \text{bracketed} \rangle$ constants in Eq. (E.12), the equation still shows a lot about how the total neutron density changes with time in the infinite-medium problem we are studying. In fact, the conclusions we stated for case 1 (about exponential growth, exponential shrinkage, steady population, or linear-increase-in-time) are still valid as long as we replace $v\nu\Sigma_f$ and $v\Sigma_a$ by their $\langle \text{average} \rangle$ values.
2. When a term in our conservation equation depends on neutron **sub-populations** for which we have not written a detailed conservation equation, we encounter a problem! In the current example, we wrote conservation for the total neutron density, but our production and loss terms depend on the neutron density in each energy interval. (Neutron density in an interval is an example of a sub-population.) This forces us to resort to approximations, such as Eq. (E.13), to obtain a quantitative solution.

As we shall see in the remainder of this chapter, these lessons apply quite generally.

E.2.3 Case 3: Population = n's with positions in V and energies $E \in \Delta E$;

Setting = energy-dependent neutrons and no delayed neutrons

Here we continue to study an infinite medium and continue to pretend that there are no delayed neutrons. Now that we are writing our conservation statement only for neutrons in a certain energy range— (E_1, E_2) or “ ΔE ” for short—we have an additional gain mechanism, because neutrons can scatter from other energies into the chosen energy interval. Now our gain mechanisms are:

1. Emission of neutron with $E \in \Delta E$ from the fixed source ,
2. Emission of neutron with $E \in \Delta E$ from **neutron-induced fission** ,
3. Scattering into ΔE from pre-scatter energies outside ΔE .

Similarly, neutrons can scatter out of the energy interval, so our loss mechanisms are:

1. Absorption of neutron with $E \in \Delta E$,
2. Scattering of a neutron whose pre-scatter energy is in ΔE and whose post-scatter energy is not in ΔE .

Again, there is no net leakage from any volume in our infinite medium, because the neutrons are distributed uniformly in position and direction, so gain from inleakage cancels with loss from outleakage.

We define:

$n(E, t)$ = energy-dependent neutron density (see previous chapter) ,

$v(E)$ = speed of neutron whose kinetic energy is E ,

$S_{\text{ext}}(E)$ = energy-dependent fixed source-rate density ,

$\chi(E)dE$ = fraction of fission neutrons emitted with energies in dE at E

The probability density function $\chi(E)$ is called the

fission spectrum .

It quantifies the energy distribution of the neutrons that are emitted from fission. (See Fig. 3.10.)

Now our conservation statement for neutrons that are in the volume \mathbf{V} and have energies in the interval ΔE is:

$$\begin{aligned}
 \frac{d}{dt} \underbrace{\left[\mathbf{V} \int_{\Delta E} dE n(E, t) \right]}_{\text{neutrons in } \mathbf{V} \text{ and } \Delta E} &= \underbrace{\left[\int_{\Delta E} dE \chi(E) \right]}_{\text{frac. of fism. n's born in } \Delta E} \underbrace{\mathbf{V} \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t)}_{\text{n/s born from fission}} \\
 &\quad - \underbrace{\mathbf{V} \int_{\Delta E} dE \Sigma_a(E) v(E) n(E, t)}_{\text{absorptions/s}} + \underbrace{\mathbf{V} \int_{\Delta E} dE S_{\text{ext}}(E)}_{\text{n/s from fixed source}} \\
 &\quad + \underbrace{\mathbf{V} \int_{\Delta E} dE \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t)}_{\text{n/s scattered into and within } \Delta E} \\
 &\quad - \underbrace{\mathbf{V} \int_{\Delta E} dE \Sigma_s(E) v(E) n(E, t)}_{\text{n/s scattered out of and within } \Delta E} . \tag{E.16}
 \end{aligned}$$

Remark: There is a kind of “error” in the inscattering term and a similar “error” in the outscattering term. The “inscattering” term, as written, includes scattering from **all** possible initial energies, including initial energies that were already in ΔE . That is, as written it includes “within-interval” scattering as well as inscattering. The outscattering term has a similar error—it includes “within-interval” scattering as well as outscattering from the interval ΔE . Both terms contain exactly the same “within-interval” error term, one with a plus sign and one with a minus sign. So the two “error” terms cancel each other, and our equation is correct. If we had written the correct terms in the first place it would have looked complicated, and in the end we would have canceled out the complicated part and gotten the same result.

The volume is constant and cancels out of the equation. Also, we know that $\Sigma_a(E) + \Sigma_s(E) = \Sigma_t(E)$, which allows us to combine the absorption and outscattering integrals:

$$\begin{aligned}
 \frac{d}{dt} \left[\int_{\Delta E} dE n(E, t) \right] &= \left[\int_{\Delta E} dE \chi(E) \right] \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t) \\
 &\quad - \int_{\Delta E} dE \Sigma_t(E) v(E) n(E, t) + \int_{\Delta E} dE S_{\text{ext}}(E) \\
 &\quad + \int_{\Delta E} dE \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t) . \tag{E.17}
 \end{aligned}$$

We recognize that every term involves an integral over the energy interval ΔE , so we group the terms into a single integral:

$$\int_{\Delta E} dE F(E) = 0 , \quad (\text{E.18})$$

where we have defined a shorthand notation for the following combination of various rate densities:

$$\begin{aligned} F(E) \equiv & \frac{\partial}{\partial t} n(E, t) + \Sigma_t(E) v(E) n(E, t) - S_{\text{ext}}(E) \\ & - \chi(E) \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t) \\ & - \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t) . \end{aligned} \quad (\text{E.19})$$

Note that Eq. (E.18) holds for every possible ΔE that we might pick—we did not say that ΔE needed to be anything special. This means that the function $F(E)$ integrates to zero over every possible range of integration.

Q: What does that tell us about $F(E)$?

A: It must be zero for all E !

Given this important observation, we can rearrange Eq. (E.19) to obtain:

$$\begin{aligned} \frac{\partial}{\partial t} n(E, t) + \Sigma_t(E) v(E) n(E, t) &= S_{\text{ext}}(E) + \chi(E) \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) v(E_i) n(E_i, t) \\ &+ \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) v(E_i) n(E_i, t) . \end{aligned} \quad (\text{E.20})$$

Because reaction-rate terms always involve the product of speed times density, which we know is the **scalar flux**, it is usually more convenient to write our conservation equations in terms of the scalar flux, $\phi(E, t) = v(E) n(E, t)$, instead of the neutron density, $n(E, t)$. This is a simple substitution:

$$\begin{aligned} \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(E, t) + \Sigma_t(E) \phi(E, t) &= S_{\text{ext}}(E) + \chi(E) \int_0^\infty dE_i \nu(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ &+ \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) \phi(E_i, t) . \end{aligned} \quad (\text{E.21})$$

This is an

integro-differential equation

for the energy-dependent scalar flux in an infinite uniform medium. This equation is well-posed and has a unique solution. If the source is known and the cross sections are known, then this problem can be solved to whatever accuracy we require.

That is, since we have now explicitly recognized conservation for every sub-population of neutrons that affects gain and loss (i.e., the neutrons in every spatial volume in every energy interval), we are not forced to make approximations about neutron distributions. This is in sharp contrast to the situation in the previous subsection (case 2), where we did not recognize conservation in every energy interval and ended up with an unsolvable equation.

E.2.4 Case 4: Population = all n's in V with $E \in \Delta E$; Setting = energy-dependent neutrons; considering delayed neutrons

Here we continue to study an infinite medium but this time we make **no significant approximations** to the physics. We account for the true distribution of neutrons in energy and for the true physics of delayed neutrons. Now our gain mechanisms are:

1. Emission from the fixed source,
2. Emission of **prompt** neutrons from neutron-induced fission,
3. Scattering into ΔE from other energies, and
4. Emission of **delayed neutrons** from neutron-induced fission

Our loss mechanisms are:

1. Absorption, and
2. Scattering from energies within ΔE to other energies

Again, there is no net leakage from any volume in our infinite medium, because the neutrons are distributed uniformly in position and direction—gain from inleakage cancels with loss from outleakage.

We define:

$$\begin{aligned} S_{dn}(E, t) &= \text{energy-dependent delayed-neutron source-rate density,} \\ \chi_p(E) &= \text{fission spectrum of prompt neutrons,} \\ \nu_p(E) &= \text{average number of prompt neutrons emitted,} \\ &\quad \text{from a fission caused by a neutron of energy } E . \end{aligned}$$

Everything proceeds just as in the previous subsection, and we end up with:

$$\begin{aligned} \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(E, t) + \Sigma_t(E) \phi(E, t) &= S_{\text{ext}}(E) + \color{red} S_{\text{dn}}(E, t) \\ &\quad + \chi_p(E) \int_0^\infty dE_i \nu_p(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ &\quad + \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) \phi(E_i, t) . \end{aligned} \quad (\text{E.22})$$

The big difference here is that the delayed-neutron source, $S_{\text{dn}}(E, t)$, is not known—it is something we have to solve for, just like the scalar flux $\phi(E, t)$. How do we do this? First we need to understand how delayed neutrons come about.

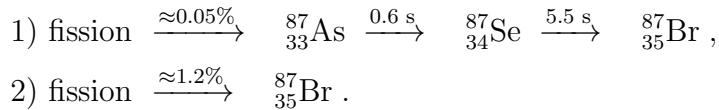
E.3 Delayed Neutrons

E.3.1 Delayed-neutron physics and math

Let us review the processes that lead to emission of delayed neutrons. Consider the decay scheme for Bromine-87, as shown in Fig. E.2. The ^{87}Br atoms **that eventually decay to metastable ^{87}Kr** (and thus yield a neutron) are called

delayed-neutron precursors

(These are only $\approx 2.6\%$ of the ^{87}Br atoms.) Notice that the precursor does not actually expel a neutron. Instead, it decays to an excited state of another nuclide, which then emits the neutron. Also note that the precursor does not have to be a direct fission fragment, but may simply be in the decay chain of a fission fragment. Two ways to produce Bromine-87, for example, are:



The delayed neutron in question is delayed by the sum of the decay times of the precursor and its parents, in the whole chain all the way back to the fission event whose fission product is the start of the chain.

Approximately 270 delayed-neutron precursors have been identified to date. Of these, only a few dozen are of much practical significance.

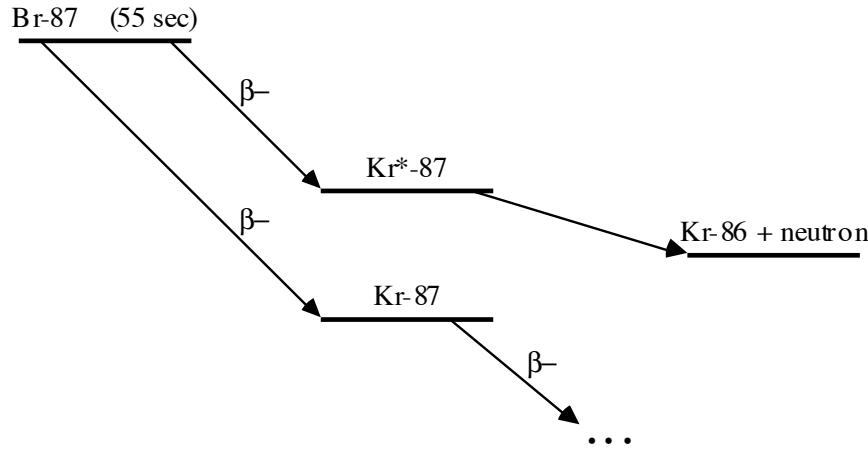


Figure E.2: Decay scheme for ^{87}Br , 2.6% of the atoms of which are **delayed-neutron precursors**. (Show 2.6% to Kr^*-87 and 97.4% to Kr-87 .)

We seek an equation or set of equations that will give us $S_{\text{dn}}(E, t)$, the energy-dependent delayed-neutron emission rate density, which appears in the equation for the energy-dependent scalar flux in our infinite-medium problem. Let us define some helpful quantities:

$C_i(t)$ = expected number density of type- i **precursors** at time t .

λ_i = decay constant of type- i precursors.

$\chi_i(E)$ = energy spectrum of neutrons emitted after decay of type- i precursors.

From these definitions it follows that

$$S_{\text{dn}}(E, t) = \sum_{i=1}^{\# \text{ of precursor types}} \chi_i(E) \lambda_i C_i(t) \quad (\text{E.23})$$

This is fine, but it doesn't help much unless we know $C_i(t)$ for each precursor type, i . How do we find these precursor densities?

As usual, we turn to our favorite equation:

$$\text{Change rate} = \text{gain rate} - \text{loss rate}.$$

For what population of things should we write this equation?

$$\text{population} = \text{type-}i \text{ precursors in volume } V$$

We already know the loss rate from decay of the precursors—it is λ_i times the precursor population. But what is the gain rate?

Note that each precursor had its ultimate origin in fission. That is, each precursor is either a fission product or originated from the decay of a fission product. So we know that the production rate density of precursors will be tied to

the fission-rate density

Also note that

each precursor ultimately produces exactly one delayed neutron

It is useful and customary to define

$\nu_{di}(E) \equiv$ expected number of type- i precursors produced from
a fission that is caused by a neutron of energy E . (E.24)

Because each precursor ultimately produces exactly one delayed neutron, ν_{di} is also the expected number of delayed neutrons that will ultimately be emitted from type- i precursors, per fission caused by a neutron of energy E .

With these definitions we can now write our conservation equation for precursors of type i :

$$\frac{d}{dt}C_i(t) = \int_0^\infty dE \nu_{di}(E) \Sigma_f(E) \phi(E, t) - \lambda_i C_i(t) . \quad (\text{E.25})$$

E.3.2 Back to Case 4: Population = all n's in V with $E \in \Delta E$; Setting = energy-dependent neutrons; considering delayed neutrons

Let us collect what we have so far:

$$\begin{aligned} \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(E, t) + \Sigma_t(E) \phi(E, t) &= S_{\text{ext}}(E) + \sum_{i=1}^{\# \text{ of precursor types}} \chi_i(E) \lambda_i C_i(t) \\ &+ \chi_p(E) \int_0^\infty dE_i \nu_p(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ &+ \int_0^\infty dE_i \Sigma_s(E_i \rightarrow E) \phi(E_i, t) . \end{aligned} \quad (\text{E.26})$$

$$\frac{d}{dt}C_i(t) = \int_0^\infty dE \nu_{di}(E) \Sigma_f(E) \phi(E, t) - \lambda_i C_i(t) . \quad (\text{E.27})$$

We see that

- The scalar flux depends on the delayed-neutron precursor concentrations .
- The delayed-neutron precursors concentrations depend on the scalar flux .

Thus, we have a

coupled system

of equations to solve. While this may look like a complicated system that could be difficult to solve, it is at least a well-posed set of equations that has a unique solution (given initial conditions for the scalar flux and the precursor concentrations).

So once again conservation has come through for us! That is, we have written “change rate = gain rate – loss rate” in mathematical terms, for both the neutron population and the precursor populations, and arrived at a well-posed system of equations with a unique solution. The solution, if we can find it, tells us in detail how the neutrons and precursors behave in the setting we have considered.

I hope you see that conservation is a powerful tool with far-reaching implications!

E.4 Reactor Kinetics Equations

Let us see what we can learn about solutions of Eqs. (E.26) and (E.27). Reminder: these are for an infinite homogeneous reactor. We begin by integrating the first equation over all neutron energies:

$$\begin{aligned} \frac{d}{dt} \left[\int_0^\infty dE \frac{\phi(E, t)}{v(E)} \right] + \int_0^\infty dE \Sigma_t(E) \phi(E, t) &= \int_0^\infty dE S_{\text{ext}}(E) + \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) \\ &\quad + \int_0^\infty dE_i \nu_p(E_i) \Sigma_f(E_i) \phi(E_i, t) \\ &\quad + \int_0^\infty dE_i \Sigma_s(E_i) \phi(E_i, t) . \end{aligned} \quad (\text{E.28})$$

Here we have recognized that the integral of probability distribution functions, such as $\chi_p(E)$, $\chi_{di}(E)$, or $P(E_i \rightarrow E)$ (which is buried in $\Sigma_s(E_i \rightarrow E)$), equals 1. Now we subtract the scattering-rate density from both sides of the equation and change dummy variables E_i to E :

$$\begin{aligned} \frac{d}{dt} \left[\int_0^\infty dE \frac{\phi(E, t)}{v(E)} \right] + \int_0^\infty dE \Sigma_a(E) \phi(E, t) &= \int_0^\infty dE S_{\text{ext}}(E) + \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) \\ &\quad + \int_0^\infty dE \nu_p(E) \Sigma_f(E) \phi(E, t) . \end{aligned} \quad (\text{E.29})$$

Rewrite in terms of neutron density:

$$\frac{d}{dt} \left[\int_0^\infty dE n(E, t) \right] + \int_0^\infty dE \Sigma_a(E) v(E) n(E, t) = \int_0^\infty dE S_{\text{ext}}(E)$$

$$+ \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) + \int_0^\infty dE \nu_p(E) \Sigma_f(E) v(E) n(E, t). \quad (\text{E.30})$$

Recall previous definitions:

$$n_{\text{tot}}(t) \equiv \int_0^\infty dE n(E, t), \quad (\text{E.31})$$

$$S_{\text{tot}}(t) \equiv \int_0^\infty dE S_{\text{ext}}(E), \quad (\text{E.32})$$

$$\langle v\Sigma_a \rangle \equiv \frac{\int_0^\infty dE [v(E) \Sigma_a(E)] n(E, t)}{\int_0^\infty dE n(E, t)}, \quad (\text{E.33})$$

$$\langle v\nu\Sigma_f \rangle \equiv \frac{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE n(E, t)}. \quad (\text{E.34})$$

and add another:

$$\langle v\nu_p\Sigma_f \rangle \equiv \frac{\int_0^\infty dE [v(E) \nu_p(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE n(E, t)}. \quad (\text{E.35})$$

Define the

Delayed-neutron fraction, β ("beta"),

as follows:

$$\beta \equiv 1 - \frac{\int_0^\infty dE [v(E) \nu_p(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)} = \frac{\int_0^\infty dE [v(E) \nu_d(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)} \quad (\text{E.36})$$

$$= \frac{\langle v \nu_d \Sigma_f \rangle}{\langle v \nu \Sigma_f \rangle}$$

= fraction of fission neutrons that are born delayed.

Remark: It follows from previous definitions that

$$\nu(E) = \nu_p(E) + \sum_{i=1}^{\# \text{ of precursor types}} \nu_{di}(E) \quad (\text{E.37})$$

Rewrite the conservation equation in terms of these defined quantities:

$$\frac{d}{dt} [n_{\text{tot}}(t)] = [\langle v \nu_p \Sigma_f \rangle - \langle v \Sigma_a \rangle] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ of precursor types}} \lambda_i C_i(t) . \quad (\text{E.38})$$

(Remember: this is for an infinite homogeneous reactor.) Now define

$\ell_p \equiv$ average time from prompt-neutron birth to absorption

$$= \frac{1}{\langle v \Sigma_a \rangle} . \quad \text{only for infinite reactor!} \quad (\text{E.39})$$

That is,

ℓ_p is the prompt-neutron lifetime

Also define

$k \equiv$ multiplication factor

$$= \frac{\langle v\nu\Sigma_f \rangle}{\langle v\Sigma_a \rangle} \xrightarrow{\text{Eq. (E.36)}} = \frac{\langle v\nu_p\Sigma_f \rangle}{(1-\beta)\langle v\Sigma_a \rangle} . \quad (\text{E.40})$$

In terms of these quantities, our equation for n_{tot} can be written:

$$\begin{aligned} \frac{d}{dt} [n_{\text{tot}}(t)] &= \left[\frac{(1-\beta)k\langle v\Sigma_a \rangle - \langle v\Sigma_a \rangle}{1} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \\ &= \left[\frac{(1-\beta)k - 1}{\ell_p} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \\ &= \left[\frac{k - \beta k - 1}{\ell_p} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \\ &= \left[\frac{\frac{k-1}{k} - \beta}{\ell_p/k} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \end{aligned} \quad (\text{E.41})$$

We define

$$\rho \equiv \text{reactivity} = \frac{k-1}{k} , \quad (\text{E.42})$$

$$\Lambda \equiv \text{mean generation time} = \frac{\ell_p}{k} . \quad (\text{E.43})$$

= expected time for N neutrons to have N descendants .

With these definitions our equation for the neutron density can be written in a form that has become standard (and famous, or perhaps infamous from a student's point of view) for time-dependent neutronics calculations:

$$\frac{d}{dt} [n_{\text{tot}}(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) \quad (\text{E.45})$$

Similarly, we can write our precursor equation in standard form:

$$\frac{d}{dt} C_i(t) = \frac{\beta_i}{\Lambda} n_{\text{tot}}(t) - \lambda_i C_i(t) , \quad (\text{E.46})$$

where we have defined:

$$\beta_i \equiv \frac{\int_0^\infty dE [v(E) \nu_{di}(E) \Sigma_f(E)] n(E, t)}{\int_0^\infty dE [v(E) \nu(E) \Sigma_f(E)] n(E, t)} \quad (\text{E.47})$$

= fraction of fission neutrons that are type-*i* delayed.

We have now written our equations in the form of

coupled first-order ODEs .

We can learn a lot about the behavior of the neutron population by studying and solving these equations!

We observe that:

- The quantities ρ , β , β_i , and Λ are all defined in terms of $n(E, t)$, which of course depends on time. Thus, in general, these quantities all depend on time.
- It turns out that in many interesting problems, ρ , β , β_i , and Λ are all approximately constant. Thus, it is useful for us to study the solution of our coupled first-order ODEs in this interesting constant-parameter case. This case is simple enough to permit analytic solutions that are enormously helpful in building an understanding of nuclear-reactor behavior.

We repeat these very important equations here, Eqs. (E.45) and (E.46), and introduce the name by which they are known:

THE POINT-REACTOR KINETICS EQUATIONS (PRKEs):

$$\frac{d}{dt} [n_{\text{tot}}(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_i(t) , \quad (\text{E.48})$$

$$\frac{d}{dt} C_i(t) = \frac{\beta_i}{\Lambda} n_{\text{tot}}(t) - \lambda_i C_i(t) . \quad (\text{E.49})$$

The PRKEs are used extensively to study the time-dependent behavior of nuclear reactors. In Chapter 8, we study them.

E.5 Neutron Conservation in Finite Reactors

In this section we consider the Cases 5-9 as defined in the introduction. All of these involve **finite** reactors. In these settings we must address leakage in addition to production and loss from neutron-nucleus interactions. We also must worry about spatial distributions, because the reactor may have different materials in different places, which means reaction rates will depend on where the neutrons are in addition to how many there are.

E.5.1 Case 5: Population = all n's in the reactor; Setting = finite reactor; pretending that all neutrons are prompt (no delayed neutrons)

For this population our gain mechanisms are:

- Emission from the fixed source, if any
- Emission from neutron-induced fission
- **Inleakage** through the reactor surface

Our loss mechanisms are:

- Absorption
- **Outleakage** through the reactor surface

We take the difference [outleakage – inleakage] to create a **net**-outleakage loss term. Then we know how to write all of the terms in our conservation equation:

$$\begin{aligned} \frac{d}{dt} \left[\iiint_V d^3r \int_0^\infty dE n(\vec{r}, E, t) \right] &= \iiint_V d^3r \int_0^\infty dE \nu(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &\quad \iiint_V d^3r \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) - \iiint_V d^3r \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &\quad - \oint_{\partial V} d^2r \vec{e}_n(\vec{r}) \cdot \left[\int_0^\infty dE \vec{J}(\vec{r}, E, t) \right]. \end{aligned} \quad (\text{E.50})$$

Notation Notes:

1. It is common in mathematics to use the following notation to describe a surface:

$$\partial V \equiv \text{surface enclosing the volume } V .$$

2. The expected number of neutrons per fission, ν , always has the same arguments (i.e., depends on the same variables) as does Σ_f . Instead of writing both argument lists, **we henceforth adopt the shorthand notation:**

$$\nu(\vec{r}, E, t)\Sigma_f(\vec{r}, E, t) \rightarrow \nu\Sigma_f(\vec{r}, E, t) .$$

This doesn't mean ν is a constant! It still depends on \vec{r} , E , and t in general.

Now return to Eq. (E.50). We see the same difficulty that we had previously in CASE 2:

Some terms in our equation depend on neutron sub-populations

for which we haven't written a conservation equation.

For example, the absorption rate in the reactor depends on how the neutrons are distributed spatially, but we haven't enforced conservation on spatial subvolumes—only on the population in the entire reactor volume.

As we found in Case 2, we have a couple of choices. We could:

1. Define some average quantities, which we cannot calculate exactly, and continue to try to learn something;
2. Express conservation for a more refined neutron sub-population and hope this leads to a well-posed problem.

Here we will explore the first path; later we will explore the second. Learning from our previous experience, we define:

$$n_{\text{tot}}(t) \equiv \iiint_V d^3r \int_0^\infty dE n(\vec{r}, E, t), \quad (\text{E.51})$$

$$S_{\text{tot}} \equiv \iiint_V d^3r \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t). \quad (\text{E.52})$$

$$\begin{aligned} \langle v\nu\Sigma_f \rangle &\equiv \frac{\iiint_V d^3r \int_0^\infty dE \nu\Sigma_f(\vec{r}, E, t)\phi(\vec{r}, E, t)}{\iiint_V d^3r \int_0^\infty dE n(\vec{r}, E, t)} \\ &\equiv \frac{\iiint_V d^3r \int_0^\infty dE [\nu\Sigma_f(\vec{r}, E, t)v(E)] n(\vec{r}, E, t)}{\iiint_V d^3r \int_0^\infty dE n(\vec{r}, E, t)}. \end{aligned} \quad (\text{E.53})$$

$$\langle v\Sigma_a \rangle \equiv \frac{\iiint_V d^3r \int_0^\infty dE [\Sigma_a(\vec{r}, E, t)v(E)] n(\vec{r}, E, t)}{\iiint_V d^3r \int_0^\infty dE n(\vec{r}, E, t)}. \quad (\text{E.54})$$

We also need to do something with the net outleakage term. The ratio we need is clear—it is just the term itself divided by n_{tot} —but we need a name and symbol for this ratio that reminds us of its physical meaning. Note that it is a loss term, as is the absorption term, and it has the same units. The absorption-term ratio is formally an average of [speed \times Σ_a], and we give it the symbol $\langle v\Sigma_a \rangle$. Let us then define the net-outleakage term to be an average of [speed \times Σ_{esc}] $\equiv \langle v\Sigma_{\text{esc}} \rangle$, where we shall call Σ_{esc} the “escape cross section.” It is not really a cross section—it does not describe neutron-nucleus interactions—so we could give it any name we wished. However, because it has the physical meaning of expected net number of neutrons leaking from the reactor per unit distance traveled by neutrons in the reactor, like real macroscopic cross sections do, it is common practice to call it an “escape cross section.”

The averaged speed times escape cross section is:

$$\langle v\Sigma_{\text{esc}} \rangle \equiv \frac{\oint_{\partial V} d^2r \int_0^\infty dE \vec{e}_n(\vec{r}) \cdot \vec{J}(\vec{r}, E, t)}{\iiint_V d^3r \int_0^\infty dE n(\vec{r}, E, t)}. \quad (\text{E.55})$$

Question: What are the units of the ⟨bracketed⟩ averaged quantities?

Answer: inverse time

With these definitions, our conservation equation becomes:

$$\frac{d}{dt} [n_{\text{tot}}(t)] = [\langle v\nu\Sigma_f \rangle - \langle v\Sigma_a \rangle - \langle v\Sigma_{\text{esc}} \rangle] n_{\text{tot}}(t) + S_{\text{tot}}(t). \quad (\text{E.56})$$

We can easily solve this equation using an integrating factor. If we pretend for a moment that the quantities inside the [] are constant, and if S_{tot} is also constant, the solution is relatively simple to write:

$$n_{\text{tot}}(t) = n_{\text{tot}}(0)e^{\alpha t} + S_{\text{tot}} \left[\frac{e^{\alpha t} - 1}{\alpha} \right] \quad (\text{E.57})$$

where we have defined

$$\alpha \equiv \langle v\nu\Sigma_f \rangle - \langle v\Sigma_a \rangle - \langle v\Sigma_{\text{esc}} \rangle \quad (\text{E.58})$$

Several observations are in order:

1. Because α is defined in terms of $n(\vec{r}, E, t)$, which depends on t , α will in general depend on t . (But not on position or energy, because position and energy are integrated out in the definition.)
2. Our equation doesn't tell us anything about the spatial and energy distribution of the neutron density, so it doesn't tell us how to calculate the terms in the definition of α . We could approximate them somehow, but the equation we've just derived doesn't help us with that.
3. Even though we don't know the exact values of the terms in α , the equation still shows a lot about how the total neutron density changes with time in a reactor.
4. Note that Eq. (E.57) is actually the same expression as Eq. (E.12), where the definition of α was slightly altered to account for the net leakage out of the system!

Next we shall remove the only unphysical approximation we made in this case (Case 5), which is that we ignored delayed neutrons.

E.5.2 Case 6: Population = all n's in the reactor; Setting = finite reactor; proper treatment of delayed neutrons

For this population our gain mechanisms are:

- Emission from the fixed source
- Prompt emission from neutron-induced fission
- Delayed emission from neutron-induced fission
- Inleakage through the reactor surface

Our loss mechanisms are:

- Absorption
- Outleakage through the reactor surface

We recall that the emission rate of delayed neutrons is just the decay rate of the delayed-neutron precursors (d.n.p.'s). Thus, we know how to write all of the terms in our conservation equation:

$$\begin{aligned} \frac{d}{dt} \left[\iiint_V d^3r \int_0^\infty dE n(\vec{r}, E, t) \right] &= \iiint_V d^3r \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &+ \iiint_V d^3r \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i \iiint_V d^3r C_i(\vec{r}, t) \\ &- \iiint_V d^3r \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) - \iint_{\partial V} d^2r \vec{e}_n(\vec{r}) \cdot \int_0^\infty dE \vec{J}(\vec{r}, E, t) . \quad (\text{E.59}) \end{aligned}$$

We define

$$C_{i,\text{tot}} \equiv \iiint_V d^3r C_i(\vec{r}, t) . \quad (\text{E.60})$$

Then, with our previous and similar definitions, Eq. (E.59) becomes

$$\frac{d}{dt} [n_{\text{tot}}(t)] = [\langle v \nu_p \Sigma_f \rangle - \langle v \Sigma_a \rangle - \langle v \Sigma_{\text{esc}} \rangle] n_{\text{tot}}(t) + S_{\text{tot}} + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_{i,\text{tot}}(t) . \quad (\text{E.61})$$

Of course, we also need equations for the delayed-neutron precursor populations. If we follow the same logic as in Case 4, we obtain an equation very similar Eq. (E.27) from that case:

$$\frac{d}{dt} C_{i,\text{tot}}(t) = \iiint_V d^3r \int_0^\infty dE \nu_{di} \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) - \lambda_i C_{i,\text{tot}}(t) . \quad (\text{E.62})$$

Continuing to follow CASE 4, we define several quantities. The difference is that now we must account for leakage:

$$\beta \equiv 1 - \frac{\iiint_V d^3r \int_0^\infty dE [v(E)\nu_p \Sigma_f(\vec{r}, E, t)] n(\vec{r}, E, t)}{\iiint_V d^3r \int_0^\infty dE [v(E)\nu \Sigma_f(\vec{r}, E, t)] n(\vec{r}, E, t)} \quad (\text{E.63})$$

$$\ell_p = \frac{1}{\langle v\Sigma_a \rangle + \langle v\Sigma_{\text{esc}} \rangle} , \quad (\text{E.64})$$

$$k \equiv \frac{\langle v\nu\Sigma_f \rangle}{\langle v\Sigma_a \rangle + \langle v\Sigma_{\text{esc}} \rangle} = \frac{\langle v\nu_p\Sigma_f \rangle / (1 - \beta)}{\langle v\Sigma_a \rangle + \langle v\Sigma_{\text{esc}} \rangle} . \quad (\text{E.65})$$

With previous definitions we can now perform the same algebraic manipulations as in Case 4, allowing us to rewrite Eqs. (E.61) and (E.62) as follows:

$$\frac{d}{dt} [n_{\text{tot}}(t)] = \left[\frac{\rho - \beta}{\Lambda} \right] n_{\text{tot}}(t) + S_{\text{tot}} + \sum_{i=1}^{\# \text{ prec. types}} \lambda_i C_{i,\text{tot}}(t) , \quad (\text{E.66})$$

$$\frac{d}{dt} C_{i,\text{tot}}(t) = \frac{\beta_i}{\Lambda} n_{\text{tot}}(t) - \lambda_i C_{i,\text{tot}}(t) . \quad (\text{E.67})$$

We cannot help but notice that these are

exactly the same PRKES that we found for infinite reactors (case 4) !!

The only differences are buried in the definitions of the coefficients:

1. The prompt-neutron lifetime, ℓ_p is **shortened** by the possibility of leakage.
2. The multiplication factor, k , is **reduced** by loss from leakage.
3. All averaged quantities are averaged over **position** as well as **energy**.
4. The quantities n_{tot} and $C_{i,\text{tot}}$ are **total numbers** in the reactor, not densities.

Also, again several observations are in order:

1. Because the coefficients in the point-reactor kinetics equations (PRKEs) are defined in terms of n , which depends on t , they will in general depend on t . (But not on position or energy, because position and energy are integrated out in the definitions of the terms.)
2. The conservation equation we derived here doesn't tell us anything about the spatial and energy distribution of the neutron density, so it doesn't tell us how to calculate the various parameters in the PRKEs. We could approximate them somehow, but the equation we've just derived doesn't help us with that.
3. Even though we don't know the exact values of the terms in the PRKEs, the equation still tells us a lot about how the neutron population changes with time in a reactor. See PRKE solutions in Chapter 8.
4. We made no significant approximations here, and our setting is the real world. That is, the PRKEs are an essentially exact description of how the neutron population varies in time in real reactors! (Our only unresolved issue is how to find the constants in the equations.)

E.5.3 Case 7: Population = all n's in a given sub-volume of the reactor;

Setting = finite reactor with proper treatment of delayed neutrons

For this population our gain mechanisms are:

- Emission from the fixed source in sub-volume ΔV
- Prompt emission from neutron-induced fission in sub-volume ΔV
- Delayed emission from neutron-induced fission in sub-volume ΔV
- Inleakage through the sub-volume surface, $\partial\Delta V$

Our loss mechanisms are:

- Absorption in sub-volume ΔV
- Outleakage through the sub-volume surface , $\partial\Delta V$

As we have done before, we combine (outleakage – inleakage) into

net outleakage

Our conservation equation looks just like that of the previous case (in which conservation was for the neutrons in the entire reactor volume), except that the volume of integration is smaller:

$$\begin{aligned} \frac{d}{dt} \left[\iiint_{\Delta V} d^3r \int_0^\infty dE n(\vec{r}, E, t) \right] &= \iiint_{\Delta V} d^3r \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &+ \iiint_{\Delta V} d^3r \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i \iiint_{\Delta V} d^3r C_i(\vec{r}, t) \\ &- \iiint_{\Delta V} d^3r \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) - \iint_{\partial \Delta V} d^2r \int_0^\infty dE \vec{e}_n(\vec{r}) \cdot \vec{J}(\vec{r}, E, t) . \quad (\text{E.68}) \end{aligned}$$

We can use a mathematical “trick” to write the surface integral of the sub-volume as a volume integral. Recall Gauss’s divergence theorem:

$$\iiint_{\Delta V} d^3r \vec{\nabla} \cdot \vec{u} = \iint_{\partial \Delta V} d^2r \vec{e}_n \cdot \vec{u} . \quad (\text{E.69})$$

Let’s use this on the leakage term in our conservation equation:

$$\begin{aligned} \frac{d}{dt} \left[\iiint_{\Delta V} d^3r \int_0^\infty dE n(\vec{r}, E, t) \right] &= \iiint_{\Delta V} d^3r \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &+ \iiint_{\Delta V} d^3r \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i \iiint_{\Delta V} d^3r C_i(\vec{r}, t) \\ &- \iiint_{\Delta V} d^3r \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) - \iiint_{\Delta V} d^3r \int_0^\infty dE \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) . \quad (\text{E.70}) \end{aligned}$$

We recognize that

every term is an integral over the sub-volume .

Thus, we can collect everything into a single integral over the sub-volume:

$$\begin{aligned}
 & \iiint_{\Delta V} d^3r \left\{ \frac{\partial}{\partial t} \int_0^\infty dE n(\vec{r}, E, t) - \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \right. \\
 & - \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) - \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_i(\vec{r}, t) \\
 & \left. + \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) + \int_0^\infty dE \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) \right\} = 0 . \quad (\text{E.71})
 \end{aligned}$$

This equation says that the integral of some function over some sub-volume of the reactor equals zero. This must be true for any sub-volume of the reactor, because we did not choose any special sub-volume for our conservation statement. Thus, we see that the integral of the function inside the {} integrates to zero for any range of integration. Thus, the function must be zero, and we must have:

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[\int_0^\infty dE n(\vec{r}, E, t) \right] + \int_0^\infty dE \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) + \int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t) \\
 & = \int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t) \\
 & + \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_i(\vec{r}, t) \quad (\text{E.72})
 \end{aligned}$$

In this equation, every term has units of:

$$\text{n/cm}^3\text{-s}$$

That is, every term is a “something” rate density, where every “something” is either change or some kind of gain or loss, and where the “density” is in ordinary 3-D space.

We can already recognize a problem that we have encountered previously with our conservation equations: Even if we knew

the conserved quantity, $\int dE n(\vec{r}, E, t)$,

it would not give us enough information to calculate

the absorption and fission rate densities

that appear in the equation, because these rates depend on the
energy distribution of the neutrons

and not just on the total neutron density. As before, we recognize the source of this problem:

we have not stated conservation in enough detail.

Nevertheless, as we have done before, we shall press on by defining some average quantities. We shall proceed slightly differently than before, just to illustrate that there are many ways to attack these problems. This time we choose the **scalar flux** instead of the **neutron density** as the fundamental unknown. In this case the neutron speed will be included in the weight function in the averaged cross sections, $\langle \Sigma \rangle$, instead of being part of the averaged quantity, as with our previous $\langle v\Sigma \rangle$ averages. That is, we define:

$$\left\langle \frac{1}{v} \right\rangle \equiv \frac{\int_0^\infty dE \frac{\phi(\vec{r}, E, t)}{v(E)}}{\int_0^\infty dE \phi(\vec{r}, E, t)}, \quad (\text{E.73})$$

$$\langle \Sigma_a(\vec{r}, t) \rangle \equiv \frac{\int_0^\infty dE \Sigma_a(\vec{r}, E, t) \phi(\vec{r}, E, t)}{\int_0^\infty dE \phi(\vec{r}, E, t)}, \quad (\text{E.74})$$

$$\langle \nu_p \Sigma_f(\vec{r}, t) \rangle \equiv \frac{\int_0^\infty dE \nu_p(\vec{r}, E, t) \Sigma_f(\vec{r}, E, t) \phi(\vec{r}, E, t)}{\int_0^\infty dE \phi(\vec{r}, E, t)}. \quad (\text{E.75})$$

and we recall the definitions of “total” scalar flux and “total” net current density from a previous chapter:

$$\phi(\vec{r}, t) \equiv \int_0^\infty dE \phi(\vec{r}, E, t), \quad (\text{E.76})$$

$$\vec{J}(\vec{r}, t) \equiv \int_0^\infty dE \vec{J}(\vec{r}, E, t). \quad (\text{E.77})$$

We make a similar definition for the fixed source rate density:

$$S_{\text{ext}}(\vec{r}, t) \equiv \int_0^\infty dE S_{\text{ext}}(\vec{r}, E, t). \quad (\text{E.78})$$

With these definitions we can rewrite our conservation equation as:

$$\begin{aligned} \left\langle \frac{1}{v} \right\rangle \frac{\partial}{\partial t} \phi(\vec{r}, t) + \vec{\nabla} \cdot \vec{J}(\vec{r}, t) + \langle \Sigma_a(\vec{r}, t) \rangle \phi(\vec{r}, t) \\ = \langle \nu_p \Sigma_f(\vec{r}, t) \rangle \phi(\vec{r}, t) \\ + S_{\text{ext}}(\vec{r}, t) + \sum_{i=1}^{\# \text{ dnp types}} \lambda_i C_i(\vec{r}, t) \end{aligned} \quad (\text{E.79})$$

We also need equations for the delayed-neutron precursor concentrations, C_i , which appear in this neutron conservation equation. If we follow the same kind of procedure as before, but expressing precursor production in terms of the total scalar flux, we obtain:

$$\frac{\partial}{\partial t} C_i(\vec{r}, t) = \langle \nu_{di} \Sigma_f(\vec{r}, t) \rangle \phi(\vec{r}, t) - \lambda_i C_i(\vec{r}, t) . \quad (\text{E.80})$$

IMPORTANT:

Understand the physical meaning of each term in these equations!

Equation (E.79) is an exact conservation equation for the neutrons in a reactor, provided that the terms in angle brackets, $\langle \rangle$, are averaged with the exact energy-dependent scalar flux. Of course, we don't know the energy-dependent scalar flux in advance, and this equation does not give us the opportunity to calculate it. Further, we don't have any equations for

the net current density, \vec{J} ,

nor do we have any obvious way to express the divergence of \vec{J} in terms of the scalar flux. So we have an exact equation but cannot solve it without some additional knowledge. This is the difficulty with not doing conservation in enough detail.

So let's try conservation in even more detail.

E.5.4 Case 8: Population = n's in sub-volume with energies in sub-interval;

Setting = finite reactor with proper treatment of delayed neutrons

For this population our gain mechanisms are:

- Emission from the fixed source
- Prompt emission from neutron-induced fission
- Delayed emission from neutron-induced fission
- Inleakage through the sub-volume surface
- “Inscattering” from energies outside the sub-interval to energies inside it .
- “Within-interval scattering” from energies inside the sub-interval to energies inside it

Our loss mechanisms are:

- Absorption
- Outleakage through the sub-volume surface
- “Outscattering” from energies inside the sub-interval to energies outside it.
- “Within-interval scattering” from energies inside the sub-interval to energies inside it

As we have done before, we combine (outleakage – inleakage) into net outleakage.

We have previously done all of the work required to write mathematical expressions for each of the gain and loss terms; we just need to put it all together. We treated everything except in- and out-scattering in CASE 7, and we treated scattering in CASE 4. Recall the schematic illustration of in- and out-scattering depicted in Fig. E.3.

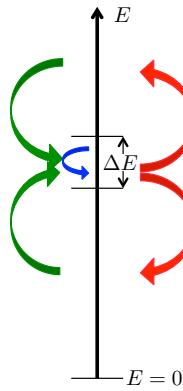


Figure E.3: Schematic illustration of “in-scattering” from energies outside of ΔE to inside it (green arrows), “out-scattering” from energies inside ΔE to energies outside it (red arrows), and “within-interval scattering” (blue arrow).

So let's put it all together. Our conservation equation for the sub-population of neutrons whose positions are in a given sub-volume (which we shall call ΔV) and whose energies are in a given sub-interval (which we shall call ΔE) is:

$$\begin{aligned}
 & \frac{d}{dt} \underbrace{\left[\iiint_{\Delta V} d^3 r \int_{\Delta E} dE n(\vec{r}, E, t) \right]}_{\text{neutrons in } \Delta V \text{ and } \Delta E} \\
 &= \underbrace{\left[\int_{\Delta E} dE \chi_p(E) \right]}_{\text{frac. of fission n's born in } \Delta E} \underbrace{\iiint_{\Delta V} d^3 r \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t)}_{\text{n/s born promptly from fission in } \Delta V} \\
 &+ \underbrace{\iiint_{\Delta V} d^3 r \int_{\Delta E} dE S_{\text{ext}}(\vec{r}, E, t)}_{\text{n/s born from fixed source in } \Delta V \text{ and } \Delta E} \\
 &+ \sum_{i=1}^{\# \text{ dnp types}} \underbrace{\left[\int_{\Delta E} dE \chi_{di}(E) \right]}_{\text{frac. of type-}i \text{ delayed n's born in } \Delta E} \underbrace{\lambda_i \iiint_{\Delta V} d^3 r C_i(\vec{r}, t)}_{\text{type-}i \text{ delayed n's emitted per s in } \Delta V} \\
 &+ \underbrace{\iiint_{\Delta V} d^3 r \int_{\Delta E} dE \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E, t) \phi(\vec{r}, E', t)}_{\text{n/s scattered into } \Delta E \text{ from initial energies inside or outside } \Delta E, \text{ in } \Delta V}
 \end{aligned}$$

Now we recognize that:

1.

$$\int_0^\infty dE_f \Sigma_s(\vec{r}, E \rightarrow E_f, t) = \Sigma_s(\vec{r}, E, t) ,$$

2.

$$\Sigma_s + \Sigma_a = \Sigma_t ,$$

3.

$$\oint_{\partial\Delta V} d^2r \vec{e}_n(\vec{r}) \cdot \vec{J}(\vec{r}, E, t) = \iiint_{\Delta V} d^3r \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) .$$

4. Given the last identity, every term in Eq. (E.81) is an integral over $\Delta V \Delta E$ (which is some arbitrarily chosen four-dimensional phase-space volume).

If we put this together we obtain:

$$\begin{aligned} & \iiint_{\Delta V} d^3r \int_{\Delta E} dE \left\{ \frac{\partial}{\partial t} [n(\vec{r}, E, t)] - S_{\text{ext}}(\vec{r}, E, t) \right. \\ & - \chi_p(E) \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) - \sum_{i=1}^{\# \text{ dnp types}} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) \\ & \left. - \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E, t) \phi(\vec{r}, E', t) + \Sigma_t(\vec{r}, E, t) \phi(\vec{r}, E, t) + \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) \right\} = 0 . \end{aligned} \quad (\text{E.82})$$

We recognize that $n = \phi/v$. Then as before, we recognize that if the integral of a function is zero no matter what the range of integration, then the function must be zero. This tells us that:

The general

ENERGY-DEPENDENT NEUTRON CONSERVATION EQUATION:

$$\begin{aligned} & \frac{1}{v(E)} \frac{\partial}{\partial t} \phi(\vec{r}, E, t) + \vec{\nabla} \cdot \vec{J}(\vec{r}, E, t) + \Sigma_t(\vec{r}, E, t) \phi(\vec{r}, E, t) \\ &= S_{\text{ext}}(\vec{r}, E, t) + \chi_p(E) \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\ &+ \sum_{i=1}^{\# \text{ dnp types}} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) + \int_0^\infty dE' \Sigma_s(\vec{r}, E' \rightarrow E, t) \phi(\vec{r}, E', t). \end{aligned} \quad (\text{E.83})$$

We also have a conservation equation for our precursors:

$$\frac{\partial}{\partial t} C_i(\vec{r}, t) = \int_0^\infty dE' \nu_{di}(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) - \lambda_i C_i(\vec{r}, t). \quad (\text{E.84})$$

This is the most detailed and informative set of equations that we have derived so far. If we could solve it, we would obtain the scalar flux as a function of position and energy in our reactor. That is, we would know the complete spatial and energy distribution of the neutrons, and we could calculate any reaction rate we wanted.

Important note: We could manipulate these equations, defining terms but not making any approximations, and obtain:

the same PRKEs as before.

Furthermore, this time the coefficients in the PRKEs (ρ, β_i, Λ) can be calculated if we know the functions that appear in our conservation equation, namely $\phi(\vec{r}, E, t)$ and $\vec{J}(\vec{r}, E, t)$.

However, we still have a problem:

Eq. (E.83) gives ϕ in terms of \vec{J} , but

we don't have an equation for \vec{J} .

Where have we gone wrong?

Previously when we encountered roadblocks we could see that our problematic terms depended on a sub-population of the neutron population for which we had written our conservation statement. It turns out that this is the case once again, although it is not as easy to see!

As before, we have two options for proceeding. One is to

approximate something.

We follow this “approximating” option in Chapter 9.

The other option is to consider conservation in more detail. We shall follow this option in the next section, which is the final and most detailed case we will consider.

**E.5.5 Case 9: Population = n's in sub-volume with energies in sub-interval and directions in solid-angle cone;
Setting = finite reactor with proper treatment of delayed neutrons)**

Our setting is the real world (finite reactor, non-uniform structure, neutrons distributed in energy, delayed neutrons present). Our conservation statement will give us information about the spatial and energy and **directional** distribution of the neutrons. This will produce a set of equations whose solution tells us all we need to know about neutron behavior in reactors! The problem is that the equation is complicated enough that it is difficult to solve, even approximately.

For this population our gain mechanisms are:

- Emission from the fixed source
- Prompt emission from neutron-induced fission
- Delayed emission from neutron-induced fission
- Inleakage through the sub-volume surface
- “Inscattering” from energies outside the energy sub-interval and/or directions outside the directional cone to energies in the sub-interval and directions in the cone.

Our loss mechanisms are:

- Absorption
- Outleakage through the sub-volume surface
- “Outscattering” from energies in the energy sub-interval and directions in the directional cone to energies outside the sub-interval and/or directions outside the cone.

As we have done before, we combine (outleakage – inleakage) into net outleakage.

Before we can write mathematical expressions for the gain and loss rates, we must develop a way to describe the neutron distribution in direction, and to describe how scattering changes a neutron’s direction. We define the direction- and energy-dependent neutron density:

$n(\vec{r}, E, \vec{\Omega}, t) d^3r dE d\Omega \equiv$ expected number of neutrons in the volume d^3r at \vec{r} ,
with lab-frame kinetic energy in the interval dE at E ,
and with direction in the solid-angle cone $d\Omega$ around the direction $\vec{\Omega}$.

This is a density in a six-dimension phase space: 3 spatial dimensions, 1 energy dimension, and 2 direction dimensions. (It takes two numbers—say a polar angle and an azimuthal angle—to define a direction.) The units are neutrons per unit volume per unit energy per unit solid angle. The usual unit of solid angle is the **steradian**. So in our usual units we have $n/(\text{cm}^3\text{-MeV-ster})$.

Analogous to the energy-dependent scalar flux, which is speed times the energy-dependent neutron density, we define the energy-dependent **angular flux**:

$$\psi(\vec{r}, E, \vec{\Omega}, t) \equiv v(E)n(\vec{r}, E, \vec{\Omega}, t). \quad (\text{E.85})$$

Note the identity relating angular flux to scalar flux:

$$\phi(\vec{r}, E, t) \equiv v(E) \iint_{4\pi} d\Omega \psi(\vec{r}, E, \vec{\Omega}, t). \quad (\text{E.86})$$

Now recall the double-differential scattering cross section:

$\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t)dE d\Omega \equiv$ expected scatters per unit neutron path length,
 for incident neutrons of energy E' and direction $\vec{\Omega}'$,
 that produce scattered neutrons with kinetic energy in the interval dE at E
 and with direction in the solid-angle cone $d\Omega$ around the direction $\vec{\Omega}$.

This has units of inverse [cm-MeV-ster]. It is a scattering cross section at the incident neutron energy, multiplied by a distribution function in outgoing neutron energy and direction.

Now our conservation equation is

$$\begin{aligned}
& \frac{d}{dt} \left[\iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega n(\vec{r}, E, \vec{\Omega}, t) \right] \\
&= \left[\iint_{\Delta \Omega} d\Omega \frac{1}{4\pi} \right] \left[\int_{\Delta E} dE \chi_p(E) \right] \iiint_{\Delta V} d^3r \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
&+ \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega S_{\text{ext}}(\vec{r}, E, \vec{\Omega}, t) \\
&+ \sum_{i=1}^{\# \text{ dnp types}} \left[\iint_{\Delta \Omega} d\Omega \frac{1}{4\pi} \right] \left[\int_{\Delta E} dE \chi_{di}(E) \right] \lambda_i \iiint_{\Delta V} d^3r C_i(\vec{r}, t) \\
&+ \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E', \vec{\Omega}', t) \\
&- \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E \rightarrow E', \vec{\Omega} \rightarrow \vec{\Omega}', t) \psi(\vec{r}, E', \vec{\Omega}', t) \\
&- \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \Sigma_a(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
&- \iint_{\partial \Delta V} d^2r \int_{\Delta E} dE \iint_{\Delta \Omega} d\Omega \vec{e}_n(\vec{r}) \cdot \vec{\Omega} \psi(\vec{r}, E, \vec{\Omega}, t) . \tag{E.87}
\end{aligned}$$

We use the divergence theorem to convert the surface integral to a volume integral, and then every term is an integral over the six-dimensional volume [$\Delta V \Delta E \Delta \Omega$]. We recognize that the integral of a double-differential scattering cross section is just a scattering cross section, much as we did in the previous case. We recognize that $\Sigma_t = \Sigma_a + \Sigma_s$. We collect all the terms into one giant integral, then recognize (as before) that the integrand must be zero. This gives:

$$\begin{aligned}
& \frac{d}{dt} \left[n(\vec{r}, E, \vec{\Omega}, t) \right] + \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}, t) + \Sigma_t(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
&= \frac{\chi_p(E)}{4\pi} \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
&+ S_{\text{ext}}(\vec{r}, E, \vec{\Omega}, t) + \sum_{i=1}^{\# \text{ dnp types}} \frac{1}{4\pi} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) \\
&+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E', \vec{\Omega}'t) . \tag{E.88}
\end{aligned}$$

We can rewrite the neutron density, n , as the angular flux divided by the speed, ψ/v :

THE TIME-DEPENDENT NEUTRON TRANSPORT EQUATION:

$$\begin{aligned}
& \frac{1}{v(E)} \frac{d}{dt} \psi(\vec{r}, E, \vec{\Omega}, t) + \vec{\Omega} \cdot \vec{\nabla} \psi(\vec{r}, E, \vec{\Omega}, t) + \Sigma_t(\vec{r}, E, t) \psi(\vec{r}, E, \vec{\Omega}, t) \\
&= \frac{\chi_p(E)}{4\pi} \int_0^\infty dE' \nu_p(\vec{r}, E', t) \Sigma_f(\vec{r}, E', t) \phi(\vec{r}, E', t) \\
&+ S_{\text{ext}}(\vec{r}, E, \vec{\Omega}, t) + \sum_{i=1}^{\# \text{ dnp types}} \frac{1}{4\pi} \chi_{di}(E) \lambda_i C_i(\vec{r}, t) \\
&+ \int_0^\infty dE' \iint_{4\pi} d\Omega' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) \psi(\vec{r}, E', \vec{\Omega}'t) . \tag{E.89}
\end{aligned}$$

This is a single integro-differential equation for the single unknown function ψ . We can show that it has a unique solution, given appropriate boundary and initial conditions. Thus,

considering conservation in detail (i.e., in every part of a six-dimensional phase space that includes position, direction, and energy) has eliminated the difficulty we found in the previous cases!

Equation (E.89) is the **neutron transport equation**. It is an essentially exact description of the behavior of neutrons in reactors. It is a relatively complicated equation that is relatively difficult to solve. Fortunately, we can gain a great deal of insight into reactor behavior by using equations that contain simplifying approximations. We do so in the next chapter. Later in your studies you will return to the transport equation to get a more precise picture of some neutronics details.

E.6 Summary

In this chapter we have delved into the details of writing conservation equations for various defined populations in various settings. We consider a range of cases from the simplest population (all the neutrons) in the simplest setting (infinite uniform medium, pretending that all neutrons have the same energy and all fission neutrons emerge immediately) to the most complicated population (the neutrons in a spatial sub-volume, energy sub-interval, and directional sub-cone) and most complicated setting (finite heterogeneous reactor, neutrons distributed in position and energy and direction, delayed neutrons treated correctly).

In one case—Case 8—we derived an essentially exact statement of neutron conservation in a reactor. The only difficulty we found with our conservation equation is that it contains the unknown net current density, \vec{J} , in addition to the unknown scalar flux, ϕ . Essentially, this conservation equation is one integro-differential equation with four unknown functions: ϕ, J_x, J_y , and J_z . We will need equations for J_x, J_y , and J_z if we are to do anything quantitative with this conservation equation.

We deduced that the difficulty found in Case 8 arose because we had not considered conservation in sufficient detail. We remedied this in Case 9, which considered a population whose directions were in some sub-cone of the full 4π steradians. This led to the neutron transport equation, which is a single equation for a single unknown function. This seems to be what we have been seeking. However, it is a complicated equation that is difficult to solve. Thus, it is worthwhile to back up to the Case-8 conservation equation and see if we can make some reasonable approximations for \vec{J} that can allow us to work with a simpler equation that is still reasonably accurate and provides insight.

We found that in every case that treated delayed neutrons we could cast our conservation equations, along with the equations for the delayed-neutron-precursor concentrations, into the form of **Point-Reactor Kinetics Equations** (PRKEs). Even when we did not know how to calculate the coefficients in these equations, we could study the characteristics of their solutions in all scenarios of interest.

In particular, if the coefficients in the PRKEs—namely ρ , the $\{\beta_i\}$, and Λ —are constant (which they often are for interesting problems), then we can write down equations for $n_{\text{tot}}(t)$ and the $\{C_i(t)\}$. This is a natural segway to Chapter 8.

Appendix F

Derivation of Fick's Law

We can derive Fick's Approximation if we look at a simple enough problem. We assume that:

1. the medium (containing the neutrons) is infinite;
2. the medium is uniform, meaning cross sections do not vary with \vec{r} ;
3. the fixed source varies only slightly over the distance of a mean-free path;
4. scattering is isotropic in the lab frame;
5. scalar flux, $\phi(\vec{r})$, is a slowly-varying function of position (does not change much over the distance of one mean free path);
6. everything is in steady state (no time dependence);
7. all neutrons have the same energy (mono-energetic case).

The idea is to express the net current density, \vec{J} , as a function of the scalar flux, ϕ . We recall that

$$\begin{aligned} \vec{e}_z \cdot \vec{J} dx dy &= J_z dx dy \\ &= \text{net n/s crossing the area } dx dy \\ &= (\text{n/s crossing upward} - \text{n/s crossing downward}). \end{aligned}$$

We can express this in terms of “**partial currents**,” which describe one-way flow rates:

$$(\text{n/s crossing upward} - \text{n/s crossing downward}) = (J_z^+ - J_z^-) dx dy$$

The scalar functions J_z^+ and J_z^- are called “partial currents” or “one-way currents” in the $+$ and $-z$ directions. Partial currents can be used to describe one-way (not net) flow rates across surfaces.

Given our simplifying assumptions, we can approximately calculate these upward and downward crossing rates. Consider Fig. F.1.

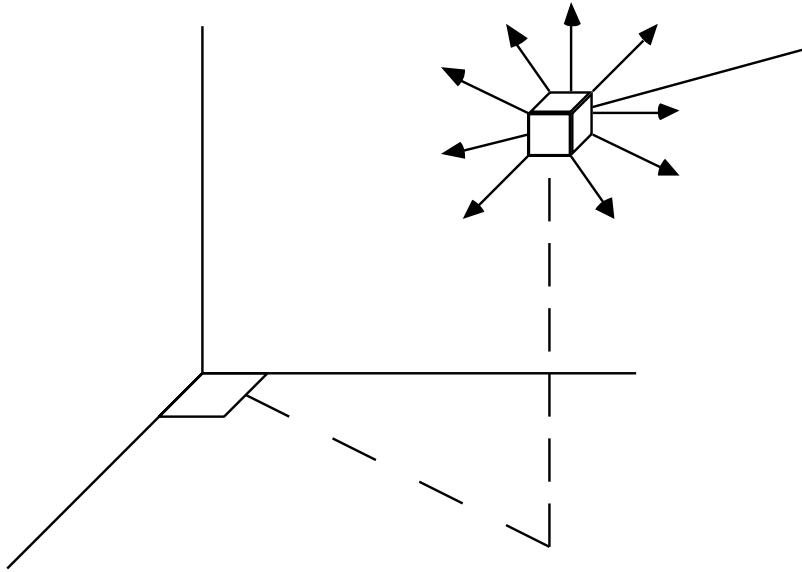


Figure F.1: Illustration of coordinate system, volume, and area used in derivation of Fick's Approximation. $dxdy$ area at origin; d^3r' volume; show x', y', z' ; show θ .

Note: all neutrons that cross $dxdy$ must have streamed to this area directly after being emitted by the extraneous source or directly from a scattering event. With this observation, we now figure out the rate at which neutrons cross $dxdy$ after either scattering or being “born” in some volume element, d^3r' . First, the rate at which neutrons in d^3r' scatter is:

$$\text{scattering rate} = \Sigma_s \phi(\vec{r}') d^3r'. \quad (\text{F.1})$$

Second, the rate at which neutrons are “born” in d^3r' from the extraneous (“fixed”) source is:

$$\text{birth rate} = S_{\text{ext}}(\vec{r}') d^3r'. \quad (\text{F.2})$$

We combine these by defining the total source:

$$S_{\text{tot}}(\vec{r}') = S_{\text{ext}}(\vec{r}') + \Sigma_s \phi(\vec{r}') \quad (\text{F.3})$$

Third, the fraction that head toward $dxdy$ is

$$\text{fraction} = \cos \theta \frac{dxdy}{4\pi|r'|^2} \quad (\text{F.4})$$

(If $dxdy$ were tangent to the surface of a sphere centered at \vec{r}' , the fraction would just be $dxdy / (\text{area of sphere})$. The $\cos \theta$ factor is because $dxdy$ is not tangent.)

Fourth, the fraction that actually gets to $dxdy$ without colliding is

$$\text{fraction} = \exp(-\Sigma_t |\vec{r}'|); \quad (\text{F.5})$$

If we put this all together, we find

$$\begin{aligned} & \text{n/s that cross } dxdy \text{ after scattering or birth in } d^3r' \\ &= S_{\text{tot}}(\vec{r}') \exp(-\Sigma_t |\vec{r}'|) \cos \theta \frac{dxdy}{4\pi|r'|^2} d^3r'. \end{aligned} \quad (\text{F.6})$$

The next step is to integrate over all differential volumes (d^3r') that are above the xy plane, thereby obtaining the total neutrons/second that flow downward through $dxdy$. When we try this, however, we encounter a difficulty: we don't know $\phi(\vec{r}')$! We get around that by invoking our assumptions that neither S_{ext} nor ϕ changes much over a distance of a few mean-free paths. We use a truncated Taylor-series expansion:

$$S_{\text{tot}}(x', y', z') = S_{\text{tot}}(0, 0, 0) + x' \left. \frac{\partial S_{\text{tot}}}{\partial x} \right|_{r=0} + y' \left. \frac{\partial S_{\text{tot}}}{\partial y} \right|_{r=0} + z' \left. \frac{\partial S_{\text{tot}}}{\partial z} \right|_{r=0} + \dots \quad (\text{F.7})$$

We then assume that: if we are so far from the origin that the truncated series is inaccurate,

then the $\exp(-\Sigma_t |\vec{r}'|)$ term makes it irrelevant.

Next is just integration over the volume above xy plane. We omit the details, which are tedious but not conceptually difficult. We find that:

$$J_z^-(0, 0, 0) \approx \frac{1}{\Sigma_t} \frac{S_{\text{tot}}(0, 0, 0)}{4} + \frac{1}{6\Sigma_t^2} \left. \frac{\partial S_{\text{tot}}}{\partial z} \right|_{r=0} \quad (\text{F.8})$$

The upward part is computed similarly:

$$J_z^+(0, 0, 0) \approx \frac{1}{\Sigma_t} \frac{S_{\text{tot}}(0, 0, 0)}{4} - \frac{1}{6\Sigma_t^2} \left. \frac{\partial S_{\text{tot}}}{\partial z} \right|_{r=0} \quad (\text{F.9})$$

Thus, we obtain the z-component of the net current density:

$$\begin{aligned} J_z(0, 0, 0) &= J_z^+(0, 0, 0) - J_z^-(0, 0, 0) \\ &\approx -\frac{1}{3\Sigma_t^2} \left. \frac{\partial S_{tot}}{\partial z} \right|_{r=0} \end{aligned} \quad (\text{F.10})$$

The other components are similar:

$$J_x(0, 0, 0) \approx -\frac{1}{3\Sigma_t^2} \left. \frac{\partial S_{tot}}{\partial x} \right|_{r=0} \quad (\text{F.11})$$

$$J_y(0, 0, 0) \approx -\frac{1}{3\Sigma_t^2} \left. \frac{\partial S_{tot}}{\partial y} \right|_{r=0} \quad (\text{F.12})$$

or, in shorthand (vector notation),

$$\vec{J}(\vec{r}) \approx -\frac{1}{3\Sigma_t^2} \vec{\nabla} S_{tot}(\vec{r}) . \quad (\text{F.13})$$

We dropped indication of evaluation at the origin, since the equation is valid at any point in the medium. (The location of the origin is arbitrary in an infinite medium.)

Now we attempt to express S_{tot} in terms of ϕ . Recall the balance equation:

$$\vec{\nabla} \cdot \vec{J}(\vec{r}) + \Sigma_a \phi(\vec{r}) = S_{ext}(\vec{r}) \quad (\text{F.14})$$

or (adding $\Sigma_s \phi$ to both sides)

$$\vec{\nabla} \cdot \vec{J}(\vec{r}) + \Sigma_t \phi(\vec{r}) = S_{tot}(\vec{r}) \quad (\text{F.15})$$

or, using what we have found for \vec{J} ,

$$S_{tot}(\vec{r}) \approx \Sigma_t \phi(\vec{r}) - \vec{\nabla} \cdot \frac{1}{3\Sigma_t^2} \vec{\nabla} S_{tot}(\vec{r}) . \quad (\text{F.16})$$

Given our assumption that S_{tot} is slowly varying, the second term in the last expression is much smaller than the first, and we have

$$S_{tot}(\vec{r}) \simeq \Sigma_t \phi(\vec{r}) \quad (\text{F.17})$$

This approximation yields:

$$\vec{J}(\vec{r}) \approx -\frac{1}{3\Sigma_t^2} \vec{\nabla} S_{tot}(\vec{r}) \approx -\frac{1}{3\Sigma_t^2} \vec{\nabla} [\Sigma_t \phi(\vec{r})] = -\frac{1}{3\Sigma_t} \vec{\nabla} \phi(\vec{r}) . \quad (\text{F.18})$$

This is a vector equation, which is the same thing as three scalar equations—one for each of the three vector components:

$$J_x(x, y, z) \approx -\frac{1}{3\Sigma_t} \frac{\partial \phi(x, y, z)}{\partial x}, \quad (\text{F.19})$$

$$J_y(x, y, z) \approx -\frac{1}{3\Sigma_t} \frac{\partial \phi(x, y, z)}{\partial y}, \quad (\text{F.20})$$

$$J_z(x, y, z) \approx -\frac{1}{3\Sigma_t} \frac{\partial \phi(x, y, z)}{\partial z}. \quad (\text{F.21})$$

We obtained this result by considering a very simple situation and making some simplifying approximations. A similar result can be shown to hold (approximately, anyway) under more general conditions:

$$\vec{J}(\vec{r}) \approx -D(\vec{r}) \vec{\nabla} \phi(\vec{r}). \quad (\text{F.22})$$

One-speed or one-group “diffusion approximation”