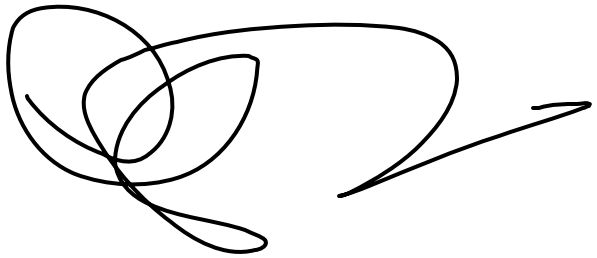
Chapter 7



Establishing Neutron Conservation Laws

7.1 Introduction

We know that a reactor's behavior depends on the gain and loss rates of neutrons in the reactor. We also know that some gain and loss is due to neutron-nucleus reactions, and that some is due to neutron leakage. In the previous chapter we developed mathematical expressions for reaction rates and leakage rates. In this chapter we put these expressions into statements of conservation to see what we can learn about how the neutron population changes with time in reactors and how the neutrons distribute themselves in position and energy.

To do this we will examine conservation statements for different neutron populations in different settings, ranging from simplistic to realistic. The populations and settings we consider are as follows.

- Conservation in an Infinite Uniform Medium:
  - Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.
    Setting = infinite uniform medium, pretending that all fission neutrons are emitted promptly.
    Goal: Establish time-dependent and energy-dependent balance of a population of neutrons, ignoring delayed neutrons. Later, we will slightly simplify this to study in depth neutron slowing-down,

  - Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.
    Setting = infinite uniform medium, accounting for delayed neutrons (which are emitted during the decay of fission products and their daughters).
    Goal: Establish the point reactor kinetics equations (time-dependent balance of a population of neutrons), **accounting** for delayed neutrons.

- Neutron Conservation in Finite Reactors:

  - Population = all neutrons in some chosen volume whose energies are in some chosen energy interval.
    Setting = finite reactor, accounting for delayed neutrons.
    Goal: The time-dependent, energy-dependent, space-dependent neutron conversation law (time-space-energy dependent).
  - Population = all neutrons in some chosen volume whose energies are in some chosen energy interval and whose directions are in some chosen directional range.
    Setting = finite reactor, accounting for delayed neutrons.
    Goal: The most detailed conservation law: the **neutron transport** equation (time-space-energy-direction dependent).

In every case we will follow the same procedure. **First** we will note that

$$\text{change rate} = \text{gain rate} - \text{loss rate}$$

for whatever population we have chosen. **Second**, for whatever neutron population we are considering, we will construct a mathematical expression for each of these three terms, consistent with the setting under consideration. **Third**, we will try to solve the resulting mathematical equation and learn what it tells us about the behavior of the neutron population.

7.2 Neutron Conservation in an Infinite Uniform Medium

In this section we consider the first two settings, which involve infinite homogeneous reactors. In these settings we do not have to worry about leakage, so we can focus our attention solely on production and loss from neutron-nucleus interactions. We also do not have to worry about spatial distributions, because everything is uniformly distributed throughout the infinite medium.

7.2.1 Case 1: Population = n's with positions in V and energies $E \in \Delta E$; Setting = energy-dependent neutrons and no delayed neutrons

Here we study an infinite medium and pretend that there are no delayed neutrons. Now that we are writing our conservation statement only for neutrons in a certain energy range—=(E_1, E_2) or "ΔE" for short —. Our gain mechanisms are:

1. Emission of neutron with $E \in \Delta E$ from the fixed (extraneous) source ,
2. Emission of prompt neutrons with $E \in \Delta E$ from neutron-induced fission ,
3. Scattering into ΔE from pre-scatter energies outside ΔE .

Our loss mechanisms are:

1. Absorption of neutron with $E \in \Delta E$,
2. Scattering of a neutron whose pre-scatter energy is in ΔE and whose post-scatter energy is not in ΔE.

There is no net leakage from any volume in our infinite medium, because the neutrons are distributed uniformly in position and direction, so gain from inleakage cancels with loss from outleakage, i.e. net leakage is zero.

We define:

$$
\begin{aligned}
n(E,t) &= \text{energy-dependent neutron density (see previous chapter)} ,\\
v(E) &= \text{speed of neutron whose kinetic energy is } E ,\\
S_{\text{ext}}(E) &= \text{energy-dependent fixed source-rate density} ,\\
\chi(E)dE &= \text{fraction of fission neutrons emitted with energies in } dE \text{ at } E
\end{aligned}
$$

The probability density function $\chi(E)$ is called the

fission spectrum .

It quantifies the energy distribution of the neutrons that are emitted from fission. (See Fig. 3.10.)

Our conservation statement for neutrons that are in the volume $\mathbf{V}$ and have energies in the interval ΔE is:

$$\underbrace{\frac{d}{dt}\left[\mathbf{V}\int_{\Delta E} dE\; n(E,t)\right]}_{\text{neutrons in } \mathbf{V} \text{ and } \Delta E} = \underbrace{\left[\int_{\Delta E} dE\; \chi(E)\right]}_{\text{frac. of fisn. n's born in } \Delta E} \underbrace{\mathbf{V}\int_0^\infty dE_i \nu(E_i)\Sigma_f(E_i)v(E_i)n(E_i,t)}_{\text{n/s born from fission}}$$

$$- \underbrace{\mathbf{V}\int_{\Delta E} dE\; \Sigma_a(E)v(E)n(E,t)}_{\text{absorptions/s}} + \underbrace{\mathbf{V}\int_{\Delta E} dE\; S_{\text{ext}}(E)}_{\text{n/s from fixed source}}$$

$$+ \underbrace{\mathbf{V}\int_{\Delta E} dE \int_0^\infty dE_i \Sigma_s(E_i \to E)v(E_i)n(E_i,t)}_{\text{n/s scattered into and within } \Delta E}$$

$$- \underbrace{\mathbf{V}\int_{\Delta E} dE\; \Sigma_s(E)v(E)n(E,t)}_{\text{n/s scattered out of and within } \Delta E} \quad . \tag{7.1}$$

Remark: There is a kind of "error" in the inscattering term and a similar "error" in the outscattering term. The "inscattering" term, as written, includes scattering from **all** possible initial energies, including initial energies that were already in ΔE. That is, as written it includes "within-interval" scattering as well as inscattering. The outscattering term has a similar error—it includes "within-interval" scattering as well as outscattering from the interval ΔE. Both terms contain exactly the same "within-interval" error term, one with a plus sign and one with a minus sign. So the two "error" terms cancel each other, and our equation is correct. If we had written the correct terms in the first place it would have looked complicated, and in the end we would have canceled out the complicated part and gotten the same result.

The volume is constant and cancels out of the equation. Also, we know that $\Sigma_a(E)+\Sigma_s(E) = \Sigma_t(E)$, which allows us to combine the absorption and outscattering integrals:

$$\frac{d}{dt}\left[\int_{\Delta E} dE\; n(E,t)\right] = \left[\int_{\Delta E} dE\; \chi(E)\right]\int_0^\infty dE_i \nu(E_i)\Sigma_f(E_i)v(E_i)n(E_i,t)$$

$$- \int_{\Delta E} dE\; \Sigma_t(E)v(E)n(E,t) + \int_{\Delta E} dE\; S_{\text{ext}}(E)$$

$$+ \int_{\Delta E} dE \int_0^\infty dE_i \Sigma_s(E_i \to E)v(E_i)n(E_i,t)\;. \tag{7.2}$$

We recognize that every term involves an integral over the energy interval ΔE, so we group the terms into a single integral:

$$\int_{\Delta E} dE\; F(E) = 0\;, \tag{7.3}$$

where we have defined a shorthand notation for the following combination of various rate densities:

$$F(E) \equiv \frac{\partial}{\partial t}n(E,t) + \Sigma_t(E)v(E)n(E,t) - S_{\text{ext}}(E)$$

$$- \chi(E)\int_0^\infty dE_i \nu(E_i)\Sigma_f(E_i)v(E_i)n(E_i,t)$$

$$- \int_0^\infty dE_i \Sigma_s(E_i \to E)v(E_i)n(E_i,t)\;. \tag{7.4}$$

Note that Eq. (7.3) holds for every possible ΔE that we might pick—we did not say that ΔE needed to be anything special. This means that the function $F(E)$ integrates to zero over every possible range of integration.

Q: What does that tell us about $F(E)$?

A: It must be zero for all E! This means that the integrand should always be zero

Given this important observation, we can rearrange Eq. (7.4) to obtain:

$$\frac{\partial}{\partial t}n(E,t) + \Sigma_t(E)\,v(E)\,n(E,t) \;= S_{\text{ext}}(E) + \chi(E)\int_0^\infty dE_i\,\nu(E_i)\Sigma_f(E_i)\,v(E_i)\,n(E_i,t)$$

$$+ \int_0^\infty dE_i\,\Sigma_s(E_i \to E)\,v(E_i)\,n(E_i,t)\;. \tag{7.5}$$

Because reaction-rate terms always involve the product of speed times density, which we know is the **scalar flux**, it is usually more convenient to write our conservation equations in terms of the scalar flux, $\phi(E,t) = v(E)n(E,t)$, instead of the neutron density, $n(E,t)$. This is a simple substitution:

$$\frac{1}{v(E)}\frac{\partial}{\partial t}\phi(E,t) + \Sigma_t(E)\,\phi(E,t) = S_{\text{ext}}(E) + \chi(E)\int_0^\infty dE_i\,\nu(E_i)\Sigma_f(E_i)\,\phi(E_i,t)$$
$$+ \int_0^\infty dE_i\,\Sigma_s(E_i \to E)\,\phi(E_i,t)\;. \tag{7.6}$$

This is an

integro-differential equation

for the energy-dependent scalar flux in an infinite uniform medium. This equation is well-posed and has a unique solution. If the source is known and the cross sections are known, then this problem can be solved to whatever accuracy we require.

That is, since we have now explicitly recognized conservation for every sub-population of neutrons that affects gain and loss (i.e., the neutrons in every spatial volume in every energy interval), we are not forced to make approximations about neutron distributions. However, there is a second source of neutrons in a reactor, which are the neutrons produced by the (β^- or n) decay of neutron precursors. This one is analyzed in the next section.

7.2.2 Case 2: Adding neutron precursors to Case 1

We add to our gain mechanism:

Emission of delayed neutrons from neutron-induced fission

We define:

$$\begin{aligned} S_{dn}(E,t) &= \text{energy-dependent delayed-neutron source-rate density,} \\ \chi_p(E) &= \text{fission spectrum of prompt neutrons,} \\ \nu_p(E) &= \text{average number of prompt neutrons emitted,} \\ &\quad \text{from a fission caused by a neutron of energy } E\;. \end{aligned}$$

Everything proceeds just as in the previous subsection, and we end up with:

$$\frac{1}{v(E)}\frac{\partial}{\partial t}\phi(E,t) + \Sigma_t(E)\,\phi(E,t) = S_{\text{ext}}(E) + S_{\text{dn}}(E,t)$$

$$+\chi_p(E)\int_0^\infty dE_i\,\nu_p(E_i)\Sigma_f(E_i)\,\phi(E_i,t)$$

$$+\int_0^\infty dE_i\,\Sigma_s(E_i \to E)\,\phi(E_i,t)\;. \qquad (7.7)$$

The big difference here is that the delayed-neutron source, $S_{\text{dn}}(E,t)$, is not known—it is something we have to solve for, just like the scalar flux $\phi(E,t)$. How do we do this? First we need to understand how delayed neutrons come about.

7.2.2.1 Delayed-neutron physics and math

Let us review the processes that lead to emission of delayed neutrons. Consider, for example, the decay scheme for Bromine-87, a long long-lived fission product with a half-life of 55 seconds, as shown in Fig. 7.1. The ^{87}Br atoms **that eventually decay to metastable ^{87}Kr** (and thus yield a neutron) are called

delayed-neutron precursors

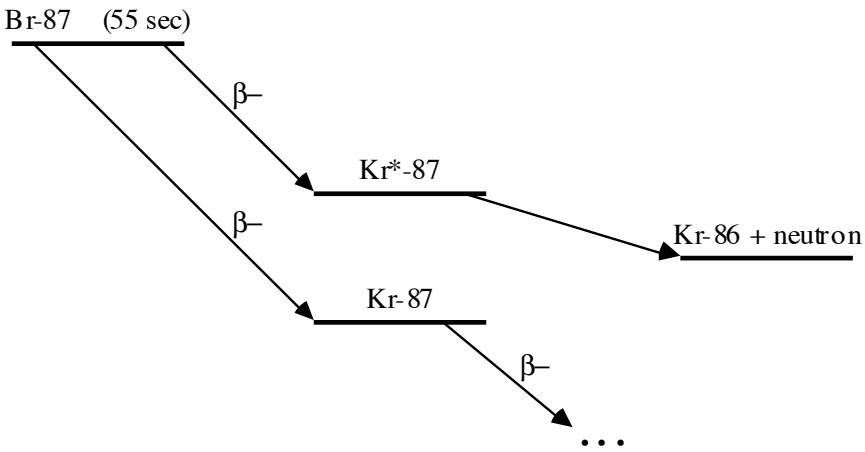


Figure 7.1: Decay scheme for ^{87}Br, 2.6% of the atoms of which are **delayed-neutron precursors**. (Frequency of outcomes for Br-87 decay: 2.6% to Kr*-87 and 97.4% to Kr-87.)

(These are only $\approx 2.6\%$ of the ^{87}Br atoms.) Notice that the precursor does not actually expel a neutron. Instead, it decays to an excited state of another nuclide, which then emits the neutron. Also note that the precursor does not have to be a direct fission fragment, but may simply be in the decay chain of a fission fragment. Two ways to produce Bromine-87, for example, are:

$$\text{1) fission} \xrightarrow{\approx 0.05\%} {}^{87}_{33}\text{As} \xrightarrow{0.6\text{ s}} {}^{87}_{34}\text{Se} \xrightarrow{5.5\text{ s}} {}^{87}_{35}\text{Br}\ ,$$

$$\text{2) fission} \xrightarrow{\approx 1.2\%} {}^{87}_{35}\text{Br}\ .$$

The delayed neutron in question is delayed by the sum of the decay times of the precursor and its parents, in the whole chain all the way back to the fission event whose fission product is the start of the chain.

Q: Why are delayed neutrons important in nuclear reactors? **A:** Because they reduce the response time of the reactor.

Approximately 270 delayed-neutron precursors have been identified to date. Of these, only a few dozen are of much practical significance.

We seek an equation or set of equations that will give us $S_{\text{dn}}(E,t)$, the energy-dependent delayed-neutron emission rate density, which appears in the equation for the energy-dependent scalar flux in our infinite-medium problem. Let us define some helpful quantities:

$$\begin{aligned} C_i(t) &= \text{expected number density of type-}i\text{ precursors at time } t. \\ \lambda_i &= \text{decay constant of type-}i\text{ precursors.} \\ \chi_i(E) &= \text{energy spectrum of neutrons emitted after decay of type-}i\text{ precursors.} \end{aligned}$$

From these definitions it follows that

$$S_{\text{dn}}(E,t) = \sum_{i=1}^{\#\text{ of precursor types}} \chi_i(E)\lambda_i C_i(t) \tag{7.8}$$

This is fine, but it doesn't help much unless we know $C_i(t)$ for each precursor type, i. How do we find these precursor densities?

As usual, we turn to our favorite equation:

Change rate = gain rate − loss rate.

For what population of things should we write this equation?

population = **type-i precursors in volume V**

We already know the loss rate from decay of the precursors—it is λ_i times the precursor population. But what is the gain rate?

Note that each precursor had its ultimate origin in fission. That is, each precursor is either a fission product or originated from the decay of a fission product. So we know that the production rate density of precursors will be tied to

the fission-rate density

Also note that

each precursor ultimately produces exactly one delayed neutron

It is useful and customary to define

$$\nu_{di}(E) \equiv \text{expected number of type-}i\text{ precursors produced from a fission that is caused by a neutron of energy } E\text{ .} \tag{7.9}$$

Because each precursor ultimately produces exactly one delayed neutron, ν_{di} is also the expected number of delayed neutrons that will ultimately be emitted from type-i precursors, per fission caused by a neutron of energy E.

With these definitions we can now write our conservation equation for precursors of type i:

$$\frac{d}{dt}C_i(t) = \int_0^\infty dE\ \nu_{di}(E)\,\Sigma_f(E)\,\phi(E,t) - \lambda_i\,C_i(t)\ . \tag{7.10}$$

Q: Why is this a conservation equation?
A: Because it is describing: "change rate = gain rate – loss rate"

7.2.2.2 Back to Finishing Case 2...

Let us collect what we have so far:

$$\begin{aligned}\frac{1}{v(E)}\frac{\partial}{\partial t}\phi(E,t) + \Sigma_t(E)\phi(E,t) = S_{\text{ext}}(E) &+ \sum_{i=1}^{\#\text{ of precursor types}} \chi_i(E)\lambda_i C_i(t) \\ &+ \chi_p(E)\int_0^\infty dE_i \nu_p(E_i)\Sigma_f(E_i)\phi(E_i,t) \\ &+ \int_0^\infty dE_i \Sigma_s(E_i \to E)\phi(E_i,t)\ .\end{aligned} \tag{7.11}$$

$$\frac{d}{dt}C_i(t) = \int_0^\infty dE\ \nu_{di}(E)\Sigma_f(E)\phi(E,t) - \lambda_i C_i(t)\ . \tag{7.12}$$

We see that

- The scalar flux depends on the delayed-neutron precursor concentrations .
- The delayed-neutron precursors concentrations depend on the scalar flux .

Thus, we have a

coupled system

of equations to solve. While this may look like a complicated system that could be difficult to solve, it is at least a well-posed set of equations that has a unique solution (given initial conditions for the scalar flux and the precursor concentrations).

So once again conservation has come through for us! That is, we have written "change rate = gain rate − loss rate" in mathematical terms, for both the neutron population and the precursor populations, and arrived at a well-posed system of equations with a unique solution. The solution, if we can find it, tells us in detail how the neutrons and precursors behave in the setting we have considered.

I hope you see that conservation is a powerful tool with far-reaching implications!

7.3 Reactor Kinetics Equations

Let us see what we can learn about solutions of Eqs. (7.11) and (7.12). Reminder: these are for an infinite homogeneous reactor. We begin by integrating the first equation over all neutron energies:

$$\frac{d}{dt}\left[\int_0^\infty dE \frac{\phi(E,t)}{v(E)}\right] + \int_0^\infty dE\, \Sigma_t(E)\,\phi(E,t) = \int_0^\infty dE\, S_{\text{ext}}(E) + \sum_{i=1}^{\text{\# of precursor types}} \lambda_i C_i(t) + \int_0^\infty dE_i\, \nu_p(E_i)\,\Sigma_f(E_i)\,\phi(E_i,t) + \int_0^\infty dE_i\, \Sigma_s(E_i)\,\phi(E_i,t)\ . \tag{7.13}$$

Here we have recognized that the integral of probability distribution functions, such as $\chi_p(E)$, $\chi_{di}(E)$, or $P(E_i \to E)$ (which is buried in $\Sigma_s(E_i \to E)$), equals 1. Now we subtract the scattering-rate density from both sides of the equation and change dummy variables E_i to E:

$$\frac{d}{dt}\left[\int_0^\infty dE \frac{\phi(E,t)}{v(E)}\right] + \int_0^\infty dE\, \Sigma_a(E)\,\phi(E,t) = \int_0^\infty dE\, S_{\text{ext}}(E) + \sum_{i=1}^{\text{\# of precursor types}} \lambda_i C_i(t) + \int_0^\infty dE\, \nu_p(E)\,\Sigma_f(E)\,\phi(E,t)\ . \tag{7.14}$$

Rewrite in terms of neutron density:

$$\frac{d}{dt}\left[\int_0^\infty dE\, n(E,t)\right] + \int_0^\infty dE\, \Sigma_a(E)\, v(E) n(E,t) = \int_0^\infty dE\, S_{\text{ext}}(E)$$

$$+ \sum_{i=1}^{\text{\# of precursor types}} \lambda_i\, C_i(t) + \int_0^\infty dE\, \nu_p(E)\, \Sigma_f(E)\, v(E) n(E,t)\ . \quad (7.15)$$

Let us introduce some definitions:

$$n_{\text{tot}}(t) \equiv \int_0^\infty dE\, n(E,t)\ , \quad (7.16)$$

$$S_{\text{tot}}(t) \equiv \int_0^\infty dE\, S_{\text{ext}}(E)\ , \quad (7.17)$$

$$\langle v\Sigma_a \rangle \equiv \frac{\int_0^\infty dE\, [v(E)\, \Sigma_a(E)]\ n(E,t)}{\int_0^\infty dE\, n(E,t)}\ , \quad (7.18)$$

$$\langle v\nu\Sigma_f \rangle \equiv \frac{\int_0^\infty dE\, [v(E)\nu(E)\, \Sigma_f(E)]\ n(E,t)}{\int_0^\infty dE\, n(E,t)}\ . \quad (7.19)$$

and add another:

$$\langle v\nu_p\Sigma_f \rangle \equiv \frac{\int_0^\infty dE\, [v(E)\, \nu_p(E)\, \Sigma_f(E)]\ n(E,t)}{\int_0^\infty dE\, n(E,t)}\ . \quad (7.20)$$

Q: What do we conserve in the $\langle . \rangle$ averaging of the cross sections?
A: The reaction rates.

Define the

Delayed-neutron fraction, β ("beta"),

as follows:

$$\beta \equiv 1 - \frac{\int_0^\infty dE\,[v(E)\,\nu_p(E)\,\Sigma_f(E)]\;n(E,t)}{\int_0^\infty dE\,[v(E)\,\nu(E)\,\Sigma_f(E)]\;n(E,t)} = \frac{\int_0^\infty dE\,[v(E)\,\nu_d(E)\,\Sigma_f(E)]\;n(E,t)}{\int_0^\infty dE\,[v(E)\,\nu(E)\,\Sigma_f(E)]\;n(E,t)} \tag{7.21}$$

$$= \frac{\langle v\nu_d\Sigma_f\rangle}{\langle v\nu\Sigma_f\rangle}$$

= fraction of fission neutrons that are born delayed.

Remark: It follows from previous definitions that

$$\nu(E) = \nu_p(E) + \sum_{i=1}^{\#\text{ of precursor types}} \nu_{di}(E) \tag{7.22}$$

Rewrite the conservation equation in terms of these defined quantities:

$$\frac{d}{dt}\left[n_{\text{tot}}(t)\right] = \left[\langle v\nu_p\Sigma_f\rangle - \langle v\Sigma_a\rangle\right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\#\text{ of precursor types}} \lambda_i C_i(t)\;. \tag{7.23}$$

(Remember: this is for an infinite homogeneous reactor.) Now define

$$\ell_p \equiv \text{ average time from prompt-neutron birth to absorption}$$

$$= \frac{1}{\langle v\Sigma_a\rangle}\;. \qquad \textbf{only for infinite reactor!} \tag{7.24}$$

That is,

ℓ_p is the prompt-neutron lifetime

Also define

$$k \equiv \text{multiplication factor}$$

$$= \frac{\langle v\nu\Sigma_f \rangle}{\langle v\Sigma_a \rangle} \xrightarrow{\text{Eq. (7.21)}} = \frac{\langle v\nu_p\Sigma_f \rangle}{(1-\beta)\langle v\Sigma_a \rangle} \,. \tag{7.25}$$

In terms of these quantities, our equation for n_{tot} can be written:

$$\begin{aligned}
\frac{d}{dt}\left[n_{\text{tot}}(t)\right] &= \left[\frac{(1-\beta)k\langle v\Sigma_a\rangle - \langle v\Sigma_a\rangle}{1}\right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\#\text{ prec. types}} \lambda_i C_i(t) \\
&= \left[\quad \frac{(1-\beta)k-1}{\ell_p} \quad\right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\#\text{ prec. types}} \lambda_i C_i(t) \\
&= \left[\frac{k-\beta k-1}{\ell_p}\right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\#\text{ prec. types}} \lambda_i C_i(t) \\
&= \left[\frac{\frac{k-1}{k}-\beta}{\ell_p/k}\right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\#\text{ prec. types}} \lambda_i C_i(t)
\end{aligned} \tag{7.26}$$

We define

$$\rho \equiv \text{reactivity} \qquad = \frac{k-1}{k} \,, \tag{7.27}$$

$$\Lambda \equiv \text{mean generation time} \qquad = \frac{\ell_p}{k} \,. \tag{7.28}$$

$$= \text{ expected time for } N \text{ neutrons to have } N \text{ descendants} \,. \tag{7.29}$$

With these definitions our equation for the neutron density can be written in a form that has become standard (and famous, or perhaps infamous from a student's point of view) for time-dependent neutronics calculations:

$$\frac{d}{dt}\left[n_{\text{tot}}(t)\right] = \left[\frac{\rho-\beta}{\Lambda}\right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\#\text{ prec. types}} \lambda_i C_i(t) \tag{7.30}$$

Similarly, we can write our precursor equation in standard form:

$$\frac{d}{dt}C_i(t) = \frac{\beta_i}{\Lambda}n_{\text{tot}}(t) - \lambda_i C_i(t) \quad , \tag{7.31}$$

where we have defined:

$$\beta_i \equiv \frac{\int_0^\infty dE\,[v(E)\,\nu_{di}(E)\,\Sigma_f(E)]\;n(E,t)}{\int_0^\infty dE\,[v(E)\,\nu(E)\,\Sigma_f(E)]\;n(E,t)} \tag{7.32}$$

$$= \text{fraction of fission neutrons that are type-}i\text{ delayed.}$$

We have now written our equations in the form of

coupled first-order ODEs .

We can learn a lot about the behavior of the neutron population by studying and solving these equations!

We observe that:

- The quantities ρ, β, β_i, and Λ are all defined in terms of $n(E,t)$, which of course depends on time. Thus, in general, these quantities all depend on time.
- It turns out that in many interesting problems, ρ, β, β_i, and Λ are all approximately constant. Thus, it is useful for us to study the solution of our coupled first-order ODEs in this interesting constant-parameter case. This case is simple enough to permit analytic solutions that are enormously helpful in building an understanding of nuclear-reactor behavior.

We repeat these very important equations here, Eqs. (7.30) and (7.31), and introduce the name by which they are known:

THE POINT-REACTOR KINETICS EQUATIONS (PRKEs):

$$\frac{d}{dt}\left[n_{\text{tot}}(t)\right] = \left[\frac{\rho-\beta}{\Lambda}\right] n_{\text{tot}}(t) + S_{\text{tot}}(t) + \sum_{i=1}^{\#\text{ prec. types}} \lambda_i C_i(t) \quad , \tag{7.33}$$

$$\frac{d}{dt} C_i(t) = \frac{\beta_i}{\Lambda} n_{\text{tot}}(t) - \lambda_i C_i(t) \quad . \tag{7.34}$$

Remember these are just two couple equations of conservation for the neutron density and the number density of neutron precursors. The PRKEs are used extensively to study the time-dependent behavior of nuclear reactors. In Chapter 8, we will study them.
Motivate them with an example.

7.4 Neutron Conservation in Finite Reactors

In this section we consider the Cases 3 and 4. All of these involve **finite** reactors. In these settings we must address leakage as a loss or gain mechanism. We also must worry about spatial distributions, because the reactor may have different materials in different places, which means reaction rates will depend on where the neutrons are in addition to how many there are.

7.4.1 Case 3: Population = all n's in a given sub-volume of the reactor; Setting = finite reactor with proper treatment of delayed neutrons

For this population our gain mechanisms are:

- Emission from the fixed source in sub-volume ΔV
- Prompt emission from neutron-induced fission in sub-volume ΔV
- Delayed emission from neutron-induced fission in sub-volume ΔV
- Inleakage through the sub-volume surface, $\partial\Delta V$

Our loss mechanisms are:

- Absorption in sub-volume ΔV
- Outleakage through the sub-volume surface , $\partial\Delta V$

For a characteristic volume ΔV in the reactor, the leakage term captures all neutrons streaming in and out of the boundary surface of this volume. For simplicity, we combine (outleakage − inleakage) into

net outleakage

We then state a conservation equation for a volume ΔV in the reactor:

$$\frac{d}{dt}\left[\iiint_{\Delta V} d^3r \int_0^\infty dE\; n(\vec{r},E,t)\right] = \iiint_{\Delta V} d^3r \int_0^\infty dE\; \nu_p(\vec{r},E,t)\Sigma_f(\vec{r},E,t)\phi(\vec{r},E,t)$$

$$+ \iiint_{\Delta V} d^3r \int_0^\infty dE\; S_{\text{ext}}(\vec{r},E,t) + \sum_{i=1}^{\#\text{ dnp types}} \lambda_i \iiint_{\Delta V} d^3r\; C_i(\vec{r},t)$$

$$- \iiint_{\Delta V} d^3r \int_0^\infty dE\; \Sigma_a(\vec{r},E,t)\phi(\vec{r},E,t) - \oiint_{\partial\Delta V} d^2r \int_0^\infty dE\; \vec{e}_n(\vec{r})\cdot\vec{J}(\vec{r},E,t)\;. \quad (7.35)$$

We can use a mathematical "trick" to write the surface integral of the sub-volume as a volume integral. Recall Gauss's divergence theorem:

$$\iiint_{\Delta V} d^3r\; \vec{\nabla}\cdot\vec{u} = \oiint_{\partial\Delta V} d^2r\; \vec{e}_n\cdot\vec{u}\;. \quad (7.36)$$

Let's use this on the leakage term in our conservation equation:

$$\frac{d}{dt}\left[\iiint_{\Delta V} d^3r \int_0^\infty dE\; n(\vec{r},E,t)\right] = \iiint_{\Delta V} d^3r \int_0^\infty dE\; \nu_p(\vec{r},E,t)\Sigma_f(\vec{r},E,t)\phi(\vec{r},E,t)$$

$$+ \iiint_{\Delta V} d^3r \int_0^\infty dE\; S_{\text{ext}}(\vec{r},E,t) + \sum_{i=1}^{\#\text{ dnp types}} \lambda_i \iiint_{\Delta V} d^3r\; C_i(\vec{r},t)$$

$$- \iiint_{\Delta V} d^3r \int_0^\infty dE\; \Sigma_a(\vec{r},E,t)\phi(\vec{r},E,t) - \iiint_{\Delta V} d^3r \int_0^\infty dE\; \vec{\nabla}\cdot\vec{J}(\vec{r},E,t)\;. \qquad (7.37)$$

We recognize that

every term is an integral over the sub-volume .

Thus, we can collect everything into a single integral over the sub-volume:

$$\iiint_{\Delta V} d^3r \left\{\frac{\partial}{\partial t}\int_0^\infty dE\; n(\vec{r},E,t) - \int_0^\infty dE\; \nu_p(\vec{r},E,t)\Sigma_f(\vec{r},E,t)\phi(\vec{r},E,t)\right.$$

$$- \int_0^\infty dE\; S_{\text{ext}}(\vec{r},E,t) - \sum_{i=1}^{\#\text{ dnp types}} \lambda_i C_i(\vec{r},t)$$

$$\left. + \int_0^\infty dE\; \Sigma_a(\vec{r},E,t)\phi(\vec{r},E,t) + \int_0^\infty dE\; \vec{\nabla}\cdot\vec{J}(\vec{r},E,t)\right\} = 0\;. \qquad (7.38)$$

This equation says that the integral of some function over some sub-volume of the reactor equals zero. This must be true for any sub-volume of the reactor, because we did not choose any special sub-volume for our conservation statement. Thus, we see that the integral of the function inside the { } integrates to zero for any range of integration. Thus, the function must be zero, and we must have:

$$\frac{\partial}{\partial t}\left[\int_0^\infty dE\; n(\vec{r},E,t)\right] + \int_0^\infty dE\; \vec{\nabla}\cdot\vec{J}(\vec{r},E,t) + \int_0^\infty dE\; \Sigma_a(\vec{r},E,t)\phi(\vec{r},E,t)$$

$$= \int_0^\infty dE\; \nu_p(\vec{r},E,t)\Sigma_f(\vec{r},E,t)\phi(\vec{r},E,t)$$

$$+ \int_0^\infty dE\; S_{\text{ext}}(\vec{r},E,t) + \sum_{i=1}^{\#\text{ dnp types}} \lambda_i C_i(\vec{r},t) \tag{7.39}$$

In this equation, every term has units of:

$$\underline{\text{n/cm}^3\text{-s}}$$

That is, every term is a "something" rate density, where every "something" is either change or some kind of gain or loss, and where the "density" is in ordinary 3-D space.

We can already recognize a problem that we have encountered previously with our conservation equations: Even if we knew

$$\underline{\text{the conserved quantity, } \int dE\, n(\vec{r},E,t),}$$

it would not give us enough information to calculate

$$\underline{\text{the absorption and fission rate densities}}$$

that appear in the equation, because these rates depend on the

$$\underline{\text{energy distribution of the neutrons}}$$

and not just on the total neutron density. We recognize the source of this problem:

we have not stated conservation in enough detail.

Nevertheless, as we have done before, we shall press on by defining some average quantities. We shall proceed slightly differently than before, just to illustrate that there are many ways to attack these problems. This time we choose the **scalar flux** instead of the **neutron density** as the fundamental unknown. In this case the neutron speed will be included in the weight function in the averaged cross sections, $\langle\Sigma\rangle$, instead of being part of the averaged quantity, as with our previous $\langle v\Sigma\rangle$ averages. That is, we define:

$$\left\langle \frac{1}{v} \right\rangle \equiv \frac{\int_0^\infty dE \, \frac{\phi(\vec{r},E,t)}{v(E)}}{\int_0^\infty dE \, \phi(\vec{r},E,t) \, ,} \tag{7.40}$$

$$\langle \Sigma_a(\vec{r},t) \rangle \equiv \frac{\int_0^\infty dE \, \Sigma_a(\vec{r},E,t)\phi(\vec{r},E,t)}{\int_0^\infty dE \, \phi(\vec{r},E,t)} \, , \tag{7.41}$$

$$\langle \nu_p \Sigma_f(\vec{r},t) \rangle \equiv \frac{\int_0^\infty dE \, \nu_p(\vec{r},E,t)\Sigma_f(\vec{r},E,t)\phi(\vec{r},E,t)}{\int_0^\infty dE \, \phi(\vec{r},E,t)} \, . \tag{7.42}$$

Q: What do we conserve in the $\langle . \rangle$ averaging of the cross sections?
A: The reaction rates.

and we recall the definitions of "total" scalar flux and "total" net current density from a previous chapter:

$$\phi(\vec{r},t) \equiv \int_0^\infty dE \, \phi(\vec{r},E,t) \, , \tag{7.43}$$

$$\vec{J}(\vec{r},t) \equiv \int_0^\infty dE \, \vec{J}(\vec{r},E,t) \, . \tag{7.44}$$

We make a similar definition for the fixed source rate density:

$$S_{\text{ext}}(\vec{r},t) \equiv \int_0^\infty dE \, S_{\text{ext}}(\vec{r},E,t) \, . \tag{7.45}$$

With these definitions we can rewrite our conservation equation as:

$$\left\langle \frac{1}{v} \right\rangle \frac{\partial}{\partial t}\phi(\vec{r},t) + \vec{\nabla}\cdot\vec{J}(\vec{r},t) + \langle \Sigma_a(\vec{r},t)\rangle\,\phi(\vec{r},t)$$

$$= \langle \nu_p \Sigma_f(\vec{r},t)\rangle\,\phi(\vec{r},t)$$

$$+S_{\text{ext}}(\vec{r},t) + \sum_{i=1}^{\#\text{ dnp types}} \lambda_i C_i(\vec{r},t) \tag{7.46}$$

We also need equations for the delayed-neutron precursor concentrations, C_i, which appear in this neutron conservation equation. If we follow the same kind of procedure as before, but expressing precursor production in terms of the total scalar flux, we obtain:

$$\frac{\partial}{\partial t}C_i(\vec{r},t) = \langle \nu_{di}\Sigma_f(\vec{r},t)\rangle\,\phi(\vec{r},t) - \lambda_i C_i(\vec{r},t)\ . \tag{7.47}$$

IMPORTANT:

Understand the physical meaning of each term in these equations!

Equation (7.46) is an exact conservation equation for the neutrons in a reactor, provided that the terms in angle brackets, $\langle\ \rangle$, are averaged with the exact energy-dependent scalar flux. Of course, we don't know the energy-dependent scalar flux in advance, and this equation does not give us the opportunity to calculate it. Further, we don't have any equations for

the net current density, $\vec{J}$,

nor do we have any obvious way to express the divergence of $\vec{J}$ in terms of the scalar flux. So we have an exact equation but cannot solve it without some additional knowledge. We can take two approaches at this point. First, we could introduce an approximation for the current as a function of the scalar flux. This is done in the next chapter in the framework of diffusion theory. Other choice is to examine the conservation equation in further detail, i.e. considering the dependencies in energy and in direction. This leads to the

transport equation

So let's try conservation in even more detail.

7.4.2 Case 4: Adding the dependence in energy and direction to Case 3

Our setting is the real world (finite reactor, non-uniform structure, neutrons distributed in energy, delayed neutrons present). Our conservation statement will give us information about the spatial and energy and **directional** distribution of the neutrons. This will produce a set of equations whose solution tells us all we need to know about neutron behavior in reactors! The problem is that the equation is complicated enough that it is difficult to solve, even approximately.

For this population our gain mechanisms are:

- Emission from the fixed source
- Prompt emission from neutron-induced fission
- Delayed emission from neutron-induced fission
- Inleakage through the sub-volume surface
- "Inscattering" from energies outside the energy sub-interval and/or directions outside the directional cone to energies in the sub-interval and directions in the cone.

Our loss mechanisms are:

- Absorption
- Outleakage through the sub-volume surface
- "Outscattering" from energies in the energy sub-interval and directions in the directional cone to energies outside the sub-interval and/or directions outside the cone.

As we have done before, we combine (outleakage − inleakage) into net outleakage.

Before we can write mathematical expressions for the gain and loss rates, we must develop a way to describe the neutron distribution in direction, and to describe how scattering changes a neutron's direction. We define the direction- and energy-dependent neutron density:

$n(\vec{r}, E, \vec{\Omega}, t) d^3r\, dE\, d\Omega \equiv$ expected number of neutrons in the volume d^3r at $\vec{r}$,
with lab-frame kinetic energy in the interval dE at E,
and with direction in the solid-angle cone $d\Omega$ around the direction $\vec{\Omega}$.

This is a density in a six-dimension phase space: 3 spatial dimensions, 1 energy dimension, and 2 direction dimensions. (It takes two numbers—say a polar angle and an azimuthal angle—to define a direction.) The units are neutrons per unit volume per unit energy per unit solid angle. The usual unit of solid angle is the **steradian**. So in our usual units we have n/(cm^3-MeV-ster).

Analogous to the energy-dependent scalar flux, which is speed times the energy-dependent neutron density, we define the energy-dependent **angular flux**:

$$\psi(\vec{r}, E, \vec{\Omega}, t) \equiv v(E) n(\vec{r}, E, \vec{\Omega}, t) \; . \tag{7.48}$$

Note the identity relating angular flux to scalar flux:

$$\phi(\vec{r}, E, t) \equiv v(E) \iint_{4\pi} d\Omega \; \psi(\vec{r}, E, \vec{\Omega}, t) \; . \tag{7.49}$$

Now recall the double-differential scattering cross section:

$\Sigma_s(\vec{r}, E' \to E, \vec{\Omega}' \to \vec{\Omega}, t) dE \; d\Omega \equiv$ expected scatters per unit neutron path length,
for incident neutrons of energy E' and direction $\vec{\Omega}'$,
that produce scattered neutrons with kinetic energy in the interval dE at E
and with direction in the solid-angle cone $d\Omega$ around the direction $\vec{\Omega}$.

This has units of inverse [cm-MeV-ster]. It is a scattering cross section at the incident neutron energy, multiplied by a distribution function in outgoing neutron energy and direction.

Now our conservation equation is

$$\frac{d}{dt}\left[\iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta\Omega} d\Omega\, n(\vec{r},E,\vec{\Omega},t)\right]$$

$$= \left[\iint_{\Delta\Omega} d\Omega\, \frac{1}{4\pi}\right]\left[\int_{\Delta E} dE \chi_p(E)\right]\iiint_{\Delta V} d^3r \int_0^\infty dE'\, \nu_p(\vec{r},E',t)\Sigma_f(\vec{r},E',t)\phi(\vec{r},E',t)$$

$$+ \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta\Omega} d\Omega\, S_{\text{ext}}(\vec{r},E,\vec{\Omega},t)$$

$$+ \sum_{i=1}^{\#\text{ dnp types}}\left[\iint_{\Delta\Omega} d\Omega\, \frac{1}{4\pi}\right]\left[\int_{\Delta E} dE \chi_{di}(E)\right]\lambda_i \iiint_{\Delta V} d^3r\, C_i(\vec{r},t)$$

$$+ \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta\Omega} d\Omega \int_0^\infty dE' \iint_{4\pi} d\Omega'\, \Sigma_s(\vec{r},E'\to E,\vec{\Omega}'\to\vec{\Omega},t)\psi(\vec{r},E',\vec{\Omega}'t)$$

$$- \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta\Omega} d\Omega \int_0^\infty dE' \iint_{4\pi} d\Omega'\, \Sigma_s(\vec{r},E\to E',\vec{\Omega}\to\vec{\Omega}',t)\psi(\vec{r},E',\vec{\Omega},t)$$

$$- \iiint_{\Delta V} d^3r \int_{\Delta E} dE \iint_{\Delta\Omega} d\Omega\, \Sigma_a(\vec{r},E,t)\psi(\vec{r},E,\vec{\Omega},t)$$

$$- \oiint_{\partial\Delta V} d^2r \int_{\Delta E} dE \iint_{\Delta\Omega} d\Omega\, \vec{e}_n(\vec{r})\cdot\vec{\Omega}\psi(\vec{r},E,\vec{\Omega},t)\ . \tag{7.50}$$

We use the divergence theorem to convert the surface integral to a volume integral, and then every term is an integral over the six-dimensional volume $[\Delta V\ \Delta E\ \Delta\Omega]$. We recognize that the integral of a double-differential scattering cross section is just a scattering cross section, much as we did in the previous case. We recognize that $\Sigma_t = \Sigma_a + \Sigma_s$. We collect all the terms into one giant integral, then recognize (as before) that the integrand must be zero. This gives:

$$\frac{d}{dt}\Big[n(\vec{r},E,\vec{\Omega},t)\Big] + \vec{\Omega}\cdot\vec{\nabla}\psi(\vec{r},E,\vec{\Omega},t) + \Sigma_t(\vec{r},E,t)\psi(\vec{r},E,\vec{\Omega},t)$$

$$= \frac{\chi_p(E)}{4\pi}\int_0^\infty dE'\ \nu_p(\vec{r},E',t)\Sigma_f(\vec{r},E',t)\phi(\vec{r},E',t)$$

$$+ S_{\text{ext}}(\vec{r},E,\vec{\Omega},t) + \sum_{i=1}^{\#\text{ dnp types}} \frac{1}{4\pi}\chi_{di}(E)\lambda_i C_i(\vec{r},t)$$

$$+ \int_0^\infty dE' \iint_{4\pi} d\Omega'\ \Sigma_s(\vec{r},E'\to E,\vec{\Omega}'\to\vec{\Omega},t)\psi(\vec{r},E',\vec{\Omega}'t)\ . \qquad (7.51)$$

We can rewrite the neutron density, n, as the angular flux divided by the speed, ψ/v:

THE TIME-DEPENDENT NEUTRON TRANSPORT EQUATION:

$$\frac{1}{v(E)}\frac{d}{dt}\psi(\vec{r},E,\vec{\Omega},t) + \vec{\Omega}\cdot\vec{\nabla}\psi(\vec{r},E,\vec{\Omega},t) + \Sigma_t(\vec{r},E,t)\,\psi(\vec{r},E,\vec{\Omega},t)$$

$$= \frac{\chi_p(E)}{4\pi}\int_0^\infty dE'\ \nu_p(\vec{r},E',t)\Sigma_f(\vec{r},E',t)\,\phi(\vec{r},E',t)$$

$$+ S_{\text{ext}}(\vec{r},E,\vec{\Omega},t) + \sum_{i=1}^{\#\text{ dnp types}} \frac{1}{4\pi}\chi_{di}(E)\lambda_i\,C_i(\vec{r},t)$$

$$+ \int_0^\infty dE' \iint_{4\pi} d\Omega'\ \Sigma_s(\vec{r},E'\to E,\vec{\Omega}'\to\vec{\Omega},t)\,\psi(\vec{r},E',\vec{\Omega}'t)\ . \qquad (7.52)$$

This is a single integro-differential equation for the single unknown function ψ. We can show that it has a unique solution, given appropriate boundary and initial conditions. Thus,

considering conservation in detail (i.e., in every part of a six-dimensional phase space that includes position, direction, and energy) has eliminated the difficulty we found in the previous cases!

Equation (7.52) is the **neutron transport equation.** It is an essentially exact description of the behavior of neutrons in reactors. It is a relatively complicated equation that is relatively difficult to solve. Fortunately, we can gain a great deal of insight into reactor behavior by using equations that contain simplifying approximations. We do so in the next chapter. Later in your studies you will return to the transport equation to get a more precise picture of some neutronics details.

7.5 Summary

In this chapter we have delved into the details of writing conservation equations for various defined populations in various settings.

We first started with an infinite reactor and derived the conservation equations in a sub-volume first considering only neutrons emitted by fission and a extraneous source. These equations are exact and no closure models are needed. However, in the next section, we added as a source the delayed neutrons emitted by neutron precursors, which may be fission products and or belong to the decay-chain of these ones. A new conservation equation for the number density of neutron precursors was necessary to close our system. Integrating the system over the energy range, the point kinetics equation was derived.

Then, we considered a finite reactor, in which net leakage plays a role in the conservation equations. Now, the energy dependence balance is not enough to model the conservation of neutrons in the reactor, due to the presence of the neutron current density in the leakage term. Two different roads can be followed. First, to introduce an approximate model for the neutron current density, which is done in the next chapter. Second, to derive a balance considering the angular dependence of the neutron flux, which leads to the neutron transport equation.

Some important remarks are:

- We found that in every case that treated delayed neutrons we could cast our conservation equations, along with the equations for the delayed-neutron-precursor concentrations, into the form of **Point-Reactor Kinetics Equations** (PRKEs). Even when we did not know how to calculate the coefficients in these equations, we could study the characteristics of their solutions in all scenarios of interest.
- In particular, if the coefficients in the PRKEs—namely ρ, the $\{\beta_i\}$, and Λ—are constant (which they often are for interesting problems), then we can write down equations for $n_{\text{tot}}(t)$ and the $\{C_i(t)\}$. This is a natural segway to Chapter 8.
- **Important:** We derived the PRKEs using an infite problem. You will obtain exactly the same form for the equations if the domain is finite. That is, the PRKEs are quite general! So our work here applies far beyond the infinite uniform medium that we started with. In case of a finite domain, the only differences are buried in the definitions of the coefficients:
  1. The prompt-neutron lifetime, ℓ_p is shortened by the possibility of leakage.
  2. The multiplication factor, k, is reduced by loss from leakage.

  If you want additional details, please see Appendix E.

- In Case 3 we derived an essentially exact statement of neutron conservation in a reactor. The only difficulty we found with our conservation equation is that it contains the unknown net current density, $\vec{J}$, in addition to the unknown scalar flux, ϕ.
- Note that this difficulty is not dependent on the energy or time. Hence, the same problem will be found is we considered a mono-energetic population of neutrons that is steady in time.

Part IV

Neutron distribution in time

Chapter 8

Time Dependence via PRKE

Foreword: Read Appendix D for a review of mathematical techniques to solve ODEs.

8.1 Introduction

In this Chapter, we study the time dependence of a neutron population. We do this using the Point Reactor Kinetic Equations, PRKEs, recalled below.

This model, although simplified because it does not address the interaction between the space+energy distribution of neutrons and their temporal variation, is a widely used model to underastand the temporal behavior of nuclear reactors (and fissile systems, in general).

THE POINT-REACTOR KINETICS EQUATIONS (PRKEs):

$$\frac{d}{dt}\left[n(t)\right] = \left[\frac{\rho-\beta}{\Lambda}\right] n(t) + S(t) + \sum_{i=1}^{\#\text{ prec. types}} \lambda_i C_i(t) \quad , \tag{8.1}$$

$$\frac{d}{dt} C_i(t) = \frac{\beta_i}{\Lambda} n(t) - \lambda_i C_i(t) \quad . \tag{8.2}$$

8.2 Analytical solution of the PRKE equations

We consider the case in which ρ, β, β_i, Λ, and S are constant. (It turns out that for much of the time during many transients of interest, this is a very good approximation, so this is an important case to understand.) In this case, which has constant coefficients, the Point-Reactor Kinetics Equations (PRKEs) are a set of coupled **first-order ODEs with constant coefficients.** We know from our previous mathematical studies that:

1. The solution of each equation is the sum of

   a "particular" solution plus a "homogeneous" (no-source) solution.

2. The no-source solution of each equation is a sum of exponentials.

Given our assumption that S and the coefficients are constants, it is easy to show that constant particular solutions satisfy the PRKEs (with the exception of one tricky case). You should verify that the following solutions do satisfy the equations:

$$C_i^{\text{particular}} = \frac{\beta_i}{\Lambda \lambda_i} n^{\text{particular}} \quad , \tag{8.3}$$

$$n^{\text{particular}} = -\left[\frac{\Lambda}{\rho}\right] S \quad . \tag{8.4}$$

You can see that there is a problem with this particular solution if the reactivity, ρ, is zero. This is the "tricky" case—the case of a critical reactor—which we will address later.

Now let us address the "homogeneous" solution—the solution that satisfies the equations with S removed. If we define

$$I = \text{ number of types of delayed-neutron precursors that we are tracking,} \tag{8.5}$$

then the PRKEs are a set of $I + 1$ coupled ODEs (with constant coefficients in the case we are studying right now). We know from our previous mathematics studies that each "homogeneous" solution $(n, C_1, C_2, \ldots, C_I)$ will be

a sum of $I + 1$ exponentials.

and that the time constants in the exponentials for a given function (such as $C_2(t)$) are

the same as the time constants in the other functions.

Thus, our PRKE solutions have the following form (excluding the tricky $\rho = 0$ case and continuing to assume that S_{tot} is constant):

$$n(t) = \sum_{j=1}^{I+1} A_j \, e^{s_j t} - \frac{\Lambda}{\rho} S \quad . \tag{8.6}$$

$$C_i(t) = \sum_{j=1}^{I+1} C_{i,j} \, e^{s_j t} + \frac{\beta_i}{\Lambda \, \lambda_i} n^{\text{particular}}$$

$$= \sum_{j=1}^{I+1} C_{i,j} \, e^{s_j t} - \frac{\beta_i}{\Lambda \, \lambda_i} \frac{\Lambda}{\rho} S \quad . \tag{8.7}$$

But how do we find the constants $s_j, j = 1, \cdots, I+1$? This is not terribly complicated: we insert the solutions into the equations and find the s values that allow the solutions to be non-trivial (i.e., the solutions that can have non-zero A_j and $C_{i,j}$ coefficients in front of the exponentials in the solutions). After a bit of algebra (okay, more than just a bit) we find that each of the s values must satisfy the following equation:

THE INHOUR EQUATION:

$$\rho = \Lambda s + \sum_{i=1}^{I} \frac{\beta_i s}{s + \lambda_i} \quad . \tag{8.8}$$

There are exactly $I + 1$ distinct values of s that satisfy this equation, which is called the **"inhour equation."** This equation gives us the s values that go into Eqs (8.6)-(8.7). These values determine how quickly or slowly each part of the solution changes with time. Note that the s values have units of

inverse time

In the early days of reactor analysis, people often used "inverse hours" as the unit for the s values. This led to the name "in-hour," which the equation still bears.

It is not difficult to show, for example by plotting the left-hand side and right-hand side of the inhour equation on the same plot as in Fig. 8.1, that the $I+1$ values of s have the following properties:

- I of them are always negative, and in fact $< -\lambda_{\min}$.
- The other one (call it s_1) has the same sign as the reactivity, ρ.
- If $\rho = 0$, then $s_1 = 0$.
- If $\rho < 0$, then s_1 is negative but is closer to 0 than the other s values:

$$s_1 \in (-\lambda_{\min}, 0) \qquad \text{if } \rho < 0 \ .$$

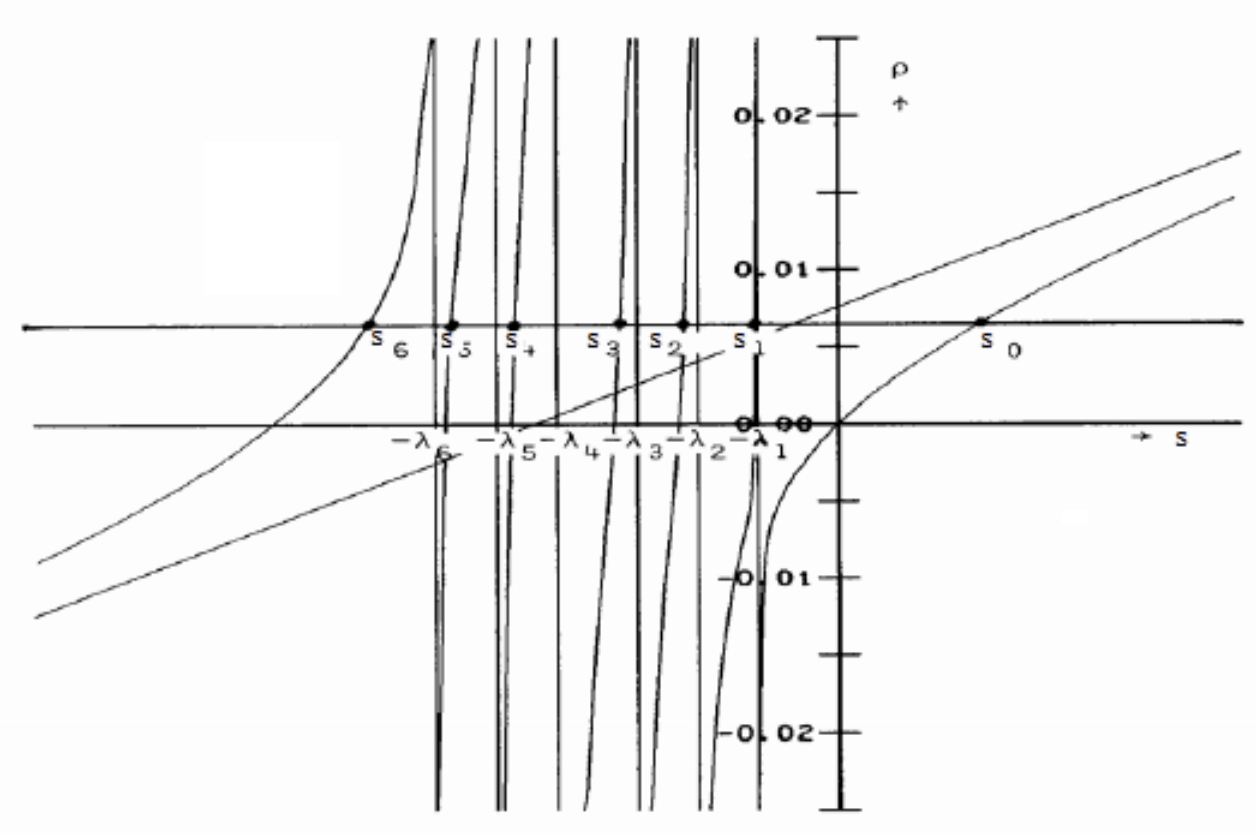


Figure 8.1: Graphical "solution" of inhour equation with $I = 6$ groups of delayed neutron precursors. There is already a horizontal constant line that is the value of $\rho > 0$; look at the signs of the $I + 1$ roots s_i. Now add another horizontal line for $\rho < 0$ and look at the signs of the $I + 1$ roots s_i in that case.

That is, I values of s will always be real and negative; the other value, which is the largest algebraically, is real and has the same sign as the reactivity, ρ. If $\rho < 0$, there is one s value that can never be more negative than $-\lambda_{\text{min}}$, where λ_{min} is the shortest of the precursor decay constants. This corresponds to the precursor with the longest half-life, which is ^{87}Br, with $T_{1/2} = 55$ s. Thus,

$$\lambda_{\text{min}} \approx \frac{1}{80s}$$

This simple information about the s values tells us a great deal about the time variation of the neutron population in a reactor. We explore this for the six possible cases in the subsections that follow.

8.2.1 Subcritical Reactor with Fixed Source

If the reactor is subcritical, then $\rho < 0$ and, according to the discussion above, all exponential time coefficients $\{s_j, j = 1...I+1\}$ are negative. It follows that for large t, the exponentials become vanishingly small, and we are left with only the "particular" part of the solution:

$$n(t) \xrightarrow{\text{large } t} -\frac{\Lambda}{\rho}S = -\frac{\ell_p/k}{(k-1)/k}S = \frac{\ell_p}{1-k}S \ . \tag{8.9}$$

That is, we have discovered that

> In a **source-driven subcritical system,**
> the neutron population eventually reaches a steady state .

The steady-state neutron density is proportional to:

- the source strength,
- the prompt-neutron lifetime, ℓ_p,
- $1/(1-k)$.

The factor of $(1-k)$ in the denominator describes the phenomenon of

"subcritical multiplication."

We see that because of subcritical multiplication, if k is arbitrarily close to 1, then the steady-state neutron population can be arbitrarily large, even with a "small" source and a subcritical reactor.

Note further that for k close to unity, ρ is close to zero and thus s_1 will be close to zero. This means that one exponential in the expression for n will be slow to decay away, which means it will take a **long time** for the neutron population to attain its steady-state level.

8.2.2 Supercritical Reactor with Fixed Source

This case is relatively simple. The particular solution is a constant, I of the exponentials decay with time, and one of the exponential terms has a positive s value. The increasing exponential eventually dominates all other terms, and

$$n(t) \xrightarrow{\text{large } t} A_1 e^{s_1 t} \ , \qquad \text{with } s_1 > 0 \ . \tag{8.10}$$

That is, we have discovered that

> In a **source-driven supercritical system,**
> the neutron population eventually increases exponentially with time .

8.2.3 Critical Reactor with Fixed Source

If the reactor is critical, then $\rho = 0$ and, according to the discussion above, all exponential time constants $\{s_j\}$ are negative except for one that is zero. **That is, one of the exponential functions is actually a constant function.** However, we stated earlier that a constant "particular" solution was also chosen. This means, that for this particular case, the particular solution needs to be something else. We try the next-simplest function we can imagine for the particular solution, namely one that is linear in t. We find then that the following particular solution does satisfy the equations with ρ=0:

$$n^{\text{particular}}(t) = tC_p \ , \tag{8.11}$$

and

$$C_i^{\text{particular}}(t) = \frac{\beta_i}{\lambda_i \Lambda}\left(t - \frac{1}{\lambda_i}\right) C_p \ , \tag{8.12}$$

where

$$C_p \equiv \frac{\Lambda}{\Lambda + \sum_{i=1}^{I} \frac{\beta_i}{\lambda_i}} S = \text{ a positive constant.} \tag{8.13}$$

Equation (8.11) says the "particular" portion of the solution for n is a linearly increasing function of time. What about the "homogeneous" portion? It has I exponentials that decay away in time, so they soon become unimportant. The other exponential is exp(0), because

$$s_1 = 0 \text{ when } \rho = 0.$$

Of course, exp(0)=1. We therefore have

$$n(t) \xrightarrow{\text{large } t} A_1 + t\left[\frac{\Lambda}{\Lambda + \sum_{i=1}^{I} \frac{\beta_i}{\lambda_i}} S\right] \ . \tag{8.14}$$

Thus, we have now found that

> In a **source-driven critical system,**
> the neutron population eventually increases linearly with time .

8.2.4 Subcritical Reactor with No Fixed Source

In a source-free reactor the particular solution is **zero** for all cases. If the reactor is subcritical, then $\rho < 0$ and, as discussed above, all exponential time constants $\{s_j\}$ are negative. If we continue to let s_1 represent the algebraically largest s (the one closest to zero, in this case), then the other exponential terms decay more rapidly than the s_1 term, and we have

$$n(t) \xrightarrow{\text{large } t} A_1 e^{-|s_1|t} \ . \tag{8.15}$$

That is,

> In a **source-free subcritical system,**
> the neutron population eventually decreases exponentially with time .

Important: At the beginning of a transient (for example, immediately after rapid insertion of control rods) the population may drop **much faster than this single exponential**, because of the other rapidly-varying exponentials that are part of the solution. But the population eventually approaches this single decaying exponential with time constant s_1.

Recall that in a subcritical system,

$$s_1 \in (-\lambda_{\min}, 0) \ . \tag{8.16}$$

Also note that $\lambda_{\min}$ is associated with the longest-living delayed-neutron precursors, which have half-lives of around **55 seconds**, so

$$\lambda_{\min} = \frac{\ln 2}{T_{1/2,\max}} \approx \frac{1}{80 \text{ s}} \ . \tag{8.17}$$

This means it takes the neutron population at least 80 seconds to decrease by a factor of e. This is not a very fast decrease! But remember:

the population may decrease
much faster than this in the early stages of a transient !

8.2.5 Supercritical Reactor with No Fixed Source

In this case one of the exponential terms has a positive s value while all other exponential terms are decaying in time. The increasing exponential eventually dominates the others, and

$$n(t) \xrightarrow{\text{large } t} A_1 e^{s_1 t} \ . \tag{8.18}$$

That is,

> In a **source-free supercritical system,**
> the neutron population eventually increases exponentially with time .

At the beginning of a transient the population may grow more slowly or more rapidly than this, or even decrease for a little while (!), but it eventually approaches this single increasing exponential.

8.2.6 Critical Reactor with No Fixed Source

If the reactor is critical, then ρ=0 and, as discussed above, all exponential time constants $\{s_j\}$ are negative except for one that is zero. It follows that

$$n(t) \xrightarrow{\text{large } t} A_1 \ . \tag{8.19}$$

That is,

> In a **source-free critical system,**
> the neutron population eventually reaches a steady state .

This is just what we would expect—the definition of a critical reactor is that it is **able to sustain** a steady chain reaction, without help from "extra" sources of neutrons. However, note that the population could change significantly while it is on its way to the steady value. That is,

> even in a source-free critical reactor, the neutron population
> may change for a while before it settles into steady state.

8.2.7 First Summary of PRKE Solutions

We have studied PRKE solutions for situations in which S (fixed source), ρ (reactivity), β (delayed-neutron fraction), β_i (type-i delayed-neutron fraction), and Λ (mean generation time) are all constants. This is a realistic case that does arise in nature! We studied six sub-cases, which are the combinations of subcritical, supercritical, and critical reactors with and without fixed sources.

Interestingly, we find that while the presence of delayed neutrons can cause interesting and complicated behavior early in a transient, eventually the neutron population behaves quite predictably:

1. If there is **no fixed source**, then the neutron population:
   (a) eventually grows exponentially if $\rho > 0$;
   (b) eventually falls exponentially if $\rho < 0$;
   (c) eventually stays steady if $\rho = 0$;
2. If there **is a fixed source**, then the neutron population:
   (a) eventually grows exponentially if $\rho > 0$;
   (b) eventually reaches a steady value if $\rho < 0$;
   (c) eventually grows linearly in time if $\rho = 0$;

If you compare this against what we found in Case 1 of the previous chapter, you see similarities, and you may be tempted to conclude that delayed neutrons are not important. Not so—**delayed neutrons are extremely important!** They have a dramatic effect on the time constants that appear in the exponentials mentioned above!

If there were no delayed neutrons, then it is easy to see (by looking at the inhour equation with β=0) that

$$s \xrightarrow{\beta=0} \frac{k-1}{\Lambda} \, . \tag{8.20}$$

In a typical commercial reactor, Λ is on the order of 0.0001 s. Now consider a slightly supercritical reactor, with k=1.001. How much would the neutron population (and thus the power) change in 1 second if there were no delayed neutrons?

$$n(t)|_{t=1\text{ s}} = n(0)\, \exp\left[\frac{0.001}{0.0001\text{ s}}(1\text{ s})\right] = n(0)\, e^{10} > 22,000\, n \, . \tag{8.21}$$

Imagine how difficult it would be to control a reactor if its power could change this quickly!

Delayed neutrons slow this down dramatically. They make it relatively easy to control reactors that are slightly supercritical or subcritical. There will be exercises in which you will quantify this for yourself.

Included here are a series of important sketches that summarize what we found above. These are Figs. 8.2-8.6. **Study these!** Be able to **explain** them to a high-school student. Practice explaining them to each other. Work all of this into your intuition about the way neutrons behave in reactors.

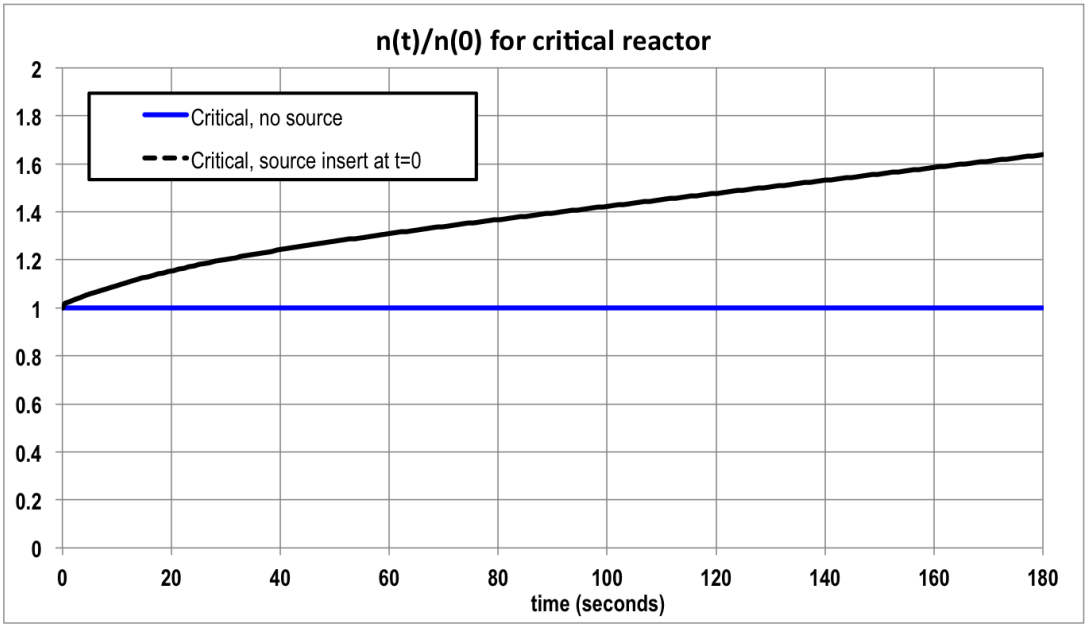


Figure 8.2: Behavior of neutron population in a **CRITICAL** reactor. The with-fixed-source figure assumes the source is inserted at t=0. Note the jump before the smooth increase that becomes linear in time.

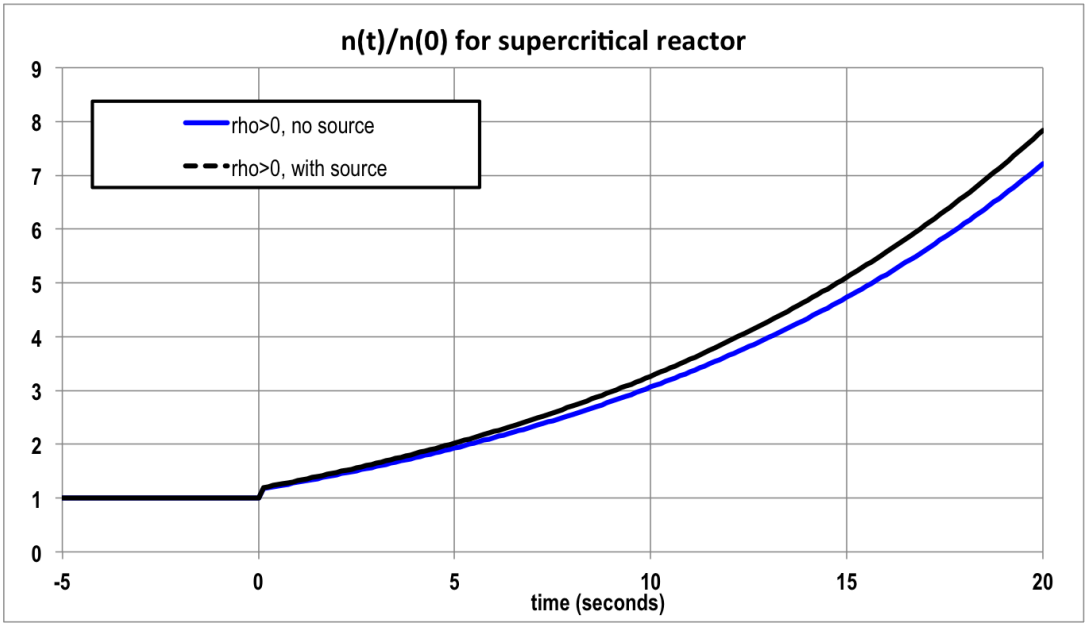


Figure 8.3: Behavior of neutron population in a **SUPERCRITICAL** reactor. The reactor is at steady state until time zero, when there is an insertion of positive reactivity (perhaps by withdrawal of control rods). Note the rapid jump in population prior to the slower exponential increase.

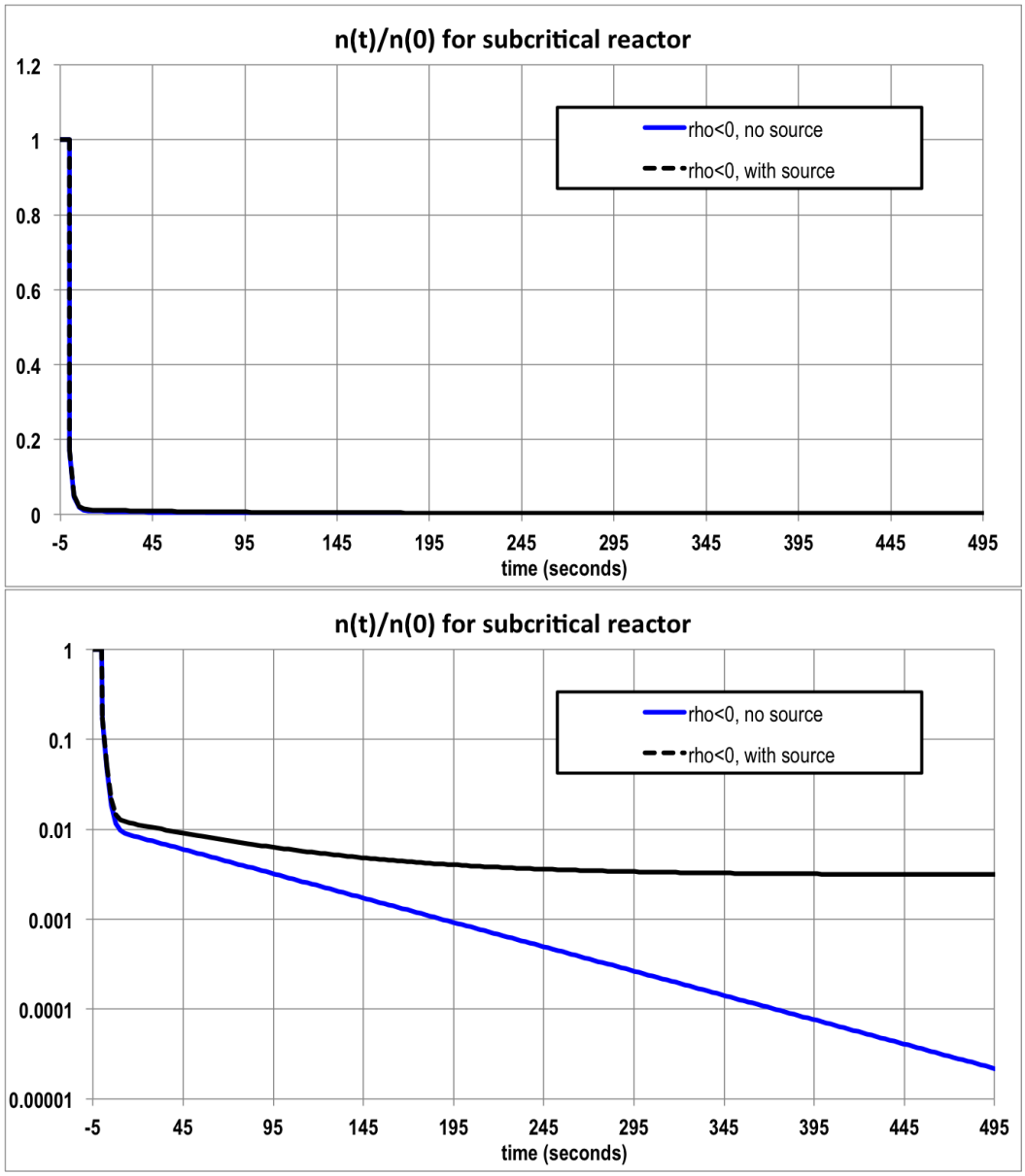


Figure 8.4: Behavior of neutron population in a **SUBCRITICAL** reactor (linear and semi-log). The reactor is at steady state until time zero, when there is an insertion of negative reactivity (perhaps by insertion of control rods). Note the rapid drop in population prior to the slower exponential decay to either zero (if no source) or a new steady value (if source is present).

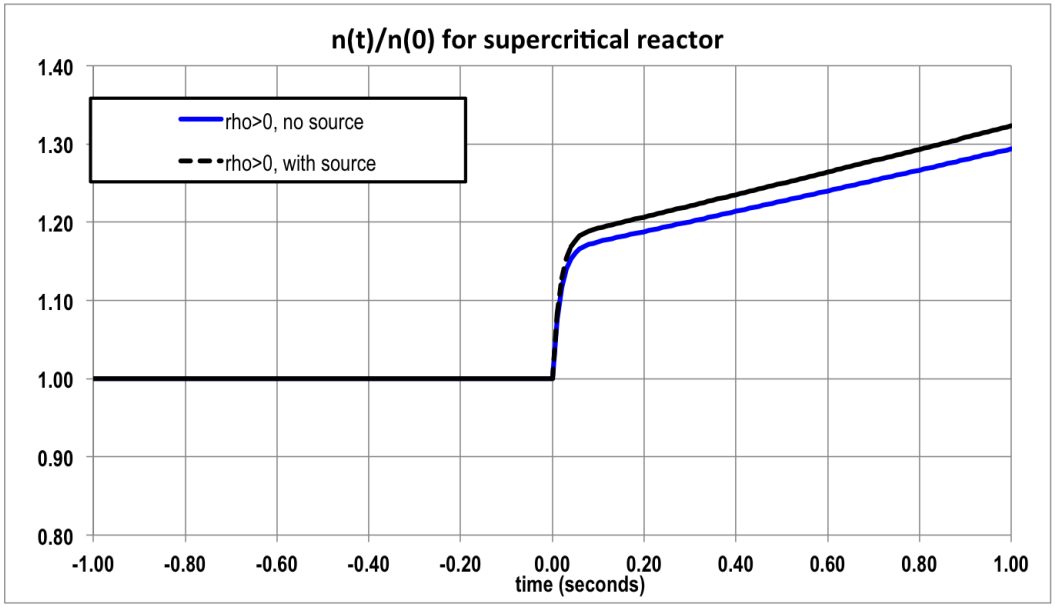


Figure 8.5: Behavior of neutron population in a **SUPERCRITICAL** reactor. The reactor is at steady state until time zero, when there is an insertion of positive reactivity (perhaps by withdrawal of control rods). This zoom-in view of early time more clearly shows the rapid jump in population prior to the slower exponential increase.

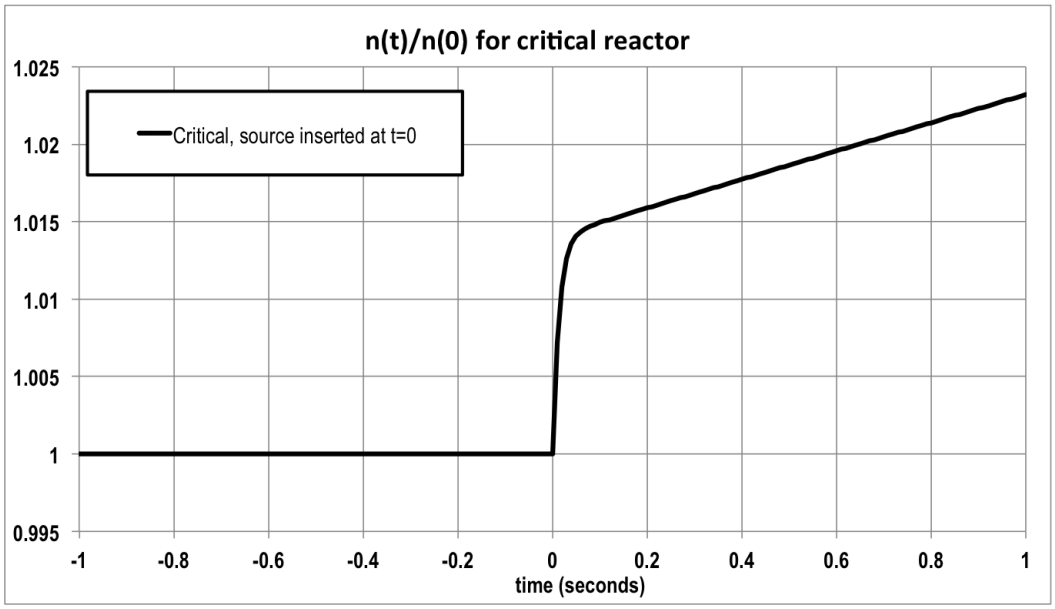


Figure 8.6: Behavior of neutron population in a **CRITICAL** reactor with source is inserted at t=0. This zoom-in view of early time more clearly shows the rapid jump before the smooth increase that becomes linear in time).

8.3 A simpler setting: only one delayed neutron precursor group

8.3.1 Analytical solution with no external source present

Let us some the PRKEs, but with one simplification to make the Math easier: lump all delayed-neutron precursors in one group.

The simplified PRKEs are:

$$\frac{d}{dt}\left[n(t)\right] = \left[\frac{\rho-\beta}{\Lambda}\right] n(t) + \lambda C(t) \quad , \tag{8.22}$$

$$\frac{d}{dt}C(t) = \frac{\beta}{\Lambda} n(t) - \lambda C(t) \quad . \tag{8.23}$$

with initial conditions (reactor initially critical):

$$n(0) = n_{\text{init}} = \text{given}\,, \tag{8.24}$$

$$C(0) = C_{\text{init}} = \frac{\beta}{\lambda\Lambda} n_{\text{init}} = \text{given}\,. \tag{8.25}$$

Assume the following exponential forms:

$$n(t) = n_0 e^{st} \qquad C(t) = C_0 e^{st}$$

Plug back in Eqs. (8.22)-(8.23) (and simplify throughout by e^{st}):

$$n_0 s = \left[\frac{\rho-\beta}{\Lambda}\right] n_0 + \lambda C_0 \quad ,$$

$$C_0 s = \frac{\beta}{\Lambda} n_0 - \lambda C_0 \quad .$$

We get from the second equation:

$$C_0 = \frac{1}{\lambda+s}\frac{\beta}{\Lambda} n_0 \tag{8.26}$$

Plug back in the first equation:

$$\cancel{n_0} s = \left[\frac{\rho-\beta}{\Lambda}\right] \cancel{n_0} + \lambda \frac{1}{\lambda+s}\frac{\beta}{\Lambda}\cancel{n_0}$$

After some algebra, we get the in-hour equation

$$\Lambda s + \frac{\beta s}{\lambda + s} = \rho \tag{8.27}$$

Eq (8.27) is also a simple quadratic equation in s and you can solve for the two roots s_1 and s_2. As before,

s_1 is of the sign of ρ

s_2 is always negative

Now, we have our final form of the solution:

$$n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad , \tag{8.28}$$
$$C(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \quad . \tag{8.29}$$

When invoking Eq (8.26), there is a relationship between A_i and C_i so that

$$C_i = \frac{1}{\lambda + s_i} \frac{\beta}{\Lambda} A_i$$

so that the final form is

$$n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad , \tag{8.30}$$
$$C(t) = \frac{\beta}{\Lambda} \left(\frac{A_1}{\lambda + s_1} e^{s_1 t} + \frac{A_2}{\lambda + s_2} e^{s_2 t} \right) \quad . \tag{8.31}$$

Now, we only need to find two constants, A_1 and A_2. We use initial conditions:

$$n(0) = A_1 + A2 = n_{\text{init}}$$
$$C(0) = \frac{\beta}{\Lambda} \left(\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2} \right) = \frac{\beta}{\lambda \Lambda} n_{\text{init}}$$

or, after some simplifications,

$$A_1 + A_2 = n_{\text{init}} \tag{8.32}$$
$$\left(\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2} \right) = \frac{n_{\text{init}}}{\lambda} \tag{8.33}$$

Solve for A_1 and A_2. Done. Plot for various values of ρ.

8.3.2 Analytical solution with external source present

The PRKE are:

$$\frac{d}{dt}\left[n(t)\right] = \left[\frac{\rho-\beta}{\Lambda}\right] n(t) + \lambda C(t) + S \quad , \tag{8.34}$$

$$\frac{d}{dt}C(t) = \frac{\beta}{\Lambda} n(t) - \lambda C(t) \quad . \tag{8.35}$$

with initial conditions (reactor initially subcritical):

$$n(0) = n_{\text{init}} = -\frac{\Lambda}{\rho_{\text{init}}} S \,, \tag{8.36}$$

$$C(0) = C_{\text{init}} = \frac{\beta}{\lambda\Lambda} n_{\text{init}} = -\frac{\beta}{\lambda\rho_{\text{init}}} S \,. \tag{8.37}$$

Note that ρ in Eq (8.34) is the final reactivity. So, you add or subtract some amount of reactivity to the initial reactivity ρ_{init} in order to get the final reactivity ρ.

8.3.2.1 Final reactivity is not zero

If the final reactivity is not zero, then all of the arguments of the exponentials in the homogeneous solution are non-zero. s_1 is of the sign of ρ. $s_2 < 0$. We just need to add to the homogeneous solution a particular solution. We seek it as a constant and finally obtain:

$$n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} - \frac{\Lambda}{\rho} S \quad , \tag{8.38}$$

$$C(t) = \frac{\beta}{\Lambda}\left(\frac{A_1}{\lambda + s_1} e^{s_1 t} + \frac{A_2}{\lambda + s_2} e^{s_2 t}\right) - \frac{\beta}{\lambda\rho} S \quad . \tag{8.39}$$

Apply initial conditions to get the constants (solve the following system):

$$A_1 + A_2 - \frac{\Lambda}{\rho} S = -\frac{\Lambda}{\rho_{\text{init}}} S \quad , \tag{8.40}$$

$$\frac{\beta}{\Lambda}\left(\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2}\right) - \frac{\beta}{\lambda\rho} S = -\frac{\beta}{\lambda\rho_{\text{init}}} S \quad . \tag{8.41}$$

or, after some simplifications, solve:

$$A_1 + A_2 = -\Lambda S\left(\frac{1}{\rho_{\text{init}}} - \frac{1}{\rho}\right), \tag{8.42}$$

$$\frac{A_1}{\lambda + s_1} + \frac{A_2}{\lambda + s_2} = -\frac{\Lambda S}{\lambda}\left(\frac{1}{\rho_{\text{init}}} - \frac{1}{\rho}\right) \quad . \tag{8.43}$$

Solve for A_1 and A_2. Done. Plot for various values of $\rho \neq 0$.

8.3.2.2 Final reactivity is zero

In this case, $s_1 = 0$ and $s_2 < 0$. Because $s_1 = 0$, the associated exponential function is actually a constant function. As a consequence, the particular cannot be a constant function but the next-simplest function (let's try a linear-in-time function again). **We are going to leave it at this . . . NUEN 304 will take over.**

8.3.3 Numerical solution

Here's the coolest part. You can view the PRKEs in matrix notation as

$$\frac{dX}{dt} = A(t)X(t) + b(t)$$

where

$$X = \begin{bmatrix} n \\ c \end{bmatrix} \qquad A(t) = \begin{bmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda \\ \frac{\beta}{\Lambda} & -\lambda \end{bmatrix} \qquad b(t) = \begin{bmatrix} S(t) \\ 0 \end{bmatrix}$$

Next, we select equally spaced times at which we wish to compute the solution: $t = \Delta t,\ 2\Delta t, \ldots N\Delta t$. Now, using Taylor series expansion, we can approximate the derivative of X over the time interval $[t_n, t_{n+1}]$ as:

$$\frac{dX}{dt} \approx \frac{X(t_{n+1}) - X(t_n)}{t_{n+1} - t_n} = \frac{X(t_{n+1}) - X(t_n)}{\Delta t}\,.$$

Note that if we assumed X to be a straight line in the interal $[t_n, t_{n+1}]$, then our expression for the derivative would be exact! If this were the case, we could say that the derivative has this value for any time in $[t_n, t_{n+1}]$. For example, we could decide whether this derivative applies at t_n, t_{n+1} or any other time in between:

$$\left.\frac{dX}{dt}\right|_{t_n} \text{ or } \left.\frac{dX}{dt}\right|_{t_{n+1}} \text{ or } \ldots$$

For reasons you will see in MATH, we pick t_{n+1}. Therefore, the other side of the PRKEs needs to be evaluated at that time too:

$$\left.\frac{dX}{dt}\right|_{t_{n+1}} = \frac{X(t_{n+1}) - X(t_n)}{\Delta t} = A(t_{n+1})X(t_{n+1}) + b(t_{n+1})\,.$$

We introduce the short-cut notation $X_n = X(t_n)$ and so forth. This yields:

$$\frac{X_{n+1} - X_n}{\Delta t} = A(t_{n+1})X_{n+1} + b(t_{n+1})\,.$$

Re-arranging:

$$(I - \Delta t A(t_{n+1}))\, X_{n+1} = X_n + \Delta b(t_{n+1})\,,$$

where I is the identity matrix. This is a small 2×2 linear system that one can solve for solution vector X_{n+1}, knowing vector X_n. We can do this many times. Well, not you, but a computer because solving 100,000 times such systems gets tiring quickly. Once you have X_{n+1}, you move to the next time step and compute X_{n+2}, knowing X_{n+1} and so forth. We jump start the process by computing X_1, knowing X_0, the initial condition values. Pretty cool.

In gory details, the 2×2 linear system matrix on the left-hand side is:

$$\begin{bmatrix} 1 - \Delta t \left(\frac{\rho(t_{n+1}) - \beta}{\Lambda} \right) & -\Delta t \lambda \\ -\Delta t \frac{\beta}{\Lambda} & 1 + \Delta t \lambda \end{bmatrix}$$

and the right-hand side vector is

$$\begin{bmatrix} n(t_n) + \Delta t S(t_{n+1}) \\ c(t_n) \end{bmatrix}$$

Or, all in all, solve this system for $n(t_{n+1})$ and $c(t_{n+1})$:

$$\begin{bmatrix} 1 - \Delta t \left(\frac{\rho(t_{n+1}) - \beta}{\Lambda} \right) & -\Delta t \lambda \\ -\Delta t \frac{\beta}{\Lambda} & 1 + \Delta t \lambda \end{bmatrix} \begin{bmatrix} n(t_{n+1}) \\ c(t_{n+1}) \end{bmatrix} = \begin{bmatrix} n(t_n) + \Delta t S(t_{n+1}) \\ c(t_n) \end{bmatrix}$$

Honestly, this is so much simpler than brainstorming about analytical solutions ... you should embrace computing ... engineers do a lot of this every single day ...

8.4 Feedback

A few words are in order about an important phenomenon that we have not yet described:

feedback

In a nuclear reactor, feedback happens when changes in the neutron population cause changes in the reactor properties. These changes in properties then alter the way that the neutron population behaves. Most of the feedback in a nuclear reactor is caused by

temperature changes

in the reactor material. Remember that temperature can be strongly influenced by the

heat-generation rate

which of course is determined by the

fission rate

(and other exothermic-reaction rates, to a lesser extent) which of course are determined by the

neutron population

Also recall that

cross sections

depend upon the

material temperature

and that

reactivity (or multiplication factor)

is a function of the cross sections. That is, we need to know the temperature to get the cross sections that allow us to calculate the neutron population, but we need to know the neutron population to calculate the fission source that drives the temperature.

Illustration:

1. Consider a reactor that is critical at a very low power level, such that fission is not causing the fuel to be noticeably hotter than its surroundings.

2. Now suppose control rods are pulled out so that the reactivity becomes positive (because the absorption cross section has been reduced). As we have seen, the neutron population will soon be on an exponential increase. Thus, the fission rate will increase exponentially with time.

3. Eventually this will cause the fuel to heat up significantly. This will change the cross sections in the fuel (remember—the cross sections are averaged over nucleus motion, and temperature determines the motion). In fact, in almost all reactors this will cause the absorption cross section to increase more than the fission cross section, which causes the multiplication factor to **decrease**.

4. As temperature increases further, the multiplication factor decreases further. This continues until k reaches 1 and the reactor settles into steady state.

The above is an example of

negative feedback .

This is what we want. Negative feedback gives us

stability.

which means that fluctuations in neutron population are damped out in time. Positive feedback, on the other hand, would amplify changes. Imagine a reactor in which increased temperature caused an increase in k (and thus ρ). In such a reactor, a small increase in neutron population would produce an increase in reactivity, which would cause the population to increase, which would further increase reactivity, etc. The reactor power would spiral upward until something (such as melting fuel) introduced negative feedback. Not good.

So we design our reactors with negative feedback.

8.5 Summary

Previously, we found that in every case that treated delayed neutrons we could cast our conservation equations, along with the equations for the delayed-neutron-precursor concentrations, into the form of **Point-Reactor Kinetics Equations** (PRKEs). Even when we did not know how to calculate the coefficients in these equations, we could study the characteristics of their solutions in all scenarios of interest.

In particular, if the coefficients in the PRKEs—namely ρ, the $\{\beta_i\}$, and Λ—are constant (which they often are for interesting problems), then we can write down equations for $n(t)$ and the $\{C_i(t)\}$. These equations contain exponentials that contain time constants (the $\{s_i\}$), which can be obtained from the **"INHOUR Equation."**

There are six scenarios that illustrate a wide variety of possible reactor behaviors. We studied each of these under the assumption that the coefficients We found that

1. If there is no fixed source, then the neutron population:
   (a) eventually grows exponentially if $\rho > 0$;
   (b) eventually falls exponentially if $\rho < 0$;
   (c) eventually stays steady if $\rho = 0$.
2. If there is a fixed source, then the neutron population:
   (a) eventually grows exponentially if $\rho > 0$;
   (b) eventually reaches a steady value if $\rho < 0$;
   (c) eventually grows linearly in time if $\rho = 0$.

We discussed the important phenomenon of **feedback**. We did not go into detail, but we noted that we design our reactors with **negative feedback**, which is a stabilizing influence. Feedback changes the behavior of the neutron population with time because it changes the reactivity, ρ, which of course strongly influences the population's behavior.